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IGCSE

Cambridge International Mathematics

(0607)
Extended

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IGCSE CAMBRIDGE INTERNATIONAL MATHEMATICS (0607)

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FOREWORD

This book has been written to cover the '*IGCSE Cambridge International Mathematics (0607) Extended*' course over a two-year period.

The new course was developed by University of Cambridge International Examinations (CIE) in consultation with teachers in international schools around the world. It has been designed for schools that want their mathematics teaching to focus more on investigations and modelling, and to utilise the powerful technology of graphics calculators.

The course springs from the principles that students should develop a good foundation of mathematical skills and that they should learn to develop strategies for solving open-ended problems. It aims to promote a positive attitude towards Mathematics and a confidence that leads to further enquiry. Some of the schools consulted by CIE were IB schools and as a result, Cambridge International Mathematics integrates exceptionally well with the approach to the teaching of Mathematics in IB schools.

This book is an attempt to cover, in one volume, the content outlined in the Cambridge International Mathematics (0607) syllabus. References to the syllabus are made throughout but the book can be used as a full course in its own right, as a preparation for GCE Advanced Level Mathematics or IB Diploma Mathematics, for example. The book has been endorsed by CIE but it has been developed independently of the Independent Baccalaureate Organization and is not connected with, or endorsed by, the IBO.

To reflect the principles on which the new course is based, we have attempted to produce a book and CD package that embraces technology, problem solving, investigating and modelling, in order to give students different learning experiences. There are non-calculator sections as well as traditional areas of mathematics, especially algebra. An introductory section 'Graphics calculator instructions' appears on p. 11. It is intended as a basic reference to help students who may be unfamiliar with graphics calculators. Two chapters of 'assumed knowledge' are accessible from the CD: 'Number' and 'Geometry and graphs' (see pp. 29 and 30). They can be printed for those who want to ensure that they have the prerequisite levels of understanding for the course. To reflect one of the main aims of the new course, the last two chapters in the book are devoted to multi-topic questions, and investigations and modelling. Review exercises appear at the end of each chapter with some 'Challenge' questions for the more able student. Answers are given at the end of the book, followed by an index.

The interactive CD contains 🔊 *Self Tutor* software (see p. 5), geometry and graphics software, demonstrations and simulations, and the two printable chapters on assumed knowledge. The CD also contains the text of the book so that students can load it on a home computer and keep the textbook at school.

The Cambridge International Mathematics examinations are in the form of three papers: one a non-calculator paper, another requiring the use of a graphics calculator, and a third paper containing an investigation and a modelling question. All of these aspects of examining are addressed in the book.

The book can be used as a scheme of work but it is expected that the teacher will choose the order of topics. There are a few occasions where a question in an exercise may require something done later in the book but this has been kept to a minimum. Exercises in the book range from routine practice and consolidation of basic skills, to problem solving exercises that are quite demanding.

In this changing world of mathematics education, we believe that the contextual approach shown in this book, with the associated use of technology, will enhance the students' understanding, knowledge and appreciation of mathematics, and its universal application.

We welcome your feedback.

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ACKNOWLEDGEMENTS

USING THE INTERACTIVE CD

The interactive Student CD that comes with this book is designed for those who want to utilise technology in teaching and learning Mathematics.

The CD icon that appears throughout the book denotes an active link on the CD. Simply click on the icon when running the CD to access a large range of interactive features that includes:

- spreadsheets
- printable worksheets
- graphing packages
- geometry software
- demonstrations
- simulations
- printable chapters
- SELF TUTOR

INTERACTIVE LINK
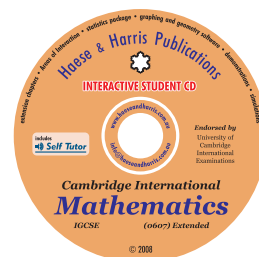
For those who want to ensure they have the prerequisite levels of understanding for this new course, printable chapters of assumed knowledge are provided for Number (see p. 29) and Geometry and Graphs (see p. 30).

SELF TUTOR is an exciting feature of this book.

The ◀ *Self Tutor* icon on each worked example denotes an active link on the CD.

Simply 'click' on the ◀ *Self Tutor* (or anywhere in the example box) to access the worked example, with a teacher's voice explaining each step necessary to reach the answer.

Play any line as often as you like. See how the basic processes come alive using movement and colour on the screen.

Ideal for students who have missed lessons or need extra help.

Example 8 ◀ *Self Tutor*

A die has the numbers 0, 0, 1, 1, 4 and 5. It is rolled *twice*. Illustrate the sample space using a 2-D grid. Hence find the probability of getting:

a a total of 5

b two numbers which are the same.
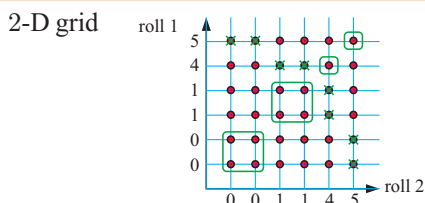
There are $6 \times 6 = 36$ possible outcomes.

a P(total of 5) $= \frac{8}{36}$ {those with a $\times$}

b P(same numbers) $= \frac{10}{36}$ {those (circled)}

See **Chapter 25**, **Probability**, p.516

GRAPHICS CALCULATORS

The course assumes that each student will have a graphics calculator. An introductory section 'Graphics calculator instructions' appears on p. 11. To help get students started, the section includes some basic instructions for the Texas Instruments TI-84 Plus and the Casio fx-9860G calculators.

SYMBOLS AND NOTATION USED IN THIS BOOK

$\mathbb{N}$ the set of positive integers and zero, $\{0, 1, 2, 3, ...\}$

$\mathbb{Z}$ the set of integers, $\{0, \pm 1, \pm 2, \pm 3, ...\}$

$\mathbb{Z}^+$ the set of positive integers, $\{1, 2, 3, ...\}$

$\mathbb{Q}$ the set of rational numbers

$\mathbb{Q}^+$ the set of positive rational numbers, $\{x \mid x > 0, \ x \in \mathbb{Q}\}$

$\mathbb{R}$ the set of real numbers

$\mathbb{R}^+$ the set of positive real numbers, $\{x \mid x > 0, \ x \in \mathbb{R}\}$

$\{x_1, x_2, ...\}$ the set with elements $x_1, x_2, ...$

$n(A)$ the number of elements in the finite set A

$\{x \mid ......$ the set of all x such that

$\in$ is an element of

$\notin$ is not an element of

$\varnothing$ or $\{\ \}$ the empty (null) set

U the universal set

$\cup$ union

$\cap$ intersection

$\subseteq$ is a subset of

$\subset$ is a proper subset of

A' the complement of the set A

$a^{\frac{1}{n}}, \ \sqrt[n]{a}$ a to the power of $\frac{1}{n}$, nth root of a (if $a \geqslant 0$ then $\sqrt[n]{a} \geqslant 0$)

$a^{\frac{1}{2}}, \ \sqrt{a}$ a to the power $\frac{1}{2}$, square root of a (if $a \geqslant 0$ then $\sqrt{a} \geqslant 0$)

$|x|$ the modulus or absolute value of x, that is $\begin{cases} x \text{ for } x \geqslant 0, \ x \in \mathbb{R} \\ -x \text{ for } x < 0, \ x \in \mathbb{R} \end{cases}$

$\equiv$ identity   or   is equivalent to

$\approx$ is approximately equal to

$\cong$ is congruent to

$\parallel$ is parallel to

$\perp$ is perpendicular to

$>$ is greater than

$\geq$ or $\geqslant$ is greater than or equal to

$<$ is less than

$\leq$ or $\leqslant$ is less than or equal to

u_n the nth term of a sequence or series

$f : x \mapsto y$ f is a function under which x is mapped to y

$f(x)$ the image of x under the function f

f^{-1} the inverse function of the function f

$\log_a x$ logarithm to the base a of x

$\sin, \cos, \tan$ the circular functions

$A(x, y)$ the point A in the plane with Cartesian coordinates x and y

AB $\begin{cases} \text{the line segment with end points A and B} \\ \text{the distance from A to B} \\ \text{the line containing points A and B} \end{cases}$

$\widehat{A}$ the angle at A

$\widehat{CAB}$ the angle between CA and AB

$\triangle ABC$ the triangle whose vertices are A, B and C

$\mathbf{v}$ the vector $\mathbf{v}$

$\overrightarrow{AB}$ the vector represented in magnitude and direction by the directed line segment from A to B

$|\mathbf{a}|$ the magnitude of vector $\mathbf{a}$

$|\overrightarrow{AB}|$ the magnitude of $\overrightarrow{AB}$

$P(A)$ probability of event A

$P(A')$ probability of the event "not A"

$x_1, x_2, ...$ observations of a variable

$f_1, f_2, ...$ frequencies with which the observations $x_1, x_2, x_3, ...$ occur

$\overline{x}$ mean of the values $x_1, x_2, ...$

Σf sum of the frequencies $f_1, f_2, ...$

r Pearson's correlation coefficient

r^2 coefficient of determination

TABLE OF CONTENTS

Graphics calculator instructions

Contents:

A Basic calculations
B Basic functions
C Secondary function and alpha keys
D Memory
E Lists
F Statistical graphs
G Working with functions
H Two variable analysis

In this course it is assumed that you have a **graphics calculator**. If you learn how to operate your calculator successfully, you should experience little difficulty with future arithmetic calculations.

There are many different brands (and types) of calculators. Different calculators do not have exactly the same keys. It is therefore important that you have an instruction booklet for your calculator, and use it whenever you need to.

However, to help get you started, we have included here some basic instructions for the **Texas Instruments TI-84 Plus** and the **Casio fx-9860G** calculators. Note that instructions given may need to be modified slightly for other models.

GETTING STARTED

Texas Instruments TI-84 Plus

The screen which appears when the calculator is turned on is the **home screen**. This is where most basic calculations are performed.

You can return to this screen from any menu by pressing 2nd MODE .

When you are on this screen you can type in an expression and evaluate it using the ENTER key.

Casio fx-9860g

Press MENU to access the Main Menu, and select **RUN·MAT**.

This is where most of the basic calculations are performed.

When you are on this screen you can type in an expression and evaluate it using the EXE key.

A   *BASIC CALCULATIONS*

Most modern calculators have the rules for **Order of Operations** built into them. This order is sometimes referred to as BEDMAS.

This section explains how to enter different types of numbers such as negative numbers and fractions, and how to perform calculations using grouping symbols (brackets), powers, and square roots. It also explains how to round off using your calculator.

NEGATIVE NUMBERS

To enter negative numbers we use the **sign change** key. On both the **TI-84 Plus** and **Casio** this looks like $\boxed{(-)}$. Simply press the sign change key and then type in the number.

For example, to enter -7, press $\boxed{(-)}$ 7.

FRACTIONS

On most scientific calculators and also the **Casio** graphics calculator there is a special key for entering fractions. No such key exists for the **TI-84 Plus**, so we use a different method.

Texas Instruments TI-84 Plus

To enter common fractions, we enter the fraction as a division.

For example, we enter $\frac{3}{4}$ by typing   3 $\boxed{\div}$ 4.   If the fraction is part of a larger calculation, it is generally wise to place this division in brackets, i.e., $\boxed{(}$ 3 $\boxed{\div}$ 4 $\boxed{)}$.

To enter mixed numbers, either convert the mixed number to an improper fraction and enter as a common fraction *or* enter the fraction as a sum.

For example, we can enter $2\frac{3}{4}$ as $\boxed{(}$ 11 $\boxed{\div}$ 4 $\boxed{)}$ *or* $\boxed{(}$ 2 $\boxed{+}$ 3 $\boxed{\div}$ 4 $\boxed{)}$.

Casio fx-9860g

To enter fractions we use the **fraction** key $\boxed{a^{b/c}}$.

For example, we enter $\frac{3}{4}$ by typing   3 $\boxed{a^{b/c}}$ 4   and $2\frac{3}{4}$ by typing   2 $\boxed{a^{b/c}}$ 3 $\boxed{a^{b/c}}$ 4.

Press $\boxed{\text{SHIFT}}$ $\boxed{a^{b/c}}$ $(a\frac{b}{c} \leftrightarrow \frac{d}{c})$ to convert between mixed numbers and improper fractions.

SIMPLIFYING FRACTIONS & RATIOS

Graphics calculators can *sometimes* be used to express fractions and ratios in simplest form.

Texas Instruments TI-84 Plus

To express the fraction $\frac{35}{56}$ in simplest form, press 35 $\boxed{\div}$ 56 $\boxed{\text{MATH}}$ **1** $\boxed{\text{ENTER}}$. The result is $\frac{5}{8}$.

To express the ratio $\frac{2}{3} : 1\frac{1}{4}$ in simplest form, press $\boxed{(}$ 2 $\boxed{\div}$ 3 $\boxed{)}$ $\boxed{\div}$ $\boxed{(}$ 1 $\boxed{+}$ 1 $\boxed{\div}$ 4 $\boxed{)}$ $\boxed{\text{MATH}}$ **1** $\boxed{\text{ENTER}}$.

The ratio is 8 : 15.
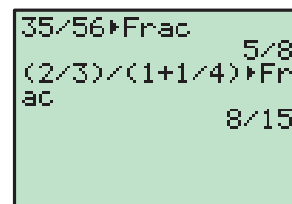
Casio fx-9860g

To express the fraction $\frac{35}{56}$ in simplest form, press 35 $\boxed{a\,b/c}$ 56 $\boxed{\text{EXE}}$. The result is $\frac{5}{8}$.

To express the ratio $\frac{2}{3} : 1\frac{1}{4}$ in simplest form, press 2 $\boxed{a\,b/c}$ 3 $\boxed{\div}$ 1 $\boxed{a\,b/c}$ 1 $\boxed{a\,b/c}$ 4 $\boxed{\text{EXE}}$. The ratio is 8 : 15.
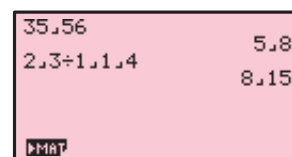
ENTERING TIMES

In questions involving time, it is often necessary to be able to express time in terms of hours, minutes and seconds.

Texas Instruments TI-84 Plus

To enter 2 hours 27 minutes, press 2 $\boxed{\text{2nd}}$ $\boxed{\text{APPS}}$ (ANGLE) **1:^{o}** 27 $\boxed{\text{2nd}}$ $\boxed{\text{APPS}}$ **2:′**. This is equivalent to 2.45 hours.

To express 8.17 hours in terms of hours, minutes and seconds, press 8.17 $\boxed{\text{2nd}}$ $\boxed{\text{APPS}}$ **4:▶DMS** $\boxed{\text{ENTER}}$.
This is equivalent to 8 hours, 10 minutes and 12 seconds.

Casio fx-9860g

To enter 2 hours 27 minutes, press $\boxed{\text{OPTN}}$ $\boxed{\text{F6}}$ $\boxed{\text{F5}}$ (ANGL) 2 $\boxed{\text{F4}}$ ($^{o'''}$) 27 $\boxed{\text{F4}}$ ($^{o'''}$) $\boxed{\text{EXE}}$. This is equivalent to 2.45 hours.

To express 8.17 hours in terms of hours, minutes and seconds, press 8.17 $\boxed{\text{OPTN}}$ $\boxed{\text{F6}}$ $\boxed{\text{F5}}$ (ANGL) $\boxed{\text{F6}}$ $\boxed{\text{F3}}$ (▶DMS) $\boxed{\text{EXE}}$. This is equivalent to 8 hours, 10 minutes and 12 seconds.
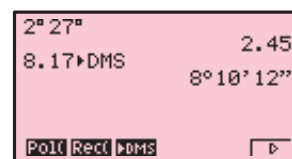
B BASIC FUNCTIONS

GROUPING SYMBOLS (BRACKETS)

Both the **TI-84 Plus** and **Casio** have bracket keys that look like $\boxed{(}$ and $\boxed{)}$.

Brackets are regularly used in mathematics to indicate an expression which needs to be evaluated before other operations are carried out.

For example, to enter $2 \times (4 + 1)$ we type   2 $\boxed{\times}$ $\boxed{(}$ 4 $\boxed{+}$ 1 $\boxed{)}$.

We also use brackets to make sure the calculator understands the expression we are typing in.

For example, to enter $\frac{2}{4+1}$ we type   2 $\boxed{\div}$ $\boxed{(}$ 4 $\boxed{+}$ 1 $\boxed{)}$.   If we typed   2 $\boxed{\div}$ 4 $\boxed{+}$ 1   the calculator would think we meant $\frac{2}{4} + 1$.

In general, it is a good idea to place brackets around any complicated expressions which need to be evaluated separately.

POWER KEYS

Both the **TI-84 Plus** and **Casio** also have power keys that look like $\boxed{\wedge}$. We type the base first, press the power key, then enter the index or exponent.

For example, to enter 25^3 we type   25 $\boxed{\wedge}$ 3.

Note that there are special keys which allow us to quickly evaluate squares.

Numbers can be squared on both **TI-84 Plus** and **Casio** using the special key $\boxed{x^2}$.

For example, to enter 25^2 we type   25 $\boxed{x^2}$.

ROOTS

To enter roots on either calculator we need to use a secondary function (see **Secondary Function and Alpha Keys**).

Texas Instruments TI-84 Plus

The **TI-84 Plus** uses a secondary function key $\boxed{\text{2nd}}$.

We enter square roots by pressing $\boxed{\text{2nd}}$ $\boxed{x^2}$.

For example, to enter $\sqrt{36}$ we press $\boxed{\text{2nd}}$ $\boxed{x^2}$ 36 $\boxed{)}$.

The end bracket is used to tell the calculator we have finished entering terms under the square root sign.

Cube roots are entered by pressing $\boxed{\text{MATH}}$ **4:** $\sqrt[3]{}$ (.

For example, to enter $\sqrt[3]{8}$ we press $\boxed{\text{MATH}}$ **4** 8 $\boxed{)}$.

Higher roots are entered by pressing $\boxed{\text{MATH}}$ **5:** $\sqrt[x]{}$.

For example, to enter $\sqrt[4]{81}$ we press 4 $\boxed{\text{MATH}}$ **5** 81 $\boxed{)}$.

Casio fx-9860g

The Casio uses a shift key | SHIFT | to get to its second functions.

We enter square roots by pressing | SHIFT | x^2 .

For example, to enter $\sqrt{36}$ we press | SHIFT | x^2 36.

If there is a more complicated expression under the square root sign you should enter it in brackets.

For example, to enter $\sqrt{18 \div 2}$ we press | SHIFT | x^2 | (| 18 | ÷ | 2 |) | .

Cube roots are entered by pressing | SHIFT | | (| . For example, to enter $\sqrt[3]{8}$ we press | SHIFT | | (| 8.

Higher roots are entered by pressing | SHIFT | | ^ | . For example, to enter $\sqrt[4]{81}$ we press 4 | SHIFT | | ^ | 81.

LOGARITHMS

We can perform operations involving logarithms in base 10 using the | log | button. For other bases the method we use depends on the brand of calculator.
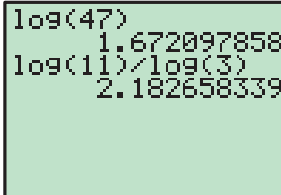
Texas Instruments TI-84 Plus

To evaluate $\log(47)$, press | log | 47 |) | | ENTER | .

Since $\log_a b = \dfrac{\log b}{\log a}$, we can use the base 10 logarithm to calculate logarithms in other bases.

To evaluate $\log_3 11$, we note that $\log_3 11 = \dfrac{\log 11}{\log 3}$, so we press

| log | 11 |) | | ÷ | | log | 3 |) | | ENTER | .
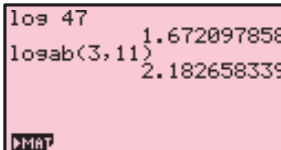
```
log(47)
          1.672097858
log(11)/log(3)
          2.182658339
```

Casio fx-9860g

To evaluate $\log(47)$ press | log | 47 | EXE | .

To evaluate $\log_3 11$, press | SHIFT | **4** (**CATALOG**), and select **logab(** . You can use the alpha keys to navigate the catalog, so in this example press | → | to jump to "L". Press 3 | , | 11 |) | | EXE | .

```
log 47
          1.672097858
logab(3,11)
          2.182658339
▶MAT
```

ROUNDING OFF
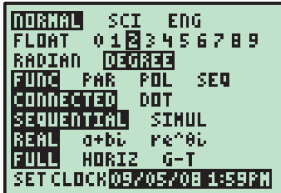
You can use your calculator to round off answers to a fixed number of decimal places.

Texas Instruments TI-84 Plus

To round to 2 decimal places, press | MODE | then | ▼ | to scroll down to Float.

Use the | ▶ | button to move the cursor over the 2 and press | ENTER | . Press | 2nd | | MODE | to return to the home screen.

If you want to unfix the number of decimal places, press | MODE | | ▼ | | ENTER | to highlight Float.

```
NORMAL  SCI  ENG
FLOAT  0123456789
RADIAN  DEGREE
FUNC  PAR  POL  SEQ
CONNECTED  DOT
SEQUENTIAL  SIMUL
REAL  a+bi  re^θi
FULL  HORIZ  G-T
SET CLOCK 09/05/08 1:59PM
```

Casio fx-9860g

To round to 2 decimal places, select **RUN·MAT** from the Main Menu, and press [SHIFT] [MENU] to enter the setup screen. Scroll down to Display, and press [F1] (**Fix**). Press 2 [EXE] to select the number of decimal places. Press [EXIT] to return to the home screen.
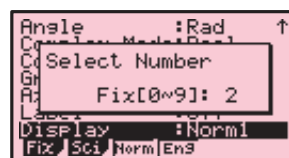
To unfix the number of decimal places, press [SHIFT] [MENU] to return to the setup screen, scroll down to Display, and press [F3] (**Norm**).

INVERSE TRIGONOMETRIC FUNCTIONS

To enter inverse trigonometric functions, you will need to use a secondary function (see **Secondary Function and Alpha Keys**).

Texas Instruments TI-84 Plus

The inverse trigonometric functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ are the secondary functions of [SIN], [COS] and [TAN] respectively. They are accessed by using the secondary function key [2nd].

For example, if $\cos x = \frac{3}{5}$, then $x = \cos^{-1}\left(\frac{3}{5}\right)$.

To calculate this, press [2nd] [COS] 3 [÷] 5 [ )] [ENTER].

Casio fx-9860g

The inverse trigonometric functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ are the secondary functions of [sin], [cos] and [tan] respectively. They are accessed by using the secondary function key [SHIFT].

For example, if $\cos x = \frac{3}{5}$, then $x = \cos^{-1}\left(\frac{3}{5}\right)$.

To calculate this, press [SHIFT] [cos] [( ] 3 [÷] 5 [ )] [EXE].

STANDARD FORM

If a number is too large or too small to be displayed neatly on the screen, it will be expressed in standard form, which is the form $a \times 10^n$ where $1 \leqslant a < 10$ and n is an integer.

Texas Instruments TI-84 Plus

To evaluate 2300^3, press 2300 [∧] 3 [ENTER].
The answer displayed is 1.2167E10, which means 1.2167×10^{10}.

To evaluate $\dfrac{3}{20\,000}$, press 3 [÷] 20 000 [ENTER].
The answer displayed is 1.5E−4, which means 1.5×10^{-4}.

You can enter values in standard form using the EE function, which is accessed by pressing [2nd] [,].

For example, to evaluate $\dfrac{2.6 \times 10^{14}}{13}$, press 2.6 [2nd] [,] 14 [÷] 13 [ENTER].
The answer is 2×10^{13}.
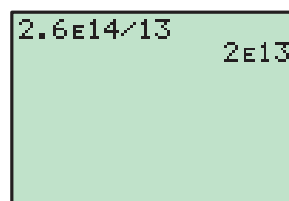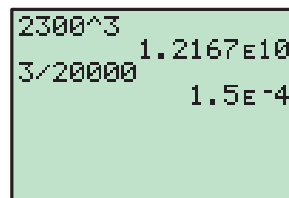
Casio fx-9860g

To evaluate 2300^3, press 2300 $\boxed{\wedge}$ 3 $\boxed{\text{EXE}}$.

The answer displayed is 1.2167E+10, which means 1.2167×10^{10}.

To evaluate $\dfrac{3}{20\,000}$, press 3 $\boxed{\div}$ 20 000 $\boxed{\text{EXE}}$.

The answer displayed is 1.5E−04, which means 1.5×10^{-4}.

You can enter values in standard form using the $\boxed{\text{EXP}}$ key. For example, to

evaluate $\dfrac{2.6 \times 10^{14}}{13}$, press 2.6 $\boxed{\text{EXP}}$ 14 $\boxed{\div}$ 13 $\boxed{\text{EXE}}$.

The answer is 2×10^{13}.
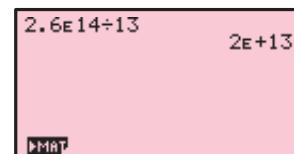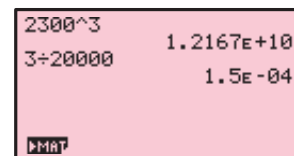
C — SECONDARY FUNCTION AND ALPHA KEYS

Texas Instruments TI-84 Plus

The **secondary function** of each key is displayed in blue above the key. It is accessed by pressing the $\boxed{\text{2nd}}$ key, followed by the key corresponding to the desired secondary function. For example, to calculate $\sqrt{36}$, press $\boxed{\text{2nd}}$ $\boxed{x^2}$ 36 $\boxed{)}$ $\boxed{\text{ENTER}}$.

The **alpha function** of each key is displayed in green above the key. It is accessed by pressing the $\boxed{\text{ALPHA}}$ key followed by the key corresponding to the desired letter. The main purpose of the alpha keys is to store values into memory which can be recalled later. Refer to the **Memory** section.

Casio fx-9860g

The **shift function** of each key is displayed in yellow above the key. It is accessed by pressing the $\boxed{\text{SHIFT}}$ key followed by the key corresponding to the desired shift function.

For example, to calculate $\sqrt{36}$, press $\boxed{\text{SHIFT}}$ $\boxed{x^2}$ 36 $\boxed{\text{EXE}}$.

The **alpha function** of each key is displayed in red above the key. It is accessed by pressing the $\boxed{\text{ALPHA}}$ key followed by the key corresponding to the desired letter. The main purpose of the alpha keys is to store values which can be recalled later.

D — MEMORY

Utilising the memory features of your calculator allows you to recall calculations you have performed previously. This not only saves time, but also enables you to maintain accuracy in your calculations.

SPECIFIC STORAGE TO MEMORY

Values can be stored into the variable letters A, B, ..., Z using either calculator. Storing a value in memory is useful if you need that value multiple times.

Texas Instruments TI-84 Plus

Suppose we wish to store the number 15.4829 for use in a number of calculations. Type in the number then press $\boxed{\text{STO}\blacktriangleright}$ $\boxed{\text{ALPHA}}$ $\boxed{\text{MATH}}$ **(A)** $\boxed{\text{ENTER}}$.

We can now add 10 to this value by pressing $\boxed{\text{ALPHA}}$ $\boxed{\text{MATH}}$ $\boxed{+}$ 10 $\boxed{\text{ENTER}}$, or cube this value by pressing $\boxed{\text{ALPHA}}$ $\boxed{\text{MATH}}$ $\boxed{\wedge}$ 3 $\boxed{\text{ENTER}}$.

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Suppose we wish to store the number 15.4829 for use in a number of calculations. Type in the number then press $\boxed{\rightarrow}$ $\boxed{\text{ALPHA}}$ $\boxed{\text{X},\theta,\text{T}}$ **(A)** $\boxed{\text{EXE}}$.

We can now add 10 to this value by pressing $\boxed{\text{ALPHA}}$ $\boxed{\text{X},\theta,\text{T}}$ $\boxed{+}$ 10 $\boxed{\text{EXE}}$, or cube this value by pressing $\boxed{\text{ALPHA}}$ $\boxed{\text{X},\theta,\text{T}}$ $\boxed{\wedge}$ 3 $\boxed{\text{EXE}}$.

ANS VARIABLE

Texas Instruments TI-84 Plus

The variable **Ans** holds the most recent evaluated expression, and can be used in calculations by pressing $\boxed{\text{2nd}}$ $\boxed{(-)}$.

For example, suppose you evaluate 3×4, and then wish to subtract this from 17. This can be done by pressing 17 $\boxed{-}$ $\boxed{\text{2nd}}$ $\boxed{(-)}$ $\boxed{\text{ENTER}}$.

If you start an expression with an operator such as $\boxed{+}$, $\boxed{-}$, etc, the previous answer **Ans** is automatically inserted ahead of the operator. For example, the previous answer can be halved simply by pressing $\boxed{\div}$ 2 $\boxed{\text{ENTER}}$.

If you wish to view the answer in fractional form, press $\boxed{\text{MATH}}$ 1 $\boxed{\text{ENTER}}$.

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The variable **Ans** holds the most recent evaluated expression, and can be used in calculations by pressing $\boxed{\text{SHIFT}}$ $\boxed{(-)}$. For example, suppose you evaluate 3×4, and then wish to subtract this from 17. This can be done by pressing 17 $\boxed{-}$ $\boxed{\text{SHIFT}}$ $\boxed{(-)}$ $\boxed{\text{EXE}}$.

If you start an expression with an operator such as $\boxed{+}$, $\boxed{-}$, etc, the previous answer Ans is automatically inserted ahead of the operator. For example, the previous answer can be halved simply by pressing $\boxed{\div}$ 2 $\boxed{\text{EXE}}$.

If you wish to view the answer in fractional form, press $\boxed{\text{F}\leftrightarrow\text{D}}$.

RECALLING PREVIOUS EXPRESSIONS

Texas Instruments TI-84 Plus

The **ENTRY** function recalls previously evaluated expressions, and is used by pressing 2nd ENTER .

This function is useful if you wish to repeat a calculation with a minor change, or if you have made an error in typing.

Suppose you have evaluated $100 + \sqrt{132}$. If you now want to evaluate $100 + \sqrt{142}$, instead of retyping the command, it can be recalled by pressing 2nd ENTER .

The change can then be made by moving the cursor over the 3 and changing it to a 4, then pressing ENTER .

If you have made an error in your original calculation, and intended to calculate $1500 + \sqrt{132}$, again you can recall the previous command by pressing 2nd ENTER .

Move the cursor to the first 0.

You can insert the digit 5, rather than overwriting the 0, by pressing 2nd DEL 5 ENTER .

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Pressing the left cursor key allows you to edit the most recently evaluated expression, and is useful if you wish to repeat a calculation with a minor change, or if you have made an error in typing.

Suppose you have evaluated $100 + \sqrt{132}$.

If you now want to evaluate $100 + \sqrt{142}$, instead of retyping the command, it can be recalled by pressing the left cursor key.

Move the cursor between the 3 and the 2, then press DEL 4 to remove the 3 and change it to a 4. Press EXE to re-evaluate the expression.

E   LISTS

Lists are used for a number of purposes on the calculator. They enable us to enter sets of numbers, and we use them to generate number sequences using algebraic rules.

CREATING A LIST

Texas Instruments TI-84 Plus

Press STAT 1 to take you to the **list editor** screen.

To enter the data $\{2, 5, 1, 6, 0, 8\}$ into **List 1**, start by moving the cursor to the first entry of L_1. Press 2 ENTER 5 ENTER ...... and so on until all the data is entered.

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Selecting **STAT** from the Main Menu takes you to the **list editor** screen.

To enter the data $\{2, 5, 1, 6, 0, 8\}$ into **List 1**, start by moving the cursor to the first entry of **List 1**. Press 2 EXE 5 EXE ...... and so on until all the data is entered.
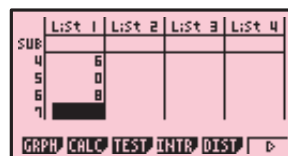
DELETING LIST DATA

Texas Instruments TI-84 Plus

Pressing STAT **1** takes you to the **list editor** screen.

Move the cursor to the heading of the list you want to delete then press CLEAR ENTER .

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Selecting **STAT** from the Main Menu takes you to the **list editor** screen.

Move the cursor to anywhere on the list you wish to delete, then press F6 (▷) F4 **(DEL-A)** F1 **(Yes)**.

REFERENCING LISTS

Texas Instruments TI-84 Plus

Lists can be referenced by using the secondary functions of the keypad numbers 1–6.

For example, suppose you want to add 2 to each element of **List 1** and display the results in **List 2**. To do this, move the cursor to the heading of **L₂** and press 2nd 1 (**L₁**) + 2 ENTER .

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Lists can be referenced using the List function, which is accessed by pressing SHIFT **1**.

For example, if you want to add 2 to each element of **List 1** and display the results in **List 2**, move the cursor to the heading of **List 2** and press SHIFT 1 **(List)** 1 + 2 EXE .

For Casio models without the List function, you can do this by pressing OPTN F1 **(LIST)** F1 **(List)** 1 + 2 EXE .

NUMBER SEQUENCES

Texas Instruments TI-84 Plus

You can create a sequence of numbers defined by a certain rule using the *seq* command.

This command is accessed by pressing 2nd STAT ▶ to enter the **OPS** section of the List menu, then selecting **5:seq**.

For example, to store the sequence of even numbers from 2 to 8 in **List 3**, move the cursor to the heading of **L₃**, then press 2nd STAT ▶ 5 to enter the *seq* command, followed by 2 X,T,θ,n , X,T,θ,n , 1 , 4) ENTER .

This evaluates $2x$ for every value of x from 1 to 4.
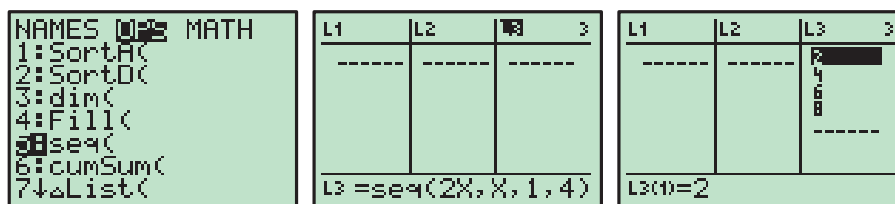
Casio fx-9860g

You can create a sequence of numbers defined by a certain rule using the *seq* command.

This command is accessed by pressing OPTN F1 (**LIST**) F5 (**Seq**).

For example, to store the sequence of even numbers from 2 to 8 in **List 3**, move the cursor to the heading of **List 3**, then press OPTN F1 F5 to enter a sequence, followed by 2 X,θ,T , X,θ,T , 1 , 4 , 1) EXE .

This evaluates $2x$ for every value of x from 1 to 4 with an increment of 1.
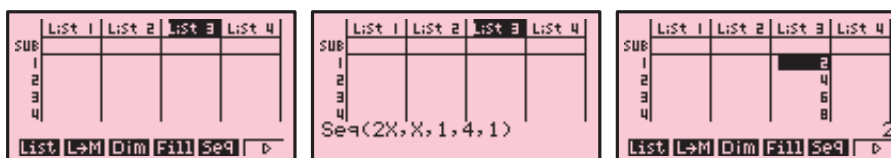
F   STATISTICAL GRAPHS

Your graphics calculator is a useful tool for analysing data and creating statistical graphs.

In this section we produce descriptive statistics and graphs for the data set: 5 2 3 3 6 4 5 3 7 5 7 1 8 9 5.

Texas Instruments TI-84 Plus

Enter the data set into **List 1** using the instructions on page **19**. To obtain descriptive statistics of the data set, press STAT ▶ **1:1-Var Stats** 2nd **1 (L₁)** ENTER .
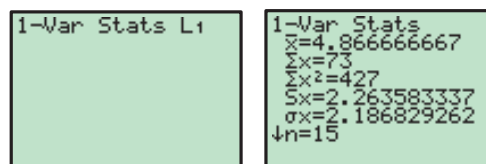
To obtain a boxplot of the data, press 2nd Y= (**STAT PLOT**) **1** and set up **Statplot1** as shown. Press ZOOM **9:ZoomStat** to graph the boxplot with an appropriate window.
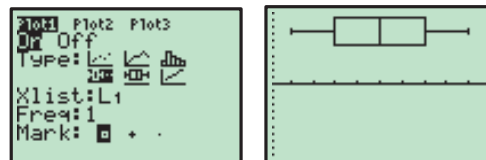
To obtain a vertical bar chart of the data, press 2nd Y= **1**, and change the type of graph to a vertical bar chart as shown. Press ZOOM **9:ZoomStat** to draw the bar chart. Press WINDOW and set the **Xscl** to 1, then GRAPH to redraw the bar chart.
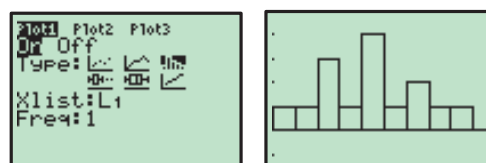
We will now enter a second set of data, and compare it to the first.

Enter the data set 9 6 2 3 5 5 7 5 6 7 6 3 4 4 5 8 4 into **List 2**, press [2nd] [Y=] **1**, and change the type of graph back to a boxplot as shown. Move the cursor to the top of the screen and select **Plot2**. Set up **Statplot2** in the same manner, except set the **XList** to **L₂**. Press [ZOOM] **9:ZoomStat** to draw the side-by-side boxplots.

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Enter the data into **List 1** using the instructions on page **19**. To obtain the descriptive statistics, press [F6] (▷) until the **GRPH** icon is in the bottom left corner of the screen, then press [F2] (**CALC**) [F1] (**1 VAR**).

To obtain a boxplot of the data, press [EXIT] [EXIT] [F1] (**GRPH**) [F6] (**SET**), and set up **StatGraph 1** as shown. Press [EXIT] [F1] (**GPH1**) to draw the boxplot.

To obtain a vertical bar chart of the data, press [EXIT] [F6] (**SET**) [F2] (**GPH2**), and set up **StatGraph 2** as shown. Press [EXIT] [F2] (**GPH2**) to draw the bar chart (set Start to 0, and Width to 1).

We will now enter a second set of data, and compare it to the first.

Enter the data set 9 6 2 3 5 5 7 5 6 7 6 3 4 4 5 8 4 into **List 2**, then press [F6] (**SET**) [F2] (**GPH2**) and set up **StatGraph 2** to draw a boxplot of this data set as shown. Press [EXIT] [F4] (**SEL**), and turn on both **StatGraph 1** and **StatGraph 2**. Press [F6] (**DRAW**) to draw the side-by-side boxplots.
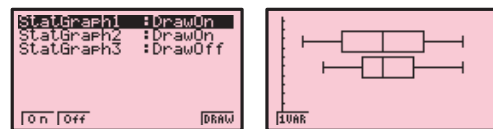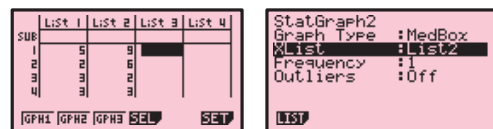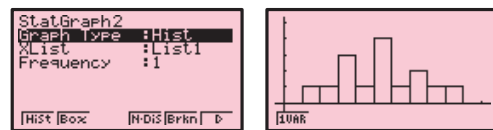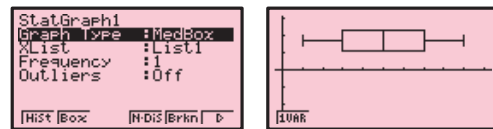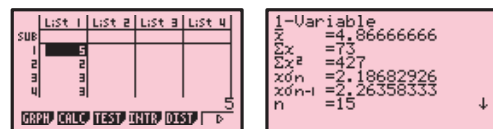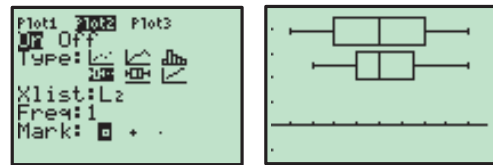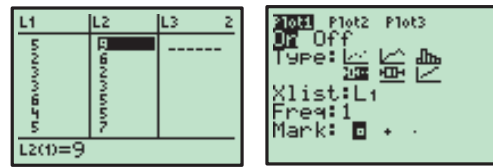
G   WORKING WITH FUNCTIONS

GRAPHING FUNCTIONS

Texas Instruments TI-84 Plus

Pressing [Y=] selects the **Y=** editor, where you can store functions to graph. Delete any unwanted functions by scrolling down to the function and pressing [CLEAR] .
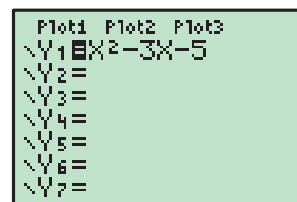
To graph the function $y = x^2 - 3x - 5$, move the cursor to **Y₁**, and press $\boxed{\text{X,T,}\theta\text{,n}}$ $\boxed{x^2}$ $\boxed{-}$ 3 $\boxed{\text{X,T,}\theta\text{,n}}$ $\boxed{-}$ 5 $\boxed{\text{ENTER}}$. This stores the function into **Y₁**. Press $\boxed{\text{GRAPH}}$ to draw a graph of the function.
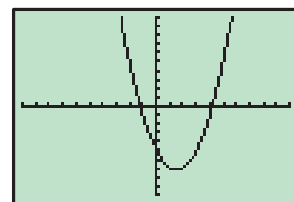
To view a table of values for the function, press $\boxed{\text{2nd}}$ $\boxed{\text{GRAPH}}$ **(TABLE)**. The starting point and interval of the table values can be adjusted by pressing $\boxed{\text{2nd}}$ $\boxed{\text{WINDOW}}$ **(TBLSET)**.

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Selecting **GRAPH** from the Main Menu takes you to the Graph Function screen, where you can store functions to graph. Delete any unwanted functions by scrolling down to the function and pressing $\boxed{\text{DEL}}$ $\boxed{\text{F1}}$ **(Yes)**.
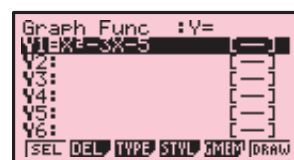
To graph the function $y = x^2 - 3x - 5$, move the cursor to **Y1** and press $\boxed{\text{X,}\theta\text{,T}}$ $\boxed{x^2}$ $\boxed{-}$ 3 $\boxed{\text{X,}\theta\text{,T}}$ $\boxed{-}$ 5 $\boxed{\text{EXE}}$. This stores the function into **Y1**. Press $\boxed{\text{F6}}$ **(DRAW)** to draw a graph of the function.
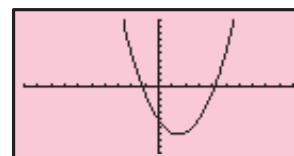
To view a table of values for the function, press $\boxed{\text{MENU}}$ and select **TABLE**. The function is stored in **Y1**, but not selected. Press $\boxed{\text{F1}}$ **(SEL)** to select the function, and $\boxed{\text{F6}}$ **(TABL)** to view the table. You can adjust the table settings by pressing $\boxed{\text{EXIT}}$ and then $\boxed{\text{F5}}$ **(SET)** from the Table Function screen.

GRAPHING ABSOLUTE VALUE FUNCTIONS

Texas Instruments TI-84 Plus

You can perform operations involving absolute values by pressing $\boxed{\text{MATH}}$ $\boxed{\blacktriangleright}$, which brings up the **NUM** menu, followed by **1: abs (** .
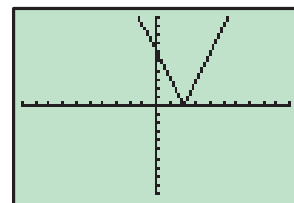
To graph the absolute value function $y = |3x - 6|$, press $\boxed{\text{Y=}}$, move the cursor to **Y₁**, then press $\boxed{\text{MATH}}$ $\boxed{\blacktriangleright}$ 1 3 $\boxed{\text{X,T,}\theta\text{,n}}$ $\boxed{-}$ 6 $\boxed{)}$ $\boxed{\text{GRAPH}}$.

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To graph the absolute value function $y = |3x - 6|$, select GRAPH from the Main Menu, move the cursor to **Y1** and press $\boxed{\text{OPTN}}$ $\boxed{\text{F5}}$ **(NUM)** $\boxed{\text{F1}}$ **(Abs)** $\boxed{(}$ 3 $\boxed{\text{X,}\theta\text{,T}}$ $\boxed{-}$ 6 $\boxed{)}$ $\boxed{\text{EXE}}$ $\boxed{\text{F6}}$ **(DRAW)**.
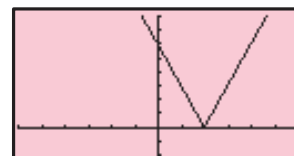
FINDING POINTS OF INTERSECTION

It is often useful to find the points of intersection of two graphs, for instance, when you are trying to solve simultaneous equations.

Texas Instruments TI-84 Plus

We can solve $y = 11 - 3x$ and $y = \dfrac{12 - x}{2}$ simultaneously by finding the point of intersection of these two lines.

Press $\boxed{\text{Y=}}$, then store $11 - 3x$ into **Y₁** and $\dfrac{12 - x}{2}$ into **Y₂**. Press $\boxed{\text{GRAPH}}$ to draw a graph of the functions.

To find their point of intersection, press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ **(CALC) 5**, which selects **5:intersect**. Press $\boxed{\text{ENTER}}$ twice to specify the functions **Y₁** and **Y₂** as the functions you want to find the intersection of, then use the arrow keys to move the cursor close to the point of intersection and press $\boxed{\text{ENTER}}$ once more.

The solution $x = 2$, $y = 5$ is given.
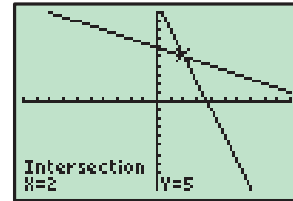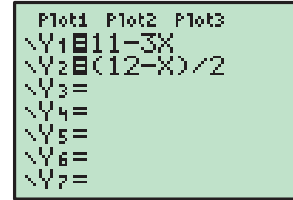
Casio fx-9860g

We can solve $y = 11 - 3x$ and $y = \dfrac{12 - x}{2}$ simultaneously by finding the point of intersection of these two lines. Select **GRAPH** from the Main Menu, then store $11 - 3x$ into **Y1** and $\dfrac{12 - x}{2}$ into **Y2**. Press $\boxed{\text{F6}}$ **(DRAW)** to draw a graph of the functions.

To find their point of intersection, press $\boxed{\text{F5}}$ **(G-Solv)** $\boxed{\text{F5}}$ **(ISCT)**. The solution $x = 2$, $y = 5$ is given.

Note: If there is more than one point of intersection, the remaining points of intersection can be found by pressing $\boxed{\blacktriangleright}$.
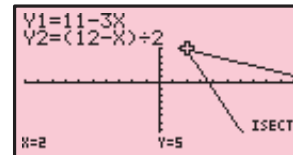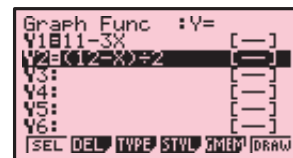
SOLVING $f(x) = 0$

In the special case when you wish to solve an equation of the form $f(x) = 0$, this can be done by graphing $y = f(x)$ and then finding when this graph cuts the x-axis.

Texas Instruments TI-84 Plus

To solve $x^3 - 3x^2 + x + 1 = 0$, press $\boxed{\text{Y=}}$ and store $x^3 - 3x^2 + x + 1$ into **Y₁**. Press $\boxed{\text{GRAPH}}$ to draw the graph.

To find where this function first cuts the x-axis, press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ **(CALC) 2**, which selects **2:zero**. Move the cursor to the left of the first zero and press $\boxed{\text{ENTER}}$, then move the cursor to the right of the first zero and press $\boxed{\text{ENTER}}$.

Finally, move the cursor close to the first zero and press $\boxed{\text{ENTER}}$ once more. The solution $x \approx -0.414$ is given.

Repeat this process to find the remaining solutions $x = 1$ and $x \approx 2.414$.
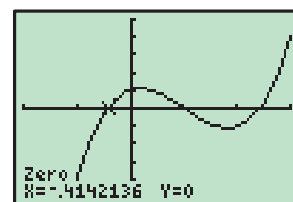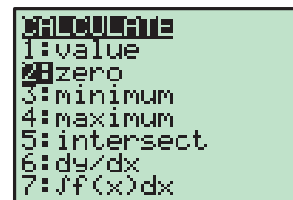
Casio fx-9860g

To solve $x^3 - 3x^2 + x + 1 = 0$, select **GRAPH** from the Main Menu and store $x^3 - 3x^2 + x + 1$ into **Y1**. Press $\boxed{\text{F6}}$ **(DRAW)** to draw the graph.

To find where this function cuts the x-axis, press $\boxed{\text{F5}}$ **(G-Solv)** $\boxed{\text{F1}}$ **(ROOT)**. The first solution $x \approx -0.414$ is given.

Press $\boxed{\blacktriangleright}$ to find the remaining solutions $x = 1$ and $x \approx 2.414$.
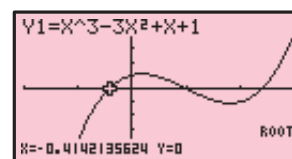
TURNING POINTS

Texas Instruments TI-84 Plus

To find the turning point (vertex) of $y = -x^2 + 2x + 3$, press $\boxed{\text{Y=}}$ and store $-x^2 + 2x + 3$ into **Y1**. Press $\boxed{\text{GRAPH}}$ to draw the graph.

From the graph, it is clear that the vertex is a maximum, so press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ **(CALC) 4** to select **4:maximum**.

Move the cursor to the left of the vertex and press $\boxed{\text{ENTER}}$, then move the cursor to the right of the vertex and press $\boxed{\text{ENTER}}$. Finally, move the cursor close to the vertex and press $\boxed{\text{ENTER}}$ once more. The vertex is $(1, 4)$.
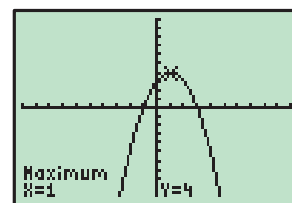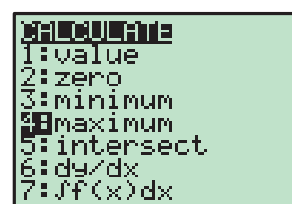
Casio fx-9860g

To find the turning point (vertex) of $y = -x^2 + 2x + 3$, select **GRAPH** from the Main Menu and store $-x^2 + 2x + 3$ into **Y1**. Press $\boxed{\text{F6}}$ **(DRAW)** to draw the graph.

From the graph, it is clear that the vertex is a maximum, so to find the vertex press $\boxed{\text{F5}}$ **(G-Solv)** $\boxed{\text{F2}}$ **(MAX)**.

The vertex is $(1, 4)$.
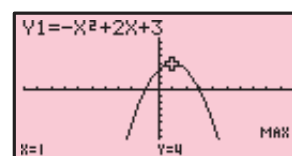
ADJUSTING THE VIEWING WINDOW

When graphing functions it is important that you are able to view all the important features of the graph. As a general rule it is best to start with a large viewing window to make sure all the features of the graph are visible. You can then make the window smaller if necessary.

Texas Instruments TI-84 Plus

Some useful commands for adjusting the viewing window include:

$\boxed{\text{ZOOM}}$ **0:ZoomFit** :　　This command scales the y-axis to fit the minimum and maximum values of the displayed graph within the current x-axis range.

ZOOM **6:ZStandard** :   This command returns the viewing window to the default setting of $-10 \leqslant x \leqslant 10$, $-10 \leqslant y \leqslant 10$.

If neither of these commands are helpful, the viewing window can be adjusted manually by pressing WINDOW and setting the minimum and maximum values for the x and y axes.
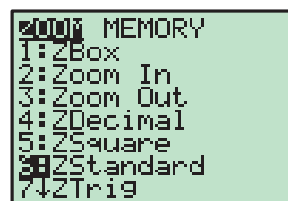
Casio fx-9860g

The viewing window can be adjusted by pressing   SHIFT F3 **(V-Window)**. You can manually set the minimum and maximum values of the x and y axes, or press   F3 **(STD)**   to obtain the standard viewing window $-10 \leqslant x \leqslant 10$, $-10 \leqslant y \leqslant 10$.

H   TWO VARIABLE ANALYSIS

LINE OF BEST FIT

We can use our graphics calculator to find the line of best fit connecting two variables. We can also find the values of Pearson's correlation coefficient r and the coefficient of determination r^2, which measure the strength of the linear correlation between the two variables.

We will examine the relationship between the variables x and y for the data:

x	1	2	3	4	5	6	7
y	5	8	10	13	16	18	20

Texas Instruments TI-84 Plus

Enter the x values into **List 1** and the y values into **List 2** using the instructions given on page **19**.

To produce a scatter diagram of the data, press 2nd Y= (**STAT PLOT**) **1**, and set up **Statplot 1** as shown.

Press ZOOM **9 : ZoomStat** to draw the scatter diagram.

We will now find the line of best fit.   Press STAT ▶ **4:LinReg(ax+b)** to select the linear regression option from the CALC menu.

Press 2nd **1 (L$_1$)** , 2nd **2 (L$_2$)** , VARS ▶ **1 1 (Y$_1$)**.   This specifies the lists **L$_1$** and **L$_2$** as the lists which hold the data, and the line of best fit will be pasted into the function **Y$_1$**. Press ENTER to view the results.
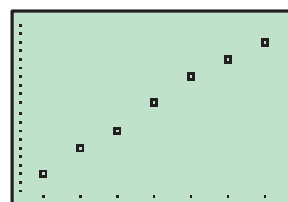
The line of best fit is given as $y \approx 2.54x + 2.71$.   If the r and r^2 values are not shown, you need to turn on the Diagnostic by pressing 2nd **0 (CATALOG)** and selecting **DiagnosticOn**.
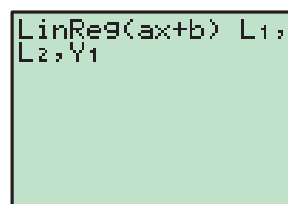
Press GRAPH to view the line of best fit.

```
LinReg
y=ax+b
a=2.535714286
b=2.714285714
r²=.9954581359
r=.9977264835
```

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Enter the x values into **List 1** and the y values into **List 2** using the instructions given on page **19**.

```
StatGraph1
Graph Type    :Scatter
XList         :List1
YList         :List2
Frequency     :1
Mark Type     :□

[GPH1][GPH2][GPH3]
```

To produce a scatter diagram for the data, press F1 (**GRPH**) F6 (**SET**), and set up **StatGraph 1** as shown. Press EXIT F1 (**GPH 1**) to draw the scatter diagram.

```
[CALC][DefG]
```

To find the line of best fit, press F1 (**CALC**) F2 (**X**).

We can see that the line of best fit is given as $y \approx 2.54x + 2.71$, and we can view the r and r^2 values.

```
LinearReg
   a =2.53571428
   b =2.71428571
   r =0.99772648
   r²=0.99545813
  MSe=0.16428571
y=ax+b
              [COPY][DRAW]
```

Press F6 (**DRAW**) to view the line of best fit.

```
[2VAR][ X ][Med][X^2][X^3][ ▷ ]
```
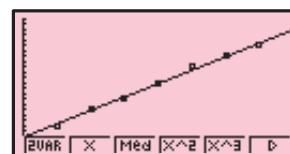
QUADRATIC AND CUBIC REGRESSION

You can use quadratic or cubic regression to find the formula for the general term of a quadratic or cubic sequence.

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To find the general term for the **quadratic sequence** $-2, 5, 16, 31, 50, ...,$ we first notice that we have been given 5 members of the sequence. We therefore enter the numbers 1 to 5 into **L1**, and the members of the sequence into **L2**.

```
QuadReg
y=ax²+bx+c
a=2
b=1
c=-5
R²=1
```

Press STAT ▶ **5: QuadReg**, then 2nd 1 (**L1**) , 2nd 2 (**L2**) ENTER .

The result is $a = 2$, $b = 1$, $c = -5$, which means the general term for the sequence is $u_n = 2n^2 + n - 5$.

To find the general term for the **cubic sequence** $-3, -9, -7, 9, 45, ...,$ we enter the numbers 1 to 5 into **L1** and the members of the sequence into **L2**.

```
CubicReg
y=ax³+bx²+cx+d
a=1
b=-2
c=-7
d=5
R²=1
```

Press STAT ▶ **6: CubicReg**, then 2nd 1 (**L1**) , 2nd 2 (**L2**) ENTER .

The result is $a = 1$, $b = -2$, $c = -7$, $d = 5$, which means the general term for the sequence is $u_n = n^3 - 2n^2 - 7n + 5$.

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To find the general term for the **quadratic sequence** -2, 5, 16, 31, 50, ..., we first notice that we have been given 5 members of the sequence. Enter the numbers 1 to 5 into **List 1**, and the members of the sequence into **List 2**.

Press $\boxed{\text{F2}}$ (**CALC**) $\boxed{\text{F3}}$ (**REG**) $\boxed{\text{F3}}$ (**X^2**).

The result is $a = 2$, $b = 1$, $c = -5$, which means the general term for the sequence is $u_n = 2n^2 + n - 5$.

To find the general term for the **cubic sequence** -3, -9, -7, 9, 45, we enter the numbers 1 to 5 into **List 1** and the members of the sequence into **List 2**.

Press $\boxed{\text{F2}}$ (**CALC**) $\boxed{\text{F3}}$ (**REG**) $\boxed{\text{F4}}$ (**X^3**).

The result is $a = 1$, $b = -2$, $c = -7$, $d = 5$ (the calculator may not always give the result exactly as is the case with c and d in this example). Therefore the general term for the sequence is $u_n = n^3 - 2n^2 - 7n + 5$.

EXPONENTIAL REGRESSION

When we have data for two variables x and y, we can use exponential regression to find the exponential model of the form $y = a \times b^x$ which best fits the data.

We will examine the exponential relationship between x and y for the data:

x	2	4	7	9	12
y	7	11	20	26	45

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Enter the x values into **L1** and the y values into **L2**.

Press $\boxed{\text{STAT}}$ $\boxed{\blacktriangleright}$ **0: ExpReg**, then $\boxed{\text{2nd}}$ 1 (**L$_1$**) $\boxed{,}$ $\boxed{\text{2nd}}$ 2 (**L$_2$**) $\boxed{\text{ENTER}}$.

So, the exponential model which best fits the data is $y \approx 5.13 \times 1.20^x$.

POWER REGRESSION

When we have data for two variables x and y, we can use power regression to find the power model of the form $y = a \times x^b$ which best fits the data.

We will examine the power relationship between x and y for the data:

x	1	3	4	6
y	3	19	35	62

Texas Instruments TI-84 Plus

Enter the x values into **L1** and the y values into **L2**.

Press $\boxed{\text{STAT}}$ $\boxed{\blacktriangleright}$, then scroll down to **A: PwrReg** and press $\boxed{\text{ENTER}}$.

Press $\boxed{\text{2nd}}$ 1 (**L$_1$**) $\boxed{,}$ $\boxed{\text{2nd}}$ 2 (**L$_2$**) $\boxed{\text{ENTER}}$.

So, the power model which best fits the data is $y \approx 3.01 \times x^{1.71}$.

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Enter the x values into **List 1** and the y values into **List 2**.

Press $\boxed{\text{F2}}$ (**CALC**) $\boxed{\text{F3}}$ (**REG**) $\boxed{\text{F6}}$ $\boxed{\text{F3}}$ (**Pwr**).

So, the power model which best fits the data is $y \approx 3.01 \times x^{1.71}$.

Assumed Knowledge (Number)

PRINTABLE CHAPTER

Contents:

Assumed Knowledge (Number)
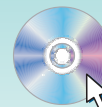
Contents:

A NUMBER TYPES [1.1]

The set of **natural** or **counting** numbers is $\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, ...\}$.

The set of natural numbers is endless, so we call it an **infinite set**.

The set of **integers** or whole numbers is $\mathbb{Z} = \{...., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...\}$.

The set $\mathbb{Z}$ includes $\mathbb{N}$ as well as all negative whole numbers.

For example: $0, 127, -15, 10\,000$ and $-34\,618$ are all members of $\mathbb{Z}$.

The set of **rational** numbers, denoted $\mathbb{Q}$, is the set of all numbers of the form $\dfrac{a}{b}$ where a and b are integers and $b \neq 0$.

For example:
- $\frac{1}{3}, \frac{15}{32}, \frac{-5}{4}$ are all rational
- 5 is rational as $5 = \frac{5}{1}$ or $\frac{20}{4}$
- $0.\overline{8}$ is rational as $0.\overline{8} = \frac{8}{9}$
- $-2\frac{1}{4}$ is rational as $-2\frac{1}{4} = \frac{-9}{4}$
- 0.431 is rational as $0.431 = \frac{431}{1000}$

All decimal numbers that terminate or recur are rational numbers.

The set of **real** numbers, denoted $\mathbb{R}$, includes all numbers which can be located on a number line.

$\frac{2}{0}$ and $\sqrt{-5}$ cannot be placed on a number, and so are not real.

The set of **irrational** numbers includes all real numbers which cannot be written in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

For example: π, $\sqrt{2}$ and $\sqrt{3}$ are all irrational.

$\sqrt{9}$ and $\sqrt{1.21}$ are rational since $\sqrt{9} = 3 = \frac{3}{1}$ and $\sqrt{1.21} = 1.1 = \frac{11}{10}$.

PRIMES AND COMPOSITES

The **factors** of a positive integer are the positive integers which divide exactly into it, leaving no remainder.

For example, the factors of 10 are: 1, 2, 5 and 10.

A positive integer is a **prime** number if it has exactly two factors, 1 and itself.

A positive integer is a **composite** number if it has more than two factors.

For example: 3 is prime as it has two factors: 1 and 3.

6 is composite as it has four factors: 1, 2, 3 and 6.

1 is neither prime nor composite.

If we are given a positive integer, we can use the following procedure to see if it is prime:

Step 1: Find the square root of the number.

Step 2: Divide the whole number in turn by all known primes less than its square root. If the division is never exact, the number is a prime.

The first few prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, ...

Example 1   ◀) *Self Tutor*

Is 131 a prime number?

$\sqrt{131} = 11.445......$, so we divide 131 by 2, 3, 5, 7 and 11.

$131 \div 2 = 65.5$

$131 \div 3 = 43.66.....$

$131 \div 5 = 26.2$

$131 \div 7 = 18.7142.....$

$131 \div 11 = 11.9090.....$

None of the divisions is exact, so 131 is a prime number.

OTHER CLASSIFICATIONS

A **perfect square** or **square number** is an integer which can be written as the square of another integer.

For example, 4 and 25 are perfect squares since $4 = 2^2$ and $25 = 5^2$.

A **perfect cube** is an integer which can be written as the cube of another integer.

For example, 8 and -125 are perfect cubes since $8 = 2^3$ and $-125 = (-5)^3$.

EXERCISE A

1 Copy and complete the table given.

Number	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
19	✓	✓	✓	✓
$-\frac{2}{3}$				
$\sqrt{7}$				
5.6389				
$\sqrt{16}$				
2π				
-11				
$\frac{6}{0}$				
$\sqrt{-2}$				

2 Show that:

    a 16, 529 and 20 802 721 are square numbers

    b 27, 343 and 13 824 are cube numbers

    c 11, 17 and 97 are prime numbers

    d 64 and 729 are both squares and cubes.

3 List all prime numbers between 50 and 70.

4 Explain why 1 is not a prime number.

5 Is 263 a prime number? Show all working.

B   OPERATIONS AND BRACKETS   [1.2]

RULES FOR THE ORDER OF OPERATIONS

- Perform the operations within **brackets** first.
- Calculate any part involving **exponents**.
- Starting from the left, perform all **divisions** and **multiplications** as you come to them.
- Starting from the left, perform all **additions** and **subtractions** as you come to them.

> The word BEDMAS may help you remember this order. Brackets are **grouping symbols** that tell us what to evaluate first.

RULES FOR THE USE OF BRACKETS

- If an expression contains *one set* of brackets or group symbols, work that part first.
- If an expression contains *two or more sets* of grouping symbols one inside the other, work the innermost first.
- The division line of fractions also behaves as a grouping symbol. This means that the numerator and the denominator must be found separately before doing the division.

Example 2   ◀)) *Self Tutor*

Simplify: **a** $3 + 7 - 5$ **b** $6 \times 3 \div 2$

a $3 + 7 - 5$ {Work left to right as only $+$ and $-$ are involved.}
$= 10 - 5$
$= 5$

b $6 \times 3 \div 2$ {Work left to right as only $\times$ and $\div$ are involved.}
$= 18 \div 2$
$= 9$

Example 3   ◀)) *Self Tutor*

Simplify: **a** $23 - 10 \div 2$ **b** $3 \times 8 - 6 \times 5$

a $23 - 10 \div 2$ **b** $3 \times 8 - 6 \times 5$
$= 23 - 5$ $\{ \div \text{ before } - \}$ $= 24 - 30$ $\{ \times \text{ before } - \}$
$= 18$ $= -6$

EXERCISE B

1 Simplify:

   a $6 + 9 - 5$ **b** $6 - 9 + 5$ **c** $6 - 9 - 5$
   d $3 \times 12 \div 6$ **e** $12 \div 6 \times 3$ **f** $6 \times 12 \div 3$

2 Simplify:

   a $5 + 8 \times 4$ **b** $9 \times 4 + 7$ **c** $17 - 7 \times 2$ **d** $6 \times 7 - 18$
   e $36 - 6 \times 5$ **f** $19 - 7 \times 0$ **g** $3 \times 6 - 6$ **h** $70 - 5 \times 4 \times 3$
   i $45 \div 3 - 9$ **j** $8 \times 5 - 6 \times 4$ **k** $7 + 3 + 5 \times 2$ **l** $17 - 6 \times 4 + 9$

Example 4   ◀)) *Self Tutor*

Simplify: $3 + (11 - 7) \times 2$

$3 + (11 - 7) \times 2$
$= 3 + 4 \times 2$ {evaluating the brackets first}
$= 3 + 8$ $\{ \times \text{ before } + \}$
$= 11$

3 Simplify:

   a $14 + (8 - 5)$ **b** $(19 + 7) - 13$ **c** $(18 \div 6) - 2$
   d $18 \div (6 - 4)$ **e** $72 - (18 \div 6)$ **f** $(72 - 18) \div 6$
   g $36 + (14 \div 2)$ **h** $36 - (7 + 13) - 9$ **i** $(22 - 5) + (15 - 11)$
   j $(18 \div 3) \div 2$ **k** $32 \div (4 \div 2)$ **l** $28 - (7 \times 3) - 9$

Example 5   ◀) *Self Tutor*

Simplify: $[12 + (9 \div 3)] - 11$

$[12 + (9 \div 3)] - 11$
$= [12 + 3] - 11$ {evaluating the inner brackets first}
$= 15 - 11$ {outer brackets next}
$= 4$

---

4 Simplify:

a $8 - [(4 - 6) + 3 \times 2]$ **b** $[22 - (11 + 4)] \times 3$ **c** $25 - [(11 - 7) + 8]$

d $[28 - (15 \div 3)] \times 4$ **e** $300 \div [6 \times (15 \div 3)]$ **f** $[(14 \times 5) \div (28 \div 2)] \times 3$

g $[24 \div (8 - 6) \times 9]$ **h** $18 - [(1 + 6) \times 2]$ **i** $[(14 \div 2) \times (14 \div 7)] \div 2$

Example 6   ◀) *Self Tutor*

Simplify: $\dfrac{12 + (5 - 7)}{18 \div (6 + 3)}$

$\dfrac{12 + (5 - 7)}{18 \div (6 + 3)}$

$= \dfrac{12 + (-2)}{18 \div 9}$ {evaluating the brackets first}

$= \dfrac{10}{2}$ {simplifying the numerator and denominator}

$= 5$

---

5 Simplify:

a $\dfrac{240}{8 \times 6}$ **b** $\dfrac{27}{17 - 8}$ **c** $\dfrac{39 \div 3}{14 + 12}$

d $\dfrac{18 + 7}{7 - 2}$ **e** $\dfrac{58 - 16}{11 - 5}$ **f** $\dfrac{6 \times 7 + 7}{7}$

g $\dfrac{54}{11 - (2 \times 4)}$ **h** $\dfrac{(6 + 9) - 5}{7 + (9 - 6)}$ **i** $\dfrac{2 - 2 \times 2}{2 \times (2 + 2)}$

6 Find:

a $7 - -8$ **b** $10 - (2 + 5)$ **c** 6×-7

d $\dfrac{-4}{8}$ **e** $(-2) \times (-2)$ **f** $4 - (-3)$

g $(-3) \times (-7)$ **h** $\dfrac{-24}{-6}$ **i** $2 - 5 \times (-3)$

j $2 \times 5 - (-3)$ **k** $\dfrac{-2 \times 3}{-4}$ **l** $\dfrac{2 \times (-3)}{-4 \times 6}$

Example 7   ◀ *Self Tutor*

Use your calculator to evaluate: **a** $12 + 32 \div (8 - 6)$ **b** $\dfrac{75}{7 + 8}$

a   Key in 12 $+$ 32 $\div$ $($ 8 $-$ 6 $)$ [ENTER]   *Answer:*   28

b   Key in 75 $\div$ $($ 7 $+$ 8 $)$ [ENTER]   *Answer:*   5

> For help using your calculator, refer to the **graphics calculator instructions** on page **11**.

Example 8   ◀ *Self Tutor*

Calculate: **a** 41×-7 **b** -18×23

a   Key in   41 $\times$ [(−)] 7 [ENTER]   *Answer:* -287

b   Key in   [(−)] 18 $\times$ 23 [ENTER]   *Answer:* -414

---

7   Evaluate each of the following using your calculator:

a $87 + 27 \times 13$	**b** $(29 + 17) \times 19$	**c** $136 \div 8 + 16$
d $136 \div (8 + 9)$	**e** 39×-27	**f** $-128 \div -32$
g $\dfrac{97 + -7}{-5 \times 3}$	**h** $-67 + 64 \div -4$	**i** $\dfrac{-25 - 15}{9 - (16 \div 4)}$
j $(-3)^4$	**k** -3^4	**l** $(-1.4)^3 \times -2.7^2$

8   Use a calculator to answer the following questions:

a   Kevin can throw a tennis ball 47.55 m, and Dean can throw a tennis ball 42.8 m. How much further than Dean can Kevin throw?

b   Find the cost of purchasing 2.7 kg of bananas at £2.45 per kilogram.

c   Chen travels 11 km on the bus to school each day, and the same distance home. He goes to school on 205 days in one year. Find the total distance Chen travels on the bus to and from school for the year.

d   June buys 3 packets of sugar weighing 1.2 kg each, 4 packets of cereal weighing 1.3 kg each, and 2 containers of ice cream weighing 1.5 kg each. Find the total weight of these items.

e   Jordan is paid €15.60 per hour to work at a petrol station. He wants to save €1170 to buy a car. How many hours will he need to work?

C   HCF AND LCM   *[1.3]*

Numbers can be expressed as products of their factors.

Factors that are prime numbers are called **prime factors**. The prime factors of any number can be found by repeated division.

For example:

2	24
2	12
2	6
3	3
	1

$\therefore$ $24 = 2 \times 2 \times 2 \times 3$

2	42
3	21
7	7
	1

$\therefore$ $42 = 2 \times 3 \times 7$

COMMON FACTORS AND HCF

Notice that 2 is a factor of both 24 and 42. We say that 2 is a **common factor** of 24 and 42.

3 is also a common factor of 24 and 42, which means the product $2 \times 3 = 6$ is another common factor.

> A **common factor** is a number that is a factor of two or more other numbers.
>
> The **highest common factor (HCF)** is the largest factor that is common to two or more numbers.

To find the highest common factor of a group of numbers it is often best to express the numbers as products of prime factors. Then the common prime factors can be found and multiplied to give the HCF.

Example 9   ◄)) Self Tutor

Find the highest common factor of 36 and 81.

So,

2	36
2	18
3	9
3	3
	1

$36 = 2 \times 2 \times 3 \times 3$ and

3	81
3	27
3	9
3	3
	1

$81 = 3 \times 3 \times 3 \times 3$

$\therefore$ HCF $= 3 \times 3 = 9$

MULTIPLES

> A **multiple** of any positive integer is obtained by multiplying it by another positive integer.

For example, the multiples of 7 are $7 \times 1 = 7$, $7 \times 2 = 14$, $7 \times 3 = 21$, $7 \times 4 = 28$, ... so we can list them as 7, 14, 21, 28, 35, ...

Example 10   ◄)) Self Tutor

Find: **a** the largest multiple of 7 which is less than 300

b the smallest multiple of 7 which is greater than 500.

a

$7 \, | \, 3 \, 0 \,^2 0$

$\quad\quad 4 \;\; 2$ with remainder 6

So, the largest multiple is
$7 \times 42 = 294$.

b

$7 \, | \, 5 \, 0 \,^1 0$

$\quad\quad 7 \;\; 1$ with remainder 3

So, the smallest multiple is
$7 \times 72 = 504$.

The **lowest common multiple** or **LCM** of two or more positive integers is the smallest multiple which is common to all of them.

Example 11   ◀ɕ *Self Tutor*

Find the LCM of: **a**   3 and 4   **b**   3, 4 and 8

a   The multiples of 3 are: 3, 6, 9, 12, 15, 18, 21, 24, 27, ...
        The multiples of 4 are: 4, 8, 12, 16, 20, 24, 28, 32, ...
        ∴   the common multiples of 3 and 4 are: 12, 24, 36, ...   of which 12 is the LCM.
b   The multiples of 3 are: 3, 6, 9, 12, 15, 18, 21, 24, 27, ...
        The multiples of 4 are: 4, 8, 12, 16, 20, 24, 28, 32, ...
        The multiples of 8 are: 8, 16, 24, 32, 40, ...
        So, the LCM is 24.

EXERCISE C

1 List all the factors of:

    a   10   **b**   16   **c**   18   **d**   17
    e   36   **f**   42   **g**   64   **h**   100

2 List the first *five* multiples of:

    a   6   **b**   11   **c**   13   **d**   29

3 Find the HCF of:

    a   3, 12   **b**   8, 12   **c**   18, 24   **d**   13, 52
    e   3, 5, 6   **f**   15, 20, 30   **g**   27, 36, 45   **h**   24, 48, 120

4 Find the largest multiple of:

    a   11 which is less than 200   **b**   17 which is less than 500.

5 Find the smallest multiple of:

    a   9 which is more than 300   **b**   23 which is more than 8000.

6 Find the LCM of:

    a   3, 8   **b**   6, 8   **c**   14, 21   **d**   9, 24
    e   2, 3, 5   **f**   2, 4, 7   **g**   3, 5, 10   **h**   9, 12, 18

7 Find the highest common multiple of:

    a   9 and 12 that is less than 40   **b**   5 and 15 that is less than 80.

8 Three bells chime once at intervals of 3, 5 and 12 seconds respectively. They first chime at exactly the same instant. After how many seconds will all three again chime simultaneously?

9 If a and b are positive integers, the LCM of a and b can be found by multiplying a and b and then dividing the result by their HCF. Check your answers to **6** using this method.

D FRACTIONS

A **common fraction** consists of two whole numbers, a **numerator** and a **denominator**, separated by a bar symbol.

$$\dfrac{4}{5}$$ ← numerator
← bar (which also means *divide*)
← denominator

TYPES OF FRACTIONS

$\frac{4}{5}$ is a **proper fraction**	{the numerator is less than the denominator}
$\frac{7}{6}$ is an **improper fraction**	{the numerator is greater than the denominator}
$2\frac{3}{4}$ is a **mixed number**	{this really means $2 + \frac{3}{4}$}
$\frac{1}{2}, \frac{3}{6}$ are **equivalent fractions**	{both fractions represent equivalent portions}

LOWEST COMMON DENOMINATOR

The **lowest common denominator (LCD)** of two or more numerical fractions is the lowest common multiple of their denominators.

ADDITION AND SUBTRACTION

To **add** (or **subtract**) two fractions we convert them to equivalent fractions with a common denominator. We then add (or subtract) the new numerators.

Example 12 ◀) *Self Tutor*

Find: $\frac{3}{4} + \frac{5}{6}$

$\quad \frac{3}{4} + \frac{5}{6}$ {LCD = 12}

$= \frac{3\times3}{4\times3} + \frac{5\times2}{6\times2}$ {to achieve a common denominator of 12}

$= \frac{9}{12} + \frac{10}{12}$

$= \frac{19}{12}$

$= 1\frac{7}{12}$

> We can multiply both the numerator and denominator of a fraction by the same number to generate an equivalent fraction.

Example 13 ◀) *Self Tutor*

Find: $1\frac{2}{3} - 1\frac{2}{5}$

$\quad 1\frac{2}{3} - 1\frac{2}{5}$

$= \frac{5}{3} - \frac{7}{5}$ {write as improper fractions}

$= \frac{5\times5}{3\times5} - \frac{7\times3}{5\times3}$ {to achieve a common denominator of 15}

$= \frac{25}{15} - \frac{21}{15}$

$= \frac{4}{15}$

MULTIPLICATION

To **multiply** two fractions, we first cancel any common factors in the numerator and denominator. We then multiply the numerators together and the denominators together.

Example 14   ◀》 *Self Tutor*

Find: **a** $\frac{1}{4} \times \frac{2}{3}$ **b** $(3\frac{1}{2})^2$

a $\frac{1}{4} \times \frac{2}{3}$

$= \frac{1}{\underset{2}{4}} \times \frac{\overset{1}{2}}{3}$

$= \frac{1}{6}$

b $(3\frac{1}{2})^2$

$= 3\frac{1}{2} \times 3\frac{1}{2}$

$= \frac{7}{2} \times \frac{7}{2}$

$= \frac{49}{4}$ or $12\frac{1}{4}$

> Cancel any common factors before completing the multiplication.

DIVISION

To **divide** by a fraction, we multiply the number by the reciprocal of the fraction we are dividing by.

Example 15   ◀》 *Self Tutor*

Find: **a** $3 \div \frac{2}{3}$ **b** $2\frac{1}{3} \div \frac{2}{3}$

a $3 \div \frac{2}{3}$

$= \frac{3}{1} \div \frac{2}{3}$

$= \frac{3}{1} \times \frac{3}{2}$

$= \frac{9}{2}$

$= 4\frac{1}{2}$

b $2\frac{1}{3} \div \frac{2}{3}$

$= \frac{7}{3} \div \frac{2}{3}$

$= \frac{7}{\underset{1}{3}} \times \frac{\overset{1}{3}}{2}$

$= \frac{7}{2}$

$= 3\frac{1}{2}$

> The reciprocal of $\frac{c}{d}$ is $\frac{d}{c}$.

EXERCISE D

1 Find:

   a $\frac{5}{13} + \frac{7}{13}$ **b** $\frac{9}{16} + \frac{2}{16}$ **c** $\frac{3}{8} + \frac{1}{4}$ **d** $\frac{2}{5} + \frac{1}{6}$

   e $\frac{3}{7} + 4$ **f** $1\frac{1}{3} + \frac{5}{6}$ **g** $2\frac{1}{3} + 1\frac{1}{6}$ **h** $1\frac{1}{2} + 4\frac{2}{3}$

2 Find:

   a $\frac{7}{11} - \frac{3}{11}$ **b** $\frac{5}{6} - \frac{2}{3}$ **c** $\frac{4}{9} - \frac{1}{3}$ **d** $1 - \frac{3}{8}$

   e $4 - 2\frac{1}{4}$ **f** $2\frac{3}{5} - 1\frac{1}{2}$ **g** $3\frac{1}{3} - 1\frac{1}{2}$ **h** $4\frac{3}{7} - 2\frac{1}{3}$

3 Calculate:

   a $\frac{2}{15} \times \frac{5}{6}$ **b** $\frac{6}{7} \times \frac{1}{3}$ **c** $3 \times \frac{3}{6}$ **d** $\frac{2}{3} \times 7$

   e $1\frac{1}{3} \times \frac{6}{7}$ **f** $1\frac{1}{8} \times \frac{4}{9}$ **g** $(2\frac{1}{2})^2$ **h** $(1\frac{1}{3})^3$

4 Evaluate:

a $\frac{3}{5} \div \frac{7}{10}$ **b** $\frac{3}{8} \div \frac{6}{11}$ **c** $\frac{4}{15} \div \frac{2}{5}$ **d** $\frac{2}{5} \div 4$

e $1 \div \frac{3}{5}$ **f** $1\frac{1}{3} \div \frac{3}{8}$ **g** $\frac{2}{3} \div 1\frac{1}{2}$ **h** $2\frac{1}{4} \div 1\frac{2}{3}$

5 Calculate:

a $3\frac{3}{7} + 1\frac{4}{5}$ **b** $4\frac{1}{2} - 2\frac{2}{3}$ **c** $7 - 6 \times \frac{3}{4}$ **d** $\frac{4}{5} \times 1\frac{1}{2} \div 3$

e $\dfrac{8 \times 3 \times \frac{1}{3}}{\frac{2}{3}}$ **f** $1 \div (\frac{1}{4} + \frac{2}{3})$ **g** $1 \div \frac{1}{4} + \frac{2}{3}$ **h** $\dfrac{3 - \frac{1}{2}}{3 \times \frac{5}{3}}$

i $\frac{2}{3} + \frac{1}{3} \times 1\frac{1}{2}$ **j** $\frac{5}{6} \times \frac{4}{5} - \frac{1}{15}$ **k** $\frac{1}{3} + \frac{1}{3} \div \frac{1}{5} + \frac{3}{5}$ **l** $1\frac{1}{2} - 2\frac{1}{3} \div 1\frac{2}{3}$

m $12 - \frac{2}{7} \times 3\frac{1}{2}$ **n** $1\frac{1}{3} + \frac{5}{6} - \frac{11}{12}$ **o** $6\frac{2}{5} - \frac{1}{4} \times 1\frac{1}{3} \div \frac{1}{6}$

Example 16   ◀)) Self Tutor

Anna scores $\frac{3}{5}$ of her team's goals in a netball match.

How many goals did she shoot if the team shot 70 goals?

Anna shot $\frac{3}{5}$ of $70 = \frac{3}{5} \times 70$

$\qquad = \dfrac{3 \times \overset{14}{\cancel{70}}}{\underset{1}{\cancel{5}}}$

$\qquad = 42$ goals.

> The word 'of' means '×'

6 **a** During a full season a football team scored 84 goals. If William scored $\frac{5}{12}$ of the goals, how many goals did William score?

b $\frac{2}{7}$ of my weekly earnings must be paid as tax to the government. If I earn £840.00 in one week, how much tax needs to be paid?

c Jade drinks $\frac{1}{4}$ of a full bottle of water. An hour later she drinks $\frac{1}{6}$ of the original volume. What fraction remains?

d In the first half of a basketball game one team scored 45 points. This was $\frac{5}{9}$ of their score for the whole game. How many points did the team score in the second half?

e Sally was paid €1830.00 as an end of year bonus for doing well at her job. She spent $\frac{1}{3}$ of it on clothing, $\frac{1}{4}$ on sporting goods, and $\frac{1}{12}$ on jewellery.

　i What fraction did she spend?　　**ii** How much of the money was left?

f Carli ate $\frac{1}{4}$ of a chocolate bar and later ate $\frac{2}{3}$ of what remained. What fraction of the original bar does she have left?

g $\frac{3}{7}$ of Fred's fortune is ¥3 240 000. How much money does Fred have in total?

FRACTIONS ON A CALCULATOR

Most scientific calculators and the **Casio fx-9860G** have a fraction key $\boxed{a\,b/c}$ that is used to enter common fractions.

There is no such key for the **TI-84 Plus**, so fractions need to be entered differently. You should consult the calculator section on page **12**.

Remember that although you may perform operations on fractions using your calculator, you **must not rely** on your calculator and forget how to manually perform operations with fractions.

Example 17   ◀ *Self Tutor*

Find, using your calculator: **a** $\frac{1}{4} - \frac{2}{3}$ **b** $1\frac{1}{12} + 2\frac{1}{4}$ **c** $\frac{1}{4} \div \frac{2}{3}$

Note the solution given is for a scientific calculator or the **Casio fx-9860G**.

a $\frac{1}{4} - \frac{2}{3}$ Key in   1 $\boxed{a^{b/c}}$ 4 $\boxed{-}$ 2 $\boxed{a^{b/c}}$ 3 $\boxed{\text{EXE}}$

Display
```
1⌐4-2⌐3
              -5⌐12
```
Answer: $-\frac{5}{12}$

b $1\frac{1}{12} + 2\frac{1}{4}$ Key in   1 $\boxed{a^{b/c}}$ 1 $\boxed{a^{b/c}}$ 12 $\boxed{+}$ 2 $\boxed{a^{b/c}}$ 1 $\boxed{a^{b/c}}$ 4 $\boxed{\text{EXE}}$ $\boxed{\text{SHIFT}}$ $\boxed{\text{F↔D}}$

Display
```
1⌐1⌐12+2⌐1⌐4
              3⌐1⌐3
```
Answer: $3\frac{1}{3}$

c $\frac{1}{4} \div \frac{2}{3}$ Key in   1 $\boxed{a^{b/c}}$ 4 $\boxed{\div}$ 2 $\boxed{a^{b/c}}$ 3 $\boxed{\text{EXE}}$

Display
```
1⌐4÷2⌐3
               3⌐8
```
Answer: $\frac{3}{8}$

---

7   Find, using your calculator:

a $\frac{1}{5} + \frac{1}{3}$ **b** $\frac{1}{3} + \frac{2}{7}$ **c** $\frac{5}{8} - \frac{2}{7}$ **d** $\frac{3}{4} - \frac{1}{3}$

e $\frac{2}{5} \times \frac{3}{2}$ **f** $\frac{4}{7} \times \frac{2}{3}$ **g** $\frac{6}{7} \times \frac{2}{5}$ **h** $\frac{3}{5} \div \frac{2}{9}$

i $2\frac{1}{4} + 2\frac{1}{2} \times \frac{3}{4}$ **j** $1\frac{3}{7} \times 2\frac{1}{8} + 1\frac{1}{7}$ **k** $2\frac{3}{7} \div (2\frac{3}{4} \times \frac{4}{7})$

E   **POWERS AND ROOTS**   **[1.4]**

Rather than writing $3 \times 3 \times 3 \times 3$, we can write such a product as 3^4.

We call this power or **index notation**.

3^4 reads "three to the power of four" or "three to the fourth".

Since $2^2 = 4$, we write $\sqrt{4} = 2$ where $\sqrt{4}$ reads "the square root of 4".

Also, since $2^3 = 8$ we write $\sqrt[3]{8} = 2$ where $\sqrt[3]{8}$ reads "the cube root of 8".

$$3^4$$

↞ power, index or exponent

↞ base

In general,   if $a^n = b$ then $\sqrt[n]{b} = a$.

For example, $2^7 = 128$ so $\sqrt[7]{128} = 2$.

EXERCISE E

The powers of 2, 3, 4 and 5 are worth memorising.

1 Write down:

    a the first 8 powers of 2

    b the first 5 powers of 3

    c the first 4 powers of 4

    d the first 4 powers of 5.

2 Find without using a calculator:

    a $\sqrt{25}$ **b** $\sqrt[3]{8}$ **c** $\sqrt[5]{32}$ **d** $\sqrt[3]{27}$ **e** $\sqrt[4]{81}$

    f $\sqrt[3]{125}$ **g** $\sqrt[3]{64}$ **h** $\sqrt[4]{625}$ **i** $\sqrt[3]{1000}$ **j** $\sqrt{2\frac{1}{4}}$

3 Use your calculator to find the exact value of:

    a $\sqrt[5]{1024}$ **b** 7^4 **c** 4^7 **d** $\sqrt[3]{729}$

    e $\sqrt{6.76}$ **f** $\sqrt{16.81}$ **g** $(0.83)^3$ **h** $\sqrt{0.5041}$

    i $\sqrt[3]{1.157\,625}$ **j** $\sqrt{7163.9296}$ **k** $(1.04)^4$ **l** $(2.3)^5$

F   RATIO AND PROPORTION   [1.5]

A **ratio** is a way of comparing two quantities.

If we have 6 apples and 4 bananas, the ratio of the number of apples to the number of bananas is 6 to 4.

We write this as   apples : bananas $= 6 : 4$

Notice that   bananas : apples $= 4 : 6$

If measurements are involved we must use the **same units** for each quantity.

For example, the ratio of lengths shown is
$20 : 7$ $\{20$ mm : 7 mm$\}$ and not $2 : 7$.

                                   2 cm   7 mm

Example 18   ◀ঠ Self Tutor

Find the ratio of the number of squares to the number of triangles.

          number of squares : number of triangles

          $= 8 : 11$

> ### *Example 19*   ◆)) *Self Tutor*
>
> Write as a ratio, without simplifying your answer:
>   **a** Jack has \$5 and Jill has 50 cents.
>   **b** Mix 200 ml of cordial with 1 litre of water.
>
> | | | |
> |---|---|---|
> | **a** | Jack : Jill $= \$5 : 50$ cents | {write in the correct order} |
> | | $= 500$ cents : 50 cents | {write in the same units} |
> | | $= 500 : 50$ | {express without units} |
> | **b** | cordial : water $= 200$ ml : 1 litre | {write in the correct order} |
> | | $= 200$ ml : 1000 ml | {write in the same units} |
> | | $= 200 : 1000$ | {express without units} |

EXERCISE F.1

1 **a** Find the ratio of squares to triangles in:
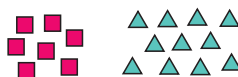
  b Find the ratio of ♣s to ♠s in:

  c
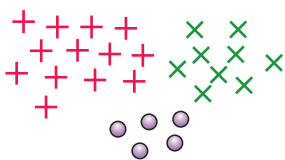
Find the ratio of:
  i + signs to × signs
  ii × signs to + signs
  iii ○s to × signs
  iv × signs to ○s.

2 Write as a ratio, without simplifying your answer:
  a \$8 is to \$3   **b** 3 litres is to 7 litres   **c** 35 kg is to 45 kg
  d £3 is to 50 pence   **e** 500 ml is to 3 litres   **f** 400 m is to 2.5 km
  g 9 km is to 150 m   **h** 12 m is to 8 km   **i** 4 h is to 40 min

SIMPLIFYING RATIOS

If we have 6 apples and 4 bananas, we have 3 apples for every 2 bananas.

So, $6 : 4$ is the same as $3 : 2$.

We say that $6 : 4$ and $3 : 2$ are **equal ratios**.

Notice that to get from $6 : 4$ to $3 : 2$ we can divide each number in the first ratio by 2.

Also, to get from $3 : 2$ to $6 : 4$ we can multiply each number in the first ratio by 2.

> To **simplify a ratio** we can multiply or divide each part by the same non-zero number.

Example 20 ◀》 Self Tutor

Express $45 : 10$ in simplest form.

$45 : 10$
$= 45 \div 5 : 10 \div 5$ {5 is the HCF of 45 and 10}
$= 9 : 2$

Example 21 ◀》 Self Tutor

Express in simplest form:

a $0.4 : 1.4$ **b** $2\frac{1}{2} : \frac{1}{2}$

a $0.4 : 1.4$
$= 0.4 \times 10 : 1.4 \times 10$
$= 4 : 14$
$= 4 \div 2 : 14 \div 2$
$= 2 : 7$

b $2\frac{1}{2} : \frac{1}{2}$
$= \frac{5}{2} : \frac{1}{2}$
$= \frac{5}{2} \times 2 : \frac{1}{2} \times 2$
$= 5 : 1$

EXERCISE F.2

1 Express as a ratio in simplest form:

a $6 : 8$	**b** $8 : 4$	**c** $4 : 12$	**d** $9 : 15$
e $3 : 6$	**f** $14 : 8$	**g** $8 : 16$	**h** $18 : 24$
i $125 : 100$	**j** $2 : 4 : 6$	**k** $1000 : 50$	**l** $6 : 12 : 24$

2 Express as a ratio in simplest form:

a $0.5 : 0.2$	**b** $0.3 : 0.7$	**c** $0.6 : 0.4$	**d** $0.4 : 0.2$
e $0.7 : 1.2$	**f** $0.03 : 0.12$	**g** $2 : 0.5$	**h** $0.05 : 1$

3 Express as a ratio in simplest form:

a $\frac{1}{3} : \frac{2}{3}$	**b** $\frac{3}{4} : \frac{1}{4}$	**c** $1\frac{1}{2} : \frac{1}{2}$	**d** $1\frac{1}{2} : 2\frac{1}{2}$
e $\frac{1}{3} : \frac{1}{2}$	**f** $1\frac{1}{3} : 2\frac{1}{2}$	**g** $\frac{2}{3} : 3\frac{1}{4}$	**h** $1\frac{3}{10} : 1\frac{2}{3}$
i $1\frac{1}{3} : \frac{2}{3}$	**j** $\frac{3}{4} : 1\frac{1}{2}$	**k** $6 : 1\frac{1}{2}$	**l** $\frac{1}{2} : \frac{1}{3} : \frac{1}{4}$

4 Write the following comparisons as ratios in simplest form. Compare the first quantity mentioned with the second quantity mentioned.

a a shirt costing €64 to another shirt costing €32

b a rockmelon of mass 3 kg to a watermelon of mass 9 kg

c the height of a 1.5 m shrub to the height of a 6 m tree

d a wetsuit costing \$175 to a wetsuit costing \$250

e the top speed of a car which is 150 km/h to the top speed of a formula one car which is 350 km/h

f a log of length 4 m to a stick of length 10 cm

g the height of an insect which is 2 mm to the height of a rat which is 10 cm.

EQUAL RATIOS

Ratios are **equal** if they can be expressed in the same simplest form.

For example, $6:4$ and $9:6$ are equal as they are both expressed in simplest form as $3:2$.

A **proportion** is a statement that two ratios are equal.

For example, $6:4 = 9:6$ is a proportion.

Sometimes we need to find one quantity given the ratio and the other quantity. To do this we use equal ratios.

Example 22   ◀) *Self Tutor*

Find x if: **a** $3:5 = 6:x$ **b** $15:20 = x:16$

a $3:5 = 6:x$ **b** $15:20 = 15 \div 5 : 20 \div 5$

$\therefore \quad x = 5 \times 2$ $= 3:4$

$\therefore \quad x = 10$ $\therefore \quad 3:4 = x:16$

$\therefore \quad x = 3 \times 4 = 12$

Example 23   ◀) *Self Tutor*

The ratio of walkers to guides on the Milford Track walk was $9:2$. How many guides were needed if there were 27 walkers?

Let the number of guides be x.   walkers : guides $= 27 : x$

$\therefore \quad 9:2 = 27:x$

$\therefore \quad x = 2 \times 3$

$\therefore \quad x = 6$

$\therefore$ 6 guides were needed.

EXERCISE F.3

1 Find x if:

   a $2:3 = 8:x$ **b** $1:4 = x:12$ **c** $3:2 = 15:x$

   d $4:3 = x:21$ **e** $5:7 = 25:x$ **f** $6:11 = x:77$

   g $5:12 = 40:x$ **h** $7:10 = x:80$ **i** $4:5 = x:45$

2 Find x if:

   a $4:5 = 12:x$ **b** $3:9 = x:18$ **c** $2:3 = 10:x$

   d $5:10 = x:18$ **e** $16:4 = 12:x$ **f** $21:28 = 12:x$

3 A recipe for tomato soup uses tomatoes and onions in the ratio $7 : 2$. If 21 kg of tomatoes are used, how many kilograms of onions are needed?

4 An orchard has apple trees and pear trees in the ratio $5 : 3$. If there are 180 pear trees, how many apple trees are there?

5 A car cleaning service increases the cost of a \$15 'standard clean' to \$18. The \$25 'deluxe clean' is increased in the same ratio as the 'standard'. How much does a 'deluxe clean' cost now?

6 Concrete is mixed in a ratio of premix to cement of $6 : 1$. If I have 540 kg of premix, how much cement do I need?

7 The mass of two bags is in the ratio $7 : 12$. The bigger bag has a mass of 48 kg.

   a Find the mass of the smaller bag.   **b** Find the combined mass of the bags.

8 **a** Increase \$12 in the ratio: **i** $5 : 4$ **ii** $2 : 1$ **iii** $15 : 2$

    b Decrease 200 kg in the ratio: **i** $3 : 4$ **ii** $4 : 5$ **iii** $1 : 20$
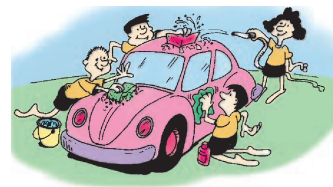
THE UNITARY METHOD FOR RATIOS

Some ratio problems are easily handled using the **unitary method**.

> The unitary method works well with harder ratio problems.

Consider **Example 23** where   walkers : guides $= 27 : x$

                  i.e., $9 : 2 = 27 : x$

The unitary method is: 9 parts is   27

        $\therefore$ 1 part is $27 \div 9 = 3$

        $\therefore$ 2 parts is $3 \times 2 = 6$

        $\therefore$ 6 guides were needed.

Example 24   🔊 *Self Tutor*

The ratio of Pam's height to Sam's height is $7 : 6$. If Pam is 1.63 m tall, how tall is Sam?

Let Sam's height be x m.	Let Sam's height be x m.

Pam : Sam $= 7 : 6$ **or**   Pam : Sam $= 7 : 6$

$\therefore$ $1.63 : x = 7 : 6$ $\therefore$ $1.63 : x = 7 : 6$

So,   7 parts is 1.63 m $\therefore$ $\dfrac{x}{1.63} = \dfrac{6}{7}$

$\therefore$ 1 part is $1.63 \div 7$ m $\therefore$ $x = \frac{6}{7} \times 1.63$

$\therefore$ 6 parts is $1.63 \div 7 \times 6$ m $\therefore$ $x \approx 1.40$

$\therefore$ Sam's height is ≈ 1.40 m. $\therefore$ Sam's height is ≈ 1.40 m.

(Page included on the Student CD only)

Example 25   ◆)) *Self Tutor*

In one year, Maria's height increases from 150 cm in the ratio 31 : 30. Find Maria's new height.

The second number in the ratio represents the original quantity. The first number represents the final quantity.

$$\text{Maria's original height} = 30 \text{ parts} = 150 \text{ cm}$$
$$\therefore \quad 1 \text{ part} = 5 \text{ cm}$$
$$\therefore \quad \text{Maria's new height} = 31 \text{ parts}$$
$$= 31 \times 5 \text{ cm}$$
$$= 155 \text{ cm}$$

Example 26   ◆)) *Self Tutor*

On a particular day, 1 Australian dollar is worth 0.94 US dollars. How many Australian dollars can be exchanged for $20 US?

Suppose $20 US is worth x Australian dollars.

$$\therefore \quad 1 : 0.94 = x : 20$$
$$\therefore \quad 100 : 94 = x : 20$$
$$\text{If } 94 \text{ parts} = 20$$
$$\text{then } 1 \text{ part} = \tfrac{20}{94}$$
$$\therefore \quad 100 \text{ parts} = \tfrac{20}{94} \times 100 \approx 21.28$$

$\therefore$ $20 US can be exchanged for $21.28 AUD.

EXERCISE F.4

Use the *unitary method* to solve these problems:

1 The ratio of Bob's weight to Colin's weight is 6 : 7. If Bob weighs 83.7 kg, how much does Colin weigh?

2 If Kayo and Sally split their profits in the ratio of 5 : 4 respectively and Kayo gets $23 672, how much does Sally get, to the nearest dollar?

3 Jack's lawn is on average 8.3 cm high. The ratio of height of Jack's lawn to Henri's lawn is 1 : 1.13. Find the average height of Henri's lawn.

4 The ratio of water to alcohol in a bottle of wine is 15 : 2. If there are 662 ml of water in the bottle, what is the quantity of alcohol in the bottle?

5 On a particular day, one British pound is worth 1.26 euros. How many pounds can be exchanged for 50 euros?

6 The map alongside has a scale of $1 : 500\,000$, which means that 1 cm on the map represents $500\,000$ cm in real life.

    a If the distance $BC = 2.1$ cm on the map, find the actual distance between B and C.

    b If E and C are 13.5 km apart, find the length of EC on the map.

7 A bakery decides to donate 5 cents to charity for every $1.95 loaf of bread bought. If the bakery sells $3217.50 worth of bread, how much is donated to charity?

Scale: $1 : 500\,000$

8 A quantity is increased in the ratio $5 : 4$, and then decreased in the ratio $3 : 4$. Find, in simplest form, the ratio of the final quantity to the original quantity.
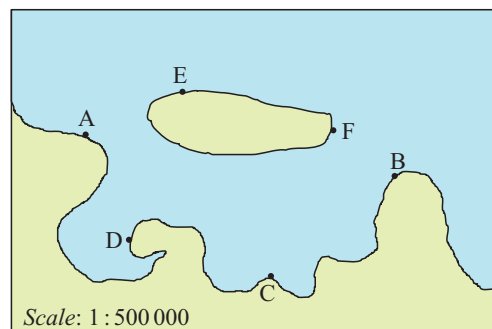
USING RATIOS TO DIVIDE QUANTITIES

Quantities can be divided in a particular ratio by considering the **number of parts** the whole is divided into.

Example 27   ◀ **Self Tutor**

An inheritance of $60\,000$ is to be divided between Donny and Marie in the ratio $2 : 3$. How much does each receive?

There are $2 + 3 = 5$ parts.

$\therefore$ Donny gets $\frac{2}{5}$ of $60\,000$ and Marie gets $\frac{3}{5}$ of $60\,000$

$\qquad\qquad\quad = \frac{2}{5} \times 60\,000 \qquad\qquad\qquad\qquad = \frac{3}{5} \times 60\,000$

$\qquad\qquad\quad = \$24\,000 \qquad\qquad\qquad\qquad\quad\ = \$36\,000$

EXERCISE F.5

1 What is the total number of parts represented by the following ratios?

    a $2 : 3$ **b** $4 : 1$ **c** $7 : 9$ **d** $12 : 5$

    e $10 : 3$ **f** $3 : 16$ **g** $7 : 4$ **h** $9 : 10$

2 Divide a 50 cm piece of string in the following ratios:

    a $1 : 1$ **b** $4 : 1$ **c** $3 : 2$ **d** $7 : 13$

3 Divide:

    a \$50 in the ratio $1 : 4$ **b** €35 in the ratio $3 : 4$ **c** 90 kg in the ratio $4 : 5$

4 Lottery winnings of $400\,000$ are to be divided in the ratio $5 : 3$. Find the larger share.

5 The ratio of girls to boys in a school is $5 : 4$. If there are 918 students at the school, how many are girls?

6 A man leaves $200\,000$ euros to his sons Aleksi and Kristo in the ratio of their ages when he dies. Aleksi is 4 years older than Kristo. When the father dies, Aleksi is 62.

    a How old is Kristo?   **b** How much does Aleksi inherit (to the nearest euro)?

    c How much does Kristo inherit (to the nearest euro)?

7 Divide an inheritance of £36 000 in the ratio 3 : 5 : 10.

8 The ratio of flour : sugar : cocoa in a cake recipe is 2 : 1.5 : 0.5. If 10 kg of flour is used, how much:

    a sugar   **b** cocoa is used?

9 At the moorings on a river, there are yachts, motorboats, houseboats and rowboats.
The ratio of yachts : motorboats : houseboats is 4 : 5 : 3.
If there are 50 watercraft on the river and two of them are rowboats, how many are:

    a yachts   **b** motorboats   **c** houseboats?
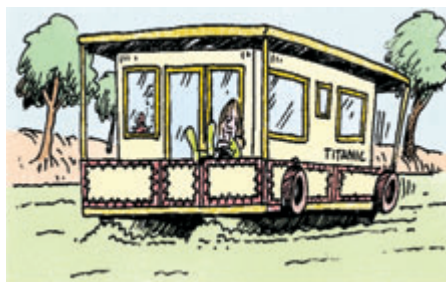
G NUMBER EQUIVALENTS   [1.7]

Consider the decimal number 0.75.

We can write it as a fraction, since $0.75 = \frac{75}{100} = \frac{3}{4}$.

We can also write it as the ratio 3 : 4, or as a percentage since $0.75 \times 100\% = 75\%$.

So, 0.75, $\frac{3}{4}$, 3 : 4 and 75% are all **equivalent**.

It is important that you can convert between these different forms of numbers.

Example 28   ◀)) *Self Tutor*

Copy and complete:

	Decimal	Fraction	Ratio	Percentage
a		$\frac{2}{5}$		
b			3 : 2	
c				68%
d	0.875			

a $\frac{2}{5} = \frac{4}{10} = 0.4$ **b** 3 : 2 is the fraction $\frac{3}{2}$ **c** $68\% = \frac{68}{100} = 0.68$

    $\frac{2}{5}$ is the ratio 2 : 5 $\frac{3}{2} = 3 \div 2 = 1.5$ $\frac{68}{100} = \frac{17}{25}$

    $\frac{2}{5} \times 100\% = 40\%$ $\frac{3}{2} \times 100\% = 150\%$ $\frac{17}{25}$ is the ratio 17 : 25

d $0.875 = \frac{875}{1000} = \frac{7}{8}$

    $\frac{7}{8}$ is the ratio 7 : 8

    $\frac{875}{1000} = \frac{87.5}{100} = 87.5\%$

	Decimal	Fraction	Ratio	Percentage
a	0.4	$\frac{2}{5}$	2 : 5	40%
b	1.5	$\frac{3}{2}$	3 : 2	150%
c	0.68	$\frac{17}{25}$	17 : 25	68%
d	0.875	$\frac{7}{8}$	7 : 8	87.5%

EXERCISE G

1 Convert:

    a 0.65 into an equivalent fraction, percentage and ratio

    b 60% into an equivalent fraction, decimal and ratio

    c 2 : 5 into an equivalent fraction, decimal and percentage

    d $\frac{5}{8}$ into an equivalent decimal, ratio and percentage.

2 Copy and complete:

Decimal	Fraction	Ratio	Percentage
	$\frac{1}{4}$		
			85%
0.36			
		4 : 5	
$0.\overline{2}$			
		5 : 4	
			200%
	$5\frac{1}{2}$		

H   ROUNDING NUMBERS   [1.11]

You are expected to be able to round numbers to various levels of accuracy.

These include:
- to a certain number of **decimal places**
- to a certain number of **significant figures**
- to the **nearest integer** or **whole number**.

DECIMAL PLACE ROUNDING

In many situations we may be given a measurement as a decimal number. Stating the *exact* value of the measurement may not be particularly important; what we want is a good *approximation* of the measurement.

For example, since 1924 the Olympic marathon has been measured as exactly 42.195 km or 26.2187 miles. The exact value is rarely quoted, however, since most people use approximations; they commonly say 42 km, 42.2 km, 26 miles, or 26.2 miles.
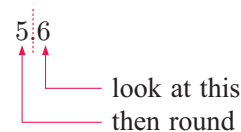
RULES FOR ROUNDING

- **Rounding to the nearest whole number**
  Look at the *first* decimal place.
  If the digit is   5, 6, 7, 8 or 9,   round **up**.
  If the digit is   0, 1, 2, 3 or 4,   round **down**.

5.6

look at this
then round

- **Rounding to the nearest one decimal place**
  Look at the *second* decimal place.
  If the digit is   5, 6, 7, 8 or 9,   round **up**.
  If the digit is   0, 1, 2, 3 or 4,   round **down**.

  5.64

  look at this
  then round
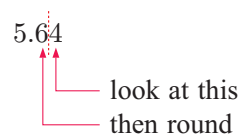
- **Rounding to the nearest two decimal places**
  Look at the *third* decimal place.
  If the digit is   5, 6, 7, 8 or 9,   round **up**.
  If the digit is   0, 1, 2, 3 or 4,   round **down**.

  5.648

  look at this
  then round

Example 29   ◀) *Self Tutor*
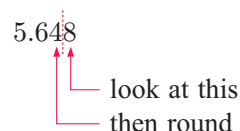
Round 39.748 to the nearest:

   a   whole number   **b**   one decimal place   **c**   two decimal places

   a $39.748 \approx 40$ to the nearest whole number
   b $39.748 \approx 39.7$ to one decimal place
   c $39.748 \approx 39.75$ to two decimal places

Notice that: $0.5464 \approx 0.546$ (to 3 decimal places)
                ≈ 0.55 (to 2 decimal places)
                ≈ 0.5 (to 1 decimal place)

EXERCISE H.1

1 Round to the nearest whole number:
   a   0.813   **b**   7.499   **c**   7.500   **d**   11.674   **e**   128.437

2 Write these numbers correct to 1 decimal place:
   a   2.43   **b**   3.57   **c**   4.92   **d**   6.38   **e**   4.275

3 Write these numbers correct to 2 decimal places:
   a   4.236   **b**   2.731   **c**   5.625   **d**   4.377   **e**   6.5237

4 Write 0.183 75 correct to:
   a   1 decimal place   **b**   2 decimal places   **c**   3 decimal places   **d**   4 decimal places

---

Example 30   ◀) *Self Tutor*

Find $\frac{2}{7}$ correct to 3 decimal places.

$$\begin{array}{r} 0 \, . \, 2 \ 8 \ 5 \ 7 \\ \hline 7 \, \big| \, 2 \, . \, 0 \, {}^{6}0 \, {}^{4}0 \, {}^{5}0 \end{array}$$

$\therefore \ \frac{2}{7} \approx 0.286$

---

5 Find the answer correct to the number of decimal places shown in square brackets:

a $\frac{17}{4}$ [1]   **b** $\frac{73}{8}$ [2]   **c** 4.3×2.6 [1]

d 0.12×0.4 [2]   **e** $\frac{8}{11}$ [2]   **f** 0.08×0.31 [3]

g $(0.7)^2$ [1]   **h** $\frac{37}{6}$ [2]   **i** $\frac{17}{7}$ [3]

6 In her maths exam Julie was asked to round 7.45 cm to one decimal place and to the nearest whole integer.

Julie's answer was that 7.45 cm ≈ 7.5 cm (to one decimal place), and that 7.5 cm ≈ 8 cm (to the nearest whole number).

Explain what Julie has done wrong and why we need to be careful when we make approximations.

SIGNIFICANT FIGURE ROUNDING

The first **significant figure** of a decimal number is the first (left-most) non-zero digit.

For example:
- the first significant figure of 1234 is 1
- the first significant figure of 0.023 45 is 2.

Every digit to the right of the first significant figure is regarded as another significant figure.

Procedure for rounding off to significant figures:

> Count off the specified number of significant figures then look at the next digit.
>
> - If the digit is less than 5, do not change the last significant figure.
> - If the digit is 5 or more then increase the last significant figure by 1.
>
> Delete all figures following the significant figures, replacing with 0s where necessary.

Notice that if 13.238 is rounded to 13.24, then it has been rounded to 2 decimal places or to 4 significant figures.

Example 31   ◀ৡ *Self Tutor*

Round: **a** 5.371 to 2 significant figures   **b** 0.0086 to 1 significant figure
       c 423 to 1 significant figure

a $5.371 \approx 5.4$ (2 s.f.)

This is the 2nd significant figure, so we look at the next digit which is 7.
The 7 tells us to round the 3 to a 4 and leave off the remaining digits.

b $0.0086 \approx 0.009$ (1 s.f.)

These zeros at the front are place holders and so must stay. The first significant figure is the 8.
The next digit is 6, which tells us to round the 8 to a 9 and leave off the remaining digits.

c $423 \approx 400$ (1 s.f.)

4 is the first significant figure so it has to be rounded. The second digit, 2, tells us to keep the original 4. We convert the 23 into 00. These two zeros are place holders; they are not 'significant figures' but need to be there to make sure the 4 still represents 4 hundreds.

EXERCISE H.2

1 Round correct to the number of significant figures shown in brackets.

a	42.3	[2]	**b**	6.237	[3]	**c**	0.0462	[2]
d	0.2461	[2]	**e**	437	[2]	**f**	2064	[2]
g	31 009	[3]	**h**	10.27	[3]	**i**	0.999	[1]
j	0.999	[2]	**k**	264 003	[4]	**l**	0.037 642	[4]
m	3699.231	[4]	**n**	0.007 639	[2]	**o**	29 999	[3]
p	69.7003	[2]						

2 The crowd at a football match was officially 26 247 people.

   a Round the crowd size to:

      i 1 significant figure

      ii 2 significant figures.

   b Which of these figures might be used by the media to indicate crowd size?

3 The newspaper stated that 2500 people attended a protest march in Paris.
If this figure had been rounded to two significant figures, what was the largest number of people that could have attended the protest?

4 During a rabbit plague there were 132 709 rabbits in South Australia. What figure would you expect to see in a newspaper headline for an article on these rabbits?

ONE FIGURE APPROXIMATIONS

A fast way of estimating a calculation is to perform a **one figure approximation**. We round each number in the calculation to one significant figure, then perform the calculation with these approximations.

Rules:
- Leave single digit numbers as they are.
- Round all other numbers to one figure approximations.
- Perform the calculation.

For example, 3785×7
$$\approx 4000 \times 7$$
$$\approx 28\,000$$

Example 32   🔊 *Self Tutor*

Estimate the product: **a** 57×8 **b** 537×6

  a Round off to the nearest 10.
$$57 \times 8$$
$$\approx 60 \times 8$$
$$\approx 480$$

  b Round off to the nearest 100.
$$537 \times 6$$
$$\approx 500 \times 6$$
$$\approx 3000$$

Example 33   ◄») Self Tutor

Estimate the product: 623×69

Round 623 to the nearest 100 and round 69 to the nearest 10.

$$623 \times 69$$
$$\approx 600 \times 70$$
$$\approx 42\,000$$

The estimate tells us the correct answer should have 5 digits in it.

Example 34   ◄») Self Tutor

Find the approximate value of $3946 \div 79$.

$$3946 \div 79 \approx 4000 \div 80$$
$$\approx 400 \div 8$$
$$\approx 50$$

EXERCISE H.3

1 Estimate the products:

   a 79×4 **b** 47×8 **c** 62×7

   d 494×6 **e** 817×8 **f** 2094×7

2 Estimate the products using one figure approximations:

   a 57×42 **b** 73×59 **c** 85×98

   d 275×54 **e** 389×73 **f** 4971×32

   g 3079×29 **h** $40\,989 \times 9$ **i** 880×750

3 Estimate using one figure approximations:

   a $397 \div 4$ **b** $6849 \div 7$ **c** $79\,095 \div 8$

   d $6000 \div 19$ **e** $80\,000 \div 37$ **f** $18\,700 \div 97$

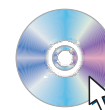   **g** $2780 \div 41$ **h** $48\,097 \div 243$ **i** $798\,450 \div 399$

DEMO

I   TIME   [1.12]

In early civilisations, time was measured by regular changes in the sky. The recurring period of daylight and darkness came to be called a **day**. The Babylonians divided the day into hours, minutes and seconds. Ancient astronomers found the time taken for the Earth to complete one orbit around the Sun. This became known as a **year**.

The base unit of time in the International System of Units is the **second**, abbreviated **s**.

UNITS OF TIME

1 **minute** = 60 seconds	1 **hour** = 60 minutes = 3600 seconds
1 **day** = 24 hours	1 **week** = 7 days
1 **year** = 12 months = 365 days	1 **decade** = 10 years
	1 **century** = 100 years
	1 **millennium** = 1000 years

Example 35   ◄)) Self Tutor

Convert 8 days 7 hours and 6 minutes to minutes.

$$
\begin{aligned}
8 \text{ days} \times 24 \times 60 = \quad & 11\,520 \text{ minutes} \\
7 \text{ hours} \times 60 = \quad & 420 \text{ minutes} \\
6 \text{ minutes} = \quad & 6 \text{ minutes} \\
\therefore \quad \text{total} = \quad & \overline{11\,946} \text{ minutes}
\end{aligned}
$$

Example 36   ◄)) Self Tutor

Convert   30 240 minutes to days.

$$
\begin{aligned}
30\,240 &= (30\,240 \div 60) \text{ hours} \qquad \{60 \text{ min in 1 hour}\} \\
&= 504 \text{ hours} \\
&= (504 \div 24) \text{ days} \qquad \{24 \text{ hours in 1 day}\} \\
&= 21 \text{ days}
\end{aligned}
$$

Example 37   ◄)) Self Tutor

Convert   3 hours, 14 minutes to seconds.

$$
\begin{aligned}
3 \text{ hours, 14 minutes} &= (3 \times 60) \text{ min} + 14 \text{ min} \qquad \{60 \text{ min in 1 hour}\} \\
&= 194 \text{ min} \\
&= (194 \times 60) \text{ s} \qquad \{60 \text{ s in 1 min}\} \\
&= 11\,640 \text{ s}
\end{aligned}
$$

EXERCISE 1.1

1 Convert the following times to minutes:

   a 5 hours   **b** 3 days   **c** 2 days 15 hours   **d** 2220 seconds

2 Convert the following times to days:

   a 1248 hours   **b** 23 040 min   **c** 3 years   **d** 6 hours

3 Convert the following times to seconds:

   a 35 minutes   **b** 3 hours 19 min   **c** 5 days   **d** 1 week 2 days

4 Calculate the following, expressing your answers in hours, minutes and seconds:

> The abbreviation for hours is **h** and for minutes is **min**.

   a 1 h 19 min + 2 h 42 min + 1 h 7 min

   b 4 h 51 min 16 s + 2 h 19 min 54 s

   c 12 h − 7 h 55 min

   d 5 h 23 min − 2 h 48 min

5 Xani has 6 science lessons a week, each of 45 minutes duration. Find the total time spent in science lessons in a twelve week term.

Example 38   ◀ﻪ *Self Tutor*

What is the time difference between 9.55 am and 1.25 pm?

$$9.55 \text{ am to } 10.00 \text{ am} = 5 \text{ min}$$
$$10.00 \text{ am to } 1.00 \text{ pm} = 3 \text{ h}$$
$$1.00 \text{ pm to } 1.25 \text{ pm} = 25 \text{ min}$$
$$\overline{3 \text{ h } 30 \text{ min}}$$

The time difference is 3 hours 30 minutes.

Example 39   ◀ﻪ *Self Tutor*

What is the time $3\frac{1}{2}$ hours after 10.40 am?

$$10.40 \text{ am} + 3\frac{1}{2} \text{ hours} = 10.40 \text{ am} + 3 \text{ h} + 30 \text{ min}$$
$$= 1.40 \text{ pm} + 30 \text{ min}$$
$$= 2.10 \text{ pm}$$

---

6 Find the time difference between:

   a 4.30 am and 6.55 am   **b** 10.08 am and 5.52 pm

   c 3.15 pm and 9.03 pm   **d** 7.54 am and 2.29 pm

7 Henry left home at 7.48 am and arrived at work at 9.02 am. How long did it take him to get to work?

8 Your time schedule shows you worked the following hours last week:

Monday	8.45 am - 5.05 pm
Tuesday	8.50 am - 5.10 pm
Wednesday	8.45 am - 4.55 pm
Thursday	8.30 am - 5.00 pm
Friday	8.35 am - 5.15 pm

   a How many hours did you work last week?

   b If you are paid €9.00 per hour, what was your income for the week?

9 Calculate the time:

    a 3 hours after 8.16 am   **b** 3 hours before 11.45 am

    c $5\frac{1}{2}$ hours after 10.15 am   **d** $3\frac{1}{2}$ hours before 1.18 pm

10 Boris caught a plane flight at 8.45 am. The flight was $6\frac{1}{2}$ hours. At what time did he arrive at his destination, assuming it was in the same time zone?

11 If a train is travelling at 36 m/s, how far will it travel in 1 hour? Give your answer in kilometres.

24-HOUR TIME

24-hour time is used by the armed forces and in train and airline timetables. It avoids the need for using **am** and **pm** to indicate morning and afternoon.

In 24-hour time, four digits are always used. For example:

- 0800 is 8.00 am

  "Oh eight hundred hours"

- 0000 is midnight

- 2359 is one minute before midnight

- 2000 is 8.00 pm

  "twenty hundred hours"

- 1200 is noon or midday

Morning times (am) are from midnight (0000) to midday (1200).

Afternoon times (pm) are from midday (1200) to midnight (0000).

Notice that: • midnight is 0000 not 2400

                • to convert afternoon 24-hour time to pm times we subtract 1200.

EXERCISE 1.2

1 Change to 24-hour time:

    a 9.57 am   **b** 11.06 am   **c** 4 o'clock pm

    d 2.25 pm   **e** 8 o'clock am   **f** 1.06 am

    g 8.58 pm   **h** noon   **i** 2 minutes past midnight

2 Change to am/pm time:

    a 1140   **b** 0346   **c** 1634   **d** 1900

    e 0800   **f** 2330   **g** 1223   **h** 2040

3 Copy and complete the following railway schedule:

	Departure	Travelling time	Arrival
a	0520	6 h 20 min	
b	0710		1405
c		56 min	1027
d		4 h 23 min	1652
e	2012		0447 (next day)

EXERCISE A

1

Number	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
19	✓	✓	✓	✓
$-\frac{2}{3}$	×	×	✓	✓
$\sqrt{7}$	×	×	×	✓
5.6389	×	×	✓	✓
$\sqrt{16}$	✓	✓	✓	✓
2π	×	×	×	✓
-11	×	✓	✓	✓
$\frac{6}{0}$	×	×	×	×
$\sqrt{-2}$	×	×	×	×

2 **a** $16 = 4^2$, $\quad 529 = 23^2$, $\quad 20\,802\,721 = 4561^2$

b $27 = 3^3$, $\quad 343 = 7^3$, $\quad 13\,824 = 24^3$

c $\quad \sqrt{11} \approx 3.32$

$11 \div 2 = 5.5$

$11 \div 3 = 3.\overline{6}$ $\quad$ So, 11 is a prime.

$\quad \sqrt{17} \approx 4.12$

$17 \div 2 = 8.5$

$17 \div 3 = 5.\overline{6}$ $\quad$ So, 17 is a prime.

$\quad \sqrt{97} \approx 9.85$

$97 \div 2 = 48.5$

$97 \div 3 = 32.\overline{3}$

$97 \div 5 = 19.4$

$97 \div 7 \approx 13.8$ $\quad$ So, 97 is a prime.

d $\quad 64 = 8^2$ $\quad$ and $\quad 64 = 4^3$

$729 = 27^2$ $\quad$ and $\quad 729 = 9^3$

3 53, 59, 61, 67

4 1 does not have two different factors.

5 $\quad \sqrt{263} \approx 16.2$

$263 \div 2 = 131.5$

$263 \div 3 = 87.\overline{6}$

$263 \div 5 = 52.6$

$263 \div 7 \approx 37.6$

$263 \div 11 \approx 23.9$

$263 \div 13 \approx 20.2$ $\quad$ So, 263 is a prime.

EXERCISE B

1 **a** 10 **b** 2 **c** -8 **d** 6 **e** 6 **f** 24

2 **a** 37 **b** 43 **c** 3 **d** 24 **e** 6 **f** 19

g 12 **h** 10 **i** 6 **j** 16 **k** 20 **l** 2

3 **a** 17 **b** 13 **c** 1 **d** 9 **e** 69 **f** 9

g 43 **h** 7 **i** 21 **j** 3 **k** 16 **l** -2

4 **a** 4 **b** 21 **c** 13 **d** 92 **e** 10 **f** 15

g 108 **h** 4 **i** 7

5 **a** 5 **b** 3 **c** $\frac{1}{2}$ **d** 5 **e** 7 **f** 7

g 18 **h** 1 **i** $-\frac{1}{4}$

6 **a** 15 **b** 3 **c** -42 **d** $-\frac{1}{2}$ **e** 4 **f** 7

g 21 **h** 4 **i** 17 **j** 13 **k** $1\frac{1}{2}$ **l** $\frac{1}{4}$

7 **a** 438 **b** 874 **c** 33 **d** 8 **e** -1053

f 4 **g** -6 **h** -83 **i** -8 **j** 81 **k** -81

l 20.003 76

8 **a** 4.75 m **b** £6.62 **c** 4510 km **d** 11.8 kg

e 75 hours

EXERCISE C

1 **a** 1, 2, 5, 10 **b** 1, 2, 4, 8, 16

c 1, 2, 3, 6, 9, 18 **d** 1, 17

e 1, 2, 3, 4, 6, 9, 12, 18, 36

f 1, 2, 3, 6, 7, 14, 21, 42 **g** 1, 2, 4, 8, 16, 32, 64

h 1, 2, 4, 5, 10, 20, 25, 50, 100

2 **a** 6, 12, 18, 24, 30 **b** 11, 22, 33, 44, 55

c 13, 26, 39, 52, 65 **d** 29, 58, 87, 116, 145

3 **a** 3 **b** 4 **c** 6 **d** 13 **e** 1 **f** 5 **g** 9 **h** 24

4 **a** 198 **b** 493 **5** **a** 306 **b** 8004

6 **a** 24 **b** 24 **c** 42 **d** 72 **e** 30 **f** 28

g 30 **h** 36

7 **a** 36 **b** 75 **8** 60 seconds

EXERCISE D

1 **a** $\frac{12}{13}$ **b** $\frac{11}{16}$ **c** $\frac{5}{8}$ **d** $\frac{17}{30}$ **e** $4\frac{3}{7}$ **f** $2\frac{1}{6}$ **g** $3\frac{1}{2}$

h $6\frac{1}{6}$

2 **a** $\frac{4}{11}$ **b** $\frac{1}{6}$ **c** $\frac{1}{9}$ **d** $\frac{5}{8}$ **e** $1\frac{3}{4}$ **f** $1\frac{1}{10}$ **g** $1\frac{5}{6}$

h $2\frac{2}{21}$

3 **a** $\frac{1}{9}$ **b** $\frac{2}{7}$ **c** $1\frac{1}{2}$ **d** $4\frac{2}{3}$ **e** $1\frac{1}{7}$ **f** $\frac{1}{2}$ **g** $6\frac{1}{4}$

h $2\frac{10}{27}$

4 **a** $\frac{6}{7}$ **b** $\frac{11}{16}$ **c** $\frac{2}{3}$ **d** $\frac{1}{10}$ **e** $1\frac{2}{3}$ **f** $3\frac{5}{9}$ **g** $\frac{4}{9}$

h $1\frac{7}{20}$

5 **a** $5\frac{8}{35}$ **b** $1\frac{5}{6}$ **c** $2\frac{1}{2}$ **d** $\frac{2}{5}$ **e** 12 **f** $1\frac{1}{11}$

g $4\frac{2}{3}$ **h** $\frac{1}{2}$ **i** $1\frac{1}{6}$ **j** $\frac{3}{5}$ **k** $2\frac{3}{5}$ **l** $\frac{1}{10}$

m 11 **n** $1\frac{1}{4}$ **o** $4\frac{2}{5}$

6 **a** 35 goals **b** £240.00 **c** $\frac{7}{12}$ remains **d** 36 points

e **i** $\frac{2}{3}$ **ii** €610.00 **f** $\frac{1}{4}$ **g** ¥7 560 000

7 **a** $\frac{8}{15}$ **b** $\frac{13}{21}$ **c** $\frac{19}{56}$ **d** $\frac{5}{12}$ **e** $\frac{3}{5}$ **f** $\frac{8}{21}$

g $\frac{12}{35}$ **h** $2\frac{7}{10}$ **i** $4\frac{1}{8}$ **j** $4\frac{5}{28}$ **k** $1\frac{6}{11}$

EXERCISE E

1 **a** 2, 4, 8, 16, 32, 64, 128, 256 **b** 3, 9, 27, 81, 243

c 4, 16, 64, 256 **d** 5, 25, 125, 625

2 **a** 5 **b** 2 **c** 2 **d** 3 **e** 3 **f** 5 **g** 4

h 5 **i** 10 **j** $1\frac{1}{2}$

3 **a** 4 **b** 2401 **c** 16 384 **d** 9 **e** 2.6

f 4.1 **g** 0.571 787 **h** 0.71 **i** 1.05

j 84.64 **k** 1.169 858 56 **l** 64.363 43

EXERCISE F.1

1 **a** 7 : 11 **b** 14 : 9

c **i** 13 : 9 **ii** 9 : 13 **iii** 5 : 9 **iv** 9 : 5

2 **a** 8 : 3 **b** 3 : 7 **c** 35 : 45 **d** 300 : 50

e 500 : 3000 **f** 400 : 2500 **g** 9000 : 150

h 12 : 8000 **i** 240 : 40

EXERCISE F.2

1 **a** 3 : 4 **b** 2 : 1 **c** 1 : 3 **d** 3 : 5

e 1 : 2 **f** 7 : 4 **g** 1 : 2 **h** 3 : 4

i 5 : 4 **j** 1 : 2 : 3 **k** 20 : 1 **l** 1 : 2 : 4

2 **a** 5 : 2 **b** 3 : 7 **c** 3 : 2 **d** 2 : 1 **e** 7 : 12

f 1 : 4 **g** 4 : 1 **h** 1 : 20

3 **a** $1:2$ **b** $3:1$ **c** $3:1$ **d** $3:5$
   e $2:3$ **f** $8:15$ **g** $8:39$ **h** $39:50$
   i $2:1$ **j** $1:2$ **k** $4:1$ **l** $6:4:3$

4 **a** $2:1$ **b** $1:3$ **c** $1:4$ **d** $7:10$ **e** $3:7$
   f $40:1$ **g** $1:50$

EXERCISE F.3

1 **a** $x=12$ **b** $x=3$ **c** $x=10$ **d** $x=28$ **e** $x=35$
   f $x=42$ **g** $x=96$ **h** $x=56$ **i** $x=36$

2 **a** $x=15$ **b** $x=6$ **c** $x=15$ **d** $x=9$
   e $x=3$ **f** $x=16$

3 6 kg of onions **4** 300 apple trees **5** $30 **6** 90 kg

7 **a** 28 kg **b** 76 kg

8 **a** **i** $15 **ii** $24 **iii** $90
   b **i** 150 kg **ii** 160 kg **iii** 10 kg

EXERCISE F.4

1 97.65 kg **2** $18 938 **3** 9.38 cm **4** 88.3 ml
5 £39.68 **6** **a** 10.5 km **b** 2.7 cm **7** $82.50
8 $15:16$

EXERCISE F.5

1 **a** 5 **b** 5 **c** 16 **d** 17 **e** 13 **f** 19 **g** 11 **h** 19

2 **a** 25 cm : 25 cm **b** 40 cm : 10 cm **c** 30 cm : 20 cm
   d 17.5 cm : 32.5 cm

3 **a** $10 : $40 **b** €15 : €20 **c** 40 kg : 50 kg

4 $250 000 **5** 510 girls

6 **a** 58 years **b** €103 333 **c** €96 667

7 £6000 : £10 000 : £20 000

8 **a** 7.5 kg **b** 2.5 kg **9** **a** 16 **b** 20 **c** 12

EXERCISE G

1 **a** $0.65 = \frac{13}{20} = 65\% = 13:20$
   b $60\% = \frac{3}{5} = 0.6 = 3:5$ **c** $2:5 = \frac{2}{5} = 0.4 = 40\%$
   d $\frac{5}{8} = 0.625 = 5:8 = 62.5\%$

2

Decimal	Fraction	Ratio	Percentage
0.25	$\frac{1}{4}$	$1:4$	25%
0.85	$\frac{17}{20}$	$17:20$	85%
0.36	$\frac{9}{25}$	$9:25$	36%
0.8	$\frac{4}{5}$	$4:5$	80%
$0.\overline{2}$	$\frac{2}{9}$	$2:9$	$22.\overline{2}\%$
1.25	$1\frac{1}{4}$	$5:4$	125%
2	$\frac{2}{1}$	$2:1$	200%
5.5	$5\frac{1}{2}$	$11:2$	550%

EXERCISE H.1

1 **a** 1 **b** 7 **c** 8 **d** 12 **e** 128
2 **a** 2.4 **b** 3.6 **c** 4.9 **d** 6.4 **e** 4.3
3 **a** 4.24 **b** 2.73 **c** 5.63 **d** 4.38 **e** 6.52
4 **a** 0.2 **b** 0.18 **c** 0.184 **d** 0.1838
5 **a** 4.3 **b** 9.13 **c** 11.2 **d** 0.05 **e** 0.73
   f 0.025 **g** 0.5 **h** 6.17 **i** 2.429

6 7.45 cm $\approx$ 7.5 cm   is correct, however Julie should have used
the original value of 7.45 cm to round to the nearest integer, i.e.,
7.45 cm $\approx$ 7 cm.

EXERCISE H.2

1 **a** 42 **b** 6.24 **c** 0.046 **d** 0.25 **e** 440 **f** 2100
   g 31 000 **h** 10.3 **i** 1 **j** 1.0 **k** 264 000
   l 0.037 64 **m** 3699 **n** 0.0076 **o** 30 000 **p** 70

2 **a** **i** 30 000 people **ii** 26 000 people **b** 26 000 people

3 2549 people **4** probably 133 000

EXERCISE H.3

1 **a** 320 **b** 400 **c** 420 **d** 3000 **e** 6400 **f** 14 000
2 **a** 2400 **b** 4200 **c** 9000 **d** 15 000 **e** 28 000
   f 150 000 **g** 90 000 **h** 360 000 **i** 720 000
3 **a** 100 **b** 1000 **c** 10 000 **d** 300 **e** 2000
   f 200 **g** 75 **h** 250 **i** 2000

EXERCISE I.1

1 **a** 300 minutes **b** 4320 minutes **c** 3780 minutes
   d 37 minutes
2 **a** 52 days **b** 16 days **c** 1095 days **d** 0.25 days
3 **a** 2100 seconds **b** 11 940 seconds
   c 432 000 seconds **d** 777 600 seconds
4 **a** 5 h 8 min **b** 7 h 11 min 10 sec
   c 4 h 5 min **d** 2 h 35 min
5 54 hours
6 **a** 2 h 25 min **b** 7 h 44 min **c** 5 h 48 min
   d 6 h 35 min
7 1 h 14 min **8** **a** 42 hours **b** €378
9 **a** 11.16 am **b** 8.45 am **c** 3.45 pm **d** 9.48 am
10 3.15 pm **11** 129.6 km

EXERCISE I.2

1 **a** 0957 **b** 1106 **c** 1600 **d** 1425 **e** 0800
   f 0106 **g** 2058 **h** 1200 **i** 0002
2 **a** 11.40 am **b** 3.46 am **c** 4.34 pm **d** 7.00 pm
   e 8.00 am **f** 11.30 pm **g** 12.23 pm **h** 8.40 pm

3

	Departure	Travelling Time	Arrival
a	0520	6 h 20 min	1140
b	0710	6 h 55 min	1405
c	0931	56 min	1027
d	1229	4 h 23 min	1652
e	2012	8 h 35 min	0447 (next day)

Assumed Knowledge (Geometry and graphs)

Contents:

Assumed Knowledge (Geometry and graphs)

Contents:
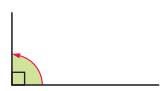
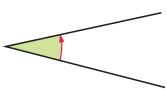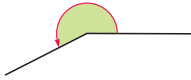## A   ANGLES   [4.1, 4.3]

Angles are described by their size or degree measure. The different types of angle are summarised in the following table:

Revolution	Straight Angle	Right Angle
One complete turn. One revolution $= 360^o$.	$\frac{1}{2}$ turn. 1 straight angle $= 180^o$.	$\frac{1}{4}$ turn. 1 right angle $= 90^o$.
Acute Angle	**Obtuse Angle**	**Reflex Angle**
Less than a $\frac{1}{4}$ turn. An acute angle has size between 0° and 90°.	Between $\frac{1}{4}$ turn and $\frac{1}{2}$ turn. An obtuse angle has size between 90° and 180°.	Between $\frac{1}{2}$ turn and 1 turn. A reflex angle has size between 180° and 360°.

There are several ways to label angles.

We can use a small case letter or a letter of the Greek alphabet.

We can also use **three point notation** to refer to an angle.

For example, the illustrated angle θ is angle PQR or $P\hat{Q}R$.
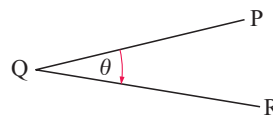
EXERCISE A

1 Draw a freehand sketch of:

    a an acute angle   **b** an obtuse angle   **c** a right angle

    d a reflex angle   **e** a straight angle

2 State whether the following angles are straight, acute, obtuse or reflex:

    a   **b**   **c**   **d**

    e   **f**   **g**   **h**
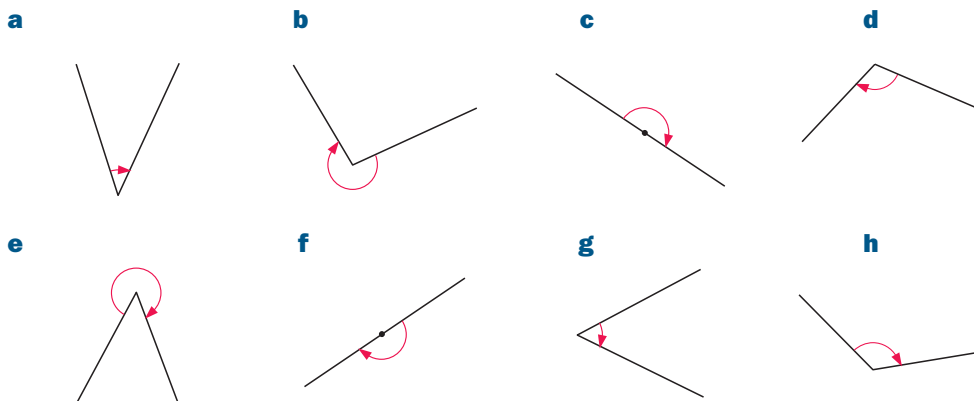
3 For the angle sizes given below, state whether the angle is:

    i a revolution   **ii** a straight angle   **iii** a right angle

    iv an acute angle   **v** an obtuse angle   **vi** a reflex angle.

    a 31° **b** 117° **c** 360° **d** 213° **e** 89° **f** 90°

    g 127° **h** 180° **i** 358° **j** 45° **k** 270° **l** 150°

4 Find the measure of angle:

    a BAC   **b** DAE

    c BAD   **d** CAE

    e CAF   **f** BAE
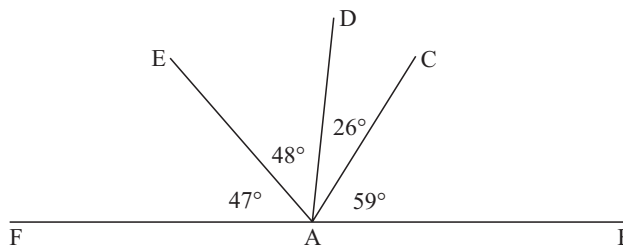
5 Which is the larger angle, $K\hat{L}M$ or $P\hat{Q}R$?
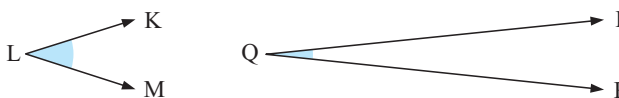
B LINES AND LINE SEGMENTS   [4.1]

When we talk about lines and line segments, it is important to state exactly what we mean.

Line AB is the endless straight line passing through the points A and B.

Line segment AB is the part of the straight line AB that connects A with B.

The **distance AB** is the length of the line segment AB.
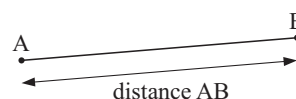
There are several other important words we use when talking about lines:

- **Concurrent lines** are three or more lines that all pass through a common point.

- **Collinear points** are points which lie in a straight line.
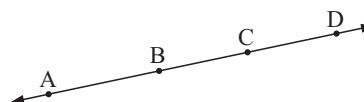
- **Perpendicular lines** intersect at right angles.
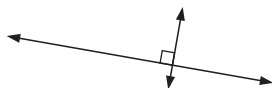
- **Parallel lines** are lines which never intersect. Arrow heads indicate parallelism.
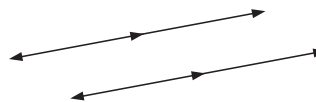
- A **transversal** is a line which crosses over two other lines.
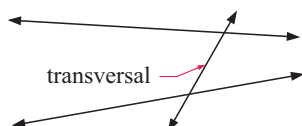
EXERCISE B

1
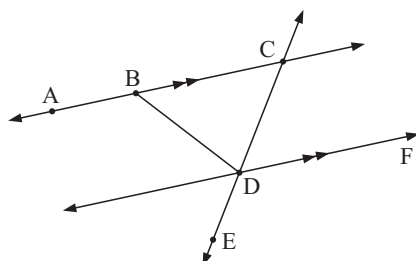
Copy and complete:

a Points A, B and C are ......

b The part of line BD we can see is ......

c The line parallel to line BC is ......

d The lines through B and D, C and E, and D and F, are ...... at D.

2 Draw a diagram which shows:

a line AB perpendicular to line CD

b X, Y and Z being collinear

c four concurrent lines

d two lines being cut by a transversal.

C   *POLYGONS*   *[4.1]*

A **polygon** is any closed figure with straight line sides which can be drawn on a flat surface.
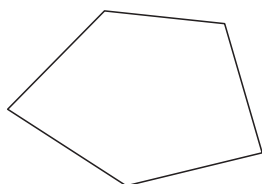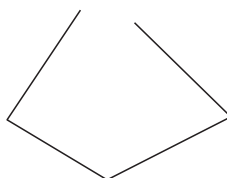
is a polygon.   are not polygons.
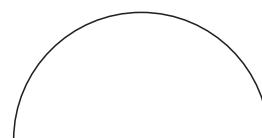
A **triangle** is a polygon with 3 sides.
A **quadrilateral** is a polygon with 4 sides.
A **pentagon** is a polygon with 5 sides.
A **hexagon** is a polygon with 6 sides.
An **octagon** is a polygon with 8 sides.

TRIANGLES

There are several types of triangle you should be familiar with:
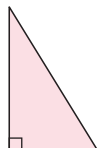
scalene   **right angled**   **isosceles**   **equilateral**

Three sides of   One angle a   Two sides   Three sides
different lengths.   right angle.   are equal.   are equal.

REGULAR POLYGONS

A **regular polygon** is one in which all sides are equal and all angles are equal.

For example, these are all regular polygons:
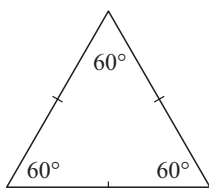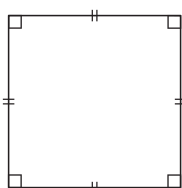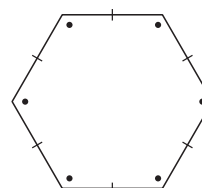
equilateral triangle   square   regular hexagon

SPECIAL QUADRILATERALS

There are six special quadrilaterals:

- A **parallelogram** is a quadrilateral which has opposite sides parallel.
- A **rectangle** is a parallelogram with four equal angles of 90°.
- A **rhombus** is a quadrilateral in which all sides are equal.
- A **square** is a rhombus with four equal angles of 90°.
- A **trapezium** is a quadrilateral which has exactly one pair of parallel sides.
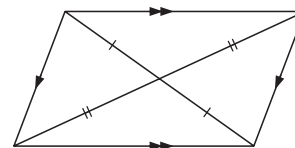- A **kite** is a quadrilateral which has two pairs of equal adjacent sides.

parallelogram   rectangle   rhombus   square   trapezium   kite
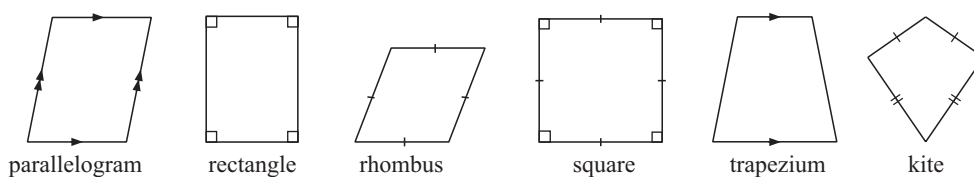
The following properties of quadrilaterals are useful:

PARALLELOGRAM

In any parallelogram:
- opposite sides are equal in length
- opposite angles are equal in size
- diagonals bisect each other.

RHOMBUS

In any rhombus:
- opposite sides are parallel
- opposite angles are equal in size
- diagonals bisect each other at right angles
- diagonals bisect the angles at each vertex.
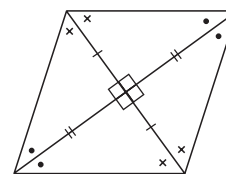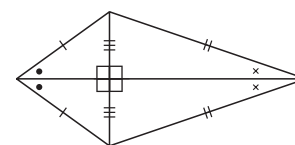
RECTANGLE

In any rectangle:
- opposite sides are equal in length
- diagonals are equal in length
- diagonals bisect each other.
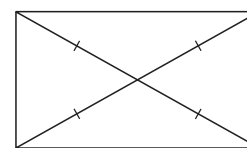
KITE

In any kite:
- two pairs of adjacent sides are equal
- the diagonals are perpendicular
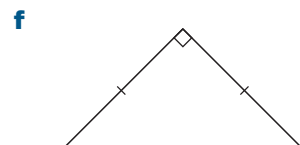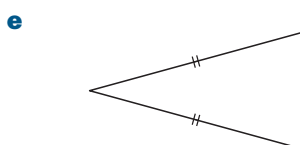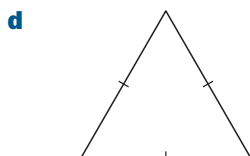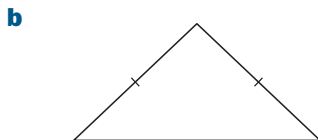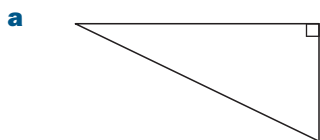- one diagonal splits the kite into two isosceles triangles.

SQUARE

In any square:
- opposite sides are parallel
- all sides are equal in length
- all angles are right angles
- diagonals bisect each other at right angles
- diagonals bisect the angles at each vertex.
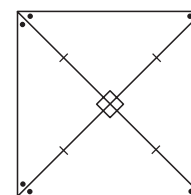
EXERCISE C

1 Classify these triangles:

a

b

c

d

e

f

2 Draw diagrams to illustrate:

 a a quadrilateral   **b** a pentagon   **c** a hexagon

 d an octagon   **e** a regular quadrilateral   **f** a regular triangle

 g a parallelogram   **h** a rhombus   **i** a trapezium

3 List with illustration:

 a the *three* properties of a parallelogram   **b** the *four* properties of a rhombus.
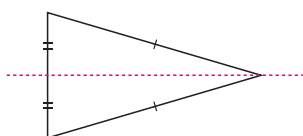
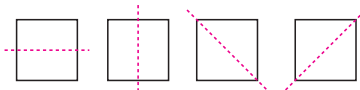## D   SYMMETRY   [4.2]

LINE SYMMETRY

A figure has **line symmetry** if it can be reflected in a line so that each half of the figure is reflected onto the other half of the figure.

For example, an isosceles triangle has one line of symmetry, which is the line from its apex to the midpoint of its base.
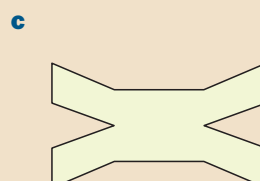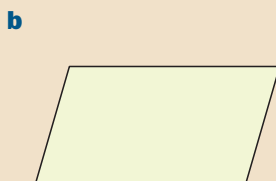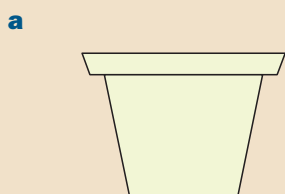
A square has 4 lines of symmetry.

> A figure has a line of symmetry if it can be folded onto itself along that line.

Example 1   ◀⏺ *Self Tutor*

For the following figures, draw all lines of symmetry:

a

b

c

a

1 line of symmetry

b

no lines of symmetry

c

2 lines of symmetry

ROTATIONAL SYMMETRY

A figure has **rotational symmetry** if it can be mapped onto itself more than once as it rotates through 360° about a point called the **centre of symmetry**.

DEMO

The flag of the Isle of Man features a symbol called a *triskelion* which has rotational symmetry.

Every time you rotate the triskelion through 120° it fits onto itself. This is done 3 times to get back to the starting position. We say that its *order* of rotational symmetry is 3.

The number of times an object will fit onto itself when rotated through 360° (one complete turn) is called its **order of rotational symmetry**.

EXERCISE D

1 How many lines of symmetry do the following have?
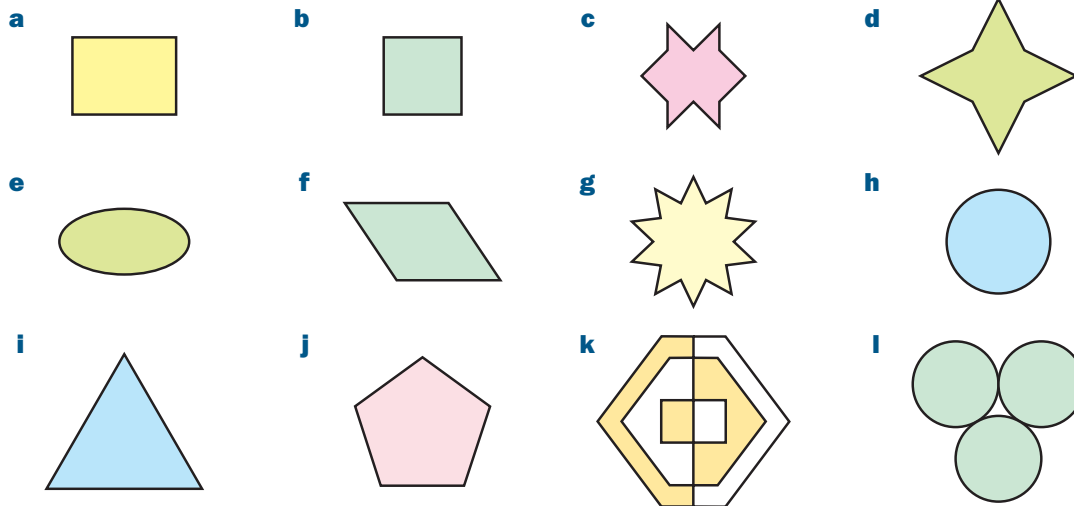
a   **b**   **c**   **d**

e   **f**   **g**   **h**

i   **j**   **k**   **l**

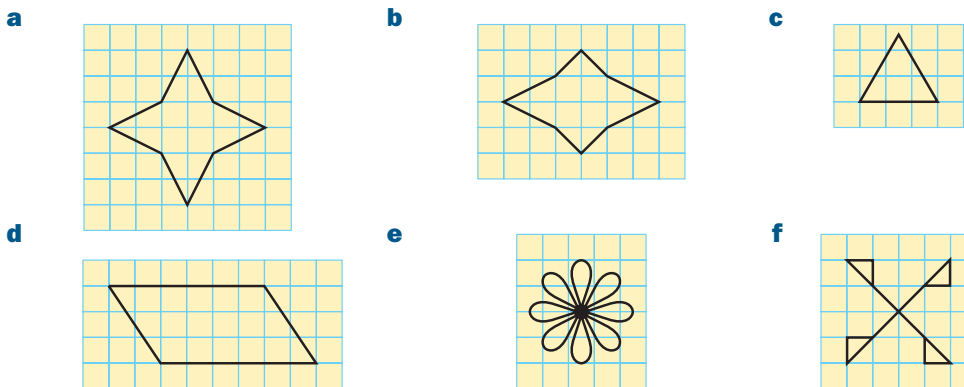**2** Which of the following alphabet letters show line symmetry?

A B C D E F G H I J K L M
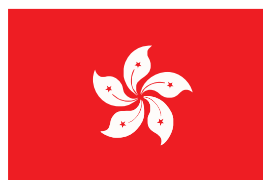N O P Q R S T U V W X Y Z

3 Draw a triangle which has:

   a no lines of symmetry   **b** one line of symmetry   **c** three lines of symmetry.

4 Draw a quadrilateral which has:

     a   no lines of symmetry   **b**   one line of symmetry   **c**   two lines of symmetry

     d   four lines of symmetry.

5 What is the order of rotational symmetry for each of these figures?

     a   **b**   **c**

     d   **e**   **f**

6 Find the order of rotational symmetry of the *Bauhinia blakeana* flower on the flag of Hong Kong.
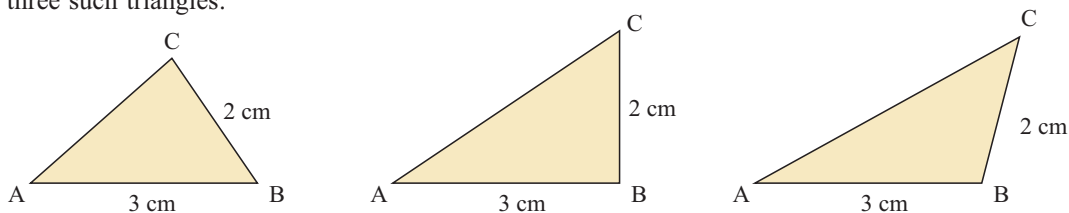
7 Which of the following letters show rotational symmetry of order greater than 1?

ABCDEFGHIJKLMNOPQRSTUVWXYZ

E   CONSTRUCTING TRIANGLES

If several people were asked to accurately draw triangle ABC in which AB = 3 cm and BC = 2 cm, many different shaped triangles would probably be drawn.

Here are three such triangles:
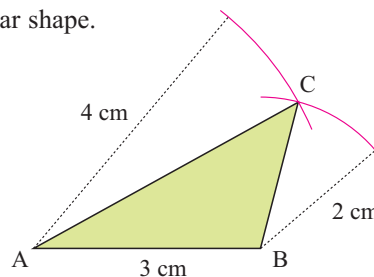
The information given is insufficient to draw a triangle of one particular shape.

However, if we are asked to accurately draw triangle ABC in which AB = 3 cm, BC = 2 cm and AC = 4 cm, one and only one triangular shape can be drawn.

The easiest way to draw this triangle is to use a ruler and compass construction.

Everyone using this construction would draw **the same** figure.

EXERCISE E

1 Construct a triangle which has sides of length:

   **a** 2 cm, 3 cm and 3 cm   **b** 2 cm, 3 cm and 4 cm

    c 2 cm, 3 cm and 5 cm   **d** 2 cm, 3 cm and 7 cm.

2 Copy and complete:

"The sum of the lengths of any two sides of a triangle must be .... the length of the third side".

3 Draw a triangle ABC with all sides greater than 6 cm in length and with the angles at A, B and C being 60°, 50° and 70° respectively.

Place a ruler and set square as shown in the figure. Slide the set square along the ruler to the left, keeping the ruler firmly in place.
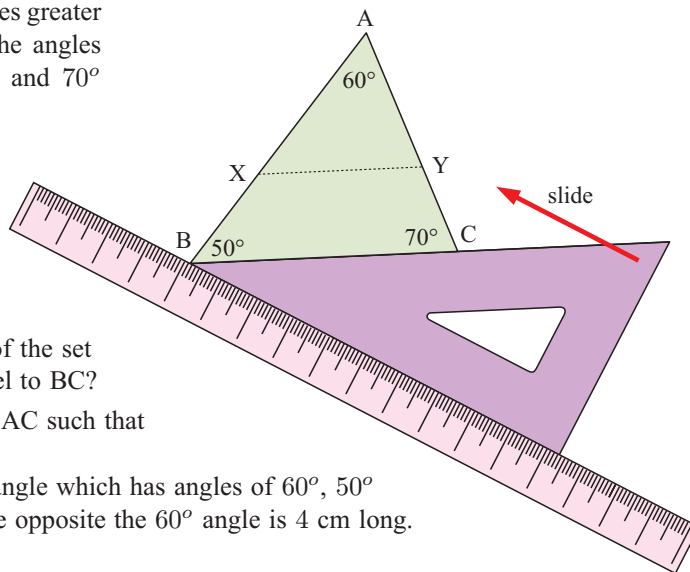
    a Why does the hypotenuse of the set square produce lines parallel to BC?

    b Locate X on AB and Y on AC such that XY = 4 cm.

    You should now have a triangle which has angles of 60°, 50° and 70°, and where the side opposite the 60° angle is 4 cm long.

F   CONGRUENCE   [4.1]

In mathematics we use the term **congruent** to describe things which have the same shape and size. The closest we get to congruence in humans is identical twins.

EXERCISE F.1

1 Which of the following figures are congruent?
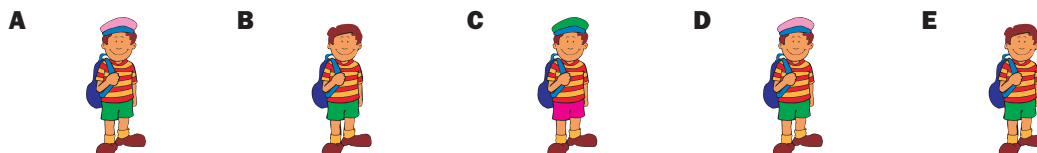
    A   **B**   **C**   **D**   **E**

2 Which of the following geometric figures are congruent?

    A   **B**   **C**   **D**   **E**   **F**   **G**   **H**

    I   **J**   **K**   **L**   **M**   **N**   **O**
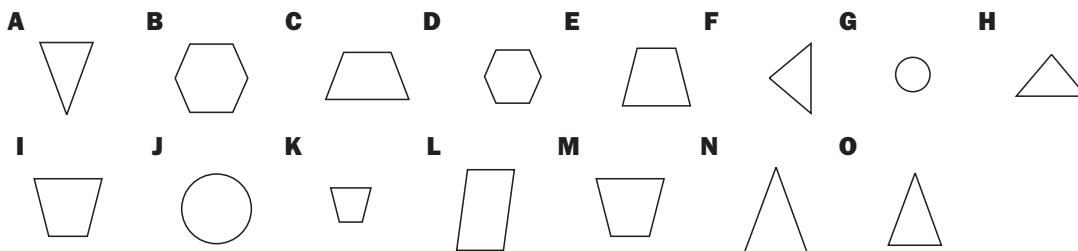
3   Here are some pairs of congruent geometric figures.

a

b

c
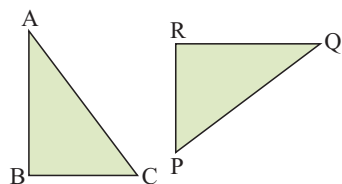
d
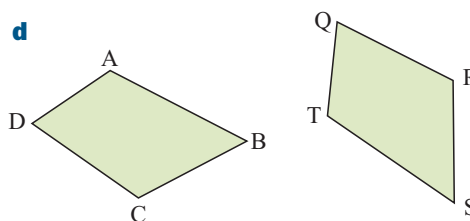
e
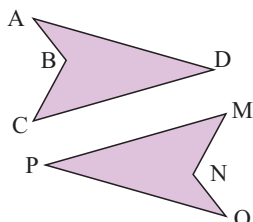
f
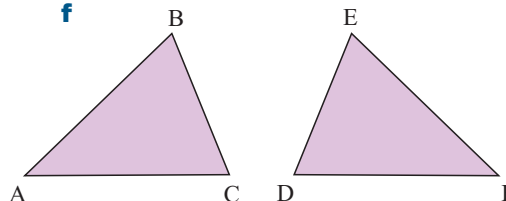
For each pair:

  i   Identify the side in the second figure corresponding to the side AB in the first figure.
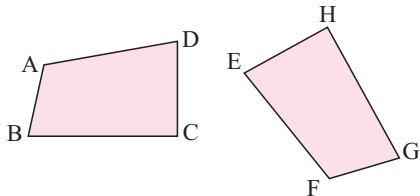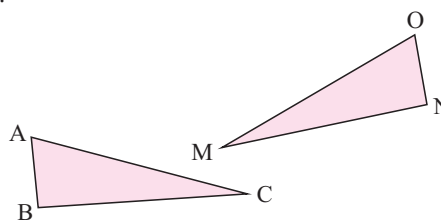
  ii   Identify the angle in the second figure corresponding to $\widehat{ABC}$ in the first figure.

CONGRUENT TRIANGLES

> Two triangles are **congruent** if they are identical in every respect except for position.
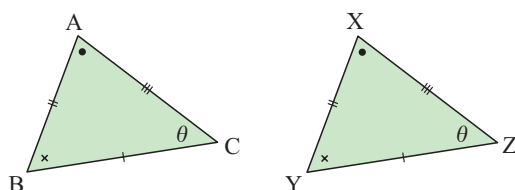
If one triangle was cut out with scissors and placed on the top of the other, they would match each other perfectly.

The above triangles are congruent.

We write $\triangle ABC \cong \triangle XYZ$,   where $\cong$ reads   "*is congruent to*".

When writing this congruence statement, we label the vertices that are in corresponding positions in the same order.
So, we write $\triangle ABC \cong \triangle XYZ$ but **not** $\triangle YXZ$ **or** $\triangle ZYX$.

If two triangles are equiangular (have all three angles equal), they are not necessarily congruent.

For example, these triangles are equiangular but clearly triangle **B** is much larger than triangle **A**.

If we are given two sides and a non-included angle, more than one triangle can be drawn.

For example, triangles **C** and **D** have two equal sides and the same non-included angle, but they are *not* the same triangle.
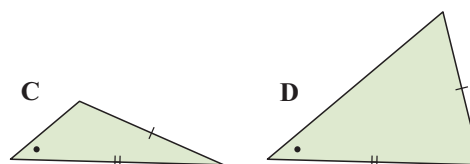
One and only one triangle can be drawn if we are given:

- two sides and the *included* angle between them
- one angle is a right angle, the hypotenuse, and one other side
- two angles and a side.

DEMO

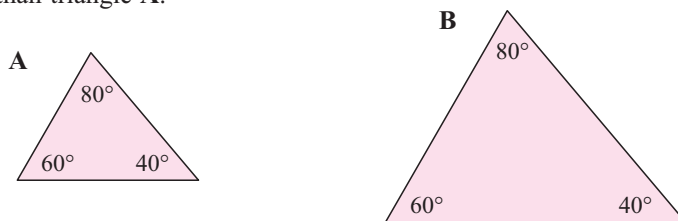There are, however, four acceptable tests for the **congruence** of two triangles.

TESTS FOR TRIANGLE CONGRUENCE

Two triangles are congruent if one of the following is true:

- All corresponding sides are equal in length. (**SSS**)

- Two sides and the **included** angle are equal. (**SAS**)

- Two angles and a pair of **corresponding sides** are equal. (**AAcorS**)

- For right angled triangles, the hypotenuses and one pair of sides are equal. (**RHS**)

The information we are given will help us decide which test to use to prove two triangles are congruent. The diagrams in the following exercise are sketches only and **are not** drawn to scale. However, the information on them is **correct**.

Example 2   ◀》 *Self Tutor*

Are these triangles congruent? If so, state the congruence relationship and give a brief reason.

a

b

c

d
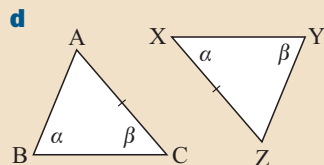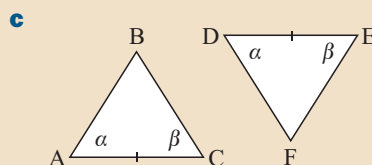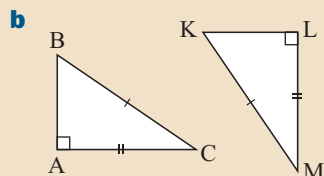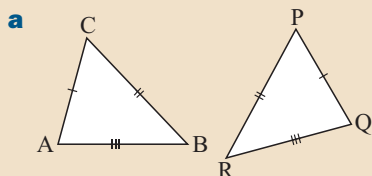
> α (alpha) and β (beta) are Greek letters and are often used to show equal angles.

a $\triangle ABC \cong \triangle QRP$ {SSS}   **b** $\triangle ABC \cong \triangle LKM$ {RHS}

c $\triangle ABC \cong \triangle DFE$ {AAcorS}

d The two angles α and β are common, but although AC equals XZ, these sides *are not* corresponding.   {AC is opposite α whereas XZ is opposite β.}
So, the triangles are not congruent.

EXERCISE F.2

1 In each set of three triangles, two are congruent. The diagrams are *not* drawn to scale. State which pair is congruent, together with a reason (SSS, SAS, AAcorS or RHS).

a

b

c

d

e

f

g

h
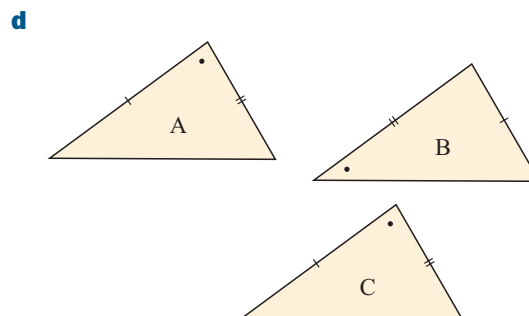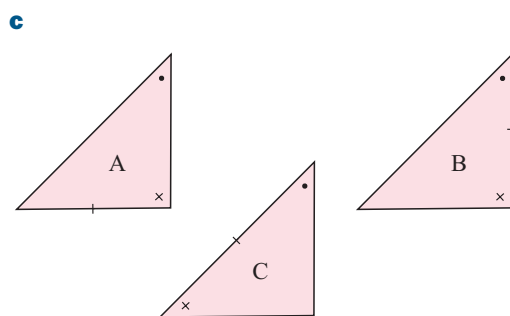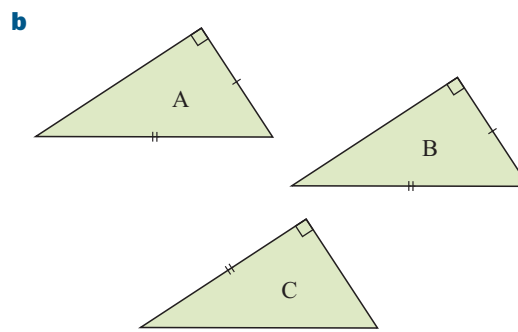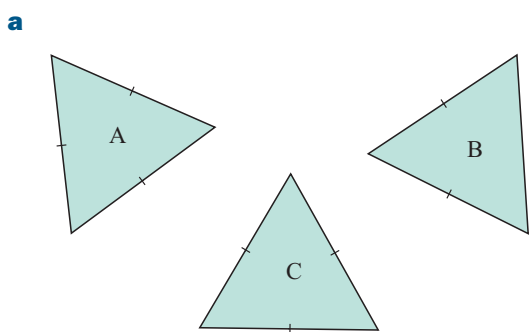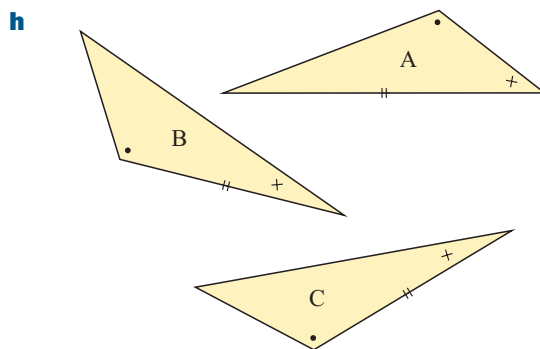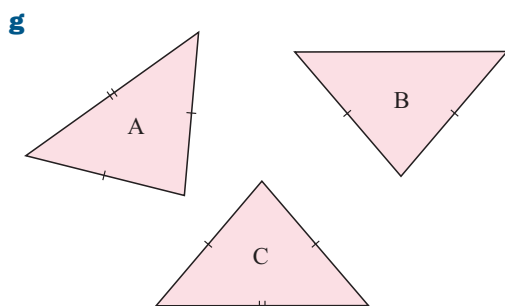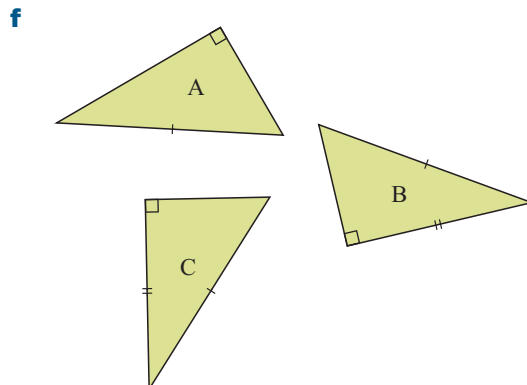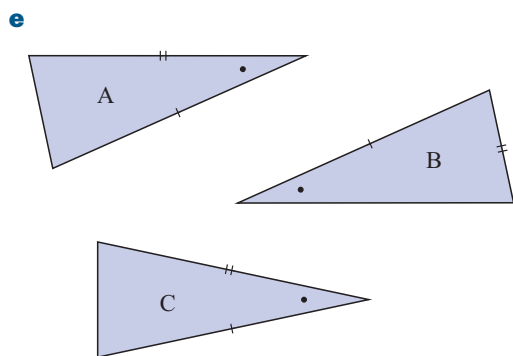
2 Are the following pairs of triangles congruent? If so, state the congruence relationship and give a brief reason.

a

b

c

d

e

f
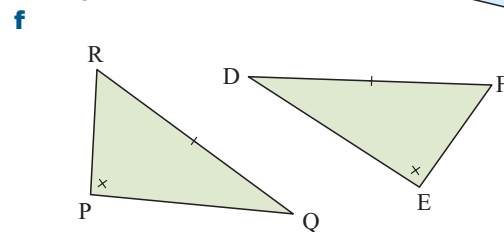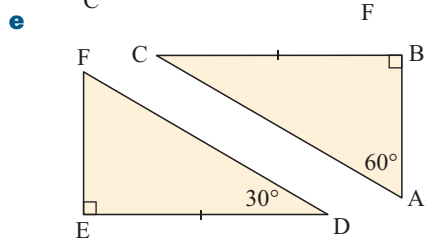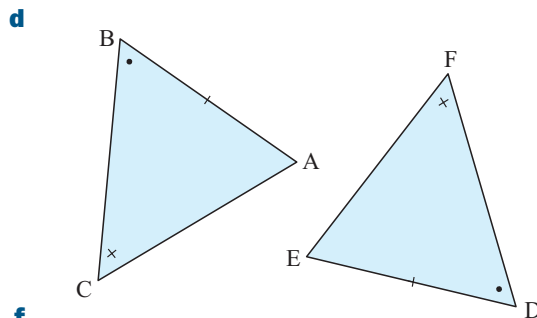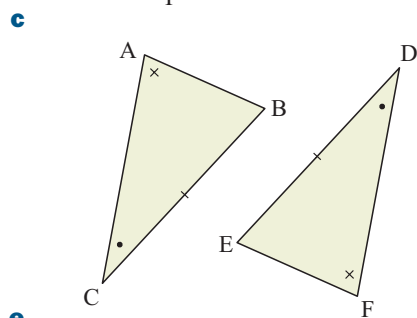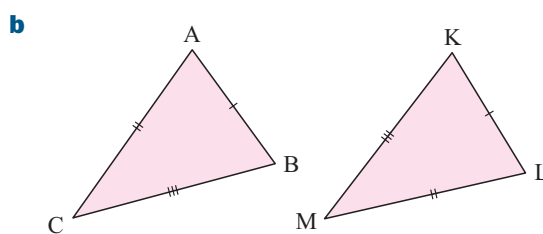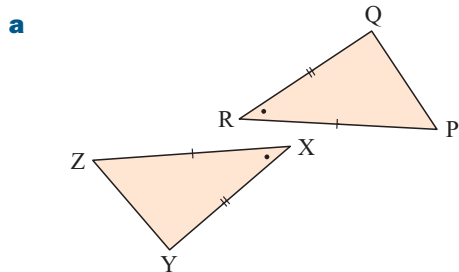
g
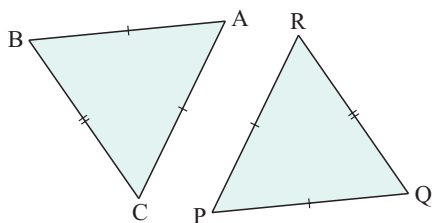
h
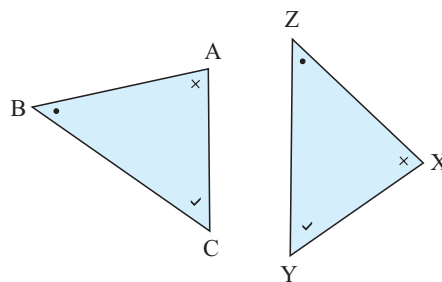
i
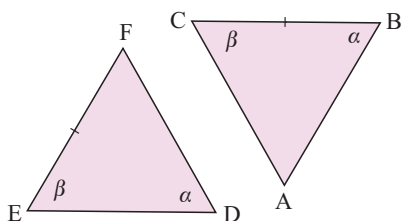
j
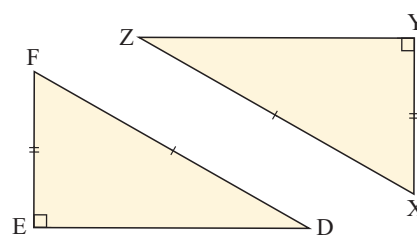
G   INTERPRETING GRAPHS AND TABLES   [11.1]

Magazines and newspapers frequently contain graphs and tables which display information. It is important that we interpret this information correctly. We often use **percentages** in our analysis.

Example 3   ◄)) *Self Tutor*

A survey was carried out amongst retired people to see if they were worried about global warming. The results were collated and a bar chart drawn.

a How many retirees were not concerned about global warming?

b How many retirees were surveyed?

c What percentage of retirees were:

   i   a little concerned

   ii very worried about global warming?
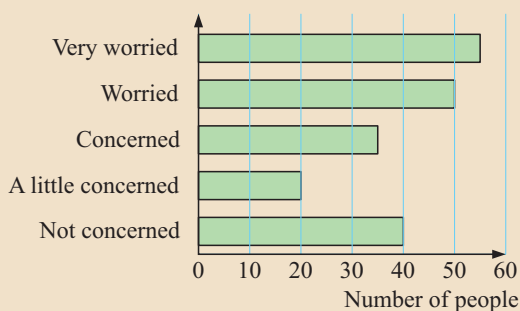
a 40 retirees were not concerned about global warming.

b $40 + 20 + 35 + 50 + 55 = 200$ retirees were surveyed.

c **i** The percentage 'a little concerned'
$$= \tfrac{20}{200} \times 100\%$$
$$= 10\%$$

   ii The percentage 'very worried'
$$= \tfrac{55}{200} \times 100\%$$
$$= 27.5\%$$

EXERCISE G

1 The graph alongside shows the number of days' supply of blood for various blood groups available in a major hospital. We can see that for blood type B+ there is enough blood for transfusions within the next 5 days of normal usage.
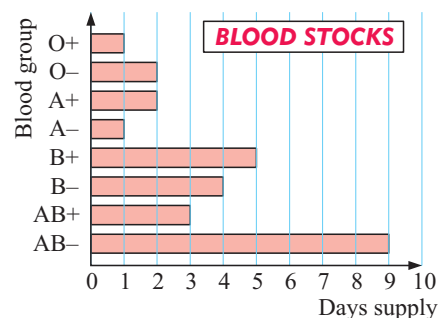
   a How many blood groups are there?

   b What blood type is in greatest supply?

   c If supply for 2 or less days is 'critical', what blood types are in critical supply?

   d What percentage of the available blood is AB+ or AB−?

2

Metric tonnes of rice production in 2007
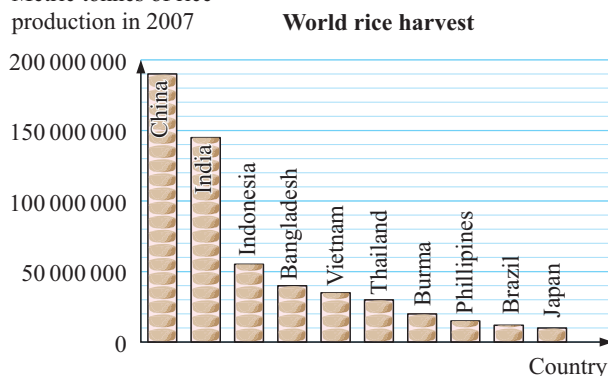
The graph shows the world's largest rice growers and the quantity of rice they harvested in 2007.

   a What tonnage of rice was harvested in:

       i Vietnam   **ii** Burma in 2007?

   b What was the total tonnage harvested from the countries included?

   c What percentage of the total rice harvest was grown in China?

   d What percentage of rice does India grow compared with China?

3 The graph shows the change in temperature at the 'Weather Centre' from 6 am to 6 pm on a particular day.

   a What was the temperature at:

       i 6 am   **ii** 6 pm   **iii** noon?

   b Over what period was the temperature:

       i decreasing   **ii** increasing?

   c What was the maximum temperature and at what time did it occur?
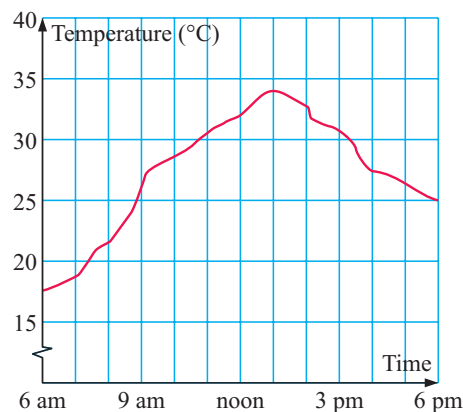
4

Children with diabetes
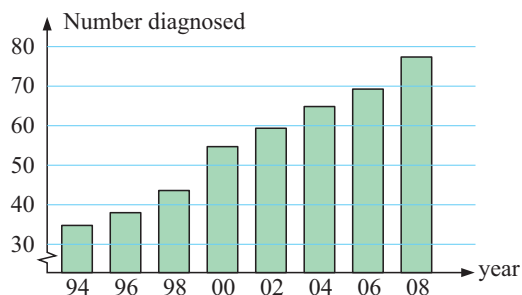
The number of children diagnosed with diabetes in a large city over time is illustrated in the graph alongside.

   a How many children were diagnosed in:

       i 1996   **ii** 2006?

   b Find the percentage increase in diagnosis of the disease from:

       i 1996 to 2006   **ii** 2000 to 2008.

5 At a school, the year 11 students were asked to nominate their favourite fruit. The following data was collected:

Type of fruit	Frequency
Apple	20
Banana	24
Grapes	3
Orange	11
Mandarin	10
Nectarine	7
Pear	2
Peach	3

   a How many year 11 students were there?

   b What was the most popular fruit?

   c What percentage of year 11 students choose mandarins as their favourite fruit?

   d The school canteen sells apples, bananas and pears. What percentage of year 11 students will be able to order their favourite fruit?

6 The table below displays the percentage of people with each blood type, categorised by country.

Blood Type	O+	O−	A+	A−	B+	B−	AB+	AB−
Austria	30%	7%	33%	8%	12%	3%	6%	1%
Canada	39%	7%	36%	7%	8%	1%	2%	1%
France	36%	6%	37%	7%	9%	1%	3%	1%
Hong Kong	40%	0.3%	26%	0.3%	27%	0.2%	6%	0.2%
South Korea	27%	0.3%	34%	0.3%	27%	0.3%	11%	0.1%
United Kingdom	37%	7%	35%	7%	8%	2%	3%	1%
USA	37%	7%	36%	6%	9%	1%	3%	1%

   a What percentage of Canadians have type B+ blood?

   b In which countries is A+ the most common blood type?

   c Which of the countries listed has the highest percentage of population with type AB+ blood?

   d Which blood type is the fourth most common in the USA?

   e In which country is B+ more common than A+?

7

Telling Statistics		
Crashes by age		
Age	Total Crashes	Fatalities
16 − 24	73 645	243
25 − 34	58 062	173
35 − 44	49 600	123
45 − 54	36 658	70
55 − 64	19 833	58
65 − 74	15 142	58
75 − 84	7 275	45
85+	937	9

The table shows car crashes by age group over a one year period in an English county.

   a Which age group has:

     i the most crashes   **ii** the least crashes?

   b What percentage of crashes does the 55-64 age group have compared with:

     i the 16-24 age group   **ii** all age groups?

8 The graph alongside shows the distance June travels on her morning jog. Use the graph to determine:

   a the distance she jogs

   b the time taken for her jog

   c the distance travelled after 15 minutes

   d the time taken to jog the first 6 km.

   e the average speed for the whole distance.
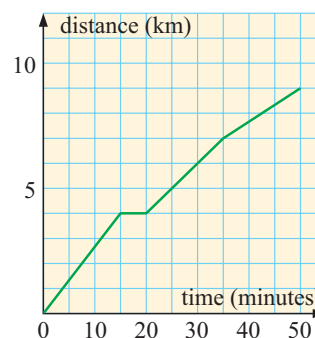
9
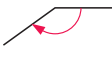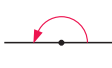
Distances are in kilometres.
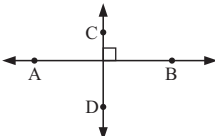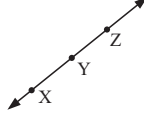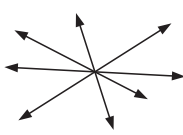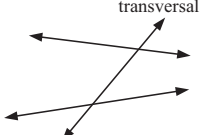
	London	Aberdeen	Birmingham	Bristol	Cambridge	Cardiff	Edinburgh	Glasgow	Liverpool	Manchester	Newcastle-on-Tyne	Oxford
London												
Aberdeen	865											
Birmingham	189	692										
Bristol	192	823	137									
Cambridge	97	753	163	287								
Cardiff	250	857	173	73	343							
Edinburgh	646	204	472	601	542	632						
Glasgow	644	240	469	599	562	633	72					
Liverpool	338	581	158	287	330	322	362	356				
Manchester	317	570	142	269	246	303	351	345	55			
Newcastle-on-Tyne	444	380	319	469	367	501	176	241	274	227		
Oxford	90	802	101	119	129	175	583	571	266	248	407	

a How far is it from:

　　i London to Aberdeen　　　**ii** Cambridge to Oxford?

b How far would you need to travel to complete the circuit from Glasgow to Cardiff to Oxford and back to Glasgow?

c If you could average 85 km/h on a trip from Cardiff to London, how long would it take, to the nearest 5 minutes?
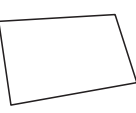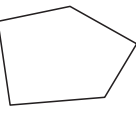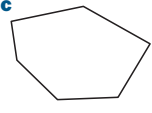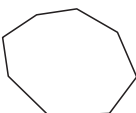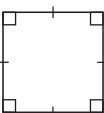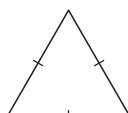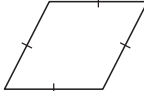
EXERCISE A

1 a **b** **c**

d **e**

2 a acute   **b** reflex   **c** straight   **d** obtuse
e reflex   **f** straight   **g** acute   **h** obtuse

3 a acute   **b** obtuse   **c** revolution   **d** reflex
e acute   **f** right angle   **g** obtuse   **h** straight
i reflex   **j** acute   **k** reflex   **l** obtuse

4 a 59° **b** 48° **c** 85° **d** 74° **e** 121° **f** 133°

5 $K\widehat{L}M$

EXERCISE B

1 a ...... are collinear.   **b** ...... is a line segment.
c ...... is line DF.   **d** ...... are concurrent at D.

2 a **b**

c **d** transversal

EXERCISE C
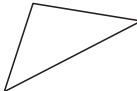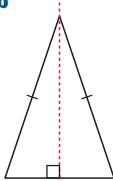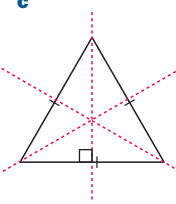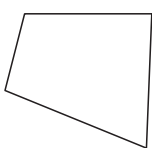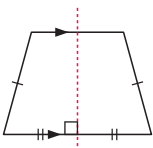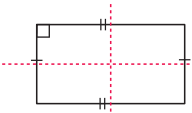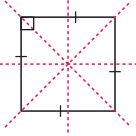
1 a right angled, scalene   **b** obtuse angled, isosceles
c obtuse angled, scalene   **d** equilateral
e acute angled, isosceles   **f** right angled, isosceles

2 a **b** **c**

d **e** **f**

g **h** **i**

3 a • Opposite sides are equal in length.
• Opposite angles are equal in size.
• Diagonals bisect each other.
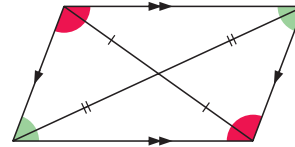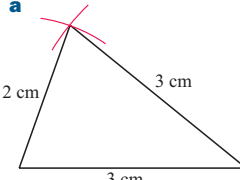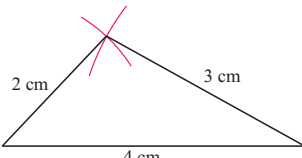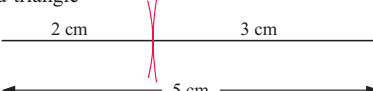
b • Opposite sides are parallel.
• Opposite angles are equal in size.
• Diagonals bisect each other at right angles.
• Diagonals bisect the angles at each vertex.
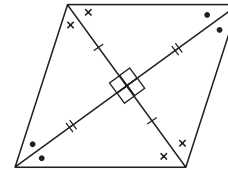
EXERCISE D

1 a 2 **b** 4 **c** 2 **d** 4 **e** 2 **f** 0 as is not a rhombus
g 10 **h** infinite **i** 3 **j** 5 **k** 1 **l** 3

2 A, B, C, D, E, H, I, K, M, O, T, U, V, W, X, Y

3 a **b** **c**

4 a **b**

c **d**

5 a 4   **b** 2   **c** 3   **d** 2   **e** 8   **f** 4

6 5   **7** H, I, N, O, S, X, Z

EXERCISE E

1 a 2 cm, 3 cm, 3 cm
b 2 cm, 3 cm, 2 cm, 4 cm

c cannot form a triangle
2 cm, 3 cm, 5 cm

d cannot form a triangle
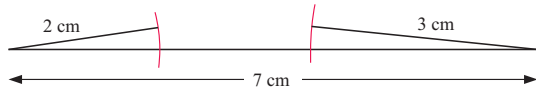
2 The sum of the lengths of any two sides of a triangle must be greater than the length of the third side.

EXERCISE F.1

1 **A** and **D**; **B** and **E**   **2** **A** and **O**; **E**, **I** and **M**; **F** and **H**

3 **a** **i** FG **ii** $F\widehat{G}H$ **b** **i** ON **ii** $O\widehat{N}M$
   c **i** QR **ii** $Q\widehat{R}P$ **d** **i** TS **ii** $T\widehat{S}R$
   e **i** ON **ii** $O\widehat{N}M$ **f** **i** FE **ii** $F\widehat{E}D$

EXERCISE F.2

1 **a** A and C {SSS}   **b** A and B {RHS}
   c B and C {AAcorS}   **d** A and C {SAS}
   e A and C {SAS}   **f** B and C {RHS}
   g A and C {SSS}   **h** B and C {AAcorS}

2 **a** ΔPRQ $\cong$ ΔZXY {SAS} **b** ΔABC $\cong$ ΔLKM {SSS}
   c ΔABC $\cong$ ΔFED {AAcorS}
   d ΔABC $\cong$ ΔEDF {AAcorS}
   e ΔABC $\cong$ ΔFED {AAcorS}
   f Only one pair of sides and one angle are the same $\therefore$ Δs may or may not be congruent (not enough information).
   g ΔABC $\cong$ ΔPQR {SSS}
   h Δs are similar {all angles equal} but may or may not be congruent (not enough information).
   i α and β are common to both, however sides EF and CB are equal but not corresponding $\therefore$ Δs are not congruent.
   j ΔDEF $\cong$ ΔZYX {RHS}

EXERCISE G

1 **a** 8 **b** AB$-$ **c** O+, O$-$, A+, A$-$ **d** 44.4%
2 **a** **i** 35 000 000 t **ii** 20 000 000 t
   b $\approx$ 550 000 000 t **c** $\approx$ 34.5% **d** $\approx$ 76.3%
3 **a** **i** 17.5^{o}C **ii** 25°C **iii** 32°C
   b **i** 1 pm to 6 pm **ii** 6 am to 1 pm **c** 34°C at 1 pm
4 **a** **i** 38 **ii** 69 **b** **i** 81.6% **ii** 40%
5 **a** 80 students **b** banana **c** 12.5% **d** 57.5%
6 **a** 8%   **b** France   **c** South Korea
   d 0$-$ **e** Hong Kong
7 **a** **i** 16 - 24 **ii** 85+ **b** **i** 23.9% **ii** 7.45%
8 **a** 9 km **b** 50 min **c** 4 km **d** 30 min **e** 10.8 km/h
9 **a** **i** 865 km **ii** 129 km **b** 1379 km **c** 2 h 55 min

Algebra (Expansion and factorisation)

1

Contents:

Opening problem

A square garden plot is surrounded by a path 50 cm wide. Each side of the path is x m long, as shown.

a Write an expression for the side length of the garden plot.

b Use the difference of two squares factorisation to show that the area of the path is $(2x - 1)$ square metres.

0.5 m

x m
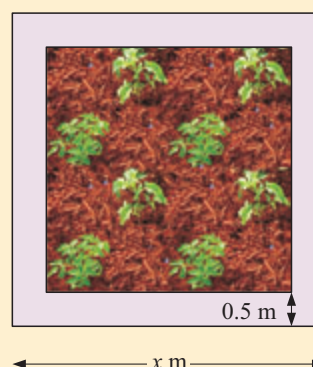
The study of **algebra** is vital for many areas of mathematics. We need it to manipulate equations, solve problems for unknown variables, and also to develop higher level mathematical theories.

In this chapter we consider the **expansion** of expressions which involve brackets, and the reverse process which is called **factorisation**.

A   *THE DISTRIBUTIVE LAW*   [2.7]

Consider the expression $2(x + 3)$. We say that 2 is the **coefficient** of the expression in the brackets. We can **expand** the brackets using the **distributive law**:

$$a(b + c) = ab + ac$$

The distributive law says that we must multiply the coefficient by each term within the brackets, and add the results.

Geometric Demonstration:
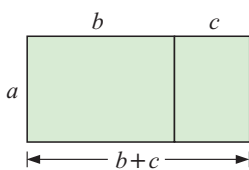
The overall area is $a(b + c)$.

However, this could also be found by adding the areas of the two small rectangles: $ab + ac$.

So, $a(b + c) = ab + ac$.   {equating areas}

Example 1   ◀) *Self Tutor*

Expand the following:

a $3(4x + 1)$ **b** $2x(5 - 2x)$ **c** $-2x(x - 3)$

a $3(4x + 1)$ **b** $2x(5 - 2x)$ **c** $-2x(x - 3)$
    $= 3 \times 4x + 3 \times 1$ $= 2x(5 + -2x)$ $= -2x(x + -3)$
    $= 12x + 3$ $= 2x \times 5 + 2x \times -2x$ $= -2x \times x + -2x \times -3$
                            $= 10x - 4x^2$ $= -2x^2 + 6x$

With practice, we do not need to write all of these steps.

Example 2   ◀) *Self Tutor*

Expand and simplify:

a $2(3x - 1) + 3(5 - x)$ **b** $x(2x - 1) - 2x(5 - x)$

a $2(3x - 1) + 3(5 - x)$ **b** $x(2x - 1) - 2x(5 - x)$
    $= 6x - 2 + 15 - 3x$ $= 2x^2 - x - 10x + 2x^2$
    $= 3x + 13$ $= 4x^2 - 11x$

> Notice in **b** that the minus sign in front of $2x$ affects *both* terms inside the following bracket.

EXERCISE 1A

1 Expand and simplify:

   a $3(x + 1)$ **b** $2(5 - x)$ **c** $-(x + 2)$ **d** $-(3 - x)$
   e $4(a + 2b)$ **f** $3(2x + y)$ **g** $5(x - y)$ **h** $6(-x^2 + y^2)$

i $\ -2(x+4)$	**j** $\ -3(2x-1)$	**k** $\ x(x+3)$	**l** $\ 2x(x-5)$
m $\ -3(x+2)$	**n** $\ -4(x-3)$	**o** $\ -(7-2x)$	**p** $\ -2(x-y)$
q $\ a(a+b)$	**r** $\ -a(a-b)$	**s** $\ x(2x-1)$	**t** $\ 2x(x^2-x-2)$

2 Expand and simplify:

a $\ 1+2(x+2)$	**b** $\ 13-4(x+3)$	**c** $\ 3(x-2)+5$
d $\ 4(3-x)-10$	**e** $\ x(x-1)+x$	**f** $\ 2x(3-x)+x^2$
g $\ 2a(b-a)+3a^2$	**h** $\ 4x-3x(x-1)$	**i** $\ 7x^2-5x(x+2)$

3 Expand and simplify:

a $\ 3(x-4)+2(5+x)$	**b** $\ 2a+(a-2b)$	**c** $\ 2a-(a-2b)$
d $\ 3(y+1)+6(2-y)$	**e** $\ 2(y-3)-4(2y+1)$	**f** $\ 3x-4(2-3x)$
g $\ 2(b-a)+3(a+b)$	**h** $\ x(x+4)+2(x-3)$	**i** $\ x(x+4)-2(x-3)$
j $\ x^2+x(x-1)$	**k** $\ -x^2-x(x-2)$	**l** $\ x(x+y)-y(x+y)$
m $\ -4(x-2)-(3-x)$	**n** $\ 5(2x-1)-(2x+3)$	**o** $\ 4x(x-3)-2x(5-x)$

B   THE PRODUCT $(a+b)(c+d)$ [2.7]

Consider the product $(a+b)(c+d)$.

It has two **factors**, $(a+b)$ and $(c+d)$.

We can evaluate this product by using the distributive law several times.

$$(a+b)(c+d) = a(c+d) + b(c+d)$$
$$= ac + ad + bc + bd$$

So, $\boxed{(a+b)(c+d) = ac + ad + bc + bd}$

> This is sometimes called the **FOIL** rule.

The final result contains four terms:

ac is the product of the   **F**irst   terms of each bracket.
ad is the product of the   **O**uter   terms of each bracket.
bc is the product of the   **I**nner   terms of each bracket.
bd is the product of the   **L**ast   terms of each bracket.

Example 3   🔊 *Self Tutor*

> In practice we do not include the second line of these examples.

Expand and simplify: $(x+3)(x+2)$.

$(x+3)(x+2)$

$= x \times x + x \times 2 + 3 \times x + 3 \times 2$
$= x^2 + 2x + 3x + 6$
$= x^2 + 5x + 6$

Example 4   ◀) *Self Tutor*

Expand and simplify: $(2x + 1)(3x - 2)$

$(2x + 1)(3x - 2)$

$= 2x \times 3x + 2x \times -2 + 1 \times 3x + 1 \times -2$

$= 6x^2 - 4x + 3x - 2$

$= 6x^2 - x - 2$

Example 5   ◀) *Self Tutor*

Expand and simplify:

   a $(x + 3)(x - 3)$ **b** $(3x - 5)(3x + 5)$

   a $(x + 3)(x - 3)$

         $= x^2 - 3x + 3x - 9$

         $= x^2 - 9$

   b $(3x - 5)(3x + 5)$

         $= 9x^2 + 15x - 15x - 25$

         $= 9x^2 - 25$

> In **Examples 5** and **6**, what do you notice about the two middle terms?

Example 6   ◀) *Self Tutor*

Expand and simplify:

   a $(3x + 1)^2$ **b** $(2x - 3)^2$

   a $(3x + 1)^2$

         $= (3x + 1)(3x + 1)$

         $= 9x^2 + 3x + 3x + 1$

         $= 9x^2 + 6x + 1$

   b $(2x - 3)^2$

         $= (2x - 3)(2x - 3)$

         $= 4x^2 - 6x - 6x + 9$

         $= 4x^2 - 12x + 9$

EXERCISE 1B

1   Consider the figure alongside:

   Give an expression for the area of:

     a   rectangle 1   **b**   rectangle 2

     c   rectangle 3   **d**   rectangle 4

     e   the overall rectangle.

   What can you conclude?
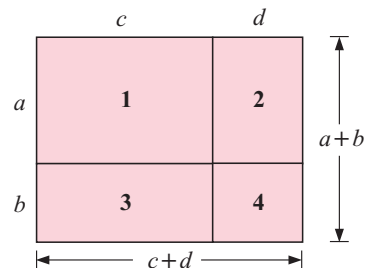
2   Expand and simplify:

     a $(x + 3)(x + 7)$ **b** $(x + 5)(x - 4)$ **c** $(x - 3)(x + 6)$ **d** $(x + 2)(x - 2)$

     e $(x - 8)(x + 3)$ **f** $(2x + 1)(3x + 4)$ **g** $(1 - 2x)(4x + 1)$ **h** $(4 - x)(2x + 3)$

     i $(3x - 2)(1 + 2x)$ **j** $(5 - 3x)(5 + x)$ **k** $(7 - x)(4x + 1)$ **l** $(5x + 2)(5x + 2)$

3 Expand and simplify:

    a $(x + 2)(x - 2)$ **b** $(a - 5)(a + 5)$ **c** $(4 + x)(4 - x)$

    d $(2x + 1)(2x - 1)$ **e** $(5a + 3)(5a - 3)$ **f** $(4 + 3a)(4 - 3a)$

4 Expand and simplify:

    a $(x + 3)^2$ **b** $(x - 2)^2$ **c** $(3x - 2)^2$

    d $(1 - 3x)^2$ **e** $(3 - 4x)^2$ **f** $(5x - y)^2$

5 A square photograph has sides of length x cm. It is surrounded by a wooden frame with the dimensions shown. Show that the area of the rectangle formed by the outside of the frame is given by $A = x^2 + 10x + 24$ cm^2.
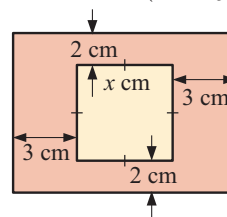
C   DIFFERENCE OF TWO SQUARES   [2.7]

a^2 and b^2 are perfect squares and so $a^2 - b^2$ is called the **difference of two squares**.

Notice that $(a + b)(a - b) = a^2 \underbrace{- ab + ab}_{} - b^2 = a^2 - b^2$

                     the middle two terms add to zero

Thus, $\boxed{(a + b)(a - b) = a^2 - b^2}$

Geometric Demonstration:

Consider the figure alongside:

The shaded area
$=$ area of large square $-$ area of small square
$= a^2 - b^2$

Cutting along the dotted line and flipping (2) over, we can form a rectangle.

The rectangle's area is $(a + b)(a - b)$.
$\therefore \quad (a + b)(a - b) = a^2 - b^2$
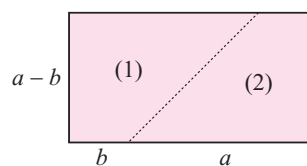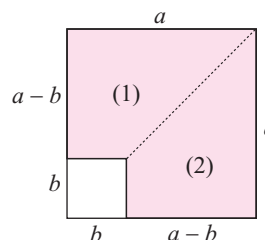
COMPUTER
DEMO
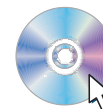
Example 7   ◀)) Self Tutor

Expand and simplify:

    a $(x + 5)(x - 5)$ **b** $(3 - y)(3 + y)$

    a $(x + 5)(x - 5)$ **b** $(3 - y)(3 + y)$

         $= x^2 - 5^2$ $= 3^2 - y^2$

         $= x^2 - 25$ $= 9 - y^2$

Example 8   ◀》 *Self Tutor*

Expand and simplify:

    a $(2x-3)(2x+3)$ **b** $(5-3y)(5+3y)$ **c** $(3x+4y)(3x-4y)$

a $(2x-3)(2x+3)$	**b** $(5-3y)(5+3y)$	**c** $(3x+4y)(3x-4y)$
$= (2x)^2 - 3^2$	$= 5^2 - (3y)^2$	$= (3x)^2 - (4y)^2$
$= 4x^2 - 9$	$= 25 - 9y^2$	$= 9x^2 - 16y^2$

EXERCISE 1C

1 Expand and simplify using the rule $(a+b)(a-b) = a^2 - b^2$:

    a $(x+2)(x-2)$ **b** $(x-2)(x+2)$ **c** $(2+x)(2-x)$

    d $(2-x)(2+x)$ **e** $(x+1)(x-1)$ **f** $(1-x)(1+x)$

    g $(x+7)(x-7)$ **h** $(c+8)(c-8)$ **i** $(d-5)(d+5)$

    j $(x+y)(x-y)$ **k** $(4+d)(4-d)$ **l** $(5+e)(5-e)$

2 Expand and simplify using the rule $(a+b)(a-b) = a^2 - b^2$:

    a $(2x-1)(2x+1)$ **b** $(3x+2)(3x-2)$ **c** $(4y-5)(4y+5)$

    d $(2y+5)(2y-5)$ **e** $(3x+1)(3x-1)$ **f** $(1-3x)(1+3x)$

    g $(2-5y)(2+5y)$ **h** $(3+4a)(3-4a)$ **i** $(4+3a)(4-3a)$

3 Expand and simplify using the rule $(a+b)(a-b) = a^2 - b^2$:

    a $(2a+b)(2a-b)$ **b** $(a-2b)(a+2b)$ **c** $(4x+y)(4x-y)$

    d $(4x+5y)(4x-5y)$ **e** $(2x+3y)(2x-3y)$ **f** $(7x-2y)(7x+2y)$

4 **a** Use the difference of two squares expansion to show that:

        i $43 \times 37 = 40^2 - 3^2$ **ii** $24 \times 26 = 25^2 - 1^2$

    b Evaluate without using a calculator:

        i 18×22 **ii** 49×51 **iii** 103×97.

Discovery 1   *The product of three consecutive integers*

Con was trying to multiply $19 \times 20 \times 21$ without a calculator. Aimee told him to 'cube the middle integer and then subtract the middle integer' to get the answer.

What to do:

1 Find $19 \times 20 \times 21$ using a calculator.

2 Find $20^3 - 20$ using a calculator. Does Aimee's rule seem to work?

3 Check that Aimee's rule works for the following products:

    a $4 \times 5 \times 6$ **b** $9 \times 10 \times 11$ **c** $49 \times 50 \times 51$

4 Let the middle integer be x, so the other integers must be $(x-1)$ and $(x+1)$.

    Find the product $(x-1) \times x \times (x+1)$ by expanding and simplifying. Have you proved Aimee's rule?

D   PERFECT SQUARES EXPANSION   [2.7]

$(a + b)^2$ and $(a - b)^2$ are called **perfect squares**.

Notice that $(a + b)^2 = (a + b)(a + b)$
$$= a^2 + ab + ab + b^2 \quad \{\text{using 'FOIL'}\}$$
$$= a^2 + 2ab + b^2$$

> Notice that the middle two terms are identical.

Thus, we can state the perfect square expansion rule:

$$(a + b)^2 = a^2 + 2ab + b^2$$

We can remember the rule as follows:

Step 1:   Square the *first term.*

Step 2:   Add twice the product of the *first* and *last terms.*

Step 3:   Add on the square of the *last term.*

Notice that $(a - b)^2 = (a + (-b))^2$
$$= a^2 + 2a(-b) + (-b)^2$$
$$= a^2 - 2ab + b^2$$

Once again, we have the square of the first term, twice the product of the first and last terms, and the square of the last term.

Example 9   ◀) Self Tutor

Expand and simplify:

a $(x + 3)^2$ **b** $(x - 5)^2$

a $(x + 3)^2$
$$= x^2 + 2 \times x \times 3 + 3^2$$
$$= x^2 + 6x + 9$$

b $(x - 5)^2$
$$= (x + -5)^2$$
$$= x^2 + 2 \times x \times (-5) + (-5)^2$$
$$= x^2 - 10x + 25$$

Example 10   ◀) Self Tutor

Expand and simplify using the perfect square expansion rule:

a $(5x + 1)^2$ **b** $(4 - 3x)^2$

a $(5x + 1)^2$
$$= (5x)^2 + 2 \times 5x \times 1 + 1^2$$
$$= 25x^2 + 10x + 1$$

b $(4 - 3x)^2$
$$= (4 + -3x)^2$$
$$= 4^2 + 2 \times 4 \times (-3x) + (-3x)^2$$
$$= 16 - 24x + 9x^2$$

Example 11　🔊 Self Tutor

Expand and simplify:　　**a**　$(2x^2 + 3)^2$　　**b**　$5 - (x+2)^2$

a　$(2x^2 + 3)^2$
$= (2x^2)^2 + 2 \times 2x^2 \times 3 + 3^2$
$= 4x^4 + 12x^2 + 9$

b　$5 - (x+2)^2$
$= 5 - [x^2 + 4x + 4]$
$= 5 - x^2 - 4x - 4$
$= 1 - x^2 - 4x$

Notice the use of square brackets in the second line. These remind us to change the signs inside them when they are removed.

EXERCISE 1D

1　Consider the figure alongside:
Give an expression for the area of:

　a　square 1　　　**b**　rectangle 2　　**c**　rectangle 3

　d　square 4　　　**e**　the overall square.

What can you conclude?
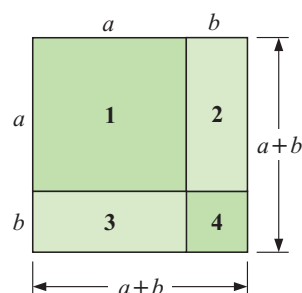
2　Use the rule　$(a+b)^2 = a^2 + 2ab + b^2$　to expand and simplify:

　a　$(x+5)^2$　　　　　　**b**　$(x+4)^2$　　　　　　**c**　$(x+7)^2$

　d　$(a+2)^2$　　　　　　**e**　$(3+c)^2$　　　　　　**f**　$(5+x)^2$

3　Expand and simplify:

　a　$(x-3)^2$　　　　　　**b**　$(x-2)^2$　　　　　　**c**　$(y-8)^2$

　d　$(a-7)^2$　　　　　　**e**　$(5-x)^2$　　　　　　**f**　$(4-y)^2$

4　Expand and simplify:

　a　$(3x+4)^2$　　　　　**b**　$(2a-3)^2$　　　　　**c**　$(3y+1)^2$

　d　$(2x-5)^2$　　　　　**e**　$(3y-5)^2$　　　　　**f**　$(7+2a)^2$

　g　$(1+5x)^2$　　　　　**h**　$(7-3y)^2$　　　　　**i**　$(3+4a)^2$

5　Expand and simplify:

　a　$(x^2+2)^2$　　　　　**b**　$(y^2-3)^2$　　　　　**c**　$(3a^2+4)^2$

　d　$(1-2x^2)^2$　　　　**e**　$(x^2+y^2)^2$　　　　**f**　$(x^2-a^2)^2$

6　Expand and simplify:

　a　$3x+1-(x+3)^2$　　　　　　　　**b**　$5x-2+(x-2)^2$

　c　$(x+2)(x-2)+(x+3)^2$　　　　**d**　$(x+2)(x-2)-(x+3)^2$

　e　$(3-2x)^2-(x-1)(x+2)$　　　**f**　$(1-3x)^2+(x+2)(x-3)$

　g　$(2x+3)(2x-3)-(x+1)^2$　　　**h**　$(4x+3)(x-2)-(2-x)^2$

　i　$(1-x)^2+(x+2)^2$　　　　　　**j**　$(1-x)^2-(x+2)^2$

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E   FURTHER EXPANSION   [2.7]

In this section we expand more complicated expressions by repeated use of the expansion laws.

Consider the expansion of $(a + b)(c + d + e)$.

Now	$(a + b)(c + d + e)$	Compare:	$\square(c + d + e)$
	$= (a+b)c + (a+b)d + (a+b)e$		$= \square c + \square d + \square e$
	$= ac + bc + ad + bd + ae + be$		

Notice that there are 6 terms in this expansion and that each term within the first bracket is multiplied by each term in the second.

2 terms in the first bracket $\times$ 3 terms in the second bracket $\longrightarrow$ 6 terms in the expansion.

Example 12   ◀ᴼ Self Tutor

Expand and simplify: $(x + 3)(x^2 + 2x + 4)$

$$(x + 3)(x^2 + 2x + 4)$$
$$= x(x^2 + 2x + 4) + 3(x^2 + 2x + 4)$$
$$= x^3 + 2x^2 + 4x \qquad \text{\{all terms in the 2nd bracket} \times x\}$$
$$\quad + 3x^2 + 6x + 12 \qquad \text{\{all terms in the 2nd bracket} \times 3\}$$
$$= x^3 + 5x^2 + 10x + 12 \qquad \text{\{collecting like terms\}}$$

Example 13   ◀ᴼ Self Tutor

Expand and simplify: $(x + 2)^3$

$$(x + 2)^3$$
$$= (x + 2) \times (x + 2)^2$$
$$= (x + 2)(x^2 + 4x + 4)$$
$$= x^3 + 4x^2 + 4x \qquad \text{\{all terms in the 2nd bracket} \times x\}$$
$$\quad + 2x^2 + 8x + 8 \qquad \text{\{all terms in the 2nd bracket} \times 2\}$$
$$= x^3 + 6x^2 + 12x + 8 \qquad \text{\{collecting like terms\}}$$

Example 14   ◀ᴼ Self Tutor

Expand and simplify:

a $x(x + 1)(x + 3)$ **b** $(x + 1)(x - 3)(x + 2)$

a $\qquad x(x + 1)(x + 3)$
$$= (x^2 + x)(x + 3) \qquad \text{\{all terms in the first bracket} \times x\}$$
$$= x^3 + 3x^2 + x^2 + 3x \qquad \text{\{expanding the remaining factors\}}$$
$$= x^3 + 4x^2 + 3x \qquad \text{\{collecting like terms\}}$$

> **b** $(x+1)(x-3)(x+2)$
> $= (x^2 - 3x + x - 3)(x+2)$ {expanding the first two factors}
> $= (x^2 - 2x - 3)(x+2)$ {collecting like terms}
> $= x^3 - 2x^2 - 3x + 2x^2 - 4x - 6$ {expanding the remaining factors}
> $= x^3 - 7x - 6$ {collecting like terms}

EXERCISE 1E

1 Expand and simplify:

   a $(x+2)(x^2+x+4)$ **b** $(x+3)(x^2+2x-3)$

   c $(x+3)(x^2+2x+1)$ **d** $(x+1)(2x^2-x-5)$

   e $(2x+3)(x^2+2x+1)$ **f** $(2x-5)(x^2-2x-3)$

   g $(x+5)(3x^2-x+4)$ **h** $(4x-1)(2x^2-3x+1)$

> Each term of the first bracket is multiplied by each term of the second bracket.

2 Expand and simplify:

   a $(x+1)^3$ **b** $(x+3)^3$ **c** $(x-4)^3$

   d $(x-3)^3$ **e** $(3x+1)^3$ **f** $(2x-3)^3$

3 Expand and simplify:

   a $x(x+2)(x+4)$ **b** $x(x-3)(x+2)$ **c** $x(x-4)(x-5)$

   d $2x(x+2)(x+5)$ **e** $3x(x-2)(3-x)$ **f** $-x(2+x)(6-x)$

   g $-3x(3x-1)(x+4)$ **h** $x(1-5x)(2x+3)$ **i** $(x-2)(x+2)(x-3)$

4 Expand and simplify:

   a $(x+4)(x+3)(x+2)$ **b** $(x-3)(x-2)(x+4)$ **c** $(x-3)(x-2)(x-5)$

   d $(2x-3)(x+3)(x-1)$ **e** $(3x+5)(x+1)(x+2)$ **f** $(4x+1)(3x-1)(x+1)$

   g $(2-x)(3x+1)(x-7)$ **h** $(x-2)(4-x)(3x+2)$

5 State how many terms you would obtain by expanding the following:

   a $(a+b)(c+d)$ **b** $(a+b+c)(d+e)$ **c** $(a+b)(c+d+e)$

   d $(a+b+c)(d+e+f)$ **e** $(a+b+c+d)(e+f)$ **f** $(a+b+c+d)(e+f+g)$

   g $(a+b)(c+d)(e+f)$ **h** $(a+b+c)(d+e)(f+g)$

F   ALGEBRAIC COMMON FACTORS

Algebraic products are products which contain variables.

For example, $6c$ and $4x^2y$ are both algebraic products.

In the same way that whole numbers have factors, algebraic products are also made up of factors.

For example, in the same way that we can write 60 as $2 \times 2 \times 3 \times 5$, we can write $2xy^2$ as $2 \times x \times y \times y$.

To find the **highest common factor** of a group of numbers, we express the numbers as products of prime factors. The common prime factors are then found and multiplied to give the highest common factor (HCF).

We can use the same technique to find the highest common factor of a group of algebraic products.

Example 15 ◀》 **Self Tutor**

Find the highest common factor of:

a $8a$ and $12b$ **b** $4x^2$ and $6xy$

a $8a = 2 \times 2 \times 2 \times a$ **b** $4x^2 = 2 \times 2 \times x \times x$
        $12b = 2 \times 2 \times 3 \times b$ $6xy = 2 \times 3 \times x \times y$
     $\therefore$ HCF $= 2 \times 2$ $\therefore$ HCF $= 2 \times x$
            $= 4$ = $2x$

> Write each term
> as a product of
> its **factors**!

Example 16 ◀》 **Self Tutor**

Find the HCF of $3(x+3)$ and $(x+3)(x+1)$.

$3(x+3) = 3 \times (x+3)$ $(x+3)(x+1) = (x+3) \times (x+1)$

$\therefore$ HCF $= (x+3)$

EXERCISE 1F

1 Find the missing factor:

a $3 \times \square = 6a$ **b** $3 \times \square = 15b$ **c** $2 \times \square = 8xy$

d $2x \times \square = 8x^2$ **e** $\square \times 2x = 2x^2$ **f** $\square \times 5x = -10x^2$

g $-a \times \square = ab$ **h** $\square \times a^2 = 4a^3$ **i** $3x \times \square = -9x^2y$

2 Find the highest common factor of the following:

a $2a$ and 6 **b** $5c$ and $8c$ **c** $8r$ and 27

d $12k$ and $7k$ **e** $3a$ and $12a$ **f** $5x$ and $15x$

g $25x$ and $10x$ **h** $24y$ and $32y$ **i** $36b$ and $54d$

3 Find the HCF of the following:

a $23ab$ and $7ab$ **b** abc and $6abc$ **c** $36a$ and $12ab$

d a^2 and a **e** $9r$ and r^3 **f** $3q$ and qr

g $3b^2$ and $9b$ **h** dp^2 and pd **i** $4r$ and $8r^2$

j $3pq$ and $6pq^2$ **k** $2a^2b$ and $6ab$ **l** $6xy$ and $18x^2y^2$

m $15a$, $20ab$ and $30b$ **n** $12wxz$, $12wz$, $24wxyz$ **o** $24p^2qr$, $36pqr^2$

4 Find the HCF of:

a $5(x+2)$ and $(x+8)(x+2)$ **b** $2(x+5)^2$ and $6(x+9)(x+5)$

c $3x(x+4)$ and $x^2(x+2)$ **d** $6(x+1)^2$ and $2(x+1)(x-2)$

e $2(x+3)^2$ and $4(x+3)(x-7)$ **f** $4x(x-3)$ and $6x(x-3)^2$

G   *FACTORISING WITH COMMON FACTORS* [2.8]

Factorisation is the process of writing an expression as a **product** of its **factors**.

Factorisation is the reverse process of expansion.

In **expansions** we have to *remove brackets*, whereas in **factorisation** we have to *insert brackets*.

Notice that $3(x + 2)$ is the *product of two factors*, 3 and $x + 2$.

The brackets are essential, since,

in $3(x + 2)$ the whole of $x + 2$ is multiplied by 3
whereas in $3x + 2$ only the x is multiplied by 3.

expansion

$$3(x + 2) = 3x + 6$$

factorisation

To factorise an algebraic expression involving a number of terms we look for the HCF of the terms and
write it in front of a set of brackets. We then find the contents of the brackets.

For example, $5x^2$ and $10xy$ have HCF $5x$.

$$\text{So,} \quad 5x^2 + 10xy = 5x \times x + 5x \times 2y$$
$$= 5x(x + 2y)$$

FACTORISE FULLY

Notice that $4a + 12 = 2(2a + 6)$ is not fully factorised as $(2a + 6)$ still has a common factor of 2
which could be removed. Although 2 is a common factor it is not the *highest* common factor. The HCF is
4 and so

$$4a + 12 = 4(a + 3) \quad \text{is fully factorised.}$$

Example 17 ◀)) *Self Tutor*

Fully factorise: **a** $3a + 6$ **b** $ab - 2bc$

$3a + 6$	**b** $ab - 2bc$
$= 3 \times a + 3 \times 2$	$= a \times b - 2 \times b \times c$
$= 3(a + 2)$ {HCF is 3}	$= b(a - 2c)$ {HCF is b}

> With practice the middle line is not necessary.

Example 18 ◀)) *Self Tutor*

Fully factorise: **a** $8x^2 + 12x$ **b** $3y^2 - 6xy$

a $8x^2 + 12x$
$= 2 \times 4 \times x \times x + 3 \times 4 \times x$
$= 4x(2x + 3)$ {HCF is $4x$}

b $3y^2 - 6xy$
$= 3 \times y \times y - 2 \times 3 \times x \times y$
$= 3y(y - 2x)$ {HCF is $3y$}

Example 19 ◀)) *Self Tutor*

Fully factorise: **a** $-2a + 6ab$ **b** $-2x^2 - 4x$

a $-2a + 6ab$
$= 6ab - 2a$ {Write with $6ab$ first.}
$= 2 \times 3 \times a \times b - 2 \times a$
$= 2a(3b - 1)$ {HCF is $2a$}

b $-2x^2 - 4x$
$= -2 \times x \times x + -2 \times 2 \times x$
$= -2x(x + 2)$ {HCF is $-2x$}

Example 20 ◀)) *Self Tutor*

Fully factorise:

a $2(x + 3) + x(x + 3)$ **b** $x(x + 4) - (x + 4)$

a $2(x + 3) + x(x + 3)$ {HCF is $(x + 3)$}
$= (x + 3)(2 + x)$

b $x(x + 4) - (x + 4)$ {HCF is $(x + 4)$}
$= x(x + 4) - 1(x + 4)$
$= (x + 4)(x - 1)$

Example 21 ◀)) *Self Tutor*

Fully factorise $(x - 1)(x + 2) + 3(x - 1)$

$(x - 1)(x + 2) + 3(x - 1)$ {HCF of $(x - 1)$}
$= (x - 1)[(x + 2) + 3]$
$= (x - 1)(x + 5)$

> Notice the use of square brackets in the second line. This helps to distinguish between the sets of brackets.

EXERCISE 1G

> You can check your factorisations by expanding back out!

1 Copy and complete:

    a $2x + 4 = 2(x + ...)$ **b** $3a - 12 = 3(a - ...)$

    c $15 - 5p = 5(... - p)$ **d** $18x + 12 = 6(... + 2)$

    e $4x^2 - 8x = 4x(x - ...)$ **f** $2m + 8m^2 = 2m(... + 4m)$

2 Copy and complete:

    a $4x + 16 = 4(... + ...)$ **b** $10 + 5d = 5(... + ...)$

    c $5c - 5 = 5(... - ...)$ **d** $cd + de = d(... + ...)$

    e $6a + 8ab = ...(3 + 4b)$ **f** $6x - 2x^2 = ...(3 - x)$

    g $7ab - 7a = ...(b - 1)$ **h** $4ab - 6bc = ...(2a - 3c)$

3 Fully factorise:

    a $3a + 3b$ **b** $8x - 16$ **c** $3p + 18$ **d** $28 - 14x$

    e $7x - 14$ **f** $12 + 6x$ **g** $ac + bc$ **h** $12y - 6a$

    i $5a + ab$ **j** $bc - 6cd$ **k** $7x - xy$ **l** $xy + y$

    m $a + ab$ **n** $xy - yz$ **o** $3pq + pr$ **p** $cd - c$

4 Fully factorise:

    a $x^2 + 2x$ **b** $5x - 2x^2$ **c** $4x^2 + 8x$ **d** $14x - 7x^2$

    e $6x^2 + 12x$ **f** $x^3 + 9x^2$ **g** $x^2y + xy^2$ **h** $4x^3 - 6x^2$

    i $9x^3 - 18xy$ **j** $a^3 + a^2 + a$ **k** $2a^2 + 4a + 8$ **l** $3a^3 - 6a^2 + 9a$

5 Fully factorise:

    a $-9a + 9b$ **b** $-3 + 6b$ **c** $-8a + 4b$ **d** $-7c + cd$

    e $-a + ab$ **f** $-6x^2 + 12x$ **g** $-5x + 15x^2$ **h** $-2b^2 + 4ab$

    i $-a + a^2$

6 Fully factorise:

    a $-6a - 6b$ **b** $-4 - 8x$ **c** $-3y - 6z$ **d** $-9c - cd$

    e $-x - xy$ **f** $-5x^2 - 20x$ **g** $-12y - 3y^2$ **h** $-18a^2 - 9ab$

    i $-16x^2 - 24x$

7 Fully factorise:

    a $2(x - 7) + x(x - 7)$ **b** $a(x + 3) + b(x + 3)$ **c** $4(x + 2) - x(x + 2)$

    d $x(x + 9) + (x + 9)$ **e** $a(b + 4) - (b + 4)$ **f** $a(b + c) + d(b + c)$

    g $a(m + n) - b(m + n)$ **h** $x(x + 3) - x - 3$

8 Fully factorise:

    a $(x + 3)(x - 5) + 4(x + 3)$ **b** $5(x - 7) + (x - 7)(x + 2)$

    c $(x + 6)(x + 4) - 8(x + 6)$ **d** $(x - 2)^2 - 6(x - 2)$

    e $(x + 2)^2 - (x + 2)(x + 1)$ **f** $5(a + b) - (a + b)(a + 1)$

    g $3(a - 2)^2 - 6(a - 2)$ **h** $(x + 4)^2 + 3(x + 4)(x - 1)$

    i $x(x - 1) - 6(x - 1)(x - 5)$ **j** $3(x + 5) - 4(x + 5)^2$

H   DIFFERENCE OF TWO SQUARES FACTORISATION   [2.8]

We know the **expansion** of $(a+b)(a-b)$ is $a^2 - b^2$.

Thus, the **factorisation** of $a^2 - b^2$ is $(a+b)(a-b)$.

$$a^2 - b^2 = (a+b)(a-b)$$

In contrast, the **sum** of two squares does not factorise into two real linear factors.

> The **difference** between a^2 and b^2 is $a^2 - b^2$ which is the difference of two squares.

Example 22   ◀)) Self Tutor

Use the rule $a^2 - b^2 = (a+b)(a-b)$ to factorise fully:

a $9 - x^2$ **b** $4x^2 - 25$

a $\quad 9 - x^2$
$= 3^2 - x^2 \qquad$ {difference of squares}
$= (3+x)(3-x)$

b $\quad 4x^2 - 25$
$= (2x)^2 - 5^2 \qquad$ {difference of squares}
$= (2x+5)(2x-5)$

Example 23   ◀)) Self Tutor

Fully factorise: **a** $2x^2 - 8$ **b** $-3x^2 + 48$

a $\quad 2x^2 - 8$
$= 2(x^2 - 4) \qquad$ {HCF is 2}
$= 2(x^2 - 2^2) \qquad$ {difference of squares}
$= 2(x+2)(x-2)$

b $\quad -3x^2 + 48$
$= -3(x^2 - 16) \qquad$ {HCF is -3}
$= -3(x^2 - 4^2) \qquad$ {difference of squares}
$= -3(x+4)(x-4)$

> Always look to remove a common factor first.

We notice that $x^2 - 9$ is the difference of two squares and therefore we can factorise it using $a^2 - b^2 = (a+b)(a-b)$.

Even though 7 is not a perfect square, we can still factorise $x^2 - 7$ by writing $7 = (\sqrt{7})^2$.

So, $x^2 - 7 = x^2 - (\sqrt{7})^2 = (x + \sqrt{7})(x - \sqrt{7})$.

We say that $x + \sqrt{7}$ and $x - \sqrt{7}$ are the **linear factors** of $x^2 - 7$.

Example 24   ◀🔊 *Self Tutor*

Factorise into linear factors: **a** $x^2 - 11$ **b** $(x+3)^2 - 5$

> Linear factors are factors of the form $ax + b$.

a
$$x^2 - 11$$
$$= x^2 - (\sqrt{11})^2$$
$$= (x + \sqrt{11})(x - \sqrt{11})$$

b
$$(x+3)^2 - 5$$
$$= (x+3)^2 - (\sqrt{5})^2$$
$$= [(x+3) + \sqrt{5}][(x+3) - \sqrt{5}]$$
$$= [x + 3 + \sqrt{5}][x + 3 - \sqrt{5}]$$

Example 25   ◀🔊 *Self Tutor*

Factorise using the difference between two squares:

a $(3x+2)^2 - 9$ **b** $(x+2)^2 - (x-1)^2$

a
$$(3x+2)^2 - 9$$
$$= (3x+2)^2 - 3^2$$
$$= [(3x+2) + 3][(3x+2) - 3]$$
$$= [3x + 5][3x - 1]$$

b
$$(x+2)^2 - (x-1)^2$$
$$= [(x+2) + (x-1)][(x+2) - (x-1)]$$
$$= [x + 2 + x - 1][x + 2 - x + 1]$$
$$= [2x + 1][3]$$
$$= 3(2x + 1)$$

EXERCISE 1H

1 Use the rule $a^2 - b^2 = (a+b)(a-b)$ to fully factorise:

   a $x^2 - 4$ **b** $4 - x^2$ **c** $x^2 - 81$ **d** $25 - x^2$

   e $4x^2 - 1$ **f** $9x^2 - 16$ **g** $4x^2 - 9$ **h** $36 - 49x^2$

2 Fully factorise:

   a $3x^2 - 27$ **b** $-2x^2 + 8$ **c** $3x^2 - 75$ **d** $-5x^2 + 5$

   e $8x^2 - 18$ **f** $-27x^2 + 75$

3 If possible, factorise into linear factors:

   a $x^2 - 3$ **b** $x^2 + 4$ **c** $x^2 - 15$ **d** $3x^2 - 15$

   e $(x+1)^2 - 6$ **f** $(x+2)^2 + 6$ **g** $(x-2)^2 - 7$ **h** $(x+3)^2 - 17$

   i $(x-4)^2 + 9$

4 Factorise using the difference of two squares:

   a $(x+1)^2 - 4$ **b** $(2x+1)^2 - 9$ **c** $(1-x)^2 - 16$

   d $(x+3)^2 - 4x^2$ **e** $4x^2 - (x+2)^2$ **f** $9x^2 - (3-x)^2$

   g $(2x+1)^2 - (x-2)^2$ **h** $(3x-1)^2 - (x+1)^2$ **i** $4x^2 - (2x+3)^2$

5 Answer the **Opening Problem** on page **31**.

I   PERFECT SQUARES FACTORISATION   [2.8]

We know the **expansion** of $(x+a)^2$ is $x^2 + 2ax + a^2$, so the **factorisation** of $x^2 + 2ax + a^2$ is $(x+a)^2$.

$$x^2 + 2ax + a^2 = (x+a)^2$$

Notice that
$$\begin{aligned}(x-a)^2 &= (x+(-a))^2 \\ &= x^2 + 2(-a)x + (-a)^2 \\ &= x^2 - 2ax + a^2\end{aligned}$$

So,
$$x^2 - 2ax + a^2 = (x-a)^2$$

> $(x+a)^2$ and $(x-a)^2$ are **perfect squares**!

Example 26   ◀)) Self Tutor

Use perfect square rules to fully factorise:

a $x^2 + 10x + 25$ **b** $x^2 - 14x + 49$

a $\quad x^2 + 10x + 25$
$= x^2 + 2 \times x \times 5 + 5^2$
$= (x+5)^2$

b $\quad x^2 - 14x + 49$
$= x^2 - 2 \times x \times 7 + 7^2$
$= (x-7)^2$

Example 27   ◀)) Self Tutor

Fully factorise:

a $9x^2 - 6x + 1$ **b** $-8x^2 - 24x - 18$

a $\quad 9x^2 - 6x + 1$
$= (3x)^2 - 2 \times 3x \times 1 + 1^2$
$= (3x-1)^2$

b $\quad -8x^2 - 24x - 18$
$= -2(4x^2 + 12x + 9) \qquad \{\text{HCF is } -2\}$
$= -2([2x]^2 + 2 \times 2x \times 3 + 3^2)$
$= -2(2x+3)^2$

EXERCISE 1I

1 Use perfect square rules to fully factorise:

a $x^2 + 6x + 9$ **b** $x^2 + 8x + 16$ **c** $x^2 - 6x + 9$

d $x^2 - 8x + 16$ **e** $x^2 + 2x + 1$ **f** $x^2 - 10x + 25$

g $y^2 + 18y + 81$ **h** $m^2 - 20m + 100$ **i** $t^2 + 12t + 36$

2 Fully factorise:

a $9x^2 + 6x + 1$ **b** $4x^2 - 4x + 1$ **c** $9x^2 + 12x + 4$

d $25x^2 - 10x + 1$ **e** $16x^2 + 24x + 9$ **f** $25x^2 - 20x + 4$

g $-x^2 + 2x - 1$ **h** $-2x^2 - 8x - 8$ **i** $-3x^2 - 30x - 75$

3 Explain why: **a** $x^2 + 12x + 36$ is never negative   **b** $x^2 + 4 \geqslant 4x$ for all real x.

J   EXPRESSIONS WITH FOUR TERMS   [2.8]

Some expressions with four terms do not have an overall common factor, but can be factorised by pairing the four terms.

For example, $\underbrace{ab + ac}\ + \underbrace{bd + cd}$

$$= a(b + c) + d(b + c) \qquad \{\text{factorising each pair separately}\}$$
$$= (b + c)(a + d) \qquad \{\text{removing the common factor } (b + c)\}$$

Note:

- Many expressions with four terms cannot be factorised using this method.
- Sometimes it is necessary to reorder the terms first.

Example 28   ◀⁕ **Self Tutor**

Factorise: **a** $3ab + d + 3ad + b$ **b** $x^2 + 2x + 5x + 10$

a $3ab + d + 3ad + b$
$= \underbrace{3ab + b}\ + \underbrace{3ad + d}$
$= b(3a + 1) + d(3a + 1)$
$= (3a + 1)(b + d)$

b $\underbrace{x^2 + 2x}\ + \underbrace{5x + 10}$
$= x(x + 2) + 5(x + 2)$
$= (x + 2)(x + 5)$

Example 29   ◀⁕ **Self Tutor**

Factorise: **a** $x^2 + 3x - 4x - 12$ **b** $x^2 + 3x - x - 3$

a $\underbrace{x^2 + 3x}\ \underbrace{- 4x - 12}$
$= x(x + 3) - 4(x + 3)$
$= (x + 3)(x - 4)$

b $\underbrace{x^2 + 3x}\ \underbrace{- x - 3}$
$= x(x + 3) - (x + 3)$
$= x(x + 3) - 1(x + 3)$
$= (x + 3)(x - 1)$

EXERCISE 1J

1 Factorise:

a $2a + 2 + ab + b$	**b** $4d + ac + ad + 4c$	**c** $ab + 6 + 2b + 3a$
d $mn + 3p + np + 3m$	**e** $x^2 + 3x + 7x + 21$	**f** $x^2 + 5x + 4x + 20$
g $2x^2 + x + 6x + 3$	**h** $3x^2 + 2x + 12x + 8$	**i** $20x^2 + 12x + 5x + 3$

2 Factorise:

a $x^2 - 4x + 5x - 20$	**b** $x^2 - 7x + 2x - 14$	**c** $x^2 - 3x - 2x + 6$
d $x^2 - 5x - 3x + 15$	**e** $x^2 + 7x - 8x - 56$	**f** $2x^2 + x - 6x - 3$
g $3x^2 + 2x - 12x - 8$	**h** $4x^2 - 3x - 8x + 6$	**i** $9x^2 + 2x - 9x - 2$

K   FACTORISING $x^2 + bx + c$ [2.8]

A **quadratic trinomial** is an algebraic expression of the form $ax^2 + bx + c$ where x is a variable and a, b, c are constants, $a \neq 0$.

In this exercise we will look at quadratic trinomials for which $a = 1$. They have the form $x^2 + bx + c$.

Consider the expansion of the product $(x + 2)(x + 5)$:

$$
\begin{aligned}
(x + 2)(x + 5) &= x^2 + 5x + 2x + 2 \times 5 \qquad \{\text{using FOIL}\} \\
&= x^2 + [5 + 2]x + [2 \times 5] \\
&= x^2 + [\textbf{sum of 2 and 5}]x + [\textbf{product of 2 and 5}] \\
&= x^2 + 7x + 10
\end{aligned}
$$

In general,

$$
\mathbf{x^2} \; + \; \underset{\substack{\text{the coefficient of} \\ x \text{ is the } \textbf{sum} \text{ of} \\ \alpha \text{ and } \beta}}{\mathbf{(\alpha + \beta)\, x}} \; + \; \underset{\substack{\text{the constant term is} \\ \text{the } \textbf{product} \text{ of } \alpha \\ \text{and } \beta}}{\boldsymbol{\alpha\beta}} \; = \; \mathbf{(x + \alpha)\,(x + \beta)}
$$

So, to factorise $x^2 + 7x + 10$ into $(x + ...)(x + ...)$, we seek two numbers which add to 7, and when multiplied give 10.

These numbers are $+2$ and $+5$, so $x^2 + 7x + 10 = (x + 2)(x + 5)$

Example 30   ◀ᴑ Self Tutor

Factorise: $x^2 + 11x + 24$

We need to find two numbers which have sum $= 11$ and product $= 24$.

Pairs of factors of 24:

Factor product	1×24	2×12	3×8	4×6
Factor sum	25	14	11	10

The numbers we want are   3 and 8.

$\therefore$ $x^2 + 11x + 24 = (x + 3)(x + 8)$

Most of the time we can find the two numbers mentally.

With practice, you should be able to perform factorisations like this in your head.

Example 31   ◀ᴑ Self Tutor

Factorise: $x^2 - 7x + 12$

sum $= -7$ and   product $= 12$
$\therefore$ the numbers are -3 and -4
$\therefore$ $x^2 - 7x + 12 = (x - 3)(x - 4)$

The sum is negative but the product is positive, so both numbers must be negative.

Example 32   ◀)) Self Tutor

Factorise: **a** $x^2 - 2x - 15$ **b** $x^2 + x - 6$

a sum $= -2$ and   product $= -15$
$\therefore$ the numbers are -5 and $+3$
$\therefore$ $x^2 - 2x - 15 = (x - 5)(x + 3)$

b sum $= 1$ and   product $= -6$
$\therefore$ the numbers are $+3$ and -2
$\therefore$ $x^2 + x - 6 = (x + 3)(x - 2)$

> The product is negative and the numbers are opposite in sign.

> Always look for **common factors** first.

Example 33   ◀)) Self Tutor

Fully factorise by first removing a common factor: $3x^2 + 6x - 72$

$3x^2 + 6x - 72$ {first look for a **common factor**}
$= 3(x^2 + 2x - 24)$ {sum $= 2$,   product $= -24$
$= 3(x + 6)(x - 4)$ $\therefore$ the numbers are 6 and -4}

Example 34   ◀)) Self Tutor

Fully factorise by first removing a common factor: $77 + 4x - x^2$

$77 + 4x - x^2$
$= -x^2 + 4x + 77$ {rewrite in descending powers of x}
$= -1(x^2 - 4x - 77)$ {remove -1 as a common factor}
$= -(x - 11)(x + 7)$ {sum $= -4$, product $= -77$
                          $\therefore$ the numbers are -11 and 7}

EXERCISE 1K

1 Find two numbers which have:
  a product 12 and sum 7   **b** product 15 and sum 8   **c** product 16 and sum 10
  d product 18 and sum 11   **e** product -21 and sum 4   **f** product -21 and sum -4
  g product -12 and sum -4 **h** product -30 and sum 13.

2 Factorise:
  a $x^2 + 4x + 3$ **b** $x^2 + 14x + 24$ **c** $x^2 + 10x + 21$
  d $x^2 + 15x + 54$ **e** $x^2 + 9x + 20$ **f** $x^2 + 8x + 15$
  g $x^2 + 10x + 24$ **h** $x^2 + 9x + 14$ **i** $x^2 + 6x + 8$

3 Factorise:
  a $x^2 - 3x + 2$ **b** $x^2 - 4x + 3$ **c** $x^2 - 5x + 6$
  d $x^2 - 14x + 33$ **e** $x^2 - 16x + 39$ **f** $x^2 - 19x + 48$
  g $x^2 - 11x + 28$ **h** $x^2 - 14x + 24$ **i** $x^2 - 20x + 36$

4 Factorise:

   a $x^2 - 7x - 8$ **b** $x^2 + 4x - 21$ **c** $x^2 - x - 2$

   d $x^2 - 2x - 8$ **e** $x^2 + 5x - 24$ **f** $x^2 - 3x - 10$

   g $x^2 + 3x - 54$ **h** $x^2 + x - 72$ **i** $x^2 - 4x - 21$

   j $x^2 - x - 6$ **k** $x^2 - 7x - 60$ **l** $x^2 + 7x - 60$

   m $x^2 + 3x - 18$ **n** $x^2 - 7x - 18$ **o** $x^2 - 12x + 35$

5 Factorise:

   a $x^2 + 7x + 6$ **b** $x^2 - 2x - 63$ **c** $x^2 - 11x + 18$

   d $x^2 + 6x - 16$ **e** $x^2 - 5x + 4$ **f** $x^2 + 12x + 35$

   g $x^2 - x - 20$ **h** $x^2 - 9x - 22$ **i** $x^2 + 8x - 48$

   j $x^2 - 3x - 28$ **k** $x^2 + 13x$ **l** $x^2 - 14x + 49$

6 Fully factorise by first removing a common factor:

   a $2x^2 + 10x + 8$ **b** $3x^2 - 21x + 18$ **c** $2x^2 + 14x + 24$

   d $2x^2 - 44x + 240$ **e** $4x^2 - 8x - 12$ **f** $3x^2 - 42x + 99$

   g $2x^2 - 2x - 180$ **h** $3x^2 - 6x - 24$ **i** $2x^2 + 18x + 40$

   j $x^3 - 7x^2 - 8x$ **k** $4x^2 - 24x + 36$ **l** $7x^2 + 21x - 70$

   m $5x^2 - 30x - 80$ **n** $x^3 - 3x^2 - 28x$ **o** $x^4 + 2x^3 + x^2$

7 Fully factorise:

   a $-x^2 - 3x + 54$ **b** $-x^2 - 7x - 10$ **c** $-x^2 - 10x - 21$

   d $4x - x^2 - 3$ **e** $-4 + 4x - x^2$ **f** $3 - x^2 - 2x$

   g $-x^2 + 2x + 48$ **h** $6x - x^2 - 9$ **i** $10x - x^2 - 21$

   j $-2x^2 + 4x + 126$ **k** $20x - 2x^2 - 50$ **l** $-x^3 + x^2 + 2x$

L   SPLITTING THE MIDDLE TERM   [2.8]

In this section we will consider quadratic trinomials of the form $ax^2 + bx + c$ where $a \neq 1$.

In the following Discovery we will learn a useful technique for their factorisation.

Discovery 2   ***Splitting the middle term***

      Consider $(2x + 3)(4x + 5)$

                     $= 8x^2 + 10x + 12x + 15$ {using FOIL}

                     $= 8x^2 + 22x + 15$

      In reverse, $8x^2 + 22x + 15$

                     $= \underline{8x^2 + 10x} + \underline{12x + 15}$

                     $= 2x(4x + 5) + 3(4x + 5)$

                     $= (4x + 5)(2x + 3)$

So, we can factorise $8x^2 + 22x + 15$ into $(2x + 3)(4x + 5)$ by splitting the $+22x$ into a suitable sum, in this case $+ 10x + 12x$

In general, if we start with a quadratic trinomial we will need a method to work out how to do the splitting.

Consider the expansion in greater detail:

$(2x + 3)(4x + 5) = 2 \times 4 \times x^2 + [2 \times 5 + 3 \times 4]x + 3 \times 5 = 8x^2 + 22x + 15$

The four numbers   2, 3, 4 and 5   are present in the *middle term*, and also in the *first* and *last* terms combined.

As 2×5 and 3×4 are factors of $2 \times 3 \times 4 \times 5 = 120$,   this gives us the method for performing the splitting.

Step 1: Multiply the coefficient of x^2 and the constant term.	In our case, $8 \times 15 = 120$.
Step 2: Look for the factors of this number which add to give the coefficient of the middle term.	What factors of 120 add to give us 22? The answer is 10 and 12.
Step 3: These numbers are the coefficients of the split terms.	So, the split is $10x + 12x$.

Consider another example, $6x^2 + 17x + 12$.

The product of the coefficient of x^2 and the constant term is $6 \times 12 = 72$.

We now need two factors of 72 whose *sum* is 17.   These numbers are   8 and 9.

So, $6x^2 + 17x + 12$
$= \underbrace{6x^2 + 8x} + \underbrace{9x + 12}$ $\{17x$ has been split into $8x$ and $9x\}$
$= 2x(3x + 4) + 3(3x + 4)$
$= (3x + 4)(2x + 3)$

or $6x^2 + 17x + 12$
$= \underbrace{6x^2 + 9x} + \underbrace{8x + 12}$
$= 3x(2x + 3) + 4(2x + 3)$
$= (2x + 3)(3x + 4)$

> When splitting the middle term it does not matter the order in which you list the two new terms.

What to do:

1 For the following quadratics, copy and complete the table below:

	quadratic	product	sum	'split'
Example	$10x^2 + 29x + 21$	210	29	$14x + 15x$
a	$2x^2 + 11x + 12$			
b	$3x^2 + 14x + 8$			
c	$4x^2 + 16x + 15$			
d	$6x^2 - 5x - 6$			
e	$4x^2 - 13x + 3$			
f	$6x^2 - 17x + 5$			

2 Use your tabled results to factorise each of the quadratics in **1**.

The following procedure is recommended for factorising $ax^2 + bx + c$ by splitting the middle term:

Step 1:   Find ac.

Step 2:   Find the factors of ac which add to b.

Step 3:   If these factors are p and q, replace bx by $px + qx$.

Step 4:   Complete the factorisation.

Example 35   ◀ŋ *Self Tutor*

Factorise $3x^2 + 17x + 10$.

For $3x^2 + 17x + 10$, $3 \times 10 = 30$

We need to find two factors of 30 which have a sum of 17.

These are 2 and 15.

$$\begin{aligned} \therefore \quad 3x^2 + 17x + 10 &= 3x^2 + 2x + 15x + 10 \\ &= x(3x + 2) + 5(3x + 2) \\ &= (3x + 2)(x + 5) \end{aligned}$$

Example 36   ◀ŋ *Self Tutor*

Factorise $6x^2 - 11x - 10$.

For $6x^2 - 11x - 10$, $6 \times -10 = -60$

We need to find two factors of -60 which have a sum of -11.

These are -15 and 4.

$$\begin{aligned} \therefore \quad 6x^2 - 11x - 10 &= 6x^2 - 15x + 4x - 10 \\ &= 3x(2x - 5) + 2(2x - 5) \\ &= (2x - 5)(3x + 2) \end{aligned}$$

As the product is negative, the two numbers must be opposite in sign. Since the sum is negative, the larger number must be negative.

EXERCISE 1L

1 Fully factorise:

a $2x^2 + 5x + 3$	**b** $2x^2 + 7x + 5$	**c** $7x^2 + 9x + 2$
d $3x^2 + 7x + 4$	**e** $3x^2 + 13x + 4$	**f** $3x^2 + 8x + 4$
g $8x^2 + 14x + 3$	**h** $21x^2 + 17x + 2$	**i** $6x^2 + 5x + 1$
j $6x^2 + 19x + 3$	**k** $10x^2 + 17x + 3$	**l** $14x^2 + 37x + 5$

2 Fully factorise:

a $2x^2 - 9x - 5$	**b** $3x^2 + 5x - 2$	**c** $3x^2 - 5x - 2$
d $2x^2 + 3x - 2$	**e** $2x^2 + 3x - 5$	**f** $5x^2 - 14x - 3$
g $5x^2 - 8x + 3$	**h** $11x^2 - 9x - 2$	**i** $3x^2 - 7x - 6$

   j $2x^2 - 3x - 9$ **k** $3x^2 - 17x + 10$ **l** $5x^2 - 13x - 6$

   m $3x^2 + 10x - 8$ **n** $2x^2 + 17x - 9$ **o** $2x^2 + 9x - 18$

   p $2x^2 + 11x - 21$ **q** $15x^2 + x - 2$ **r** $21x^2 - 62x - 3$

3 Fully factorise:

   a $15x^2 + 19x + 6$ **b** $15x^2 + x - 6$ **c** $15x^2 - x - 6$

   d $30x^2 - 38x + 12$ **e** $18x^2 - 12x + 2$ **f** $48x^2 + 72x + 27$

   g $16x^2 + 12x + 2$ **h** $16x^2 + 4x - 2$ **i** $40x^2 - 10x - 5$

   j $32x^2 - 24x + 4$ **k** $25x^2 + 25x + 6$ **l** $25x^2 - 25x + 6$

   m $25x^2 - 10x - 8$ **n** $25x^2 - 149x - 6$ **o** $36x^2 + 24x - 5$

   p $36x^2 + 11x - 5$ **q** $36x^2 + 9x - 10$ **r** $36x^2 + 52x - 3$

M   MISCELLANEOUS FACTORISATION   [2.8]

The following flowchart may prove useful:

> **Expression** to be factorised.
>
> Remove any **common factors**.

Look for the **difference of two squares**	Look for **perfect squares**.	For four terms, look for **grouping in pairs**.	Look for the **sum and product** type.	Look for **splitting the middle term**.
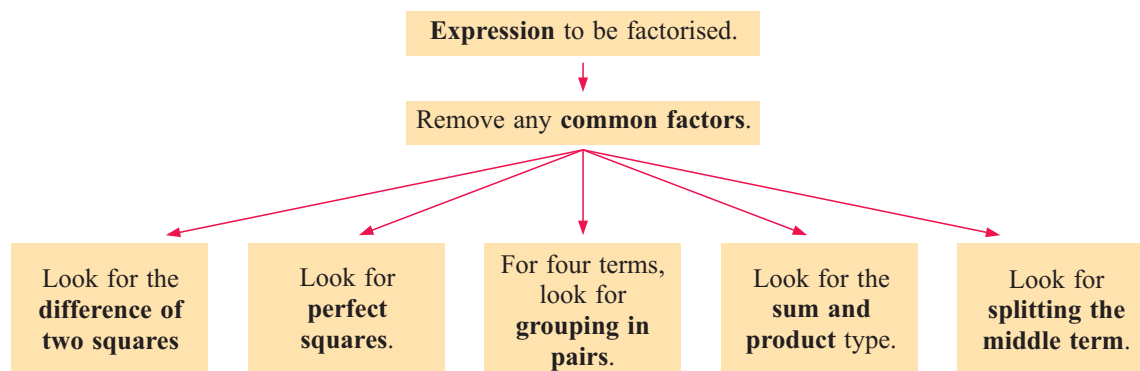
EXERCISE 1M

1 Fully factorise:

   a $3x^2 + 2x$ **b** $x^2 - 81$ **c** $2p^2 + 8$

   d $3b^2 - 75$ **e** $2x^2 - 32$ **f** $n^4 - 4n^2$

   g $x^2 - 8x - 9$ **h** $d^2 + 6d - 7$ **i** $x^2 + 8x - 9$

   j $4t + 8t^2$ **k** $3x^2 - 108$ **l** $2g^2 - 12g - 110$

   m $4a^2 - 9d^2$ **n** $5a^2 - 5a - 10$ **o** $2c^2 - 8c + 6$

   p $x^4 - x^2$ **q** $d^4 + 2d^3 - 3d^2$ **r** $x^3 + 4x^2 + 4x$

2 Find the pattern in the following expressions and hence factorise:

   a $x^2 - 6x + 9$ **b** $x^2 - 121$ **c** $x^2 - 2x + 1$

   d $y^2 + 10y + 25$ **e** $x^2 + 22x + 121$ **f** $x^2 - 2xy + y^2$

   g $1 - x^2$ **h** $25y^2 - 1$ **i** $49y^2 - 36z^2$

   j $4d^2 + 28d + 49$ **k** $4ab^2 - ac^2$ **l** $2\pi R^2 - 2\pi r^2$

3 Fully factorise:

   a $ab + ac - 2a$ **b** $a^2b^2 - 2ab$ **c** $18x - 2x^3$

   d $x^2 + 14x + 49$ **e** $4a^3 - 4ab^2$ **f** $x^3y - 4xy$

   g $4x^4 - 4x^2$ **h** $(x - 2)y - (x - 2)z$ **i** $(x + 1)a + (x + 1)b$

   j $(x - y)a + (x - y)$ **k** $x(x + 2) + 3(x + 2)$ **l** $x^3 + x^2 + x + 1$

4 Factorise completely:

   a $7x - 35y$ **b** $2g^2 - 8$ **c** $-5x^2 - 10x$

   d $m^2 + 3mp$ **e** $a^2 + 8a + 15$ **f** $m^2 - 6m + 9$

   g $5x^2 + 5xy - 5x^2y$ **h** $xy + 2x + 2y + 4$ **i** $y^2 + 5y - 9y - 45$

   j $2x^2 + 10x + x + 5$ **k** $3y^2 - 147$ **l** $3p^2 - 3q^2$

   m $4c^2 - 1$ **n** $3x^2 + 3x - 36$ **o** $2bx - 6b + 10x - 30$

5 Fully factorise:

   a $12 - 11x - x^2$ **b** $-2x^2 - 6 + 8x$ **c** $14 - x^2 - 5x$

   d $4x^2 - 2x^3 - 2x$ **e** $(a + b)^2 - 9$ **f** $(x + 2)^2 - 4$

6 Fully factorise:

   a $2x^2 + 17x + 21$ **b** $2x^2 + 11x + 15$ **c** $4x^2 + 12x + 5$

   d $12x^2 + 13x + 3$ **e** $6x^2 - 29x - 5$ **f** $16x^2 + 8x + 1$

   g $25x^2 - 16$ **h** $12x^2 - 71x - 6$ **i** $12x^2 - 38x + 6$

   j $9x^2 + 3x - 12$ **k** $12x^2 - 29x + 15$ **l** $36x^2 + 3x - 14$

Review set 1A

1 Expand and simplify:

   a $3x(x - 2)$ **b** $-3x(x - 5)$ **c** $(x + 3)(x - 8)$

   d $(x + 3)^2$ **e** $-(x - 2)^2$ **f** $(4x + 1)(4x - 1)$

   g $(4x + 1)(3x - 2)$ **h** $(x + 3)(x - 1) - (3 - x)(x + 4)$

2 Expand and simplify:

   a $(x^2 + 3)^2$ **b** $(2 - 3d)(2 + 3d)$ **c** $(x - 5)^3$

   d $(x + 4)(x^2 - x + 2)$ **e** $(2x - 5)(4 - x)$ **f** $(4x + y)(4x - y)$

3 Find the HCF of:

   a $6c$ and $15c^2$ **b** $4pq$ and $8p$ **c** $18r^2s$ and $15rs^2$.

4 Fully factorise:

   a $3x^2 - 12x$ **b** $15x - 6x^2$ **c** $2x^2 - 98$

   d $x^2 - 6x + 9$ **e** $a^2 + 2ab + b^2$ **f** $(x + 2)^2 - 3(x + 2)$

5 Fully factorise:

   a $5x - 5 + xy - y$ **b** $3x + 7 + 6bx + 14b$

6 Fully factorise:

    a $x^2 + 10x + 21$ **b** $x^2 + 4x - 21$ **c** $x^2 - 4x - 21$

    d $6 - 5x + x^2$ **e** $4x^2 - 8x - 12$ **f** $-x^2 - 13x - 36$

7 Fully factorise:

    a $8x^2 + 22x + 15$ **b** $12x^2 - 20x + 3$ **c** $12x^2 - 7x - 10$

8 If possible, factorise into linear factors:

    a $x^2 - 10$ **b** $x^2 + 16$ **c** $(x - 4)^2 - 13$

Review set 1B

1 Expand and simplify:

    a $(3x - y)^2$ **b** $-2a\,(b - a)$ **c** $(4x + 1)\,(1 - 3x)$

    d $(2x + 7)^2$ **e** $-(5 - x)^2$ **f** $(1 - 7x)\,(1 + 7x)$

    g $(4x + 1)\,(5x - 4)$ **h** $2\,(x + 3)\,(x + 2) - 3\,(x + 2)\,(x - 1)$

2 Expand and simplify:

    a $4(x - 2) + 3(2x - 1)$ **b** $(3x + 2)^2$ **c** $(8 + q)(8 - q)$

    d $(x + 6)(4x - 3)$ **e** $(2x - 1)^3$ **f** $(x - 3)(x^2 - 4x + 2)$

3 Fully factorise:

    a $5ab + 10b^2$ **b** $3x^2 - 12$ **c** $x^2 + 8x + 16$

    d $2a^2 - 4ab + 2b^2$ **e** $3x^3 + 6x^2 - 9x$ **f** $(x - 3)^2 - 3x + 9$

4 Fully factorise: $2xy - z - 2xz + y$

5 Fully factorise:

    a $x^2 + 12x + 35$ **b** $x^2 + 2x - 35$ **c** $x^2 - 12x + 35$

    d $2x^2 - 4x - 70$ **e** $30 - 11x + x^2$ **f** $-x^2 + 12x - 20$

6 If possible, factorise into linear factors:

    a $x^2 - 81$ **b** $2x^2 - 38$ **c** $x^2 + 25$

7 Fully factorise:

    a $12x^2 + 5x - 2$ **b** $12x^2 + x - 6$ **c** $24x^2 + 28x - 12$

8 Fully factorise:

    a $cd + 9 + 3d + 3c$ **b** $(4 - x)(x + 2) - 3(4 - x)$

    c $6x^2 - 17x + 12$

9   **a** By expanding $(3x + 2)^2$ and $(2x + 5)^2$, show that $(3x + 2)^2 - (2x + 5)^2 = 5x^2 - 8x - 21$.

      b Factorise $5x^2 - 8x - 21$.

      c Factorise $(3x + 2)^2 - (2x + 5)^2$ using $a^2 - b^2 = (a + b)(a - b)$. What do you notice?
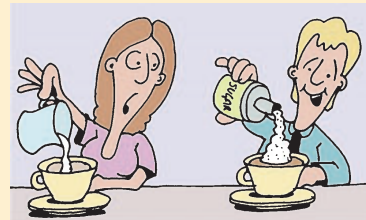
Sets

2

Contents:

Opening problem

A survey of 50 tea-drinkers found that 32 people surveyed put milk in their tea, 19 put sugar in their tea, and 10 put both milk and sugar in their tea.

How many of the people surveyed have their tea with:

- milk but not sugar
- milk or sugar
- neither milk nor sugar?

A   SET NOTATION   [9.1, 9.2]

A set is a collection of objects or things.

A set is usually denoted by a capital letter.

For example: E is the set of all year 11 students who study English.

C is the set of all cars parked in the school's car park

P is the set of all prime numbers less than 10.

These sets could be written in the form: $E = \{$year 11 students who study English$\}$

$C = \{$cars parked in the school's car park$\}$

$P = \{2, 3, 5, 7\}$

where the curly brackets are read as 'the set of'.

The **elements** of a set are the objects or members which make up the set.

We use $\in$ to mean '*is an element of*' or '*is a member of*'
and $\notin$ to mean '*is not an element of*' or '*is not a member of*'.

So, for $P = \{2, 3, 5, 7\}$ we can write $2 \in P$ and $4 \notin P$.

There are 4 elements in the set P, so we write $n(P) = 4$.

$n(A)$ reads 'the number of elements in set A'.

SUBSETS

The elements of the set $S = \{2, 5, 7\}$ are also elements of the set $P = \{2, 3, 5, 7\}$.

We say that S is a **subset** of P, and write $S \subseteq P$.

A is a **subset** of B if all elements of A are also elements of B.
We write $A \subseteq B$.

For example, $\{1, 3\} \subseteq \{1, 2, 3\}$ but $\{1, 2, 3\} \nsubseteq \{1, 3\}$.

Two sets A and B are **equal** if their elements are exactly the same.

For example, $\{2, 3, 5, 7\} = \{5, 3, 7, 2\}$

A is a **proper subset** of B if every element of A is also an element of B, but $A \neq B$.
We write $A \subset B$.

THE UNIVERSAL SET

Associated with any set is a universal set, denoted U.

The **universal set** U contains all of the elements under consideration.

For example, if we are considering the positive integers less than 20, then

$$U = \{1, 2, 3, 4, 5, 6, 7, \ldots\ldots, 19\}.$$

These dots indicate the continuation of the pattern up to the final element.

In U, the set of prime numbers is $P = \{2, 3, 5, 7, 11, 13, 17, 19\}$ and the set of composite numbers is $Q = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18\}$.

Notice that $P \subseteq U$ and $Q \subseteq U$.

THE EMPTY SET

Sometimes we find that a set has no elements. Such a set is called the **empty set**, and is denoted $\varnothing$ or $\{\ \}$.
The empty set $\varnothing$ is a proper subset of any other set.

THE COMPLEMENT OF A SET

Suppose $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $A = \{2, 4, 5, 7, 8\}$

The set of elements $\{1, 3, 6\}$ includes all elements of U that are *not* elements of A. We call this set the *complement* of A, and denote it A'.

No element of A is in A', and no element of A' is in A.

> The **complement** of A is the set of all elements of U which are *not* elements of A.
> We denote the complement A'.

EXERCISE 2A

1 $U = \{1, 2, 3, 4, 5, ..., 12\}$, $S = \{2, 4, 7, 9, 11\}$ and $T = \{4, 11\}$.

    a Find: **i** $n(U)$ **ii** $n(S)$ **iii** $n(T)$.

    b List the sets S' and T'.

    c True or false?   **i** $S \subseteq U$ **ii** $S \subseteq T$ **iii** $T \subset S$

    d True or false?   **i** $5 \in S$ **ii** $5 \notin T$

    e If $R = \{4, 7, 11, 9, x\}$ and $S \subseteq R$, find x.

    f Is S finite or infinite? Explain your answer.

2 The subsets of $\{a, b\}$ are $\varnothing$, $\{a\}$, $\{b\}$ and $\{a, b\}$.

    a List all subsets of $\{a\}$.

    b List all subsets of $\{a, b, c\}$.

    c Predict the number of subsets of $\{a, b, c, d\}$ without listing them.

3 List all proper subsets of $S = \{2, 4, 7, 9\}$.

4 List the elements of the set S which contains the:

    a factors of 6   **b** multiples of 6

    c factors of 17   **d** multiples of 17

    e prime numbers less than 20

    f composite numbers between 10 and 30.

> Prime numbers have exactly two factors. Composite numbers have more than 2 factors.

5 Find $n(S)$ for each set in **4**.

6 Suppose $A = \{$prime numbers between 20 and 30$\}$, $B = \{$even numbers between 20 and 30$\}$,
$C = \{$composite numbers between 20 and 30$\}$, and $D = \{$multiples of 18 between 20 and 30$\}$.

    a List the elements of each set.

    b Find: **i** $n(A)$ **ii** $n(D)$.

    c Which of the sets listed are:

        i subsets of A **ii** proper subsets of C?

    d True or false?   **i** $23 \in C$ **ii** $27 \notin A$ **iii** $25 \in B$

7 **a** Suppose $U = \{2, 3, 4, 5, 6, 7, 8\}$, $A = \{2, 3, 4, 7\}$, and $B = \{2, 5\}$. Find:

        i $n(U)$ **ii** $n(A)$ **iii** $n(A')$ **iv** $n(B)$ **v** $n(B')$

    b Copy and complete: For any set $S \subseteq U$ when U is the universal set, $n(S) + n(S') = ...$

B   SPECIAL NUMBER SETS   [9.2]

There are some number sets we refer to frequently and so we give them special symbols. We use:

- $\mathbb{N}$ to represent the set of all **natural** or **counting** numbers $\{0, 1, 2, 3, 4, 5, 6\ ......\}$
  The set of natural numbers is endless, so we say $n(\mathbb{N})$ is infinite.

- $\mathbb{Z}$ to represent the set of all **integers** $\{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\ ......\}$
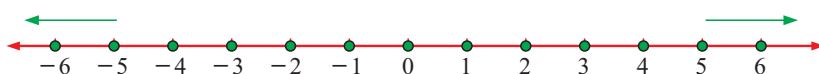
- $\mathbb{Z}^+$ to represent the set of all **positive integers** $\{1, 2, 3, 4, 5, 6\ ......\}$

- $\mathbb{Q}$ to represent the set of all **rational numbers** which have the form $\dfrac{p}{q}$
  where p and q are integers and $q \neq 0$.

- $\mathbb{R}$ to represent the set of all **real numbers**, which are numbers which can be placed on a number line.

Example 1   ◀)) *Self Tutor*

True or false?   Give reasons for your answers.

  a $2 \in \mathbb{Z}$ **b** $2\frac{1}{2} \in \mathbb{Q}$ **c** $5 \notin \mathbb{Q}$ **d** $\pi \in \mathbb{Q}$ **e** $-2 \notin \mathbb{R}$

  a $2 \in \mathbb{Z}$ is true as $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, ...\}$

  b $2\frac{1}{2} \in \mathbb{Q}$ is true as $2\frac{1}{2} = \frac{5}{2}$ where 5 and 2 are integers.

  c $5 \notin \mathbb{Q}$ is false as $5 = \frac{5}{1}$ where 5 and 1 are integers.

  d $\pi \in \mathbb{Q}$ is false as π is a known irrational number.

  e $-2 \notin \mathbb{R}$ is false as -2 can be put on the number line.

Example 2   ◀)) *Self Tutor*

Show that $0.\overline{36}$, which is $0.36363636....$, is a rational number.

    Let $x = 0.\overline{36} = 0.36363636....$

$\therefore$ $100x = 36.363636.... = 36 + x$

$\therefore$ $99x = 36$ and so $x = \frac{36}{99} = \frac{4}{11}$

So, $0.\overline{36}$ is actually the rational number $\frac{4}{11}$.

EXERCISE 2B

1 True or false?

   a $3 \in \mathbb{Z}^+$ **b** $6 \in \mathbb{Z}$ **c** $\frac{3}{4} \in \mathbb{Q}$ **d** $\sqrt{2} \notin \mathbb{Q}$

   e $-\frac{1}{4} \notin \mathbb{Q}$ **f** $2\frac{1}{3} \in \mathbb{Z}$ **g** $0.3684 \in \mathbb{R}$ **h** $\dfrac{1}{0.1} \in \mathbb{Z}$

2 Which of these are rational?

   a 8 **b** -8 **c** $2\frac{1}{3}$ **d** $-3\frac{1}{4}$

   e $\sqrt{3}$ **f** $\sqrt{400}$ **g** 9.176 **h** $\pi - \pi$

3 Show that these numbers are rational: **a** $0.\overline{7}$ **b** $0.\overline{41}$ **c** $0.\overline{324}$

4 **a** Explain why 0.527 is a rational number.

   b $0.\overline{9}$ is a rational number. In fact, $0.\overline{9} \in \mathbb{Z}$. Give evidence to support this statement.

5 Explain why these statements are false:

   a The sum of two irrationals is irrational.   **b** The product of two irrationals is irrational.

6 True or false?   **a** $\mathbb{N} \subseteq \mathbb{Z}$ **b** $\mathbb{R} \subseteq \mathbb{Q}$ **c** $\mathbb{Z} \subseteq \mathbb{Q}$.

C   INTERVAL NOTATION   [9.2]

Interval notation allows us to quickly describe sets of numbers using mathematical symbols only.

For example: $\{x \mid -3 < x \leqslant 2, \ x \in \mathbb{R}\}$ reads 'the set of all real x such that x lies between negative 3 and 2, including 2'.

                   Unless stated otherwise, we assume we are dealing with *real* numbers. Thus, the set can also be written as $\{x \mid -3 < x \leqslant 2\}$.

                   We can represent the set on a number line as:
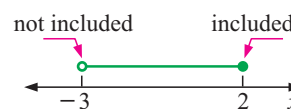
Sometimes we want to restrict a set to include only integers or rationals.

For example: $\{x \mid -5 < x < 5, \ x \in \mathbb{Z}\}$

                   reads 'the set of all integers x such that x lies between negative 5 and 5'.

                   We can represent the set on a number line as:

Example 3   ◀)) *Self Tutor*

Write in interval notation:

  a

  b
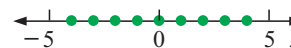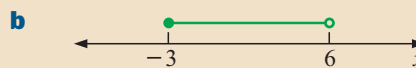
   a $\{x \mid 1 \leqslant x \leqslant 5, \ x \in \mathbb{N}\}$ **b** $\{x \mid -3 \leqslant x < 6\}$
      or $\{x \mid 1 \leqslant x \leqslant 5, \ x \in \mathbb{Z}\}$

Example 4　　　　　　　　　　　　　　　　　　　　　🔊 *Self Tutor*

a List the elements of the set:
　　i $\{x \mid -2 \leqslant x \leqslant 1, \ x \in \mathbb{Z}\}$　　　　　　**ii** $\{x \mid 0 \leqslant x < 4, \ x \in \mathbb{N}\}$

b Write in interval notation:　$\{-3, -2, -1, 0, 1, 2, 3, 4, ...\}$

a　**i**　$\{x \mid -2 \leqslant x \leqslant 1, \ x \in \mathbb{Z}\}$　　　**ii**　$\{x \mid 0 \leqslant x < 4, \ x \in \mathbb{N}\}$
　　　　$= \{-2, -1, 0, 1\}$　　　　　　　　　　$= \{0, 1, 2, 3\}$

b　We first notice that the elements of the set are all integers, so we are dealing with $x \in \mathbb{Z}$.
　　$\{-3, -2, -1, 0, 1, 2, ...\} = \{x \mid x \geqslant -3, \ x \in \mathbb{Z}\}$

EXERCISE 2C

1 Write verbal statements for the meaning of:

　a $\{x \mid x > 4\}$　　　　　　**b** $\{x \mid x \leqslant 5\}$　　　　　　**c** $\{y \mid 0 < y < 8\}$

　d $\{x \mid 1 \leqslant x \leqslant 4\}$　　　　**e** $\{t \mid 2 < t < 7\}$　　　　**f** $\{n \mid n \leqslant 3 \ \text{or} \ n > 6\}$

2 List the elements of the set:

　a $\{x \mid 1 < x \leqslant 6, \ x \in \mathbb{N}\}$　　**b** $\{x \mid x \geqslant 5, \ x \in \mathbb{Z}\}$　　**c** $\{x \mid x < 3, \ x \in \mathbb{Z}\}$

　d $\{x \mid -3 \leqslant x \leqslant 5, \ x \in \mathbb{N}\}$　**e** $\{x \mid x \geqslant -4, \ x \in \mathbb{Z}\}$　**f** $\{x \mid x \leqslant 6, \ x \in \mathbb{Z}^{+}\}$

3 Write these sets in interval notation:

　a $\{-5, -4, -3, -2, -1\}$　**b** $\{0, 1, 2, 3, 4, 5\}$　　**c** $\{4, 5, 6, 7, 8, ...\}$

　d $\{......, -3, -2, -1, 0, 1\}$　**e** $\{-5, -4, -3, -2, -1, 0, 1\}$　**f** $\{......, 41, 42, 43, 44\}$

4 Write in interval notation:

　a

　b

　c

　d

　e

　f
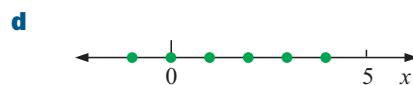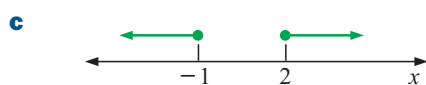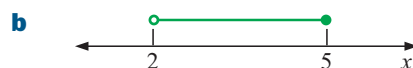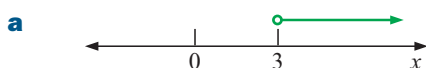
5 Sketch the following number sets:

　a $\{x \mid 4 \leqslant x < 8, x \in \mathbb{N}\}$　**b** $\{x \mid -5 < x \leqslant 4, x \in \mathbb{Z}\}$　**c** $\{x \mid -3 < x \leqslant 5, x \in \mathbb{R}\}$

　d $\{x \mid x > -5, x \in \mathbb{Z}\}$　　**e** $\{x \mid x \leqslant 6\}$　　　　**f** $\{x \mid -5 \leqslant x \leqslant 0\}$

6 Are the following sets finite or infinite?

　a $\{x \mid 1 \leqslant x \leqslant 10, x \in \mathbb{Z}\}$　**b** $\{x \mid x < 4, x \in \mathbb{Z}\}$　　**c** $\{x \mid x > 5, x \in \mathbb{Z}^{+}\}$

　d $\{x \mid x < 7, x \in \mathbb{Z}^{+}\}$　　**e** $\{x \mid x \leqslant 9, x \in \mathbb{N}\}$　　**f** $\{x \mid 2 \leqslant x \leqslant 6, x \in \mathbb{R}\}$

D   VENN DIAGRAMS   [9.3]

A **Venn diagram** consists of a universal set U represented by a rectangle, and sets within it that are generally represented by circles.

For example, consider the universal set $U = \{x \mid x \leqslant 10, \ x \in \mathbb{Z}^+\}$.

We can display the set $S = \{2, 4, 6, 7\}$ on the Venn diagram using a circle. The elements of S are placed within the circle, while the elements of S' are placed outside the circle.
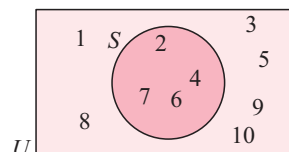
Example 5   ◀) Self Tutor

If $U = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and $E = \{2, 3, 5, 7\}$, list the set E' and illustrate E and E' on a Venn diagram.
Hence find:

a $n(E)$ **b** $n(E')$ **c** $n(U)$

$E' = \{0, 1, 4, 6\}$
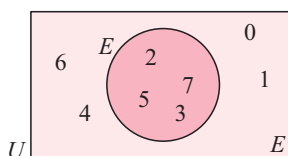
a E contains 4 elements, so $n(E) = 4$

b $n(E') = 4$ **c** $n(U) = 8$

Consider the sets $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{2, 3, 5, 7\}$ and $B = \{2, 7\}$.

We notice that $B \subset A$, so the circle representing B lies entirely within the circle representing A.
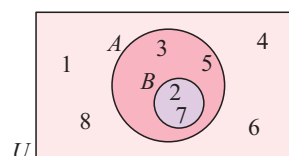
We can use this property to draw a Venn diagram for the special number sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$. In this case $\mathbb{R}$ is the universal set, and $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.
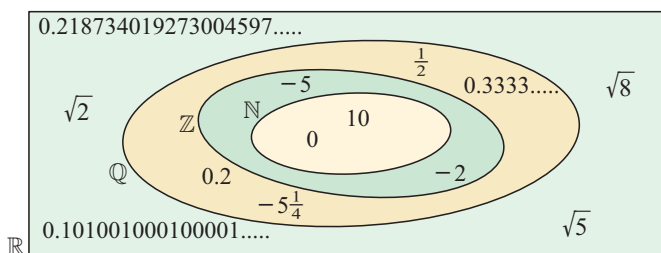
Example 6   ◀)) *Self Tutor*

Illustrate on a Venn diagram: $\sqrt{3}, 8\frac{1}{2}, -2, 7.1, 16, 0.115$
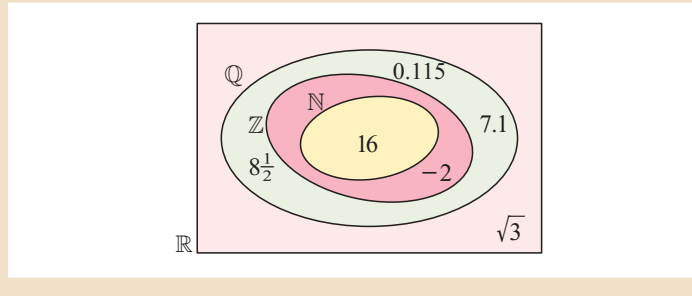
If two sets A and B have elements in common, but $A \nsubseteq B$ and $B \nsubseteq A$, the circles for these sets overlap.

Example 7   ◀)) *Self Tutor*

Consider the sets $U = \{x \mid 0 \leqslant x \leqslant 12, x \in \mathbb{Z}\}$, $A = \{2, 3, 5, 7, 11\}$
and $B = \{1, 3, 6, 7, 8\}$.   Show A and B on a Venn diagram.
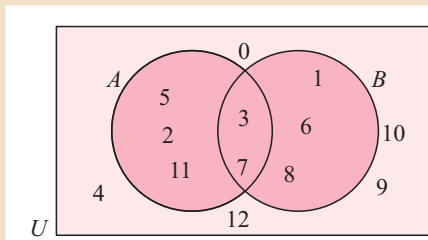
We notice that 3 and 7 are in both A and B so the circles representing A and B must overlap.

We place 3 and 7 in the overlap, then fill in the rest of A, then fill in the rest of B.

The remaining elements of U go outside the two circles.

EXERCISE 2D

1 Suppose $U = \{x \mid x \leqslant 8, x \in \mathbb{Z}^+\}$ and $A = \{$prime numbers $\leqslant 8\}$.
   a Show set A on a Venn diagram.   **b** List the set A'.   **c** Find $n(A)$ and $n(A')$.

2 Suppose $U = \{$letters of the English alphabet$\}$ and
            $V = \{$letters of the English alphabet which are vowels$\}$.

   a Show these two sets on a Venn diagram.   **b** List the set V'.

3
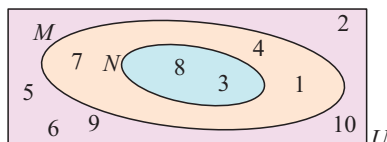
   a List the elements of:
           i U **ii** N **iii** M
   b Find $n(N)$ and $n(M)$.
   c Is $M \subseteq N$?

4 Show A and B on a Venn diagram if:
   a $U = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$
   b $U = \{4, 5, 6, 7, 8, 9, 10\}$, $A = \{6, 7, 9, 10\}$, $B = \{5, 6, 8, 9\}$
   c $U = \{3, 4, 5, 6, 7, 8, 9\}$, $A = \{3, 5, 7, 9\}$, $B = \{4, 6, 8\}$

5 Suppose the universal set is $U = \mathbb{R}$, the set of all real numbers.

$\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ are all subsets of $\mathbb{R}$.

   a Copy the given Venn diagram and label the sets U, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ on it.

   b Place these numbers on the Venn diagram:

      $\frac{1}{2}$, $\sqrt{2}$, $0.\overline{3}$, -5, $5\frac{1}{4}$, 0, 10,   and

      $0.2137005618.....$ which does not terminate or recur.

   c True or false?   **i** $\mathbb{N} \subseteq \mathbb{Z}$ **ii** $\mathbb{Z} \subseteq \mathbb{Q}$ **iii** $\mathbb{N} \subseteq \mathbb{Q}$

   d Shade the region representing the set of irrationals $\mathbb{Q}'$.
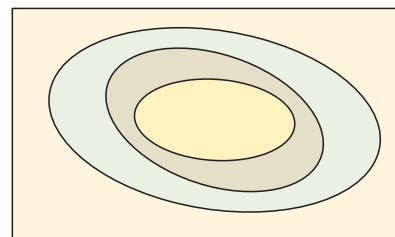
6 Show the following information on a Venn diagram:

   a $U = \{\text{triangles}\}$, $E = \{\text{equilateral triangles}\}$, $I = \{\text{isosceles triangles}\}$

   b $U = \{\text{quadrilaterals}\}$, $P = \{\text{parallelograms}\}$, $R = \{\text{rectangles}\}$

   c $U = \{\text{quadrilaterals}\}$, $P = \{\text{parallelograms}\}$, $R = \{\text{rectangles}\}$, $H = \{\text{rhombuses}\}$

   d $U = \{\text{quadrilaterals}\}$, $P = \{\text{parallelograms}\}$, $T = \{\text{trapezia}\}$

   e $U = \{\text{triangles}\}$, $I = \{\text{isosceles triangles}\}$, $R = \{\text{right angled triangles}\}$,
      $E = \{\text{equilateral triangles}\}$

7 Suppose $U = \{x \mid x \leqslant 30,\ x \in \mathbb{Z}^+\}$,
          $A = \{\text{prime numbers} \leqslant 30\}$,
          $B = \{\text{multiples of } 5 \leqslant 30\}$
   and $C = \{\text{odd numbers} \leqslant 30\}$.

Use the Venn diagram shown to display the elements of the sets.
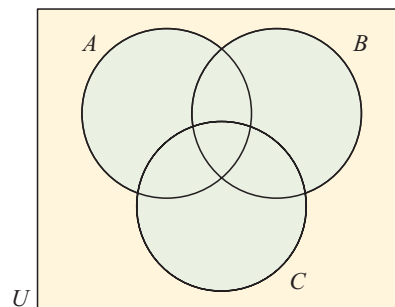
E   UNION AND INTERSECTION   [9.4]

THE UNION OF TWO SETS

$A \cup B$ denotes the **union** of sets A and B.

This set contains all elements belonging to A **or** B **or both** A and B.

$A \cup B = \{x \mid x \in A \ \textbf{or} \ x \in B\}$
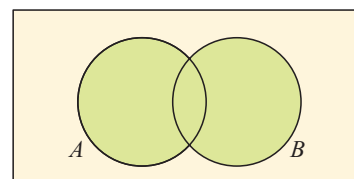
$A \cup B$ is shaded green.

THE INTERSECTION OF TWO SETS

$A \cap B$ denotes the **intersection** of sets A and B.

This is the set of all elements common to both sets.

$A \cap B = \{x \mid x \in A \ \textbf{and} \ x \in B\}$
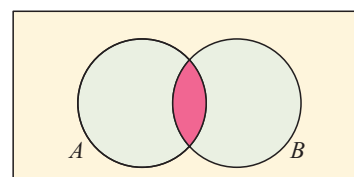
$A \cap B$ is shaded red.

In the Venn diagram alongside,

$A = \{2, 3, 4, 7\}$ and $B = \{1, 3, 7, 8, 10\}$.

We can see that $A \cap B = \{3, 7\}$

and $A \cup B = \{1, 2, 3, 4, 7, 8, 10\}$.
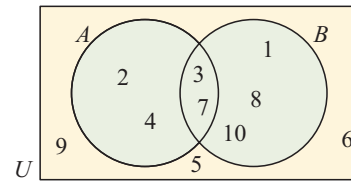
DISJOINT SETS

Two sets A and B are **disjoint** if they have no elements in common, or in other words if $A \cap B = \varnothing$.

If A and B have elements in common then they are **non-disjoint**.

Example 8   ◀ッ **Self Tutor**

If $U = \{\text{positive integers} \leqslant 12\}$, $A = \{\text{primes} \leqslant 12\}$ and $B = \{\text{factors of } 12\}$:

  a List the elements of the sets A and B.

  b Show the sets A, B and U on a Venn diagram.

  c List the elements in: **i** A' **ii** $A \cap B$ **iii** $A \cup B$

  d Find: **i** $n(A \cap B)$ **ii** $n(A \cup B)$ **iii** $n(B')$

  a $A = \{2, 3, 5, 7, 11\}$ and $B = \{1, 2, 3, 4, 6, 12\}$

  b
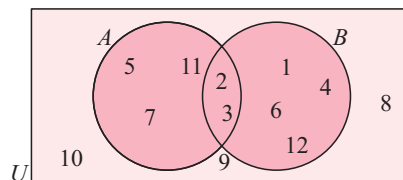
  c   **i** $A' = \{1, 4, 6, 8, 9, 10, 12\}$ **ii** $A \cap B = \{2, 3\}$

      iii $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 11, 12\}$

  d   **i** $n(A \cap B) = 2$ **ii** $n(A \cup B) = 9$

      iii $B' = \{5, 7, 8, 9, 10, 11\}$,   so $n(B') = 6$

EXERCISE 2E.1

1
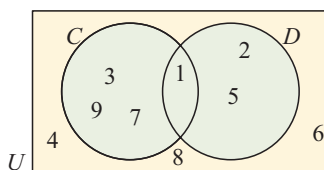
  a List:

      i set C **ii** set D **iii** set U

      iv set $C \cap D$ **v** set $C \cup D$

  b Find:

      i $n(C)$ **ii** $n(D)$ **iii** $n(U)$

      iv $n(C \cap D)$ **v** $n(C \cup D)$

2
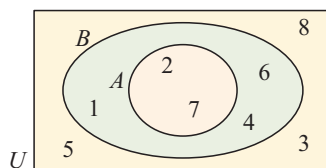
a List:

    i set A **ii** set B **iii** set U

    iv set $A \cap B$ **v** set $A \cup B$

b Find:

    i $n(A)$ **ii** $n(B)$ **iii** $n(U)$

    iv $n(A \cap B)$ **v** $n(A \cup B)$

3 Consider $U = \{x \mid x \leqslant 12,\ x \in \mathbb{Z}^+\}$,

          $A = \{2,\ 7,\ 9,\ 10,\ 11\}$ and $B = \{1,\ 2,\ 9,\ 11,\ 12\}$.

  a Show these sets on a Venn diagram.

  b List the elements of: **i** $A \cap B$ **ii** $A \cup B$ **iii** B'

  c Find: **i** $n(A)$ **ii** $n(B')$ **iii** $n(A \cap B)$ **iv** $n(A \cup B)$

4 If A is the set of all factors of 36 and B is the set of all factors of 63, find:

  a $A \cap B$ **b** $A \cup B$

5 If $X = \{$A, B, D, M, N, P, R, T, Z$\}$ and $Y = \{$B, C, M, T, W, Z$\}$, find:

  a $X \cap Y$ **b** $X \cup Y$

6 Suppose $U = \{x \mid x \leqslant 30,\ x \in \mathbb{Z}^+\}$,

          $A = \{$factors of 30$\}$ and $B = \{$prime numbers $\leqslant 30\}$.

  a Find: **i** $n(A)$ **ii** $n(B)$ **iii** $n(A \cap B)$ **iv** $n(A \cup B)$

  b Use **a** to verify that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

7
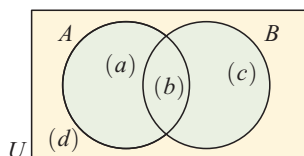
a Use the Venn diagram given to show that:

    $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

b Suppose A and B are disjoint events.

    Explain why $n(A \cup B) = n(A) + n(B)$.

> (a) means that there are a elements in this region, so $n(A) = a + b$.

8 Simplify:

  a $X \cap Y$ for $X = \{1,\ 3,\ 5,\ 7\}$ and $Y = \{2,\ 4,\ 6,\ 8\}$

  b $A \cup A'$ for any set $A \in U$.

  c $A \cap A'$ for any set $A \in U$.

USING VENN DIAGRAMS TO ILLUSTRATE REGIONS

We can use a Venn diagram to help illustrate regions such as the union or intersection of sets.

Shaded regions of a Venn diagram can be used to verify **set identities**. These are equations involving sets which are true for *all* sets.

Examples of set identities include:

$$A \cup A' = U \qquad\qquad\qquad A \cap A' = \varnothing$$
$$(A \cup B)' = A' \cap B' \qquad\qquad (A \cap B)' = A' \cup B'$$

Example 9   ◆) *Self Tutor*

On separate Venn diagrams, shade the region representing:

a in A or in B but not in both   **b** $A' \cap B$
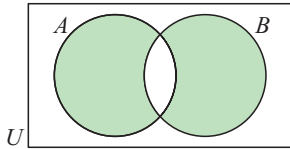
a   **b** We look for where the outside of
                                                     A intersects (overlaps) with B.
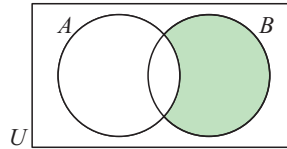
Example 10   ◆) *Self Tutor*

Verify that $(A \cup B)' = A' \cap B'$.

this shaded region is $(A \cup B)$

∴   this shaded region is $(A \cup B)'$

represents A'

represents B'
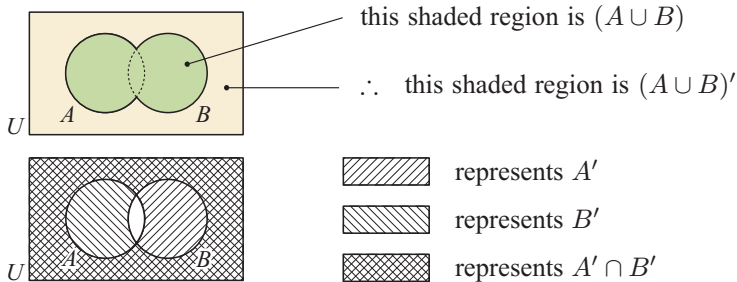
represents $A' \cap B'$

$(A \cup B)'$ and $A' \cap B'$ are represented by the same regions, verifying that
$(A \cup B)' = A' \cap B'$.

EXERCISE 2E.2

1 On separate Venn diagrams like the one given, shade the region representing:

    a not in A **b** in both A and B

    c $A \cap B'$ **d** in either A or B

    e $A \cup B'$ **f** $(A \cup B)'$

    g $(A \cap B)'$ **h** in exactly one of A or B.
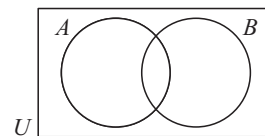
2 Describe in words, the shaded region of:
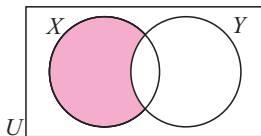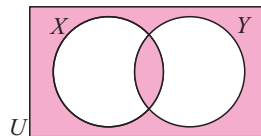
    a   **b**   **c**
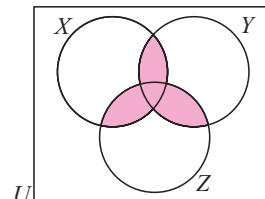
3 If A and B are two non-disjoint sets, shade the region of a Venn diagram representing:

    a A' **b** $A' \cap B$ **c** $A' \cup B$ **d** $A' \cap B'$

4 The Venn diagram alongside is the most general case for three events in the same sample space U. On separate Venn diagrams shade:

a A **b** B' **c** $B \cap C$

d $A \cup C$ **e** $A \cap B \cap C$ **f** $(A \cup B) \cap C$

g $(A \cup C) \cap B$ **h** $(A \cap C) \cup B$ **i** $(A \cup B)' \cap C$
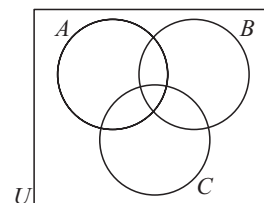
5 Verify that:

a $(A \cap B)' = A' \cup B'$

b $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

c $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

DEMO
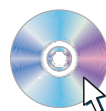
F PROBLEM SOLVING   [9.3, 9.4]

When we solve problems with Venn diagrams, we generally do not deal with individuals. Instead, we simply record the *number* of individuals in each region.

Example 11   ◀) Self Tutor

The Venn diagram alongside illustrates the number of people in a sporting club who play tennis (T) and hockey (H).

Determine the number of people:

a in the club   **b** who play hockey

c who play both sports   **d** who play neither sport

e who play at least one sport.
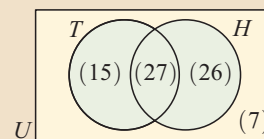
a Number in the club $= 15 + 27 + 26 + 7 = 75$ **b** Number who play hockey $= 27 + 26 = 53$

c Number who play both sports $= 27$ **d** Number who play neither sport $= 7$

e Number who play at least one sport $= 15 + 27 + 26 = 68$

Example 12   ◀) Self Tutor

In a class of 24 boys, 16 play football and 11 play baseball. If two play neither game, how many play both games?

Method 1:

Let x be the number who play both games.

∴ $16 - x$ play football and $11 - x$ play baseball.

2 boys play neither sport.

∴ $(16 - x) + x + (11 - x) + 2 = 24$

∴ $29 - x = 24$

∴ $x = 5$

So, 5 play both games.
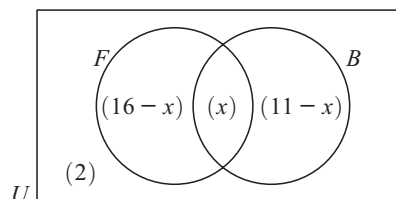
Method 2:

2 boys play neither game, so $24 - 2 = 22$ must be in the union $F \cup B$.

However, 16 are in the F circle, so 6 must go in the rest of B. —————————

But B contains 11 in total, so 5 go in the intersection $F \cap B$.

So, 5 play both games.
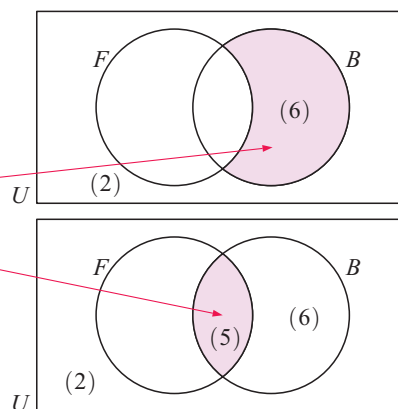
Example 13   ◀⟩ *Self Tutor*

In a senior class all 29 students take one or more of biology, chemistry and physics. The headmaster is informed that 15 take biology, 15 take chemistry, and 18 take physics.

10 take biology and chemistry, 5 take chemistry and physics, and 7 take biology and physics.

How many students take all three subjects?

Let x be the number taking all three subjects.

$\therefore$ $(10 - x)$ are in B and C but not P
      $(5 - x)$ are in C and P but not B
      $(7 - x)$ are in B and P but not C.

But 15 go into set B

$\therefore$ in the rest of B we have

$$15 - x - (10 - x) - (7 - x)$$
$$= 15 - x - 10 + x - 7 + x$$
$$= x - 2$$

Also, 15 go in set C

$\therefore$ in the rest of C we have

$$15 - x - (10 - x) - (5 - x)$$
$$= 15 - x - 10 + x - 5 + x$$
$$= x$$

and 18 go in set P

$\therefore$ in the rest of P we have

$$18 - x - (7 - x) - (5 - x)$$
$$= 18 - x - 7 + x - 5 + x$$
$$= x + 6$$

But $x - 2 + 10 - x + x + 7 - x + x + 5 - x + x + 6 = 29$

                   $\therefore$ $x + 26 = 29$

                      $\therefore$ $x = 3$

So, 3 study all of the subjects.
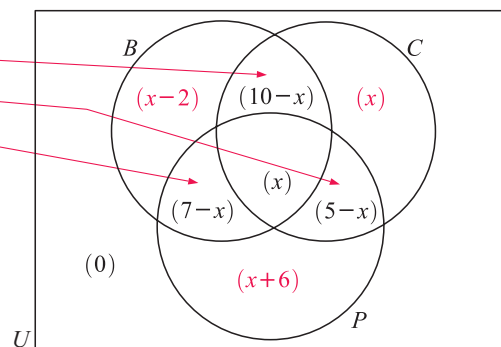
EXERCISE 2F

1 A survey was conducted with a group of teenagers to see how many liked going to the cinema (C) and ice-skating (I). The results are shown in the Venn diagram. Determine the number of teenagers:
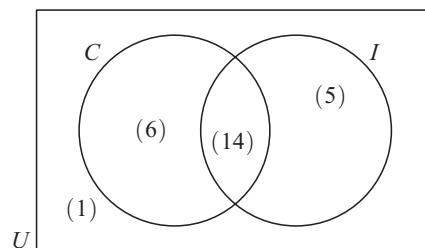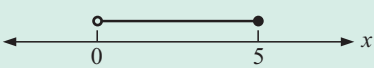
    a in the group

    b who like going to the cinema

    c who like at least one of these activities

    d who only like ice-skating

    e who do not like going ice-skating.

2 The Venn diagram alongside describes the member participation of an outdoor adventure club in rock-climbing (R) and orienteering (O). Determine the number of members:
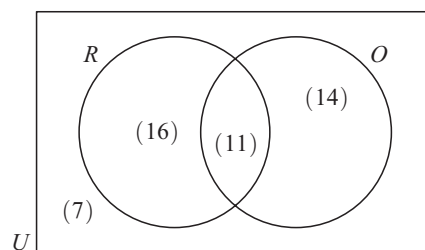
    a in the club

    b who go rock-climbing

    c who do not go orienteering

    d who rock-climb but do not orienteer

    e who do exactly one of these activities.

3 A team of 24 swimmers took part in a competition. 15 competed in freestyle, 11 competed in backstroke, and 6 competed in both of these strokes. Display this information on a Venn diagram, and hence determine the number of swimmers who competed in:

    a backstroke but not freestyle   **b** at least one of these strokes

    c freestyle but not backstroke   **d** neither stroke

    e exactly one of these strokes.

4 In a building with 58 apartments, 45 households have children, 19 have pets, and 5 have neither children nor pets. Draw a Venn diagram to display this information, and hence determine the number of households which:

    a do not have children   **b** have children or pets or both

    c have children or pets but not both   **d** have pets but not children

    e have children but not pets.

5 In a class of 36 girls, 18 play volleyball, 13 play badminton, and 11 play neither sport. Determine the number of girls who play both volleyball and badminton.

6 At their beachhouse, Anna and Ben have 37 books. Anna has read 21 of them and Ben has read 26 of them. 2 of the books have not been read at all. Find the number of books which have been read by:

    a both Anna and Ben   **b** Ben but not Anna.

7 On a particular day, 500 people visited a carnival. 300 people rode the ferris wheel and 350 people rode the roller coaster. Each person rode at least one of these attractions. Using a Venn diagram, find how many people rode:

    a both attractions   **b** the ferris wheel but not the roller coaster.

8 There are 46 shops in the local mall. 25 shops sell clothes, 16 sell shoes, and 34 sell at least one of these items. With the aid of a Venn diagram, determine how many shops sell:

    a both clothes and shoes   **b** neither clothes nor shoes   **c** clothes but not shoes.

9 Joe owns an automotive garage which does car services and mechanical repairs. In one week 18 cars had services or repairs, 9 had services, and 5 had both services and repairs. How many cars had repairs?

10 All the guests at a barbecue ate either sausages or chicken shashliks. 18 people ate sausages, 15 ate sausages and chicken shashliks, and 24 ate exactly one of sausages or chicken shashliks. How many guests attended the barbecue?

11 At a dance school each member studies at least one of classical ballet or modern dance. 72% study classical ballet and 35% study modern dance. What percentage of the students study both classical ballet and modern dance?

12 A group of 28 workers are repairing a road. 9 use machinery, 15 do not use shovels, and 7 do not use either machinery or shovels. How many workers use both machinery and shovels?

13 In a small country town there are three restaurants. 42% of the population eat at A, 45% at B, and 41% at C; 15% eat at both A and B, 9% at A and C, and 17% at B and C; 4% eat at all three restaurants. Display this information on a Venn diagram, and hence find the percentage of the population who eat at:

   a none of the restaurants   **b** at least one of the restaurants

   c exactly one of the restaurants   **d** either A or B **e** C only.

14 There are three hairdressing salons, S, X and Z, located within a large suburban shopping complex. A survey is made of female shoppers within the complex, to determine whether they are clients of S, X or Z. 83 women are clients of at least one of S, X, or Z. 31 are clients of S, 32 are clients of X, and 49 are clients of Z. Of the women who are clients of S, 10 are also clients of X and 14 are also clients of Z. 12 women are clients of both X and Z. How many women are clients of all three salons?

15 75 supermarkets are surveyed to determine which brands of detergent they sell. All sell at least one of brands A, B, or C. 55 sell brand A, 57 sell brand B, and 50 sell brand C. 33 supermarkets sell both A and C, while 17 supermarkets sell both A and B but not C. 22 supermarkets sell all three brands. Construct a Venn diagram which represents this situation. Use this diagram to determine how many supermarkets sell:

   a both A and C but not B **b** at least two of these brands.

16 In a school of 145 students, each was asked to choose one or more activities from sport, music, and drama. 76 students chose sport, 64 students chose music, and 40 students chose drama. 12 students chose both music and sport, 7 chose both sport and drama, and 21 chose both music and drama. How many students chose all three activities?

Review set 2A

1   **a** Explain why 1.3 is a rational number.   **b** True or false? $\sqrt{4000} \in \mathbb{Q}$

    c List the set of all prime numbers between 20 and 40.

    d Write a statement describing the meaning of $\{t \mid -1 \leqslant t < 3\}$.

    e Write in interval notation:

    f Sketch the number set $\{x \mid -2 \leqslant x \leqslant 3, \, x \in \mathbb{Z}\}$.

2 Suppose $U = \{x \mid x \leqslant 12, \ x \in \mathbb{Z}^+\}$ and $A = \{\text{multiples of } 3 \leqslant 12\}$.

    a Show A on a Venn diagram.   **b** List the set A'.

    c Find $n(A')$.   **d** True or false? If $C = \{1, 2, 4\}$ then $C \subseteq A$.

3 List the proper subsets of $M = \{1, 3, 6, 8\}$.

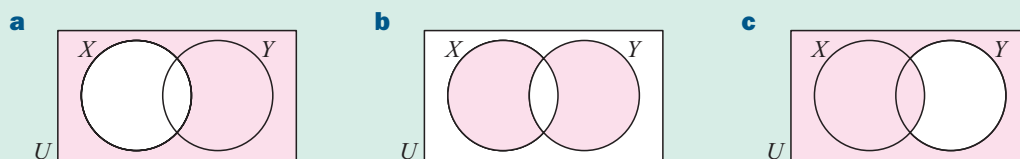**4** True or false?   **a** $\mathbb{N} \subseteq \mathbb{Z}^+$ **b** $\mathbb{Q} \subseteq \mathbb{Z}$

5
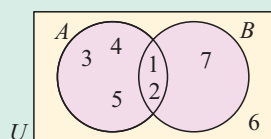
    a List:

        i set A **ii** set B **iii** set U

        iv set $A \cup B$ **v** set $A \cap B$

    b Find:

        i $n(A)$ **ii** $n(B)$ **iii** $n(A \cup B)$

6 Consider $U = \{x \mid x \leqslant 10, \ x \in \mathbb{Z}^+\}$, $P = \{2, 3, 5, 7\}$ and $Q = \{2, 4, 6, 8\}$.

    a Show these sets on a Venn diagram.

    b List the elements of: **i** $P \cap Q$ **ii** $P \cup Q$ **iii** Q'

    c Find: **i** $n(P')$ **ii** $n(P \cap Q)$ **iii** $n(P \cup Q)$

    d True or false? $P \cap Q \subseteq P$

7 Describe in words the shaded region of:

    a   **b**   **c**

8 In a survey at an airport, 55 travellers said that last year they had been to Spain, 53 to France, and 49 to Germany. 18 had been to Spain and France, 15 to Spain and Germany and 25 to France and Germany, while 10 had been to all three countries. Draw a Venn diagram to illustrate this information, and use it to find how many travellers took part in the survey.

Review set 2B

1   **a** True or false?   **i** $-2 \in \mathbb{Z}^+$ **ii** $\frac{1}{\sqrt{7}} \in \mathbb{Q}$

    b Show that $0.\overline{51}$ is a rational number.

    c Write in interval notation:

    d Sketch the number set $\{x \mid x \leqslant 3 \quad \text{or} \quad x > 7, \ x \in \mathbb{R}\}$.

2 Illustrate these numbers on a Venn diagram like the one shown:

    $-1, \ \sqrt{2}, \ 2, \ 3.1, \ \pi, \ 4.\overline{2}$
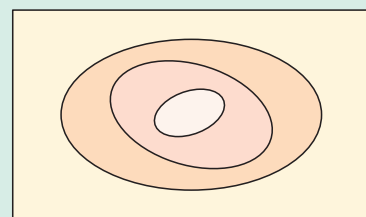
3 Show this information on a Venn diagram:

   a $U = \{10, 11, 12, 13, 14, 15\}, \quad A = \{10, 12, 14\}, \quad B = \{11, 12, 13\}$

   b $U = \{\text{quadrilaterals}\}, \quad S = \{\text{squares}\}, \quad R = \{\text{rectangles}\}$

   c $U = \mathbb{N}, \ A = \{\text{multiples of 2}\}, \quad B = \{\text{multiples of 3}\}, \quad C = \{\text{multiples of 4}\}.$

4 If A is the set of all factors of 24 and B is the set of all factors of 18, find:

   a $A \cap B$ **b** $A \cup B$

5 Suppose $U = \{x \mid x \leqslant 10, \ x \in \mathbb{Z}^+\}, \ A = \{\text{primes less than 10}\}, \quad$ and $B = \{\text{odd numbers between 0 and 10}\}.$

   a Show these sets on a Venn diagram.

   b List: **i** A' **ii** $A \cap B$

   c True or false?   **i** $A \subset B$ **ii** $A \cap B \subseteq A$

   d Find: **i** $n(A)$ **ii** $n(B')$ **iii** $n(A \cup B).$

6 On separate Venn diagrams like the one shown, shade the region representing:

   a B' **b** in A and in B **c** $(A \cup B)'$
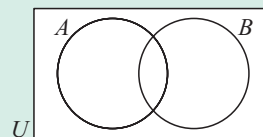
7
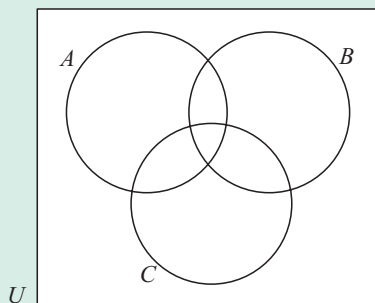
Using separate Venn diagrams like the one shown, shade regions to verify that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$

8 A survey was conducted with 120 students in a school to see how many students were members of extra-curricular clubs. The following results were recorded for the jazz band, drama club and rowing team.

       10 students were not members of any of the three clubs;

       50 were members of the jazz band;

       50 were members of the drama club;

       50 were members of the rowing team;

       15 were members of the jazz band and drama club;

       10 were members of the drama club and rowing team;

       20 were members of the jazz band and rowing team.

   Draw a Venn diagram and use it to find how many students were members of all three clubs.

Algebra (Equations and inequalities)

3

Contents:

Opening problem

Dinesh has an older brother named Mandar who weighs 87 kg, and a younger brother named Ravi who weighs 63 kg. Dinesh is heavier than Ravi, but lighter than Mandar.

Suppose Dinesh weighs w kilograms.

Can you represent the possible values for w using an inequality?

A   SOLVING LINEAR EQUATIONS   [2.3]

Many problems in mathematics can be solved by using **equations**. We convert the worded problem into an algebraic equation by representing an unknown quantity with a variable such as x. We then follow a formal procedure to **solve** the equation, and hence find the solution to the problem.

A **linear equation** is an equation which contains a variable which is not raised to any power other than 1.

For example, $3x + 4 = 2$, $\frac{2}{3}x + 1 = 6$,   and $\dfrac{x - 1}{4} = 8$ are all linear equations.

In this section we review the formal procedure for solving linear equations.

SIDES OF AN EQUATION

> The **left hand side** (LHS) of an equation is on the left of the $=$ sign.
> The **right hand side** (RHS) of an equation is on the right of the $=$ sign.

For example, $\underbrace{3x + 7}_{\text{LHS}} = \underbrace{13}_{\text{RHS}}$

THE SOLUTIONS OF AN EQUATION

> The **solutions** of an equation are the values of the variable which make the equation true,
> i.e., make the **left hand side (LHS)** equal to the **right hand side (RHS)**.

In the example $3x + 7 = 13$ above, the only value of the variable x which makes the equation true is
$x = 2$.

Notice that when $x = 2$, $\begin{aligned} \text{LHS} &= 3x + 7 \\ &= 3 \times 2 + 7 \\ &= 6 + 7 \\ &= 13 \\ &= \text{RHS} \qquad \therefore \quad \text{LHS} = \text{RHS} \end{aligned}$

MAINTAINING BALANCE

The **balance** of an equation is maintained provided we perform the same operation on **both sides** of the
equals sign. We can compare equations to a set of scales.
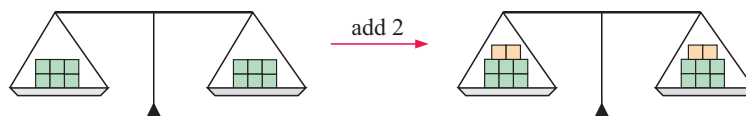
Adding to, subtracting from, multiplying by, and dividing by the same quantity on **both sides** of an equation
will **maintain the balance** or **equality**.

When we use the "$=$" sign between two algebraic expressions we have an equation which is in balance.
Whatever we do to one side of the equation, we must do the same to the other side to **maintain the balance**.

Compare the balance of weights:
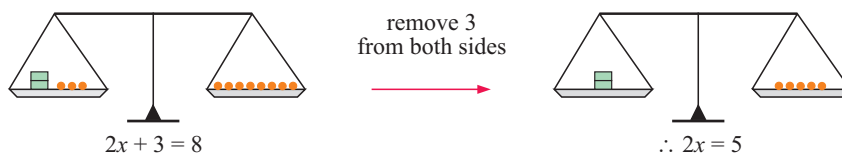
We perform operations on both sides of each equation in order to **isolate the unknown**. We consider how
the expression has been **built up** and then **isolate the unknown** by using **inverse operations** in **reverse
order**.

For example, for the equation $2x + 3 = 8$,   the LHS is built up
by starting with x, multiplying by 2, then adding 3.

So, to isolate x, we first subtract 3 from both sides, then divide
both sides by 2.
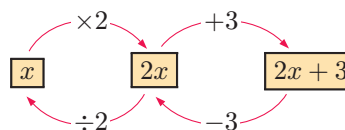
Example 1   ◀》 *Self Tutor*

Solve for x: $3x + 7 = 22$

$\therefore \quad 3x + 7 = 22$

$\therefore \quad 3x + 7 - 7 = 22 - 7$ {subtracting 7 from both sides}

$\therefore \quad 3x = 15$ {simplifying}

$\therefore \quad \dfrac{3x}{3} = \dfrac{15}{3}$ {dividing both sides by 3}

$\therefore \quad x = 5$ {simplifying}

Check: LHS $= 3 \times 5 + 7 = 22$ $\therefore$ LHS = RHS ✓

> The inverse operation for $+7$ is -7.

Example 2   ◀》 *Self Tutor*

Solve for x: $11 - 5x = 26$

$11 - 5x = 26$

$\therefore \quad 11 - 5x - 11 = 26 - 11$ {subtracting 11 from both sides}

$\therefore \quad -5x = 15$ {simplifying}

$\therefore \quad \dfrac{-5x}{-5} = \dfrac{15}{-5}$ {dividing both sides by -5}

$\therefore \quad x = -3$ {simplifying}

Check: LHS $= 11 - 5 \times -3 = 11 + 15 = 26$ $\therefore$ LHS = RHS   ✓

Example 3   ◀》 *Self Tutor*

Solve for x: $\dfrac{x}{3} + 2 = -2$

$\dfrac{x}{3} + 2 = -2$

$\therefore \quad \dfrac{x}{3} + 2 - 2 = -2 - 2$ {subtracting 2 from both sides}

$\therefore \quad \dfrac{x}{3} = -4$

$\therefore \quad \dfrac{x}{3} \times 3 = -4 \times 3$ {multiplying both sides by 3}

$\therefore \quad x = -12$

Check: LHS $= -\frac{12}{3} + 2 = -4 + 2 = -2 =$ RHS.   ✓

> $\dfrac{x}{3}$ is really $x \div 3$. The inverse operation of $\div 3$ is $\times 3$.

Example 4

◀) **Self Tutor**

Solve for x: $\dfrac{4x + 3}{5} = -2$

$$\frac{4x + 3}{5} = -2$$

$$\therefore \quad 5 \times \frac{(4x + 3)}{5} = -2 \times 5 \qquad \text{\{multiplying both sides by 5\}}$$

$$\therefore \quad 4x + 3 = -10$$

$$\therefore \quad 4x + 3 - 3 = -10 - 3 \qquad \text{\{subtracting 3 from both sides\}}$$

$$\therefore \quad 4x = -13$$

$$\therefore \quad \frac{4x}{4} = -\frac{13}{4} \qquad \text{\{dividing both sides by 4\}}$$

$$\therefore \quad x = -3\tfrac{1}{4}$$

EXERCISE 3A.1

1 Solve for x:

   a $x + 11 = 0$ **b** $4x = -12$ **c** $5x + 35 = 0$ **d** $4x - 5 = -17$

   e $5x + 3 = 28$ **f** $3x - 9 = 18$ **g** $8x - 1 = 7$ **h** $3x + 5 = -10$

   i $13 + 7x = -1$ **j** $14 = 3x + 5$ **k** $4x - 7 = -13$ **l** $-3 = 2x + 9$

2 Solve for x:

   a $8 - x = -3$ **b** $-4x = 22$ **c** $3 - 2x = 11$ **d** $6 - 4x = -8$

   e $3 - 7x = -4$ **f** $17 - 2x = -5$ **g** $15 = 3 - 2x$ **h** $24 - 3x = -9$

   i $4 = 3 - 2x$ **j** $13 = -1 - 7x$ **k** $-21 = 3 - 6x$ **l** $23 = -4 - 3x$

3 Solve for x:

   a $\dfrac{x}{4} = 7$ **b** $\dfrac{2x}{5} = -6$ **c** $\dfrac{x}{2} + 3 = -5$ **d** $\dfrac{x}{4} - 2 = -5$

   e $\dfrac{x - 1}{3} = 6$ **f** $\dfrac{x + 5}{6} = -1$ **g** $4 = \dfrac{2 + x}{3}$ **h** $-1 + \dfrac{x}{3} = 7$

4 Solve for x:

   a $\dfrac{2x + 11}{3} = 0$ **b** $\tfrac{1}{2}(3x + 1) = -4$ **c** $\dfrac{1 + 2x}{5} = 7$ **d** $\dfrac{1 - 2x}{5} = 3$

   e $\tfrac{1}{4}(1 - 3x) = -2$ **f** $\tfrac{1}{4}(5 - 2x) = -3$

EQUATIONS WITH A REPEATED UNKNOWN

Equations where the unknown appears more than once need to be solved systematically. Generally, we:

- expand any brackets
- collect like terms
- use inverse operations to isolate the unknown while at the same time maintaining the balance of the equation.

Example 5   ◀) *Self Tutor*

Solve for x: $5(x+1) - 2x = -7$

$$5(x+1) - 2x = -7$$
$$\therefore \quad 5x + 5 - 2x = -7 \qquad \text{\{expanding the brackets\}}$$
$$\therefore \quad 3x + 5 = -7 \qquad \text{\{collecting like terms\}}$$
$$\therefore \quad 3x + 5 - 5 = -7 - 5 \qquad \text{\{subtracting 5 from both sides\}}$$
$$\therefore \quad 3x = -12$$
$$\therefore \quad \frac{3x}{3} = \frac{-12}{3} \qquad \text{\{dividing both sides by 3\}}$$
$$\therefore \quad x = -4$$

When the unknown appears on both sides of the equation, remove it from one side. Aim to do this so the unknown is left with a **positive** coefficient.

Example 6   ◀) *Self Tutor*

Solve for x: $5x + 2 = 3x - 5$

$$5x + 2 = 3x - 5$$
$$\therefore \quad 5x + 2 - 3x = 3x - 5 - 3x \qquad \text{\{subtracting } 3x \text{ from both sides\}}$$
$$\therefore \quad 2x + 2 = -5$$
$$\therefore \quad 2x + 2 - 2 = -5 - 2 \qquad \text{\{subtracting 2 from both sides\}}$$
$$\therefore \quad 2x = -7$$
$$\therefore \quad \frac{2x}{2} = \frac{-7}{2} \qquad \text{\{dividing both sides by 2\}}$$
$$\therefore \quad x = -3\tfrac{1}{2}$$

Example 7   ◀) *Self Tutor*

Solve for x: $15 - 2x = 11 + x$

$$15 - 2x = 11 + x$$
$$\therefore \quad 15 - 2x + 2x = 11 + x + 2x \qquad \text{\{adding } 2x \text{ to both sides\}}$$
$$\therefore \quad 15 = 11 + 3x$$
$$\therefore \quad 15 - 11 = 11 + 3x - 11 \qquad \text{\{subtracting 11 from both sides\}}$$
$$\therefore \quad 4 = 3x$$
$$\therefore \quad \frac{4}{3} = \frac{3x}{3} \qquad \text{\{dividing both sides by 3\}}$$
$$\therefore \quad x = 1\tfrac{1}{3}$$

Summary

Step 1:   If necessary, **expand** any **brackets** and **collect like terms**.

Step 2:   If necessary, remove the unknown from one side of the equation. Aim to do this so the unknown is left with a **positive** coefficient.

Step 3:   Use inverse operations to **isolate the unknown** and maintain balance.

Step 4:   **Check** that your solution satisfies the equation,   i.e., LHS = RHS.

EXERCISE 3A.2

1   Solve for x:

    a $3(x-2) - x = 12$ **b** $4(x+2) - 2x = -16$ **c** $5(x-3) + 4x = -6$

    d $2(3x+2) - x = -6$ **e** $5(2x-1) - 4x = 11$ **f** $-2(4x+3) + 2x = 12$

2   Solve for x:

    a $3(x+2) + 2(x+4) = -1$ **b** $5(x+1) - 3(x+2) = 11$ **c** $4(x-3) - 2(x-1) = -6$

    d $3(3x+1) - 4(x+1) = 14$ **e** $2(3+2x) + 3(x-4) = 8$ **f** $4(5x-3) - 3(2x-5) = 17$

3   Solve for x:

    a $5x + 2 = 3x + 14$ **b** $8x + 7 = 4x - 5$ **c** $7x + 3 = 2x + 9$

    d $3x - 8 = 5x - 2$ **e** $x - 3 = 5x + 11$ **f** $3 + x = 15 + 4x$

4   Solve for x:

    a $6 + 2x = 15 - x$ **b** $3x + 7 = 15 - x$ **c** $5 + x = 11 - 2x$

    d $17 - 3x = 4 - x$ **e** $8 - x = x + 6$ **f** $9 - 2x = 3 - x$

5   Solve for x:

    a $2(x+4) - x = 8$ **b** $5(2-3x) = -8 - 6x$ **c** $3(x+2) - x = 12$

    d $2(x+1) + 3(x-4) = 5$ **e** $4(2x-1) + 9 = 3x$ **f** $11x - 2(x-1) = -5$

    g $3x - 2(x+1) = -7$ **h** $8 - (2-x) = 2x$ **i** $5x - 4(4-x) = x + 12$

    j $4(x-1) = 1 - (3-x)$ **k** $3(x-6) + 7x = 5(2x-1)$ **l** $3(2x-4) = 5x - (12-x)$

B   SOLVING EQUATIONS WITH FRACTIONS *[2.3]*

More complicated fractional equations can be solved by:

- writing all fractions with the **lowest common denominator (LCD)** and then
- **equating numerators**.

To solve equations involving fractions, we make the denominators the same so that we can equate the numerators.

Example 8

◀)) *Self Tutor*

Solve for x: $\dfrac{2x+3}{4} = \dfrac{x-2}{3}$

$$\dfrac{2x+3}{4} = \dfrac{x-2}{3} \qquad \{\text{LCD} = 12\}$$

$$\therefore \quad \dfrac{3 \times (2x+3)}{3 \times 4} = \dfrac{4 \times (x-2)}{4 \times 3} \qquad \{\text{to achieve a common denominator}\}$$

$$\therefore \quad 3(2x+3) = 4(x-2) \qquad \{\text{equating numerators}\}$$

$$\therefore \quad 6x + 9 = 4x - 8 \qquad \{\text{expanding brackets}\}$$

$$\therefore \quad 6x + 9 - 4x = 4x - 8 - 4x \qquad \{\text{subtracting } 4x \text{ from both sides}\}$$

$$\therefore \quad 2x + 9 = -8$$

$$\therefore \quad 2x + 9 - 9 = -8 - 9 \qquad \{\text{subtracting 9 from both sides}\}$$

$$\therefore \quad 2x = -17$$

$$\therefore \quad \dfrac{2x}{2} = -\dfrac{17}{2} \qquad \{\text{dividing both sides by 2}\}$$

$$\therefore \quad x = -8\tfrac{1}{2}$$

Example 9

◀)) *Self Tutor*

Solve for x: $\dfrac{x}{3} - \dfrac{1-2x}{6} = -4$

$$\dfrac{x}{3} - \dfrac{1-2x}{6} = -4 \qquad \{\text{LCD} = 6\}$$

$$\therefore \quad \dfrac{x}{3} \times \dfrac{2}{2} - \left(\dfrac{1-2x}{6}\right) = -4 \times \dfrac{6}{6} \qquad \{\text{to create a common denominator}\}$$

$$\therefore \quad 2x - (1 - 2x) = -24 \qquad \{\text{equating numerators}\}$$

$$\therefore \quad 2x - 1 + 2x = -24 \qquad \{\text{expanding}\}$$

$$\therefore \quad 4x - 1 = -24$$

$$\therefore \quad 4x - 1 + 1 = -24 + 1 \qquad \{\text{adding 1 to both sides}\}$$

$$\therefore \quad 4x = -23$$

$$\therefore \quad x = -\tfrac{23}{4} \qquad \{\text{dividing both sides by 4}\}$$

UNKNOWN IN THE DENOMINATOR

If the unknown appears as part of the denominator, we still solve by:

- writing the equations with the **lowest common denominator (LCD)** and then
- **equating numerators**.

Example 10

◀ Self Tutor

Solve for x: $\dfrac{3x+1}{x-1} = -2$

$$\dfrac{3x+1}{x-1} = \dfrac{-2}{1} \qquad \{\text{LCD} = x-1\}$$

$$\therefore \quad \dfrac{3x+1}{x-1} = \dfrac{-2 \times (x-1)}{1 \times (x-1)} \qquad \{\text{to achieve a common denominator}\}$$

$$\therefore \quad 3x+1 = -2(x-1) \qquad \{\text{equating numerators}\}$$

$$\therefore \quad 3x+1 = -2x+2 \qquad \{\text{expanding brackets}\}$$

$$\therefore \quad 3x+1+2x = -2x+2+2x \qquad \{\text{adding } 2x \text{ to both sides}\}$$

$$\therefore \quad 5x+1 = 2$$

$$\therefore \quad 5x+1-1 = 2-1 \qquad \{\text{subtracting 1 from both sides}\}$$

$$\therefore \quad 5x = 1$$

$$\therefore \quad x = \tfrac{1}{5} \qquad \{\text{dividing both sides by 5}\}$$

EXERCISE 3B

1 Solve for x:

a $\dfrac{2x+3}{5} = \dfrac{1}{2}$

b $\dfrac{x+6}{2} = \dfrac{x}{3}$

c $\dfrac{2x-11}{7} = \dfrac{3x}{5}$

d $\dfrac{x+4}{2} = \dfrac{2x-3}{3}$

e $\dfrac{3x+2}{2} = \dfrac{x-1}{4}$

f $\dfrac{1-x}{2} = \dfrac{x+2}{3}$

g $\dfrac{x+5}{2} = 1-x$

h $\dfrac{2x+7}{3} = x+4$

i $\dfrac{2x+9}{2} = x-8$

2 Solve for x:

a $\dfrac{3}{x} = \dfrac{1}{5}$

b $\dfrac{3}{x} = \dfrac{2}{3}$

c $\dfrac{2}{7} = \dfrac{5}{x}$

d $\dfrac{4}{9} = \dfrac{1}{x}$

e $\dfrac{1}{2x} = \dfrac{4}{3}$

f $\dfrac{7}{3x} = -4$

g $\dfrac{4}{5x} = 3$

h $-5 = \dfrac{2}{3x}$

3 Solve for x:

a $\dfrac{3x-11}{4x} = -2$

b $\dfrac{2x+7}{x-4} = -1$

c $\dfrac{2x+1}{x-4} = 4$

d $\dfrac{2x}{x+4} = 3$

e $\dfrac{-3}{2x-1} = 5$

f $\dfrac{4x+1}{x+2} = -3$

4 Solve for x:

a $\dfrac{x}{2} - \dfrac{x}{6} = 4$

b $\dfrac{x}{4} - 3 = \dfrac{2x}{3}$

c $\dfrac{x}{8} + \dfrac{x+2}{2} = -1$

d $\dfrac{x+2}{3} + \dfrac{x-3}{4} = 1$

e $\dfrac{2x-1}{3} - \dfrac{5x-6}{6} = -2$

f $\dfrac{x}{4} = 4 - \dfrac{x+2}{3}$

g $\dfrac{2x-7}{3} - 1 = \dfrac{x-4}{6}$ **h** $\dfrac{x+1}{3} - \dfrac{x}{6} = \dfrac{2x-3}{2}$ **i** $\dfrac{x}{5} - \dfrac{2x-5}{3} = \dfrac{3}{4}$

j $\dfrac{x+1}{3} + \dfrac{x-2}{6} = \dfrac{x+4}{12}$ **k** $\dfrac{x-6}{5} - \dfrac{2x-1}{10} = \dfrac{x-1}{2}$ **l** $\dfrac{2x+1}{4} - \dfrac{1-4x}{2} = \dfrac{3x+7}{6}$

C   FORMING EQUATIONS

> **Algebraic equations** are mathematical sentences which indicate that two expressions have the same value. They always contain the "=" sign.

Many problems we are given are stated in words. Before we can solve a worded problem, we need to translate the given statement into a mathematical equation. We then solve the equation to find the solution to the problem.

The following **steps** should be followed:

Step 1:   Decide what the unknown quantity is and choose a variable such as x to represent it.

Step 2:   Look for the operation(s) involved in the problem. For example, consider the key words in the table opposite.

Statement	*Translation*
decreased by	subtract
more than	add
double	multiply by 2
halve	divide by 2

Step 3:   Form the equation with an "=" sign. These phrases indicate equality:
"the answer is", "will be", "the result is", "is equal to", or simply "is"

Example 11   ◄)) Self Tutor

Translate into an equation:

a "When a number is added to 6, the result is 15."

b "Twice a certain number is 7 more than the number."

a *In words*   *Indicates*

"a number"   We let x be the number

"a number is added to 6" $6 + x$

"the result is" $6 + x =$

So, $6 + x = 15$

b *In words*   *Indicates*

"a certain number"   Let x be the number

"twice a certain number" $2x$

"7 more than the number" $x + 7$

"is"   So, $2x = x + 7$

> In the following exercise, you do not have to set out your answers like those given in the example.

With practice you will find that you can combine the steps, but you should note:

- the mathematical sentence you form must be an accurate translation of the information
- for these types of problems, you must have only one variable in your equation.

Example 12   ◀) Self Tutor

Translate into an equation: "The sum of 2 consecutive even integers is 34."

Let the smaller even integer be x.

$\therefore$ the next even integer is $x + 2$.

So, $x + (x + 2) = 34$ is the equation.

Example 13   ◀) Self Tutor

Apples cost 13 cents each and oranges cost 11 cents each.

If I buy 5 more apples than oranges and the total cost of the apples and oranges is $2.33, write a linear equation involving the total cost.

Type of fruit	Number of pieces of fruit	Cost per piece of fruit	Total cost
oranges	x	11 cents	$11x$ cents
apples	$x + 5$	13 cents	$13(x + 5)$ cents
			233 cents

From the table we know the total cost, and so $11x + 13(x + 5) = 233$.

EXERCISE 3C

1 Translate into linear equations, but *do not solve*:

   a When a number is increased by 6, the answer is 13.

   b When a number is decreased by 5, the result is -4.

   c A number is doubled and 7 is added. The result is 1.

   d When a number is decreased by 1 and the resulting number is halved, the answer is 45.

   e Three times a number is equal to 17 minus the number.

   f Five times a number is 2 more than the number.

2 Translate into equations, but *do not solve:*

   a The sum of two consecutive integers is 33.

   b The sum of 3 consecutive integers is 102.

   c The sum of two consecutive odd integers is 52.

   d The sum of 3 consecutive odd integers is 69.

3 Write an equation for each of the following:

a Peter is buying some outdoor furniture for his patio. Tables cost $40 each and chairs cost $25 each. Peter buys 10 items of furniture at a total cost of $280. (Let the number of tables purchased be t.)

b Pencils cost 40 pence each and erasers cost 70 pence each. If I purchase three fewer erasers than pencils, the total cost will be £3.40. (Let the variable p represent the number of pencils purchased.)

c A group of friends went to a cafe for tea and coffee. Tea costs €2.50 and coffee costs €3.60. The number of people who ordered coffee was twice the number who ordered tea, and the total bill was €29.10. (Let the number of people who ordered tea be t.)

D ‖ PROBLEM SOLVING USING EQUATIONS

PROBLEM SOLVING METHOD

- Identify the unknown quantity and allocate a variable to it.
- Decide which operations are involved.
- Translate the problem into a linear equation and check that your translation is correct.
- Solve the linear equation by isolating the variable.
- Check that your solution does satisfy the original problem.
- Write your answer in sentence form.

Example 14   ◀) Self Tutor

The sum of 3 consecutive even integers is 132. Find the smallest integer.

Let x be the smallest even integer

$\therefore$ the next is $x + 2$ and the largest is $x + 4$.

So, $x + (x + 2) + (x + 4) = 132$ {their sum is 132}

$\therefore \quad 3x + 6 = 132$

$\therefore \quad 3x + 6 - 6 = 132 - 6$ {subtracting 6 from both sides}

$\therefore \quad 3x = 126$

$\therefore \quad \dfrac{3x}{3} = \dfrac{126}{3}$ {dividing both sides by 3}

$\therefore \quad x = 42$ $\therefore$ the smallest integer is 42.

Example 15   ◀) *Self Tutor*

If twice a number is subtracted from 11, the result is 4 more than the number. What is the number?

Let x be the number,

LHS algebraic expression is $11 - 2x$

RHS algebraic expression is $x + 4$.

$$\therefore \quad 11 - 2x = x + 4 \qquad \text{\{the equation\}}$$
$$\therefore \quad 11 - 2x + 2x = x + 4 + 2x \qquad \text{\{adding } 2x \text{ to both sides\}}$$
$$\therefore \quad 11 = 3x + 4$$
$$\therefore \quad 11 - 4 = 3x + 4 - 4 \qquad \text{\{subtracting 4 from both sides\}}$$
$$\therefore \quad 7 = 3x$$
$$\therefore \quad \frac{7}{3} = \frac{3x}{3} \qquad \text{\{dividing both sides by 3\}}$$
$$\therefore \quad x = 2\tfrac{1}{3} \qquad \text{So, the number is } 2\tfrac{1}{3}.$$

Example 16   ◀) *Self Tutor*

Cans of sardines come in two sizes. Small cans cost \$2 each and large cans cost \$3 each. If 15 cans of sardines are bought for a total of \$38, how many small cans were purchased?

Size	Cost per can	Number bought	Value
small	\$2	x	\$2x$
large	\$3	$15 - x$	\$3(15 - x)$
		15	\$38

$$\text{So,} \quad 2x + 3(15 - x) = 38$$
$$\therefore \quad 2x + 45 - 3x = 38 \qquad \text{\{expanding brackets\}}$$
$$\therefore \quad 45 - x = 38$$
$$\therefore \quad 45 - x - 45 = 38 - 45 \qquad \text{\{subtracting 45 from both sides\}}$$
$$\therefore \quad -x = -7$$
$$\therefore \quad x = 7 \qquad \text{So, 7 small cans were bought.}$$

EXERCISE 3D

1 When a number is doubled and the result is increased by 6, the answer is 20. Find the number.

2 The sum of two consecutive integers is 75. Find the integers.

3 The sum of three consecutive even integers is 54. Find the largest of them.

4 When a number is subtracted from 40, the result is 14 more than the original number. Find the number.

5 When 22 is subtracted from a number and the result is doubled, the answer is 6 more than the original number. Find the number.

6 When one quarter of a number is subtracted from one third of the number, the result is 7. Find the number.

7 Roses cost £5 each and geraniums cost £3 each. Michelle bought 4 more geraniums than roses, and in total she spent £52. How many roses did she buy?

8 Nick has 40 coins in his collection, all of which are either 5-cent or 10-cent coins. If the total value of his coins is $3.15, how many of each coin type does he have?

9 A store sells batteries in packets of 6 or 10. In stock they have 25 packets which contain a total of 186 batteries. How many of each packet size are in stock?

E   POWER EQUATIONS

Equations of the form $x^n = k$ where $n = 2, 3, 4, ...$ are **not** linear equations. They are in fact called **power equations**.

We have already seen that: If $x^2 = k$ where $k > 0$, then $x = \pm\sqrt{k}$.

If we know that the solution has to be positive then $x = \sqrt{k}$.

In general,
- if n is odd and $x^n = k$, then $x = \sqrt[n]{k}$
- if n is even and $x^n = k$ where $k > 0$, then $x = \pm\sqrt[n]{k}$.

Example 17   ◀)) Self Tutor

Use your calculator to solve for x, giving answers correct to 3 significant figures.

a $x^2 = 6$ **b** $x^2 = 13$, $x > 0$ **c** $x^3 = 31$

a $x^2 = 6$

$\therefore$ $x = \pm\sqrt{6}$

$\therefore$ $x \approx \pm 2.45$

b $x^2 = 13$, $x > 0$

$\therefore$ $x = \sqrt{13}$

$\therefore$ $x \approx 3.61$

c $x^3 = 31$

$\therefore$ $x = \sqrt[3]{31}$

$\therefore$ $x \approx 3.14$

See graphics calculator instructions for finding cube roots (page **14**).

Example 18   ◀)) Self Tutor

Solve for x: $\dfrac{x}{2} = \dfrac{5}{x}$

$\dfrac{x}{2} = \dfrac{5}{x}$ {LCD $= 2x$}

$\therefore$ $\dfrac{x \times x}{2 \times x} = \dfrac{5 \times 2}{x \times 2}$ {to get a common denominator}

$\therefore$ $x^2 = 10$ {equating numerators}

$\therefore$ $x = \pm\sqrt{10}$

EXERCISE 3E

1 Solve for x, giving your answers correct to 3 significant figures:

    a $x^2 = 11$ **b** $x^2 = -5$ **c** $x^2 = 71$ **d** $x^2 = 89, \quad x > 0$

    e $x^3 = 8$ **f** $x^3 = 11$ **g** $x^3 = -11$ **h** $x^4 = 81$

    i $x^4 = -1$ **j** $x^5 = 23$ **k** $x^5 = -113$ **l** $x^6 = 39.2, \quad x > 0$

2 Solve for x:

    a $\dfrac{x}{3} = \dfrac{4}{x}$ **b** $\dfrac{x}{6} = \dfrac{6}{x}$ **c** $\dfrac{1}{x} = \dfrac{x}{3}$ **d** $\dfrac{x}{7} = \dfrac{7}{x}$

    e $\dfrac{2}{x} = \dfrac{x}{5}$ **f** $\dfrac{7}{x} = \dfrac{x}{5}$ **g** $\dfrac{x}{2} = \dfrac{8}{x}$ **h** $\dfrac{x}{5} = \dfrac{-2}{x}$

F   INTERPRETING LINEAR INEQUALITIES   [2.1]

The speed limit when passing roadworks is often 25 kilometres per hour.

This can be written as a **linear inequality** using the variable s to represent the speed of a car in km per h. $s \leqslant 25$ reads 's is less than or equal to 25'.

We can also represent the allowable speeds on a **number line**:

The number line shows that any speed of 25 km per h or less is an acceptable speed.
We say that these are **solutions** of the inequality.

REPRESENTING INEQUALITIES ON A NUMBER LINE

Suppose our solution to an inequality is $x \geqslant 4$, so *every* number which is 4 or greater than 4 is a possible value for x. We could represent this on a number line by:

The filled-in circle indicates
that 4 **is** included.

The arrowhead indicates that all numbers on
the number line in this direction are included.

Likewise if our solution is $x < 5$ our representation would be:
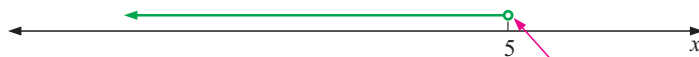
The hollow circle indicates that 5 **is not** included.

Example 19   ◆) *Self Tutor*

Represent the following inequalities on a number line:

    a $1 \leqslant x < 5$ **b** $x < 0$ or $x \geqslant 4$

    a   **b**

EXERCISE 3F

1 Represent the following inequalities on a number line:

   a $x > 5$ **b** $x \geqslant 1$ **c** $x \leqslant 2$ **d** $x < -1$

   e $-2 \leqslant x \leqslant 2$ **f** $-3 < x \leqslant 4$ **g** $1 \leqslant x < 6$ **h** $-1 < x < 0$

   i $x < 0$ or $x \geqslant 3$ **j** $x \leqslant -1$ or $x \geqslant 2$ **k** $x < 2$ or $x > 5$ **l** $x \leqslant -2$ or $x > 0$

2 Write down the inequality used to describe the set of numbers:
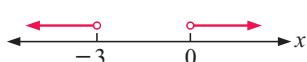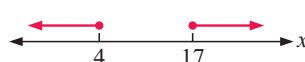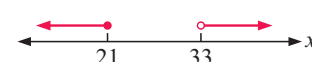
G | SOLVING LINEAR INEQUALITIES [2.2]

Notice that $5 > 3$ and $3 < 5,$
and that $-3 < 2$ and $2 > -3.$

This suggests that if we **interchange** the LHS and RHS of an inequation, then we must **reverse** the inequality sign. $>$ is the reverse of $<$, $\geqslant$ is the reverse of $\leqslant$, and so on.

You may also remember from previous years that:

- If we **add** or **subtract** the same number to both sides, the inequality sign is *maintained*.
  For example, if $5 > 3$ then $5 + 2 > 3 + 2$.

- If we **multiply** or **divide** both sides by a **positive** number, the inequality sign is *maintained*.
  For example, if $5 > 3$ then $5 \times 2 > 3 \times 2$.

- If we **multiply** or **divide** both sides by a **negative** number, the inequality sign is *reversed*.
  For example, if $5 > 3$ then $5 \times -1 < 3 \times -1$.

The method of solution of linear inequalities is thus identical to that of linear equations with the exceptions that:

- **interchanging** the sides **reverses** the inequality sign
- **multiplying** or **dividing** both sides by a **negative** number **reverses** the inequality sign.

Example 20   ◆) Self Tutor

Solve for x and graph the solutions: **a** $3x - 4 \leqslant 2$ **b** $3 - 2x < 7$
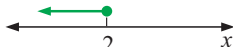
a $3x - 4 \leqslant 2$

$\therefore \quad 3x - 4 + 4 \leqslant 2 + 4$ {adding 4 to both sides}

$\therefore \quad 3x \leqslant 6$

$\therefore \quad \dfrac{3x}{3} \leqslant \dfrac{6}{3}$ {dividing both sides by 3}

$\therefore \quad x \leqslant 2$

Check: If $x = 1$ then $3x - 4 = 3 \times 1 - 4 = -1$ and $-1 \leqslant 2$ is true.
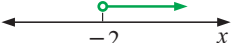
b $3 - 2x < 7$

$\therefore \quad 3 - 2x - 3 < 7 - 3$ {subtracting 3 from both sides}

$\therefore \quad -2x < 4$

$\therefore \quad \dfrac{-2x}{-2} > \dfrac{4}{-2}$ {dividing both sides by -2, so reverse the sign}

$\therefore \quad x > -2$

Check: If $x = 3$ then $3 - 2x = 3 - 2 \times 3 = -3$
                   and $-3 < 7$ is true.

> Notice the reversal of the inequality sign in **b** line 4 as we are dividing by -2.

Example 21   ◆) Self Tutor

Solve for x and graph the solutions: $-5 < 9 - 2x$

$\qquad\qquad -5 < 9 - 2x$

$\therefore \quad -5 + 2x < 9 - 2x + 2x$ {adding $2x$ to both sides}

$\therefore \quad 2x - 5 < 9$

$\therefore \quad 2x - 5 + 5 < 9 + 5$ {adding 5 to both sides}

$\therefore \quad 2x < 14$

$\therefore \quad \dfrac{2x}{2} < \dfrac{14}{2}$ {dividing both sides by 2}

$\therefore \quad x < 7$

Check: If $x = 5$ then $-5 < 9 - 2 \times 5$, i.e., $-5 < -1$ which is true.

Example 22

Self Tutor

Solve for x and graph the solutions: $3 - 5x \geqslant 2x + 7$

$$3 - 5x \geqslant 2x + 7$$

$\therefore \quad 3 - 5x - 2x \geqslant 2x + 7 - 2x$ {subtracting $2x$ from both sides}

$\therefore \quad 3 - 7x \geqslant 7$

$\therefore \quad 3 - 7x - 3 \geqslant 7 - 3$ {subtracting 3 from both sides}

$\therefore \quad -7x \geqslant 4$

$\therefore \quad \dfrac{-7x}{-7} \leqslant \dfrac{4}{-7}$ {dividing both sides by -7, so reverse the sign}

$\therefore \quad x \leqslant -\frac{4}{7}$
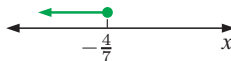
Check: If $x = -1$ then $3 - 5 \times (-1) \geqslant 2 \times (-1) + 7$, i.e., $8 \geqslant 5$ which is true.

EXERCISE 3G

1 Solve for x and graph the solutions:

a $4x \leqslant 12$	**b** $-3x < 18$	**c** $-5x \geqslant -35$
d $3x + 2 < 0$	**e** $5x - 7 > 2$	**f** $2 - 3x \geqslant 1$
g $5 - 2x \leqslant 11$	**h** $2(3x - 1) < 4$	**i** $5(1 - 3x) \geqslant 8$

2 Solve for x and graph the solutions:

a $7 \geqslant 2x - 1$	**b** $-13 < 3x + 2$	**c** $20 > -5x$
d $-3 \geqslant 4 - 3x$	**e** $3 < 5 - 2x$	**f** $2 \leqslant 5(1 - x)$

3 Solve for x and graph the solutions:

a $3x + 2 > x - 5$	**b** $2x - 3 < 5x - 7$	**c** $5 - 2x \geqslant x + 4$
d $7 - 3x \leqslant 5 - x$	**e** $3x - 2 > 2(x - 1) + 5x$	**f** $1 - (x - 3) \geqslant 2(x + 5) - 1$

4 Solve for x:

a $3x + 1 > 3(x + 2)$	**b** $5x + 2 < 5(x + 1)$	**c** $2x - 4 \geqslant 2(x - 2)$

d Comment on your solutions to **a**, **b** and **c**.

Review set 3A

1 Solve for x: **a** $9 + 2x = -11$ **b** $\dfrac{3 - 2x}{7} = -5$

2 Solve for x: **a** $\dfrac{x}{5} = \dfrac{4}{7}$ **b** $\dfrac{4x + 5}{3} = \dfrac{x}{2}$

3 Solve for x: **a** $\dfrac{1}{3x} = 5$ **b** $\dfrac{x + 6}{3 - 2x} = -1$

4 Solve for x: **a** $7x - 5 = 4(x + 4)$ **b** $\dfrac{x}{2} + 2x = \dfrac{3x - 1}{4}$

5 Represent the following inequalities on a number line:

     a $-1 \leqslant x < 3$ **b** $x < 0$ or $x \geqslant 3$

6 Solve for x and show the solutions on a number line:

     a $2x + 7 < 22 - 3x$ **b** $5(x + 4) \geqslant 5 - 2(3 - x)$

7 Translate into linear equations but *do not solve*:

     a   When a number is increased by 11 and the result is doubled, the answer is 48.

     b   The sum of three consecutive integers is 63.

8 When 7 times a certain number is decreased by 11, the result is 31 more than the number. Find the number.

9 I have 25 coins consisting of 5-cent and 50-cent pieces. If the total value is $7.10, how many 5-cent coins do I have?

10 Solve for x: **a** $x^4 = 16$ **b** $\dfrac{7}{x} = \dfrac{x}{3}$

Review set 3B

1 Solve for x: **a** $10 - 3x = -14$ **b** $\dfrac{3x + 5}{4} = 8$

2 Solve for x: **a** $\dfrac{x}{2} = \dfrac{3}{8}$ **b** $\dfrac{1 - 3x}{4} = \dfrac{x - 2}{2}$

3 Solve for x: **a** $2(x - 2) - 5(x + 3) = -5$ **b** $3(2 - x) = x - 11$

4 Solve for x: **a** $\dfrac{5}{3x} = \dfrac{3}{2}$ **b** $\dfrac{2x + 1}{3} - \dfrac{4 - x}{6} = -2$

5 Represent the following inequalities on a number line:

     a $-2 < x < 2$ **b** $x \leqslant -5$ or $x \geqslant -1$

6 Solve for x and show the solutions on a number line:

     a $3(x - 4) \leqslant x + 6$ **b** $7 - 2(x - 3) > 5(3 - 2x)$

7 Translate into linear equations, but *do not solve*.

     a   Four times a number is equal to the number plus 15.

     b   The sum of two consecutive odd integers is 36.

8 Five more than a certain number is nine less than three times the number. Find the number.

9 Solve for x: **a** $x^3 = 64$ **b** $\dfrac{x}{11} = \dfrac{-5}{x}$

10 Clara, Dean and Elaine were candidates in an election in which 1000 people voted. Elaine won the election, receiving 95 more votes than Dean, and 186 more votes than Clara. How many votes did Dean receive?

Lines, angles and polygons

4

Contents:

Opening problem

On his birthday, Billy receives a cake in the shape of a regular hexagon. He divides the cake into 4 pieces by making cuts from one corner to each of the other corners as shown.

Things to think about:

a When Billy cuts the cake in this way, are the four angles at the top of the hexagon equal in size? Can you find the size of each angle?

b When a regular hexagon is cut in this way, four triangles are formed. Do any of these triangles contain right angles?
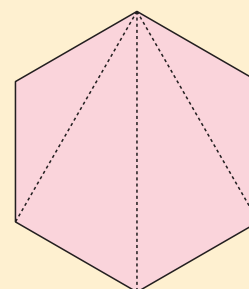
A   ANGLE PROPERTIES   [4.4]

There is some important terminology we use when talking about how two angles are related. You should become familiar with these terms:

- Two angles with sizes which add to 90° are called **complementary angles**.
- Two angles with sizes which add to 180° are called **supplementary angles**.

- Two angles which have the same vertex and share a common arm are called **adjacent angles**.

  $\hat{PAQ}$ and $\hat{QAR}$ are adjacent angles.
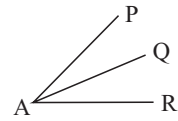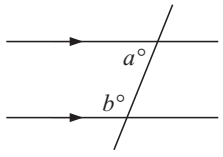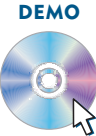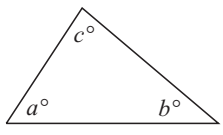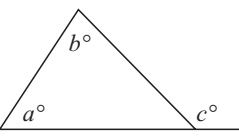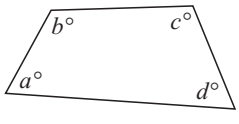
- For intersecting lines, angles which are directly opposite each other are called **vertically opposite angles**.
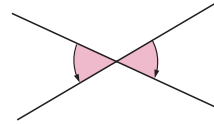
The following properties can be used to solve problems involving angles:

Title	Property	Figure
Angles centred at a point	The sum of the sizes of the angles at a point is 360°.	$a + b + c = 360$
Adjacent angles on a straight line	The sum of the sizes of the angles on a line is 180°. The angles are supplementary.	$a + b = 180$
Adjacent angles in a right angle	The sum of the sizes of the angles in a right angle is 90°. The angles are complementary.	$a + b = 90$
Vertically opposite angles	Vertically opposite angles are equal in size.	$a = b$
Corresponding angles	When two *parallel* lines are cut by a third line, then angles in corresponding positions are equal in size.	$a = b$
Alternate angles	When two *parallel* lines are cut by a third line, then angles in alternate positions are equal in size.	$a = b$

Title	Property	Figure
Co-interior angles (also called allied angles)	When two *parallel* lines are cut by a third line, then co-interior angles are supplementary.	$a + b = 180$
Angles of a triangle	The sum of the interior angles of a triangle is 180°. **DEMO**	$a + b + c = 180$
Exterior angle of a triangle	The size of the exterior angle of a triangle is equal to the sum of the interior opposite angles. **DEMO**	$c = a + b$
Angles of a quadrilateral	The sum of the interior angles of a quadrilateral is 360°. **DEMO**	$a + b + c + d = 360$

Example 1
◄) *Self Tutor*

Find, giving brief reasons, the value of the unknown in:
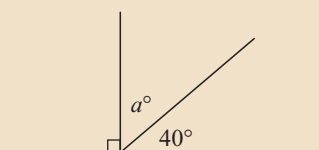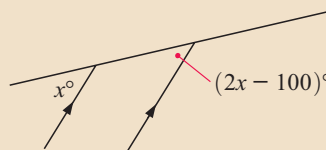
a

b

a	$90 + a + 40 = 180$	{angles on a line}
	$\therefore \quad a + 130 = 180$	
	$\therefore \quad a = 50$	
b	$2x - 100 = x$	{equal corresponding angles}
	$\therefore \quad 2x - 100 - x = x - x$	{subtracting x from both sides}
	$\therefore \quad x - 100 = 0$	{simplifying}
	$\therefore \quad x = 100$	

EXERCISE 4A

1   Use the figure illustrated to answer the following questions:
AB is a fixed line and OP can rotate about O between OA and OB.
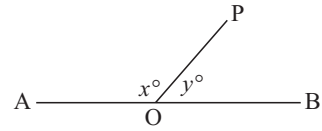
    a   If $x = 136$, find y.   **b**   If $y = 58$, find x.

    c   What is x if y is 39?   **d**   If x is 0, what is y?

    e   If $x = 81$, find y.   **f**   If $x = y$, what is the value of each?

2   Find the values of the unknowns, giving brief reasons. You should **not** need to set up an equation.

    a
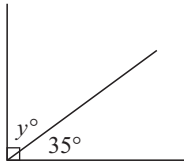
    b
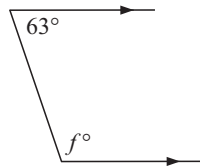
    c
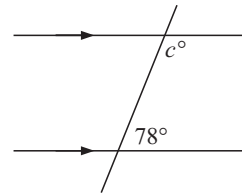
    d
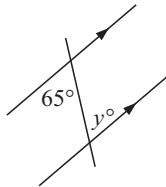
    e
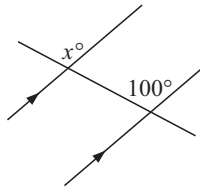
    f
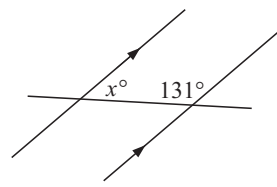
    g
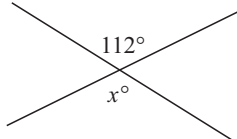
    h
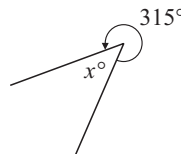
    i
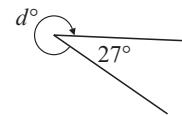
    j
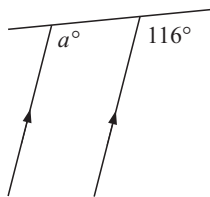
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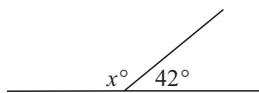
    l
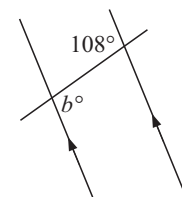
    m
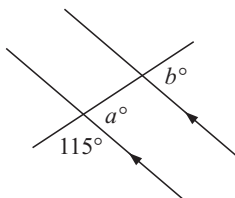
    n
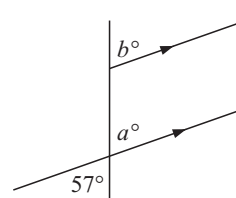
    o

    p

    q

    r

s

$d°$ $c°$ $b°$ $a°$ 122°

t

155°
68° $b°$

u

$s°$

81° $r°$

3 Find, giving brief reasons, the value of the unknown in:

a

145°
$d°$

b

$e°$ $e°$
$e°$

c

$3f°$
$3f°$

d

$3h°$ $2h°$

e

$2g°$ $g°$

f

$(2x - 75)°$

$x°$

g

$(2x + 10)°$ $(x + 35)°$

h

$(3x + 15)°$
$2x°$

i

$3x°$ $(3x - 23)°$
45°
$(x + 44)°$
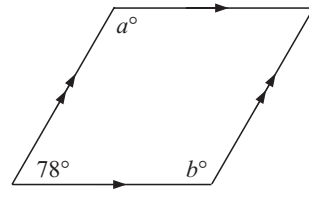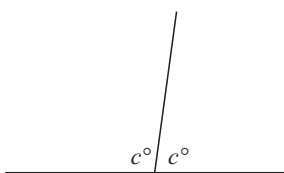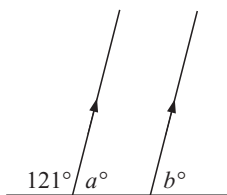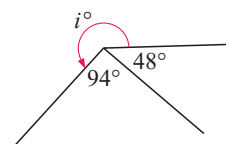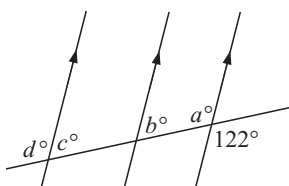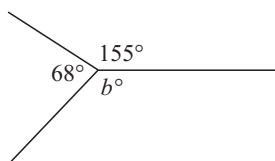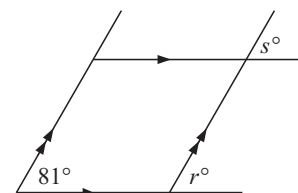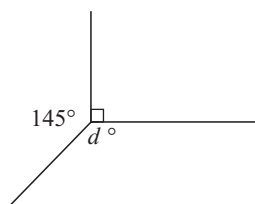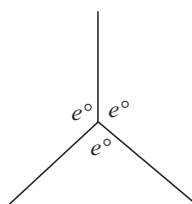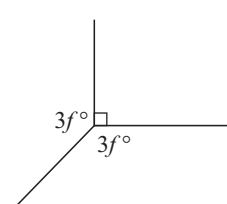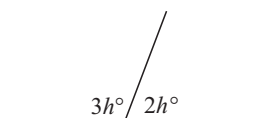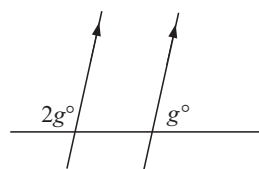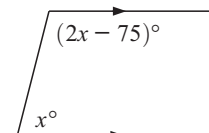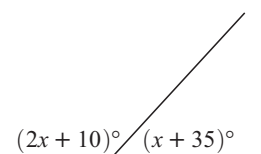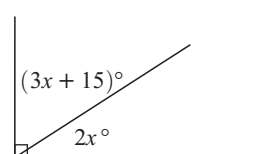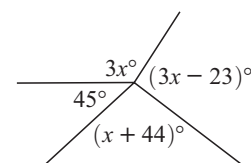
4 State whether KL is parallel to MN, giving a brief reason for your answer. Note that these diagrams are sketches only and have not been drawn accurately.

a

L   N
85°
85°
K   M

b

K   L
91°
M   91°
N

c

L   N
96°   96°
K   M

5 Find the value of the unknowns in:

a

120° $y°$

$x°$

b

$a°$ $b°$

35°

c

$q°$
65°

$r°$ $p°$
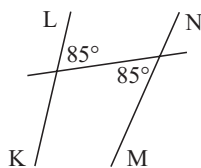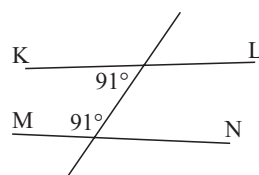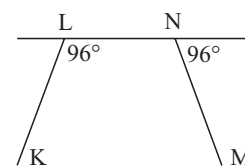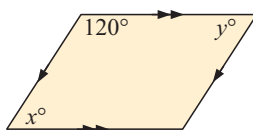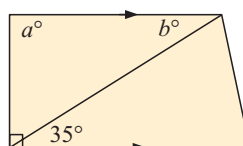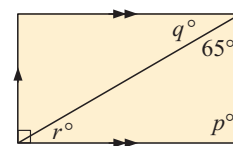
| **B** | *TRIANGLES* | **[4.4]** |

A **triangle** is a polygon which has three sides.

All triangles have the following properties:

- The sum of the interior angles of a triangle is 180°.
- Any exterior angle is equal to the sum of the interior opposite angles.
- The longest side is opposite the largest angle.
- The triangle is the only **rigid** polygon.

Discussion

Bridges and other specialised structures often have triangular supports rather than rectangular ones. The reason for this is that "*the triangle is the only rigid polygon* ".

1 Explain what is meant by "rigid polygon"?

2 Why is this important?

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Example 2 ◄)) *Self Tutor*

Find the unknown in the following, giving brief reasons:

a

b
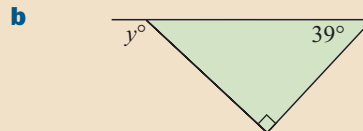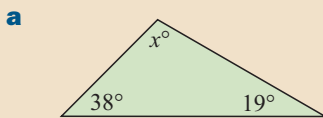
a　$x + 38 + 19 = 180$　　　　　　{angle sum of a triangle}
　　$\therefore$　$x = 180 - 38 - 19$
　　$\therefore$　$x = 123$

b　　$y = 39 + 90$　　　　　　{exterior angle of a triangle}
　$\therefore$　$y = 129$

Example 3 ◄)) *Self Tutor*

Find the values of the unknowns, giving brief reasons for your answers.

a

b

a $2x + x + (x + 20) = 180$ {angles of a triangle}

$\therefore \quad 4x + 20 = 180$

$\therefore \quad 4x = 160$

$\therefore \quad x = 40$

b $a = 180 - 140 = 40$ {angles on a line}

Likewise $b = 180 - 120 = 60$

But $a + b + c = 180$ {angles of a triangle}

$\therefore \quad 40 + 60 + c = 180$

$\therefore \quad 100 + c = 180$

$\therefore \quad c = 80$

EXERCISE 4B

1   Find the unknown in the following, giving brief reasons:

a

b

c

d

e

f
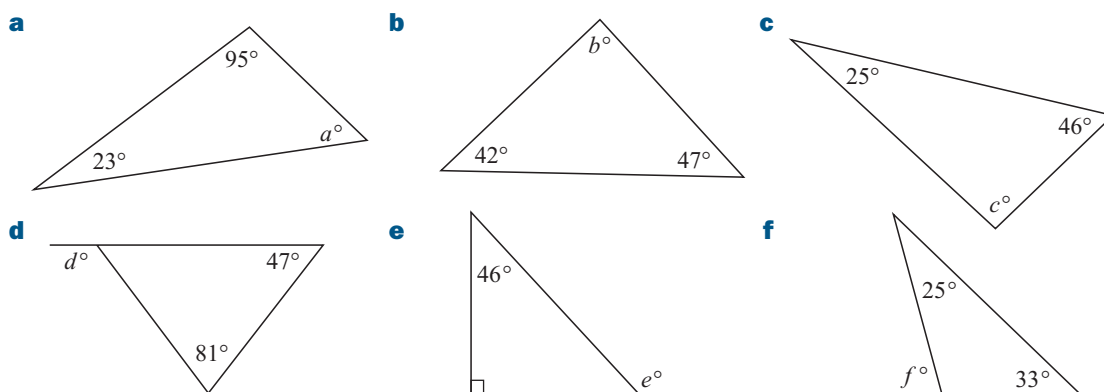
2   The following triangles are *not* drawn to scale. State the longest side of each triangle.

a

b

c

d

e

f

The longest side is opposite the largest angle.

g

h

i
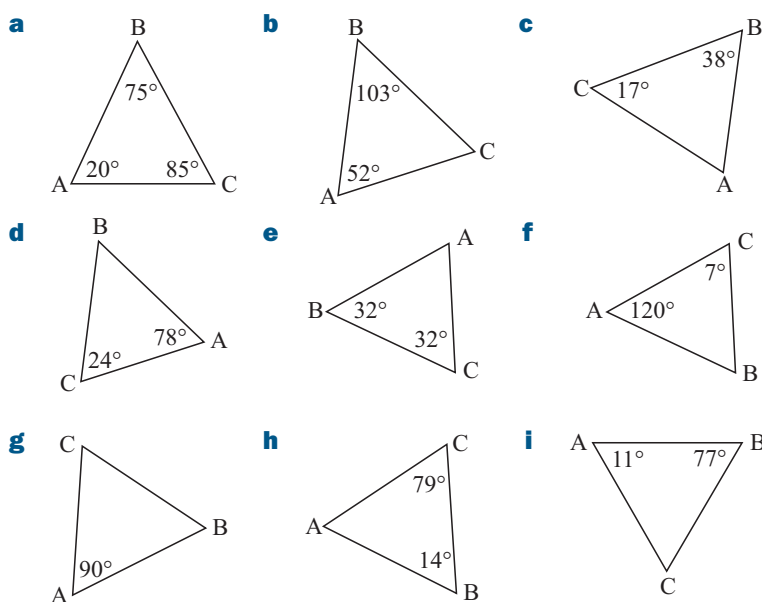
3 State whether the following statements are *true* or *false*:

  a The sum of the angles of a triangle is equal to two right angles.

  b A right angled triangle can contain an obtuse angle.

  c The sum of two angles of a triangle is always greater than the third angle.

  d The two smaller angles of a right angled triangle are supplementary.

  e A concave triangle is impossible.

4 Find the values of the unknowns in each triangle, giving a brief reason for each answer:

  a

    $a°$

    $4a°$ $4a°$

  b

    $(b+5)°$

    $b°$ $(b-5)°$

  c

    $48°$ $d°$

    $76°$

    $c°$

  d

    $48°$

    $a°$ $84°$

    $b°$

  e

    $40°$

    $120°$ $a°$ $b°$

  f

    $65°$

    $b°$

    $a°$ $c°$

    $72°$ $d°$
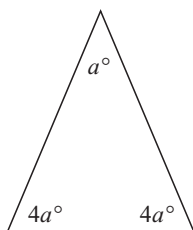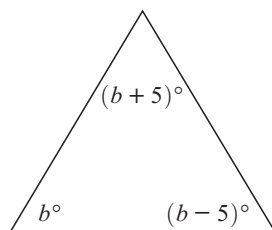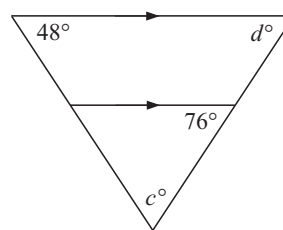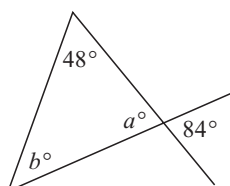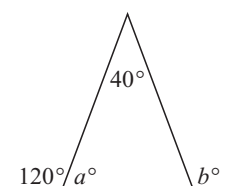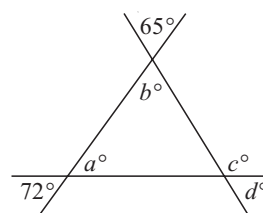
5 The three angles of a scalene triangle are $x°$, $(x-12)°$ and $(2x+6)°$. What are the sizes of these angles?

C   *ISOSCELES TRIANGLES*   *[4.4]*

> An **isosceles triangle** is a triangle in which two sides are equal in length.

The angles opposite the two equal sides are called the **base angles**.

The vertex where the two equal sides meet is called the **apex**.

apex
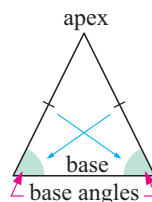
base

base angles

GEOMETRY PACKAGE
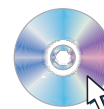
THE ISOSCELES TRIANGLE THEOREM

In an isosceles triangle:

- base angles are equal
- the line joining the apex to the midpoint of the base bisects the vertical angle and meets the base at right angles.

β β

α α
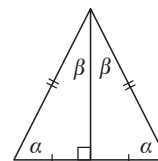
PROPERTIES

The following properties are associated with the isosceles triangle theorem:

> *Property 1:*   If a triangle has two equal angles then it is isosceles.
>
> *Property 2:*   The angle bisector of the apex of an isosceles triangle bisects the base at right angles.
>
> *Property 3:*   The perpendicular bisector of the base of an isosceles triangle passes through its apex.
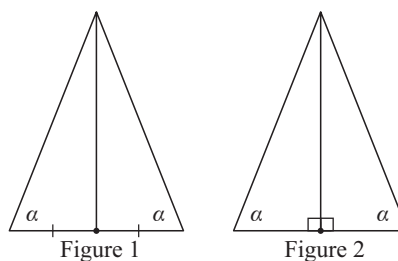
VIDEO CLIP
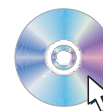
GEOMETRY PACKAGE
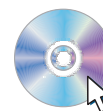
- To prove *Property 1*, Sam tries to use Figure 1 and triangle congruence.

  Will he be successful?

  Why or why not? Could Sam be successful using Figure 2?

- Can you prove *Property 2* using triangle congruence?

Figure 1   Figure 2

Example 4   ◀) *Self Tutor*

Find x, giving brief reasons:

a
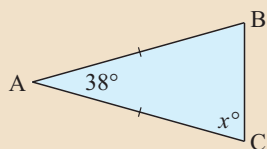
b
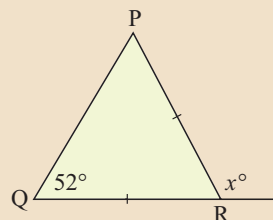
a
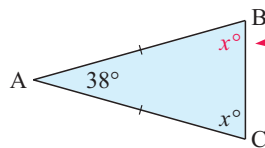
As $AB = AC$, the triangle is isosceles

$\therefore$ $A\widehat{B}C = x^o$ also

Now $x + x + 38 = 180$ {angles of a Δ}

$\therefore$ $2x = 180 - 38$

$\therefore$ $2x = 142$

$\therefore$ $x = 71$

b
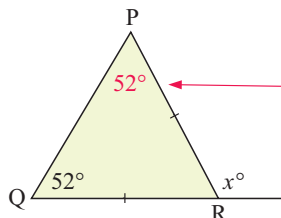
As $PR = QR$, the triangle is isosceles

$\therefore$ $Q\widehat{P}R = 52^o$ {isosceles Δ theorem}

$\therefore$ $x = 52 + 52$ {exterior angle theorem}

$\therefore$ $x = 104$

EXERCISE 4C

1 Find x, giving reasons:

a
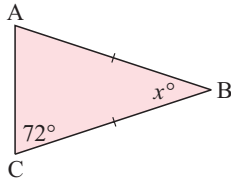
b
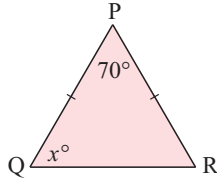
c
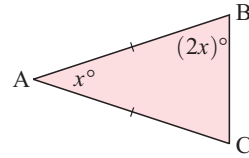
d
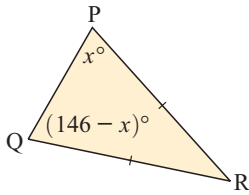
e
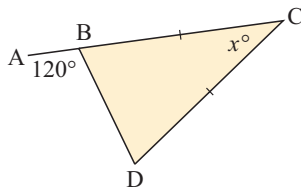
f
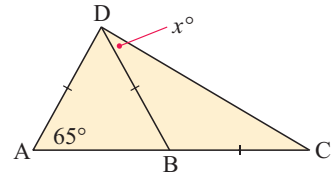
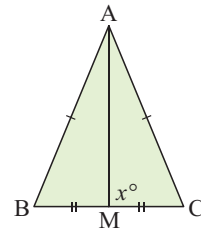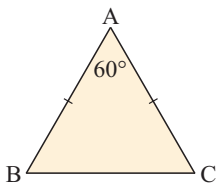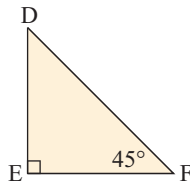**2** Find x, giving brief reasons:

a
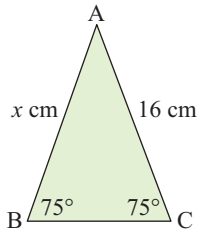
b
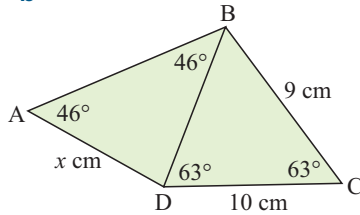
c

3 Classify the following triangles as equilateral, isosceles or scalene. They are not drawn to scale, but the information marked on them is correct.
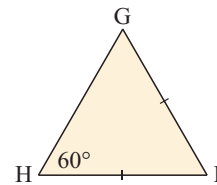
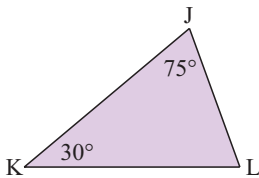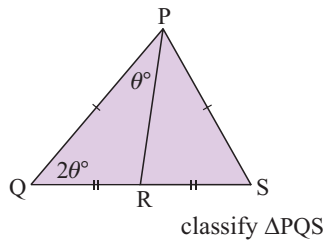**a**
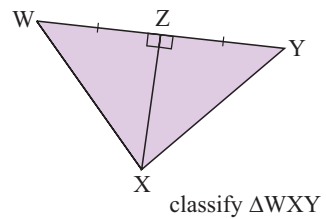

b

c

d

e

classify ΔPQS
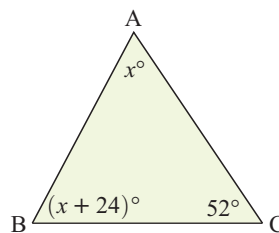
f

classify ΔWXY

4 The figure alongside has not been drawn to scale.
   a Find x.
   b What can be deduced about the triangle?

5 Because of its symmetry, a regular pentagon can be constructed
    from five isosceles triangles.

    a Find the size of angle θ at the centre O.

    b Hence, find ϕ.

    c Hence, find the measure of one interior angle such as $A\hat{B}C$.

6 Repeat question **5** but use a regular decagon. Remember that a
    decagon has 10 sides.
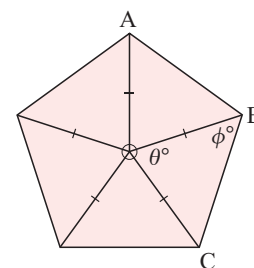
D   THE INTERIOR ANGLES OF A POLYGON   [4.4]

We have already seen that the sum of the interior angles of a triangle is 180°.
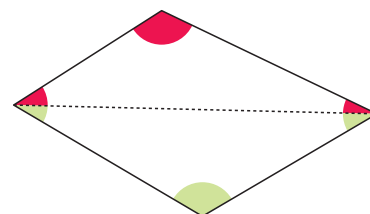
Now consider finding the sum of the angles of a quadrilateral.

We can construct a diagonal of the quadrilateral as shown.

The red angles must add to 180° and so must the green angles.

But these 6 angles form the 4 angles of the quadrilateral.

So,   the sum of the interior angles of a quadrilateral is 360°.

We can generalise this process to find the sum of the interior angles of any polygon.

Discovery 1   Angles of an n-sided polygon

What to do:

1 Draw any pentagon (5-sided polygon) and label one of its
    vertices A. Draw in all the diagonals from A.

2 Repeat **1** for a hexagon, a heptagon (7-gon), an octagon,
    and so on, drawing diagonals from one vertex only.

3 Copy and complete the following table:
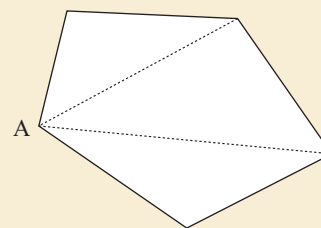
Polygon	Number of sides	Number of diagonals from A	Number of triangles	Angle sum of polygon
quadrilateral	4	1	2	$2 \times 180^o = 360^o$
pentagon				
hexagon				
octagon				
20-gon				

4 Copy and complete the following statement:
    "*The sum of the interior angles of any n-sided polygon is* ...... $\times 180^o$."

You should have discovered that:

> The sum of the interior angles of any n-sided polygon is $(n-2) \times 180^o$.

Example 5
◀)) *Self Tutor*

Find x, giving a brief reason:
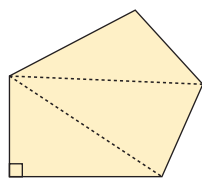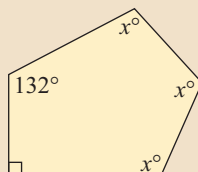
The pentagon has 5 sides

∴   the sum of interior angles is $3 \times 180^o = 540^o$

∴ $x + x + x + 132 + 90 = 540$

∴ $3x + 222 = 540$

∴ $3x = 318$

∴ $x = 106$

EXERCISE 4D

1 Find the sum of the interior angles of:

   a a quadrilateral   **b** a pentagon   **c** a hexagon   **d** an octagon.

2 Find the value of the unknown in:
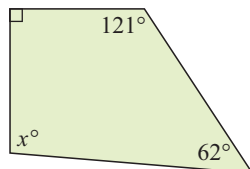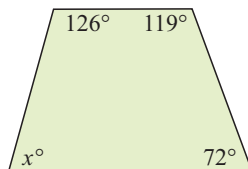
   a   **b**   **c**
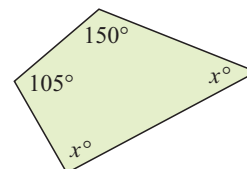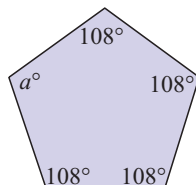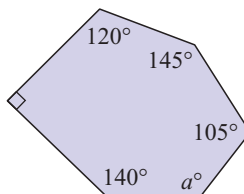
   d   **e**   **f**
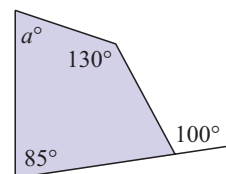
3 Find the value of x in each of the following, giving a reason:

   a   **b**   **c**

d

e

f
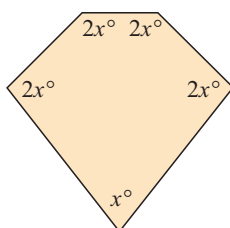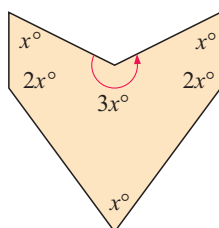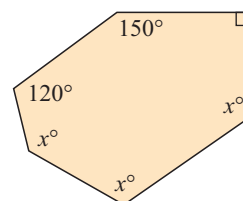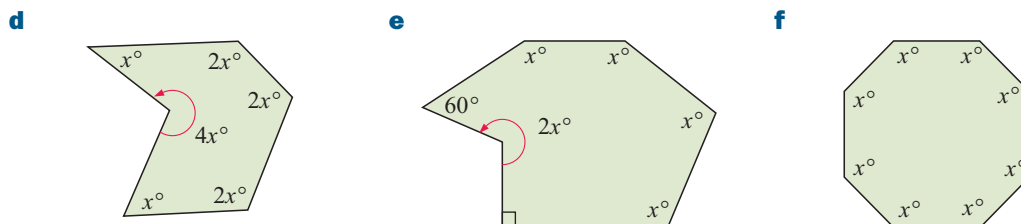
4 A pentagon has three right angles and two other equal angles.
What is the size of each of the two equal angles?

5 Find the size of each interior angle within a regular:

    a pentagon   **b** hexagon   **c** octagon   **d** decagon

6 The sum of the angles of a polygon is $1800°$. How many angles has the polygon?

7 Joanna has found a truly remarkable polygon which has interior angles with a sum of $2060°$. Comment on Joanna's finding.

8 Copy and complete the following table for regular polygons:

Regular polygon	Number of sides	Number of angles	Size of each angle
equilateral triangle			
square			
pentagon			
hexagon			
octagon			
decagon			

9 Copy and complete:

- the sum of the angles of an n-sided polygon is ......
- the size of each angle θ, of a regular n-sided polygon, is $\theta = $ ......

10 Answer the **Opening Problem** on page **93**.

11 The figure alongside is a regular heptagon.

    a Find the size of each interior angle.

    b Hence, find the value of each of the unknowns.
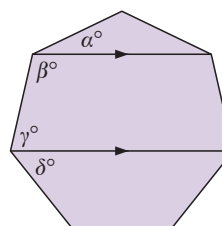
12
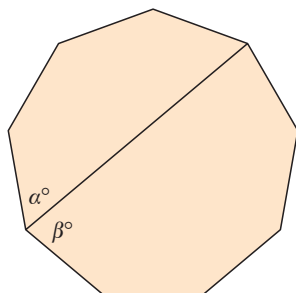
The figure alongside is a regular nonagon.
Find α and β.

13 A **tessellation** is a pattern made with a number of objects of the same shape and size, which can cover an area without leaving any gaps. Which regular polygons tessellate?

Hint: For a regular polygon to tessellate, copies of its shape must be able to meet at a point with no gaps. What property must the size of its interior angle have?
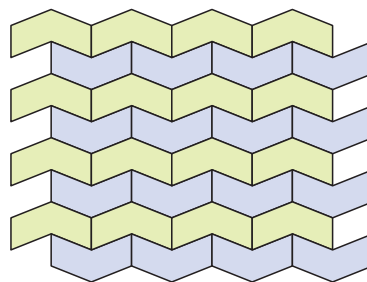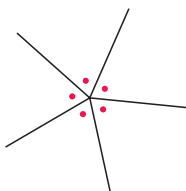
14
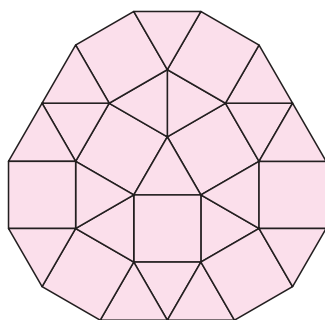
We can cover a region with tiles which are equilateral triangles and squares with sides of equal length.

a Copy this pattern and add to it the next outer layer.

b Can you construct a pattern without gaps, using a regular octagon and a square?

E THE EXTERIOR ANGLES OF A POLYGON [4.4]

The **exterior angles** of a polygon are formed by extending the sides in either direction.

Discovery 2   Exterior angles of a polygon

The shaded angle is said to be an exterior angle of quadrilateral ABCD at vertex B.

The purpose of this Discovery is to find the sum of all exterior angles of a polygon.
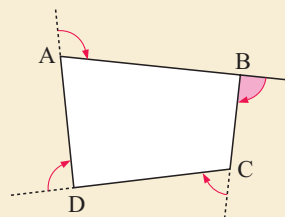
What to do:

1 In the school grounds, place four objects on the ground no more than 10 m apart, forming the vertices of an imaginary quadrilateral. Start at one vertex, and looking towards the next vertex, walk directly to it and turn to face the next vertex. Measure the angle that you have turned through.

2 Repeat this process until you are back to where you started from, and turn in the same way to face your original direction of sight, measuring each angle that you turn through.

3 Through how many degrees have you turned from start to finish?

4 Would your answer in **3** change if an extra object was included to form a pentagon?

5 Write a statement indicating what you have learnt about the sum of the exterior angles of any polygon.

From the Discovery, you should have found that:

> The sum of the exterior angles of any polygon is always 360°.

This fact is useful for finding the size of an interior angle of a regular polygon.

Example 6 ◀)) Self Tutor

A regular polygon has 15 sides. Calculate the size of each interior angle.

For a 15-sided polygon, each exterior angle is $360^o \div 15 = 24^o$

$\therefore$ each interior angle is $180^o - 24^o = 156^o$

EXERCISE 4E

1 Solve for x:

a
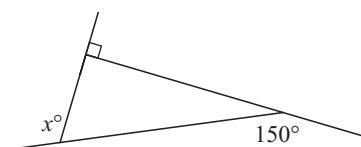
b
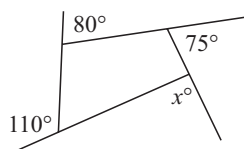
c
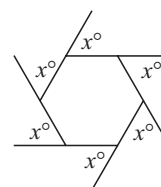
2 Calculate the size of each interior angle of these regular polygons:
  - **a** with 5 sides
  - **b** with 8 sides
  - **c** with 10 sides
  - **d** with 20 sides
  - **e** with 100 sides
  - **f** with n sides

3 Calculate the number of sides of a regular polygon given that an exterior angle is:
  - **a** 45°
  - **b** 15°
  - **c** 2°
  - **d** $\frac{1}{2}^o$

4 Calculate the number of sides of a regular polygon with an interior angle of:
  - **a** 120°
  - **b** 150°
  - **c** 175°
  - **d** 179°

Review set 4A

1 Copy and complete: If two parallel lines are cut by a third line then:
  - **a** the alternate angles are ......
  - **b** co-interior angles are ......

2 Find the value of the unknown, giving reasons for your answer:

a

b

c

d

125°

$(x-10)°$

e

$(x+20)°$

$3x°$
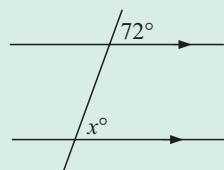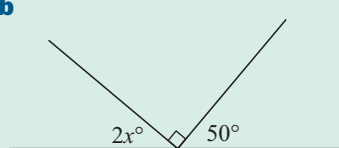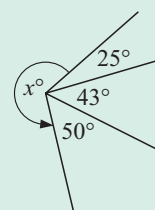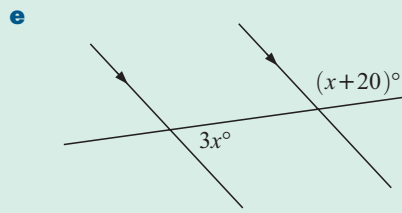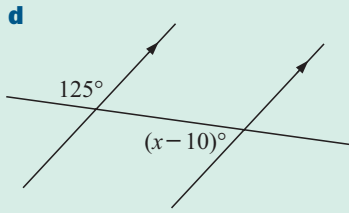
3 Decide if the figure contains parallel lines,
      giving a brief reason for your answer:

130°

50°
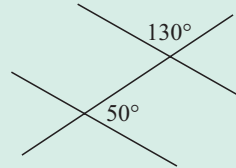
4 Find the value of x in:

a

$x°$

60°

58°

b

62°

$x°$

c

44°

$x°$
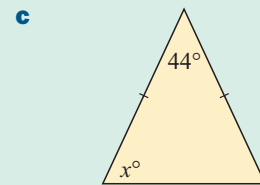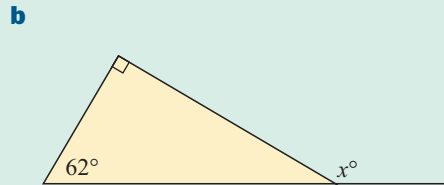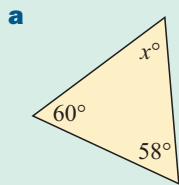
5 What can be deduced about triangle ABC shown?
      Give reasons for your answer.

A

56°

B

118°

C
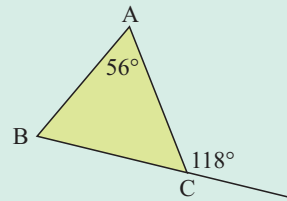
6 Find, giving reasons, the values of the unknowns in:

a

76°

$x°$

72°

b

5 m

60°

$x°$

y m

c

55°

$x°$

$y°$
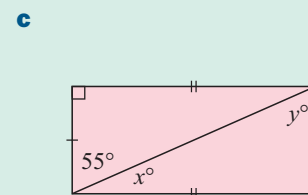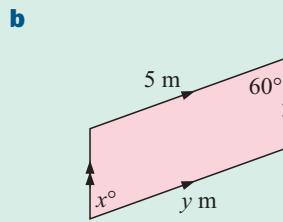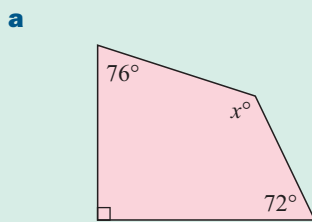
7 Find the value of x in each of the following, giving reasons:

a

85°

145°

$x°$

$x°$

b

$x°$ $x°$

$x°$ $x°$

$x°$ $x°$

c

120°

140°

$x°$

d

$x°$

105°
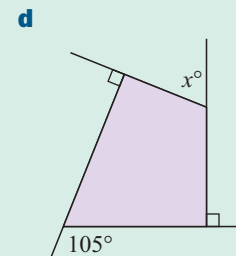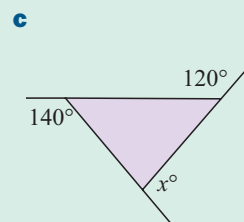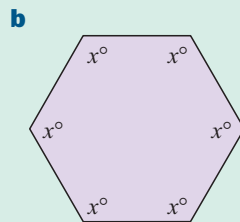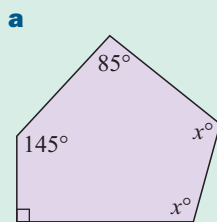
Review set 4B

1 State the values of the unknowns in each figure, giving a brief reason for each answer:

a

$a°$ $107°$

b

$b°$ $b°$

c

$c°$ $56°$

d

$b°$
$a°$
$110°$

e

$(x + 15)°$ $(4x − 35)°$
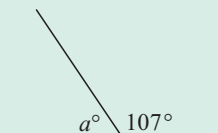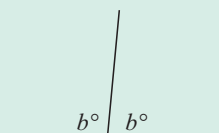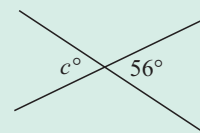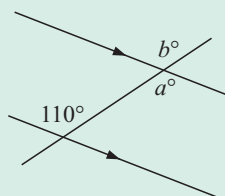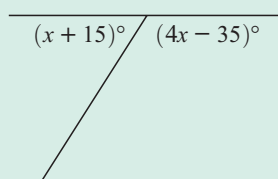
2 Find the value of the unknown in each figure, giving a brief reason for your answer:

a

$x°$
$x°$ $x°$

b

$69°$
$62°$ $a°$

c

$3x$ cm
$60°$
3 cm
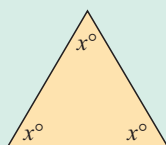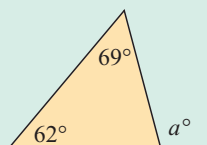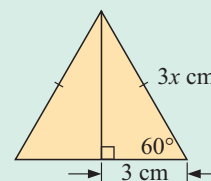
3 Find the values of x and y, giving brief reasons for your answers:

$y°$
$x°$
$42°$
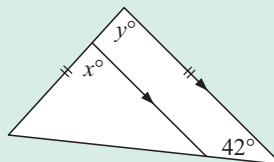
4 Classify each triangle in as much detail as possible:

a

$50°$
$40°$

b

$50°$
$110°$

c

$3x°$
$x°$ $x°$

5 Find the values of the unknowns:

a

$2x°$ $x°$
$112°$ $124°$

b
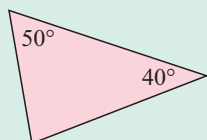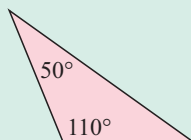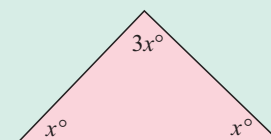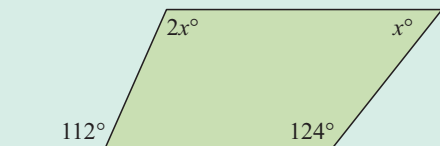
$c°$ $46°$
$a°$
$b°$
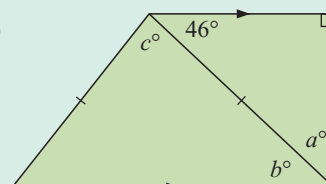
6 Copy and complete:

Polygon	Number of sides	Sum of interior angles
pentagon		
hexagon		
octagon		

7 Calculate the size of each interior angle for a regular polygon with:

    a 12 sides   **b** 18 sides

8 Solve for x:
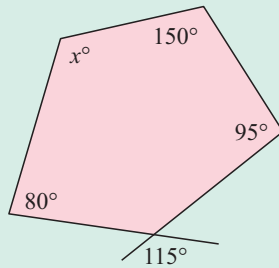
    a   **b**
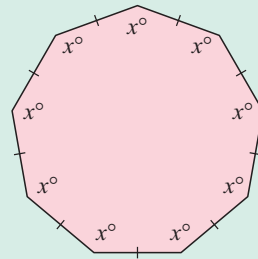
Challenge

1 **a** Measure the angles at A, B, C, D and E and find their sum.

    b Repeat with two other '5-point star' diagrams of your choosing.

    c What do you suspect about the angle sum for all '5-point star' diagrams?

    d Use deductive geometry to confirm your suspicion.

    e Repeat **a** to **d** but with a '7-point star' diagram.
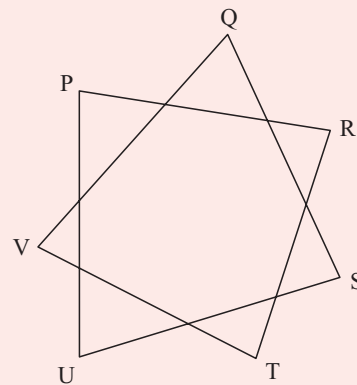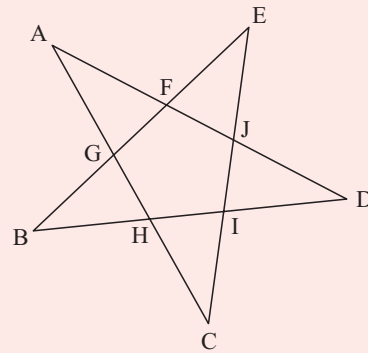
Graphs, charts and tables

5

Contents:

Opening problem

Amon and Maddie were practicing their golf drives on a driving range. Amon hit 20 drives, which travelled the following distances in metres:

152	183	194	172	160	148	177	159	192	188
165	174	181	191	188	165	180	174	147	191

Maddie suggested that Amon should have a try with her driver. While using Maddie's driver, Amon's drives travelled the following distances:

150	152	169	195	189	177	188	196	182	175
180	167	174	182	179	197	180	168	178	194

Things to think about:

1 Just by looking at the data, is there a noticeable difference between the data sets?

2 How can we represent the data in a way that makes it easier to compare the data sets?

A *STATISTICAL GRAPHS*   [11.3]

When we construct a statistical investigation, we collect information called **data**.

There are several different types of data that we can collect:

> **Categorical data** is data which is sorted into categories.
>
> **Numerical data** is data which can be written in numerical form. It can either be **discrete** or **continuous**.
> Discrete data takes exact number values, and is often a result of **counting**.
> Continuous data takes numerical values within a continuous range, and is usually a result of **measuring**.

Graphs and charts are used to display data in a form that is not only more visually appealing, but also easier to understand.

In this chapter we will look at different kinds of graphs and charts which can be used to both analyse and compare data. In particular, we will see that categorical data is usually displayed using a **bar chart** or a **pie chart**, and numerical data is usually displayed using a **line graph** or a **stem-and-leaf plot**.

BAR CHART

A **bar chart** is a popular method of displaying statistical data and is probably the easiest statistical graph to construct. The information may be displayed either vertically or horizontally. The height (if vertical) or length (if horizontal) of each bar is proportional to the quantity it represents. All bars are of the same width. If the data is discrete then the bars are separated by a space.

Vertical bar chart   **Horizontal bar chart**
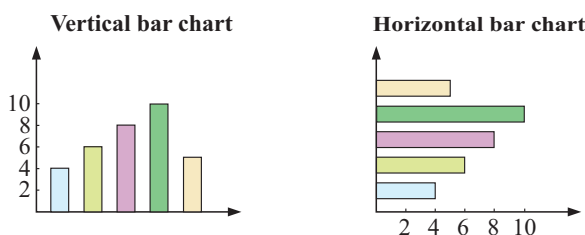
Example 1   ◀️ *Self Tutor*

Teachers at a local school were asked what mode of transport they used that day to travel to school. The results are summarised in the table:

Mode of transport	Car	Bicycle	Bus	Walk
Number of teachers	15	3	7	5

a Display the data on a vertical bar chart.

b What percentage of teachers used a bus that morning?

a   **Mode of transport to school**
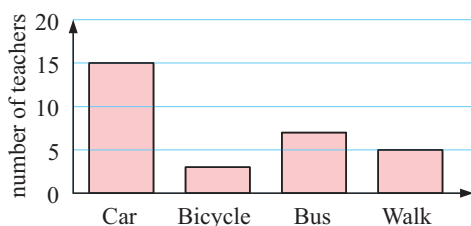
b Total number of teachers
$$= 15 + 3 + 7 + 5$$
$$= 30$$
Percentage who used a bus
$$= \tfrac{7}{30} \times 100\%$$
$$\approx 23.3\%$$

PIE CHART

Pie chart

A **pie chart** presents data in a circle. The circle is divided into sectors which represent the categories. The size of each sector is proportional to the quantity it represents, so its sector angle can be found as a fraction of 360°.

Example 2   ◀》 *Self Tutor*

For the data on teachers' mode of transport in **Example 1**:

a calculate the sector angles   **b** construct a pie chart.

a For car, $\frac{15}{30} \times 360^o = 180^o$

For bicycle, $\frac{3}{30} \times 360^o = 36^o$

For bus, $\frac{7}{30} \times 360^o = 84^o$

For walk, $\frac{5}{30} \times 360^o = 60^o$.

b **Mode of transport to school**

Car 180°

Walk 60° 84° 36° Bicycle

Bus
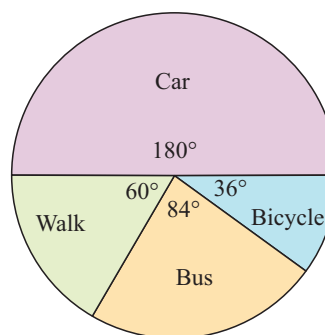
SCATTER DIAGRAMS AND LINE GRAPH

We are often given data which shows how one quantity *varies* with another. The data will be given as ordered pairs which we can plot on a set of axes.

If the data is *discrete*, it does not make sense to join the points and we have a **scatter diagram**.

For example, if each apple at a market stall costs 20 pence, we obtain a scatter diagram. It does not make sense to buy a part of an apple.

cost (pence)

number of apples
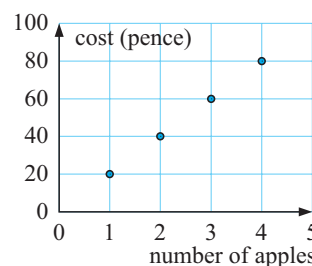
If the data is *continuous* we can join the points with straight lines to form a **line graph**. We often obtain line graphs when we observe a quantity that varies with time. You will see this in the following example:

Example 3   ◀》 *Self Tutor*

The temperature at Geneva airport was measured each hour and the results recorded in the table below. Draw a line graph to illustrate this data.

Time	0800	0900	1000	1100	1200	1300	1400	1500	1600	1700	1800	1900
Temperature (^oC)	5	8	11	12	15	16	18	14	12	11	11	9

Temperature at Geneva airport
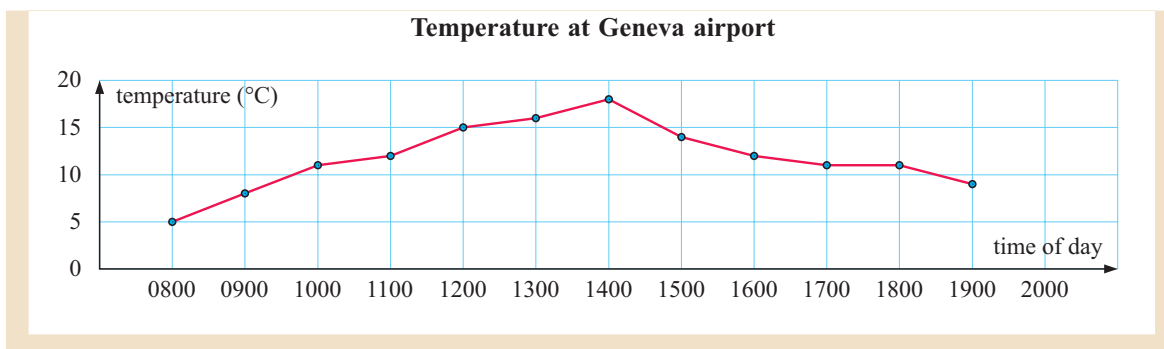
STEM-AND-LEAF PLOTS

A **stem-and-leaf plot** (often called a stem-plot) is a way of writing down the data in groups and is used for small data sets. It shows actual data values and gives a visual comparison of frequencies.

For numbers with two digits, the first digit forms part of the **stem** and the second digit forms a **leaf**.

For example, for the data value 17,   1 would be recorded on the stem, and the 7 would be the leaf value.

Example 4   ◄) *Self Tutor*

Construct a stem-and-leaf plot for the following data:

25 38 17 33 24 16 32 17 22 35 30 44 20 39 42 37 26 31 28 33

The **stem-and-leaf plot** is: The **ordered stem-and-leaf plot** is:

Stem	Leaf
1	7 6 7
2	5 4 2 0 6 8
3	8 3 2 5 0 9 7 1 3
4	4 2

Stem	Leaf
1	6 7 7
2	0 2 4 5 6 8
3	0 1 2 3 3 5 7 8 9
4	2 4 **Key:** 1	7 means 17

The ordered stem-plot arranges all data from smallest to largest.

Notice the following features:

- All the actual data is shown
- The minimum (smallest) data value is 16.
- The maximum (largest) data value is 44.
- The 'thirties' interval (30 to 39) occurred most often, and is the **modal class**.
- The key indicates the place value of the stem. For example, if the key was   1|7 means 1.7   then 2|3 would represent the data value 2.3.

EXERCISE 5A

1 Of the 30 teachers in a school, 6 teach Maths, 7 teach English, 4 teach Science, 5 teach Humanities subjects, 4 teach Modern languages, 2 teach Theatre Arts, and 2 teach Physical education.

   a Draw a vertical bar chart to display this information.

   b What percentage of the teachers teach Maths or Science?

2   A factory produces three types of toy cars: Astons, Bentleys, and Corvettes. The annual sales are:

Toy car make	Aston	Bentley	Corvette
Sales (thousands of £)	65	30	25

    a   Find the sector angles for each category.   **b**   Construct a pie chart for the data.

3   The midday temperature and daily rainfall were measured every day for one week in Kingston. The results are summarised in the table below:

Day	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Temperature (°C)	23	20	18	25	27	24	26
Rainfall (mm)	8	4	2	12	10	9	1

Draw a vertical bar chart to illustrate the temperature readings and a horizontal bar chart to illustrate the rainfall readings.

4   **a**   Draw a stem-and-leaf plot using stems 2, 3, 4, and 5 for the following data:
        29, 27, 33, 30, 46, 40, 35, 24, 21, 58, 27, 34, 25, 36, 57, 34, 42, 51, 50, 48

    b   Redraw the stem-and-leaf plot from **a** to make it ordered.

5   The data set below is the test scores (out of 100) for a Science test for 50 students.

    92 29 78 67 68 58 80 89 92 69 66 56 88
    81 70 73 63 55 67 64 62 74 56 75 90 56
    59 64 89 39 51 87 89 76 59 47 38 88 62
    72 80 95 68 80 64 53 43 61 71 44

    a   Construct a stem-and-leaf plot for this data.

    b   What percentage of the students scored 80 or more for the test?

    c   What percentage of students scored less than 50 for the test?

6   In a pie chart on *leisure activities* the sector angle for *ice skating* is 34°. This sector represents 136 students.

    a   The sector angle for *watching television* is 47°. How many students watch television for leisure?

    b   If 38 students *visit friends*, what sector angle would represent them?

    c   How many students were used in the sample?

7   The length of the shadow cast by a tree was measured at hourly intervals:

Time	0800	0900	1000	1100	1200	1300	1400	1500	1600
Shadow length (m)	30	20	12	6	2	5	10	17	25

    a   Is it more appropriate to use a scatter diagram or a line graph to display this data?

    b   Display this data on your chosen graph.

8   For the ordered stem-and-leaf plot given, find:

    a   the minimum value

    b   the maximum value

    c   the number of data with a value greater than 25

    d   the number of data with a value of at least 40

    e   the percentage of the data which is less than 15.

Stem	Leaf
0	1 3 7
1	0 3 4 7 8 8 9
2	0 0 1 2 2 3 5 5 6 8 9
3	2 4 4 5 8 9
4	3

Key: $3 | 2$ means 32

9 Following weekly lessons, Guy's golf scores on successive Saturdays were:

98   96   92   93   89   90   88   85   and   84

   a Which is more appropriate to draw for this data;   a line graph or scatterplot?

   b Graph the data appropriately.

10 A test score out of 60 marks is recorded for a group of 45 students:

34	37	44	51	53	39	33	58	40	42	43	43	47	37	35
41	43	48	50	55	44	44	52	54	59	39	31	29	44	57
45	34	29	27	18	49	41	42	37	42	43	43	45	34	51

   a Construct a stem-and-leaf plot for this data using 0, 1, 2, 3, 4, and 5 as the stems.

   b Redraw the stem-and-leaf plot so that it is ordered.

   c What advantage does a stem-and-leaf plot have over a frequency table?

   d What is the   **i** highest   **ii** lowest   mark scored for the test?

   e If an 'A' is awarded to students who scored 50 or more for the test, what percentage of students scored an 'A'?

   f What percentage of students scored less than half marks for the test?

B   GRAPHS WHICH COMPARE DATA   [11.3]

Two distributions can be compared by using:

- side-by-side bar charts or **compound bar charts**
- back-to-back bar charts
- back-to-back stem-and-leaf plots


```
                    12 | 6
         4          11 | 2 3 7
         7 1        10 | 6 7 8 8
 9 8 7 6 6 5 3 3 2 0  9 | 0 3 4 5 5 7 9
     7 5 3 2 1       8 | 2 2 4 5 8
         2 0         7 | 0 5 6
                     6 | 1
```
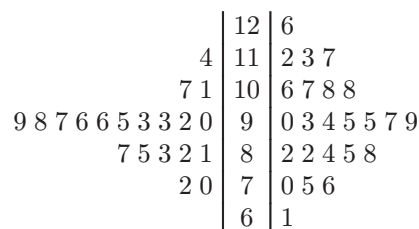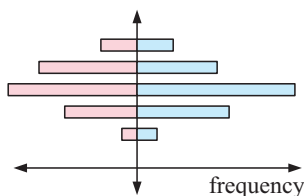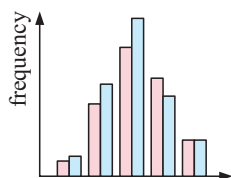
Key: 6|1 means 61

Suppose two machines A and B in a factory were tested each day for 60 days to see how many faulty screw caps they produced.   The results can be summarised in a **frequency table**.   The **frequency** of each outcome is the number of times it occurs.

Machine A

Number faulty	Frequency
2	1
3	0
4	4
5	16
6	24
7	6
8	5
9	2
10	0
11	0
12	1
13	1

Machine B

Number faulty	Frequency
3	2
4	3
5	5
6	5
7	10
8	11
9	20
10	3
11	1

We can compare the data for the two machines using a compound bar chart or back-to-back bar chart:

Compound bar chart of screw cap data   **Back-to-back bar chart of screw cap data**
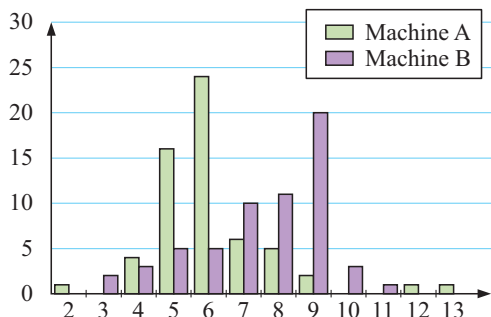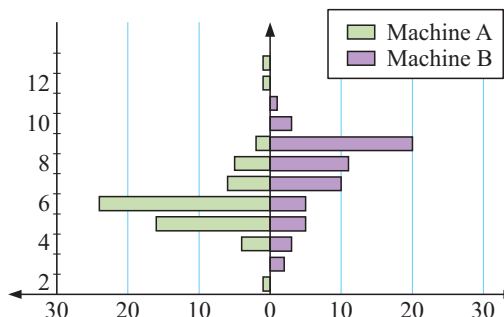
We can see from the graphs that machine B generally produces more faulty screw caps than machine A.

Example 5   🔊 *Self Tutor*

15 students were examined in Geography and Mathematics. The results were:

Geography	65	70	56	67	49	82	79	88	47	76	69	58	90	45	82
Mathematics	75	67	81	88	84	66	77	72	60	58	67	71	74	82	89

Use an ordered back-to-back stem-and-leaf plot to compare the results.

Before ordering: **After ordering:**

Geography   *Mathematics*   *Geography*   *Mathematics*

```
  5 7 9 | 4 |                                         9 7 5 | 4 |
    8 6 | 5 | 8                                         8 6 | 5 | 8
  9 7 5 | 6 | 7 6 0 7                                 9 7 5 | 6 | 0 6 7 7
  6 9 0 | 7 | 5 7 2 1 4                               9 6 0 | 7 | 1 2 4 5 7
  2 8 2 | 8 | 1 8 4 2 9                               8 2 2 | 8 | 1 2 4 8 9
      0 | 9 |        Key:  8|2 means 82                    0 | 9 |        Key:   8|2 means 82
```

From the plot we observe that:
- the Geography marks are more spread than those for Mathematics
- the Mathematics marks are generally higher than those for Geography.

EXERCISE 5B

1 In each of the following graphs, two data sets of the same size are compared. What can be deduced from each graph?

a

b

c

d

e

f
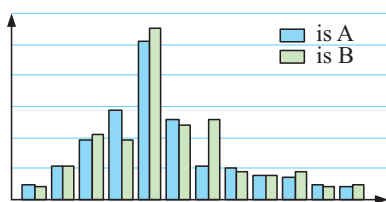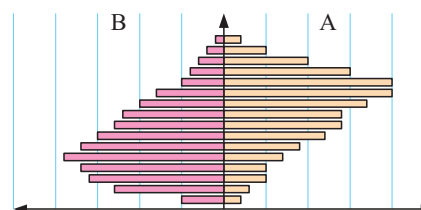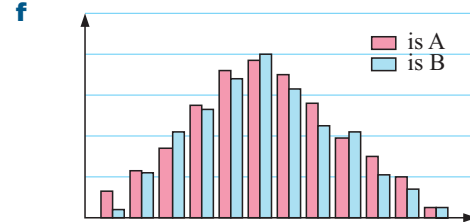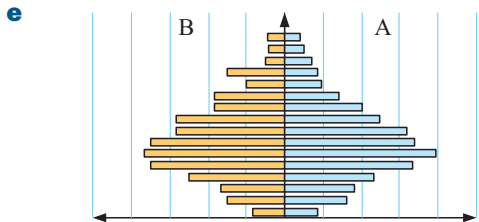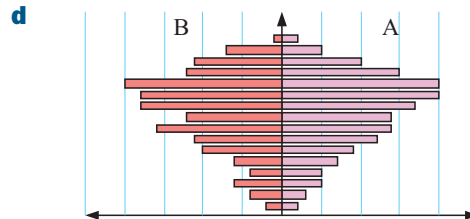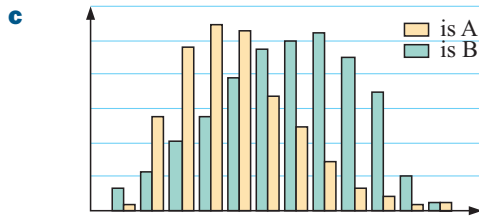
2 Pedro sells small ceramic items on an internet auction site. He lives in the USA and sells within his country and overseas. He uses the standard postal service for local deliveries and a private freight company for international ones. Pedro records the number of items that are broken each month over a 70 month period. Which delivery service is Pedro happier with? Explain your answer.
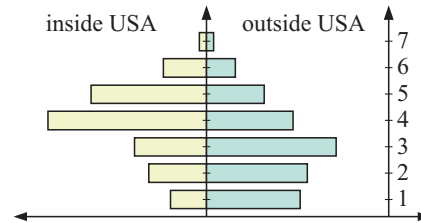
3 This stem-and-leaf plot shows the one day international cricket scores for a batsman in the 2007 and 2008 seasons.

   a In which season was the batsman more consistent?

   b In which season did the batsman score more runs?

Season 2007		Season 2008
4 3 0	0	
8 8 6 1	1	4
9 6 5 5	2	0 6
9 7 6 2 0	3	1 3 3 6 9
7 4	4	2 2 4 5 8 9
7	5	2 3 6 7
2	6	1 1

Key: 2|0 means 20 runs

4 The house sales of a real estate agency for 2007 and 2008 are summarised in the table below:

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2007	2	8	8	11	25	26	31	34	28	13	10	4
2008	5	3	6	14	21	25	28	29	15	13	7	3

   a Construct a back-to-back bar chart of the data.

   b Compare the sales in the two years.

5 Two brothers living together travel by different means to university; Alex travels by train and Stan travels by bus. Over a three week period, their travel times in minutes were:

Alex	17	22	34	60	41	15	55	30	36	23	27	48	34	25	45
Stan	31	19	28	42	24	18	30	36	34	25	38	31	22	29	32

   a Construct a back-to-back stem-and-leaf plot of the data.

   b Which mode of transport is more reliable? Explain your answer.

6 **a** Construct a side-by-side stem-and-leaf plot for the data in the **Opening Problem** on page **111**.

   b Is there a noticeable difference between the data sets?

C   USING TECHNOLOGY TO GRAPH DATA [11.3]

Many special computer programs are used to help us organise and graph data.

STATISTICS PACKAGE

Click on the icon to run the statistical graphing software.
Change the data in the table and see the effect on the graph.

- Notice that the graph's heading and the labels on the axes can be changed.
- The type of graph can be changed by clicking on the icon to give the type that you want.

Experiment with the package and use it whenever possible.
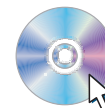
Discovery   *Spreadsheets for graphing data*

The colours of cars in a supermarket carpark are recorded alongside. Suppose you want to draw a frequency bar chart of this data.

Colour	white	red	blue	green	other
Frequency	38	27	19	18	11

The following steps using a computer spreadsheet enable you to do this quickly and easily.

Step 1: Start a new spreadsheet, type in the table, and then highlight the area as shown.

Step 2: Click on [icon] from the menu bar.

Chart Wizard

Step 3: Choose

Standard Types | Custom Types

Chart type:
- Column
- Bar
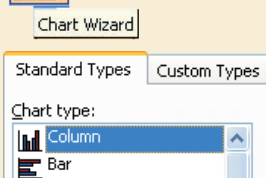
	A	B
1	Colour	Frequency
2	white	38
3	red	27
4	blue	19
5	green	18
6	other	11

SPREADSHEET

This is probably already highlighted. Click [Finish] .

You should get:
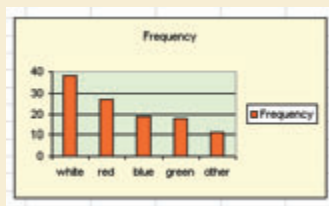
This demonstration takes you through all the steps.

DEMO

Now suppose the colours of cars in the neighbouring car park were also recorded. The results are summarised in the following table.

Colour	white	red	blue	green	other
Frequency 1	38	27	19	18	11
Frequency 2	15	18	21	9	3

Step 4: Into the C column type the *Frequency 2* data and highlight the three columns as shown.

	A	B	C	D
1	Colour	Frequency 1	Frequency 2	
2	white	38	15	
3	red	27	18	
4	blue	19	21	
5	green	18	9	
6	other	11	3	
7				

Step 5:   Click on   [Chart Wizard]   then   Standard Types   Custom Types

Chart type:

Column
Bar
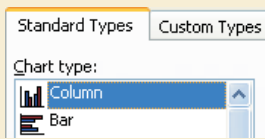
Step 6:   Choose   [Chart sub-type]   then   [Finish]

Step 7:   Experiment with other types of graphs and data.
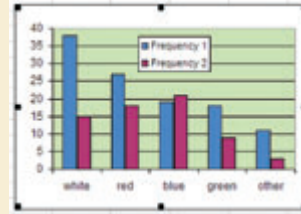
What to do:

1   Gather statistics of your own or use data from questions in the previous exercise. Use the spreadsheet to draw an appropriate statistical graph of the data.

2   Find out how to adjust labels, scales, legends, and other features of the graph.

EXERCISE 5C

Use technology to answer the following questions. Make sure each graph is fully labelled.

1   The given data is to be displayed on a pie chart.

    a   Find the sector angles to the nearest degree for each category.

    b   Draw the pie chart for this data.

Favourite colour	Frequency
Green	38
Blue	34
Red	29
Purple	21
Other	34

2   After leaving school, four graduates compare their weekly incomes in dollars at 5 year intervals. Their incomes are shown in the table alongside.

	After 5 years	After 10 years
Kaylene	685	2408
Harry	728	1345
Wei	820	2235
Matthew	1056	1582

    a   Draw side-by-side bar charts to represent the data.

    b   Draw back-to-back bar charts to represent the data.

    c   Find the percentage increase for each person for the data given.

    d   Two of the four gained university qualifications. Which ones are they likely to be?

3   Consider again the real estate agency data from question **4** of the previous exercise.

    a   Construct: **i**   a side-by-side bar chart   **ii**   a back-to-back chart   for the data.

    b   Which chart provides a better means of comparing the two data sets?

Review set 5A

1   The table alongside shows the money spent by a business over the course of a year.

    a   How much did the business spend?

    b   Calculate the sector angle for each category.

    c   Construct a pie chart to display the data.

Administration	£144 000
Production	£200 000
Marketing	£180 000
Distribution	£120 000
Advertising	£76 000

2 Farmer Jane owns the assortment of animals shown in the table alongside.

  a Display this data on a vertical bar chart.

  b What percentage of animals on the farm are chickens?

  c Name the three most common animals on the farm.

Animal	Number
Chickens	12
Cows	35
Dogs	4
Ducks	10
Geese	5
Goats	24
Pigs	11
Sheep	19

3 The table alongside shows the number of males and females participating in the school's drama class each year for the last five years.

	2004	2005	2006	2007	2008
Males	10	8	5	10	14
Females	15	18	17	20	18

  a Display the data on a side-by-side bar chart.

  b In which year was there the greatest difference between the number of males and females?

4 For the given data on calf weights:

  a Explain what 5|1 means.

  b Find the minimum calf weight.

  c Find the maximum calf weight.

  d Find which weight occurs most frequently.

Weight of calves (kg)

```
2 | 8
3 | 1 4 9
4 | 0 3 3 6 7
5 | 1 4 4 4 5 5 8
6 | 3 5        Key:  6|3 means 63 kg
```

5 Huw owns a flower shop. The table below shows the total cost of supplying, as well as the sales for each flower type, for the past year.

Flower	Roses	Carnations	Tulips	Orchids	Azaleas	Lilies
Cost (£)	17 000	15 000	8000	7000	5000	8000
Sales (£)	35 000	25 000	16 000	15 000	15 000	14 000

  a Draw two pie charts to display the data.

  b Which two flower types account for more than half of the total costs?

  c Which two flower types account for half of the total sales?

  d Huw needs to reduce his workload, so he decides to stop selling one type of flower. Which flower type would you recommend he stop selling? Explain your answer.

6 To determine if a weight-loss program is effective, a group of 25 men trialled the program for three months. Their weights in kilograms before starting the program were:

> 95   104   93   86   82   111   100   125   117   121   119   97   120
> 104   132   111   98   83   123   79   101   135   114   99   122

Their weights after completing the program were:

> 79   92   84   78   68   99   85   106   99   107   102   83   110
> 93   125   96   79   74   113   68   89   122   95   83   106

  a Construct a back-to-back stem-and-leaf plot of the data.

  b Would you say that the weight-loss program is effective? Explain your answer.

Review set 5B

1 The number of drivers caught speeding in each month last year were:

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
80	65	60	95	50	85	70	40	55	80	60	90

Display this data on a vertical bar chart.

2 The prices of IBM shares over a 10 day period were:

Day	1	2	3	4	5	6	7	8	9	10
Share Price ($)	116.00	116.50	118.00	116.80	116.50	117.00	118.40	118.50	118.50	119.80

    a Is the line graph or a scatter diagram more appropriate for this data?

    b Display the data on your chosen graph. Use a range of $115 to $120 on the vertical axis.

    c On what day was the share price: **i** highest   **ii** lowest?

3 The weights of 16 new-born babies in kilograms are:

    2.3, 3.1, 3.8, 4.1, 3.6, 3.5, 4.2, 2.8, 3.4, 3.9, 4.0, 5.1, 4.4, 3.9, 4.0, 3.3

Draw a stem-and-leaf plot for this data.

4 Over the course of a 35 game season, basketballers Rhys and Evan kept a record of the number of rebounds they made each game. The results were:

Rhys									**Evan**								
8	6	4	9	6	10	9	9	8	7	7	6	8	6	7	6	6	8
7	5	7	8	7	4	7	8	5	6	5	7	7	8	5	6	6	5
8	9	6	3	9	10	8	7	10	7	9	7	6	6	5	8	7	7
9	7	8	8	10	9	6	8		8	5	7	7	6	8	7	8	

    a Construct a back-to-back bar chart for the data.

    b Which player was the most consistent?

    c Which player generally collected the most rebounds?

5 A pet store owner wants to know whether men or women own more pets. He surveys 50 men and 50 women, each of whom live by themselves, and collects data on how many pets they own. The results are as follows:

    a Display the data on a side-by-side bar chart.

    b In general, do men or women own more pets?

Men

No. of pets	Frequency
0	15
1	20
2	10
3	3
4	2
5	0

Women

No. of pets	Frequency
0	7
1	10
2	15
3	10
4	5
5	3

6 A selection of students in their final year of high school were asked what they planned to do next year. The responses are displayed in the table alongside:

Use technology to construct a pie chart for this data.

Response	Frequency
Work	89
Apprenticeship	47
University	71
Travel	38

Exponents and surds

6

Contents:

Opening problem

Amadeo Avogadro (1776-1856) established that one gram of hydrogen contains 6.02×10^{23} atoms.

Things to think about:

- How can we write this number as an ordinary number?
- How many atoms would be in one tonne of hydrogen gas?
- Can you find the mass of 10^{30} atoms of hydrogen?

A   EXPONENT OR INDEX NOTATION   [1.4, 1.9]

The use of **exponents**, also called **powers** or **indices**, allows us to write products of factors and also to write very large or very small numbers quickly.

We have seen previously that $2 \times 2 \times 2 \times 2 \times 2$, can be written as 2^5.

2^5 reads "two to the power of five" or "two with index five".

In this case 2 is the **base** and 5 is the **exponent**, **power** or **index**.

We say that 2^5 is written in **exponent** or **index notation**.

2^5 — exponent, power or index
— base

Example 1   ◀) *Self Tutor*

Find the integer equal to: **a** 3^4 **b** $2^4 \times 3^2 \times 7$

a 3^4

$= 3 \times 3 \times 3 \times 3$

$= 9 \times 9$

$= 81$

b $2^4 \times 3^2 \times 7$

$= 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 7$

$= 16 \times 9 \times 7$

$= 1008$

Example 2   ◀) *Self Tutor*

Write as a product of prime factors in index form:

a   144   **b**   4312

a

2	144
2	72
2	36
2	18
3	9
3	3
	1

$\therefore \quad 144 = 2^4 \times 3^2$

b

2	4312
2	2156
2	1078
7	539
7	77
11	11
	1

$\therefore \quad 4312 = 2^3 \times 7^2 \times 11$

EXERCISE 6A.1

1 Find the integer equal to:

   a 2^3 **b** 3^3 **c** 2^5 **d** 5^3

   e $2^2 \times 3^3 \times 5$ **f** $2^3 \times 3 \times 7^2$ **g** $3^2 \times 5^2 \times 13$ **h** $2^4 \times 5^2 \times 11$

2 By dividing continuously by the primes 2, 3, 5, 7, ..., write as a product of prime factors in index form:

   a   50   **b**   98   **c**   108   **d**   360

   e   1128   **f**   784   **g**   952   **h**   6500

3 The following numbers can be written in the form 2^n. Find n.

   a   32   **b**   256   **c**   4096

4 The following numbers can be written in the form 3^n. Find n.

   a   27   **b**   729   **c**   59 049

5 By considering $3^1, 3^2, 3^3, 3^4, 3^5$ .... and looking for a pattern, find the last digit of 3^{33}.

6 What is the last digit of 7^{77}?

7 Find n if:

   a $5^4 = n$ **b** $n^3 = 343$ **c** $11^n = 161\,051$ **d** $(0.6)^n = 0.046\,656$

Historical note

Nicomachus of Gerasa lived around 100 AD. He discovered an interesting number pattern involving cubes and sums of odd numbers:

$$1 = 1^3$$
$$3 + 5 = 8 = 2^3$$
$$7 + 9 + 11 = 27 = 3^3 \quad \text{etc.}$$

NEGATIVE BASES

So far we have only considered **positive** bases raised to a power.

We will now briefly look at **negative** bases. Consider the statements below:

$(-1)^1 = -1$ $(-2)^1 = -2$
$(-1)^2 = -1 \times -1 = 1$ $(-2)^2 = -2 \times -2 = 4$
$(-1)^3 = -1 \times -1 \times -1 = -1$ $(-2)^3 = -2 \times -2 \times -2 = -8$
$(-1)^4 = -1 \times -1 \times -1 \times -1 = 1$ $(-2)^4 = -2 \times -2 \times -2 \times -2 = 16$

From the pattern above it can be seen that:

- a **negative** base raised to an **odd** power is **negative**
- a **negative** base raised to an **even** power is **positive**.

Example 3   ◀⑨ Self Tutor

Evaluate:

  a $(-2)^4$ **b** -2^4 **c** $(-2)^5$ **d** $-(-2)^5$

Notice the effect of the brackets in these examples.

a $(-2)^4$
      $= 16$

b -2^4
      $= -1 \times 2^4$
      $= -16$

c $(-2)^5$
      $= -32$

d $-(-2)^5$
      $= -1 \times (-2)^5$
      $= -1 \times -32$
      $= 32$

CALCULATOR USE

Different calculators have different keys for entering powers, but in general they perform raising to powers in a similar manner.

Not all calculators will use these key sequences. If you have problems, refer to the calculator instructions on page **12**.

Power keys

$\boxed{x^2}$ squares the number in the display.

$\boxed{\wedge}$ raises the number in the display to whatever power is required. On some calculators this key is $\boxed{y^x}$, $\boxed{a^x}$ or $\boxed{x^y}$.

Example 4   ◀)) *Self Tutor*

Find, using your calculator: **a** 6^5 **b** $(-5)^4$ **c** -7^4

Answer

a   Press: 6 $\boxed{\wedge}$ 5 $\boxed{\text{ENTER}}$ 7776

b   Press: $\boxed{(}$ $\boxed{(-)}$ 5 $\boxed{)}$ $\boxed{\wedge}$ 4 $\boxed{\text{ENTER}}$ 625

c   Press: $\boxed{(-)}$ 7 $\boxed{\wedge}$ 4 $\boxed{\text{ENTER}}$ -2401

EXERCISE 6A.2

1 Simplify:

  a $(-1)^4$ **b** $(-1)^5$ **c** $(-1)^{10}$ **d** $(-1)^{15}$ **e** $(-1)^8$ **f** -1^8

  g $-(-1)^8$ **h** $(-3)^3$ **i** -3^3 **j** $-(-3)^3$ **k** $-(-6)^2$ **l** $-(-4)^3$

2 Simplify:

  a $2^3 \times 3^2 \times (-1)^5$ **b** $(-1)^4 \times 3^3 \times 2^2$ **c** $(-2)^3 \times (-3)^4$

3 Use your calculator to find the value of the following, recording the entire display:

  a 2^8 **b** $(-5)^4$ **c** -3^5 **d** 7^4 **e** 8^3 **f** $(-7)^6$

  g -7^6 **h** 1.05^{12} **i** -0.623^{11} **j** $(-2.11)^{17}$

B   EXPONENT OR INDEX LAWS   [1.9, 2.4]

Notice that:
  - $2^3 \times 2^4 = 2 \times 2 \times 2 \ \times \ 2 \times 2 \times 2 \times 2 = 2^7$

  - $\dfrac{2^5}{2^2} = \dfrac{2 \times 2 \times 2 \times \cancel{2} \times \cancel{2}^{\,1}}{\cancel{2} \times \cancel{2}_{\,1}} = 2^3$

  - $(2^3)^2 = 2 \times 2 \times 2 \ \times \ 2 \times 2 \times 2 = 2^6$

  - $(3 \times 5)^2 = 3 \times 5 \times 3 \times 5 = 3 \times 3 \times 5 \times 5 = 3^2 5^2$

  - $\left(\dfrac{2}{5}\right)^3 = \dfrac{2}{5} \times \dfrac{2}{5} \times \dfrac{2}{5} = \dfrac{2 \times 2 \times 2}{5 \times 5 \times 5} = \dfrac{2^3}{5^3}$

These examples can be generalised to the exponent or index laws:

$\bullet$ $a^m \times a^n = a^{m+n}$	To **multiply** numbers with the **same base**, keep the base and **add** the indices.
$\bullet$ $\dfrac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$	To **divide** numbers with the **same base**, keep the base and **subtract** the indices.
$\bullet$ $(a^m)^n = a^{m \times n}$	When **raising** a **power** to a **power**, keep the base and **multiply** the indices.
$\bullet$ $(ab)^n = a^n b^n$	The power of a product is the product of the powers.
$\bullet$ $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}, \quad b \neq 0$	The power of a quotient is the quotient of the powers.

Example 5 ◀》 *Self Tutor*

Simplify using the laws of indices:

a $2^3 \times 2^2$ **b** $x^4 \times x^5$

a $2^3 \times 2^2 = 2^{3+2}$ **b** $x^4 \times x^5 = x^{4+5}$
$\qquad = 2^5$ $\qquad = x^9$
$\qquad = 32$

> To multiply, keep the base and add the indices.

> To divide, keep the base and subtract the indices.

Example 6 ◀》 *Self Tutor*

Simplify using the index laws: **a** $\dfrac{3^5}{3^3}$ **b** $\dfrac{p^7}{p^3}$

a $\dfrac{3^5}{3^3} = 3^{5-3}$ **b** $\dfrac{p^7}{p^3} = p^{7-3}$
$\qquad = 3^2$ $\qquad = p^4$
$\qquad = 9$

Example 7 ◀》 *Self Tutor*

Simplify using the index laws:

a $(2^3)^2$ **b** $(x^4)^5$

a $(2^3)^2$ **b** $(x^4)^5$
$= 2^{3 \times 2}$ $= x^{4 \times 5}$
$= 2^6$ $= x^{20}$
$= 64$

> To raise a power to a power, keep the base and multiply the indices.

> Each factor within the brackets has to be raised to the power outside them.

Example 8 ◀》 *Self Tutor*

Remove the brackets of:

a $(3a)^2$ **b** $\left(\dfrac{2x}{y}\right)^3$

a $(3a)^2$ **b** $\left(\dfrac{2x}{y}\right)^3$
$= 3^2 \times a^2$ $= \dfrac{2^3 \times x^3}{y^3}$
$= 9a^2$ $= \dfrac{8x^3}{y^3}$

Example 9   ◀») *Self Tutor*

Express the following in simplest form, without brackets:

a $(3a^3b)^4$ **b** $\left(\dfrac{x^2}{2y}\right)^3$

a $(3a^3b)^4$ **b** $\left(\dfrac{x^2}{2y}\right)^3$

$= 3^4 \times (a^3)^4 \times b^4$

$= 81 \times a^{3\times4} \times b^4$ $= \dfrac{(x^2)^3}{2^3 \times y^3}$

$= 81a^{12}b^4$ $= \dfrac{x^{2\times3}}{8 \times y^3}$

$= \dfrac{x^6}{8y^3}$

EXERCISE 6B

1 Simplify using the index laws:

  a $2^3 \times 2^1$ **b** $2^2 \times 2^2$ **c** $3^5 \times 3^4$ **d** $5^2 \times 5^3$

  e $x^2 \times x^4$ **f** $a^3 \times a$ **g** $n^4 \times n^6$ **h** $b^3 \times b^5$

2 Simplify using the index laws:

  a $\dfrac{2^4}{2^3}$ **b** $\dfrac{3^5}{3^2}$ **c** $\dfrac{5^7}{5^3}$ **d** $\dfrac{4^9}{4^5}$

  e $\dfrac{x^6}{x^3}$ **f** $\dfrac{y^7}{y^4}$ **g** $a^8 \div a^7$ **h** $b^9 \div b^5$

3 Simplify using the index laws:

  a $(2^2)^3$ **b** $(3^4)^3$ **c** $(2^3)^6$ **d** $(10^2)^5$

  e $(x^3)^2$ **f** $(x^5)^3$ **g** $(a^5)^4$ **h** $(b^6)^4$

4 Simplify using the index laws:

  a $a^5 \times a^2$ **b** $n^3 \times n^5$ **c** $a^7 \div a^3$ **d** $a^5 \times a$

  e $b^9 \div b^4$ **f** $(a^3)^6$ **g** $a^n \times a^5$ **h** $(b^2)^4$

  i $b^6 \div b^3$ **j** $m^4 \times m^3 \times m^7$ **k** $(a^3)^3 \times a$ **l** $(g^2)^4 \times g^3$

5 Remove the brackets of:

  a $(ab)^3$ **b** $(ac)^4$ **c** $(bc)^5$ **d** $(abc)^3$

  e $(2a)^4$ **f** $(5b)^2$ **g** $(3n)^4$ **h** $(2bc)^3$

  i $\left(\dfrac{2}{p}\right)^3$ **j** $\left(\dfrac{a}{b}\right)^3$ **k** $\left(\dfrac{m}{n}\right)^4$ **l** $\left(\dfrac{2c}{d}\right)^5$

6 Express the following in simplest form, without brackets:

a $(2b^4)^3$ **b** $\left(\dfrac{3}{x^2y}\right)^2$ **c** $(5a^4b)^2$ **d** $\left(\dfrac{m^3}{2n^2}\right)^4$

e $\left(\dfrac{3a^3}{b^5}\right)^3$ **f** $(2m^3n^2)^5$ **g** $\left(\dfrac{4a^4}{b^2}\right)^2$ **h** $(5x^2y^3)^3$

C   ZERO AND NEGATIVE INDICES   [1.9, 2.4]

Consider $\dfrac{2^3}{2^3}$ which is obviously 1.

Using the exponent law for division, $\dfrac{2^3}{2^3} = 2^{3-3} = 2^0$

We therefore conclude that $2^0 = 1$.

In general, we can state the **zero index law**: $a^0 = 1$ for all $a \neq 0$.

Now consider $\dfrac{2^4}{2^7}$ which is $\dfrac{2 \times 2 \times 2 \times 2}{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2} = \dfrac{1}{2^3}$

Using the exponent law of division, $\dfrac{2^4}{2^7} = 2^{4-7} = 2^{-3}$

Consequently, $2^{-3} = \dfrac{1}{2^3}$, which means that 2^{-3} and 2^3 are **reciprocals** of each other.

In general, we can state the **negative index law**:

> If a is any non-zero number and n is an integer, then $a^{-n} = \dfrac{1}{a^n}$.
>
> This means that a^n and a^{-n} are **reciprocals** of one another.
>
> In particular notice that $a^{-1} = \dfrac{1}{a}$.

Using the negative index law,
$$\left(\tfrac{2}{3}\right)^{-4} = \dfrac{1}{\left(\tfrac{2}{3}\right)^4}$$
$$= 1 \div \dfrac{2^4}{3^4}$$
$$= 1 \times \dfrac{3^4}{2^4}$$
$$= \left(\tfrac{3}{2}\right)^4$$

So, in general we can see that: $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$ provided $a \neq 0, \ b \neq 0$.

Example 10   ◀ᴽ *Self Tutor*

Simplify, giving answers in simplest rational form:

a 7^0 **b** 3^{-2} **c** $3^0 - 3^{-1}$ **d** $\left(\frac{5}{3}\right)^{-2}$

a $7^0 = 1$ **b** $3^{-2} = \dfrac{1}{3^2} = \frac{1}{9}$

c $3^0 - 3^{-1} = 1 - \frac{1}{3} = \frac{2}{3}$ **d** $\left(\frac{5}{3}\right)^{-2} = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$

EXERCISE 6C

1 Simplify, giving answers in simplest rational form:

a 3^0 **b** 6^{-1} **c** 4^{-1} **d** 5^0

e 3^2 **f** 3^{-2} **g** 5^3 **h** 5^{-3}

i 7^2 **j** 7^{-2} **k** 10^3 **l** 10^{-3}

2 Simplify, giving answers in simplest rational form:

a $\left(\frac{1}{2}\right)^0$ **b** $\dfrac{5^4}{5^4}$ **c** $2t^0$ **d** $(2t)^0$

e 7^0 **f** 3×4^0 **g** $\dfrac{5^3}{5^5}$ **h** $\dfrac{2^6}{2^{10}}$

i $\left(\frac{1}{4}\right)^{-1}$ **j** $\left(\frac{3}{8}\right)^{-1}$ **k** $\left(\frac{2}{3}\right)^{-1}$ **l** $\left(\frac{1}{5}\right)^{-1}$

m $2^0 + 2^1$ **n** $5^0 - 5^{-1}$ **o** $3^0 + 3^1 - 3^{-1}$ **p** $\left(\frac{1}{3}\right)^{-2}$

q $\left(\frac{2}{3}\right)^{-3}$ **r** $\left(1\frac{1}{2}\right)^{-3}$ **s** $\left(\frac{4}{5}\right)^{-2}$ **t** $\left(2\frac{1}{2}\right)^{-2}$

3 Write the following without brackets or negative indices:

a $(3b)^{-1}$ **b** $3b^{-1}$ **c** $7a^{-1}$ **d** $(7a)^{-1}$

e $\left(\dfrac{1}{t}\right)^{-2}$ **f** $(4t)^{-2}$ **g** $(5t)^{-2}$ **h** $(5t^{-2})^{-1}$

i xy^{-1} **j** $(xy)^{-1}$ **k** xy^{-3} **l** $(xy)^{-3}$

m $(3pq)^{-1}$ **n** $3(pq)^{-1}$ **o** $3pq^{-1}$ **p** $\dfrac{(xy)^3}{y^{-2}}$

4 Write as powers of 2, 3 or 5:

a 25   **b** $\frac{1}{25}$ **c** 27   **d** $\frac{1}{27}$

e 16   **f** $\frac{1}{16}$ **g** $\frac{2}{3}$ **h** $\frac{3}{5}$

i $\frac{9}{125}$ **j** $\frac{32}{81}$ **k** $2\frac{2}{5}$ **l** $9\frac{3}{8}$

5 Write as powers of 10:

a 1000   **b** 0.01   **c** 100 000   **d** 0.000 001

D   STANDARD FORM   [1.9]

Consider the pattern alongside. Notice that each time we divide by
10, the **exponent** or **power** of 10 decreases by one.

$$
\begin{array}{l}
\div 10 \left(\begin{array}{l} 10\,000 = 10^4 \\ 1000 = 10^3 \end{array}\right) -1 \\
\div 10 \left(\begin{array}{l} 1000 = 10^3 \\ 100 = 10^2 \end{array}\right) -1 \\
\div 10 \left(\begin{array}{l} 100 = 10^2 \\ 10 = 10^1 \end{array}\right) -1 \\
\div 10 \left(\begin{array}{l} 10 = 10^1 \\ 1 = 10^0 \end{array}\right) -1 \\
\div 10 \left(\begin{array}{l} 1 = 10^0 \\ \frac{1}{10} = 10^{-1} \end{array}\right) -1 \\
\div 10 \left(\begin{array}{l} \frac{1}{10} = 10^{-1} \\ \frac{1}{100} = 10^{-2} \end{array}\right) -1 \\
\div 10 \left(\begin{array}{l} \frac{1}{100} = 10^{-2} \\ \frac{1}{1000} = 10^{-3} \end{array}\right) -1
\end{array}
$$

We can use this pattern to simplify the writing of very large and very small numbers.

For example, $5\,000\,000$ and $0.000\,003$

$$= 5 \times 1\,000\,000$$ $$= \dfrac{3}{1\,000\,000}$$

$$= 5 \times 10^6$$ $$= \dfrac{3}{1} \times \dfrac{1}{1\,000\,000}$$

$$= 3 \times 10^{-6}$$

STANDARD FORM

> **Standard form** (or **scientific notation**) involves writing any given number as *a number between* 1
> *and* 10, multiplied by an *integer power of* 10,
> $$\text{i.e.,} \quad a \times 10^n \quad \text{where} \quad 1 \leqslant a < 10 \quad \text{and} \quad n \in \mathbb{Z}.$$

Example 11   ◀)) *Self Tutor*

Write in standard form: **a** $37\,600$ **b** $0.000\,86$

a $37\,600 = 3.76 \times 10\,000$ {shift decimal point 4 places to the left and $\times 10\,000$}

$ = 3.76 \times 10^4$

b $0.000\,86 = 8.6 \div 10^4$ {shift decimal point 4 places to the right and $\div 10\,000$}

$ = 8.6 \times 10^{-4}$

Example 12   ◀)) *Self Tutor*

Write as an ordinary number:

a 3.2×10^2 **b** 5.76×10^{-5}

a 3.2×10^2 **b** 5.76×10^{-5}

$= 3.20 \times 100$ $= 000005.76 \div 10^5$

$= 320$ $= 0.000\,057\,6$

Example 13
◀) *Self Tutor*

Simplify the following, giving your answer in standard form:

a $(5 \times 10^4) \times (4 \times 10^5)$ **b** $(8 \times 10^5) \div (2 \times 10^3)$

a $(5 \times 10^4) \times (4 \times 10^5)$
$= 5 \times 4 \times 10^4 \times 10^5$
$= 20 \times 10^{4+5}$
$= 2 \times 10^1 \times 10^9$
$= 2 \times 10^{10}$

b $(8 \times 10^5) \div (2 \times 10^3)$
$= \dfrac{8 \times 10^5}{2 \times 10^3}$
$= \frac{8}{2} \times 10^{5-3}$
$= 4 \times 10^2$

To help write numbers in standard form:

- If the original number is > 10, the power of 10 is **positive** $(+)$.
- If the original number is < 1, the power of 10 is **negative** $(-)$.
- If the original number is between 1 and 10, leave it as it is and multiply it by 10^0.

EXERCISE 6D.1

1 Write the following as powers of 10:

   a 100   **b** 1000   **c** 10   **d** 100 000

   e 0.1   **f** 0.01   **g** 0.0001   **h** 100 000 000

2 Express the following in standard form:

   a 387   **b** 38 700   **c** 3.87   **d** 0.0387

   e 0.003 87   **f** 20.5   **g** 205   **h** 0.205

   **i** 20 500   **j** 20 500 000   **k** 0.000 205

3 Express the following in standard form:

   a The circumference of the Earth is approximately 40 075 kilometres.

   b The distance from the Earth to the Sun is 149 500 000 000 m.

   c Bacteria are single cell organisms, some of which have a diameter of 0.0004 mm.

   d There are typically 40 million bacteria in a gram of soil.

   e The probability that your six numbers will be selected for Lotto on Saturday night is 0.000 000 141 62.

   f Superfine sheep have wool fibres as low as 0.01 mm in diameter.
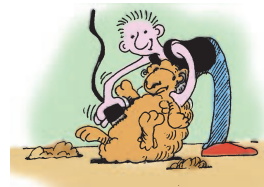
4 Write as an ordinary decimal number:

   a 3×10^2 **b** 2×10^3 **c** 3.6×10^4 **d** 9.2×10^5

   e 5.6×10^6 **f** 3.4×10^1 **g** 7.85×10^6 **h** 9×10^8

5 Write as an ordinary decimal number:

   a 3×10^{-2} **b** 2×10^{-3} **c** 4.7×10^{-4} **d** 6.3×10^{-5}

   e 1.7×10^0 **f** 9.5×10^{-4} **g** 3.49×10^{-1} **h** 7×10^{-6}

6 Write as an ordinary decimal number:

    a The wavelength of visible light is 9×10^{-7} m.

    b In 2007, the world population was approximately 6.606×10^9.

    c The diameter of our galaxy, the Milky Way, is 1×10^5 light years.

    d The smallest viruses are 1×10^{-5} mm in size.

    e 1 atomic mass unit is approximately 1.66×10^{-27} kg.

7 Write in standard form:

    a 18.17×10^6 **b** 0.934×10^{11} **c** 0.041×10^{-2}

8 Simplify the following, giving your answer in standard form:

    a $(8 \times 10^3) \times (2 \times 10^4)$ **b** $(8 \times 10^3) \times (4 \times 10^5)$

    c $(5 \times 10^4) \times (3 \times 10^5)$ **d** $(2 \times 10^3)^3$

    e $(6 \times 10^3)^2$ **f** $(7 \times 10^{-2})^2$

    g $(9 \times 10^4) \div (3 \times 10^3)$ **h** $(8 \times 10^5) \div (4 \times 10^6)$

STANDARD FORM ON A CALCULATOR

Scientific and graphics calculators are able to display very large and very small numbers in standard form.

If you perform $2\,300\,000 \times 400\,000$ your calculator might display $\boxed{9.2\text{E}+11}$ or $\boxed{9.2^{11}}$ or $\boxed{9.2\text{E}11}$, all of which actually represent 9.2×10^{11}.

Likewise, if you perform $0.0024 \div 10\,000\,000$ your calculator might display $\boxed{2.4^{-10}}$ or $\boxed{2.4\text{E}\text{-}10}$, which actually represent 2.4×10^{-10}.

You will find instructions for graphics calculators on page **16**.

EXERCISE 6D.2

1 Write each of the following as it would appear on the display of *your* calculator:

    a $4\,650\,000$ **b** $0.000\,051\,2$ **c** 5.99×10^{-4}

    d 3.761×10^{10} **e** $49\,500\,000$ **f** $0.000\,008\,44$

2 Calculate each of the following, giving your answers in standard form. The decimal part should be correct to 2 decimal places:

    a $0.06 \times 0.002 \div 4000$ **b** $426 \times 760 \times 42\,000$ **c** $627\,000 \times 74\,000$

    d $320 \times 600 \times 51\,400$ **e** $0.004\,28 \div 120\,000$ **f** $0.026 \times 0.00\,42 \times 0.08$

Example 14   ◀๙ *Self Tutor*

Use your calculator to find:

    a $(1.42 \times 10^4) \times (2.56 \times 10^8)$ **b** $(4.75 \times 10^{-4}) \div (2.5 \times 10^7)$

Instructions are given for the **Casio fx-6890G**:   *Answer:*

    a 1.42 $\boxed{\text{EXP}}$ 4 $\boxed{\times}$ 2.56 $\boxed{\text{EXP}}$ 8 $\boxed{\text{EXE}}$ 3.6352×10^{12}

    b 4.75 $\boxed{\text{EXP}}$ $\boxed{(-)}$ 4 $\boxed{\div}$ 2.5 $\boxed{\text{EXP}}$ 7 $\boxed{\text{EXE}}$ 1.9×10^{-11}

---

3 Find, in standard form, with decimal part correct to 2 places:

    a $(5.31 \times 10^4) \times (4.8 \times 10^3)$
    b $(2.75 \times 10^{-3})^2$
    c $\dfrac{8.24 \times 10^{-6}}{3 \times 10^4}$

    d $(7.2 \times 10^{-5}) \div (2.4 \times 10^{-6})$
    e $\dfrac{1}{4.1 \times 10^4}$
    f $(3.2 \times 10^3)^2$

4 For the following give answers in standard form correct to 3 significant figures:
    a How many millimetres are there in 479.8 kilometres?
    b How many seconds are there in one year?
    c How many seconds are there in a millennium?
    d How many kilograms are there in 0.5 milligrams?

5 If a missile travels at 3600 km/h, how far will it travel in:

    a 1 day   **b** 1 week   **c** 2 years?

    Give your answers in standard form with decimal part
    correct to 2 places. Assume that 1 year = 365 days.

6 Light travels at a speed of 3×10^8 metres per second. How far will light travel in:
    a 1 minute   **b** 1 day   **c** 1 year?

    Give your answers in standard form with decimal part correct to 2 decimal places.
    Assume that 1 year = 365 days.

E   SURDS   *[1.10]*

For the remainder of this chapter we consider **surds** and **radicals**, which are numbers that are written using the **radical** or **square root sign** $\sqrt{}$.

Surds and radicals occur frequently in mathematics, often as solutions to equations involving squared terms. We will see a typical example of this in **Chapter 8** when we study Pythagoras' theorem.

RATIONAL AND IRRATIONAL RADICALS

Some radicals are rational, but most are irrational.

For example, some rational radicals include:
$$\sqrt{1} = \sqrt{1^2} = 1 \quad \text{or} \quad \tfrac{1}{1}$$
$$\sqrt{4} = \sqrt{2^2} = 2 \quad \text{or} \quad \tfrac{2}{1}$$
$$\sqrt{\tfrac{1}{4}} = \sqrt{\left(\tfrac{1}{2}\right)^2} = \tfrac{1}{2}$$

Two examples of irrational radicals are $\sqrt{2} \approx 1.414\,214$
                     and $\sqrt{3} \approx 1.732\,051$.

Strictly speaking, a **surd** is an **irrational radical**. However, in this and many other courses, the term *surd* is used to describe *any* radical. It is reasonable to do so because the properties of surds and radicals are the same.

Historical note

The name **surd** and the **radical** sign $\sqrt{}$ both had a rather absurd past. Many centuries after Pythagoras, when the Golden Age of the Greeks was past, the writings of the Greeks were preserved, translated, and extended by Arab mathematicians.

The Arabs thought of a square number as growing out of its roots. The roots had to be extracted. The Latin word for "root" is **radix**, from which we get the words **radical** and radish! The printed symbol for radix was first **R**, then **r**, which was copied by hand as $\sqrt{}$.

The word **surd** actually came about because of an error of translation by the Arab mathematician Al-Khwarizmi in the 9th century AD. The Greek word **a-logos** means "irrational" but also means "deaf". So, the Greek **a-logos** was interpreted as "deaf" which in Latin is **surdus**. Hence to this day, **irrational radicals** like $\sqrt{2}$ are called **surds**.
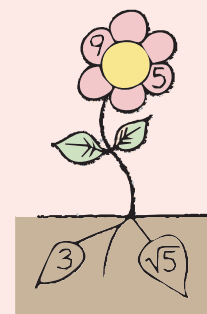
BASIC OPERATIONS WITH SURDS

We have seen square roots and cube roots in previous courses. We can use their properties to help with some simplifications.

Example 15
🔊 **Self Tutor**

Simplify:

a $(\sqrt{5})^2$

b $\left(\dfrac{1}{\sqrt{5}}\right)^2$

a $(\sqrt{5})^2$
$= \sqrt{5} \times \sqrt{5}$
$= 5$

b $\left(\dfrac{1}{\sqrt{5}}\right)^2$
$= \dfrac{1}{\sqrt{5}} \times \dfrac{1}{\sqrt{5}}$
$= \frac{1}{5}$

Example 16
🔊 **Self Tutor**

Simplify:

a $(2\sqrt{5})^3$

b $-2\sqrt{5} \times 3\sqrt{5}$

a $(2\sqrt{5})^3$
$= 2\sqrt{5} \times 2\sqrt{5} \times 2\sqrt{5}$
$= 2 \times 2 \times 2 \times \sqrt{5} \times \sqrt{5} \times \sqrt{5}$
$= 8 \times 5 \times \sqrt{5}$
$= 40\sqrt{5}$

b $-2\sqrt{5} \times 3\sqrt{5}$
$= -2 \times 3 \times \sqrt{5} \times \sqrt{5}$
$= -6 \times 5$
$= -30$

ADDING AND SUBTRACTING SURDS

'Like surds' can be added and subtracted in the same way as 'like terms' in algebra.

Consider $2\sqrt{3} + 4\sqrt{3}$, which has the same form as $2x + 4x$.

If we interpret this as 2 'lots' of $\sqrt{3}$ plus 4 'lots' of $\sqrt{3}$, we have 6 'lots' of $\sqrt{3}$.

So, $2\sqrt{3} + 4\sqrt{3} = 6\sqrt{3}$, and we can compare this with $2x + 4x = 6x$.

Example 17   ◆)) Self Tutor

Simplify:

  a $3\sqrt{2} + 4\sqrt{2}$ **b** $5\sqrt{3} - 6\sqrt{3}$

  a $3\sqrt{2} + 4\sqrt{2}$ **b** $5\sqrt{3} - 6\sqrt{3}$
        $= 7\sqrt{2}$ $= -1\sqrt{3}$
                                  $= -\sqrt{3}$

     {Compare: $3x + 4x = 7x$}   {Compare: $5x - 6x = -x$}

EXERCISE 6E

1 Simplify:

  a $(\sqrt{7})^2$ **b** $(\sqrt{13})^2$ **c** $(\sqrt{15})^2$ **d** $(\sqrt{24})^2$

  e $\left(\dfrac{1}{\sqrt{3}}\right)^2$ **f** $\left(\dfrac{1}{\sqrt{11}}\right)^2$ **g** $\left(\dfrac{1}{\sqrt{17}}\right)^2$ **h** $\left(\dfrac{1}{\sqrt{23}}\right)^2$

2 Simplify:

  a $(\sqrt[3]{2})^3$ **b** $(\sqrt[3]{-5})^3$ **c** $\left(\dfrac{1}{\sqrt[3]{5}}\right)^3$

3 Simplify:

  a $3\sqrt{2} \times 4\sqrt{2}$ **b** $-2\sqrt{3} \times 5\sqrt{3}$ **c** $3\sqrt{5} \times (-2\sqrt{5})$

  d $-2\sqrt{2} \times (-3\sqrt{2})$ **e** $(3\sqrt{2})^2$ **f** $(3\sqrt{2})^3$

  g $(2\sqrt{3})^2$ **h** $(2\sqrt{3})^3$ **i** $(2\sqrt{2})^4$

4 Simplify:

  a $\sqrt{2} + \sqrt{2}$ **b** $\sqrt{2} - \sqrt{2}$ **c** $3\sqrt{2} - 2\sqrt{2}$

  d $2\sqrt{3} - \sqrt{3}$ **e** $5\sqrt{7} + 2\sqrt{7}$ **f** $3\sqrt{5} - 6\sqrt{5}$

  g $3\sqrt{2} + 4\sqrt{2} - \sqrt{2}$ **h** $6\sqrt{2} - 9\sqrt{2}$ **i** $\sqrt{5} + 7\sqrt{5}$

  j $3\sqrt{2} - 5\sqrt{2} - \sqrt{2}$ **k** $3\sqrt{3} - \sqrt{3} + 2\sqrt{3}$ **l** $3\sqrt{5} + 7\sqrt{5} - 10$

5 Simplify:

  a $3\sqrt{2} + 2\sqrt{3} - \sqrt{2} + 5\sqrt{3}$ **b** $7\sqrt{2} - 4\sqrt{3} - 2\sqrt{2} + 3\sqrt{3}$

  c $-6\sqrt{2} - 2\sqrt{3} - \sqrt{2} + 6\sqrt{3}$ **d** $2\sqrt{5} + 4\sqrt{2} + 9\sqrt{5} - 9\sqrt{2}$

  e $3\sqrt{2} - 5\sqrt{7} - \sqrt{2} - 5\sqrt{7}$ **f** $3\sqrt{2} + 4\sqrt{11} + 6 - \sqrt{2} - \sqrt{11} - 3$

  g $6\sqrt{6} - 2\sqrt{2} - \sqrt{2} - 5\sqrt{6} + 4$ **h** $5\sqrt{3} - 6\sqrt{7} - 5 + 4\sqrt{3} + \sqrt{7} - 8$

F   PROPERTIES OF SURDS   [1.10]

Discovery   Properties of surds

Notice that $\sqrt{4 \times 9} = \sqrt{36} = 6$ and $\sqrt{4} \times \sqrt{9} = 2 \times 3 = 6$, which suggests that $\sqrt{4} \times \sqrt{9} = \sqrt{4 \times 9}$.

Also, $\sqrt{\dfrac{36}{4}} = \sqrt{9} = 3$ and $\dfrac{\sqrt{36}}{\sqrt{4}} = \dfrac{6}{2} = 3$,   which suggests that $\dfrac{\sqrt{36}}{\sqrt{4}} = \sqrt{\dfrac{36}{4}}$.

What to do:

Test the following possible properties or rules for surds by substituting different values of a and b. Use your calculator to evaluate the results.

1 $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for all $a \geqslant 0, b \geqslant 0$.

2 $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$ for all $a \geqslant 0, b > 0$.

3 $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$ for all $a \geqslant 0, b \geqslant 0$.

4 $\sqrt{a - b} = \sqrt{a} - \sqrt{b}$ for all $a \geqslant 0, b \geqslant 0$.

You should have discovered the following properties of surds:

- $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$ for $a \geqslant 0, \quad b \geqslant 0$

- $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ for $a \geqslant 0, \quad b > 0$

However, in general it is not true that $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$ or that $\sqrt{a - b} = \sqrt{a} - \sqrt{b}$.

Example 18   🔊 Self Tutor

Write in simplest form:

a $\sqrt{3} \times \sqrt{2}$ **b** $2\sqrt{5} \times 3\sqrt{2}$

a $\sqrt{3} \times \sqrt{2}$ **b** $2\sqrt{5} \times 3\sqrt{2}$
  $= \sqrt{3 \times 2}$ $= 2 \times 3 \times \sqrt{5} \times \sqrt{2}$
  $= \sqrt{6}$ $= 6 \times \sqrt{5 \times 2}$
                                         $= 6\sqrt{10}$

Example 19
◀) **Self Tutor**

Simplify: **a** $\dfrac{\sqrt{32}}{\sqrt{2}}$ **b** $\dfrac{\sqrt{12}}{2\sqrt{3}}$

a $\dfrac{\sqrt{32}}{\sqrt{2}}$

$= \sqrt{\dfrac{32}{2}}$

$= \sqrt{16}$

$= 4$

b $\dfrac{\sqrt{12}}{2\sqrt{3}}$

$= \tfrac{1}{2}\sqrt{\dfrac{12}{3}}$ $\left\{\text{using} \quad \dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}\right\}$

$= \tfrac{1}{2}\sqrt{4}$

$= \tfrac{1}{2} \times 2$

$= 1$

Example 20
◀) **Self Tutor**

Write $\sqrt{32}$ in the form $k\sqrt{2}$.

$\sqrt{32}$

$= \sqrt{16 \times 2}$

$= \sqrt{16} \times \sqrt{2}$ $\{\text{using} \quad \sqrt{ab} = \sqrt{a} \times \sqrt{b}\}$

$= 4\sqrt{2}$

SIMPLEST SURD FORM

A surd is in **simplest form** when the number under the radical sign is the smallest integer possible.

Example 21
◀) **Self Tutor**

Write $\sqrt{28}$ in simplest surd form.

$\sqrt{28}$

$= \sqrt{4 \times 7}$ $\{4 \text{ is the largest perfect square factor of } 28\}$

$= \sqrt{4} \times \sqrt{7}$

$= 2\sqrt{7}$

Look for the largest perfect square factor.

EXERCISE 6F

1   Simplify:

    a $\sqrt{2} \times \sqrt{5}$ **b** $\sqrt{3} \times \sqrt{7}$ **c** $\sqrt{3} \times \sqrt{11}$

    d $\sqrt{7} \times \sqrt{7}$ **e** $\sqrt{3} \times 2\sqrt{3}$ **f** $2\sqrt{2} \times \sqrt{5}$

2 Simplify:

    a $3\sqrt{3} \times 2\sqrt{2}$ **b** $2\sqrt{3} \times 3\sqrt{5}$ **c** $\sqrt{2} \times \sqrt{3} \times \sqrt{5}$

    d $\sqrt{3} \times \sqrt{2} \times 2\sqrt{2}$ **e** $-3\sqrt{2} \times (\sqrt{2})^3$ **f** $(3\sqrt{2})^3 \times (\sqrt{3})^3$

3 Simplify:

    a $\dfrac{\sqrt{8}}{\sqrt{2}}$ **b** $\dfrac{\sqrt{2}}{\sqrt{8}}$ **c** $\dfrac{\sqrt{18}}{\sqrt{2}}$ **d** $\dfrac{\sqrt{2}}{\sqrt{18}}$ **e** $\dfrac{\sqrt{20}}{\sqrt{5}}$ **f** $\dfrac{\sqrt{5}}{\sqrt{20}}$

    g $\dfrac{\sqrt{27}}{\sqrt{3}}$ **h** $\dfrac{\sqrt{18}}{\sqrt{3}}$ **i** $\dfrac{\sqrt{3}}{\sqrt{30}}$ **j** $\dfrac{\sqrt{50}}{\sqrt{2}}$ **k** $\dfrac{2\sqrt{6}}{\sqrt{24}}$ **l** $\dfrac{5\sqrt{75}}{\sqrt{3}}$

4 Write the following in the form $k\sqrt{2}$:

    a $\sqrt{8}$ **b** $\sqrt{18}$ **c** $\sqrt{50}$ **d** $\sqrt{98}$

    e $\sqrt{200}$ **f** $\sqrt{288}$ **g** $\sqrt{20\,000}$ **h** $\sqrt{\tfrac{1}{2}}$

5 Write the following in the form $k\sqrt{3}$:

    a $\sqrt{12}$ **b** $\sqrt{27}$ **c** $\sqrt{75}$ **d** $\sqrt{\tfrac{1}{3}}$

6 Write the following in the form $k\sqrt{5}$:

    a $\sqrt{20}$ **b** $\sqrt{45}$ **c** $\sqrt{125}$ **d** $\sqrt{\tfrac{1}{5}}$

7 **a** Find:

      i $\sqrt{16} + \sqrt{9}$ **ii** $\sqrt{16 + 9}$ **iii** $\sqrt{25} - \sqrt{9}$ **iv** $\sqrt{25 - 9}$

    b Copy and complete: In general, $\sqrt{a + b} \neq$ ......   and $\sqrt{a - b} \neq$ ......

8 Write the following in simplest surd form:

    a $\sqrt{24}$ **b** $\sqrt{50}$ **c** $\sqrt{54}$ **d** $\sqrt{40}$ **e** $\sqrt{56}$ **f** $\sqrt{63}$

    g $\sqrt{52}$ **h** $\sqrt{44}$ **i** $\sqrt{60}$ **j** $\sqrt{90}$ **k** $\sqrt{96}$ **l** $\sqrt{68}$

    m $\sqrt{175}$ **n** $\sqrt{162}$ **o** $\sqrt{128}$ **p** $\sqrt{700}$

9 Write the following in simplest surd form:

    a $\sqrt{\tfrac{5}{9}}$ **b** $\sqrt{\tfrac{18}{4}}$ **c** $\sqrt{\tfrac{12}{16}}$ **d** $\sqrt{\tfrac{75}{36}}$

G   *MULTIPLICATION OF SURDS*   *[1.10]*

The rules for expanding brackets involving surds are identical to those for ordinary algebra.

We can thus use:

$$a(b + c) = ab + ac$$
$$(a + b)(c + d) = ac + ad + bc + bd$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$
$$(a + b)(a - b) = a^2 - b^2$$

Example 22   ◀》 *Self Tutor*

Expand and simplify:

   a $\sqrt{2}(\sqrt{2} + \sqrt{3})$ **b** $\sqrt{3}(6 - 2\sqrt{3})$

  a $\sqrt{2}(\sqrt{2} + \sqrt{3})$
       $= \sqrt{2} \times \sqrt{2} \ + \ \sqrt{2} \times \sqrt{3}$
       $= 2 + \sqrt{6}$

  b $\sqrt{3}(6 - 2\sqrt{3})$
       $= (\sqrt{3})(6 + -2\sqrt{3})$
       $= (\sqrt{3})(6) \ + \ (\sqrt{3})(-2\sqrt{3})$
       $= 6\sqrt{3} + -6$
       $= 6\sqrt{3} - 6$

Example 23   ◀》 *Self Tutor*

Expand and simplify:

   a $-\sqrt{2}(\sqrt{2} + 3)$ **b** $-\sqrt{3}(7 - 2\sqrt{3})$

  a $-\sqrt{2}(\sqrt{2} + 3)$
       $= -\sqrt{2} \times \sqrt{2} \ + \ -\sqrt{2} \times 3$
       $= -2 - 3\sqrt{2}$

  b $-\sqrt{3}(7 - 2\sqrt{3})$
       $= (-\sqrt{3})(7 - 2\sqrt{3})$
       $= (-\sqrt{3})(7) \ + \ (-\sqrt{3})(-2\sqrt{3})$
       $= -7\sqrt{3} + 6$

Example 24   ◀》 *Self Tutor*

Expand and simplify: $(3 - \sqrt{2})(4 + 2\sqrt{2})$

      $(3 - \sqrt{2})(4 + 2\sqrt{2})$
  $= (3 - \sqrt{2})(4) + (3 - \sqrt{2})(2\sqrt{2})$
  $= 12 - 4\sqrt{2} + 6\sqrt{2} - 4$
  $= 8 + 2\sqrt{2}$

Example 25   ◀》 *Self Tutor*

Expand and simplify:

   a $(\sqrt{3} + 2)^2$ **b** $(\sqrt{3} - \sqrt{7})^2$

  a $(\sqrt{3} + 2)^2$
       $= (\sqrt{3})^2 \ + \ 2 \times \sqrt{3} \times 2 \ + \ 2^2$
       $= 3 + 4\sqrt{3} + 4$
       $= 7 + 4\sqrt{3}$

  b $(\sqrt{3} - \sqrt{7})^2$
       $= (\sqrt{3})^2 \ - \ 2 \times \sqrt{3} \times \sqrt{7} \ + \ (\sqrt{7})^2$
       $= 3 - 2\sqrt{21} + 7$
       $= 10 - 2\sqrt{21}$

Example 26   ◄》 *Self Tutor*

Expand and simplify:

a $(3 + \sqrt{2})(3 - \sqrt{2})$ **b** $(2\sqrt{3} - 5)(2\sqrt{3} + 5)$

a $(3 + \sqrt{2})(3 - \sqrt{2})$
$= 3^2 - (\sqrt{2})^2$
$= 9 - 2$
$= 7$

b $(2\sqrt{3} - 5)(2\sqrt{3} + 5)$
$= (2\sqrt{3})^2 - 5^2$
$= (4 \times 3) - 25$
$= 12 - 25$
$= -13$

Did you notice that these answers are **integers**?

EXERCISE 6G

1 Expand and simplify:

   a $\sqrt{2}(\sqrt{5} + \sqrt{2})$ **b** $\sqrt{2}(3 - \sqrt{2})$ **c** $\sqrt{3}(\sqrt{3} + 1)$ **d** $\sqrt{3}(1 - \sqrt{3})$

   e $\sqrt{7}(7 - \sqrt{7})$ **f** $\sqrt{5}(2 - \sqrt{5})$ **g** $\sqrt{11}(2\sqrt{11} - 1)$ **h** $\sqrt{6}(1 - 2\sqrt{6})$

   i $\sqrt{3}(\sqrt{3} + \sqrt{2} - 1)$ **j** $2\sqrt{3}(\sqrt{3} - \sqrt{5})$ **k** $2\sqrt{5}(3 - \sqrt{5})$ **l** $3\sqrt{5}(2\sqrt{5} + \sqrt{2})$

2 Expand and simplify:

   a $-\sqrt{2}(3 - \sqrt{2})$ **b** $-\sqrt{2}(\sqrt{2} + \sqrt{3})$ **c** $-\sqrt{2}(4 - \sqrt{2})$ **d** $-\sqrt{3}(1 + \sqrt{3})$

   e $-\sqrt{3}(\sqrt{3} + 2)$ **f** $-\sqrt{5}(2 + \sqrt{5})$ **g** $-(\sqrt{2} + 3)$ **h** $-\sqrt{5}(\sqrt{5} - 4)$

   i $-(3 - \sqrt{7})$ **j** $-\sqrt{11}(2 - \sqrt{11})$ **k** $-(\sqrt{3} - \sqrt{7})$ **l** $-2\sqrt{2}(1 - \sqrt{2})$

   m $-3\sqrt{3}(5 - \sqrt{3})$ **n** $-7\sqrt{2}(\sqrt{2} + \sqrt{6})$ **o** $(-\sqrt{2})^3(3 - \sqrt{2})$

3 Expand and simplify:

   a $(1 + \sqrt{2})(2 + \sqrt{2})$ **b** $(2 + \sqrt{3})(2 + \sqrt{3})$ **c** $(\sqrt{3} + 2)(\sqrt{3} - 1)$

   d $(4 - \sqrt{2})(3 + \sqrt{2})$ **e** $(1 + \sqrt{3})(1 - \sqrt{3})$ **f** $(5 + \sqrt{7})(2 - \sqrt{7})$

   g $(\sqrt{5} + 2)(\sqrt{5} - 3)$ **h** $(\sqrt{7} - \sqrt{3})(\sqrt{7} + \sqrt{3})$ **i** $(2\sqrt{2} + \sqrt{3})(2\sqrt{2} - \sqrt{3})$

   j $(4 - \sqrt{2})(3 - \sqrt{2})$

4 Expand and simplify:

   a $(1 + \sqrt{2})^2$ **b** $(2 - \sqrt{3})^2$ **c** $(\sqrt{3} + 2)^2$ **d** $(1 + \sqrt{5})^2$

   e $(\sqrt{2} - \sqrt{3})^2$ **f** $(5 - \sqrt{2})^2$ **g** $(\sqrt{2} + \sqrt{7})^2$ **h** $(4 - \sqrt{6})^2$

   i $(\sqrt{6} - \sqrt{2})^2$ **j** $(\sqrt{5} + 2\sqrt{2})^2$ **k** $(\sqrt{5} - 2\sqrt{2})^2$ **l** $(6 + \sqrt{8})^2$

   m $(5\sqrt{2} - 1)^2$ **n** $(3 - 2\sqrt{2})^2$ **o** $(1 + 3\sqrt{2})^2$

5 Expand and simplify:

   a $(4 + \sqrt{3})(4 - \sqrt{3})$ **b** $(5 - \sqrt{2})(5 + \sqrt{2})$ **c** $(\sqrt{5} - 2)(\sqrt{5} + 2)$

   d $(\sqrt{7} + 4)(\sqrt{7} - 4)$ **e** $(3\sqrt{2} + 2)(3\sqrt{2} - 2)$ **f** $(2\sqrt{5} - 1)(2\sqrt{5} + 1)$

   g $(5 - 3\sqrt{3})(5 + 3\sqrt{3})$ **h** $(2 - 4\sqrt{2})(2 + 4\sqrt{2})$ **i** $(1 + 5\sqrt{7})(1 - 5\sqrt{7})$

6 Expand and simplify:

   a $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})$ **b** $(\sqrt{7} + \sqrt{11})(\sqrt{7} - \sqrt{11})$ **c** $(\sqrt{x} - \sqrt{y})(\sqrt{y} + \sqrt{x})$

H   *DIVISION BY SURDS*   *[1.10]*

When an expression involves division by a surd, we can write the expression with an **integer denominator** which does **not** contain surds.

If the denominator contains a simple surd such as $\sqrt{a}$ then we use the rule $\sqrt{a} \times \sqrt{a} = a$.

For example: $\dfrac{6}{\sqrt{3}}$ can be written as $\dfrac{6}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$ since we are really just multiplying the original fraction by 1.

$\dfrac{6}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$ then simplifies to $\dfrac{6\sqrt{3}}{3}$ or $2\sqrt{3}$.

Example 27   ◀) *Self Tutor*

Express with integer denominator: **a** $\dfrac{7}{\sqrt{3}}$ **b** $\dfrac{10}{\sqrt{5}}$ **c** $\dfrac{10}{2\sqrt{2}}$

a $\dfrac{7}{\sqrt{3}}$

$= \dfrac{7}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$

$= \dfrac{7\sqrt{3}}{3}$

b $\dfrac{10}{\sqrt{5}}$

$= \dfrac{10}{\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}}$

$= \dfrac{10}{5}\sqrt{5}$

$= 2\sqrt{5}$

c $\dfrac{10}{2\sqrt{2}}$

$= \dfrac{10}{2\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$

$= \dfrac{10\sqrt{2}}{4}$

$= \dfrac{5\sqrt{2}}{2}$

If the denominator has the form $a + \sqrt{b}$ then we can remove the surd from the denominator by multiplying both the numerator and the denominator by its **radical conjugate** $a - \sqrt{b}$. This produces a rational denominator, so the process is called **rationalisation** of the denominator.

Example 28   ◀) *Self Tutor*

Express $\dfrac{1}{3 + \sqrt{2}}$ with integer denominator.

$\dfrac{1}{3 + \sqrt{2}} = \left(\dfrac{1}{3 + \sqrt{2}} \right) \left(\dfrac{3 - \sqrt{2}}{3 - \sqrt{2}} \right)$

$= \dfrac{3 - \sqrt{2}}{3^2 - (\sqrt{2})^2}$ {using $(a+b)(a-b) = a^2 - b^2$}

$= \dfrac{3 - \sqrt{2}}{7}$

> We are really multiplying by one, which does not change the value of the original expression.

Example 29

🔊 *Self Tutor*

Write $\dfrac{1 - 2\sqrt{3}}{1 + \sqrt{3}}$ in simplest form.

$$\dfrac{1 - 2\sqrt{3}}{1 + \sqrt{3}} = \left(\dfrac{1 - 2\sqrt{3}}{1 + \sqrt{3}}\right)\left(\dfrac{1 - \sqrt{3}}{1 - \sqrt{3}}\right)$$

$$= \dfrac{1 - \sqrt{3} - 2\sqrt{3} + 6}{1 - 3}$$

$$= \dfrac{7 - 3\sqrt{3}}{-2}$$

$$= \dfrac{3\sqrt{3} - 7}{2}$$

EXERCISE 6H

1 Express with integer denominator:

a $\dfrac{1}{\sqrt{2}}$ **b** $\dfrac{2}{\sqrt{2}}$ **c** $\dfrac{4}{\sqrt{2}}$ **d** $\dfrac{10}{\sqrt{2}}$ **e** $\dfrac{\sqrt{7}}{\sqrt{2}}$

f $\dfrac{1}{\sqrt{3}}$ **g** $\dfrac{3}{\sqrt{3}}$ **h** $\dfrac{4}{\sqrt{3}}$ **i** $\dfrac{18}{\sqrt{3}}$ **j** $\dfrac{\sqrt{11}}{\sqrt{3}}$

k $\dfrac{1}{\sqrt{5}}$ **l** $\dfrac{3}{\sqrt{5}}$ **m** $\dfrac{\sqrt{3}}{\sqrt{5}}$ **n** $\dfrac{15}{\sqrt{5}}$ **o** $\dfrac{125}{\sqrt{5}}$

p $\dfrac{\sqrt{10}}{\sqrt{2}}$ **q** $\dfrac{1}{2\sqrt{3}}$ **r** $\dfrac{2\sqrt{2}}{\sqrt{3}}$ **s** $\dfrac{15}{2\sqrt{5}}$ **t** $\dfrac{1}{(\sqrt{2})^3}$

2 Rationalise the denominator:

a $\dfrac{1}{3 - \sqrt{5}}$ **b** $\dfrac{1}{2 + \sqrt{3}}$ **c** $\dfrac{1}{4 - \sqrt{11}}$ **d** $\dfrac{\sqrt{2}}{5 + \sqrt{2}}$

e $\dfrac{\sqrt{3}}{3 + \sqrt{3}}$ **f** $\dfrac{5}{2 - 3\sqrt{2}}$ **g** $\dfrac{-\sqrt{5}}{3 + 2\sqrt{5}}$ **h** $\dfrac{3 - \sqrt{7}}{2 + \sqrt{7}}$

3 Write in the form $a + b\sqrt{2}$ where $a, b \in \mathbb{Q}$:

a $\dfrac{4}{2 - \sqrt{2}}$ **b** $\dfrac{-5}{1 + \sqrt{2}}$ **c** $\dfrac{1 - \sqrt{2}}{1 + \sqrt{2}}$ **d** $\dfrac{\sqrt{2} - 2}{3 - \sqrt{2}}$

e $\dfrac{\frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}}$ **f** $\dfrac{1 + \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}}$ **g** $\dfrac{1}{1 - \frac{\sqrt{2}}{3}}$ **h** $\dfrac{\frac{\sqrt{2}}{2} + 1}{1 - \frac{\sqrt{2}}{4}}$

Review set 6A

1 Find the integer equal to: **a** 3^4 **b** 5×2^3

2 Write as a product of primes in index form: **a** 36   **b** 242

3 Simplify:

   a $(-2)^3$ **b** $-(-1)^8$ **c** $(-1)^3 \times (-2)^2 \times -3^3$

4 Simplify, giving your answers in simplest rational form:

   a 3^{-3} **b** $\left(\frac{4}{3}\right)^{-2}$ **c** $3^0 - 3^1$

5 Write $\frac{1}{16}$ as a power of 2.

6 Simplify, using the exponent laws:

   a $5^6 \times 5$ **b** $b^7 \div b^2$ **c** $(x^4)^3$

7 Express in simplest form, without brackets:

   a $(4c^3)^2$ **b** $(2a^2b)^3$ **c** $\left(\dfrac{s}{3t^2}\right)^4$

8 Write in standard form:

   a 9 **b** $34\,900$ **c** 0.0075

9 Write as an ordinary decimal number:

   a 2.81×10^6 **b** 2.81×10^0 **c** 2.81×10^{-3}

10 Simplify, giving your answer in standard form:

   a $(6 \times 10^3) \times (7.1 \times 10^4)$ **b** $(2.4 \times 10^6) \div (4 \times 10^2)$

11 The Earth orbits around the Sun at a speed of approximately 1.07×10^5 km/h. How far does the Earth move, relative to the Sun, in:

   a 1 day   **b** 1 week   **c** 1 year?

   Give your answers in standard form with decimal part correct to 2 decimal places. Assume that 1 year = 365 days.

12 **a** Simplify $(3\sqrt{2})^2$.   **b** Simplify $-2\sqrt{3} \times 4\sqrt{3}$.

    c Simplify $3\sqrt{2} - \sqrt{8}$.   **d** Write $\sqrt{48}$ in simplest surd form.

13 Expand and simplify:

   a $2\sqrt{3}(4 - \sqrt{3})$ **b** $(3 - \sqrt{7})^2$ **c** $(2 - \sqrt{3})(2 + \sqrt{3})$

   d $(3 + 2\sqrt{5})(2 - \sqrt{5})$ **e** $(4 - \sqrt{2})(3 + 2\sqrt{2})$

14 Rationalise the denominator:

   a $\dfrac{8}{\sqrt{2}}$ **b** $\dfrac{15}{\sqrt{3}}$ **c** $\dfrac{\sqrt{3}}{\sqrt{2}}$ **d** $\dfrac{5}{6 - \sqrt{3}}$

15 Write $\sqrt{\frac{1}{7}}$ in the form $k\sqrt{7}$.

16 Write in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Q}$:

   a $\dfrac{\frac{\sqrt{3}}{2} + 1}{1 - \frac{\sqrt{3}}{2}}$ **b** $\dfrac{2 + \sqrt{3}}{2 - \sqrt{3}} - \dfrac{2\sqrt{3}}{2 + \sqrt{3}}$

Review set 6B

1 Find the integer equal to: **a** 7^3 **b** $3^2 \times 5^2$

2 Write as a product of primes in index form: **a** 42   **b** 144

3 Simplify:

    a $-(-1)^7$ **b** -4^3 **c** $(-2)^5 \times (-3)^2$

4 Simplify, giving your answers in simplest rational form:

    a 6^{-2} **b** $\left(1\frac{1}{2}\right)^{-1}$ **c** $\left(\frac{3}{5}\right)^{-2}$

5 Simplify, using the exponent laws:

    a $3^2 \times 3^6$ **b** $a^5 \div a^5$ **c** $(y^3)^5$

6 Write as powers of 2, 3 or 5:

    a $\frac{16}{25}$ **b** $\frac{40}{81}$ **c** 180   **d** $11\frac{1}{9}$

7 Express in simplest form, without brackets or negative indices:

    a $(5c)^{-1}$ **b** $7k^{-2}$ **c** $(4d^2)^{-3}$

8 Write in standard form:

    a 263.57   **b** 0.000 511   **c** 863 400 000

9 Write as an ordinary decimal number:

    a 2.78×10^0 **b** 3.99×10^7 **c** 2.081×10^{-3}

10 Simplify, giving your answer in standard form:

    a $(8 \times 10^3)^2$ **b** $(3.6 \times 10^5) \div (6 \times 10^{-2})$

11 How many kilometres are there in 0.21 millimetres? Give your answer in standard form.

12 Simplify:

    a $2\sqrt{3} \times 3\sqrt{5}$ **b** $(2\sqrt{5})^3$ **c** $5\sqrt{2} - 7\sqrt{2}$

    d $-\sqrt{2}(2 - \sqrt{2})$ **e** $(\sqrt{3})^4$ **f** $\sqrt{3} \times \sqrt{5} \times \sqrt{15}$

13 Write in simplest surd form: **a** $\sqrt{75}$ **b** $\sqrt{\frac{20}{9}}$

14 Expand and simplify:

    a $(5 - \sqrt{3})(5 + \sqrt{3})$ **b** $-(2 - \sqrt{5})^2$

    c $2\sqrt{3}(\sqrt{3} - 1) - 2\sqrt{3}$ **d** $(2\sqrt{2} - 5)(1 - \sqrt{2})$

15 Express with integer denominator:

    a $\dfrac{14}{\sqrt{2}}$ **b** $\dfrac{\sqrt{2}}{\sqrt{3}}$ **c** $\dfrac{\sqrt{2}}{3 + \sqrt{2}}$ **d** $\dfrac{-5}{4 - \sqrt{3}}$

16 Write in the form $a + b\sqrt{5}$ where $a, b \in \mathbb{Q}$:

    a $\dfrac{1 + \sqrt{5}}{2 - \sqrt{5}}$ **b** $\dfrac{3 - \sqrt{5}}{3 + \sqrt{5}} - \dfrac{4}{3 - \sqrt{5}}$

Discovery   *Continued square roots*

$$X = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots\ldots}}}}} \quad \text{is an example of a}$$
continued square root.

Some continued square roots have actual values which are integers.

What to do:

1 Use your calculator to show that
$$\sqrt{2} \approx 1.41421$$
$$\sqrt{2 + \sqrt{2}} \approx 1.84776$$
$$\sqrt{2 + \sqrt{2 + \sqrt{2}}} \approx 1.96157.$$

2 Find the values, correct to 6 decimal places, of:

a $\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$ **b** $\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}$

3 Continue the process and hence predict the actual value of X.

4 Use algebra to find the exact value of X.
Hint: Find X^2 in terms of X, and solve by inspection.

5 Can you find a continued square root whose actual value is 3?

Challenge

1 Find $\sqrt{\dfrac{3 + 2\sqrt{2}}{3 - 2\sqrt{2}}}$ giving your answer in the form $a + b\sqrt{2}$ where $a, b \in \mathbb{Q}$.

2 If $x = \sqrt{5} - \sqrt{3}$, find x^2 and x^4. Hence find the value of $x^4 - 16x^2$.
Copy and complete: $x = \sqrt{5} - \sqrt{3}$ is one of the solutions of the equation $x^4 - 16x^2 \boxed{} = 0$

3 **a** We know that in general, $\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$
Deduce that if $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$ then at least one of a or b is 0.
b What can be deduced about a and b if $\sqrt{a - b} = \sqrt{a} - \sqrt{b}$?

4 **a** Find the value of $\left(\dfrac{1 + \sqrt{5}}{2}\right)^n - \left(\dfrac{1 - \sqrt{5}}{2}\right)^n$ for $n = 1, 2, 3$ and 4.

b What do you suspect about $\left(\dfrac{1 + \sqrt{5}}{2}\right)^n - \left(\dfrac{1 - \sqrt{5}}{2}\right)^n$ for all $n \in \mathbb{Z}^+$?

Formulae and simultaneous equations

7

Contents:

Opening problem

Toby owns a fleet of trucks, and the trucks come in two sizes. The small trucks can carry a maximum of 3 tonnes, while the large trucks can carry a maximum of 5 tonnes.

There are 25 trucks in Toby's fleet, and the trucks can carry a combined total of 95 tonnes.

How many trucks of each size does Toby own?
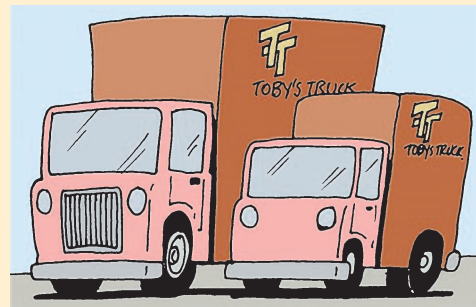
In this chapter we expand on our knowledge of algebra to consider equations with more than one unknown or variable.

We will first consider **formulae** which describe the relationship between variables.

We will then consider pairs of equations with two variables which can be solved **simultaneously** to find the values of the unknowns.

| **A** | *FORMULA SUBSTITUTION* | *[2.5]* |

> A **formula** is an equation which connects two or more variables.
> The plural of formula is **formulae** or **formulas**.

For example, the formula $s = \dfrac{d}{t}$ relates the three variable quantities speed (s), distance travelled (d), and time taken (t).

We usually write formulae with one variable on its own on the left hand side. The other variable(s) and constants are written on the right hand side.

The variable on its own is called the **subject** of the formula.

If a formula contains two or more variables and we know the value of all but one of them, we can **substitute** the known values into the formula and hence find the value of the unknown variable.

> *Step 1:*   Write down the formula.
>
> *Step 2:*   State the values of the known variables.
>
> *Step 3:*   Substitute into the formula to form a one variable equation.
>
> *Step 4:*   Solve the equation for the unknown variable.

Example 1   ◀)) *Self Tutor*

When a stone is dropped from a cliff into the sea, the total distance fallen, D metres, is given by the formula $D = \frac{1}{2}gt^2$ where t is the time of fall in seconds and g is the gravitational constant of 9.8 m/s^2. Find:

a the distance fallen after 4 seconds

b the time (to the nearest $\frac{1}{100}$th second) taken for the stone to fall 200 metres.

a $D = \frac{1}{2}gt^2$ where $g = 9.8$ and $t = 4$

$\therefore$ $D = \frac{1}{2} \times 9.8 \times 4^2$ *Calculator:*

$ = 78.4$ $0.5 \boxed{\times} 9.8 \boxed{\times} 4 \boxed{x^2} \boxed{\text{ENTER}}$

$\therefore$ the stone has fallen 78.4 metres.

b $D = \frac{1}{2}gt^2$ where $D = 200$ and $g = 9.8$

$\therefore$ $\frac{1}{2} \times 9.8 \times t^2 = 200$

$\therefore$ $4.9t^2 = 200$

$\therefore$ $t^2 = \dfrac{200}{4.9}$

$\therefore$ $t = \pm\sqrt{\dfrac{200}{4.9}}$ *Calculator:*

$\therefore$ $t \approx 6.39$ $\{$as $t > 0\}$ $\boxed{\sqrt{\ }} 200 \boxed{\div} 4.9 \boxed{)} \boxed{\text{ENTER}}$

$\therefore$ the time taken is about 6.39 seconds.

EXERCISE 7A

1 The formula for finding the circumference C of a circle with radius r
  is $C = 2\pi r$. Find:

  a the circumference of a circle of radius 4.2 cm

  b the radius of a circle with circumference 112 cm

  c the diameter of a circle with circumference 400 metres.
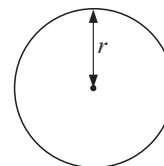
2 When a stone is dropped from the top of a cliff, the total distance fallen
  is given by the formula $D = \frac{1}{2}gt^2$ where D is the distance in metres
  and t is the time taken in seconds. Given that $g = 9.8$ m/s^2, find:

  a the total distance fallen in the first 2 seconds of fall

  b the height of the cliff, to the nearest metre, if the stone takes 4.8
    seconds to hit the ground.

3 When a car travels a distance d kilometres in time t hours, the average speed for the journey is given
  by the formula $s = \dfrac{d}{t}$ km/h. Find:

  a the average speed of a car which travels 250 km in $3\frac{1}{2}$ hours

  b the distance travelled by a car in $2\frac{3}{4}$ hours if its average speed is 80 km/h

  c the time taken, to the nearest minute, for a car to travel 790 km at an average speed of 95 km/h.

4 A circle's area A is given by $A = \pi r^2$ where r is the length of its radius. Find:

  a the area of a circle of radius 6.4 cm

  b the radius of a circular swimming pool which has an area of 160 m^2.

5 A cylinder of radius r and height h has volume given by $V = \pi r^2 h$. Find:

  a the volume of a cylindrical tin can of radius 8 cm and height 21.2 cm

  b the height of a cylinder of radius 6 cm and volume 120 cm^3

  c the radius, in mm, of a copper pipe of volume 470 cm^3 and length 6 m.
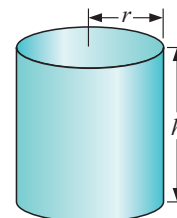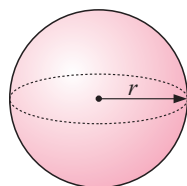
6 The formula for calculating the total surface area A of a sphere of
  radius r is $A = 4\pi r^2$. Find:

  a the total surface area of a sphere of radius 7.5 cm

  b the radius, in cm, of a spherical balloon which has a surface
    area of 2 m^2.

7 A sphere of radius r has volume given by $V = \frac{4}{3}\pi r^3$. Find:

  a the volume of a sphere of radius 2.37 m

  b the radius of a sphere that has volume 2500 cm^3.

8 The formula $D = 3.56\sqrt{h}$ km gives the approximate distance
  to the horizon which can be seen by a person with eye level
  h metres above sea level. Find:

  a the distance to the horizon when a person's eye level is
    20 m above sea level

  b how far above sea level a person's eye must be for the
    person to be able to see for 25 km.
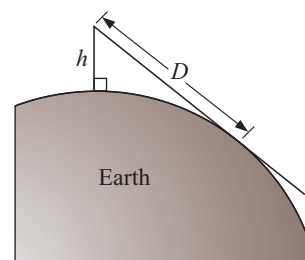
9 The *period* or time taken for one complete swing of a simple pendulum is given approximately by $T = \frac{1}{5}\sqrt{l}$ seconds, where l is the length of the pendulum in cm. Find:

     a the time for one complete swing of the pendulum if its length is 45 cm

     b the length of a pendulum which has a period of 1.8 seconds.

point of support

l

pendulum
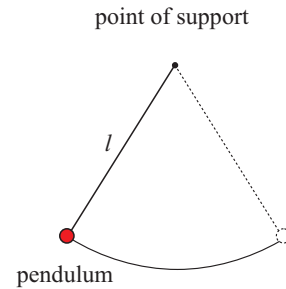
Activity   *Pizza pricing*

Luigi's Pizza Parlour has a 'Seafood Special' pizza advertised this week.

22 cm	30 cm	38 cm	46 cm
Small	Medium	Large	Family
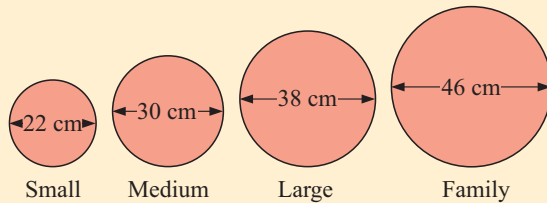
LUIGI'S PIZZAS

Free Delivery!

"Seafood Special"
Small	€8.00
Medium	€10.60
Large	€14.00
Family	€18.20

Sasha, Enrico and Bianca decide to find Luigi's formula for determining his price €P for each size pizza. Letting r cm be the radius of a pizza, the formulae they worked out were:

Sasha: $P = \dfrac{17r - 27}{20}$ Enrico: $P = \sqrt{\dfrac{33r - 235}{2}}$ Bianca: $P = 5 + \dfrac{r^2}{40}$.

What to do:

1 Investigate the suitability of each formula.

2 Luigi is introducing a Party size pizza of diameter 54 cm. What do you think his price will be?

B   *FORMULA REARRANGEMENT*   [2.5]

In the formula $D = xt + p$, D is expressed in terms of the other variables, x, t and p. We therefore say that D is the **subject** of the formula.

We can rearrange formulae to make one of the other variables the subject. However, we must do this carefully to ensure the formulae are still true.

> We **rearrange** formulae using the same processes which we use to solve equations. Anything we do to one side we must also do to the other.

Example 2

Make y the subject of $2x + 3y = 12$.

$$2x + 3y = 12$$
$$\therefore \quad 2x + 3y - 2x = 12 - 2x \qquad \{-2x \text{ from both sides}\}$$
$$\therefore \quad 3y = 12 - 2x$$
$$\therefore \quad \frac{3y}{3} = \frac{12 - 2x}{3} \qquad \{\div \text{ both sides by 3}\}$$
$$\therefore \quad y = \frac{12}{3} - \frac{2x}{3}$$
$$= 4 - \tfrac{2}{3}x$$

Example 3

Make y the subject of $x = 5 - cy$.

$$x = 5 - cy$$
$$\therefore \quad x + cy = 5 - cy + cy \qquad \{+cy \text{ to both sides}\}$$
$$\therefore \quad x + cy = 5$$
$$\therefore \quad x + cy - x = 5 - x \qquad \{-x \text{ from both sides}\}$$
$$\therefore \quad cy = 5 - x$$
$$\therefore \quad \frac{cy}{c} = \frac{5 - x}{c} \qquad \{\div \text{ both sides by } c\}$$
$$\therefore \quad y = \frac{5 - x}{c}$$

Example 4

Make z the subject of $c = \dfrac{m}{z}$.

$$c = \frac{m}{z}$$
$$\therefore \quad c \times z = \frac{m}{z} \times z \qquad \{\times \text{ both sides by } z\}$$
$$\therefore \quad cz = m$$
$$\therefore \quad \frac{cz}{c} = \frac{m}{c} \qquad \{\div \text{ both sides by } c\}$$
$$\therefore \quad z = \frac{m}{c}$$

EXERCISE 7B.1

1 Make y the subject of:

a $2x + 5y = 10$ **b** $3x + 4y = 20$ **c** $2x - y = 8$
d $2x + 7y = 14$ **e** $5x + 2y = 20$ **f** $2x - 3y = -12$

2 Make x the subject of:

a $p + x = r$ **b** $xy = z$ **c** $3x + a = d$
d $5x + 2y = d$ **e** $ax + by = p$ **f** $y = mx + c$
g $2 + tx = s$ **h** $p + qx = m$ **i** $6 = a + bx$

3 Make y the subject of:

   a $mx - y = c$ **b** $c - 2y = p$ **c** $a - 3y = t$

   d $n - ky = 5$ **e** $a - by = n$ **f** $p = a - ny$

4 Make z the subject of:

   a $az = \dfrac{b}{c}$ **b** $\dfrac{a}{z} = d$ **c** $\dfrac{3}{d} = \dfrac{2}{z}$ **d** $\dfrac{z}{2} = \dfrac{a}{z}$ **e** $\dfrac{b}{z} = \dfrac{z}{n}$ **f** $\dfrac{m}{z} = \dfrac{z}{a - b}$

5 Make:

   a a the subject of $F = ma$ **b** r the subject of $C = 2\pi r$

   c d the subject of $V = ldh$ **d** K the subject of $A = \dfrac{b}{K}$

   e h the subject of $A = \dfrac{bh}{2}$ **f** T the subject of $I = \dfrac{PRT}{100}$

   g M the subject of $E = MC^2$ **h** a the subject of $M = \dfrac{a + b}{2}$

REARRANGEMENT AND SUBSTITUTION

In the previous section on formula substitution, the variables were replaced by numbers and then the equation was solved. However, often we need to substitute several values for the unknowns and solve the equation for each case. In this situation it is quicker to **rearrange** the formula **before substituting**.

Example 5   ◀๗) *Self Tutor*

The circumference of a circle is given by $C = 2\pi r$,   where r is the circle's radius. Rearrange this formula to make r the subject, and hence find the radius when the circumference is: **a**   10 cm   **b**   20 cm   **c**   50 cm.

$2\pi r = C$,   so $r = \dfrac{C}{2\pi}$

  a   When $C = 10$, $r = \dfrac{10}{2\pi} \approx 1.59$ $\therefore$ the radius is about 1.59 cm.

  b   When $C = 20$, $r = \dfrac{20}{2\pi} \approx 3.18$ $\therefore$ the radius is about 3.18 cm.

  c   When $C = 50$, $r = \dfrac{50}{2\pi} \approx 7.96$ $\therefore$ the radius is about 7.96 cm.

EXERCISE 7B.2

1 The equation of a straight line is $5x + 3y = 18$.

   Rearrange this formula into the form $y = mx + c$.

   Hence, state the value of: **a**   the gradient m **b**   the y-intercept c.

2 **a**   Make a the subject of the formula $K = \dfrac{d^2}{2ab}$.

   b   Find the value of a when:

      i $K = 112$, $d = 24$, $b = 2$ **ii** $K = 400$, $d = 72$, $b = 0.4$.

3   When a car travels a distance d kilometres in time t hours, the average speed s for the journey is given

by the formula $s = \dfrac{d}{t}$ km/h.

a   Make d the subject of the formula. Hence find the distance travelled by a car if:

    i   the average speed is 60 km/h and the time travelled is 3 hours

    ii   the average speed is 80 km/h and the time travelled is $1\frac{1}{2}$ hours

    iii   the average speed is 95 km/h and the time travelled is 1 h 20 min.

b   Make t the subject of the formula. Hence find the time required for a car to travel:

    i   180 km at an average speed of 60 km/h

    ii   140 km at an average speed of 35 km/h

    iii   220 km at an average speed of 100 km/h.

4   The simple interest $\$I$ paid on an investment of $\$P$ is determined by the annual rate of interest r
(as a decimal) and the duration of the investment, n years. The interest is given by the formula
$I = P \times r \times n$.

a   Make n the subject of the formula.

b   **i**   Find the time required to generate $1050 interest on an investment of $6400 at an interest rate
of 8% per annum.

    ii   Find the time required for an investment of $1000 to double at an interest rate of 10% per
annum.

C   FORMULA DERIVATION   [2.5]

When we construct or **derive** a formula it is often useful to first consider an example with numbers. We can
then generalise the result.

For example,   the perimeter of the rectangle is given by

$$P = 3 + 6 + 3 + 6 \text{ metres}$$
$$\therefore \quad P = (2 \times 3) + (2 \times 6) \text{ metres}$$
$$\therefore \quad P \text{ is double the width plus double the length.}$$

Thus, in general, $P = 2a + 2b$

       or $P = 2(a + b)$.
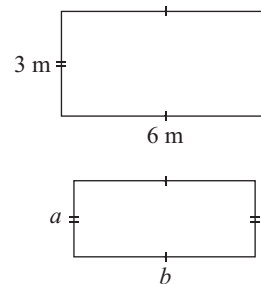
Example 6   ◀)) Self Tutor

Write the formula for the total cost RM C of a taxi trip given a fixed charge of:

a   RM 3 and RM 0.55 per km for 12 km   **b**   RM 3 and RM 0.55 per km for k km

c   RM 3 and RM d per km for k km   **d**   RM F and RM d per km for k km

a $C = 3 + (0.55 \times 12)$	**b** $C = 3 + (0.55 \times k)$
	$\therefore \quad C = 3 + 0.55k$
c $C = 3 + d \times k$	**d** $C = F + dk$
$\therefore \quad C = 3 + dk$	

Example 7   ◀) *Self Tutor*

Write a formula for the amount A in a person's bank account if initially the balance was:

   a $5000, and $200 was withdrawn each week for 10 weeks

   b $5000, and $200 was withdrawn each week for w weeks

   c $5000, and x was withdrawn each week for w weeks

   d B, and x was withdrawn each week for w weeks.

   a $A = 5000 - 200 \times 10$

   b $A = 5000 - 200 \times w$
      $\therefore \quad A = 5000 - 200w$

   c $A = 5000 - x \times w$
    $\therefore \quad A = 5000 - xw$

   d $A = B - x \times w$
    $\therefore \quad A = B - xw$

EXERCISE 7C

1 Write a formula for the amount €A in a new savings account given monthly deposits of:

   a €200 over 17 months   **b** €200 over m months   **c** €D over m months

2 Write a formula for the amount A in a bank account if the initial balance was:

   a $2000, and then $150 was deposited each week for 8 weeks

   b $2000, and then $150 was deposited each week for w weeks

   c $2000, and then d was deposited each week for w weeks

   d P, and then d was deposited each week for w weeks.

3 Write a formula for the total cost £C of hiring a plumber given a fixed call out fee of:

   a £40 plus £60 per hour for 5 hours of work

   b £40 plus £60 per hour for t hours of work

   c £40 plus £x per hour for t hours of work

   d £F plus £x per hour for t hours of work.

4 Write a formula for the amount €A in Leon's wallet if initially he had:

   a €200, and he bought 8 presents costing €5 each

   b €200, and he bought x presents costing €5 each

   c €200, and he bought x presents costing €b each

   d €P, and he bought x presents costing €b each.

5 Write a formula for the capacity, C litres, of a tank if initially the tank held:

   a 5000 litres, and 10 litres per minute for 200 minutes ran out through a tap

   b 5000 litres, and r litres per minute for 200 minutes ran out through a tap

   c 5000 litres, and r litres per minute for m minutes ran out through a tap

   d L litres, and r litres per minute for m minutes ran out through a tap.

6 The perimeter of a polygon is the sum of the lengths of its sides. Write a formula for the perimeter P of the following shapes:

a **i**

5 m

4 m

ii

a m

4 m

iii

a m

b m

b **i**

3 cm

4 cm

5 cm

ii

x cm

4 cm

5 cm

iii

x cm

y cm

5 cm

iv

x cm

y cm

z cm
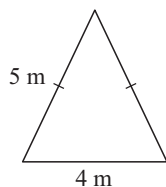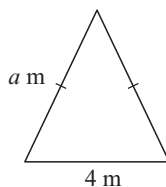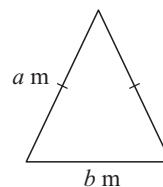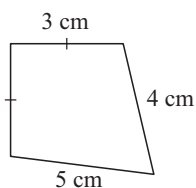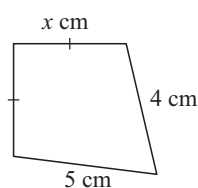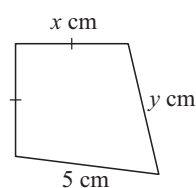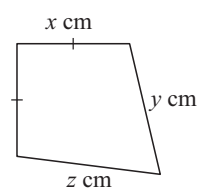
D MORE DIFFICULT REARRANGEMENTS   [2.5]

Example 8   ◀) Self Tutor

Make x the subject of $ax + 3 = bx + d$.

$$ax + 3 = bx + d$$
$$\therefore \quad ax + 3 - bx = bx + d - bx \qquad \{\text{subtracting } bx \text{ from both sides}\}$$
$$\therefore \quad ax + 3 - bx = d$$
$$\therefore \quad ax + 3 - bx - 3 = d - 3 \qquad \{\text{subtracting 3 from both sides}\}$$
$$\therefore \quad ax - bx = d - 3 \qquad \{\text{writing terms containing } x \text{ on LHS}\}$$
$$\therefore \quad x(a - b) = d - 3 \qquad \{x \text{ is a common factor on LHS}\}$$
$$\therefore \quad \frac{x(a - b)}{(a - b)} = \frac{d - 3}{(a - b)} \qquad \{\text{dividing both sides by } (a - b)\}$$
$$\therefore \quad x = \frac{d - 3}{a - b}$$

Example 9   ◀) Self Tutor

Make t the subject of $s = \frac{1}{2}gt^2$ where $t > 0$.

$$\frac{1}{2}gt^2 = s \qquad \{\text{rewrite with } t^2 \text{ on LHS}\}$$
$$\therefore \quad gt^2 = 2s \qquad \{\text{multiplying both sides by 2}\}$$
$$\therefore \quad t^2 = \frac{2s}{g} \qquad \{\text{dividing both sides by } g\}$$
$$\therefore \quad t = \sqrt{\frac{2s}{g}} \qquad \{\text{as } t > 0\}$$

Example 10   ◄ⁱ⁾ Self Tutor

Make x the subject of $T = \dfrac{a}{\sqrt{x}}$.

$$T = \dfrac{a}{\sqrt{x}}$$

$$\therefore \quad T^2 = \left(\dfrac{a}{\sqrt{x}}\right)^2 \qquad \text{\{squaring both sides\}}$$

$$\therefore \quad T^2 = \dfrac{a^2}{x}$$

$$\therefore \quad T^2 x = a^2 \qquad \text{\{multiplying both sides by } x\}$$

$$\therefore \quad x = \dfrac{a^2}{T^2} \qquad \text{\{dividing both sides by } T^2\}$$

Example 11   ◄ⁱ⁾ Self Tutor

Make x the subject of $T = \dfrac{a}{x - b}$.

$$T = \dfrac{a}{x - b}$$

$$\therefore \quad T(x - b) = a \qquad \text{\{multiplying both sides by } (x - b)\}$$

$$\therefore \quad Tx - Tb = a$$

$$\therefore \quad Tx = a + Tb \qquad \text{\{writing term containing } x \text{ on LHS\}}$$

$$\therefore \quad x = \dfrac{a + Tb}{T} \qquad \text{\{dividing both sides by } T\}$$

Example 12   ◄ⁱ⁾ Self Tutor

Make x the subject of $y = \dfrac{3x + 2}{x - 1}$.

$$y = \dfrac{3x + 2}{x - 1}$$

$$\therefore \quad y(x - 1) = 3x + 2 \qquad \text{\{multiplying both sides by } (x - 1)\}$$

$$\therefore \quad xy - y = 3x + 2$$

$$\therefore \quad xy - 3x = y + 2 \qquad \text{\{writing terms containing } x \text{ on LHS\}}$$

$$\therefore \quad x(y - 3) = y + 2 \qquad \text{\{} x \text{ is a common factor\}}$$

$$\therefore \quad x = \dfrac{y + 2}{y - 3} \qquad \text{\{dividing each side by } (y - 3)\}$$

EXERCISE 7D

1 Make x the subject of:

   a $3x + a = bx + c$ **b** $ax = c - bx$ **c** $mx + a = nx - 2$

   d $8x + a = -bx$ **e** $a - x = b - cx$ **f** $rx + d = e - sx$

2 Make:

   a r the subject of $A = \pi r^2, \quad r > 0$ **b** x the subject of $N = \dfrac{x^5}{a}$

   c r the subject of $V = \frac{4}{3}\pi r^3$ **d** x the subject of $D = \dfrac{n}{x^3}$

   e x the subject of $y = 4x^2 - 7$ **f** Q the subject of $P^2 = Q^2 + R^2$

3 Make:

   a a the subject of $d = \dfrac{\sqrt{a}}{n}$ **b** l the subject of $T = \frac{1}{5}\sqrt{l}$

   c a the subject of $c = \sqrt{a^2 - b^2}$ **d** d the subject of $\dfrac{k}{a} = \dfrac{5}{\sqrt{d}}$

   e l the subject of $T = 2\pi\sqrt{\dfrac{l}{g}}$ **f** b the subject of $A = 4\sqrt{\dfrac{a}{b}}$

4 Make:

   a a the subject of $P = 2(a + b)$ **b** h the subject of $A = \pi r^2 + 2\pi rh$

   c r the subject of $I = \dfrac{E}{R + r}$ **d** q the subject of $A = \dfrac{B}{p - q}$

   e x the subject of $A = \dfrac{3}{2x + y}$ **f** y the subject of $M = \dfrac{4}{x^2 + y^2}, \quad y > 0$

5 Make x the subject of:

   a $y = \dfrac{x}{x + 1}$ **b** $y = \dfrac{x - 3}{x + 2}$ **c** $y = \dfrac{3x - 1}{x + 3}$

   d $y = \dfrac{5x - 2}{x - 1}$ **e** $y = \dfrac{4x - 1}{2 - x}$ **f** $y = \dfrac{3x + 7}{3 - 2x}$

   g $y = 1 + \dfrac{2}{x - 3}$ **h** $y = -2 + \dfrac{5}{x + 4}$ **i** $y = -3 - \dfrac{6}{x - 2}$

6 The formula for determining the volume V of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

   a   Make r the subject of the formula.

   b   Find the radius of a sphere which has volume:

      i 40 cm^3 **ii** $1\,000\,000$ cm^3.

7 The force of attraction between two bodies with masses m_1 kg and m_2 kg which are d metres apart, is given by $F = G\,\dfrac{m_1 m_2}{d^2}$ Newtons,   where $G = 6.7 \times 10^{-11}$ is the gravitational force constant.

   a   Make d the subject of this formula, assuming $d > 0$.

   b   Find the distance between two bodies of mass 4.2×10^{18} kg   if the force of attraction between them is:

      i 2.4×10^{10} Newtons   **ii** 1.3×10^6 Newtons.

8 According to Einstein's theory of relativity, the mass of a particle is given by the formula

$$m = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$ where m_0 is the mass of the particle at rest,

v is the speed of the particle, and

c is the speed of light.

a Make v the subject of the formula.

b Find the speed necessary to increase the mass of a particle to three times its rest mass, i.e., so that $m = 3m_0$. Give the value for v as a fraction of c.

c A cyclotron increased the mass of an electron to $30m_0$. At what speed was the electron travelling, given that $c \approx 3 \times 10^8$ m/s?

E   SIMULTANEOUS EQUATIONS   [2.6]

If we have two equations and we wish to make both equations true at the same time, we require values for the variables which satisfy both equations. These values are the **simultaneous solution** to the pair of equations.

Discovery   *The coin problem*

In my pocket I have 8 coins. They are \$1 and \$2 coins, and their total value is \$11. How many of each type of coin do I have?

What to do:

1 Copy and complete the following table:

Number of \$1 coins	0	1	2	3	4	5	6	7	8
Value of \$1 coins									
Number of \$2 coins	8	7							
Value of \$2 coins									
Total value of coins									

2 Use the table to find the solution to the problem.

3 Suppose I have x \$1 coins and y \$2 coins in my pocket.

a By considering the total number of coins, explain why $x + y = 8$.

b By considering the total value of the coins, explain why $x + 2y = 11$.

4 You should have found that there were five \$1 coins and three \$2 coins.

a Substitute $x = 5$ and $y = 3$ into $x + y = 8$. What do you notice?

b Substitute $x = 5$ and $y = 3$ into $x + 2y = 11$. What do you notice?

5 My friend has 12 coins in her pocket. They are all either £1 or £2. If the total value of her coins is £17, how many of each type does she have? Can you find the solution by algebraic means?

In this course we will consider **linear simultaneous equations** containing two unknowns, usually x and y. There are infinitely many values of x and y which satisfy the first equation. Likewise, there are infinitely many values of x and y which satisfy the second equation. In general, however, only one combination of values for x and y satisfies **both** equations at the same time.

For example, consider the simultaneous equations $\begin{cases} x + y = 9 \\ 2x + 3y = 21 \end{cases}$.

If $x = 6$ and $y = 3$ then:
- $x + y = (6) + (3) = 9$ ✓   The first equation is satisfied
- $2x + 3y = 2(6) + 3(3) = 12 + 9 = 21$ ✓ The second equation is satisfied.

So, $x = 6$ and $y = 3$ is the **solution** to the simultaneous equations $\begin{cases} x + y = 9 \\ 2x + 3y = 21 \end{cases}$.

The solutions to linear simultaneous equations can be found by **trial and error** as in the **Discovery**, but this can be quite tedious. They may also be found by drawing **graphs**, but this can be slow and also inaccurate if the solutions are not integers.

We thus consider **algebraic methods** for finding the simultaneous solution.

EQUATING VALUES OF y

If both equations are given with y as the **subject**, we can find the simultaneous solution by equating the right hand sides of the equations.

Example 13   ◀》 *Self Tutor*

Find the simultaneous solution to the equations: $y = 2x - 1$, $y = x + 3$

If $y = 2x - 1$ and $y = x + 3$, then

$\qquad 2x - 1 = x + 3$ {equating ys}
$\therefore \quad 2x - 1 - x = x + 3 - x$ {subtracting x from both sides}
$\therefore \quad x - 1 = 3$
$\therefore \quad x = 4$ {adding 1 to both sides}
and so $y = 4 + 3$ {using $y = x + 3$}
$\therefore \quad y = 7$

So, the simultaneous solution is $x = 4$ and $y = 7$.

Check: In $y = 2x - 1$, $y = 2 \times 4 - 1 = 8 - 1 = 7$ ✓
In $y = x + 3$, $y = 4 + 3 = 7$ ✓

Always check your solution in both equations.

EXERCISE 7E.1

1 Find the simultaneous solution to the following pairs of equations:

a $y = x - 2$
$y = 3x + 6$

b $y = x + 2$
$y = 2x - 3$

c $y = 6x - 6$
$y = x + 4$

d $y = 2x + 1$
$y = x - 3$

e $y = 5x + 2$
$y = 3x - 2$

f $y = 3x - 7$
$y = 3x - 2$

g $y = 3x + 2$
    $y = 2x + 3$

h $y = 3x + 1$
    $y = 3x + 5$

i $y = 5x - 2$
    $y = 10x - 4$

2 Find the simultaneous solution to the following pairs of equations:

a $y = x + 4$
    $y = 5 - x$

b $y = x + 1$
    $y = 7 - x$

c $y = 2x - 5$
    $y = 3 - 2x$

d $y = x - 4$
    $y = -2x - 4$

e $y = 3x + 2$
    $y = -2x - 3$

f $y = 4x + 6$
    $y = 6 - 2x$

SOLUTION BY SUBSTITUTION

The method of **solution by substitution** is used when at least one equation is given with either x or y as the **subject** of the formula, or if it is easy to make x or y the subject.

Example 14   ◀ᴗ) *Self Tutor*

Solve simultaneously, by substitution: $y = 9 - x$
                        $2x + 3y = 21$

$y = 9 - x$ ..... (1) $2x + 3y = 21$ ..... (2)

Since $y = 9 - x$,   then $2x + 3(9 - x) = 21$
                    $\therefore \quad 2x + 27 - 3x = 21$
                    $\therefore \quad\quad 27 - x = 21$
                    $\therefore \quad\quad\quad x = 6$

Substituting $x = 6$ into (1) gives $y = 9 - 6 = 3$.

The solution is: $x = 6$, $y = 3$.

Check: (1) $3 = 9 - 6$ ✓   (2) $2(6) + 3(3) = 12 + 9 = 21$ ✓

> We substitute $9 - x$ for y in the other equation.

Example 15   ◀ᴗ) *Self Tutor*

Solve simultaneously, by substitution: $2y - x = 2$
                            $x = 1 + 8y$

$2y - x = 2$ ..... (1) $x = 1 + 8y$ ..... (2)

Substituting (2) into (1) gives $2y - (1 + 8y) = 2$
                    $\therefore \quad 2y - 1 - 8y = 2$
                    $\therefore \quad\quad\quad -6y = 3$
                    $\therefore \quad\quad\quad\quad y = -\frac{1}{2}$

Substituting $y = -\frac{1}{2}$ into (2) gives $x = 1 + 8 \times -\frac{1}{2} = -3$

The solution is: $x = -3$, $y = -\frac{1}{2}$.

Check: (1) $2(-\frac{1}{2}) - (-3) = -1 + 3 = 2$ ✓
       (2) $1 + 8(-\frac{1}{2}) = 1 - 4 = -3$ ✓

> x is the subject of the second equation, so we substitute $1 + 8y$ for x in the first equation.

EXERCISE 7E.2

1 Solve simultaneously, using substitution:

a $y = 3 + x$
$5x - 2y = 0$

b $y = x - 2$
$x + 3y = 6$

c $y = 5 - x$
$4x + y = 5$

d $y = 2x - 1$
$3x - y = 6$

e $y = 3x + 4$
$2x + 3y = 12$

f $y = 5 - 2x$
$5x - 2y = 8$

2 Use the substitution method to solve simultaneously:

a $x = y + 2$
$3x - 2y = 9$

b $x = -1 + 5y$
$x = 3 - 5y$

c $x = 6 - 3y$
$3x - 3y = 2$

d $x = 1 - 2y$
$2x + 3y = 4$

e $x = -4 - 2y$
$2y - 3x = 8$

f $x = -y - 8$
$2x - 4y = 5$

3 a Try to solve by substitution: $y = 2x + 5$ and $y = 2x + 7$.

b What is the simultaneous solution for the equations in **a**? Explain your answer.

4 a Try to solve by substitution: $y = 4x + 3$ and $2y = 8x + 6$.

b How many simultaneous solutions do the equations in **a** have? Explain your answer.

SOLUTION BY ELIMINATION

In many problems which require the simultaneous solution of linear equations, each equation will be of the form $ax + by = c$. Solution by substitution is often tedious in such situations and the method of **elimination** of one of the variables is preferred.

One method is to make the coefficients of x (or y) the **same size** but **opposite in sign** and then **add** the equations. This has the effect of **eliminating** one of the variables.

Example 16   ◀) *Self Tutor*

Solve simultaneously, by elimination: $4x + 3y = 2$ ...... (1)
                                            $x - 3y = 8$ ...... (2)

Notice that coefficients of y are the same size but opposite in sign.

We **add** the LHSs and the RHSs to get an equation which contains x only.

$$\begin{aligned} 4x + 3y &= 2 \\ + \quad x - 3y &= 8 \\ \hline 5x &= 10 \qquad \text{\{adding the equations\}} \\ \therefore \quad x &= 2 \qquad \text{\{dividing both sides by 5\}} \end{aligned}$$

Substituting $x = 2$ into (1) gives $4(2) + 3y = 2$
$$\therefore \quad 8 + 3y = 2$$
$$\therefore \quad 3y = -6$$
$$\therefore \quad y = -2$$

The solution is $x = 2$ and $y = -2$.

Check: in (2): $(2) - 3(-2) = 2 + 6 = 8$ ✓

In problems where the coefficients of x (or y) are **not** the **same size** or **opposite in sign**, we may first have to **multiply** each equation by a number.

Example 17   ◄⑴ Self Tutor

Solve simultaneously, by elimination: $3x + 2y = 7$
$2x - 5y = 11$

$3x + 2y = 7$ ...... (1) $2x - 5y = 11$ ...... (2)

We can eliminate y by multiplying (1) by 5 and (2) by 2.

$$\therefore \quad 15x + 10y = 35$$
$$+ \quad 4x - 10y = 22$$
$$\overline{\quad\quad 19x \quad\quad\quad = 57 \quad\quad \text{\{adding the equations\}}}$$
$$\therefore \quad x = 3 \quad\quad \text{\{dividing both sides by 19\}}$$

Substituting $x = 3$ into equation (1) gives

$$3(3) + 2y = 7$$
$$\therefore \quad 9 + 2y = 7$$
$$\therefore \quad 2y = -2$$
$$\therefore \quad y = -1$$

So, the solution is: $x = 3, \ y = -1$.

Check: $3(3) + 2(-1) = 9 - 2 = 7$ ✓ $2(3) - 5(-1) = 6 + 5 = 11$ ✓

Example 18   ◄⑴ Self Tutor

Solve by elimination: $3x + 4y = 14$
$4x + 5y = 17$

$$3x + 4y = 14 \quad ...... \ (1)$$
$$4x + 5y = 17 \quad ...... \ (2)$$

To eliminate x, multiply both sides of

(1) by 4: $12x + 16y = 56$ ...... (3)
(2) by -3: $\underline{-12x - 15y = -51}$ ...... (4)
$\qquad\qquad\qquad\quad y = 5$ {adding (3) and (4)}

Substituting $y = 5$ into (2) gives

$$4x + 5(5) = 17$$
$$\therefore \quad 4x + 25 = 17 \qquad\qquad \textit{Check:}$$
$$\therefore \quad 4x = -8 \qquad\qquad \text{(1)} \quad 3(-2) + 4(5) = (-6) + 20 = 14 \ \checkmark$$
$$\therefore \quad x = -2 \qquad\qquad \text{(2)} \quad 4(-2) + 5(5) = (-8) + 25 = 17 \ \checkmark$$

Thus $x = -2$ and $y = 5$.

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WHICH VARIABLE TO ELIMINATE

There is always a choice whether to eliminate x or y, so our choice depends on which variable is easier to eliminate.

In **Example 18**, try to solve by multiplying (1) by 5 and (2) by -4. This eliminates y rather than x. The final solution should be the same.

EXERCISE 7E.3

1 What equation results when the following are added vertically?

a $3x + 4y = 6$
   $8x - 4y = 5$

b $2x - y = 7$
   $-2x + 5y = 5$

c $7x - 3y = 2$
   $2x + 3y = 7$

d $6x - 11y = 12$
   $3x + 11y = -6$

e $-7x + 2y = 5$
   $7x - 3y = 6$

f $2x - 3y = -7$
   $-2x - 8y = -4$

2 Solve the following using the method of elimination:

a $5x - y = 4$
   $2x + y = 10$

b $3x - 2y = 7$
   $3x + 2y = -1$

c $-5x - 3y = 14$
   $5x + 8y = -29$

d $4x + 3y = -11$
   $-4x - 2y = 6$

e $2x - 5y = 14$
   $4x + 5y = -2$

f $-6x - y = 17$
   $6x + 5y = -13$

3 Give the equation that results when both sides of the equation:

a $2x + 5y = 1$ are multiplied by 5

b $3x - y = 4$ are multiplied by -1

c $x - 7y = 8$ are multiplied by 3

d $5x + 4y = 9$ are multiplied by -2

e $-3x - 2y = 2$ are multiplied by 6

f $4x - 2y = 3$ are multiplied by -4

4 Solve the following using the method of elimination:

a $2x + y = 8$
   $x - 3y = 11$

b $3x + 2y = 7$
   $x + 3y = 7$

c $5x - 2y = 17$
   $3x - y = 9$

d $2x + 3y = 13$
   $3x + 2y = 17$

e $4x - 3y = 1$
   $2x + 5y = 7$

f $2x + 5y = 14$
   $5x - 3y + 27 = 0$

g $7x - 2y = 20$
   $4x + 3y = -1$

h $3x - 2y = 5$
   $5x - 3y = 8$

i $3x - 4y = -15$
   $2x + 3y = 7$

j $4x + 3y = 14$
   $5x - 2y = 29$

k $2x - 3y = 1$
   $5x + 7y = 3$

l $8x - 5y = -9$
   $3x + 4y = -21$

m $3x + 8y + 34 = 0$
   $5x - 4y = 0$

n $5x - 6y = 8$
   $6x - 7y - 9 = 0$

o $2x - 7y - 18 = 0$
   $3x - 5y - 5 = 0$

5 Use the method of elimination to attempt to solve:

a $2x - y = 3$
   $4x - 2y = 6$

b $3x + 4y = 6$ Comment on your results.
   $6x + 8y = 7$

| **F** | **PROBLEM SOLVING** | **[2.6]** |

Many problems can be described mathematically by a **pair of linear equations**, or two equations of the form $ax + by = c$, where x and y are the two variables or unknowns.

We have already seen an example of this in the **Discovery** on page **158**.

Once the equations are formed, they can then be solved simultaneously and thus the original problem solved. The following method is recommended:

> *Step 1:* Decide on the two unknowns; call them x and y, say. Do not forget the units.
>
> *Step 2:* Write down **two** equations connecting x and y.
>
> *Step 3:* Solve the equations simultaneously.
>
> *Step 4:* Check your solutions with the original data given.
>
> *Step 5:* Give your answer in sentence form.

The form of the original equations will help you decide whether to use the substitution method, or the elimination method.

Example 19   ◀)) *Self Tutor*

Two numbers have a sum of 45 and a difference of 13. Find the numbers.

Let x and y be the unknown numbers, where $x > y$.

Then $x + y = 45$ ...... (1)   {'sum' means add}
and $x - y = 13$ ...... (2)   {'difference' means subtract}
$\therefore$ $2x = 58$ {adding (1) and (2)}
$\therefore$ $x = 29$ {dividing both sides by 2}

Substituting into (1) gives: $29 + y = 45$
$\therefore$ $y = 16$

The numbers are 29 and 16.

Check: (1) $29 + 16 = 45$ ✓   (2) $29 - 16 = 13$ ✓

> When solving problems with simultaneous equations we must find two equations containing two unknowns.

Example 20   ◀)) *Self Tutor*

When shopping in Jamaica, 5 coconuts and 14 bananas cost me $8.70, and 8 coconuts and 9 bananas cost $9.90.

Find the cost of each coconut and each banana.

Let each coconut cost x cents and each banana cost y cents.

$$\therefore \quad 5x + 14y = 870 \quad \text{...... (1)}$$
$$8x + 9y = 990 \quad \text{...... (2)}$$

To eliminate x, we multiply (1) by 8 and (2) by -5.

> The units must be the same on both sides of each equation.

$$\therefore \quad 40x + 112y = 6960 \qquad \text{..... (3)}$$
$$\underline{-40x - 45y = -4950} \qquad \text{..... (4)}$$
$$67y = 2010 \qquad \{\text{adding (3) and (4)}\}$$
$$\therefore \quad y = 30 \qquad \{\text{dividing both sides by 67}\}$$

Substituting in (2) gives $\quad 8x + 9 \times 30 = 990$
$$\therefore \quad 8x = 990 - 270$$
$$\therefore \quad 8x = 720$$
$$\therefore \quad x = 90 \qquad \{\text{dividing both sides by 8}\}$$

Check: $\qquad 5 \times 90 + 14 \times 30 = 450 + 420 = 870 \quad \checkmark$
$$8 \times 90 + 9 \times 30 = 720 + 270 = 990 \quad \checkmark$$

Thus coconuts cost 90 cents each and bananas cost 30 cents each.

Example 21   ◀)) *Self Tutor*

In my pocket I have only 5-cent and 10-cent coins. How many of each type of coin do I have if I have 24 coins altogether and their total value is $1.55?

Let x be the number of 5-cent coins and y be the number of 10-cent coins.

$$\therefore \quad x + y = 24 \qquad \text{...... (1)} \qquad \{\text{the total number of coins}\}$$
$$\text{and} \quad 5x + 10y = 155 \qquad \text{...... (2)} \qquad \{\text{the total value of coins}\}$$

Multiplying (1) by -5 gives $\quad -5x - 5y = -120 \qquad \text{...... (3)}$
$$\underline{5x + 10y = 155} \qquad \text{...... (2)}$$
$$\therefore \quad 5y = 35 \qquad \{\text{adding (3) and (2)}\}$$
$$\therefore \quad y = 7 \qquad \{\text{dividing both sides by 5}\}$$

Substituting into (1) gives $\quad x + 7 = 24$
$$\therefore \quad x = 17$$

Check: $\quad 17 + 7 = 24 \quad \checkmark$
$$5 \times 17 + 10 \times 7 = 85 + 70 = 155 \quad \checkmark$$

Thus I have 17 five cent coins and 7 ten cent coins.

EXERCISE 7F

1 The sum of two numbers is 49 and their difference is 19. Find the numbers.

2 The average of two numbers is 43 and their difference is 16. Find the numbers.

3 The average of two numbers is 20. If one of the numbers is doubled and the other is trebled, the average increases to 52. Find the numbers.

4 Four nectarines and three peaches cost £2.90, and three nectarines and a peach cost £1.90. Find the cost of each fruit.

5   A visit to the cinemas costs €64 for a group of 2 adults and 5 children, and €68 for a group of 3 adults and 4 children. Find the cost of each adult ticket and each children's ticket.

6   Answer the **Opening Problem** on page **147**.

7   Martin collects 20-cent and 50-cent coins. He has 37 coins, and the total value of the coins is $11.30. How many coins of each type does Martin have?

8   A tailor made costumes for students in a school play. Each male costume took 2 hours and cost $40 to make. Each female costume took 3 hours and cost $80 to make. In total the tailor spent 30 hours and $680 on the costumes. How many male costumes and how many female costumes did the tailor make?

9

19 cm

11 cm

$(x + 3y)$ cm

$(3x + 2y)$ cm
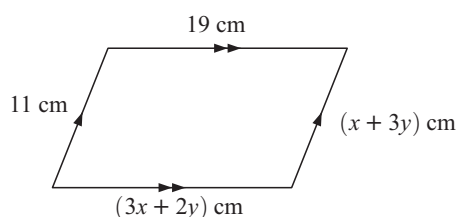
For the parallelogram alongside, find x and y.

10   Ron the painter charges an hourly fee, as well as a fixed call-out fee, for his work. He charges $165 for a 2 hour job, and $345 for a 5 hour job. Find Ron's call-out fee and hourly rate.

11   On the **Celsius** temperature scale, ice melts at 0^oC and water boils at 100^oC. On the **Fahrenheit** temperature scale, ice melts at 32^oF and water boils at 212^oF.

There is a linear relationship between temperature in degrees Celsius (C) and temperature in degrees Fahrenheit (F). It has the form $C = aF + b$ where a and b are constants.

    a   Find the values of a and b in simplest fractional form.

    b   Convert 77^oF to degrees Celsius.

Review set 7A

1   The formula for the density D of a substance with mass M and volume V is $D = \dfrac{M}{V}$.

    a   Find the density of lead if 350 g of lead occupies 30.7 cm^3.

    b   Find the mass of a lump of uranium with density 18.97 g/cm^3 and volume 2 cm^3.

    c   Find the volume of a piece of pine timber with mass 6 kg and density 0.65 g/cm^3.

2   Make t the subject of: **a** $3t - s = 13$ **b** $\dfrac{t}{r} = \dfrac{4}{t}$

3   Make x the subject of the formula $y = \dfrac{2x - 3}{x - 2}$.

4   **a**   Write a formula for the volume of water V in a trough if it is empty initially, then:

      i   six 8-litre buckets of water are poured into it

      ii n 8-litre buckets of water are poured into it

      iii n l-litre buckets of water are poured into it.

    b   Write a formula for the volume of water V in a trough that initially contained 25 litres if n buckets of water, each containing l litres, are poured into it.

5   Let $A = \dfrac{b}{b+c}.$ Find the value of c when:

    a $A = 2,\quad b = 6$ **b** $A = 3,\quad b = -9$ **c** $A = -1,\quad b = 5$

6   Solve simultaneously, by substitution:

    a $y - 5x = 8$ and $y = 3x + 6$ **b** $3x + 2y = 4$ and $2x - y = 5.$

7   Solve simultaneously: $y = 4x - 1$ and $3x - 2y = -8.$

8   Solve simultaneously:

    a $2x + 3y = 5$ **b** $4x - 3y - 2 = 0$

       $3x - y = -9$ $5x - 2y = 13$

9   3 sausages and 4 chops cost \$12.40, and 5 sausages and 3 chops cost \$11.50. Find the cost of each item.

10   Drivers are fined £200 for exceeding the speed limit by up to 15 km/h, and £450 for exceeding the speed limit by more than 15 km/h. In one night, a policeman gave out 22 speeding fines worth a total of £6650. How many drivers were caught exceeding the speed limit by more than 15 km/h?

Review set 7B

1
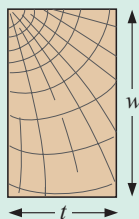
The strength S of a wooden beam is given by $S = 200w^2 t$ units, where w cm is its width and t cm is its thickness.

Find:

    a   the strength of a beam of width 16 cm and thickness 4 cm

    b   the width of a 5 cm thick beam of strength 60 000 units.

2   **a**   Make y the subject of $6x + 5y = 20.$

    b   Make r the subject of $C = 2\pi r.$

3   The surface area of a sphere is given by the formula $A = 4\pi r^2.$ If the surface area of a sphere is 250 cm^2, find its radius correct to 2 decimal places.

4   Make x the subject of: **a** $K = \dfrac{6}{y - 2x}$ **b** $M = \dfrac{5c}{\sqrt{2x}}$

5   Write a formula for the total cost £C of packing parcels of books for dispatch if the charge is:

    a   £2 per parcel plus £1.20 for one book

    b   £2 per parcel plus £1.20 per book for 5 books

    c   £p per parcel plus £1.20 per book for b books

    d   £p per parcel plus £x per book for b books.

6   Solve simultaneously:

    a $y = 3x + 4$ and $y = -2x - 6$ **b** $y = 3x - 4$ and $2x - y = 8.$

7   Solve simultaneously: $2x - 5y = -9$ and $y = 3x - 3$

8   Solve the following using the method of elimination: $3x - 2y = 3$ and $4x + 3y = 4.$

9   The larger of two numbers is 2 more than three times the smaller number. If their difference is 12, find the numbers.

10 Eric and Jordan were each given $8 to spend at a candy
    store. Eric bought 2 chocolate bars and 5 lolly bags,
    while Jordan bought 4 chocolate bars and 2 lolly bags.
    Neither child had any money left over. Find the cost of
    a chocolate bar.

Challenge

1 Pat thinks of three numbers. Adding them in pairs he obtains answers of 11, 17 and 22. What are
    the numbers?

2 Solve for x: $373x + 627y = 2492$
                 $627x + 373y = 3508$

    Hint: There is no need to use the technique of 'elimination'. Look at the two equations carefully.

3 Find a, b and c given that: $ab = c$
                              $bc = a$
                              $ca = b$

4 How many four digit numbers can be found in which

- the first digit is three times the last digit
- the second digit is the sum of the first and third digits
- the third digit is twice the first digit?

5 What can be deduced about a, b and c if $a + b = 2c$ and $b + c = 2a$?

6 Find x, y and z if:

    a $x + y - z = 10$ **b** $x + y - z = 8$
             $x - y + z = -4$ $2x + y + z = 9$
             $2x + y + 3z = 5$ $x + 3y + z = 10$

7 If $a + b = 1$ and $a^2 + b^2 = 3$, find the value of ab.

The theorem of Pythagoras

8

Contents:

Opening problem

The Louvre Pyramid in Paris, France has a square base with edges 35 m long. The pyramid is 20.6 m high.

Can you find the length of the slant edges of the pyramid?

Right angles (90° angles) are used when constructing buildings and dividing areas of land into rectangular regions.

The ancient **Egyptians** used a rope with 12 equally spaced knots to form a triangle with sides in the ratio $3 : 4 : 5$.

This triangle has a right angle between the sides of length 3 and 4 units.

In fact, this is the simplest right angled triangle with sides of integer length.

The Egyptians used this procedure to construct their right angles:

take hold of knots at arrows   make rope taut   line of one side of building
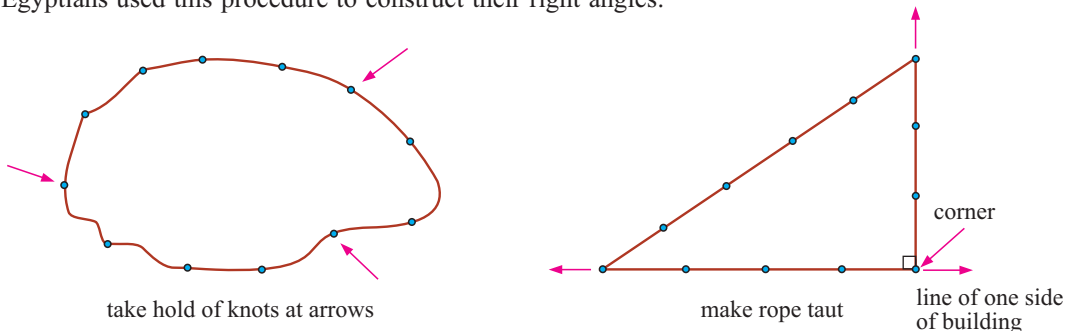
A   *PYTHAGORAS' THEOREM*   [4.6]

A **right angled triangle** is a triangle which has a right angle as one of its angles.

The side **opposite** the **right angle** is called the **hypotenuse** and is the **longest** side of the triangle.

The other two sides are called the **legs** of the triangle.

Around 500 BC, the Greek mathematician **Pythagoras** discovered a rule which connects the lengths of the sides of all right angled triangles. It is thought that he discovered the rule while studying tessellations of tiles on bathroom floors. Such patterns, like the one illustrated, were common on the walls and floors of bathrooms in ancient **Greece**.
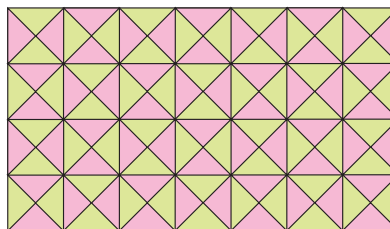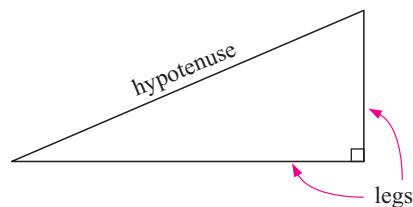
PYTHAGORAS' THEOREM

In a right angled triangle with hypotenuse c and legs a and b, $c^2 = a^2 + b^2$.

By looking at the tile pattern above, can you see how Pythagoras may have discovered the rule?

In geometric form, **Pythagoras' theorem** is:

In any right angled triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides.
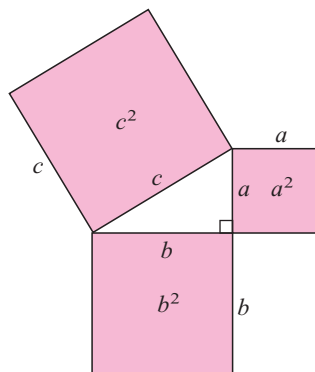
GEOMETRY PACKAGE
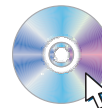
There are over 400 different proofs of Pythagoras' theorem. Here is one of them:

On a square we draw 4 identical (congruent) right angled triangles, as illustrated. A smaller square is formed in the centre.

Suppose the legs are of length a and b and the hypotenuse has length c.

The total area of the large square
$= 4\times$ area of one triangle $+$ area of smaller square,

$$\therefore \quad (a+b)^2 = 4(\tfrac{1}{2}ab) + c^2$$
$$\therefore \quad a^2 + 2ab + b^2 = 2ab + c^2$$
$$\therefore \quad a^2 + b^2 = c^2$$

Example 1   ◀) *Self Tutor*

Find the length of the hypotenuse in:

2 cm

x cm

3 cm
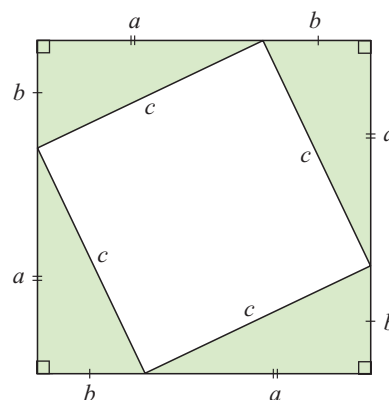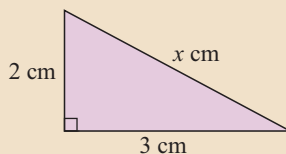
The hypotenuse is opposite the right angle and has length x cm.

$$\therefore \quad x^2 = 3^2 + 2^2$$
$$\therefore \quad x^2 = 9 + 4$$
$$\therefore \quad x^2 = 13$$
$$\therefore \quad x = \sqrt{13} \qquad \{\text{as } x > 0\}$$

$\therefore$ the hypotenuse is about 3.61 cm long.

> If $x^2 = k$, then $x = \pm\sqrt{k}$, but we reject $-\sqrt{k}$ as lengths must be positive.

Example 2   ◀) *Self Tutor*

Find the length of the third side of this triangle:

6 cm

x cm

5 cm
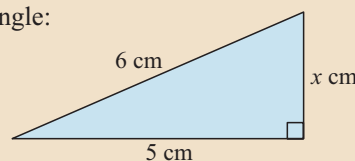
The hypotenuse has length 6 cm.

$$\therefore \quad x^2 + 5^2 = 6^2 \qquad \{\text{Pythagoras}\}$$
$$\therefore \quad x^2 + 25 = 36$$
$$\therefore \quad x^2 = 11$$
$$\therefore \quad x = \sqrt{11} \qquad \{\text{as } x > 0\}$$

$\therefore$ the third side is about 3.32 cm long.

Example 3

🔊 Self Tutor

Find x in surd form:

a

$\sqrt{10}$ cm

x cm

2 cm

b

$2x$ m

x m

6 m
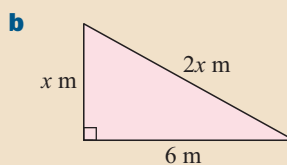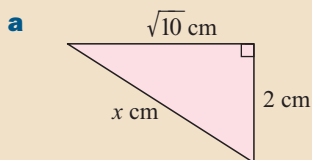
a The hypotenuse has length x cm.

$\therefore \quad x^2 = 2^2 + (\sqrt{10})^2 \qquad \{\text{Pythagoras}\}$

$\therefore \quad x^2 = 4 + 10$

$\therefore \quad x^2 = 14$

$\therefore \quad x = \sqrt{14} \qquad \{\text{as } x > 0\}$

b $\quad (2x)^2 = x^2 + 6^2 \qquad \{\text{Pythagoras}\}$

$\therefore \quad 4x^2 = x^2 + 36$

$\therefore \quad 3x^2 = 36$

$\therefore \quad x^2 = 12$

$\therefore \quad x = \sqrt{12} \qquad \{\text{as } x > 0\}$

$\therefore \quad x = 2\sqrt{3}$

Example 4

🔊 Self Tutor

Find the value of y, giving your answer correct to 3 significant figures.

A   5 cm   B

y cm x cm

1 cm

D   6 cm   C
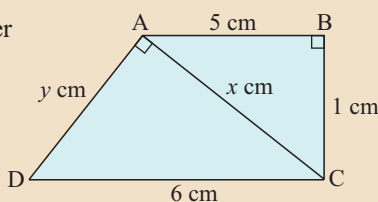
In triangle ABC, the hypotenuse is x cm.

$\therefore \quad x^2 = 5^2 + 1^2 \qquad \{\text{Pythagoras}\}$

$\therefore \quad x^2 = 26$

$\therefore \quad x = \sqrt{26} \qquad \{\text{as } x > 0\}$

In triangle ACD, the hypotenuse is 6 cm.

$\therefore \quad y^2 + (\sqrt{26})^2 = 6^2 \qquad \{\text{Pythagoras}\}$

$\therefore \quad y^2 + 26 = 36$

$\therefore \quad y^2 = 10$

$\therefore \quad y = \sqrt{10} \qquad \{\text{as } y > 0\}$

$\therefore \quad y \approx 3.16$

> Since we must find the value of y, we leave x in surd form. Rounding it will reduce the accuracy of our value for y.

EXERCISE 8A.1

1 Find the length of the hypotenuse in the following triangles, giving your answers correct to 3 significant figures:

a

4 cm

7 cm x cm

b

x cm   5 cm

c

x km   8 km

13 km
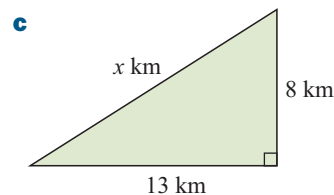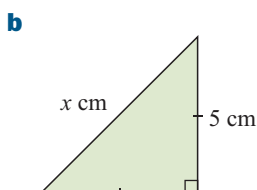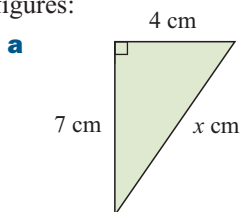
2 Find the length of the third side these triangles, giving your answers correct to 3 significant figures:
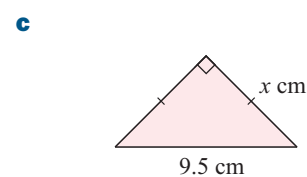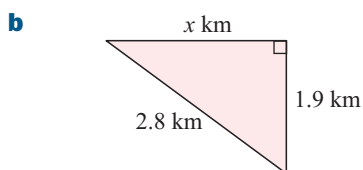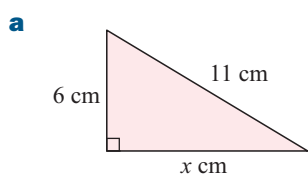
a

11 cm

6 cm

x cm

b

x km

2.8 km

1.9 km

c

x cm

9.5 cm

3 Find x in the following, giving your answers in simplest surd form:

a

3 cm

x cm

$\sqrt{2}$ cm

b

$\sqrt{7}$ cm

x cm

$\sqrt{5}$ cm

c

x cm

$\sqrt{10}$ cm

4 Solve for x, giving your answers in simplest surd form:

a

1 cm

$\frac{1}{2}$ cm

x cm

b

x cm

$\frac{1}{2}$ cm

$\frac{3}{2}$ cm

c

$\frac{\sqrt{3}}{2}$ m

x m
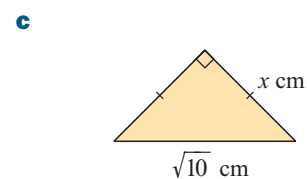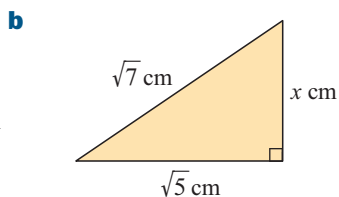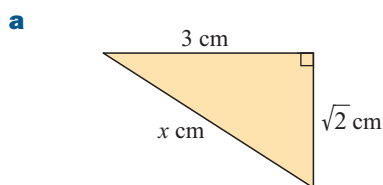
1 m
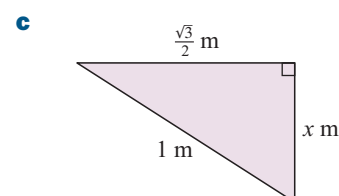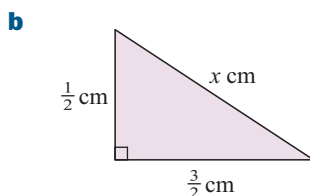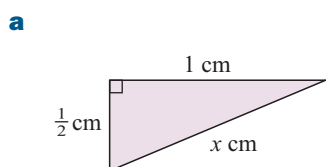
5 Find the values of x, giving your answers correct to 3 significant figures:

a

9 cm

$2x$ cm

x cm

b

26 cm

$2x$ cm

$3x$ cm

c

$2x$ m

$3x$ m

$\sqrt{20}$ m
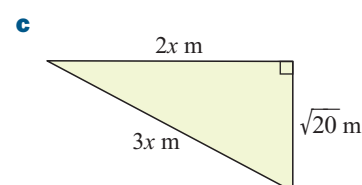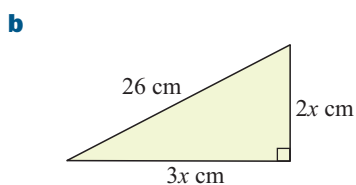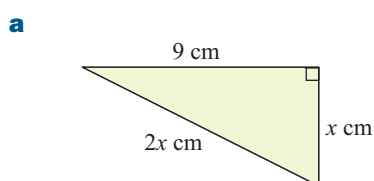
6 Find the value of any unknowns, giving answers in surd form:

a

x cm

y cm

3 cm

2 cm

b

2 cm

4 cm

7 cm

y cm

x cm

c

1 cm

3 cm

x cm

y cm

2 cm

7 Find x, correct to 3 significant figures:

a

3 cm

4 cm

x cm

b

$(x-2)$ cm

5 cm

13 cm
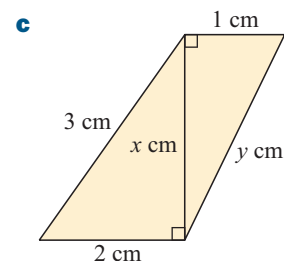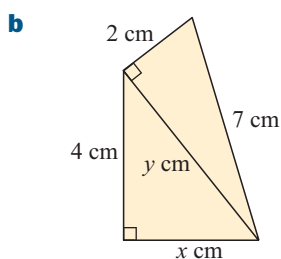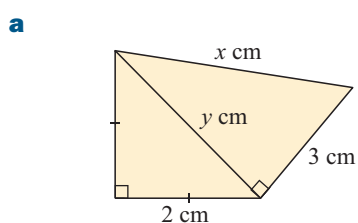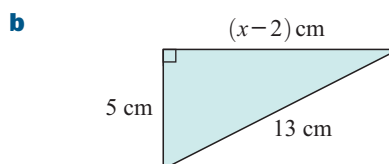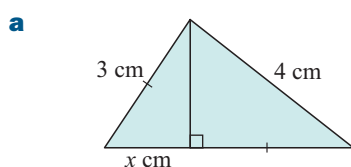
8 Find the length of side AC correct to 3 significant figures:

A

5 m

9 m

B

C

D
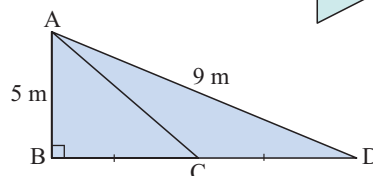
9 Find the distance AB in the following:

a
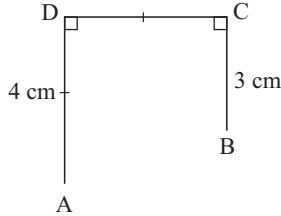
b
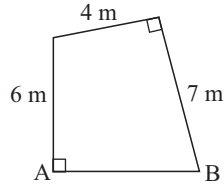
c
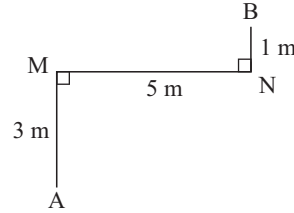
Challenge

10 In 1876, President Garfield of the USA published a proof of the theorem of Pythagoras. Alongside is the figure he used. Write out the proof.
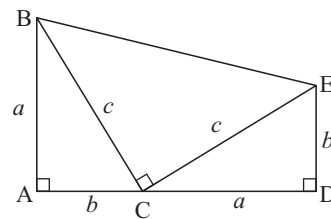
11 You are given a rectangle in which there is a point that is 3 cm, 4 cm and 5 cm from three of the vertices. How far is the point from the fourth vertex?
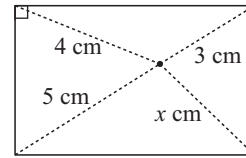
PYTHAGOREAN TRIPLES

The simplest right angled triangle with sides of **integer** length is the **3-4-5 triangle**.

The numbers 3, 4, and 5 satisfy the rule $3^2 + 4^2 = 5^2$.
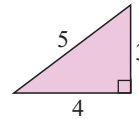
> The set of positive integers $\{a,\, b,\, c\}$ is a **Pythagorean triple** if it obeys the rule $a^2 + b^2 = c^2$.

Other examples are: $\{5,\, 12,\, 13\}$, $\{7,\, 24,\, 25\}$, $\{8,\, 15,\, 17\}$.

Example 5　　　　　　　　　◄) *Self Tutor*

Show that $\{5,\, 12,\, 13\}$ is a Pythagorean triple.

We find the square of the **largest** number first.

$$13^2 = 169$$
$$\text{and}\quad 5^2 + 12^2 = 25 + 144 = 169$$
$$\therefore\quad 5^2 + 12^2 = 13^2$$

So, $\{5,\, 12,\, 13\}$ is a Pythagorean triple.

Example 6
◀)) *Self Tutor*

Find k if $\{9,\ k,\ 15\}$ is a Pythagorean triple.

Let $9^2 + k^2 = 15^2$ {Pythagoras}

$\therefore\quad 81 + k^2 = 225$

$\therefore\qquad k^2 = 144$

$\therefore\qquad k = \sqrt{144}$ {as $k > 0$}

$\therefore\qquad k = 12$

EXERCISE 8A.2

1 Determine if the following are Pythagorean triples:

   a $\{8,\ 15,\ 17\}$ **b** $\{6,\ 8,\ 10\}$ **c** $\{5,\ 6,\ 7\}$

   d $\{14,\ 48,\ 50\}$ **e** $\{1,\ 2,\ 3\}$ **f** $\{20,\ 48,\ 52\}$

2 Find k if the following are Pythagorean triples:

   a $\{8,\ 15,\ k\}$ **b** $\{k,\ 24,\ 26\}$ **c** $\{14,\ k,\ 50\}$

   d $\{15,\ 20,\ k\}$ **e** $\{k,\ 45,\ 51\}$ **f** $\{11,\ k,\ 61\}$

3 Explain why there are infinitely many Pythagorean triples of the form $\{3k,\ 4k,\ 5k\}$ where $k \in \mathbb{Z}^+$.

Discovery   *Pythagorean triples spreadsheet*

Well known Pythagorean triples include $\{3,\ 4,\ 5\}$, $\{5,\ 12,\ 13\}$, $\{7,\ 24,\ 25\}$ and $\{8,\ 15,\ 17\}$.

SPREADSHEET

Formulae can be used to generate Pythagorean triples.

An example is $2n+1$, $2n^2+2n$, $2n^2+2n+1$ where n is a positive integer.

A spreadsheet can quickly generate sets of Pythagorean triples using such formulae.

What to do:

1 Open a new spreadsheet and enter the following:

   a in column A, the values of n for $n = 1,\ 2,\ 3,$ $4,\ 5,\ ......$

   b in column B, the values of $2n+1$

   c in column C, the values of $2n^2+2n$

   d in column D, the values of $2n^2+2n+1$.

	A	B	C	D
1	n	a	b	c
2	1	=2*A2+1	=2*A2^2+2*A2	=C2+1
3	=A2+1			
4				
5			fill down	

2 Highlight the appropriate formulae and **fill down** to Row 11 to generate the first 10 sets of triples.

	A	B	C	D
1	n	a	b	c
2	1	3	4	5
3	2	5	12	13
4	3	7	24	25
5	4	9	40	41

3 Check that each set of numbers is indeed a triple by adding columns to find a^2+b^2 and c^2.

4 Your final task is to prove that the formulae $\{2n+1,\ 2n^2+2n,\ 2n^2+2n+1\}$ will produce sets of Pythagorean triples for all positive integer values of n.

   Hint: Let $a = 2n+1$, $b = 2n^2+2n$ and $c = 2n^2+2n+1$, then simplify $c^2 - b^2 = (2n^2+2n+1)^2 - (2n^2+2n)^2$ using the *difference of two squares* factorisation.

B   THE CONVERSE OF PYTHAGORAS' THEOREM   [4.6]

If we have a triangle whose three sides have known lengths, we can use the **converse** of **Pythagoras' theorem** to test whether it is right angled.

THE CONVERSE OF PYTHAGORAS' THEOREM

GEOMETRY PACKAGE

If a triangle has sides of length a, b and c units and $a^2 + b^2 = c^2$,

then the triangle is right angled.

Example 7   ◀)) Self Tutor

Is the triangle with sides 6 cm, 8 cm and 5 cm right angled?

The two shorter sides have lengths 5 cm and 6 cm.

Now $5^2 + 6^2 = 25 + 36 = 61$,   but $8^2 = 64$.

∴ $5^2 + 6^2 \neq 8^2$ and hence the triangle is not right angled.

EXERCISE 8B

1 The following figures are not drawn to scale. Which of the triangles are right angled?

a

7 cm, 5 cm, 4 cm

b

9 cm, 12 cm, 15 cm

c

9 cm, 5 cm, 8 cm

d

3 cm, $\sqrt{7}$ cm, $\sqrt{12}$ cm

e

$\sqrt{27}$ m, $\sqrt{48}$ m, $\sqrt{75}$ m

f

8 m, 15 m, 17 m
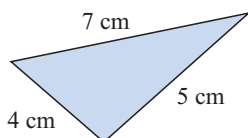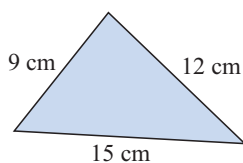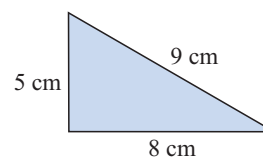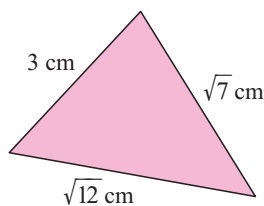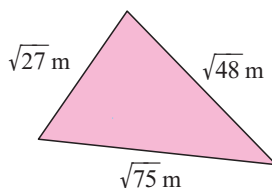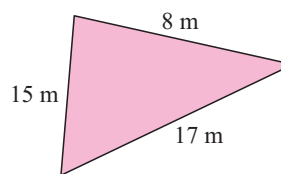
2 The following triangles are not drawn to scale. If any of them is right angled, find the right angle.

a

A, 2 cm, 1 cm, B, C, $\sqrt{5}$ cm

b

A, 8 m, $\sqrt{208}$ m, B, 12 m, C

c

C, 5 km, $\sqrt{24}$ km, B, A, 7 km
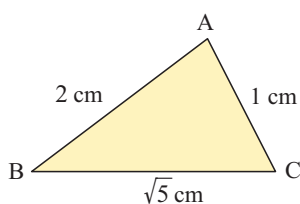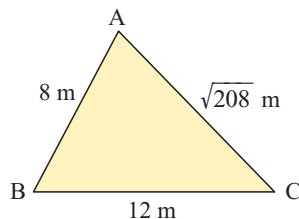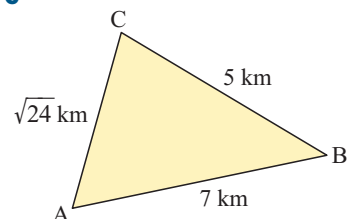
3 Ted has two planks 6800 mm long, and two planks 3500 mm long. He lays them down as borders for the concrete floor of his new garage.

To check that the shape is rectangular, Ted measures a diagonal length. He finds it to be 7648 mm. Is Ted's floor rectangular?
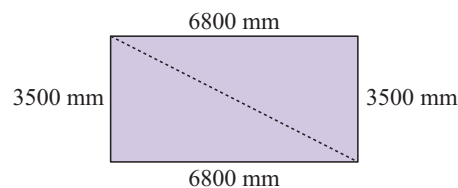
4
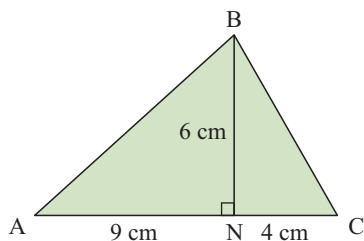
Triangle ABC has altitude BN which is 6 cm long.

AN = 9 cm   and   NC = 4 cm.

Is triangle ABC right angled at B?

C   *PROBLEM SOLVING*   *[4.6]*

Many practical problems involve triangles. We can apply Pythagoras' theorem to any triangle that is right angled, or use the converse of the theorem to test whether a right angle exists.

SPECIAL GEOMETRICAL FIGURES

The following special figures contain right angled triangles:
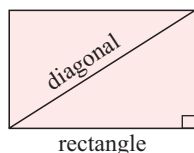
In a **rectangle**, right angles exist between adjacent sides.

Construct a **diagonal** to form a right angled triangle.

In a **square** and a **rhombus**, the diagonals bisect each other at right angles.
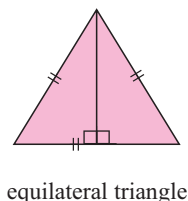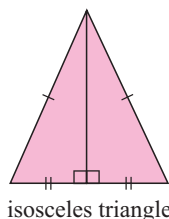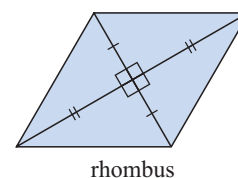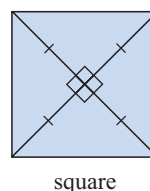
In an **isosceles triangle** and an **equilateral triangle**, the altitude bisects the base at right angles.

Things to remember

- Draw a neat, clear diagram of the situation.
- Mark on known lengths and right angles.
- Use a symbol such as x to represent the unknown length.
- Write down Pythagoras' theorem for the given information.
- Solve the equation.
- Where necessary, write your answer in sentence form.

Example 8
◀)) *Self Tutor*

A rectangular gate is 3 m wide and has a 3.5 m diagonal. How high is the gate?

Let x m be the height of the gate.

Now $(3.5)^2 = x^2 + 3^2$ {Pythagoras}

$\therefore$ $12.25 = x^2 + 9$

$\therefore$ $3.25 = x^2$

$\qquad \therefore$ $x = \sqrt{3.25}$ {as $x > 0$}

$\qquad \therefore$ $x \approx 1.80$

Thus the gate is approximately 1.80 m high.
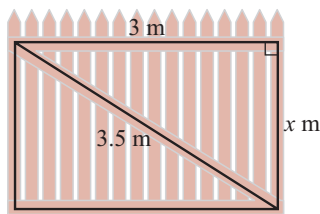
Example 9
◀)) *Self Tutor*

A rhombus has diagonals of length 6 cm and 8 cm.

Find the length of its sides.
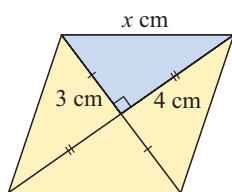
The diagonals of a rhombus bisect at right angles.

Let each side of the rhombus have length x cm.

$\therefore$ $x^2 = 3^2 + 4^2$ {Pythagoras}

$\therefore$ $x^2 = 25$

$\therefore$ $x = \sqrt{25}$ {as $x > 0$}

$\therefore$ $x = 5$

Thus the sides are 5 cm in length.

Example 10
◀)) *Self Tutor*

Two towns A and B are illustrated on a grid which has grid lines 5 km apart. How far is it from A to B?
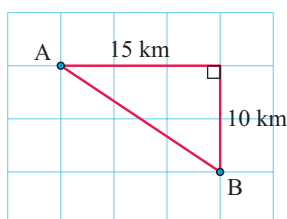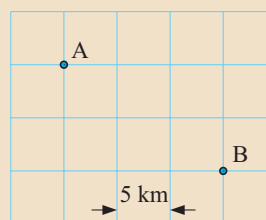
$AB^2 = 15^2 + 10^2$ {Pythagoras}

$\therefore$ $AB^2 = 225 + 100 = 325$

$\therefore$ $AB = \sqrt{325}$ {as $AB > 0$}

$\therefore$ $AB \approx 18.0$

So, A and B are about 18.0 km apart.

Example 11 ◀) *Self Tutor*

An equilateral triangle has sides of length 6 cm. Find its area.

The altitude bisects the base at right angles.

$\therefore \quad a^2 + 3^2 = 6^2$ {Pythagoras}

$\therefore \quad a^2 + 9 = 36$

$\therefore \quad a^2 = 27$

$\therefore \quad a = \sqrt{27}$ {as $a > 0$}

Now, area $= \frac{1}{2} \times$ base $\times$ height

$= \frac{1}{2} \times 6 \times \sqrt{27}$

$= 3\sqrt{27}$ cm^2

≈ 15.6 cm^2

So, the area is about 15.6 cm^2.
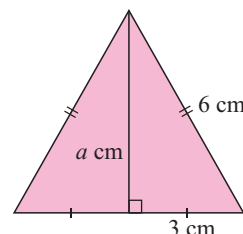
Example 12 ◀) *Self Tutor*

A helicopter travels from base station S for 112 km to outpost A. It then turns 90° to the right and travels 134 km to outpost B. How far is outpost B from base station S?

Let SB be x km.

From the diagram alongside, we see in triangle SAB that $\widehat{SAB} = 90^o$.

$x^2 = 112^2 + 134^2$ {Pythagoras}

$\therefore \quad x^2 = 30\,500$

$\therefore \quad x = \sqrt{30\,500}$ {as $x > 0$}

$\therefore \quad x \approx 175$

So, outpost B is 175 km from base station S.
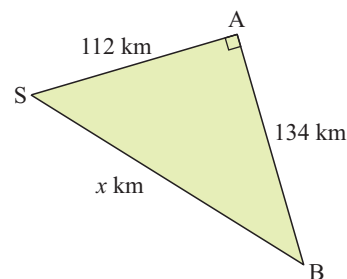
EXERCISE 8C

1 A rectangle has sides of length 8 cm and 3 cm. Find the length of its diagonals.

2 The longer side of a rectangle is three times the length of the shorter side. If the length of the diagonal is 10 cm, find the dimensions of the rectangle.

3 A rectangle with diagonals of length 20 cm has sides in the ratio 2 : 1. Find the:
   a perimeter
   b area of the rectangle.

4 A rhombus has sides of length 6 cm. One of its diagonals is 10 cm long. Find the length of the other diagonal.

5 A square has diagonals of length 10 cm. Find the length of its sides.

6   A rhombus has diagonals of length 8 cm and 10 cm. Find its perimeter.

7   On the grid there are four towns A, B, C and D. The grid
     lines are 5 km apart. How far is it from:

      a   A to B   **b**   B to C   **c**   C to D

      d   D to A   **e**   A to C   **f**   B to D?

     Give all answers correct to 3 significant figures.
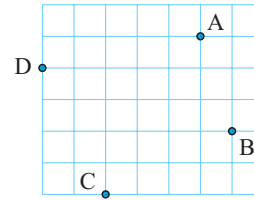
8   A yacht sails 5 km due west and then 8 km due south.
     How far is it from its starting point?

9   A cyclist at C is travelling towards B. How far will he
                                                have to cycle before he is equidistant from A and B?

10   A street is 8 m wide, and there are street lights positioned either side of the street every 20 m.
      How far is street light X from street light:

       a   A   **b**   B   **c**   C   **d**   D?
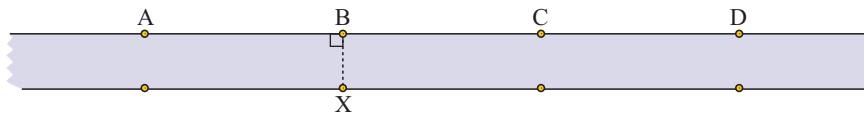
11   Find any unknowns in the following:

      a   **b**   **c**
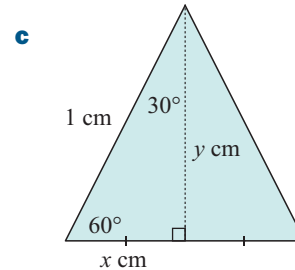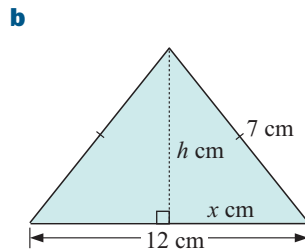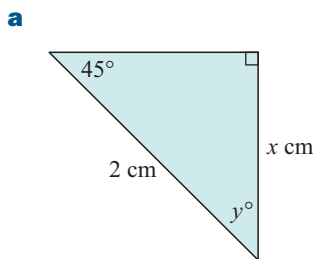
12   An equilateral triangle has sides of length 12 cm. Find the length of one of its altitudes.

13   The area of a triangle is given by the formula $A = \frac{1}{2}bh$.

      a   An isosceles triangle has equal sides of length 8 cm and a base
          of length 6 cm. Find the area of the triangle.

      b   An equilateral triangle has area $16\sqrt{3}$ cm^2. Find the length of
          its sides.
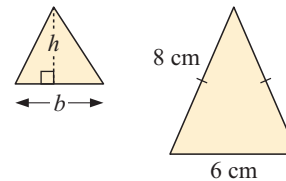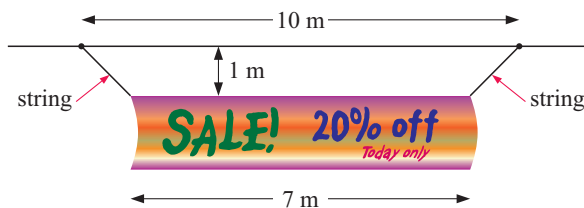
14   Heather wants to hang a 7 m long
                                             banner from the roof of her shop. The
                                             hooks for the strings are 10 m apart, and
                                               Heather wants the top of the banner to
                                               hang 1 m below the roof. How long
                                               should each of the strings be?

15 Two bushwalkers set off from base camp at the same time, walking at right angles to one another. One walks at an average speed of 5 km/h, and the other at an average speed of 4 km/h. Find their distance apart after 3 hours.

16 To get to school from her house, Ella walks down Bernard Street, then turns 90° and walks down Thompson Road until she reaches her school gate. She walks twice as far along Bernard Street as she does along Thompson Road. If Ella's house is 2.5 km in a straight line from her school gate, how far does Ella walk along Bernard Street?

17 Boat A is 10 km east of boat B. Boat A travels 6 km north, and boat B travels 2 km west. How far apart are the boats now?

D   CIRCLE PROBLEMS   [4.6, 4.7]

There are certain properties of circles which involve right angles. In these situations we can apply Pythagoras' theorem. The properties will be examined in more detail in **Chapter 27**.

ANGLE IN A SEMI-CIRCLE

> The angle in a semi-circle is a right angle.

No matter where C is placed on the arc AB, $\widehat{ACB}$ is always a right angle.
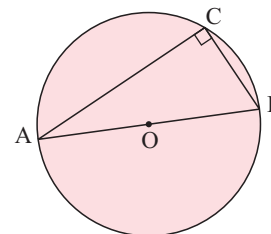
Example 13   ◀ⁱ) *Self Tutor*

A circle has diameter XY of length 13 cm. Z is a point on the circle such that XZ is 5 cm. Find the length YZ.

From the angle in a semi-circle theorem, we know $X\widehat{Z}Y$ is a right angle.
Let the length YZ be x cm.

$$\therefore \quad 5^2 + x^2 = 13^2 \qquad \{\text{Pythagoras}\}$$
$$\therefore \quad x^2 = 169 - 25 = 144$$
$$\therefore \quad x = \sqrt{144} \qquad \{\text{as} \quad x > 0\}$$
$$\therefore \quad x = 12$$

So,   YZ has length 12 cm.
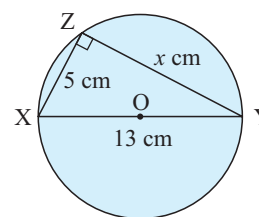
A CHORD OF A CIRCLE

> The line drawn from the centre of a circle at right angles to a chord bisects the chord.

This follows from the **isosceles triangle theorem**. The construction of radii from the centre of the circle to the end points of the chord produces two right angled triangles.
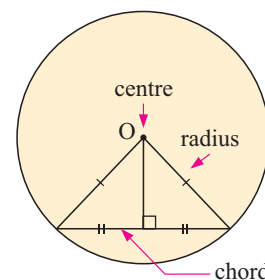
Example 14　　　　　　　　　　　　　　　　　　◀) *Self Tutor*

A circle has a chord of length 10 cm. If the radius of the circle is 8 cm, find the shortest distance from the centre of the circle to the chord.

The shortest distance is the 'perpendicular distance'. The line drawn from the centre of a circle, perpendicular to a chord, bisects the chord, so

$$AB = BC = 5 \text{ cm.}$$

In　$\triangle AOB$,　$5^2 + x^2 = 8^2$　　　　　{Pythagoras}

$$\therefore \quad x^2 = 64 - 25 = 39$$
$$\therefore \quad x = \sqrt{39} \qquad \{\text{as} \quad x > 0\}$$
$$\therefore \quad x \approx 6.24$$

So, the shortest distance is about 6.24 cm.
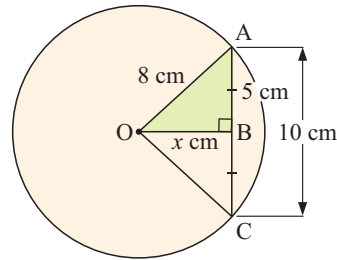
TANGENT-RADIUS PROPERTY

A tangent to a circle and a radius at the point of contact meet at right angles.

Notice that we can now form a right angled triangle.
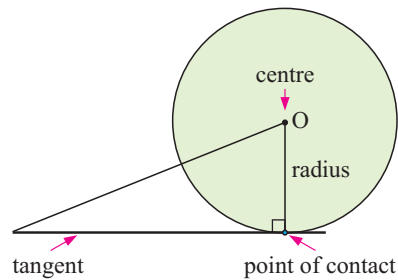
Example 15　　　　　　　　　　　　　　　　　　◀) *Self Tutor*

A tangent of length 10 cm is drawn to a circle with radius 7 cm. How far is the centre of the circle from the end point of the tangent?

Let the distance be d cm.

$$\therefore \quad d^2 = 7^2 + 10^2 \qquad \{\text{Pythagoras}\}$$
$$\therefore \quad d^2 = 149$$
$$\therefore \quad d = \sqrt{149} \qquad \{\text{as} \quad d > 0\}$$
$$\therefore \quad d \approx 12.2$$

So, the centre is 12.2 cm from the end point of the tangent.
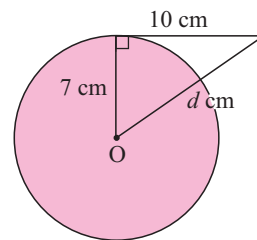
Example 16　　　　　　　　　　　　　　　　　　◀) *Self Tutor*

Two circles have a common tangent with points of contact at A and B. The radii are 4 cm and 2 cm respectively. Find the distance between the centres given that AB is 7 cm.
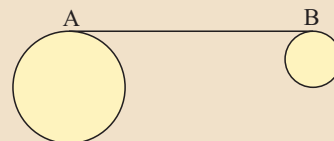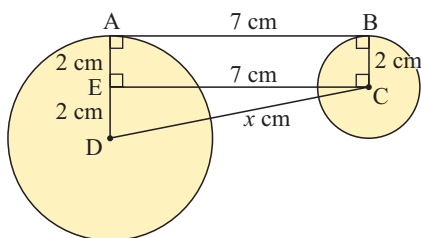
For centres C and D, we draw BC, AD, CD and CE ∥ AB.

∴   ABCE is a rectangle

∴   CE = 7 cm   {as CE = AB}
and   DE = 4 − 2 = 2 cm
Now $x^2 = 2^2 + 7^2$ {Pythagoras in ΔDEC}
∴ $x^2 = 53$
∴ $x = \sqrt{53}$ {as $x > 0$}
∴ $x \approx 7.28$

∴   the distance between the centres is about 7.28 cm.

EXERCISE 8D

1   AT is a tangent to a circle with centre O. The circle has radius 5 cm and AB = 7 cm. Find the length of the tangent.
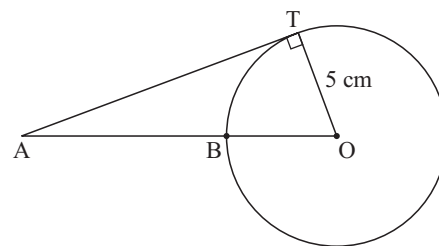
2   A circle has centre O and a radius of 8 cm. Chord AB is 13 cm long. Find the shortest distance from the chord to the centre of the circle.
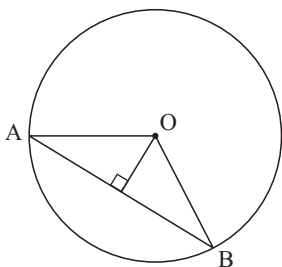
3   AB is a diameter of a circle and AC is half the length of AB. If BC is 12 cm long, what is the radius of the circle?
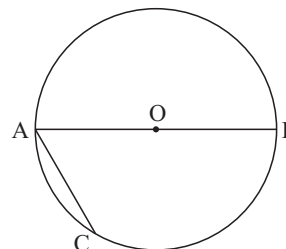
4   A rectangle with side lengths 11 cm and 6 cm is inscribed in a circle. Find the radius of the circle.
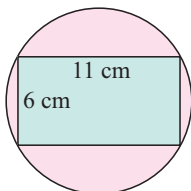
5   A circle has diameter AB of length 10 cm. C is a point on the circle such that AC is 8 cm. Find the length BC.

6   A square is inscribed in a circle of radius 6 cm. Find the length of the sides of the square, correct to 3 significant figures.
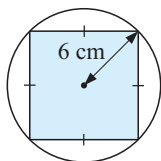
7   A chord of a circle has length 3 cm. If the circle has radius 4 cm, find the shortest distance from the centre of the circle to the chord.

8   A chord of length 6 cm is 3 cm from the centre of a circle. Find the length of the circle's radius.

9   A chord is 5 cm from the centre of a circle of radius 8 cm. Find the length of the chord.

10   A circle has radius 3 cm. A tangent is drawn to the circle from point P which is 9 cm from O, the circle's centre. How long is the tangent? Leave your answer in surd form.

11   Find the radius of a circle if a tangent of length 12 cm has its end point 16 cm from the circle's centre.

12   Two circular plates of radius 15 cm are placed in opposite corners of a rectangular table as shown.
Find the distance between the centres of the plates.

13
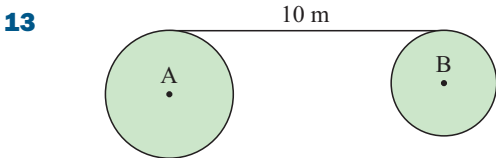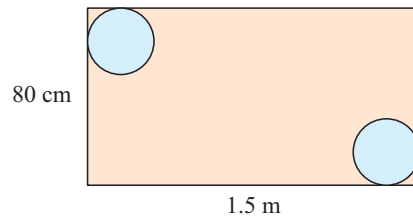
A and B are the centres of two circles with radii 4 m and 3 m respectively. The illustrated common tangent has length 10 m. Find the distance between the centres correct to 2 decimal places.

14   Two circles are drawn so they do not intersect. The larger circle has radius 6 cm. A common tangent is 10 cm long and the centres are 11 cm apart. Find the radius of the smaller circle, correct to 3 significant figures.
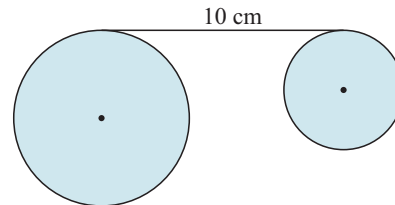
15   The following figures have not been drawn to scale, but the information marked on them is correct. What can you deduce from each figure?

a
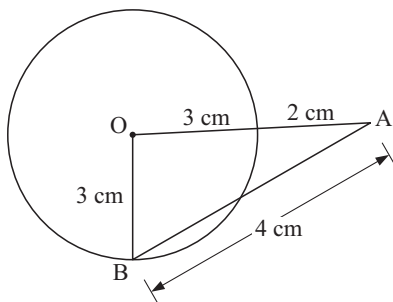
b
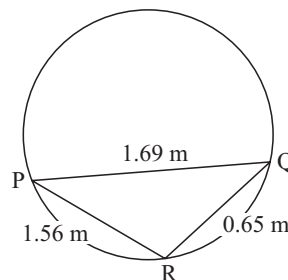
16
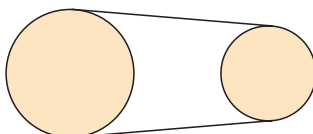
Any two circles which do not intersect have two common external tangents as illustrated. The larger circle has radius b and the smaller one has radius a. The circles are $2a$ units apart. Show that each common tangent has length $\sqrt{8a(a+b)}$ units.

17 A chord AB of length 2 cm is drawn in a circle of radius 3 cm. A diameter BC is constructed, and the tangent from C is drawn. The chord AB is extended to meet the tangent at D. Find the length of AD.
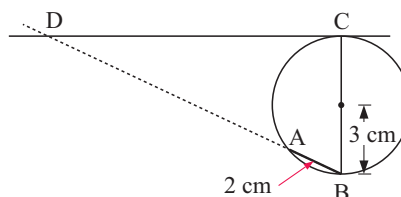
E   THREE-DIMENSIONAL PROBLEMS   [4.6]

Pythagoras' theorem is often used to find lengths in **three-dimensional** problems. In these problems we sometimes need to apply it *twice*.

Example 17   ◄⁾ Self Tutor

A 50 m rope is attached inside an empty cylindrical wheat silo of diameter 12 m as shown. How high is the wheat silo?
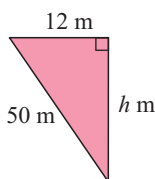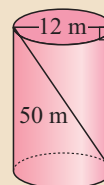
Let the height be h m.

$\therefore \quad h^2 + 12^2 = 50^2 \qquad$ {Pythagoras}

$\therefore \quad h^2 + 144 = 2500$

$\therefore \qquad h^2 = 2356$

$\therefore \qquad h = \sqrt{2356} \qquad$ {as $h > 0$}

$\therefore \qquad h \approx 48.5$

So,   the wheat silo is approximately 48.5 m high.

Example 18   ◄⁾ Self Tutor

The floor of a room is 6 m by 4 m, and its height is 3 m. Find the distance from a corner point on the floor to the opposite corner point on the ceiling.

The required distance is AD. We join BD.

In $\triangle$BCD, $x^2 = 4^2 + 6^2 \qquad$ {Pythagoras}

In $\triangle$ABD, $y^2 = x^2 + 3^2 \qquad$ {Pythagoras}

$\therefore \quad y^2 = 4^2 + 6^2 + 3^2$

$\therefore \quad y^2 = 61$

$\therefore \quad y = \sqrt{61} \qquad$ {as $y > 0$}

$\therefore \quad y \approx 7.81$

$\therefore$ the distance is about 7.81 m.

Example 19   ◀)) *Self Tutor*
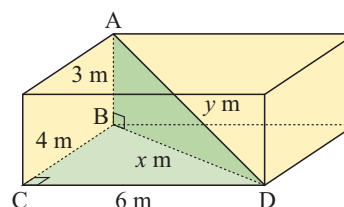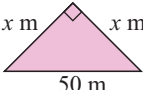
A pyramid of height 40 m has a square base with edges 50 m.
Determine the length of the slant edges.
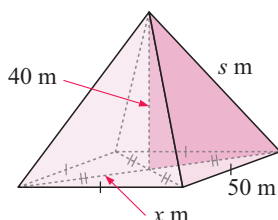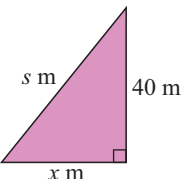
Let a slant edge have length s m.

Let half a diagonal have length x m.

Using

$$x^2 + x^2 = 50^2 \qquad \{\text{Pythagoras}\}$$
$$\therefore \quad 2x^2 = 2500$$
$$\therefore \quad x^2 = 1250$$

Using

$$s^2 = x^2 + 40^2 \qquad \{\text{Pythagoras}\}$$
$$\therefore \quad s^2 = 1250 + 1600$$
$$\therefore \quad s^2 = 2850$$
$$\therefore \quad s = \sqrt{2850} \qquad \{\text{as } s > 0\}$$
$$\therefore \quad s \approx 53.4$$

So, each slant edge is approximately 53.4 m long.

EXERCISE 8E

1 A cone has a slant height of 17 cm and a base radius of 8 cm. How high is the cone?

2 A cylindrical drinking glass has radius 3 cm and height 10 cm. Can a 12 cm long thin stirrer be placed in the glass so that it will stay entirely within the glass?

3 A 20 cm nail just fits inside a cylindrical can. Three identical spherical balls need to fit entirely within the can. What is the maximum radius of each ball?

4 A cubic die has sides of length 2 cm. Find the
distance between opposite corners of the die.
Leave your answer in surd form.
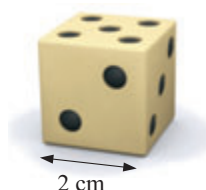
5 A room is 5 m by 3 m and has a height of 3.5 m. Find the distance from a corner point on the floor to the opposite corner of the ceiling.

6 Determine the length of the longest metal rod which could be stored in a rectangular box 20 cm by 50 cm by 30 cm.

7 A tree is 8 m north and 6 m east of another tree. One of the trees is 12 m tall, and the other tree is 17 m tall. Find the distance between:
   a the trunks of the trees   **b** the tops of the trees.

8 A rainwater tank is cylindrical with a conical top.
The slant height of the top is 5 m, and the height of the cylinder is 9 m.
Find the distance between P and Q, to the nearest cm.
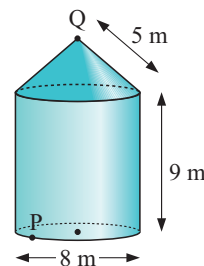
9
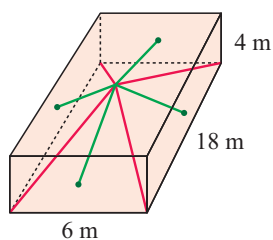
A 6 m by 18 m by 4 m hall is to be decorated with streamers for a party.

4 streamers are attached to the corners of the floor, and 4 streamers are attached to the centres of the walls as illustrated. All 8 streamers are then attached to the centre of the ceiling. Find the total length of streamers required.

10 Answer the **Opening Problem** on page **169**.

Review set 8A

1 Find the lengths of the unknown sides in the following triangle. Give your answers correct to 3 significant figures.

a

b

c
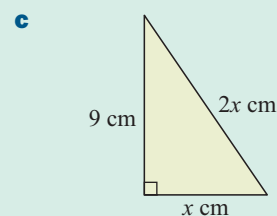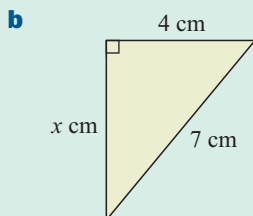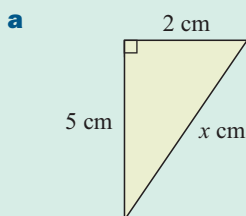
2 Is the following triangle right angled? Give evidence.
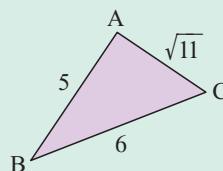
3 Show that $\{5, 11, 13\}$ is not a Pythagorean triple.

4
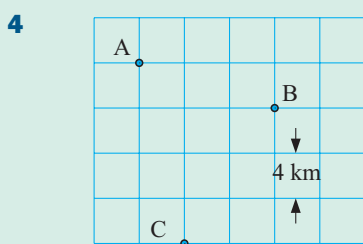
Find, correct to 3 significant figures, the distance from:

a A to B   **b** B to C   **c** A to C.

5 A rhombus has diagonals of length 12 cm and 18 cm. Find the length of its sides.

6 A circle has a chord of length 10 cm. The shortest distance from the circle's centre to the chord is 5 cm. Find the radius of the circle.

7 The diagonal of a cube is 10 m long. Find the length of the sides of the cube.
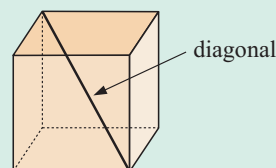
8 Two circles have the same centre. The tangent drawn
from a point P on the smaller circle cuts the larger circle
at Q and R.
Find Show that PQ and PR are equal in length.

9 Find x, correct to 3 significant figures:

a

2x cm

x cm

10 cm

b

x cm　　　tangent

9 cm

O

5 cm

10

A

1.5 m

3 m

B

5 m

2 m
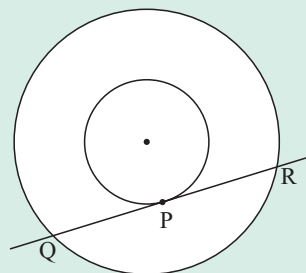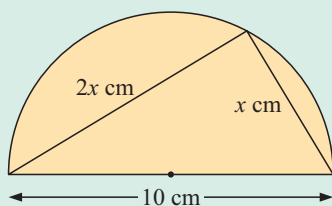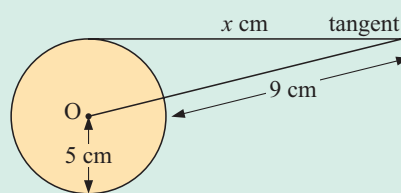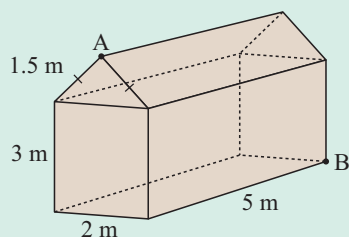
A barn has the dimensions given.
Find the shortest distance from A to B.

Review set 8B

1 Find the value of x:

a

x cm

5 cm

$\sqrt{7}$ cm

b

x m

5 m

6 m

c

2x

$\sqrt{42}$

5x
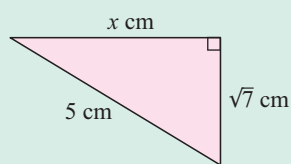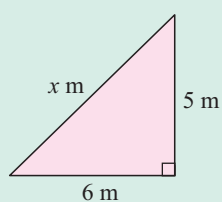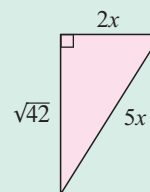
2 Show that the following triangle is right angled
and state which vertex is the right angle:

B

2

5

A

$\sqrt{29}$

C
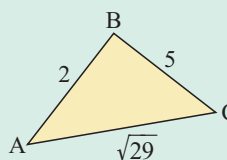
3 Is triangle ABC right angled?
Give evidence to support your answer.

A

10 m

C

5 m

4 m　B
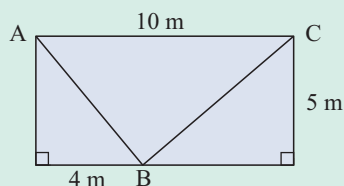
4
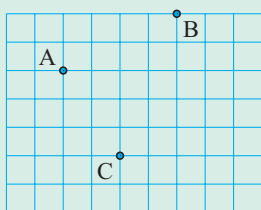
The grid lines on the map are 3 km apart. A, B and C are farm houses. How far is it from:

  a   A to B   **b**   B to C   **c**   C to A?

5   If the diameter of a circle is 20 cm, find the shortest distance from a chord of length 16 cm to the centre of the circle.

6   Find the length of plastic coated wire required to make this clothes line:

3 m

3 m
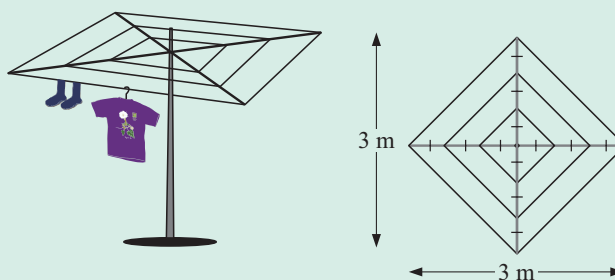
7
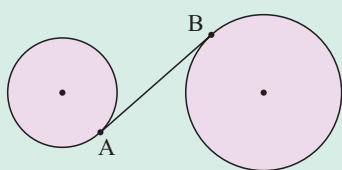
The circles illustrated have radii of length 5 cm and 7 cm respectively.

Their centres are 18 cm apart. Find the length of the common tangent AB.

8   A 20 cm chopstick just fits inside a rectangular box with base 10 cm by 15 cm. Find the height of the box.

9   Find y in the following, giving your answers correct to 3 significant figures:

  a

y cm

10 cm

  b

8 cm

y cm

tangent

$(y - 6)$ cm
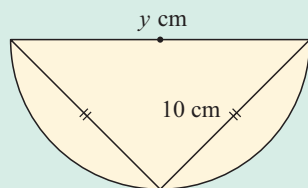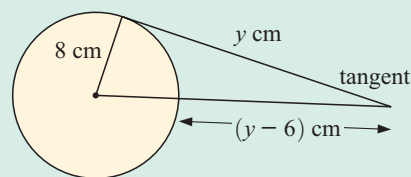
10   Marvin the Magnificent is attempting to walk a tightrope across an intersection from one building to another as illustrated. Using the dimensions given, find the length of the tightrope.

18 m

12 m

12 m

13 m
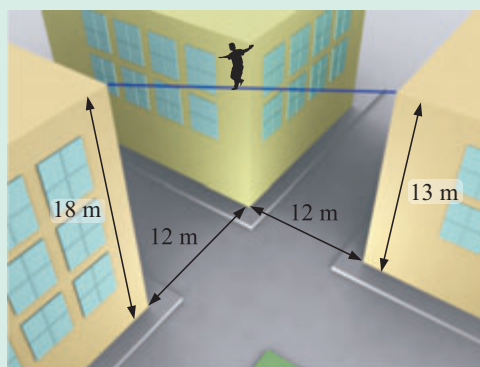
Challenge

1 A highway runs in an East-West direction joining towns C and B, which are 25 km apart. Town A lies directly north from C, at a distance of 15 km. A straight road is built from A to the highway and meets the highway at D, which is equidistant from A and B. Find the position of D on the highway.

2 Annabel Ant at A wishes to visit Bertie Beetle at B on the opposite vertex of a block of cheese which is 30 cm by 20 cm by 20 cm. What is the shortest distance that Annabel must travel if

    a her favourite food is cheese

    b she hates eating cheese?
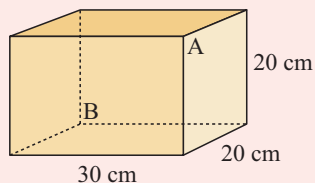
3
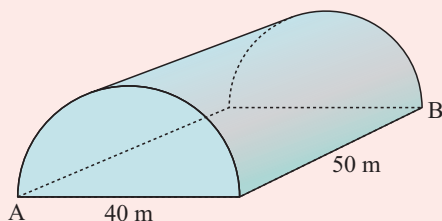
An aircraft hanger is semi-cylindrical, with diameter 40 m and length 50 m. A helicopter places an inelastic rope across the top of the hanger and one end is pinned to a corner at A. The rope is then pulled tight and pinned at the opposite corner B. Determine the length of the rope.

4 The radius of the small circle is r and the radius of the semi-circle is R. Find the ratio $r : R$, given that the semi-circles pass through each other's centres.

Note: When circles touch, their centres and their point of contact lie in a straight line, that is, they are collinear.
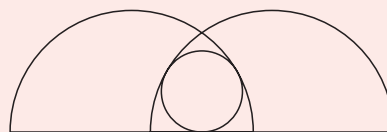
5
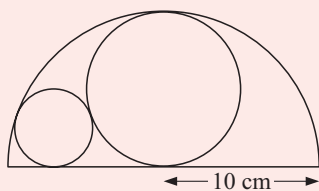
The larger circle touches the diameter of the semi-circle at its centre. The circles touch each other and the semi-circle. The semi-circle has radius 10 cm. Find the radius of the small circle.

Mensuration (length and area)

9

Contents:

Opening problem
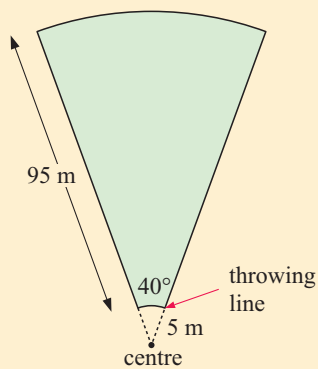
A javelin throwing arena is illustrated alongside. It has the shape of a sector of a circle of radius 100 m. The throwing line is 5 m from the centre. A white line is painted on the two 95 m straights and on the two circular arcs.

a Find the total length of the painted white line.

b If the shaded landing area is grassed, what is the total area of grass?

95 m

40°

throwing line

5 m

centre

The measurement of **length**, **area**, **volume** and **capacity** is of great importance.

Constructing a tall building or a long bridge across a river, joining the circuits of a microchip, and rendezvousing in space to repair a satellite all require the use of measurement with skill and precision.

Builders, architects, engineers and manufacturers need to measure the sizes of objects to considerable accuracy.

The most common system of measurement is the **Système International (SI)**.

Important units that you should be familiar with include:

Measurement of	Standard unit	What it means
Length	metre	How long or how far.
Mass	kilogram	How heavy an object is.
Capacity	litre	How much liquid or gas is contained.
Time	hours, minutes, seconds	How long it takes.
Temperature	degrees Celsius and Fahrenheit	How hot or cold.
Speed	metres per second (m/s)	How fast it is travelling.

The SI uses prefixes to indicate an increase or decrease in the size of a unit.

Prefix	Symbol	Meaning	Prefix	Symbol	Meaning
terra	T	1 000 000 000 000	centi	c	0.01
giga	G	1 000 000 000	milli	m	0.001
mega	M	1 000 000	micro	μ	0.000 001
kilo	k	1000	nano	n	0.000 000 001
hecto	h	100	pico	p	0.000 000 000 001

In this course we will work primarily with the prefixes kilo, centi and milli.

A   LENGTH   [6.1]

The base unit of length in the SI is the **metre** (m). Other units of length based on the metre are:

- **millimetres** (mm)   used to measure   the length of a bee
- **centimetres** (cm)   used to measure   the width of your desk
- **kilometres** (km)   used to measure   the distance between two cities.

The table below summarises the connection between these units of length:

> 1 **kilometre** (km) = 1000 **metres** (m)
> 1 **metre** (m) = 100 **centimetres**
> 1 **centimetre** (cm) = 10 **millimetres** (mm)

LENGTH UNITS CONVERSIONS
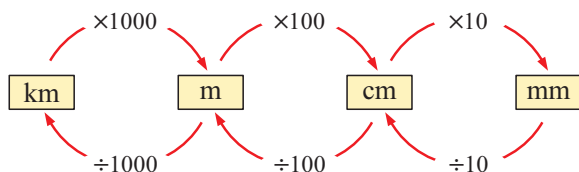
So, to convert cm into km we $\div$ 100 and then $\div$ 1000.

Notice that, when converting from:

- smaller units to larger units we **divide** by the conversion factor
- larger units to smaller units we **multiply** by the conversion factor.

Example 1   ◀ঠ Self Tutor

Convert: **a**   4.5 km to m   **b**   1.25 m to mm

a   4.5 km
$= (4.5 \times 1000)$ m
$= 4500$ m

b   1.25 m
$= (1.25 \times 100)$ cm
$= (1.25 \times 100 \times 10)$ mm
$= 1250$ mm

Example 2   ◀ঠ Self Tutor

Convert: **a**   350 cm to m   **b**   23 000 mm to m

a   350 cm
$= (350 \div 100)$ m
$= 3.5$ m

b   23 000 mm
$= (23\,000 \div 10)$ cm
$= (23\,000 \div 10 \div 100)$ m
$= 23$ m

EXERCISE 9A

1 Suggest an appropriate unit of length for measuring the following:

   a the length of a baby   **b** the width of an eraser

   c the distance from London to Cambridge   **d** the height of an old oak tree

   e the length of an ant   **f** the length of a pen

2 Estimate the following and then check by measuring:

   a the length of your desk   **b** the width of your pencil case

   c the height of a friend   **d** the dimensions of your classroom

   e the length of a tennis court   **f** the width of a hockey pitch

3 Convert:

   a 52 km to m   **b** 115 cm to mm   **c** 1.65 m to cm

   d 6.3 m to mm   **e** 0.625 km to cm   **f** 8.1 km to mm

4 Convert:

   a 480 cm to m   **b** 54 mm to cm   **c** 5280 m to km

   d 2000 mm to m   **e** 580 000 cm to km   **f** 7 000 000 mm to km

5 Convert the following lengths:

   a 42.1 km to m   **b** 210 cm to m   **c** 75 mm to cm

   d 1500 m to km   **e** 1.85 m to cm   **f** 42.5 cm to mm

   g 2.8 km to cm   **h** 16 500 mm to m   **i** 0.25 km to mm

6 A packet contains 120 wooden skewers, each of which is 15 cm long. If the skewers are placed in a line end to end, how far will the line stretch?

7 The length of a marathon is about 42 km. If the distance around your school's track is 400 m, how many laps must you complete to run the length of a marathon?

8 When Lucy walks, the length of her stride is 80 cm. Every morning she walks 1.7 km to school. How many steps does she take?

9 The height of a pack of 52 cards is 1.95 cm. Find:

    a the thickness, in millimetres, of a single card.

    b the height, in metres, of 60 packs stacked on top of one another

    c the number of cards required to form a stack 12 cm high.

B   PERIMETER   [6.2]

The **perimeter** of a figure is the measurement of the distance around its boundary.

For a **polygon**, the perimeter is obtained by adding the lengths of all sides.
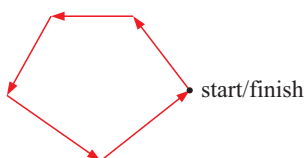
One way of thinking about perimeter is to imagine walking around a property. Start at one corner and walk around the boundary. When you arrive back at your starting point the **perimeter** is the distance you have walked.

You should remember from previous years these perimeter formulae:

Square
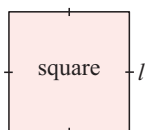
$$P = 4l$$

Rectangle
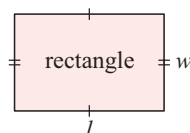
$$P = 2l + 2w$$
$$\text{or } P = 2(l + w)$$

Example 3   ◄⁾ *Self Tutor*

Find the perimeter of:

a

9.7 m

13.2 m

b

4.2 cm

6.7 cm
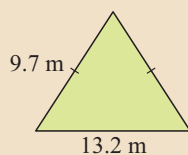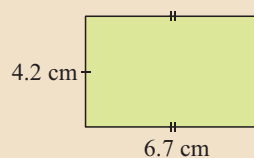
a Perimeter $= 2 \times 9.7 + 13.2$ m
$\quad = 32.6$ m

b $P = 2 \times 4.2 + 2 \times 6.7$
$\quad = 8.4 + 13.4$
$\quad = 21.8$ cm

EXERCISE 9B

1 Measure with your ruler the lengths of the sides of each given figure and then find its perimeter.

a **b** **c**

2 Find the perimeter of:

a

9.7 cm

b

3.2 cm

5.7 cm

c

4.3 km

3.8 km

d

7 m

6 m   5 m

11 m

e

2.9 cm

f

5 m   13 m

12 m

g

2.2 m

h

8.3 km

i

3.4 m   4.1 m
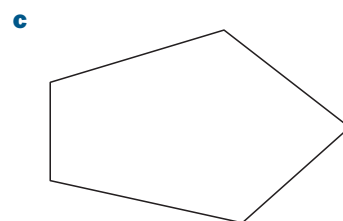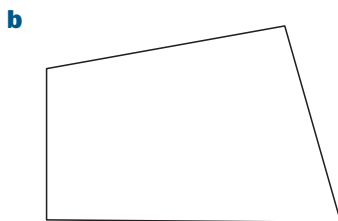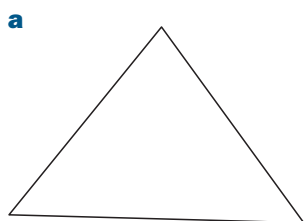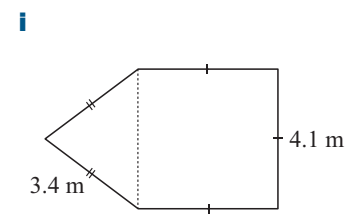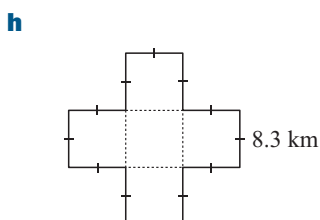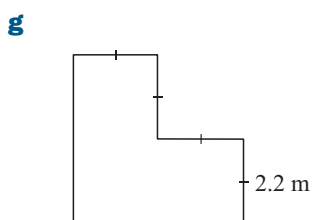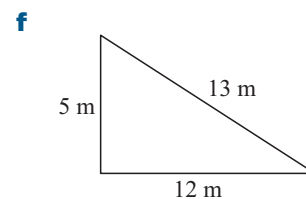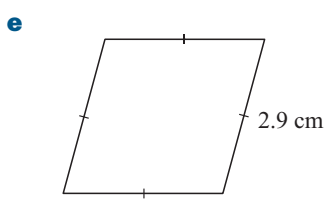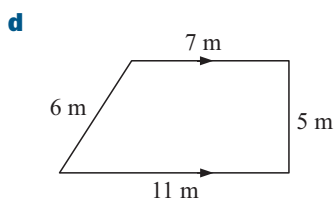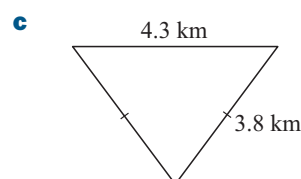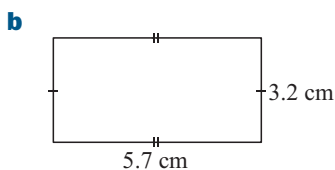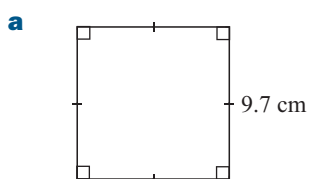
3 Find a formula for the perimeter P of:

a

b

a

b

k l

m

c

$4a$

$3a$
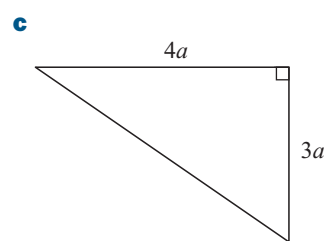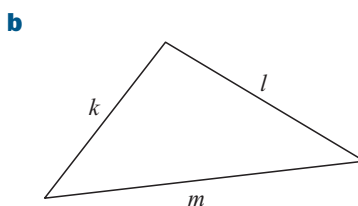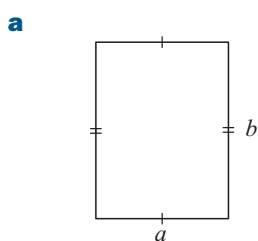
4 An isosceles triangle has a perimeter of 30 cm. If the base is 7.8 cm long, find the length of the equal sides.

5 A rectangle has one side of length 11.2 m and its perimeter is 39.8 m. Find the width of the rectangle.

6 A rectangle is 16.4 cm by 11.8 cm and has the same perimeter as an equilateral triangle. Find the length of the sides of the triangle.

7
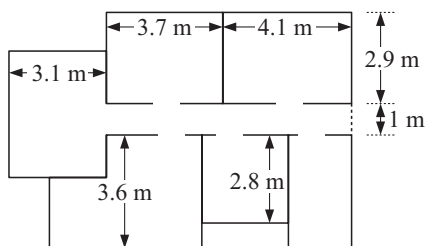
Find the perimeter of the house in the plan alongside.

8 An octagonal area of lawn is created by removing 2 m by 2 m corners from a rectangular area. Find the new perimeter of the lawn.

9
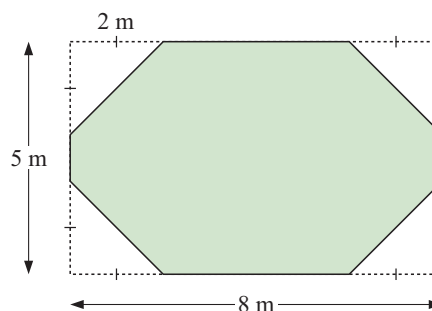
Calculate the length of wire required to construct the frame for the model house illustrated alongside.
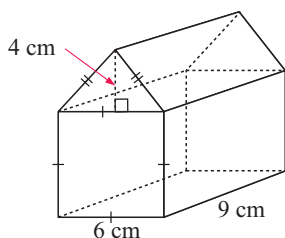
C   AREA   [6.1, 6.2, 6.5]

All around us we see surfaces such as walls, ceilings, paths and ovals. All of these surfaces have boundaries that help to define the surface.

> An **area** is the amount of *surface* within specified boundaries.
>
> The **area** of the surface of a closed figure is measured in terms of the number of square units it encloses.

UNITS OF AREA

Area can be measured in square millimetres, square centimetres, square metres and square kilometres; there is also another unit called a hectare (ha).

Since 1 m = 100 cm, these squares have the same area.
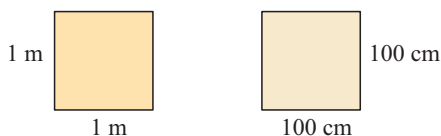
So, $1 \text{ m}^2 = 100 \text{ cm} \times 100 \text{ cm}$
                $= 10\,000 \text{ cm}^2$

$$1 \textbf{ mm}^2 = 1 \text{ mm} \times 1 \text{ mm}$$
$$1 \textbf{ cm}^2 = 10 \text{ mm} \times 10 \text{ mm} \quad = 100 \text{ mm}^2$$
$$1 \textbf{ m}^2 = 100 \text{ cm} \times 100 \text{ cm} \quad = 10\,000 \text{ cm}^2$$
$$1 \textbf{ ha} = 100 \text{ m} \times 100 \text{ m} \quad = 10\,000 \text{ m}^2$$
$$1 \textbf{ km}^2 = 1000 \text{ m} \times 1000 \text{ m} \quad = 1\,000\,000 \text{ m}^2 \text{ or } 100 \text{ ha}$$

AREA UNITS CONVERSIONS

$$\boxed{km^2} \quad \boxed{ha} \quad \boxed{m^2} \quad \boxed{cm^2} \quad \boxed{mm^2}$$

×100 ×10 000 ×10 000 ×100

÷100 ÷10 000 ÷10 000 ÷100

Example 4   ◀) **Self Tutor**
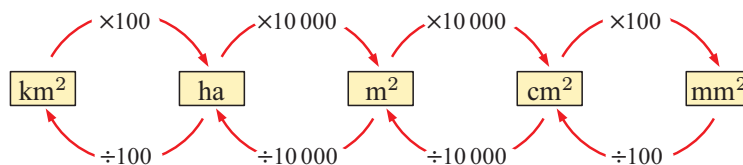
Convert: **a** 6 m^2 to cm^2 **b** $18\,500 \text{ m}^2$ to ha

a $6 \text{ m}^2 = (6 \times 10\,000) \text{ cm}^2$

$= 60\,000 \text{ cm}^2$

b $18\,500 \text{ m}^2 = (18\,500 \div 10\,000)$ ha

$= 18\,500. \div 10\,000$ ha

$= 1.85$ ha

EXERCISE 9C.1

1   Suggest an appropriate area unit for measuring the following:

   a   the area of a postage stamp   **b**   the area of your desktop

   c   the area of a vineyard   **d**   the area of your bedroom floor

   e   the area of Ireland   **f**   the area of a toe-nail

2   Convert:

   a 23 mm^2 to cm^2 **b** 3.6 ha to m^2 **c** 726 cm^2 to m^2

   d 7.6 m^2 to mm^2 **e** 8530 m^2 to ha   **f** 0.354 ha to cm^2

   g 13.54 cm^2 to mm^2 **h** 432 m^2 to cm^2 **i** $0.004\,82 \text{ m}^2$ to mm^2

   j 3 km^2 into m^2 **k** 0.7 km^2 into ha   **l** 660 ha into m^2

   m 660 ha into km^2 **n** 0.05 m^2 into cm^2 **o** 25 cm^2 into m^2

   p 5.2 mm^2 into cm^2 **q** 0.72 km^2 into mm^2

3   Calculate the area of the following rectangles:

   a 50 cm by 0.2 m in cm^2 **b** 0.6 m by 0.04 m in cm^2

   c 30 mm by 4 mm in cm^2 **d** 0.2 km by 0.4 km in m^2

> Make sure you change both length units into the units required in the answer.

4   A 2 ha block of land is to be divided into 32 blocks of equal size. Find the area for each block in m^2.

5   The area of a square coaster is 64 cm^2. How many coasters can be made from a sheet of wood with area 4 m^2?

AREA FORMULAE

You should remember these area formulae from previous years:

Rectangles
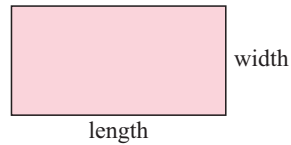
Area = length × width

Triangles
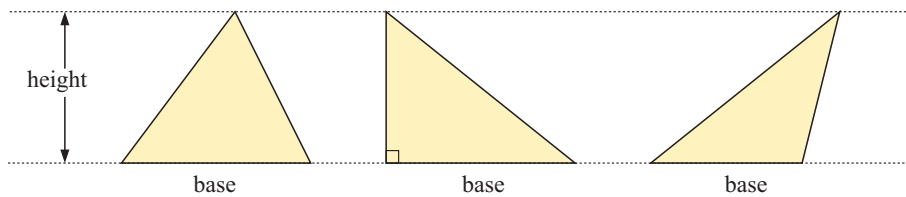
**COMPUTER
DEMO**

Area = $\frac{1}{2}$ (base × height)
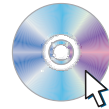
Parallelograms
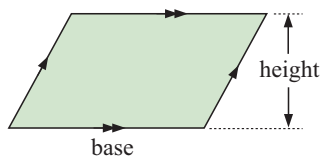
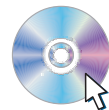**COMPUTER
DEMO**

Area = base × height
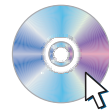
Trapezia
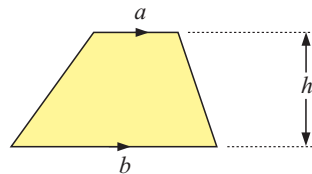
**COMPUTER
DEMO**

$$\text{Area} = \left[\begin{array}{c} \textbf{The average of the} \\ \textbf{lengths of the two} \\ \textbf{parallel sides} \end{array}\right] \times \left[\begin{array}{c} \textbf{the distance between} \\ \textbf{the parallel sides} \end{array}\right]$$

$$A = \left(\frac{a+b}{2}\right) \times h \quad \text{or} \quad A = \tfrac{1}{2}(a+b) \times h$$

Example 5   ◀)) *Self Tutor*

Find the area of:

a

5 m
12 m

b

5 cm
10 cm

c
11 m

4 m
16 m
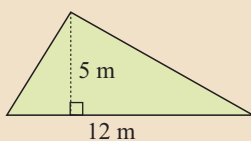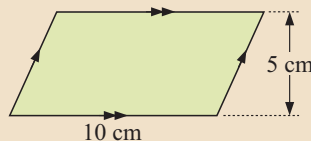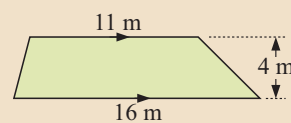
a   Area
$= \frac{1}{2}$ base $\times$ height
$= \frac{1}{2} \times 12 \times 5$
$= 30 \text{ m}^2$

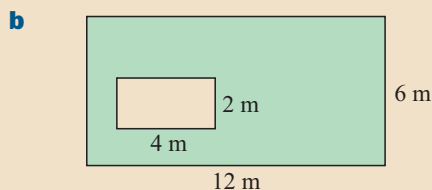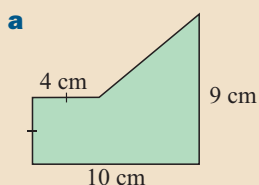**b**   Area
$=$ base $\times$ height
$= 10 \times 5$
$= 50 \text{ cm}^2$

c   Area
$= \left(\dfrac{a+b}{2}\right) \times h$
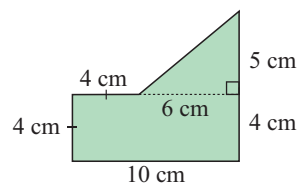$= \left(\dfrac{11+16}{2}\right) \times 4$
$= 54 \text{ m}^2$

Example 6 ◀)) *Self Tutor*

Find the green shaded area:

a

4 cm
9 cm
10 cm

b

2 m
4 m
6 m
12 m

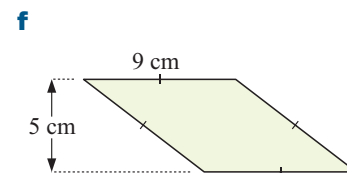**a**   We divide the figure into a rectangle and a triangle as shown below:
Area $=$ area of rectangle $+$ area of triangle
$= 10 \times 4 + \frac{1}{2} \times 6 \times 5$
$= 40 + 15$
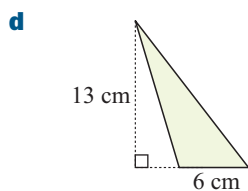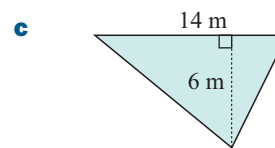$= 55 \text{ cm}^2$

4 cm
4 cm
6 cm
5 cm
4 cm
10 cm

b   Area $=$ area large rectangle $-$ area small rectangle
$= 12 \times 6 - 4 \times 2$
$= 64 \text{ m}^2$

EXERCISE 9C.2

1   Find the area of the shaded region:

a

5 cm
8 cm

b

25 m

c

14 m
6 m

d

13 cm
6 cm

e

6 cm
5 cm

f

9 cm
5 cm

g

12 m
15 m

h

7 cm
4 cm
5 cm

i

3.8 cm
5 cm
6.2 cm
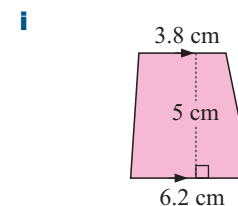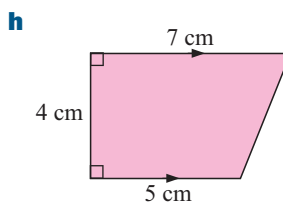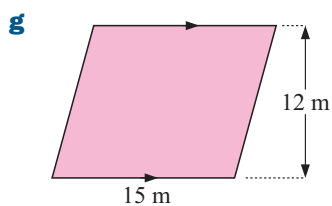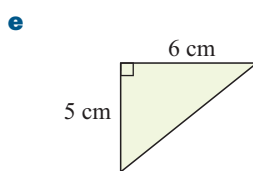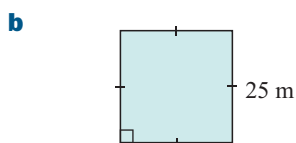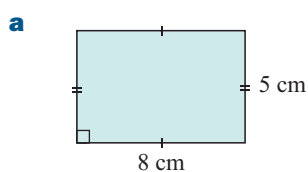
2 Calculate the height h of the following figures:

a

h cm

8 cm

Area $= 24$ cm^2

b

h cm

10 cm

Area $= 36$ cm^2

c

7.6 m

h m

11.2 m
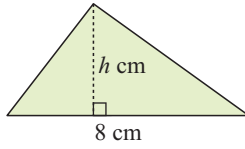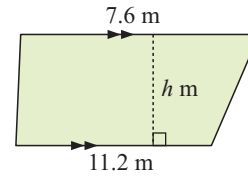
Area $= 47$ m^2

3 Find the area shaded:
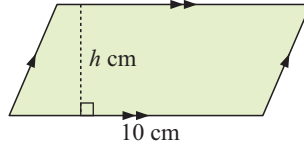
a

10 m

14 m

6 m

b

3 m

5 m

6 m

9 m

c

10 m

4 m

18 m

d

6 cm

12 cm

10 cm

e

3 cm

3 cm

7 cm

f

3 cm

14 cm

18 cm

g

2 m

6 m

7 m

h

6 cm

2 cm

5 cm

3 cm

10 cm

i

4 cm

2 cm   6 cm
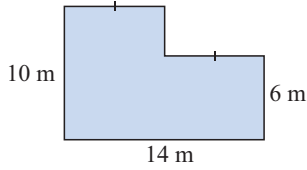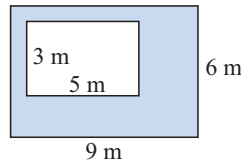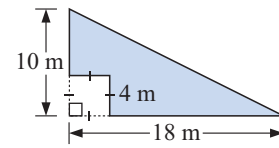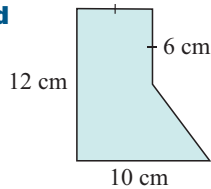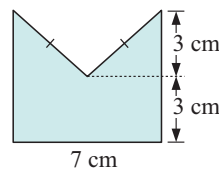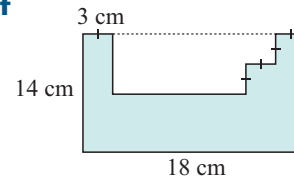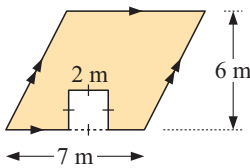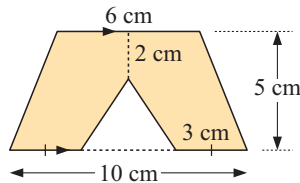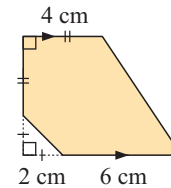
4 A photograph is 6 cm by 4 cm and its border is 12 cm by 10 cm. Calculate the visible area of the border.

5 Instant lawn costs \$15 per square metre. Find the cost of covering a 5.2 m by 3.6 m area with instant lawn.

6 A 4.2 m by 3.5 m tablecloth is used to cover a square table with sides of length 3.1 m. Find the area of the tablecloth which overhangs the edges.

7 A square tile has an area of 256 cm^2. How many tiles are needed for a floor 4 m $\times$ 2.4 m?

8   **a** Find the area of a rhombus which has diagonals of length 12 cm and 8 cm.

     b One diagonal of a rhombus is twice as long as the other diagonal. If the rhombus has area 32 cm^2, find the length of the shorter diagonal.

9 The area of trapezium ABCD is 204 cm^2.
Find the area of triangle DBC.

A   15 cm   B

17 cm   10 cm

D   36 cm   C
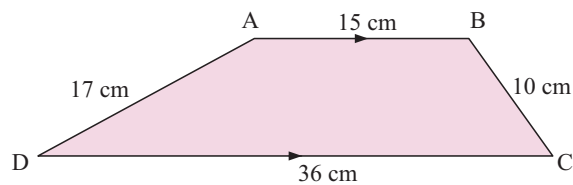
10 **a** A kite has diagonals of length 16 cm and 10 cm. Find its area.

   b Find the area of a kite with diagonals of length a cm and b cm.

11 Parallelogram ABCD has AB $= 10$ cm and diagonal DB $= 15$ cm. If the shortest distance from C to line AB is 8 cm, find the shortest distance from A to DB.

12 Find the area of this trapezium:
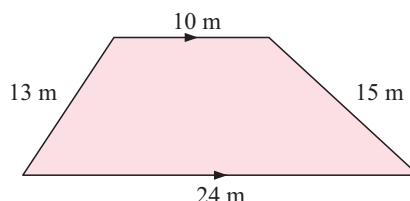
D   CIRCLES AND SECTORS   [6.3, 6.5]

You should already be familiar with these terms relating to circles:

A **circle** is the set of all points a fixed distance from a point called the circle's **centre**.

A line segment from the centre to any point on the circle is called a **radius**.

We denote the length of the radius by r. The perimeter of a circle is called its **circumference**.

A line segment which joins any two points on the circle is called a **chord**.
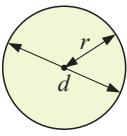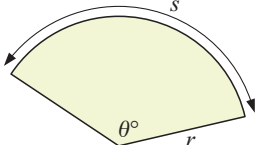
A chord which passes through the centre of the circle is called a **diameter**. We denote the length of the diameter by d.

An **arc** is a continuous part of the circle. The length of an arc is called its **arclength**.

Every arc has a corresponding **sector**, which is the portion of the circle subtended by the same angle θ as the arc.
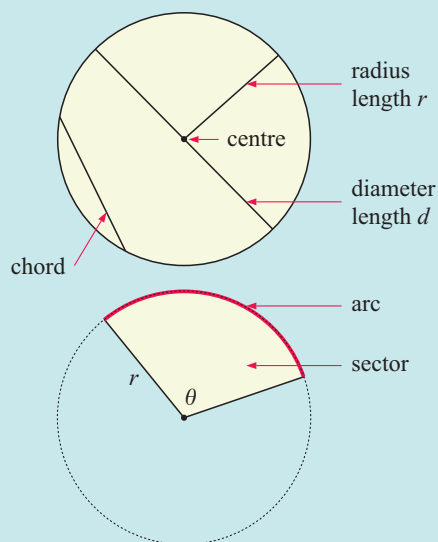
The formulae for the circumference and area of a circle both involve the number π or "pi". π is an irrational number, and $\pi \approx 3.14$.

Circle		
		Circumference $C = \pi d$ or $C = 2\pi r$ Area $A = \pi r^2$
Sector		Arclength $s = \left(\frac{\theta}{360}\right) \times 2\pi r$ Area $A = \left(\frac{\theta}{360}\right) \times \pi r^2$

Example 7
◀》 *Self Tutor*

Find the perimeter of:

a

3.25 m

b

60°　12 cm

> The length of an arc is a fraction of the circumference of a circle.
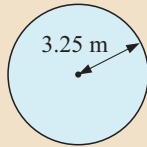
a　Perimeter
$= 2\pi r$
$= 2 \times \pi \times 3.25$ m
≈ 20.4 m
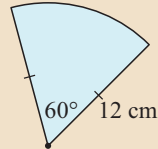
b　Perimeter
$= 12 + 12 + \text{length of arc}$
$= 24 + \left(\frac{60}{360}\right) \times 2 \times \pi \times 12$
≈ 36.6 cm

Example 8
◀》 *Self Tutor*

Find the area of each of the following figures:

a

8.96 m

b

60°
8 cm

> The area of a sector is a fraction of the area of a circle.
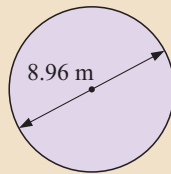
a　$r = \dfrac{8.96}{2} = 4.48$ m

$A = \pi r^2$
$= \pi \times (4.48)^2$
≈ 63.1 m^2
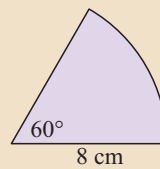
b　Area $= \left(\frac{\theta}{360}\right) \times \pi r^2$
$= \frac{60}{360} \times \pi \times 8^2$
≈ 33.5 cm^2

Example 9
◀》 *Self Tutor*

A sector has area 25 cm^2 and radius 6 cm. Find the angle subtended at the centre.

25 cm^2

$\theta°$　6 cm
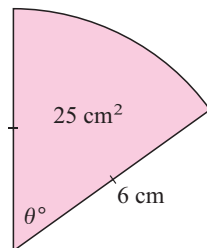
$$\text{Area} = \left(\frac{\theta}{360}\right) \times \pi r^2$$

$$\therefore \quad 25 = \frac{\theta}{360} \times \pi \times 6^2$$

$$\therefore \quad 25 = \frac{\theta\pi}{10}$$

$$\therefore \quad \frac{250}{\pi} = \theta$$

$$\therefore \quad \theta \approx 79.6$$

$\therefore$　the angle measures $79.6°$.

Example 10   ◀) *Self Tutor*

Find a formula for the area A of:
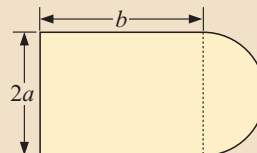
Area = area of rectangle + area of semi-circle

$\therefore \quad A = 2a \times b + \frac{1}{2} \times \pi \times a^2 \qquad$ {the radius of the semi-circle is a units}

$\therefore \quad A = \left(2ab + \dfrac{\pi a^2}{2} \right) \quad$ units^2

EXERCISE 9D

1 Calculate, correct to 3 significant figures, the circumference of a circle with:

   a radius 8 cm   **b** radius 0.54 m   **c** diameter 11 cm.

2 Calculate, correct to 3 significant figures, the area of a circle with:

   a radius 10 cm   **b** radius 12.2 m   **c** diameter 9.7 cm.

3 Calculate the length of the arc of a circle if:

   a the radius is 12.5 cm and the angle at the centre is 60°

   b the radius is 8.4 m and the angle at the centre is 120°.

4 Calculate the area of a sector of:

   a radius 5.62 m and angle 80° **b** radius 8.7 cm and angle 210°.

5 Find the area shaded:
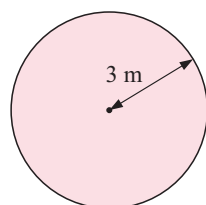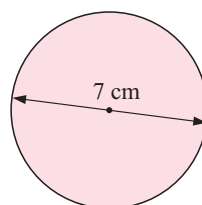
   a   **b**   **c**
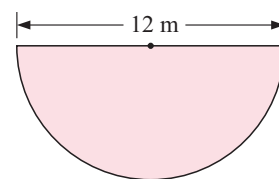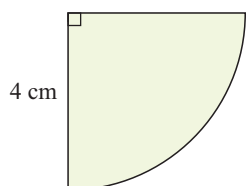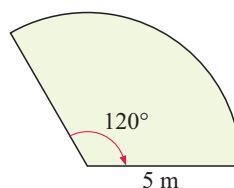
   d   **e**   **f**
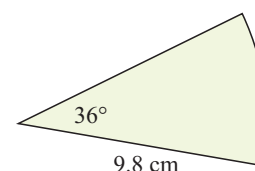
6 Calculate the radius of a circle with:

   a circumference 20 cm   **b** area 20 cm^2 **c** area 9π m^2.

7 Find, in terms of π:

   a the length of arc AB

   b the perimeter of sector OAB

   c the area of sector OAB.
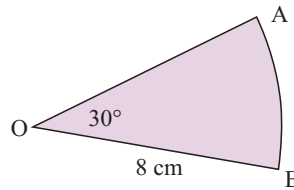
8   **a** Find the circumference of a circle of radius 13.4 cm.

   b Find the length of an arc of a circle of radius 8 cm and angle 120°.

   c Find the perimeter of a sector of a circle of radius 9 cm and sector angle 80°.

9 Find the perimeter and area of the following shapes:

   a

   b

   c

   d

   e

   f

   g

   h

   i
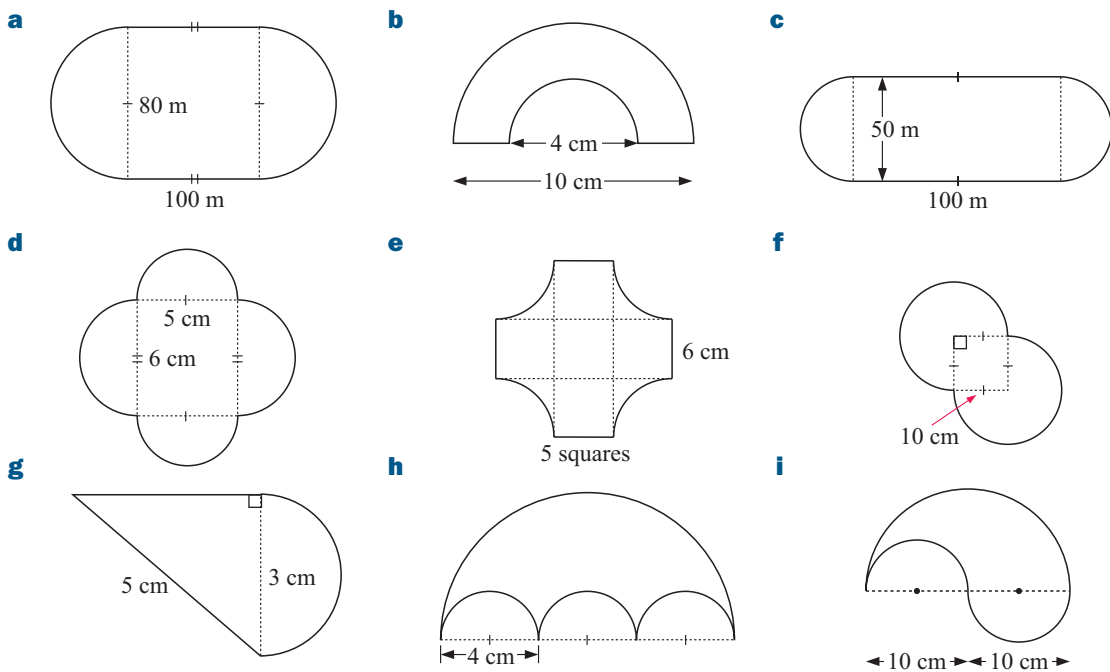
10 Find the radius of a trundle wheel with circumference 1 m.
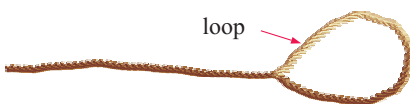
11 A lasso is made from a rope that is 10 m long. The loop of the lasso has a radius of 0.6 m when circular. Find the length of the rope that is not part of the loop.

12 A circular golfing green has a diameter of 20 m. The pin must be positioned on the green at least 3 metres from its edge. Find the area of the green on which the pin is allowed to be positioned.
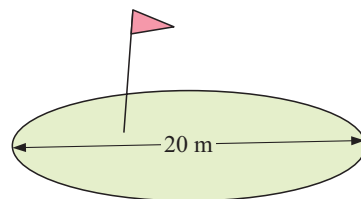
13 The second hand of a clock is 10 cm long. How far does the tip of the second hand travel in 20 seconds?

14 A square slice of bread has sides of length 10 cm. A semi-circular piece of ham with diameter 10 cm is placed on the bread, and the bread is cut in half diagonally as illustrated. Find the area of the ham that is on:

   a side A   **b** side B   of the bread.
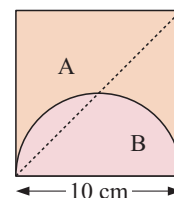
15 Find the angle of a sector with area 30 cm² and radius 12 cm.

16 Brothers Jason and Neil go out for pizza. Jason chooses a rectangular piece of pizza measuring 10 cm by 15 cm. Neil's piece will be cut from a circular pizza with diameter 30 cm. Neil wants his piece to be the same size as Jason's. What angle should be made at the centre of the pizza?

17 Find a formula for the area A of the following regions:

   a   **b**   **c**
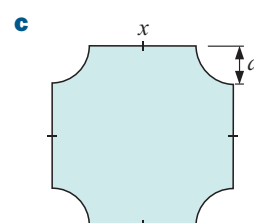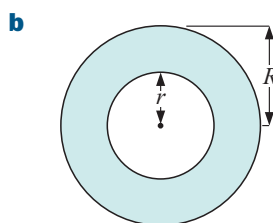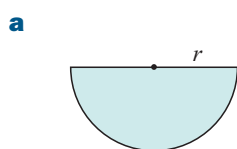
   You may need to use Pythagoras' theorem!

   d   **e**   **f**
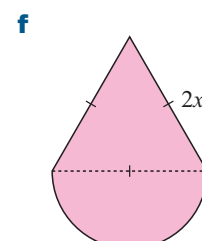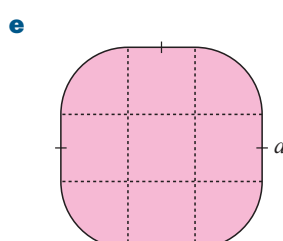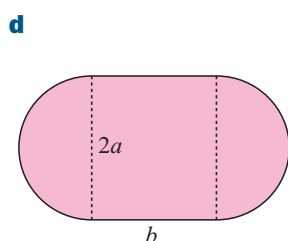
18 Answer the questions of the **Opening Problem** on page **191**.

19 A dartboard is divided into 20 equal sections numbered from 1 to 20. Players throw darts at the board, and score points given by the section number that the dart lands in. If a dart lands in the outer double ring, the points value of the throw is doubled. If a dart lands in the middle treble ring, the points value of the throw is trebled. The central bulls eye has a diameter of 32 mm, and is worth 25 points. The red centre of the bulls eye has a diameter of 12 mm and is worth 50 points.

   a Find the circumference of the dartboard.

   b Find the outer circumference of the "triple 9" section.

   c Find the area in cm², of the region of the dartboard which results in a score of:

      i 50   **ii** 25   **iii** 5
      iv 14   **v** 12
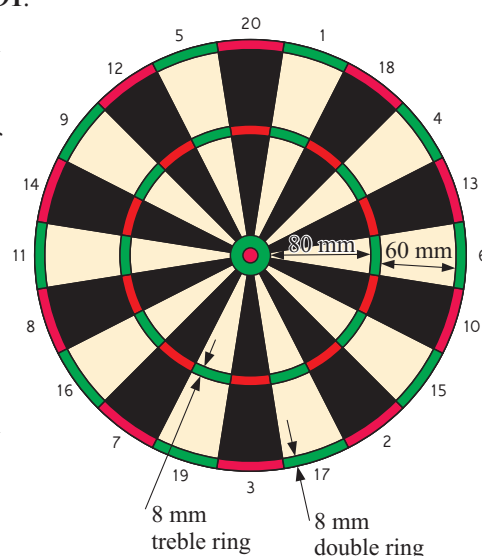
Review set 9A

1   **a**   Convert: **i**   3.28 km to m   **ii**   755 mm to cm   **iii**   32 cm to m

     b   A staple is made from a piece of wire 3 cm long. How many staples can be made from a roll of wire 1200 m long?

2   Find the perimeter of:

     a   **b**   **c**

3   Find the area of:

     a   **b**   **c**
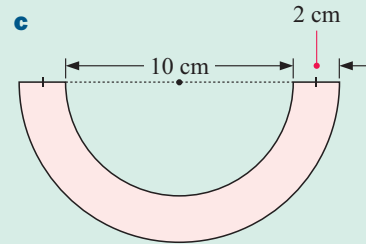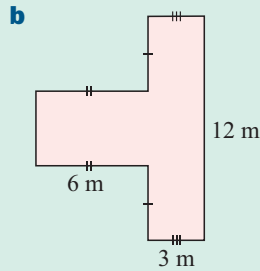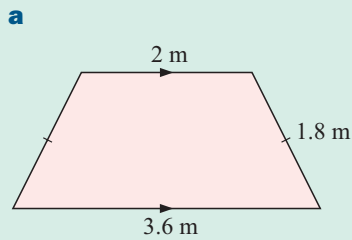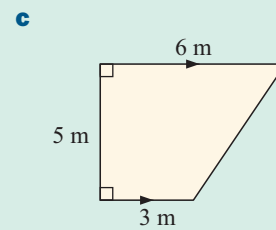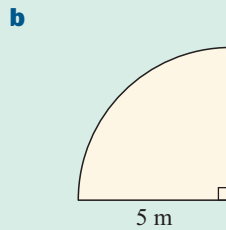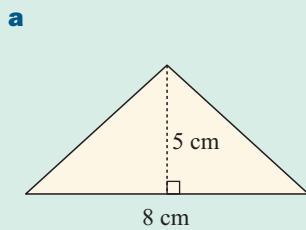
4   The diameter of a car tyre is 50 cm.

     a   How far does the car need to travel for the tyre to complete one revolution?

     b   How many revolutions does the tyre complete if the car travels 2 km?

5   Determine the angle of a sector with arc length 32 cm and radius 7 cm.

6   Find a formula for the   **i**   perimeter P **ii**   area A of:

     a   **b**
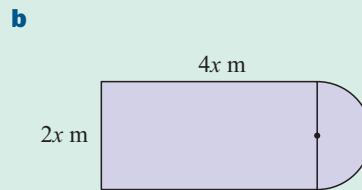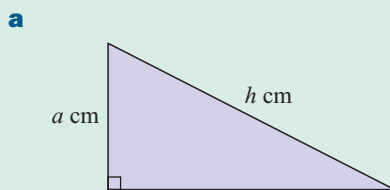
7   A circle has area 8 m². Find:

     a   its radius   **b**   its circumference.
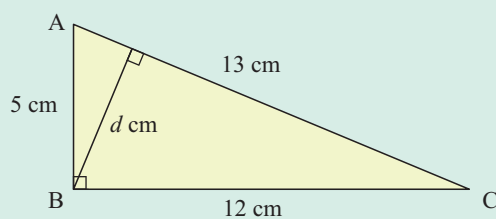
8   By finding the area of triangle ABC in two different ways, show that $d = \frac{60}{13}$.

Review set 9B

1 **a** Convert:

    i 3000 mm² to cm²   **ii** 40 000 cm² to m²   **iii** 600 000 m² to ha

  b How many 80 cm by 40 cm rectangles can be cut from curtain material that is 10 m by 2 m?

2 Calculate the area in hectares of a rectangular field with sides 300 m and 0.2 km.

3 Find the perimeter of:

a

b

c

4 Find the area of:

a

b

c
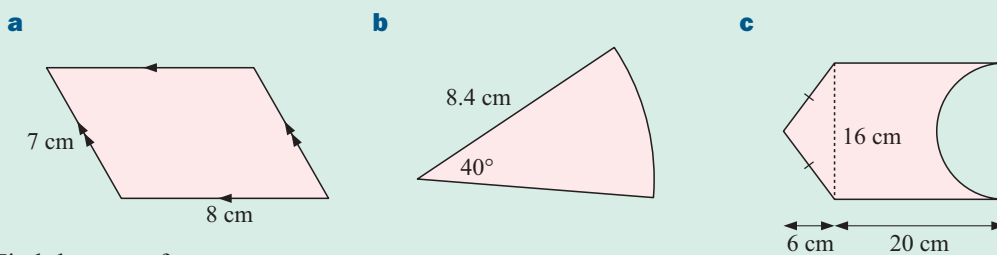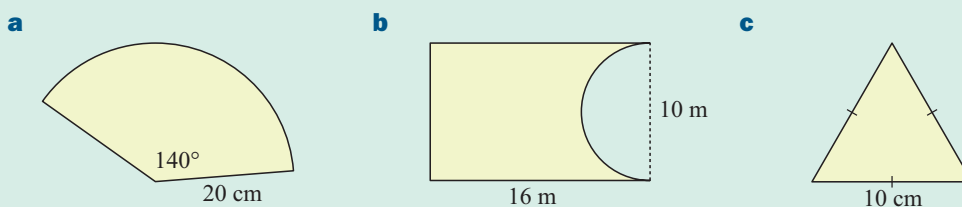
5 A sector of a circle has radius 12 cm and angle $135°$.

  a What fraction of a whole circle is this sector?

  b Find the perimeter of the sector.

  c Find the area of the sector.

6 A circle has circumference of length 40.8 m. Find its:

  a radius   **b** area.

7 Find the formula for the area A of:

a

b

c
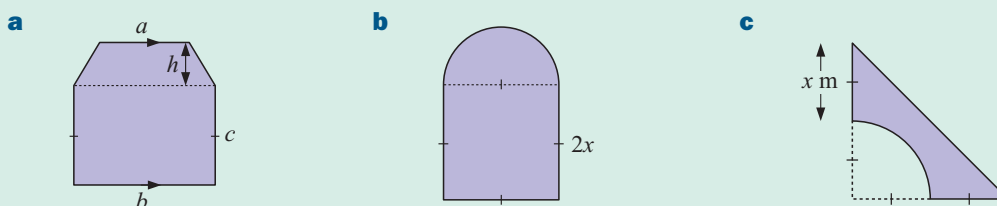
8 Find in terms of π the perimeter and area of the shaded region:

Y:\HAESE\IGCSE01\IG01_09\207IGCSE01_09.CDR Thursday, 2 October 2008 11:56:16 AM PETER
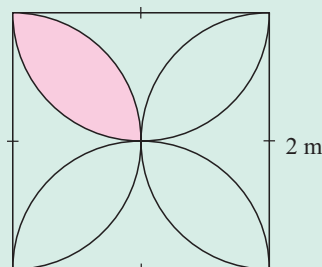
Challenge

1　The rectangle contains 7 semi-circles and 2 quarter circles.
These circle parts are touching.
What percentage of the rectangle is shaded?
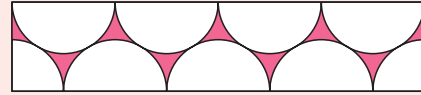
2　A regular hexagon and an equilateral triangle have the same perimeter. What is the ratio of their areas?

3
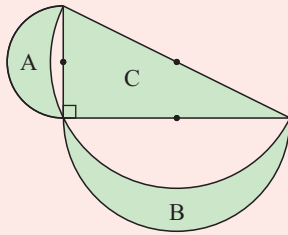
Show that Area A + Area B = Area C.

4　In the figure shown, prove that the area of the annulus (shaded) is $\frac{1}{4}\pi l^2$, where l is the length of a tangent to the inner circle which meets the outer circle at A and B.
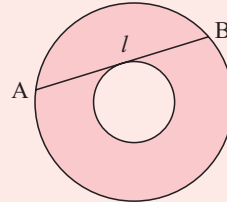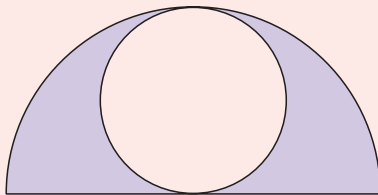
5　Prove that the shaded area of the semi-circle is equal to the area of the inner circle.

6　The figure given represents a rectangular box.   The areas of 3 touching faces are 2 cm^2, 4 cm^2 and 8 cm^2. Determine the volume of the box.

4 cm^2

2 cm^2

8 cm^2
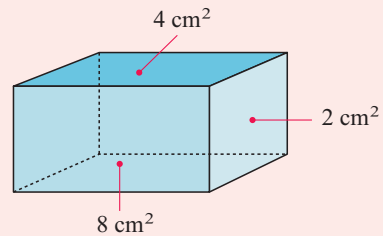
7　Show that the two shaded regions have equal areas.
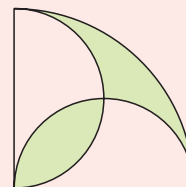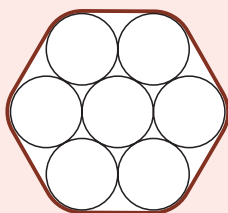
8　A thin plastic band is used to tie the seven plastic pipes together. If the band is 50 cm long, what is the radius of each pipe?

Topics in arithmetic

10

Contents:

Opening problem

80% of the profits from a cake stall will be given to charity.

The ingredients for the cakes cost £100, and the total value of the cakes sold was £275.

- How much profit did the cake stall make?
- How much will be given to charity?

A   PERCENTAGE   [1.8]

You should have seen percentages in previous years, and in particular that:

- $x\%$ means $\dfrac{x}{100}$.

- $x\%$ of a quantity is $\dfrac{x}{100} \times$ the quantity.

EXERCISE 10A

1 Write as a percentage:

    a $\frac{7}{10}$ **b** $\frac{13}{50}$ **c** $\frac{3}{25}$ **d** $\frac{11}{200}$ **e** 2

    f 0.2   **g** 0.05   **h** 1   **i** 0.98   **j** 0.003

2 Write as a fraction in simplest form and also as a decimal:

    a 75%   **b** 7%   **c** 1.5%   **d** 160%   **e** 113%

3 Express as a percentage:

    a 35 marks out of 50 marks   **b** 3 km out of 20 km

    c 8 months out of 2 years   **d** 40 min out of 2.5 hours

4 Tomas reduces his weight from 85.4 kg to 78.7 kg. What was his percentage weight loss?

5 If 17% of Helen's assets amount to €43 197, find:

    a 1% of her assets   **b** the total value of all her assets.

6 Find:

    a 20% of 3 kg   **b** 4.2% of \$26 000   **c** 105% of 80 kg

    d $1\frac{1}{4}$% of 2000 litres   **e** 0.7% of 2670 tonnes   **f** 46.7% of £35 267.20

7 Alex scored 72% for an examination out of 150 marks. How many marks did Alex score?

8 **a** A marathon runner starts a race with a mass of 72.0 kg. Despite continual rehydration he loses 3% of this mass during the race. Calculate his mass at the end of the race.

    b Another runner had a starting mass of 68.3 kg and a finishing mass of 66.9 kg. Calculate her percentage loss in mass.

9 Petra has a monthly income of £4700. She does not have to pay any tax on the first £2400 she earns, but she has to pay 15% of the remainder as tax.

    a How much tax does Petra have to pay?

    b How much does Petra have left after tax?

    c What percentage of the £4700 does Petra actually pay in tax?

10 If 3 students in a class of 24 are absent, what percentage are present?

11 After 2 hours a walker completed 8 km of a journey of 11.5 km. Calculate the percentage of the journey which remains.

12 The side lengths of the rectangle are increased by 20%. What is the percentage increase in the area of the rectangle?

    8 cm

    10 cm
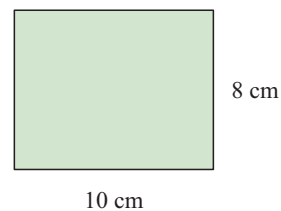
13 A circle's radius is increased by 10%. By what percentage does its area increase?

B   PROFIT AND LOSS   [1.8]

We use money nearly every day, so we need to understand profit, loss and discount.

Profit is an example of an **increase**. Loss and discount are examples of a **decrease**.

> A **profit** occurs if the selling price is *higher* than the cost price.
>
> **Profit = selling price − cost price**
>
> A **loss** occurs if the selling price is *lower* than the cost price.
>
> **Loss = cost price − selling price**

MARK UP AND MARK DOWN

If a purchase price is **marked up** then it is increased, and a *profit* will be made.

If a purchase price is **marked down** then it is decreased, and a *loss* will be made.

Example 1   ◂)) *Self Tutor*

A camera is purchased for €650 and is marked up by 20%.
Find: **a**   the profit   **b**   the selling price.

a   Profit = 20% of cost price   **b**   Selling price
    = 20% of €650   = cost price + profit
    = $\frac{20}{100} \times$ €650   = €650 + €130
    = €130   = €780

Marked up by 20% means *increased* by 20%.

Example 2   ◂)) *Self Tutor*

A pair of board shorts was bought for $35. They were marked down by 20% and sold in an end-of-summer clearance. Find:

a   the loss   **b**   the selling price.

a   Loss   **b**   Selling price
    = 20% of cost price   = cost price − loss
    = 20% of $35   = $35 − $7
    = $\frac{20}{100} \times$ $35   = $28
    = $7

Marked down 20% means *decreased* by 20%.

EXERCISE 10B.1

1   Estelle bakes loaves of bread for her bakery. If a loaf of bread costs €1.40 to make, and sells for €2.90, find her profit on the sale.

2   Greg built a wooden table which he sold for €148. He calculated that the cost of building the table was €176. Find his profit or loss on the sale.

3 Brad bought an old car for £600. He spent £1038 restoring it and sold it for £3500. Find his profit or loss on the sale.

4 Ed bought 180 caps at $10 each to sell at a baseball game. Unfortunately he only sold 128 caps at $15 each. Find his profit or loss on the sale of the caps.

5 Find **i** the profit **ii** the selling price for the following items:

　　a a shirt is purchased for $20 and marked up 10%

　　b a DVD player is purchased for $250 and marked up 80%

　　c a rugby ball is purchased for £50 and sold at a 15% profit

　　d a house is purchased for €255 000 and sold at a 21% profit.

6 Find **i** the loss **ii** the selling price for the following items:

　　a a jumper is purchased for €55 and marked down 30% as it is shop-soiled

　　b a heater is purchased for £175 and marked down 35% as winter is almost over

　　c a microwave is purchased for $105 and is sold at a 25% loss in a stock-clearance

　　d a car is purchased for €9600 and sold at a 14% loss as the car dealer has too many used cars.

7 A contractor buys his materials from a wholesaler and sells them at a 12% mark up. For one particular job the materials cost him $920. What profit does he make on the materials?

8 A pair of shoes has a marked price of $160. However, the store is having a clearance sale, so a 28% discount is offered.

　　a How much discount will be deducted?　　**b** What is the selling price of the shoes?

　　c If a 12.5% sales tax must be paid on the selling price, what will be the final price paid by the customer?
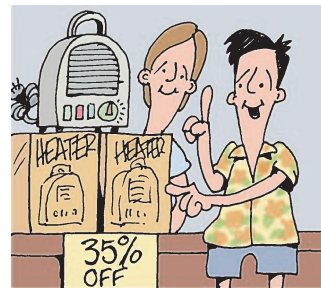
PERCENTAGE PROFIT AND LOSS

Sometimes it is important for a retailer to express profit or loss as a **percentage of the cost price**.

Profit and loss correspond to a percentage increase or decrease in the price respectively.

Example 3 ◀⑨ Self Tutor

A bicycle was bought for $240 and sold for $290.
Find the profit as a percentage of cost price.

Profit= $290 − $240 = $50

$\therefore$　profit as a *percentage* of cost price $= \dfrac{\text{profit}}{\text{cost price}} \times 100\%$

$= \dfrac{50}{240} \times 100\%$

$\approx 20.8\%$

We are really calculating the *percentage increase* in the price!

Example 4

🔊 *Self Tutor*

Monika bought shares in Woolworths at €21.00 per share but was forced to sell them at €18.60 each.
Find: **a**   her loss per share   **b**   the loss per share as a percentage of the cost price.

a   Loss
= cost price − selling price
= €21.00 − €18.60
= €2.40
∴   the loss made was €2.40 per share.

b Loss as a percentage of the cost price

$= \dfrac{\text{loss}}{\text{cost price}} \times 100\%$

$= \dfrac{€2.40}{€21.00} \times 100\%$

$\approx 11.4\%$

EXERCISE 10B.2

1 A tennis racquet bought for €95 was then sold for €132. Find the profit as a percentage of the cost price.

2 A 25 m roll of carpet was bought wholesale for $435. If the whole roll is sold at $32.50 per metre, find:
  a the selling price   **b** the profit
  c the profit as a percentage of the wholesale (cost) price.

3 Athos bought a crate of apricots for $17.60. There were 11 kg of apricots in the crate. He sold the apricots in his shop in 1 kilogram bags for $2.85 each.
  a How much did 1 kg of apricots cost Athos?
  b What was his profit per kilogram?
  c Find his profit as a percentage of his cost price.

4 A furniture store has a clearance sale. If a sofa costing €1450 is marked down to €980, find:
  a the loss made on the sale
  b the loss as a percentage of the cost price.

We are really calculating the *percentage decrease* in the price!

5 Felipe pays £18 200 for a boat, but because of financial difficulties he is soon forced to sell it for $13 600.
  a Find the loss on this sale.   **b** Express this loss as a percentage of the cost price.

6 Sue bought a concert ticket for $55 but was unable to go to the concert. She sold the ticket to a friend for $40, as that was the best price she could get.
  a Find her loss.   **b** Express her loss as a percentage of her cost price.

7 A newly married couple purchased a two-bedroom unit for £126 000. They spent another £14 300 putting in a new kitchen and bathroom. Two years later they had twins and were forced to sell the unit so they could buy a bigger house. Unfortunately, due to a down-turn in the market they received only £107 500 for the sale. What was:
  a the total cost of the unit   **b** the loss on the sale
  c the loss as a percentage of their total costs?

C   *SIMPLE INTEREST*   *[1.8]*

Whenever money is lent, the person lending the money is known as the **lender** and the person receiving the money is known as the **borrower**. The amount borrowed from the lender is called the **principal**.

The lender usually charges a fee called **interest** to the borrower. This fee represents the cost of using the other person's money. The borrower has to repay the principal borrowed plus the interest charged for using that money.

The amount of interest charged on a loan depends on the **principal**, the **time** the amount is borrowed for, and the **interest rate**.

SIMPLE INTEREST

Under the simple interest method, interest is calculated on the initial amount borrowed for the entire period of the loan.

Example 5   ◀)) *Self Tutor*

Calculate the simple interest on a $4000 loan at a rate of 7% per annum over 3 years. Hence find the total amount to be repaid.

$$
\begin{aligned}
\text{The interest payable for 1 year} &= 7\% \text{ of } \$4000 \\
&= 0.07 \times \$4000 \\
\therefore \quad \text{the interest payable over 3 years} &= 0.07 \times \$4000 \times 3 \\
&= \$840 \\
\text{So, the total amount to be repaid} &= \$4000 + \$840 \\
&= \$4840
\end{aligned}
$$

Example 6   ◀)) *Self Tutor*

John wants to earn £2000 in interest on a 4 year loan of £15 000. What rate of simple interest will he need to charge?

$$
\begin{aligned}
\text{The interest needed each year} &= £2000 \div 4 = £500 \\
\therefore \quad \text{the interest rate} &= \frac{£500}{£15\,000} \times 100\% \\
&= \tfrac{10}{3}\% \\
\therefore \quad \text{the rate would need to be } 3\tfrac{1}{3}\% \text{ p.a.}
\end{aligned}
$$

> p.a. reads *per annum* or *per year*.

Example 7   ◀)) *Self Tutor*

How long would it take a €12 000 loan to generate €3000 simple interest if it is charged at a rate of 8.2% p.a.?

The interest each year $\therefore$ the period $= \dfrac{3000}{984}$ years

$= 8.2\%$ of €12 000

$= 0.082 \times$ €12 000 ≈ 3.0488 years

$=$ €984 ≈ 3 years and 18 days.

EXERCISE 10C.1

1 Calculate the simple interest on a:

   a $8500 loan at 6.8% simple interest p.a. for 3 years

   b $17 250 loan at 7.5% simple interest p.a. for 1 year 3 months.

2 Calculate the total amount to be repaid on a loan of:

   a $2250 at 5.7% p.a. simple interest for 5 years

   b €5275 at 7.9% p.a. simple interest for 240 days.

3 Find the rate of simple interest per annum charged on a loan if:

   a $368 is charged on a $4280 loan over 17 months

   b £1152 is charged on an £11 750 loan over $3\frac{1}{2}$ years.

4 How long will it take to earn:

   a €2500 on a loan of €20 000 at 6.7% p.a. simple interest

   b £4000 on a loan of £16 000 at 8.3% p.a. simple interest?

THE SIMPLE INTEREST FORMULA

In **Example 5**, the interest payable on a $4000 loan at 7% p.a. simple interest for 3 years was $4000 $\times$ 0.07 $\times$ 3.

From this observation we construct the **simple interest formula**:

$$I = Prn \quad \text{where} \quad I \text{ is the \textbf{simple interest}}$$
$$P \text{ is the \textbf{principal} or amount borrowed}$$
$$r \text{ is the \textbf{rate of interest per annum} as a \textbf{decimal}}$$
$$n \text{ is the \textbf{time} or \textbf{duration} of the loan in \textbf{years}.}$$

Example 8 ◀》 Self Tutor

Calculate the simple interest on a $6000 loan at a rate of 8% p.a. over 4 years. Hence find the total amount to be repaid.

$P = 6000$ Now $I = Prn$

$r = 8 \div 100 = 0.08$ $\therefore$ $I = 6000 \times 0.08 \times 4$

$n = 4$ $\therefore$ $I = \$1920$

The total amount to be repaid is $\$6000 + \$1920$

                              $= \$7920$

Example 9
◀) *Self Tutor*

If you wanted to earn $5000 in interest on a 4 year loan of $17 000, what rate of simple interest per annum would you need to charge?

$I = 5000$ Now $I = Prn$

$n = 4$ $\therefore$ $5000 = 17\,000 \times r \times 4$

$P = 17\,000$ $\therefore$ $68\,000r = 5000$

$$\therefore \quad r = \frac{5000}{68\,000} \quad \text{\{dividing both sides by } 68\,000\}$$

$$\therefore \quad r \approx 0.0735$$

$\therefore$ you would need to charge a rate of 7.35% p.a. simple interest.

Example 10
◀) *Self Tutor*

How long would it take to earn interest of $4000 on a loan of $15 000 if a rate of 7.5% p.a. simple interest is charged?

$I = 4000$ Now $I = Prn$

$P = 15\,000$ $\therefore$ $4000 = 15\,000 \times 0.075 \times n$

$r = 7.5 \div 100 = 0.075$ $\therefore$ $4000 = 1125n$

$$\therefore \quad n \approx 3.56$$

So, it would take 3 years 7 months to earn the interest.

EXERCISE 10C.2

1 Calculate the simple interest on a loan of:

    a $2000 at a rate of 6% p.a. over 4 years

    b   £9600 at a rate of 7.3% p.a. over a 17 month period

    c $30 000 at a rate of 6.8% p.a. over a 5 year 4 month period

    d   €7500 at a rate of 7.6% p.a. over a 278 day period.

2 Which loan charges less simple interest?

    £25 000 at a rate of 7% p.a. for 4 years   *or*   £25 000 at a rate of 6.75% p.a. for $4\frac{1}{2}$ years

3 What rate of simple interest per annum must be charged if you want to earn:

    a $800 after 5 years on $6000   **b**   €1000 after 20 months on €8800?

4 What rate of simple interest per annum would need to be charged on a loan of £20 000 if you wanted to earn £3000 in interest over 2 years?

5 Monique wants to buy a television costing $1500 in 18 months' time. She has already saved $1300 and deposits this in an account that pays simple interest. What annual rate of interest must the account pay to enable the student to reach her target?

6 How long would it take to earn interest of:

    a $3000 on a loan of $10 000 at a rate of 8% p.a. simple interest

    b ¥82 440 on a loan of ¥229 000 at a rate of 6% p.a. simple interest?

7 If you deposited $8000 in an investment account that paid a rate of 7.25% p.a. simple interest, how long would it take to earn $1600 in interest?

Activity   Simple interest calculator

Click on the icon to obtain a simple interest calculator.

What to do:

Use the software to check the answers to **Examples 5** to **10**.

SIMPLE INTEREST

D   REVERSE PERCENTAGE PROBLEMS   [1.8]

In many situations we are given the final amount after a percentage has been added or subtracted. For example, when Sofia was given a 20% bonus, her total pay was €800. To solve such problems we need to work in reverse. We let a variable represent the original amount, then construct an equation which relates it to the final amount.

Example 11   ◀» Self Tutor

Colleen receives 5% commission on the items she sells. In one week she earns €140 commission. Find the value of the items she sold.

Let the value of the items sold be €x.

We know that 5% of €x is €140, so

$$5\% \times x = 140$$
$$\therefore \quad 0.05 \times x = 140$$
$$\therefore \quad x = 2800$$

So, the value of the items sold was €2800.

Example 12   ◀» Self Tutor

The value of an old book has increased by 35% to $540. What was its original value?

Let the original value be $$x$.

The value has increased by 35% to $540, so we know that 135% of $$x$ is $540.

$$\therefore \quad 135\% \times x = 540$$
$$\therefore \quad 1.35 \times x = 540$$
$$\therefore \quad x = 400$$

So, the original value of the book was $400.

EXERCISE 10D

1   Linda scored 80% in her French test. If she received 60 marks, find the total number of marks possible for the test.

2   45% of the students at a school are boys. If there are 279 boys, how many students are at the school?

3   The legs account for 40% of the total weight of a table. If the legs weigh 16 kg, find the total weight of the table.

4   A walker has travelled 9 km along a trail. If he has completed 80% of the trail, how much further does he still have to go?

5   Heidi pays 22% of her monthly income in tax, and spends 5% on petrol. If Heidi pays £1034 each month in tax, find:

     a   her monthly income   **b**   the amount she spends on petrol each month.

6   The population of a city has increased by 8% over the last 10 years to 151 200. What was the population 10 years ago?

7   On a hot day a metal rod expands by 1.25% to become 1.263 m long. What was its original length?

8   Twenty years ago, Maria invested a sum of money. It has increased by 119% to reach a value of $1095. What was her original investment?

9   A car is sold for €7560 at a loss of 10%. What was the original cost of the car?

E   MULTIPLIERS AND CHAIN PERCENTAGE   [1.8]

In Susan's Casualwear business she buys items at a certain price and has to increase this price by 40% to make a profit and pay tax.

Suppose she buys a pair of slacks for $80. At what price should she mark them for sale?

One method is to find 40% of $80 and add this on to $80,

     40% of $80 $= \frac{40}{100} \times \$80 = \$32$

     So, the marked price would be $80 + $32 = $112.

This method needs **two steps**.

A **one-step** method is to use a **multiplier**.

     Increasing by 40% is the same as multiplying by 100% + 40% or 140%.

     So, $80 × 140% = $80 × 1.4 = $112

Example 13   ◀) *Self Tutor*

What multiplier corresponds to:

   a   a 25% increase   **b**   a 15% decrease?

   a   100% + 25% = 125%   ∴   multiplier is 1.25
   b   100% − 15% = 85%   ∴   multiplier is 0.85

> A **multiplier** is a **one-step** method for increasing or decreasing quantities.

Example 14

🔊 *Self Tutor*

A house is bought for €120 000 and soon after is sold for €156 000. What is the percentage increase on the investment?

Method 1:

multiplier

$$= \frac{\text{new value}}{\text{old value}}$$

$$= \frac{156\,000}{120\,000}$$

$$= 1.30$$

$$= 130\%$$

∴   a 30% increase occurred

Method 2:

percentage increase

$$= \frac{\text{increase}}{\text{original}} \times 100\%$$

$$= \frac{156\,000 - 120\,000}{120\,000} \times 100\%$$

$$= \frac{36\,000}{120\,000} \times 100\%$$

$$= 30\%$$

∴   a 30% increase occurred

> percentage change
> $$= \frac{\text{change}}{\text{original}} \times 100\%$$

EXERCISE 10E.1

1 What multiplier corresponds to a:

    a 10% increase
    b 10% decrease
    c 33% increase
    d 21% decrease
    e 7.2% increase
    f 8.9% decrease?

2 Use a multiplier to calculate the following:

    a increase $80 by 6%
    b increase £68 by 20%
    c increase 50 kg by 14%
    d decrease €27 by 15%
    e decrease £780 by 16%
    f decrease 35 m by 10%

3 **a** Jason was being paid a wage of €25 per hour. His employer agreed to increase his wage by 4%. What is Jason's new wage per hour?

    b At the school athletics day Sadi increased her previous best javelin throw of 29.5 m by 8%. How far did she throw the javelin?

    c The lawn in Oscar's back garden is 8.7 cm high. When Oscar mows the lawn he reduces its height by 70%. How high is the lawn now?

4 Find the percentage change that occurs when:

    a $80 increases to $120
    b £9000 decreases to £7200
    c €95 reduces to €80
    d €90 increases to €118
    e 16 kg increases to 20 kg
    f 8 m reduces to 6.5 m

5 A block of land is bought for €75 000 and sold later for €110 000. Calculate the percentage increase in the investment.

6 A share trader buys a parcel of shares for $4250 and sells them for $3800. Calculate the percentage decrease in the investment.

7 Terry found that after a two week vacation his weight had increased from 85 kg to 91 kg. What percentage increase in weight was this?

8 Frederik left a wet piece of timber, originally 3.80 m long, outside to dry. In the sun it shrank to a length of 3.72 m. What percentage reduction was this?

9 Shelley was originally farming 250 ha of land. However, she now farms 270 ha. What percentage increase is this?

CHAIN PERCENTAGE PROBLEMS

When two or more percentage changes occur in succession, we have a **chain percentage**.

We can use a multiplier more than once within a problem to change the value of a quantity.

Example 15　　　　　　　　　　◆) *Self Tutor*

Increase $3500 by 10% and then decrease the result by 14%.

Increasing by 10% has a multiplier of 110% = 1.1

Decreasing by 14% has a multipler of 86% = 0.86

So, the final amount $= \$3500 \times 1.1 \times 0.86$

$\qquad\qquad\qquad\quad = \$3311$

Example 16　　　　　　　　　　◆) *Self Tutor*

A 1.25 litre soft drink is bought by a deli for $0.80. The deli owner adds 60% mark up then 15% goods tax. What price does the customer pay (to the nearest 5 cents)?

A 60% mark up means we multiply by   160% = 1.6

A 15% goods tax indicates we multiply by a further 115% = 1.15

$\therefore$ cost to customer $= \$0.80 \times 1.6 \times 1.15$

$\qquad\qquad\qquad\qquad = \$1.472$

$\qquad\qquad\qquad\qquad \approx \$1.45$ {to the nearest 5 cents}

Example 17　　　　　　　　　　◆) *Self Tutor*

An investment of ¥600 000 attracts interest rates of 6.8%, 7.1% and 6.9% over 3 successive years. What is it worth at the end of this period?

A 6.8% increase uses a multiplier of 106.8% = 1.068
A 7.1% increase uses a multiplier of 107.1% = 1.071
A 6.9% increase uses a multiplier of 106.9% = 1.069

$\therefore$ the final value $= $ ¥600 000 $\times 1.068 \times 1.071 \times 1.069$

$\qquad\qquad\qquad\qquad = $ ¥733 651

Example 18   ◄⁑ *Self Tutor*

Over a three year period the value of housing increases by 6%, decreases by 5%, and then increases by 8%. What is the overall effect of these changes?

Let $\$x$ be the original value of a house.

$$\therefore \quad \text{value after one year} = \$x \times 1.06 \qquad \{6\% \text{ increase}\}$$
$$\text{value after two years} = \$x \times 1.06 \times 0.95 \qquad \{5\% \text{ decrease}\}$$
$$\text{value after three years} = \$x \times 1.06 \times 0.95 \times 1.08 \qquad \{8\% \text{ increase}\}$$
$$= \$x \times 1.087\,56$$
$$\approx \$x \times 108.76\%$$

So, an 8.76% increase has occurred.

EXERCISE 10E.2

1 a Increase $\$2000$ by 20% and then by 20%.

   b Increase $£3000$ by 10% and then decrease the result by 15%.

   c Decrease $€4000$ by 9% and then decrease the result by 11%.

   d Decrease $\$5000$ by 6% and then increase the result by 10%.

2 True or false?
"If we increase an amount by a certain percentage and then decrease the result by the same percentage, we get back to the original amount."

3 Jarrod buys a wetsuit for $\$55$ to be sold in his shop. He adds 40% for profit and also adds 12% goods tax. What price will he write on the sales tag?

4 Game consoles are bought by an electronics store owner for $£120$. They are marked up by 55% in order for profit to be made. After a few weeks a discount of 10% is given to encourage more sales. A goods tax of 8% is applied at the point of sale. What does the customer now pay?

5 A cutlery set costs a retail shop $€65$. In order to make a profit, a 45% mark up is made. As the item does not sell, two months later the price is reduced by 30%. When it is sold a goods tax of 17.5% is added. What is the price paid by the customer to the nearest euro?

6 A motorcycle today costs $£3750$. The inflation rates over the next four years are predicted to be 3%, 4%, 5% and 5%. If this occurs, what is the expected cost of the motorcycle at the end of this period?

7 If the rate of inflation is expected to remain constant at 3% per year for the next 5 years, what would you expect a $€35\,000$ car to cost in 5 years' time?

8 An investment of $\$30\,000$ is left to accumulate interest over a 4-year period. During the first year the interest paid was 8.7%. In successive years the rates paid were 8.4%, 7.6% and 5.9%. Find the value of the investment after 4 years.

9 Jian invests $\$34\,000$ in a fund which accumulates interest at 8.4% per annum. If the money is left in the fund for a 6-year period, what will be its maturing value?
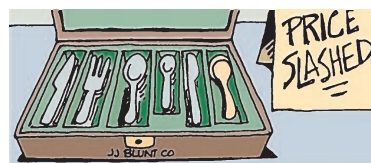
10 What is the overall effect of:

    a increases of 8%, 9% and 12% over three consecutive years

    b decreases of 3%, 8% and 6% over three consecutive years

    c an increase of 5% over four consecutive years?

11 Joshua's wages increase by 3.2%, 4.8% and 7.5% over three consecutive years. What is his overall percentage increase over this period?

12 Jasmin's income increases by 11%, decreases by 7%, increases by 2%, and then increases by 14% over four consecutive years. What is her overall percentage increase for this four year period?

F   COMPOUND GROWTH   [1.8]

If you bank $1000, then you are actually lending the money to the bank. The bank in turn uses your money to lend to other people. While banks pay you interest to encourage your custom, they charge interest to borrowers at a higher rate. That way the banks make a profit.

If you leave the money in the bank for a period of time, the interest is automatically added to your account and so the principal is increased. The next lot of interest will then be calculated on the higher principal. This creates a **compounding** effect on the interest as you are getting **interest on interest**.

Consider an investment of $1000 with interest of 6% p.a. paid each year and compounded.

After year	Interest paid	Value
0		$1000.00
1	6% of $1000.00 = $60.00	$1000.00 + $60.00 = $1060.00
2	6% of $1060.00 = $63.60	$1060.00 + $63.60 = $1123.60
3	6% of $1123.60 = $67.42	$1123.60 + $67.42 = $1191.02

We can use **chain percentage increases** to calculate the account balance after 3 years.

Each year, the account balance is 106% of its previous value.

$$\therefore \quad \text{future value after 3 years} = \$1000 \times 1.06 \times 1.06 \times 1.06$$
$$= \$1000 \times (1.06)^3$$
$$= \$1191.02$$

Example 19   ◀ᴺ *Self Tutor*

$5000 is invested at 8% p.a. compound interest with interest calculated annually.

    a What will it amount to after 3 years?   **b** Find the interest earned.

    a The multiplier is $108\% = 1.08$ **b** Interest earned $= \$6298.56 - \$5000$
      $\therefore \quad$ value after 3 years $= \$5000 \times (1.08)^3$ $= \$1298.56$
                                $= \$6298.56$

Example 20

◀) **Self Tutor**

How much do I have to invest now at 9% p.a. compound interest, if I want it to amount to €10 000 in 8 years' time?

Suppose the amount to be invested now is €x.

$$\therefore \quad x \times (1.09)^8 = 10\,000$$

$$\therefore \quad x = \frac{10\,000}{(1.09)^8} \approx 5018.6628$$

So, I need to invest about €5020 now.

Example 21

◀) **Self Tutor**

An investment of £5500 amounted to £8000 after 4 years of compound growth. What was the annual rate of growth?

If the annual multiplier was x, then $5500 \times x^4 = 8000$

$$\therefore \quad x^4 = \frac{8000}{5500}$$

$$\therefore \quad x = \sqrt[4]{\frac{8000}{5500}} \approx 1.098\,201$$

$$\therefore \quad x \approx 109.82\%$$

So, the annual growth rate was 9.82%.

EXERCISE 10F

1 Find the final value of a compound interest investment of:

   a $2500 after 3 years at 6% p.a. with interest calculated annually

   b £4000 after 4 years at 7% p.a. with interest calculated annually

   c €8250 after 4 years at 8.5% p.a. with interest calculated annually.

2 Find the total interest earned for the following compound interest investments:

   a €750 after 2 years at 6.8% p.a. with interest calculated annually

   b $3350 after 3 years at 7.25% p.a. with interest calculated annually

   c £12 500 after 4 years at 8.1% p.a. with interest calculated annually.

3 Xiao Ming invests 12 000 Yuan into an account which pays 7% p.a. compounded annually. Find:

   a the value of her account after 2 years   **b** the total interest earned after 2 years.

4 Kiri places $5000 in a fixed term investment account which pays 5.6% p.a. compounded annually.

   a How much will she have in her account after 3 years?

   b What interest has she earned over this period?

5 Which investment would earn you more interest on an 8000 peso investment for 5 years:

one which pays 8% p.a. simple interest *or* one which pays 7.5% p.a. compound interest?

6 How much do I need to invest now at a fixed rate of interest if I need it to amount to:

a $2000 in 4 years at 7.2% p.a. compounded

b $20 000 in $3\frac{1}{2}$ years at 6.8% p.a. compounded?

7 Calculate the rate at which compound interest is paid if:

a $1000 becomes $1973.82 after 6 years   **b** €5000 becomes €12 424 after 17 years.

8 Molly invests $6000 at 5% p.a. fixed *simple* interest. Max invests $6000 at 4.5% p.a. fixed *compound* interest.

a Which is the better investment for 4 years and by how much?

b Which investment is better after 30 years and by how much?

9 The value of a car halves in 3 years. Find its annual rate of *depreciation* or loss in value.

10 After 4 years a tractor purchased for €58 500 has a resale value of €35 080. Find its annual rate of depreciation.

G   SPEED, DISTANCE AND TIME   [1.13]

We are all familiar with the concept of speed, or how fast something is moving.

> The **average speed** s is calculated by dividing the total distance travelled d by the total time taken t. It is the distance travelled *per unit of time*. $s = \dfrac{d}{t}$

For example, if I cycle 60 km in 3 hours then $d = 60$ km and $t = 3$ hours, and my average speed $s = \dfrac{60 \text{ km}}{3 \text{ h}} = 20$ km/h.

Notice that $s = \dfrac{d}{t}$ can be rearranged to $t = \dfrac{d}{s}$ and $d = st$.

The following triangle may help you with these rearrangements:

Cover the variable you are trying to find to see how it can be expressed in terms of the other two variables.
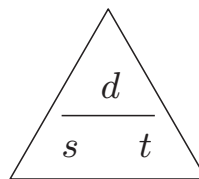
Example 22   ◀) *Self Tutor*

Clark runs a 42.2 km marathon in 3 hours 5 mins and 17 s.
Find his average speed in km/h.

3 hours 5 min 17 s $= 3 + \frac{5}{60} + \frac{17}{3600}$ hours

$\qquad\qquad\qquad\qquad = 3.088\,055\,......$ hours

$\therefore \quad s = \dfrac{d}{t} = \dfrac{42.2 \text{ km}}{3.088\,055\,...... \text{ h}} \approx 13.7$ km/h

Clark's average speed is about 13.7 km/h.

Example 23
◀)) *Self Tutor*

Biathlete Jo cycles 60 km at a speed of 30 km/h, and then runs another 15 km at a speed of 10 km/h. Find:

  a   the total time   **b**   the average speed for Jo's training session.

a   1st leg: $t = \dfrac{d}{s}$ 2nd leg: $t = \dfrac{d}{s}$

$$= \frac{60 \text{ km}}{30 \text{ km/h}} \qquad\qquad\qquad = \frac{15 \text{ km}}{10 \text{ km/h}}$$

$$= 2 \text{ hours} \qquad\qquad\qquad\quad = 1.5 \text{ hours}$$

$\therefore$ total time $= 2 \text{ hours} + 1.5 \text{ hours} = 3.5 \text{ hours}$

b   Total distance travelled $= 60 \text{ km} + 15 \text{ km} = 75 \text{ km}$

$\therefore$ average speed $s = \dfrac{d}{t}$

$$= \frac{75 \text{ km}}{3.5 \text{ h}}$$

$$\approx 21.4 \text{ km/h}$$

EXERCISE 10G

1   An Olympic sprinter runs 100 m in 10.05 seconds. Calculate his average speed in:

     a   m/s   **b**   km/h

2   A car travels for 2 h 20 min at an average speed of 65 km/h. Calculate the distance travelled.

3   How long does it take, in seconds, for a 160 m long train to enter a tunnel when the train is travelling at 170 km/h?

4   Jane can run 11.4 km in 49 min 37 sec. Calculate her average speed in km/h.

5   Find the time taken, to the nearest second, to:

     a   drive 280 km at an average speed of 95 km/h

     b   run 10 000 m at an average speed of 11 km/h.

6   Find the distance travelled when:

     a   walking at an average speed of 3.5 km/h for 2 h 15 min

     b   flying at an average speed of 840 km/h for 6 h 35 min.

7   A student walks for 2 hours at 3.5 km/h and then for 1 hour at 2 km/h. Find:

     a   the total distance walked   **b**   the average speed.

8   In a 42.2 km marathon, Kuan runs the first 35 km at an average speed of 11 km/h. He walks the next 6 km at 4.5 km/h, and staggers the final 1.2 km at 1.5 km/h. Find Kuan's average speed for the marathon.

9   A family drives 775 km for their holiday. The first part of the trip is 52 km and takes 1 h 10 min. The remainder is covered at an average speed of 100 km/h. Find the average speed for the whole journey.

10   Convert: **a**   15 m/s into km/h   **b**   120 km/h into m/s.

11 Two cyclists are travelling side by side along a long straight cycle track, one at 30 km/h and the other at 31 km/h. The faster cyclist wishes to overtake the slower one, and will need to gain 8 m relative to the slower cyclist in order to do this safely. Find:

   a the time taken for the faster cyclist to overtake the slower cyclist

   b the distance travelled by the faster cyclist in this time.
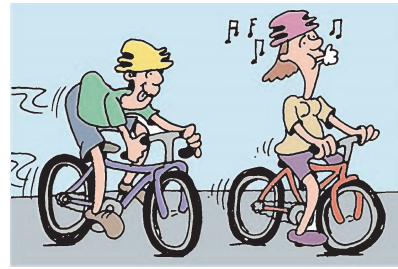
H   *TRAVEL GRAPHS*   *[1.13]*

Suppose a car travels 150 kilometres in 1.5 hours.

$$\text{Average speed} = \frac{\text{distance}}{\text{time}}$$
$$= \frac{150}{1.5}$$
$$= 100 \text{ km/h}$$

Notice that the point A on the graph indicates we have travelled 100 km in one hour.
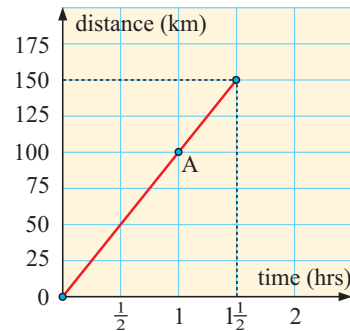
Example 24   ◆) *Self Tutor*

The graph alongside indicates the distance a homing pigeon travelled from its point of release until it reached its home. Use the graph to determine:

   a the total length of the flight

   b the time taken for the pigeon to reach home

   c the time taken to fly the first 200 km

   d the time taken to fly from the 240 km mark to the 400 km mark

   e the average speed for the first 4 hours

---
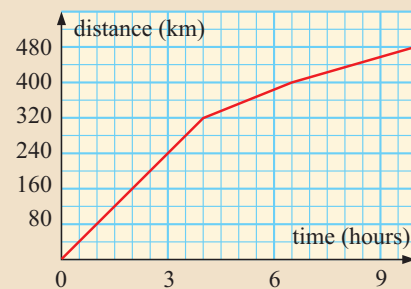
   a Length of flight is 480 km.

   b Time to reach home is 10 hours.

   c Time for first 200 km is $2\frac{1}{2}$ hours.

   d It takes 3 hours to fly 240 km. It takes $6\frac{1}{2}$ hours to fly 400 km.

      ∴ it takes $3\frac{1}{2}$ hours to fly from 240 km to 400 km.

   e In the first 4 hours it flies 320 km   ∴ average speed $= \dfrac{320}{4} = 80$ km/h.

EXERCISE 10H

1 The graph alongside shows the distance Frances walks to work. Use the graph to determine:

    a the distance to work

    b the time taken to get to work

    c the distance walked after

        i 12 minutes   **ii** 20 minutes

    d the time taken to walk

        i 0.4 km   **ii** 1.3 km

    e the average speed for the whole distance.
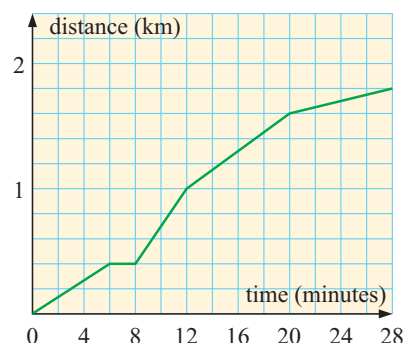
2 Two cyclists took part in a handicap time trial. The distance-time graph indicates how far each has travelled. Use the graph to find:

    a the handicap time given to cyclist B

    b the distance travelled by each cyclist

    c how far both cyclists had travelled when A caught B

    d how long it took each cyclist to travel 80 km

    e how much faster A completed the time trial than B

    f the average speed of each cyclist.
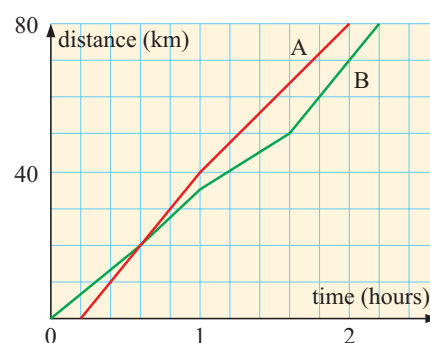
3 The Reynolds and Smith families live next door to each other in San Francisco. They are taking a vacation to their favourite beach, 150 km from where they live.

    a Who left first?

    b Who arrived first?

    c Who travelled fastest?

    d How long after the first family left did they pass each other on the road?

    e How long had the second family been driving when they passed the first family?

    f Approximately how far from San Francisco is this "passing point"?
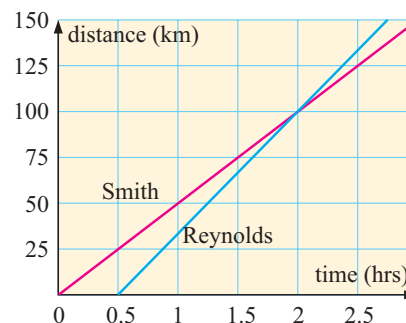
4 Patricia drives from home to pick up her children from school. She draws a graph which can be used to explain her journey. The vertical axis shows her distance from home in kilometres. The horizontal axis measures the time from when she left home in minutes. Her first stop is at traffic lights.

    a When did she stop for the red traffic light?

    b How long did the light take to change?

    c How long did she spend at the school?

    d How far away is the school from her home?

    e When was her rate of travel (speed) greatest?
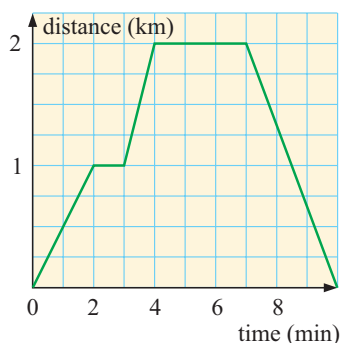
Review set 10A

1 What multiplier corresponds to:

 a a 13% decrease   **b** a 10.9% increase?

2 **a** Increase $2500 by 16%.   **b** Decrease 65 kg by 10%.

3 In the long jump, Tran jumps 5.65 m and Lim beats this distance by 7%. How far did Lim jump?

4 Eito purchases a pair of shoes for ¥6100 and marks them up 45% for sale. What is:

 a the selling price   **b** the profit?

5 Moira bought a car for £4500 but had to sell it for £4000 a few weeks later. What was her:

 a loss   **b** percentage loss?

6 A store has an item for $80 and discounts it by 15%. Find:

 a the discount   **b** the sale price.

7 David purchased a stamp collection for €860. Two years later it was valued at €2410. Calculate the percentage increase in the value of the investment.

8 A publisher sells a book for $20 per copy to a retailer. The retailer marks up the price by 75% and then adds 10% for a goods tax. What price does the customer pay?

9 The annual rate of inflation is predicted to be 3% next year, then 3.5% in the year after that. What will be the cost in two years' time of an item that currently costs $50 if the cost rises in line with inflation?

10 How much is borrowed if a simple interest rate of 8% p.a. results in an interest charge of $3600 after 3 years?

11 A person wants to earn £3000 interest on an investment of £17 000 over 3 years. What is the minimum simple interest rate that will achieve this target?

12 How long would it take to earn €5000 interest on an investment of €22 500 at a rate of 9.5% p.a. simple interest?

13 Calculate the simple interest on a $6000 loan at a rate of 8.5% p.a. over 3 years.

14 Find the final value of a compound interest investment of $20 000 after 3 years at 7.5% p.a. with interest calculated annually.

15 Which of the following would earn more interest on a $7500 investment for 4 years:

 • 9% p.a. simple interest calculated annually   *or*
 • 8% p.a. compounded interest calculated annually?

16 Find the time taken to drive 305 km at an average speed of 70 km/h.

17 A motorist drives for 30 minutes at 90 km/h, and then for 1 hour at 60 km/h. Find:

 a the total distance travelled   **b** the average speed for the whole trip.

Review set 10B

1 What multiplier corresponds to:
   a a 10% increase   **b** an 11.7% decrease?

2 **a** Increase £3625 by 8%.   **b** Decrease 387 km by 1.8%.

3 Adam bought a bicycle for €165 and sold it soon after for €130. What was Adam's:
   a loss   **b** percentage loss?

4 A furniture store bought a chair for €380, marked it up by 35% and then discounted it by 15%.
   What was:
   a the marked-up price   **b** the discounted price?

5 A company cut its advertising budget by 12%. If the company previously spent $80 000 on advertising, what was the new advertising budget?

6 A toaster is sold to a retailer for €38. The retailer marks it up by 40%, discounts it by 15%, and then sells it to a customer after adding on a goods tax of 16%. What did the customer pay?

7 For the next three years the annual inflation rate is predicted to be 3.2%, 4.1% and 4.8%. If this occurs, what should be the value of a house currently at $325 000?

8 What simple interest is earned on an investment of $6500 for 4 years at 6.8% p.a.?

9 How much is borrowed if a simple interest rate of 7.2% p.a. results in an interest charge of $216 after $2\frac{1}{2}$ years?

10 How long would it take for a loan of €45 000 to earn €15 120 interest at a rate of 8% p.a. simple interest?

11 An investment of $25 000 is made for 4 years at 8.2% p.a. compounded yearly. Find:
   a the final value of the investment   **b** the interest earned.

12 €8000 is invested for 10 years at 8% p.a. compound interest. Find:
   a the final value of the investment   **b** the amount of interest earned
   c the simple interest rate needed to be paid for the same return on the investment.

13 Find the distance travelled by flying at an average speed of 780 km/h for 1 hour 40 minutes.

14 The graph shows the distance travelled by two families between New York and Washington DC. Use the graph to find:
   a the distance from New York to Washington DC
   b how much quicker the Maple family completed the trip than the Johnson family
   c the average speed for each family over the first two hours
   d the average speed for the Johnsons over the whole trip.

15 How much must be invested now at 9.3% compound interest p.a. if it is to amount to £6000 in 10 years' time?
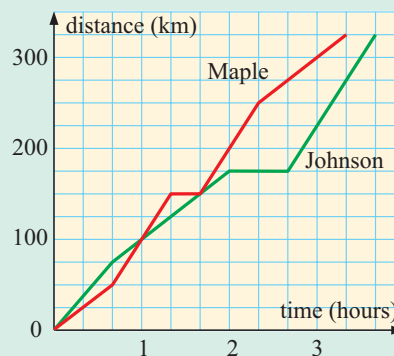
16   After 5 years a house costing $175 000 was sold for $240 000. What was the annual rate of compound growth?

17   Jimmy is driving 200 km to his holiday destination. He drives the first 100 km at a speed of 60 km per hour, and the next 100 km at a speed of 100 km per hour. Find his average speed for the journey.

Challenge

1   In numbering the pages of a book, 408 digits were used. How many pages has the book?

2   A job can be completed in 8 hours by 5 men working equal amounts. How long would 3 men take to do the job working at a $33\frac{1}{3}\%$ more effective rate?

3   The average pulse rate of a person is 60 beats per minute. How many times does the average person's heart beat in an average lifetime of 70 years?

4   In an electrical shop, the manager buys a TV set from the distributor. He then marks the set up 60%. When the set did not sell at this price he put it into a "$37\frac{1}{2}\%$ off" sale. Did he make a profit or a loss?

5   A 10% solution of salt in water means "10 grams of salt for every 100 grams of salt water solution". You are given 100 grams of a 10% solution of salt in water and wish to change it to a 30% solution of salt in water. How many grams of salt would you need to add to the solution?

6   Use 1, 2, 3, ... 9 once each to fill the nine spaces so that, when the three numbers

- in any row, or
- in any column

are multiplied together, the result matches the number at the end of the row or column.

			72
			120
			42
70	32	162	

7   Complete the following number square so that every row, column and diagonal adds up to 34.
The missing numbers are 4, 6, 7, 8, 10, 11, 12, 13, 14, 15.

	1		
	5		
2	16	9	
3			

8   The following is a **cross-number** puzzle. All of the answers are numbers. There is only one set of answers which fits all of the clues. Enough clues are given to solve the problem.

A	B	C
D		
E		

Down   A   no clue is necessary   **Across**   B   the sum of the digits of B down
               B   a multiple of 99   D   a prime number
               C   the square of D across   E   the result of (A down) + (B across) + (C down)

Mensuration (solids and containers)

11

Contents:

Opening problem

Chun's roof is leaking. 10 ml of water is dripping onto her floor every minute. She places a 10 cm by 8 cm by 3 cm container under the leak to catch the drops.

How often will Chun need to empty the container?

In this chapter we deal with measurements associated with 3-dimensional objects. These may be solids which have **mass**, **surface area**, and **volume**, or containers which can hold a certain **capacity**.

The shapes we deal with include **prisms** and **pyramids** which have all flat or **plane faces**, and **cylinders**, **cones** and **spheres** which have **curved surfaces**.

A   SURFACE AREA   [6.4]

SOLIDS WITH PLANE FACES

The **surface area** of a three-dimensional figure with plane faces is the sum of the areas of the faces.

To help find the surface area of a solid, it is often helpful to draw a **net**. This is a two-dimensional plan which can be folded to construct the solid.

Software that demonstrates **nets** can be found at *http://www.peda.com/poly/*

Example 1 🔊 *Self Tutor*

Find the total surface area of the rectangular box:
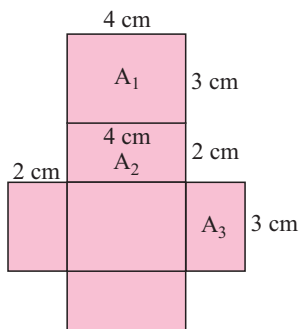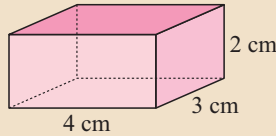
$A_1 = 4 \times 3 = 12$ cm² (bottom and top)

$A_2 = 4 \times 2 = 8$ cm² (front and back)

$A_3 = 2 \times 3 = 6$ cm² (sides)

$\therefore$ total surface area $= 2 \times A_1 + 2 \times A_2 + 2 \times A_3$
$= 2 \times 12 + 2 \times 8 + 2 \times 6$
$= 52$ cm²

So, the total surface area of the box is 52 cm².

Example 2 🔊 *Self Tutor*

What is the total surface area of this wedge?
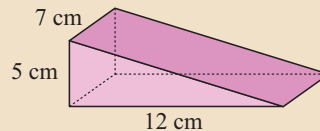
We draw a net of the solid:

We next find h using Pythagoras:

$h^2 = 12^2 + 5^2$
$\therefore \quad h^2 = 169$
$\therefore \quad h = \sqrt{169} = 13$ {as $h > 0$}

Sometimes we need to use Pythagoras' theorem to find a missing length.

Now, $A_1 = \frac{1}{2}bh$
$\quad = \frac{1}{2} \times 12 \times 5$
$\quad = 30$ cm²

$A_2 = 7 \times 5$
$\quad = 35$ cm²

$A_3 = 12 \times 7$
$\quad = 84$ cm²

$A_4 = 13 \times 7$
$\quad = 91$ cm²

$\therefore$ total surface area $= 2 \times A_1 + A_2 + A_3 + A_4$
$= 2 \times 30 + 35 + 84 + 91$
$= 270$ cm²

Example 3   ◀ッ *Self Tutor*

Find the surface area of the
square-based pyramid:
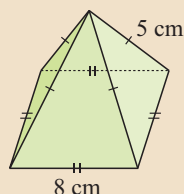
The figure has:

- 1 square base
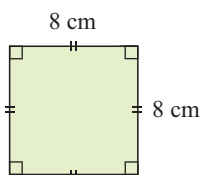
- 4 triangular faces
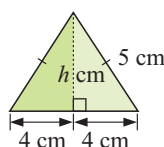
$h^2 + 4^2 = 5^2$ {Pythagoras}

$\therefore \quad h^2 + 16 = 25$

$\therefore \quad h^2 = 9$

$\therefore \quad h = 3$ {as $h > 0$}

Total surface area $= 8 \times 8 \; + \; 4 \times (\frac{1}{2} \times 8 \times 3)$

$= 64 + 48$

$= 112 \text{ cm}^2$

Example 4   ◀ッ *Self Tutor*

Find the cost of erecting a 6 m by 4 m rectangular garden shed that is 2 m high
if the metal sheeting costs \$15 per square metre.

The shed: Net:
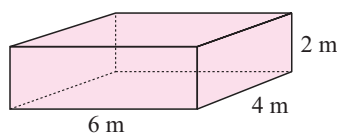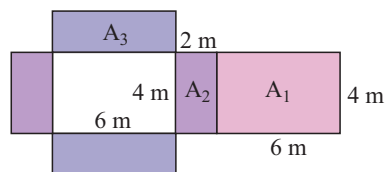
$A_1 = 6 \times 4$ $A_2 = 4 \times 2$ $A_3 = 6 \times 2$

$= 24 \text{ m}^2$ $= 8 \text{ m}^2$ $= 12 \text{ m}^2$

$\therefore$ total surface area $= A_1 + 2 \times A_2 + 2 \times A_3$

$= 24 + 2 \times 8 + 2 \times 12$

$= 64 \text{ m}^2$

$\therefore$ cost $= 64 \times \$15$

$= \$960$

EXERCISE 11A.1

1 Find the surface area of a cube with sides:

    a 3 cm   **b** 4.5 cm   **c** 9.8 mm

2 Find the surface area of the following rectangular prisms:
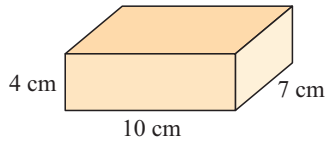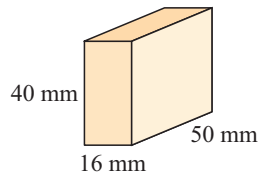
    a   **b**   **c**
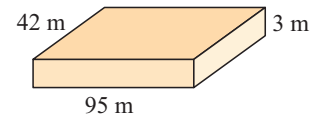
3 Find the surface area of the following triangular prisms:
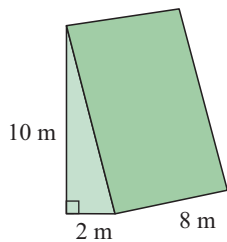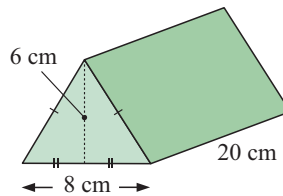
    a   **b**   **c**
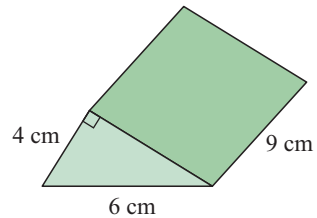
4 Find the surface area of the following square-based pyramids:
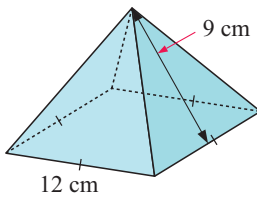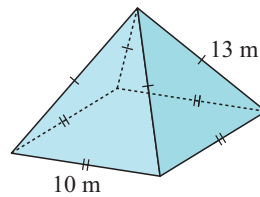
    a   **b**   **c**
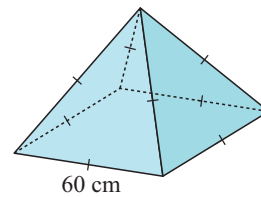
5 Find the surface area of the following prisms:
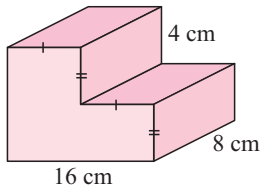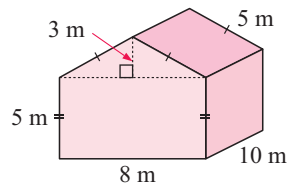
    a   **b**   **c**
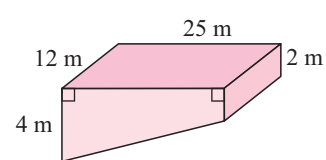
6 A metal pencil box is 20 cm by 15 cm by 8 cm high. Find the total area of metal used to make the pencil box.

7
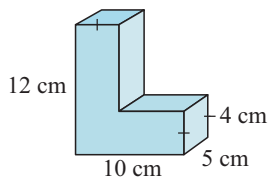
Tracy owns 8 wooden bookends like the one illustrated.

    a Calculate the total surface area of the bookends.

    b If 50 ml of varnish covers an area of 2000 cm^2, how much varnish is needed to coat all 8 bookends?

8 Calculate the area of material needed to make this tent. Do not forget the floor.
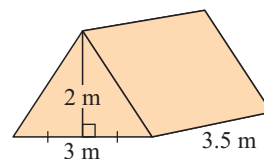
9
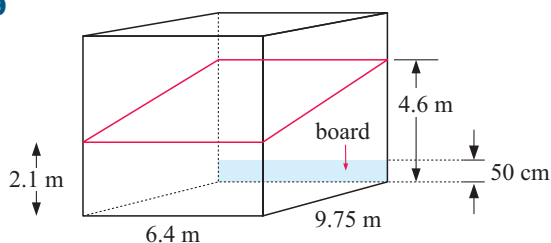
A squash court has the dimensions given. Each shot may strike the floor, or the walls below the red line and excluding the board on the front wall. Calculate the total surface playing area of the walls and the floor. Give your answer correct to 4 significant figures.

4.6 m

board

2.1 m

50 cm

9.75 m

6.4 m

SOLIDS WITH CURVED SURFACES

We will consider the outer surface area of three types of object with curved surfaces. These are **cylinders**, **cones** and **spheres**.

h

r

Cylinders

Consider the cylinder shown alongside. If the cylinder is cut, opened out and flattened onto a plane, it takes the shape of a rectangle.

hollow $2\pi r$ $2\pi r$

h h h

cut here   open out   flattened surface area
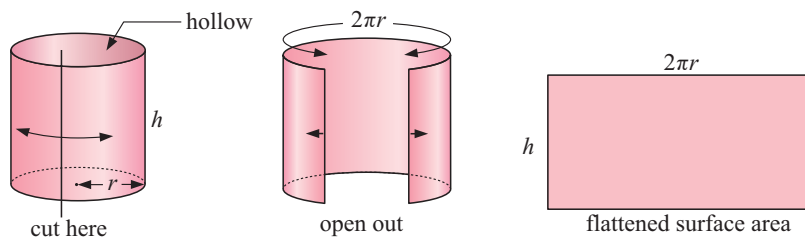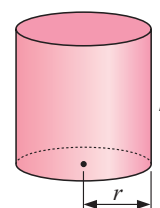
You can verify that the curved surface produces a rectangle by peeling the label off a cylindrical can and noticing the shape when the label is flattened. The length of the rectangle is the same as the circumference of the cylinder.

So, for a hollow cylinder,   the outer surface area A = area of rectangle

$\therefore \quad A = \text{length} \times \text{width}$

$\therefore \quad A = 2\pi r \times h$

$\therefore \quad A = 2\pi rh$

Object	Figure	Outer surface area
Hollow cylinder	hollow, h, r, hollow	$A = 2\pi rh$ (no ends)
Hollow can	hollow, h, r, solid	$A = 2\pi rh + \pi r^2$ (one end)
Solid cylinder	solid, h, r, solid	$A = 2\pi rh + 2\pi r^2$ (two ends)

Cones

The curved surface of a cone is made from a sector of a circle with radius equal to the slant height of the cone. The circumference of the base equals the arc length of the sector.
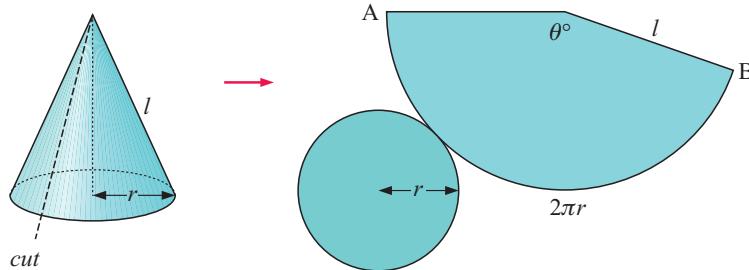
$$\text{arc AB} = \left(\frac{\theta}{360}\right) 2\pi l$$

But $\text{arc AB} = 2\pi r$

$$\therefore \quad \left(\frac{\theta}{360}\right) 2\pi l = 2\pi r$$

$$\therefore \quad \frac{\theta}{360} = \frac{r}{l}$$

The area of curved surface $=$ area of sector
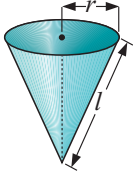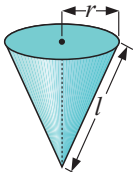
$$= \left(\frac{\theta}{360}\right) \pi l^2$$

$$= \left(\frac{r}{l}\right) \pi l^2$$

$$= \pi r l$$

The area of the base $= \pi r^2$

$$\therefore \quad \text{the total area} = \pi r l + \pi r^2$$

Object	Figure	Outer surface area
Hollow cone		$A = \pi r l$ (no base)
Solid cone		$A = \pi r l + \pi r^2$ (solid)

Spheres
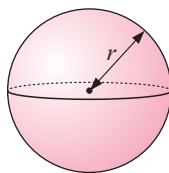
Surface area $A = 4\pi r^2$

The mathematics required to prove this formula is beyond the scope of this course.

Example 5
◀ঠ) *Self Tutor*

Find the surface areas of the following solids:

a solid cylinder

15 cm

6 cm

b

8 cm
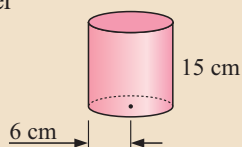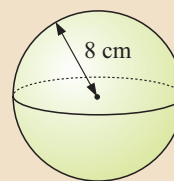
a Surface area	**b** Surface area
$= 2\pi r^2 + 2\pi r h$	$= 4\pi r^2$
$= 2 \times \pi \times 6^2 + 2 \times \pi \times 6 \times 15$	$= 4 \times \pi \times 8^2 \text{ cm}^2$
$= 252\pi$	$= 256\pi \text{ cm}^2$
$\approx 792 \text{ cm}^2$	$\approx 804 \text{ cm}^2$

Example 6
◀ঠ) *Self Tutor*

Find the surface area of a solid cone of base radius 5 cm and height 12 cm.

Let the slant height be l cm.

l cm

12 cm

5 cm
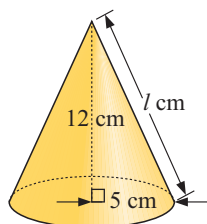
$$l^2 = 5^2 + 12^2 \qquad \{\text{Pythagoras}\}$$
$$\therefore \quad l^2 = 169$$
$$\therefore \quad l = \sqrt{169} = 13 \qquad \{\text{as } l > 0\}$$

Now $A = \pi r^2 + \pi r l$
$$\therefore \quad A = \pi \times 5^2 + \pi \times 5 \times 13$$
$$\therefore \quad A = 90\pi$$
$$\therefore \quad A \approx 283$$

Thus the surface area is approximately 283 cm^2.

EXERCISE 11A.2

1 Find the outer surface area of the following:

a
solid

8 cm

3 cm

b
can (no top)

6 cm

12 cm

c
solid

10 cm

5 cm

d
tank (no top)

3.8 m

3.8 m

e
solid

2.2 m

6.8 m

f
hollow throughout

5.5 cm

2.3 cm
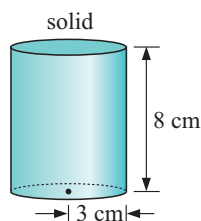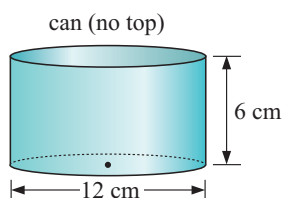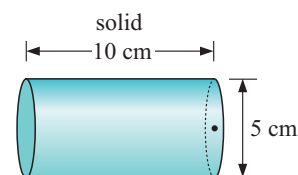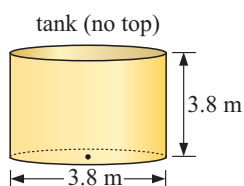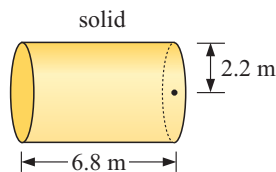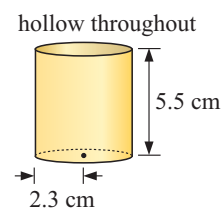
2   Find the total surface area of the following cones, giving your answers in terms of π:

a
solid

12 cm

4 cm

b
hollow
(no base)

8 m

6 m

c
solid

9 cm
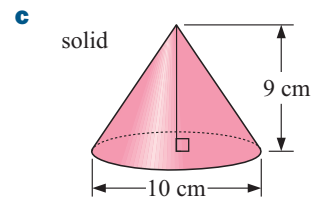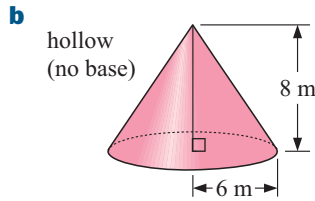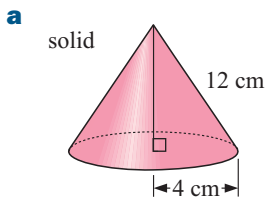
10 cm

3   Find the total surface area of the following:

a

20 cm

b

6.8 km

c

3 cm
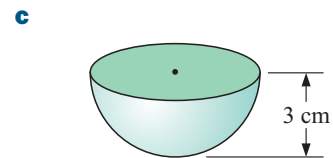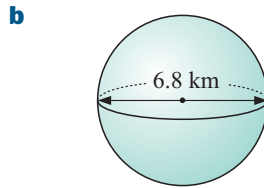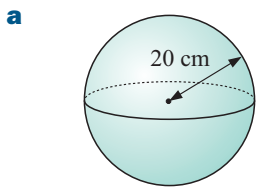
4   Find the total surface area of:

   a   a cylinder of base radius 9 cm and height 20 cm

   b   a cone of base radius and perpendicular height both 10 cm

   c   a sphere of radius 6 cm

   d   a hemisphere of base radius 10 m

   e   a cone of base radius 8 cm and vertical angle 60°.

60°

8 cm
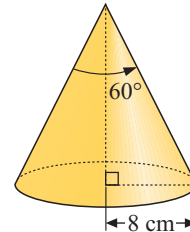
5   A ball bearing has a radius of 1.2 cm. Find the surface area of the ball bearing.

6   Find the area of metal required to make the can illustrated alongside.
Include the top and bottom in your answer.

8 cm

6 cm
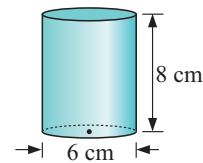
7   How many spheres of 15 cm diameter can be covered by 10 m^2 of material?

8   A conical piece of filter paper has a base radius of 2 cm, and
is 5 cm high.
Find the surface area of the filter paper, correct to 3 significant
figures.

5 cm

2 cm
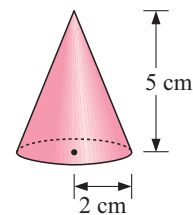
9   Find: **a**   the radius of a sphere of surface area 400 m^2

   b   the height of a solid cylinder of radius 10 cm and surface area 2000 cm^2

   c   the slant height of a solid cone of base radius 8 m and surface area 850 m^2.

10   Find a formula for the surface area A of the following solids:

a

x

x

$4x$

b

$x+3$

x

x

c

x

$2x$

$x+2$

d

e

$2x + 1$

$2x$

f

$3x$

x

$5x$

$2x$

$4x$
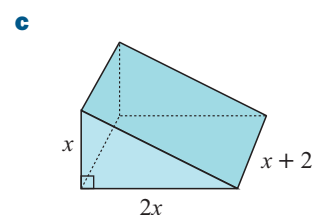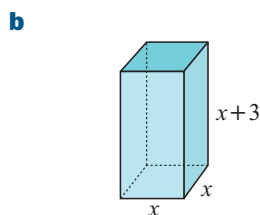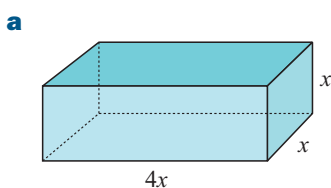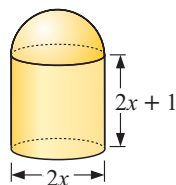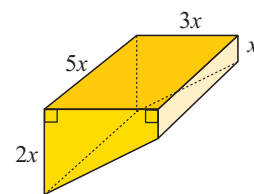
11 Find a formula for the total surface area of:

a

c

a

b

b

b

a

c
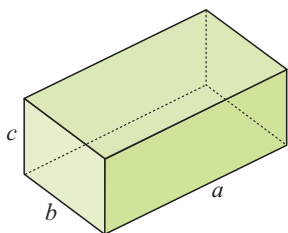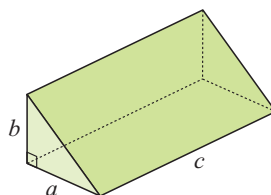
Challenge:

12 The figure shown is half a cone, formed by cutting a cone from its apex directly down to the centre of its base. Write a formula for the total surface area A of the object in terms of r and h.

h

r
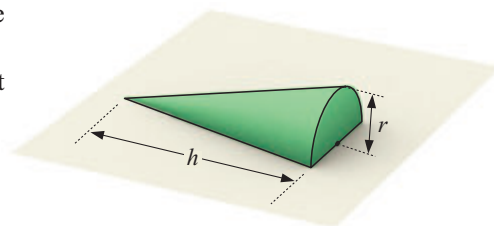
B   VOLUME   [6.1, 6.4]

> The **volume** of a solid is the amount of space it occupies. It is measured in *cubic* units.

UNITS OF VOLUME

Volume can be measured in cubic millimetres, cubic centimetres or cubic metres.

Since 1 cm = 10 mm, we can see that:

$$1 \text{ cm}^3$$
$$= 10 \text{ mm} \times 10 \text{ mm} \times 10 \text{ mm}$$
$$= 1000 \text{ mm}^3$$

1 cm   1 cm   1 cm

10 mm   10 mm   10 mm
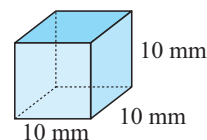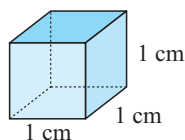
Likewise, since 1 m = 100 cm, we can see that:

$$1 \text{ m}^3$$
$$= 100 \text{ cm} \times 100 \text{ cm} \times 100 \text{ cm}$$
$$= 1\,000\,000 \text{ cm}^3$$

1 m   1 m   1 m

100 cm   100 cm   100 cm
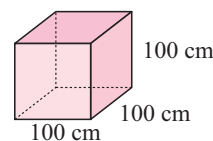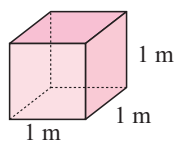
VOLUME UNITS CONVERSIONS

$\times 1\,000\,000$ $\times 1000$

$$\boxed{m^3} \qquad \boxed{cm^3} \qquad \boxed{mm^3}$$
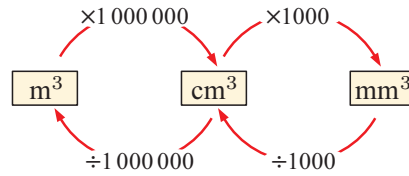
$\div 1\,000\,000$ $\div 1000$

Example 7   ◀)) Self Tutor

Convert the following: **a** 5 m^3 to cm^3 **b** $25\,000 \text{ mm}^3$ to cm^3

a 5 m^3
$= (5 \times 100^3) \text{ cm}^3$
$= (5 \times 1\,000\,000) \text{ cm}^3$
$= 5\,000\,000 \text{ cm}^3$

b $25\,000 \text{ mm}^3$
$= (25\,000 \div 10^3) \text{ cm}^3$
$= (25\,000 \div 1000) \text{ cm}^3$
$= 25 \text{ cm}^3$

EXERCISE 11B.1

1 Convert the following:

    a 8.65 cm^3 to mm^3 **b** $86\,000 \text{ mm}^3$ to cm^3 **c** $300\,000 \text{ cm}^3$ to m^3

    d 124 cm^3 to mm^3 **e** 300 mm^3 to cm^3 **f** 3.7 m^3 to cm^3

2 1.85 cm^3 of copper is required to make one twenty-cent coin. How many twenty-cent coins can be made from a cubic metre of copper?

3 A toy store has 400 packets of marbles on display, and each packet contains 80 marbles. There are 850 mm^3 of glass in each marble. Find the total volume of glass in the marbles on display, giving your answer in cm^3.

VOLUME FORMULAE

Rectangular prism or cuboid
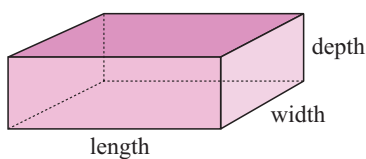
depth
width
length

Volume = length $\times$ width $\times$ depth

Solids of uniform cross-section

Notice in the triangular prism alongside, that vertical slices parallel to the front triangular face will all be the same size and shape as that face. We say that solids like this are solids of *uniform cross-section*. The cross-section in this case is a triangle.

solid   cross-section

Another example is the hexagonal prism shown opposite.
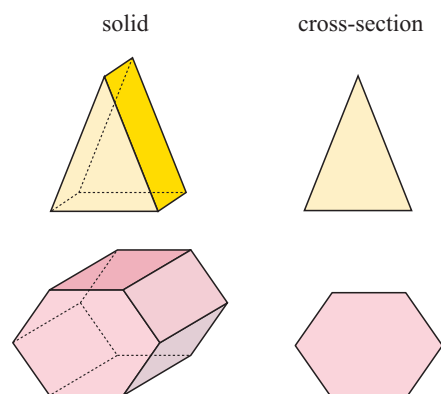
For any solid of uniform cross-section:

Volume = area of cross-section × length

In particular, for a **cylinder**, the cross-section is a circle and so:
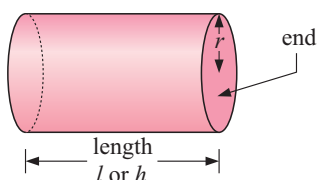
end

$$\text{Volume} = \text{area of circle} \times \text{length}$$
$$= \pi r^2 \times l$$

i.e., $V = \pi r^2 l$ or $V = \pi r^2 h$

length
l or h

PYRAMIDS AND CONES

These **tapered solids** have a flat base and come to a point called the **apex**. They **do not** have identical cross-sections. The cross-sections always have the same shape, but not the same size.

For example,

square-based pyramid   triangular-based pyramid   cone
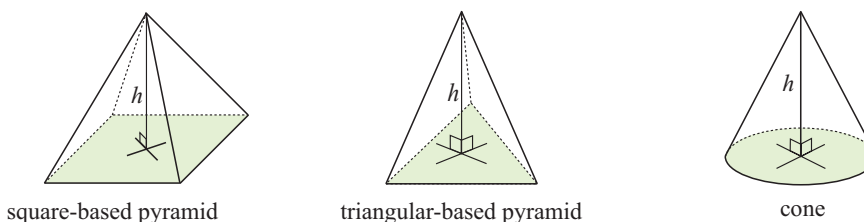
Volume = $\frac{1}{3}$ (area of base × height)

A formal proof of this formula is beyond the scope of this course. It may be demonstrated using water displacement. Compare tapered solids with solids of uniform cross-section with identical bases and the same heights.

For example: • a cone and a cylinder
               • a square-based pyramid and a square-based prism.

SPHERES

The Greek philosopher **Archimedes** was born in Syracuse in 287 BC. Amongst many other important discoveries, he found that the volume of a sphere is equal to two thirds of the volume of the smallest cylinder which encloses it.
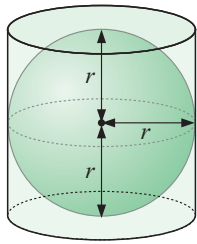
$$\text{Volume of cylinder} = \pi r^2 \times h$$
$$= \pi r^2 \times 2r$$
$$= 2\pi r^3$$
$$\therefore \quad \text{volume of sphere} = \tfrac{2}{3} \times \text{volume of cylinder}$$
$$= \tfrac{2}{3} \times 2\pi r^3$$
$$= \tfrac{4}{3}\pi r^3$$

Thus $\quad \boxed{V = \tfrac{4}{3}\pi r^3}$

Archimedes' tomb was marked by a sphere inscribed in a cylinder.

SUMMARY

Object	Figure	Formula
Solids of uniform cross-section	height ... end ... height ... end	**Volume of uniform solid = area of end × height**
Pyramids and cones	height ... base ... height ... h ... base	**Volume of a pyramid or cone = $\tfrac{1}{3}$(area of base × height)**
Spheres	r	**Volume of a sphere $= \tfrac{4}{3}\pi r^3$**

Example 8
◀) *Self Tutor*

Find, correct to 3 significant figures, the volume of the following solids:

a

4.5 cm

7.5 cm   6 cm

b

10 cm

←10 cm→
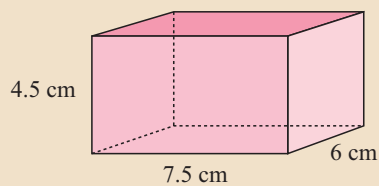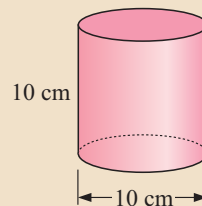
a   Volume

$= \text{length} \times \text{width} \times \text{depth}$

$= 7.5 \text{ cm} \times 6 \text{ cm} \times 4.5 \text{ cm}$

$\approx 203 \text{ cm}^3$

b   Volume

$= \text{area of cross-section} \times \text{height}$

$= \pi r^2 \times h$

$= \pi \times 5^2 \times 10$

$\approx 785 \text{ cm}^3$

Example 9 🔊 *Self Tutor*

Find the volumes
of these solids:

a

12 cm

10 cm

b

10 cm

◄6 cm►
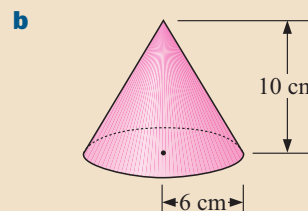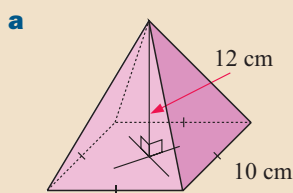
a   Volume

$= \frac{1}{3} \times \text{area of base} \times \text{height}$

$= \frac{1}{3} \times 10 \times 10 \times 12$

$= 400 \text{ cm}^3$

b   Volume

$= \frac{1}{3} \times \text{area of base} \times \text{height}$

$= \frac{1}{3} \times \pi \times 6^2 \times 10$

$\approx 377 \text{ cm}^3$

Example 10 🔊 *Self Tutor*

Find the volume of the sphere in
cubic centimetres, to the nearest
whole number:
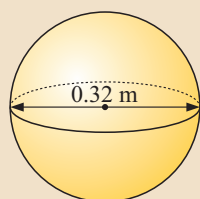
0.32 m

First, convert 0.32 m to cm.

$0.32 \text{ m} = 32 \text{ cm}$

$V = \frac{4}{3}\pi r^3$

$\therefore \quad V = \frac{4}{3}\pi \times 16^3$

$\therefore \quad V \approx 17\,157 \text{ cm}^3$

Change the units
to centimetres
before calculating
the volume.

EXERCISE 11B.2

1   Find the volume of the following:

a

5 m

7 m

11 m

b

12 cm

6 cm

c

5 cm

8 cm

16 cm

d

area 15 cm²

3 cm

e

10 cm

8 cm

12 cm

8 cm

f

8 cm　9 cm

g

10 cm

8 cm

h

6 cm

4 cm

i

11 cm

10 cm

j

6.2 m

k

7 cm

l

5 cm
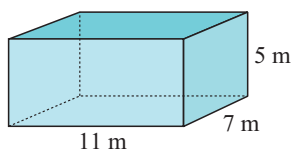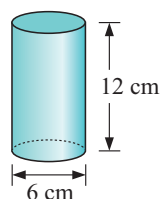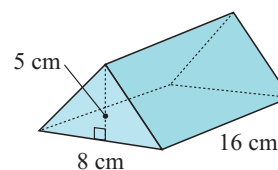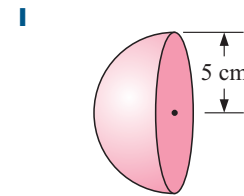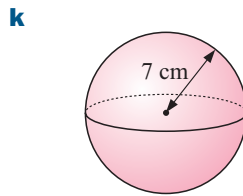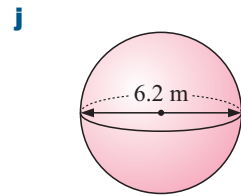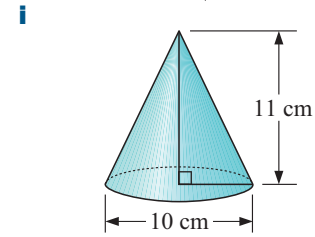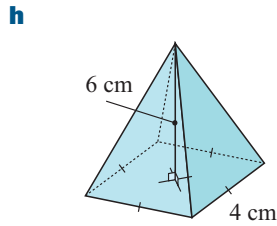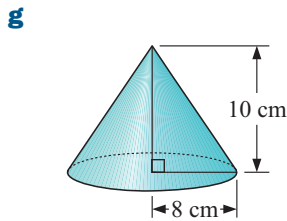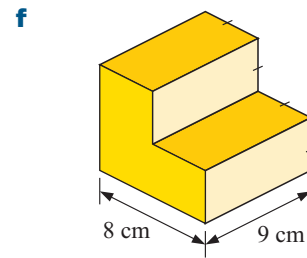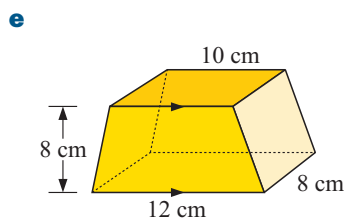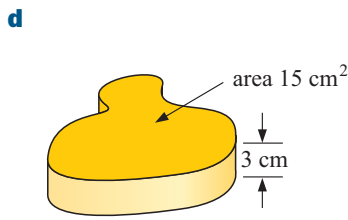
2 Find formula for the volume V of the following objects:

a

area, A

l

b

b

a

c
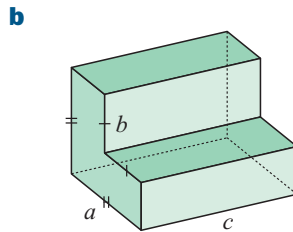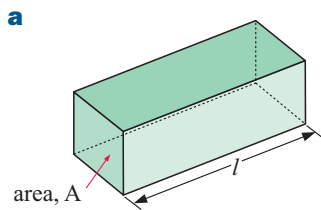
3 A beach ball has a diameter of 1.2 m. Find the volume of air inside the ball.

4 The roof timber at the end of a building has the dimensions given. If the timber is 10 cm thick, find the volume of timber used for making one end.

10 m

0.6 m

5

15 m

30 m
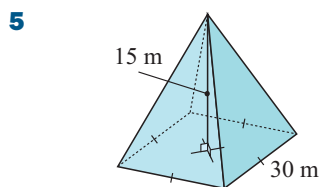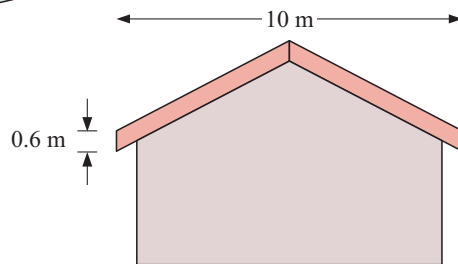
The conservatory for tropical plants at the Botanic gardens is a square-based pyramid with sides 30 metres long and height 15 metres. Calculate the volume of air in this building.

6 Calculate the volume of wood needed to make the podium illustrated.

50 cm

60 cm

90 cm

3 m

30 cm
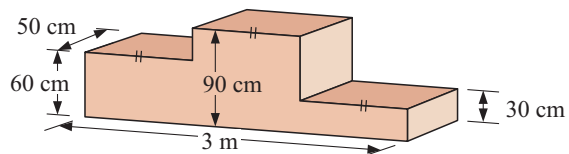
7 How many spherical fishing sinkers with diameter 1 cm could be made by melting a rectangular block of lead 20 cm by 5 cm by 6 cm and casting the molten product?

8 A conical heap of garden soil is dumped on a flat surface. If the diameter of the heap equals its height, and its volume is 1.5 m³, how high is the heap?

9 A half-pipe for skating is made with the dimensions shown. Show that the volume of concrete used is given by the formula: $V = l(2h^2 - \frac{1}{2}\pi r^2)$
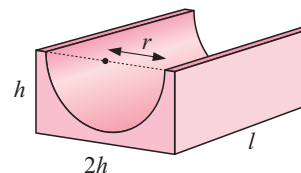
C   CAPACITY   [6.1, 6.4]

> The **capacity** of a container is the quantity of fluid or gas used to fill it.

The basic unit of capacity is the litre.

1 **centilitre** (cl) = 10 millilitres (ml)
1 **litre** = 1000 millilitres (ml)
1 **litre** = 100 centilitres (cl)
1 **kilolitre** (kl) = 1000 litres

> To avoid confusion with the number 1, we write out the full word litre.

CAPACITY UNITS CONVERSION
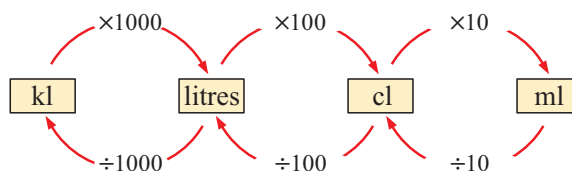
Example 11   ◀)) Self Tutor

Convert: **a** 4.2 litres to ml   **b** 36 800 litres to kl   **c** 25 cl to litres

a   4.2 litres
= (4.2 × 1000) ml
= 4200 ml

b   36 800 litres
= (36 800 ÷ 1000) kl
= 36.8 kl

c   25 cl
= (25 ÷ 100) litres
= 0.25 litres

EXERCISE 11C.1

1 Give the most appropriate units of capacity for measuring the amount of water in a:

   a test tube   **b** small drink bottle   **c** swimming pool   **d** laundry tub

2 Convert:

   a 68 cl into ml   **b** 3.76 litres into cl   **c** 375 ml into cl

   d 47 320 litres into kl   **e** 3.5 kl into litres   **f** 0.423 litres into ml

   g 0.054 kl into litres   **h** 58 340 cl into kl

3 An evaporative air conditioner runs on water. It needs 12 litres to run effectively. How many 80 cl jugs of water are needed to run the air conditioner?

4 In one city the average driver uses 30 litres of petrol per week. If there are 700 000 drivers in the city, calculate the total petrol consumption of the city's drivers in a 52 week year. Give your answer in kl.

CONNECTING VOLUME AND CAPACITY

≡ reads 'is equivalent to'.

1 millilitre (ml) of fluid fills a container of size 1 cm^3.

We say: $1 \text{ ml} \equiv 1 \text{ cm}^3$, $1 \text{ litre} \equiv 1000 \text{ cm}^3$ and $1 \text{ kl} = 1000 \text{ litres} \equiv 1 \text{ m}^3$.

Example 12
◀) Self Tutor

How many kl of water would a 3 m by 2.4 m by 1.8 m tank hold when full?
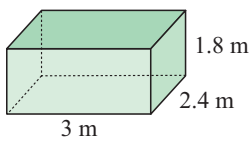
V = area of cross-section × height
$= (3 \times 2.4) \times 1.8 \text{ m}^3$
$= 12.96 \text{ m}^3$

∴ capacity is 12.96 kl.

Example 13
◀) Self Tutor

Water pours into a cylindrical tank of diameter 4 m at a constant rate of 1 kl per hour. By how much does the water level rise in 5 hours (to the nearest mm)?
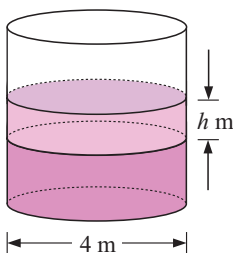
In 5 hours the capacity of water that flows in
$= 1 \text{ kl per hour} \times 5 \text{ hours}$
$= 5 \text{ kl}$

Since $1 \text{ kl} \equiv 1 \text{ m}^3$, the volume of water that flows in is 5 m^3.

If h metres is the height increase, then the volume of water that must have entered the tank is:

$$V = \pi r^2 h$$
$$\therefore \quad V = \pi \times 2^2 \times h$$
$$\therefore \quad V = 4\pi h$$
$$\therefore \quad 4\pi h = 5 \qquad \{\text{equating volumes}\}$$
$$\therefore \quad h = \frac{5}{4\pi} \approx 0.398 \qquad \{\text{dividing both sides by } 4\pi\}$$

∴ the water level rises 0.398 m.

EXERCISE 11C.2

1 The size of a car engine is often given in litres. Convert the following into cubic centimetres:

    a 2.3 litres   **b** 0.8 litres   **c** 1.8 litres   **d** 3.5 litres

2 **a** Find the capacity (in ml) of a bottle of volume 25 cm^3.

    b Find the volume of a tank (in m^3) if its capacity is 3200 kl.

    c How many litres are there in a tank of volume 7.32 m^3?

3 Find the capacity (in kl) of the following tanks:
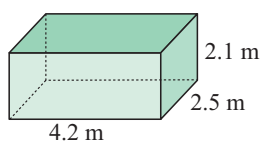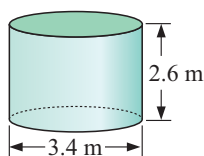
    a   **b**   **c**
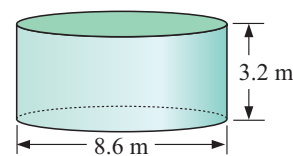
4 A spherical snow globe is made of glass that is 1 cm thick. If the outer diameter of the globe is 12 cm, find the capacity of its interior.

5 How many cylindrical bottles 12 cm high and with 6 cm diameter could be filled from a tank containing 125 litres of detergent?

6 A 1 litre cylindrical can of paint has a base radius of 5 cm. Find the height of the can.

7 The area of the bottom of a pool is 20 m^2, and the pool is 1.5 m deep. A hose fills the pool at a rate of 50 litres per minute. How long will it take to fill the pool?
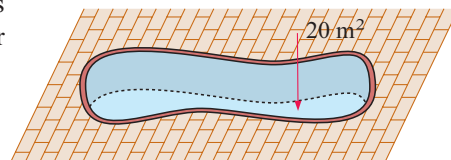
8 Rachael is drinking orange juice from a cylindrical glass with base diameter 6 cm. She takes a sip, drinking 30 ml. By how much does the level of the juice in her glass fall?

9 Consider the following 3 containers:
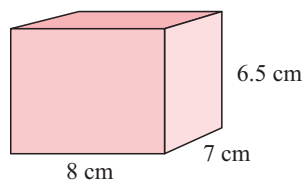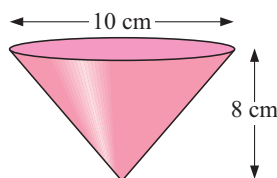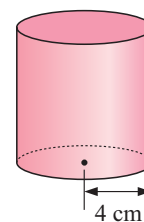
    Container A is filled with water. The water from container A is then poured into container B until it is full, and the remainder is poured into container C. Find the height of the water level in container C.

10 Water flows along a cylindrical pipe of radius 1.5 cm at a rate of 12 cm/s. It fills a tank measuring 1.2 m by 1.1 m by 0.8 m. Calculate the time required to fill the tank, giving your answer in hours and minutes to the nearest minute.

11 Answer the **Opening Problem** on page **231**.

D MASS [6.1]

The **mass** of an object is the amount of matter in it.

In the SI system of units, the primary unit of mass is the **kilogram**. The units of mass are connected in the following way:

1 **gram** (g) is the mass of 1 ml of pure water.	1 g = 1000 mg
1 **kilogram** (kg) is the mass of 1 litre of pure water.	1 kg = 1000 g
1 **tonne** (t) is the mass of 1 kl of pure water.	1 t = 1000 kg

MASS UNITS CONVERSION
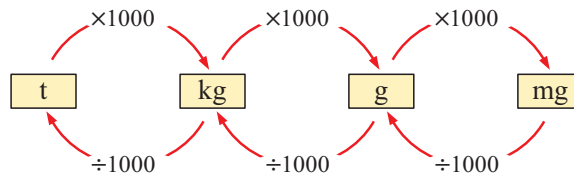
DENSITY

The **density** of an object is its mass divided by its volume.

$$\text{density} = \frac{\text{mass}}{\text{volume}}$$

For example, water has a density of 1 g per cm^3.

EXERCISE 11D

1 Convert:

   a 3200 g to kg   **b** 1.87 t to kg   **c** 47 835 mg to kg

   d 4653 mg to g   **e** 2.83 t to g   **f** 0.0632 t to g

   g 74 682 g to t   **h** 1.7 t to mg   **i** 91 275 g to kg

2 Kelly has 1200 golf balls each weighing 45 grams. What is the total weight of the balls in kilograms?

3 Estia has 75 kg of timber. She wants to cut the timber into cubes for a children's game. Each cube is to weigh 8 g. If 20% of the timber is wasted due to saw cuts, how many cubes can be made?

4 Find the density of these objects in g per cm^3:

   a
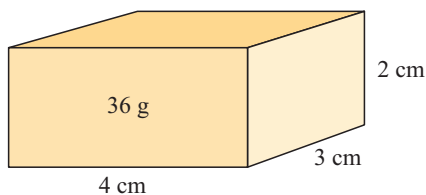
   b
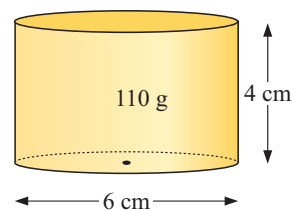
5 Laura buys a 1 kg packet of flour. Her rectangular canister is 12 cm by 10 cm by 15 cm high. If flour weighs 0.53 grams per cm^3, will the flour fit in the canister?

6 A cubic centimetre of liquid has mass 1.05 grams. Calculate, in kilograms, the mass of 1 litre of liquid.

7 A rectangular block of metal measures 40 cm by 20 cm by 20 cm. It has a density of 4.5 g/cm^3. Calculate the mass of the block in kilograms.

E   COMPOUND SOLIDS   [6.5]

We can find the surface area and volume of more complicated solids by separating the shape into objects that we are familiar with.

Example 14   🔊 *Self Tutor*

The diagram consists of a cone joined to a cylinder.
Find, in terms of π:

a the surface area of the solid

b the volume of the solid.

8 cm

10 cm

12 cm
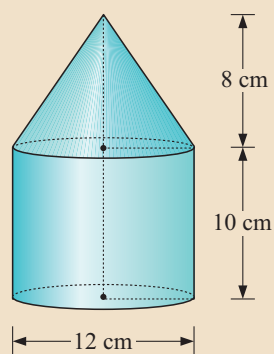
a $l^2 = 6^2 + 8^2$ {Pythagoras}

∴ $l^2 = 100$

∴ $l = 10$

8 cm

l cm

6 cm

Total area = area of cone + area of cylinder + area of base

$\qquad = \pi r l + 2\pi r h + \pi r^2$

$\qquad = \pi \times 6 \times 10 + 2\pi \times 6 \times 10 + \pi \times 6^2$

$\qquad = 60\pi + 120\pi + 36\pi$

$\qquad = 216\pi \ cm^2$

b   Volume = volume of cone + volume of cylinder

$\qquad = \frac{1}{3}\pi r^2 h_{cone} + \pi r^2 h_{cyl}$

$\qquad = \frac{1}{3} \times \pi \times 6^2 \times 8 + \pi \times 6^2 \times 10$

$\qquad = 96\pi + 360\pi$

$\qquad = 456\pi \ cm^3$
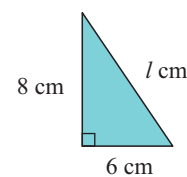
Example 15

◀) **Self Tutor**

A concrete tank has an external diameter of 10 m and an internal height of 3 m. If the walls and bottom of the tank are 30 cm thick, how many cubic metres of concrete are required to make the tank?
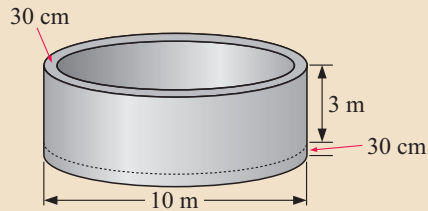
The tank's walls form a hollow cylinder with outer radius 5 m and inner radius 4.7 m. Its bottom is a cylinder with radius 5 m and height 30 cm.

$$\begin{aligned} \textit{walls of tank:} \quad \text{volume} &= \text{base area} \times \text{height} \\ &= [\pi \times 5^2 - \pi \times (4.7)^2] \times 3 \\ &\approx 27.43 \text{ m}^3 \end{aligned}$$

$$\begin{aligned} \textit{bottom of tank:} \quad \text{volume} &= \text{base area} \times \text{height} \\ &= \pi \times 5^2 \times 0.3 \\ &\approx 23.56 \text{ m}^3 \end{aligned}$$

Total volume of concrete required $\approx (27.43 + 23.56)$ m$^3 \approx 51.0$ m^3.

EXERCISE 11E

1 An observatory is built as a cylinder with a hemisphere above it.

   a Suppose $r = 4$ m and $h = 5$ m. Find to 3 significant figures, the observatory's:

     i above ground surface area

     ii volume.

   b Suppose $h = 2r$.

     i Find a formula for the volume of the solid in simplest form.

     ii If the volume is 1944π, find r.

     iii If the surface area is 384π, find r.
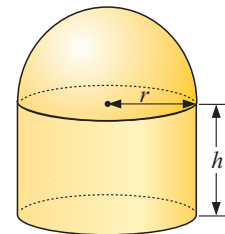
2
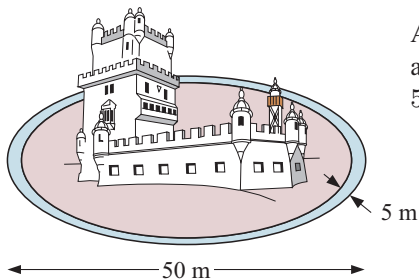
A castle is surrounded by a circular moat which is 5 m wide and 2 m deep. The diameter of the outer edge of the moat is 50 m. Find, in kilolitres, the quantity of water in the moat.

3 **a** Find the volume of the dumbbell illustrated.

   b If the material used to make the dumbbell weighs 2.523 grams per cm^3, find the total mass of the dumbbell.
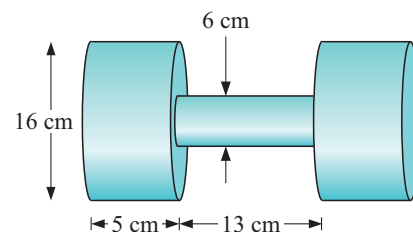
4 **a** Consider the icecream cone opposite, filled with icecream:
Suppose $r = 3.3$ cm and $h = 10.5$ cm. Find, correct to 3
significant figures, the:

    i surface area   **ii** volume of the icecream.

b Suppose $r = 2.9$ cm and the volume is 130 cm^3. Find h.

c Suppose $h = 3r$ and the volume is 115 cm^3. Find r.

d Suppose $r = 4.5$ cm and the total surface area of the cone and
icecream is 320 cm^2. Find the slant length of the cone.
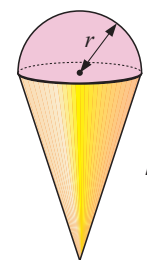
5 For the given solid, calculate the:

    a surface area

    b volume

    c mass given a density of 6.7 g per cm^3.
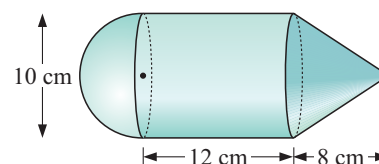
6 Assuming these solids have a density of d g per cm^3, find formulae for their:

    i surface area A **ii** volume V **iii** mass M.
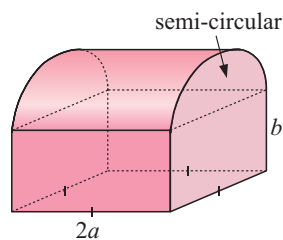
    a   **b**
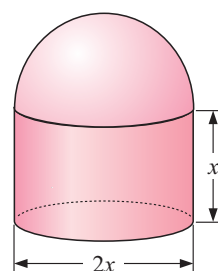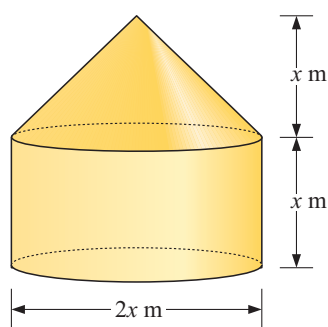
7 A large grain container is built as a cone over a cylinder.

    a Find the capacity of the container in terms of x.

    b If the capacity of the container is 500 kl, find x correct to 4
significant figures.

    c State, in terms of x, the surface area of the container. Do not
forget the base.

    d The cone is made from a sector of a circle with sector angle θ^o.
Find the value of θ to the nearest degree.
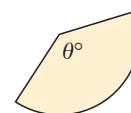
8 Regulation army huts have the dimensions shown. They sleep
36 soldiers. Each soldier must have access to 11 m^3 of air
space.

    a Find the internal volume of air space.

    b Hence, find the dimensions of the hut.

    c If each bed and surrounds requires 4 m^2 of floor space,
can the 36 soldiers be accommodated? How can this be achieved?

9
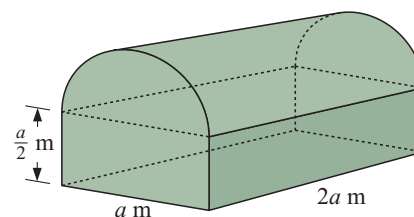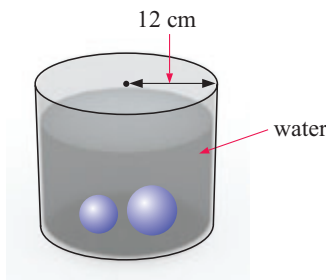
12 cm

water

A cylindrical container with radius 12 cm is partly filled with water. Two spheres with radii 3 cm and 4 cm are dropped into the water and are fully submerged. Find the *exact* increase in height of water in the cylinder.

Discovery 1   *Making cylindrical bins*

Your business has won a contract to make $40\,000$ cylindrical bins, each to contain $\frac{1}{20}$ m^3.

To minimise costs (and therefore maximise profits) you need to design the bin of minimum surface area.

What to do:

no top

h

$2x$

1 Find the formula for the volume V and the outer surface area A in terms of the base radius x and the height h.

2 Convert $\frac{1}{20}$ m^3 into cm^3.

3 Show that the surface area can be written as
$A = \pi x^2 + \dfrac{100\,000}{x}$ cm^2.

4 Use the **graphing package** or a **graphics calculator** to obtain a sketch of the function
$Y = \pi X^2 + 100\,000/X$
Find the minimum value of Y and the value of X when this occurs.

GRAPHING PACKAGE

5 Draw the bin made from a minimum amount of material. Make sure you fully label your diagram.

6 Investigate the dimensions of a cylindrical can which is to hold exactly 500 ml of soft drink. Your task is to minimise the surface area of material required. Remember your container will need two ends.
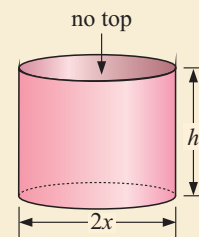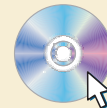
Discovery 2   *Constructing a lampshade*

Click on the icon to obtain a printable copy of instructions on how to make a lampshade which is of truncated cone shape.

LAMPSHADE

Discovery 3   *The turkey problem*

Click on the icon to obtain a printable copy of instructions on minimising materials needed to fence an enclosure needed to keep turkeys.

TURKEY PROBLEM

Review set 11A

1   Convert:

   a   2600 mm^3 to cm^3 **b**   8 000 000 cm^3 to m^3 **c**   1.2 m^3 to cm^3

   d   5.6 litres to ml   **e**   250 litres to kl   **f**   56 cm^3 to ml

2   Find the surface area of the following solids:
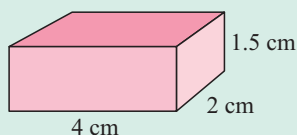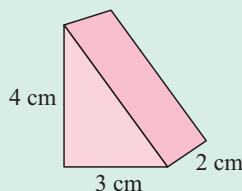
   a   **b**   **c**
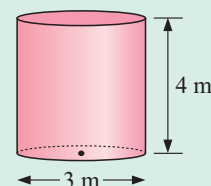
3   Li has a 10 cm × 8 cm × 5 cm block of clay. She wants to mould 50 clay spheres to use as heads for her figurines. Assuming there is no wastage, find the radius of the spheres Li should make.

4   Find the volume of the following, correct to 2 decimal places:
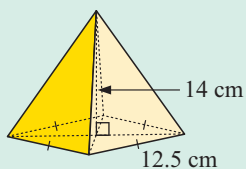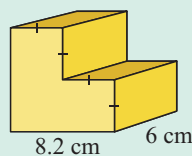
   a   **b**   **c**
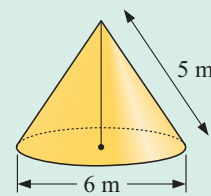
5   Bananas at a market cost $3.80 per kg. A bunch of 8 bananas is bought for $5.32. Find the average mass of a banana.

6   Find the density of these objects in g per cm^3.
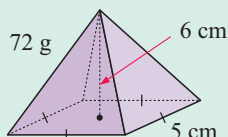
   a   **b**
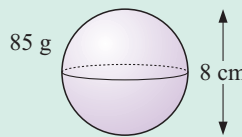
7   Marie has bought a plant which she needs to transfer from its cylindrical pot to the ground. The pot has diameter 10 cm and is 20 cm high. Marie needs to dig a hole 4 cm wider and 5 cm deeper than the pot. She then inserts the plant, and fills in the rest of the hole with extra soil.

   a   Find the volume of the hole Marie needs to dig.

   b   Before inserting the plant, Marie fills the hole with water. How much water will she need, assuming none of it soaks away?

   c   Once the water soaks in, she puts the plant in the hole. How much extra soil will she need to fill the hole?
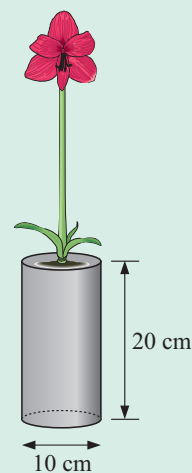
8   A conical heap of sand is twice as wide as it is high. If the volume of the sand is 2 m^3, find the height of the heap.

9   A wedge with angle θ as shown is cut from the centre of a cylindrical cake of radius r and height h. Find an expression for the:

   a   volume

   b   surface area of the wedge.
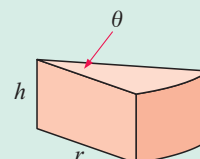
Review set 11B

1 Convert:
   a 350 mg to g
   b 250 kg to t
   c 16.8 kg to g
   d 150 ml to litres
   e 260 litres to cl
   f 0.8 litres to ml

2 Find a formula for the surface area A, in terms of x, of the following solids:
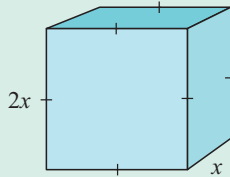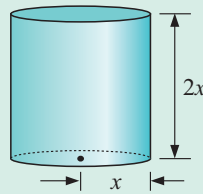
   a　　　　　　　　**b**　　　　　　　　**c**
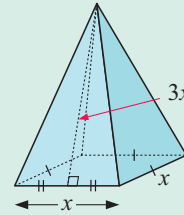
3 **a** Find the radius of a sphere with surface area 200 cm^2.

   b A solid cone has a base radius of 2 m and a surface area of 10π m^2. Find the slant length of the cone.

4 A teapot contains 1.3 litres of tea. 4 cups measuring 270 ml each are poured from the teapot. How much tea remains in the teapot?

5 Find the volume of the following, correct to 2 decimal places:
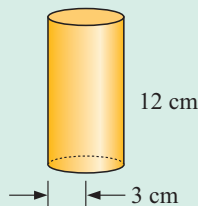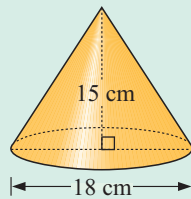
   a　　　　　　　　**b**　　　　　　　　**c**
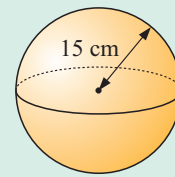
6 A sphere has radius 11.4 cm and density 5.4 g per cm^3. Calculate the mass of the sphere in kg.

7 Gavin uses a cylindrical bucket to water his plants. The bucket is 30 cm wide and 30 cm high. If Gavin fills his bucket 8 times to water his plants, how much water does he use?

8 Calculate the volume of wood in the model train tunnel illustrated.
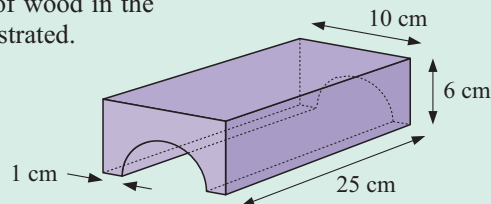
9 A clock tower has the dimensions shown. On each side of the tower there is a clock which is 2 m in diameter. The highest point of the tower is 24 m above ground level.

   a Find the height of the roof pyramid.

   b Find the volume of the tower.

   c The whole tower, excluding the clock faces, is to be painted. Find, correct to 3 significant figures, the surface area to be painted.
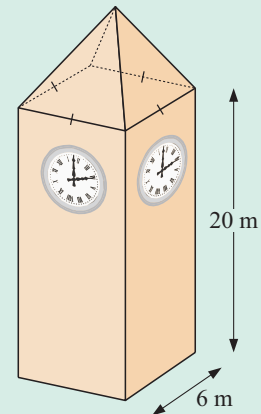
Coordinate geometry

12

Contents:

Opening problem

On the given map, Peta's house is located at point P, and Russell lives at point R.

a State ordered pairs of numbers which exactly specify the positions of P and R.

b State the coordinates of the eastern end of the oval.

c Find the shortest distance from Peta's house to Russell's house.

d Locate the point which is midway between Peta's house and Russell's house.
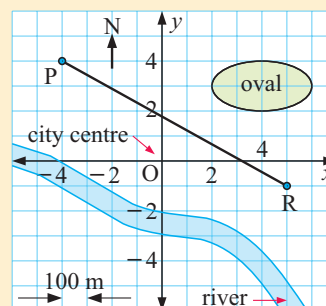
Historical note

History now shows that the two Frenchmen René Descartes and Pierre de Fermat arrived at the idea of **analytical geometry** at about the same time. Descartes' work "*La Geometrie*", however, was published first, in 1637, while Fermat's "*Introduction to Loci*" was not published until after his death.

Today, they are considered the co-founders of this important branch of mathematics, which links algebra and geometry
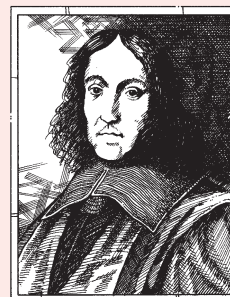
René Descartes   **Pierre de Fermat**

The initial approaches used by these mathematicians were quite opposite. Descartes began with a line or curve and then found the equation which described it. Fermat, to a large extent, started with an equation and investigated the shape of the curve it described. This interaction between algebra and geometry shows the power of **analytical geometry** as a branch of mathematics.

Analytical geometry and its use of coordinates provided the mathematical tools which enabled Isaac Newton to later develop another important branch of mathematics called calculus.

Newton humbly stated: "*If I have seen further than Descartes, it is because I have stood on the shoulders of giants.*"

THE NUMBER PLANE

The position of any point in the **number plane** can be specified in terms of an **ordered pair** of numbers (x, y), where:

> x is the **horizontal step** from a fixed point or **origin** O, and
> y is the **vertical step** from O.

Once the origin O has been given, two perpendicular axes are drawn. The **x-axis** is horizontal and the **y-axis** is vertical.

The **number plane** is also known as either:

- the **2-dimensional plane**, or
- the **Cartesian plane**, named after **René Descartes**.

In the diagram, the point P is at (a, b).

> a and b are referred to as the **coordinates** of P.
> a is called the **x-coordinate**, and
> b is called the **y-coordinate**.
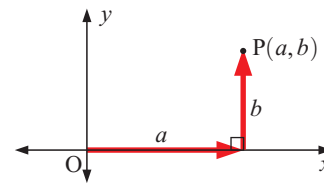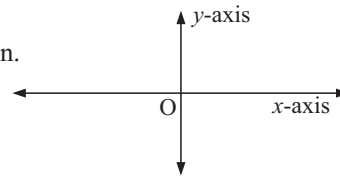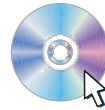
A	## PLOTTING POINTS	[7.1]

To plot the point A(3, 4):

- start at the origin O
- move right along the x-axis 3 units
- then move upwards 4 units.
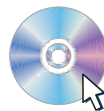
To plot the point B(5, −2):

- start at the origin O
- move right along the x-axis 5 units
- then move downwards 2 units.

To plot the point C(−4, 1):

- start at the origin O
- move left along the x-axis 4 units
- then move upwards 1 unit.
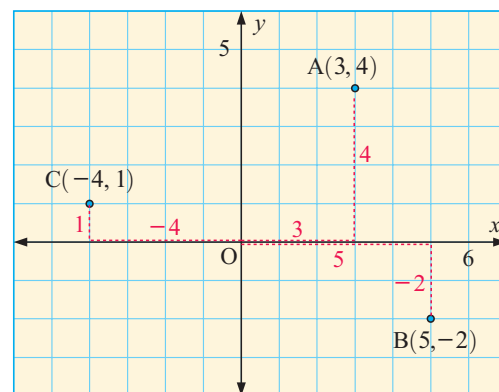
The x-coordinate is always given first. It indicates the movement away from the origin in the horizontal direction.

For A(3, 4) we say that:
3 is the **x-coordinate** of A.
4 is the **y-coordinate** of A.

QUADRANTS

The x and y-axes divide the Cartesian plane into four regions referred to as **quadrants**. These quadrants are numbered in an **anti-clockwise direction** as shown alongside.
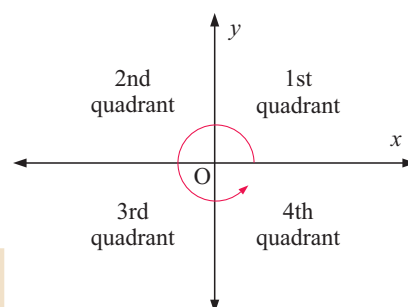
Example 1   ◀) Self Tutor

Plot the points A(3, 5), B(-1, 4), C(0, -3), D(-3, -2) and E(4, -2) on the same set of axes.
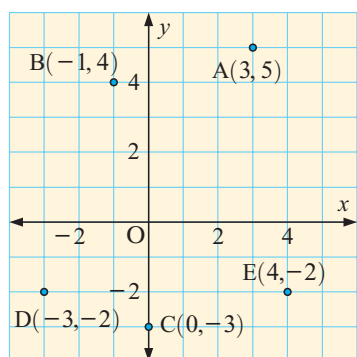
Start at O and move horizontally first, then vertically.
$\rightarrow$ is positive
$\leftarrow$ is negative
$\uparrow$ is positive
$\downarrow$ is negative.

Example 2   ◀) Self Tutor

On a Cartesian plane, show all the points with positive x-coordinate and negative y-coordinate.
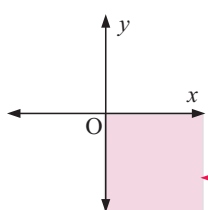
This shaded region contains all points where x is positive and y is negative. The points on the axes are not included.

EXERCISE 12A

1 State the coordinates of the points J, K, L, M and N:
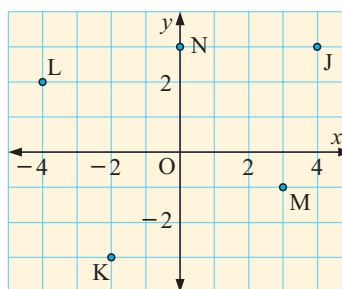
2 On the same set of axes plot the following points:

   a P$(2, 1)$ **b** Q$(2, -3)$ **c** R$(-3, -1)$ **d** S$(-2, 3)$

   e T$(-4, 0)$ **f** U$(0, -1)$ **g** V$(-5, -3)$ **h** W$(4, -2)$

3 State the quadrant in which each of the points in question **2** lies.

4 On different sets of axes show all points with:

   a x-coordinate equal to -2 **b** y-coordinate equal to -3

   c x-coordinate equal to 0 **d** y-coordinate equal to 0

   e negative x-coordinate   **f** positive y-coordinate

   g negative x and y-coordinates   **h** positive x and negative y-coordinates

5 On separate axes plot the following sets of points:

   a $\{(0, 0), (1, -1), (2, -2), (3, -3), (4, -4)\}$

   b $\{(-2, 3), (-1, 1), (0, -1), (1, -3), (2, -5)\}$

      i Are the points collinear?

     ii Do any of the following rules fit the set of points?

        A $y = 2x + 1$ **B** $y = 2x - 1$ **C** $y = x$

        D $y = -2x - 1$ **E** $x + y = 0$

B   DISTANCE BETWEEN TWO POINTS   [7.1, 7.2]

Consider the points A$(1, 3)$ and B$(4, 1)$. We can join the points by a straight line segment of length d units. Suppose we draw a right angled triangle with hypotenuse AB and with sides parallel to the axes.

It is clear that $d^2 = 3^2 + 2^2$ {Pythagoras}

$$\therefore \quad d^2 = 13$$
$$\therefore \quad d = \sqrt{13} \qquad \{\text{as } d > 0\}$$

$\therefore$ the distance from A to B is $\sqrt{13}$ units.
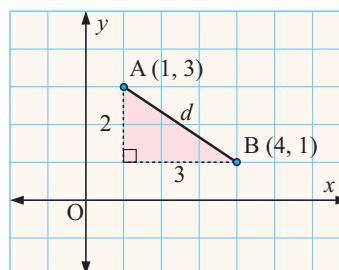
Example 3   ◀)) *Self Tutor*

Find the distance between P$(-2, 1)$ and Q$(3, 3)$.

We construct a right angled triangle with shorter sides on the grid lines.

$$PQ^2 = 5^2 + 2^2 \qquad \{\text{Pythagoras}\}$$
$$\therefore \quad PQ = 29$$
$$\therefore \quad PQ = \sqrt{29} \text{ units} \qquad \{PQ > 0\}$$
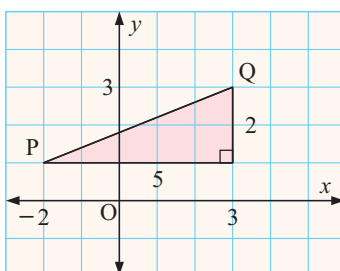
> With practice you will not need to use grid paper. A neat sketch will do.

EXERCISE 12B.1

1 If necessary, use Pythagoras' theorem to find the distance between:

    a A and B   **b** A and D   **c** C and A

    d F and C   **e** G and F   **f** C and G

    g E and C   **h** E and D   **i** B and G.
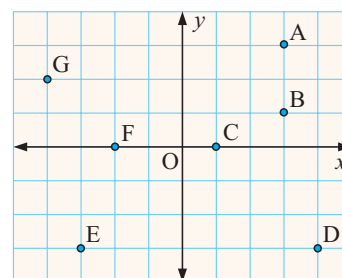
2 Plot the following pairs of points and use Pythagoras' theorem to find the distances between them. Give your answers correct to 3 significant figures:

    a A(3, 5) and B(2, 6)   **b** P(2, 4) and Q(-3, 2)   **c** R(0, 6) and S(3, 0)

    d L(2, -7) and M(1, -2)   **e** C(0, 5) and D(-4, 0)   **f** A(5, 1) and B(-1, -1)

    g P(-2, 3) and Q(3, -2)   **h** R(3, -4) and S(-1, -3)   **i** X(4, -1) and Y(3, -3)

THE DISTANCE FORMULA

To avoid drawing a diagram each time we wish to find a distance, a **distance formula** can be developed.

In going from A to B, the x-step $= x_2 - x_1$, and

                the y-step $= y_2 - y_1$.

Now, using Pythagoras' theorem,

$$(AB)^2 = (x\text{-step})^2 + (y\text{-step})^2$$

$$\therefore \quad AB = \sqrt{(x\text{-step})^2 + (y\text{-step})^2}$$

$$\therefore \quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$
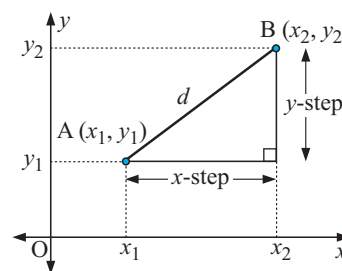
If A(x_1, y_1) and B(x_2, y_2) are two points in a plane, then the distance between these points is given by:

$$\mathbf{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

or $d = \sqrt{(x\text{-step})^2 + (y\text{-step})^2}.$

> The distance formula saves us having to graph the points each time we want to find a distance. However, you can still use a sketch and Pythagoras if you need.

Example 4   🔊 Self Tutor

Find the distance between A(-2, 1) and B(3, 4).

A(-2, 1) B(3, 4)

    $x_1 \; y_1$ $x_2 \; y_2$

$$AB = \sqrt{(3 - -2)^2 + (4 - 1)^2}$$
$$= \sqrt{5^2 + 3^2}$$
$$= \sqrt{25 + 9}$$
$$= \sqrt{34} \text{ units}$$

Example 5

◀》 *Self Tutor*

Consider the points A(-2, 0), B(2, 1) and C(1, -3).
Determine if the triangle ABC is equilateral, isosceles or scalene.

B($2,1$)

A($-2,0$)

C($1,-3$)
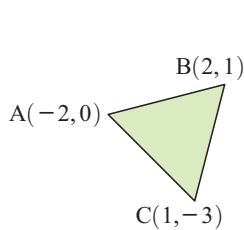
$$AB = \sqrt{(2 - -2)^2 + (1 - 0)^2}$$
$$= \sqrt{4^2 + 1^2}$$
$$= \sqrt{17} \text{ units}$$

$$AC = \sqrt{(1 - -2)^2 + (-3 - 0)^2}$$
$$= \sqrt{3^2 + (-3)^2}$$
$$= \sqrt{18} \text{ units}$$

$$BC = \sqrt{(1 - 2)^2 + (-3 - 1)^2}$$
$$= \sqrt{(-1)^2 + (-4)^2}$$
$$= \sqrt{17} \text{ units}$$

As   AB = BC,   triangle
ABC is isosceles.

Example 6

◀》 *Self Tutor*

Use the distance formula to show that triangle ABC is right angled
if A is (1, 2),   B is (2, 5),   and   C is (4, 1).

$$AB = \sqrt{(2 - 1)^2 + (5 - 2)^2}$$
$$= \sqrt{1^2 + 3^2}$$
$$= \sqrt{10} \text{ units}$$

$$AC = \sqrt{(4 - 1)^2 + (1 - 2)^2}$$
$$= \sqrt{3^2 + (-1)^2}$$
$$= \sqrt{10} \text{ units}$$

$$BC = \sqrt{(4 - 2)^2 + (1 - 5)^2}$$
$$= \sqrt{2^2 + (-4)^2}$$
$$= \sqrt{20} \text{ units}$$

So, $AB^2 + AC^2 = 10 + 10 = 20$
      and $BC^2 = 20$

$\therefore$ triangle ABC is right angled at A.

B $\sqrt{20}$ C

$\sqrt{10}$ $\sqrt{10}$

A
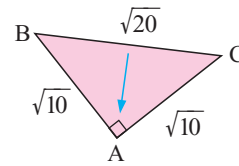
The right angle
is opposite the
longest side.

Example 7

◀》 *Self Tutor*

Find b given that   A(3, -2)   and   B(b, 1)   are $\sqrt{13}$ units apart.

From A to B, x-step $= b - 3$
                 y-step $= 1 - -2 = 3$

$\therefore$ $\sqrt{(b - 3)^2 + 3^2} = \sqrt{13}$
$\therefore$ $(b - 3)^2 + 9 = 13$
$\therefore$ $(b - 3)^2 = 4$
$\therefore$ $b - 3 = \pm 2$
$\therefore$ $b = 3 \pm 2$
$\therefore$ $b = 5$ or 1.

There are two
possible solutions in
this example. Draw
a diagram to see
why this is so.

EXERCISE 12B.2

1   Find the distance between the following pairs of points:

    a   A(3, 1) and B(5, 3)
    b   C(−1, 2) and D(6, 2)

    c   O(0, 0) and P(−2, 4)
    d   E(8, 0) and F(2, 0)

    e   G(0, −2) and H(0, 5)
    f   I(2, 0) and J(0, −1)

    g   R(1, 2) and S(−2, 3)
    h   W(5, −2) and Z(−1, −5)

2   Classify triangle ABC as either equilateral, isosceles or scalene:

    a   A(3, −1), B(1, 8), C(−6, 1)
    b   A(1, 0), B(3, 1), C(4, 5)

    c   A(−1, 0), B(2, −2), C(4, 1)
    d   A($\sqrt{2}$, 0), B(−$\sqrt{2}$, 0), C(0, −$\sqrt{5}$)

    e   A($\sqrt{3}$, 1), B(−$\sqrt{3}$, 1), C(0, −2)
    f   A(a, b), B(−a, b), C(0, 2)

3   Show that the following triangles are right angled. In each case state the right angle.

    a   A(−2, −1), B(3, −1), C(3, 3)
    b   A(−1, 2), B(4, 2), C(4, −5)

    c   A(1, −2), B(3, 0), C(−3, 2)
    d   A(3, −4), B(−2, −5), C(2, 1)

4   Find a given that:

    a   P(2, 3) and Q(a, −1) are 4 units apart

    b   P(−1, 1) and Q(a, −2) are 5 units apart

    c   X(a, a) is $\sqrt{8}$ units from the origin

    d   A(0, a) is equidistant from P(3, −3) and Q(−2, 2).

C   MIDPOINT OF A LINE SEGMENT   [7.3]

THE MIDPOINT FORMULA

If point M is halfway between points A and B then M is the **midpoint** of AB.

Consider the points A(1, 2) and B(5, 4).

It is clear from the diagram alongside that the midpoint M of AB is (3, 3).

We notice that: $\dfrac{1+5}{2} = 3$ and $\dfrac{2+4}{2} = 3$.

So,   the x-coordinate of M is the *average* of the
      x-coordinates of A and B,

and   the y-coordinate of M is the *average* of the
      y-coordinates of A and B.
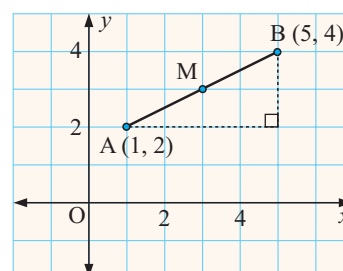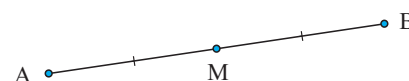
In general,   if   A(x_1, y_1)   and   B(x_2, y_2)   are two points then the **midpoint** M of AB has coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

Example 8   ◀⑴ *Self Tutor*

Find the coordinates of the midpoint of AB for A$(-1, 3)$ and B$(4, 7)$.

x-coordinate of midpoint	y-coordinate of midpoint
$= \dfrac{-1+4}{2}$	$= \dfrac{3+7}{2}$
$= \dfrac{3}{2}$	$= 5$
$= 1\frac{1}{2}$	$\therefore$ the midpoint of AB is $(1\frac{1}{2}, 5)$.

> You may also be able to find the midpoint from a sketch.

Example 9   ◀⑴ *Self Tutor*

M is the midpoint of AB. Find the coordinates of B if A is $(1, 3)$ and M is $(4, -2)$.

Let B be (a, b)

$\therefore \quad \dfrac{a+1}{2} = 4 \quad$ and $\quad \dfrac{b+3}{2} = -2$

$\therefore \quad a+1 = 8 \quad$ and $\quad b+3 = -4$

$\therefore \quad a = 7 \quad$ and $\quad b = -7$

$\therefore \quad$ B is $(7, -7)$.

A $(1, 3)$

M $(4, -2)$

B (a, b)
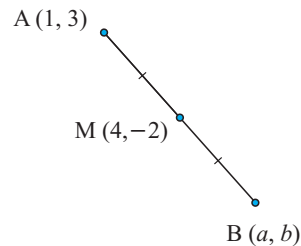
Example 10   ◀⑴ *Self Tutor*

Suppose A is $(-2, 4)$ and M is $(3, -1)$, where M is the midpoint of AB.
Use *equal steps* to find the coordinates of B.

x-step: $\quad -2 \xrightarrow{+5} 3 \xrightarrow{+5} 8$

y-step: $\quad 4 \xrightarrow{-5} -1 \xrightarrow{-5} -6$

$\therefore \quad$ B is $(8, -6)$.

A$(-2, 4)$ $\quad$ +5

M$(3, -1)$ $\quad$ -5 $\quad$ +5

B$(8, -6)$ $\quad$ -5
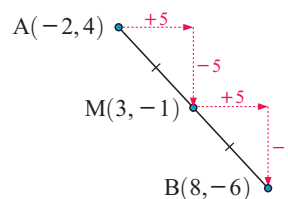
EXERCISE 12C

1 Use this diagram only to find the coordinates of the
midpoint of the line segment:

  a GA   **b** ED

  c AC   **d** AD

  e CD   **f** GF

  g EG   **h** GD
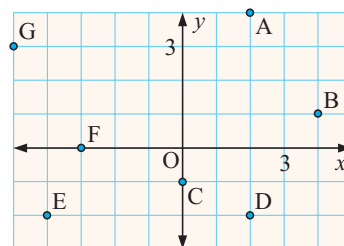
2 Find the coordinates of the midpoint of the line segment joining the pairs of points:

   a $(8, 1)$ and $(2, 5)$ **b** $(2, -3)$ and $(0, 1)$ **c** $(3, 0)$ and $(0, 6)$

   d $(-1, 4)$ and $(1, 4)$ **e** $(5, -3)$ and $(-1, 0)$ **f** $(-2, 4)$ and $(4, -2)$

   g $(5, 9)$ and $(-3, -4)$ **h** $(3, -2)$ and $(1, -5)$

3 M is the midpoint of AB. Find the coordinates of B for:

   a $A(6, 4)$ and $M(3, -1)$ **b** $A(-5, 0)$ and $M(0, -1)$ **c** $A(3, -2)$ and $M(1\frac{1}{2}, 2)$

   d $A(-1, -2)$ and $M(-\frac{1}{2}, 2\frac{1}{2})$ **e** $A(7, -3)$ and $M(0, 0)$ **f** $A(3, -1)$ and $M(0, -\frac{1}{2})$

   Check your answers using the *equal steps* method given in **Example 10**.

4 If T is the midpoint of PQ, find the coordinates of P for:

   a $T(-3, 4)$ and $Q(3, -2)$ **b** $T(2, 0)$ and $Q(-2, -3)$.

5 AB is the diameter of a circle with centre C. If A is $(3, -2)$ and B is $(-1, -4)$, find the coordinates of C.

6 PQ is a diameter of a circle with centre $(3, -\frac{1}{2})$. Find the coordinates of P given that Q is $(-1, 2)$.

7 The diagonals of parallelogram PQRS bisect each other at X. Find the coordinates of S.
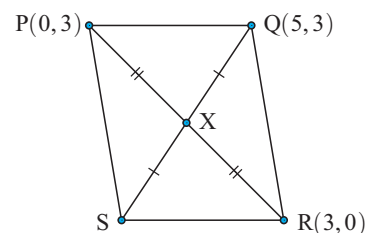
8 Triangle ABC has vertices $A(-1, 3)$, $B(1, -1)$, and $C(5, 2)$. Find the length of the line segment from A to the midpoint of BC.

9 A, B, C and D are four points on the same straight line. The distances between successive points are equal, as shown. If A is $(1, -3)$, C is $(4, a)$ and D is $(b, 5)$, find the values of a and b.
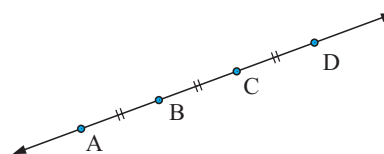
D   GRADIENT OF A LINE SEGMENT   [7.1, 7.4]

When looking at line segments drawn on a set of axes, it is clear that different line segments are inclined to the horizontal at different angles. Some appear to be *steeper* than others.

> The **gradient** of a line is a measure of its steepness.

If we choose any two distinct (different) points on the line, the **horizontal step** and **vertical step** between them may be determined.

Case 1:
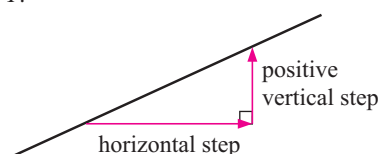
Case 2:
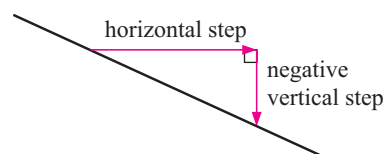
> The **gradient** of a line may be found by using: $\dfrac{\textbf{vertical step}}{\textbf{horizontal step}}$ or $\dfrac{y\textbf{-step}}{x\textbf{-step}}$ or $\dfrac{\textbf{rise}}{\textbf{run}}$.

We can see that:
- in *Case 1* both steps are positive and so the gradient is positive.
- in *Case 2* the steps are opposite in sign and so the gradient is negative.

Lines like
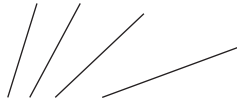
are forward sloping and have **positive gradients**.

Lines like
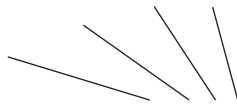
are backward sloping and have **negative gradients**.

Have you ever wondered why gradient is measured by y-step divided by x-step rather than x-step divided by y-step?

Perhaps it is because horizontal lines have no gradient and zero (0) should represent this. Also, as lines become steeper we want their numerical gradients to increase.

Example 11　　　　　　　　　　　　　◀)) *Self Tutor*

Find the gradient of each line segment:
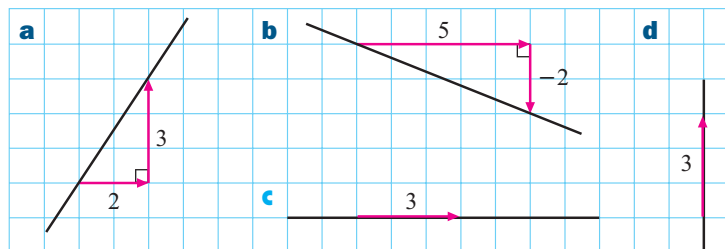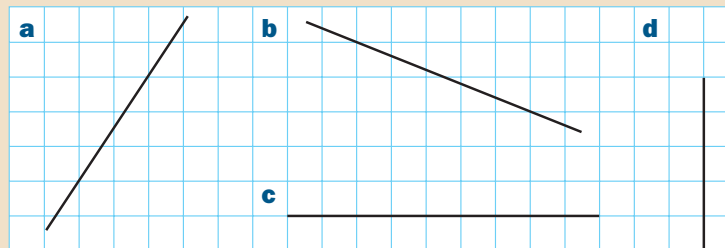
a gradient $= \frac{3}{2}$　　　　　　　　　　　**b** gradient $= \frac{-2}{5} = -\frac{2}{5}$

c gradient $= \frac{0}{3} = 0$　　　　　　　　　　**d** gradient $= \frac{3}{0}$　which is undefined

We can see that:

> The gradient of any **horizontal** line is **0**, since the vertical step (numerator) is 0.
>
> The gradient of any **vertical** line is **undefined**, since the horizontal step (denominator) is 0.

THE GRADIENT FORMULA

If A is (x_1, y_1) and B is (x_2, y_2) then the **gradient** of AB is $\dfrac{y_2 - y_1}{x_2 - x_1}$.
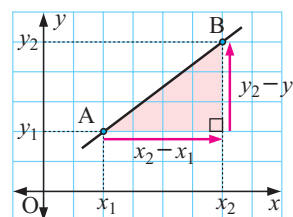
Example 12   ◀) Self Tutor

Find the gradient of the line through $(3, -2)$ and $(6, 4)$.

$(3, -2)$ $(6, 4)$

x_1 y_1 x_2 y_2

$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{4 - -2}{6 - 3}$$

$$= \frac{6}{3}$$

$$= 2$$

Example 13   ◀) Self Tutor

Through $(2, 4)$ draw a line with gradient $-\frac{2}{3}$.

Plot the point $(2, 4)$

$$\text{gradient} = \frac{y\text{-step}}{x\text{-step}} = \frac{-2}{3}$$

$\therefore$ let y-step $= -2$, x-step $= 3$.

Use these steps to find another point and draw the line through these points.
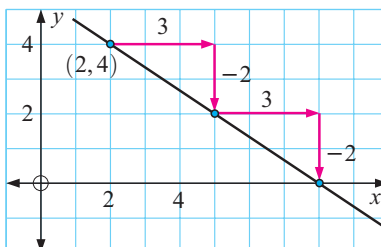
It is a good idea to use a positive x-step.

EXERCISE 12D.1

1 Find the gradient of each line segment:
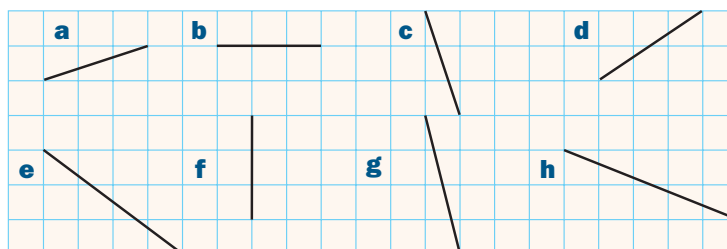
2 On grid paper draw a line segment with gradient:

    a $\frac{3}{4}$ **b** $-\frac{1}{2}$ **c** 2 **d** -3 **e** 0 **f** $-\frac{2}{5}$

3   Find the gradient of the line segment joining the following pairs of points:

   a $(2, 3)$ and $(7, 4)$ **b** $(5, 7)$ and $(1, 6)$ **c** $(1, -2)$ and $(3, 6)$

   d $(5, 5)$ and $(-1, 5)$ **e** $(3, -1)$ and $(3, -4)$ **f** $(5, -1)$ and $(-2, -3)$

   g $(-5, 2)$ and $(2, 0)$ **h** $(0, -1)$ and $(-2, -3)$

4   On the same set of axes draw lines through $(1, 2)$ with gradients of $\frac{3}{4}$, $\frac{1}{2}$, 1, 2 and 3.

5   On the same set of axes draw lines through $(-2, -1)$ with gradients of 0, $-\frac{1}{2}$, -1 and -3.

USING GRADIENTS

In real life gradients occur in many situations, and can be interpreted in a variety of ways.

For example, the sign alongside would indicate to motor vehicle drivers that there is an uphill climb ahead.

Consider the situation in the graph alongside where a motor vehicle travels at a constant speed for a distance of 600 km in 8 hours.

Clearly, the gradient of the line $= \dfrac{\text{vertical step}}{\text{horizontal step}}$

$$= \dfrac{600}{8}$$

$$= 75$$

However,   speed $= \dfrac{\text{distance}}{\text{time}} = \dfrac{600 \text{ km}}{8 \text{ hours}} = 75$ km/h.
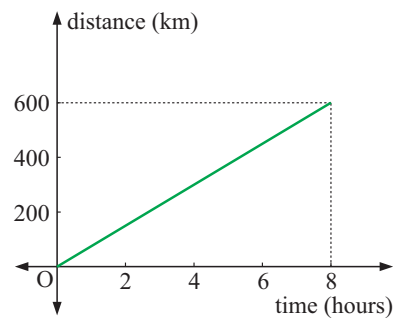
So, in a graph of distance against time, the *gradient* can be interpreted as the *speed*.

In the following exercise we will consider a number of problems where gradient can be interpreted as a rate.

EXERCISE 12D.2

1
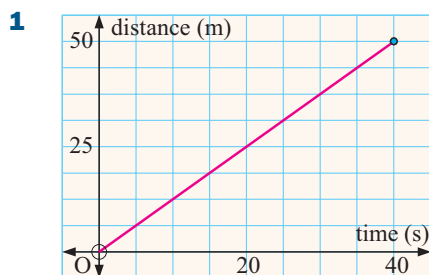
The graph alongside indicates the distances and corresponding times as Tan walks a distance of 50 metres.

   a   Find the gradient of the line.

   b   Interpret the gradient found in **a**.

   c   Is the speed of the walker constant or variable? What evidence do you have for your answer?

2   The graph alongside indicates the distances travelled by a train. Determine:

   a   the average speed for the whole trip

   b   the average speed from

       i   A to B   **ii**   B to C

   c   the time interval over which the speed was greatest.

3

wage (€)

(12, 256)

(4, 112)   (7, 166)

40

O   5   10   15
                          hours
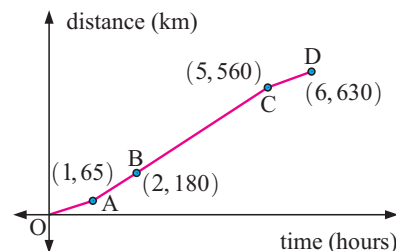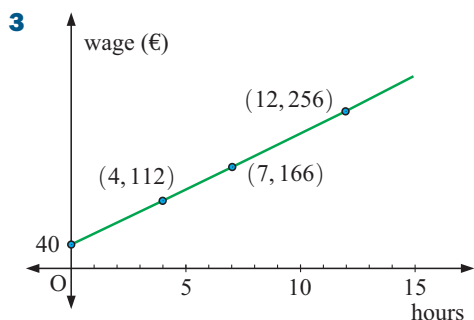
The graph alongside indicates the wages paid to security guards.

a What does the intercept on the vertical axis mean?

b Find the gradient of the line. What does this gradient mean?

c Determine the wage for working:

  i 6 hours   **ii** 15 hours.

d If no payment is made for not working, but the same payment shown in the graph is made for 8 hours' work, what is the new rate of pay?

4 The graphs alongside indicate the fuel consumption and distance travelled at speeds of 60 km/h (graph A) and 90 km/h (graph B).

  a Find the gradient of each line.

  b What do these gradients mean?

  c If fuel costs $1.40 per litre, how much more would it cost to travel 1000 km at 90 km/h compared with 60 km/h?

distance travelled (km)

(29, 350)

A

(33, 320)

B

O   fuel consumption (litres)

5

charge ($)

C(20, 27)

B(8, 15)

3 A
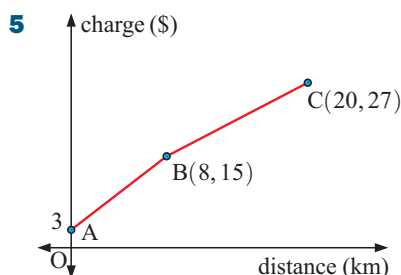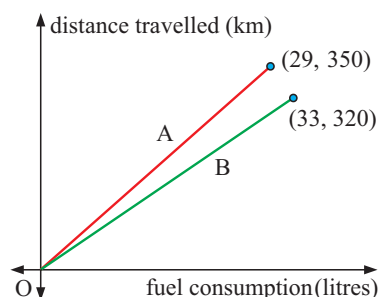
O   distance (km)

The graph alongside indicates the courier charge for different distances travelled.

  a What does the value at A indicate?

  b Find the gradients of the line segments AB and BC. What do these gradients indicate?

  c If a straight line segment was drawn from A to C, find its gradient. What would this gradient mean?

E GRADIENT OF PARALLEL AND PERPENDICULAR LINES   [7.5]

PARALLEL LINES
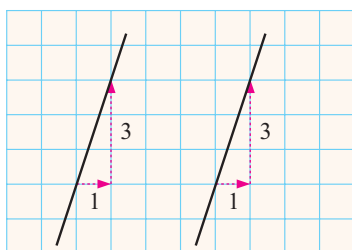
Notice that the given lines are parallel and both of them have a gradient of 3.

In fact:

- if two lines are **parallel**, then they have **equal gradient**, and
- if two lines have **equal gradient**, then they are **parallel**.

PERPENDICULAR LINES
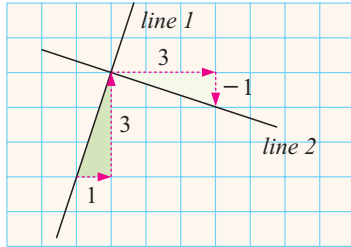
Notice that *line 1* and *line 2* are perpendicular.

Line 1 has gradient $\frac{3}{1} = 3$.

Line 2 has gradient $\frac{-1}{3} = -\frac{1}{3}$.

We see that the gradients are *negative reciprocals* of each other and their product is $3 \times -\frac{1}{3} = -1$.

For lines which are not horizontal or vertical:

> - if the lines are **perpendicular** then their gradients are **negative reciprocals**
> - if the gradients are **negative reciprocals** then the lines are **perpendicular**.

Proof:
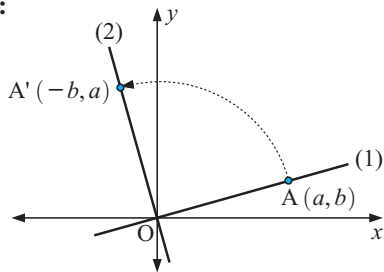
Suppose the two perpendicular lines are translated so that they intersect at the origin O. If A(a, b) lies on one line, then under an anticlockwise rotation about O of 90° it finishes on the other line and its coordinates are A$'(-b, a)$.

The gradient of line (1) is $\dfrac{b - 0}{a - 0} = \dfrac{b}{a}$.

The gradient of line (2) is $\dfrac{a - 0}{-b - 0} = -\dfrac{a}{b}$.

Example 14   ◀》 *Self Tutor*

If a line has gradient $\frac{2}{3}$, find the gradient of:

  a   all lines parallel to the given line

  b   all lines perpendicular to the given line.

  a   Since the original line has gradient $\frac{2}{3}$, the gradient of all parallel lines is also $\frac{2}{3}$.

  b   The gradient of all perpendicular lines is $-\frac{3}{2}$.
       {the negative reciprocal}

> $\frac{b}{a}$ and $-\frac{a}{b}$ are negative reciprocals of each other.

Example 15   ◀》 *Self Tutor*

Find a given that the line joining A$(2, 3)$ to   B$(a, -1)$ is parallel to a line with gradient -2.

gradient of AB $= -2$ {parallel lines have equal gradient}

$$\therefore \quad \frac{-1 - 3}{a - 2} = -2$$

$$\therefore \quad \frac{-4}{a - 2} = \frac{-2}{1}$$

$$\therefore \quad \frac{-4}{a-2} = \frac{-2}{1}\left(\frac{a-2}{a-2}\right) \quad \text{\{achieving a common denominator\}}$$

$$\therefore \quad -4 = -2(a-2) \quad \text{\{equating numerators\}}$$

$$\therefore \quad -4 = -2a + 4$$

$$\therefore \quad 2a = 8$$

$$\therefore \quad a = 4$$

Example 16 ◀) Self Tutor

Find t given that the line joining D$(-1, -3)$ to C$(1, t)$ is perpendicular to a line with gradient 2.

gradient of DC $= -\frac{1}{2}$ \{perpendicular to line of gradient 2\}

$$\therefore \quad \frac{t - -3}{1 - -1} = -\frac{1}{2}$$

$$\therefore \quad \frac{t+3}{2} = \frac{-1}{2} \quad \text{\{simplifying\}}$$

$$\therefore \quad t + 3 = -1 \quad \text{\{equating numerators\}}$$

$$\therefore \quad t = -4$$

EXERCISE 12E.1

1 Find the gradient of all lines perpendicular to a line with a gradient of:

  a $\frac{1}{2}$ **b** $\frac{2}{5}$ **c** 3 **d** 7 **e** $-\frac{2}{5}$ **f** $-2\frac{1}{3}$ **g** -5 **h** -1

2 The gradients of two lines are listed below. Which of the line pairs are perpendicular?

  a $\frac{1}{3}, 3$ **b** $5, -5$ **c** $\frac{3}{7}, -2\frac{1}{3}$ **d** $4, -\frac{1}{4}$

  e $6, -\frac{5}{6}$ **f** $\frac{2}{3}, -\frac{3}{2}$ **g** $\frac{p}{q}, \frac{q}{p}$ **h** $\frac{a}{b}, -\frac{b}{a}$

3 Find a given that the line joining:

  a A$(1, 3)$ to B$(3, a)$ is parallel to a line with gradient 3

  b P$(a, -3)$ to Q$(4, -2)$ is parallel to a line with gradient $\frac{1}{3}$

  c M$(3, a)$ to N$(a, 5)$ is parallel to a line with gradient $-\frac{2}{5}$.

4 Find t given that the line joining:

  a A$(2, -3)$ to B$(-2, t)$ is perpendicular to a line with gradient $1\frac{1}{4}$

  b C$(t, -2)$ to D$(1, 4)$ is perpendicular to a line with gradient $\frac{2}{3}$

  c P$(t, -2)$ to Q$(5, t)$ is perpendicular to a line with gradient $-\frac{1}{4}$.

5 Given the points   A$(1, 4)$, B$(-1, 0)$, C$(6, 3)$ and D$(t, -1)$, find t if:

  a AB is parallel to CD   **b** AC is parallel to DB

  c AB is perpendicular to CD   **d** AD is perpendicular to BC.

COLLINEAR POINTS

> Three or more points are **collinear** if they lie on the same straight line.

If three points A, B and C are collinear, the gradient of AB is equal
to the gradient of BC and also the gradient of AC.
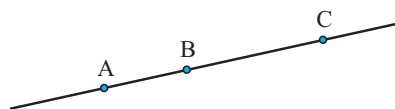
Example 17   🔊 *Self Tutor*

Show that the following points are collinear: A(1, −1), B(6, 9), C(3, 3).

$$\text{gradient of AB} = \frac{9 - -1}{6 - 1} \qquad \text{gradient of BC} = \frac{3 - 9}{3 - 6}$$

$$= \frac{10}{5} \qquad\qquad\qquad\qquad = \frac{-6}{-3}$$

$$= 2 \qquad\qquad\qquad\qquad\qquad = 2$$

∴   AB is parallel to BC, and as point B is common to both line segments,
    A, B and C are *collinear*.

EXERCISE 12E.2

1 Determine whether or not the following sets of three points are collinear:

    a A(1, 2), B(4, 6) and C(−4, −4)   **b** P(−6, −6), Q(−1, 0) and R(4, 6)

    c R(5, 2), S(−6, 5) and T(0, −4)   **d** A(0, −2), B(−1, −5) and C(3, 7)

2 Find c given that:

    a A(−4, −2), B(0, 2) and C(c, 5) are collinear

    b P(3, −2), Q(4, c) and R(−1, 10) are collinear.

F   USING COORDINATE GEOMETRY   [7.2 - 7.5]

Coordinate geometry is a powerful tool which can be used:

- to **check** the truth of a geometrical fact
- to **prove** a geometrical fact by using general cases.

In these problems we find distances, midpoints, and gradients either from a sketch or by using the appropriate
formulae.

Example 18   🔊 *Self Tutor*

P(3, −1), Q(1, 7) and R(−1, 5) are the vertices of triangle PQR.
M is the midpoint of PQ and N is the midpoint of PR.

    a Find the coordinates of M and N.   **b** Find the gradients of MN and QR.

    c What can be deduced from **b**?   **d** Find distances MN and QR.

    e What can be deduced from **d**?

---

a M is $\left(\dfrac{3+1}{2}, \dfrac{-1+7}{2}\right)$ which is $(2, 3)$.   N is $\left(\dfrac{3+-1}{2}, \dfrac{-1+5}{2}\right)$ which is $(1, 2)$.

b gradient of MN $= \dfrac{2-3}{1-2}$ gradient of QR $= \dfrac{5-7}{-1-1}$

$\qquad\qquad\qquad = 1$ $\qquad\qquad\qquad\qquad\quad = 1$

c Equal gradients implies that   MN $\parallel$ QR.

d MN $= \sqrt{(1-2)^2 + (2-3)^2}$ QR $= \sqrt{(-1-1)^2 + (5-7)^2}$

$\qquad = \sqrt{1+1}$ $\qquad\qquad\qquad = \sqrt{4+4}$

$\qquad = \sqrt{2}$ $\qquad\qquad\qquad\quad = \sqrt{8}$

$\qquad \approx 1.41$ units $\qquad\qquad\quad = 2\sqrt{2}$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad \approx 2.83$ units

e From **d**, QR is twice as long as MN. $\{2\sqrt{2}$ compared with $\sqrt{2}\}$
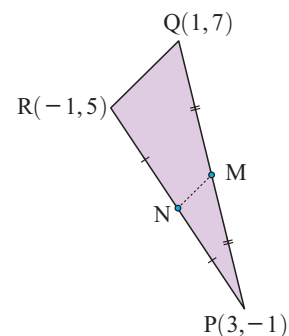
EXERCISE 12F

1 Given A$(0, 4)$, B$(5, 6)$ and C$(4, 1)$, where M is the midpoint of AB and N is the midpoint of BC:

a Illustrate the points A, B, C, M and N on a set of axes.

b Show that MN is parallel to AC, using gradients.

c Show that MN is half the length of AC.

2 Given K$(2, 5)$, L$(6, 7)$, M$(4, 1)$:

a Illustrate the points on a set of axes.

b Show that triangle KLM is isosceles.

c Find the midpoint P of LM.

d Use gradients to verify that KP is perpendicular to LM.

e Illustrate what you have found in **b**, **c** and **d** on your sketch.

For figures named ABCD, etc. the labelling is in cyclic order.

or
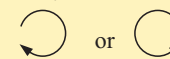
3 Given A$(3, 4)$, B$(5, 8)$, C$(13, 5)$ and D$(11, 1)$:

a Plot A, B, C and D on a set of axes.

b Use gradients to show that:

   i AB is parallel to DC   **ii** BC is parallel to AD.

c What kind of figure is ABCD?

d Check that   AB $=$ DC   and   BC $=$ AD   using the distance formula.

e Find the midpoints of diagonals: **i** AC **ii** BD.

f What property of parallelograms has been checked in **e**?

4 Given A$(3, 5)$, B$(8, 5)$, C$(5, 1)$ and D$(0, 1)$:

a Plot A, B, C and D on a set of axes.   **b** Show that ABCD is a rhombus.

c Find the midpoints of AC and BD.   **d** Show that AC and BD are perpendicular.

5 Consider the points A$(-3, 7)$, B$(1, 8)$, C$(4, 0)$ and D$(-7, -1)$. P, Q, R and S are the midpoints of AB, BC, CD and DA respectively.

   a Find the coordinates of: **i** P   **ii** Q   **iii** R   **iv** S.

   b Find the gradient of: **i** PQ   **ii** QR   **iii** RS   **iv** SP.

   c What can be deduced about quadrilateral PQRS from **b**?

6 S$(6, a)$ lies on a semi-circle as shown.

   a Find a.

   b Using this value of a, find the gradient of:

       i AC   **ii** CB.

   c Use **b** to show that angle ACB is a right angle.
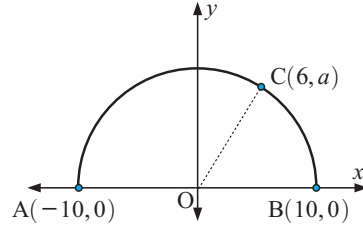
Review set 12A

1 Plot the following points on the number plane:

      A$(1, 3)$ B$(-2, 0)$ C$(-2, -3)$ D$(2, -1)$

2 Find the distance between the following sets of points:

   a P$(4, 0)$ and Q$(0, -3)$ **b** R$(2, -5)$ and S$(-1, -3)$

3 Find the coordinates of the midpoint of the line segment joining A$(8, -3)$ and B$(2, 1)$.

4 Find the gradients of the lines in the following graphs:
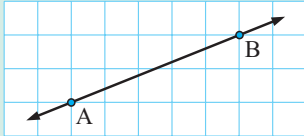
   a   **b**
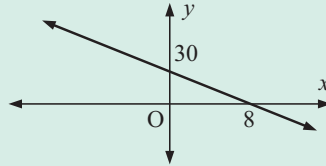
5 The graph alongside shows the distance travelled by a train over a 2 hour journey between two cities.

   a Find the average speed from:

       i O to A   **ii** A to B   **iii** B to C

   b Compare your answers to **a** with the gradients of the line segments:

       i OA   **ii** AB   **iii** BC

   c Find the average speed for the whole journey.
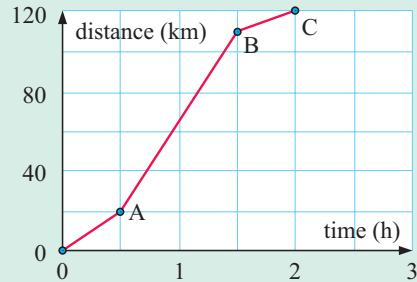
6 Given A$(2, -1)$, B$(-5, 3)$, C$(3, 4)$, classify triangle ABC as equilateral, isosceles or scalene.

7 Find k if the line joining X$(2, -3)$ and Y$(-1, k)$ is:

   a parallel to a line with gradient $\frac{1}{2}$

   b perpendicular to a line with gradient $-\frac{1}{4}$.

8 Show that A$(1, -2)$, B$(4, 4)$ and C$(5, 6)$ are collinear.

9 Find b given that A$(-6, 2)$, B$(b, 0)$ and C$(3, -4)$ are collinear.

10 Given A$(-3, 1)$, B$(1, 4)$ and C$(4, 0)$:

   a Show that triangle ABC is isosceles.

   b Find the midpoint X of AC.

   c Use gradients to verify that BX is perpendicular to AC.

Review set 12B

1   **a** Find the midpoint of the line segment joining A$(-2, 3)$ to B$(-4, 3)$.

   b Find the distance from C$(-3, -2)$ to D$(0, 5)$.

   c Find the gradient of all lines perpendicular to a line with gradient $\frac{2}{3}$.

2 On different sets of axes, show all points with:

   a x-coordinates equal to -3 **b** y-coordinates equal to 5

   c positive x-coordinates and negative y-coordinates.

3 K$(-3, 2)$ and L$(3, m)$ are 9 units apart. Find m.

4 If M$(1, -1)$ is the midpoint of AB, and A is $(-3, 2)$, find the coordinates of B.

5 Find the gradient of the line segment joining:

   a $(5, -1)$ and $(-2, 6)$ **b** $(5, 0)$ and $(5, -2)$

6 The graph alongside shows the amount charged by a plumber according to the time he takes to do a job.

   a What does the value at A indicate?

   b Find the gradients of the line segments AB and BC. What do these gradients indicate?

   c If a straight line segment was drawn from A to C, what would be its gradient? What would this gradient mean?

7 AB and CD are both diameters of the circle. Find:

   a the coordinates of D

   b the radius of the circle.
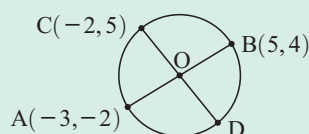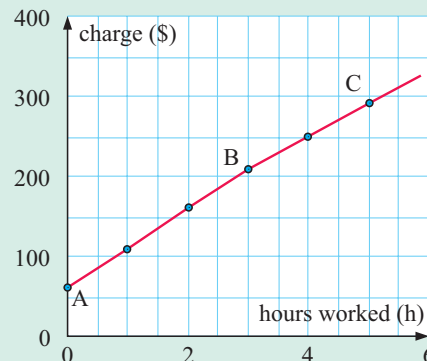
8 Find c if the line joining A$(5, 3)$ to B$(c, -2)$ is perpendicular to the line with gradient 3.

9 A$(-1, 2)$, B$(3, a)$ and C$(-3, 7)$ are collinear. Find a.

10 Given A$(-3, 2)$, B$(2, 3)$, C$(4, -1)$ and D$(-1, -2)$ are the vertices of quadrilateral ABCD:

   a Find the gradient of AB and DC.

   b Find the gradient of AD and BC.

   c What do you deduce from your answers to **a** and **b**?

   d Find the midpoints of the diagonals of the quadrilateral. What property of parallelograms does this check?

Challenge

1   Triangle ABC sits on the x-axis so that vertices A and B are equidistant from O.

   a   Find the length of AC.

   b   Find the length of BC.

   c   If   AC = BC,   deduce that $ab = 0$.

   d   Copy and complete the following statement based on the result of **c**.

   "*The perpendicular bisector of the base of an isosceles triangle ......*"

2
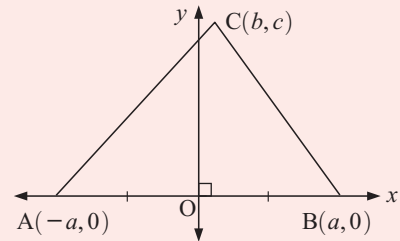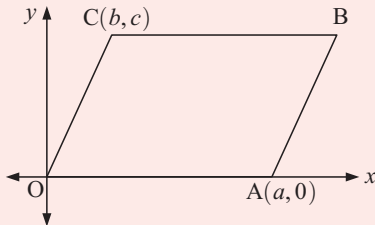
OABC is a parallelogram. You may assume that the opposite sides of the parallelogram are equal in length.

   a   Find the coordinates of B.

   b   Find the midpoints of AC and OB.

   c   What property of parallelograms has been deduced in **b**?

3   By considering the figure alongside:

   a   Find the equations of the perpendicular bisectors of OA and AB (these are the lines PS and SQ respectively).

   b   Use **a** to find the x-coordinate of S.

   c   Show that RS is perpendicular to OB.

   d   Copy and complete:

   "*The perpendicular bisectors of the sides of a triangle ......*"

Analysis of discrete data

13

Contents:

Historical note

- Florence Nightingale (1820-1910), the famous "lady with the lamp", developed and used graphs to represent data relating to hospitals and public health.

- Today about 92% of all nations conduct a census at regular intervals. The UN gives assistance to developing countries to help them with census procedures, so that accurate and comparable worldwide statistics can be collected.

Opening problem

A stretch of highway was notoriously dangerous, being the site of a large number of accidents each month.

In an attempt to rectify this, the stretch was given a safety upgrade. The road was resurfaced, and the speed limit was reduced.

To determine if the upgrade has made a difference, data is analysed for the number of accidents occurring each month for the three years before and the three years after the upgrade.
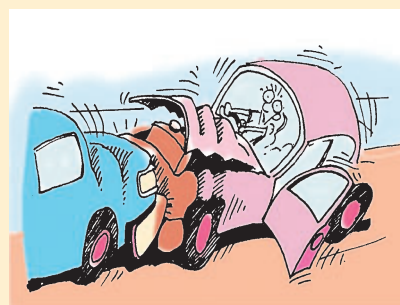
The results are:

Before upgrade   **After upgrade**

  8 4 9 7   11 10 8 8   6 10 10 9   4 7 8 3   8 4 6 7   8 4 7 7
  5 9 6 7   5   7 7 9   8   6   5 7   9 8 6 5   6 8 6 7   7 3 5 5
  8 7 2 6   9   7 10 6   8   7 10 8   9 7 8 7   6 6 7 8   8 9 7 4

Things to think about:

- Can you state clearly the problem that needs to be solved?
- What is the best way of organising this data?
- What are suitable methods for displaying the data?
- How can we best indicate what happens in a typical month on the highway?
- How can we best indicate the spread of the data?
- Can a satisfactory conclusion be made?

STATISTICS

Statistics is the art of solving problems and answering questions by collecting and analysing data.

The facts or pieces of information we collect are called **data**. Data is the plural of the word *datum*, which means a single piece of information.

A list of information is called a **data set** and because it is not in an organised form it is called **raw data**.

The process of **statistical enquiry** (or **investigation**) includes the following steps:

Step 1:   Examining a problem which may be solved using data and posing the correct question(s).
Step 2:   Collecting data.
Step 3:   Organising the data.
Step 4:   Summarising and displaying the data.
Step 5:   Analysing the data, making a conclusion in the form of a conjecture.
Step 6:   Writing a report.

CENSUS OR SAMPLE

The two ways to collect data are by census or sample.

A **census** is a method which involves collecting data about every individual in a *whole population*.

The individuals in a population may be people or objects. A census is detailed and accurate but is expensive, time consuming, and often impractical.

A **sample** is a method which involves collecting data about a *part of the population* only.

A sample is cheaper and quicker than a census but is not as detailed or as accurate. Conclusions drawn from samples always involve some error.

A sample must truly reflect the characteristics of the whole population. It must therefore be **unbiased** and **sufficiently large**.
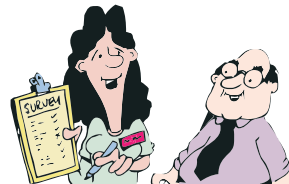
A **biased sample** is one in which the data has been unfairly influenced by the collection process and is not truly representative of the whole population.

A   VARIABLES USED IN STATISTICS   [11.2]

There are two types of variables that we commonly deal with: **categorical** variables and **quantitative** variables.

> A **categorical variable** is one which describes a particular quality or characteristic.
> It can be divided into **categories**. The information collected is called **categorical data**.

Examples of categorical variables are:

- *Getting to school:*   the categories could be train, bus, car and walking.
- *Colour of eyes:*   the categories could be blue, brown, hazel, green, and grey.

We saw examples of categorical variables in **Chapter 5**.

> A **quantitative variable** is one which has a numerical value, and is often called a **numerical variable**. The information collected is called **numerical data**.

Quantitative variables can be either **discrete** or **continuous**.

> A **quantitative discrete variable** takes exact number values and is often a result of **counting**.

Examples of discrete quantitative variables are:

- *The number of people in a household:*   the variable could take the values 1, 2, 3, ...
- *The score out of 30 for a test:*   the variable could take the values 0, 1, 2, 3, ..., 30.
- *The times on a digital watch:* $12:15$

> A **quantitative continuous variable** takes numerical values within a certain continuous range. It is usually a result of **measuring**.

Examples of quantitative continuous variables are:

- *The weights of newborn babies:*   the variable could take any positive value on the number line but is likely to be in the range 0.5 kg to 7 kg.
- *The heights of Year 10 students:*   the variable would be measured in centimetres. A student whose height is recorded as 145 cm could have exact height anywhere between 144.5 cm and 145.5 cm.
- *The times on an analogue watch:*

In this chapter we will focus on **discrete** variables. Continuous variables will be covered in **Chapter 17**.

INTERNET STATISTICS

There are thousands of sites worldwide which display statistics for everyone to see. Sites which show statistics that are important on a global scale include:

- *www.un.org*   for the United Nations

- *www.who.int*   for the World Health Organisation

EXERCISE 13A

1 Classify the following variables as either categorical or numerical:

    a the brand of shoes a person wears   **b** the number of cousins a person has

    c voting intention at the next election   **d** the number of cars in a household

    e the temperature of coffee in a mug   **f** favourite type of apple

    g town or city where a person was born   **h** the cost of houses on a street

2 Write down the possible categories for the following categorical variables:

    a gender   **b** favourite football code   **c** hair colour

3 State whether a census or a sample would be used for these investigations:

    a the reasons for people using taxis

    b the heights of the basketballers at a particular school

    c finding the percentage of people in a city who suffer from asthma

    d the resting pulse rates of members of your favourite sporting team

    e the number of pets in Canadian households

    f the amount of daylight each month where you live

4 Discuss any possible bias in the following situations:

    a Only Year 12 students are interviewed about changes to the school uniform.

    b Motorists stopped in peak hour are interviewed about traffic problems.

    c A phone poll where participants must vote by text message.

    d A 'who will you vote for' survey at an expensive city restaurant.

5 For each of the following possible investigations, classify these quantitative variables as quantitative discrete or quantitative continuous:

    a the number of clocks in each house

    b the weights of the members of a basketball team

    c the number of kittens in each litter

    d the number of bread rolls bought each week by a family

    e the number of leaves on a rose plant stem

    f the amount of soup in each can

    g the number of people who die from heart attacks each year in a given city

    h the amount of rainfall in each month of the year

    i the stopping distances of cars travelling at 80 km/h

    j the number of cars passing through an intersection each hour

B   ORGANISING AND DESCRIBING DISCRETE DATA   *[11.2, 11.3]*

In the **Opening Problem** on page **275**, the quantitative discrete variable is *the number of accidents per month*.

To organise the data a **tally-frequency table** could be used. We count the data systematically and use a '|' to indicate each data value. We use |||| to represent 5.

Below is the table for *Before upgrade*:

No. of accidents/month	Tally	Frequency
2	|	1
3		0
4	|	1
5	|||	3
6	||||	5
7	|||| |||	8
8	|||| ||	7
9	||||	5
10	||||	5
11	|	1
	Total	36

A **vertical bar chart** could be used to display this data:
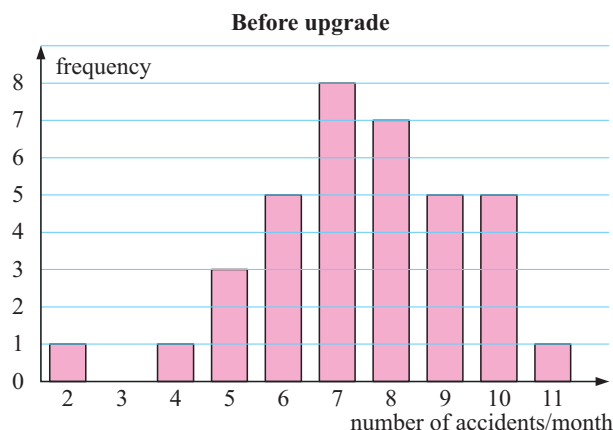
DESCRIBING THE DISTRIBUTION OF A DATA SET

The **mode** of a data set is the most frequently occurring value(s). Many data sets show **symmetry** or **partial symmetry** about the mode.

If we place a curve over the vertical bar chart we see that this curve shows symmetry. We say that we have a **symmetrical distribution**.
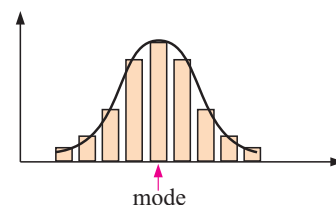
The distribution alongside is said to be **negatively skewed** because, by comparison with the symmetrical distribution, it has been 'stretched' on the left (or negative) side of the mode.
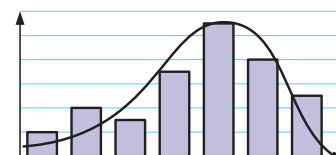
So, we have:
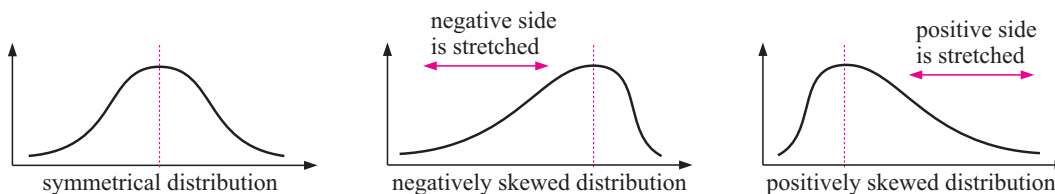
EXERCISE 13B.1

1 20 students were asked "How many pets do you have in your household?" and the following data was collected: 2 1 0 3 1 2 1 3 4 0 0 2 2 0 1 1 0 1 0 1

  a What is the variable in this investigation?

  b Is the data discrete or continuous? Why?

  c Construct a vertical bar chart to display the data. Use a heading for the graph, and add an appropriate scale and label to each axis.

  d How would you describe the distribution of the data? Is it symmetrical, positively skewed or negatively skewed?

  e What percentage of the households had no pets?

  f What percentage of the households had three or more pets?

2 A randomly selected sample of shoppers was asked, 'How many times did you shop at a supermarket in the past week?' A column graph was constructed for the results.

Supermarket shoppers

  a How many shoppers gave data in the survey?

  b How many of the shoppers shopped once or twice?

  c What percentage of the shoppers shopped more than four times?

  d Describe the distribution of the data.
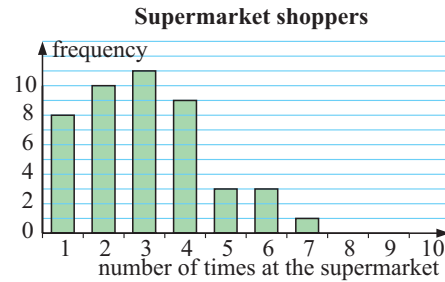
3 The number of toothpicks in a box is stated as 50 but the actual number of toothpicks has been found to vary. To investigate this, the number of toothpicks in a box was counted for a sample of 60 boxes:

50 52 51 50 50 51 52 49 50 48 51 50 47 50 52 48 50 49 51 50
49 50 52 51 50 50 52 50 53 48 50 51 50 50 49 48 51 49 52 50
49 49 50 52 50 51 49 52 52 50 49 50 49 51 50 50 51 50 53 48

  a What is the variable in this investigation?

  b Is the data continuous or discrete?

  c Construct a frequency table for this data.

  d Display the data using a bar chart.

  e Describe the distribution of the data.

  f What percentage of the boxes contained exactly 50 toothpicks?

4 Revisit the **Opening Problem** on page **275**. Using the *After upgrade* data:

  a Organise the data in a tally-frequency table.

  b Is the data skewed?

  c Draw a side-by-side vertical bar chart of the data. (Use the graph on page **279**.)

  d What evidence is there that the safety upgrade has made a difference?

GROUPED DISCRETE DATA

In situations where there are lots of different numerical values recorded, it may not be practical to use an ordinary tally-frequency table. In these cases, it is often best to **group** the data into **class intervals**. We can then display the grouped data in a bar chart.

For example, a local hardware store is concerned about the number of people visiting the store at lunch time.

Over 30 consecutive week days they recorded data.

The results were:

37 30 17 13 46 23 40 28 38 24 23 22 18 29 16
35 24 18 24 44 32 54 31 39 32 38 41 38 24 32

In this case we group the data into class intervals of length 10. The tally-frequency table is shown below. We use the table to construct the vertical bar chart below.

The first column represents the values from 10 to 19, the second from 20 to 29, and so on.

Number of people	Tally	Frequency
10 to 19	⊬⊬	5
20 to 29	⊬⊬ ||||	9
30 to 39	⊬⊬ ⊬⊬ |	11
40 to 49	||||	4
50 to 59	|	1
	Total	30

Lunch-time customers
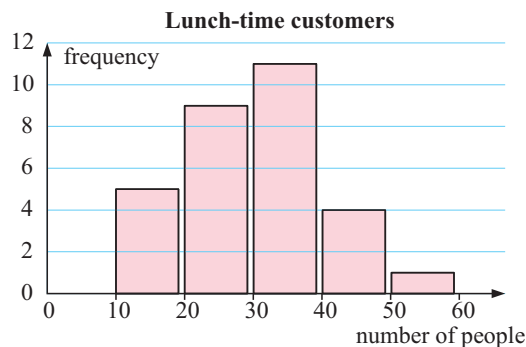
EXERCISE 13B.2

1 Forty students were asked to count the number of red cars they saw on the way to school one morning. The results are shown alongside.

23	7	15	32	13	44	28	15	20	48
16	34	32	5	22	11	31	10	24	16
33	45	20	24	13	9	24	16	5	47
37	23	27	19	21	29	37	11	30	28

a Construct a tally-frequency table for this data using class intervals 0 - 9, 10 - 19, ..., 40 - 49.

b Display the data on a vertical bar chart.

c How many students saw less than 20 red cars?

d What percentage of students saw at least 30 red cars?

e What is the modal class for the data?

2 The data gives the number of chairs made each day by a furniture production company over 26 days:

38	27	29	33	18	22	42	27	36	30	16	23	34
22	19	37	28	44	37	25	33	40	22	16	39	25

a Construct a tally and frequency table for this data.

b Draw a vertical bar chart to display the data.

c On what percentage of days were less than 40 chairs made?

d On how many days were at least 30 chairs made?

e Find the modal class for the data.

3 Over a 6 week period, a museum keeps a record of the number of visitors it receives each day. The results are:

515	432	674	237	445	510	585	411	605	332	196
432	527	421	318	570	640	298	554	611	458	322
584	232	674	519	174	377	543	630	490	501	
139	549	322	612	222	625	582	381	459	609	

a Construct a tally and frequency table for this data using class intervals 0 - 99, 100 - 199, 200 - 299, ..., 600 - 699.

b Draw a vertical bar chart to display the data.

c On how many days did the museum receive at least 500 visitors?

d What is the modal class for the data?

e Describe the distribution of the data.

4 Thirty high schools were surveyed to find out how many Year 12 students there were at each school. The following data was collected:

82	57	75	71	66	80	92	74	89	70
74	60	98	70	83	55	67	85	78	62
83	71	77	69	62	83	94	88	64	75

   a Construct a tally and frequency table for this data.

   b Draw a vertical bar chart to display the data.

   c How many schools have at least 80 Year 12 students?

   d What percentage of the schools have less than 70 Year 12 students?

   e Find the modal class for the data.

   f Describe the distribution of the data.

C   THE CENTRE OF A DISCRETE DATA SET [11.4]

We can get a better understanding of a data set if we can locate the **middle** or **centre** of the data and get an indication of its **spread**. Knowing one of these without the other is often of little use.

There are *three statistics* that are used to measure the **centre** of a data set.
These are: the **mean**, the **median** and the **mode**.

THE MEAN

The **mean** of a data set is the statistical name for the arithmetic average.

$$\text{mean} = \frac{\text{the sum of all data values}}{\text{the number of data values}}$$

or $\overline{x} = \dfrac{\sum x}{n}$ where $\sum x$ is the sum of the data.

The mean gives us a single number which indicates a centre of the data set. It is not necessarily a member of the data set.

For example, a mean test mark of 67% tells us that there are several marks below 67% and several above it. 67% is at the centre, but it does not mean that one of the students scored 67%.

THE MEDIAN

The **median** is the *middle value* of an ordered data set.

An ordered data set is obtained by listing the data, usually from smallest to largest.
The median splits the data in halves. Half of the data are less than or equal to the median and half are greater than or equal to it.

For example, if the median mark for a test is 67% then you know that half the class scored less than or equal to 67% and half scored greater than or equal to 67%.

For an **odd number** of data, the median is one of the data.

For an **even number** of data, the median is the average of the two middle values and may not be one of the original data.

If there are n data values, find the value of $\dfrac{n+1}{2}$.

The median is the $\left(\dfrac{n+1}{2}\right)$th data value.

For example:

If $n = 13$, $\frac{13+1}{2} = 7$, so the median = 7th ordered data value.

If $n = 14$, $\frac{14+1}{2} = 7.5$, so the median = average of 7th and 8th ordered data values.

THE MODE

The **mode** is the most frequently occurring value in the data set.

Example 1   ◀) Self Tutor

The number of small aeroplanes flying into a remote airstrip over a 15-day period
is 5 7 0 3 4 6 4 0 5 3 6 9 4 2 8.   For this data set, find:

  a the mean   **b** the median   **c** the mode.

a mean $= \dfrac{5+7+0+3+4+6+4+0+5+3+6+9+4+2+8}{15}$ $\longleftarrow \Sigma x$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \longleftarrow n$

$\qquad\quad\ = \frac{66}{15}$

$\qquad\quad\ = 4.4$ aeroplanes

b The ordered data set is: $\overline{0\,0\,2\,3\,3\,4\,4}\ 4\ \overline{5\,5\,6\,6\,7\,8\,9}$ {as $n = 15$, $\frac{n+1}{2} = 8$}

$\qquad\therefore$ median $= 4$ aeroplanes

c 4 is the score which occurs the most often $\therefore$ mode $= 4$ aeroplanes

Suppose that on the next day, 6 aeroplanes land on the airstrip in **Example 1**. We need to recalculate the
measures of the centre to see the effect of this new data value.

We expect the mean to rise as the new data value is greater than the old mean.

In fact, the new mean $= \dfrac{66+6}{16} = \dfrac{72}{16} = 4.5$ aeroplanes.

The new ordered data set is: $\overline{0\,0\,2\,3\,3\,4\,4}\ \underbrace{4\ 5}\ \overline{5\,6\,6\,6\,7\,8\,9}$
$\qquad\qquad\qquad\qquad\qquad\qquad\quad$ two middle scores

$\therefore$ median $= \dfrac{4+5}{2} = 4.5$ aeroplanes

This new data set has two modes, 4 and 6 aeroplanes, and we say that the data set is **bimodal**.

If a data set has three or more modes, we do not use the mode as a measure of the middle.

Note that equal or approximately equal values of the mean, mode and median *may* indicate a **symmetrical
distribution** of data. However, we should always check using a graph before calling a data set symmetric.

Example 2
◀)) *Self Tutor*

Solve the following problems:

a The mean of six scores is 78.5. What is the sum of the scores?

b Find x if 10, 7, 3, 6 and x have a mean of 8.

a $\dfrac{\text{sum}}{6} = 78.5$

$\therefore \quad \text{sum} = 78.5 \times 6$

$\qquad\qquad = 471$

$\therefore \quad$ the sum of the scores is 471.

b   There are 5 scores.

$\therefore \quad \dfrac{10 + 7 + 3 + 6 + x}{5} = 8$

$\therefore \quad \dfrac{26 + x}{5} = 8$

$\therefore \quad 26 + x = 40$

$\therefore \quad x = 14$

EXERCISE 13C

1 Find the   **i**   mean   **ii**   median   **iii**   mode   for each of the following data sets:

   a   12, 17, 20, 24, 25, 30, 40

   b   8, 8, 8, 10, 11, 11, 12, 12, 16, 20, 20, 24

   c   7.9, 8.5, 9.1, 9.2, 9.9, 10.0, 11.1, 11.2, 11.2, 12.6, 12.9

   d   427, 423, 415, 405, 445, 433, 442, 415, 435, 448, 429, 427, 403, 430, 446, 440, 425, 424, 419, 428, 441

2 Consider the following two data sets:

     Data set A: 5, 6, 6, 7, 7, 7, 8, 8, 9, 10, 12   *Data set B:*   5, 6, 6, 7, 7, 7, 8, 8, 9, 10, 20

   a   Find the mean for each data set.

   b   Find the median for each data set.

   c   Explain why the mean of *Data set A* is less than the mean of *Data set B*.

   d   Explain why the median of *Data set A* is the same as the median of *Data set B*.

3 The selling price of nine houses are:

         $158 000, $290 000, $290 000, $1.1 million, $900 000, $395 000,
         $925 000, $420 000, $760 000

   a   Find the mean, median and modal selling prices.

   b   Explain why the mode is an unsatisfactory measure of the middle in this case.

   c   Is the median a satisfactory measure of the middle of this data set?

4 The following raw data is the daily rainfall (to the nearest millimetre) for the month of February 2007 in a city in China: 0, 4, 1, 0, 0, 0, 2, 9, 3, 0, 0, 0, 8, 27, 5, 0, 0, 0, 0, 8, 1, 3, 0, 0, 15, 1, 0, 0

   a   Find the mean, median and mode for the data.

   b   Give a reason why the median is not the most suitable measure of centre for this set of data.

   c   Give a reason why the mode is not the most suitable measure of centre for this set of data.

5 A basketball team scored 38, 52, 43, 54, 41 and 36 points in their first six matches.

   a Find the mean number of points scored for the first six matches.

   b What score does the team need to shoot in their next match to maintain the same mean score?

   c The team scores only 20 points in the seventh match. What is the mean number of points scored for the seven matches?

   d If the team scores 42 points in their eighth and final match, will their previous mean score increase or decrease? Find the mean score for all eight matches.

6 The mean of 12 scores is 8.8. What is the sum of the scores?

7 While on a camping holiday, Lachlan drove on average, 214 km per day for a period of 8 days. How far did Lachlan drive in total while on holiday?

8 The mean monthly sales for a CD store are $216 000. Calculate the total sales for the store for the year.

9 Find x if 7, 15, 6, 10, 4 and x have a mean of 9.

10 Find a, given that 10, a, 15, 20, a, a, 17, 7 and 15 have a mean of 12.

11 Over a semester, Jamie did 8 science tests. Each was marked out of 30 and Jamie averaged 25. However, when checking his files, he could only find 7 of the 8 tests. For these he scored 29, 26, 18, 20, 27, 24 and 29. Determine how many marks out of 30 he scored for the eighth test.

12 A sample of 12 measurements has a mean of 8.5 and a sample of 20 measurements has a mean of 7.5. Find the mean of all 32 measurements.

13 In the United Kingdom, the months of autumn are September, October and November. If the mean temperature was S^oC for September, O^oC for October and N^oC for November, find an expression for the mean temperature A^oC for the whole of autumn.

14 The mean, median and mode of seven numbers are 8, 7 and 6 respectively. Two of the numbers are 8 and 10. If the smallest of the seven numbers is 4, find the largest of the seven numbers.
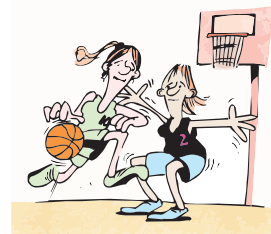
D MEASURING THE SPREAD OF DISCRETE DATA [11.4]

Knowing the middle of a data set can be quite useful, but for a more accurate picture of the data set we also need to know its spread.

For example,   2, 3, 4, 5, 6, 7, 8, 9, 10   has a mean value of 6 and so does

4, 5, 5, 6, 6, 6, 7, 7, 8.   However, the first data set is more widely spread than the second one.

Three commonly used statistics that indicate the spread of a set of data are the

   ● **range**   ● **interquartile range**   ● **standard deviation**.

The standard deviation, which is the spread about the mean, will not be covered in this course.

THE RANGE

The **range** is the difference between the **maximum** (largest) data value and the **minimum** (smallest) data value.

range = maximum data value − minimum data value

Example 3   ◀》 *Self Tutor*

Find the range of the data set: 5, 3, 8, 4, 9, 7, 5, 6, 2, 3, 6, 8, 4.

range = maximum value − minimum value = $9 - 2 = 7$

THE QUARTILES AND THE INTERQUARTILE RANGE

The **median** divides an ordered data set into halves, and these halves are divided in half again by the **quartiles**.

The middle value of the lower half is called the **lower quartile**. One quarter, or 25%, of the data have values less than or equal to the lower quartile. 75% of the data have values greater than or equal to the lower quartile.

The middle value of the upper half is called the **upper quartile**. One quarter, or 25%, of the data have values greater than or equal to the upper quartile. 75% of the data have values less than or equal to the upper quartile.

The **interquartile range** is the range of the middle half (50%) of the data.

interquartile range = upper quartile − lower quartile

The data is thus divided into quarters by the lower quartile Q_1, the median Q_2, and the upper quartile Q_3.

So, the interquartile range,   **IQR = Q_3 − Q_1**.

Example 4   ◀》 *Self Tutor*

For the data set: 7, 3, 4, 2, 5, 6, 7, 5, 5, 9, 3, 8, 3, 5, 6   find the:
   a   median   **b**   lower and upper quartiles   **c**   interquartile range.

The ordered data set is: 2 3 3 3 4 5 5 5 5 6 6 7 7 8 9   (15 of them)

a   As $n = 15$, $\dfrac{n+1}{2} = 8$ ∴   the median = 8th score = 5

b   As the median is a data value, we now ignore it and split the remaining data into two:

             lower   upper Q_1 = median of lower half = 3

      2 3 3 3 4 5 5   5 6 6 7 7 8 9 Q_3 = median of upper half = 7

c   IQR = Q_3 − Q_1 = $7 - 3 = 4$

Example 5

◀) *Self Tutor*

For the data set: 6, 10, 7, 8, 13, 7, 10, 8, 1, 7, 5, 4, 9, 4, 2, 5, 9, 6, 3, 2 find the:

a   median   **b**   lower and upper quartiles   **c**   interquartile range.

The ordered data set is: 1 2 2 3 4 4 5 5 6 6 7 7 7 8 8 9 9 10 10 13   (20 of them)

a   As $n = 20$, $\dfrac{n+1}{2} = 10.5$

$\therefore$ median $= \dfrac{\text{10th value } + \text{ 11th value}}{2} = \dfrac{6+7}{2} = 6.5$

b   As the median is not a data value we split the data into two:

$$\begin{array}{cc} \text{lower} & \text{upper} \\ \underbrace{1\,2\,2\,3\ \underbrace{4\,4}\ 5\,5\,6\,6} & \underbrace{7\,7\,7\,8\ \underbrace{8\,9}\ 9\,10\,10\,13} \\ Q_1 = 4 & Q_3 = 8.5 \end{array}$$

c   IQR $= Q_3 - Q_1$ **Note:**   Some computer packages (for example, **MS Excel**)
   $= 8.5 - 4$ calculate quartiles in a different way from this example.
   $= 4.5$

EXERCISE 13D

1   For each of the following data sets, make sure the data is ordered and then find:

   i   the median   **ii**   the upper and lower quartiles
   iii   the range   **iv**   the interquartile range.

a   5, 6, 6, 6, 7, 7, 7, 8, 8, 8, 8, 9, 9, 9, 9, 9, 10, 10, 11, 11, 11, 12, 12

b   11, 13, 16, 13, 25, 19, 20, 19, 19, 16, 17, 21, 22, 18, 19, 17, 23, 15

c   23.8, 24.4, 25.5, 25.5, 26.6, 26.9, 27, 27.3, 28.1, 28.4, 31.5

2   The times spent (in minutes) by 24 people in a queue at a supermarket, waiting to be served at the checkout, were:

1.4   5.2   2.4   2.8   3.4   3.8   2.2   1.5
0.8   0.8   3.9   2.3   4.5   1.4   0.5   0.1
1.6   4.8   1.9   0.2   3.6   5.2   2.7   3.0

a   Find the median waiting time and the upper and lower quartiles.

b   Find the range and interquartile range of the waiting time.

c   Copy and complete the following statements:

   i   "50% of the waiting times were greater than .... minutes."
   ii   "75% of the waiting times were less than ...... minutes."
   iii   "The minimum waiting time was .... minutes and the maximum waiting time was ..... minutes. The waiting times were spread over ...... minutes."

3   For the data set given, find:

Stem	Leaf
2	0 1 2 2
3	0 0 1 4 4 5 8
4	0 2 3 4 6 6 9
5	1 1 4 5 8

5|1 represents 51

- **a** the minimum value
- **b** the maximum value
- **c** the median
- **d** the lower quartile
- **e** the upper quartile
- **f** the range
- **g** the interquartile range.

E   DATA IN FREQUENCY TABLES   [11.4]

When the same data appears several times we often summarise it in a **frequency table**. For convenience we denote the data values by x and the frequencies of these values by f.

The mode

There are 14 of data value 6 which is more than any other data value.

The mode is therefore 6.

Data value	Frequency	Product
3	1	$1 \times 3 = 3$
4	2	$2 \times 4 = 8$
5	4	$4 \times 5 = 20$
6	14	$14 \times 6 = 84$
7	11	$11 \times 7 = 77$
8	6	$6 \times 8 = 48$
9	2	$2 \times 9 = 18$
Total	40	258

The mean

A '*Product*' column helps to add all scores.

We know the mean $= \dfrac{\text{the sum of all data values}}{\text{the number of data values}}$, so for data in a frequency table, $$\bar{x} = \frac{\sum fx}{\sum f}$$

In this case the mean $= \dfrac{\sum fx}{\sum f} = \dfrac{258}{40} = 6.45$.

The median

There are 40 data values, an even number, so there are *two middle* data values.

As the sample size $n = 40$, $\dfrac{n+1}{2} = \dfrac{41}{2} = 20.5$

$\therefore$ the median is the average of the 20th and 21st data values.

In the table, the blue numbers show us accumulated values.

Remember that the median is the middle of the *ordered* data set.

Data Value (x)	Frequency (f)	
3	1	1 one number is 3
4	2	3 3 numbers are 4 or less
5	4	7 7 numbers are 5 or less
6	14	21 21 numbers are 6 or less
7	11	32 32 numbers are 7 or less
8	6	
9	2	
Total	40	

We can see that the 20th and 21st data values (in order) are both 6s.

$\therefore$ median $= \dfrac{6+6}{2} = 6$.

Notice that we have a skewed distribution for which the mean, median and mode are nearly equal. This is why we need to be careful when we use measures of the middle to call distributions symmetric.

Example 6   ◀ *Self Tutor*

Each student in a class of 20 is assigned a number between 1 and 10 to indicate his or her fitness.

Calculate the: **a** mean   **b** median
                 c mode   **d** range

of the scores.

Score	Number of students
5	1
6	2
7	4
8	7
9	4
10	2
Total	20

a

Score (x)	Frequency (f)	Product (fx)
5	1	$5 \times 1 = 5$
6	2	$6 \times 2 = 12$
7	4	$7 \times 4 = 28$
8	7	$8 \times 7 = 56$
9	4	$9 \times 4 = 36$
10	2	$10 \times 2 = 20$
Total	20	157

The mean score $= \dfrac{\sum fx}{\sum f}$

$= \dfrac{157}{20}$

$= 7.85$

b There are 20 scores, and so the median is the average of the 10th and 11th.

Score	Number of students
5	1	← 1st student
6	2	← 2nd and 3rd student
7	4	← 4th, 5th, 6th and 7th student
8	7	← 8th, 9th, **10th**, **11th**, 12th, 13th, 14th student
9	4	
10	2	

STATISTICS PACKAGE
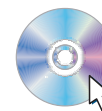
The 10th and 11th students both scored 8   ∴   median $= 8$.

c Looking down the 'number of students' column, the highest frequency is 7. This corresponds to a score of 8, so the mode $= 8$.

d The range $=$ highest data value $-$ lowest data value $= 10 - 5 = 5$

EXERCISE 13E

1 The members of a school band were each asked how many musical instruments they played. The results were:

Number of instruments	Frequency
1	18
2	14
3	6
4	4
Total	42

Calculate the: **a** mode   **b** median   **c** mean   **d** range.

2 The following frequency table records the number of books read in the last year by 50 fifteen-year-olds.

No. of books	Frequency
0	2
1	4
2	7
3	4
4	2
5	0
6	1
7	8
8	13
9	7
10	2

  a For this data, find the:
    i mean   **ii** median   **iii** mode   **iv** range.
  b Construct a vertical bar chart for the data and show the position of the measures of centre (mean, median and mode) on the horizontal axis.
  c Describe the distribution of the data.
  d Why is the mean smaller than the median for this data?
  e Which measure of centre would be most suitable for this data set?

3 Hui breeds ducks. The number of ducklings surviving for each pair after one month is recorded in the table.

Number of survivors	Frequency
0	1
1	2
2	5
3	9
4	20
5	30
6	9
Total	76

  a Calculate the:
    i mean   **ii** mode   **iii** median.
  b Calculate the range of the data.
  c Is the data skewed?
  d How does the skewness of the data affect the measures of the middle of the distribution?

4 Participants in a survey were asked how many foreign countries they had visited. The results are displayed in the vertical bar chart alongside:

  a Construct a frequency table from the graph.
  b How many people took part in the survey?
  c Calculate the:
    i mean   **ii** median
    iii mode   **iv** range   of the data.
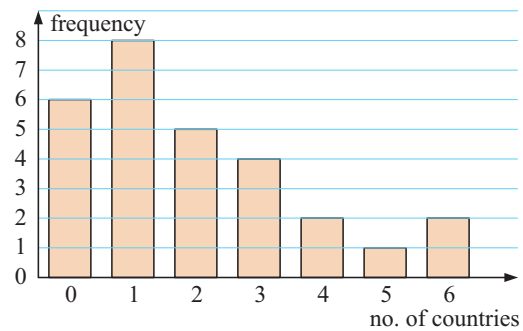
F   GROUPED DISCRETE DATA   [11.4]

One issue to consider when grouping data into class intervals is that the original data is lost. This means that calculating the exact mean and median becomes impossible. However, we can still *estimate* these values, and in general this is sufficient.

THE MEAN

When information has been grouped in classes we use the **midpoint** of the class to represent all scores within that interval.

We are assuming that the scores within each class are evenly distributed throughout that interval. The mean calculated will therefore be an **estimate** of the true value.

Example 7

◀) **Self Tutor**

The table summarises the marks received by students for a Physics examination out of 50.

Class interval	Frequency
0 - 9	2
10 - 19	31
20 - 29	73
30 - 39	85
40 - 49	28

a Estimate the mean mark.

b What is the modal class?

c Can the range of the data be found?

Class interval	f	Mid-point (x)	fx
0 - 9	2	4.5	9.0
10 - 19	31	14.5	449.5
20 - 29	73	24.5	1788.5
30 - 39	85	34.5	2932.5
40 - 49	26	44.5	1157.0
Totals	217		6336.5

a Mean $= \dfrac{\sum fx}{\sum f}$

$= \dfrac{6336.5}{217}$

≈ 29.2

b The modal class is 30 - 39 marks.

c No, as we do not know the smallest and largest score.

THE MEDIAN

We can estimate the median of a grouped data set by using a **cumulative frequency graph** or **ogive**. This is done in **Chapter 17**.

The approximate median can also be calculated using a formula:

This formula is **not examinable**.

$$\text{median} \approx L + \frac{N}{F} \times I$$

where L = the lower boundary for the class interval containing the median

N = the number of scores in the median class needed to arrive at the middle score

F = the frequency of the class interval containing the median

I = the class interval length.

For example, in **Example 7** we notice that $n = 217$, so the median is the $\dfrac{217 + 1}{2} = 109$th score.

There are $2 + 31 + 73 = 106$ scores in the first 3 classes, so the median class is 30 - 39.

For discrete data, the lower and upper boundaries for the class interval 30 - 39 are 29.5 and 39.5, $\therefore$ $L = 29.5$.

$\frac{3}{85} \times 10$ is the fraction of the interval in which the median is found.

$N = 109 - 106 = 3$, $F = 85$ and $I = 10$

$\therefore$ the median $\approx 29.5 + \frac{3}{85} \times 10 \approx 29.9$

EXERCISE 13F

1 40 students receive marks out of 100 for an examination in Chemistry. The results were:

   70 65 50 40 58 72 39 85 90 65 53 75 83 92 66 78 82 88 56 68
   43 90 80 85 78 72 59 83 71 75 54 68 75 89 92 81 77 59 63 80

   a Find the exact mean of the data.

   b Find the median of the data.

   c Group the data into the classes 0 - 9, 10 - 19, 20 - 29, ..., 90 - 99, forming a frequency table. Include columns for the midpoint (x), and fx.

   d Estimate from the grouped data of **c** the:

       i mean   **ii** median.

   e How close are your answers in **d** to exact values in **a** and **b**?

2 A sample of 100 students was asked how many times they bought lunch from the canteen in the last four weeks. The results were:

Number of lunches	0-5	6-10	11-15	16-20
Frequency	28	45	15	12

   a Estimate the mean number of bought lunches for each student.

   b What is the modal class?

   c Estimate the median number of bought lunches.

3 The percentage marks for boys and girls in a science test are given in the table:

   a Estimate the mean mark for the girls.

   b Estimate the mean mark for the boys.

   c What can you deduce by comparing **a** and **b**?

Marks	Girls	Boys
21 - 30	2	0
31 - 40	6	5
41 - 50	10	8
51 - 60	8	11
61 - 70	4	5
71 - 80	5	6
81 - 90	3	4
91 - 100	1	0

G   STATISTICS FROM TECHNOLOGY   [11.8]

GRAPHICS CALCULATOR

A **graphics calculator** can be used to find descriptive statistics and to draw some types of graphs.

Consider the data set: 5 2 3 3 6 4 5 3 7 5 7 1 8 9 5

No matter what brand of calculator you use you should be able to:

- Enter the data as a list.
- Enter the statistics calculation part of the menu and obtain the descriptive statistics like these shown.

Instructions for these tasks can be found at the front of the book in the **Graphics Calculator Instructions** section.

$\overline{x}$ is the mean

```
1-Var Stats
x̄=4.866666667
Σx=73
Σx²=427
Sx=2.263583337
σx=2.186829262
↓n=15
```

```
1-Var Stats
↑n=15
minX=1
Q₁=3
Med=5
Q₃=7
maxX=9
```
5-number summary

COMPUTER PACKAGE

Various statistical packages are available for computer use, but many are expensive and often not easy to use. Click on the icon to use the statistics package on the CD.

Enter the data sets: Set 1:   5 2 3 3 6 4 5 3 7 5 7 1 8 9 5

                           Set 2:   9 6 2 3 5 5 7 5 6 7 6 3 4 4 5 8 4

STATISTICS PACKAGE

Examine the side-by-side bar charts.

Click on the Statistics tab to obtain the descriptive statistics.

Select Print... from the File menu to print all of these on one sheet of paper.

EXERCISE 13G

1 Helen and Jessica both play for the Lightning Basketball Club. The numbers of points scored by each of them over a 12-game period were:

   Helen: 20, 14, 0, 28, 38, 25, 26, 17, 24, 24, 6, 3

   Jessica: 18, 20, 22, 2, 18, 31, 7, 15, 17, 16, 22, 29

   a Calculate the mean and median number of points for both of them.
   b Calculate the range and interquartile range for both of them.
   c Which of the girls was the higher scorer during the 12-game period?
   d Who was more consistent?

2 Enter the **Opening Problem** data on page **275** for the *Before upgrade* data in Set 1 and the *After upgrade* data in Set 2 of the computer package. Print out the page of graphs and descriptive statistics. Write a brief report on the effect of the safety upgrade.

3 Use your graphics calculator to check the answers to **Example 6** on page **289**.

4 Use your graphics calculator to check the answers to **Example 7** on page **291**.

5 The heights (to the nearest centimetre) of boys and girls in a Year 10 class in Norway are as follows:

   Boys   165 171 169 169 172 171 171 180 168 168 166 168 170 165 171 173 187
            181 175 174 165 167 163 160 169 167 172 174 177 188 177 185 167 160
   Girls 162 171 156 166 168 163 170 171 177 169 168 165 156 159 165 164 154
            171 172 166 152 169 170 163 162 165 163 168 155 175 176 170 166

   a Use your calculator to find measures of centre (mean and median) and spread (range and IQR) for each data set.
   b Write a brief comparative report on the height differences between boys and girls in the class.

Review set 13A

1 Classify the following numerical variables as either discrete or continuous:
   a the number of oranges on each orange tree
   b the heights of seedlings after two weeks
   c the scores of team members in a darts competition.

2 A randomly selected sample of small businesses has been asked, "How many full-time employees are there in your business?". A bar chart has been constructed for the results.
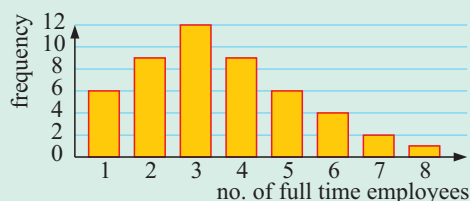
   a How many small businesses gave data in the survey?

   b How many of the businesses had only one or two full-time employees?

   c What percentage of the businesses had five or more full-time employees?

   d Describe the distribution of the data.

   e Find the mean of the data.

3 The data alongside are the number of call-outs each day for a city fire department over a period of 25 days:

29	14	23	32	28	30	9	24	31
30	18	22	27	15	32	26	22	19
16	23	38	12	8	22	31		

   a Construct a tally and frequency table for the data using class intervals 0 - 9, 10 - 19, 20 - 29, and 30 - 39.

   b Display the data on a vertical bar chart.

   c On how many days were there are least 20 call-outs?

   d On what percentage of days were there less than 10 call-outs?

   e Find the modal class for the data.

4 For the following data set of the number of points scored by a rugby team, find:

   a the mean   **b** the mode   **c** the median   **d** the range

   e the upper and lower quartiles   **f** the interquartile range.

   28, 24, 16, 6, 46, 34, 43, 16, 36, 49, 30, 28, 4, 31, 47, 41, 26, 25, 20, 29, 42

5 The test score out of 40 marks was recorded for a group of 30 students:

25	18	35	32	34	28	24	39	29	33
22	34	39	31	36	35	36	33	35	27
26	25	20	18	9	19	32	23	28	27

   a Construct a tally and frequency table for this data.

   b Draw a bar chart to display the data.

   c How many students scored less than 20 for the test?

   d If an 'A' was awarded to students who scored 30 or more for the test, what percentage of students scored an 'A'?

6 Eight scores have an average of six. Scores of 15 and x increase the average to 7. Find x.

7 For the data set given, find the:

   a minimum value   **b** maximum value

   c median   **d** lower quartile

   e upper quartile   **f** range

   g interquartile range.

Stem	Leaf
1	2 4 4 6 8
2	2 3 7 7 7 9 9
3	0 0 2 4 5 8
4	0 2 3

4|0 represents 40

8 Using the bar chart alongside:

   a construct a frequency table

   b determine the total number of:

   i games played   **ii** goals scored

   c find the:

   i mean   **ii** median

   iii mode   **iv** range.

Goals per game by a footballer
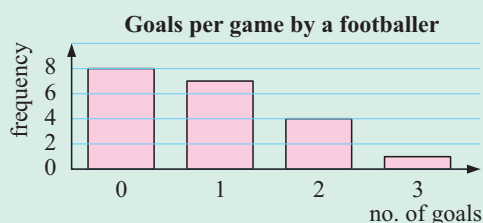
9 The numbers of potatoes growing on each of 100 potato plants were recorded and summarised in the table below:

No. of potatoes	0 - 2	3 - 5	6 - 8	9 - 11	12 - 14	15 - 17
Frequency	7	11	25	40	15	2

Estimate the mean number of potatoes per plant.

10 Use technology to find the

   a mean   **b** median   **c** lower quartile   **d** upper quartile

of the following data set:

$$\begin{array}{cccccccccc}
147 & 152 & 164 & 159 & 174 & 140 & 155 & 166 & 180 & 177 \\
156 & 182 & 174 & 168 & 170 & 165 & 143 & 185 & 126 & 148 \\
138 & 141 & 156 & 142 & 130 & 182 & 156 & 182 & 141 & 158
\end{array}$$

Review set 13B

1 A class of 20 students was asked "How many children are there in your household?" and the following data was collected:

      1 2 3 3 2 4 5 4 2 3 8 1 2 1 3 2 1 2 1 2

   a What is the variable in the investigation?

   b Is the data discrete or continuous? Explain your answer.

   c Construct a frequency table for the data.

   d Construct a vertical bar chart to display the data.

   e How would you describe the distribution of the data? Is it symmetrical, or positively or negatively skewed?

   f What is the mode of the data?

2 A class of thirty students were asked how many emails they had sent in the last week. The results were:

         12   6   21   15   18   4   28   32   17   44   9   32   26   18   11

         24   31   17   52   7   42   37   19   6   20   15   27   8   36   28

   a Construct a tally and frequency table for this data, using intervals 0 - 9, 10 - 19,......,50 - 59.

   b Draw a vertical bar chart to display the data.

   c Find the modal class.

   d What percentage of students sent at least 30 emails?

   e Describe the distribution of the data.

3 The local transport authority recorded the number of vehicles travelling along a stretch of road each day for 40 days. The data is displayed in the bar chart alongside:

   a On how many days were there at least 180 vehicles?

   b On what percentage of days were there less than 160 vehicles?

   c What is the modal class?

   d Estimate the mean number of vehicles on the stretch of road each day.
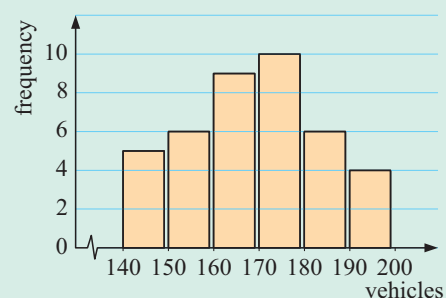
4 A sample of 15 measurements has a mean of 14.2 and a sample of 10 measurements has a mean of 12.6. Find the mean of the total sample of 25 measurements.

5 Determine the mean of the numbers 7, 5, 7, 2, 8 and 7. If two additional numbers, 2 and x, reduce the mean by 1, find x.

6 Jenny's golf scores for her last 20 rounds were:

    90 106 84 103 112 100 105 81 104 98
   107 95 104 108 99 101 106 102 98 101

    a Find the median, lower quartile and upper quartile of the data set.

    b Find the interquartile range of the data set and explain what it represents.

7 For the data displayed in the stem-and-leaf plot find the:

    a mean   **b** median

    c lower quartile   **d** upper quartile

    e range   **f** interquartile range

Stem	Leaf
3	8 9
4	2 2 5
5	0 1 1 7 9
6	1 3 4
7	0

7 | 0 represents 70

8 The given table shows the distribution of scores for a Year 10 spelling test in Austria.

    a Calculate the: **i** mean   **ii** mode

                 iii median   **iv** range of the scores

    b The average score for all Year 10 students across Austria in this spelling test was 6.2. How does this class compare to the national average?

    c The data set is skewed. Is the skewness positive or negative?

Score	Frequency
6	2
7	4
8	7
9	12
10	5
Total	30

9 Sixty people were asked: "How many times have you been to the cinema in the last twelve months?". The results are given in the table alongside.

Estimate the mean and median of the data.

No. of times	Frequency
0 - 4	19
5 - 9	24
10 - 14	10
15 - 19	5
20 - 24	2

10 The data below shows the number of pages contained in a random selection of books from a library:

295 612 452 182 335 410 256 715 221 375
508 310 197 245 411 162 95 416 372 777
411 236 606 192 487

    a Use technology to find the:

        i mean   **ii** median

        iii lower quartile   **iv** upper quartile

    b Find the range and interquartile range for the data.

LIBRARY

Straight lines

14

Contents:

Opening problem

On the coordinate axes alongside, three lines are drawn.

How can we describe using algebra the set of all points on:

- line (1)
- line (2)
- line (3)?
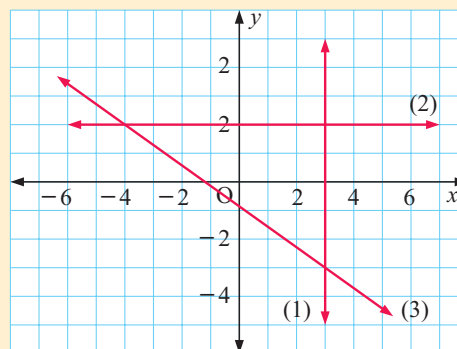
In **Chapter 12** we learnt how to find the gradient of a line segment. In this chapter we will see how the gradient can be used to find the **equation** of a line.

> The **equation of a line** is an equation which connects the x and y values for every point on the line.

A   VERTICAL AND HORIZONTAL LINES   [7.6]

To begin with we consider horizontal and vertical lines which are parallel to either the x or y-axis.

Discovery 1   *Vertical and horizontal lines*

What to do:

1 Using graph paper, plot the following sets of points on the Cartesian plane. Rule a line through each set of points.

   a $(3, 4)$, $(3, 2)$, $(3, 0)$, $(3, -2)$, $(3, -4)$

   b $(6, -1)$, $(6, -3)$, $(6, 1)$, $(6, 5)$, $(6, 3)$

   c $(0, -5)$, $(0, -2)$, $(0, 1)$, $(0, 4)$, $(0, -3)$

   d $(-3, -1)$, $(5, -1)$, $(-1, -1)$, $(4, -1)$, $(0, -1)$

   e $(-2, 6)$, $(-2, -3)$, $(-2, 0)$, $(-2, -2)$, $(-2, 2)$

   f $(4, 0)$, $(0, 0)$, $(7, 0)$, $(-1, 0)$, $(-3, 0)$

2 Can you state the gradient of each line? If so, what is it?

3 What do all the points on a vertical line have in common?

4 What do all the points on a horizontal line have in common?

5 Can you state the equation of each line?

VERTICAL LINES

> All **vertical** lines have **equations** of the form $x = a$.
> The gradient of a vertical line is **undefined**.

A sketch of the vertical lines $x = -2$ and $x = 1$ is shown alongside.

For all points on a vertical line, regardless of the value of the y-coordinate, the value of the x-coordinate is always the same.
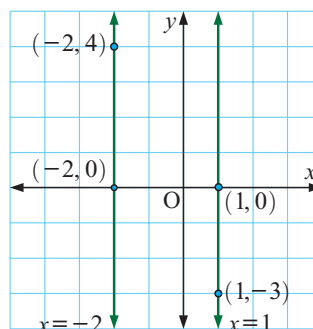
HORIZONTAL LINES

> All **horizontal** lines have **equations** of the form $y = b$.
> The gradient of a horizontal line is **zero**.

A sketch of the horizontal lines $y = -3$ and $y = 2$ is shown alongside.

For all points on a horizontal line, regardless of the value of the x-coordinate, the value of the y-coordinate is always the same.
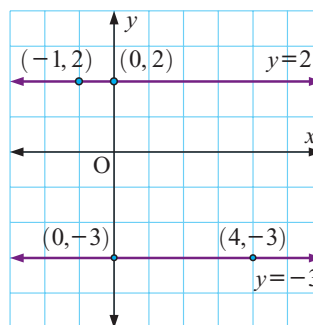
EXERCISE 14A

1 Identify as either a vertical or horizontal line and hence plot the graph of:

   a $y = 6$ **b** $x = -3$ **c** $x = 2$ **d** $y = -4$

2 Identify as either a vertical or horizontal line:

   a a line with zero gradient   **b** a line with undefined gradient

3 Find the equation of:

    a   the x-axis   **b**   the y-axis

    c   a line parallel to the x-axis and three units below it

    d   a line parallel to the y-axis and 4 units to the right of it.

4 Find the equation of:

    a   the line with zero gradient that passes through $(-1, \, 3)$

    b   the line with undefined gradient that passes through $(4, \, -2)$.

B   GRAPHING FROM A TABLE OF VALUES

In this section we will graph some straight lines from tables of values. Our aim is to identify some features of the graphs and determine which part of the equation of the line controls them.

GRAPHING FROM A TABLE OF VALUES

Consider the equation $y = 2x + 1$. We can choose any value we like for x and use our equation to find the corresponding value for y.

We can hence construct a **table of values** for points on the line.

x	-3	-2	-1	0	1	2	3
y	-5	-3	-1	1	3	5	7

For example: $\begin{aligned} y &= 2 \times -3 + 1 \\ &= -6 + 1 \\ &= -5 \end{aligned}$ $\begin{aligned} y &= 2 \times 2 + 1 \\ &= 4 + 1 \\ &= 5 \end{aligned}$

From this table we plot the points $(-3, \, -5), \ (-2, \, -3),$ $(-1, \, -1), \ (0, \, 1),$ and so on.

The tabled points are collinear and we can connect them with a straight line.

We can use the techniques from **Chapter 12** to find the gradient of the line. Using the points $(0, \, 1)$ and $(1, \, 3)$, the gradient is

$$\frac{y\text{-step}}{x\text{-step}} \quad \text{or} \quad \frac{\text{rise}}{\text{run}} = \frac{2}{1} = 2.$$
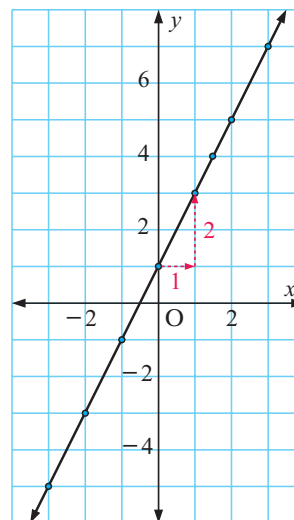
AXES INTERCEPTS

> The x-**intercept** of a line is the value of x where the line meets the x-axis.
>
> The y-**intercept** of a line is the value of y where the line meets the y-axis.

We can see that for the graph above, the x-intercept is $-\frac{1}{2}$ and the y-intercept is 1.

Example 1

◀ᴅ) *Self Tutor*

Consider the equation $y = x - 2$.

 a Construct a table of values using $x = -3, -2, -1, 0, 1, 2$ and 3.
 b Draw the graph of $y = x - 2$.
 c Find the gradient and axes intercepts of the line.

 a

x	-3	-2	-1	0	1	2	3
y	-5	-4	-3	-2	-1	0	1

 b

 c Using the points $(0, -2)$ and $(2, 0)$,

the gradient $= \dfrac{y\text{-step}}{x\text{-step}} = \dfrac{2}{2} = 1$.

The x-intercept is 2.
The y-intercept is -2.
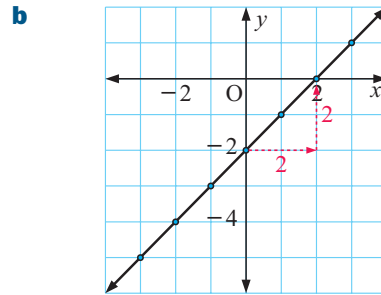
EXERCISE 14B

 1 For the following equations: **i** Construct a table of values for values of x from -3 to 3.
   ii Plot the graph of the line.
   iii Find the gradient and axes intercepts of the line.

 a $y = x$ **b** $y = 3x$ **c** $y = \frac{1}{3}x$ **d** $y = -3x$

 e $y = 2x + 1$ **f** $y = -2x + 1$ **g** $y = \frac{1}{2}x + 3$ **h** $y = -\frac{1}{2}x + 3$

 2 Arrange the graphs **1a**, **1b** and **1c** in order of steepness. What part of the equation controls the degree of steepness of a line?

 3 Compare the graphs of **1b** and **1d**. What part of the equation controls whether the graph is forward sloping or backward sloping?

 4 Compare the graphs of **1b**, **1e** and **1g**. What part of the equation controls where the graph cuts the y-axis?

Discovery 2   *Graphs of the form* $y = mx + c$

The use of a graphics calculator or suitable graphing package is recommended for this Discovery.

What to do:

 1 On the same set of axes graph the family of lines of the form $y = mx$:

   a where $m = 1, 2, 4, \frac{1}{2}, \frac{1}{5}$ **b** where $m = -1, -2, -4, -\frac{1}{2}, -\frac{1}{5}$

 GRAPHING
 PACKAGE

 2 What are the gradients of the lines in question **1**?

 3 What is your interpretation of m in the equation $y = mx$?

 4 On the same set of axes, graph the family of lines of the form $y = 2x + c$ where $c = 0, 2, 4, -1, -3$.

 5 What is your interpretation of c for the equation $y = 2x + c$?

IGCSE01
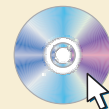
C EQUATIONS OF LINES (GRADIENT-INTERCEPT FORM)   [7.6]

$y = mx + c$ is the **gradient-intercept form** of the equation of a line with gradient m and y-intercept c.

For example:
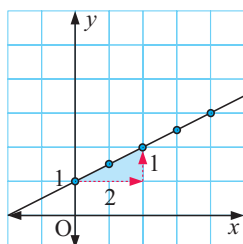
The illustrated line has gradient $= \dfrac{y\text{-step}}{x\text{-step}} = \frac{1}{2}$

and the y-intercept is 1

$\therefore$ its equation is $y = \frac{1}{2}x + 1$.

We can find the equation of a line if we are given:

- its gradient and the coordinates of one point on the line, **or**
- the coordinates of two points on the line.

Example 2   ◀)) Self Tutor

Find the equation of these lines:

a

b

a
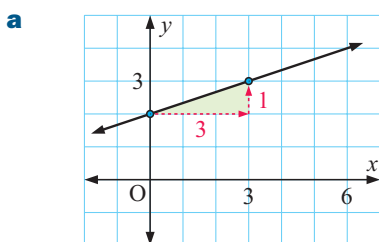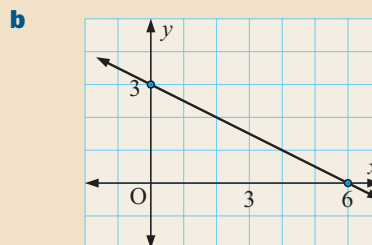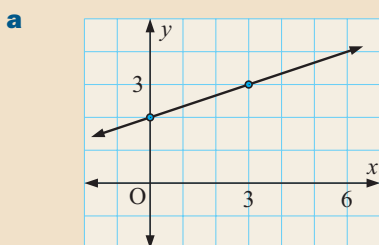
The gradient $m = \dfrac{y\text{-step}}{x\text{-step}} = \frac{1}{3}$.

The y-intercept $c = 2$.

$\therefore$ the equation is $y = \frac{1}{3}x + 2$

b
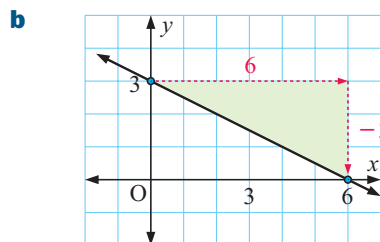
The gradient $m = \dfrac{y\text{-step}}{x\text{-step}} = \frac{-3}{6} = -\frac{1}{2}$.

The y-intercept $c = 3$.

$\therefore$ the equation is $y = -\frac{1}{2}x + 3$

Example 3   ◀) *Self Tutor*

Find the equation of a line:

a   with gradient 2 and y-intercept 5

b   with gradient $\frac{2}{3}$ which passes through $(6, -1)$.

a $m = 2$ and $c = 5$, so the equation is $y = 2x + 5$.

b $m = \frac{2}{3}$ so the equation is $y = \frac{2}{3}x + c$

But when $x = 6$, $y = -1$ {the point $(6, -1)$ lies on the line}

$$\therefore \quad -1 = \tfrac{2}{3}(6) + c$$
$$\therefore \quad -1 = 4 + c$$
$$\therefore \quad -5 = c$$

So, the equation is $y = \frac{2}{3}x - 5$.

Example 4   ◀) *Self Tutor*

Find the equation of a line passing through $(-1, 5)$ and $(3, -2)$.
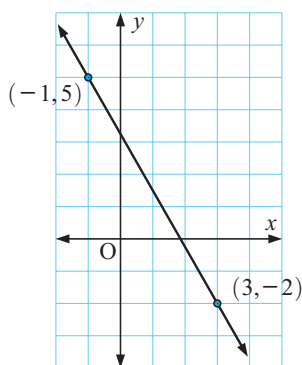
The gradient of the line $m = \dfrac{-2 - 5}{3 - -1} = -\dfrac{7}{4}$

$\therefore$ the equation is $y = -\frac{7}{4}x + c$

But when $x = -1$, $y = 5$

$$\therefore \quad 5 = -\tfrac{7}{4}(-1) + c$$
$$\therefore \quad 5 = \tfrac{7}{4} + c$$
$$\therefore \quad \tfrac{20}{4} = \tfrac{7}{4} + c$$
$$\therefore \quad c = \tfrac{13}{4}$$

So, the equation is $y = -\frac{7}{4}x + \frac{13}{4}$.

EXERCISE 14C

1   Find the equation of a line:

   a   with gradient 3 and y-intercept -2 **b**   with gradient -4 and y-intercept 8

   c   with gradient $\frac{1}{2}$ and y-intercept $\frac{2}{3}$ **d**   with gradient $-\frac{2}{3}$ and y-intercept $\frac{3}{4}$.

2   Find the gradient and y-intercept of a line with equation:

   a $y = 3x + 11$ **b** $y = -2x + 6$ **c** $y = \frac{1}{2}x$

   d $y = -\frac{1}{3}x - 2$ **e** $y = 3$ **f** $x = 8$

   g $y = 3 - 2x$ **h** $y = -1 + \frac{1}{2}x$ **i** $y = \dfrac{3x + 1}{2}$

   j $y = \dfrac{2x - 1}{3}$ **k** $y = \dfrac{1 - x}{4}$ **l** $y = \dfrac{3 - 2x}{5}$

3 Find the equations of these lines:

a

b

c

d

e

f
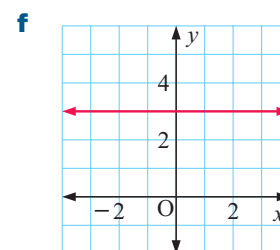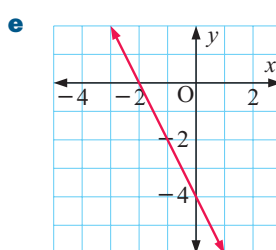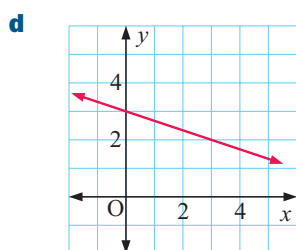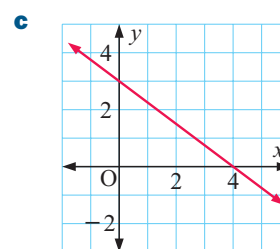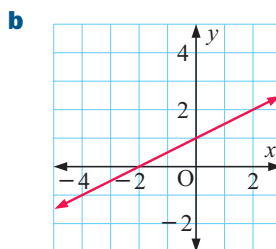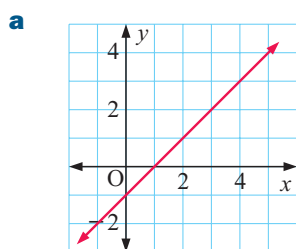
4 Find the equation of a line with:

  a gradient 2 which passes through the point $(1, 4)$

  b gradient -3 which passes through the point $(3, 1)$

  c gradient $\frac{2}{3}$ which passes through the point $(3, 0)$

  d gradient $-\frac{1}{4}$ which passes through the point $(2, -3)$

  e gradient $\frac{4}{5}$ which passes through the point $(10, -4)$

  f gradient $-\frac{1}{6}$ which passes through the point $(-12, -5)$.

5 Find the equation of a line passing through:

a $(-1, 1)$ and $(2, 7)$	**b** $(2, 0)$ and $(6, 2)$	**c** $(-3, 3)$ and $(6, 0)$
d $(-1, 7)$ and $(2, -2)$	**e** $(5, 2)$ and $(-1, 2)$	**f** $(3, -1)$ and $(3, 4)$
g $(3, -1)$ and $(0, 4)$	**h** $(2, 6)$ and $(-2, 1)$	**i** $(4, -1)$ and $(-1, -2)$
j $(3, 4)$ and $(-1, -3)$	**k** $(4, -5)$ and $(-3, 1)$	**l** $(-1, 3)$ and $(-2, -2)$.

6 Find the equation connecting the variables given:

a

b

c
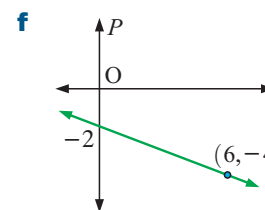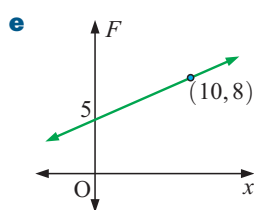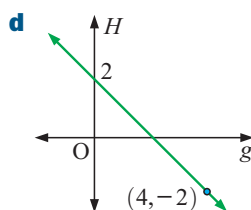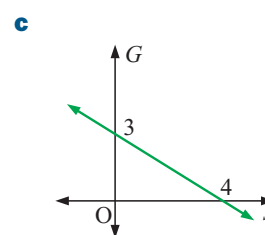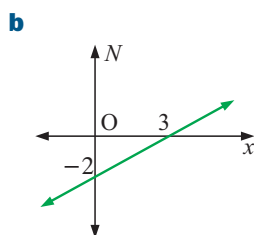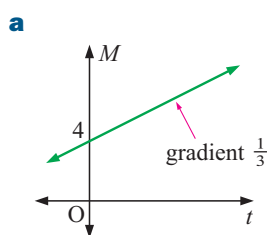
Notice the variable on each axis.

d

e

f

7   Find the equation of a line:

    a   which has gradient $\frac{1}{2}$ and cuts the y-axis at 3

    b   which is parallel to a line with gradient 2, and passes through the point $(-1,\ 4)$

    c   which cuts the x-axis at 5 and the y-axis at -2

    d   which cuts the x axis at -1, and passes through $(-3,\ 4)$

    e   which is perpendicular to a line with gradient $\frac{3}{4}$, and cuts the x-axis at 5

    f   which is perpendicular to a line with gradient -2, and passes through $(-2,\ 3)$.

8   Find a given that:

    a $(3,\ a)$ lies on $y = \frac{1}{2}x + \frac{1}{2}$ **b** $(-2,\ a)$ lies on $y = -3x + 7$

    c $(a,\ 4)$ lies on $y = 2x - 6$ **d** $(a,\ -1)$ lies on $y = -x + 3$

D   EQUATIONS OF LINES (GENERAL FORM) [7.6]

> The **general form** of the equation of a line is $ax + by = d$ where a, b and d are integers.

The general form allows us to write the equation of a line without the use of fractions. We can rearrange equations given in gradient-intercept form so that they are in general form.

For example, to convert $y = \frac{2}{3}x + 4$ into the general form, we first multiply each term by 3 to remove the fraction.

$$\therefore \quad 3y = 3(\tfrac{2}{3})x + 12 \quad \{\text{multiply by } 3\}$$
$$\therefore \quad 3y = 2x + 12$$
$$\therefore \quad -12 = 2x - 3y \qquad \{\text{subtract } 12 \text{ and } 3y \text{ from both sides}\}$$
$$\therefore \quad 2x - 3y = -12$$

Likewise, an equation given in general form can be rearranged into the gradient-intercept form. This is done by making y the subject of the equation.

Example 5   ◀) *Self Tutor*

  a   Convert $y = -\frac{3}{4}x + 1\frac{1}{2}$ into general form.

  b   Convert $3x - 5y = 8$ into gradient-intercept form.

  a $y = -\frac{3}{4}x + 1\frac{1}{2}$ **b** $3x - 5y = 8$

      $\therefore \quad 4y = 4\left(-\frac{3}{4}\right)x + 4\left(\frac{3}{2}\right)$ $\therefore \quad -5y = -3x + 8$

      $\therefore \quad 4y = -3x + 6$ $\therefore \quad 5y = 3x - 8$

  $\therefore \quad 3x + 4y = 6$ $\therefore \quad y = \frac{3}{5}x - \frac{8}{5}$

EXERCISE 14D.1

1 Convert the following straight line equations into general form:

a $y = \frac{1}{2}x + 3$ **b** $y = -2x + 7$ **c** $y = -\frac{1}{3}x - 2$ **d** $y = \dfrac{2x + 1}{3}$

2 Convert the following into gradient-intercept form:

a $x + 2y = 8$ **b** $2x + 5y = 10$ **c** $3x - y = 11$ **d** $x + 4y = 6$

e $6x - 5y = 15$ **f** $4x + 3y = 12$ **g** $9x - 2y = 18$ **h** $5x + 6y = 30$

3 Find the gradient and y-intercept of the lines with equations:

a $x + 3y = 6$ **b** $x - 4y = -8$ **c** $3x + 5y = 5$ **d** $4x - y = 8$

e $x + 5y = 4$ **f** $7x - 5y = -15$ **g** $4x + 3y = 24$ **h** $ax - by = d$

4 Find, in general form, the equation of a line with:

a gradient 2 and y-intercept 4   **b** gradient -3 and y-intercept -2

c gradient $\frac{1}{2}$ and y-intercept $\frac{1}{4}$ **d** gradient $-\frac{1}{3}$ and y-intercept 0

e gradient $-\frac{3}{4}$ and y-intercept 2   **f** gradient $\frac{2}{5}$ and y-intercept -1.

5 Find, in general form, the equations of the illustrated lines:
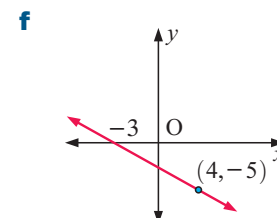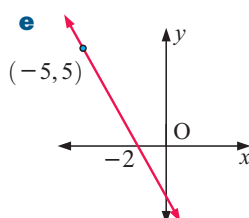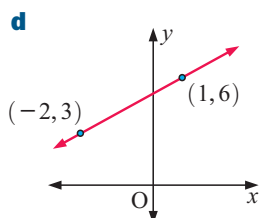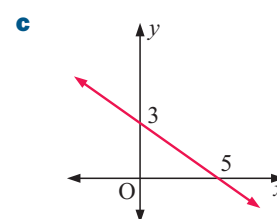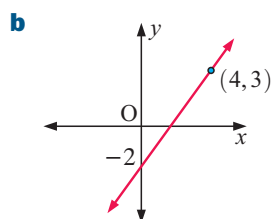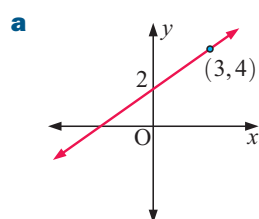
6 Find k if:

a $(2, k)$ lies on $2x + y = 7$ **b** $(3, k)$ lies on $x + 2y = -1$

c $(k, 1)$ lies on $3x + 2y = 8$ **d** $(k, -3)$ lies on $2x - 3y = 5$.

FINDING THE GENERAL FORM EQUATION OF A LINE QUICKLY

Suppose we are given the gradient of a line and a point which lies on it. Rather than finding the equation in gradient-intercept form and then converting it to general form, there is a faster method.

If a line has gradient $\frac{3}{4}$, it must have form $y = \frac{3}{4}x + c$.

$\therefore$ $4y = 3x + 4c$
$\therefore$ $3x - 4y = d$ where d is a constant.

If a line has gradient $-\frac{3}{4}$, using the same working we would obtain $3x + 4y = d$.

This suggests that:

- for gradient $\dfrac{a}{b}$ the general form of the line is $ax - by = d$
- for gradient $-\dfrac{a}{b}$ the general form of the line is $ax + by = d$.

The constant term d on the RHS is obtained by substituting the coordinates of any point which lies on the line.

Example 6 ◀) *Self Tutor*

Find the equation of a line:

a with gradient $\frac{3}{4}$, that passes through $(5, -2)$

b with gradient $-\frac{3}{4}$, that passes through $(1, 7)$.

a The equation is $\quad 3x - 4y = 3(5) - 4(-2)$

$\qquad\qquad\quad \therefore \quad 3x - 4y = 23$

b The equation is $\quad 3x + 4y = 3(1) + 4(7)$

$\qquad\qquad\quad \therefore \quad 3x + 4y = 31$

> With practice you can write down the equation very quickly, but you can still use the $y = mx + c$ method if you need.

EXERCISE 14D.2

1 Find the equation of a line:

a through $(4, 1)$ with gradient $\frac{1}{2}$ **b** through $(-2, 5)$ with gradient $\frac{2}{3}$

c through $(5, 0)$ with gradient $\frac{3}{4}$ **d** through $(3, -2)$ with gradient 3

e through $(1, 4)$ with gradient $-\frac{1}{3}$ **f** through $(2, -3)$ with gradient $-\frac{3}{4}$

g through $(3, -2)$ with gradient -2 **h** through $(0, 4)$ with gradient -3.

2 We can use the reverse process to question **1** to write down the gradient of a line given in general form. Find the gradient of the line with equation:

a $2x + 3y = 8$ **b** $3x - 7y = 11$ **c** $6x - 11y = 4$

d $5x + 6y = -1$ **e** $3x + 6y = -1$ **f** $15x - 5y = 17$

3 Explain why:

a any line parallel to $\quad 3x + 5y = 2 \quad$ has the form $\quad 3x + 5y = d$

b any line perpendicular to $\quad 3x + 5y = 2 \quad$ has the form $\quad 5x - 3y = d$.

4 Find the equation of a line which is:

a parallel to the line $\quad 3x + 4y = 6 \quad$ and passes through $(2, 1)$

b perpendicular to the line $\quad 5x + 2y = 10 \quad$ and passes through $(-1, -1)$

c perpendicular to the line $\quad x - 3y + 6 = 0 \quad$ and passes through $(-4, 0)$

d parallel to the line $\quad x - 3y = 11 \quad$ and passes through $(0, 0)$.

5 $2x - 3y = 6 \quad$ and $\quad 6x + ky = 4 \quad$ are two straight lines.

a Write down the gradient of each line. **b** Find k if the lines are parallel.

c Find k if the lines are perpendicular.

E | GRAPHING LINES FROM EQUATIONS   [7.6]

In this section we see how to graph straight lines given equations in either gradient-intercept or general form.

GRAPHING FROM THE GRADIENT-INTERCEPT FORM

Lines with equations given in the gradient-intercept form are easily graphed by finding two points on the graph, one of which is the y-intercept.

The other can be found by substitution or using the gradient.

Example 7   ◀)) Self Tutor

Graph the line with equation $y = \frac{1}{3}x + 2$.

Method 1:

The y-intercept is 2.

When $x = 3$, $y = 1 + 2 = 3$.

$\therefore$ $(0, 2)$ and $(3, 3)$ lie on the line.
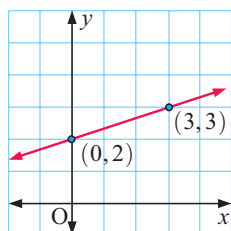
Method 2:

The y-intercept is 2

and the gradient $= \frac{1}{3}$ ◀— y-step
◀— x-step

So, we start at $(0, 2)$ and move to another point by moving across 3, then up 1.
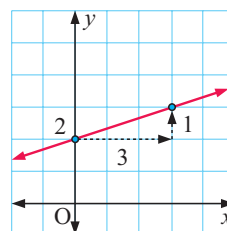
GRAPHING FROM THE GENERAL FORM

Remember that the form $ax + by = d$ is called the **general form** of a line.

The easiest way to graph lines in general form is to use axes intercepts.

> The x-intercept is found by letting $y = 0$.
> The y-intercept is found by letting $x = 0$.

Example 8   ◀)) Self Tutor
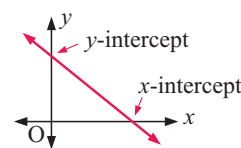
Graph the line with equation $2x - 3y = 12$ using axes intercepts.

For $2x - 3y = 12$:

when $x = 0$, $-3y = 12$
$\therefore$ $y = -4$
when $y = 0$, $2x = 12$
$\therefore$ $x = 6$
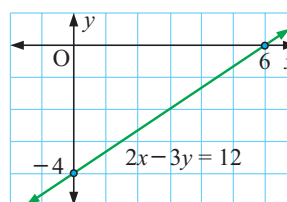
EXERCISE 14E

1 Draw the graph of the line with equation:

     a $y = 2x + 3$ **b** $y = \frac{1}{2}x - 3$ **c** $y = -x + 5$

     d $y = -4x - 2$ **e** $y = -\frac{1}{3}x$ **f** $y = -3x + 4$

     g $y = \frac{3}{4}x$ **h** $y = \frac{1}{3}x - 1$ **i** $y = -\frac{3}{2}x + 2$

2 **a** The line with equation $y = 2x - 1$ is reflected in the x-axis. Graph the line and draw its image. Find the equation of the reflected line.

     b The line with equation $y = \frac{1}{2}x + 2$ is reflected in the y-axis. Graph the line and draw its image. Find the equation of the reflected line.

3 Use axes intercepts to draw sketch graphs of:

     a $2x + y = 4$ **b** $3x + y = 6$ **c** $3x - 2y = 12$

     d $3x + 4y = 12$ **e** $x - y = 2$ **f** $x + y = -2$

   **g** $2x - 3y = -9$ **h** $4x + 5y = 20$ **i** $5x - 2y = -10$

4 **a** Graph the line with equation $3x + 2y = 1$ and show that $(-1, 2)$ lies on it.

     b If the line with equation $3x + 2y = 1$ is rotated clockwise about the point $(-1, 2)$ through an angle of 90°, find the equation of the rotated line.

5 Graph the line with equation $3x - 5y = 15$. If the line is rotated anticlockwise about the origin through an angle of 180°, find the equation of this new line.

F   LINES OF SYMMETRY   [7.8]

Many geometrical shapes have **lines of symmetry**. If the shape is drawn on the Cartesian plane, we can use the information learnt in this chapter to find the equations of the lines of symmetry.

Example 9   ◀)) *Self Tutor*

Consider the points A(1, 3), B(6, 3) and C(6, 1).

     a Plot the points and locate D such that ABCD is a rectangle.

     b State the coordinates of D.

     c Write down the equations of any lines of symmetry.

a
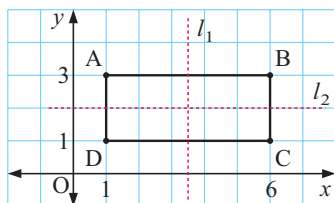
b D is at $(1, 1)$

c l_1 is a vertical line with x-coordinate midway between 1 and 6.

     ∴ its equation is $x = \frac{1+6}{2} = 3.5$

     l_2 is a horizontal line with y-coordinate midway between 1 and 3.

     ∴ its equation is $y = \frac{3+1}{2} = 2$.

So, the lines of symmetry have equations $x = 3.5$ and $y = 2$.

Example 10   ◆〉 *Self Tutor*
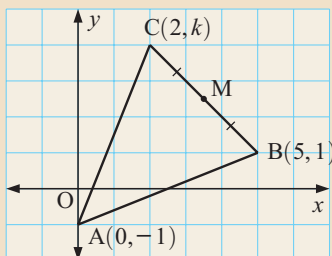
ABC is an isosceles triangle with AB = AC.

A(0, −1), B(5, 1) and C(2, k) are the coordinates of A, B and C, with $k > 0$.

a Explain why $k = 4$.

b Find the coordinates of M, the midpoint of line segment BC.

c Show that line segments AM and BC are perpendicular.

d Find the equation of the line of symmetry of triangle ABC.

a Since AB = AC, $\sqrt{(5 - 0)^2 + (1 - -1)^2} = \sqrt{(2 - 0)^2 + (k - -1)^2}$

$$\therefore \quad \sqrt{25 + 4} = \sqrt{4 + (k + 1)^2}$$
$$\therefore \quad (k + 1)^2 = 25$$
$$\therefore \quad k + 1 = \pm 5$$
$$\therefore \quad k = 4 \text{ or } -6$$

But $k > 0$, so $k = 4$.

b The midpoint of BC is M$\left(\frac{5+2}{2}, \frac{1+4}{2}\right)$.

So, M is $\left(\frac{7}{2}, \frac{5}{2}\right)$.

c gradient of BC $= \dfrac{4 - 1}{2 - 5}$ gradient of AM $= \dfrac{\frac{5}{2} - -1}{\frac{7}{2} - 0}$ As these gradients are negative reciprocals of each other, AM ⊥ BC.

$$= \frac{3}{-3}$$ $$= \frac{\frac{7}{2}}{\frac{7}{2}}$$

$$= -1$$ $$= 1$$

d The line of symmetry is AM.

Its gradient $m = 1$ and its y-intercept $c = -1$.

∴   its equation is $y = x - 1$.

EXERCISE 14F

1 **a** Plot the points A(1, 0), B(9, 0), C(8, 3) and D(2, 3).

   b Classify the figure ABCD.

   c Find the equations of any lines of symmetry of ABCD.

2 **a** Plot the points A(−1, 2), B(3, 2), C(3, −2) and D(−1, −2).

   b Classify the figure ABCD.

   c Find the equations of any lines of symmetry of ABCD.

3 P, Q, R and S are the vertices of a rectangle. Given P(1, 3), Q(7, 3) and S(1, −2) find:

   a the coordinates of R

   b the midpoints of line segments PR and QS

   c the equations of the lines of symmetry.

   d What geometrical fact was verified by **b**?

4 Plot the points O(0, 0), A(1, 4), B(9, 2) and C(8, −2).

    a Prove that line segments OA and BC are parallel, and OA = BC.

    b Prove that line segments OA and AB are perpendicular.

    c What have you established about OABC from **a** and **b**?

    d Find the midpoints of OA and AB.

    e Find the equations of any lines of symmetry of OABC.

5 OABC is a quadrilateral with A(5, 0), B(8, 4) and C(3, 4).

    a Plot the points A, B and C and complete the figure OABC.

    b Find the equation of line segment BC.

    c By finding lengths only, show that OABC is a rhombus.

    d Find the midpoints of line segments OB and AC.

    e What has been verified in **d**?

    f Find the equations of any lines of symmetry.

6
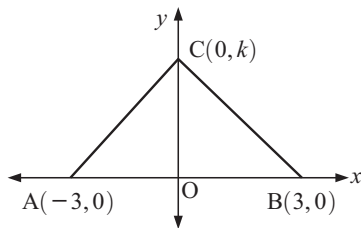
    a Find the gradient of line segments AC and BC.

    b If $A\widehat{C}B$ is a right angle, use **a** to find the value of k.

    c Classify $\triangle ABC$.

    d State the equation of the line of symmetry of $\triangle ABC$.

7 ABC is an equilateral triangle.

    a Find the coordinates of C.

    b If M and N are the midpoints of line segments BC and AC respectively, find the coordinates of M and N.

    c Find the equations of all lines of symmetry of $\triangle ABC$.
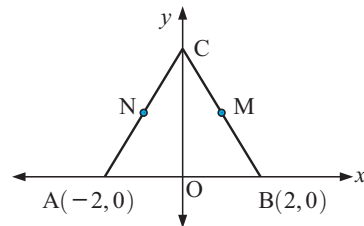
Review set 14A
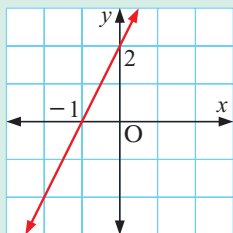
1   **a** Find the equation of a vertical line through (−1, 5).

    b Determine the gradient of a line with equation $4x + 5y = 11$.

    c Find the axes intercepts and gradient of a line with equation $2x + 3y = 6$.

    d Find, in general form, the equation of a line passing through (−2, −3) and (1, 5).

2 Determine the equation of the illustrated line:
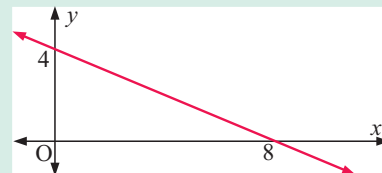
3 Find the equation of a line through (1, −2) and (3, 4).

4 Find the equation of a line with gradient −2 and y-intercept 3.

5 Find the equation of a line with gradient $\frac{2}{3}$ which passes through $(-3, 4)$.

6 Use axes intercepts to draw a sketch graph of $3x - 2y = 6$.

7 Find k if $(-3, -1)$ lies on the line $4x - y = k$.

8 Find the equation of a line with zero gradient that passes through $(5, -4)$.

9 Find, in general form, the equation of a line parallel to $2x - 3y = 10$ which passes through $(3, -4)$.

10 Draw the graph of the line with equation $y = \frac{3}{4}x - 2$.

11 Given A$(-3, 1)$, B$(1, 4)$ and C$(4, 0)$:

    a Show that triangle ABC is isosceles.

    b Find the midpoint X of line segment AC.

    c Use gradients to verify that line segments BX and AC are perpendicular.

    d Find the equations of any lines of symmetry of triangle ABC.

12 Consider the points A$(3, -2)$, B$(8, 0)$, C$(6, 5)$ and D$(1, 3)$.

    a Prove that line segments AB and DC are parallel and equal in length.

    b Prove that line segments AB and BC are perpendicular and equal in length.

    c Classify the quadrilateral ABCD.

    d Find the equations of all lines of symmetry of ABCD.

Review set 14B

1   **a** Find the equation of the x-axis.

    b Write down the gradient and y-intercept of a line with equation $y = 5 - 2x$.

    c Find, in general form, the equation of a line with gradient $-\frac{4}{7}$ which passes through $(-2, 2)$.

2 Determine the equation of the illustrated line:

3 Find, in gradient-intercept form, the equation of a line:

    a with gradient -2 and y-intercept 7

    b passing through $(-1, 3)$ and $(2, 1)$

    c parallel to a line with gradient $\frac{3}{2}$ and passing through $(5, 0)$.

4 If $(k, 5)$ lies on a line with equation $3x - y = -8$, find k.

5 Find the equation connecting the variables for the graph given.
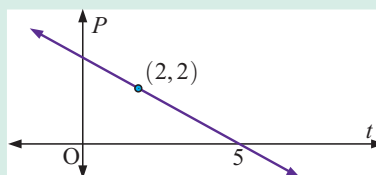
6 Find the axes intercepts for a line with equation $2x - 5y = 10$, and hence draw a graph of this line.

7 Find the equations of the following graphs:

a
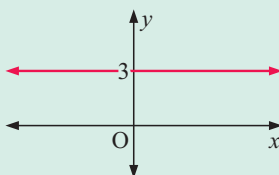
b
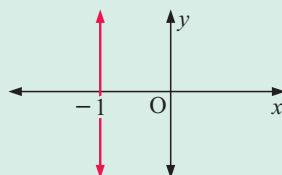
c
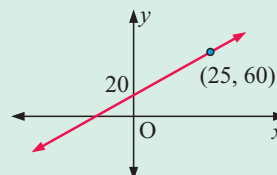
8 Find the gradient of a line with equation $4x + 3y = 5$.

9 **a** Find the gradient of a line with equation $y = 2x - 3$.

    b Find, in gradient-intercept form, the equation of a line perpendicular to $y = 2x - 3$ which passes through $(4, 1)$.

10 Draw the graph of the line with equation $y = -2x + 7$.

11 Find, in general form, the equation of the line perpendicular to $3x - 5y = 4$ which passes through $(5, -2)$.

12 Consider the points A$(-11, 2)$, B$(-5, -6)$, C$(3, 0)$ and D$(-3, 8)$.

    a Plot A, B, C and D on a set of axes.

    b Show that all the sides of quadrilateral ABCD are equal in length.

    c Show that line segment AB is perpendicular to line segment BC. Hence classify ABCD.

    d ABCD has four lines of symmetry. Find their equations in general form.

    e Find the midpoint of line segment AC.

    f Check that each of the lines of symmetry passes through the point found in **e**.

Trigonometry

15

Contents:

Opening problem

For safety reasons, ramps for aiding wheelchair access must not have an angle of incline θ exceeding 5°.
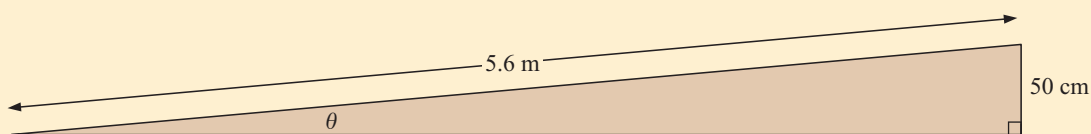
A ramp outside a post office was found to be 5.6 m long, with a vertical rise of 50 cm. Does this ramp comply with the safety regulations?

Trigonometry is a branch of mathematics that deals with triangles. In particular, it considers the relationship between their side lengths and angles.

We can apply trigonometry in engineering, astronomy, architecture, navigation, surveying, the building industry, and in many other branches of applied science.

We can see in the **Opening Problem** there is a right angled triangle to consider. The trigonometry of right angled triangles will be discussed in this chapter and the principles extended to other triangles in **Chapter 29**.

A   *LABELLING SIDES OF A RIGHT ANGLED TRIANGLE*   [8.1]
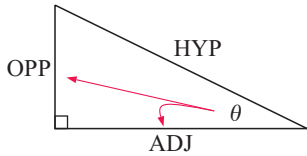
For the right angled triangle with angle θ:

- the **hypotenuse (HYP)** is the longest side
- the **opposite (OPP)** side is opposite θ
- the **adjacent (ADJ)** side is adjacent to θ.

Given a right angled triangle ABC with angles of θ and ϕ:

For angle θ,   BC is the **opposite side**
                      AB is the **adjacent side**.

For angle ϕ,   AB is the **opposite side**
                    BC is the **adjacent side**.
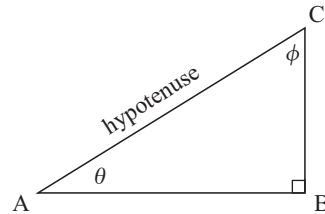
Example 1   ◄)) *Self Tutor*

For the triangle shown, name:

a   the hypotenuse

b   the side opposite θ

c   the side adjacent to θ.
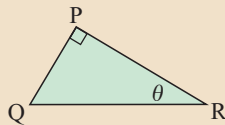
a   The hypotenuse is QR.
        (the longest side, opposite the right angle)

b   PQ
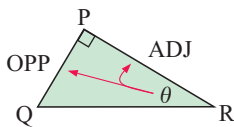
c   PR

In any right angled triangle, locate the hypotenuse first. Then locate the opposite side for the angle you are working with.

θ is theta
ϕ is phi
α is alpha
β is beta

Example 2   ◄)) *Self Tutor*

For this triangle name the:

a   hypotenuse   **b**   side opposite angle α

c   side adjacent α **d**   side opposite β
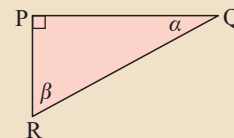
e   side adjacent β

a   QR   (opposite the right angle)

b   PR

c   PQ

d   PQ
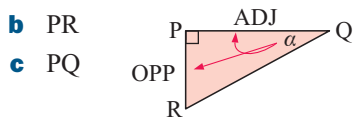
e   PR
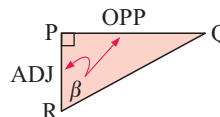
EXERCISE 15A

1 For the triangles given, name:

      i the hypotenuse   **ii** the side opposite angle θ **iii** the side adjacent to θ

a
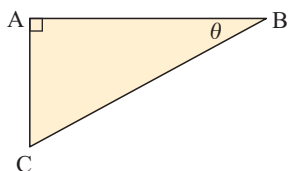
b
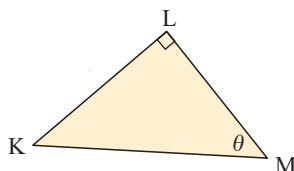
c
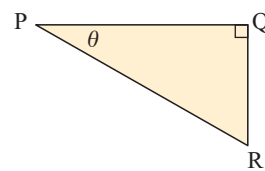
d
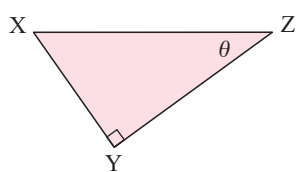
e
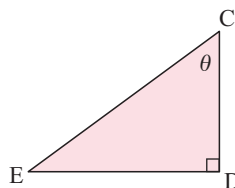
f
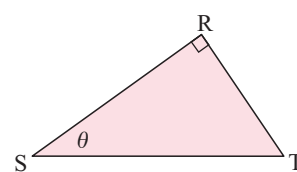
2 For the triangle given, name the:

      i hypotenuse   **ii** side opposite α **iii** side adjacent to α

      iv side opposite β **v** side adjacent to β

a
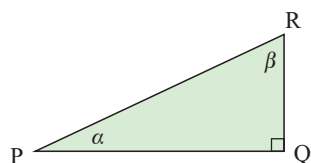
b
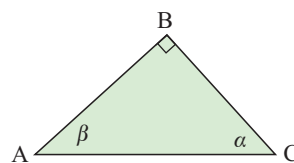
Discovery 1   Ratio of sides of right angled triangles

In this discovery we will find the ratios $\dfrac{\text{OPP}}{\text{HYP}}$, $\dfrac{\text{ADJ}}{\text{HYP}}$ and $\dfrac{\text{OPP}}{\text{ADJ}}$ in a series of triangles which are enlargements of each other.

What to do:

1 Consider four right angled triangles ABC where $\hat{CAB}$ is 30° in each case but the sides vary in length.

By accurately measuring to the nearest millimetre, complete a table like the one following:
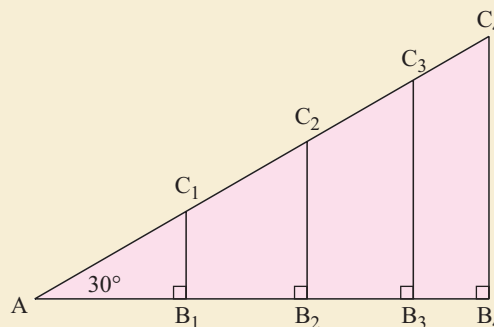
Triangle	AB	BC	AC	$\dfrac{AB}{AC}$	$\dfrac{BC}{AC}$	$\dfrac{BC}{AB}$
1						
2						
3						
4						

Convert all fractions to 2 decimal places.

PRINTABLE WORKSHEET
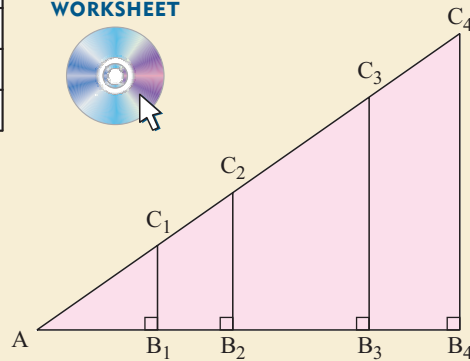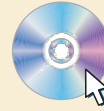
2   Repeat **1** for the set of triangles alongside.

3   What have you discovered from **1** and **2**?

Notice that $\dfrac{AB}{AC} = \dfrac{ADJ}{HYP},$ $\dfrac{BC}{AC} = \dfrac{OPP}{HYP}$ and $\dfrac{BC}{AB} = \dfrac{OPP}{ADJ}.$

From the **Discovery** you should have found that:

> For a fixed angled right angled triangle, the ratios $\dfrac{OPP}{HYP},$ $\dfrac{ADJ}{HYP}$ and $\dfrac{OPP}{ADJ}$ are constant no matter how much the triangle is enlarged.

B   THE TRIGONOMETRIC RATIOS   [8.1]

For a particular angle of a right angled triangle the ratios $\dfrac{OPP}{HYP},$ $\dfrac{ADJ}{HYP}$ and $\dfrac{OPP}{ADJ}$ are fixed.

These ratios have the traditional names **sine**, **cosine** and **tangent** respectively.

We abbreviate them to **sin**, **cos** and **tan**.

> In any right angled triangle with one angle θ we have:
>
> $$\sin\theta = \frac{\mathbf{OPP}}{\mathbf{HYP}}, \quad \cos\theta = \frac{\mathbf{ADJ}}{\mathbf{HYP}}, \quad \tan\theta = \frac{\mathbf{OPP}}{\mathbf{ADJ}}$$
>
>
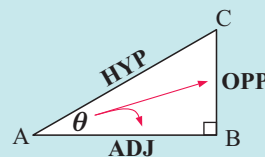
Notice that $\dfrac{\sin\theta}{\cos\theta} = \sin\theta \div \cos\theta = \dfrac{OPP}{HYP} \times \dfrac{HYP}{ADJ} = \dfrac{OPP}{ADJ} = \tan\theta$

So, $$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

We can use these ratios to find unknown sides and angles of right angled triangles.

FINDING TRIGONOMETRIC RATIOS

Example 3   ◀ׄ) *Self Tutor*

For the given triangle find
$\sin\theta$, $\cos\theta$ and $\tan\theta$.

$$\sin\theta = \frac{\text{OPP}}{\text{HYP}} \qquad \cos\theta = \frac{\text{ADJ}}{\text{HYP}} \qquad \tan\theta = \frac{\text{OPP}}{\text{ADJ}}$$

$$= \tfrac{4}{5} \qquad\qquad\qquad = \tfrac{3}{5} \qquad\qquad\qquad = \tfrac{4}{3}$$
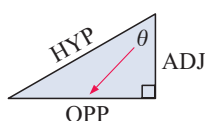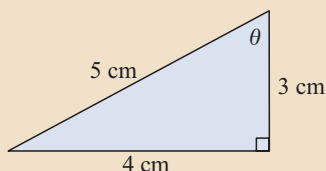
FINDING SIDES

In a right angled triangle, if we are given another angle and a side we can find:

- the third angle using the 'angle sum of a triangle is 180°'
- the other sides using trigonometry.

> *Step 1:* Redraw the figure and mark on it HYP, OPP, ADJ relative to the given angle.
>
> *Step 2:* Choose the correct trigonometric ratio and use it to set up an equation.
>
> *Step 3:* Solve to find the unknown.

Example 4   ◀ׄ) *Self Tutor*

Find the unknown length in the following triangles:

a

b

a
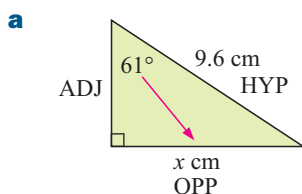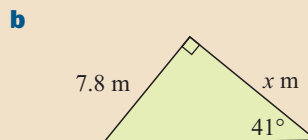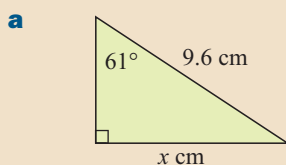
Now $\sin 61^o = \dfrac{x}{9.6}$ $\left\{\sin\theta = \dfrac{\text{OPP}}{\text{HYP}}\right\}$

$\therefore$ $\sin 61^o \times 9.6 = x$ $\{\times$ both sides by 9.6$\}$

$\therefore$ $x \approx 8.40$ $\{\boxed{\text{SIN}}\ 61\ \boxed{)}\ \boxed{\times}\ 9.6\ \boxed{\text{ENTER}}\}$

The length of the side is about 8.40 cm.

b
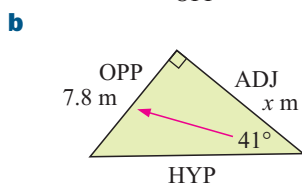
Now $\tan 41^o = \dfrac{7.8}{x}$ $\left\{\tan\theta = \dfrac{\text{OPP}}{\text{ADJ}}\right\}$

$\therefore$ $x \times \tan 41^o = 7.8$ $\{\times$ both sides by $x\}$

$\therefore$ $x = \dfrac{7.8}{\tan 41^o}$ $\{\div$ both sides by $\tan 41^o\}$

$\therefore$ $x \approx 8.97$ $\{7.8\ \boxed{\div}\ \boxed{\text{TAN}}\ 41\ \boxed{)}\ \boxed{\text{ENTER}}\}$

The length of the side is about 8.97 m.

Unless otherwise stated, all lengths should be given correct to 3 significant figures and all angles correct to 1 decimal place.

EXERCISE 15B.1

1 For each of the following triangles find:

    i $\sin\theta$ **ii** $\cos\theta$ **iii** $\tan\theta$ **iv** $\sin\phi$ **v** $\cos\phi$ **vi** $\tan\phi$

a
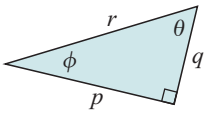
b
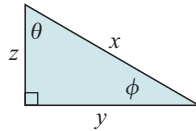
c
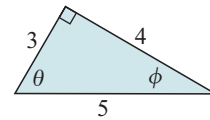
d
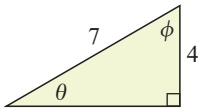
e
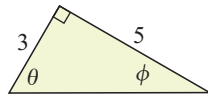
f
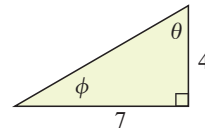
2 Construct a trigonometric equation connecting the angle with the sides given:

a
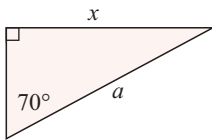
b
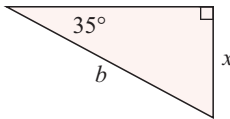
c
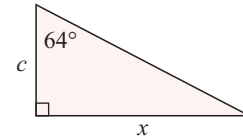
d
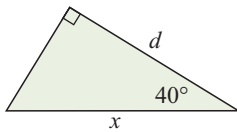
e
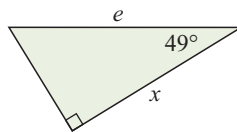
f
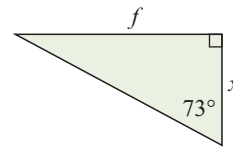
g
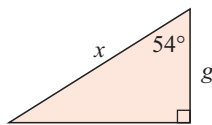
h
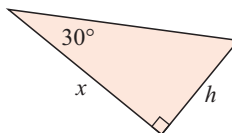
i
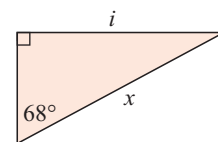
3 Find, correct to 2 decimal places, the value of x in:

a

b

c

d

5 km

42°

x km

e

34°

8 m

x m

f

80 cm

x cm

74°

g

28°

x m

52 m

h

x km

45 km   68°

i

x mm

27°

11 mm

j

5.6 cm

70°

x cm

k

50°

x m   27 m

l

12 km

x km

49°
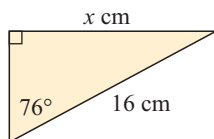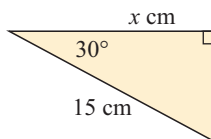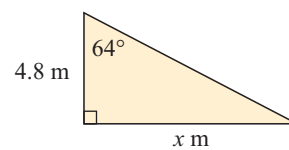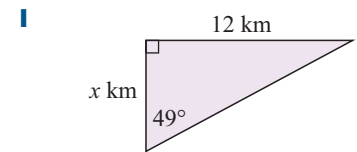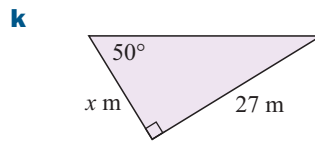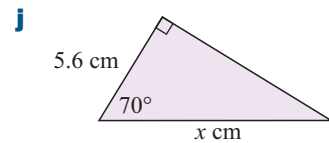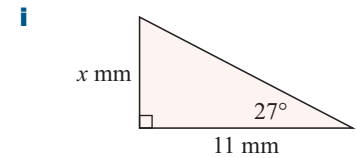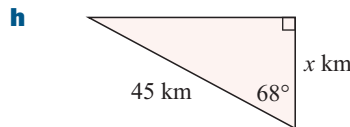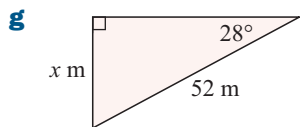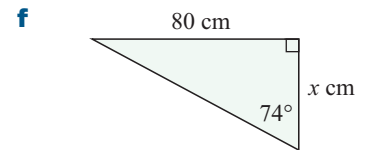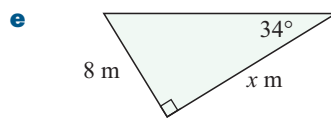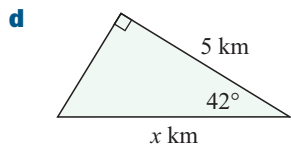
4   Find *all* the unknown angles and sides of:

a

a cm

28°

b cm

θ   12 cm

b

15 m

θ

b m

a m   63°

c

45 cm   b cm

25°   θ

a cm

FINDING ANGLES

HYP
5 cm

OPP
3 cm

θ
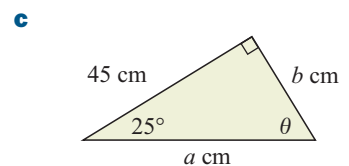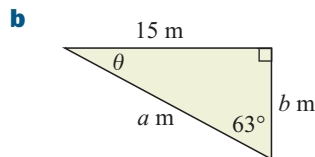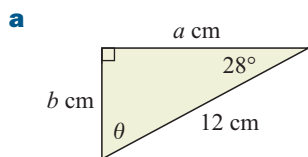
In the right angled triangle shown, $\sin \theta = \frac{3}{5}$.
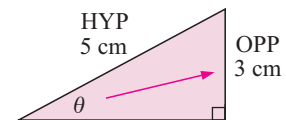
So, we are looking for the angle θ with a sine of $\frac{3}{5}$.

If $\sin^{-1}(\ldots\ldots)$ reads   "the angle with a sine of ......",   we can write $\theta = \sin^{-1}\left(\frac{3}{5}\right)$.

Another way of describing this is to say   "θ is the *inverse sine* of $\frac{3}{5}$".

> If $\sin \theta = x$ then θ is the **inverse sine** of x.

You can find graphics calculator instructions for finding these inverse trigonometric functions on page **16**.

We can define **inverse cosine** and **inverse tangent** in a similar way.
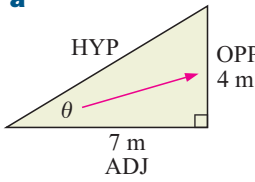
Example 5   ◀ *Self Tutor*

Find the measure of the angle marked θ in:

a

4 m

θ

7 m

b

2.67 km

5.92 km

θ

a

$$\tan \theta = \tfrac{4}{7} \qquad \{\tan \theta = \dfrac{\text{OPP}}{\text{ADJ}}\}$$

$$\therefore \quad \theta = \tan^{-1}\left(\tfrac{4}{7}\right)$$
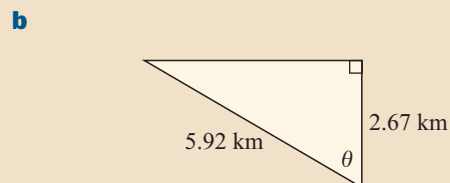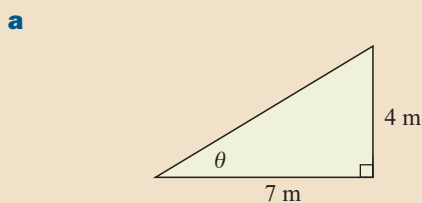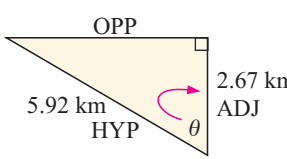
$$\therefore \quad \theta \approx 29.7^{o} \qquad \{\boxed{\text{SHIFT}}\ \boxed{\text{tan}}\ \boxed{(}\ 4\ \boxed{\div}\ 7\ \boxed{)}\ \boxed{\text{EXE}}\}$$

So, the angle measure is about 29.7^{o} .

b

$$\cos \theta = \dfrac{2.67}{5.92} \qquad \{\text{as } \cos \theta = \dfrac{\text{ADJ}}{\text{HYP}}\}$$

$$\therefore \quad \theta = \cos^{-1}\left(\dfrac{2.67}{5.92}\right)$$

$$\therefore \quad \theta \approx 63.2^{o} \qquad \{\boxed{\text{SHIFT}}\ \boxed{\text{cos}}\ \boxed{(}\ 2.67\ \boxed{\div}\ 5.92\ \boxed{)}\ \boxed{\text{EXE}}\}$$

So, the angle measure is about 63.2^{o} .

EXERCISE 15B.2

1 Find the measure of the angle marked θ in:

a
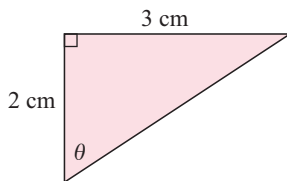
b
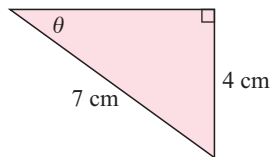
c
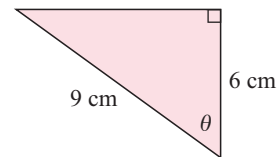
d
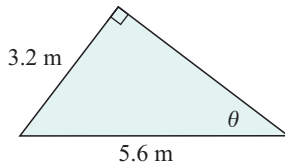
e
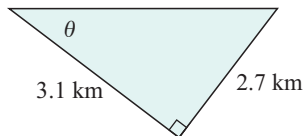
f
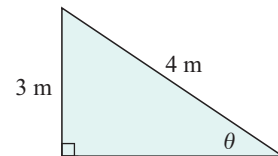
g
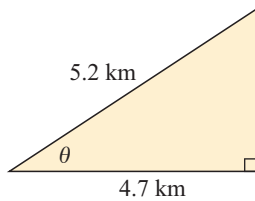
h
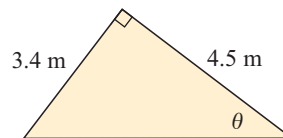
i
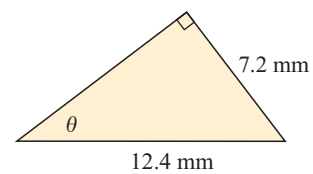
2 Find all the unknown sides and angles in the following:

a
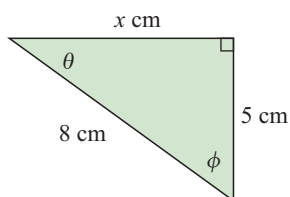
b
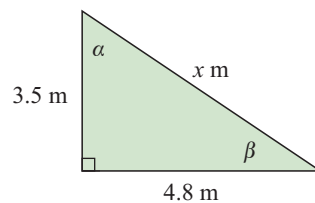
c
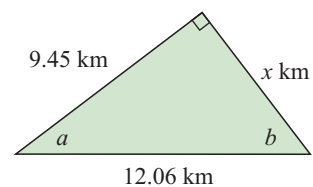
3 Check your answers for x in question **2** using Pythagoras' theorem.

4 Find the unknowns correct to 3 significant figures:

a

32°
5 m
x m

b

6 cm
θ
8 cm

c

12 cm
50°
x cm

d

7 m
31°
x m
θ

e

6 cm
φ
4 cm
θ

f

11 m
x m
42°
y m

g

10 cm
8 cm
θ
x cm
15 cm

h

x m
12 m
39°
θ
20 m

i

9 cm
7 cm
θ
x cm
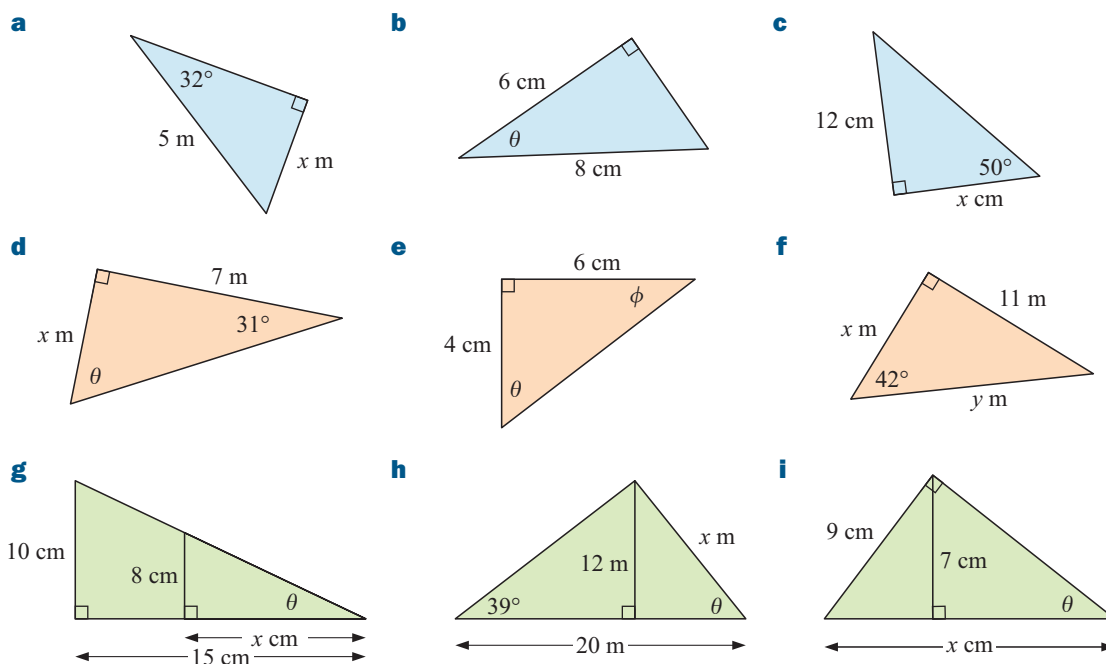
5 Find θ using trigonometry in the following. What conclusions can you draw?

a

8.1 m
7.9 m
θ

b

14 mm
θ
12 mm

c

8.6 km
θ
8.6 km
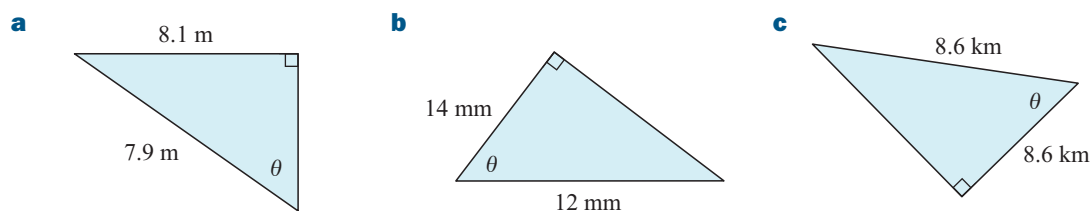
Discovery 2   *Hipparchus and the universe*

Hipparchus was a Greek astronomer and mathematician born in Nicaea in the 2nd century BC. He is considered among the greatest astronomers of antiquity.

Part 1: How Hipparchus measured the distance to the moon

Consider two towns A and B on the earth's equator. The moon is directly overhead town A. From B the moon is just visible, since MB is a tangent to the earth and is therefore perpendicular to BC. Angle C is the difference in longitude between towns A and B, which Hipparchus calculated to be approximately 89° in the 2nd century BC.

B
r
M
A
r
C
Moon
Earth
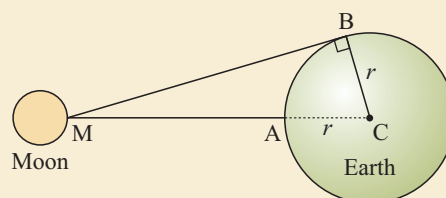
We know today that the radius of the earth is approximately 6378 km. Hipparchus would have used a less accurate figure, probably based on Eratosthenes' measure of the earth's circumference.

What to do:

1 Use $r = 6378$ km and $\widehat{\text{BCM}} = 89^o$ to estimate the distance from the centre of the earth C to the moon.

2 Now calculate the distance AM **between** the earth and the moon.

3 In calculating just one distance between the earth and the moon, Hipparchus was assuming that the orbit of the moon was circular. In fact it is not. Research the shortest and greatest distances to the moon. How were these distances determined? How do they compare with Hipparchus' method?

Part 2:   How Hipparchus measured the radius of the moon

From town A on the earth's surface, the angle between an imaginary line to the centre of the moon and an imaginary line to the edge of the moon (a tangent to the moon) is about 0.25^o.

The average distance from the earth to the moon is about $384\,403$ km.
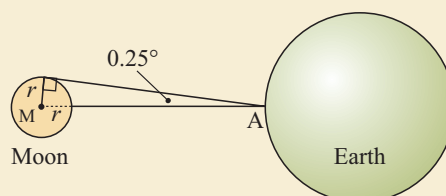
What to do:

1 Confirm from the diagram that $\sin 0.25^o = \dfrac{r}{r + 384\,403}$.

2 Solve this equation to find r, the radius of the moon.

3 Research the actual radius of the moon, and if possible find out how it was calculated. How does your answer to **2** compare?

C   PROBLEM SOLVING   [8.1]

The trigonometric ratios can be used to solve a wide variety of problems involving right angled triangles. When solving such problems it is important to follow the steps below:

Step 1:   Read the question carefully.

Step 2:   Draw a diagram, not necessarily to scale, with the given information clearly marked.

Step 3:   If necessary, label the vertices of triangles in the figure.

Step 4:   State clearly any assumptions you make which will enable you to use right angled triangles or properties of other geometric figures.

Step 5:   Choose an appropriate trigonometric ratio and use it to generate an equation connecting the quantities. On some occasions more than one equation may be needed.

Step 6:   Solve the equation(s) to find the unknown.

Step 7:   Answer the question in words.

ANGLES OF ELEVATION AND DEPRESSION

The angle between the horizontal and your line of sight is called the **angle of elevation** if you are looking upwards, or the **angle of depression** if you are looking downwards.
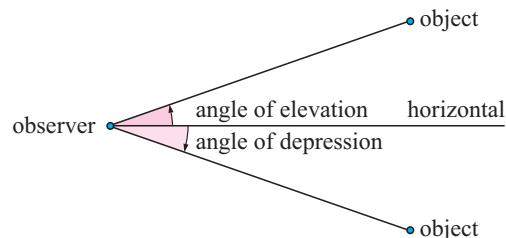
If the angle of elevation from A to B is θ^o, then the angle of depression from B to A is also θ^o.
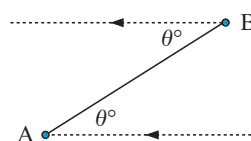
When using trigonometry to solve problems we often use:

• the properties of isosceles and right angled triangles
• the properties of circles and tangents
• angles of elevation and depression.

Example 6   ◀) *Self Tutor*

Determine the length of the horizontal roofing beam required to support a roof of pitch 16° as shown alongside:

$$\cos \theta = \frac{\text{ADJ}}{\text{HYP}}$$

$$\therefore \quad \cos 16^o = \frac{x}{9.4}$$
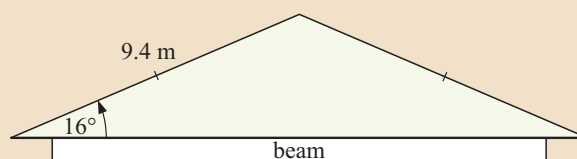
$$\therefore \quad x = 9.4 \times \cos 16^o$$

$$\therefore \quad x \approx 9.036$$

{*Calculator:* 9.4 [×] [COS] 16 [)] [ENTER] }

$$\therefore \quad \text{the length of the beam} = 2 \times 9.036 \text{ m}$$

$$\approx 18.1 \text{ m}$$

Example 7   ◀) *Self Tutor*

A ladder 4.1 m in length rests against a vertical wall and reaches 3.5 m up from ground level. Find:

a the angle the ladder makes with the ground

b the distance from the foot of the ladder to the wall using trigonometry.
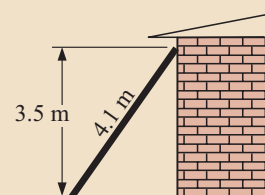
a $\sin \theta = \dfrac{\text{OPP}}{\text{HYP}} = \dfrac{3.5}{4.1}$

$\therefore \quad \theta = \sin^{-1} \left(\dfrac{3.5}{4.1} \right)$ 4.1 m   3.5 m

$\therefore \quad \theta \approx 58.6^o$

θ x m
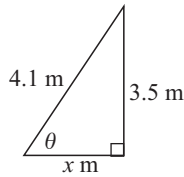
{ SHIFT sin (3.5 ÷ 4.1) EXE }

$\therefore$ the ladder makes an angle of about
58.6^o with the ground.

b $\cos \theta = \dfrac{\text{ADJ}}{\text{HYP}}$

$\therefore \quad \cos 58.61^o = \dfrac{x}{4.1}$

$\therefore \quad 4.1 \times \cos 58.61^o = x$

$\therefore \quad 2.14 \approx x$

$\therefore$ the foot of the ladder is about
2.14 m from the wall.

Example 8   ◀) *Self Tutor*

The angle between a tangent from point P to a circle and the line from P to the centre of the circle is
27°. Determine the length of the line from P to the centre of the circle if the radius is 3 cm.

$\sin \theta = \dfrac{\text{OPP}}{\text{HYP}}$

$\therefore \quad \sin 27^o = \dfrac{3}{x}$

$\therefore \quad x = \dfrac{3}{\sin 27^o}$
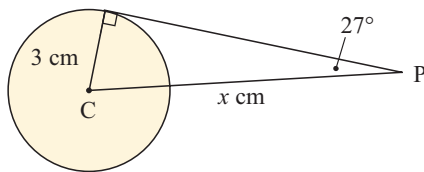
$\therefore \quad x \approx 6.61$

$\therefore$ CP has length approximately   6.61 cm.

EXERCISE 15C

1   From a point 235 m from the base of a cliff, the angle of elevation to the cliff top is 25°. Find the
height of the cliff.

2   What angle will a 5 m ladder make with a wall if it reaches 4.2 m up the wall?

3   The angle of elevation from a fishing boat to the top of a lighthouse 25 m above sea-level is 6°.
Calculate the horizontal distance from the boat to the lighthouse.

4   A rectangular gate has a diagonal strut of length 3 m.
The angle between the diagonal and a side is 28°. Find
the length of the longer side of the gate.

5   A model helicopter takes off from the horizontal ground
with a constant vertical speed of 5 m/s. After 10 seconds
the angle of elevation from Sam to the helicopter is 62°.
Sam is 1.8 m tall. How far is Sam's head from the
helicopter at this time?
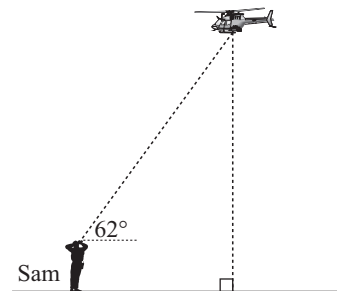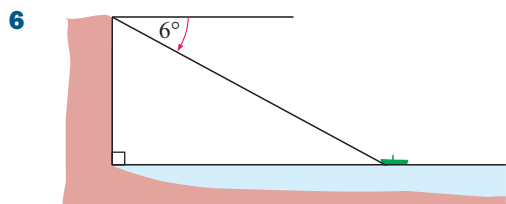
6   From a vertical cliff 80 m above sea level, a fishing boat is observed at an angle of depression of 6°. How far out to sea is the boat?

7   A railway line goes up an incline of constant angle 4° over a horizontal distance of 4 km. How much altitude has the train gained by the end of the incline?

8   A kite is attached to a 50 m long string. The other end of the string is secured to the ground. If the kite is flying 35 m above ground level, find the angle θ that the string makes with the ground.
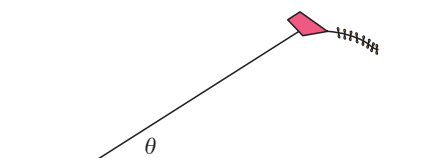
9   Antonio drew a margin along the edge of his 30 cm long page. At the top of the page the margin was 2 cm from the edge of the page, but at the bottom the margin was 3 cm from the edge of the page. How many degrees off parallel was Antonio's margin?

10   A goal post was hit by lightning and snapped in two. The top of the post is now resting 15 m from its base at an angle of 25°. Find the height of the goal post before it snapped.
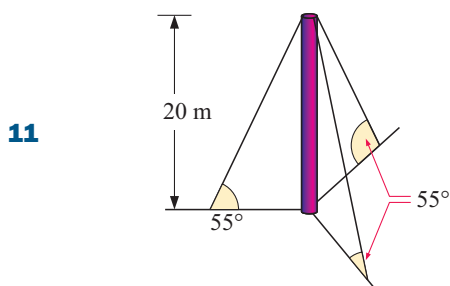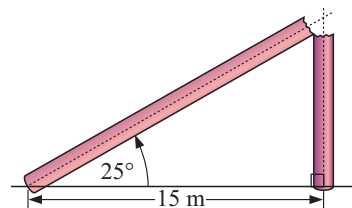
11   Three strong cables are used to brace a 20 m tall pole against movement due to the wind. Each rope is attached so that the angle of elevation to the top of the pole is 55°. Find the total length of cable.

12   A rectangle has length 6 m and width 4 m. Find the acute angle formed where the diagonals intersect.

13   A tangent from point P to a circle of radius 4 cm is 10 cm long. Find:

   a   the distance of P from the centre of the circle

   b   the size of the angle between the tangent and the line joining P to the centre of the circle.

14   AB is a chord of a circle with centre O and radius of length 5 cm. AB has length 8 cm. What angle does AB subtend at the centre of the circle, i.e., what is the size of angle AOB?
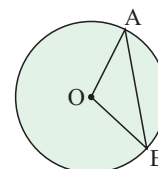
15   Find the area of the parallelogram:
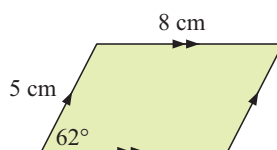
16   A rhombus has sides of length 10 cm, and the angle between two adjacent sides is 76°. Find the length of the longer diagonal of the rhombus.

17 For the circle given, find:

    a the radius of the circle

    b the distance between A and B.
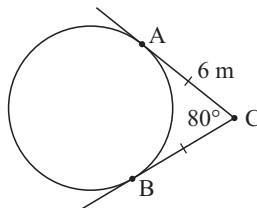
18 An aeroplane takes off from the ground at an angle of 27° and its average speed in the first 10 seconds is 200 km/h. What is the altitude of the plane at the end of this time?

19 An observer notices that an aeroplane flies directly overhead. Two minutes later the aeroplane is at an angle of elevation of 27°. Assuming the aeroplane is travelling with constant speed, what will be its angle of elevation after another two minutes?

20 Find the size of angle ABC.
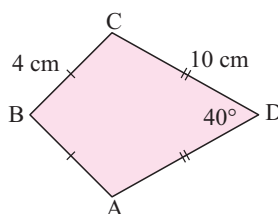
21 An isosceles triangle is drawn with base angles 24° and base 28 cm. Find the base angles of the isosceles triangle with the same base length but with treble the area.

22 The angle of elevation from a point on level ground to the top of a building 100 m high is 22°. Find:

    a the distance of the point from the base of the building

    b the distance the point must be moved towards the building in order that the angle of elevation becomes 40°.

23 From a point A which is 30 m from the base of a building B, the angle of elevation to the top of the building C is 56°, and to the top of the flag pole CD is 60°.

Find the length of the flag pole.
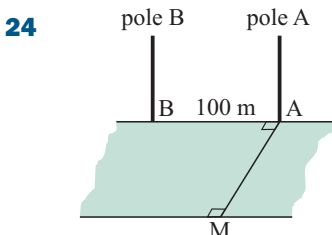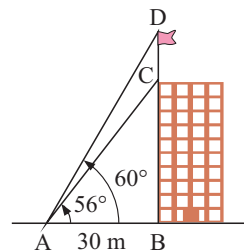
24 A man, M, positions himself on a river bank as in the diagram alongside, so he can observe two poles A and B of equal height on the opposite bank of the river.

He finds the angle of elevation to the top of pole A is 22°, and the angle of elevation to the top of pole B is 19°.

Show how he could use these facts to determine the width of the river, if he knows that A and B are 100 m apart.

25 A surveyor standing on a horizontal plain can see a volcano in the distance. The angle of elevation of the top of the volcano is 23°. If the surveyor moves 750 m closer, the angle of elevation is now 37°. Determine the height of the volcano.

26 Find the shortest distance between the two parallel lines.
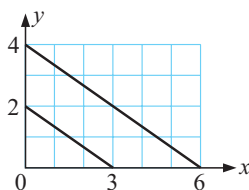
27
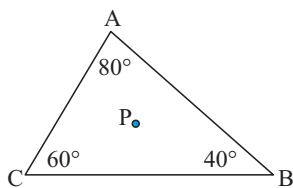
In the triangle alongside, P is 5 m from each of the vertices. Find the length of each side of the triangle.

D | THE FIRST QUADRANT OF THE UNIT CIRCLE   [8.2]

The **unit circle** is the circle with centre O(0, 0) and radius 1 unit.

Consider point P(a, b) in the first quadrant.

Notice that $\sin\theta = \dfrac{b}{1} = b$ and $\cos\theta = \dfrac{a}{1} = a$

So,   P has coordinates $(\cos\theta,\ \sin\theta)$.
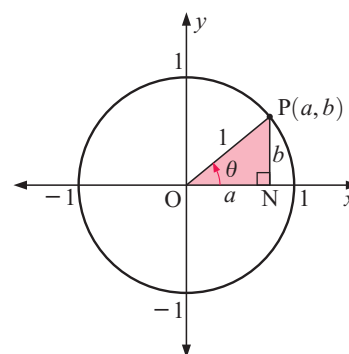
Example 9   ◀) Self Tutor

a State exactly the coordinates of point P.

b Find the coordinates of P correct to 3 decimal places.
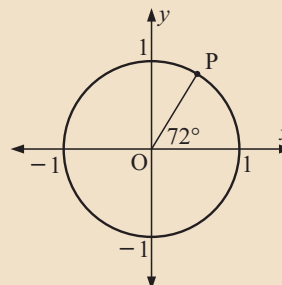
a P is $(\cos 72^o,\ \sin 72^o)$ **b** P is $(0.309,\ 0.951)$

EXERCISE 15D.1

1
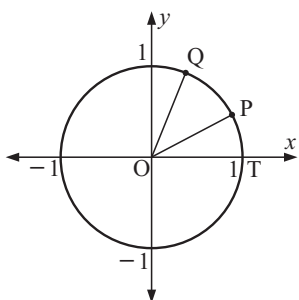
If angle TOP measures 28° and angle TOQ measures 68°, find:

a the exact coordinates of P and Q

b the coordinates of P and Q correct to 3 decimal places.

2 Point A has coordinates $(0.2588, 0.9659)$.
Without finding the size of θ, state:

a $\cos\theta$ **b** $\sin\theta$ **c** $\tan\theta$
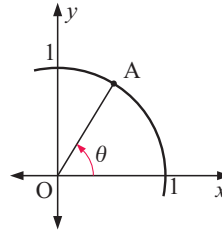
3
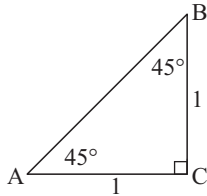
ABC is a right angled isosceles triangle with equal sides 1 unit long.

a   Find the exact length of AB.

b   Find the exact values of:

   i $\cos 45^o$ **ii** $\sin 45^o$ **iii** $\tan 45^o$.

4 P is the point on the unit circle where angle NOP measures 45°.

a   Classify triangle ONP.

b   Find the exact lengths of ON and NP.

c   State exactly the coordinates of P.

d   Find the exact values of:

   i $\cos 45^o$ **ii** $\sin 45^o$ **iii** $\tan 45^o$.
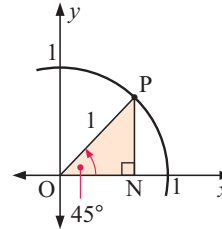
5
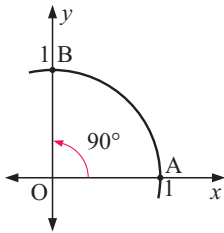
Using the unit circle shown:

a   find the coordinates of points A and B

b   find the values of:

   i $\cos 90^o$ **ii** $\sin 90^o$ **iii** $\tan 90^o$

c   find the values of:

   i $\cos 0^o$ **ii** $\sin 0^o$ **iii** $\tan 0^o$.

6 Consider the equilateral triangle ABC with sides of length 2 units.

a   Explain why $\theta = 60^o$ and $\phi = 30^o$.

b   Find the length of: **i**   MC   **ii**   MB

c   Find the exact values of:

   i $\cos 60^o$ **ii** $\sin 60^o$ **iii** $\tan 60^o$

d   Find the exact values of:

   i $\cos 30^o$ **ii** $\sin 30^o$ **iii** $\tan 30^o$
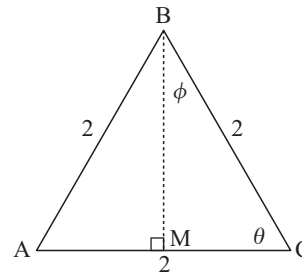
7
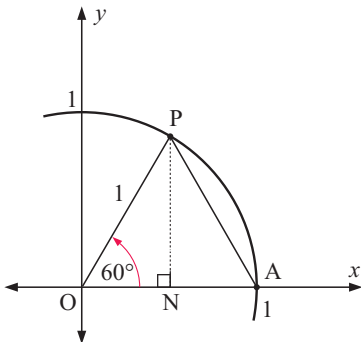
P is the point on the unit circle such that angle AOP is 60°.

a   Explain why triangle AOP is equilateral.

b   PN is drawn perpendicular to the x-axis. State the exact length of:

   i   ON   **ii**   PN.

c   State the exact coordinates of P.

d   Find the exact value of:

   i $\cos 60^o$ **ii** $\sin 60^o$ **iii** $\tan 60^o$

e   Find the size of OP̂N.

f   Find the exact value of:

   i $\cos 30^o$ **ii** $\sin 30^o$ **iii** $\tan 30^o$

8   **a**   If θ is any acute angle as shown, find the length of:
        i OQ   **ii** PQ   **iii** AT
    b   Explain how $\tan \theta$ or tangent θ may have been given its
name.
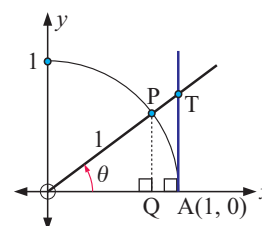
IMPORTANT ANGLES

From the previous exercise you should have discovered trigonometric ratios of some important angles:

θ	$\cos \theta$	$\sin \theta$	$\tan \theta$
0°	1	0	0
30°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
90°	0	1	undefined

> You should memorise these results or be able to quickly deduce them from diagrams.

In the following exercise you will see some new notation. It is customary to write:
$\sin^2 \theta$ to represent $(\sin \theta)^2$, $\cos^2 \theta$ to represent $(\cos \theta)^2$, and $\tan^2 \theta$ to represent $(\tan \theta)^2$.

EXERCISE 15D.2

1   Show that:

    a $\sin^2 30^o + \cos^2 30^o = 1$
    b $\cos^2 45^o + \sin^2 45^o = 1$
    c $\sin 30^o \cos 60^o + \sin 60^o \cos 30^o = 1$
    d $\sin^2 30^o + \sin^2 45^o + \sin^2 60^o = \frac{3}{2}$

2   Without using a calculator find the value of:

    a $\sin^2 60^o$
    b $\dfrac{\sin 30^o}{\cos 30^o}$
    c $\tan^2 60^o$

    d $\cos 0^o + \sin 90^o$
    e $\cos^2 30^o$
    f $1 - \tan^2 30^o$

    g $\dfrac{\sin 60^o}{\cos 60^o}$
    h $1 - \cos 60^o$
    i $2 + \sin 30^o$

3   Find the exact value of the unknown in:

    a
    12 cm 60° a cm

    b
    10 cm 30° h cm

    c
    c m 60° 8 m

    d
    120° 15 cm d cm

    e
    6 cm x cm 60°

    f
    $\sqrt{2}$ cm 45° y cm
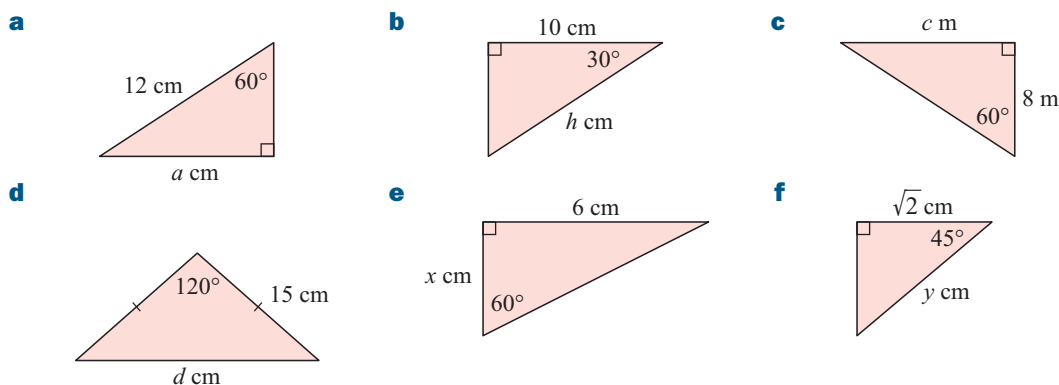
| **E** | **_TRUE BEARINGS_** | **[8.7]** |

We can measure a direction by comparing it with the **true north direction**. We call this a **true bearing**. Measurements are always taken in the **clockwise** direction.

Imagine you are standing at point A, facing north. You turn **clockwise** through an angle until you face B. The **bearing of B from A** is the angle through which you have turned.

So, the bearing of B from A is the clockwise measure of the angle between AB and the 'north' line through A.

In the diagram above, the bearing of B from A is 72° from true north. We write this as 72^oT or 072°.

To find the true **bearing of A from B**, we place ourselves at point B and face north. We then measure the clockwise angle through which we have to turn so that we face A. The true bearing of A from B is 252°.
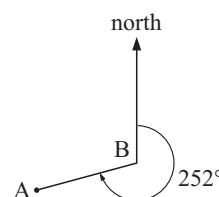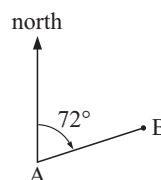
Note: • A true bearing is always written using three digits. For example, we write 072° rather than 72°.
   • The bearings of A from B and B from A always differ by 180°.
     You should be able to explain this using angle pair properties for parallel lines.

Example 10   ◀) **_Self Tutor_**

An aeroplane departs A and flies on a 143° course for 368 km. It then changes direction to a 233° course and flies a further 472 km to town C. Find:

a   the distance of C from A   **b**   the bearing of C from A.

First, we draw a fully labelled diagram of the flight. On the figure, we show angles found using parallel lines. Angle ABC measures 90°.

a $AC = \sqrt{368^2 + 472^2}$ {Pythagoras}
        ≈ 598.5
     So, C is about 599 km from A.

b   To find the required angle we first need to find θ.

     Now $\tan \theta = \dfrac{\text{OPP}}{\text{ADJ}} = \dfrac{472}{368}$

        $\therefore \quad \theta = \tan^{-1}\left(\dfrac{472}{368}\right)$

        $\therefore \quad \theta \approx 52.1^o$

     The required angle is $143^o + 52.1^o \approx 195.1^o$
     $\therefore$ the bearing of C from A is about 195.1^o.
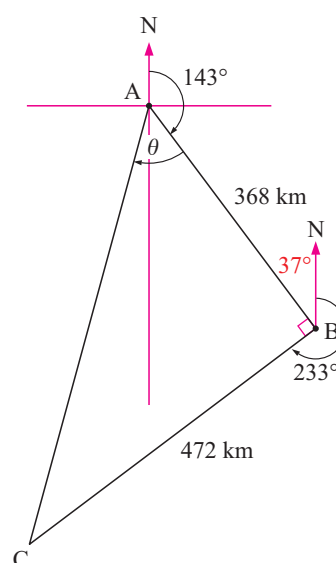
EXERCISE 15E

1 Draw diagrams to represent bearings from O of:

    a 136° **b** 240° **c** 051° **d** 327°

2 Find the bearing of Q from P if the bearing of P from Q is:

    a 054° **b** 113° **c** 263° **d** 304°

3 A, B and C are the checkpoints of a triangular orienteering course. For each of the following courses, find the bearing of:

    i B from A   **ii** C from B   **iii** B from C
    iv C from A   **v** A from B   **vi** A from C.

a
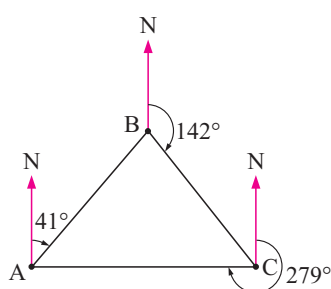
b
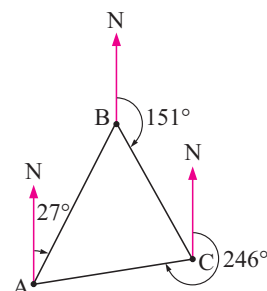
4 A bushwalker walks 14 km east and then 9 km south. Find the bearing of his finishing position from his starting point.

5 Runner A runs at 10 km/h due north. Runner B leaves the same spot and runs at 12 km/h due east. Find the distance and bearing of runner B from runner A after 30 minutes.

6 A hiker walks in the direction 153° and stops when she is 20 km south of her starting point. How far did she walk?

7 A ship sails for 60 km on a bearing 040°. How far east of its starting point is the ship?

8 Natasha is 50 m due east of Michelle. Natasha walks 20 m due north, and Michelle walks 10 m due south. Find the distance and bearing of Michelle from Natasha now.

9 An aeroplane travels on a bearing of 295° so that it is 200 km west of its starting point. How far has it travelled on this bearing?

10 A fishing trawler sails from port P in the direction 024° for 30 km, and then in the direction 114° for 20 km. Calculate:

    a the distance and bearing of the trawler from P

    b the direction in which the trawler must sail in order to return to P.

F   3-DIMENSIONAL PROBLEM SOLVING   [8.7]

Right angled triangles occur frequently in 3-dimensional figures. We can use Pythagoras' theorem and trigonometry to find unknown angles and lengths.

Example 11   ◀) Self Tutor

A cube has sides of length 10 cm. Find the angle between the diagonal AB of the cube and one of the edges at B.
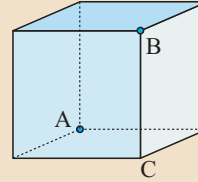
The angle between AB and *any* of the edges at B is the same.

∴   the required angle is $\widehat{ABC}$.
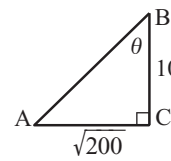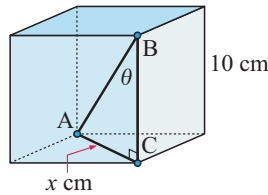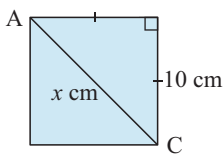
By Pythagoras:

$$x^2 = 10^2 + 10^2$$
$$\therefore \quad x^2 = 200$$
$$\therefore \quad x = \sqrt{200}$$

$$\tan \theta = \frac{\text{OPP}}{\text{ADJ}} = \frac{\sqrt{200}}{10}$$
$$\therefore \quad \theta = \tan^{-1}\left(\frac{\sqrt{200}}{10}\right)$$
$$\therefore \quad \theta \approx 54.7^o \qquad \text{The required angle is about } 54.7^o.$$

EXERCISE 15F.1

1 The figure alongside is a cube with sides of length 15 cm. Find:

    a EG   **b** $\widehat{AGE}$.
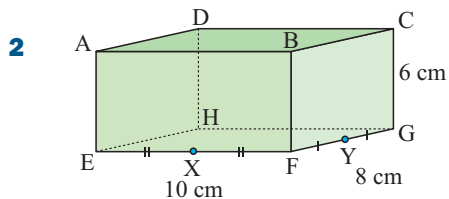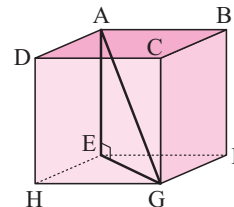
2 The figure alongside is a rectangular prism. X and Y are the midpoints of the edges EF and FG respectively. Find:

    a HX   **b** $\widehat{DXH}$ **c** HY   **d** $\widehat{DYH}$.

3 In the triangular prism alongside, find:

    a DF

    b $\widehat{AFD}$.
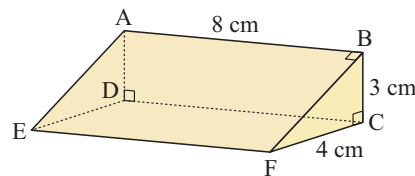
4 AB and BC are wooden support struts on a crate. Find the total length of wood required to make the two struts.
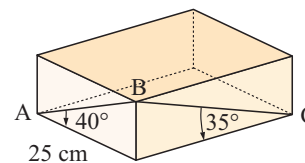
5 All edges of a square-based pyramid are 12 m in length.

  a Find the angle between a slant edge and a base diagonal.

  b Show that this angle is the same for any square-based pyramid with all edge lengths equal.

SHADOW LINES (PROJECTIONS)

Consider a wire frame in the shape of a cube as shown in the diagram alongside. Imagine a light source shining down directly on this cube from above.
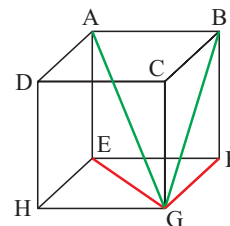
The shadow cast by wire AG would be EG. This is called the **projection** of AG onto the base plane EFGH.

Similarly, the projection of BG onto the base plane is FG.

Example 12
🔊 *Self Tutor*

Find the shadow or projection of the following onto the base plane if a light is shone from directly above the figure:

  a UP

  b WP

  c VP

  d XP.
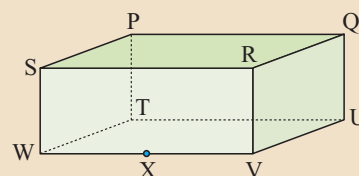
  a The projection of UP onto the base plane is UT.

  b The projection of WP onto the base plane is WT.

  c The projection of VP onto the base plane is VT.

  d The projection of XP onto the base plane is XT.
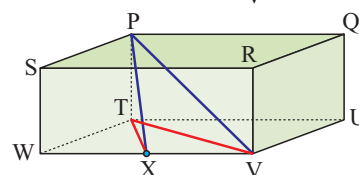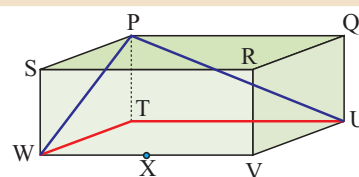
THE ANGLE BETWEEN A LINE AND A PLANE

> The angle between a line and a plane is the angle between the line and its projection on the plane.

Example 13
🔊 *Self Tutor*

Name the angle between the following line segments and the base plane EFGH:

  a AH

  b AG.
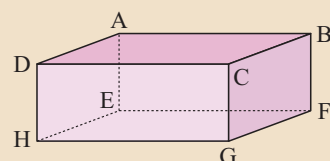
a   The projection of AH onto the base plane EFGH is EH

∴   the required angle is $A\widehat{H}E$.

b   The projection of AG onto the base plane EFGH is EG

∴   the required angle is $A\widehat{G}E$.
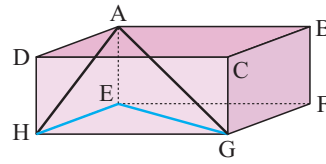
Example 14   ◀◎ *Self Tutor*

Find the angle between the following line segments
and the base plane EFGH:

a   DG   **b**   BH
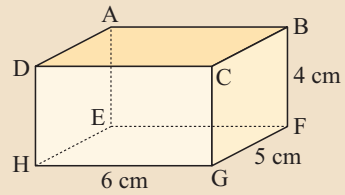
a   The required angle is $D\widehat{G}H$.   **b**   The required angle is $B\widehat{H}F$.

∴ $\tan \theta = \dfrac{\text{OPP}}{\text{ADJ}} = \dfrac{4}{6}$

∴ $\theta = \tan^{-1}\left(\dfrac{4}{6}\right)$

∴ $\theta \approx 33.69^{o}$
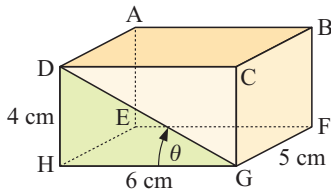
∴   the angle is about 33.7^{o}
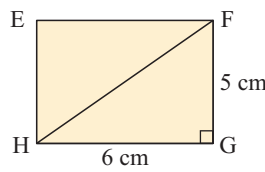
By Pythagoras,

$(HF)^2 = 6^2 + 5^2$

∴ $(HF)^2 = 61$

∴ $HF = \sqrt{61}$ cm
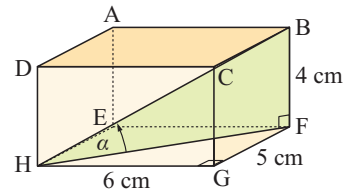
$\tan \alpha = \dfrac{\text{OPP}}{\text{ADJ}} = \dfrac{4}{\sqrt{61}}$

∴ $\alpha = \tan^{-1}\left(\dfrac{4}{\sqrt{61}}\right)$

∴ $\alpha \approx 27.12^{o}$

∴   the angle is about 27.1^{o}.

EXERCISE 15F.2

1   Find the following projections onto the base planes of the given figures:

a   **i**   CF
        ii DG
        iii DF
        iv CM
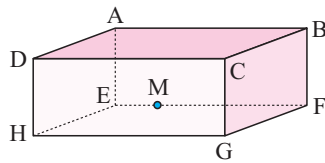
b   **i**   PA
        ii PN

c   **i**   BD
        ii AE
        iii AF
        iv AX
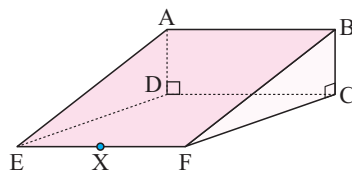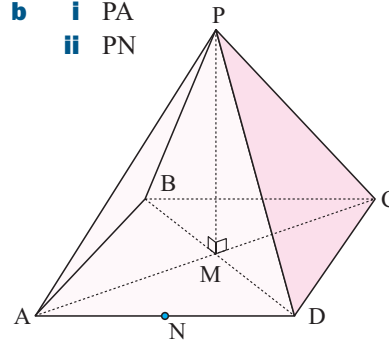
2 For each of the following figures, name the angle between the given line segment and the base plane:

a **i** DE   **ii** CE
     iii AG   **iv** BX
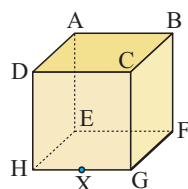
b **i** PY   **ii** QW
     iii QX   **iv** YQ
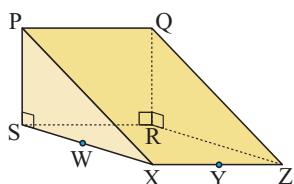
c **i** AQ   **ii** AY
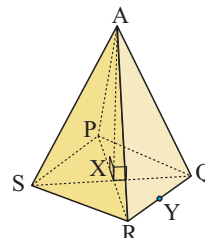
3 For each of the following figures, find the angle between the given line segments and the base plane:

a **i** DE
     ii DF
     iii DX
     iv AX
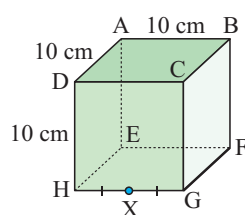
b **i** PU
     ii PV
     iii SX
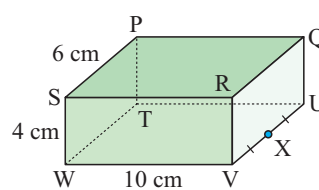
c **i** KO
     ii JX
     iii KY
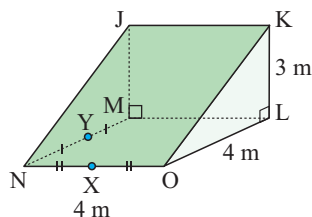
d **i** XD
     ii XY
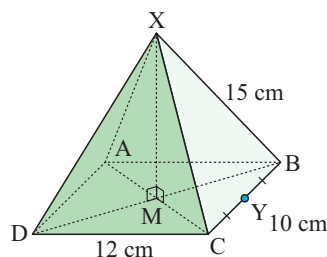
THE ANGLE BETWEEN TWO PLANES

Suppose two planes meet in a line.

We choose a point P on the line, and draw AP in one plane so it is perpendicular to the line. We then draw BP in the other plane so that BP is perpendicular to the line and so that $\widehat{APB} \leqslant 90^o$.

The angle $\theta = \widehat{APB}$ is defined as the angle between the two planes.
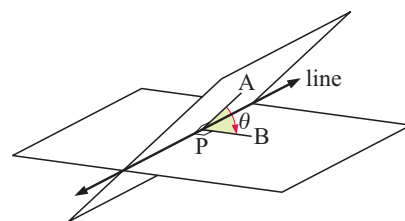
Example 15   ◀�**Self Tutor**

A square-based pyramid has base lengths 6 cm and height 8 cm.
Find the angle between a triangular face and the base.
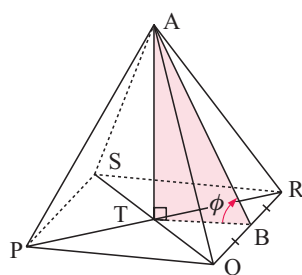
The base:
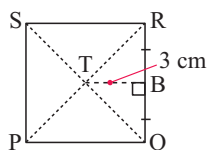
AB is perpendicular to QR and TB is perpendicular to QR

$\therefore$ $\triangle$ABT is the required angle.
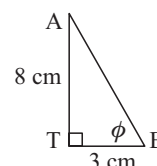
Now $\tan \phi = \frac{8}{3}$, so $\phi = \tan^{-1}\left(\frac{8}{3}\right) \approx 69.4^o$.

The angle is about 69.4^o.

EXERCISE 15F.3

1 A cube has sides of length 10 cm. Find the angle between the plane defined by BGD and plane ABCD.
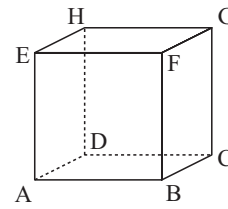
2   Find the angle that plane ASC makes with plane ABCD.
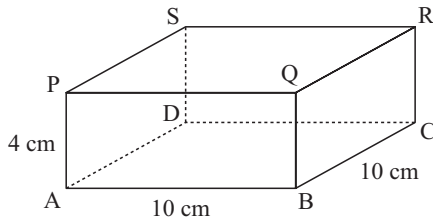
3 The slant edges of the pyramid opposite are 5 cm long. Find the angle between planes BCE and ABCD.
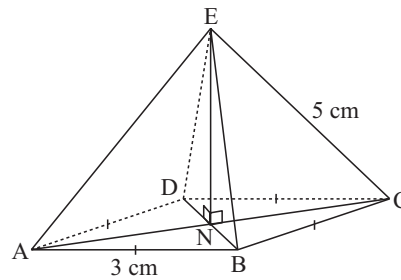
4 The pyramid of Cheops in Egypt has a height of 145 m and has base lengths 230 m. Find the angle between a sloping face and the base.

Review set 15A

1 Find $\sin\theta$, $\cos\theta$ and $\tan\theta$ for the triangle:
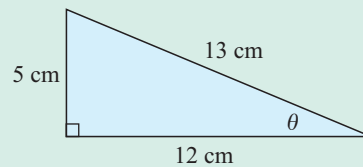
2 Find the value of x:

a

b
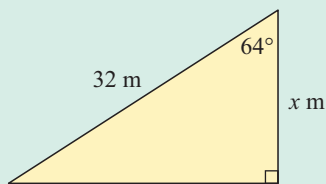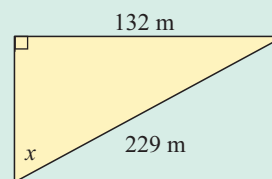
3 Find the measure of all unknown sides and angles in triangle CDE:
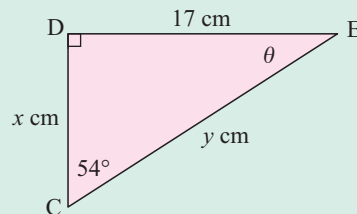
4 From a point 120 m horizontally from the base of a building, the angle of elevation to the top of the building is 34°. Find the height of the building.

5 Find the exact value of $\sin^2 60^o + \tan^2 45^o$.

6 Find:

   a $\sin\theta$

   b $\tan\theta$

   c the exact coordinates of P.
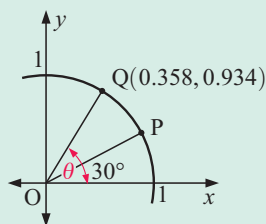
7 Find the diameter of this circle.
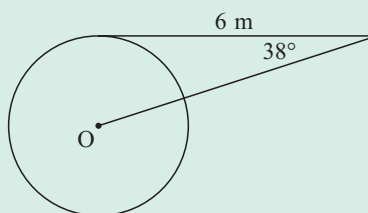
8 A ship sails 40 km on the bearing 056°. How far is it north of its starting point?

9 Find the angle that:

   a BG makes with FG

   b AG makes with the base plane EFGH.
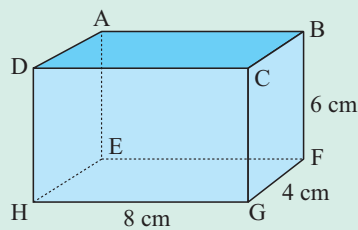
10 Two aeroplanes leave from an airport at the same time. Aeroplane A flies on a bearing of 124° at a speed of 450 km/h. Aeroplane B flies on a bearing of 214° at a speed of 380 km/h. Find the distance and bearing of B from A after 1 hour.

Review set 15B

1 Find: **i** $\sin\theta$ **ii** $\cos\phi$ **iii** $\tan\theta$ for the following triangles:

   a
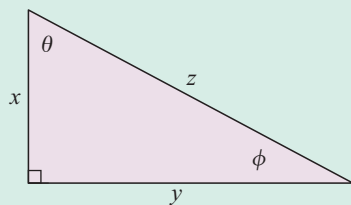
   b
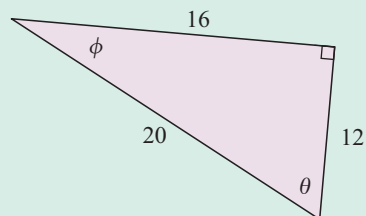
2 Find the value of x in the following:

   a

   b
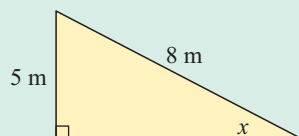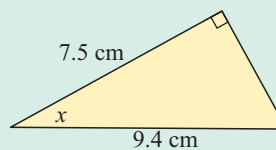
3   Find the measure of all unknown sides and angles in
      triangle KLM:
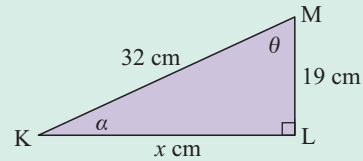
4   The angle of elevation from a point 2 km from the base of the vertical cliff to the top of the cliff
      is 17.7^o. Find the height of the cliff, in metres.

5   A tangent to a circle from a point 13 cm from the centre is 11 cm in length. Find the angle between
      the tangent and the line from the point to the centre of the circle.

6   Without using a calculator, find the value of:

   a $\dfrac{\sin^2 30^o}{\cos^2 30^o}$

   b $\sin^2 60^o + \tan 45^o - \cos 0^o$

7   Two cyclists depart from A at the same time. X cycles in a direction
      145° for two hours at a speed of 42 km per hour. Y cycles due east
      and at the end of the two hours is directly north of X.
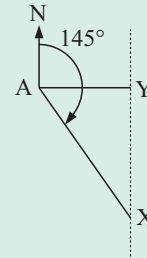
   a   How far did X travel in 2 hours?

   b   How far did Y travel in 2 hours?

   c   Determine the average speed at which Y has travelled.

8   Three towns P, Q and R are such that Q lies 10.8 km southeast of P and R lies
      15.4 km southwest of P.

   a   Draw a labelled diagram of the situation.

   b   Find the distance from R to Q.

   c   Find the bearing of Q from R.

9
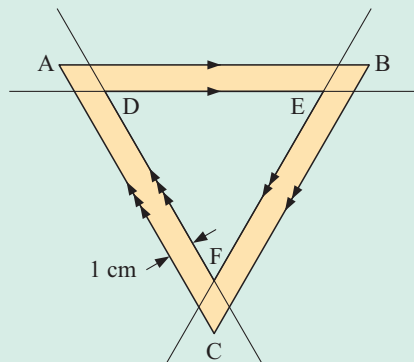
Triangle ABC is equilateral with sides 10 cm long.
Triangle DEF is formed by drawing lines parallel to, and
1 cm away from, the sides of triangle ABC, as illustrated.
Find the perimeter of triangle DEF.

10   The figure alongside is a square-based pyramid in which
       all edges are 20 cm in length. Find the angle that:

   a   AD makes with plane EBCD

   b   plane ADC makes with plane EBCD.
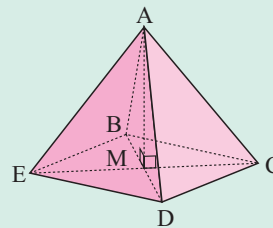
Algebraic fractions

16

Contents:

Opening problem

For the right angled triangle given, can you:

a write down expressions for $\cos\theta$, $\sin\theta$ and $\tan\theta$ in terms of x

b use these expressions to verify that $\sin\theta \div \cos\theta = \tan\theta$?

(triangle with hypotenuse x^2, vertical side $x-1$, base $x+2$, angle θ)
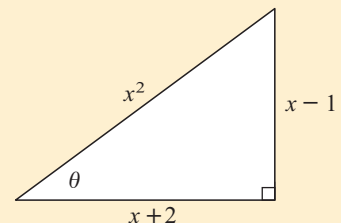
Fractions which involve unknowns are called **algebraic fractions**.

Algebraic fractions occur in many areas of mathematics. We see them in problems involving trigonometry and similar triangles.

A   SIMPLIFYING ALGEBRAIC FRACTIONS   [2.9]

CANCELLATION

We have observed previously that number fractions can be simplified by cancelling common factors.

For example, $\frac{12}{28} = \frac{\overset{1}{\cancel{4}}\times 3}{\underset{1}{\cancel{4}}\times 7} = \frac{3}{7}$ where the common factor 4 was cancelled.

The same principle can be applied to algebraic fractions:

> If the numerator and denominator of an algebraic fraction are both written in factored form and common factors are found, we can simplify by **cancelling the common factors**.

For example, $\dfrac{4ab}{2a} = \dfrac{2 \times \overset{1}{\cancel{2}} \times \overset{1}{\cancel{a}} \times b}{_1\cancel{2} \times \cancel{a}\,_1}$ {fully factorised}

$= \dfrac{2b}{1}$ {after cancellation}

$= 2b$

For algebraic fractions, check both the numerator and denominator to see if they can be expressed as the product of factors, then look for common factors which can be cancelled.

ILLEGAL CANCELLATION

Take care with fractions such as $\dfrac{a+3}{3}$.

> When cancelling in algebraic fractions, only factors can be cancelled, not terms.

The expression in the numerator, $a+3$,   **cannot** be written as the product of factors other than $1 \times (a+3)$. a and 3 are *terms* of the expression, not factors.

A typical **error** in **illegal cancellation** is: $\dfrac{a + \overset{1}{\cancel{3}}}{_1\cancel{3}} = \dfrac{a + 1}{1} = a + 1.$

You can check that this cancellation of terms is incorrect by substituting a value for a.

For example, if $a = 3$ then LHS $= \dfrac{a + 3}{3} = \dfrac{3 + 3}{3} = 2,$ whereas RHS $= a + 1 = 4$.

Example 1   ◀》 Self Tutor

Simplify: **a** $\dfrac{2x^2}{4x}$ **b** $\dfrac{6xy}{3x^3}$ **c** $\dfrac{x+y}{x}$

a $\dfrac{2x^2}{4x}$

$= \dfrac{\overset{1}{\cancel{2}} \times \overset{1}{\cancel{x}} \times x}{_2\cancel{4} \times \cancel{x}_1}$

$= \dfrac{x}{2}$

b $\dfrac{6xy}{3x^3}$

$= \dfrac{\overset{2}{\cancel{6}} \times \overset{1}{\cancel{x}} \times y}{_1\cancel{3} \times \cancel{x}_1 \times x \times x}$

$= \dfrac{2y}{x^2}$

c $\dfrac{x+y}{x}$

cannot be simplified as $x+y$ is a sum, not a product.

Example 2   🔊 *Self Tutor*

Simplify: **a** $\dfrac{(x+3)(x-2)}{4(x+3)}$ **b** $\dfrac{2(x+3)^2}{x+3}$

> In these examples $(x+3)$ is the common factor.

a $\dfrac{{}_1\cancel{(x+3)}(x-2)}{4\cancel{(x+3)}_1}$ **b** $\dfrac{2(x+3)^2}{x+3}$

$= \dfrac{x-2}{4}$ $= \dfrac{2\cancel{(x+3)}^{\,1}(x+3)}{{}_1\cancel{(x+3)}}$

$= 2(x+3)$

EXERCISE 16A.1

1 Simplify if possible:

a $\dfrac{6a}{3}$ **b** $\dfrac{10b}{5}$ **c** $\dfrac{3}{6x}$ **d** $\dfrac{8t}{t}$ **e** $\dfrac{t+2}{t}$

f $\dfrac{8a^2}{4a}$ **g** $\dfrac{2b}{4b^2}$ **h** $\dfrac{2x^2}{x^2}$ **i** $\dfrac{4a}{12a^3}$ **j** $\dfrac{4x^2}{8x}$

k $\dfrac{t^2+8}{t}$ **l** $\dfrac{a^2b}{ab^2}$ **m** $\dfrac{a+b}{a-c}$ **n** $\dfrac{15x^2y^3}{3xy^4}$ **o** $\dfrac{8abc^2}{4bc}$

p $\dfrac{(2a)^2}{a}$ **q** $\dfrac{(2a)^2}{4a^2}$ **r** $\dfrac{(3a^2)^2}{3a}$ **s** $\dfrac{(3a^2)^2}{9a^2}$ **t** $\dfrac{(3a^2)^2}{18a^3}$

2 Split the following expressions into two parts and simplify if possible.

For example, $\dfrac{x+9}{x} = \dfrac{x}{x} + \dfrac{9}{x} = 1 + \dfrac{9}{x}$.

a $\dfrac{x+3}{3}$ **b** $\dfrac{4a+1}{2}$ **c** $\dfrac{a+b}{c}$ **d** $\dfrac{a+2b}{b}$

e $\dfrac{2a+4}{2}$ **f** $\dfrac{3a+6b}{3}$ **g** $\dfrac{4m+8n}{4}$ **h** $\dfrac{4m+8n}{2m}$

3 Which of the expressions in **2** could be simplified and which could not? Explain why this is so.

4 Simplify:

a $\dfrac{3(x+2)}{3}$ **b** $\dfrac{4(x-1)}{2}$ **c** $\dfrac{7(b+2)}{14}$

d $\dfrac{2(n+5)}{12}$ **e** $\dfrac{10}{5(x+2)}$ **f** $\dfrac{15}{5(3-a)}$

g $\dfrac{6(x+2)}{(x+2)}$ **h** $\dfrac{x-4}{2(x-4)}$ **i** $\dfrac{2(x+2)}{x(x+2)}$

j $\dfrac{x(x-5)^2}{3(x-5)}$ **k** $\dfrac{(x+2)(x+3)}{2(x+2)^2}$ **l** $\dfrac{(x+2)(x+5)}{5(x+5)}$

m $\dfrac{(x+2)(x-1)}{(x-1)(x+3)}$ **n** $\dfrac{(x+5)(2x-1)}{3(2x-1)}$ **o** $\dfrac{(x+6)^2}{3(x+6)}$

p $\dfrac{x^2(x+2)}{x(x+2)(x-1)}$ **q** $\dfrac{(x+2)^2(x+1)}{4(x+2)}$ **r** $\dfrac{(x+2)^2(x-1)^2}{(x-1)^2x^2}$

FACTORISATION AND SIMPLIFICATION

It is often necessary to factorise either the numerator or denominator before simplification can be done.

Example 3 🔊 **Self Tutor**

Simplify: **a** $\dfrac{4a + 8}{4}$ **b** $\dfrac{3}{3a - 6b}$

a $\quad \dfrac{4a + 8}{4}$

$\quad = \dfrac{^1\!\!\cancel{4}(a + 2)}{_1\,\cancel{4}}$

$\quad = \dfrac{(a + 2)}{1}$

$\quad = a + 2$

b $\quad \dfrac{3}{3a - 6b}$

$\quad = \dfrac{^1\cancel{3}}{_1\,\cancel{3}(a - 2b)}$

$\quad = \dfrac{1}{a - 2b}$

Example 4 🔊 **Self Tutor**

Simplify: **a** $\dfrac{ab - ac}{b - c}$ **b** $\dfrac{2x^2 - 4x}{4x - 8}$

a $\quad \dfrac{ab - ac}{b - c}$

$\quad = \dfrac{a\cancel{(b - c)}^{\,1}}{\cancel{b - c}^{\,1}}$

$\quad = \dfrac{a}{1}$

$\quad = a$

b $\quad \dfrac{2x^2 - 4x}{4x - 8}$

$\quad = \dfrac{^1\cancel{2}x\cancel{(x - 2)}^{\,1}}{_2\cancel{4}\cancel{(x - 2)}^{\,1}}$

$\quad = \dfrac{x}{2}$

It is sometimes useful to use the property: $\boxed{b - a = -1(a - b)}$

Example 5 🔊 **Self Tutor**

Simplify: **a** $\dfrac{3a - 3b}{b - a}$ **b** $\dfrac{ab^2 - ab}{1 - b}$

a $\quad \dfrac{3a - 3b}{b - a}$

$\quad = \dfrac{3\cancel{(a - b)}^{\,1}}{-1\cancel{(a - b)}^{\,1}}$

$\quad = -3$

b $\quad \dfrac{ab^2 - ab}{1 - b}$

$\quad = \dfrac{ab\cancel{(b - 1)}^{\,1}}{-1\cancel{(b - 1)}^{\,1}}$

$\quad = -ab$

Example 6   🔊 *Self Tutor*

Simplify: $\dfrac{x^2 - x - 6}{x^2 - 4x + 3}$

$$\dfrac{x^2 - x - 6}{x^2 - 4x + 3}$$

$$= \dfrac{(x+2)\cancel{(x-3)}^{\,1}}{(x-1)\cancel{(x-3)}^{\,1}}$$

$$= \dfrac{x+2}{x-1}$$

Don't forget to expand your factorisations to check them.

EXERCISE 16A.2

1 Simplify by cancelling common factors:

a $\dfrac{6}{2(x+2)}$

b $\dfrac{2x+6}{2}$

c $\dfrac{3x+12}{3}$

d $\dfrac{3x+6}{6}$

e $\dfrac{5x+20}{10}$

f $\dfrac{3a+12}{9}$

g $\dfrac{xy+xz}{x}$

h $\dfrac{xy+xz}{z+y}$

i $\dfrac{ab+bc}{ab-bc}$

j $\dfrac{3x-12}{6(x-4)^2}$

k $\dfrac{(x+3)^2}{6x+18}$

l $\dfrac{2(x-y)^2}{6(x-y)}$

2 Simplify:

a $\dfrac{4x+8}{2x+4}$

b $\dfrac{mx+nx}{2x}$

c $\dfrac{mx+nx}{m+n}$

d $\dfrac{x+y}{mx+my}$

e $\dfrac{2x+4}{2}$

f $\dfrac{x^2+2x}{x}$

g $\dfrac{x^2+2x}{x+2}$

h $\dfrac{x}{bx+cx}$

i $\dfrac{3x^2+6x}{x+2}$

j $\dfrac{2x^2+6x}{2x}$

k $\dfrac{2x^2+6x}{x+3}$

l $\dfrac{ax^2+bx}{ax+b}$

3 Simplify, if possible:

a $\dfrac{2a-2b}{b-a}$

b $\dfrac{3a-3b}{6b-6a}$

c $\dfrac{a-b}{b-a}$

d $\dfrac{a+b}{a-b}$

e $\dfrac{x-2y}{4y-2x}$

f $\dfrac{3m-6n}{2n-m}$

g $\dfrac{3x-3}{x-x^2}$

h $\dfrac{xy^2-xy}{3-3y}$

i $\dfrac{6x^2-3x}{1-2x}$

j $\dfrac{4x+6}{4}$

k $\dfrac{12x-6}{2x-x^2}$

l $\dfrac{x^2-4}{x-2}$

m $\dfrac{x^2-4}{x+2}$

n $\dfrac{x^2-4}{2-x}$

o $\dfrac{x+3}{x^2-9}$

p $\dfrac{m^2-n^2}{m+n}$

q $\dfrac{m^2-n^2}{n-m}$

r $\dfrac{3x+6}{4-x^2}$

s $\dfrac{16-x^2}{x^2-4x}$

t $\dfrac{x^2-4}{4-x^2}$

u $\dfrac{5x^2-5y^2}{10xy-10y^2}$

v $\dfrac{2d^2-2a^2}{a^2-ad}$

w $\dfrac{4x^2-8x}{x^2-4}$

x $\dfrac{3x^2-6x}{4-x^2}$

4 Simplify:

a $\dfrac{x^2 - x}{x^2 - 1}$ **b** $\dfrac{x^2 + 2x + 1}{x^2 + 3x + 2}$ **c** $\dfrac{x^2 - 4x + 4}{2x^2 - 4x}$ **d** $\dfrac{x^2 + 4x + 3}{x^2 + 5x + 4}$

e $\dfrac{x^2 - 4}{x^2 - 3x - 10}$ **f** $\dfrac{x^2 + 7x + 12}{2x^2 + 6x}$ **g** $\dfrac{x^2 + 4x - 5}{2x^2 + 6x - 20}$ **h** $\dfrac{x^2 + 6x + 9}{x^2 + 3x}$

i $\dfrac{2x^2 - 7x - 4}{x^2 - 2x - 8}$ **j** $\dfrac{3x^2 + 5x - 2}{3x^2 - 4x + 1}$ **k** $\dfrac{2x^2 - 3x - 20}{x^2 - x - 12}$ **l** $\dfrac{8x^2 + 14x + 3}{2x^2 - x - 6}$

B   MULTIPLYING AND DIVIDING ALGEBRAIC FRACTIONS   [2.9]

The rules for multiplying and dividing algebraic fractions are identical to those used with numerical fractions. These are:

To **multiply** two or more fractions, we multiply the numerators to form the new numerator, and we multiply the denominators to form the new denominator.

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} = \frac{ac}{bd}$$

To **divide** by a fraction we multiply by its reciprocal.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

MULTIPLICATION

Step 1:	Multiply numerators and multiply denominators.
Step 2:	Separate the factors.
Step 3:	Cancel any common factors.
Step 4:	Write in simplest form.

For example, $\dfrac{n^2}{3} \times \dfrac{6}{n} = \dfrac{n^2 \times 6}{3 \times n}$ {*Step 1*}

$$= \frac{n \times \cancel{n}^{\,1} \times 2 \times \cancel{6}^{\,1}}{{}_1\cancel{3} \times \cancel{n}_{\,1}} \qquad \{Steps\ 2\ and\ 3\}$$

$$= \frac{2n}{1}$$

$$= 2n \qquad\qquad\qquad \{Step\ 4\}$$

Example 7   ◀⑨ *Self Tutor*

Simplify: **a** $\dfrac{4}{d} \times \dfrac{d}{8}$ **b** $\dfrac{5}{g} \times g^3$

a $\dfrac{4}{d} \times \dfrac{d}{8} = \dfrac{{}^1\cancel{4} \times \cancel{d}^{\,1}}{{}_1\cancel{d} \times \cancel{8}_{\,2}}$ **b** $\dfrac{5}{g} \times g^3 = \dfrac{5}{g} \times \dfrac{g^3}{1}$

$\qquad\qquad = \dfrac{1}{2}$ $= \dfrac{5 \times \cancel{g}^{\,1} \times g \times g}{\cancel{g}^{\,1}}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 5g^2$

DIVISION

Step 1:	To divide by a fraction, multiply by its reciprocal.
Step 2:	Multiply numerators and multiply denominators.
Step 3:	Cancel any common factors.
Step 4:	Write in simplest form.

For example,

$$\frac{m}{2} \div \frac{n}{6} = \frac{m}{2} \times \frac{6}{n} \qquad \{\textit{Step 1}\}$$

$$= \frac{m \times 6}{2 \times n} \qquad \{\textit{Step 2}\}$$

$$= \frac{m \times \cancel{6}^{\,3}}{{}_{1}\cancel{2} \times n} \qquad \{\textit{Step 3}\}$$

$$= \frac{3m}{n} \qquad \{\textit{Step 4}\}$$

Example 8 ◆)) Self Tutor

Simplify: **a** $\dfrac{6}{x} \div \dfrac{2}{x^2}$ **b** $\dfrac{8}{p} \div 2$

a $\dfrac{6}{x} \div \dfrac{2}{x^2} = \dfrac{6}{x} \times \dfrac{x^2}{2}$

$\qquad = \dfrac{3 \times \cancel{2}^{\,1} \times \cancel{x}^{\,1} \times x}{{}_{1}\,\cancel{x} \times \cancel{2}\,{}_{1}}$

$\qquad = 3x$

b $\dfrac{8}{p} \div 2 = \dfrac{8}{p} \times \dfrac{1}{2}$

$\qquad = \dfrac{{}^{4}\cancel{8} \times 1}{p \times \cancel{2}\,{}_{1}}$

$\qquad = \dfrac{4}{p}$

EXERCISE 16B

1 Simplify:

a $\dfrac{a}{2} \times \dfrac{b}{3}$ **b** $\dfrac{x}{4} \times \dfrac{2}{x}$ **c** $\dfrac{c}{4} \times \dfrac{2}{c}$ **d** $\dfrac{a}{2} \times \dfrac{a}{3}$

e $\dfrac{a}{b} \times \dfrac{x}{y}$ **f** $\dfrac{x}{y} \times \dfrac{y}{x}$ **g** $\dfrac{x}{3} \times x$ **h** $\dfrac{x}{4} \times \dfrac{8}{y}$

i $\dfrac{n}{2} \times \dfrac{1}{n^2}$ **j** $\dfrac{6}{p} \times \dfrac{p}{2}$ **k** $\dfrac{m}{x} \times \dfrac{x}{n}$ **l** $x \times \dfrac{2}{x}$

m $\dfrac{5}{t} \times t^2$ **n** $\left(\dfrac{x}{y}\right)^2$ **o** $\left(\dfrac{4}{d}\right)^2$ **p** $\dfrac{a}{b} \times \dfrac{b}{c} \times \dfrac{c}{a}$

2 Simplify:

a $\dfrac{x}{3} \div \dfrac{x}{2}$ **b** $\dfrac{3}{y} \div \dfrac{6}{y}$ **c** $3 \div \dfrac{1}{x}$ **d** $6 \div \dfrac{2}{y}$

e $\dfrac{3}{p} \div \dfrac{1}{p}$ **f** $\dfrac{c}{n} \div n$ **g** $d \div \dfrac{5}{d}$ **h** $x \div \dfrac{x}{3}$

i $1 \div \dfrac{a}{b}$ **j** $\dfrac{3}{d} \div 2$ **k** $\dfrac{4}{x} \div \dfrac{x^2}{2}$ **l** $\dfrac{4}{x} \div \dfrac{8}{x^2}$

m $a \div \dfrac{a^2}{3}$ **n** $\dfrac{x}{y} \div \dfrac{x^2}{y}$ **o** $\dfrac{5}{a} \div \dfrac{a}{2}$ **p** $\dfrac{a^2}{5} \div \dfrac{a}{3}$

C   ADDING AND SUBTRACTING ALGEBRAIC FRACTIONS   [2.9]

The rules for addition and subtraction of algebraic fractions are identical to those used with numerical fractions.

To **add** two or more fractions we obtain the *lowest common denominator* and then add the resulting numerators.

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

To **subtract** two or more fractions we obtain the *lowest common denominator* and then subtract the resulting numerators.

$$\frac{a}{c} - \frac{d}{c} = \frac{a-d}{c}$$

To find the lowest common denominator, we look for the **lowest common multiple of the denominators**.

For example: when adding $\frac{3}{4} + \frac{2}{3}$,   the lowest common denominator is 12,

when adding $\frac{2}{3} + \frac{1}{6}$,   the lowest common denominator is 6.

The same method is used when there are variables in the denominator.

For example: when adding $\dfrac{4}{x} + \dfrac{5}{y}$,   the lowest common denominator is xy,

when adding $\dfrac{4}{x} + \dfrac{3}{2x}$,   the lowest common denominator is $2x$,

when adding $\dfrac{1}{3a} + \dfrac{2}{5b}$,   the lowest common denominator is $15ab$.

To find $\dfrac{x}{2} + \dfrac{3x}{5}$ we notice the LCD is 10. We then proceed in the same manner as for ordinary fractions:

$$\frac{x}{2} + \frac{3x}{5} = \frac{x \times 5}{2 \times 5} + \frac{3x \times 2}{5 \times 2}$$
$$= \frac{5x}{10} + \frac{6x}{10}$$
$$= \frac{11x}{10}$$

Example 9   ◀) Self Tutor

Simplify: **a** $\dfrac{x}{3} + \dfrac{5x}{6}$ **b** $\dfrac{3b}{4} - \dfrac{2b}{3}$

a $\dfrac{x}{3} + \dfrac{5x}{6}$

$= \dfrac{x \times 2}{3 \times 2} + \dfrac{5x}{6}$ {LCD $= 6$}

$= \dfrac{2x}{6} + \dfrac{5x}{6}$

$= \dfrac{2x + 5x}{6}$

$= \dfrac{7x}{6}$

b $\dfrac{3b}{4} - \dfrac{2b}{3}$

$= \dfrac{3b \times 3}{4 \times 3} - \dfrac{2b \times 4}{3 \times 4}$ {LCD $= 12$}

$= \dfrac{9b}{12} - \dfrac{8b}{12}$

$= \dfrac{b}{12}$

Example 10   ◀⬤ *Self Tutor*

Simplify: **a** $\dfrac{2}{a} + \dfrac{3}{c}$ **b** $\dfrac{7}{x} - \dfrac{5}{2x}$

a $\dfrac{2}{a} + \dfrac{3}{c}$

$= \dfrac{2 \times c}{a \times c} + \dfrac{3 \times a}{c \times a}$ {LCD $= ac$}

$= \dfrac{2c}{ac} + \dfrac{3a}{ac}$

$= \dfrac{2c + 3a}{ac}$

b $\dfrac{7}{x} - \dfrac{5}{2x}$

$= \dfrac{7 \times 2}{x \times 2} - \dfrac{5}{2x}$ {LCD $= 2x$}

$= \dfrac{14}{2x} - \dfrac{5}{2x}$

$= \dfrac{9}{2x}$

Example 11   ◀⬤ *Self Tutor*

Simplify: **a** $\dfrac{b}{3} + 1$ **b** $\dfrac{a}{4} - a$

a $\dfrac{b}{3} + 1 = \dfrac{b}{3} + \dfrac{3}{3}$

$= \dfrac{b + 3}{3}$

b $\dfrac{a}{4} - a = \dfrac{a}{4} - \dfrac{a \times 4}{1 \times 4}$

$= \dfrac{a}{4} - \dfrac{4a}{4}$

$= \dfrac{-3a}{4}$ or $-\dfrac{3a}{4}$

EXERCISE 16C

1 Simplify by writing as a single fraction:

a $\dfrac{x}{2} + \dfrac{x}{5}$ **b** $\dfrac{x}{3} - \dfrac{x}{6}$ **c** $\dfrac{x}{4} + \dfrac{3x}{5}$ **d** $\dfrac{x}{2} - \dfrac{x}{5}$

e $\dfrac{2t}{3} - \dfrac{7t}{12}$ **f** $\dfrac{11n}{21} - \dfrac{n}{7}$ **g** $\dfrac{a}{2} + \dfrac{a}{3}$ **h** $\dfrac{a}{2} + \dfrac{b}{4}$

i $\dfrac{n}{3} + \dfrac{2n}{15}$ **j** $\dfrac{5g}{6} - \dfrac{g}{3}$ **k** $\dfrac{4s}{5} - \dfrac{2s}{3}$ **l** $a - \dfrac{3a}{5}$

m $\dfrac{x}{3} + \dfrac{x}{2} + \dfrac{x}{6}$ **n** $\dfrac{y}{2} + \dfrac{y}{4} - \dfrac{y}{3}$ **o** $\dfrac{z}{4} + \dfrac{z}{6} - \dfrac{z}{3}$ **p** $2q - \dfrac{q}{3} + \dfrac{2q}{7}$

2 Simplify:

a $\dfrac{3}{a} + \dfrac{2}{b}$ **b** $\dfrac{4}{a} + \dfrac{3}{d}$ **c** $\dfrac{5}{a} - \dfrac{3}{b}$ **d** $\dfrac{3a}{m} - \dfrac{2a}{m}$

e $\dfrac{a}{y} + \dfrac{b}{3y}$ **f** $\dfrac{4}{a} - \dfrac{5}{2a}$ **g** $\dfrac{3}{a} - \dfrac{2}{ab}$ **h** $\dfrac{c}{a} + \dfrac{b}{d}$

i $\dfrac{4}{b} + \dfrac{a}{b}$ **j** $\dfrac{2}{a} - \dfrac{c}{d}$ **k** $\dfrac{5}{x} + \dfrac{x}{3}$ **l** $\dfrac{p}{6} - \dfrac{2}{d}$

m $\dfrac{m}{3} + \dfrac{n}{m}$ **n** $\dfrac{2m}{p} - \dfrac{m}{n}$ **o** $\dfrac{3b}{5} + \dfrac{b}{4}$ **p** $\dfrac{5b}{3} - \dfrac{3b}{5}$

3 Simplify:

a $\dfrac{x}{3} + 2$ b $\dfrac{m}{2} - 1$ c $\dfrac{a}{3} + a$ d $\dfrac{b}{5} - 2$

e $\dfrac{x}{6} - 3$ f $3 + \dfrac{x}{4}$ g $5 - \dfrac{x}{6}$ h $2 + \dfrac{3}{x}$

i $6 - \dfrac{3}{x}$ j $b + \dfrac{3}{b}$ k $\dfrac{5}{x} + x$ l $\dfrac{y}{6} - 2y$

4 Simplify:

a $\dfrac{x}{3} + \dfrac{3x}{5}$ b $\dfrac{3x}{5} - \dfrac{2x}{7}$ c $\dfrac{5}{a} + \dfrac{1}{2a}$ d $\dfrac{6}{y} - \dfrac{3}{4y}$

e $\dfrac{3}{b} + \dfrac{4}{c}$ f $\dfrac{5}{4a} - \dfrac{6}{b}$ g $\dfrac{x}{10} + 3$ h $4 - \dfrac{x}{3}$

D   MORE COMPLICATED FRACTIONS   [2.9]

Addition and subtraction of more complicated algebraic fractions can be made relatively straightforward if we adopt a consistent approach.

For example: $\dfrac{x+2}{3} + \dfrac{5-2x}{2} = \dfrac{2}{2}\left(\dfrac{x+2}{3}\right) + \dfrac{3}{3}\left(\dfrac{5-2x}{2}\right)$ {achieves LCD of 6}

$$= \dfrac{2(x+2)}{6} + \dfrac{3(5-2x)}{6}$$ {simplify each fraction}

We can then write the expression as a single fraction and simplify the numerator.

Example 12   ◀)) Self Tutor

Write as a single fraction: **a** $\dfrac{x}{12} + \dfrac{x-1}{4}$ **b** $\dfrac{x-1}{3} - \dfrac{x+2}{7}$

a $\dfrac{x}{12} + \dfrac{x-1}{4}$

$= \dfrac{x}{12} + \dfrac{3}{3}\left(\dfrac{x-1}{4}\right)$

$= \dfrac{x + 3(x-1)}{12}$

$= \dfrac{x + 3x - 3}{12}$

$= \dfrac{4x - 3}{12}$

b $\dfrac{x-1}{3} - \dfrac{x+2}{7}$

$= \dfrac{7}{7}\left(\dfrac{x-1}{3}\right) - \dfrac{3}{3}\left(\dfrac{x+2}{7}\right)$

$= \dfrac{7(x-1)}{21} - \dfrac{3(x+2)}{21}$

$= \dfrac{7(x-1) - 3(x+2)}{21}$

$= \dfrac{7x - 7 - 3x - 6}{21}$

$= \dfrac{4x - 13}{21}$

Example 13

$\blacktriangleleft)$ *Self Tutor*

Write as a single fraction: **a** $\dfrac{2}{x} + \dfrac{1}{x+2}$ **b** $\dfrac{5}{x+2} - \dfrac{1}{x-1}$

a $\dfrac{2}{x} + \dfrac{1}{x+2}$

$= \dfrac{2}{x}\left(\dfrac{x+2}{x+2}\right) + \left(\dfrac{1}{x+2}\right)\dfrac{x}{x}$

$\qquad \{\text{LCD} = x(x+2)\}$

$= \dfrac{2(x+2) + x}{x(x+2)}$

$= \dfrac{2x + 4 + x}{x(x+2)}$

$= \dfrac{3x + 4}{x(x+2)}$

b $\dfrac{5}{x+2} - \dfrac{1}{x-1}$

$= \left(\dfrac{5}{x+2}\right)\left(\dfrac{x-1}{x-1}\right) - \left(\dfrac{1}{x-1}\right)\left(\dfrac{x+2}{x+2}\right)$

$\qquad \{\text{LCD} = (x+2)(x-1)\}$

$= \dfrac{5(x-1) - 1(x+2)}{(x+2)(x-1)}$

$= \dfrac{5x - 5 - x - 2}{(x+2)(x-1)}$

$= \dfrac{4x - 7}{(x+2)(x-1)}$

Example 14

$\blacktriangleleft)$ *Self Tutor*

Simplify:

a $\dfrac{x^2 + 2x}{x^2 + 3x + 2} \times \dfrac{x+1}{2x^2}$ **b** $\dfrac{x^2 - 3x - 4}{x^2 + x} \div \dfrac{x^2 - x - 12}{x^2 - x}$

a $\dfrac{x^2 + 2x}{x^2 + 3x + 2} \times \dfrac{x+1}{2x^2}$

$= \dfrac{\overset{1}{x}(x+2)^{1}}{(x+1)(x+2)} \times \dfrac{(x+1)^{1}}{2x^{2}_{1}}$

$= \dfrac{1}{2x}$

b $\dfrac{x^2 - 3x - 4}{x^2 + x} \div \dfrac{x^2 - x - 12}{x^2 - x}$

$= \dfrac{x^2 - 3x - 4}{x^2 + x} \times \dfrac{x^2 - x}{x^2 - x - 12}$ {reciprocating}

$= \dfrac{\overset{1}{(x-4)}(x+1)}{x(x+1)} \times \dfrac{\overset{1}{x}(x-1)}{(x-4)(x+3)}$ {factorising}

$= \dfrac{x-1}{x+3}$ {on cancelling}

EXERCISE 16D.1

1 Write as a single fraction:

a $\dfrac{x}{4} + \dfrac{x-1}{5}$

b $\dfrac{2x+5}{3} + \dfrac{x}{6}$

c $\dfrac{x}{7} + \dfrac{2x-1}{6}$

d $\dfrac{a+b}{2} + \dfrac{b-a}{3}$

e $\dfrac{x-1}{4} + \dfrac{2x-1}{5}$

f $\dfrac{x+1}{2} + \dfrac{2-x}{7}$

g $\dfrac{x}{5} - \dfrac{x-3}{6}$

h $\dfrac{x-1}{6} - \dfrac{x}{7}$

i $\dfrac{x}{10} - \dfrac{2x-1}{5}$

j $\dfrac{x}{6} - \dfrac{1-x}{12}$

k $\dfrac{x-1}{3} - \dfrac{x-2}{5}$

l $\dfrac{2x+1}{3} - \dfrac{1-3x}{8}$

2   Write as a single fraction:

a $\dfrac{2}{x+1} + \dfrac{3}{x-2}$ **b** $\dfrac{5}{x+1} + \dfrac{7}{x+2}$ **c** $\dfrac{5}{x-1} - \dfrac{4}{x+2}$

d $\dfrac{2}{x+2} - \dfrac{4}{2x+1}$ **e** $\dfrac{3}{x-1} + \dfrac{4}{x+4}$ **f** $\dfrac{7}{1-x} - \dfrac{8}{x+2}$

g $\dfrac{1}{x+1} + \dfrac{3}{x}$ **h** $\dfrac{5}{x} - \dfrac{2}{x+3}$ **i** $\dfrac{x}{x+2} + \dfrac{3}{x-4}$

j $2 + \dfrac{4}{x-3}$ **k** $\dfrac{3x}{x+2} - 1$ **l** $\dfrac{x}{x+3} + \dfrac{x-1}{x+2}$

m $\dfrac{2}{2x-1} - \dfrac{1}{x+3}$ **n** $\dfrac{7}{x} - \dfrac{4}{3x-1}$ **o** $\dfrac{x}{x+2} - \dfrac{x+1}{x-2}$

p $\dfrac{2}{x(x+1)} + \dfrac{1}{x+1}$ **q** $\dfrac{1}{x-1} - \dfrac{1}{x} + \dfrac{1}{x+1}$ **r** $\dfrac{2}{x+1} - \dfrac{1}{x-1} + \dfrac{3}{x+2}$

s $\dfrac{x}{x-1} - \dfrac{1}{x} + \dfrac{x}{x+1}$

3   Simplify:

a $\dfrac{3x-9}{6} \times \dfrac{12}{x^2-9}$ **b** $\dfrac{5x-20}{2x} \div \dfrac{2x-8}{x}$

c $\dfrac{2x+8}{5} \times \dfrac{10}{x^2-16}$ **d** $\dfrac{x^2-x-2}{x+1} \div \dfrac{x-2}{5}$

e $\dfrac{x^2+x-2}{x^2-4} \times \dfrac{x^2+3x-10}{x^2+7x+10}$ **f** $\dfrac{x^2-4}{x^2+3x} \div \dfrac{x^2+7x+10}{x^2+8x+15}$

g $\dfrac{2x^2-x-3}{6x^2+x-15} \times \dfrac{4x^2-7x+3}{4x^2+x-3}$ **h** $\dfrac{4x^2-9}{x^2+4x-5} \div \dfrac{2x^2-3x}{x^2+5x}$

4   Answer the **Opening Problem** on page **339**.

PROPERTIES OF ALGEBRAIC FRACTIONS

Writing expressions as a single fraction can help us to find when the expression is zero.

However, we need to be careful when we cancel common factors, as we can sometimes lose values when an expression is undefined.

Example 15   ◀)) *Self Tutor*

Write as a single fraction: **a** $\dfrac{3}{(x+2)(x-1)} + \dfrac{x}{x-1}$ **b** $\dfrac{-3}{(x+2)(x-1)} + \dfrac{x}{x-1}$

a $\dfrac{3}{(x+2)(x-1)} + \dfrac{x}{x-1}$

    $= \dfrac{3}{(x+2)(x-1)} + \left(\dfrac{x}{x-1}\right)\left(\dfrac{x+2}{x+2}\right)$ $\{\text{LCD} = (x+2)(x-1)\}$

    $= \dfrac{3 + x(x+2)}{(x+2)(x-1)}$

    $= \dfrac{x^2+2x+3}{(x+2)(x-1)}$ which we cannot simplify further.

---

b $\quad \dfrac{-3}{(x+2)(x-1)} + \dfrac{x}{x-1}$

$\quad = \dfrac{-3}{(x+2)(x-1)} + \left(\dfrac{x}{x-1}\right)\left(\dfrac{x+2}{x+2}\right) \qquad \{\text{LCD} = (x+2)(x-1)\}$

$\quad = \dfrac{-3 + x(x+2)}{(x+2)(x-1)}$

$\quad = \dfrac{x^2 + 2x - 3}{(x+2)(x-1)}$

$\quad = \dfrac{(x+3)(x-1)^{\,1}}{(x+2)(x-1)^{\,1}}$

$\quad = \dfrac{x+3}{x+2}$

The expression is zero when $x = -3$. The expression is undefined when $x = -2$ and also when $x = 1$. We can see this from the original expression.

EXERCISE 16D.2

1 Write as a single fraction:

a $\quad \dfrac{2}{x(x+1)} + \dfrac{1}{x+1}$

b $\quad \dfrac{2}{x(x+1)} + \dfrac{x}{x+1}$

c $\quad \dfrac{2x}{x-3} + \dfrac{4}{(x+2)(x-3)}$

d $\quad \dfrac{2x}{x-3} - \dfrac{30}{(x+2)(x-3)}$

e $\quad \dfrac{3}{(x-2)(x+3)} + \dfrac{x}{x+3}$

f $\quad \dfrac{x}{x+3} - \dfrac{15}{(x-2)(x+3)}$

g $\quad \dfrac{2x}{x+4} - \dfrac{40}{(x-1)(x+4)}$

h $\quad \dfrac{x+5}{x-2} - \dfrac{63}{(x-2)(x+7)}$

2 a Write $\quad \dfrac{2}{(x+2)(x-3)} + \dfrac{2x}{x-3} \quad$ as a single fraction.

b Hence, find the values of x when this expression is: **i** undefined   **ii** zero.

3 Simplify: **a** $\quad \dfrac{\frac{x}{x-2} - 3}{x-3}$ **b** $\quad \dfrac{\frac{3x}{x+4} - 1}{x-2}$ **c** $\quad \dfrac{\frac{x^2}{x+2} - 1}{x+1}$

             d $\quad \dfrac{\frac{x^2}{2-x} + 9}{x-3}$ **e** $\quad \dfrac{\frac{1}{x^2} - \frac{1}{4}}{x-2}$ **f** $\quad \dfrac{\frac{x-3}{x^2} - \frac{1}{16}}{x-4}$

4 a Simplify: $\quad \dfrac{\frac{2}{x+1} - \frac{x}{3}}{2-x}$

b Hence, find the values of x when this expression is: **i** undefined   **ii** zero.

Review set 16A

1 Simplify:

a $\quad \dfrac{6x^2}{2x}$ **b** $\quad 6 \times \dfrac{n}{2}$ **c** $\quad \dfrac{x}{2} \div 3$ **d** $\quad \dfrac{8x}{(2x)^2}$

2 Simplify, if possible:

a $\quad \dfrac{8}{4(c+3)}$ **b** $\quad \dfrac{3x+8}{4}$ **c** $\quad \dfrac{4x+8}{4}$ **d** $\quad \dfrac{x(x+1)}{3(x+1)(x+2)}$

3 Write as a single fraction:

     a $\dfrac{2x}{3} + \dfrac{3x}{5}$ **b** $\dfrac{2x}{3} \times \dfrac{3x}{5}$ **c** $\dfrac{2x}{3} \div \dfrac{3x}{5}$ **d** $\dfrac{2x}{3} - \dfrac{3x}{5}$

4 Simplify by factorisation:

     a $\dfrac{4x + 8}{x + 2}$ **b** $\dfrac{5 - 10x}{2x - 1}$ **c** $\dfrac{4x^2 + 6x}{2x + 3}$

5 Write as a single fraction:

     a $\dfrac{x + 3}{4} + \dfrac{2x - 2}{3}$ **b** $\dfrac{x - 1}{7} - \dfrac{1 - 2x}{2}$ **c** $\dfrac{2}{x + 2} + \dfrac{1}{x}$

6 Simplify by factorisation:

     a $\dfrac{8 - 2x}{x^2 - 16}$ **b** $\dfrac{x^2 + 7x + 12}{x^2 + 4x}$ **c** $\dfrac{2x^2 - 3x - 2}{3x^2 - 4x - 4}$

7   **a**   Write $\dfrac{2x}{x + 3} - \dfrac{12}{(x + 1)(x + 3)}$ as a single fraction.

     b   Hence, find the values of x when this expression is: **i**   undefined   **ii**   zero.

Review set 16B

1 Simplify:

     a $\dfrac{4a}{6a}$ **b** $\dfrac{x}{3} \times 6$ **c** $3 \div \dfrac{1}{n}$ **d** $\dfrac{12x^2}{6x}$

2 Simplify, if possible:

     a $\dfrac{3x + 15}{5}$ **b** $\dfrac{3x + 15}{3}$ **c** $\dfrac{2(a + 4)}{(a + 4)^2}$ **d** $\dfrac{abc}{2ac(b - a)}$

3 Write as a single fraction:

     a $\dfrac{3x}{4} + 2x$ **b** $\dfrac{3x}{4} - 2x$ **c** $\dfrac{3x}{4} \times 2x$ **d** $\dfrac{3x}{4} \div 2x$

4 Simplify by factorisation:

     a $\dfrac{3 - x}{x - 3}$ **b** $\dfrac{5x + 10}{2x + 4}$ **c** $\dfrac{3x^2 - 9x}{ax - 3a}$

5 Write as a single fraction:

     a $\dfrac{x}{5} + \dfrac{2x - 1}{3}$ **b** $\dfrac{x}{6} - \dfrac{1 + 2x}{2}$ **c** $\dfrac{3}{2x} - \dfrac{1}{x + 2}$

6 Simplify by factorisation:

     a $\dfrac{2x^2 - 8}{x + 2}$ **b** $\dfrac{x^2 - 5x - 14}{x^2 - 4}$ **c** $\dfrac{3x^2 - 5x - 2}{4x^2 - 7x - 2}$

7   **a**   Write $\dfrac{24}{x^2 - 4} - \dfrac{3x}{x + 2}$ as a single fraction.

     b   Hence, find the values of x when this expression is: **i**   undefined   **ii**   zero.

8 Simplify: **a** $\dfrac{\frac{3x + 7}{x - 1} - 13}{x - 2}$ **b** $\dfrac{2x^2 + 5x + 2}{x^2 - 4} \div \dfrac{2x^2 + x}{x^2 + x - 6}.$

Continuous data

17

Contents:

Opening problem

Andriano collected data for the rainfall from the last month for 90 towns in Argentina. The results are displayed in the frequency table alongside:

Rainfall (r mm)	Frequency
$50 \leqslant r < 60$	7
$60 \leqslant r < 70$	20
$70 \leqslant r < 80$	32
$80 \leqslant r < 90$	22
$90 \leqslant r < 100$	9
Total	90

Things to think about:

- Is the data discrete or continuous?
- What does the interval $60 \leqslant r < 70$ actually mean?
- How can the shape of the distribution be described?
- Is it possible to calculate the exact mean of the data?

In **Chapter 13** we saw that a **continuous numerical variable** can theoretically take any value on part of the number line. A continuous variable often has to be **measured** so that data can be recorded.

Examples of continuous numerical variables are:

The height of year 10 students: the variable can take any value from about 100 cm to 200 cm.

The speed of cars on a stretch of highway: the variable can take any value from 0 km/h to the fastest speed that a car can travel, but is most likely to be in the range 50 km/h to 150 km/h.

A   *THE MEAN OF CONTINUOUS DATA*   [11.5]

Continuous data is placed into **class intervals** which are usually represented by **inequalities**.

For example,   for heights in the 140s of centimetres we could write $140 \leqslant h < 150$.

To find the mean of continuous data, we use the same method as for grouped discrete data described in **Chapter 13** section **F**.

Since the data is given in intervals, our answer will only be an *estimate* of the mean.

Example 1   ◀️ *Self Tutor*

The heights of students (h cm) in a hockey training squad were measured and the results tabled:

Height (h cm)	Frequency (f)
$130 \leqslant h < 140$	2
$140 \leqslant h < 150$	4
$150 \leqslant h < 160$	12
$160 \leqslant h < 170$	20
$170 \leqslant h < 180$	9
$180 \leqslant h < 190$	3

a   Estimate the mean height.

b   State the modal class.

Height (h cm)	Mid-value (x)	Frequency (f)	fx
$130 \leqslant h < 140$	135	2	270
$140 \leqslant h < 150$	145	4	580
$150 \leqslant h < 160$	155	12	1860
$160 \leqslant h < 170$	165	20	3300
$170 \leqslant h < 180$	175	9	1575
$180 \leqslant h < 190$	185	3	555
Total		50	8140

a   Mean $= \dfrac{\sum fx}{\sum f}$

$\approx \dfrac{8140}{50}$

≈ 163 cm

b   The modal class is $160 \leqslant h < 170$.

EXERCISE 17A

1   A frequency table for the weights of a volleyball squad is given alongside.

Weight (kg)	Frequency
$75 \leqslant w < 80$	2
$80 \leqslant w < 85$	5
$85 \leqslant w < 90$	8
$90 \leqslant w < 95$	7
$95 \leqslant w < 100$	5
$100 \leqslant w < 105$	1

   a   Explain why 'weight' is a continuous variable.

   b   What is the modal class? Explain what this means.

   c   Describe the distribution of the data.

   d   Estimate the mean weight of the squad members.

2 A plant inspector takes a random sample of ten week old plants from a nursery and measures their height in millimetres.
The results are shown in the table alongside.

Height (mm)	Frequency
$20 \leqslant h < 40$	4
$40 \leqslant h < 60$	17
$60 \leqslant h < 80$	15
$80 \leqslant h < 100$	8
$100 \leqslant h < 120$	2
$120 \leqslant h < 140$	4

   a What is the modal class?
   b Estimate the mean height.
   c How many of the seedlings are 40 mm or more?
   d What percentage of the seedlings are between 60 and 80 mm?
   e If the total number of seedlings in the nursery is 857, estimate the number which measure:
       i less than 100 mm   **ii** between 40 and 100 mm.

3 The distances travelled to school by a random sample of students were:

Distance (km)	$0 \leqslant d < 1$	$1 \leqslant d < 2$	$2 \leqslant d < 3$	$3 \leqslant d < 4$	$4 \leqslant d < 5$
Number of students	76	87	54	23	5

   a What is the modal class?
   b Estimate the mean distance travelled by the students.
   c What percentage of the students travelled at least 2 km to school?
   d If there are 28 students in Josef's class, estimate the number who travelled less than 1 km to school.

4 The times taken in minutes for players to finish a computer game were:

Time (t)	$5 \leqslant t < 10$	$10 \leqslant t < 15$	$15 \leqslant t < 20$	$20 \leqslant t < 25$	$25 \leqslant t < 30$	$30 \leqslant t < 35$
Frequency	2	8	16	20	24	10

   a What percentage of the players finished the game in less than 20 minutes?
   b Estimate the mean time to finish the game.
   c If 2589 other people play the game, estimate the number who will complete it in less than 25 minutes.

B   HISTOGRAMS   [11.6]

When data is recorded for a continuous variable there are likely to be many different values. This data is therefore organised using **class intervals**. A special type of graph called a **histogram** is used to display the data.

A histogram is similar to a bar chart but, to account for the continuous nature of the variable, the bars are joined together.

Column Graph   **Histogram**

discrete data   continuous data
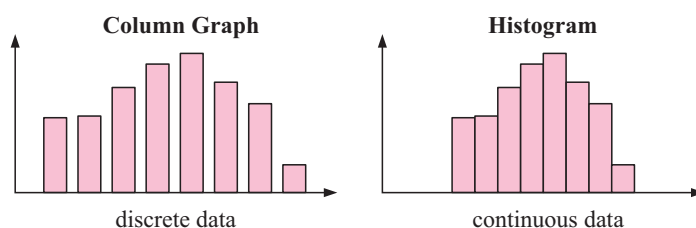
Consider the continuous data opposite which summarises the heights of girls in year 9. The data is continuous, so we can graph it using a histogram.

In this case the width of each class interval is the same, so we can construct a **frequency histogram**. The height of each column is the frequency of the class, and the **modal class** is simply the class with the highest column.
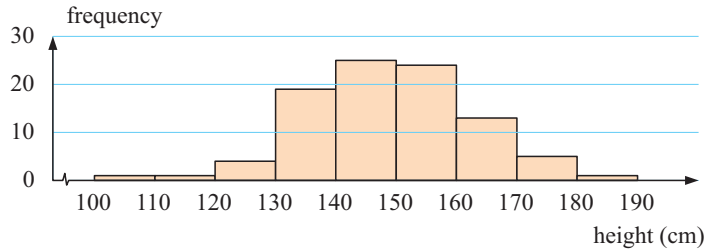
Height (cm)	Frequency (f)
$100 \leqslant h < 110$	1
$110 \leqslant h < 120$	1
$120 \leqslant h < 130$	4
$130 \leqslant h < 140$	19
$140 \leqslant h < 150$	25
$150 \leqslant h < 160$	24
$160 \leqslant h < 170$	13
$170 \leqslant h < 180$	5
$180 \leqslant h < 190$	1

In some situations we may have class intervals with different widths. There are a couple of reasons why this may happen:

- We may wish to collect small numbers of data at the extremities of our data range.

  For example, to make the table of girls' heights easier to display, we may combine the smallest three classes and also the tallest two classes:

Height (cm)	Frequency (f)
$100 \leqslant h < 130$	6
$130 \leqslant h < 140$	19
$140 \leqslant h < 150$	25
$150 \leqslant h < 160$	23
$160 \leqslant h < 170$	13
$170 \leqslant h < 190$	6

- The data may be naturally grouped in the context of the problem.

  For example, a post office will charge different rates depending on the weight of the parcel being sent. The weight intervals will probably not be equally spaced, but it makes sense for the post office to record the number of parcels sent in each class.

  So, the post office may collect the following data of parcels sent over a week:

Mass (m kg)	$0 \leqslant m < 1$	$1 \leqslant m < 2$	$2 \leqslant m < 5$	$5 \leqslant m < 20$
Number of parcels	18	22	24	11

In either case, the histogram we draw is *not* a frequency histogram. The frequency of each class is not represented by the *height* of its bar, but rather by its *area*. The height of each bar is called the **frequency density** of the class.

Since   frequency = frequency density $\times$ class interval width,

$$\textbf{frequency density} = \frac{\textbf{frequency}}{\textbf{class interval width}}$$

The **modal class** is the class with the highest frequency density, and so it is the highest bar on the histogram. It is not necessarily the class with the highest frequency.

Example 2   ◀》 Self Tutor

The table below shows masses of parcels received by a company during one week.

Mass (kg)	$0 \leqslant m < 1$	$1 \leqslant m < 2$	$2 \leqslant m < 3$	$3 \leqslant m < 6$	$6 \leqslant m < 10$
Number of parcels	20	30	15	60	40

a Draw a histogram to represent the data.

b Find the modal class interval.

c Use a graphics calculator to estimate the mean mass of the parcels received.

CIW means **class interval width**.

Mass (m kg)	Frequency	CIW	Frequency density
$0 \leqslant m < 1$	20	1	20
$1 \leqslant m < 2$	30	1	30
$2 \leqslant m < 3$	15	1	15
$3 \leqslant m < 6$	60	3	20
$6 \leqslant m < 10$	40	4	10

a Histogram showing masses of parcels   b The modal class is $1 \leqslant m < 2$.
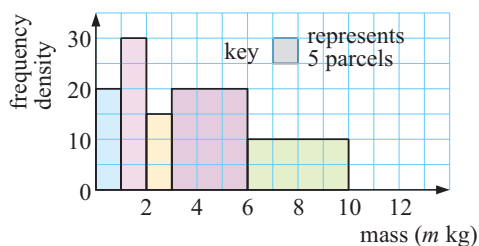
c

Middle value (x)	Frequency (f)
0.5	20
1.5	30
2.5	15
4.5	60
8	40

mean ≈ 4.14 kg (calculator)

Example 3   ◀》 Self Tutor

The times taken for students to complete a cross-country run were measured. The results were:

Time (t min)	$20 \leqslant t < 23$	$23 \leqslant t < 26$	$26 \leqslant t < 31$	$31 \leqslant t < 41$
Frequency	27	51	100	75

a Draw a histogram to represent the data.

b Find the modal class.

c Estimate the mean time for students to run the event.

Time (t min)	Frequency	CIW	Frequency density
$20 \leqslant t < 23$	27	3	9
$23 \leqslant t < 26$	51	3	17
$26 \leqslant t < 31$	100	5	20
$31 \leqslant t < 41$	75	10	7.5

a Histogram showing cross-country times

key ▢ represents 10 students
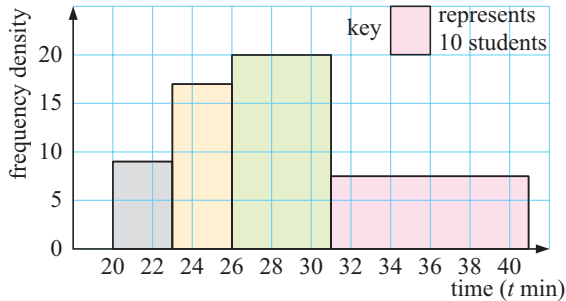
b The modal class is $26 \leqslant t < 31$.

c

Middle value (x)	Frequency (f)
21.5	27
24.5	51
28.5	100
36	75

The mean ≈ 29.2 mins.

EXERCISE 17B

1 Students were asked to draw a sketch of their favourite fruit. The time taken (t min) was measured for each student, and the results summarised in the table below.

Time (t min)	$0 \leqslant t < 2$	$2 \leqslant t < 4$	$4 \leqslant t < 8$	$8 \leqslant t < 12$
Frequency	10	15	30	60

   a Construct a histogram to illustrate the data.

   b State the modal class.

   c Use a graphics calculator to estimate the mean time.

2 When the masses of people in a Singapore fitness club were measured, the results were:

Mass (m kg)	$40 \leqslant m < 50$	$50 \leqslant m < 60$	$60 \leqslant m < 65$	$65 \leqslant m < 70$	$70 \leqslant m < 80$
Frequency	25	75	60	70	30

   a Represent this data on a histogram.

   b Find the modal class.

   c Use technology to estimate the mean mass.

3 A group of students was asked to throw a baseball as far as they could in a given direction. The results were recorded and tabled. They were:

Distance (d m)	$20 \leqslant d < 25$	$25 \leqslant d < 35$	$35 \leqslant d < 45$	$45 \leqslant d < 55$	$55 \leqslant d < 85$
Frequency	15	30	35	25	15

   a Draw a histogram of the data.

   b What is the modal class?

   c Use a graphics calculator to estimate the mean distance thrown.

Travel times to work

4 The histogram shows the times spent travelling to work by a sample of employees of a large corporation.

Given that 60 people took between 10 and 20 minutes to get to work, find the sample size used.
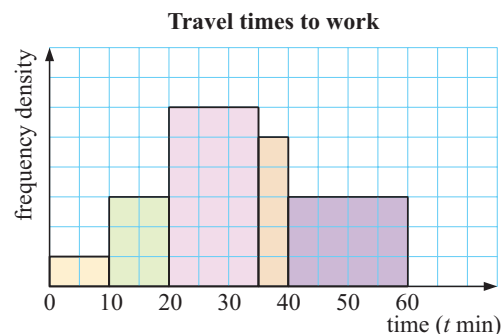
C   CUMULATIVE FREQUENCY   [11.7]

Sometimes it is useful to know the number of scores that lie above or below a particular value. In such situations it is convenient to construct a **cumulative frequency distribution table** and a **cumulative frequency graph** to represent the data.

> The cumulative frequency gives a *running total* of the scores up to a particular value.
> It is the total frequency up to a particular value.

From a frequency table we can construct a cumulative frequency column and then graph this data on a **cumulative frequency curve**. The cumulative frequencies are plotted on the vertical axis.

From the cumulative frequency graph we can find:

- the median Q_2
- the quartiles Q_1 and Q_3 } These divide the ordered data into quarters.
- percentiles

The **median** Q_2 splits the data into two halves, so it is 50% of the way through the data.
The **first quartile** Q_1 is the score value 25% of the way through the data.
The **third quartile** Q_3 is the score value 75% of the way through the data.
The n**th percentile** P_n is the score value $n\%$ of the way through the data.

So, $P_{25} = Q_1$, $P_{50} = Q_2$ and $P_{75} = Q_3$.

Example 4   ◆)) Self Tutor

The data shown gives the weights of 80 male basketball players.

a Construct a cumulative frequency distribution table.

b Represent the data on a cumulative frequency graph.

c Use your graph to estimate the:

   i   median weight

   ii number of men weighing less than 83 kg

  iii number of men weighing more than 92 kg

  iv 85th percentile.

Weight (w kg)	Frequency
$65 \leqslant w < 70$	1
$70 \leqslant w < 75$	2
$75 \leqslant w < 80$	8
$80 \leqslant w < 85$	16
$85 \leqslant w < 90$	21
$90 \leqslant w < 95$	19
$95 \leqslant w < 100$	8
$100 \leqslant w < 105$	3
$105 \leqslant w < 110$	1
$110 \leqslant w < 115$	1

a

Weight (w kg)	frequency	cumulative frequency
$65 \leqslant w < 70$	1	1
$70 \leqslant w < 75$	2	3
$75 \leqslant w < 80$	8	11
$80 \leqslant w < 85$	16	27
$85 \leqslant w < 90$	21	48
$90 \leqslant w < 95$	19	67
$95 \leqslant w < 100$	8	75
$100 \leqslant w < 105$	3	78
$105 \leqslant w < 110$	1	79
$110 \leqslant w < 115$	1	80

this is $1 + 2$
this is $1 + 2 + 8$

this 48 means that there are 48 players who weigh less than 90 kg, so (90, 48) is a point on the cumulative frequency graph.

b　　　**Cumulative frequency graph of basketballers' weights**

median is ≈ 88 kg
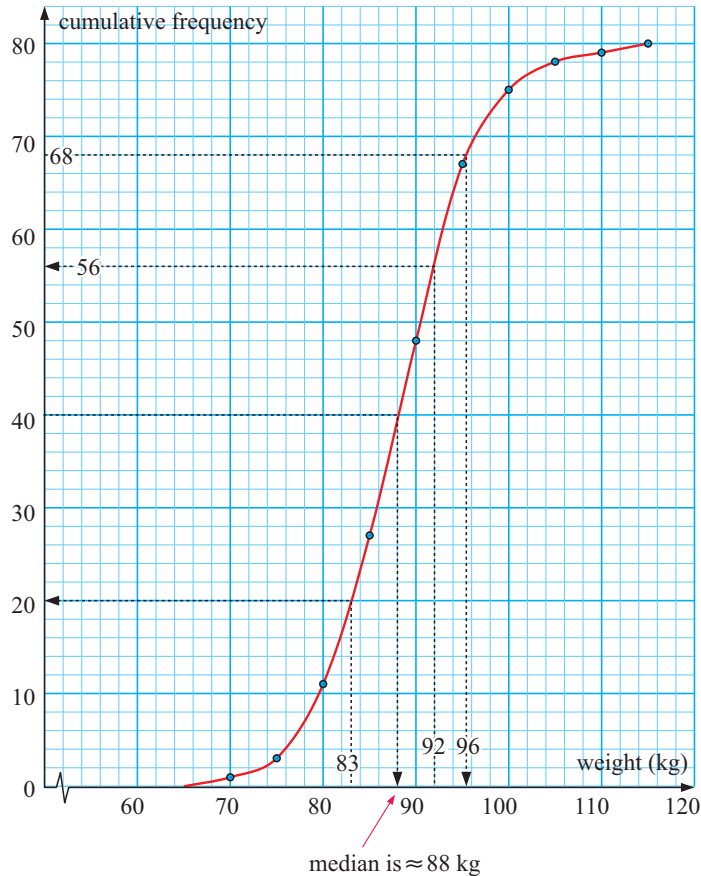
c　**i**　50% of 80 = 40,
　　　∴ median ≈ 88 kg

　　ii　There are 20 men who
　　　weigh less than 83 kg.

　iii　There are 80 − 56 = 24
　　　men who weigh more
　　　than 92 kg.

　iv　85% of 80 = 68, so the
　　　85th percentile ≈ 96 kg.

Cumulative frequency graphs are very useful for **comparing** two distributions of **unequal sizes**. In such cases we use percentiles on the vertical axis. This effectively scales each graph so that they both range from 0 to 100 on the vertical axis.

Example 5　　　　　　　　　　　　　　　　　　◀) *Self Tutor*

The heights of 100 14-year-old girls and 200 14-year-old boys were measured and the results tabled.

a　Draw on the same axes the cumulative frequency curve for both the boys and the girls.

b　Estimate for both the boys and the girls:

　i　the median

　ii　the interquartile range (IQR).

c　Compare the two distributions.

Frequency (girls)	Height (h cm)	Frequency (boys)
5	$140 \leqslant h < 145$	4
10	$145 \leqslant h < 150$	10
15	$150 \leqslant h < 155$	20
30	$155 \leqslant h < 160$	26
20	$160 \leqslant h < 165$	40
10	$165 \leqslant h < 170$	60
8	$170 \leqslant h < 175$	30
2	$175 \leqslant h < 180$	10

a

CF (girls)	Freq. (girls)	Height (h cm)	Freq. (boys)	CF (boys)
5	5	$140 \leqslant h < 145$	4	4
15	10	$145 \leqslant h < 150$	10	14
30	15	$150 \leqslant h < 155$	20	34
60	30	$155 \leqslant h < 160$	26	60
80	20	$160 \leqslant h < 165$	40	100
90	10	$165 \leqslant h < 170$	60	160
98	8	$170 \leqslant h < 175$	30	190
100	2	$175 \leqslant h < 180$	10	200

$\frac{14}{200} = 7\%$,

so $(150, 7)$ is a point on the boys' cumulative frequency graph.
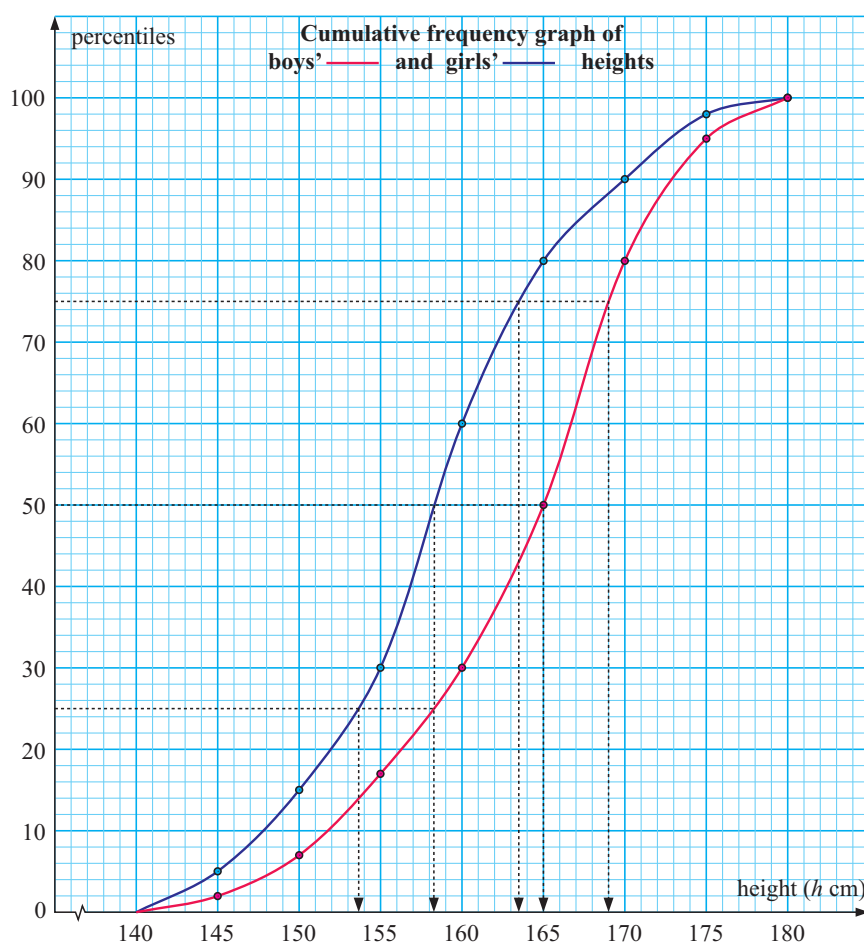
b For girls: **i** median ≈ 158 cm   **ii** IQR $\approx 163.5 - 153.7 \approx 10$ cm
   For boys: **i** median ≈ 165 cm   **ii** IQR $\approx 169 - 158 \approx 11$ cm

c The two distributions are similar in shape, but the boys' heights are further right than the girls'. The median height for the boys is 7 cm more than for the girls. They are considerably taller. As the IQRs are nearly the same, the spread of heights is similar for each gender.

EXERCISE 17C

1 In a running race, the times (in minutes) of 160 competitors were recorded as follows:

Draw a cumulative frequency graph of the data and use it to estimate:

   a the median time

   b the approximate number of runners whose time was not more than 32 minutes

   c the approximate time in which the fastest 40 runners completed the course

   d the interquartile range.

Times (min)	Frequency
$20 \leqslant t < 25$	18
$25 \leqslant t < 30$	45
$30 \leqslant t < 35$	37
$35 \leqslant t < 40$	33
$40 \leqslant t < 45$	19
$45 \leqslant t < 50$	8

2
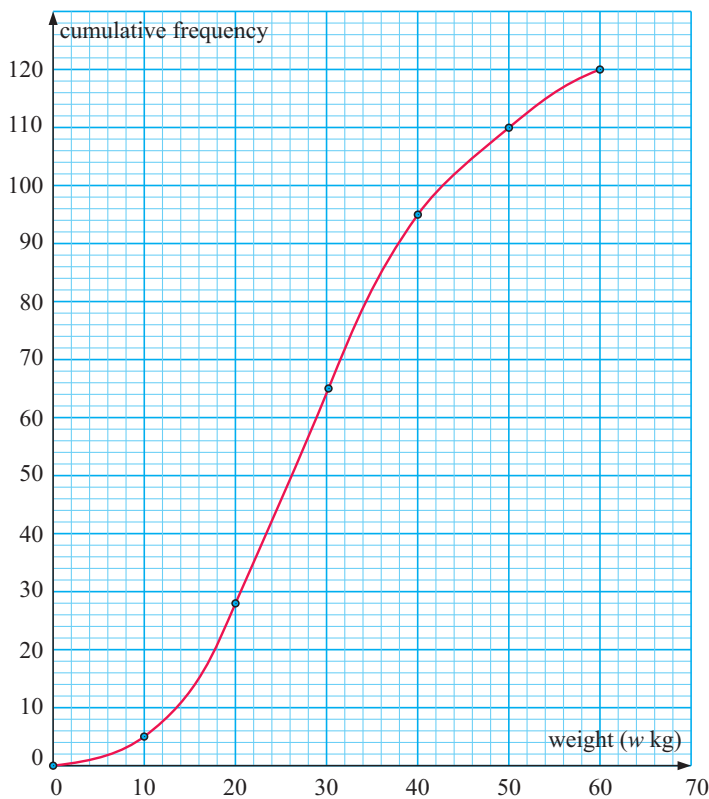
The cumulative frequency curve shows the weights of Sam's goat herd in kilograms.

   a How many goats does Sam have?

   b Estimate the median goat weight.

   c Any goats heavier than the 60th percentile will go to market. How many goats will go to market?

   d What is the IQR for Sam's herd?

3 The lengths of 30 trout (l cm) were measured. The following data was obtained:

Length (cm)	$30 \leqslant l < 32$	$32 \leqslant l < 34$	$34 \leqslant l < 36$	$36 \leqslant l < 38$	$38 \leqslant l < 40$	$40 \leqslant l < 42$	$42 \leqslant l < 44$
Frequency	1	1	3	7	11	5	2

   a Construct a cumulative frequency curve for the data.

   b Estimate the percentage of trout with length less than 39 cm.

   c Estimate the median length of trout caught.

   d Estimate the interquartile range of trout length and explain what this represents.

   e Estimate the 35th percentile and explain what this represents.

   f Use a calculator to estimate the mean of the data.

   g Comment on the shape of the distribution of trout lengths.

4 The weights of cabbages grown by two brothers on separate properties were measured for comparison. The results are shown in the table:

  a Draw, on the same axes, cumulative frequency curves for both cabbage samples.

  b Estimate for each brother:
    i the median weight
    ii the IQR

  c Compare the 60th percentile weights.

  d Compare the two distributions.

Frequency (Alan)	Weight (w grams)	Frequency (John)
4	$400 \leqslant w < 550$	5
32	$550 \leqslant w < 700$	60
44	$700 \leqslant w < 850$	70
52	$850 \leqslant w < 1000$	60
44	$1000 \leqslant w < 1150$	35
24	$1150 \leqslant w < 1300$	20
200	totals	250

5 The times taken for trampers to climb Ben Nevis were recorded and the results tabled.

Time (t min)	$175 \leqslant t < 190$	$190 \leqslant t < 205$	$205 \leqslant t < 220$	$220 \leqslant t < 235$	$235 \leqslant t < 250$
Frequency	11	35	74	32	8

  a Construct a cumulative frequency curve for the walking times.

  b Estimate the median time for the walk.

  c Estimate the IQR and explain what it means.

  d Guides on the walk say that anyone who completes the walk in 3 hours 15 min or less is extremely fit. Estimate the number of extremely fit trampers.

  e Comment on the shape of the distribution of walking times.

6
Cumulative frequency curve of watermelon weight data
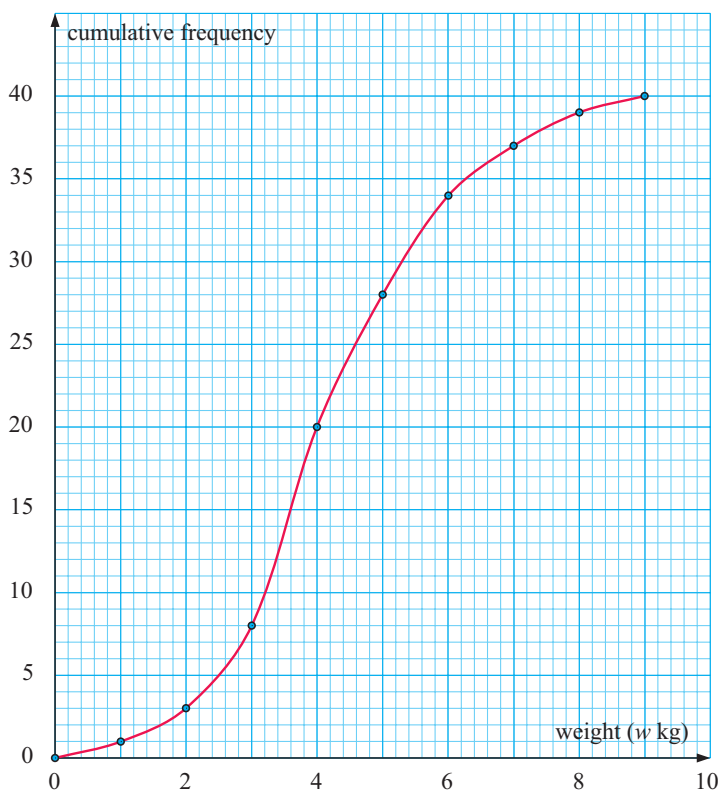
The given graph describes the weight of 40 watermelons.

  a Estimate the:
    i median weight
    ii IQR

  for the weight of the watermelons.

  b Construct a cumulative frequency table for the data including a frequency column.

  c Estimate the mean weight of the watermelons.

Review set 17A

1 A frequency table for the masses of eggs (m grams) in a carton marked '50 g eggs' is given below.

Mass (g)	Frequency
$48 \leqslant m < 49$	1
$49 \leqslant m < 50$	1
$50 \leqslant m < 51$	16
$51 \leqslant m < 52$	4
$52 \leqslant m < 53$	3

    a Explain why 'mass' is a continuous variable.

    b What is the modal class? Explain what this means.

    c Estimate the mean of the data.

    d Describe the distribution of the data.

2 The speeds of vehicles (v km/h) travelling along a stretch of road are recorded over a 60 minute period. The results are given in the table alongside.

Speed (v km/h)	Frequency
$40 \leqslant v < 45$	14
$45 \leqslant v < 50$	22
$50 \leqslant v < 55$	35
$55 \leqslant v < 60$	38
$60 \leqslant v < 65$	25
$65 \leqslant v < 70$	10

    a Estimate the mean speed of the vehicles.

    b Find the modal class.

    c What percentage of drivers exceeded the speed limit of 60 km/h?

    d Describe the distribution of the data.

3 A selection of measuring bottles were examined and their capacities were noted. The results are given in the table below:

Capacity (C litres)	$0 \leqslant C < 0.5$	$0.5 \leqslant C < 1$	$1 \leqslant C < 2$	$2 \leqslant C < 3$	$3 \leqslant C < 5$
Frequency	13	18	24	18	16

    a Draw a histogram to illustrate this information.   **b** What is the modal class?

4 The heights of plants in a field were measured and the results recorded alongside:

Height (h cm)	Frequency
$0 \leqslant h < 10$	11
$10 \leqslant h < 20$	14
$20 \leqslant h < 30$	20
$30 \leqslant h < 40$	15
$40 \leqslant h < 60$	18
$60 \leqslant h < 100$	10

    a Represent this data on a histogram.

    b Find the modal class.

    c Estimate the mean height of the plants.

5 The weekly wages of employees in a factory are recorded in the table below.

Weekly wage (£w)	$0 \leqslant w < 400$	$400 \leqslant w < 800$	$800 \leqslant w < 1200$	$1200 \leqslant w < 1600$	$1600 \leqslant w < 2000$
Frequency	20	60	120	40	10

    a Draw a cumulative frequency graph to illustrate this information.

    b Use the graph to estimate:

        i the median wage

        ii the wage that is exceeded by 20% of the employees.

6 The cumulative frequency curve shows the time spent by people in a supermarket on a given day.

  a Construct a cumulative frequency table for the data, using the intervals $0 \leqslant t < 5$, $5 \leqslant t < 10$, and so on.

  b Use the graph to estimate:

  i the median time

  ii the IQR

  iii the 80th percentile.

  c Copy and complete:

  i 60% of the people spent less than ...... minutes in the supermarket

  ii 80% of the people spent at least ...... minutes in the supermarket.
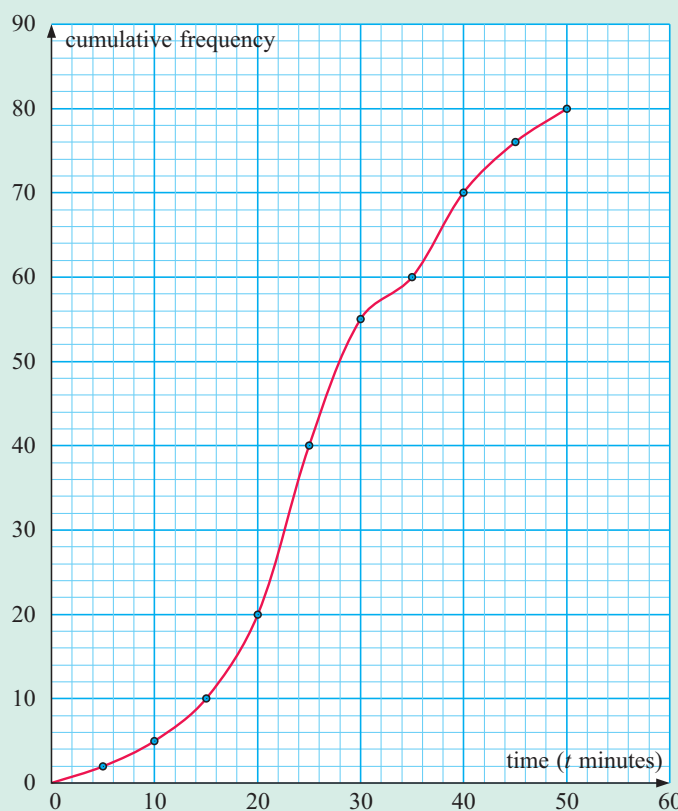
Review set 17B

1 The table alongside summarises the masses of 50 domestic cats chosen at random.

  a What is the length of each class interval?

  b What is the modal class?

  c Find the approximate mean.

  d Draw a frequency histogram of the data.

  e From a random selection of 428 cats, how many would you expect to weigh at least 8 kg?

Mass (m kg)	Frequency
$0 \leqslant m < 2$	5
$2 \leqslant m < 4$	18
$4 \leqslant m < 6$	12
$6 \leqslant m < 8$	9
$8 \leqslant m < 10$	5
$10 \leqslant m < 12$	1

2 The table alongside summarises the best times of 100 swimmers who swim 50 m.

  a Estimate the mean time.

  b What is the modal class?

Time (t sec)	Frequency
$25 \leqslant t < 30$	5
$30 \leqslant t < 35$	17
$35 \leqslant t < 40$	34
$40 \leqslant t < 45$	29
$45 \leqslant t < 50$	15

3 The table below displays the distances jumped by 50 year 10 students in a long jump competition:

Distance (d m)	$3 \leqslant d < 4$	$4 \leqslant d < 5$	$5 \leqslant d < 5.5$	$5.5 \leqslant d < 6$	$6 \leqslant d < 7$
Frequency	8	16	12	9	5

   a Display this information on a histogram.

   b What is the modal class?

4 The histogram alongside shows the areas of land blocks on a street. If 20 land blocks were between 300 m^2 to 500 m^2 in size:
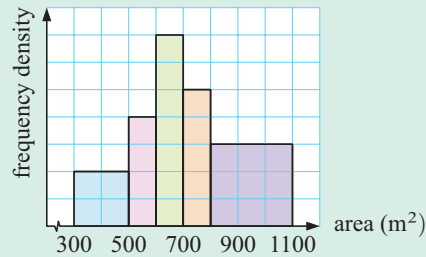
   a construct a frequency table for the data

   b estimate the mean area.

5 The percentage scores in a test were recorded. The results were categorised by gender.

   a Draw the cumulative frequency graphs for boys and girls on the same set of axes. Use percentiles on the vertical axis.

   b Estimate the median and interquartile range of each data set.

   c Compare the distributions.

Frequency (boys)	Percentage score (s)	Frequency (girls)
5	$0 \leqslant s < 10$	0
8	$10 \leqslant s < 20$	4
12	$20 \leqslant s < 30$	8
10	$30 \leqslant s < 40$	10
30	$40 \leqslant s < 50$	15
50	$50 \leqslant s < 60$	25
20	$60 \leqslant s < 70$	40
10	$70 \leqslant s < 80$	10
5	$80 \leqslant s < 90$	5
0	$90 \leqslant s < 100$	3

6 In a one month period at a particular hospital the lengths of newborn babies were recorded. The results are shown in the table given.

   a Represent the data on a frequency histogram.

   b How many babies are 52 cm or more?

   c What percentage of babies have lengths in the interval 50 cm $\leqslant l < 53$ cm?

   d Construct a cumulative frequency distribution table.

   e Represent the data on a cumulative frequency graph.

   f Use your graph to estimate the:

     i median length

     ii number of babies with length less than 51.5 cm.

Length (l cm)	Frequency
$48 \leqslant l < 49$	1
$49 \leqslant l < 50$	3
$50 \leqslant l < 51$	9
$51 \leqslant l < 52$	10
$52 \leqslant l < 53$	16
$53 \leqslant l < 54$	4
$54 \leqslant l < 55$	5
$55 \leqslant l < 56$	2

Similarity

18

Contents:

Opening problem

Jacob makes cylindrical tanks of different sizes. His smallest tank has a height of 2 m and a radius of 0.8 m. If he wants to manufacture another tank in the same proportions but with a base radius of 1 m, find the:

a height of the new tank

b ratio of the   **i** surface areas   **ii** capacities.

height

radius
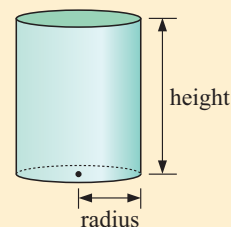
We have seen previously that two figures are **congruent** if they are identical in every respect apart from position. In this chapter we discuss **similarity**, which is a closely related topic. It deals with figures that differ in size but have the same proportions.

A   SIMILARITY   [4.5]

> Two figures are **similar** if one is an enlargement of the other, regardless of their orientation.

If two figures are similar then their corresponding sides are *in proportion*. This means that the lengths of sides will be increased (or decreased) by the same ratio from one figure to the next. This ratio is called the **scale factor** of the enlargement.

Discussion

- Discuss whether the following pairs of figures are similar:
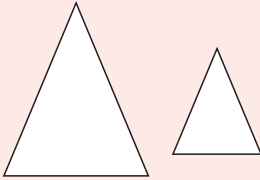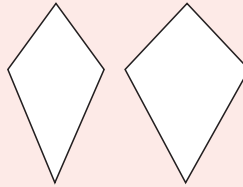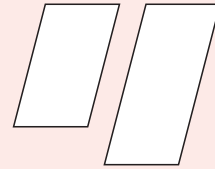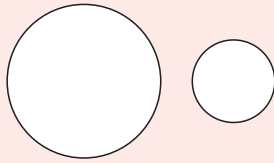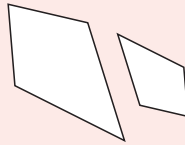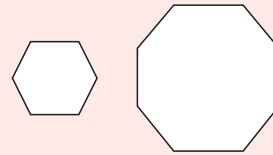
- Are congruent figures similar?

Consider the enlargement alongside for which the scale factor k is 1.5.
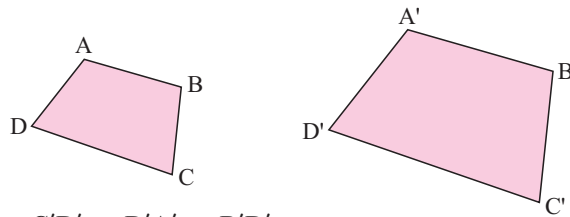
Since $k = 1.5$,   notice that $\dfrac{A'B'}{AB} = \dfrac{B'C'}{BC} = \dfrac{C'D'}{CD} = \dfrac{D'A'}{DA} = \dfrac{B'D'}{BD} = ... = 1.5$

Angle sizes do not change under enlargements.

> If two polygons are **similar** then:
> - the figures are equiangular   **and**   • the corresponding sides are *in proportion*.

Example 1   ◆ *Self Tutor*

These figures are similar.
Find x correct to 2 decimal places.
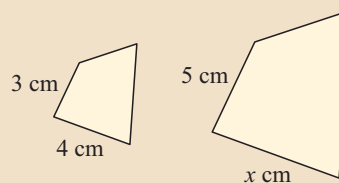
Since the figures are similar, their corresponding sides are in the same ratio.

$\therefore \quad \dfrac{x}{4} = \dfrac{5}{3}$

$\therefore \quad x = \dfrac{5}{3} \times 4$

$\therefore \quad x = \dfrac{20}{3}$

$\therefore \quad x \approx 6.67$

EXERCISE 18A

1 Solve for x:

    a $x : 6 = 2 : 15$ **b** $x : 5 = 7 : 31$ **c** $x : 8 = 9 : 102$

2 Find x given that the figures are similar:

    a

4 cm

3 cm

x cm

6 cm

    b

4 cm

5 cm

7 cm

x cm

    c

5 cm

3 cm

x cm

8 cm

    d

5 m

x m

7 m

11 m
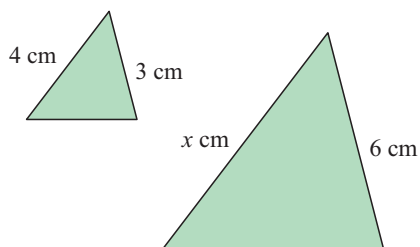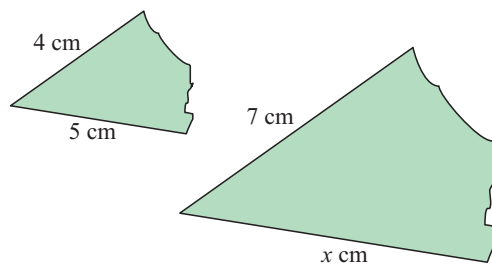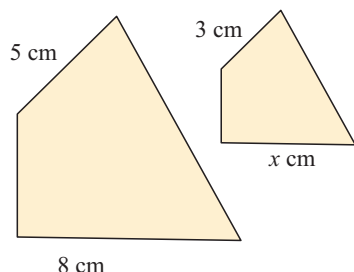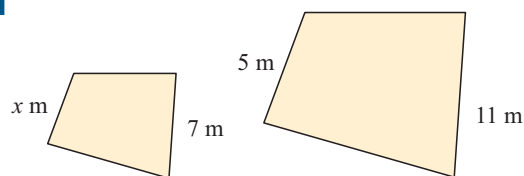
3   **a** If a line of length 2 cm is enlarged with scale factor 3, find its new length.

    b A 3 cm length has been enlarged to 4.5 cm. Find the scale factor k.

4 Comment on the truth of the following statements. For any statement which is false you should justify your answer with an illustration.

    a All circles are similar.   **b** All ellipses are similar.

    c All squares are similar.   **d** All rectangles are similar.

5

2 m

8 m

12 m
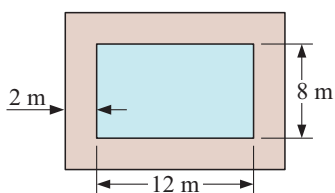
The diagram shows a 2 m wide path around a 12 m by 8 m swimming pool.

Are the two rectangles similar? Justify your answer.

6 Rectangles ABCD and FGHE are similar. Find the length of FG.

B   C

4 m

A   6 m   D

F   G

3.6 m

E   H
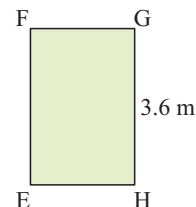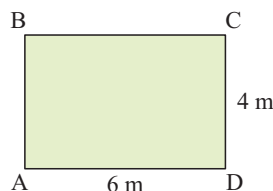
7 Sketch two polygons that:

    a are equiangular, but not similar

    b have sides in proportion, but are not similar.

8 Can you draw two triangles which are equiangular but not similar?

'In proportion' means 'in the same ratio'.

B   SIMILAR TRIANGLES   **[4.5]**

In the previous exercise, you should have found that:

> If two triangles are equiangular then they are **similar**.
>
> Similar triangles have corresponding sides in the same ratio.

If two triangles are equiangular then one of them must be an enlargement of the other.

For example, ΔABC is *similar* to ΔPQR.
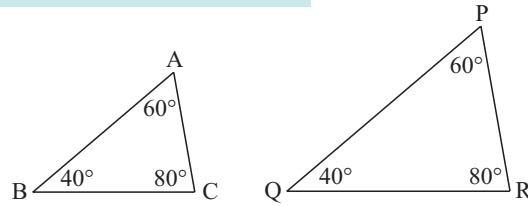
So, $\dfrac{QR}{BC} = \dfrac{RP}{CA} = \dfrac{PQ}{AB}$ where each fraction equals the scale factor of the enlargement.

> To establish that two triangles are **similar**, we need to show that they are **equiangular** or that their **sides are in proportion**.

You should note that:

- either of these properties is sufficient to prove that two triangles are similar.
- if two angles of one triangle are equal in size to two angles of the other triangle then the remaining angles of the triangles are equal.

Example 2   🔊 *Self Tutor*

Show that the following figures possess similar triangles:

a   **b**
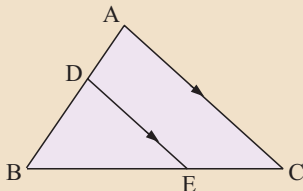
a Δs ABC and DBE are equiangular as:

- $\alpha_1 = \alpha_2$ {equal corresponding angles}
- angle B is common to both triangles

∴ the triangles are similar.
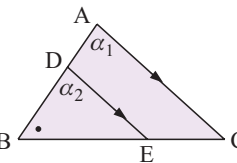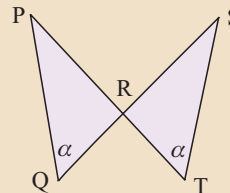
b Δs PQR and STR are equiangular as:

- $\alpha_1 = \alpha_2$ {given}
- $\beta_1 = \beta_2$ {vertically opposite angles}

∴ the triangles are similar.
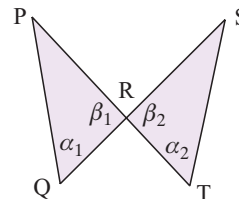
EXERCISE 18B.1

1 Show that the following figures possess similar triangles:

a
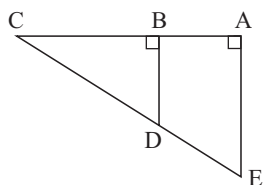
b
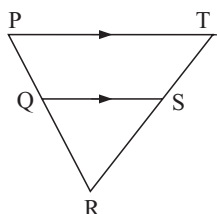
c
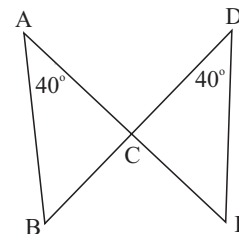
d
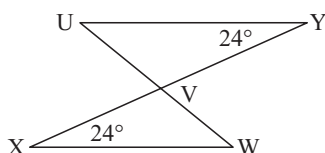
e
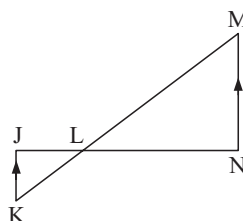
f
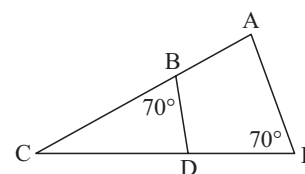
FINDING SIDE LENGTHS

Once we have established that two triangles are similar, we may use the fact that corresponding sides are in the same ratio to find unknown lengths.

Example 3
🔊 *Self Tutor*

Establish that a pair of triangles is similar and find x:
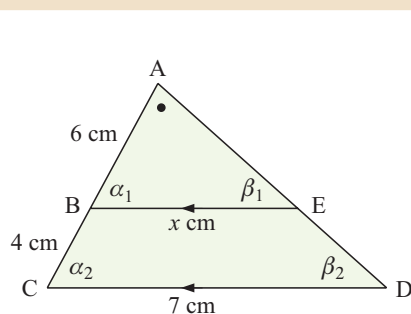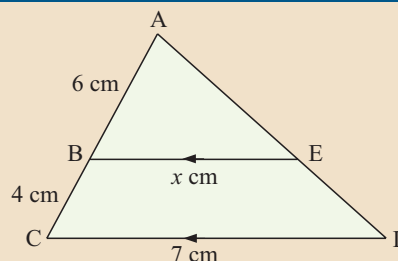
$\alpha_1 = \alpha_2$ {corresponding angles}

$\beta_1 = \beta_2$ {corresponding angles}

So, Δs ABE and ACD are similar and

$$\frac{BE}{CD} = \frac{AB}{AC} = \frac{AE}{AD} \quad \text{\{same ratio\}}$$

$\therefore \quad \dfrac{x}{7} = \dfrac{6}{10}$

$\therefore \quad x = \dfrac{6}{10} \times 7 = 4.2$

When solving similar triangle problems, it may be useful to use the following method, written in the context of the example above:

Step 1:	Label equal angles.
Step 2:	Put the information in table form, showing the equal angles and the sides opposite these angles.
Step 3:	Since the triangles are equiangular, they are similar.
Step 4:	Use the columns to write down the equation for the ratio of the corresponding sides.
Step 5:	Solve the equation.

α	β	$\bullet$	
-	6	x	small Δ
-	10	7	large Δ

from which $\dfrac{6}{10} = \dfrac{x}{7}$

$\therefore \quad x = 4.2$

Example 4 ◀)) *Self Tutor*

Establish that a pair of triangles is similar, then find x if BD $= 20$ cm:

α	β	$\bullet$	
-	$x+2$	x	small Δ
-	20	12	large Δ
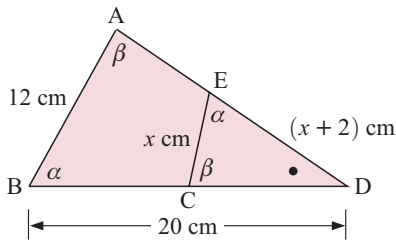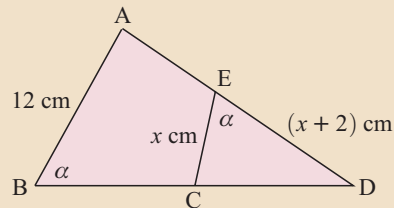
The triangles are equiangular and hence similar.

$\therefore \quad \dfrac{x+2}{20} = \dfrac{x}{12} \qquad$ {same ratio}

$\therefore \quad 12(x+2) = 20x$

$\therefore \quad 12x + 24 = 20x$

$\therefore \qquad 24 = 8x$

$\therefore \qquad x = 3$

EXERCISE 18B.2

1 In the following, establish that a pair of triangles is similar, then find x:

a

b

c

d

e

f

g

A, D, B, 4 cm, x cm, α, 3 cm, α, 5 cm, E, C

h

X, 5 cm, Y, Z, 2 cm, U, x cm, V

i

S, 10 cm, 8 cm, 50°, R, x cm, Q, P, 50°, T, 9 cm
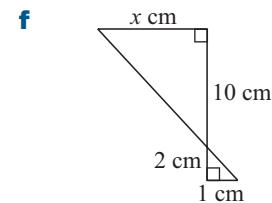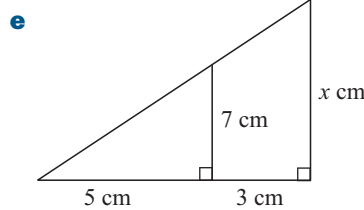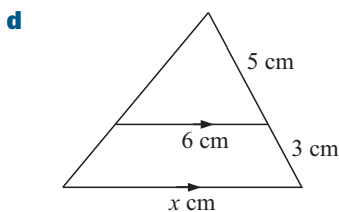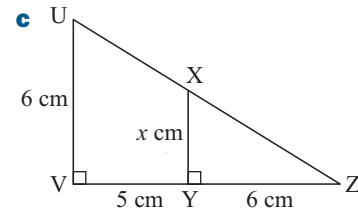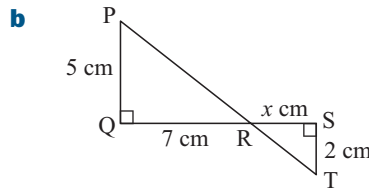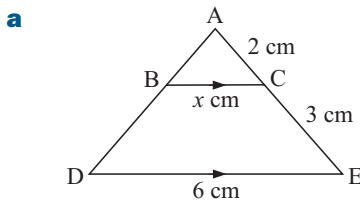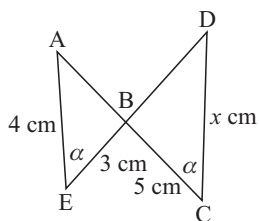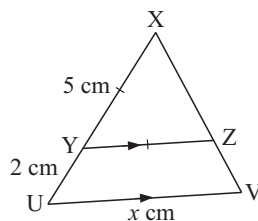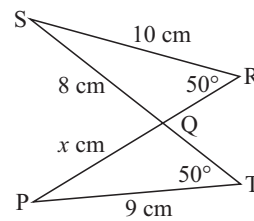
C   PROBLEM SOLVING   [4.5]

The properties of similar triangles have been known since ancient times. But even with the technologically advanced measuring instruments available today, similar triangles are important for finding heights and distances which would otherwise be difficult to measure.

Step 1:   Read the question carefully and draw a sketch showing all the given information.

Step 2:   Introduce a variable or unknown such as x, to represent the quantity to be found.

Step 3:   Set up an equation involving the unknown and then solve for the unknown.

Step 4:   Answer the question in a sentence.

Diagrams are very important. They often help you to solve the problem. Make sure your diagrams are large enough.

Example 5   ◀⁍ Self Tutor

Find the height of the pine tree:

x m

1 m stick

2 m shadow

10 m
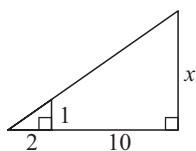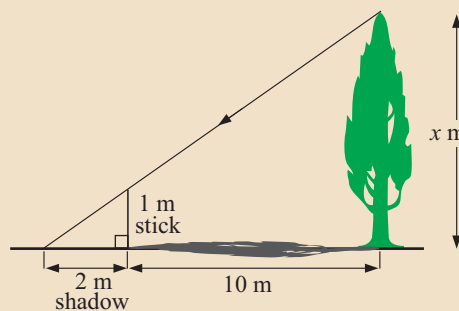
contains similar triangles

1, 2

and

x, 12
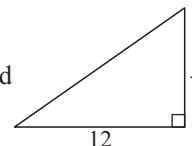
$$\therefore \quad \frac{x}{1} = \frac{12}{2}$$

$$\therefore \quad x = 6$$

So, the pine is 6 m high.

Example 6
◀) **Self Tutor**

When a 30 cm ruler is stood vertically on the ground it casts a 24 cm shadow.
At the same time a man casts a shadow of length 152 cm. How tall is the man?
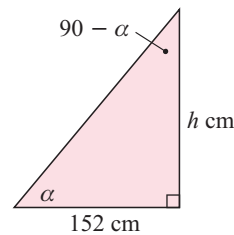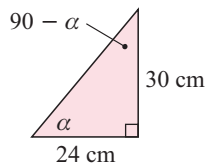
The triangles are equiangular and ∴ similar.

$$\therefore \quad \frac{h}{30} = \frac{152}{24} \quad \text{\{same ratio\}}$$

$$\therefore \quad h = \frac{152}{24} \times 30$$

$$\therefore \quad h = 190 \quad \text{i.e., the man is 190 cm tall.}$$

α	$90 - \alpha$	90	
30 cm	24 cm	-	small △
h cm	152 cm	-	large △

EXERCISE 18C

1 Find the height of the pine tree:

a

1 m stick
1.2 m shadow   7.2 m

b

1 m stick
2.4 m shadow   15.6 m
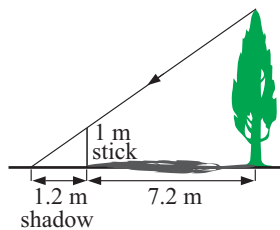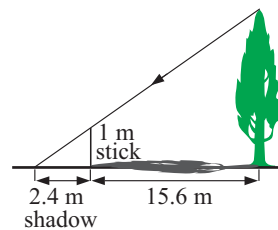
2 A ramp is built to enable wheelchair access to a building that is 24 cm above ground level. The ramp has a constant slope of 2 in 15, which means that for every 15 cm horizontally it rises 2 cm. Calculate the length of the base of the ramp.

building
24 cm
ground level
2 cm
15 cm
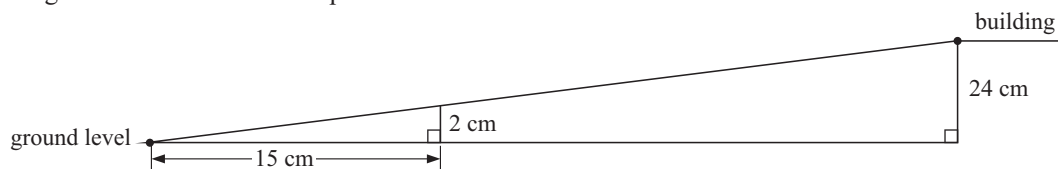
3 A piece of timber leaning against a wall, just touches the top of a fence, as shown. Find how far up the wall the timber reaches.

timber
wall
1.6 m fence
3 m   2 m

4

25 m

1.2 m

2 m
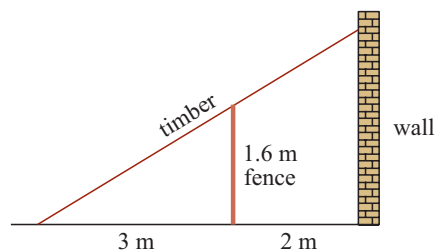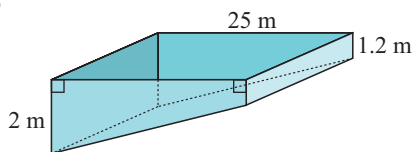
A swimming pool is 1.2 m deep at one end, and 2 m deep at the other end. The pool is 25 m long. Isaac jumps into the pool 10 metres from the shallow end. How deep is the pool at this point?

5   Ryan is standing on the edge of the shadow formed by a 5 m tall building. Ryan is 4 m from the building, and is 1.8 m tall. How far must Ryan walk towards the building so that he is completely shaded from the Sun?

5 m

6

T

2 m

6 m

10 m

building

K

3 m

view from above
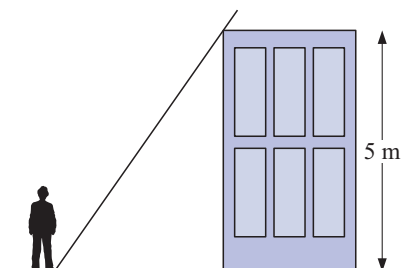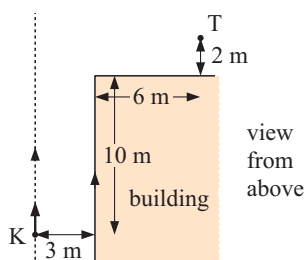
Kalev is currently at K, walking parallel to the side of a building at a speed of 1 m/s. A flag pole is located at T. How long will it be before Kalev will be able to see the flag pole?

7   A, B, C and D are pegs on the bank of a canal which has parallel straight sides. C and D are directly opposite each other. AB = 30 m and BC = 140 m.

When I walk from A directly away from the bank, I reach a point E, 25 m from A, where E, B and D line up. How wide is the canal?

E

B   C

A

D

8

A   400 m   B

C

70 m

E   D

30 m
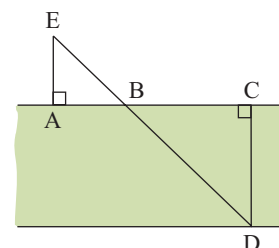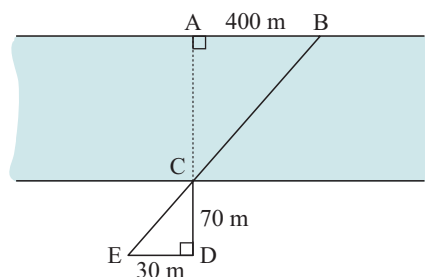
An engineer was asked to construct a bridge across a river. He noticed that if he started at C and walked 70 m away from the river to D and 30 m parallel to the river to E, then C and E formed a straight line with a statue at B.

Determine the length of the bridge to be built to span the river if it must extend 40 m from the river bank in both directions. Give your answer correct to the nearest metre.

9   The dimensions of a tennis court are given in the diagram alongside.

Samantha hits a shot from the base line corner at S. The ball passes over her service line at T such that UT = 1.92 m.

The ball then travels over the net and lands on the opposite baseline AD.
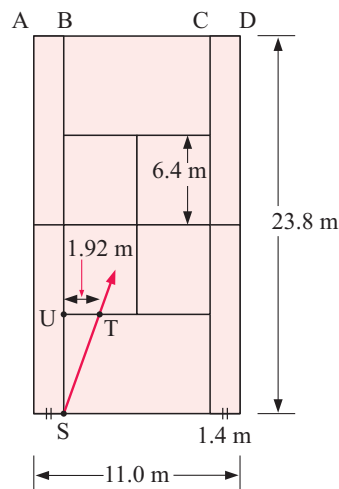
  a   Find the length of SU and BC.

  b   A ball landing on the baseline is "in" if it lands between points B and C.

Assuming the ball continues along the same trajectory, will it land "in"?

A   B   C   D

6.4 m

23.8 m

1.92 m

U

T

S

1.4 m

11.0 m

D | AREA AND VOLUME OF SIMILAR SHAPES [4.5]

AREAS
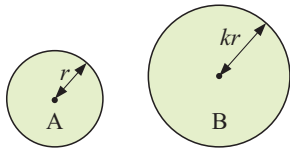
The two circles shown are similar. Circle B is an enlargement of circle A with scale factor k.

$$\begin{aligned}
\text{Area of B} &= \pi(kr)^2 \\
&= \pi \times k^2 r^2 \\
&= k^2(\pi r^2) \\
&= k^2 \times \text{area of A}
\end{aligned}$$
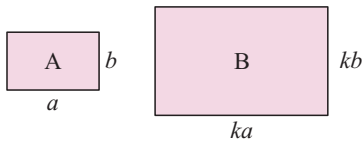
We can perform a similar comparison for these similar rectangles.

$$\begin{aligned}
\text{Area of B} &= ka \times kb \\
&= k^2 ab \\
&= k^2 \times \text{area of A}
\end{aligned}$$

Using examples like this we can conclude that:

> If a figure is enlarged with scale factor k to produce a similar figure then the new area $= k^2 \times$ the old area.

Example 7
◄) Self Tutor
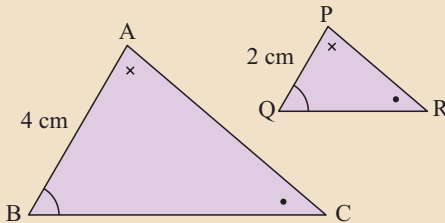
Triangles ABC and PQR are similar with AB = 4 cm and PQ = 2 cm.

The area of $\triangle$ABC is 20 cm^2.

What is the area of $\triangle$PQR?

Suppose we enlarge $\triangle$PQR to give $\triangle$ABC with scale factor k.

$$k = \frac{4}{2} = 2$$
$$\therefore \quad k^2 = 4$$

So, area $\triangle$ABC $= k^2 \times$ area $\triangle$PQR

$$\therefore \quad 20 \text{ cm}^2 = 4 \times \text{area } \triangle\text{PQR}$$
$$\therefore \quad 5 \text{ cm}^2 = \text{area of } \triangle\text{PQR}$$

Example 8
◄) Self Tutor
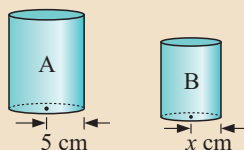
Cylinders A and B have surface areas of 1600 cm^2 and 900 cm^2 respectively.

Given that the cylinders are similar, find x.

Consider the reduction of cylinder A to give cylinder B.

$$\text{Surface area of B} = k^2 \times \text{surface area of A}$$
$$\therefore \quad 900 = k^2 \times 1600$$
$$\therefore \quad \tfrac{9}{16} = k^2$$
$$\therefore \quad k = \tfrac{3}{4} \quad \{k > 0\}$$

Now the radius of B $= k \times$ the radius of A
$$\therefore \quad x = \tfrac{3}{4} \times 5$$
$$\therefore \quad x = 3.75$$

VOLUMES
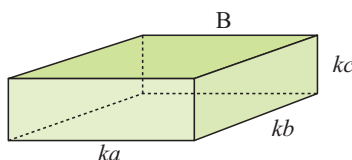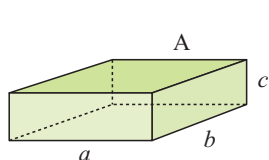
Volume of box B
$$= ka \times kb \times kc$$
$$= k^3 abc$$
$$= k^3 \times \text{volume of box A}$$

In general:

> If the lengths of a solid are enlarged with scale factor k to produce a similar figure then the new volume $= k^3 \times$ the old volume.

Example 9   ◀)) Self Tutor
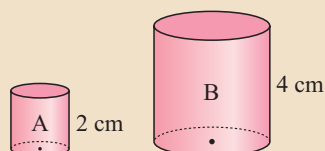
These two cylinders are similar with heights 2 cm and 4 cm respectively.

Cylinder A has volume 10 cm^3.

Find the volume of cylinder B.

Suppose we enlarge cylinder A to give cylinder B.

$$\therefore \quad k = \tfrac{4}{2} = 2$$
$$\therefore \quad \text{volume of B} = k^3 \times \text{volume of A}$$
$$= 8 \times 10 \text{ cm}^3$$
$$= 80 \text{ cm}^3$$

Example 10   ◀)) Self Tutor
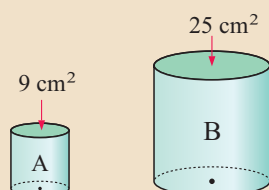
A and B are similar cylinders with areas of ends 9 cm^2 and 25 cm^2.

Find the ratio of their volumes.

Suppose we enlarge cylinder A to give cylinder B.

End area of B $= k^2 \times$ end area of A

$$\therefore \quad 25 = k^2 \times 9$$
$$\therefore \quad \tfrac{25}{9} = k^2$$
$$\therefore \quad k = \sqrt{\tfrac{25}{9}} = \tfrac{5}{3} \quad \{k > 0\}$$
$$\therefore \quad k^3 = \tfrac{125}{27}$$

$\therefore$ volume of A : volume of B $= 27 : 125$

EXERCISE 18D

1 For each of the following *similar* shapes, find the unknown length or area:

a

8 cm²   6 cm x cm²   9 cm

b

2 cm x cm²   4 cm   24 cm²

c

40 cm²   10 cm   10 cm² x cm

d

20 cm²   8 cm   6 cm² x cm
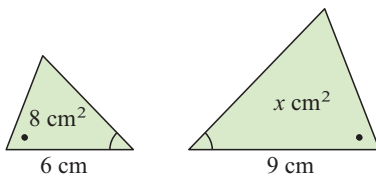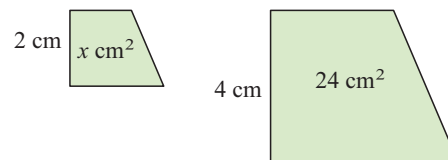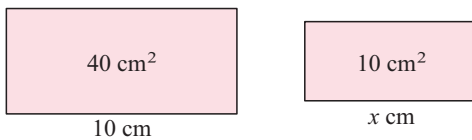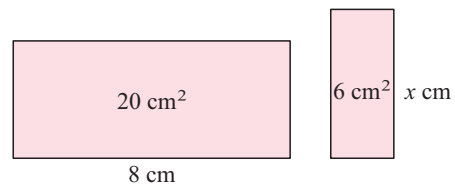
2 The side lengths of triangle A are 5 cm, 6 cm and 7 cm. The longest side of a similar triangle B is 28 cm. Find:

    a the scale factor when enlarging triangle A to give triangle B

    b the length of the other two sides of triangle B

    c the ratio of their areas.

3

C   D   A   E   B
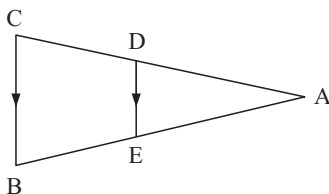
In the given figure, BE $= 6$ cm, AE $= 4$ cm and $\triangle$ADE has area 16 cm². Find:

    a the scale factor to enlarge $\triangle$ADE into $\triangle$ACB

    b the area of $\triangle$ABC

    c the area of DEBC.

4 In the given figure, AE $= 3$ cm, $\triangle$ABE has area 5 cm², and BEDC has area 6 cm².

Find the length of ED to 3 significant figures.

A   B   E   C   D
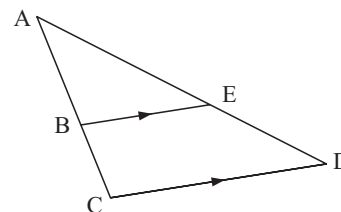
5 Consider the following *similar* solids. Find the unknown length or volume:

a

6 cm

V cm^3

3 cm 10 cm^3

b

6 cm

12 cm^3

9 cm

V cm^3

c

9 cm

50 cm^3

x cm 10 cm^3

d

64 cm^3

27 cm^3

6 cm

x cm
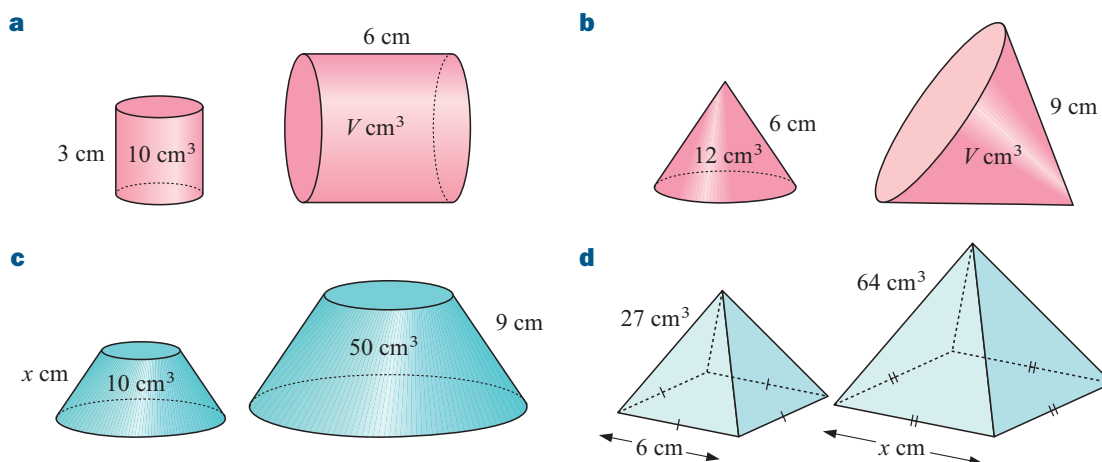
6 Two solid wooden spheres have radii a cm and $3a$ cm respectively. If the smaller one has volume 250 cm^3, find the volume of the larger one.

7 The surface areas of two similar cylinders are 6 cm^2 and 54 cm^2 respectively.

   a If the larger cylinder has height 12 cm, find the height of the smaller one.

   b If the volume of the smaller cylinder is 24 cm^3, find the volume of the larger one.

8 Two similar cones have volumes 4 cm^3 and 108 cm^3 respectively. Find the surface area of the smaller one if the larger has surface area 54 cm^2.

9

flat

Two buckets are similar in shape. The smaller one is 30 cm tall and the larger one is 45 cm tall.

Both have water in them to a depth equal to half of their heights. The volume of water in the small bucket is 4400 cm^3.
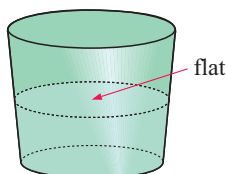
   a What is the scale factor in comparing the smaller bucket to the larger one?

   b What is the volume of water in the larger bucket?

   c The surface area of the water in the larger bucket is 630 cm^2. What is the surface area of the water in the smaller bucket?

10 Two cylindrical containers have capacities 875 ml and 2240 ml respectively. Thin straws are placed in the cylinders as shown.

These have length 10 cm and 16 cm respectively. Are the cylinders similar? Give evidence to support your answer.

10 cm

16 cm
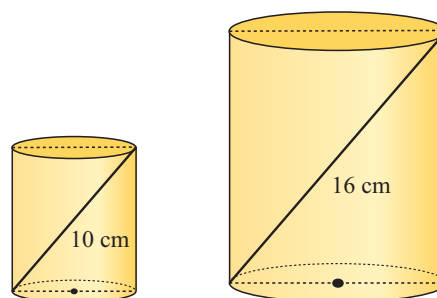
11 Answer the questions of the **Opening Problem** on page **367**.

Review set 18A

1   Draw two equiangular quadrilaterals which are not similar.

2   Find the value of x:

a

b
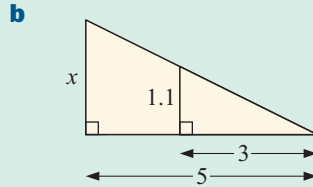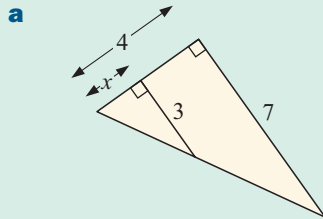
3   Find x, given that the figures are similar:
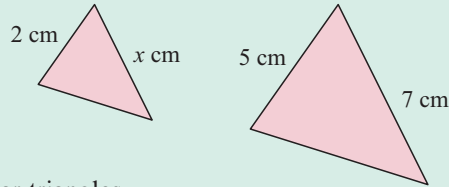
4   Show that the following figures possess similar triangles.

a

b

c

5   Find the value of the unknown in:

a

b
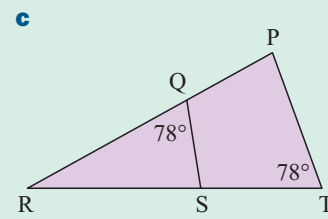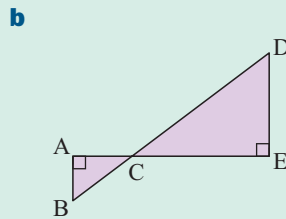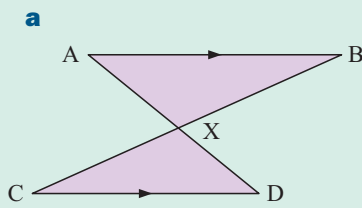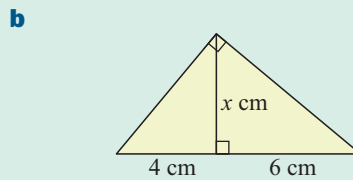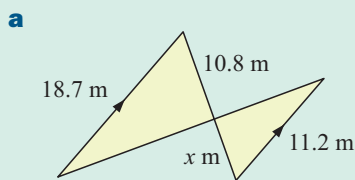
6   Find the unknowns for the following similar figures:

a

b
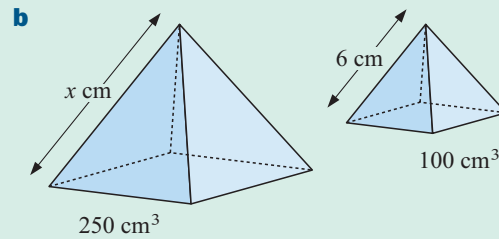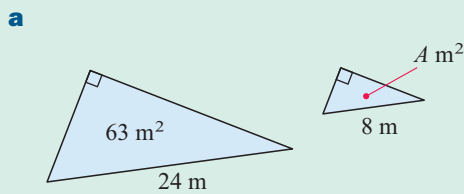
7   Parallelograms ABCD and EFGH are similar.
Find: **a** x **b** y
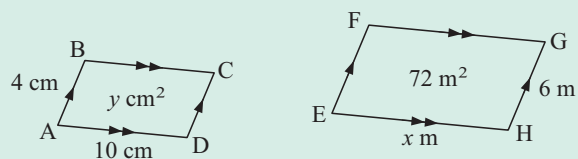
8   P and Q are markers on the banks of a canal which has parallel sides. R and S are telegraph poles which are directly opposite each other. PQ = 30 m and QR = 100 m.

When I walked 20 m from P directly away from the bank, I reached a point T such that T, Q and S lined up. How wide is the canal?
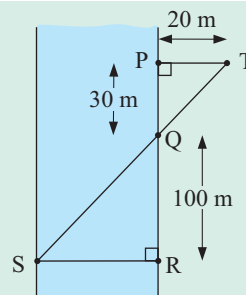
9   The surface areas of two solid similar cones are 4.2 m² and 67.2 m² respectively.

a   Find the scale factor k to enlarge the smaller cone into the larger cone.

b   If the larger cone has a height of 3.96 m, find the height of the smaller cone.

c   If the larger cone has a volume of 32 m³, find the volume of the smaller one.

Review set 18B

1   Draw two quadrilaterals which have sides in proportion but are not similar.

2   Two solids are similar and the ratio of their volumes is 343 : 125.

a   What is the scale factor?

b   What is the ratio of their surface areas?

3   In the following figures, establish that a pair of similar triangles exists and hence find x:

a

b

c
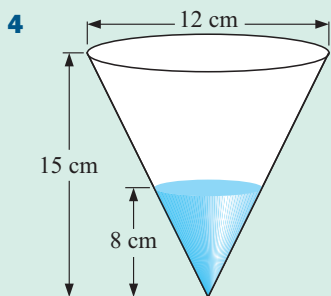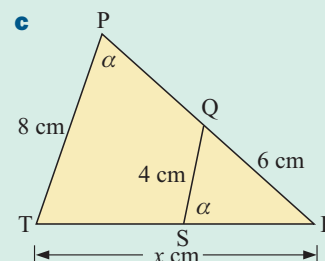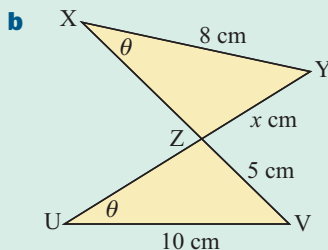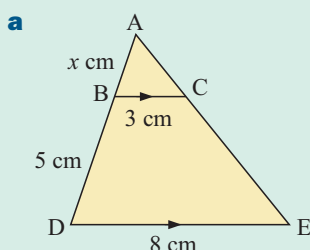
4   A conical flask has height 15 cm and base diameter 12 cm. Water is poured into the flask to a depth of 8 cm.

a   Show that triangles ABC and MNC are similar.

b   Hence, show that $x = 3.2$.

c   Find the diameter of the surface of the water.
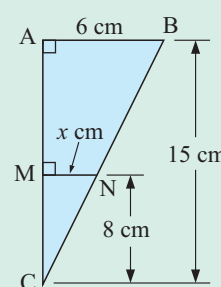
5   Find x for these similar figures:

a

b
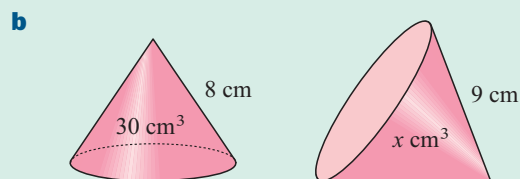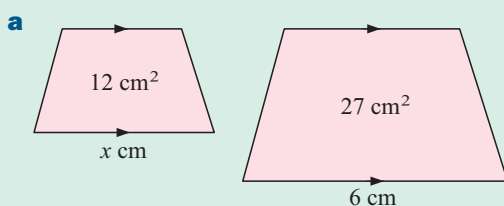
6 Triangles ABC and PQR are similar.
Find: **a** x **b** y
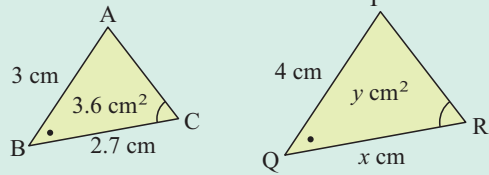
7
   a Explain why triangles ABE and ACD are similar.
   b Find the length of CD given that BE has length 4 cm.
   c If triangle ABE has area 10 cm^2, find the area of quadrilateral BEDC.
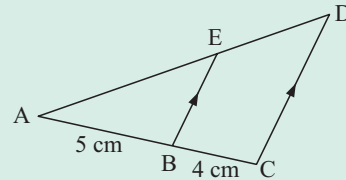
8 Rectangles ABCD and EFGH are similar. Find the dimensions of rectangle EFGH.
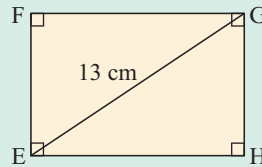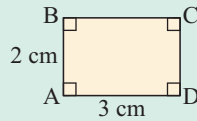
9 Two similar cylinders have surface areas of 250 cm^2 and 360 cm^2 respectively. Find the volume of the larger cylinder if the volume of the smaller cylinder is 375 cm^3.

Challenge
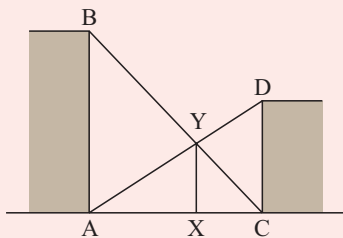
1 The vertical walls of two buildings are 40 m and 30 m tall. A vertical flag pole XY is between the buildings such that B, Y and C are collinear and A, Y and D are collinear.
How high is the flag pole?

2 A rectangular piece of sailcloth 3 m by 4 m is folded so that one corner is placed over the diagonally opposite corner. How long is the crease?
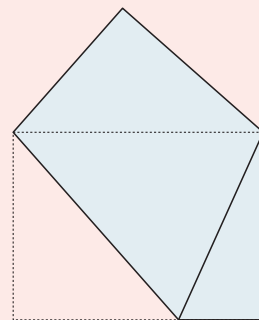
Introduction to functions

19

Contents:

Opening problem

A dot is placed at the 20 cm mark of a ruler as illustrated.

Things to think about:

- How far from the dot is the:
  - ▶ 5 cm mark   ▶ 16 cm mark   ▶ 20 cm mark   ▶ 24 cm mark   ▶ 30 cm mark?
- Is there a formula to find the distance d cm between the dot and the x cm mark?

A   MAPPING DIAGRAMS   [3.1]

Consider the family of Mr and Mrs Schwarz. Their sons are Hans and Gert and their daughter is Alex.

There are many relationships between members of the family. For example, "is a brother of", "is older than", "is the parent of", and so on.

Alongside is a **mapping diagram** which maps the set of children C onto the set of parents, P.

The connection for this mapping is: "is a child of".
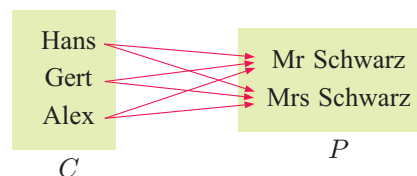
A **mapping** is used to map the members or **elements** of one set called the **domain**, onto the members of another set called the **range**.

In particular we can define:

- The **domain** of a mapping is the set of elements which are to be mapped.
- The **range** of a mapping is the set of elements which are the result of mapping the elements of the domain.

Consider these two mappings:

For the mapping $y = x + 3$:

x (domain) y (range)
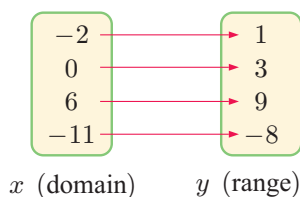
For the mapping $y = x^2 + 1$:

x (domain) y (range)
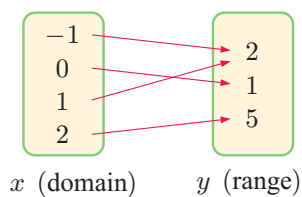
$y = x + 3$ or 'add 3 onto x' is called a **one-one** mapping because every element in the domain maps onto one and only one element in the range.

$y = x^2 + 1$ or 'square x and then add 1' is called a **many-one** mapping because more than one element in the domain maps onto the same element in the range.

In the example of "is a child of", the mapping is **many-many**.

EXERCISE 19A

1 Copy and complete the following 'sets and mappings' diagrams, and state whether the mapping is one-one, many-one, one-many or many-many.

a mapping '$y = 2x - 5$'

x y
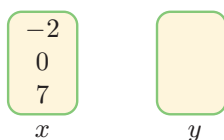
b mapping 'is not equal to'

x y

c mapping '$x = y^2$'

x y
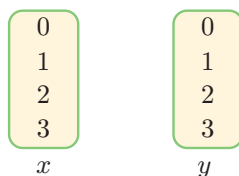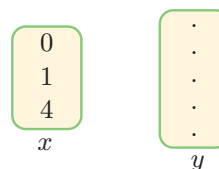
d mapping 'is greater than'

x y
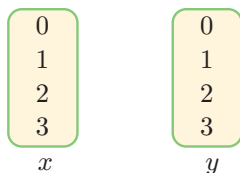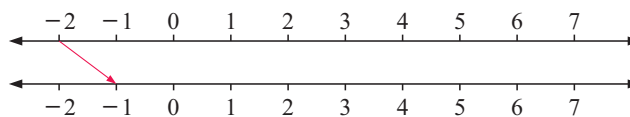
e mapping 'add 1'

2 For these domains and mappings, describe the corresponding range:

a domain {real numbers}   mapping: 'subtract 20'

b domain {odd numbers}   mapping: 'double'

c domain {positive real numbers}   mapping: 'find the square root'

d domain {real numbers $\geqslant 0$}   mapping: 'add 10'

e domain {even numbers}   mapping: 'divide by 2'

B   FUNCTIONS   [3.1, 3.2]

We can also use set notation to describe mappings.

For example, consider the set $\{0, \pm 1, \pm 2, \pm 3\}$ under the mapping 'square the number'.
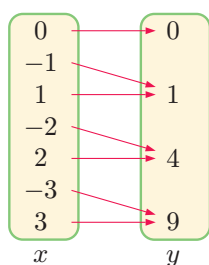
$\{0, \pm 1, \pm 2, \pm 3\}$ maps onto $\{0, 1, 4, 9\}$

We say that: $\{0, \pm 1, \pm 2, \pm 3\}$ is the domain and
$\{0, 1, 4, 9\}$ is the range.

We could write this mapping as $x \mapsto x^2$

This is a many-one mapping, and is an example of a *function*.

A **function** is a mapping in which each element of the domain maps onto *exactly one* element of the range.
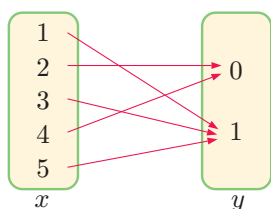
is a **many-one** mapping and is a function.
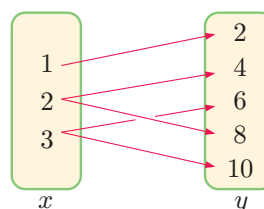
is a **one-many** mapping and is **not a function**.

We can see that functions can only be one-one or many-one mappings. One-many and many-many mappings are *not* functions.

Example 1   ◀ Self Tutor

For the domain $\{0, 1, 2, 3\}$ and the function 'subtract 2', find the range.
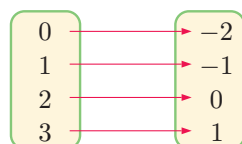
So, the range is $\{-2, -1, 0, 1\}$

Suppose a function maps set A onto set B. We say that:

- A is the **domain** of the function
- B is the **range** of the function.

To help describe the domain and range of a function, we can use interval notation:

 x For numbers *between* a and b we write $a < x < b$.

 x For numbers '*outside*' a and b we write $x < a$ or $x > b$.

 x would be written as $a \leqslant x < b$.

A filled in circle indicates the inclusion of the end point.
An open circle indicates the non-inclusion of that point.
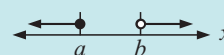
Consider these examples:

- All values of x and y are possible.
  $\therefore$ the domain is $\{x \mid x \in \mathbb{R}\}$
  and the range is $\{y \mid y \in \mathbb{R}\}$.

> $\mathbb{R}$ represents the set of all real numbers, or all numbers on the number line.
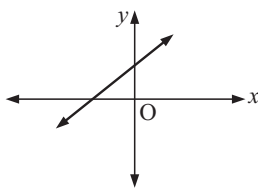
- All values of $x < 2$ are possible.
  $\therefore$ the domain is $\{x \mid x < 2, \ x \in \mathbb{R}\}$.
  All values of $y > -1$ are possible.
  $\therefore$ the range is $\{y \mid y > -1, \ y \in \mathbb{R}\}$.
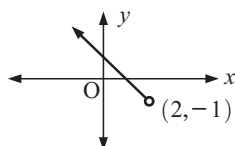
- x can take any value.
  $\therefore$ the domain is $\{x \mid x \in \mathbb{R}\}$.
  y cannot be < -2.
  $\therefore$ the range is $\{y \mid y \geqslant -2, \ y \in \mathbb{R}\}$.

  $y = \frac{1}{2}x^2 - 2$
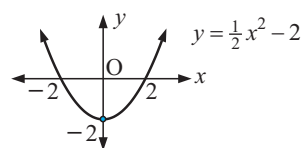
- x can take all values except $x = 0$.
  $\therefore$ the domain is $\{x \mid x \neq 0, \ x \in \mathbb{R}\}$.
  y can take all values except $y = 0$.
  $\therefore$ the range is $\{y \mid y \neq 0, \ y \in \mathbb{R}\}$.

  $y = \dfrac{3}{x}$

Example 2   ◀⬤ *Self Tutor*

For each of the following graphs, state the domain and range:

a

$(3, 4)$

b

$(-4, -2)$
$(4, -4)$

c

$(4, 3)$
$(-1, -2)$
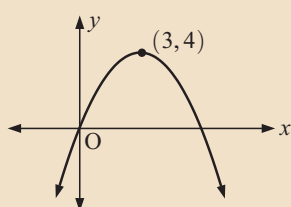
a   Domain:
    $\{x \mid x \in \mathbb{R}\}$.
   Range:
    $\{y \mid y \leqslant 4, y \in \mathbb{R}\}$.
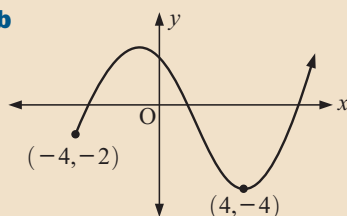
b   Domain:
    $\{x \mid x \geqslant -4, x \in \mathbb{R}\}$.
   Range:
    $\{y \mid y \geqslant -4, y \in \mathbb{R}\}$.
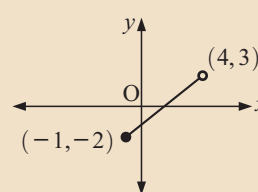
c   Domain:
    $\{x \mid -1 \leqslant x < 4, x \in \mathbb{R}\}$.
   Range:
    $\{y \mid -2 \leqslant y < 3, y \in \mathbb{R}\}$.

It is common practice when dealing with graphs on the Cartesian plane to assume the domain and range are real. So, it is common to write $\{x \mid x \leqslant 3, \ x \in \mathbb{R}\}$ as just $\{x \mid x \leqslant 3\}$.

EXERCISE 19B.1

1   State the domain and range for these sets of points:

    a $\{(-1, 5), (-2, 3), (0, 4), (-3, 8), (6, -1), (-2, 3)\}$

    b $\{(5, 4), (-3, 4), (4, 3), (2, 4), (-1, 3), (0, 3), (7, 4)\}$.

2 For each of the following graphs, find the domain and range and decide whether it is the graph of a function:

a

$(-4,-2)$

b

c

d

e

f

$(-2,1)$

g

$(6,4)$

h

i
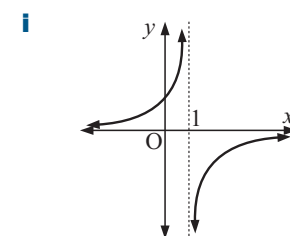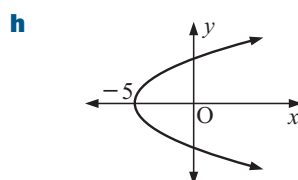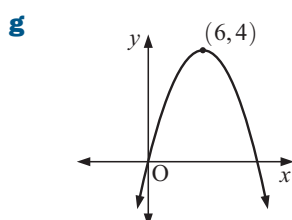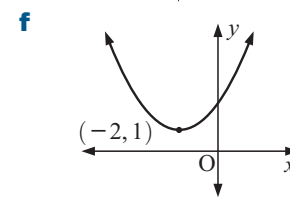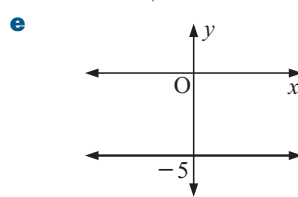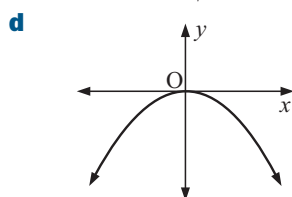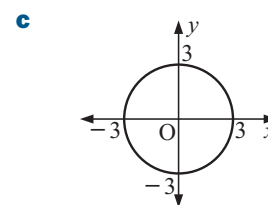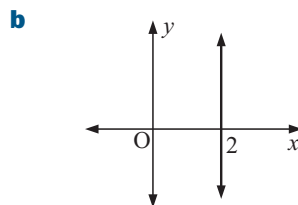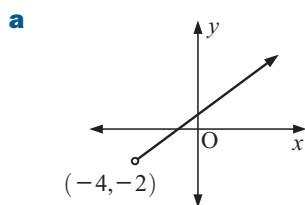
3 Find the range for the functions with domain D:

> **a** $D = \{-1, 0, 2, 7, 9\}$,   function: 'add 3'.
>
> **b** $D = \{-2, -1, 0, 1, 2\}$,   function: 'square and then divide by 2'.
>
> **c** $D = \{x \mid -2 < x < 2\}$,   function: 'multiply x by 2 then add 1'.
>
> **d** $D = \{x \mid -3 \leqslant x \leqslant 4\}$,   function: 'cube x'.

*In **c** and **d** we assume that $x \in \mathbb{R}$.*

4 For each of these functions:

> **i** use a graphics calculator to help sketch the function
>
> **ii** find the range.

> **a** $y = 3x + 1$ on the domain $\{x \mid -2 \leqslant x \leqslant 2\}$
>
> **b** $y = x^2$ on the domain $\{x \mid -3 \leqslant x \leqslant 4\}$
>
> **c** $y = 4x - 1$ on the domain $\{x \mid -2 \leqslant x \leqslant 3\}$
>
> **d** $y = \dfrac{1}{x - 1}$ on the domain $\{x \mid 0 \leqslant x \leqslant 3, \quad x \neq 1\}$
>
> **e** $y = x + \dfrac{1}{x}$ on the domain $\{x \mid -4 \leqslant x \leqslant 4, \quad x \neq 0\}$
>
> **f** $y = x^2 + 1$ on the domain $\{x \mid x \in \mathbb{R}\}$
>
> **g** $y = x^3$ on the domain $\{x \mid x \in \mathbb{R}\}$

GEOMETRIC TEST FOR FUNCTIONS: "VERTICAL LINE TEST"

If we draw all possible vertical lines on the graph of a relation:

- the relation is a function if each line cuts the graph no more than once
- the relation is *not* a function if *any* line cuts the graph more than once.

Example 3
🔊 *Self Tutor*

Which of these relations are functions?

a

b

c
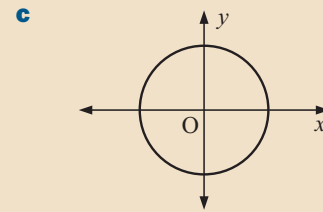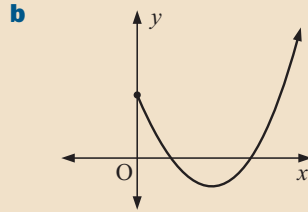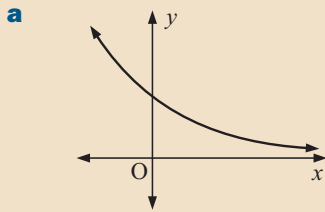
a Every vertical line we could draw cuts the graph only once.
  ∴ we have a function.

b Every vertical line we could draw cuts the graph at most once.
  ∴ we have a function.

c This vertical line cuts the graph twice. So, the relation is not a function.
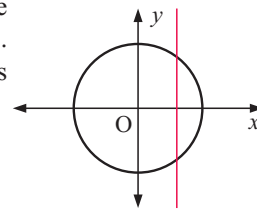
EXERCISE 19B.2

1 Which of the following sets of ordered pairs are functions? Give reasons for your answers.

  a (1, 1), (2, 2), (3, 3), (4, 4) **b** (−1, 2), (−3, 2), (3, 2), (1, 2)

  c (2, 5), (−1, 4), (−3, 7), (2, −3) **d** (3, −2), (3, 0), (3, 2), (3, 4)

  e (−7, 0), (−5, 0), (−3, 0), (−1, 0) **f** (0, 5), (0, 1), (2, 1), (2, −5)

2 Use the vertical line test to determine which of the following are graphs of functions:

a

b

c

d

e

f

g

h

i
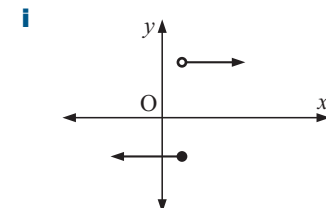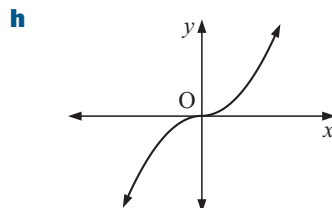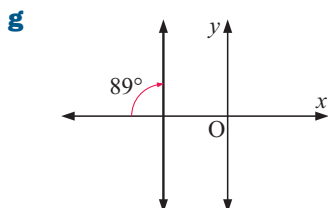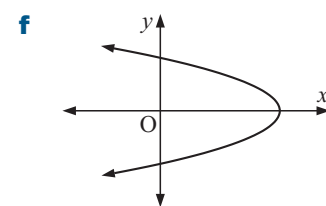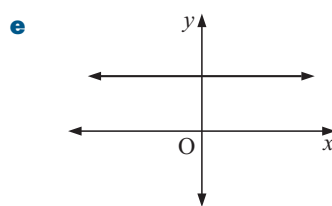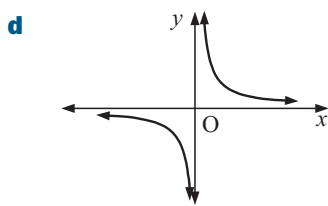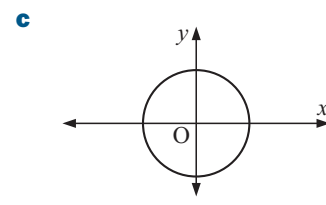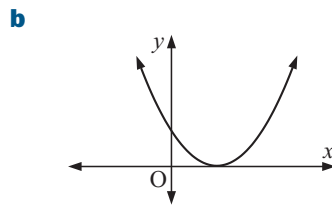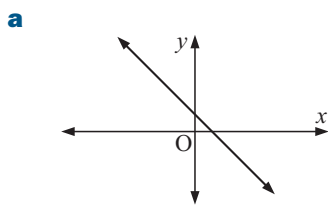
3 Will the graph of a straight line always be a function? Give evidence.

C   FUNCTION NOTATION   [3.1, 3.6]

The machine alongside has been programmed to perform a particular function.

If f is used to represent this function, we say that 'f is the function that will convert x into $2x - 1$'.

I double the input and then subtract 1

$2x - 1$
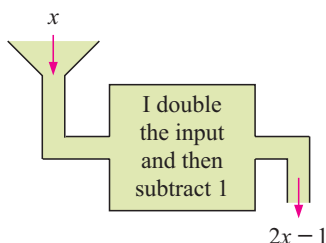
If 3 is fed into the machine,
$2(3) - 1 = 5$
comes out.

> $f(x)$ is read as 'f of x'. It is sometimes called the **image** of x.

So, f would convert   2 into $2(2) - 1 = 3$ and
                        -4 into $2(-4) - 1 = -9$.

This function can be written as: $f : x \mapsto 2x - 1$ or as $f(x) = 2x - 1$.

function f such that x maps onto $2x - 1$

> If $f(x)$ is the value of y for a given value of x, then $y = f(x)$.

Notice that for $f(x) = 2x - 1$, $f(2) = 2(2) - 1 = 3$ and
$f(-4) = 2(-4) - 1 = -9$.

Consequently, $f(2) = 3$ indicates that the point $(2, 3)$ lies on the graph of the function.

Likewise, $f(-4) = -9$ indicates that the point $(-4, -9)$ also lies on the graph.
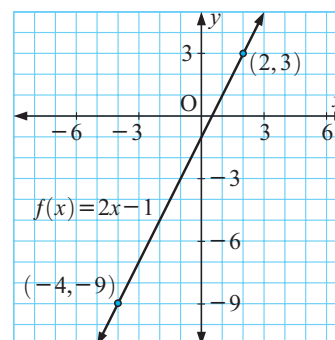
Example 4   ◀)) Self Tutor

If $f : x \mapsto 3x^2 - 4x$,   find the value of: **a** $f(2)$ **b** $f(-5)$

a $f(2)$
$= 3(2)^2 - 4(2)$ {replacing x by (2)}
$= 3 \times 4 - 8$
$= 4$

b $f(-5)$
$= 3(-5)^2 - 4(-5)$ {replacing x by (-5)}
$= 3(25) + 20$
$= 95$

Example 5   ◀)) Self Tutor

If $f(x) = 4 - 3x - x^2$,   find in simplest form: **a** $f(-x)$ **b** $f(x + 2)$

a $f(-x)$
$= 4 - 3(-x) - (-x)^2$
       {replacing x by $(-x)$}
$= 4 + 3x - x^2$

b $f(x + 2)$
$= 4 - 3(x + 2) - (x + 2)^2$
       {replacing x by $(x + 2)$}
$= 4 - 3x - 6 - \left(x^2 + 4x + 4\right)$
$= -2 - 3x - x^2 - 4x - 4$
$= -x^2 - 7x - 6$

EXERCISE 19C

1 **a** If $f(x) = 3x - 7$, find and interpret $f(5)$.

    b If $g : x \mapsto x - x^2$, find and interpret $g(3)$.

    c If $H(x) = \dfrac{2x + 5}{x - 1}$, find and interpret $H(4)$.

2 **a** If $f(x) = 5 - 4x$, find: **i** $f(0)$ **ii** $f(3)$ **iii** $f(-4)$ **iv** $f(100)$

    b If $E(x) = 2(3 - x)$, find: **i** $E(0)$ **ii** $E(1)$ **iii** $E(5)$ **iv** $E(-2)$

    c If $h : x \mapsto \dfrac{x}{x - 3}$, find: **i** $h(2)$ **ii** $h(5)$ **iii** $h(10)$ **iv** $h(-7)$

3 **a** If $f : x \mapsto 5 - x^2$, find: **i** $f(4)$ **ii** x when $f(x) = 1$.

    b If $g(x) = \dfrac{x + 1}{10}$, find: **i** $g(4)$ **ii** a when $g(a) = 2$.

    c If $m(x) = x^2 - 3$, find: **i** x when $m(x) = 0$ **ii** x when $m(x) = 1$.

    d If $f(x) = 3x + 5$ and $g(x) = 2x - 3$, find x when $f(x) = g(x)$.

4 The value of a car t years after purchase is given by
$V(t) = 28\,000 - 4000t$ dollars.

    a Find $V(4)$ and state what this value means.

    b Find t when $V(t) = 8000$ and explain what this
represents.

    c Find the original purchase price of the car.

    d Do you think this formula is valid for all $t > 0$?
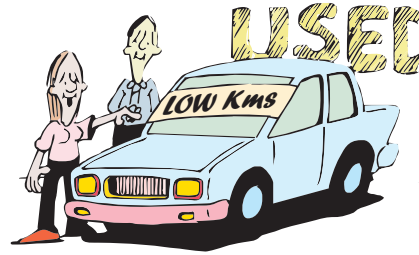
5 Sketch the graph of $y = f(x)$ where $f(x) = 2x - 1$ on the domain $\{x \mid -3 \leqslant x \leqslant 1\}$.
State the range of this function.

6 Sketch the graph of $y = g(x)$ where $g(x) = 6 - 5x$ on the domain $\{x \mid -2 \leqslant x \leqslant 2\}$.
State the range of this function.

7 The graph of a function is given alongside. Use the
graph to:

    a find $f(2)$

    b estimate x, to 1 decimal place, when
$f(x) = -3$.
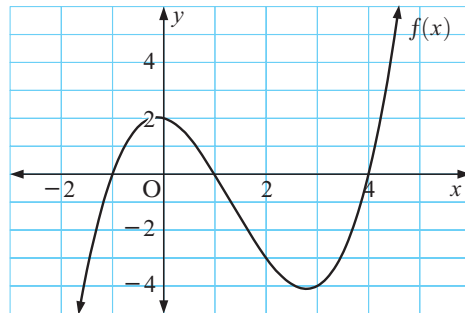
8 If $f(x) = 5 - 2x$, find in simplest form:

    a $f(a)$ **b** $f(-a)$ **c** $f(a + 1)$ **d** $f(x - 3)$ **e** $f(2x)$

9 If $P(x) = x^2 + 4x - 3$, find in simplest form:

    a $P(x + 2)$ **b** $P(1 - x)$ **c** $P(-x)$ **d** $P(x^2)$ **e** $P(x^2 + 1)$

Discovery 1 — Fluid filling functions

When water is added to a container, the depth of water is given by a function of time. If the water is added at a *constant rate*, the volume of water added is directly proportional to the time taken to add it.

DEMO

So, for a container with a uniform cross-section, the graph of depth against time is a straight line, or *linear*. We can see this in the following depth-time graph.

The question arises:
'How does the shape of the container affect the appearance of the graph?'
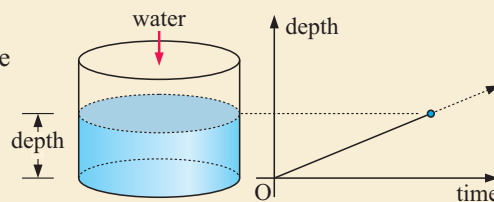
For example, consider the graph shown for a vase of conical shape.
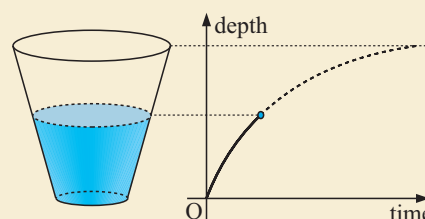
What to do:

1 For each of the following containers, draw a depth-time graph as water is added at a constant rate.

a   b   c   d

e   f   g   h
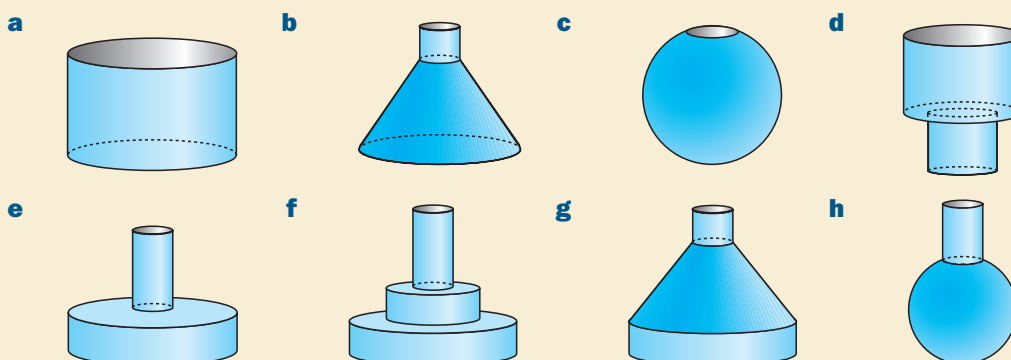
2 Use the water filling demonstration to check your answers to question **1**.

D   COMPOSITE FUNCTIONS   [3.7]

Consider a process where a value such as 2 is applied to a function f, and then the result is applied to another function g:

$$2 \xrightarrow{\;f\;} f(2) \xrightarrow{\;g\;} g(f(2))$$

The resulting value is $g(f(2))$.

For example, suppose 2 is applied to the function $f(x) = x^2$, and then the result is applied to $g(x) = x+3$:

$$2 \xrightarrow{\;f(x)=x^2\;} 4 \xrightarrow{\;g(x)=x+3\;} 7$$

We write $g(f(2)) = 7$.

Similarly, $f(g(2))$ is obtained by applying g first, then applying f to the result:

$$2 \xrightarrow{\ g(x) = x + 3\ } 5 \xrightarrow{\ f(x) = x^2\ } 25$$

$$3 \xrightarrow{\ g(x) = x + 3\ } 6 \xrightarrow{\ f(x) = x^2\ } 36$$

We write $f(g(2)) = 25$ and $f(g(3)) = 36$.

More generally, we can define a **composite function** $f(g(x))$ which allows us to perform calculations like these in one step.

> Given two functions $f(x)$ and $g(x)$, the **composite function** of f and g is the function which maps x onto $f(g(x))$.

In the above example where $f(x) = x^2$ and $g(x) = x + 3$, $f(g(x)) = f(x + 3)$
$$= (x + 3)^2$$

Notice that $f(g(2)) = 25$ and $f(g(3)) = 36$ are both true for this function.

Also notice that $g(f(x)) = g(x^2)$
$$= x^2 + 3, \quad \text{so in general} \quad f(g(x)) \neq g(f(x)).$$

Example 6   ◀)) *Self Tutor*

Consider the functions $f(x) = x^2 + 1$ and $g(x) = 2x - 3$.

a Find the value of $f(g(2))$ and $g(f(3))$.

b Find, in simplest form, $f(g(x))$ and $g(f(x))$.

c Use **b** to check your answers to **a**.

a
$$g(x) = 2x - 3 \qquad\qquad f(x) = x^2 + 1$$
$$\therefore \quad g(2) = 2(2) - 3 = 1 \qquad \therefore \quad f(3) = 3^2 + 1 = 10$$
$$\therefore \quad f(g(2)) = f(1) \qquad\qquad \therefore \quad g(f(3)) = g(10)$$
$$= 1^2 + 1 \qquad\qquad\qquad\qquad = 2(10) - 3$$
$$= 2 \qquad\qquad\qquad\qquad\qquad\quad = 17$$

> To find $f(g(x))$ we look at the f function, and whenever we see x we replace it by $g(x)$ within brackets.

b $f(g(x)) = f(2x - 3) \qquad\qquad g(f(x)) = g(x^2 + 1)$
$$\qquad = (2x - 3)^2 + 1 \qquad\qquad = 2(x^2 + 1) - 3$$
$$\qquad = 4x^2 - 12x + 9 + 1 \qquad\quad = 2x^2 + 2 - 3$$
$$\qquad = 4x^2 - 12x + 10 \qquad\qquad = 2x^2 - 1$$

c When $x = 2$, $f(g(2)) = 4(2)^2 - 12(2) + 10$
$$= 16 - 24 + 10$$
$$= 2 \quad \text{which checks with } \textbf{a} \quad \checkmark$$

When $x = 3$, $g(f(3)) = 2(3)^2 - 1$
$$= 18 - 1$$
$$= 17 \quad \text{which checks with } \textbf{a} \quad \checkmark$$

EXERCISE 19D

1 If $f(x) = x^2 + 1$ and $g(x) = 2x - 3$ find:

   a $f(3)$ **b** $g(3)$ **c** $f(g(3))$ **d** $g(f(3))$ **e** $f(f(3))$ **f** $g(g(3))$

2 Let $f(x) = 5x + 1$ and $g(x) = 4 - 2x$.

   a Find the values of $f(g(0))$ and $g(f(1))$.

   b Find in simplest form: **i** $f(g(x))$ **ii** $g(f(x))$

   c Use **a** to check your answers to **b**.

3 Let $f(x) = x^2 + 2x$ and $g(x) = x - 2$. Find, in simplest form:

   a $f(g(2))$ and $f(g(x))$ **b** $g(f(2))$ and $g(f(x))$.

4 If $f(x) = 3x - 4$ and $g(x) = 2 - x$, find in simplest form:

   a $f(g(x))$ **b** $g(f(x))$ **c** $f(f(x))$ **d** $g(g(x))$

5 Let $f(x) = x^2 + x - 6$ and $g(x) = x + 2$.

   a Find $f(g(x))$ in simplest form.

   b Use **a** to find:

      i $f(g(1))$ **ii** $f(g(4))$ **iii** $f(g(-2))$.

6 If $f(x) = \sqrt{x}$ and $g(x) = 4x - 3$, find in simplest form:

   a $f(g(x))$ **b** $g(g(x))$

7 Find an f and a g function such that:

   a $f(g(x)) = \sqrt{x - 3}$ **b** $f(g(x)) = (x + 5)^3$ **c** $f(g(x)) = \dfrac{5}{x + 7}$

   d $g(f(x)) = \dfrac{1}{\sqrt{3 - 4x}}$ **e** $g(f(x)) = 3^{2x+1}$ **f** $g(f(x)) = \dfrac{4}{(x - 1)^2}$

8 If $f(x) = 3x + 1$ and $g(x) = 2x - 3$, find x when $g(f(x)) = 17$.

9 If $f(x) = 2x + 1$ and $g(f(x)) = 4x^2 + 4x + 3$, find $g(x)$, given that $g(x) = ax^2 + bx + c$.

10 If $g(x) = 1 - 3x$ and $f(g(x)) = 9x^2 - 6x - 2$, find $f(x)$, given that $f(x) = ax^2 + bx + c$.

11 If $f(x) = ax + b$ and $f(f(x)) = 4x - 15$, find a and b.

E   RECIPROCAL FUNCTIONS   [3.2, 3.5]

A **reciprocal function** has an equation of the form $y = \dfrac{k}{x}$ where k is a constant.

It has a graph which is called a **rectangular hyperbola**.

There are examples of hyperbolae in the world around us.

When an aeroplane flies faster than the speed of sound, which is around 1200 km/h, we say it breaks the sound barrier. It sets up a shock wave in the shape of a **cone**, and when this intersects the ground it does so in the shape of a **hyperbola**. The **sonic boom** formed hits every point on the curve at the same time, so that people in different places along the curve on the ground all hear it at the same time. No sound is heard outside the curve but the boom eventually covers every place inside it.
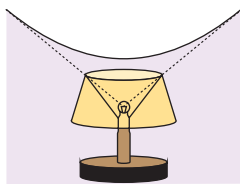
The hyperbolic shape is also noticed in the home when a lamp is close to a wall. The light and shadow form part of a hyperbola on the wall.

There are many situations in which two quantities vary **inversely**. They form a relationship which can be described using a reciprocal function.

For example, the pressure and volume of a gas at room temperature vary inversely according to the equation $P = \dfrac{77.4}{V}$.

If P is graphed against V, the curve is one branch of a hyperbola.

For 1 kg of O_2 at 25°C
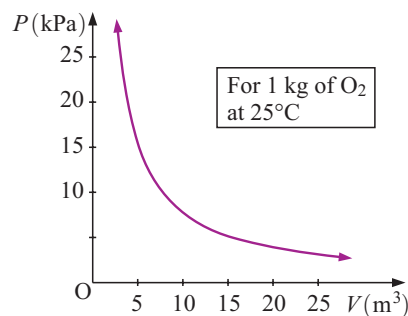
Discovery 2   The family of curves $y = \dfrac{k}{x}, \quad k \neq 0$

In this discovery you should use a **graphing package** or **graphics calculator** to draw curves of the form $y = \dfrac{k}{x}$ where $k \neq 0$.

GRAPHING PACKAGE

What to do:

1 On the same set of axes, draw the graphs of $y = \dfrac{1}{x}, \quad y = \dfrac{4}{x}$ and $y = \dfrac{8}{x}$.

2 Describe the effect of the value of k on the graph for $k > 0$.

3 Repeat **1** for $y = \dfrac{-1}{x}, \quad y = \dfrac{-4}{x}$ and $y = \dfrac{-8}{x}$.

4 Comment on the change in shape of the graph in **3**.

5 Explain why there is no point on the graph when $x = 0$.

6 Explain why there is no point on the graph when $y = 0$.

You should have noticed that functions of the form $y = \frac{k}{x}$ are undefined when $x = 0$.

On the graph we see that the function is defined for values of x getting closer and closer to $x = 0$, but the function never reaches the line $x = 0$. We say that $x = 0$ is a **vertical asymptote**.

Likewise, as the values of x get larger, the values of y get closer to 0, but never quite reach 0. We say that $y = 0$ is a **horizontal asymptote**.

$y = \dfrac{k}{x}$
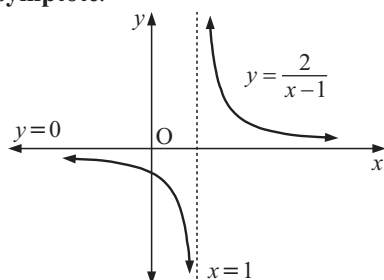
$y = \dfrac{2}{x-1}$

The graph of $y = \dfrac{2}{x-1}$ alongside is undefined when $x - 1 = 0$, which is when $x = 1$. It has the vertical asymptote $x = 1$.

As the values of x get larger, the values of y approach, but never quite reach, 0.

The graph has the horizontal asymptote $y = 0$.
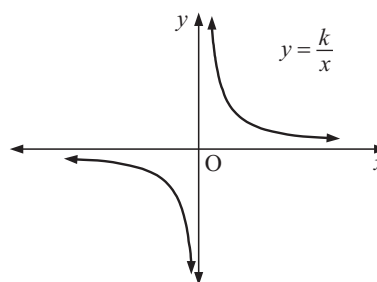
EXERCISE 19E

1 Consider the function $y = \dfrac{5}{x}$, which can be written as $xy = 5$.

   a Explain why both x and y can take all real values except 0.

   b What are the asymptotes of the function $y = \dfrac{5}{x}$?

   c Find y when: **i** $x = 500$ **ii** $x = -500$

   d Find x when: **i** $y = 500$ **ii** $y = -500$

   e By plotting points or by using technology, graph $y = \dfrac{5}{x}$.

   f Without calculating new values, sketch the graph of $y = \dfrac{-5}{x}$.

2 Kate has to make 40 invitations for her birthday party. How fast she can make the invitations will affect how long the job takes her.

Suppose Kate can make n invitations per hour and the job takes her t hours.

   a Complete a table of values:

   b Draw a graph of n versus t with n on the horizontal axis.

Invitations per hour (n)	4	8	12	....
Time taken (t)				

   c Is it reasonable to draw a smooth curve through the points plotted in **b**? What shape is the curve?

   d State a formula for the relationship between n and t.

3 Determine the equations of the following reciprocal graphs:

a (4, 2)   **b** (−3, −1)   **c** (2, −6)
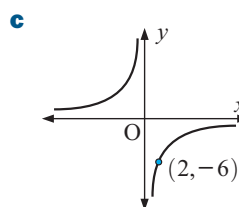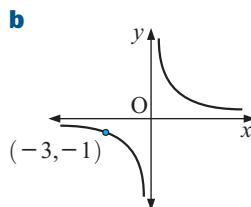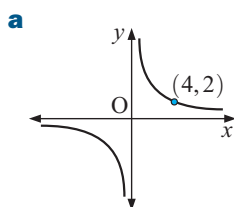
> Instructions for graphing a function can be found on page **22**.

4 For the following functions:

   i use technology to graph the function

   ii find the equation of any vertical or horizontal asymptotes.

   a $y = \dfrac{3}{x - 2}$ **b** $y = \dfrac{2}{x + 1}$ **c** $y = \dfrac{1}{x - 3} + 1$

F   THE ABSOLUTE VALUE FUNCTION   [1.6, 3.2]

Discovery 3   *The absolute value function*

What to do:

1 Suppose $y = x$ if $x \geqslant 0$ and $y = -x$ if $x < 0$.

Find y for $x = 5,\ 7,\ \frac{1}{2},\ 0,\ -2,\ -8,$ and -10.

2 Draw the graph of $y = \begin{cases} x & \text{if} \quad x \geqslant 0 \\ -x & \text{if} \quad x < 0 \end{cases}$ using the results of **1**.

3 Consider $M(x) = \sqrt{x^2}$ where x^2 must be found before finding the square root.
Find the values of $M(5)$, $M(7)$, $M(\frac{1}{2})$, $M(0)$, $M(-2)$, $M(-8)$ and $M(-10)$.

4 What conclusions can be made from **1** and **3**?

THE ABSOLUTE VALUE OF A NUMBER

> The **absolute value** or **modulus** of a real number is its size, ignoring its sign.
> We denote the absolute value of x by $|x|$.

For example,   the absolute value of 7 is 7,   and
the absolute value of -7 is also 7.

Geometric definition of absolute value

> $|x|$ is the distance of x from 0 on the number line.
> Because the modulus is a distance, it cannot be negative.

If $x > 0$:
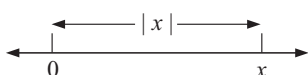
If $x < 0$:
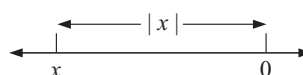
For example:

Algebraic definition of absolute value

From **Discovery 3**,

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases} \quad \text{or} \quad |x| = \sqrt{x^2}$$

$y = |x|$ has graph:

This branch
is $y = -x$, $x < 0$.

This branch
is $y = x$, $x \geqslant 0$.
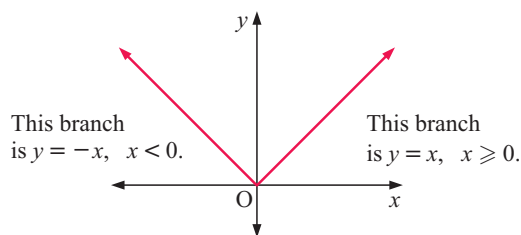
> The vertical line $x = 0$ is the line of symmetry of the graph.

Example 7   ◄)) Self Tutor

If $a = -7$ and $b = 3$ find: **a** $|a + b|$ **b** $|ab|$

a $|a + b|$
$= |-7 + 3|$
$= |-4|$
$= 4$

b $|ab|$
$= |-7 \times 3|$
$= |-21|$
$= 21$

> The absolute value behaves as a grouping symbol. Perform all operations within it first.

Example 8   ◀ᴑ) *Self Tutor*

Draw the graph of $f(x) = x + 2|x|$.

If $x \geqslant 0$,
$f(x) = x + 2(x) = 3x$

If $x < 0$,
$f(x) = x + 2(-x) = -x$
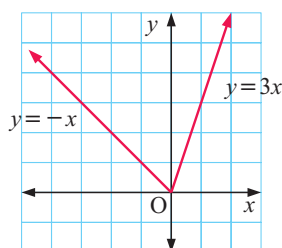
EXERCISE 19F.1

1 Write down the values of:

    a $|7|$
    b $|-7|$
    c $|0.93|$
    d $\left|-2\tfrac{1}{4}\right|$
    e $|-0.0932|$

2 If $x = -4$, find the value of:

    a $|x + 6|$
    b $|x - 6|$
    c $|2x + 3|$
    d $|7 - x|$

    e $|x - 7|$
    f $|x^2 - 6x|$
    g $|6x - x^2|$
    h $\dfrac{|x|}{x + 2}$

3 **a** Find the value of a^2 and $|a|^2$ if a is:

       i 3
       ii 0
       iii -2
       iv 9
       v -9
       vi -20

    b What do you conclude from the results in **a**?

4 Solve for x:

    a $|x| = 4$
    b $|x| = 1.4$
    c $|x| = -2$
    d $|x| + 1 = 7$

    e $|x + 1| = 3$
    f $|2 - x| = 5$
    g $5 - |x| = 1$
    h $|5 - x| = 1$

5 **a** Copy and complete:

    b What can you conclude from the results in **a**?

a	b	$|ab|$	$|a|\,|b|$	$\left|\dfrac{a}{b}\right|$	$\dfrac{|a|}{|b|}$
12	3				
12	-3				
-12	3				
-12	-3				

6 By replacing $|x|$ with x for $x \geqslant 0$ and $(-x)$ for $x < 0$, write the following functions without the modulus sign and hence graph each function:

    a $y = -|x|$
    b $y = |x| + x$
    c $y = \dfrac{|x|}{x}$
    d $y = x - 2|x|$

7 **a** Copy and complete:

    b What can you conclude from the results in **a**?

a	b	$|a + b|$	$|a| + |b|$	$|a - b|$	$|a| - |b|$
2	5				
2	-5				
-2	5				
-2	-5				

THE GRAPH OF $y = |ax + b|$

One way of graphing $y = |ax + b|$ is to first graph $y = ax + b$.

Whatever part of the graph is below the x-axis we then reflect in the x-axis.

Example 9   ◀» **Self Tutor**

Graph $y = |2x - 3|$. Comment on any symmetry in the graph.

We begin by graphing $y = 2x - 3$.

x	0	1	2
y	-3	-1	1

The graph cuts the x-axis when $y = 0$.

$$\therefore \quad 2x - 3 = 0$$
$$\therefore \quad x = \tfrac{3}{2}$$

To obtain the graph of $y = |2x - 3|$ we reflect all points with $x < \tfrac{3}{2}$ in the x-axis.

$x = \tfrac{3}{2}$ is a vertical line of symmetry.
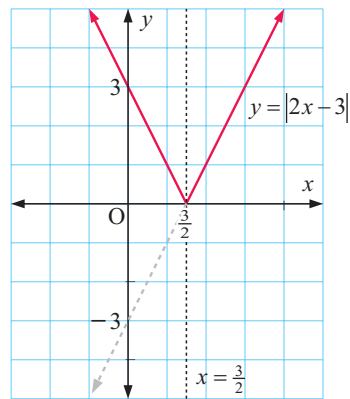
EXERCISE 19F.2

1 Sketch the graphs of:

    a $f(x) = |x + 1|$ **b** $f(x) = |x - 1|$ **c** $f(x) = |2x - 1|$

    d $f(x) = |4 - x|$ **e** $f(x) = |2 - 3x|$ **f** $f(x) = -|3x + 2|$

2 What is the equation of the line of symmetry of $f(x) = |ax + b|$?

3 Find the function $f(x) = |ax + b|$ which has the graph:

    a   **b**   **c**
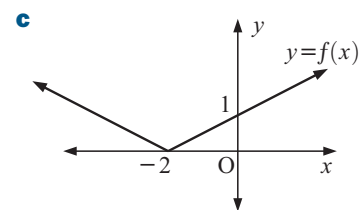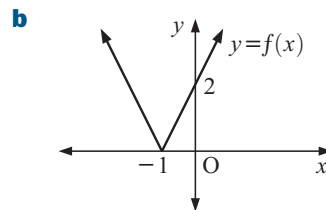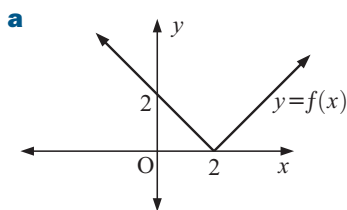
Review set 19A

1 For these functions, find the domain and range:

    a   **b**   **c**
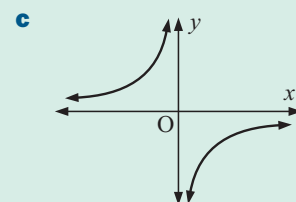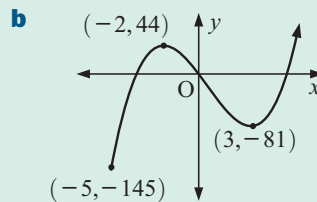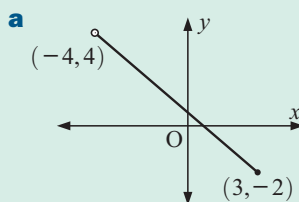
2 a If $f(x) = 5x + 2$, find and interpret $f(-3)$.

b Given $f : x \mapsto 4 - x^2$, find: **i** $f(2)$ **ii** $f(-5)$ **iii** $f(x+1)$.

3 For the following graphs, find the domain and range and decide whether it is the graph of a function:

a
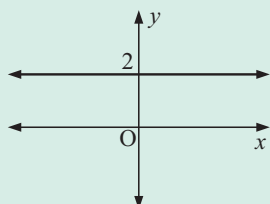
b
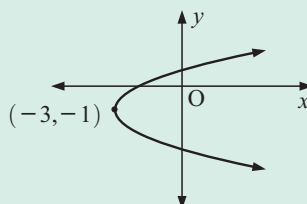
c
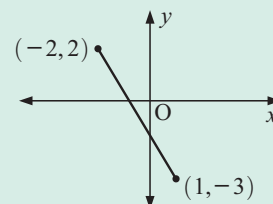
4 If $f(x) = 5 - 2x$ and $g(x) = x^2 + 2x$ find:

  a $f(g(x))$ **b** $g(f(x))$

5 If $f(x) = 5x - 4$, find:

  a $f(-3)$ **b** $f(x-1)$ **c** $f(f(x))$

6 Determine the equation of the following reciprocal graphs:

a
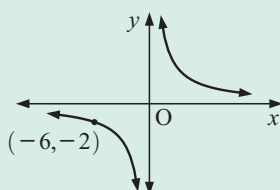
b
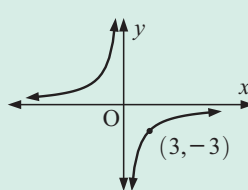
7 Consider the function $f(x) = \dfrac{4}{x+3} - 1$.

  a Use technology to graph $f(x)$.

  b Find the equations of any vertical or horizontal asymptotes.

8 Sketch the graphs of:

  a $f(x) = |x - 4|$ **b** $f(x) = |3x + 4|$

Review set 19B

1 Copy and complete the mapping diagram alongside, and state whether the mapping is one-one, many-one, one-many or many-many.

mapping '$y = x^2 - 2$'
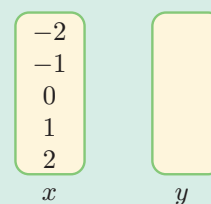
2 For the following graphs, determine:

     i the range and domain   **ii** whether it is the graph of a function

a
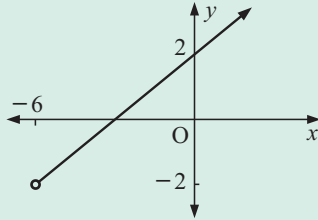
b
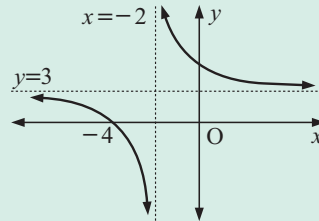
c
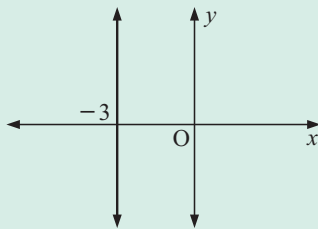
d
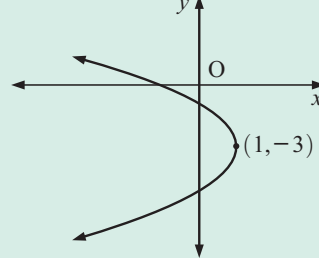
3 If $f(x) = 5x - x^2$, find in simplest form:

     a $f(-3)$ **b** $f(-x)$ **c** $f(x+1)$

4 Sketch the graph of $y = g(x)$ where $g(x) = 3 - 2x$ on the domain $\{x \mid -3 \leqslant x \leqslant 3\}$.
State the range of this function.

5 For $f(x) = x^2 + x - 2$ and $g(x) = 2x + 1$, find the simplest form:

     a $f(g(x))$ **b** $g(g(x))$ **c** $f(g(-2))$

6 Determine the equations of the following reciprocal graphs:

a
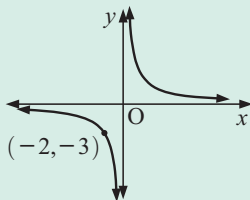
b
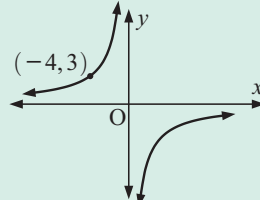
7 Find the function $f(x) = |ax + b|$ given:
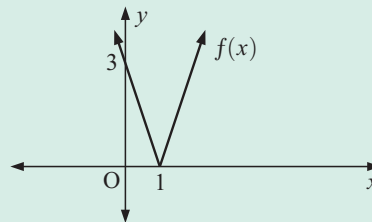
8 By replacing $|x|$ with x for $x \geqslant 0$ and $(-x)$ for $x < 0$, write $y = |x| + 4$ without the modulus sign. Hence graph the function.

Transformation geometry

20

Contents:

Opening problem

Consider the red triangle on the illustrated plane.

a What transformation would map the triangle onto:

   i triangle A   **ii** triangle B

   iii triangle C   **iv** triangle D?

b What single transformation would map triangle A onto triangle C?
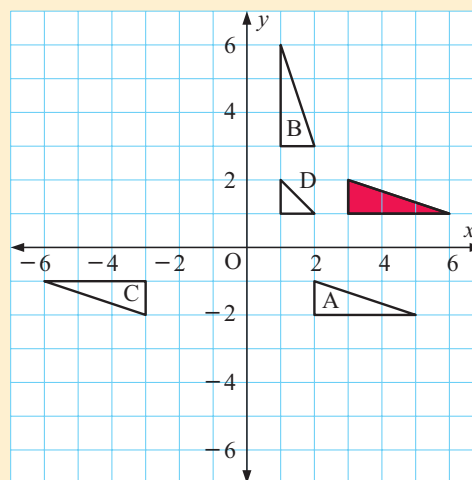
TRANSFORMATIONS

A change in the size, shape, orientation or position of an object is called a **transformation**.

Reflections, rotations, translations and enlargements are all examples of transformations. We can describe these transformations mathematically using **transformation geometry**.

Many trees, plants, flowers, animals and insects are **symmetrical** in some way. Such symmetry results from a reflection, so we can describe symmetry using transformations.
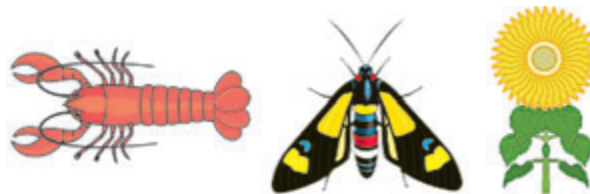
In **transformation geometry** figures are changed (or transformed) in size, shape, orientation or position according to certain rules.

The original figure is called the **object** and the new figure is called the **image**.

We will consider the following **transformations**:

- **Translations** where every point moves a fixed distance in a given direction
- **Reflections** or mirror images
- **Rotations** about a point throught a given angle
- **Enlargements** and **reductions** about a point with a given factor
- **Stretches** with a given invariant line and a given factor.

Here are some examples:

translation (slide)
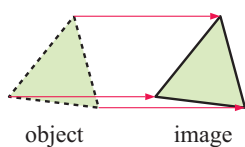
reflection
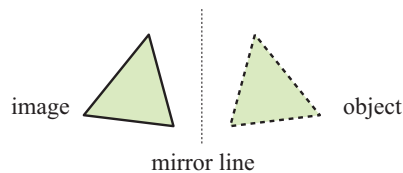
rotation about O **through angle** θ
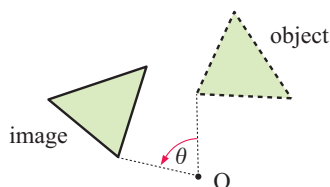
enlargement
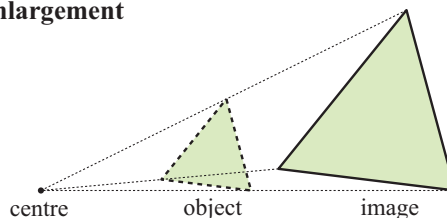
reduction $(k = \frac{1}{2})$
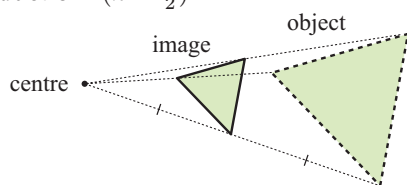
stretch
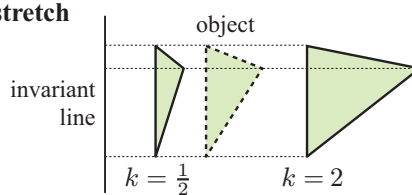
Click on the icon to obtain computer demonstrations of these transformations.

COMPUTER DEMO

A   TRANSLATIONS   *[5.4]*

A **translation** moves an object from one place to another. Every point on the object moves the same distance in the same direction.

If $P(x, y)$ is **translated** h units in the x-direction and k units in the y-direction to become $P'(x', y')$, then $x' = x + h$ and $y' = y + k$.

We write $P(x, y) \xrightarrow{\binom{h}{k}} P'(x + h, y + k)$

where P' is called the **image** of the object P and $\binom{h}{k}$ is called the **translation vector**.

$\left. \begin{array}{l} x' = x + h \\ y' = y + k \end{array} \right\}$ are called the **transformation equations**.
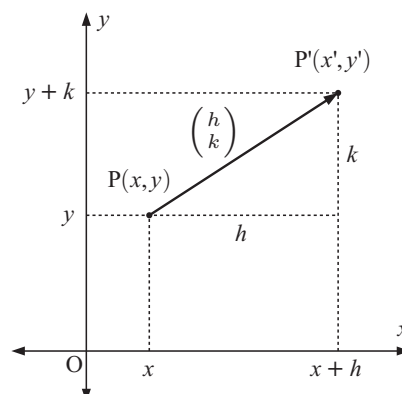
Example 1
◀) *Self Tutor*

Triangle OAB with vertices $O(0, 0)$, $A(2, 3)$ and $B(-1, 2)$ is translated $\binom{3}{2}$.

Find the image vertices and illustrate the object and image.

$O(0, 0) \xrightarrow{\binom{3}{2}} O'(3, 2)$

$A(2, 3) \xrightarrow{\binom{3}{2}} A'(5, 5)$

$B(-1, 2) \xrightarrow{\binom{3}{2}} B'(2, 4)$
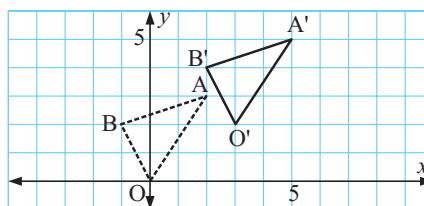
Example 2
◀) *Self Tutor*

On a set of axes draw the line with equation $y = \frac{1}{2}x + 1$.

Find the equation of the image when the line is translated through $\binom{3}{-1}$.

Under a translation, the image of a line is a parallel line, and so will have the same gradient.

$\therefore$ the image of $y = \frac{1}{2}x + 1$ has the form $y = \frac{1}{2}x + c$.

The object contains the point $(0, 1)$ since the y-intercept is 1.

Since $(0, 1) \xrightarrow{\binom{3}{-1}} (3, 0)$, $(3, 0)$ lies on the image.

$\therefore \quad 0 = \frac{1}{2}(3) + c$

$\therefore \quad c = -\frac{3}{2}$

$\therefore$ the equation of the image is $y = \frac{1}{2}x - \frac{3}{2}$.
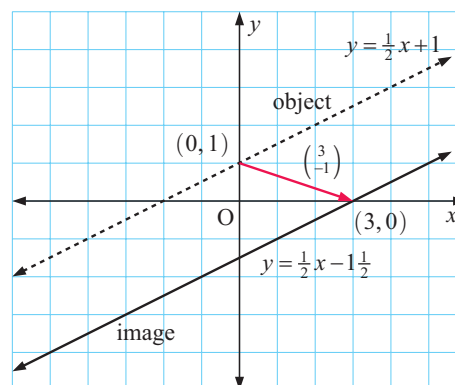
EXERCISE 20A

1 Find the image point when:

     a $(2, -1)$ is translated through $\binom{3}{4}$ **b** $(5, 2)$ is translated through $\binom{-1}{4}$.

2 If $(3, -2)$ is translated to $(3, 1)$, what is the translation vector?

3 What point has image $(-3, 2)$ under the translation $\binom{-3}{1}$?

4 Find the translation vector which maps:

     a A onto E   **b** E onto A

     c A onto C   **d** C onto A

     e B onto E   **f** D onto E

     g E onto C   **h** E onto D

     i D onto B   **j** A onto D.
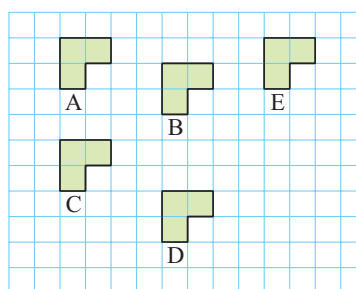
5 Triangle ABC has vertices $A(-1, 3)$, $B(4, 1)$ and $C(0, -2)$.

     a Draw triangle ABC on a set of axes.

     b Translate the figure by the translation vector $\binom{4}{-2}$.

     c State the coordinates of the image vertices A', B' and C'.

     d Through what distance has each point moved?

> When we translate point A, we often label its image A′.

6 What single transformation is equivalent to a translation of $\binom{2}{1}$ followed by a translation of $\binom{3}{4}$?

7 Find the equation of the image line when:

     a $y = 2x + 3$ is translated $\binom{-1}{2}$ **b** $y = \frac{1}{3}x + 2$ is translated $\binom{3}{0}$

     c $y = -x + 2$ is translated $\binom{2}{3}$ **d** $y = -\frac{1}{2}x$ is translated $\binom{-2}{-5}$.

B   **ROTATIONS**   *[5.4, 5.6]*

When $P(x, y)$ moves under a **rotation** about O through an angle of θ to a new position $P'(x', y')$, then $OP = OP'$ and $\widehat{POP'} = \theta$ where **positive** θ is measured **anticlockwise**.

O is the only point which does not move under the rotation.

We will concentrate on rotations of $90°$ (both clockwise and anticlockwise) and $180°$.
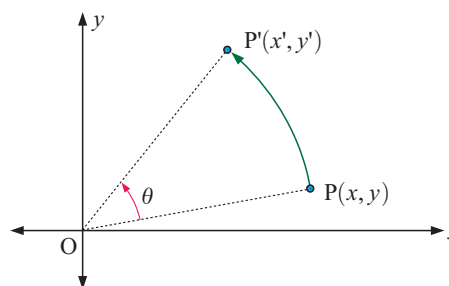
Example 3

◀)) **Self Tutor**

Find the image of the point $(3, 1)$ under a rotation about $O(0, 0)$ which is:

   a 90° anticlockwise   **b** 90° clockwise   **c** 180°.

   a $(3, 1) \rightarrow (-1, 3)$

   b $(3, 1) \rightarrow (1, -3)$

   c $(3, 1) \rightarrow (-3, -1)$

DEMO
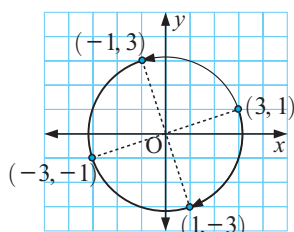
Example 4

◀)) **Self Tutor**

Triangle ABC has vertices $A(-1, 2)$, $B(-1, 5)$ and $C(-3, 5)$. It is rotated clockwise through 90° about $(-2, 0)$. Draw the image of triangle ABC and label it $A'B'C'$.

$A(-1, 2) \rightarrow A'(0, -1)$

$B(-1, 5) \rightarrow B'(3, -1)$

$C(-3, 5) \rightarrow C'(3, 1)$
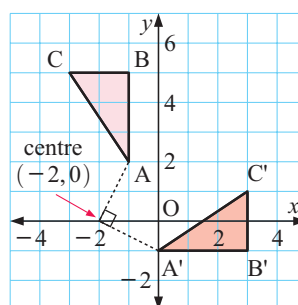
Example 5

◀)) **Self Tutor**

Find the image equation of the line $2x - 3y = -6$ under a clockwise rotation about $O(0, 0)$ through 90°.

$2x - 3y = -6$

   has x-intercept -3 (when $y = 0$)

   and y-intercept 2 (when $x = 0$)

We hence graph $2x - 3y = -6$ {dashed}

Next we rotate these intercepts clockwise through 90°.

The image has x-intercept 2 and y-intercept 3.

The gradient of the image is $m = -\frac{3}{2}$ and the y-intercept $c = 3$.

$\therefore$ the image equation is $y = -\frac{3}{2}x + 3$.
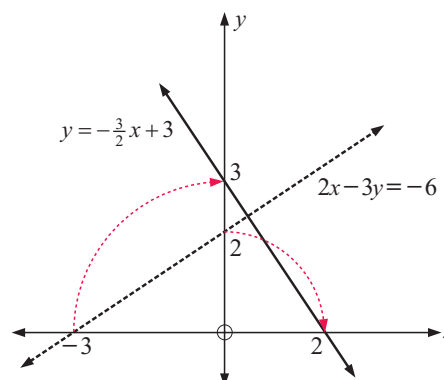
EXERCISE 20B

1 Find the image of the point $(-2, 3)$ under these rotations about the origin O$(0, 0)$:

    a clockwise through 90° **b** anticlockwise through 90° **c** through 180°.

2 Find the image of $(4, -1)$ under these rotations about $(0, 2)$:

    a 90° anticlockwise   **b** through 180° **c** 90° clockwise.

3 Triangle ABC has vertices A$(2, 4)$, B$(4, 1)$ and C$(1, -1)$. It is rotated anticlockwise through 90° about $(0, 3)$.

    a Draw triangle ABC and draw and label its image A$'$B$'$C$'$.

    b Write down the coordinates of the vertices of triangle A$'$B$'$C$'$.

4 Triangle PQR with P$(3, -2)$, Q$(1, 4)$ and R$(-1, 1)$ is rotated about R through 180°.

    a Draw triangle PQR and its image P$'$Q$'$R$'$ **b** Write down the coordinates of P$'$, Q$'$ and R$'$.

5 Find the image equation when:

    a $y = 2x$ is rotated clockwise through 90° about O$(0, 0)$

    b $y = -3$ is rotated anticlockwise through 90° about O$(0, 0)$.

6 Find the single transformation equivalent to a rotation about O$(0, 0)$ through θ^o followed by a rotation about O$(0, 0)$ through ϕ^o.

7 Find the image equation of the line $3x + 2y = 3$ under an anticlockwise rotation of 90° about O$(0, 0)$.

C   REFLECTIONS   [5.4, 5.6]
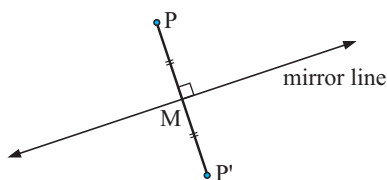
When P(x, y) is **reflected** in the **mirror line** to become P$'(x', y')$, the mirror line perpendicularly bisects PP$'$. This means that PM = P$'$M.

Thus, the mirror line perpendicularly bisects the line segment joining every point on an object with its image.

We will concentrate on reflections:

- in the x-axis or y-axis
- in lines parallel to the axes
- in the lines $y = x$ and $y = -x$.

Example 6   Self Tutor

Find the image of the point $(3, 1)$ under a reflection in:

    a the x-axis   **b** the y-axis   **c** $y = x$ **d** $y = -x$
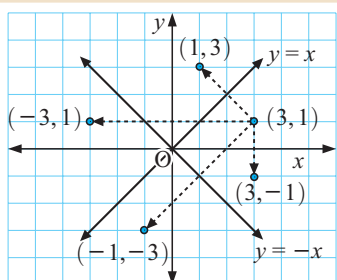
a $(3, 1) \to (3, -1)$
b $(3, 1) \to (-3, 1)$
c $(3, 1) \to (1, 3)$
d $(3, 1) \to (-1, -3)$

> ### Example 7   ◀)) *Self Tutor*
>
> Find the image of the line $y = 2x + 2$ when it is reflected in the line $x = 1$.
>
> Using the axes intercepts, two points which lie on the line $y = 2x + 2$ are $(0, 2)$ and $(-1, 0)$.
>
> When reflected in the line $x = 1$, these points are mapped to $(2, 2)$ and $(3, 0)$ respectively.
>
> $\therefore$ $(2, 2)$ and $(3, 0)$ lie on the image line.
>
> $\therefore$ the image line has gradient $\dfrac{0 - 2}{3 - 2} = -2$
>
>   so its equation is $\quad 2x + y = 2(3) + (0) \quad$ {using $(3, 0)$}
>   or $\quad 2x + y = 6$.
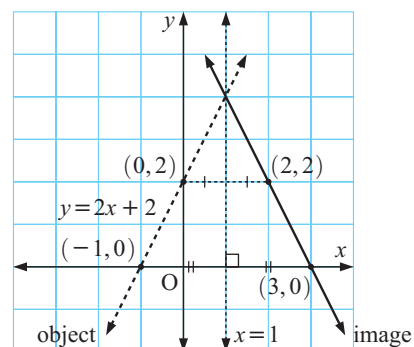
EXERCISE 20C

1   Copy and reflect in the given line:
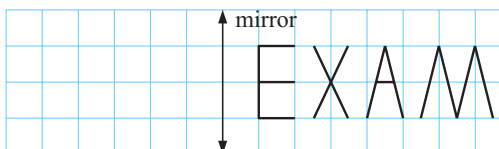
2   Find, by graphical means, the image of the point $(4, -1)$ under a reflection in:

    a   the x-axis   **b**   the y-axis   **c**   the line $y = x$ **d**   the line $y = -x$.

3   Find, by graphical means, the image of the point $(-1, -3)$ under a reflection in:

    a   the y-axis   **b**   the line $y = -x$ **c**   the line $x = 2$ **d**   the line $y = -1$

    e   the x-axis   **f**   the line $x = -3$ **g**   the line $y = x$ **h**   the line $y = 2$.

4   Copy the graph given. Reflect T in:

    a   the y-axis and label it U

    b   the line $y = -1$ and label it V

    c   the line $y = -x$ and label it W.
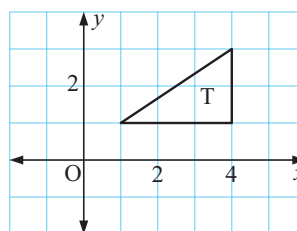
5   Find the image of:

    a $(2, 3)$ under a reflection in the x-axis followed by a translation of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

    b $(4, -1)$ under a reflection in $y = -x$ followed by a translation of $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

    c $(-1, 5)$ under a reflection in the y-axis followed by a reflection in the x-axis followed by a translation of $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

    d $(3, -2)$ under a reflection in $y = x$ followed by a translation of $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

    e $(4, 3)$ under a translation of $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ followed by a reflection in the x-axis.

6   Find the image of the line $y = -x + 3$ when it is reflected in the line $y = -1$.

7 Construct a set of coordinate axes with x and y ranging from -5 to 5.

   a Draw the lines $y = x$ and $y = -x$.

   b Draw triangle T with vertices $(1, 1)$, $(3, 1)$ and $(2, 2)$.

   c   **i** Reflect T in the x-axis and label its image U.

       ii Reflect U in the line $y = -x$ and label its image V.

      iii Describe fully the single transformation which maps T onto V directly.

   d   **i** Reflect T in the line $y = x$ and label it G.

       ii Reflect G in the line $y = -x$ and label it H.

      iii Describe fully the single transformation that maps T onto H directly.

8 Find the image of:

   a $(2, 3)$ under a clockwise 90° rotation about $O(0, 0)$ followed by a reflection in the x-axis

   b $(-2, 5)$ under a reflection in $y = -x$ followed by a translation of $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

   c $(4, -1)$ under a reflection in $y = x$ followed by a rotation of 180°.

9   **a** Draw the image of the line $y = 2x + 3$ under a translation of $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ followed by a reflection in $y = -1$.

   b State the equation of the image.

10 Find the equation of the image of $y = 2x$ when it is reflected in:

   a the x-axis   **b** the line $y = x$ **c** the line $x = 1$ **d** the line $y = 3$.

D *ENLARGEMENTS AND REDUCTIONS* **[5.4]**

The diagram below shows the **enlargement** of triangle PQR with centre point C and scale factor $k = 3$.
P′, Q′ and R′ are located such that $CP' = 3 \times CP$, $CQ' = 3 \times CQ$, and $CR' = 3 \times CR$. The image
P′Q′R′ has sides which are 3 times longer than those of the object PQR.
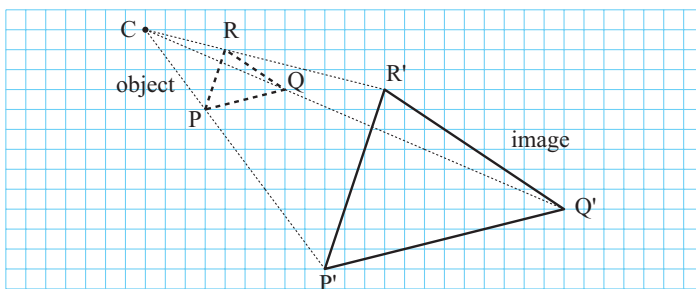
Alongside is a **reduction** of triangle KLM with centre C
and scale factor $k = \frac{1}{2}$.

To obtain the image, the distance from C to each point on
the object is halved.
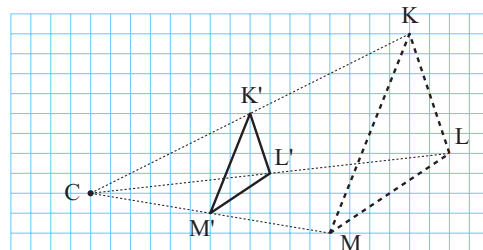
ENLARGEMENTS WITH CENTRE THE ORIGIN

Suppose $P(x, y)$ moves to $P'(x', y')$ such that P' lies on the line OP, and $OP' = kOP$.

We call this **an enlargement with centre O(0, 0) and scale factor k**.
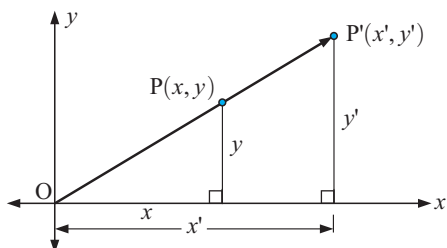
From the similar triangles

$$\frac{x'}{x} = \frac{y'}{y} = \frac{OP'}{OP} = k$$

$$\therefore \quad \begin{cases} x' = kx \\ y' = ky \end{cases}$$

Under an enlargement with centre $O(0, 0)$ and scale factor k, $(x, y) \rightarrow (kx, ky)$.

Example 8   🔊 Self Tutor

Consider the triangle ABC with vertices $A(1, 1)$, $B(4, 1)$ and $C(1, 4)$.

Find the position of the image of $\triangle ABC$ under:

a an enlargement with centre $O(0, 0)$ and scale factor $k = 2$

b a reduction with centre $O(0, 0)$ and scale factor $k = \frac{1}{2}$.
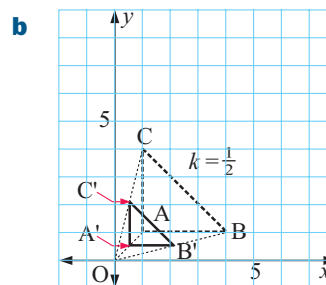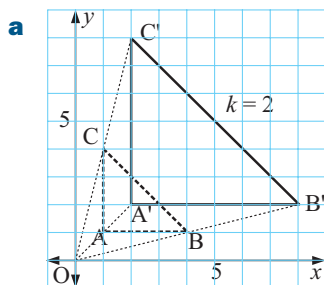
We can see from the examples above that:

If $k > 1$, the image figure is an **enlargement** of the object.

If $0 < k < 1$, the image figure is a **reduction** of the object.

EXERCISE 20D

1 Copy each diagram onto squared paper and enlarge or reduce with centre C and the scale factor k given:

a $k = 2$ **b** $k = 3$ **c** $k = \frac{1}{2}$
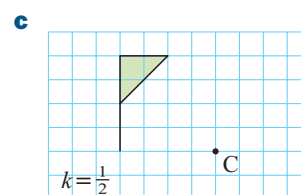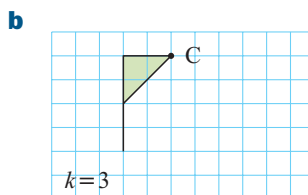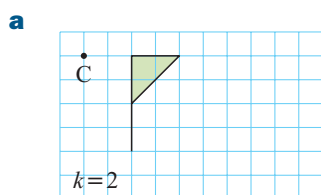
2 Copy triangle T onto squared paper.

   a Enlarge T about centre C$(7, 2)$ with scale factor $k = 2$.

   b Reduce T about centre D$(4, -3)$ with scale factor $k = \frac{1}{2}$.
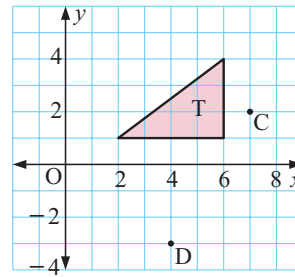
3 Find the image of the point:

   a $(3, 4)$ under an enlargement with centre O$(0, 0)$ and scale factor $k = 1\frac{1}{2}$

   b $(-1, 4)$ under a reduction with centre C$(2, -2)$ and scale factor $k = \frac{2}{3}$.

4 Find the equation of the image when:

   a $y = 2x$ is: **i** enlarged with centre O$(0, 0)$ and scale factor $k = 3$

                   ii reduced with centre O$(0, 0)$ and scale factor $k = \frac{1}{3}$.

   b $y = -x + 2$ is: **i** enlarged with centre O$(0, 0)$ and scale factor $k = 4$

                   ii reduced with centre O$(0, 0)$ and scale factor $k = \frac{2}{3}$.

   c $y = 2x + 3$ is: **i** enlarged with centre $(2, 1)$ and scale factor $k = 2$

                   ii reduced with centre $(2, 1)$ and scale factor $k = \frac{1}{2}$.

E   STRETCHES   [5.4]

In a **stretch** we enlarge or reduce an object in one direction only.

Stretches are defined in terms of a **stretch factor** and an **invariant line**.
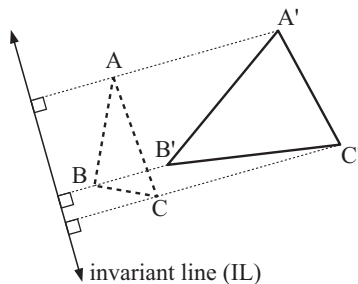
In the diagram alongside, triangle A′B′C′ is a stretch of triangle ABC with scale factor $k = 3$ and invariant line IL.

For every point on the image triangle A′B′C′, the distance from the invariant line is 3 times further away than the corresponding point on the object.

The invariant line is so named because any point along it will not move under a stretch.

STRETCHES WITH INVARIANT x-AXIS

Suppose P(x, y) moves to P′(x', y') such that P′ lies on the line through N$(x, 0)$ and P, and NP′ $= k$NP.

We call this a **stretch with invariant x-axis** and scale factor k.

> For a stretch with invariant x-axis and scale factor k,
>
> $(x, y) \rightarrow (x, ky)$.

Example 9
🔊 *Self Tutor*

Consider the triangle ABC with A(1, 1), B(5, 1) and C(1, 4) under a stretch with invariant x-axis and scale factor: **a** $k = 2$ **b** $k = \frac{1}{2}$.

Find the position of the image of $\triangle$ABC under each stretch.

a

b
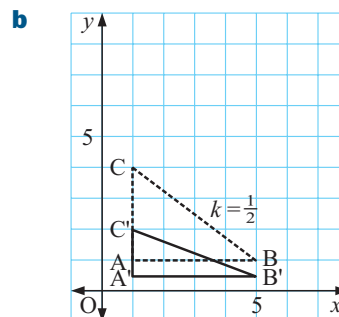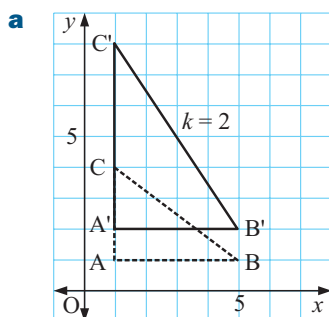
STRETCHES WITH INVARIANT y-AXIS

Suppose P(x, y) moves to P$'(x', y')$ such that P$'$ lies on the line through N$(0, y)$ and P, and   NP$' = k$NP.

We call this a **stretch with invariant y-axis** and scale factor k.

For a stretch with invariant y-axis and scale factor k,

$$(x, y) \rightarrow (kx, y).$$
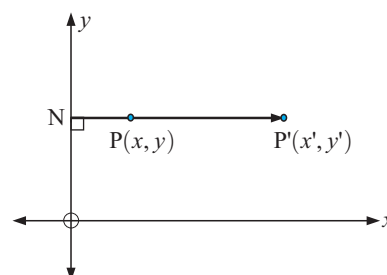
Example 10
🔊 *Self Tutor*

Consider the triangle ABC with A(1, 1), B(5, 1) and C(1, 4) under a stretch with invariant y-axis and scale factor: **a** $k = 2$ **b** $k = \frac{1}{2}$.

Find the position of the image of $\triangle$ABC under each stretch.

a

b
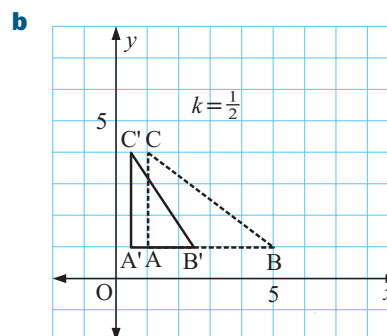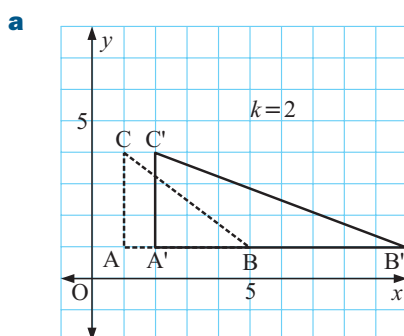
EXERCISE 20E

1 Copy these diagrams and perform the stretch with the given invariant line IL and scale factor k:

a
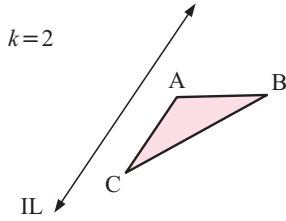
b
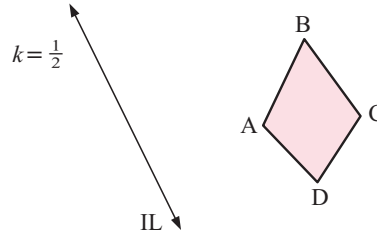
c
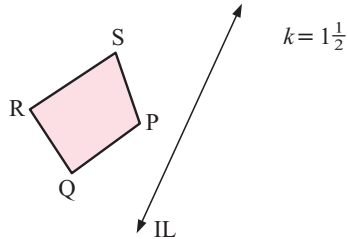
d
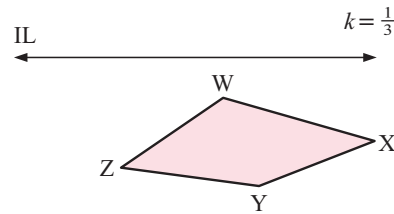
2 Copy and perform the stretch with given invariant line IL and scale factor k:

a
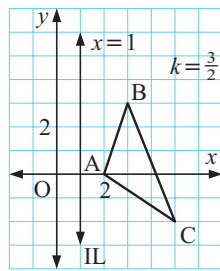
b
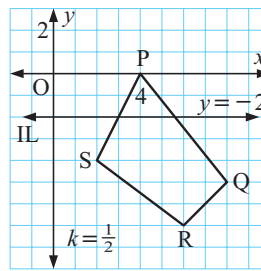
3 Find the image of:

     a $(3, -1)$ under a stretch with invariant x-axis and scale factor 4

     b $(4, 5)$ under a stretch with invariant x-axis and scale factor 2

     c $(-2, 1)$ under a stretch with invariant y-axis and scale factor $\frac{1}{2}$

     d $(3, -4)$ under a stretch with invariant y-axis and scale factor $\frac{3}{2}$

     e $(2, 3)$ under a stretch with invariant line $y = -x$ and scale factor $\frac{1}{2}$.

4 Find the image of triangle ABC if A$(1, 2)$, B$(4, 1)$ and C$(2, 5)$ are its vertices as it is stretched with:

     a invariant line $y = 1$ and scale factor $k = 2$

     b invariant line $x = -1$ and scale factor $k = \frac{1}{2}$.

5   **a** The object rectangle OABC is mapped onto the image rectangle OA′B′C′.

       Describe fully the single transformation which has occurred.

b
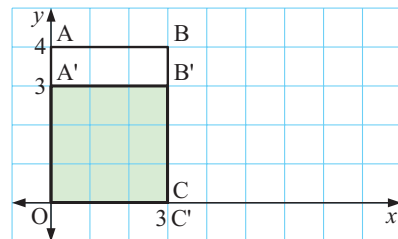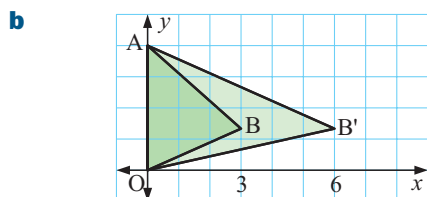
The object triangle OAB is mapped onto the image triangle OAB′.

Describe fully the single transformation which has occurred.

c If ABCD is mapped onto A′B′C′D′, describe fully the single transformation which has occurred.
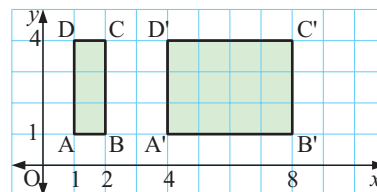
d

If ABCD is mapped onto A′B′C′D′ describe fully the single transformation which has occurred.
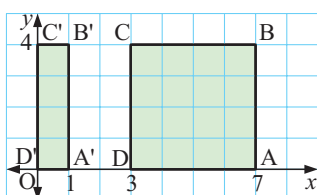
6 Find the image equation when:

a $y = 2x$ is subjected to a stretch with invariant x-axis and scale factor $k = 3$

b $y = \frac{3}{2}x$ is subjected to a stretch with invariant y-axis and scale factor $k = \frac{2}{3}$

c $y = \frac{1}{2}x + 2$ is subjected to a stretch with invariant line $x = 1$ and scale factor $k = 2$.

Discussion   *Invariant points*

Invariant points are points which do not move under a transformation.

What points would be invariant under:

- a translation
- a reflection in a mirror line
- an enlargement or reduction about O(0, 0) with scale factor k?

- a rotation about O(0, 0)
- a stretch

F TRANSFORMING FUNCTIONS   [3.8]

In this section we consider the effect of transforming the graph of $y = f(x)$ into $y = f(x) + k$, $y = f(x + k)$ and $y = kf(x)$ where $k \in \mathbb{Z}$, $k \neq 0$.

Discovery

In this discovery we will graph many different functions. To help with this you can either click on the icon and use the graphing package, or else follow the instructions on page **22** to graph the functions on your calculator.

GRAPHING PACKAGE

What to do:

1 On the same set of axes graph:

 a $y = \dfrac{1}{x}$ **b** $y = \dfrac{1}{x} + 2$ **c** $y = \dfrac{1}{x} - 3$ **d** $y = \dfrac{1}{x} + 5$

 What transformation maps $y = \dfrac{1}{x}$ onto $y = \dfrac{1}{x} + k$?

2 On the same set of axes graph:

 a $y = \dfrac{1}{x}$ **b** $y = \dfrac{1}{x+2}$ **c** $y = \dfrac{1}{x-3}$ **d** $y = \dfrac{1}{x+4}$

 What transformation maps $y = \dfrac{1}{x}$ onto $y = \dfrac{1}{x+k}$?

3 On the same set of axes graph:

 a $y = \dfrac{1}{x}$ **b** $y = \dfrac{2}{x}$ **c** $y = \dfrac{3}{x}$ **d** $y = \dfrac{-1}{x}$ **e** $y = \dfrac{-4}{x}$

 What transformation maps $y = \dfrac{1}{x}$ onto $y = \dfrac{k}{x}$?

You should have discovered that:

- $y = f(x)$ maps onto $y = f(x) + k$ under a **vertical translation** of $\binom{0}{k}$
- $y = f(x)$ maps onto $y = f(x + k)$ under a **horizontal translation** of $\binom{-k}{0}$
- $y = f(x)$ maps onto $y = kf(x)$ under a stretch with invariant x-axis and scale factor k.

Example 11   ◄ʲ) *Self Tutor*

Consider $f(x) = \frac{1}{2}x + 1$. On separate sets of axes graph:

 a $y = f(x)$ and $y = f(x + 2)$ **b** $y = f(x)$ and $y = f(x) + 2$
 c $y = f(x)$ and $y = 2f(x)$ **d** $y = f(x)$ and $y = -f(x)$
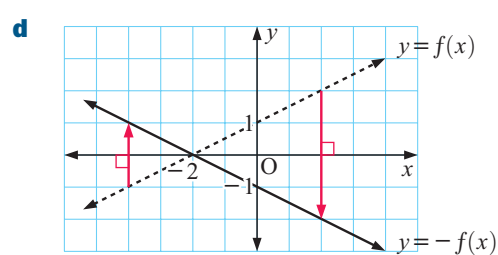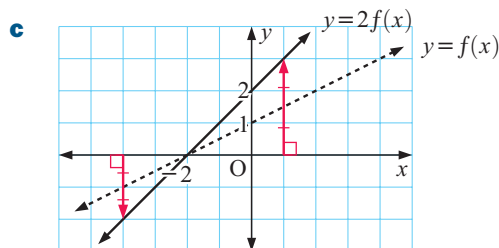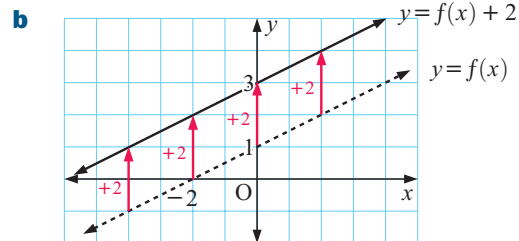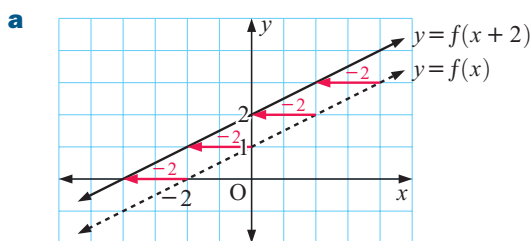
To help draw the graphs in the following exercise, you may wish to use the **graphing package** or your **graphics calculator**.
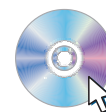
EXERCISE 20F

1 Consider $f(x) = 3x - 2$.

    a On the same grid, graph $y = f(x)$, $y = f(x) + 4$ and $y = f(x + 4)$. Label each graph.

    b What transformation on $y = f(x)$ has occurred in each case in **a**?

2 Consider $f(x) = 2^x$.

    a On the same grid, graph $y = f(x)$, $y = f(x) - 1$ and $y = f(x - 3)$. Label each graph.

    b Describe fully the single transformation which maps the graph of:

        i $y = f(x)$ onto $y = f(x - 3)$ **ii** $y = f(x) - 1$ onto $y = f(x - 3)$.

3 Consider $g(x) = \left(\frac{1}{2}\right)^x$.

    a On the same set of axes, graph $y = g(x)$ and $y = g(x) - 1$.

    b Write down the equation of the asymptote of $y = g(x) - 1$.

    c Repeat **a** and **b** with $g(x) = \left(\frac{1}{2}\right)^{x-1}$.

4 Consider $f(x) = 2x - 1$.

    a Graph $y = f(x)$ and $y = 3f(x)$ on the same set of axes.

    b What point(s) are invariant under this transformation?

5 Consider $h(x) = x^3$.

    a On the same set of axes, graph $y = h(x)$, $y = 2h(x)$ and $y = \frac{1}{2}h(x)$, labelling each graph clearly.

    b Describe fully the single transformation which maps the graph of $y = 2h(x)$ on $y = \frac{1}{2}h(x)$.

6 Consider $f(x) = x^2 - 1$.

    a Graph $y = f(x)$ and state its axes intercepts.

    b Graph the functions:

        i $y = f(x) + 3$ **ii** $y = f(x - 1)$ **iii** $y = 2f(x)$ **iv** $y = -f(x)$

    c What transformation on $y = f(x)$ has occurred in each case in **b**?

    d On the same set of axes graph $y = f(x)$ and $y = -2f(x)$. Describe the transformation.

    e What points on $y = f(x)$ are invariant when $y = f(x)$ is transformed to $y = -2f(x)$?

7 On each of the following $f(x)$ is mapped onto $g(x)$ using a single transformation.

        i Describe the transformation fully.   **ii** Write $g(x)$ in terms of $f(x)$.
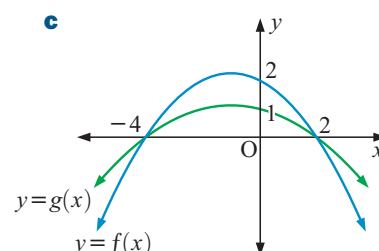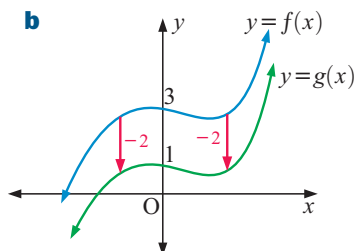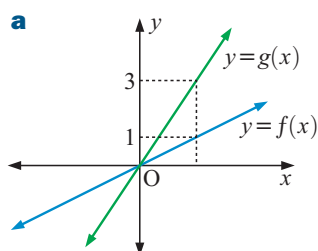
8 For the following, copy and draw the required function:

a
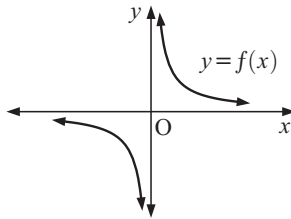
sketch $y = f(x-2)$

b
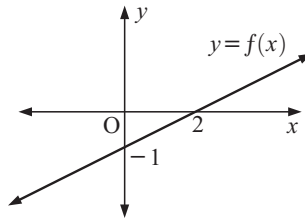
sketch $y = f(x) + 2$

c
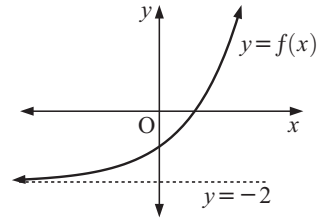
sketch $y = \frac{1}{2} f(x)$

d
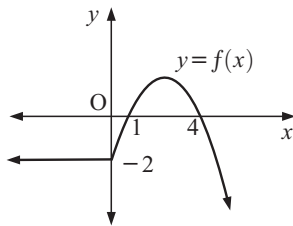
sketch $y = 2f(x)$

e
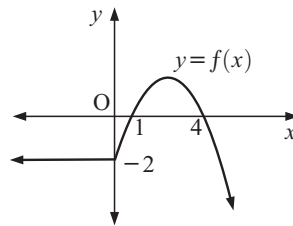
sketch $y = -2f(x)$

f
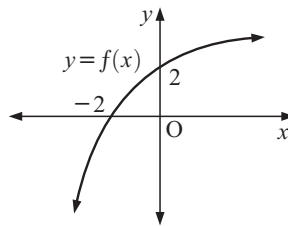
sketch $y = \frac{1}{2} f(x)$

9 The graph of $y = f(x)$ is shown alongside.
On the same set of axes, graph:

    a $y = f(x)$ **b** $y = -f(x)$ **c** $y = \frac{3}{2} f(x)$

    d $y = f(x) + 2$ **e** $y = f(x-2)$

Label each graph clearly.
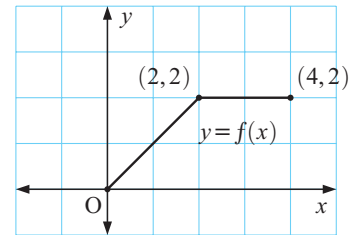
10 Consider $f(x) = x^2 - 4$, $g(x) = 2f(x)$ and $h(x) = f(2x)$.

    a Find $g(x)$ and $h(x)$ in terms of x.

    b Graph $y = f(x)$, $y = g(x)$ and $y = h(x)$ on the same set of axes, using a graphics calculator if necessary.

    c Describe fully the single transformation which maps the graph of $y = f(x)$ onto the graph of $y = g(x)$.

    d Under the mapping in **c**, which points are invariant?

    e Find the *zeros* of $h(x)$, which are the values of x for which $h(x)$ is zero.

    f Describe fully the single transformation which maps the graph of $y = f(x)$ onto the graph of $y = h(x)$.

G   *THE INVERSE OF A TRANSFORMATION*   *[5.5]*

If a transformation maps an object onto its image, then the **inverse transformation** maps the image back onto the object.

An enlargement with centre O(0, 0) and scale factor $k = 2$ maps $\triangle ABC$ onto $\triangle A'B'C'$.

$\triangle A'B'C'$ is mapped back onto $\triangle ABC$ under a reduction with centre O(0, 0) and scale factor $k = \frac{1}{2}$.

This is an *inverse transformation*.
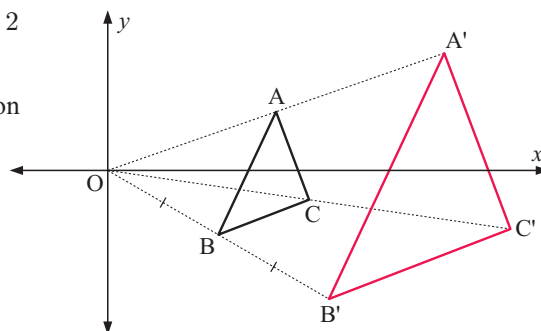
EXERCISE 20G

1 Describe fully the inverse transformation for each of the following transformations. You may wish to draw a triangle ABC with vertices A(3, 0), B(4, 2) and C(1, 3) to help you.

   a a reflection in the y-axis   **b** a rotation about O(0, 0) through 180°

   c a translation of $\binom{3}{0}$ **d** a translation of $\binom{0}{-2}$

   e a translation of $\binom{3}{-1}$ **f** a 90° clockwise rotation about O(0, 0)

   g an enlargement, centre O(0, 0), scale factor 4

   h a reduction, centre O(0, 0), scale factor $\frac{1}{3}$

   i a reflection in $y = -x$

   j a stretch with invariant x-axis and scale factor $\frac{3}{2}$

   k a reflection in $y = -2$

   l a stretch with invariant y-axis and scale factor $\frac{1}{2}$

   m a rotation about point P, clockwise through 43°.

H   COMBINATIONS OF TRANSFORMATIONS [5.6]

In previous exercises we have already looked at the single transformation equivalent to one transformation followed by another.

We now take a more formal approach to a combination of transformations.

We refer to a particular transformation using a capital letter and use the following notation:

> We represent 'transformation G **followed by** transformation H' as HG.

Notice the reversal of order here. We have seen similar notation to this in composite functions, where $f(g(x))$ is found by first finding $g(x)$, then applying f to the result.

Example 12   ◀ঠ) *Self Tutor*

Consider triangle ABC with vertices A(2, 1), B(4, 1) and C(4, 2).

Suppose R is a reflection in the line $y = -x$ and S is a rotation of 90° clockwise about O(0, 0).

Use $\triangle ABC$ to help find the single transformation equivalent to:

   a RS   **b** SR

a S maps $\triangle$ABC to (1),
R maps (1) to (2).
So, RS maps $\triangle$ABC to (2).
This is a reflection in the x-axis, so
RS is a reflection in the x-axis.
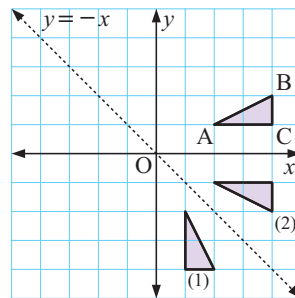
b R maps $\triangle$ABC to (3),
S maps (3) to (4).
So, SR maps $\triangle$ABC to (4).
This is a reflection in the y-axis, so
SR is a reflection in the y-axis.
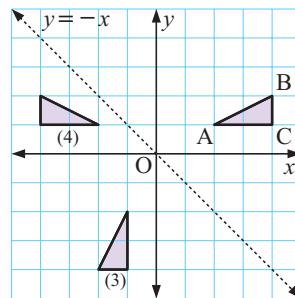
EXERCISE 20H

1 For triangle ABC with A(2, 1), B(4, 2), C(4, 1):

a If R is a 90° anticlockwise rotation about O(0, 0) and S is a stretch with invariant y-axis and scale factor $k = 2$, draw the images of: **i** RS **ii** SR.

b If M is a reflection in the line $y = x$ and E is an enlargement with centre O(0, 0) and scale factor 2, draw the images of: **i** ME **ii** EM.

c If R is a reflection in the x-axis and M is a reflection in the line $y = -x$, what single transformation is equivalent to: **i** RM **ii** MR?

d If T_1 is a clockwise rotation about O(0, 0) through 180° and T_2 is a reflection in the y-axis, what single transformation is equivalent to: **i** T_2T_1 **ii** T_1T_2?

2 Use triangle 0 as the object shape and consider the following transformations:

T_0: leave unchanged
T_1: reflect in the line $y = x$
T_2: rotate 90° anticlockwise about O(0, 0)
T_3: reflect in the y-axis
T_4: rotate 180° about O(0, 0)
T_5: reflect in the line $y = -x$
T_6: rotate 90° clockwise about O(0, 0)
T_7: reflect in the x-axis.

a Find the single transformation equivalent to:

i T_1T_2 **ii** T_2T_1 **iii** T_4T_2 **iv** T_7T_7 **v** T_5T_4
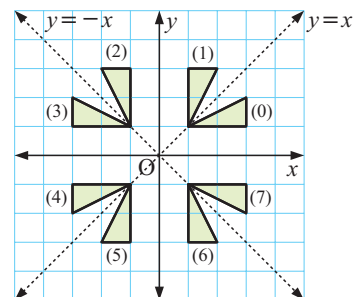
b Copy and complete the table to indicate the result of combining the first transformation with the second transformation.

second transformation

	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7
T_0	T_0		T_2	T_3				
T_1				T_3				
T_2								
T_3								
T_4			T_6					
T_5								
T_6								
T_7	T_7							

first transformation

This is T_4 followed by T_2, or $T_2 T_4$.

Review set 20A

1 Find the image of:

a $(2, -5)$ under a reflection in **i** the x-axis **ii** the line $y = x$ **iii** the line $x = 3$

b $(-1, 4)$ under a clockwise rotation of 90° about **i** $O(0, 0)$ **ii** $A(2, 1)$

c $(5, -2)$ under a translation of $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$

d $(3, -1)$ under an enlargement with centre $O(0, 0)$ and scale factor 2

e $(3, -1)$ under a stretch with invariant x-axis and scale factor $2\frac{1}{2}$

f $(3, -1)$ under a stretch with invariant line $x = 2$ and scale factor $\frac{1}{2}$.

2 Find the image of $(6, 2)$ under a 180° rotation about $O(0, 0)$ followed by a translation of $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

3 Find the equation of the image when $y = 2x - 1$ is:

a translated $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

b reflected in the x-axis

c rotated 90° clockwise about $O(0, 0)$

d stretched with invariant y-axis, scale factor of 2.

4
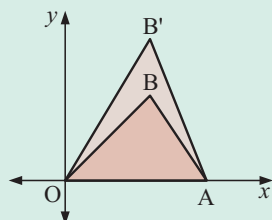
The object OAB is mapped onto OAB′. Describe fully the transformation if B is $(3, 3)$ and B′ is $(3, 5)$.

5 Consider $f(x) = 2x - 1$. On separate axes, graph:

a $y = f(x)$ and $y = f(x - 2)$

b $y = f(x)$ and $y = f(x) - 2$

c $y = f(x)$ and $y = 2f(x)$

d $y = f(x)$ and $y = -f(x)$.

6 Copy the graph alongside.
Draw on the same axes, the graphs of:

a $y = f(x) + 3$

b $y = -2f(x)$.
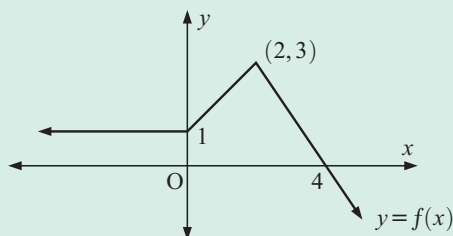
7 Find the *inverse* transformation of:

a a translation of $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

b a reflection in the line $y = -x$

c a stretch with invariant y-axis and scale factor 2.

8 Suppose R is a clockwise rotation of 90° about O(0, 0) and M
is a reflection in the line $y = -x$.
Use a triangle on a set of axes like the one given to find the
single transformation equivalent to:

    a RM   **b** MR.
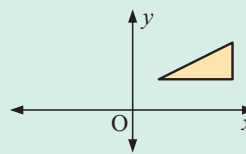
Review set 20B

1 Find the image of $(3, -2)$ under a reflection in:

    a the x-axis   **b** the line $y = -x$ **c** the line $y = 4$.

2 Find the image of $(3, -7)$ under:

    a a translation of $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ followed by a reflection in the y-axis

    b a reflection in the x-axis followed by a reflection in the line $y = -x$.

3 Find the image of:

    a $(3, 5)$ under an enlargement with centre O(0, 0) and scale factor 3

    b $(-2, 3)$ under a stretch with invariant y-axis and scale factor 2

    c $(-5, -3)$ under a stretch with invariant x-axis and scale factor $\frac{1}{2}$.

4 Find the equation of the image of $y = -2x + 1$ under:

    a a translation of $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ **b** a reflection in the line $y = x$

    c a 90° anticlockwise rotation about O(0, 0)

    d a stretch with invariant x-axis and scale factor $\frac{1}{2}$.

5
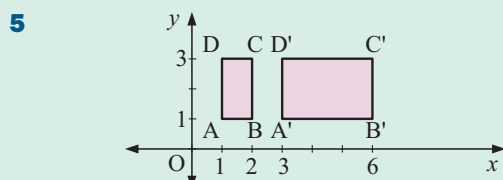
The object ABCD is mapped onto A′B′C′D′.
Describe the transformation fully.

6 Consider $f(x) = \frac{1}{2}x + 3$. On separate axes, graph:

    a $y = f(x)$ and $y = f(x + 1)$ **b** $y = f(x)$ and $y = f(x) + 1$

    c $y = f(x)$ and $y = -f(x)$ **d** $y = f(x)$ and $y = \frac{1}{2}f(x)$

7 $y = f(x)$ is mapped onto $y = g(x)$ by a single
transformation.
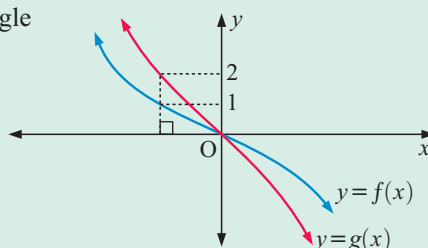
    a Describe the transformation fully.

    b Write $g(x)$ in terms of $f(x)$.

8 Find the *inverse* transformation of:

    a a reflection in the x-axis   **b** a 180° rotation about O(0, 0)

    c an enlargement about O(0, 0) with scale factor $k = 3$.

9 M is a reflection in the y-axis and R is an anticlockwise rotation through 90° about the origin
O(0, 0). Find the single transformation which is equivalent to: **a** MR   **b** RM.

Quadratic equations and functions

21

Contents:

Opening problem

A cannonball fired vertically upwards from ground level has height given by the relationship $H = 36t - 3t^2$ metres, where t is the time in seconds after firing.

Things to think about:

1 If we sketch a graph of the height H against the time t after firing, what shape will result?

2 How long would it take for the cannonball to reach its maximum height?

3 What would be the maximum height reached?

4 How long would the person who fired the cannonball have to clear the area?

This chapter is devoted to **quadratics**, which are expressions of the form $ax^2 + bx + c$ where x is a variable and a, b and c are constants.

We will consider **quadratic equations** and also **quadratic functions** which take the shape of a **parabola**.

A   QUADRATIC EQUATIONS   [2.10]

Equations of the form $ax + b = 0$ where $a \neq 0$ are called **linear equations** and have *only one* solution. For example, $3x - 2 = 0$ is the linear equation with $a = 3$ and $b = -2$. It has the solution $x = \frac{2}{3}$.

> Equations of the form $ax^2 + bx + c = 0$ where $a \neq 0$ are called **quadratic equations**. They may have *two, one* or *zero* solutions.

Here are some simple quadratic equations which clearly show the truth of this statement:

Equation	$ax^2 + bx + c = 0$ form	a	b	c	Solutions	
$x^2 - 4 = 0$	$x^2 + 0x - 4 = 0$	1	0	-4	$x = 2$ or $x = -2$	**two**
$(x - 2)^2 = 0$	$x^2 - 4x + 4 = 0$	1	-4	4	$x = 2$	**one**
$x^2 + 4 = 0$	$x^2 + 0x + 4 = 0$	1	0	4	none as x^2 is always $\geqslant 0$	**zero**

Now consider the example $x^2 + 3x - 10 = 0$.

$$\begin{array}{llll}
\text{If} \quad x = 2, & x^2 + 3x - 10 & \text{and if} \quad x = -5, & x^2 + 3x - 10 \\
& = 2^2 + 3 \times 2 - 10 & & = (-5)^2 + 3 \times (-5) - 10 \\
& = 4 + 6 - 10 & & = 25 - 15 - 10 \\
& = 0 & & = 0
\end{array}$$

$x = 2$ and $x = -5$ both satisfy the equation $x^2 + 3x - 10 = 0$, so we say they are both **solutions**.

We will discuss several methods for solving quadratic equations, and apply them to practical problems.

EQUATIONS OF THE FORM $x^2 = k$

$x^2 = k$ is the simplest form of a quadratic equation.

Consider the equation $x^2 = 7$.

$$\begin{array}{ll}
\text{Now} & \sqrt{7} \times \sqrt{7} = 7, \quad \text{so} \quad x = \sqrt{7} \quad \text{is one solution,} \\
\text{and} & (-\sqrt{7}) \times (-\sqrt{7}) = 7, \quad \text{so} \quad x = -\sqrt{7} \quad \text{is also a solution.} \\
\text{Thus,} & \text{if} \quad x^2 = 7, \quad \text{then} \quad x = \pm\sqrt{7}.
\end{array}$$

> $\pm\sqrt{7}$ is read as 'plus or minus the square root of 7'

> If $x^2 = k$ then $\begin{cases} x = \pm\sqrt{k} & \text{if } k > 0 \\ x = 0 & \text{if } k = 0 \\ \text{there are } \textbf{no real solutions} & \text{if } k < 0. \end{cases}$

Example 1   ◀)) Self Tutor

Solve for x: **a** $2x^2 + 1 = 15$ **b** $2 - 3x^2 = 8$

a $2x^2 + 1 = 15$
$\therefore \quad 2x^2 = 14$ {subtracting 1 from both sides}
$\therefore \quad x^2 = 7$ {dividing both sides by 2}
$\therefore \quad x = \pm\sqrt{7}$

---

b $2 - 3x^2 = 8$

$\therefore \quad -3x^2 = 6$ {subtracting 2 from both sides}

$\therefore \quad x^2 = -2$ {dividing both sides by -3}

which has no solutions as x^2 cannot be < 0.

Example 2   ◀ɔ) *Self Tutor*

Solve for x:

a $(x - 3)^2 = 16$ **b** $(x + 2)^2 = 11$

a $(x - 3)^2 = 16$ **b** $(x + 2)^2 = 11$

$\therefore \quad x - 3 = \pm\sqrt{16}$ $\therefore \quad x + 2 = \pm\sqrt{11}$

$\therefore \quad x - 3 = \pm 4$ $\therefore \quad x = -2 \pm \sqrt{11}$

$\therefore \quad x = 3 \pm 4$

$\therefore \quad x = 7 \text{ or } -1$

> For equations of the form $(x \pm a)^2 = k$ we do not expand the LHS.

EXERCISE 21A

1 Solve for x:

a $x^2 = 100$ **b** $2x^2 = 50$ **c** $5x^2 = 20$

d $6x^2 = 54$ **e** $5x^2 = -45$ **f** $7x^2 = 0$

g $3x^2 - 2 = 25$ **h** $4 - 2x^2 = 12$ **i** $4x^2 + 2 = 10$

2 Solve for x:

a $(x - 1)^2 = 9$ **b** $(x + 4)^2 = 16$ **c** $(x + 2)^2 = -1$

d $(x - 4)^2 = 5$ **e** $(x - 6)^2 = -4$ **f** $(x + 2)^2 = 0$

g $(2x - 5)^2 = 0$ **h** $(3x + 2)^2 = 4$ **i** $\frac{1}{3}(2x + 3)^2 = 2$

B   THE NULL FACTOR LAW   [2.10]

For quadratic equations which are not of the form $x^2 = k$, we need an alternative method of solution. One method is to **factorise** the quadratic into the product of linear factors and then apply the **Null Factor law**:

> When the product of two (or more) numbers is zero, then *at least one* of them must be zero.
>
> If $ab = 0$ then $a = 0$ or $b = 0$.

Example 3   ◀ɔ) *Self Tutor*

Solve for x using the Null Factor law:

a $3x(x - 5) = 0$ **b** $(x - 4)(3x + 7) = 0$

a $3x(x - 5) = 0$ **b** $(x - 4)(3x + 7) = 0$

$\therefore \quad 3x = 0 \text{ or } x - 5 = 0$ $\therefore \quad x - 4 = 0 \text{ or } 3x + 7 = 0$

$\therefore \quad x = 0 \text{ or } 5$ $\therefore \quad x = 4 \text{ or } 3x = -7$

$\therefore \quad x = 4 \text{ or } -\frac{7}{3}$

EXERCISE 21B.1

1 Solve for the unknown using the Null Factor law:

a $3x = 0$ **b** $a \times 8 = 0$ **c** $-7y = 0$

d $ab = 0$ **e** $2xy = 0$ **f** $abc = 0$

g $a^2 = 0$ **h** $pqrs = 0$ **i** $a^2b = 0$

2 Solve for x using the Null Factor law:

a $x(x - 5) = 0$ **b** $2x(x + 3) = 0$ **c** $(x + 1)(x - 3) = 0$

d $3x(7 - x) = 0$ **e** $-2x(x + 1) = 0$ **f** $4(x + 6)(2x - 3) = 0$

g $(2x + 1)(2x - 1) = 0$ **h** $11(x + 2)(x - 7) = 0$ **i** $-6(x - 5)(3x + 2) = 0$

j $x^2 = 0$ **k** $4(5 - x)^2 = 0$ **l** $-3(3x - 1)^2 = 0$

STEPS FOR SOLVING QUADRATIC EQUATIONS

To use the **Null Factor** law when solving equations, we must have one side of the equation *equal to zero*.

> *Step 1:* If necessary, rearrange the equation so one side is **zero**.
> *Step 2:* **Fully factorise** the other side (usually the LHS).
> *Step 3:* Use the **Null Factor** law.
> *Step 4:* **Solve** the resulting linear equations.
> *Step 5:* **Check** at least one of your solutions.

Example 4 ◀⁍) Self Tutor

Solve for x: $x^2 = 3x$

$$x^2 = 3x$$
$$\therefore \quad x^2 - 3x = 0 \qquad \text{\{rearranging so RHS} = 0\}$$
$$\therefore \quad x(x - 3) = 0 \qquad \text{\{factorising the LHS\}}$$
$$\therefore \quad x = 0 \ \text{ or } \ x - 3 = 0 \qquad \text{\{Null Factor law\}}$$
$$\therefore \quad x = 0 \ \text{ or } \ x = 3$$
$$\therefore \quad x = 0 \ \text{ or } \ 3$$

If $a \times b = 0$ then either $a = 0$ or $b = 0$.

ILLEGAL CANCELLING

Let us reconsider the equation $x^2 = 3x$ from **Example 4**.

We notice that there is a common factor of x on both sides.

If we cancel x from both sides, we will have $\dfrac{x^2}{x} = \dfrac{3x}{x}$ and thus $x = 3$.

Consequently, we will 'lose' the solution $x = 0$.

From this example we conclude that:

> We must never cancel a variable that is a common factor from both sides of an equation unless we know that the factor cannot be zero.

Example 5   ◀)) *Self Tutor*

Solve for x: **a** $x^2 + 3x = 28$ **b** $5x^2 = 3x + 2$

a
$$x^2 + 3x = 28$$
$$\therefore \quad x^2 + 3x - 28 = 0 \qquad \text{\{rearranging so RHS } = 0\}$$
$$\therefore \quad (x+7)(x-4) = 0 \qquad \text{\{sum} = +3 \text{ and product } = -28$$
$$\therefore \quad \text{the numbers are } +7 \text{ and } -4\}$$
$$\therefore \quad x + 7 = 0 \quad \text{or} \quad x - 4 = 0 \qquad \text{\{Null Factor law\}}$$
$$\therefore \quad x = -7 \text{ or } 4$$

b
$$5x^2 = 3x + 2$$
$$\therefore \quad 5x^2 - 3x - 2 = 0 \qquad \text{\{rearranging so RHS } = 0\}$$
$$\therefore \quad 5x^2 - 5x + 2x - 2 = 0 \qquad \text{\{splitting the middle term\}}$$
$$\therefore \quad 5x(x-1) + 2(x-1) = 0$$
$$\therefore \quad (x-1)(5x+2) = 0$$
$$\therefore \quad x - 1 = 0 \quad \text{or} \quad 5x + 2 = 0 \qquad \text{\{Null Factor law\}}$$
$$\therefore \quad x = 1 \text{ or } -\tfrac{2}{5}$$

Example 6   ◀)) *Self Tutor*

Solve for x: **a** $\dfrac{x+5}{4} = \dfrac{9-x}{x}$ **b** $\dfrac{4}{x} - \dfrac{1}{x+1} = -1$

a
$$\frac{x+5}{4} = \frac{9-x}{x}$$
$$\therefore \quad x(x+5) = 4(9-x) \qquad \text{\{removing the fractions\}}$$
$$\therefore \quad x^2 + 5x = 36 - 4x$$
$$\therefore \quad x^2 + 9x - 36 = 0$$
$$\therefore \quad (x-3)(x+12) = 0$$
$$\therefore \quad x = 3 \text{ or } -12$$

Check: if $x = 3$, LHS $= \frac{8}{4} = 2$

RHS $= \frac{6}{3} = 2$ ✓

if $x = -12$, LHS $= -\frac{7}{4}$

RHS $= \frac{21}{-12} = -\frac{7}{4}$ ✓

b
$$\frac{4}{x} - \frac{1}{x+1} = \frac{-1}{1}$$
$$\therefore \quad \frac{4}{x}\left(\frac{x+1}{x+1}\right) - \frac{1}{x+1}\left(\frac{x}{x}\right) = \frac{-1}{1}\frac{x(x+1)}{x(x+1)} \qquad \text{\{LCD is } x(x+1)\}$$
$$\therefore \quad 4(x+1) - x = -x(x+1) \qquad \text{\{equating numerators\}}$$
$$\therefore \quad 4x + 4 - x = -x^2 - x$$
$$\therefore \quad 3x + 4 = -x^2 - x$$
$$\therefore \quad x^2 + 4x + 4 = 0$$
$$\therefore \quad (x+2)^2 = 0$$
$$\therefore \quad x = -2$$

Check: when $x = -2$,

LHS $= \dfrac{4}{(-2)} - \dfrac{1}{(-2)+1}$
$$= -2 - \frac{1}{-1}$$
$$= -1 \quad ✓$$

EXERCISE 21B.2

1 Solve for x:

    a $x^2 - 7x = 0$ **b** $x^2 - 5x = 0$ **c** $x^2 = 8x$

    d $x^2 = 4x$ **e** $3x^2 + 6x = 0$ **f** $2x^2 + 5x = 0$

    g $4x^2 - 3x = 0$ **h** $4x^2 = 5x$ **i** $3x^2 = 9x$

2 Solve for x:

    a $x^2 - 1 = 0$ **b** $x^2 - 9 = 0$ **c** $(x - 5)^2 = 0$

    d $(x + 2)^2 = 0$ **e** $x^2 + 3x + 2 = 0$ **f** $x^2 - 3x + 2 = 0$

    g $x^2 + 5x + 6 = 0$ **h** $x^2 - 5x + 6 = 0$ **i** $x^2 + 7x + 6 = 0$

    j $x^2 + 9x + 14 = 0$ **k** $x^2 + 11x = -30$ **l** $x^2 + 2x = 15$

    m $x^2 + 4x = 12$ **n** $x^2 = 11x - 24$ **o** $x^2 = 14x - 49$

3 Solve for x:

    a $x^2 + 9x + 20 = 0$ **b** $x^2 + 11x + 28 = 0$ **c** $x^2 + 2x = 8$

    d $x^2 + x = 12$ **e** $x^2 + 6 = 5x$ **f** $x^2 + 4 = 4x$

    g $x^2 = x + 6$ **h** $x^2 = 7x + 60$ **i** $x^2 = 3x + 70$

    j $10 - 3x = x^2$ **k** $x^2 + 12 = 7x$ **l** $9x + 36 = x^2$

4 Solve for x:

    a $2x^2 + 2 = 5x$ **b** $3x^2 + 8x = 3$ **c** $3x^2 + 17x + 20 = 0$

    d $2x^2 + 5x = 3$ **e** $2x^2 + 5 = 11x$ **f** $2x^2 + 7x + 5 = 0$

    g $3x^2 + 13x + 4 = 0$ **h** $5x^2 = 13x + 6$ **i** $2x^2 + 17x = 9$

    j $2x^2 + 3x = 5$ **k** $3x^2 + 2x = 8$ **l** $2x^2 + 9x = 18$

5 Solve for x:

    a $6x^2 + 13x = 5$ **b** $6x^2 = x + 2$ **c** $6x^2 + 5x + 1 = 0$

    d $21x^2 = 62x + 3$ **e** $10x^2 + x = 2$ **f** $10x^2 = 7x + 3$

6 Solve for x by first expanding brackets and then making one side of the equation zero:

    a $x(x + 5) + 2(x + 6) = 0$ **b** $x(1 + x) + x = 3$ **c** $(x - 1)(x + 9) = 8x$

    d $3x(x + 2) - 5(x - 3) = 17$ **e** $4x(x + 1) = -1$ **f** $2x(x - 6) = x - 20$

7 Solve for x by first eliminating the algebraic fractions:

    a $\dfrac{x}{3} = \dfrac{2}{x}$ **b** $\dfrac{4}{x} = \dfrac{x}{2}$ **c** $\dfrac{x}{5} = \dfrac{2}{x}$

    d $\dfrac{x - 1}{4} = \dfrac{3}{x}$ **e** $\dfrac{x - 1}{x} = \dfrac{x + 11}{5}$ **f** $\dfrac{x}{x + 2} = \dfrac{1}{x}$

    g $\dfrac{2x}{3x + 1} = \dfrac{1}{x + 2}$ **h** $\dfrac{2x + 1}{x} = 3x$ **i** $\dfrac{x + 2}{x - 1} = \dfrac{x}{2}$

8 Solve for x:

    a $\dfrac{6}{x + 1} + \dfrac{4}{x} = 4$ **b** $\dfrac{3}{x} + \dfrac{5}{x - 2} = -2$ **c** $\dfrac{1}{x} - \dfrac{5}{x + 2} = -6$ **d** $\dfrac{3}{x - 2} - \dfrac{4}{x} = -7$

9 Solve for x:

    a $x^4 - 5x^2 + 4 = 0$ **b** $x^4 - 7x^2 + 12 = 0$ **c** $x^4 = 4x^2 + 5$

    Hint: Treat them as quadratics in the variable x^2.

C   THE QUADRATIC FORMULA   [2.10]

Try as much as we like, we will not be able to solve quadratic equations such as $x^2 + 4x - 7 = 0$ using the factorisation methods already practised. This is because the solutions are not rationals.

Not all quadratics have simple factors.

Consequently, the **quadratic formula** has been developed:

> If $ax^2 + bx + c = 0$ where $a \neq 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Proof: If $ax^2 + bx + c = 0$

then $x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0$ {dividing each term by a, as $a \neq 0$}

$\therefore \quad x^2 + \dfrac{b}{a}x \quad = -\dfrac{c}{a}$

$\therefore \quad x^2 + \dfrac{b}{a}x + \left(\dfrac{b}{2a}\right)^2 = -\dfrac{c}{a} + \left(\dfrac{b}{2a}\right)^2$ {completing the square on LHS}

$\therefore \quad \left(x + \dfrac{b}{2a}\right)^2 = -\dfrac{c}{a}\left(\dfrac{4a}{4a}\right) + \dfrac{b^2}{4a^2}$

$\therefore \quad \left(x + \dfrac{b}{2a}\right)^2 = \dfrac{b^2 - 4ac}{4a^2}$

$\therefore \quad x + \dfrac{b}{2a} = \pm\sqrt{\dfrac{b^2 - 4ac}{4a^2}}$

$\therefore \quad x = -\dfrac{b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$

$\therefore \quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

To demonstrate the validity of this formula, consider the equation $x^2 - 3x + 2 = 0$.

By factorisation: $x^2 - 3x + 2 = 0$

$\therefore \quad (x - 1)(x - 2) = 0$

$\therefore \quad x = 1$ or 2

By formula: $a = 1$, $b = -3$, $c = 2$

$\therefore \quad x = \dfrac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(2)}}{2}$

$\therefore \quad x = \dfrac{3 \pm \sqrt{9 - 8}}{2}$

$\therefore \quad x = \dfrac{3 \pm 1}{2}$

$\therefore \quad x = 2$ or 1

We can see that factorisation is quicker if the quadratic can indeed be factorised, but the quadratic formula provides an alternative for when it cannot be factorised.

USE OF THE QUADRATIC FORMULA

If $b^2 - 4ac$ is a rational perfect square then $\sqrt{b^2 - 4ac}$ will be rational, and so the solutions of the quadratic will also be rational. In such instances, it is preferable to solve the quadratic by factorisation.

For example, $6x^2 - 13x - 8 = 0$ has $b^2 - 4ac = 169 - 4(6)(-8) = 361 = 19^2$, so we should solve this equation by factorising $6x^2 - 13x - 8$ into $(3x - 8)(2x + 1)$.

Example 7   ◀》 *Self Tutor*

Solve for x:

  a $x^2 - 2x - 2 = 0$ **b** $2x^2 + 3x - 4 = 0$

a $x^2 - 2x - 2 = 0$ has
    $a = 1, \quad b = -2, \quad c = -2$

$$\therefore \quad x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)}$$

$$\therefore \quad x = \frac{2 \pm \sqrt{4 + 8}}{2}$$

$$\therefore \quad x = \frac{2 \pm \sqrt{12}}{2}$$

$$\therefore \quad x = \frac{2 \pm 2\sqrt{3}}{2}$$

$$\therefore \quad x = 1 \pm \sqrt{3} \quad \{\text{exact form}\}$$

b $2x^2 + 3x - 4 = 0$ has
    $a = 2, \quad b = 3, \quad c = -4$

$$\therefore \quad x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-4)}}{2(2)}$$

$$\therefore \quad x = \frac{-3 \pm \sqrt{9 + 32}}{4}$$

$$\therefore \quad x = \frac{-3 \pm \sqrt{41}}{4}$$

So, $x \approx 0.85$ or ≈ -2.35

                $\{$correct to 2 decimal places$\}$

EXERCISE 21C.1

1   Use the quadratic formula to solve for x, giving exact answers::

  a $x^2 + 4x - 3 = 0$ **b** $x^2 + 6x + 1 = 0$ **c** $x^2 + 4x - 7 = 0$

  d $x^2 + 2x = 2$ **e** $x^2 + 2 = 6x$ **f** $x^2 = 4x + 1$

  g $x^2 + 1 = 3x$ **h** $x^2 + 8x + 5 = 0$ **i** $2x^2 = 2x + 1$

  j $9x^2 = 6x + 1$ **k** $25x^2 + 1 = 20x$ **l** $2x^2 + 6x + 1 = 0$

2   Use the quadratic formula to solve for x, giving answers correct to 2 decimal places:

  a $x^2 - 6x + 4 = 0$ **b** $2x^2 + 4x - 1 = 0$ **c** $5x^2 + 2x - 4 = 0$

  d $3x^2 + 2x - 2 = 0$ **e** $x + \dfrac{1}{x} = 3$ **f** $x - \dfrac{3}{x} = 1$

3   Use the quadratic formula to solve for x:

  a $(x + 2)(x - 1) = 5$ **b** $(x + 1)^2 = 3 - x^2$ **c** $\dfrac{x + 1}{x} = \dfrac{x}{2}$

  d $x + \dfrac{1}{x + 2} = 4$ **e** $3x - \dfrac{4}{x + 1} = 10$ **f** $\dfrac{x + 2}{x - 1} = \dfrac{3x}{x + 1}$

QUADRATIC EQUATIONS WITH NO REAL SOLUTIONS

Consider $x^2 + 2x + 5 = 0$.

Using the quadratic formula, the solutions are:
$$x = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2(1)}$$
$$= \frac{-2 \pm \sqrt{-16}}{2}$$

However, in the real number system, $\sqrt{-16}$ does not exist. We therefore say that $x^2 + 2x + 5 = 0$ has no real solutions.

If we graph $y = x^2 + 2x + 5$ we get:

The graph does not cut the x-axis, and this further justifies the fact that $x^2 + 2x + 5 = 0$ has no real solutions.

We will discuss this more when we turn our attention to quadratic functions.
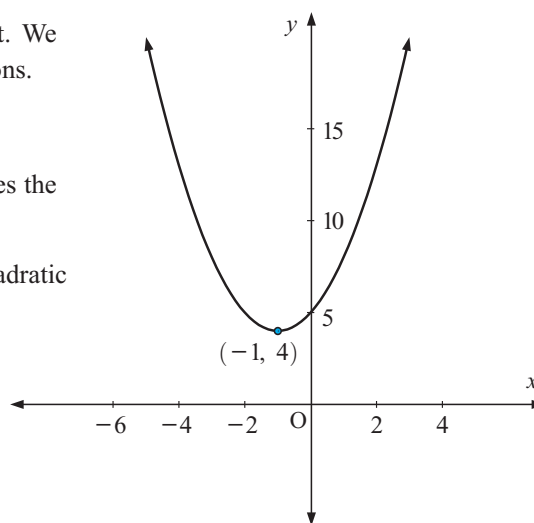
EXERCISE 21C.2

1 Show that the following quadratic equations have no real solutions:

a $x^2 - 3x + 12 = 0$ **b** $x^2 + 2x + 4 = 0$ **c** $-2x^2 + x - 1 = 0$

2 Solve for x, where possible:

a $x^2 - 25 = 0$ **b** $x^2 + 25 = 0$ **c** $x^2 - 7 = 0$

d $x^2 + 7 = 0$ **e** $4x^2 - 9 = 0$ **f** $4x^2 + 9 = 0$

g $x^2 - 4x + 5 = 0$ **h** $x^2 - 4x - 5 = 0$ **i** $x^2 - 10x + 29 = 0$

j $x^2 + 6x + 25 = 0$ **k** $2x^2 - 6x - 5 = 0$ **l** $2x^2 + x - 2 = 0$

D   QUADRATIC FUNCTIONS   [3.2]

A **quadratic function** is a relationship between two variables which can be written in the form $y = ax^2 + bx + c$ where x and y are the variables and a, b, and c are constants, $a \neq 0$.

Using function notation, $y = ax^2 + bx + c$ can be written as $f(x) = ax^2 + bx + c$.

FINDING y GIVEN x

For any value of x, the corresponding value of y can be found by substitution into the function equation.

Example 8   🔊 *Self Tutor*

If $y = 2x^2 + 4x - 5$ find the value of y when: **a** $x = 0$ **b** $x = 3$.

a When $x = 0$,

$y = 2(0)^2 + 4(0) - 5$

$= 0 + 0 - 5$

$= -5$

b When $x = 3$,

$y = 2(3)^2 + 4(3) - 5$

$= 2(9) + 12 - 5$

$= 18 + 12 - 5$

$= 25$

FINDING x GIVEN y

When we substitute a value for y, we are left with a quadratic equation which we need to solve for x. Since the equation is quadratic, there may be 0, 1 or 2 possible values for x for any one value of y.

Example 9   🔊 *Self Tutor*

If $y = x^2 - 6x + 8$ find the value(s) of x when: **a** $y = 15$ **b** $y = -1$

a If $y = 15$ then

$x^2 - 6x + 8 = 15$

$\therefore \quad x^2 - 6x - 7 = 0$

$\therefore \quad (x + 1)(x - 7) = 0$

$\therefore \quad x = -1 \quad \text{or} \quad x = 7$

So, there are 2 solutions.

b If $y = -1$ then

$x^2 - 6x + 8 = -1$

$\therefore \quad x^2 - 6x + 9 = 0$

$\therefore \quad (x - 3)^2 = 0$

$\therefore \quad x = 3$

So, there is only one solution.

EXERCISE 21D

1 Which of the following are quadratic functions?

a $y = 2x^2 - 4x + 10$ **b** $y = 15x - 8$ **c** $y = -2x^2$

d $y = \frac{1}{3}x^2 + 6$ **e** $3y + 2x^2 - 7 = 0$ **f** $y = 15x^3 + 2x - 16$

2 For each of the following functions, find the value of y for the given value of x:

a $y = x^2 + 5x - 14$ when $x = 2$ **b** $y = 2x^2 + 9x$ when $x = -5$

c $y = -2x^2 + 3x - 6$ when $x = 3$ **d** $y = 4x^2 + 7x + 10$ when $x = -2$

3 State whether the following quadratic functions are satisfied by the given ordered pairs:

a $f(x) = 6x^2 - 10$ $(0, 4)$ **b** $y = 2x^2 - 5x - 3$ $(4, 9)$

c $y = -4x^2 + 6x$ $(-\frac{1}{2}, -4)$ **d** $y = -7x^2 + 9x + 11$ $(-1, -6)$

e $f(x) = 3x^2 - 11x + 20$ $(2, -10)$ **f** $f(x) = -3x^2 + x + 6$ $(\frac{1}{3}, 4)$

4 For each of the following quadratic functions, find the value(s) of x for the given value of y:

a $y = x^2 + 6x + 10$ when $y = 1$ **b** $y = x^2 + 5x + 8$ when $y = 2$

c $y = x^2 - 5x + 1$ when $y = -3$ **d** $y = 3x^2$ when $y = -3$.

5 Find the value(s) of x for which:

    a $f(x) = 3x^2 - 3x + 6$ takes the value 6

    b $f(x) = x^2 - 2x - 7$ takes the value -4

    c $f(x) = -2x^2 - 13x + 3$ takes the value -4

    d $f(x) = 2x^2 - 10x + 1$ takes the value -11.

E   GRAPHS OF QUADRATIC FUNCTIONS   [3.2]

The graphs of all quadratic functions are **parabolas**. The parabola is one of the *conic sections*.

Conic sections are curves which can be obtained by cutting a cone with a plane. The Ancient Greek mathematicians were fascinated by conic sections.

The name parabola comes from the Greek word for **thrown** because when an object is thrown, its path makes a parabolic arc.

There are many other examples of parabolas in every day life. For example, parabolic mirrors are used in car headlights, heaters, radar discs, and radio telescopes because of their special geometric properties.
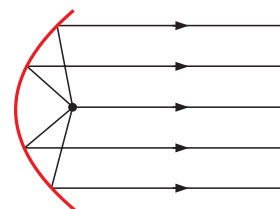
Alongside is a single span parabolic bridge. Other suspension bridges, such as the Golden Gate bridge in San Francisco, also form parabolic curves.

THE SIMPLEST QUADRATIC FUNCTION

The simplest quadratic function is $y = x^2$.   Its graph can be drawn from a table of values.

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

Notice that:

- The curve is a **parabola** and it opens upwards.
- There are no negative y values, i.e., the curve does not go below the x-axis.
- The curve is **symmetrical** about the y-axis because, for example, when $x = -3$, $y = (-3)^2 = 9$ and when $x = 3$, $y = 3^2 = 9$.
- The curve has a **turning point** or **vertex** at $(0, 0)$.
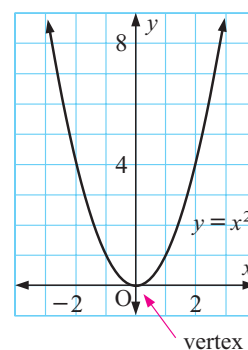
The **vertex** is the point where the graph is at its maximum or minimum.

Example 10   ◀)) *Self Tutor*

Draw the graph of $y = x^2 + 2x - 3$ from a table of values from $x = -3$ to $x = 3$.

Consider $f(x) = x^2 + 2x - 3$

Now, $\begin{aligned} f(-3) &= (-3)^2 + 2(-3) - 3 \\ &= 9 - 6 - 3 \\ &= 0 \end{aligned}$

We can do the same for the other values of x.

Tabled values are:

x	-3	-2	-1	0	1	2	3
y	0	-3	-4	-3	0	5	12
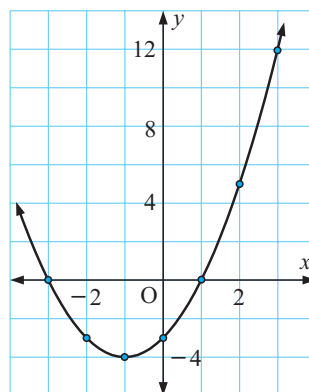
EXERCISE 21E.1

1 From a table of values for $x = -3, -2, -1, 0, 1, 2, 3$ draw the graph of:

GRAPHING PACKAGE

    a $y = x^2 - 2x + 8$ **b** $y = -x^2 + 2x + 1$ **c** $y = 2x^2 + 3x$

    d $y = -2x^2 + 4$ **e** $y = x^2 + x + 4$ **f** $y = -x^2 + 4x - 9$

2 Use a **graphing package** or **graphics calculator** to check your graphs in question **1**.
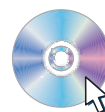
Discovery 1   *Graphs of quadratic functions*

In this discovery we consider different forms of quadratic functions, and how the form of the quadratic affects its graph.

Part 1:   Graphs of the form $y = x^2 + k$ where k is a constant

GRAPHING PACKAGE

What to do:

1 Using a **graphing package** or **graphics calculator**:

    i graph the two functions on the same set of axes

    ii state the coordinates of the vertex of each function.

    a $y = x^2$ and $y = x^2 + 2$ **b** $y = x^2$ and $y = x^2 - 2$

    c $y = x^2$ and $y = x^2 + 4$ **d** $y = x^2$ and $y = x^2 - 4$

2 What effect does the value of k have on:

    a the position of the graph   **b** the shape of the graph?

3 What transformation is needed to graph $y = x^2 + k$ from $y = x^2$?

Part 2: Graphs of the form $y = (x - h)^2$

What to do:

1 Using a **graphing package** or **graphics calculator**:

      i graph the two functions on the same set of axes

      ii state the coordinates of the vertex of each function.

  a $y = x^2$ and $y = (x - 2)^2$ **b** $y = x^2$ and $y = (x + 2)^2$

  c $y = x^2$ and $y = (x - 4)^2$ **d** $y = x^2$ and $y = (x + 4)^2$

2 What effect does the value of h have on:

  a the position of the graph   **b** the shape of the graph?

3 What transformation is needed to graph $y = (x - h)^2$ from $y = x^2$?

Part 3: Graphs of the form $y = (x - h)^2 + k$

What to do:

1 *Without the assistance of technology*, sketch the graph of $y = (x - 2)^2 + 3$.
State the coordinates of the vertex and comment on the shape of the graph.

2 Use a **graphing package** or **graphics calculator** to draw, on the same set of axes, the graphs of
$y = x^2$ and $y = (x - 2)^2 + 3$.

3 Repeat steps **1** and **2** for $y = (x + 4)^2 - 1$.

4 Copy and complete:

  • The graph of $y = (x - h)^2 + k$ is the same shape as the graph of ......

  • The graph of $y = (x - h)^2 + k$ is a .... of the graph of $y = x^2$
through a translation of ....

Part 4: Graphs of the form $y = ax^2, \ a \neq 0$

What to do:

1 Using a **graphing package** or **graphics calculator**:

      i graph the two functions on the same set of axes

      ii state the coordinates of the vertex of each function.

  a $y = x^2$ and $y = 2x^2$ **b** $y = x^2$ and $y = 4x^2$

  c $y = x^2$ and $y = \frac{1}{2}x^2$ **d** $y = x^2$ and $y = -x^2$

  e $y = x^2$ and $y = -2x^2$ **f** $y = x^2$ and $y = -\frac{1}{2}x^2$

2 These functions are all members of the family $y = ax^2$ where a is the coefficient of the x^2
term. What effect does a have on:

  a the position of the graph   **b** the shape of the graph

  c the direction in which the graph opens?
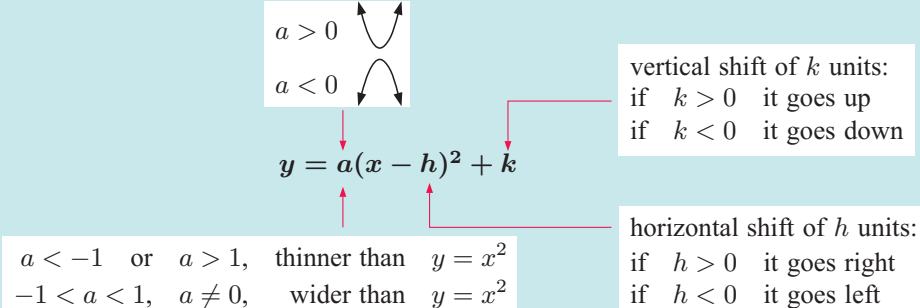
Part 5:   Graphs of the form $y = a(x - h)^2 + k, \quad a \neq 0$

What to do:

1 *Without the assistance of technology*, sketch the graphs of $y = 2x^2$ and $y = 2(x - 1)^2 + 3$ on the same set of axes. State the coordinates of the vertices and comment on the shape of the two graphs.

2 Use a **graphing package** or **graphics calculator** to check your graphs in step **1**.

3 Repeat steps **1** and **2** for:

    a $y = -x^2$ and $y = -(x + 2)^2 + 3$ **b** $y = \frac{1}{2}x^2$ and $y = \frac{1}{2}(x - 2)^2 - 4$

4 Copy and complete:

- The graph of $y = a(x - h)^2 + k$ has the same shape and opens in the same direction as the graph of ......
- The graph of $y = a(x - h)^2 + k$ is a .... of the graph of $y = ax^2$ through a translation of ......

You should have discovered the following important facts:

- Graphs of the form $y = x^2 + k$ have exactly the same shape as the graph of $y = x^2$.

  Every point on the graph of $y = x^2$ is translated $\begin{pmatrix} 0 \\ k \end{pmatrix}$ to give the graph of $y = x^2 + k$.

- Graphs of the form $y = (x - h)^2$ have exactly the same shape as the graph of $y = x^2$.

  Every point on the graph of $y = x^2$ is translated $\begin{pmatrix} h \\ 0 \end{pmatrix}$ to give the graph of $y = (x - h)^2$.

- Graphs of the form $y = (x - h)^2 + k$ have the same shape as the graph of $y = x^2$ and can be

  obtained from $y = x^2$ by a **translation** of $\begin{pmatrix} h \\ k \end{pmatrix}$. The **vertex** is at (h, k).

- If $a > 0$, $y = ax^2$ opens upwards   i.e., $\smile$

  If $a < 0$, $y = ax^2$ opens downwards   i.e., $\frown$

  If $a < -1$ or $a > 1$ then $y = ax^2$ is 'thinner' than $y = x^2$.
  If $-1 < a < 1$, $a \neq 0$ then $y = ax^2$ is 'wider' than $y = x^2$.

- $a > 0$ $\smile$

  $a < 0$ $\frown$

  $y = a(x - h)^2 + k$

  vertical shift of k units:
  if $k > 0$ it goes up
  if $k < 0$ it goes down

  horizontal shift of h units:
  if $h > 0$ it goes right
  if $h < 0$ it goes left

  $a < -1$ or $a > 1$, thinner than $y = x^2$
  $-1 < a < 1$, $a \neq 0$, wider than $y = x^2$

Example 11   ◀) *Self Tutor*

Sketch $y = x^2$ and $y = x^2 + 3$ on the same set of axes.
Mark the vertex of $y = x^2 + 3$.

We draw $y = x^2$ and translate

it 3 units upwards,   i.e., $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

$\therefore$ the vertex is now at $(0, 3)$.
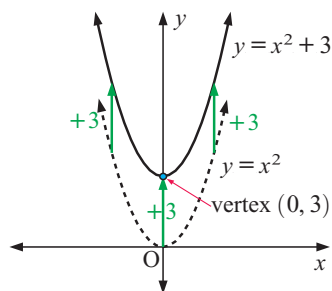
Example 12   ◀) *Self Tutor*

Sketch each of the following functions on the same set of axes as $y = x^2$.   In each case state the
coordinates of the vertex.

a $y = (x - 2)^2 + 3$ **b** $y = (x + 2)^2 - 5$

a   We draw $y = x^2$ and translate it by $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.
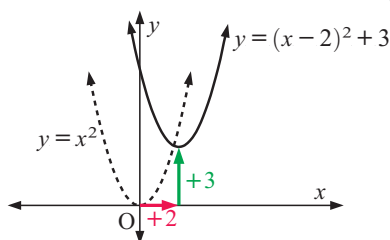
The vertex is at $(2, 3)$.

b   We draw $y = x^2$ and translate it by $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$.
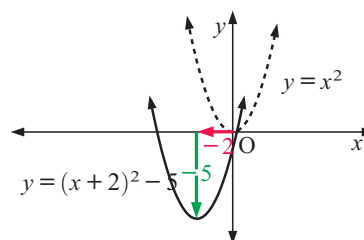
The vertex is at $(-2, -5)$.

EXERCISE 21E.2

1   Sketch each of the following functions on the same set of axes as $y = x^2$.   Use a
separate set of axes for each part, and in each case state the coordinates of the vertex.

GRAPHING PACKAGE

   a $y = x^2 - 3$ **b** $y = x^2 - 1$ **c** $y = x^2 + 1$

   d $y = x^2 - 5$ **e** $y = x^2 + 5$ **f** $y = x^2 - \frac{1}{2}$
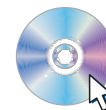
2   Sketch each of the following functions on the same set of axes as $y = x^2$.   Use a separate set of axes
for each part, and in each case state the coordinates of the vertex.

   a $y = (x - 3)^2$ **b** $y = (x + 1)^2$ **c** $y = (x - 1)^2$

   d $y = (x - 5)^2$ **e** $y = (x + 5)^2$ **f** $y = (x - \frac{3}{2})^2$

3   Sketch each of the following functions on the same set of axes as $y = x^2$.   Use a separate set of axes
for each part, and in each case state the coordinates of the vertex.

   a $y = (x - 1)^2 + 3$ **b** $y = (x - 2)^2 - 1$ **c** $y = (x + 1)^2 + 4$

   d $y = (x + 2)^2 - 3$ **e** $y = (x + 3)^2 - 2$ **f** $y = (x - 3)^2 + 3$

GRAPHS WHEN THE LEADING COEFFICIENT $\neq 1$

Example 13   ◀》 Self Tutor

Sketch $y = x^2$ on a set of axes and hence sketch:

a $y = 3x^2$ **b** $y = -3x^2$

a $y = 3x^2$ is 'thinner' than $y = x^2$.

b $y = -3x^2$ is the same shape as
$y = 3x^2$ but opens downwards.
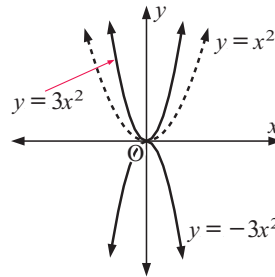
Example 14   ◀》 Self Tutor

Sketch the graph of $y = -(x - 2)^2 - 3$ from the graph of $y = x^2$ and
hence state the coordinates of its vertex.

$$y = -(x - 2)^2 - 3$$

reflect in   horizontal shift   vertical shift
x-axis   2 units right   3 units down

The vertex is at $(2, -3)$.
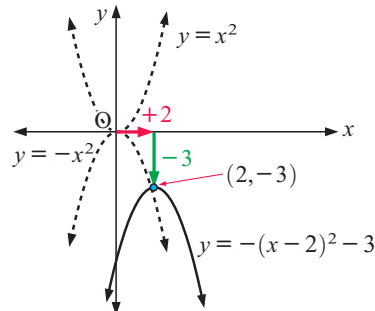
Consider the quadratic function $y = 3(x - 1)^2 + 2$.

$$y = 3(x - 1)^2 + 2$$

$a = 3 \qquad h = 1 \qquad k = 2$

This graph has the same shape as the graph of
$y = 3x^2$ but with vertex $(1, 2)$.

On expanding: $\quad y = 3(x - 1)^2 + 2$
$\qquad \therefore \quad y = 3(x^2 - 2x + 1) + 2$
$\qquad \therefore \quad y = 3x^2 - 6x + 3 + 2$
$\qquad \therefore \quad y = 3x^2 - 6x + 5$

From this we can see that:

the graph of a quadratic of the form $y = ax^2 + bx + c$ has the same shape as the graph of $y = ax^2$.

IGCSE01
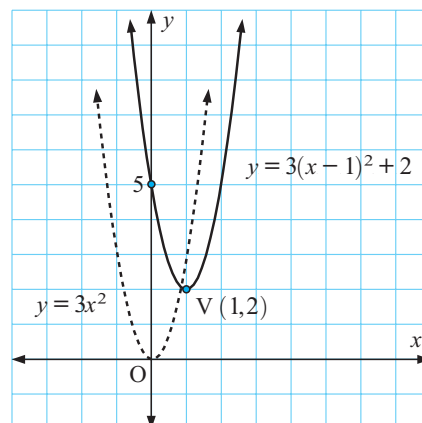
EXERCISE 21E.3

1 On separate sets of axes sketch $y = x^2$ and each of the following functions. Comment on:

    i the shape of the graph   **ii** the direction in which the graph opens.

  a $y = 5x^2$ **b** $y = -5x^2$ **c** $y = \frac{1}{3}x^2$

  d $y = -\frac{1}{3}x^2$ **e** $y = -4x^2$ **f** $y = \frac{1}{4}x^2$
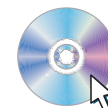
2 Sketch the graphs of the following functions without using tables of values and state the coordinates of their vertices:

  a $y = -(x-1)^2 + 3$ **b** $y = 2x^2 + 4$ **c** $y = -(x-2)^2 + 4$

  d $y = 3(x+1)^2 - 4$ **e** $y = \frac{1}{2}(x+3)^2$ **f** $y = -\frac{1}{2}(x+3)^2 + 1$

  g $y = -2(x+4)^2 + 3$ **h** $y = 2(x-3)^2 + 5$ **i** $y = \frac{1}{2}(x-2)^2 - 1$

3 Match each quadratic function with its corresponding graph:

  a $y = -1(x+1)^2 + 3$ **b** $y = -2(x-3)^2 + 2$ **c** $y = x^2 + 2$

  d $y = -1(x-1)^2 + 1$ **e** $y = (x-2)^2 - 2$ **f** $y = \frac{1}{3}(x+3)^2 - 3$

  g $y = -x^2$ **h** $y = -\frac{1}{2}(x-1)^2 + 1$ **i** $y = 2(x+2)^2 - 1$
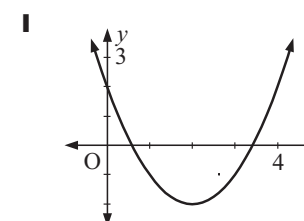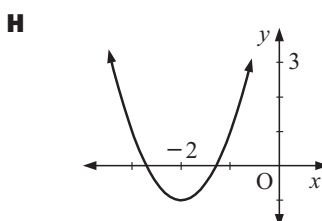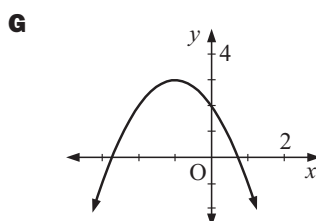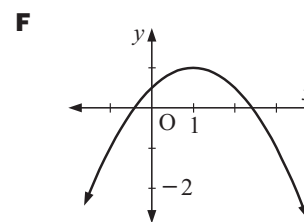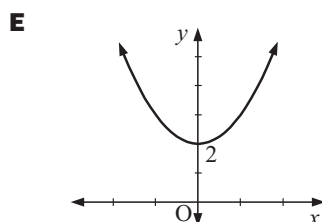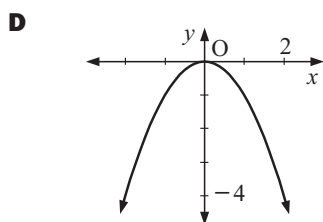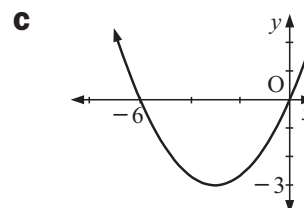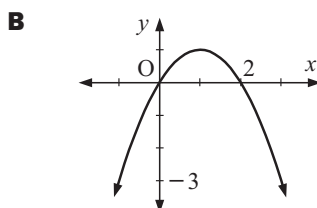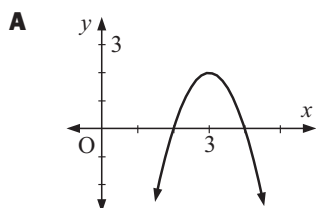
4 Use a **graphing package** or **graphics calculator** to graph each pair of quadratic functions on the same set of axes. Compare the shapes of the two graphs.

  a $y = 2x^2$ and $y = 2x^2 - 3x + 1$ **b** $y = -x^2$ and $y = -x^2 - 6x + 4$

  c $y = 3x^2$ and $y = 3x^2 - 5x$ **d** $y = -2x^2$ and $y = -2x^2 + 5$

F AXES INTERCEPTS [3.2]

Given the equation of any curve:

An **x-intercept** is a value of x where the graph meets the x-axis.

x-intercepts are found by letting y be 0 in the equation of the curve.

A **y-intercept** is a value of y where the graph meets the y-axis.

y-intercepts are found by letting x be 0 in the equation of the curve.
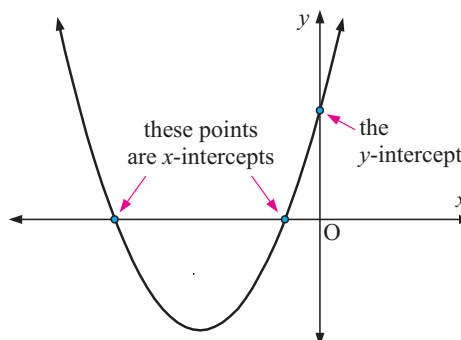
Discovery 2   Axes intercepts

What to do:

GRAPHING PACKAGE

1 For the following functions, use a graphing package or graphics calculator to:

    i draw the graph   **ii** find the y-intercept   **iii** find any x-intercepts.

    a $y = x^2 - 3x - 4$ **b** $y = -x^2 + 2x + 8$ **c** $y = 2x^2 - 3x$

    d $y = -2x^2 + 2x - 3$ **e** $y = (x - 1)(x - 3)$ **f** $y = -(x + 2)(x - 3)$

    g $y = 3(x + 1)(x + 4)$ **h** $y = 2(x - 2)^2$ **i** $y = -3(x + 1)^2$

2 From your observations in question **1**:

    a State the y-intercept of a quadratic function in the form $y = ax^2 + bx + c$.

    b State the x-intercepts of quadratic function in the form $y = a(x - \alpha)(x - \beta)$.

    c What do you notice about the x-intercepts of quadratic functions in the form $y = a(x - \alpha)^2$?

THE y-INTERCEPT

You will have noticed that for a quadratic function of the form $y = ax^2 + bx + c$, the y-intercept is the constant term c. This is because any curve cuts the y-axis when $x = 0$.

For example,   if $y = x^2 - 2x - 3$ and we let $x = 0$

                then $y = 0^2 - 2(0) - 3$

                $\therefore$ $y = -3$ (the constant term)

THE x-INTERCEPTS

You should have noticed that for a quadratic function of the form $y = a(x - \alpha)(x - \beta)$, the x-intercepts are α and β. This is because any curve cuts the x-axis when $y = 0$.

So, if we substitute $y = 0$ into the function we get $a(x - \alpha)(x - \beta) = 0$

$\therefore$ $x = \alpha$ or β {by the Null Factor law}

This suggests that x-intercepts are easy to find when the quadratic is in **factorised** form.

Example 15

◀) **Self Tutor**

Find the x-intercepts of:

a $y = 2(x - 3)(x + 2)$ **b** $y = -(x - 4)^2$

a When $y = 0$,
$2(x - 3)(x + 2) = 0$
$\therefore \quad x = 3 \quad \text{or} \quad x = -2$
$\therefore \quad$ the x-intercepts are 3 and -2.

b When $y = 0$,
$-(x - 4)^2 = 0$
$\therefore \quad x = 4$
$\therefore \quad$ the x-intercept is 4.

> If a quadratic function has only one x-intercept then its graph must **touch** the x-axis.

FACTORISING TO FIND x-INTERCEPTS

> For any quadratic function of the form $y = ax^2 + bx + c$, the x-intercepts can be found by solving the equation $ax^2 + bx + c = 0$.

We saw earlier in the chapter that quadratic equations may have *two solutions*, *one solution*, or *no solutions*. These solutions correspond to the *two x-intercepts*, *one x-intercept*, or *no x-intercepts* found when the graphs of the quadratic functions are drawn.

$y = f(x)$ $y = f(x)$ $y = f(x)$

2 solutions   1 solution   0 solutions
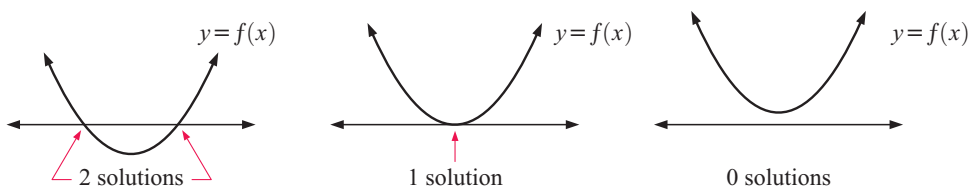
Example 16

◀) **Self Tutor**

Find the x-intercept(s) of the quadratic functions:

a $y = x^2 - 6x + 9$ **b** $y = -x^2 - x + 6$

a When $y = 0$, $\quad x^2 - 6x + 9 = 0$
$\therefore \quad (x - 3)^2 = 0$
$\therefore \quad x = 3$
$\therefore \quad$ the x-intercept is 3.

b When $y = 0$, $\quad -x^2 - x + 6 = 0$
$\therefore \quad x^2 + x - 6 = 0$
$\therefore \quad (x + 3)(x - 2) = 0$
$\therefore \quad x = -3$ or 2
$\therefore \quad$ the x-intercepts are -3 and 2.

EXERCISE 21F.1

1 For the following functions, state the y-intercept:

a $y = x^2 + 3x + 3$ **b** $y = x^2 - 5x + 2$ **c** $y = 2x^2 + 7x - 8$

d $y = 3x^2 - x + 1$ **e** $y = -x^2 + 3x + 6$ **f** $y = -2x^2 + 5 - x$

g $y = 6 - x - x^2$ **h** $y = 8 + 2x - 3x^2$ **i** $y = 5x - x^2 - 2$

2 For the following functions, find the x-intercepts:

a $y = (x - 3)(x + 1)$ **b** $y = -(x - 2)(x - 4)$ **c** $y = 2(x + 3)(x + 2)$

d $y = -3(x - 4)(x - 5)$ **e** $y = 2(x + 3)^2$ **f** $y = -5(x - 1)^2$

3 For the following functions, find the x-intercepts:

a $y = x^2 - 9$ **b** $y = 2x^2 - 6$ **c** $y = x^2 + 7x + 10$ **d** $y = x^2 + x - 12$

e $y = 4x - x^2$ **f** $y = -x^2 - 6x - 8$ **g** $y = -2x^2 - 4x - 2$ **h** $y = x^2 - 6x + 9$

i $y = x^2 - 4x + 1$ **j** $y = x^2 + 4x - 3$ **k** $y = x^2 - 6x - 2$ **l** $y = x^2 + 8x + 11$

GRAPHS FROM AXES INTERCEPTS

If we know the x and y-intercepts of a quadratic function then we can use them to draw its graph.

Example 17
🔊 *Self Tutor*

Sketch the graphs of the following functions by considering:

   i the value of a **ii** the y-intercept   **iii** the x-intercepts.

a $y = x^2 - 2x - 3$ **b** $y = -2(x + 1)(x - 2)$

a **i** $a = 1$, so the parabola opens upwards

   ii y-intercept occurs when $x = 0$

   $\therefore$ the y-intercept is -3

   iii x-intercepts occur when $y = 0$

   $\therefore$ $x^2 - 2x - 3 = 0$

   $\therefore$ $(x - 3)(x + 1) = 0$

   $\therefore$ $x = 3$ or $x = -1$

   $\therefore$ the x-intercepts are 3 and -1

   Sketch:
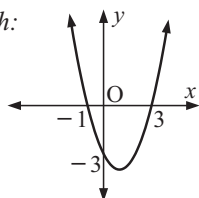
b **i** $a = -2$, so the parabola opens downwards

   ii y-intercept occurs when $x = 0$

   $\therefore$ $y = -2(0 + 1)(0 - 2)$

   $= -2 \times 1 \times -2 = 4$

   $\therefore$ the y-intercept is 4

   iii x-intercepts occur when $y = 0$

   $\therefore$ $-2(x + 1)(x - 2) = 0$

   $\therefore$ $x = -1$ or $x = 2$

   $\therefore$ the x-intercepts are -1 and 2

   Sketch:
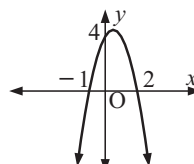
Example 18
🔊 *Self Tutor*

Sketch the graph of $y = 2(x - 3)^2$ by considering:

   a the value of a **b** the y-intercept   **c** the x-intercepts.

a $a = 2$,   so the parabola opens upwards

b y-intercept occurs when $x = 0$

   $\therefore$ $y = 2(0 - 3)^2 = 18$

   $\therefore$ the y-intercept is 18

c x-intercepts occur when $y = 0$

   $\therefore$ $2(x - 3)^2 = 0$

   $\therefore$ $x = 3$

   $\therefore$ the x-intercept is 3
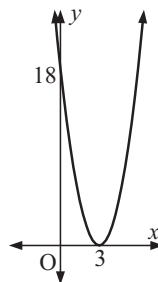
There is only one x-intercept, which means the graph *touches* the x-axis.

EXERCISE 21F.2

1 Sketch the graph of the quadratic function with:

    a x-intercepts -1 and 1, and y-intercept -1 **b** x-intercepts -3 and 1, and y-intercept 2

    c x-intercepts 2 and 5, and y-intercept -4 **d** x-intercept 2 and y-intercept 4.

2 Sketch the graphs of the following by considering:

    i the value of a **ii** the y-intercept   **iii** the x-intercepts.

    a $y = x^2 - 4x + 4$ **b** $y = (x-1)(x+3)$ **c** $y = 2(x+2)^2$

    d $y = -(x-2)(x+1)$ **e** $y = -3(x+1)^2$ **f** $y = -3(x-4)(x-1)$

    g $y = 2(x+3)(x+1)$ **h** $y = 2x^2 + 3x + 2$ **i** $y = -2x^2 - 3x + 5$

G   *LINE OF SYMMETRY AND VERTEX*   *[3.2]*

We have seen from the previous exercise that the graph of any quadratic function:

- is a **parabola**
- has a **turning point** or **vertex**.
- is symmetrical about a **line of symmetry**

If the graph has two x-intercepts then the line of symmetry must be mid-way between them. We will use this property in the following Discovery to establish an equation for the line of symmetry.

Discovery 3   *Line of symmetry and vertex*

Consider the quadratic function $y = ax^2 + bx + c$ whose graph cuts the x-axis at A and B. Let the equation of the line of symmetry be $x = h$.

What to do:

1 Use the quadratic formula to find the coordinates of A and B.

2 Since A and B are the same distance d from the line of symmetry, h must be the average of the x-coordinates of A and B. Use this property to find the line of symmetry in terms of a, b and c.

3 The vertex of the parabola lies on the line of symmetry. By considering graphs for different values of a, discuss the values of a for which the vertex of a quadratic is a maximum value or a minimum value.
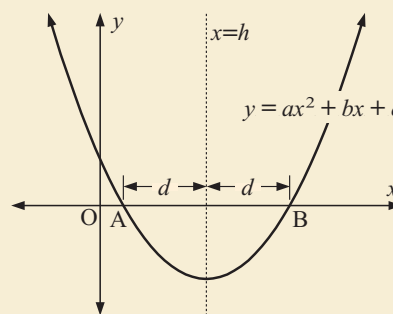
You should have discovered that:

> the equation of the **line of symmetry** of $y = ax^2 + bx + c$ is $x = \dfrac{-b}{2a}$.

This equation is true for all quadratic functions, not just those with two x-intercepts.

Proof: We know the equation $y = a(x-h)^2+k$ has vertex $(h, k),$ so its line of symmetry
is $x = h.$

Expanding $y = a(x-h)^2 + k,$ we find $y = a(x^2 - 2hx + h^2) + k$
$$\therefore \quad y = ax^2 - 2ahx + [ah^2 + k].$$

Comparing the coefficients of x with those of $y = ax^2+bx+c,$ we find $-2ah = b.$

$\therefore \quad h = \dfrac{-b}{2a},$ and so the line of symmetry is $x = \dfrac{-b}{2a}.$

Example 19   ◀) *Self Tutor*

Find the equation of the line of symmetry of $y = 2x^2 + 3x + 1.$

$y = 2x^2 + 3x + 1$ has $a = 2, \quad b = 3, \quad c = 1$

$\therefore$ the line of symmetry has equation $x = \dfrac{-b}{2a} = \dfrac{-3}{2 \times 2}$ i.e., $x = -\frac{3}{4}$

VERTEX

The vertex of any parabola lies on its line of symmetry, so its x-coordinate will be $x = \dfrac{-b}{2a}.$

The y-coordinate can be found by substituting the value for x into the function.

Example 20   ◀) *Self Tutor*

Determine the coordinates of the vertex of $y = 2x^2 + 8x + 3.$

$y = 2x^2 + 8x + 3$ has $a = 2, \quad b = 8,$ and $c = 3,$ so $\dfrac{-b}{2a} = \dfrac{-8}{2 \times 2} = -2$

$\therefore$ the equation of the line of symmetry is $x = -2$

When $x = -2, \quad y = 2(-2)^2 + 8(-2) + 3 = 8 - 16 + 3 = -5$

$\therefore$ the vertex has coordinates $(-2, -5).$

Example 21   ◀) *Self Tutor*

For the quadratic function $y = -x^2 + 2x + 3:$
  a   find its axes intercepts   **b**   find the equation of the line of symmetry
  c   find the coordinates of the vertex   **d**   sketch the function showing all important features.

a   When $x = 0, y = 3,$ so the y-intercept is 3.   **b** $a = -1, \quad b = 2, \quad c = 3$
    When $y = 0, \quad -x^2 + 2x + 3 = 0$ $\therefore \quad \dfrac{-b}{2a} = \dfrac{-2}{-2} = 1$
         $\therefore \quad x^2 - 2x - 3 = 0$ $\therefore$ the line of symmetry is $x = 1.$
         $\therefore \quad (x - 3)(x + 1) = 0$
              $\therefore \quad x = 3$ or -1
  $\therefore$ the x-intercepts are 3 and $-1.$

c From **b**, when $x = 1$,
$$y = -(1)^2 + 2(1) + 3$$
$$= -1 + 2 + 3$$
$$= 4$$
$\therefore$ the vertex is $(1, 4)$.

d
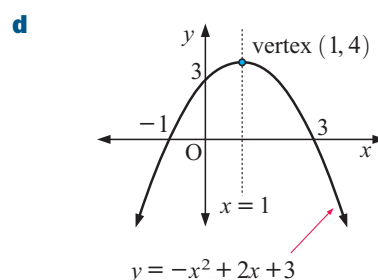
Example 22

a Sketch the graph of $y = 2(x - 2)(x + 4)$ using axes intercepts.
b Find the equation of the line of symmetry and the coordinates of the vertex.

a Since $a = 2$ the parabola opens upwards
When $x = 0$,
$$y = 2 \times -2 \times 4 = -16$$
$\therefore$ y-intercept is -16.

When $y = 0$,
$$2(x - 2)(x + 4) = 0$$
$\therefore$ $x = 2$ or $x = -4$
$\therefore$ the x-intercepts are 2 and -4.

b The line of symmetry is halfway between the x-intercepts
$\therefore$ since -1 is the average of -4 and 2,
the line of symmetry is $x = -1$
When $x = -1$, $y = 2(-1 - 2)(-1 + 4)$
$$= 2 \times -3 \times 3$$
$$= -18$$
$\therefore$ the vertex is $(-1, -18)$.
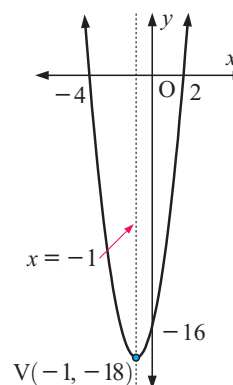
Example 23

Sketch the parabola which has x-intercepts -2 and 6, and y-intercept 2.
Find the equation of the line of symmetry.

The line of symmetry lies half-way between the x-intercepts.

The average of -2 and 6 is 2, so the line of symmetry is $x = 2$.
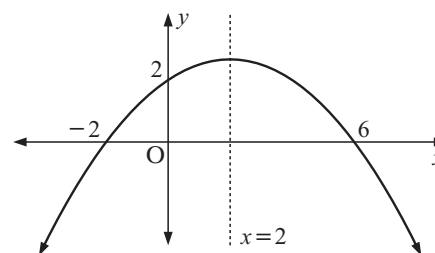
EXERCISE 21G

1 Determine the equation of the line of symmetry of:

    a $y = x^2 + 4x + 1$ **b** $y = 2x^2 - 6x + 3$ **c** $y = 3x^2 + 4x - 1$

    d $y = -x^2 - 4x + 5$ **e** $y = -2x^2 + 5x + 1$ **f** $y = \frac{1}{2}x^2 - 10x + 2$

    g $y = \frac{1}{3}x^2 + 4x$ **h** $y = 100x - 4x^2$ **i** $y = -\frac{1}{10}x^2 + 30x$

2 Find the turning point or vertex for the following quadratic functions:

    a $y = x^2 - 4x + 2$ **b** $y = x^2 + 2x - 3$ **c** $y = 2x^2 + 4$

    d $y = -3x^2 + 1$ **e** $y = 2x^2 + 8x - 7$ **f** $y = -x^2 - 4x - 9$

    g $y = 2x^2 + 6x - 1$ **h** $y = 2x^2 - 10x + 3$ **i** $y = -\frac{1}{2}x^2 + x - 5$

3 For each of the following quadratic functions find:

        i the axes intercepts   **ii** the equation of the line of symmetry
      iii the coordinates of the vertex   **iv** and hence sketch the graph.

    a $y = x^2 - 2x - 8$ **b** $y = x^2 + 3x$ **c** $y = 4x - x^2$

    d $y = x^2 + 4x + 4$ **e** $y = x^2 + 3x - 4$ **f** $y = -x^2 + 2x - 1$

    g $y = -x^2 - 6x - 8$ **h** $y = -x^2 + 3x - 2$ **i** $y = 2x^2 + 5x - 3$

    j $y = 2x^2 - 5x - 12$ **k** $y = -3x^2 - 4x + 4$ **l** $y = -\frac{1}{4}x^2 + 5x$

4 For each of the following, find the equation of the line of symmetry:
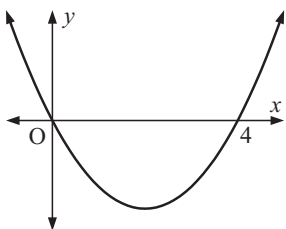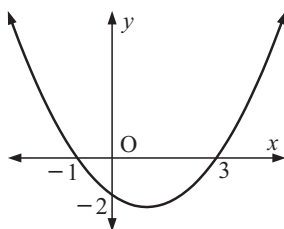
    a   **b**   **c**
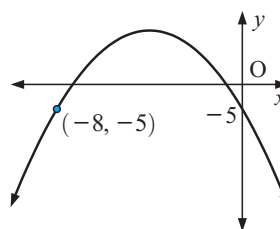
5 For each of the following quadratic functions:

        i sketch the graph using axes intercepts and *hence* find
       ii the equation of the line of symmetry
      iii the coordinates of the vertex.

    a $y = x^2 + 4x + 4$ **b** $y = x(x - 2)$ **c** $y = 2(x - 2)^2$

    d $y = -(x - 1)(x + 3)$ **e** $y = -2(x - 1)^2$ **f** $y = -5(x + 2)(x - 2)$

    g $y = 2(x + 1)(x + 4)$ **h** $y = 2x^2 - 3x - 2$ **i** $y = -2x^2 - x + 3$

6 For each of the following:

        i sketch the parabola   **ii** find the equation of the line of symmetry.

    a x-intercepts 2 and -1, y-intercept -3 **b** x-intercepts 3 and -3, y-intercept 6

    c x-intercept -2 (touching), y-intercept 4   **d** x-intercept 2 (touching), y-intercept -6

7 Find all x-intercepts of the quadratic function which:

    a cuts the x-axis at 1, and has line of symmetry $x = 2$

    b cuts the x-axis at -1, and has line of symmetry $x = -1\frac{1}{2}$

    c touches the x-axis at 2.

H   FINDING A QUADRATIC FUNCTION   [3.3, 3.4]

Having studied the graphs and properties of quadratics in some detail, we should be able to use facts about a graph to determine the corresponding function.

Example 24   ◀)) Self Tutor

Find the quadratic function with:

a   vertex $(1, 2)$ and y-intercept 3   **b**   vertex $(2, 11)$ which passes through the point $(-1, -7)$.

a

$$\text{As } h = 1 \text{ and } k = 2,$$
$$f(x) = a(x - 1)^2 + 2$$
$$\text{But } f(0) = 3$$
$$\therefore \quad a(-1)^2 + 2 = 3$$
$$\therefore \quad a + 2 = 3$$
$$\therefore \quad a = 1$$
$$\text{So, } f(x) = (x - 1)^2 + 2$$
$$\textbf{or} \quad f(x) = x^2 - 2x + 3$$
$$\{\text{in expanded form}\}$$

b

$$\text{As } h = 2 \text{ and } k = 11,$$
$$f(x) = a(x - 2)^2 + 11$$
$$\text{But } f(-1) = -7$$
$$\therefore \quad a(-3)^2 + 11 = -7$$
$$\therefore \quad 9a + 11 = -7$$
$$\therefore \quad 9a = -18$$
$$\therefore \quad a = -2$$
$$\text{So, } f(x) = -2(x - 2)^2 + 11$$
$$\textbf{or} \quad f(x) = -2(x^2 - 4x + 4) + 11$$
$$= -2x^2 + 8x + 3$$

Example 25   ◀)) Self Tutor

The graph of a quadratic function has x-intercepts $-\frac{5}{2}$ and $\frac{1}{3}$, and it passes through $(1, 42)$. Find the function.

The x-intercept $-\frac{5}{2}$ comes from the linear factor $x + \frac{5}{2}$ or $2x + 5$.

The x-intercept $\frac{1}{3}$ comes from the linear factor $x - \frac{1}{3}$ or $3x - 1$.

$$\therefore \quad f(x) = a(2x + 5)(3x - 1)$$

But $f(1) = 42$,   so $a(7)(2) = 42$

$$\therefore \quad 14a = 42$$
$$\therefore \quad a = 3$$

Thus $f(x) = 3(2x + 5)(3x - 1)$

EXERCISE 21H

1   If $f(x) = x^2 + bx + c$, find $f(x)$ given that its vertex is at:

   a $(1, 3)$ **b** $(0, 2)$ **c** $(3, 0)$ **d** $(-3, -2)$ **e** $(\frac{1}{2}, 1)$

2   If $f(x) = x^2 + bx + c$, find $f(x)$ given that it has x-intercepts:

   a   0 and 2   **b** -4 and 1   **c** -5 and 2   **d** -7 and 0.

In questions **1** and **2** we are given that $a = 1$.

3 For each of the following graphs, find the function they represent in the form $y = a\,(x - h)^2 + k$.

a

b

c
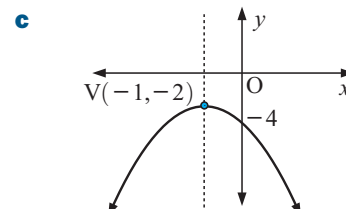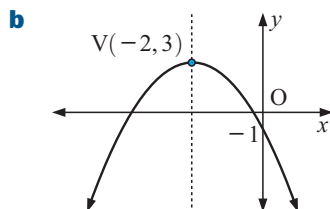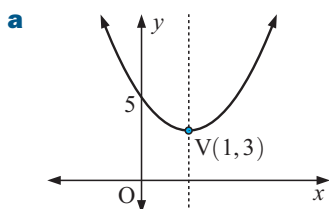
4 Find the quadratic function with:

    a vertex $(2,\, -5)$ and y-intercept 3   **b** vertex $(-4,\, 19)$ and y-intercept 3

    c vertex $(1,\, 8)$ and y-intercept 7   **d** vertex $(-2,\, 11)$ and y-intercept 3

    Give your answers in the form $f(x) = a(x - h)^2 + k$.

5 Find the quadratic function with:

    a vertex $(1,\, -4)$ and y-intercept -7 **b** vertex $(-2,\, 3)$ and y-intercept 15

    c vertex $(-3,\, -5)$ and y-intercept 7   **d** vertex $(3,\, 8)$ and y-intercept -10

    Give your answers in the form $f(x) = ax^2 + bx + c$.

6 Find the quadratic function which has:

    a vertex $(-2,\, -5)$ and passes through $(1,\, 13)$ **b** vertex $(3,\, -19)$ and passes through $(-2,\, 31)$

    Give your answers in the form $f(x) = a(x - h)^2 + k$.

7 Find the quadratic function which has:

    a vertex $(-\frac{3}{2},\, -2\frac{3}{4})$ and passes through $(1,\, -9)$

    b vertex $(-8,\, 135)$ and passes through $(1,\, -27)$.

    Give your answers in the form $f(x) = ax^2 + bx + c$.

8 Find the quadratic function which has:

    a x-intercepts -2 and 2 and passes through the point $(0,\, 8)$

    b x-intercepts 1 and 4 and passes through the point $(0,\, -12)$

    c x-intercepts -2 and 3 and passes through the point $(4,\, 18)$

    d x-intercepts -4 and 5 and passes through the point $(-1,\, 36)$

    e x-intercepts $1\frac{1}{2}$ and 3 and passes through the point $(1,\, 2)$

    f x-intercepts $-\frac{3}{4}$ and $\frac{5}{4}$ and passes through the point $(2,\, 33)$.

I   *USING TECHNOLOGY*   **[2.10]**

We have seen that the x-intercepts of the quadratic function $f(x) = ax^2 + bx + c$ correspond to the solutions of the equation $ax^2 + bx + c = 0$.

GRAPHING PACKAGE

So, we can find the solution to a quadratic equation by graphing the corresponding function and finding its x-intercepts. In this section we do this using technology. You can either use the graphing package provided, or else use the graphics calculator instructions beginning on page **22**.
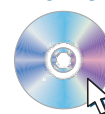
Example 26 ◀)) *Self Tutor*

Solve $2x^2 - 3x - 4 = 0$ using technology, giving your answers correct to 3 decimal places.

> Obviously you cannot provide screen dumps in your answer. You should sketch the graph instead.

Consider $y = 2x^2 - 3x - 4$.

The following graphics calculator screen dumps show successive steps:

So, $x \approx -0.851$ or 2.351.

EXERCISE 21I

1 Use technology to solve, correct to 3 decimal places:

    a $x^2 + 4x + 2 = 0$
    b $x^2 + 6x - 2 = 0$
    c $2x^2 - 3x - 7 = 0$

    d $3x^2 - 7x - 11 = 0$
    e $4x^2 - 11x - 13 = 0$
    f $5x^2 + 6x - 17 = 0$

2 Use technology to solve, correct to 3 decimal places:

    a $\frac{1}{2}x^2 - \frac{1}{3}x - \frac{1}{4} = 0$
    b $x^2 + \sqrt{2}x - 3 = 0$
    c $\sqrt{3}x^2 - 3x + 1 = 0$

J PROBLEM SOLVING [2.10, 3.2]

The problems in this section can all be converted to algebraic form as **quadratic equations**. They can all be solved using **factorisation**, the **quadratic formula** or **technology**.

However, if an equation can be solved by factorisation, it is expected that you will use this method.

PROBLEM SOLVING METHOD

Step 1: Carefully read the question until you understand the problem.
           A rough sketch may be useful.

Step 2: Decide on the unknown quantity and label it x, say.

Step 3: Use the information given to find an equation which contains x.

Step 4: Solve the equation.

Step 5: Check that any solutions satisfy the equation and are realistic to the problem.

Step 6: Write your answer to the question in sentence form.

Example 27   ◀) *Self Tutor*

The sum of a number and its square is 42. Find the number.

Let the number be x.   Therefore its square is x^2.

$$x + x^2 = 42$$
$$\therefore \quad x^2 + x - 42 = 0 \qquad \{\text{rearranging}\}$$
$$\therefore \quad (x+7)(x-6) = 0 \qquad \{\text{factorising}\}$$
$$\therefore \quad x = -7 \quad \text{or} \quad x = 6$$

So, the number is -7 or 6.   *Check:* If $x = -7$, $-7 + (-7)^2 = -7 + 49 = 42$ ✓

                                              If $x = 6$, $6 + 6^2 = 6 + 36 = 42$ ✓

Example 28   ◀) *Self Tutor*

A rectangle has length 5 cm greater than its width. If it has an area of 80 cm^2, find the dimensions of the rectangle, to the nearest mm.

If x cm is the width, then $(x+5)$ cm is the length.

$$\text{Now} \quad \text{area} = 80 \text{ cm}^2$$
$$\therefore \quad x(x+5) = 80$$
$$\therefore \quad x^2 + 5x = 80$$
$$\therefore \quad x^2 + 5x - 80 = 0$$

Using a graphics calculator, the graph of $y = x^2 + 5x - 80$ is:

As x is clearly positive, $x \approx 6.8$ cm (to the nearest mm).

$\therefore$ the rectangle is approx. 11.8 cm by 6.8 cm.
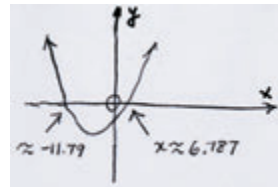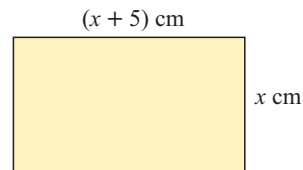
Example 29   ◀) *Self Tutor*
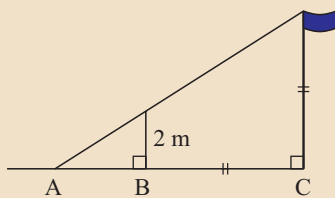
Given that BC is 3 m longer than AB, find the height of the flag pole.

Let the height of the pole be x m.

$\therefore$ BC $= x$ m   and   AB $= (x-3)$ m

The triangles are equiangular, so they are similar.

Hence $\dfrac{2}{x-3} = \dfrac{x}{(x-3)+x}$
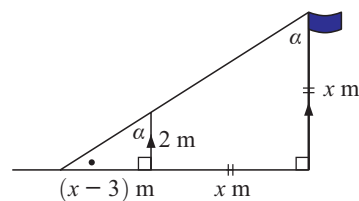
$$\therefore \quad 2(2x - 3) = x(x - 3)$$
$$\therefore \quad 4x - 6 = x^2 - 3x$$
$$\therefore \quad 0 = x^2 - 7x + 6$$
$$\therefore \quad (x - 1)(x - 6) = 0$$
$$\therefore \quad x = 1 \text{ or } 6$$

However, x cannot be 1 as $x - 3 > 0$
$$\therefore \quad x = 6$$

So, the flagpole is 6 m high.

Example 30   ◄)) *Self Tutor*

A stone is thrown into the air. Its height above the ground is given by the function
$h(t) = -5t^2 + 30t + 2$ metres   where t is the time in seconds from when the stone is thrown.

a How high is the stone above the ground at time $t = 3$ seconds?

b From what height above the ground was the stone released?

c At what time is the stone's height above the ground 27 m?

a $h(3) = -5(3)^2 + 30(3) + 2$
　　　$= -45 + 90 + 2$
　　　$= 47$
$\therefore$　the stone is 47 m above the ground.

b The stone was released when $t = 0$ s.
$\therefore \quad h(0) = -5(0)^2 + 30(0) + 2 = 2$
$\therefore$　the stone was released from 2 m above ground level.

c When $h(t) = 27$,
$$-5t^2 + 30t + 2 = 27$$
$$\therefore \quad -5t^2 + 30t - 25 = 0$$
$$\therefore \quad t^2 - 6t + 5 = 0 \qquad \{\text{dividing each term by } -5\}$$
$$\therefore \quad (t - 1)(t - 5) = 0 \qquad \{\text{factorising}\}$$
$$\therefore \quad t = 1 \text{ or } 5$$
$\therefore$　the stone is 27 m above the ground after 1 second and after 5 seconds.

EXERCISE 21J

1 The sum of a number and its square is 110. Find the number.

2 The square of a number is equal to 12 more than four times the number. Find the number.

3 The sum of two numbers is 6 and the sum of their squares is 90. Find the numbers.

4 When a number is subtracted from 2, the result is equal to the reciprocal of the original number. Find the number.

5 The base of a triangle is 5 m longer than its altitude. If its area is 33 m^2, find the length of the base.

6
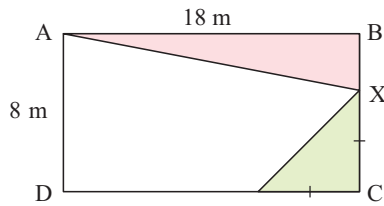
In the figure alongside, the two shaded triangles have equal area. Find the length of BX.

7 A rectangular enclosure is made from 45 m of fencing. The area enclosed is 125 m². Find the dimensions of the enclosure.

8 Two numbers have a sum of 5, and the sum of their reciprocals is 1. Find the exact numbers.

9 Find the exact value of x in:

a
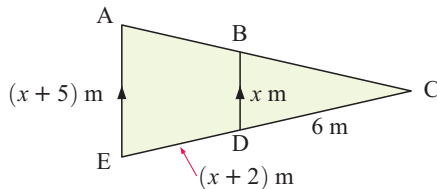
b
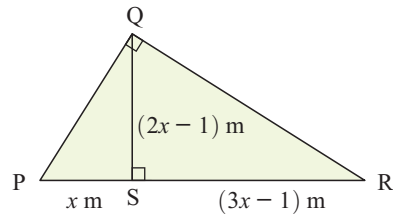
10 ABCD is a rectangle in which AB = 21 cm.
The square AXYD is removed and the remaining rectangle has area 80 cm². Find the length of BC.
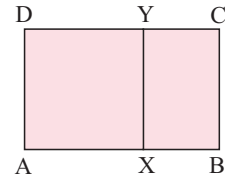
11 A right angled triangle has sides 2 cm and 16 cm respectively shorter than its hypotenuse. Find the length of each side of the triangle.

12 Nathan is swimming across a river from A to B. He is currently at N, having swum 30 m. If he was to change course and head directly for the opposite bank, he will save himself 20 m of swimming. Given that the river is 50 m wide, how much further must Nathan swim to get to B?
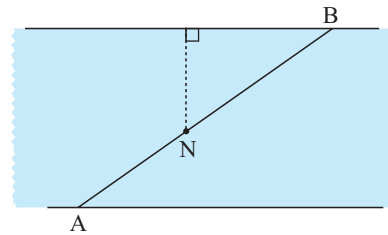
13
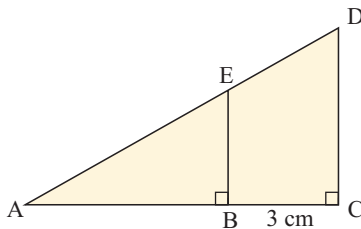
AB is 2 cm longer than BE. DC is 3 cm less than twice the length of BE.

a Explain why triangles ABE and ACD are similar.

b If BE = x cm, show that $x^2 - 4x - 6 = 0$.

c Hence, show that BE = $2 + \sqrt{10}$ cm.

14 In a 180 km bicycle race, a cyclist took $(t - 14)$ hours to complete the race, cycling at a constant speed of $(t + 10)$ km/h. Find:

a the value of t **b** the time the cyclist took to complete the race

c the speed of the cyclist.

15 The numerator of a fraction is 5 less than the denominator. If both the numerator and denominator are increased by 4, the fraction is tripled in value. Find the original fraction.

16 Two flagpoles are 6 m high and 13 m apart. Wires supporting the flagpoles are connected to a hook on the ground at X as illustrated. If the wires are perpendicular to each other, find the distance between the hook and the nearer flagpole.
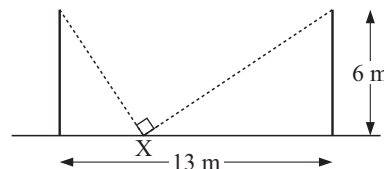
17 The sum of a number and twice its reciprocal is $2\frac{5}{6}$. Find the number.

18 An object is projected into the air with a velocity of 80 m/s. Its height after t seconds is given by the function $h(t) = 80t - 5t^2$ metres.

   a Calculate the height after: **i** 1 second **ii** 3 seconds **iii** 5 seconds.

   b Calculate the time(s) at which the height is: **i** 140 m **ii** 0 m.

   c Explain your answers in part **b**.

19 A cake manufacturer finds that the profit from making x cakes per day is given by the function $P(x) = -\frac{1}{2}x^2 + 36x - 40$ dollars.

   a Calculate the profit if: **i** 0 cakes **ii** 20 cakes are made per day.

   b How many cakes per day are made if the profit is $270?

20 Delivery boys Max and Sam each have 1350 newspapers to deliver. Max can deliver 75 more newspapers each hour than Sam, and finishes his job 1.5 hours faster. How long does each boy take to deliver their newspapers?

21 A sheet of cardboard is 15 cm long and 10 cm wide. It is to be made into an open box which has a base area of 66 cm^2, by cutting out equal squares from the four corners and then bending the edges upwards.
Find the size of the squares to be cut out.
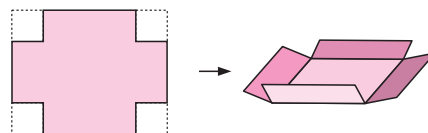
22 A doorway which is 1.5 m wide and 2 m high is to be surrounded by timber framing. The timber framing has a total area of 0.5 m^2.

   a If the timber framing is x m wide, show that $4x^2 + 11x - 1 = 0$.

   b Find the width of the timber framing to the nearest millimetre.
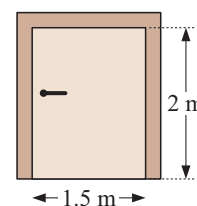
23 A right angled triangle has perimeter 40 m and area 60 m^2. Find the lengths of the sides of the triangle.

Review set 21A

1 Solve for x:

   a $2x^2 = 4$ **b** $3x^2 + 18 = 0$ **c** $5x(x - 3) = 0$

   d $x^2 + 24 = 11x$ **e** $10x^2 - 11x - 6 = 0$ **f** $3x^2 = 2x + 21$

2 Solve for x:

   a $x^2 = x$ **b** $(x + 3)^2 = -1$ **c** $3(x - 2)^2 = 15$

   d $\dfrac{x}{x - 3} = \dfrac{18}{x + 15}$ **e** $\dfrac{x + 3}{2} = \dfrac{x + 5}{3x}$ **f** $\dfrac{7}{x - 2} - \dfrac{2}{x + 1} = 2$

3 Use the quadratic formula to solve:

    a $2x^2 + 2x - 1 = 0$
    b $\dfrac{1}{x} - \dfrac{1}{1-x} = 2$

4 Use technology to solve:

    a $7x^2 + 9x - 4 = 0$
    b $9 - 2x^2 = 0$
    c $-2x + 5 - x^2 = 0$

5 If $g(x) = x^2 - 3x - 15$ find:

    a $g(0)$
    b $g(1)$
    c x such that $g(x) = 3$.

6 On the same set of axes, sketch $y = x^2$ and the function:

    a $y = 3x^2$
    b $y = (x-2)^2 + 1$
    c $y = -(x+3)^2 - 2$

7 For $y = -2(x-1)(x+3)$ find the:

    a **i** direction the parabola opens
              ii y-intercept
        iii x-intercepts
              iv equation of the line of symmetry.
    b Sketch a graph of the function showing all of the above features.

8 For $y = x^2 - 2x - 15$ find the:

    a **i** y-intercept
              ii x-intercepts
        iii equation of the line of symmetry
              iv coordinates of the vertex.
    b Sketch a graph of the function showing all of the above features.

9 If the graph of $f(x) = x^2 + bx + c$ has its vertex at $(-3, -11)$, find $f(x)$.

10 Given the function $f(x) = 3(x-2)^2 - 1$:

    a find the coordinates of the vertex
    b find the y-intercept
    c sketch the graph of $f(x)$.

11 Find the equation of the quadratic function with vertex $(-1, -5)$ and y-intercept -3. Give your answer in the form $f(x) = a(x-h)^2 + k$.

12 The graph of a quadratic function has x-intercepts -4 and $-\frac{1}{3}$, and passes through the point $(-1, -18)$. Find the quadratic function in expanded form.

13 Find the quadratic function which has vertex $(6, -2)$ and passes through the point $(4, 16)$. Give your answer in the form $f(x) = ax^2 + bx + c$.

14 The length of a rectangle is three times its width, and its area is 9 cm². Find the dimensions of the rectangle.

15 In a right angled triangle, the second to longest side is 5 cm longer than the shortest side, and the hypotenuse is three times longer than the shortest side. Find the exact length of the hypotenuse.

16 Find x in:
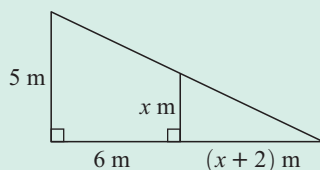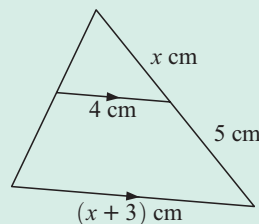
    a
    b
    c
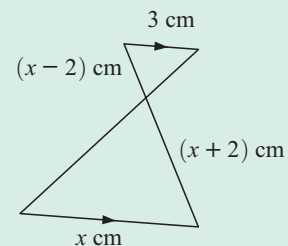
17 A stone was thrown from the top of a cliff 60 metres above sea level. The height of the stone above sea level t seconds after it was released is given by $H(t) = -5t^2 + 20t + 60$ metres.

    a Find the time taken for the stone to reach its maximum height.

    b What was the maximum height above sea level reached by the stone?

    c How long did it take before the stone struck the water?

Review set 21B

1 Solve for x:

    a $-2(x-3)^2 = 0$ **b** $(x+5)(x-4) = 0$ **c** $(2-x)^2 = -1$

    d $x^2 - 5x = 24$ **e** $2x^2 = 8$ **f** $6x^2 - x - 2 = 0$

2 Use the quadratic formula to solve: **a** $x^2 - 4x = 10$ **b** $x^2 + x - 9 = 0$

3 Solve for x:

    a $\dfrac{x}{x+3} = \dfrac{5}{x+7}$ **b** $\dfrac{2x-3}{10} = \dfrac{x-2}{x}$ **c** $\dfrac{1}{x+1} + \dfrac{3}{3x-5} = 1$

4 If $f(x) = 2x^2 + x - 2$, find:

    a $f(1)$ **b** $f(-3)$ **c** x such that $f(x) = 4$.

5 On the same set of axes, sketch $y = x^2$ and the function:

    a $y = -\frac{1}{2}x^2$ **b** $y = (x+2)^2 + 5$ **c** $y = -(x-1)^2 - 3$

6 For $y = 3(x-2)^2$ find the:

    a **i** direction the parabola opens   **ii** y-intercept

        iii x-intercepts   **iv** equation of the line of symmetry.

    b Sketch a graph of the function showing all of the above features.

7 For $y = -x^2 + 7x - 10$ find the:

    a **i** y-intercept   **ii** x-intercepts

        iii equation of the line of symmetry   **iv** coordinates of the vertex.

    b Sketch a graph of the function showing all of the above features.

8 The graph of $y = a\,(x-h)^2 + k$ is shown alongside.

    a Find the value of h.

    b Find the values of a and k by solving simultaneous equations.

9 Use axes intercepts to sketch the graphs of:

    a $f(x) = 2(x-3)(x+1)$ **b** $g(x) = -x(x+4)$
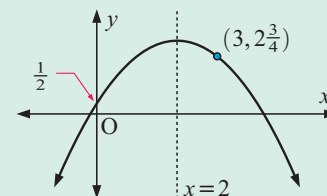
10 The quadratic function $f(x) = x^2 + bx + c$ has x-intercepts -5 and 3.
Find $f(x)$ in expanded form.

11 The graph of a quadratic function has vertex $(2, 2)$, and passes through the point $(-1, -7)$. Find the function in the form $f(x) = a(x-h)^2 + k$.

12 Find the quadratic function with vertex $(-4, 15)$ and y-intercept -17. Give your answer in the form $f(x) = ax^2 + bx + c$.

13 The sum of a number and five times its square, is equal to four. Find the number.

14 Two numbers differ by 3 and the difference between their reciprocals is 4. Find the exact values of the numbers given that they are both positive.

15 Kuan has £180 to share amongst his grandchildren. However, three of his grandchildren have misbehaved recently, so they will not receive any red packets with money in them. As a result, the remaining grandchildren receive an extra £5 each. How many grandchildren does Kuan have?

16 A 90 m $\times$ 90 m block of land is to be divided into three paddocks using two straight pieces of fencing as shown.

Given that all three paddocks must have the same area, determine where the fencing needs to be placed. To do this you need to find the lengths of CX and AY.

(Give AY to the nearest mm.)
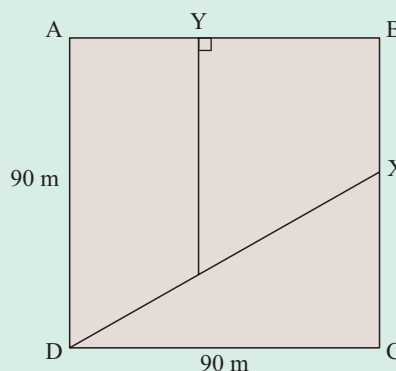
Challenge

1 **a** Show that $y = x^2 - x$ meets $y = 3 - x^2$ when $x = -1$ and $x = 1\frac{1}{2}$.

    b Find the maximum vertical separation between the two curves for $-1 \leqslant x \leqslant 1\frac{1}{2}$.

2 A manufacturer of refrigerators knows that if x of them are made, the cost of making each one of them will be $\left(100 + \dfrac{2000}{x}\right)$ dollars.

The receipts for selling all of them will be $\left(1300x - 4x^2\right)$ dollars.

How many should be produced to maximise his profits?

3 800 m of fencing is available for constructing six identical rectangular enclosures for horses.

What dimensions would maximise the area of each enclosure?
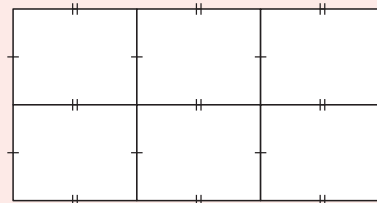
Two variable analysis

22

Contents:

Opening problem

The relationship between the *height* and *weight* of members of a football team is to be investigated. The raw data for each player is given below:

Player	Height	Weight
1	203	106
2	189	93
3	193	95
4	187	86
5	186	85
6	197	92

Player	Height	Weight
7	180	78
8	186	84
9	188	93
10	181	84
11	179	86
12	191	92

Player	Height	Weight
13	178	80
14	178	77
15	186	90
16	190	86
17	189	95
18	193	89

Things to think about:

- Are the variables categorical or quantitative?
- What is the dependent variable?
- What would the scatter diagram look like? Are the points close to being linear?
- Does an increase in the independent variable generally cause an increase or a decrease in the dependent variable?
- How can we indicate the strength of the linear connection between the variables?
- How can we find the equation of the 'line of best fit' and how can we use it?

TWO VARIABLE ANALYSIS

We often want to know how two variables are **associated** or **related**. We want to know whether an increase in one variable results in an increase or a decrease in the other.

To analyse the relationship between two variables, we first need to decide which is the **dependent** variable and which is the **independent** variable.

The value of the dependent variable *depends* on the value of the independent variable.

Next we plot known points on a **scatter diagram**. The independent variable is placed on the horizontal axis, and the dependent variable is placed on the vertical axis.

Consider the following two typical scatter diagrams:

In the first scatter diagram the points are quite random. It is hard to tell how they could be related.

In the second scatter diagram the points are all close to the red line shown. We say that there is a strong linear connection or **linear correlation** between these two variables. The red line is called the **line of best fit** because it best represents the data.
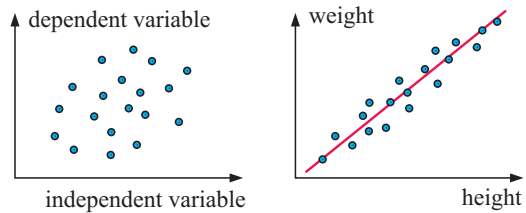
The scatter diagram for the **Opening Problem** is drawn alongside. *Height* is the independent variable and is represented on the horizontal axis. We see that in general, as the height increases, the weight increases also.
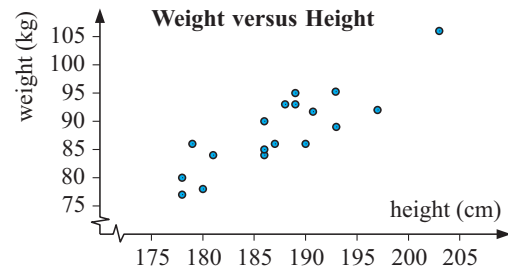
> The weight of a person is usually **dependent** on their height.

A   *CORRELATION*   [11.9]

> **Correlation** is a measure of the strength of the relationship or association between two variables.

When we analyse the correlation between two variables, we should follow these steps:

Step 1:   Look at the scatter diagram for any **pattern**.

For a generally *upward* shape we say that the correlation is **positive**.
As the independent variable increases, the dependent variable generally increases.
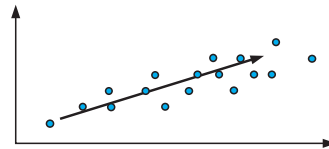
For a generally *downward* shape we say that the correlation is **negative**.
As the independent variable increases, the dependent variable generally decreases.
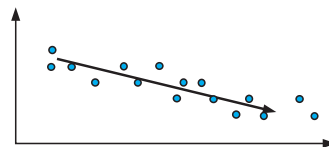
For *randomly scattered* points with no upward or downward trend, we say there is **no correlation**.
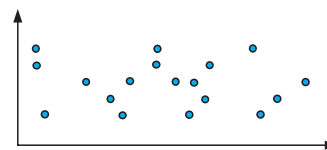
Step 2: Look at the spread of points to make a judgement about the **strength** of the correlation. For **positive relationships** we would classify the following scatter diagrams as:

strong   moderate   weak
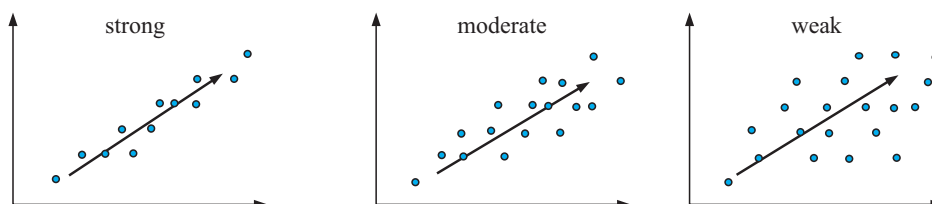
We classify the strengths for **negative relationships** in the same way:

strong   moderate   weak
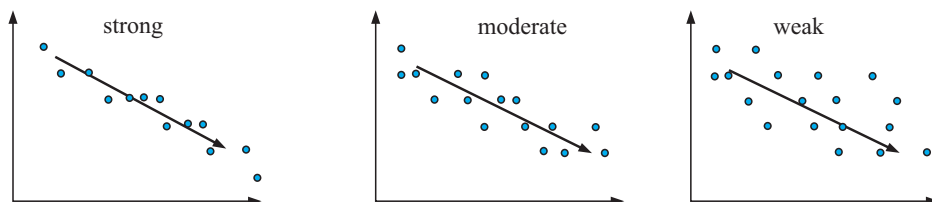
Step 3: Look at the pattern of points to see if the relationship is **linear**.

The relationship is approximately linear.   The relationship is not linear.

In the scatter diagram for the **Opening Problem** data, there appears to be a moderate positive correlation between the footballers' heights and weights. The relationship appears to be linear.

EXERCISE 22A

1 For each of the scatter diagrams below state:
   i whether there is positive, negative, or no association between the variables
   ii whether the relationship between the variables appears to be linear
   iii the strength of the association (zero, weak, moderate or strong).

a

b

c

d

e

f
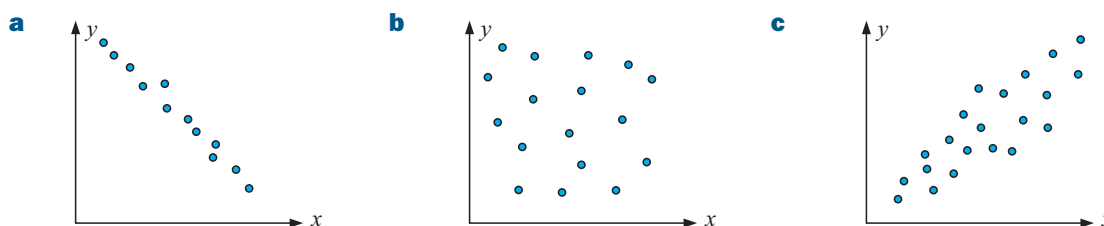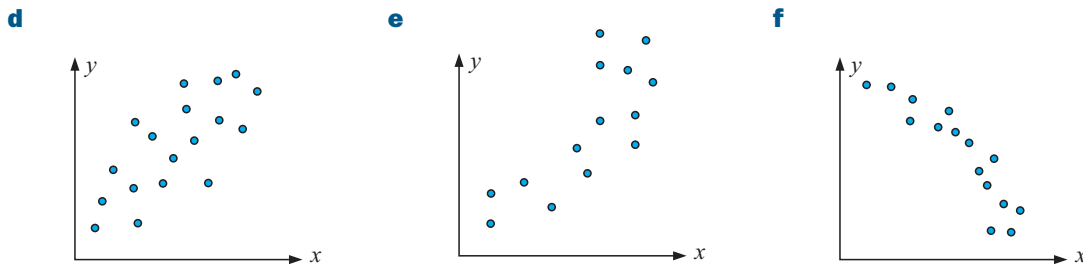
2 Copy and complete the following:

a If there is a positive association between the variables x and y, then as x increases, y ....

b If there is a negative correlation between the variables T and d, then as T increases, d ....

c If there is no association between two variables then the points on the scatter diagram are ....
....

3 **a** 10 students were asked for their exam marks in Physics and Mathematics. Their percentages are given in the table below.

Student	A	B	C	D	E	F	G	H	I	J
Physics	75	83	45	90	70	78	88	50	55	95
Maths	68	70	50	65	60	72	75	40	45	80

i Draw a scatter diagram with the Physics marks on the horizontal axis.

ii Comment on the relationship between the Physics and Mathematics marks.

b The same students were asked for their Art exam results. These were:

Student	A	B	C	D	E	F	G	H	I	J
Art	75	70	80	85	82	70	60	75	78	65

Draw a scatter diagram to see is there is any relationship between the Physics marks and the Art marks of each student.

4 The following table shows the sales of soft drinks in a shop each month, along with the average daily temperature for the month.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temp (^{o}C)	7	10	11	13	15	19	22	25	23	18	11	8
Sales (thousands £)	12	8	10	15	16	18	22	25	20	15	16	12

Draw a scatter diagram with the independent variable *temperature* along the horizontal axis. Comment on the relationship between the sales and the temperature.

5 Students were asked to measure their height in centimetres and their shoe size. The results are recorded in the table below:

Height (cm)	165	155	140	145	158	148	160	164	160	155	150	160
Shoe size	6.5	4.5	4	5.5	6	5.5	6	6.5	5.5	5	5	5.5

Comment on any relationship between height and shoe size.

| B | **LINE OF BEST FIT BY EYE** | **[11.9]** |

Consider again the **Opening problem**.

The scatter diagram for this data is shown alongside. We can see there is a moderate positive linear correlation between the variables, so it is reasonable to use a line of best fit to model the data.
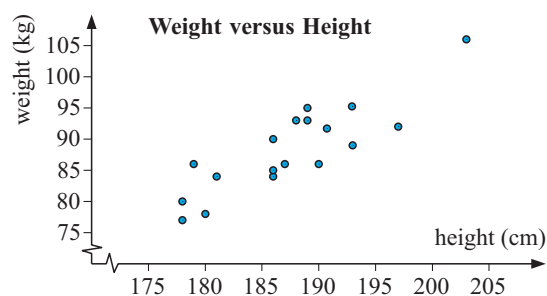
One way to do this is to draw a straight line through the data points which:

- includes the **mean point** $(\overline{x}, \overline{y})$
- has about as many points above the line as are below it.

For the **Opening problem**, the mean point is approximately (187, 88).
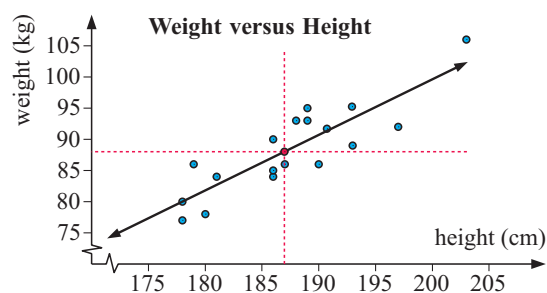
After plotting the mean on the scatter diagram, we draw in the line of best fit by eye.

As this line is an estimate only, lines drawn by eye will vary from person to person.

Having found our line of best fit, we can then use this linear model to estimate a value of y for any given value of x.

Example 1   ◆)) *Self Tutor*

Ten students were surveyed to find the number of marks they received in a pre-test for a module of work, and a test after it was completed.

The results were:

Pre-test (x)	40	79	60	65	30	73	56	67	45	85
Post-test (y)	48	91	70	71	50	85	65	75	60	95

a Find the mean point $(\overline{x}, \overline{y})$.

b Draw a scatter diagram of the data. Mark the point $(\overline{x}, \overline{y})$ on the scatter diagram and draw in the line of best fit.

c Estimate the mark for another student who was absent for the post-test but scored 70 for the pre-test.

a $\overline{x} = \dfrac{40 + 79 + 60 + ... + 85}{10} = 60$

$\overline{y} = \dfrac{48 + 91 + 70 + ... + 95}{10} = 71$ So, $(\overline{x}, \overline{y})$ is (60, 71).

b
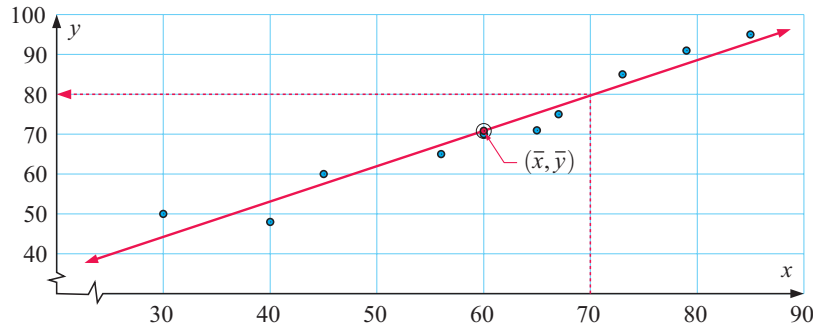
c   When $x = 70$, $y \approx 80$.

    $\therefore$ we estimate the post-test score to be 80.

EXERCISE 22B

1   A class of 13 students was asked to record their times spent preparing for a test. The table below gives their recorded preparation times and the scores that they achieved for the test.

Minutes spent preparing	75	30	35	65	110	60	40	80	56	70	50	110	18
Test score	25	31	30	38	55	20	39	47	35	45	32	33	34

   a   Which of the two variables is the independent variable?

   b   Calculate the mean time $\overline{x}$ and the mean score $\overline{y}$.

   c   Construct a scatter diagram for the data.

   d   Comment on the correlation between the variables.

   e   Copy and complete the following statements about the scatter diagram:

      There appears to be a .... , ... correlation between the *minutes spent preparing* and the *test score*. This means that as the time spent preparing increases the scores ....

   f   Plot $(\overline{x}, \overline{y})$ on the scatter diagram.

   g   Draw the line of best fit on the scatter diagram.

   h   If another student spent 25 minutes preparing for the test, what would you predict his test score to be?

2   The table alongside shows the *percentage of unemployed adults* and the *number of major thefts per day* in eight large cities.

City	Percentage	Number
A	7	113
B	6	67
C	10	117
D	7	82
E	9	120
F	6	32
G	3	61
H	7	76

   a   Find the mean percentage $\overline{x}$ and the mean number of thefts $\overline{y}$.

   b   Draw a scatter diagram for this data.

   c   Describe the association between the *percentage of unemployed adults* and the *number of major thefts per day*.

   d   Plot $(\overline{x}, \overline{y})$ on the scatter diagram.

   e   Draw the line of best fit on the scatter diagram.

   f   Another city has 8% unemployment. Estimate the number of major thefts per day for that city.

3 A café manager believes that during April the *number of people wanting dinner* is related to the *temperature at noon*. Over a 13 day period, the number of diners and the noon temperature were recorded.

Temperature (x °C)	18	20	23	25	25	22	20	23	27	26	28	24	22
Number of diners (y)	63	70	74	81	77	65	75	87	91	75	96	82	88

a Find the mean point $(\overline{x}, \overline{y})$.

b Draw a scatter diagram for this data.

c Comment on the correlation between the variables.

d Plot $(\overline{x}, \overline{y})$ on the scatter diagram.

e Draw the line of best fit on the scatter diagram.

f Estimate the number of diners at the café when it is April and the temperature is:

    i 19°C   **ii** 29°C.

C   LINEAR REGRESSION   [11.9]

The problem with drawing a line of best fit by eye is that the answer will vary from one person to another and the equation of the line may not be very accurate.

> **Linear regression** is a formal method of finding a line which best fits a set of data.

We can use technology to perform linear regression and hence find the equation of the line. Most graphics calculators and computer packages use the method of 'least squares' to determine the gradient and the y-intercept.

THE 'LEAST SQUARES' REGRESSION LINE

The mathematics behind this method is generally established in university mathematics courses.

However, in brief, we find the vertical distances d_1, d_2, d_3, ...   to the line of best fit.

We then add the squares of these distances, giving
$d_1^2 + d_2^2 + d_3^2 + ...$

The **least squares regression line** is the one which makes this sum as small as possible.

Click on the icon. Use trial and error to try to find the least squares line of best fit for the data provided in the software.
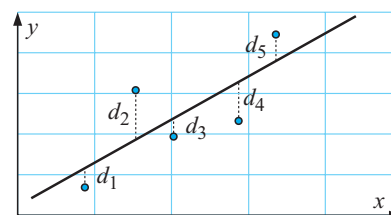
COMPUTER DEMO

Consider the following data which was collected by a milkbar owner over ten consecutive days:

Max daily temperature (t°C)	29	40	35	30	34	34	27	27	19	37
Number of icecreams sold (N)	119	164	131	152	206	169	122	143	63	208

Using a graphics calculator we can obtain a scatter diagram. We then find the equation of the least squares regression line in the form $y = ax + b$. For instructions on how to do this, see page **26** of the graphics calculator instructions.

```
LinReg
y=ax+b
a=5.64149578
b=-28.39746683
r²=.6313367927
r=.7945670474
```
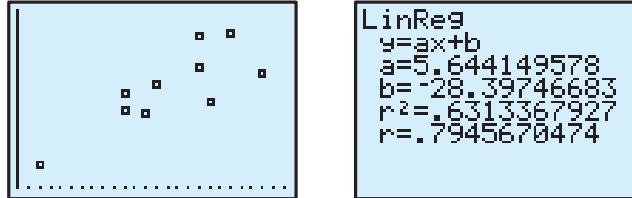
So, the linear regression model is $y \approx 5.64x - 28.4$ or $N \approx 5.64t - 28.4$.

THE CORRELATION COEFFICIENT AND COEFFICIENT OF DETERMINATION

Notice in the screen dump above that it also contains $r^2 \approx 0.631$ and $r \approx 0.795$.

r is **Pearson's correlation coefficient** and r^2 is called the **coefficient of determination**.

These values are important because they tell us how close to linear a set of data is. There is no point in fitting a linear relationship between two variables if they are clearly not linearly related.

All values of r lie between -1 and $+1$.

If $r = +1$, the data is **perfectly positively correlated**. This means the data lie exactly in a straight line with positive gradient.

If $0 < r \leqslant 1$, the data is positively correlated.

If $r = 0$, the data shows **no correlation**.

If $-1 \leqslant r < 0$, the data is negatively correlated.

If $r = -1$, the data is **perfectly negatively correlated**. This means the data lie exactly in a straight line with negative gradient.

r and r^2 are not required for this course, but they are useful and easily available from a calculator or statistics software.

Scatter diagram examples for positive correlation:

The scales on each of the four graphs are the same.
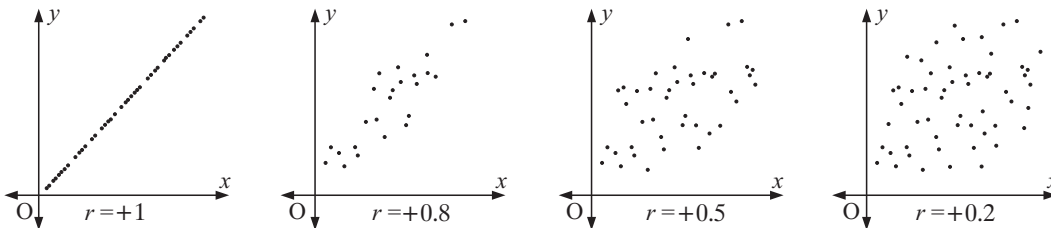
Scatter diagram examples for negative correlation:

IGCSE01
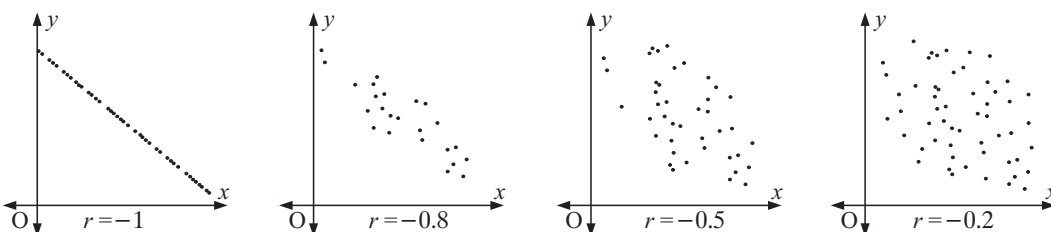
The following table is a guide for describing the strength of linear association using the coefficient of determination:

Value	Strength of association
$r^2 = 0$	no correlation
$0 < r^2 < 0.25$	very weak correlation
$0.25 \leqslant r^2 < 0.50$	weak correlation
$0.50 \leqslant r^2 < 0.75$	moderate correlation
$0.75 \leqslant r^2 < 0.90$	strong correlation
$0.90 \leqslant r^2 < 1$	very strong correlation
$r^2 = 1$	perfect correlation

For example, for the daily temperature data, as $r^2 \approx 0.631$ and $r > 0$, the two variables N and t show moderate positive correlation only.

LINEAR REGRESSION BY COMPUTER

STATISTICS PACKAGE

Click on the icon to obtain a computer statistics package. This will enable you to find the equation of the linear regression line, as well as r and r^2.
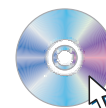
INTERPOLATION AND EXTRAPOLATION

STATISTICS PACKAGE

Suppose we have gathered data to investigate the association between two variables. We obtain the scatter diagram shown below. The data values with the lowest and highest values of x are called the **poles**.
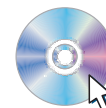
We use least squares regression to obtain a line of best fit. We can use the line of best fit to estimate values of one variable given a value for the other.

If we use values of x **in between** the poles, we say we are **interpolating** between the poles.

If we use values of x **outside** the poles, we say we are **extrapolating** outside the poles.

The accuracy of an interpolation depends on how linear the original data was. This can be gauged by determining the correlation coefficient and ensuring that the data is randomly scattered around the line of best fit.
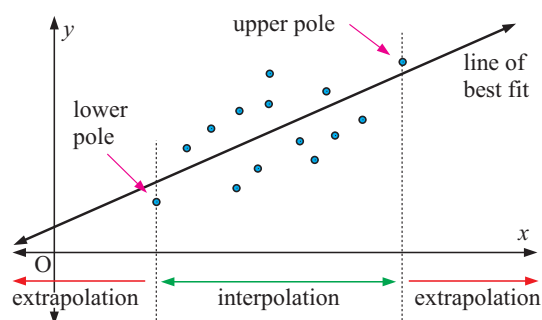
The accuracy of an extrapolation depends not only on how linear the original data was, but also on the assumption that the linear trend will continue past the poles. The validity of this assumption depends greatly on the situation under investigation.

As a general rule, it is reasonable to interpolate between the poles, but unreliable to extrapolate outside them.

CARE MUST BE TAKEN WHEN EXTRAPOLATING

We need to be careful when extrapolating.

For example, in the 30 years prior to the 1968 Mexico City Olympic Games, there was a steady, regular increase in the long jump world record. However, due to the high altitude and a perfect jump, the USA competitor Bob Beamon shattered the record by a huge amount, not in keeping with previous increases.
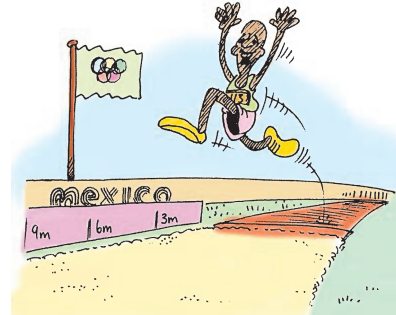
Example 2   ◀)) *Self Tutor*

The table below shows the sales for Yong's Computer supplies established in late 2001.

Year	2002	2003	2004	2005	2007	2008
Sales ($10 000s)	6.1	7.6	17.3	19.7	20.6	30.9

a Draw a scatter diagram to illustrate this data.

b Find the equation of the line of best fit using a graphics calculator. Give your answers correct to 4 significant figures.

c Predict the sales figures for 2006, giving your answer to the nearest $1000.
Comment on whether this prediction is reasonable.

d Predict the sales figures for 2010, giving your answer to the nearest $1000.
Comment on whether this prediction is reasonable.

a Let t be the time in years from 2001 and S be the sales in $10 000s.

t	S
1	6.1
2	7.6
3	17.3
4	19.7
6	20.6
7	30.9
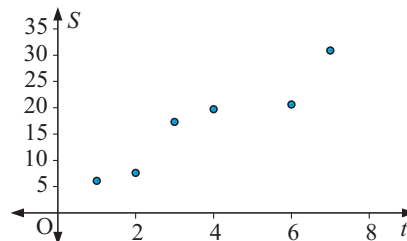
b The line of best fit is $S \approx 3.732t + 2.729$ {from a graphics calculator}

c In 2006, $t = 5$ ∴ $S \approx 3.732 \times 5 + 2.729 \approx 21.4$
So, we the estimate the sales for 2006 to be $214 000.
The scatter diagram suggests that the linear relationship between sales and the year is strong and positive. Since the prediction is interpolation between the poles, this estimate is reasonable.

d In 2010, $t = 9$ ∴ $S \approx 3.732 \times 9 + 2.729 \approx 36.3$
So, we estimate the sales for 2010 to be $363 000.
Since this prediction is an extrapolation, it will only be reasonable if the trend evident from 2002 to 2008 continues to the year 2010. This may or may not occur.

EXERCISE 22C

1 Revisit the **Opening Problem** on page **455**.

    a Calculate the equation of the line of best fit using linear regression.

    b Use the equation of the line to predict the weight of a 200 cm tall football player.

2 Tomatoes are sprayed with a pesticide-fertiliser mix. The figures below give the yield of tomatoes per bush for various spray concentrations.

Spray concentration, x ml per 2 litres	2	4	6	8	10	12
Yield of tomatoes per bush, y	45	76	93	105	119	124

    a What is the independent variable?

    b Draw a scatter diagram for the data.

    c Calculate the equation of the linear regression line using technology.

    d Interpret the gradient and vertical intercept of this line.

    e Use the equation of the linear regression line to predict the yield if the spray concentration was 7 ml. Comment on whether this prediction is reasonable.

3 The data below show the number of surviving lawn beetles in a square metre of lawn two days after it was sprayed with a new chemical.
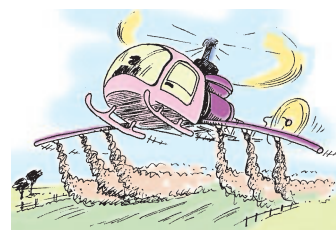
Amount of chemical, x g	2	3	5	6	9
Number of lawn beetles, y	11	6	6	4	3

    a What is the dependent variable?   **b** Draw a scatter diagram for the data.

    c Use technology to determine the equation of the linear regression line.

    d Interpret the gradient and vertical intercept of this line.

    e Use the regression line to predict the number of beetles surviving if 7 g of spray were used. Comment on whether this prediction is reasonable.

4 The yield of cherries depends on the number of frosty mornings experienced by the tree. The following table shows the yield of cherries from an orchard over several years with different numbers of frosty mornings.

Frosty mornings (n)	18	29	23	38	35	27
Yield (Y)	29.4	34.6	32.1	36.9	36.1	32.5

    a Produce a scatter diagram of Y against n.

    b Find the linear model which best fits the data.

    c Estimate the yield from the orchard if the number of frosty mornings is:

        i 31   **ii** 42.

    d Complete: "The greater the number of frosty mornings, the .... the yield of cherries."

5 Carbon dioxide (CO_2) is a chemical linked to acid rain and global warming. The concentration of CO_2 in the atmosphere has been recorded over a 40 year period. It is measured in parts per million or ppm found in Law Dome Ice Cores in Antarctica.

Year	1960	1970	1980	1990	2000
CO_2 concentration (ppm)	313	321	329	337	345

Let t be the number of years since 1960 and C be the CO_2 concentration.

  a Obtain a scatter diagram for the data. Is a linear model appropriate?

  b Find the equation of the linear regression line.

  c Estimate the CO_2 concentration for 1987.

  d If CO_2 emission continues at the same rate, estimate the concentration in 2020.
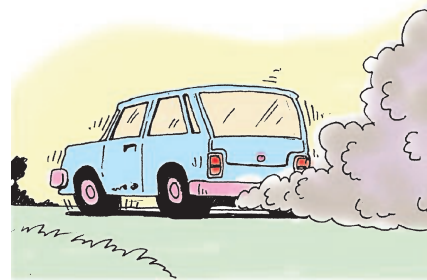
6 Safety authorities advise drivers to travel 3 seconds behind the car in front of them. This provides the driver with a greater chance of avoiding a collision if the car in front has to brake quickly or is itself involved in an accident. A test was carried out to find out how long it would take a driver to bring a car to rest from the time a red light was flashed. This *stopping time* includes both the reaction time of the driver and the braking time for the car. The following results are for one driver in the same car under the same test conditions:

Speed (v km/h)	10	20	30	40	50	60	70	80	90
Stopping time (t s)	1.23	1.54	1.88	2.20	2.52	2.83	3.15	3.45	3.83

  a Produce a scatter diagram for the data.

  b Find the linear model which best fits the data.

  c Use the model to estimate the stopping time for a speed of:

    i 55 km/h   **ii** 110 km/h

  d Comment on the reliability of your results in **c**.

  e Interpret the vertical intercept of the line of best fit.

  f Explain why the 3 second rule applies at all speeds, with a good safety margin.

Review set 22A

1 The scatter diagram shows the number of defective items made by each employee of a factory, plotted against the employee's number of weeks of experience.

  a What are the independent and dependent variables?

  b Is the association between the variables:

    i weak or strong   **ii** positive or negative?
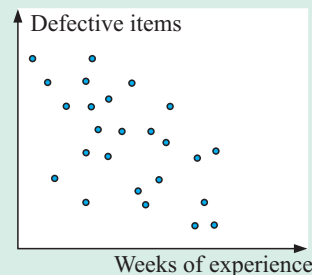
2 The maximum speed of a Chinese dragonboat with different numbers of paddlers is recorded in the table below:

Number of paddlers, x	4	6	10	18	30
Maximum speed, y km/h	8	11	13	16	25

  a Draw a scatter diagram for the data.

  b Find $\overline{x}$ and $\overline{y}$.

  c Plot the point $(\overline{x}, \overline{y})$ on the scatter diagram.

  d Draw the line of best fit by eye on the scatter diagram.

  e Predict the maximum speed of a dragonboat with: **i** 24 paddlers   **ii** 40 paddlers.

3 Strawberry plants are sprayed with a pesticide-fertiliser mix. The
data below give the yield of strawberries per plant for various
spray concentrations:

Spray concentration (x ml per litre)	0	2	4	5	8
Yield ofstrawberries per plant, (y)	8	10	21	20	35

   a Draw a scatter diagram for the data.

   b Comment on the correlation between the variables.

   c What is the significance of your answers in **b**?

   d Find the equation of regression line for the data.

   e Predict the number of strawberries per plant if the spray concentration is:

      i 3 ml per litre   **ii** 10 ml per litre.

   f Give one reason why your answer to **e ii** may be invalid.

4 The whorls on a cone shell get broader as you go from the top of the shell towards the bottom.
Measurements from a shell are summarised in the following table:

Position of whorl, p	1	2	3	4	5	6	7	8
Width of whorl, w cm	0.7	1.2	1.4	2.0	2.0	2.7	2.9	3.5

   a What are the dependent and independent variables?

   b Obtain a scatter diagram for the data.

   c Comment on the corrrelation between the variables.

   d Find the linear regression model which best fits the data.

   e If a cone shell has 14 whorls, what width do you expect
the 14th whorl to have? How reliable do you expect this
prediction to be?

Review set 22B

1 Traffic controllers want to find the association between the
average speed of cars in a city and the *age of drivers*. Devices
for measuring average speed were fitted to the cars of drivers
participating in a survey.

The results are shown in the scatter diagram.

   a What is the independent variable?

   b Describe the association between the variables.

   c Is it sensible to find the linear regression line for these
variables? Why or why not?
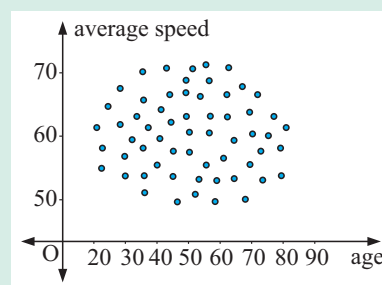
2 Following an outbreak of the *Ebola* virus, a rare and deadly haemorrhagic fever, medical authorities
begin taking records of the number of cases of the fever. Their records are shown below.

Days after outbreak, n	2	3	4	5	6	7	8	9	10	11
Diagnosed cases, d	8	14	33	47	80	97	118	123	139	153

  a Produce a scatter diagram for d against n. Does a linear model seem appropriate for this data?

  b Find $\overline{n}$ and $\overline{d}$.

  c Plot the point $(\overline{n}, \overline{d})$ on the scatter diagram.

  d Draw the line of best fit by eye.

  e Use the graph to predict the number of diagnosed cases on day 14. Is this predicted value reliable? Give reasons for your answer.

3 The following table gives peptic ulcer rates per 1000 people for differing family incomes in the year 1998.

Income (I €1000s)	10	15	20	25	30	40	50	60	80
Peptic ulcer rate (R)	8.3	7.7	6.9	7.3	5.9	4.7	3.6	2.6	1.2

  a Draw a scatter diagram for the data.

  b Find the equation of the regression line for the data.

  c Estimate the peptic ulcer rate in families with an income of €45 000.

  d Explain why the model is inadequate for families with income in excess of €100 000.

  e Later it is realised that one of the figures was written incorrectly.

     i Which is it likely to be? Explain your answer.

     ii Repeat **b** and **c** without the incorrect data value.

4 The average value of a circulated 1930 Australian penny sold at auction over the period from 1962 to 2006 is shown in dollars in the table below.

Year	1962	1972	1977	1985	1991	1996	2002	2006
Value	120	335	615	1528	2712	4720	8763	16 250

Suppose x is the number of years *since* 1962 and V is the value of the penny in dollars.

  a Draw a scatter diagram of V against x.

  b Comment on the association between the variables.

  c Find a linear model for V in terms of x using technology.

  d Explain why this model is not appropriate and we should not use it.

  e How could you estimate the value of the penny in 2012?

Further functions

23

Contents:

Opening problem

Phone calls from Jamie's mobile phone cost \$1.10 per minute, plus a 30 cent connection fee. So, a phone call which lasts x minutes will cost Jamie $C(x) = 1.1x + 0.3$ dollars.

Jamie wants to find the *inverse* of this function, which is the function which tells him how long he can talk, for a certain amount of money.

Can you find this function?

A   CUBIC FUNCTIONS   [3.2, 3.3]

A **cubic function** has the form $f(x) = ax^3 + bx^2 + cx + d$ where $a \neq 0$ and a, b, c and d are constants.

Discovery   *Cubic functions*

To discover the shape of different cubics you can either use the graphing package or your graphics calculator.

GRAPHING PACKAGE

What to do:

1   **a**   Use technology to help sketch graphs of:

        i $y = x^3$, $y = 2x^3$, $y = \frac{1}{2}x^3$, $y = \frac{1}{3}x^3$ **ii** $y = x^3$ and $y = -x^3$

        iii $y = -x^3$, $y = -2x^3$, $y = -\frac{1}{2}x^3$, $y = -\frac{1}{10}x^3$

   b   Discuss the geometrical significance of a in $y = ax^3$.
       You should comment on both the sign and the size of a.

2   **a**   Use technology to help sketch graphs of:

        i $y = x^3$, $y = x^3 + 2$, $y = x^3 - 3$

        ii $y = x^3$, $y = (x + 2)^3$, $y = (x - 3)^3$

        iii $y = (x - 1)^3 + 2$, $y = (x + 2)^3 + 1$, $y = (x - 2)^3 - 2$

   b   Discuss the geometrical significance of:

      • k in the family of cubics of the form $y = x^3 + k$

      • h in the family of cubics of the form $y = (x - h)^3$

      • h and k in the family of cubics of the form $y = (x - h)^3 + k$.

3   **a**   Use technology to help sketch graphs of:

       $y = (x - 1)(x + 1)(x + 3)$, $y = 2(x - 1)(x + 1)(x + 3)$,
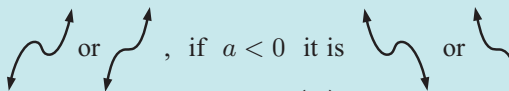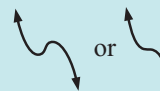
       $y = \frac{1}{2}(x - 1)(x + 1)(x + 3)$, $y = -2(x - 1)(x + 1)(x + 3)$.

   b   For each graph in **a** state the x-intercepts and the y-intercept.

   c   Discuss the geometrical significance of a in $y = a(x - \alpha)(x - \beta)(x - \gamma)$.

4   **a**   Use technology to help sketch graphs of:

       $y = 2x(x + 1)(x - 2)$, $y = 2(x + 3)(x - 1)(x - 2)$ and $y = 2x(x + 2)(x - 1)$.

   b   For each graph in **a** state the x-intercepts and y-intercept.

   c   Discuss the geometrical significance of α, β and γ for the cubic
       $y = a(x - \alpha)(x - \beta)(x - \gamma)$.

5   **a**   Use technology to help sketch graphs of:

       $y = (x - 2)^2(x + 1)$, $y = (x + 1)^2(x - 3)$, $y = 2(x - 3)^2(x - 1)$,

       $y = -x(x - 2)^2$, $y = -2(x + 1)(x - 2)^2$.

   b   For each graph in **a**, state the x-intercepts and the y-intercept.

   c   Discuss the geometrical significance of α and β for the cubic $y = a(x - \alpha)^2(x - \beta)$.

You should have discovered that:

- if $a > 0$, the graph's shape is or , if $a < 0$ it is or

- $y = (x - h)^3 + k$ is the translation of $y = x^3$ through $\begin{pmatrix} h \\ k \end{pmatrix}$

- for a cubic in the form $y = a(x - \alpha)(x - \beta)(x - \gamma)$ the graph has x-intercepts α, β and γ and the graph crosses over or **cuts** the x-axis at these points

- for a cubic in the form $y = a(x - \alpha)^2(x - \beta)$ the graph **touches** the x-axis at α and **cuts** it at β
- cubic functions have a **point of rotational symmetry** called the **point of inflection**.

Example 1   ◀)) *Self Tutor*

Use axes intercepts only to sketch the graphs of:

a $f(x) = \frac{1}{2}(x + 2)(x - 1)(x - 2)$ **b** $f(x) = 2x(x - 2)^2$

a $f(x) = \frac{1}{2}(x + 2)(x - 1)(x - 2)$
has x-intercepts -2, 1, and 2
$f(0) = \frac{1}{2}(2)(-1)(-2) = 2$
∴ the y-intercept is 2

b $f(x) = 2x(x - 2)^2$
cuts the x-axis when $x = 0$ and
touches the x-axis when $x = 2$
$f(0) = 2(0)(-2)^2 = 0$
∴ the y-intercept is 0
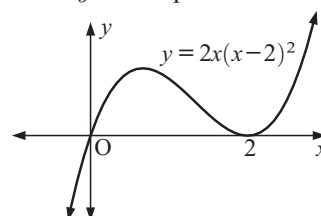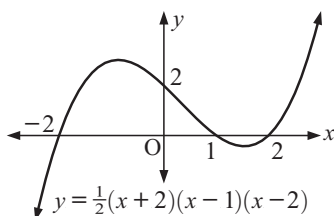
EXERCISE 23A.1

1 By expanding out the following, show that they are cubic functions.

a $f(x) = (x + 3)(x - 2)(x - 1)$ **b** $f(x) = (x + 4)(x - 1)(2x + 3)$

c $f(x) = (x + 2)^2(2x - 5)$ **d** $f(x) = (x + 1)^3 + 2$

2 Use axes intercepts only to sketch the graphs of:

a $y = (x + 1)(x - 2)(x - 3)$ **b** $y = -2(x + 1)(x - 2)(x - \frac{1}{2})$

c $y = \frac{1}{2}x(x - 4)(x + 3)$ **d** $y = 2x^2(x - 3)$

e $y = -\frac{1}{4}(x - 2)^2(x + 1)$ **f** $y = -3(x + 1)^2(x - \frac{2}{3})$

FINDING A CUBIC FUNCTION

If we are given the graph of a cubic with sufficient information, we can determine the form of the function.
We do this using the same techniques we used for quadratic functions.

Example 2   ◀)) *Self Tutor*

Find the form of the cubic with graph:

a

b
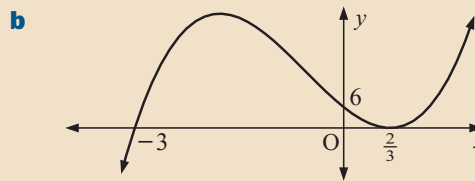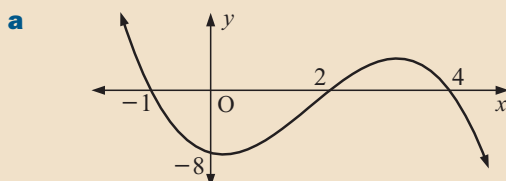
a The x-intercepts are -1, 2, and 4

$\therefore \quad f(x) = a(x+1)(x-2)(x-4)$

But when $x = 0$, $y = -8$

$\therefore \quad a(1)(-2)(-4) = -8$

$\therefore \quad 8a = -8$

$\therefore \quad a = -1$

So, $f(x) = -(x+1)(x-2)(x-4)$

b The graph touches the x-axis at $\frac{2}{3}$, indicating a squared factor $(3x-2)^2$. Its other x-intercept is -3, so

$f(x) = a(3x-2)^2(x+3)$

But when $x = 0$, $y = 6$

$\therefore \quad a(-2)^2(3) = 6$

$\therefore \quad 12a = 6$

$\therefore \quad a = \frac{1}{2}$

So, $f(x) = \frac{1}{2}(3x-2)^2(x+3)$

Example 3   ◀) **Self Tutor**

Find the equation of the cubic with graph:
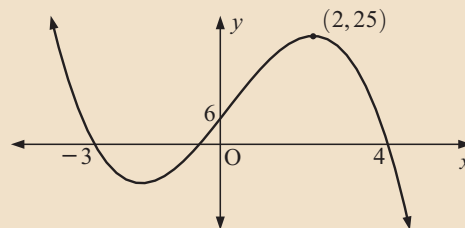
We only know two of the x-intercepts, -3 and 4, so $(x+3)$ and $(x-4)$ are linear factors.

We suppose the third linear factor is $(ax+b)$, so $f(x) = (x+3)(x-4)(ax+b)$

But $f(0) = 6$ and $f(2) = 25$

$\therefore \quad (3)(-4)(b) = 6$ $\therefore \quad (5)(-2)(2a+b) = 25$

$\therefore \quad -12b = 6$ $\therefore \quad 2a - \frac{1}{2} = -\frac{5}{2}$ {using (1)}

$\therefore \quad b = -\frac{1}{2}$ ...... (1) $\therefore \quad 2a = -2$

$\therefore \quad a = -1$

Thus $f(x) = (x+3)(x-4)(-x-\frac{1}{2})$ or $f(x) = -(x+3)(x-4)(x+\frac{1}{2})$

EXERCISE 23A.2

1 Find the form of the cubic function with graph:

a
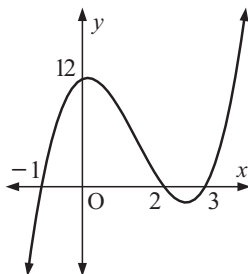
b
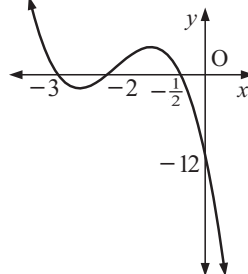
c
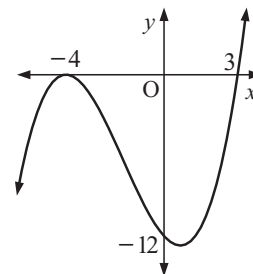
2 Find the equation of the cubic function which:

  a has x-intercepts 1 and 3, y-intercept 9 and passes through $(-1, 8)$

  b touches the x-axis at 3, has y-intercept 18 and passes through $(1, 20)$.

3 The graph alongside has the form $y = 2x^3 + bx^2 + cx - 12$.
Find the values of b and c.
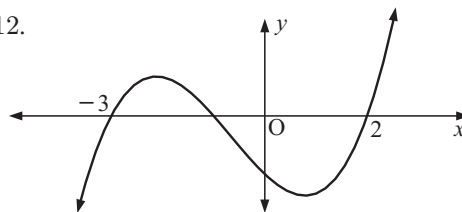
4 The graph alongside has the form $y = -x^3 + bx^2 + 4x + d$.
Find the values of b and d.
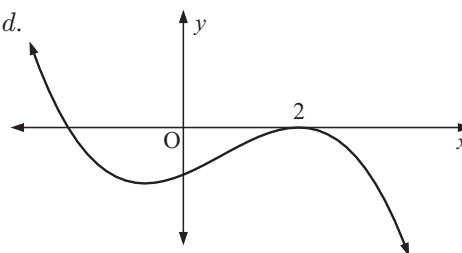
B   INVERSE FUNCTIONS   [3.9]

Consider the mapping "add 5":
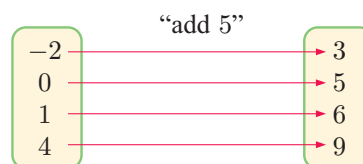
Suppose we wanted to reverse this mapping, so we want to map 3 back to -2, 5 back to 0, and so on.

To achieve this we would use the reverse
or **inverse** operation of "add 5", which is
"subtract 5":
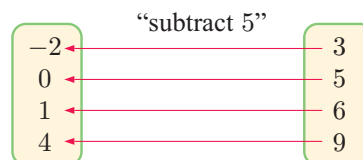
> The **inverse function** f^{-1} of a function f is the function such that, for every value of x that f maps
> to $f(x)$, f^{-1} maps $f(x)$ back to x.

From the above example, we can see that the inverse of $f(x) = x + 5$ or "add 5" is $f^{-1}(x) = x - 5$
or "subtract 5".

Consider the function $f(x) = x^3 + 4$. The process performed by this function is to "cube x, then add 4".
To find $f^{-1}(x)$, we need to reverse this process. Using inverse operations, we "subtract 4, then take the
cube root of the result", and so $f^{-1}(x) = \sqrt[3]{x - 4}$.

Unfortunately, it is not always so easy to reverse the process in a given function. However, there is an
algebraic method we can use to find the inverse function.

> The inverse of $y = f(x)$ can be found algebraically by interchanging x and y, and then making y the
> subject of the resulting formula. The new y is $f^{-1}(x)$.

The **horizontal line test** says that *'for a function to have an inverse function, no horizontal line can cut it more than once.'*

Example 4

◀) **Self Tutor**

Find $f^{-1}(x)$ for: **a** $f(x) = 8 - 3x$ **b** $f(x) = \dfrac{10}{x + 1}$

a   By interchanging x and y, the inverse of
$$y = 8 - 3x$$
is $x = 8 - 3y$
$$\therefore \quad 3y = 8 - x$$
$$\therefore \quad y = \frac{8 - x}{3}$$
$$\therefore \quad f^{-1}(x) = \frac{8 - x}{3}$$

b   By interchanging x and y, the inverse of
$$y = \frac{10}{x + 1}$$
is $x = \dfrac{10}{y + 1}$
$$\therefore \quad x(y + 1) = 10$$
$$\therefore \quad xy + x = 10$$
$$\therefore \quad xy = 10 - x$$
$$\therefore \quad y = \frac{10 - x}{x}$$
$$\therefore \quad f^{-1}(x) = \frac{10 - x}{x}$$

EXERCISE 23B

1   Find $f^{-1}(x)$ for each of the following functions:

   a $f(x) = x - 7$ **b** $f(x) = 3x + 2$ **c** $f(x) = \dfrac{3 - 2x}{4}$

   d $f(x) = x^3$ **e** $f(x) = 2x^3 + 1$ **f** $f(x) = \dfrac{4x - 1}{3}$

   g $f(x) = \sqrt{x + 1}$ **h** $f(x) = \sqrt{3x - 5}$ **i** $f(x) = \dfrac{1}{x - 2}$

2   **a**   Find the inverse function of: **i** $f(x) = 8 - x$ **ii** $f(x) = \dfrac{9}{x}$

    b   What do you observe from your answers in **a**?

3   **a**   Show that the inverse of a linear function is also linear.

    b   What is the relationship between the gradient of a linear function and the gradient of its inverse?

    c   Explain why the following statement is true:

          "If (a, b) lies on $y = f(x) = mx + c$, then (b, a) lies on $y = f^{-1}(x)$."

    d   Find the inverse function of:

      i

      ii
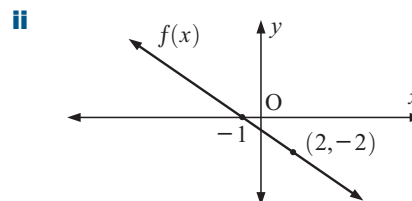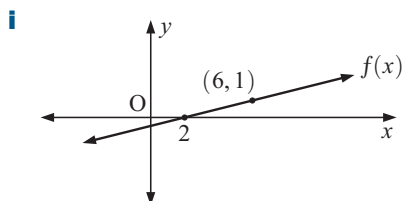
4 **a** Explain why the horizontal line test is a valid test for the existence of an inverse function.

   b Which of the following functions have an inverse function?
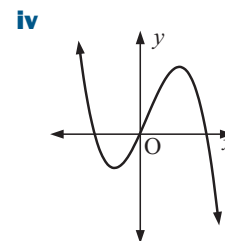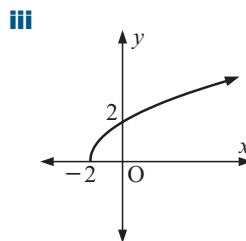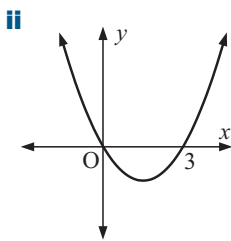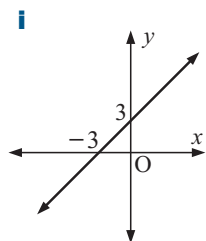
5 Show that these functions do not have an inverse function:

   a $f(x) = x^2$ **b** $f(x) = \dfrac{1}{x^2}$ **c** $f(x) = x^2 + 4x + 4$

6 **a** On the same set of axes graph $y = f(x)$ and $y = f^{-1}(x)$ for:

   i $f(x) = 2x + 1$ **ii** $f(x) = \dfrac{2}{x}$ **iii** $f(x) = \sqrt{x}, \ x \geqslant 0$

   b Copy and complete:

   The graph of $y = f^{-1}(x)$ is a reflection of $y = f(x)$ in ....

7 If $f(x)$ has a vertical asymptote of $x = k$, explain why $f^{-1}(x)$ will have a horizontal asymptote $y = k$.

8 For the following functions:

   i find $f^{-1}(x)$ **ii** graph $f(x)$ and $f^{-1}(x)$ on a set of axes.

   a $f(x) = \dfrac{2}{x - 3}$ **b** $f(x) = -\dfrac{3}{x + 1}$ **c** $f(x) = \dfrac{x}{x - 2}$

   d $f(x) = \dfrac{x + 1}{x - 1}$ **e** $f(x) = \dfrac{1}{x^3 - 1}$ **f** $f(x) = \dfrac{2x + 1}{x - 3}$

   What do you notice about the graphs of $y = f(x)$ and $y = f^{-1}(x)$ in each case?

C   *USING TECHNOLOGY*   *[2.11, 3.6]*

A graphics calculator or computer graphing package are useful tools for gaining knowledge about a function, in particular one with an unfamiliar form.

We can use a graphics calculator to obtain:

GRAPHING PACKAGE

- a table of values for a function
- a sketch of the function
- the **zeros** or x-**intercepts** of the function
- the y-intercept of the function
- any asymptotes of the function
- the turning points of the function where it is a **local maximum** or **local minimum**
- the points of intersection of two functions.
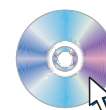
Instructions for using your calculator are found beginning on page **22**.

Example 5   🔊 *Self Tutor*

Consider $f(x) = \dfrac{3x - 9}{x^2 - x - 2}$.

a Use your graphics calculator to obtain a sketch of the function.
b State the equations of the asymptotes of the function.
c State the axes intercepts of the function.
d Describe the turning points of the function.

a

b horizontal asymptote is $y = 0$
vertical asymptotes are $x = -1, \ x = 2$

c x-intercept is 3, y-intercept is $4\frac{1}{2}$

d local maximum $(5, \ \frac{1}{3})$,
local minimum $(1, \ 3)$
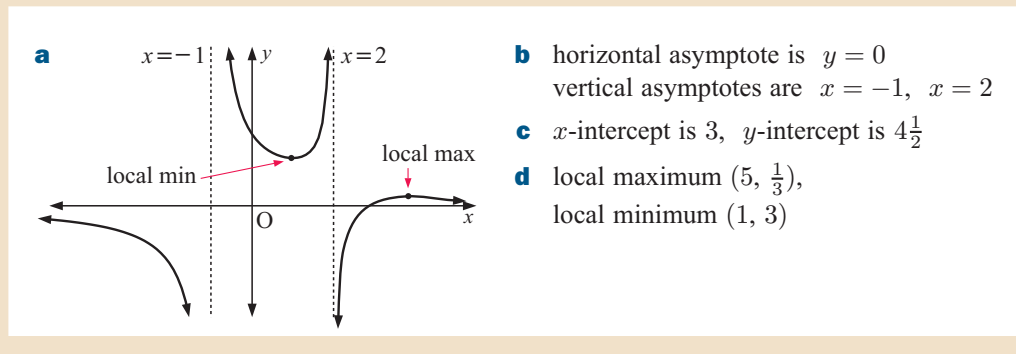
EXERCISE 23C.1

1 For these quadratic functions, find:

> **i** the turning point   **ii** the y-intercept   **iii** the x-intercepts

> **a** $y = x^2 - 3$ **b** $f(x) = 2x^2 - 2x - 1$ **c** $f(x) = 9x^2 + 6x - 4$

2 Use your graphics calculator to sketch the graphs of these modulus functions:

> **a** $y = |2x - 1| + 2$ **b** $y = |x(x - 3)|$ **c** $y = |(x - 2)(x - 4)|$
> **d** $y = |x| + |x - 2|$ **e** $y = |x| - |x + 2|$ **f** $y = \left|9 - x^2\right|$

If the graph possesses a line of symmetry, state its equation.

3 Consider $f(x) = x^3 - 4x^2 + 5x - 3$ for $-1 \leqslant x \leqslant 4$.

a Sketch the graph with help from your graphics calculator.
b Find the x- and y-intercepts of the graph.
c Find and classify any turning points of the function.
d State the range of the function.
e Create a table of values for $f(x)$ on $-1 \leqslant x \leqslant 4$ with x-steps of 0.5.

4 Consider $f(x) = x^4 - 3x^3 - 10x^2 - 7x + 3$ for $-4 \leqslant x \leqslant 6$.

a Set your calculator window to show y from -150 to 350. Hence sketch the graph of $f(x)$.
b Find the largest zero of $f(x)$.
c Find the turning point of the function near $x = 4$.
d Adjust the window to $-2 \leqslant x \leqslant 1$, $-2 \leqslant y \leqslant 7$. Hence sketch the function for $-2 \leqslant x \leqslant 1$.
e Find the other two turning points and classify them.
f Make a table of values for $f(x)$ on $0 \leqslant x \leqslant 1$ with x-steps of 0.1.

5 For these functions:

      i Use your graphics calculator to obtain a sketch of the function.
      ii State the equations of any asymptotes.
      iii Find the axes intercepts.
      iv Find and classify any turning points.

a $f(x) = \dfrac{4}{x-2}$ **b** $f(x) = 2 - \dfrac{3}{x+1}$ **c** $f(x) = 2^x - 3$

d $f(x) = 2x + \dfrac{1}{x}$ **e** $f(x) = \dfrac{4x}{x^2 - 4x - 5}$ **f** $f(x) = 3^{-x} + 2$

g $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$ **h** $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$ **i** $f(x) = \dfrac{2^x + 3}{2^x + 1}$

6 Consider $f(x) = 2^x - x^2$

    a Find the values of $f(x)$ for $x = -2, -1, 0, 1, 2, 3, 4, 5$.
    b Use **a** to set a suitable window on your graphics calculator and hence obtain a sketch graph for $f(x)$ on $-2 \leqslant x \leqslant 5$.
    c Find the zeros of $f(x)$.
    d Find the turning points of $f(x)$.

> Did you notice in **5** that vertical asymptotes can be found by letting the denominator be 0?

7 If $f(x) = 2^x$, $g(x) = x^2 - 1$ and $h(x) = \dfrac{x-1}{2x+1}$,

copy and complete, giving answers correct to 3 decimal places where necessary.

x	$f(x)$	$g(x)$	$h(x)$
-2			
-1.5			
-0.5			
0			
0.5			
1			
2			
2.7			
3.61			

8   **a** Use your graphics calculator to draw, on the same axes,
the functions $f(x) = x - \dfrac{1}{x}$ and $g(x) = 2^{-x} - 1$.

    b State the equations of the asymptotes of $f(x)$.
    c Find the coordinates of any points where $y = f(x)$ and $y = g(x)$ meet.

9 Consider $f(x) = \dfrac{x^2 + 4}{x^2 + 1}$.

    a Sketch the graph of $y = f(x)$.
    b Find the domain and range of $f(x)$.
    c Write down the equations of any asymptotes of $y = f(x)$.
    d Find the coordinates of any points where $y = f(x)$ meets $y = 5$.
    e Suppose $y = f(x)$ meets $y = k$ at exactly two points.
What possible values could k have?

10   Consider $f(x) = \dfrac{x+2}{x-1}$ and $g(x) = 2^x$.

     a   Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same set of axes for $-5 \leqslant x \leqslant 5$.

     b   Find the coordinates of the points of intersection of the two graphs.

     c   Find the values of x for which $2^x > \dfrac{x+2}{x-1}$.

SOLVING UNFAMILIAR EQUATIONS

Technology allows us to solve equations with expressions we are unfamiliar with.

Suppose we are given an equation of the form $f(x) = g(x)$.

If we subtract $g(x)$ from both sides, we have $f(x) - g(x) = 0$.

So, given $f(x) = g(x)$,   there are two different approaches we can take to find the solutions.

Method 1:   Graph $y = f(x)$ and $y = g(x)$ on the same set of axes and find the x-coordinates where they meet.

Method 2:   Graph $y = f(x) - g(x)$ and find the x-intercepts.

As an example we will consider an equation that we can solve algebraically:

$$2x^2 = 3x + 2$$
$$\therefore \quad 2x^2 - 3x - 2 = 0$$
$$\therefore \quad (2x + 1)(x - 2) = 0$$
$$\therefore \quad x = -\tfrac{1}{2} \ \text{ or } \ 2$$

Consider $f(x) = 2x^2$ and $g(x) = 3x + 2$.

Method 1:

We graph $y = f(x)$ and $y = g(x)$ on the same set of axes.
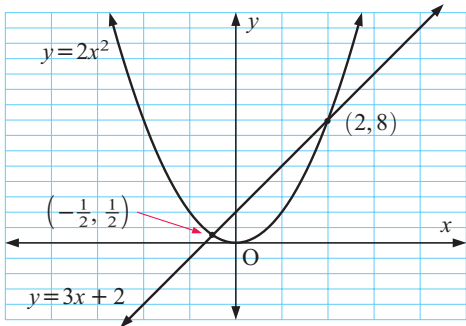
The graphs intersect at the points with x-coordinates $-\tfrac{1}{2}$ and 2.

Method 2:

We graph $y = f(x) - g(x)$ which is $y = 2x^2 - 3x - 2$
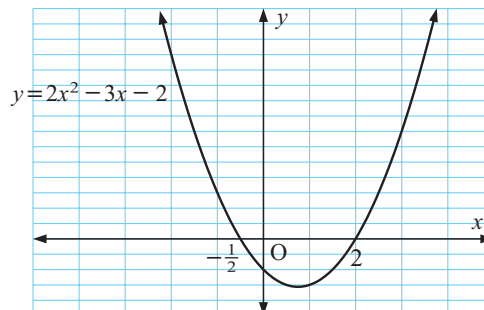
The x-intercepts are $-\tfrac{1}{2}$ and 2.

Example 6   ◀) *Self Tutor*

Solve $2^x = 4 - x$ by graphing $y = 2^x$ and $y = 4 - x$ on the same set of axes.

Give your answer correct to 3 significant figures.

Let $Y_1 = 2^x$ and $Y_2 = 4 - x$

Using a graphics calculator, we find that they meet when $x \approx 1.39$.

∴   the solution is $x \approx 1.39$
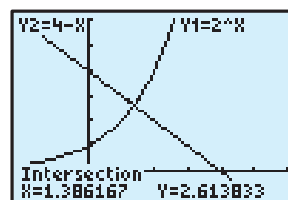
Example 7   ◀) *Self Tutor*

Solve $2^x = 4 - x$ by drawing one graph. Give your answer correct to 3 significant figures.

$$2^x = 4 - x$$
$$\therefore \quad 2^x - 4 + x = 0$$

We graph $y = 2^x - 4 + x$ using a graphics calculator to help.

The x-intercept is ≈ 1.39
$$\therefore \quad x \approx 1.39$$
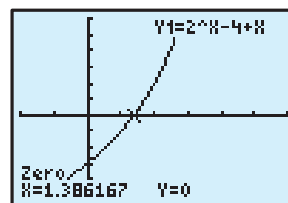
EXERCISE 23C.2

1 Use technology to solve:

     a $-2x^2 = x - 3$ **b** $\frac{1}{2}x^2 = x + 2$ **c** $x(x - 2) = 3 - x$

     d $(x + 2)(x - 1) = 2 - 3x$ **e** $(2x + 1)^2 = 3 - x$ **f** $(x - 2)^2 = 1 + x$

2 Using *Method 1* to solve, correct to 3 significant figures:

     a $2^x = 3x$ **b** $\sqrt{x} = 3 - x$ **c** $x^2 = \sqrt{x + 2}$

     d $3^x = 5$ **e** $3^x = x^2$ **f** $x^3 + 2 = 3x - x^2$

3 Use *Method 2* to solve, correct to 3 significant figures:

     a $x^3 - x + 3 = 0$ **b** $5 - x = \sqrt{x}$ **c** $3^x = x^3 + 1$

     d $x^2 - 3 = \sqrt[3]{x}$ **e** $3x^2 + 1 = \dfrac{12}{x - 4}$ **f** $\dfrac{x + 7}{x - 3} = 2^{-x}$

4 Find the coordinates of the points, correct to 2 decimal places, where these graphs meet:

     a $y = x^2 + 2x - 3$ and $y = \dfrac{4}{x}$ **b** $y = x^3$ and $y = 5^x$

     c $y = \dfrac{5}{x}$ and $y = \dfrac{1}{\sqrt{x}} + 1$ **d** $y = 2^x - 1$ and $y = \dfrac{1}{x^3}$

D   *TANGENTS TO CURVES*   *[3.5]*

A **tangent** to a curve is a straight line which **touches** the curve.

tangent
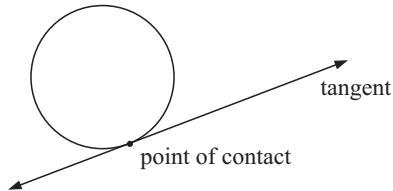
point of contact

We are familiar with a tangent to a circle which touches the circle at a single **point of contact**.

Oblique means at an angle to horizontal and vertical.

Other curves can also have tangents. For example, the quadratic function shown has a horizontal tangent (1) at the vertex and an oblique tangent (2) at point A.

(2)

A

(1)

vertex
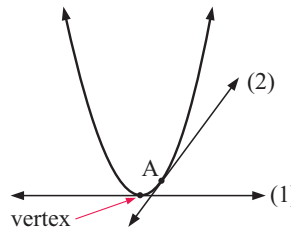
Example 8   ◄) *Self Tutor*

From the accurate graph of $y = x^2$, estimate the gradient of the tangent at the point:

a   O(0, 0)   **b**   A(1, 1)
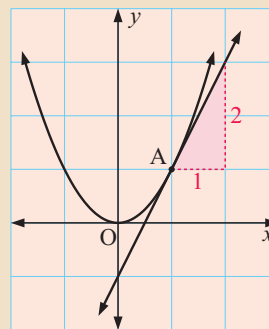
a   At O, the tangent is horizontal and so the gradient is 0.

b   At A(1, 1), the tangent has gradient $\approx \frac{2}{1} \approx 2$.

EXERCISE 23D

Print off the worksheet to answer this exercise.

1   As accurately as possible, find the gradient of the tangent to:

   a $y = x^2$ at the point   A(−1, 1)   **b** $y = x^2$ at the point   B(2, 4)

   c $y = x^3$ at the point   C(0, 0)   **d** $y = \dfrac{2}{x}$ at the point   D(1, 2)

   e $y = \dfrac{6}{x}$ at the point   E(2, 3)   **f** $y = 2^x$ at the point   F(0, 1).

PRINTABLE WORKSHEET

2   Copy and complete:

   a   The gradient of the tangent to a curve at a turning point is ....

   b   The tangent to $y = x^3$ at the origin tells us that tangents to curves can .... the curve. This occurs at special points called points of inflection.
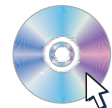
GRAPHING PACKAGE
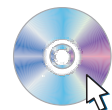
Review set 23A

1   Sketch the graphs of the following cubics, showing all axes intercepts:

    a $y = x(x - 2)(x + 3)$ **b** $y = -2(x + 1)^2(x - 3)$

2   Find the form of the cubic function which has:

    a x-intercept -1, 0 and 2 and y-intercept 6

    b x-intercepts 1 and 4, y-intercept -4, and passes through $(3, -4)$.

3   Find the equation of the cubic with
    graph given alongside.
    Give your answer in factor form and
    in expanded form.
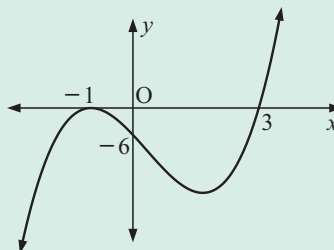
4   Find the inverse function of: **a** $f(x) = 4x - 1$ **b** $g(x) = \dfrac{1}{x + 3}$

5   For the function $f(x) = \dfrac{2x - 5}{3}$:

    a   find $f^{-1}(x)$

    b   sketch $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ on the same axes.

6   Solve for x, correct to 3 significant figures:

    a $7^x = 50$ **b** $5x^3 - \sqrt{x} = 12$ **c** $3x^2 - 5 = 2^{-x}$

7   Find the coordinates of the points of intersection for:

    a $y = (1.5)^{-x}$ and $y = 2x^2 - 7$ **b** $y = 6 - 2^x$ and $y = (x - 3)^2 - \dfrac{1}{\sqrt{x}}$

8   Suppose $f(x) = \dfrac{x^2 + 4}{x^2 - 1}$.

    a   Use technology to sketch the graph of $y = f(x)$.

    b   Find the equations of the three asymptotes.

    c   Find the domain and range of $f(x)$.

    d   If $\dfrac{x^2 + 4}{x^2 - 1} = k$ has 2 solutions, find the range of possible values of k.

9   Find, as accurately as possible, the gradient of the tangent to $y = \dfrac{4}{x}$ at the point $(2, 2)$.

Review set 23B

1   Use axes intercepts to sketch the graphs of:

    a $y = (x + 3)(x - 4)(x - 2)$ **b** $y = 3x^2(x + 2)$

2 The graph alongside has the form
$y = x^3 + 6x^2 + cx + d$.
Find the values of c and d.
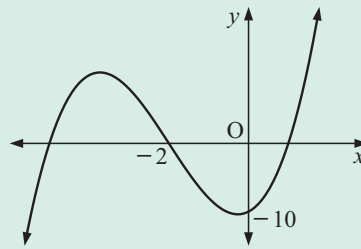
3 Find $f^{-1}(x)$ for:

   a $f(x) = 8x$ **b** $f(x) = \dfrac{2}{x-1}$ **c** $f(x) = \sqrt{x+3}$

4 Which of the following functions have an inverse function?

   a   **b**   **c**
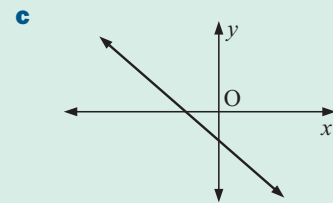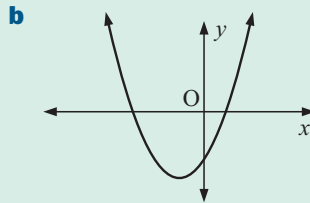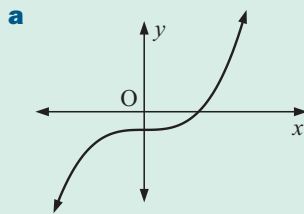
5 For $f(x) = \dfrac{x-3}{x^2 + 3x - 4}$:

   a Use a graphics calculator to help graph the function.

   b State the equations of any asymptotes.

   c Find the axes intercepts.

   d Find and classify any turning points.

6 Solve for x, correct to 3 significant figures:

   a $3^x = 11$ **b** $x^3 - 6x = 5 + x^2$ **c** $5^x = x^2 + 2$

7 Find the coordinates of the points of intersection for:

   a $y = x^3$ and $y = \dfrac{5}{x} - 2$ **b** $y = 3^x + 2$ and $y = \dfrac{1}{x^2}$

8 Suppose $f(x) = (1.2)^{\frac{x}{10}}$ and $g(x) = (0.8)^{\frac{x}{10}}$

   a Copy and complete the table of values alongside.

   b For the domain given in **a**, write down the largest and smallest values of $f(x)$ and $g(x)$.

   c Use technology and parts **a** and **b** to sketch $y = f(x)$ and $y = g(x)$ on the same set of axes.

   d Find the point of intersection of $f(x)$ and $g(x)$.

   e Find a linear function which:

       • passes through the point of intersection of $f(x)$ and $g(x)$

       • has a negative gradient

       • does not meet either graph again in the given domain.

x	$f(x)$	$g(x)$
-10	0.833	1.25
-5		
0		
5		
10		
15		
20		

9 Find as accurately as possible, the gradient of the tangent to $y = x^3$ at the point $(1, 1)$.

Vectors

24

Contents:

Opening problem

Holger can kayak in calm water at a speed of 20 km/h. However, today he needs to kayak directly across a river in which the water is flowing at a constant speed of 10 km/h to his right.

Things to think about:

- What effect does the current in the river have on the speed and direction in which Holger kayaks?
- How can we accurately find the speed and direction that Holger will travel if he tries to kayak directly across the river?
- In what direction must Holger face so that he kayaks directly across the river?

VECTORS AND SCALARS

To solve questions like those in the **Opening Problem**, we need to examine the **size** or **magnitude** of the quantities under consideration as well as the **direction** in which they are acting.

To achieve this we use quantities called **vectors** which have both size or magnitude and also direction.

> Quantities which only have magnitude are called **scalars**.
>
> Quantities which have both magnitude and direction are called **vectors**.

For example, **velocity** is a vector since it deals with speed (a scalar) in a particular direction.
Other examples of vector quantities are acceleration, force, displacement, and momentum.

A   DIRECTED LINE SEGMENT REPRESENTATION [5.1, 5.3]

Consider a bus which is travelling at 100 km/h in a south east direction.

A good way of representing this situation is to use an arrow on a scale diagram.

The **length of the arrow** represents the size or magnitude of the velocity and the **arrowhead** shows the direction of travel.
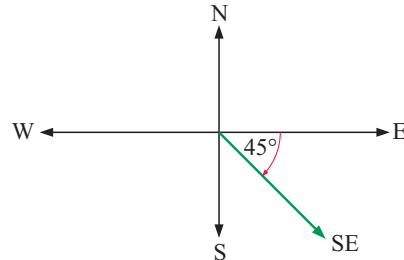
Scale: 1 cm represents 50 km/h

Consider the vector represented by the line segment from O to A.
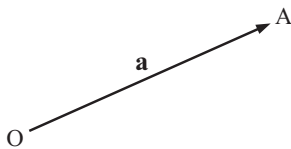
- This **vector** could be represented by

  $\overrightarrow{OA}$ or   **a**   or $\underset{\sim}{a}$.

  bold used   used by
  in text books   students

- The **magnitude** or **length** could be represented by

  $|\overrightarrow{OA}|$ or   OA   or $|\mathbf{a}|$ or $|\underset{\sim}{a}|$.

For

we say that $\overrightarrow{AB}$ is the vector which **emanates** from A and **terminates** at B, and that $\overrightarrow{AB}$ is the **position vector** of B relative to A.

Example 1   ◀) *Self Tutor*

On a scale diagram, sketch the vector which represents a velocity of:

a   15 m/s in a westerly direction   **b**   40 m/s on a bearing 075°.
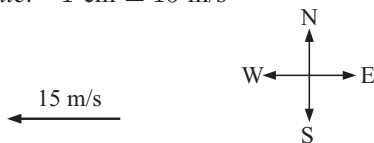
a   *Scale:*   1 cm ≡ 10 m/s   **b**   *Scale:*   1 cm ≡ 10 m/s
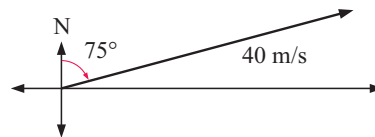
EXERCISE 24A

1   Using a scale of 1 cm represents 10 units, sketch a vector to represent:

    a   40 km/h in a SW direction

    b   35 m/s in a northerly direction

    c   a displacement of 25 m in a direction 120°

    d   an aeroplane taking off at an angle of 12° to the runway with a speed of 60 m/s.

2 If ⟶ represents a force of 45 Newtons due east, draw a directed line segment representing a force of:

    a 75 N due west   **b** 60 N south west.

3 Draw a scaled arrow diagram representing the following vectors:

    a a velocity of 60 km/h in a NE direction

    b a momentum of 45 kg m/s in the direction 250°

    c a displacement of 25 km in the direction 055°

    d an aeroplane taking off at an angle of 10° to the runway at a speed of 90 km/h.

B   *VECTOR EQUALITY*   *[5.1, 5.2]*

Two vectors are **equal** if they have the same magnitude *and* direction.

If arrows are used to represent vectors, then equal vectors are **parallel** and **equal in length**.

This means that equal vector arrows are translations of one another.
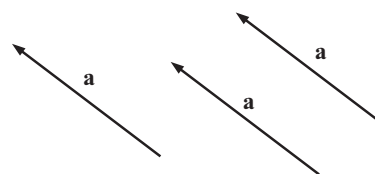
THE ZERO VECTOR

The **zero vector**, **0**, is a vector of length 0.
It is the only vector with no direction.

NEGATIVE VECTORS
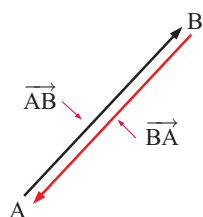
Notice that $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have the same length but opposite directions.

We say that $\overrightarrow{BA}$ is *the negative of* $\overrightarrow{AB}$ and write $\overrightarrow{BA} = -\overrightarrow{AB}$.

Given the vector **a** shown, we can draw the vector $-$**a**.
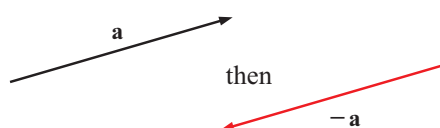
then

a and $-$**a** are parallel, equal in length, but opposite in direction.

Example 2

ABCD is a parallelogram in which $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{b}$.

Find vector expressions for:

  a $\overrightarrow{BA}$ **b** $\overrightarrow{CB}$ **c** $\overrightarrow{AD}$ **d** $\overrightarrow{CD}$
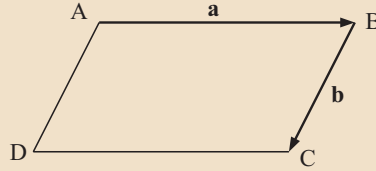
  a $\overrightarrow{BA} = -\mathbf{a}$ {the negative vector of $\overrightarrow{AB}$}   **b** $\overrightarrow{CB} = -\mathbf{b}$ {the negative vector of $\overrightarrow{BC}$}

  c $\overrightarrow{AD} = \mathbf{b}$ {parallel to and the same length as $\overrightarrow{BC}$}

  d $\overrightarrow{CD} = -\mathbf{a}$ {parallel to and the same length as $\overrightarrow{BA}$}

EXERCISE 24B

1 State the vectors which are:

  a equal in magnitude   **b** parallel

  c in the same direction   **d** equal

  e negatives of one another.
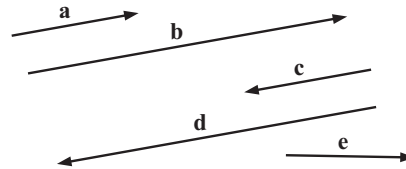
2
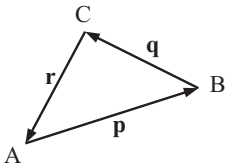
Write in terms of vectors **p**, **q** and **r**:

  a $\overrightarrow{AB}$ **b** $\overrightarrow{BA}$ **c** $\overrightarrow{BC}$

  d $\overrightarrow{CB}$ **e** $\overrightarrow{CA}$ **f** $\overrightarrow{AC}$

3 The figure alongside consists of two isosceles triangles with PQ ∥ SR and $\overrightarrow{PQ} = \mathbf{p}$, $\overrightarrow{PS} = \mathbf{q}$.

Which of the following statements are true?

  a $\overrightarrow{RS} = \mathbf{p}$ **b** $\overrightarrow{QR} = \mathbf{q}$ **c** $\overrightarrow{QS} = \mathbf{q}$

  d QS = PS   **e** $\overrightarrow{PS} = -\overrightarrow{RQ}$
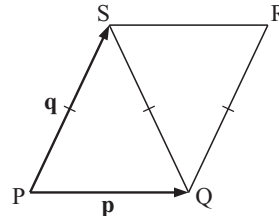
C   VECTOR ADDITION   [5.2]

We have already been operating with vectors without realising it.

Bearing problems are an example of this. The vectors in this case are **displacements**.

A typical problem could be:

"A girl runs from A in a northerly direction for 3 km and then in a westerly direction for 2 km to B. How far is she from her starting point and in what direction?"

We can use trigonometry and Pythagoras' theorem to answer such problems as we need to find θ and x.
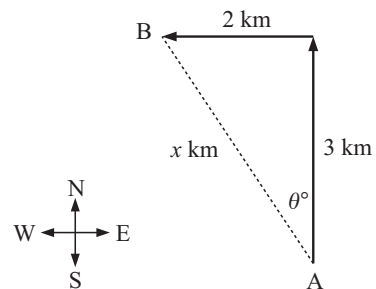
DISPLACEMENT VECTORS

Suppose we have three towns A, B and C.

A trip from A to B followed by a trip from B to C is equivalent to a trip from A to C.

This can be expressed in a vector form as the sum $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ where the $+$ sign could mean '*followed by*'.
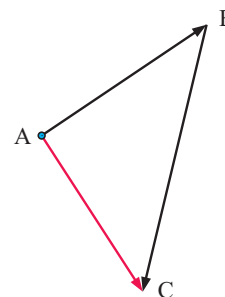
VECTOR ADDITION

After considering displacements in diagrams like those above, we can now define vector addition geometrically:

To add **a** and **b** :

Step 1:   Draw **a**.

Step 2:   At the arrowhead end of **a**, draw **b**.

Step 3:   Join the beginning of **a** to the arrowhead end of **b**. This is vector   **a** + **b**.

So, given   we have
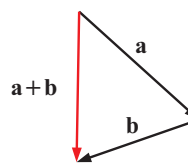
COMPUTER DEMO
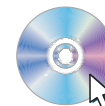
Example 3   ◀》 *Self Tutor*

Find a single vector which is equal to:

a $\overrightarrow{AB} + \overrightarrow{BE}$

b $\overrightarrow{DC} + \overrightarrow{CA} + \overrightarrow{AE}$

c $\overrightarrow{CB} + \overrightarrow{BD} + \overrightarrow{DC}$
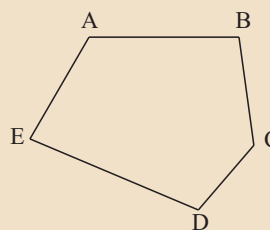
a $\overrightarrow{AB} + \overrightarrow{BE} = \overrightarrow{AE}$ {as shown}

b $\overrightarrow{DC} + \overrightarrow{CA} + \overrightarrow{AE} = \overrightarrow{DE}$

c $\overrightarrow{CB} + \overrightarrow{BD} + \overrightarrow{DC} = \overrightarrow{CC} = \mathbf{0}$ {zero vector}
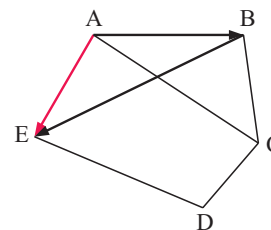
Example 4   ◀)) Self Tutor

Sonya can swim at 3 km/h in calm water. She swims in a river where the current is 1 km/h in an easterly direction. Find Sonya's resultant velocity if she swims:

   a with the current   **b** against the current   **c** northwards, across the river.

Scale: 1 cm ≡ 1 km/h

The velocity vector of the river is
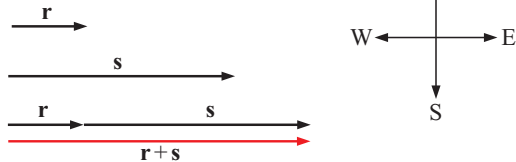
a Sonya's velocity vector is

    The net result is **r** + **s**.

    ∴ Sonya swims at 4 km/h in the direction of the current.

b Sonya's velocity vector is
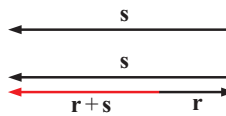
    The net result is **r** + **s**.

    ∴ Sonya swims at 2 km/h against the current.

c

Sonya's velocity vector is   and the net result is **r** + **s**.

∴ $|\mathbf{r} + \mathbf{s}| = \sqrt{10} \approx 3.16$

$\tan\theta = \frac{1}{3}$ so $\theta = \tan^{-1}(\frac{1}{3}) \approx 18.4^o$

∴ Sonya swims at about 3.16 km/h in the direction 018.4°.
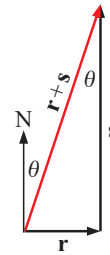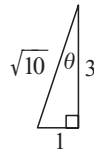
EXERCISE 24C

1 Copy the given vectors **p** and **q** and hence show how to find **p** + **q**:

   a

   b

   c

   d

e

f
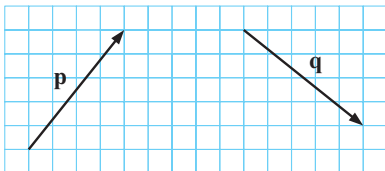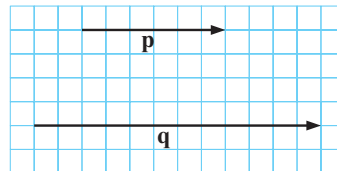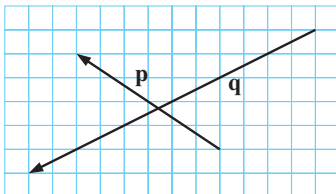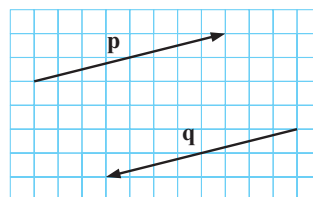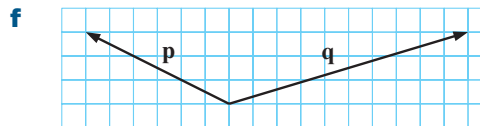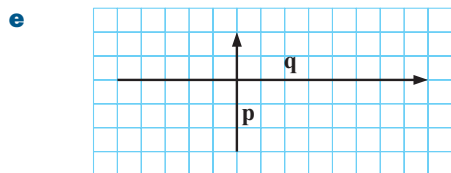
2 Find a single vector which is equal to:

> **a** $\overrightarrow{QR} + \overrightarrow{RS}$
>
> **b** $\overrightarrow{PQ} + \overrightarrow{QR}$
>
> **c** $\overrightarrow{PS} + \overrightarrow{SR} + \overrightarrow{RQ}$
>
> **d** $\overrightarrow{PR} + \overrightarrow{RQ} + \overrightarrow{QS}$
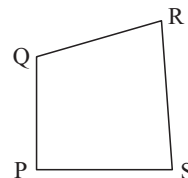
3 Paolo rides for 20 km in the direction 310° and then for 15 km in the direction 040°. Find Paolo's displacement from his starting point.

4 Consider an aeroplane trying to fly at 500 km/h due north. Find the actual speed and direction of the aeroplane if a gale of 100 km/h is blowing:

> **a** from the south
> **b** from the north
> **c** from the west.

5 A ship travelling at 23 knots on a course 124° encounters a current of 4 knots in the direction 214°. Find the actual speed and direction of the ship.

D   VECTOR SUBTRACTION   [5.2]

VECTOR SUBTRACTION

To subtract one vector from another, we simply **add its negative**.

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

For example:

for $\mathbf{a}$ and $\mathbf{b}$ then
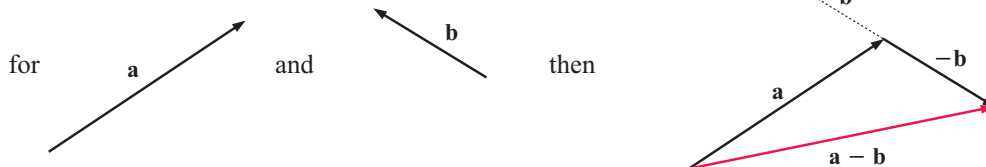
Example 5   ◄) Self Tutor
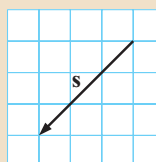
Find $\mathbf{s} - \mathbf{t}$ given: and
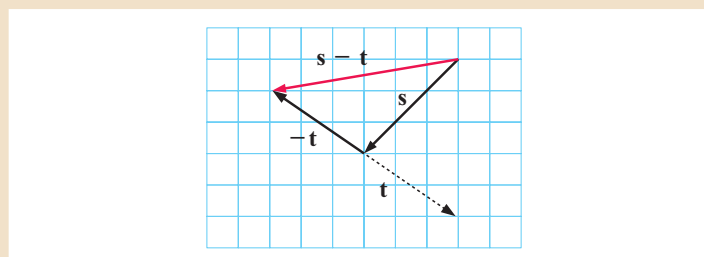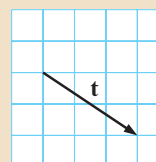
Example 6   ◀) *Self Tutor*

For points P, Q, R and S, simplify the following vector expressions:

a $\overrightarrow{QR} - \overrightarrow{SR}$ 　　　　　　　**b** $\overrightarrow{QR} - \overrightarrow{SR} - \overrightarrow{PS}$

a 　$\overrightarrow{QR} - \overrightarrow{SR}$
　　$= \overrightarrow{QR} + \overrightarrow{RS}$ 　　$\{$as $\overrightarrow{RS} = -\overrightarrow{SR}\}$
　　$= \overrightarrow{QS}$
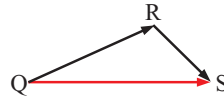
b 　$\overrightarrow{QR} - \overrightarrow{SR} - \overrightarrow{PS}$
　　$= \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{SP}$
　　$= \overrightarrow{QP}$
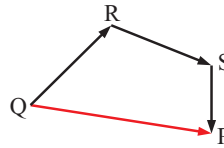
Example 7   ◀) *Self Tutor*

Xiang Zhu is about to fire an arrow at a target. In still conditions, the arrow would travel at 18 m/s. Today, however, there is a wind of 6 m/s blowing from the left directly across the arrow's path.
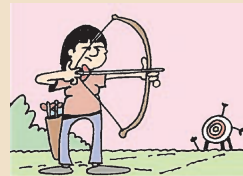
a In what direction should Zhu fire the arrow?
b What will be its actual speed?

Suppose Zhu is at Z and the target is at T. Let **a** be the arrow's velocity in still conditions, **w** be the velocity of the wind, and **x** be the vector $\overrightarrow{ZT}$.

Now 　　　　$\mathbf{a} + \mathbf{w} = \mathbf{x}$
　∴　$\mathbf{a} + \mathbf{w} - \mathbf{w} = \mathbf{x} - \mathbf{w}$
　　∴　　$\mathbf{a} = \mathbf{x} - \mathbf{w}$

a Now $|\mathbf{a}| = 18$ m/s 　and 　$|\mathbf{w}| = 6$ m/s
　∴　$\sin\theta = \frac{6}{18} = \frac{1}{3}$
　∴　$\theta = \sin^{-1}\left(\frac{1}{3}\right) \approx 19.47^o$
　∴　Zhu should fire about 19.5^o to the left of the target.

b By Pythagoras' theorem, $|\mathbf{x}|^2 + 6^2 = 18^2$
　∴　$|\mathbf{x}| = \sqrt{18^2 - 6^2} \approx 16.97$ m/s
　∴　the arrow will travel at about 17.0 m/s.
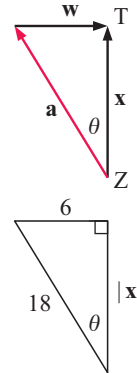
EXERCISE 24D

1 For the following vectors **p** and **q**, show how to construct **p** − **q**:

a

b

c

d

e

f
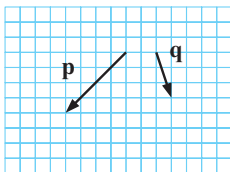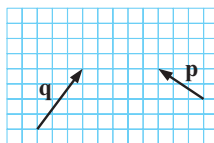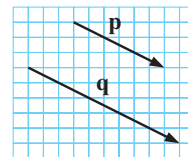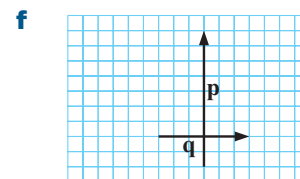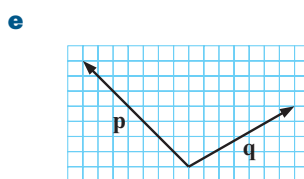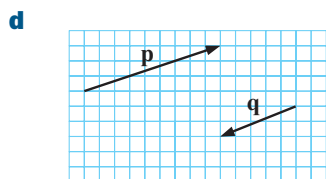
2 For points P, Q, R and S, simplify the following vector expressions:

   a $\overrightarrow{QR} + \overrightarrow{RS}$ **b** $\overrightarrow{PS} - \overrightarrow{RS}$ **c** $\overrightarrow{RS} + \overrightarrow{SR}$

   d $\overrightarrow{RS} + \overrightarrow{SP} + \overrightarrow{PQ}$ **e** $\overrightarrow{QP} - \overrightarrow{RP} + \overrightarrow{RS}$ **f** $\overrightarrow{RS} - \overrightarrow{PS} - \overrightarrow{QP}$

3 An aeroplane needs to fly due north at a speed of 500 km/h. However, it is affected by a 40 km/h wind blowing constantly from the west. What direction must it head towards and at what speed?

4 A motorboat wishes to travel NW towards a safe haven before an electrical storm arrives. In still water the boat can travel at 30 km/h. However, a strong current is flowing at 10 km/h from the north east.

   a In what direction must the boat head?

   b At what speed will the boat be travelling?

E   VECTORS IN COMPONENT FORM   [5.1 - 5.3]

When vectors are drawn on a coordinate grid, we can describe them in terms of their components in the x and y directions.

$\begin{pmatrix} x \\ y \end{pmatrix}$ is a **column vector** and is the vector in **component form**.

For example, given $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ we could draw

where -1 is the x-component or x-step
and 2 is the y-component or y-step.
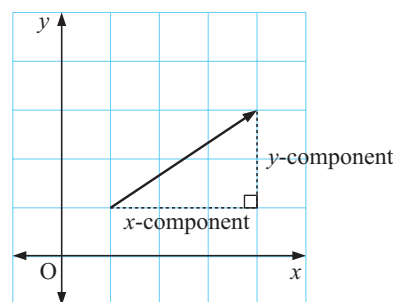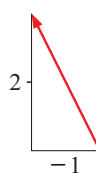
Example 8   ◀ŷ **Self Tutor**

L$(-2, 3)$ and M$(4, 1)$ are two points.   Find   **a** $\overrightarrow{LM}$ **b** $\overrightarrow{ML}$.
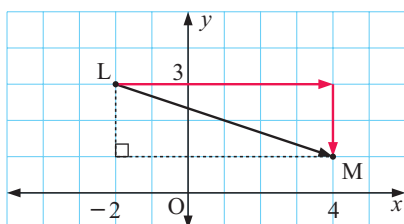
a The x-component goes from -2 to 4, which is $+6$.
The y-component goes from 3 to 1, which is -2.

$$\therefore \quad \overrightarrow{LM} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$$

b $\overrightarrow{ML} = \begin{pmatrix} -6 \\ 2 \end{pmatrix}$

An alternative to plotting points as in **Example 8** is to use a formula.
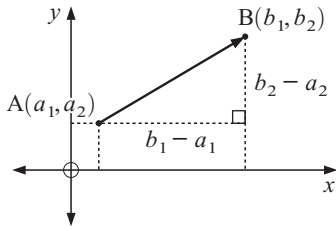
If A is (a_1, a_2) and B is (b_1, b_2) then

$$\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} \begin{matrix} \longleftarrow\ x\text{-step} \\ \longleftarrow\ y\text{-step} \end{matrix}$$

Example 9   ◀⑴ *Self Tutor*

If A is at $(2, -3)$ and B at $(4, 2)$ find: **a** $\overrightarrow{AB}$ **b** $\overrightarrow{BA}$

a $\overrightarrow{AB} = \begin{pmatrix} 4 - 2 \\ 2 - -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ **b** $\overrightarrow{BA} = \begin{pmatrix} 2 - 4 \\ -3 - 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$

From **Examples 8** and **9** you may have noticed that $\overrightarrow{BA} = -\overrightarrow{AB}.$

VECTOR EQUALITY

Vectors are equal if and only if their x-components are equal *and* their y-components are equal.

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix} \quad \text{if and only if} \quad p = r \quad \text{and} \quad q = s.$$

EXERCISE 24E.1

1 Draw arrow diagrams to represent the vectors:

  a $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ **c** $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ **d** $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

2 Write the illustrated vectors in component form:

  a   **b**   **c**

  d   **e**   **f**
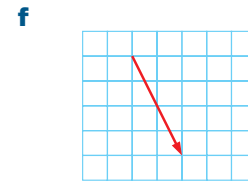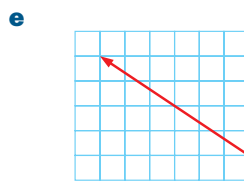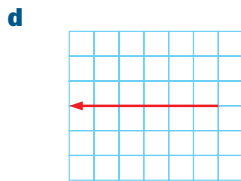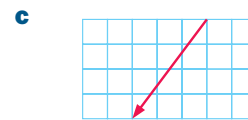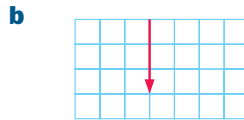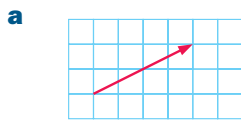
3 Write in component form:

  a $\overrightarrow{BA}$ **b** $\overrightarrow{CA}$ **c** $\overrightarrow{CD}$

  d $\overrightarrow{DA}$ **e** $\overrightarrow{BD}$ **f** $\overrightarrow{DC}$

  g $\overrightarrow{AB}$ **h** $\overrightarrow{AC}$ **i** $\overrightarrow{DB}$

A(4, 2)
B(−5, −1)
C(2, −3)
D(6, −2)
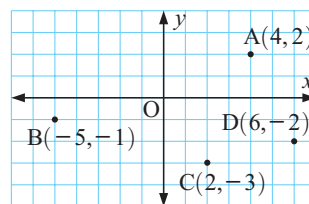
4 If A is at $(3, 4)$, B is at $(-1, 2)$, and C is at $(2, -1)$, find:

    a $\overrightarrow{OA}$
    b $\overrightarrow{AB}$
    c $\overrightarrow{CO}$
    d $\overrightarrow{BC}$
    e $\overrightarrow{CA}$

VECTOR ADDITION

Consider adding vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.
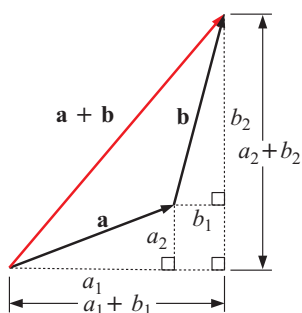
The horizontal step for $\mathbf{a} + \mathbf{b}$ is $a_1 + b_1$.

The vertical step for $\mathbf{a} + \mathbf{b}$ is $a_2 + b_2$.

$$\text{If} \quad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \text{then} \quad \mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}.$$

Example 10
    ◀) **Self Tutor**

If $\mathbf{a} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, find $\mathbf{a} + \mathbf{b}$.

Check your answer graphically.

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$
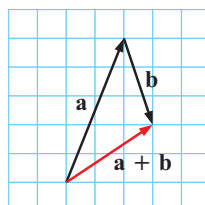
Start at the non-arrow end and move horizontally then vertically to the arrow end.

NEGATIVE VECTORS

Consider the vector $\mathbf{a} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

Notice that $-\mathbf{a} = \begin{pmatrix} -5 \\ -2 \end{pmatrix}$.
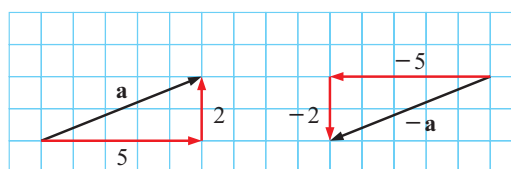
In general,
    if $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ then $-\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix}$.

ZERO VECTOR

The zero vector is $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

For any vector $\mathbf{a}$:

$\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{0}$.

$\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$.

VECTOR SUBTRACTION

To subtract one vector from another, we simply **add its negative**.   So, $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ then $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$

$$= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} -b_1 \\ -b_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}$$

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ then $\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}$.

Example 11   ◀⑴ Self Tutor

Given $\mathbf{p} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ find: **a** $\mathbf{p} - \mathbf{q}$ **b** $\mathbf{q} - \mathbf{p}$

a $\mathbf{p} - \mathbf{q} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$\qquad = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$

b $\mathbf{q} - \mathbf{p} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$\qquad = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$

THE MAGNITUDE OF A VECTOR

Using the theorem of Pythagoras,

the **magnitude** or **length** of $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is $|\mathbf{a}| = \sqrt{a_1{}^2 + a_2{}^2}$.
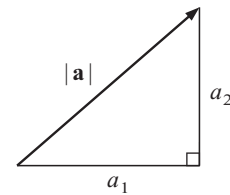
Example 12   ◀⑴ Self Tutor

Find the length of $\mathbf{a} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$.

$$|\mathbf{a}| = \sqrt{5^2 + (-2)^2}$$
$$= \sqrt{25 + 4}$$
$$= \sqrt{29} \text{ units}$$

EXERCISE 24E.2

1 If $\mathbf{a} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$ find:

    a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{b} + \mathbf{a}$ **c** $\mathbf{b} + \mathbf{c}$ **d** $\mathbf{c} + \mathbf{b}$

    e $\mathbf{a} + \mathbf{c}$ **f** $\mathbf{c} + \mathbf{a}$ **g** $\mathbf{a} + \mathbf{a}$ **h** $\mathbf{b} + \mathbf{a} + \mathbf{c}$

2 Given $\mathbf{p} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ find exactly:

    a $\mathbf{p} - \mathbf{q}$ **b** $\mathbf{q} - \mathbf{r}$ **c** $\mathbf{p} + \mathbf{q} - \mathbf{r}$ **d** $|\mathbf{p}|$ **e** $|\mathbf{q} - \mathbf{r}|$ **f** $|\mathbf{r} + \mathbf{q}|$

3 **a** Given $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, find $\overrightarrow{BC}$.

    b Given $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, $\overrightarrow{BD} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ and $\overrightarrow{CD} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, find $\overrightarrow{AC}$.

4 Find the exact magnitude of these vectors:

    a $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$ **c** $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ **d** $\begin{pmatrix} -1 \\ -5 \end{pmatrix}$ **e** $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$ **f** $\begin{pmatrix} -12a \\ 5a \end{pmatrix}$

5 For the following pairs of points, find: **i** $\overrightarrow{AB}$ **ii** the exact distance AB, i.e., $\left|\overrightarrow{AB}\right|$

    a A(3, 5) and B(1, 2)   **b** A(−2, 1) and B(3, −1)

    c A(3, 4) and B(0, 0)   **d** A(11, −5) and B(−1, 0)

6 Alongside is a hole at Hackers Golf Club.

    a Jack tees off from T and his ball finishes at A. Write a vector
        to describe the displacement of the ball from T to A.

    b He plays his second stroke from A to B. Write a vector to
        describe the displacement with this shot.

    c By great luck, Jack's next shot finishes in the hole H. Write
        a vector which describes this shot.

    d Use vector lengths to find the distance, correct to 3
        significant figures, from:

          i T to H   **ii** T to A

          iii A to B   **iv** B to H

    e Find the sum of all three vectors for the ball travelling from
        T to A to B to H. What information does the sum give about
        the golf hole?

7
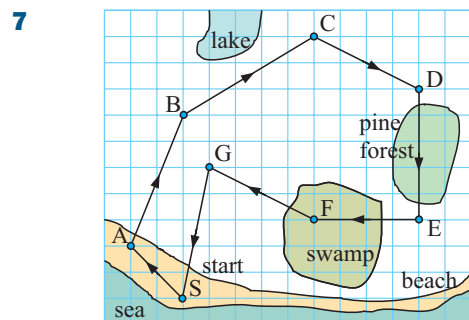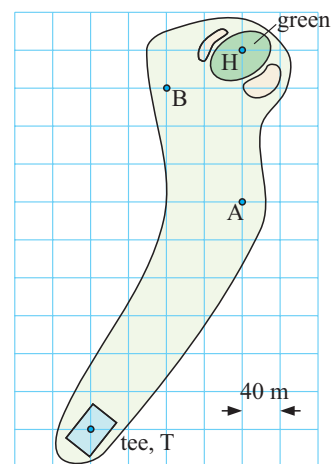
The diagram alongside shows an orienteering course run
by Kahu.

    a Write a column vector to describe each leg of the
        course.

    b Find the sum of all of the vectors.

    c What does the sum in **b** tell us?

8 Find the two values for k for which $\mathbf{v} = \begin{pmatrix} k \\ 3 \end{pmatrix}$ and $|\mathbf{v}| = 5$.

9 $\mathbf{u} = \begin{pmatrix} k \\ k \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} k + 2 \\ 1 \end{pmatrix}$ are two vectors.

  a Find k given that $|\mathbf{u}| = |\mathbf{v}|$ and $\mathbf{v} \neq -\mathbf{u}$.
  b Check your answers by finding $|\mathbf{u}|$ and $|\mathbf{v}|$ using your value(s) of k.

F   SCALAR MULTIPLICATION   [5.2]

Numbers such as 1 and -2 are called *scalars* because they have size but no direction.

$\mathbf{a}$ and $-2\mathbf{a}$ are examples of multiplying a vector by a scalar.

$2\mathbf{a}$ is the short way of writing $\mathbf{a} + \mathbf{a}$ and $-2\mathbf{a} = (-\mathbf{a}) + (-\mathbf{a})$

For   we have   and
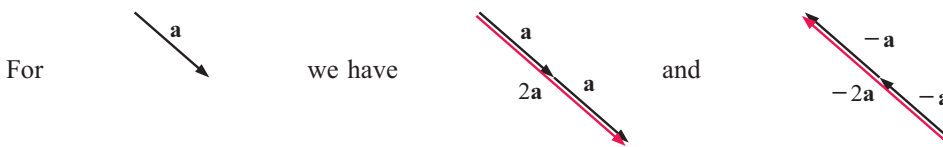
So, $2\mathbf{a}$ has the same direction as $\mathbf{a}$ and is twice as long as $\mathbf{a}$, and

   $-2\mathbf{a}$ is in the opposite direction to $\mathbf{a}$ and is twice as long as $\mathbf{a}$.

Example 13   ◀)) *Self Tutor*

For $\mathbf{r} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$,   find   **a** $2\mathbf{r} + \mathbf{s}$ **b** $\mathbf{r} - 2\mathbf{s}$ geometrically.

a
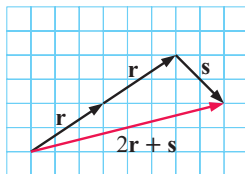
So, $2\mathbf{r} + \mathbf{s} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$.

b
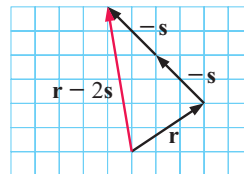
So, $\mathbf{r} - 2\mathbf{s} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$.

If k is a scalar then $k \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix}$,   so each component is multiplied by k.

We can now check the results of **Example 13** algebraically:

In **a**, $2\mathbf{r} + \mathbf{s} = 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and in **b**, $\mathbf{r} - 2\mathbf{s} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 2 \\ -2 \end{pmatrix}$

$\qquad\qquad = \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix} \qquad\qquad = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ -4 \end{pmatrix}$

$\qquad\qquad = \begin{pmatrix} 8 \\ 2 \end{pmatrix} \qquad\qquad\qquad = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$

Example 14 ◀)) *Self Tutor*

Draw sketches of any two vectors **p** and **q** such that: **a** $\mathbf{p} = 2\mathbf{q}$ **b** $\mathbf{p} = -\frac{1}{2}\mathbf{q}$.

Let **q** be
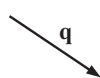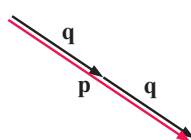
EXERCISE 24F

1 For $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, find geometrically:

a $2\mathbf{r}$ **b** $-3\mathbf{s}$ **c** $\frac{1}{2}\mathbf{r}$ **d** $\mathbf{r} - 2\mathbf{s}$

e $3\mathbf{r} + \mathbf{s}$ **f** $2\mathbf{r} - 3\mathbf{s}$ **g** $\frac{1}{2}\mathbf{s} + \mathbf{r}$ **h** $\frac{1}{2}(2\mathbf{r} + \mathbf{s})$

2 Check your answers to **1** using component form arithmetic.

3 Draw sketches of any two vectors **p** and **q** such that:

a $\mathbf{p} = \mathbf{q}$ **b** $\mathbf{p} = -\mathbf{q}$ **c** $\mathbf{p} = 3\mathbf{q}$ **d** $\mathbf{p} = \frac{3}{4}\mathbf{q}$ **e** $\mathbf{p} = -\frac{3}{2}\mathbf{q}$

4 For $\mathbf{p} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, find $\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ such that:

a $\mathbf{r} = \mathbf{p} - 3\mathbf{q}$ **b** $\mathbf{p} + \mathbf{r} = \mathbf{q}$ **c** $\mathbf{q} - 3\mathbf{r} = 2\mathbf{p}$ **d** $\mathbf{p} + 2\mathbf{r} - \mathbf{q} = 0$

5 If **a** is any vector, prove that $|k\mathbf{a}| = |k||\mathbf{a}|$. **Hint:** Write **a** in component form.

G PARALLEL VECTORS [5.1, 5.2]

Two vectors are **parallel** if one is a scalar multiple of the other.

If two vectors are parallel then one vector is a scalar multiple of the other.

If **a** is parallel to **b** then we write $\mathbf{a} \parallel \mathbf{b}$.

Thus,
- if $\mathbf{a} = k\mathbf{b}$ for some non-zero scalar k, then $\mathbf{a} \parallel \mathbf{b}$
- if $\mathbf{a} \parallel \mathbf{b}$ there exists a non-zero scalar k such that $\mathbf{a} = k\mathbf{b}$.
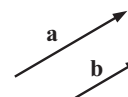
Notice that $\mathbf{a} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ are such that $\mathbf{a} = 3\mathbf{b}$.
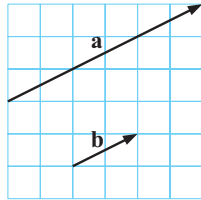
We can see that $\mathbf{a} \parallel \mathbf{b}$.

Notice also that
$$\begin{aligned} |\mathbf{a}| &= \sqrt{36 + 9} \\ &= \sqrt{45} \\ &= 3\sqrt{5} \\ &= 3\,|\mathbf{b}|\,. \end{aligned}$$

Consider the vector $k\mathbf{a}$ which is parallel to $\mathbf{a}$.

- If $k > 0$ then $k\mathbf{a}$ has the same direction as $\mathbf{a}$.
- If $k < 0$ then $k\mathbf{a}$ has the opposite direction to $\mathbf{a}$.
- $|k\mathbf{a}| = |k|\,|\mathbf{a}|$ i.e.,   the length of $k\mathbf{a}$ is the **modulus** of k times the length of $\mathbf{a}$.

If two vectors are parallel and have a point in common then all points on the vectors are collinear.

Example 15   ◀⏵ *Self Tutor*

What two facts can be deduced about $\mathbf{p}$ and $\mathbf{q}$ if:

a $\mathbf{p} = 5\mathbf{q}$ **b** $\mathbf{q} = -\frac{3}{4}\mathbf{p}$?

a $\mathbf{p} = 5\mathbf{q}$
$\therefore$ $\mathbf{p}$ is parallel to $\mathbf{q}$ and $|\mathbf{p}| = |5|\,|\mathbf{q}| = 5\,|\mathbf{q}|$
$\therefore$ $\mathbf{p}$ is 5 times longer than $\mathbf{q}$, and they have the same direction.

b $\mathbf{q} = -\frac{3}{4}\mathbf{p}$
$\therefore$ $\mathbf{q}$ is parallel to $\mathbf{p}$ and $|\mathbf{q}| = \left|-\frac{3}{4}\right|\,|\mathbf{p}| = \frac{3}{4}\,|\mathbf{p}|$
$\therefore$ $\mathbf{q}$ is $\frac{3}{4}$ as long as $\mathbf{p}$, but has the opposite direction.

EXERCISE 24G

1 What two facts can be deduced if:

  a $\mathbf{p} = 2\mathbf{q}$ **b** $\mathbf{p} = \frac{1}{2}\mathbf{q}$ **c** $\mathbf{p} = -3\mathbf{q}$ **d** $\mathbf{p} = -\frac{1}{3}\mathbf{q}$?

2 $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} k \\ -4 \end{pmatrix}$ are parallel.   Find k.

3 Use vector methods only to show that   P$(-2, 5)$,   Q$(3, 1)$,   R$(2, -1)$
and   S$(-3, 3)$,   form the vertices of a parallelogram.

4 Use vector methods to find the remaining vertex of parallelogram ABCD:

Vertices are always listed in order, so PQRS is

either

or

a
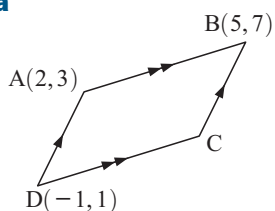
b
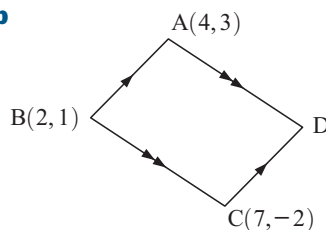
5 A(2, 3), B(−1, 5), C(−1, 1) and D(−7, 5) are four points in the Cartesian plane.

    a Find $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

    b Explain why $\overrightarrow{CD}$ is parallel to $\overrightarrow{AB}$.

    c E is the point $(k, 1)$ and $\overrightarrow{AC}$ is parallel to $\overrightarrow{BE}$. Find k.

6
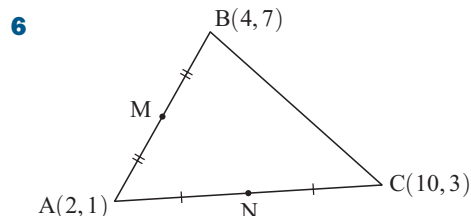
In triangle ABC, M and N are the midpoints of AB and AC respectively.

    a Is MN parallel to BC?

    b Prove your answer to **a** using vector methods.

7 Consider the points A(2, 3), B(4, 7) and C(−2, −5).

    a Find $\overrightarrow{AB}$ and $\overrightarrow{BC}$.   **b** Is there a number k such that $\overrightarrow{BC} = k\overrightarrow{AB}$?

    c Explain what your result in **b** means.

8
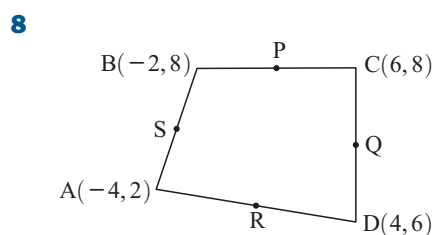
Consider the quadrilateral ABCD for which P, Q, R and S are the midpoints of its sides.

    a Find the coordinates of P, Q, R and S.

    b Find $\overrightarrow{PQ}$ and $\overrightarrow{SR}$.

    c Find $\overrightarrow{SP}$ and $\overrightarrow{RQ}$.

    d What can be deduced about quadrilateral PQRS?

H   VECTORS IN GEOMETRY   [5.2]

Vectors can be used to establish relationships between the line segments in geometric shapes. We can use these relationships to prove geometrical facts.

Example 16   ◀◎ Self Tutor

Find, in terms of   **r**, **s** and **t**:

    a $\overrightarrow{RS}$ **b** $\overrightarrow{SR}$ **c** $\overrightarrow{ST}$
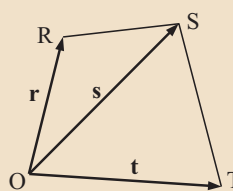
a $\overrightarrow{RS}$	**b** $\overrightarrow{SR}$	**c** $\overrightarrow{ST}$
$= \overrightarrow{RO} + \overrightarrow{OS}$	$= \overrightarrow{SO} + \overrightarrow{OR}$	$= \overrightarrow{SO} + \overrightarrow{OT}$
$= -\overrightarrow{OR} + \overrightarrow{OS}$	$= -\overrightarrow{OS} + \overrightarrow{OR}$	$= -\overrightarrow{OS} + \overrightarrow{OT}$
$= -\mathbf{r} + \mathbf{s}$	$= -\mathbf{s} + \mathbf{r}$	$= -\mathbf{s} + \mathbf{t}$
$= \mathbf{s} - \mathbf{r}$	$= \mathbf{r} - \mathbf{s}$	$= \mathbf{t} - \mathbf{s}$

Example 17   ◀ッ *Self Tutor*
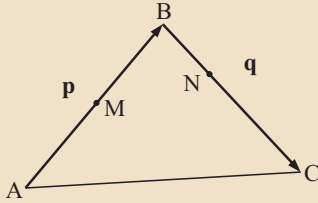
In the diagram, $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{BC} = \mathbf{q}$, M is the midpoint of AB and N divides BC in the ratio 1 : 2. Find in terms of $\mathbf{p}$ and $\mathbf{q}$, vector expressions for:

| **a** $\overrightarrow{AM}$ | **b** $\overrightarrow{BN}$ | **c** $\overrightarrow{AN}$ |
| **d** $\overrightarrow{MC}$ | **e** $\overrightarrow{MN}$ |

a $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$	**b** $\overrightarrow{BN} = \frac{1}{3}\overrightarrow{BC}$	**c** $\overrightarrow{AN} = \overrightarrow{AB} + \overrightarrow{BN}$
$\quad = \frac{1}{2}\mathbf{p}$	$\quad = \frac{1}{3}\mathbf{q}$	$\quad = \mathbf{p} + \frac{1}{3}\mathbf{q}$
d $\overrightarrow{MC} = \overrightarrow{MB} + \overrightarrow{BC}$	**e** $\overrightarrow{MN} = \overrightarrow{MB} + \overrightarrow{BN}$	
$\quad = \frac{1}{2}\mathbf{p} + \mathbf{q}$	$\quad = \frac{1}{2}\mathbf{p} + \frac{1}{3}\mathbf{q}$	

EXERCISE 24H

1 Find, in terms of $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{t}$:

    a $\overrightarrow{OB}$ **b** $\overrightarrow{CA}$ **c** $\overrightarrow{OC}$
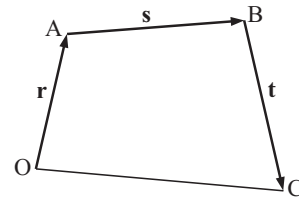
2 Find, in terms of $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$:

    a $\overrightarrow{AD}$ **b** $\overrightarrow{BC}$ **c** $\overrightarrow{AC}$
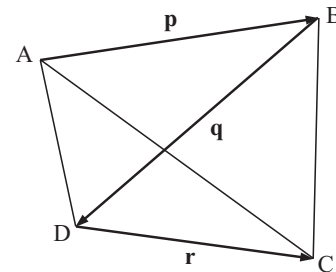
3 In the diagram, $\overrightarrow{OA} = \mathbf{p}$, $\overrightarrow{AB} = \mathbf{q}$, and M is the midpoint of AB.

    Find, in terms of $\mathbf{p}$ and $\mathbf{q}$, vector expressions for:

    a $\overrightarrow{AM}$ **b** $\overrightarrow{OB}$ **c** $\overrightarrow{BM}$ **d** $\overrightarrow{OM}$.
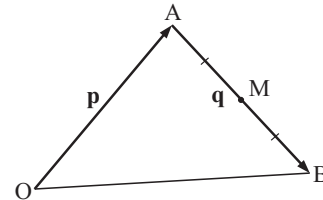
4 In the diagram $\overrightarrow{OA} = \mathbf{r}$, $\overrightarrow{AB} = \mathbf{s}$, $\overrightarrow{OC} = \mathbf{t}$, and M is the midpoint of BC.

    Find, in terms of $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{t}$, vector expressions for:

    a $\overrightarrow{BC}$ **b** $\overrightarrow{BM}$ **c** $\overrightarrow{AM}$ **d** $\overrightarrow{OM}$.
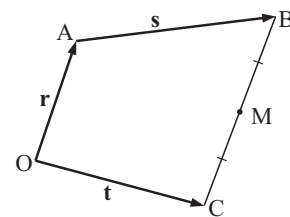
5 A, M and B are collinear and M is the midpoint of AB.

If $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$, show that $\overrightarrow{OM} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$.
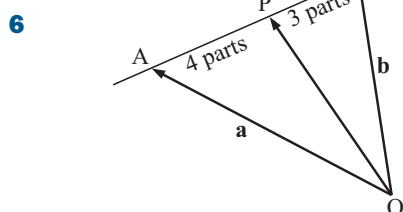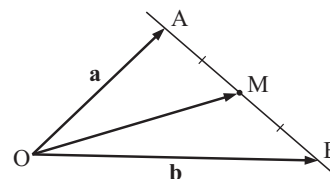
6 In the given figure, AP : PB = 4 : 3. Deduce a vector equation for $\overrightarrow{OP}$ in terms of $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$.
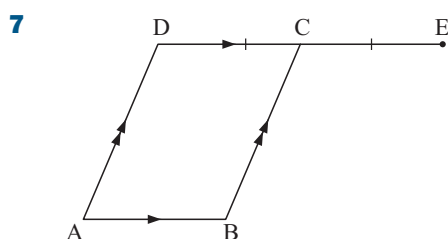
7 ABCD is a parallelogram and side DC is extended to E such that DC = CE.

If $\overrightarrow{AD} = \mathbf{p}$ and $\overrightarrow{AB} = \mathbf{q}$, find in terms of $\mathbf{p}$ and $\mathbf{q}$:

a $\overrightarrow{DC}$ **b** $\overrightarrow{DE}$ **c** $\overrightarrow{AC}$

d $\overrightarrow{AE}$ **e** $\overrightarrow{EA}$ **f** $\overrightarrow{BE}$.

8 In triangle OAB, let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. M and N are the midpoints of sides OB and AB respectively.

a Find vector expressions for:

i $\overrightarrow{BA}$ **ii** $\overrightarrow{MN}$.

b What can be deduced from **a ii**?

c If O is the origin, find the position vectors of:

i M **ii** N **iii** the midpoint of MN.
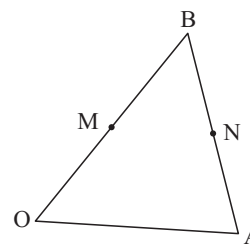
9 ABCD is a parallelogram and M is the midpoint of side DC. P is located on line segment AM such that AP : PM = 2 : 1.

a If $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{AD} = \mathbf{q}$, find vector expressions for:

i $\overrightarrow{DM}$ **ii** $\overrightarrow{AP}$ **iii** $\overrightarrow{DB}$ **iv** $\overrightarrow{DP}$

b What can be deduced about the points D, P and B?
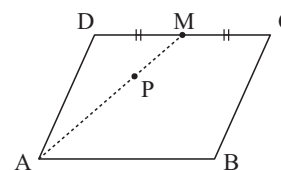
Review set 24A

1 On grid paper draw the vectors:

a $\mathbf{a} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ **b** $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ **c** $\mathbf{c} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$

2 Write in the form $\begin{pmatrix} x \\ y \end{pmatrix}$:
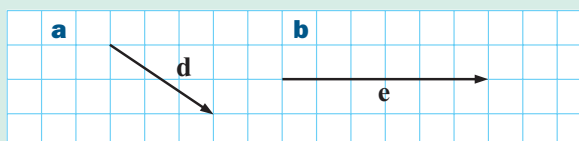
3　Draw a vector diagram to represent a velocity vector of 50 km/h in a NE direction.

4　An aeroplane is flying in an easterly direction at 300 km/h. It encounters a wind from the north at 20 km/h.

　　a　Draw a vector diagram of the plane's velocity while being affected by the wind.

　　b　What is the direction of flight during this period?

　　c　How fast is the plane now travelling?

5　Suppose　$\mathbf{a} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$,　$\mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$　and　$\mathbf{c} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$.

　　a　Draw a diagram which shows how to find　$\mathbf{a} + \mathbf{b}$.

　　b　Find:　**i**　$\mathbf{c} + \mathbf{b}$　　**ii**　$\mathbf{a} - \mathbf{b}$　　**iii**　$\mathbf{a} + \mathbf{b} - \mathbf{c}$

6　Consider　$\mathbf{p} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$　and　$\mathbf{q} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$.

　　a　Sketch　$3\mathbf{p}$.　　　　　**b**　Calculate:　**i**　$2\mathbf{q}$　　**ii**　$3\mathbf{p} + 2\mathbf{q}$　　**iii**　$\mathbf{p} - 2\mathbf{q}$

　　c　Draw a diagram which shows how to find　$\mathbf{q} + 2\mathbf{p}$.

7　Consider the vector　$\mathbf{m} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

　　a　Illustrate the vector on grid paper.　　　　**b**　Find the vector's length.

　　c　Find the bearing of the vector.

8
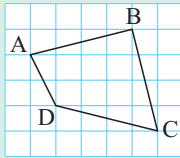
　　a　Find in component form:

　　　　i　$\overrightarrow{BC}$　　　　　　**ii**　$\overrightarrow{BD}$

　　b　Simplify　$\overrightarrow{AD} + \overrightarrow{DC}$.

　　c　Find　$\left| \overrightarrow{AC} \right|$.

9　If　$\begin{pmatrix} -3 \\ -5 \end{pmatrix}$　and　$\begin{pmatrix} -6 \\ k \end{pmatrix}$　are parallel vectors, find k.

10　If P is $(-3, 2)$ and Q is $(1, -1)$, find:

　　a　$\overrightarrow{PQ}$　　　　　　**b**　$\left| \overrightarrow{PQ} \right|$.

11
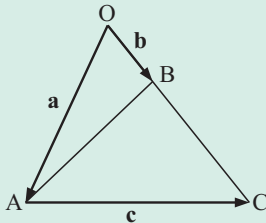
In the figure alongside,　$\overrightarrow{OA} = \mathbf{a}$,　$\overrightarrow{OB} = \mathbf{b}$,　and　$\overrightarrow{AC} = \mathbf{c}$.
Find, in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$:

　　a　$\overrightarrow{CA}$　　　**b**　$\overrightarrow{AB}$　　　**c**　$\overrightarrow{OC}$　　　**d**　$\overrightarrow{BC}$

12　In the given figure　$\overrightarrow{AD} = \mathbf{p}$　and　$\overrightarrow{AB} = 2\mathbf{q}$.
Line segment DC is parallel to line segment AB and is $1\frac{1}{2}$ times its length. T is on line segment BC such that BT : TC = 2 : 1. Find, in terms of $\mathbf{p}$ and $\mathbf{q}$:

　　a　$\overrightarrow{DC}$　　　**b**　$\overrightarrow{CB}$　　　**c**　$\overrightarrow{BT}$
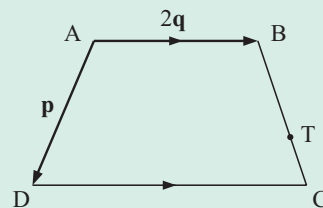
Review set 24B

1 On grid paper draw the vectors:

a $\mathbf{p} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ **b** $\mathbf{q} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ **c** $\mathbf{r} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

2 Write in the form $\begin{pmatrix} x \\ y \end{pmatrix}$:
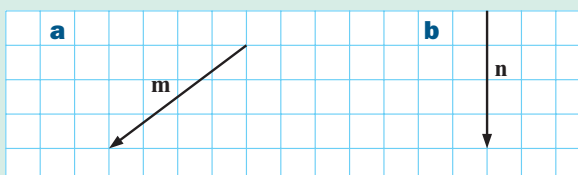
3 Draw a vector diagram to represent a displacement of 200 km in a westerly direction.

4 An aeroplane needs to fly due east to get to its destination. In still air it can travel at 400 km/h. However, a 40 km/h wind is blowing from the south.

    a Draw a vector diagram which shows clearly the direction that the aeroplane must head in.

    b What will be the actual speed of the aeroplane and its bearing to the nearest degree?

5 If K is $(3, -1)$ and L is $(2, 5)$, find:

    a $\overrightarrow{LK}$ **b** $\left| \overrightarrow{LK} \right|$

6 Find n given that $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} n \\ 9 \end{pmatrix}$ are parallel vectors.

7 Suppose $\mathbf{d} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{e} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$.

    a Draw a vector diagram to illustrate $\mathbf{d} - \mathbf{e}$.   **b** Find $\mathbf{d} - \mathbf{e}$ in component form.

    c Find: **i** $2\mathbf{e} + 3\mathbf{d}$ **ii** $4\mathbf{d} - 3\mathbf{e}$

8 What results when opposite vectors are added?

9
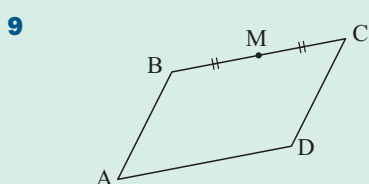
If $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{BC} = \mathbf{q}$ and ABCD is a parallelogram, find vector expressions for:

    a $\overrightarrow{CD}$ **b** $\overrightarrow{BM}$ **c** $\overrightarrow{MD}$ **d** $\overrightarrow{AD}$

10 Draw a scale diagram of a velocity vector of 5 km/h with a bearing of 315°.

11 P and Q are the midpoints of sides AB and BC.

Let $\overrightarrow{AP} = \mathbf{p}$ and $\overrightarrow{BQ} = \mathbf{q}$.

    a Find vector expressions for:

        i $\overrightarrow{PB}$ **ii** $\overrightarrow{QC}$ **iii** $\overrightarrow{PQ}$ **iv** $\overrightarrow{AC}$
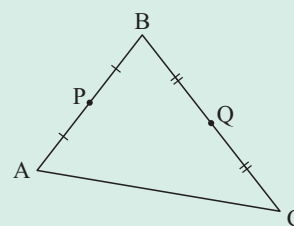
    b How are $\overrightarrow{PQ}$ and $\overrightarrow{AC}$ related?

    c Copy and complete:

    "the line joining the midpoints of two sides of a triangle is .... to the third side and .... its length".

12 Use vectors to find the remaining vertex of:

$B(3,-2)$

A

$C(2,-3)$

$D(-3,-5)$
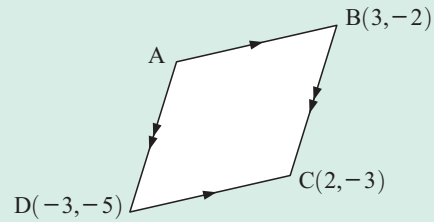
13 In the given figure, O is the origin and OUWV is a
       parallelogram.

$\overrightarrow{OU} = \mathbf{u}$, $\overrightarrow{OV} = \mathbf{v}$, and X lies on UW such that
UX : XW = 4 : 1.

a Find, in terms of **u** and **v**:

   i $\overrightarrow{UX}$ **ii** the position vector of X.

b OX is extended to Y so that $\overrightarrow{OY} = \frac{5}{4}\overrightarrow{OX}$.

   i Find $\overrightarrow{VY}$ in terms of **u** and **v**.

   ii What can be deduced about V, W and Y? Explain your answer.

V W

X

$\mathbf{v}$

O $\mathbf{u}$ U
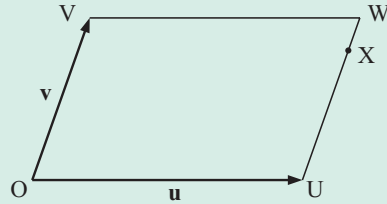
Challenge

1 **a** Draw any quadrilateral, not necessarily one of the special types. Accurately locate the
        midpoints of the sides of the quadrilateral and join them to form another quadrilateral. What
        do you suspect?

   b Repeat **a** with a different quadrilateral two more times.
        Make a detailed statement of what you suspect about the internal quadrilateral.

   c Using *only vector methods*, prove that your suspicion in **b** is correct.

2 **a** Suppose **a** and **b** are two non-zero vectors which are not parallel, and $r\mathbf{a} + s\mathbf{b} = m\mathbf{a} + n\mathbf{b}$
        where r, s, m and n are constants. Show that $r = m$ and $s = n$.

   b

   A

   C

   $\mathbf{a}$ P

   O $\mathbf{b}$ B
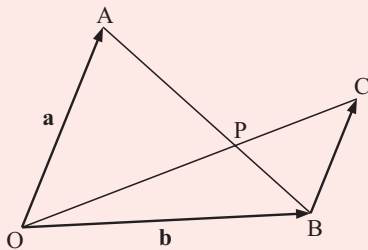
   In the figure, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and BC is parallel
   to OA and half its length.

   i Explain why $\overrightarrow{OP}$ could be written as $k(\mathbf{b} + \frac{1}{2}\mathbf{a})$
       where k is a constant.

   ii Explain why $\overrightarrow{OP}$ could also be written as
       $\mathbf{a} + t(\mathbf{b} - \mathbf{a})$ where t is another constant.

   iii Use **a** to deduce the value of k and hence write $\overrightarrow{OP}$
       in terms of vectors **a** and **b**.

Probability

25

Contents:

Opening problem

Jenaro and Marisa are playing a game at the local fair. They are given a bag containing an equal number of red balls and blue balls, and must each draw one ball from the bag at the same time. Before doing so, they must try to guess whether the balls they select will be the same colour or different colours.

Jenaro thinks it is more than likely that the balls will be the same colour. Marisa thinks it is more likely that the balls will be different colours. Their friend Pia thinks that both outcomes are equally likely.

Who is correct?

A   *INTRODUCTION TO PROBABILITY*   *[10.1]*

Consider these statements:

"The Wildcats will probably beat the Tigers on Saturday."
"It is unlikely that it will rain today."
"I will probably make the team."
"It is almost certain that I will understand this chapter."

Each of these statements indicates a **likelihood** or **chance** of a particular event happening.

We can indicate the likelihood of an event happening in the future by using a percentage.

>   0%   indicates we believe the event **will not occur**.
> 100%   indicates we believe the event **is certain to occur**.

All events can therefore be assigned a percentage between 0% and 100% (inclusive).

A number close to 0% indicates the event is **unlikely** to occur, whereas a number close to 100% means that it is **highly likely** to occur.

In mathematics, we usually write probabilities as either decimals or fractions rather than percentages. However, as 100% = 1, comparisons or conversions from percentages to fractions or decimals are very simple.

> An **impossible** event which has 0% chance of happening is assigned a probability of 0.
>
> A **certain** event which has 100% chance of happening is assigned a probability of 1.
>
> All other events can be assigned a probability between 0 and 1.

For example, when tossing a coin the probability that it falls 'heads' is 50% or $\frac{1}{2}$ or 0.5.

We can write $P(head) = \frac{1}{2}$ or $P(H) = \frac{1}{2}$,   both of which read 'the probability of getting a head is one half'.

So,   a **probability value** is a measure of the chance of a particular event happening.

The assigning of probabilities is usually based on either:

- observing past data or the results of an experiment (experimental probability),   or
- using arguments of symmetry (theoretical probability).

> If A is an event with probability $P(A)$ then $0 \leqslant P(A) \leqslant 1$.

If $P(A) = 0$, the event cannot occur.

If $P(A) = 1$, the event is certain to occur.

If $P(A)$ is very close to 1, it is highly likely that the event will occur.

If $P(A)$ is very close to 0, it is highly unlikely that the event will occur.

The probability of an event cannot be **negative**, or **greater than 1**. It does not make sense to be "less than impossible" or "more than certain".

The probability line below shows words which can be used to describe the chance of an event occurring.
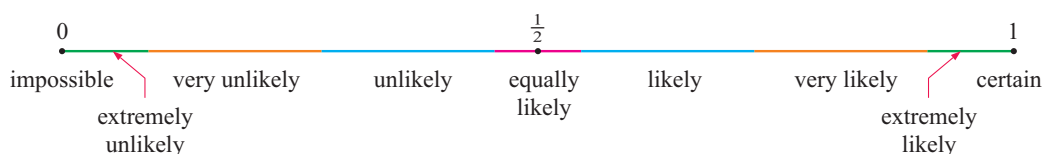
EXERCISE 25A

1   Assign suitable words or phrases to these probability calculations:

    a 0   **b** 0.51   **c** $\frac{1}{1000}$ **d** 0.23   **e** 1   **f** $\frac{1}{2}\%$

    g 0.77   **h** 0.999   **i** $\frac{15}{26}$ **j** $\frac{500}{1999}$ **k** 0.002   **l** $\frac{17}{20}$

2   Suppose that $P(A) = \frac{1}{3}$, $P(B) = 60\%$ and $P(C) = 0.54$.
    Which event is: **a**   most   **b**   least   likely?

3   Use words to describe the probability that:

    a   the maximum temperature in London tomorrow will be negative

    b   you will sleep in the next 48 hours

    c   Manchester United will win its next football match

    d   you will be eaten by a dinosaur

    e   it will rain in Singapore some time this week.

B   ESTIMATING PROBABILITY   [10.2, 10.6]

Sometimes the only way of finding the probability of a particular event occurring is by experimentation or using data that has been collected over time.

In a probability experiment:

- The **number of trials** is the total number of times the experiment is repeated.
- The **outcomes** are the different results possible for one trial of the experiment.
- The **frequency** of a particular outcome is the number of times that this outcome is observed.
- The **relative frequency** of an outcome is the frequency of that outcome divided by the total number of trials.

$$\text{relative frequency} = \frac{\text{frequency}}{\text{number of trials}}$$

For example, when tossing a tin can in the air 250 times, it comes to rest on an end 37 times. We say:

- the number of trials is 250
- the outcomes are *ends* and *sides*
- the frequency of *ends* is 37 and *sides* is 213
- the relative frequency of *ends* $= \frac{37}{250} \approx 0.148$
- the relative frequency of *sides* $= \frac{213}{250} \approx 0.852$.

> The **relative frequency** of an event is an estimate of its **probability**.

We write   estimated P(end) ≈ 0.148 and   estimated P(side) ≈ 0.852.

Suppose in one year an insurance company receives 9573 claims from its 213 829 clients. The probability of a client making a claim in the next year can be predicted by the relative frequency:

$$\frac{9537}{213\,829} \approx 0.0446 \approx 4.46\%.$$

Knowing this result will help the company calculate its charges or premiums for the following year.

Activity   *Rolling a pair of dice*

In this experiment you will roll a pair of dice and add the numbers on the uppermost faces. When this is repeated many times the sums can be recorded in a table like this one:

Sum	2	3	4	5	6	7	8	9	10	11	12
Frequency											
Relative Frequency											

What to do:

1 Roll two dice 100 times and record the results in a table.

2 Calculate the relative frequency for each possible outcome.

3 Combine the results of everyone in your class. Calculate the overall relative frequency for each outcome.

4 Discuss your results.

> The larger the number of trials, the more confident we are that the estimated probability obtained is accurate.

Example 1   ◀)) *Self Tutor*

Estimate the probability of:

a tossing a head with one toss of a coin if it falls heads 96 times in 200 tosses

b rolling a *six* with a die given that when it was rolled 300 times, a *six* occurred 54 times.

a   Estimated P(getting a head)   **b**   Estimated P(rolling a *six*)
     = relative frequency of getting a head   = relative frequency of rolling a *six*
     $= \frac{96}{200}$ $= \frac{54}{300}$
     $= 0.48$ $= 0.18$

Example 2

◀) **Self Tutor**

A marketing company surveys 80 randomly selected people to discover what brand of shoe cleaner they use. The results are shown in the table alongside:

Brand	Frequency
Shine	27
Brite	22
Cleano	20
No scuff	11

a Based on these results, estimate the probability of a community member using: **i** Brite   **ii** Cleano.

b Would you classify the estimate of **a** to be very good, good, or poor? Why?

a We start by calculating the relative frequency for each brand.

i Experimental P(Brite) = 0.275

ii Experimental P(Cleano) = 0.250

b Poor, as the sample size is very small.

Brand	Frequency	Relative Frequency
Shine	27	0.3375
Brite	22	0.2750
Cleano	20	0.2500
No scuff	11	0.1375

EXERCISE 25B

1 Estimate the probability of rolling *an odd number* with a die if *an odd number* occurred 33 times when the die was rolled 60 times.

2 Clem fired 200 arrows at a target and hit the target 168 times. Estimate the probability of Clem hitting the target.

3 Ivy has free-range hens. Out of the first 123 eggs that they laid she found that 11 had double-yolks. Estimate the probability of getting a double-yolk egg from her hens.

4 Jackson leaves for work at the same time each day. Over a period of 227 working days, on his way to work he had to wait for a train at the railway crossing on 58 days. Estimate the probability that Jackson has to wait for a train on his way to work.

5 Ravi has a circular spinner marked P, Q and R on 3 equal sectors. Estimate the probability of getting a Q if the spinner was twirled 417 times and finished on Q on 138 occasions.

6 Each time Claude shuffled a pack of cards before a game, he recorded the suit of the top card of the pack. His results for 140 games were 34 hearts, 36 diamonds, 38 spades and 32 clubs. Estimate the probability that the top card of a shuffled pack is:

a a heart   **b** a club or diamond.

7 Estimate probabilities from these observations:

a Our team has won 17 of its last 31 games.

b There are 23 two-child families in our street of 64 families.

8 A marketing company was commissioned to investigate brands of products usually found in the bathroom. The results of a soap survey are shown alongside:

a How many people were randomly selected in this survey?

b Calculate the relative frequency of use of each brand of soap, correct to 3 significant figures.

Brand	Freq	Rel Freq
Silktouch	125	
Super	107	
Just Soap	93	
Indulgence	82	
Total		

   c Using the results obtained by the marketing company, estimate the probability that the soap used by a randomly selected person is:

      i Just Soap   **ii** Indulgence   **iii** Silktouch?

9 Two coins were tossed 489 times and the *number of heads* occurring at each toss was recorded. The results are shown opposite:

   a Copy and complete the table given.

   b Estimate the chance of the following events occurring:

      i 0 heads   **ii** 1 head   **iii** 2 heads.

Outcome	Freq	Rel Freq
0 heads	121	
1 head		
2 heads	109	
Total		

10 At the Annual Show the toffee apple vendor estimated that three times as many people preferred red toffee apples to green toffee apples.

   a If 361 people wanted green toffee apples, estimate how many wanted red.

   b Copy and complete the table given.

   c Estimate the probability that the next customer will ask for:

      i a green toffee apple   **ii** a red toffee apple.

Colour	Freq	Rel Freq
Green	361	
Red		
Total		

11 The tickets sold for a tennis match were recorded as people entered the stadium. The results are shown:

   a How many tickets were sold in total?

   b Copy and complete the table given.

   c If a person in the stadium is selected at random, estimate the probability that the person bought a Concession ticket.

Ticket Type	Freq	Rel Freq
Adult	3762	
Concession	1084	
Child	389	
Total		

12 The results of a local Council election are shown in the table. It is known that 6000 people voted in the election.

   a Copy and complete the table given.

   b Estimate the chance that a randomly selected person from this electorate voted for a female councillor.

Councillor	Freq	Rel Freq
Mr Tony Trimboli	2167	
Mrs Andrea Sims	724	
Mrs Sara Chong	2389	
Mr John Henry		
Total		

C PROBABILITIES FROM TWO-WAY TABLES
[10.2, 10.6]

Two-way tables are tables which compare two categorical variables. They usually result from a survey.

For example, the year 10 students in a small school were tested to determine their ability in mathematics. The results are summarised in the two-way table shown:

	Boy	Girl
Good at maths	17	19
Not good at maths	8	12

there are 12 girls who are not good at maths.

In this case the variables are *ability in maths* and *gender*.
We can use these tables to estimate probabilities.

Example 3   ◀) *Self Tutor*

To investigate the breakfast habits of teenagers, a survey was conducted amongst the students of a high school. The results were:

	Male	Female
Regularly eats breakfast	87	53
Does not regularly eat breakfast	68	92

Use this table to estimate the probability that a randomly selected student from the school:

a is male   **b** is male *and* regularly eats breakfast

c is female *or* regularly eats breakfast

d is male, given that the student regularly eats breakfast

e regularly eats breakfast, given that the student is female.

We extend the table to include totals:

	Male	Female	Total
Regularly eats breakfast	87	53	140
Does not regularly eat breakfast	68	92	160
Total	155	145	300

a There are 155 males out of the 300 students surveyed.

∴   P(male) $= \frac{155}{300} \approx 0.517$

b 87 of the 300 students are male and regularly eat breakfast.

∴   P(male *and* regularly eats breakfast) $= \frac{87}{300} \approx 0.29$

c $53 + 92 + 87 = 232$ out of the 300 are female or regularly eat breakfast.

∴   P(female *or* regularly eats breakfast) $= \frac{232}{300} \approx 0.773$

d Of the 140 students who regularly eat breakfast, 87 are male.

∴   P(male given that regularly eats breakfast) $= \frac{87}{140} \approx 0.621$

e Of the 145 females, 53 regularly eat breakfast

∴   P(regularly eats breakfast, given female) $= \frac{53}{145} \approx 0.366$

EXERCISE 25C

1 Adult workers were surveyed and asked if they had a problem with the issue of wage levels for men and women doing the same job. The results are summarised in the two-way table shown.

	Problem	No Problem
Men	146	175
Women	188	134

Assuming that the results are representative of the whole community, estimate the probability that the next randomly chosen adult worker:

a is a woman

b has a problem with the issue

c is a male with no problem with the issue

d is a female, given the person has a problem with the issue

e has no problem with the issue, given that the person is female.

2 310 students at a high school were surveyed on the question "Do you like watching basketball being played on TV?". The results are shown in the two-way table alongside.

	Like	Dislike
Junior students	87	38
Senior students	129	56

  a Copy and complete the table to include 'totals'.
  b Estimate the probability that a randomly selected student:
      i   likes watching basketball on TV and is a junior student
      ii likes watching basketball on TV and is a senior student
      iii likes watching basketball on TV, given that the student is a senior
      iv is a senior, given that the student likes watching basketball on TV.

3 The two-way table shows the students who can and cannot swim in three different year groups at a school.

	Can swim	Cannot swim
Year 4	215	85
Year 7	269	31
Year 10	293	7

If a student is randomly selected from these year groups, estimate the probability that:

  a the student can swim   **b** the student cannot swim
  c the student is from year 7   **d** the student is from year 7 and cannot swim
  e the student is from year 7 or cannot swim
  f the student cannot swim, given that the student is from year 7 or 10
  g the student is from year 4, given that the student cannot swim.

D   EXPECTATION   [10.3]

The probability of an event can be used to predict the number of times the event will occur in a number of trials.

For example, when rolling an ordinary die, the probability of rolling a '4' is $\frac{1}{6}$.

If we roll the die 120 times, we expect $120 \times \frac{1}{6} = 20$ of the outcomes to be '4's.

> Suppose the probability of an event occurring is p. If the trial is repeated n times, the **expectation** of the event, or the number of times we expect it to occur, is np.

Example 4   ◀)) *Self Tutor*

In one week, 79 out of 511 trains were late to the station at Keswick. In the next month, 2369 trains are scheduled to pass through the station. How many of these would you expect to be late?

We estimate the probability of a train being late to be $p = \frac{79}{511}$.

We expect $2369 \times \frac{79}{511} \approx 366$ trains to be late.

EXERCISE 25D

1 In a particular region in Africa, the probability that it will rain on any one day is 0.177. On how many days of the year would you expect it to rain?

2 At practice, Tony kicked 53 out of 74 goals from the penalty goal spot. If he performs as well through the season and has 18 attempts to kick penalty goals, how many is he expected to score?

3 A certain type of drawing pin, when tossed 400 times, landed on its back 144 times.

    a Estimate the probability that it will land on its back if it is tossed once.

    b If the drawing pin is tossed 72 times, how many "backs" would you expect?

4 A bag contains 5 red and 3 blue discs. A disc is chosen at random and then replaced. This is repeated 200 times. How many times would you expect a red disc to be chosen?

5 A die has the numbers 0, 1, 2, 2, 3 and 4 on its faces. The die is rolled 600 times. How many times might we expect a result of:

    a 0   **b** 2   **c** 1, 2 or 3   **d** not a 4?

6 **a** If 2 coins are tossed, what is the chance that they both fall heads?

    b If the 2 coins are tossed 300 times, on how many occasions would you expect them to both fall heads?

7 On the last occasion Annette threw darts at the target shown, she hit the inner circle 17% of the time and the outer circle 72% of the time.

    a Estimate the probability of Annette missing the target with her next throw.

    b Suppose Annette throws the dart 100 times at the target. She receives 100 points if she hits the inner circle and 20 points if she hits the outer circle. Find:

        i the total number of points you would expect her to get

       ii the mean number of points you would expect per throw.
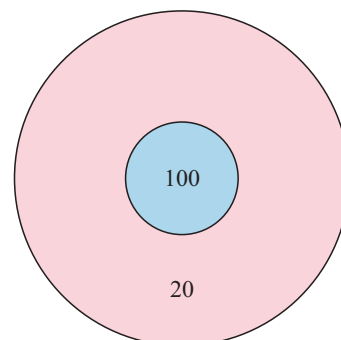
E   REPRESENTING COMBINED EVENTS
[10.4, 10.6]

The possible outcomes for tossing two coins are listed below:

    two heads   head and tail   tail and head   two tails

These results are the **combination** of two events: tossing coin 1 and tossing coin 2.

If H represents a 'head' and T a 'tail', the sample space of possible outcomes is   HH, HT, TH and TT.

> A **sample space** is the set of all possible outcomes of an experiment.

Possible ways of representing sample spaces are:
- listing them
- using a 2-dimensional grid
- using a tree diagram
- using a Venn diagram.

Example 5
◄)) *Self Tutor*

Represent the sample space for tossing two coins using:

a a list **b** a 2-D grid **c** a tree diagram

a {HH, HT, TH, TT}

b
coin 2

T

H

H   T   coin 1

c
coin 1   coin 2

H   H
       T

T   H
       T
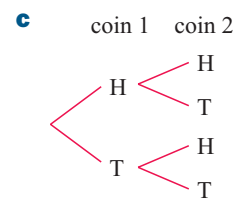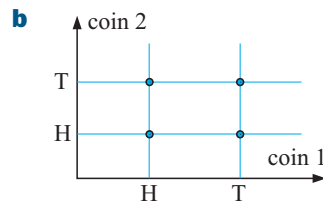
Example 6
◄)) *Self Tutor*

Illustrate, using a tree diagram, the possible outcomes when drawing two marbles from a bag containing several marbles of each of the colours red, green and yellow.

Let   R be the event of getting a red
      G be the event of getting a green
      Y be the event of getting a yellow.

marble 1   marble 2

R   R
       G
       Y

G   R
       G
       Y

Y   R
       G
       Y
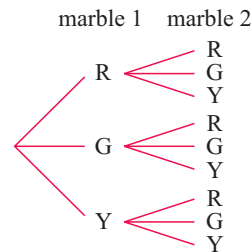
We have already seen Venn diagrams in **Chapter 2**.

If two events have common outcomes, a Venn diagram may be a suitable way to display the sample space.

For example, the Venn diagram opposite shows that of the 27 students in a class, 11 play tennis, 17 play basketball, and 3 play neither of these sports.

T

B

7   4   13

3

U

T ≡ tennis   B ≡ basketball
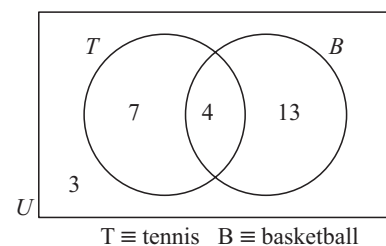
EXERCISE 25E

1 List the sample space for the following:

  a twirling a square spinner labelled A, B, C, D

  b the sexes of a 2-child family

  c the order in which 4 blocks A, B, C and D can be lined up

  d the 8 different 3-child families.

  e spinning a coin **i** twice **ii** three times **iii** four times.

2   Illustrate on a 2-dimensional grid the sample space for:

  a   rolling a die and tossing a coin simultaneously

  b   rolling two dice

  c   rolling a die and spinning a spinner with sides A, B, C, D

  d   twirling two square spinners: one labelled A, B, C, D and the other 1, 2, 3, 4.

3   Illustrate on a tree diagram the sample space for:

  a   tossing a 5-cent and 10-cent coin simultaneously

  b   tossing a coin and twirling an equilateral triangular spinner labelled A, B and C

  c   twirling two equilateral triangular spinners labelled 1, 2 and 3 and X, Y and Z

  d   drawing two tickets from a hat containing a number of pink, blue and white tickets.

  e   drawing two beads from a bag containing 3 red and 4 blue beads.

4   Draw a Venn diagram to show a class of 20 students where 10 study History, 15 study Geography, and 2 study neither subject.

F   THEORETICAL PROBABILITY   [10.4, 10.6]

From the methods of showing sample spaces in the previous section, we can find the probabilities of combined events.

These are theoretical probabilities which are calculated using

$$P(\text{event happens}) = \frac{\text{number of ways the event can happen}}{\text{total number of possible outcomes}}.$$

Example 7   ◀) *Self Tutor*

Three coins are tossed. Write down a list of all possible outcomes. Find the probability of getting:

  a   3 heads   **b**   at least one head

  c   3 heads if it is known that there is at least one head.

The sample space is: HHH   HHT   TTH   TTT
                                    HTH   THT
                                    THH   HTT

  a   P(3 heads) $= \frac{1}{8}$

  b   P(at least one H) $= \frac{7}{8}$ {all except TTT}

  c   P(HHH knowing at least one H) $= \frac{1}{7}$
        {The sample space now excludes TTT}

Notice how we list the outcomes in a systematic way.

Example 8   ◀)) *Self Tutor*

A die has the numbers 0, 0, 1, 1, 4 and 5. It is rolled *twice*. Illustrate the sample
space using a 2-D grid. Hence find the probability of getting:

    a a total of 5   **b** two numbers which are the same.

2-D grid   roll 1
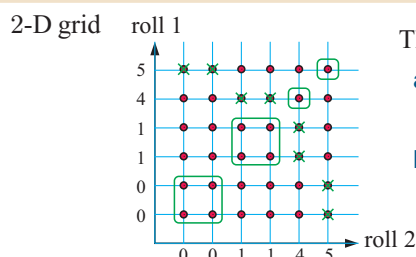
There are $6 \times 6 = 36$ possible outcomes.

  a P(total of 5)

      $= \frac{8}{36}$ {those with a $\times$}

  b P(same numbers)

      $= \frac{10}{36}$ {those circled}

Example 9   ◀)) *Self Tutor*

In a class of 30 students, 19 play sport, 8 play the piano, and 3 both play sport and the piano. Display
this information on a Venn diagram and hence determine the probability that a randomly selected class
member plays:

    a both sport and the piano   **b** at least one of sport and the piano

    c sport, but not the piano   **d** exactly one of sport and the piano

    e neither sport nor the piano   **f** the piano if it is known that the student plays sport.
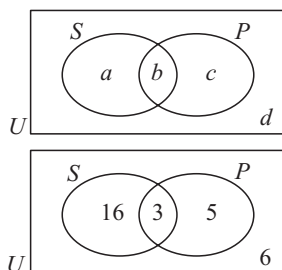
Let S represent the event of 'playing sport',
and P represent the event of 'playing the piano'.

  Now $a + b = 19$ {as 19 play sport}

           $b + c = 8$ {as 8 play the piano}

               $b = 3$ {as 3 play both}

  $a + b + c + d = 30$ {as there are 30 in the class}

  $\therefore$ $b = 3$, $a = 16$, $c = 5$, $d = 6$.

a   P(S and P)   **b**   P(at least one of S and P)

    $= \frac{3}{30}$ or $\frac{1}{10}$ $= \frac{16+3+5}{30}$

                              $= \frac{24}{30}$ (or $\frac{4}{5}$)

> In **f**, since we know
> that the student
> plays sport, we look
> only at the sport set S.
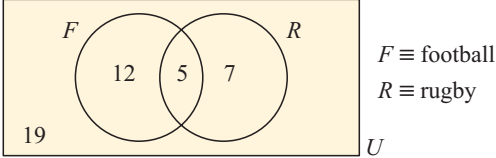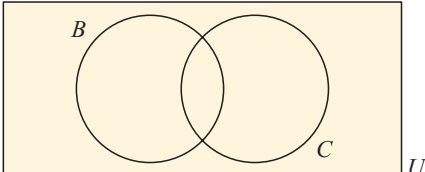
c   P(S but not P)   **d**   P(exactly one of S and P)

    $= \frac{16}{30}$ $= \frac{16+5}{30}$

    $= \frac{8}{15}$ $= \frac{7}{10}$

e   P(neither S nor P)   **f**   P(P given S)

    $= \frac{6}{30}$ $= \frac{3}{16+3}$

    $= \frac{1}{5}$ $= \frac{3}{19}$

EXERCISE 25F

1 a List all possible orderings of the letters O, D and G.

   b If these three letters are placed at random in a row, what is the probability of:

       i spelling DOG   **ii** O appearing first

       iii O not appearing first   **iv** spelling DOG or GOD?

2 The Venn diagram shows the sports played by boys at the local high school.
A student is chosen at random.
Find the probability that he:

   a plays football   **b** plays both codes

   c plays football or rugby   **d** plays exactly one of these sports

   e plays neither of these sports

   f plays football, given that he is in at least one team

   g plays rugby, given that he plays football.

$F \equiv$ football
$R \equiv$ rugby

3 Draw the grid of the sample space when a 10-cent and a 50-cent coin are tossed simultaneously. Hence determine the probability of getting:

   a two heads   **b** two tails

   c exactly one head   **d** at least one head.

4 A coin and a pentagonal spinner with sectors 1, 2, 3, 4 and 5 are tossed and spun respectively.

   a Draw a grid to illustrate the sample space of possible outcomes.

   b How many outcomes are possible?

   c Use your grid to determine the chance of getting:

       i a head and a 4   **ii** a tail and an odd number

      iii an even number   **iv** a tail or a 3.

5 List the six different orders in which Alex, Bodi and Kek may sit in a row. If the three of them sit randomly in a row, determine the probability that:

   a Alex sits in the middle   **b** Alex sits at the left end

   c Alex sits at the right end   **d** Bodi and Kek are seated together.

6 a List the 8 possible 3-child families, according to the gender of the children.
For example, BGB means *"the first is a boy, the second is a girl, and the third is a boy"*.

   b Assuming that each of these is equally likely to occur, determine the probability that a randomly selected 3-child family consists of:

       i all boys   **ii** all girls

      iii boy, then girl, then girl   **iv** two girls and a boy

       v a girl for the eldest   **vi** at least one boy.

7 In a class of 24 students, 10 take Biology, 12 take Chemistry, and 5 take neither Biology nor Chemistry. Find the probability that a student picked at random from the class takes:

   a Chemistry but not Biology

   b Chemistry or Biology.

8   **a**   List, in systematic order, the 24 different orders in which four people P, Q, R and S may sit in a row.

   b   Hence, determine the probability that when the four people sit at random in a row:

   i   P sits on one end

   ii   Q sits on one of the two middle seats

   iii   P and Q are seated together

   iv   P, Q and R are seated together, not necessarily in that order.

9   A pair of dice is rolled.

   a   Show that there are 36 members in the sample space of possible outcomes by displaying them on a grid.

   b   Hence, determine the probability of a result with:

   i   one die showing a 4 and the other a 5   **ii**   both dice showing the same result

   iii   at least one die showing a result of 3   **iv**   either a 4 or 6 being displayed

   v   both dice showing even numbers   **vi**   the sum of the values being 7.

10   60 married men were asked whether they gave their wife flowers or chocolates for their last birthday. The results were: 26 gave chocolates, 21 gave flowers, and 5 gave both chocolates and flowers. If one of the married men was chosen at random, determine the probability that he gave his wife:

   a   flowers but not chocolates   **b**   neither chocolates nor flowers

   c   chocolates or flowers.

11   List the possible outcomes when four coins are tossed simultaneously. Hence determine the probability of getting:

   a   all heads   **b**   two heads and two tails   **c**   more tails than heads

   d   at least one tail   **e**   exactly one head.

12   **a**   Copy and complete the grid alongside for the sample space of drawing one card from an ordinary pack.
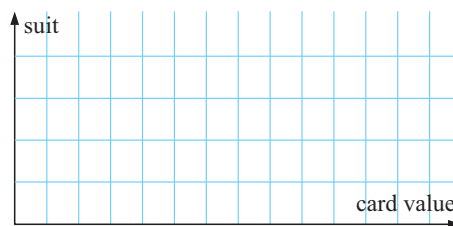
   b   Use your grid to determine the probability of getting:

   i   a Queen   **ii**   the Jack of hearts   **iii**   a spade

   iv   a picture card   **v**   a red 7   **vi**   a diamond or a club

   vii   a King or a heart   **viii**   a Queen and a 3.

13   The medical records for a class of 28 children show whether they had previously had measles or mumps. The records show 22 have had measles, 13 have had measles and mumps, and 27 have had measles or mumps. If one child from the class is selected at random, determine the probability that he or she has had:

   a   measles   **b**   mumps but not measles   **c**   neither mumps nor measles.

G   COMPOUND EVENTS   [10.4]

We have previously used two-dimensional grids to represent sample spaces and hence find answers to certain probability problems.

Consider again a simple example of tossing a coin and rolling a die simultaneously.

To determine the probability of getting a head and a '5', we can illustrate the sample space on the two-dimensional grid shown. We can see that there are 12 possible outcomes but only one with the property that we want, so the answer is $\frac{1}{12}$.

However, notice that $P(\text{a head}) = \frac{1}{2}$, $P(\text{a '5'}) = \frac{1}{6}$ and $\frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$.

This suggests that $P(\text{a head and a '5'}) = P(\text{a head}) \times P(\text{a '5'})$,
i.e., we multiply the separate probabilities.
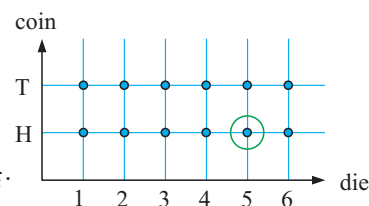
INDEPENDENT EVENTS

It seems that if A and B are two events for which the occurrence of each one does not affect the occurrence of the other, then $P(A \text{ and } B) = P(A) \times P(B)$.

The two events 'getting a head' and 'rolling a 5' are events with this property, as the occurrence or non-occurrence of either one of them cannot affect the occurrence of the other. We say they are **independent**.

> If two events A and B are **independent** then $P(A \text{ and } B) = P(A) \times P(B)$.

Example 10   ◄ Self Tutor

A coin is tossed and a die rolled simultaneously. Find the probability that a tail and a '2' result.

'Getting a tail' and 'rolling a 2' are independent events.

$\therefore \quad P(\text{a tail and a '2'}) = P(\text{a tail}) \ \times \ P(\text{a '2'})$
$$= \tfrac{1}{2} \times \tfrac{1}{6}$$
$$= \tfrac{1}{12}$$

COMPLEMENTARY EVENTS

Two events are **complementary** if exactly one of them *must* occur.

The probabilities of complementary events sum to 1.

The **complement** of event E is denoted E'. It is the event when E fails to occur.

> For any event E with **complementary** event E',
> $P(E) + P(E') = 1$ or $P(E') = 1 - P(E)$.

Example 11   ◆》 *Self Tutor*

Sunil has probability $\frac{4}{5}$ of hitting a target and Monika has probability $\frac{5}{6}$.

If they both fire simultaneously at the target, determine the probability that:

  a they both hit it   **b** they both miss it.

Let S be the event of Sunil hitting and M be the event of Monika hitting.

a P(both hit)	**b** P(both miss)
$= \text{P}(S \text{ and } M \text{ hits})$	$= \text{P}(S' \text{ and } M')$
$= \text{P}(S) \times \text{P}(M)$	$= \text{P}(S') \times \text{P}(M')$
$= \frac{4}{5} \times \frac{5}{6}$	$= \frac{1}{5} \times \frac{1}{6}$
$= \frac{2}{3}$	$= \frac{1}{30}$

EXERCISE 25G.1

1 A coin and a pentagonal spinner with edges marked A, B, C, D and E are tossed and twirled simultaneously. Find the probabilities of getting:

  a a head and a D   **b** a tail and either an A or a D.

2 A spinner with 6 equal sides has 3 red, 2 blue and 1 yellow edge. A second spinner with 7 equal sides has 4 purple and 3 green edges. Both spinners are twirled simultaneously. Find the probability of getting:

  a a red and a green   **b** a blue and a purple.

3 Janice and Lee take set shots at a netball goal from 3 m. From past experience, Janice throws a goal on average 2 times in every 3 shots, whereas Lee throws a goal 4 times in every 7. If they both shoot for goals, determine the probability that:

  a both score a goal   **b** both miss

  c Janice scores a goal but Lee misses.

4 When a nut was tossed 400 times it finished on its edge 84 times and on its side for the rest. Use this information to estimate the probability that when two identical nuts are tossed:

                                                                         edge   side

  a they both fall on their edges   **b** they both fall on their sides.

5 Tei has probability $\frac{1}{3}$ of hitting a target with an arrow, while See has probability $\frac{2}{5}$. If they both fire at the target, determine the probability that:

  a both hit the target   **b** both miss the target

  c Tei hits the target and See misses   **d** Tei misses the target and See hits.

6 A certain brand of drawing pin was tossed into the air 600 times. It landed on its back ⚓ 243

times and on its side ⟩⟩ for the remainder. Use this information to estimate the probability that:

  a one drawing pin, when tossed, will fall on its   **i** back   **ii** side

  b two drawing pins, when tossed, will both fall on their backs

  c two drawing pins, when tossed, will both fall on their sides.

DEPENDENT EVENTS

Suppose a cup contains 4 red and 2 green marbles. One marble is randomly chosen, its colour is noted, and it is then put aside. A second marble is then randomly selected. What is the chance that it is red?

If the first marble was red,   P(second is red) $= \frac{3}{5}$ ← 3 reds remaining
← 5 to choose from

If the first marble was green,   P(second is red) $= \frac{4}{5}$ ← 4 reds remaining
← 5 to choose from

So, the probability of the second marble being red **depends** on what colour the first marble was. We therefore have **dependent events**.

Two or more events are **dependent** if they are **not independent**.

Dependent events are events for which the occurrence of one of the events *does affect* the occurrence of the other event.

For compound events which are dependent, a similar product rule applies as to that for independent events:

If A and B are dependent events then   P(A **then** B) = P(A) × P(B given that A has occurred).

Example 12   ◀ೊ Self Tutor

A box contains 4 blue and 3 yellow buttons of the same size. Two buttons are randomly selected from the box without replacement. Find the probability that:

a   both are yellow   **b**   the first is yellow and the second is blue.

a   P(both are yellow)
= P(first is yellow *and* second is yellow)
= P(first is yellow) × P(second is yellow given that the first is yellow)

$= \frac{3}{7} \times \frac{2}{6}$ ← 2 yellows remaining
← 6 to choose from

$= \frac{1}{7}$

b   P(first is Y and second is B)
= P(first is Y) × P(second is B given that the first is Y)

$= \frac{3}{7} \times \frac{4}{6}$ ← 4 blues remaining
← 6 to choose from

$= \frac{2}{7}$

EXERCISE 25G.2

1   A packet contains 8 identically shaped jelly beans. 5 are green and 3 are yellow. Two jelly beans are randomly selected without replacing the first before the second is drawn.

   a   Determine the probability of getting:

      i   two greens   **ii**   a green then a yellow

      iii   a yellow then a green   **iv**   two yellows.

   b   Why do your answers in **a** add up to 1?

2 A pocket in a golf bag contains 6 white and 4 yellow golf balls. Two of them are selected at random without replacement.

  a Determine the probability that:

    i both are white   **ii** the first is white and the second is yellow

    iii one of each colour is selected.

  b Why do your answers in **a** not add up to 1?

3 A container has 4 purple, 3 blue and 1 gold ticket. Three tickets are selected without replacement. Find the probability that:

  a all are purple   **b** all are blue   **c** the first two are purple and the third is gold.

H   USING TREE DIAGRAMS   [10.5]

Tree diagrams can be used to illustrate sample spaces, provided that the alternatives are not too numerous. Once the sample space is illustrated, the tree diagram can be used for determining probabilities.

Consider **Example 11** again. The tree diagram for this information is:

S means Sunil hits
M means Monika hits

	Monika's results	outcome	probability
	M	S and M	$\frac{4}{5} \times \frac{5}{6} = \frac{20}{30}$
S	M'	S and M'	$\frac{4}{5} \times \frac{1}{6} = \frac{4}{30}$
S'	M	S' and M	$\frac{1}{5} \times \frac{5}{6} = \frac{5}{30}$
	M'	S' and M'	$\frac{1}{5} \times \frac{1}{6} = \frac{1}{30}$
		total	1

Sunil's results: S branch $\frac{4}{5}$, S' branch $\frac{1}{5}$; Monika's results: $\frac{5}{6}$, $\frac{1}{6}$.
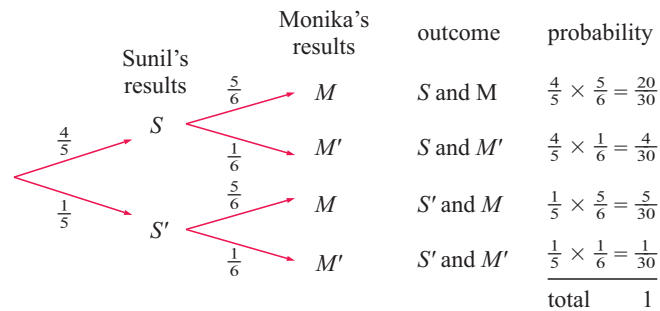
Notice that:

- The probabilities for hitting and missing are marked on the branches.
- There are *four* alternative paths and each path shows a particular outcome.
- All outcomes are represented and the probabilities of each of the outcomes are obtained by **multiplying** the probabilities along that path.

Example 13   ◀) *Self Tutor*

Stephano is having problems. His desktop computer will only boot up 90% of the time and his laptop will only boot up 70% of the time.

  a Draw a tree diagram to illustrate this situation.

  b Use the tree diagram to determine the chance that:

    i both will boot up   **ii** Stephano has no choice but to use his desktop computer.

  a D = desktop computer boots up
    L = laptop boots up

	laptop	outcome	probability
desktop D	0.7 L	D and L	$0.9 \times 0.7 = 0.63$
0.9	0.3 L'	D and L'	$0.9 \times 0.3 = 0.27$
0.1 D'	0.7 L	D' and L	$0.1 \times 0.7 = 0.07$
	0.3 L'	D' and L'	$0.1 \times 0.3 = 0.03$
		total	1.00

b **i**   P(both boot up)   **ii**   P(desktop boots up but laptop does not)

= P(D and L)   = P(D and L')

= 0.9×0.7 = 0.9×0.3

= 0.63 = 0.27

Example 14   ◀)) *Self Tutor*

Bag A contains 4 red jelly beans and 1 yellow jelly bean. Bag B contains 2 red and 3 yellow jelly beans. A bag is randomly selected by tossing a coin, and one jelly bean is removed from it. Determine the probability that it is yellow.
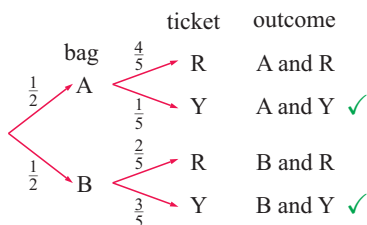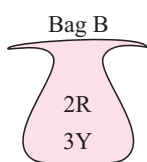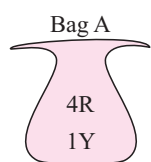
To get a yellow we take either the first branch ticked **or** the second one ticked. We **add** the probabilities for these outcomes.

P(yellow) = P(A and Y) + P(B and Y)

$= \frac{1}{2} \times \frac{1}{5} + \frac{1}{2} \times \frac{3}{5}$ {branches marked ✓}

$= \frac{4}{10}$

$= \frac{2}{5}$

EXERCISE 25H

1 Suppose this spinner is spun twice:
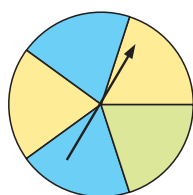
a Copy and complete the branches on the tree diagram shown.
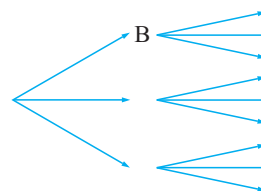
b What is the probability that blue appears on both spins?

c What is the probability that green appears on both spins?

d What is the probability that different colours appear on both spins?

e What is the probability that blue appears on *either* spin?

2 In a particular board game there are nine tiles: five are green and the remainder are brown. The tiles start face down on the table so they all look the same.

a If a player is required to pick a tile at random, determine the probability that it is:

   i green   **ii** brown.
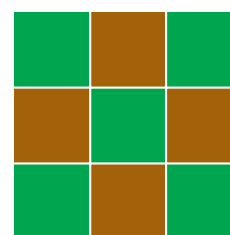
b Suppose a player has to pick two tiles in a row, replacing the first and shuffling them before the second is selected. Copy and complete the tree diagram illustrating the possible outcomes.

c Using **b**, determine the probability that:

   i both tiles are green

   ii both tiles are brown

   iii tile 1 is brown and tile 2 is green

   iv one tile is brown and the other is green.
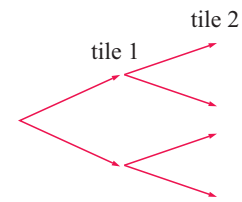
3 The probability of the race track being muddy next week is estimated to be $\frac{1}{4}$. If it is muddy, Rising Tide will start favourite with probability $\frac{2}{5}$ of winning. If it is dry he has a $\frac{1}{20}$ chance of winning.

  a Display the sample space of possible results on a tree diagram.

  b Determine the probability that Rising Tide will win next week.
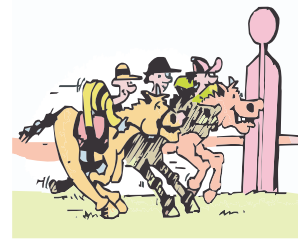
4 Machine A cans 60% of the fruit at a factory. Machine B cans the rest. Machine A spoils 3% of its product, while Machine B spoils 4%. Determine the probability that the next can inspected at this factory will be spoiled.

5 Box A contains 2 blue and 3 red blocks and Box B contains 5 blue and 1 red block. A box is chosen at random (by the flip of a coin) and one block is taken at random from it. Determine the probability that the block is red.

6

4B 3R	3B 4R	5B 2R
A	B	C

Three bags contain different numbers of blue and red tickets. A bag is selected using a die which has three A faces, two B faces, and one C face.

One ticket is selected randomly from the chosen bag. Determine the probability that it is: **a** blue   **b** red.
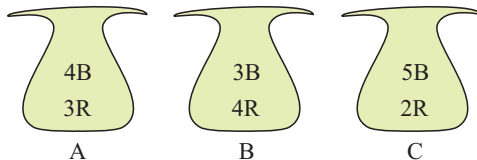
I   SAMPLING WITH AND WITHOUT REPLACEMENT   [10.5]

Sampling is the process of selecting one object from a large group and inspecting it for some particular feature. The object is then either **put back** (sampling **with replacement**) or **put to one side** (sampling **without replacement**).

Sometimes the inspection process makes it impossible to return the object to the large group.
Such processes include:

- Is the chocolate hard- or soft-centred? Bite it or squeeze it to see.
- Does the egg contain one or two yolks? Break it open and see.
- Is the object correctly made? Pull it apart to see.

The sampling process is used for quality control in industrial processes.

Example 15

◀》 *Self Tutor*

A bin contains 4 blue and 5 green marbles. A marble is selected from this bin and its colour is noted. It is then *replaced*. A second marble is then drawn and its colour is noted. Determine the probability that:

a both are blue   **b** the first is blue and the second is green

c there is one of each colour.

Tree diagram:
  B = blue
  G = green

2nd marble

1st marble

	outcome	probability
B	B and B	$\frac{4}{9} \times \frac{4}{9} = \frac{16}{81}$
G	B and G	$\frac{4}{9} \times \frac{5}{9} = \frac{20}{81}$
B	G and B	$\frac{5}{9} \times \frac{4}{9} = \frac{20}{81}$
G	G and G	$\frac{5}{9} \times \frac{5}{9} = \frac{25}{81}$
	total	1
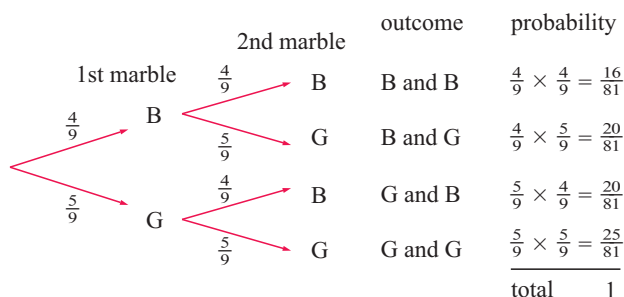
a   P(both blue)

$= \frac{4}{9} \times \frac{4}{9}$

$= \frac{16}{81}$

b   P(first is B and second is G)

$= \frac{4}{9} \times \frac{5}{9}$

$= \frac{20}{81}$

c   P(one of each colour)

$= $ P(B then G or G then B)

$= $ P(B then G) $+$ P(G then B)

$= \frac{4}{9} \times \frac{5}{9} + \frac{5}{9} \times \frac{4}{9}$

$= \frac{40}{81}$

Example 16

◀》 *Self Tutor*

Sylke has bad luck with the weather when she takes her summer holidays. She estimates that it rains 60% of the time and it is cold 70% of the time.

a Draw a tree diagram to illustrate this situation.

b Use the tree diagram to determine the chance that for Sylke's holidays:

    i it is cold and raining   **ii** it is fine and cold.

a $C = $ the weather is cold $R = $ it is raining

temperature   rain

	outcome	probability
R	C and R	$0.7 \times 0.6 = 0.42$
R'	C and R'	$0.7 \times 0.4 = 0.28$
R	C' and R	$0.3 \times 0.6 = 0.18$
R'	C' and R'	$0.3 \times 0.4 = 0.12$
	total	1.00

b **i**   P(it is cold and raining)

$= $ P(C and R)

$= 0.7 \times 0.6$

$= 0.42$

ii   P(it is fine and cold)

$= $ P(R' and C)

$= 0.4 \times 0.7$

$= 0.28$

EXERCISE 25I

1 A box contains 6 red and 3 yellow tickets. Two tickets are drawn at random (the first being *replaced* before the second is drawn). Draw a tree diagram to represent the sample space and use it to determine the probability that:

    a both are red
    b both are yellow

    c the first is red and the second is yellow
    d one is red and the other is yellow.

2 7 tickets numbered 1, 2, 3, 4, 5, 6 and 7 are placed in a hat. Two of the tickets are taken from the hat at random *without replacement*. Determine the probability that:

    a both are odd
    b both are even

    c the first is even and the second is odd
    d one is even and the other is odd.

3 Jessica has a bag of 9 acid drops which are all identical in shape. 5 are raspberry flavoured and 4 are orange flavoured. She selects one acid drop at random, eats it, and then takes another, also at random. Determine the probability that:

    a both acid drops were orange flavoured
    b both acid drops were raspberry flavoured

    c the first was raspberry and the second was orange

    d the first was orange and the second was raspberry.

    Add your answers to **a**, **b**, **c** and **d**. Explain why this sum is 1.

4 A cook selects an egg at random from a carton containing 7 ordinary eggs and 5 double-yolk eggs. She cracks the egg into a bowl and sees whether it has two yolks or not. She then selects another egg at random from the carton and checks it.

Let S represent "a single yolk egg" and D represent "a double yolk egg".

    a Draw a tree diagram to illustrate this sampling process.

    b What is the probability that both eggs had two yolks?

    c What is the probability that both eggs had only one yolk?

5 Freda selects a chocolate at random from a box containing 8 hard-centred and 11 soft-centred chocolates. She bites it to see whether it is hard-centred or not. She then selects another chocolate at random from the box and checks it.

Let H represent "a hard-centred chocolate" and S represent "a soft-centred chocolate".

    a Draw a tree diagram to illustrate this sampling process.

    b What is the probability that both chocolates have hard centres?

    c What is the probability that both chocolates have soft centres?

6 A sporting club runs a raffle in which 200 tickets are sold. There are two winning tickets which are drawn at random, in succession, without replacement. If Adam bought 8 tickets in the raffle, determine the probability that he:

    a wins first prize
    b does not win first prize
    c wins both prizes

    d wins neither prize
    e wins second prize *given that* he did not win first prize.

J   MUTUALLY EXCLUSIVE AND NON-MUTUALLY EXCLUSIVE EVENTS   [10.5]

Suppose we select a card at random from a normal pack of 52 playing cards. Consider carefully these events:

Event X: the card is a heart   *Event Y:*   the card is an ace   *Event Z:*   the card is a 7

Notice that:

- X and Y have a common outcome: the Ace of hearts
- X and Z have a common outcome: the 7 of hearts
- Y and Z do not have a common outcome.

When considering a situation like this:

- if two events have no common outcomes we say they are **mutually exclusive** or **disjoint**
- if two events have common outcomes they are **not mutually exclusive**.
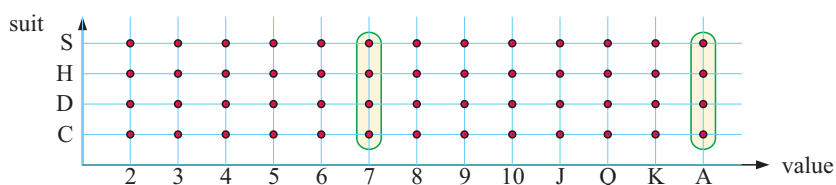
Notice that: P(ace **or** seven) $= \frac{8}{52}$ and   P(ace) + P(seven) $= \frac{4}{52} + \frac{4}{52} = \frac{8}{52}$

If two events A and B are **mutually exclusive** then   P(A **or** B) = P(A) + P(B)
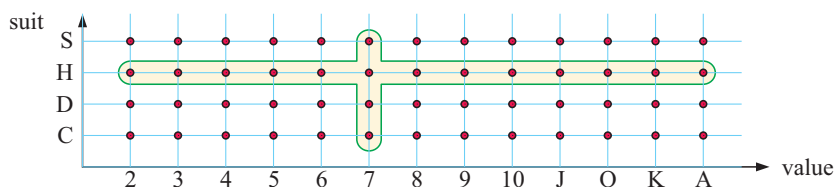
Notice that: P(heart **or** seven) $= \frac{16}{52}$ and   P(heart) + P(seven) $= \frac{13}{52} + \frac{4}{52} = \frac{17}{52}$.

Actually,   P(heart **or** seven) = P(heart) + P(seven) − P(heart **and** seven).

If two events A and B are **not mutually exclusive** then   P(A **or** B) = P(A) + P(B) − P(A **and** B).

EXERCISE 25J

1   An ordinary die with faces 1, 2, 3, 4, 5 and 6 is rolled once. Consider these events:

   A: getting a 1 B: getting a 3 C: getting an odd number

   D: getting an even number E: getting a prime number F: getting a result greater than 3.

   a List all possible pairs of events which are mutually exclusive.
   b Find: **i** P(B or D)   **ii** P(D or E)   **iii** P(A or E)
                  iv P(B or E)   **v** P(C or D)   **vi** P(A or B or F).

2 A committee consists of 4 accountants, 2 managers, 5 lawyers, and 6 engineers. A chairperson is randomly selected. Find the probability that the chairperson is:

　　a a lawyer　　　　　**b** a manager or an engineer　　　**c** an accountant or a manager

3 A jar contains 3 red balls, 2 green balls, and 1 yellow ball. Two balls are selected at random from the jar without replacement. Find the probability that the balls are either both red or both green.

4 A coin and an ordinary die are tossed simultaneously.

　　a Draw a grid showing the 12 possible outcomes.

　　b Find the probability of getting:　　**i** a head and a 5　　**ii** a head or a 5.

　　c Check that:　　P(H or 5) = P(H) + P(5) − P(H **and** 5).

5 Two ordinary dice are rolled.

　　a Draw a grid showing the 36 possible outcomes.

　　b Find the probability of getting:　　**i** a 3 and a 4　　**ii** a 3 or a 4.

　　c Check that:　　P(3 or 4) = P(3) + P(4) − P(3 **and** 4).

K MISCELLANEOUS PROBABILITY QUESTIONS [10.4 - 10.6]

In this section you will encounter a variety of probability questions. You will need to select the appropriate technique for each problem, and are encouraged to use tools such as tree and Venn diagrams.

EXERCISE 25K

1 50 students went on a 'thrill seekers' holiday. 40 went white-water rafting, 21 went paragliding, and each student did at least one of these activities.

　　a From a Venn diagram, find how many students did both activities.

　　b If a student from this group is randomly selected, find the probability that he or she:

　　　　i went white-water rafting but not paragliding

　　　　ii went paragliding given that he or she went white-water rafting.

2 A bag contains 7 red and 3 blue balls. Two balls are randomly selected without replacement. Find the probability that:

　　a the first is red and the second is blue　　　**b** the balls are different in colour.

3 In a class of 25 students, 19 have fair hair, 15 have blue eyes, and 22 have fair hair, blue eyes or both. A child is selected at random. Determine the probability that the child has:

　　a fair hair and blue eyes　　　　**b** neither fair hair nor blue eyes

　　c fair hair but not blue eyes　　　**d** blue eyes given that the child has fair hair.

4 Abdul cycles to school and must pass through a set of traffic lights. The probability that the lights are red is $\frac{1}{4}$. When they are red, the probability that Abdul is late for school is $\frac{1}{10}$. When they are not red the probability is $\frac{1}{50}$.

　　a Calculate the probability that Abdul is late for school.

　　b There are 200 days in the school year. How many days in the school year would you expect Abdul to be late?

5 28 students go tramping. 23 get sunburn, 8 get blisters, and 5 get both sunburn and blisters. Determine the probability that a randomly selected student:

    a did not get blisters

    b either got blisters or sunburn

    c neither got blisters nor sunburn

    d got blisters, given that the student was sunburnt

    e was sunburnt, given that the student did not get blisters.

6 An examination in French has two parts: aural and written. When 30 students sit for the examination, 25 pass aural, 26 pass written, and 3 fail both parts. Determine the probability that a student who:

    a passed aural also passed written   **b** passed aural, failed written.

7 Three coins are tossed. Find the probability that:

    a all of them are tails   **b** two are heads and the other is a tail.

8 Marius has 2 bags of peaches. Bag A has 4 ripe and 2 unripe peaches, and bag B has 5 ripe and 1 unripe peaches. Ingrid selects a bag by tossing a coin, and takes a peach from that bag.

    a Determine the probability that the peach is ripe.

    b Given that the peach is ripe, what is the probability it came from B?

9 Two coins are tossed and a die is rolled.

    a Illustrate the sample space on a grid.

    b Find the probability of getting:

        i two heads and a '6'   **ii** two tails and an odd number

        iii a head and a tail, or a '6'.

10 In a country town there are 3 supermarkets: P, Q and R. 60% of the population shop at P, 36% shop at Q, 34% shop at R, 18% shop at P and Q, 15% shop at P and R, 4% shop at Q and R, and 2% shop at all 3 supermarkets. A person is selected at random.

Determine the probability that the person shops at:

    a none of the supermarkets

    b at least one of the supermarkets

    c exactly one of the supermarkets

    d either P or Q

    e P, given that the person shops at at least one supermarket

    f R, given that the person shops at either P or Q or both.

11 On a given day, Claude's car has an 80% chance of starting first time and André's car has a 70% chance of the same. Given that at least one of the cars has started first time, what is the chance that André's car started first time?

Review set 25A

1 Donna kept records of the number of clients she interviewed over a period of consecutive days.

   a For how many days did the survey last?

   b Estimate Donna's chances of interviewing:

   i no clients on a day

   ii four or more clients on a day

   iii less than three clients on a day.
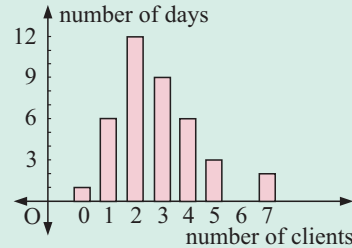
2 Illustrate on a 2-dimensional grid the possible outcomes when a coin and a pentagonal spinner with sides labelled A, B, C, D and E are tossed and spun simultaneously.

3 University students were surveyed to find who owns a motor vehicle (MV) and who owns a computer. The results are shown in the two-way table.

Estimate the probability that a randomly selected university student has:

	MV	$no\ MV$
computer	124	168
no computer	16	22

   a a computer   **b** a motor vehicle   **c** a computer and a motor vehicle

   d a motor vehicle given that the student does not have a computer.

4 What is meant by saying that two events are "independent"?

5 Use a tree diagram to illustrate the sample space for the possible four-child families. Hence determine the probability that a randomly chosen four-child family:

   a is all boys   **b** has exactly two boys   **c** has more girls than boys.

6 In a shooting competition, Louise has 80% chance of hitting her target and Kayo has 90% chance of hitting her target. If they both have a single shot, determine the probability that:

   a both hit their targets   **b** neither hits her target
   c at least one hits her target   **d** only Kayo hits her target.

7 Two fair six-sided dice are rolled simultaneously. Determine the probability that the result is a 'double', i.e., both dice show the same number.

8 A bag contains 4 green and 3 red marbles. Two marbles are randomly selected from the bag without replacement. Determine the probability that:

   a both are green   **b** they are different in colour.

9 A circle is divided into 5 sectors with equal angles at the centre. It is made into a spinner, and the sectors are numbered 1, 2, 3, 4, and 5. A coin is tossed and the spinner is spun.

   a Use a 2-dimensional grid to show the sample space.

   b What is the chance of getting: **i** a head and a 5   **ii** a head or a 5?

10 Bag X contains three white and two red marbles. Bag Y contains one white and three red marbles. A bag is randomly chosen and two marbles are drawn from it. Illustrate the given information on a tree diagram and hence determine the probability of drawing two marbles of the same colour.

11 At a local girls school, 65% of the students play netball, 60% play tennis, and 20% play neither sport. Display this information on a Venn diagram, and hence determine the likelihood that a randomly chosen student plays:

   a netball   **b** netball but not tennis   **c** at least one of these two sports
   d exactly one of these two sports   **e** tennis, given that she plays netball.

Review set 25B

1 Pierre conducted a survey to determine the ages of people walking through a shopping mall. The results are shown in the table alongside. Estimate, to 3 decimal places, the probability that the next person Pierre meets in the shopping mall is:

Age	Frequency
0 - 19	22
20 - 39	43
40 - 59	39
60+	14

  a between 20 and 39 years of age

  b less than 40 years of age   **c** at least 20 years of age.

2 **a** List the sample space of possible results when a tetrahedral die with four faces labelled A, B, C and D is rolled and a 20-cent coin is tossed simultaneously.

  b Use a tree diagram to illustrate the sample spaces for the following:

   i Bags A, B and C contain green or yellow tickets. A bag is selected and then a ticket taken from it.

   ii Martina and Justine play tennis. The first to win three sets wins the match.

3 When a box of drawing pins was dropped onto the floor, it was observed that 49 pins landed on their backs and 32 landed on their sides. Estimate, to 2 decimal places, the probability of a drawing pin landing:

  a on its back   **b** on its side.

back   side

4 The letters A, B, C, D, ..., N are put in a hat.

  a Determine the probability of drawing a vowel (A, E, I, O or U) if one of the letters is chosen at random.

  b If two letters are drawn without replacement, copy and complete the following tree diagram including all probabilities:
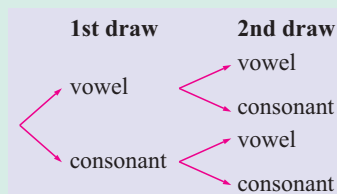
  c Use your tree diagram to determine the probability of drawing:

   i a vowel and a consonant

   ii at least one vowel.

1st draw   **2nd draw**

vowel → vowel

vowel → consonant

consonant → vowel

consonant → consonant

5 A farmer fences his rectangular property into 9 rectangular paddocks as shown alongside.
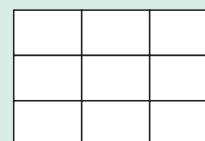
If a paddock is selected at random, what is the probability that it has:

  a no fences on the boundary of the property

  b one fence on the boundary of the property

  c two fences on the boundary of the property?

6 Bag X contains 3 black and 2 red marbles. Bag Y contains 4 black and 1 red marble. A bag is selected at random and then two marbles are selected without replacement. Determine the probability that:

  a both marbles are red   **b** two black marbles are picked from Bag Y.

7 Two dice are rolled simultaneously. Illustrate this information on a 2-dimensional grid. Determine the probability of getting:

  a a double 5   **b** at least one 4

  c a sum greater than 9   **d** a sum of 7 or 11.

8 A class consists of 25 students. 15 have blue eyes, 9 have fair hair, and 3 have both blue eyes and fair hair. Represent this information on a Venn diagram.

Hence find the probability that a randomly selected student from the class:

   a has neither blue eyes nor fair hair

   b has blue eyes, but not fair hair

   c has fair hair given that he or she has blue eyes

   d does not have fair hair given that he or she does not have blue eyes.

9 The two-way table alongside shows the results from asking the question "Do you like the school uniform?".

If a student is randomly selected from these year groups, estimate the probability that the student:

	Likes	Dislikes
Year 8	129	21
Year 9	108	42
Year 10	81	69

   a likes the school uniform

   b dislikes the school uniform

   c is in year 8 and dislikes the uniform

   d is in year 9 given the student likes the uniform

   e likes the uniform given the student is in year 10.

10 The probability of a delayed flight on a foggy day is $\frac{9}{10}$. When it is not foggy the probability of a delayed flight is $\frac{1}{12}$. If the probability of a foggy day is $\frac{1}{20}$, find the probability of:

   a a foggy day and a delayed flight

   b a delayed flight

   c a flight which is not delayed.

   d Comment on your answers to **b** and **c**.

Sequences

26

Contents:

Opening problem

contains one square
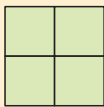
contains 5 squares
(four 1 by 1 and one 2 by 2).
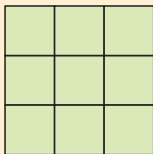
contains $1 + 4 + 9$ squares
(nine 1 by 1, four 2 by 2 and one 3 by 3).

Things to think about:

a How many squares are contained in a 4 by 4 square, a 5 by 5 square and a 6 by 6 square?

b How many squares are contained in a 100×100 square?

c Is there an easy way of finding the answer to **b**?

A   NUMBER SEQUENCES   [2.12]

A **number sequence** is a set of numbers listed in a specific order, where the numbers can be found by a specific rule.

The first term is denoted by u_1, the second by u_2, the third by u_3, and so on.

The **nth term** is written as u_n.

We often describe a number sequence in words, giving a rule which connects one term with the next.

For example,   15, 11, 7, 3, -1, ...   can be described by the rule:

Start with 15 and each term thereafter is 4 less than the previous one.

The next two terms are $u_6 = -1 - 4 = -5$
    and $u_7 = -5 - 4 = -9$.

We say that this sequence is **linear** because each term differs from the previous one by a constant value.

Example 1   ◀)) *Self Tutor*

Write down a rule to describe the sequence and hence find its next two terms:

a   3, 7, 11, 15, 19, ...   **b**   2, 6, 18, 54, ...   **c**   0, 1, 1, 2, 3, 5, 8, ...

a   Start with 3 and each term thereafter is 4 more than the previous term.
    $u_6 = 23$ and $u_7 = 27$.
b   Start with 2 and each term thereafter is 3 times the previous term.
    $u_5 = 54 \times 3 = 162$ and $u_6 = 162 \times 3 = 486$.
c   The first two terms are 0 and 1, and each term thereafter is the sum of the previous two terms.
    $u_8 = 5 + 8 = 13$ and $u_9 = 8 + 13 = 21$.
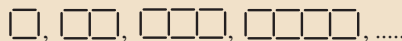
Example 2   ◀)) *Self Tutor*

Draw the next two matchstick figures in these sequences and write the number of matchsticks used as a number sequence:

a

b
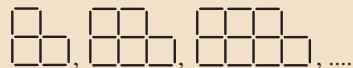
a

4, 7, 10, 13, 16, 19, ...

b

10, 15, 20, 25, 30, ...

EXERCISE 26A

1 Write down a rule to describe the sequence and hence find its next *two* terms:

    a 5, 8, 11, 14, 17, ...   **b** 2, 9, 16, 23, 30, ...   **c** 8, 19, 30, 41, 52, ...

    d 38, 34, 30, 26, 22, ...   **e** 3, -2, -7, -12, -17, ...   **f** $\frac{1}{2}$, 2, $3\frac{1}{2}$, 5, $6\frac{1}{2}$, ...

2 Write down a rule to describe the sequence and hence find its next *two* terms:

    a 3, 6, 12, 24, 48, ...   **b** 1, 2, 4, 8, 16, ...   **c** 2, 10, 50, 250, ...

    d 36, 18, 9, $4\frac{1}{2}$, ...   **e** 162, 54, 18, 6, ...   **f** 405, 135, 45, 15, ...

3 Find the next *two* terms of:

    a 0, 1, 4, 9, 16, ...   **b** 1, 4, 9, 16, 25, ...   **c** 0, 1, 8, 27, 64, ...

    d $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$, $\frac{1}{25}$, ...   **e** 1, 2, 4, 7, 11, ...   **f** 2, 6, 12, 20, 30, ...

4 Write down a rule to describe the sequences and hence find its next *three* terms:

    a 1, 1, 2, 3, 5, 8, ...   **b** 1, 3, 4, 7, 11, ...   **c** 5, 8, 12, 18, 24, 30, ...

5 Draw the next *two* matchstick figures in these sequences and write the number of matchsticks used as a number sequence:

6 Draw the next *two* figures in these sequences and write the number of dots used as a number sequence:

B   ALGEBRAIC RULES FOR SEQUENCES   [2.12]

An alternative way to describe a sequence is to write an algebraic rule or formula for its *n*th term u_n.

For example: $u_n = 3n + 2$, $u_n = n^2 + n$, $u_n = \dfrac{1}{n}$

Since these are general formulae for all terms of their sequences, u_n is often called the **general term**.

The possible substitutions for n are: $n = 1, 2, 3, 4, 5, 6, \ldots$ so an algebraic rule for u_n is valid for $n \in \mathbb{Z}^+$ only.

Example 3   ◀)) *Self Tutor*

Find the first 5 terms of the sequence with the rule:

a $u_n = 5n - 3$ **b** $u_n = n(n + 2)$ **c** $u_n = 3 \times 2^n$

a $u_1 = 5(1) - 3 = 2$
   $u_2 = 5(2) - 3 = 7$
   $u_3 = 5(3) - 3 = 12$
   $u_4 = 5(4) - 3 = 17$
   $u_5 = 5(5) - 3 = 22$
   The sequence is:
   2, 7, 12, 17, 22, ...

b $u_1 = 1(3) = 3$
   $u_2 = 2(4) = 8$
   $u_3 = 3(5) = 15$
   $u_4 = 4(6) = 24$
   $u_5 = 5(7) = 35$
   The sequence is:
   3, 8, 15, 24, 35, ...

c $u_1 = 3(2)^1 = 6$
   $u_2 = 3(2)^2 = 12$
   $u_3 = 3(2)^3 = 24$
   $u_4 = 3(2)^4 = 48$
   $u_5 = 3(2)^5 = 96$
   The sequence is:
   6, 12, 24, 48, 96, ...

Discussion   *Properties of sequences*

- Consider the sequence in **Example 3** part **a**. The sequence is *linear* because each term differs from the previous one by the same constant 5.
  What part of the formula for u_n indicates this fact?

- What can be said about the formula for a linear sequence where each term differs from the previous one by: **a** 7   **b** -4 ?

- Consider the sequence in **Example 3** part **c**. Notice that each term is double the previous term. What part of the formula for u_n causes this?

Example 4   ◀)) *Self Tutor*

a Find the next two terms and an expression for the nth term u_n of 3, 6, 9, 12, 15, ...
b *Hence* find a formula for the general term u_n of:
   i 4, 7, 10, 13, 16, ... **ii** 1, 4, 7, 10, 13, ... **iii** $\frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \frac{1}{14}$, ...

a $u_1 = 3 \times 1, \quad u_2 = 3 \times 2, \quad u_3 = 3 \times 3, \quad u_4 = 3 \times 4, \quad u_5 = 3 \times 5$
   $\therefore \quad u_n = 3 \times n = 3n$
   $\therefore \quad u_6 = 3 \times 6 = 18$ and $u_7 = 3 \times 7 = 21$.

b **i** $u_1 = 3 + 1, \quad u_2 = 6 + 1, \quad u_3 = 9 + 1, \quad u_4 = 12 + 1, \quad u_5 = 15 + 1$
   Each term is 1 more than in the sequence in **a**.
   $\therefore \quad u_n = 3n + 1$

   ii $u_1 = 3 - 2, \quad u_2 = 6 - 2, \quad u_3 = 9 - 2, \quad u_4 = 12 - 2, \quad u_5 = 15 - 2$
   Each term is 2 less than in the sequence in **a**.
   $\therefore \quad u_n = 3n - 2$

   iii $u_1 = \dfrac{1}{3 + 2}, \quad u_2 = \dfrac{1}{6 + 2}, \quad u_3 = \dfrac{1}{9 + 2}, \quad u_4 = \dfrac{1}{12 + 2}, \quad u_5 = \dfrac{1}{15 + 2}$

   By comparison with the sequence in **a**, $u_n = \dfrac{1}{3n + 2}$

EXERCISE 26B

1 Find the first *four* terms of the sequence with nth term:

a $u_n = 2n + 3$ **b** $u_n = 2n + 5$ **c** $u_n = 3n + 2$

d $u_n = 3n + 5$ **e** $u_n = -2n + 1$ **f** $u_n = -2n + 3$

g $u_n = 6 - 3n$ **h** $u_n = 17 - 4n$ **i** $u_n = 76 - 7n$

2 Find the first *four* terms of the sequence with nth term:

a $u_n = n^2 + 1$ **b** $u_n = n^2 - 1$ **c** $u_n = n^2 + n$

d $u_n = n(n + 2)$ **e** $u_n = n^3 + 1$ **f** $u_n = n^3 + 2n^2 - 1$

3 **a** Find a formula for the general term u_n of the sequence: 2, 4, 6, 8, 10, 12, ...

  b *Hence* find a formula for the general term u_n of:

    i 4, 6, 8, 10, 12, 14, ...   **ii** 1, 3, 5, 7, 9, ...

4 **a** Find a formula for the general term u_n of: 5, 10, 15, 20, 25, ...

  b *Hence* find a formula for the general term u_n of:

    i 6, 11, 16, 21, 26, ...   **ii** 3, 8, 13, 18, 23, ...

5 **a** Find a formula for the general term u_n of: 2, 4, 8, 16, 32, ...

  b *Hence* find a formula for the general term u_n of: 6, 12, 24, 48, 96, ...

6 **a** Find a general formula for: 1, 2, 3, 4, 5, 6, ...

  b *Hence* find a general formula for:

    i 2, 3, 4, 5, 6, 7, ...   **ii** 3, 4, 5, 6, 7, 8, ...

    iii $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, ...   **iv** $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, ...

    v $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$, ...   **vi** $\frac{3}{1}, \frac{4}{2}, \frac{5}{3}, \frac{6}{4}, \frac{7}{5}, \frac{8}{6}$, ...

    vii $1 \times 2, 2 \times 3, 3 \times 4, 4 \times 5$, ...   **viii** $2 \times 3, 3 \times 4, 4 \times 5, 5 \times 6$, ...

    ix $1 \times 3, 2 \times 4, 3 \times 5, 4 \times 6$, ...   **x** $\frac{1}{3}, \frac{4}{6}, \frac{7}{9}, \frac{10}{12}$, ...

7 **a** Find a general formula for: 1, 4, 9, 16, 25, ...

  b *Hence* find a general formula for:

    i 4, 9, 16, 25, 36, ...   **ii** 0, 3, 8, 15, 24, ...

    iii $\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}$, ...   **iv** $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}$, ...

8 Find a general formula for:

a 1, 8, 27, 64, 125, ...   **b** 0, 7, 26, 63, 124, ...

c 3, 6, 12, 24, 48, ...   **d** 24, 12, 6, 3, $1\frac{1}{2}$, ...

C   GEOMETRIC SEQUENCES   *[2.12]*

In a **geometric sequence**, each term is found by multiplying the previous one by the same constant.

For example: $2, 2 \times 3, 2 \times 3^2, 2 \times 3^3, 2 \times 3^4$, ... is geometric as each term is three times the previous term.

We see that
$$u_1 = 2 \times 3^0$$
$$u_2 = 2 \times 3^1$$
$$u_3 = 2 \times 3^2$$
$$u_④ = 2 \times 3^③ \longleftarrow \text{so each time the power of 3 is one less than the } \textit{term number.}$$

So, a general formula for the sequence is $u_n = 2 \times 3^{n-1}$.

Example 5   ◀) Self Tutor

List the first *five* terms of the geometric sequence defined by:

a $u_n = 5 \times 2^n$ **b** $u_n = 5 \times 2^{n-1}$

a $u_1 = 5 \times 2^1 = 10$ **b** $u_1 = 5 \times 2^0 = 5$
     $u_2 = 5 \times 2^2 = 20$ $u_2 = 5 \times 2^1 = 10$
     $u_3 = 5 \times 2^3 = 40$ $u_3 = 5 \times 2^2 = 20$
     $u_4 = 5 \times 2^4 = 80$ $u_4 = 5 \times 2^3 = 40$
     $u_5 = 5 \times 2^5 = 160$ $u_5 = 5 \times 2^4 = 80$

Example 6   ◀) Self Tutor

Find the next *two* terms and a formula for the nth term of:

a   2, 6, 18, 54, ...   **b**   2, −6, 18, −54, ...

a   To get each term we multiply the previous one by 3.
$$u_1 = 2 \times 3^0$$
$$u_2 = 2 \times 3^1$$
$$u_3 = 2 \times 3^2$$
$$u_4 = 2 \times 3^3$$
$$u_5 = 2 \times 3^4 = 162$$
$$u_6 = 2 \times 3^5 = 486$$
$$\therefore \quad u_n = 2 \times 3^{n-1}$$

b   To get each term we multiply the previous one by −3.
$$u_1 = 2 \times (-3)^0$$
$$u_2 = 2 \times (-3)^1$$
$$u_3 = 2 \times (-3)^2$$
$$u_4 = 2 \times (-3)^3$$
$$u_5 = 2 \times (-3)^4 = 162$$
$$u_6 = 2 \times (-3)^5 = -486$$
$$\therefore \quad u_n = 2 \times (-3)^{n-1}$$

> $2 \times (-3)^{n-1}$ cannot be simplified unless we know the value of n.

EXERCISE 26C

1   List the first *five* terms of the geometric sequence defined by:

   a $u_n = 3 \times 2^n$ **b** $u_n = 3 \times 2^{n-1}$ **c** $u_n = 3 \times 2^{n+1}$

   d $u_n = 5 \times 2^{n-1}$ **e** $u_n = 24 \times (\frac{1}{2})^{n-1}$ **f** $u_n = 36 \times (\frac{1}{3})^{n-1}$

   g $u_n = 24 \times (-2)^n$ **h** $u_n = 8(-1)^{n-1}$ **i** $u_n = 8(-\frac{1}{2})^n$

2   Find the next *two* terms and a formula for the nth term of:

   a   1, −1, 1, −1, 1, ...   **b**   −1, 1, −1, 1, −1, ...   **c**   2, 4, 8, 16, 32, ...

   d   2, −4, 8, −16, 32, ...   **e**   6, 18, 54, 162, ...   **f**   6, −18, 54, −162, ...

   g   4, 12, 36, 108, ...   **h**   2, −14, 98, −686, ...   **i**   3, −6, 12, −24, ...

   j   −16, 8, −4, 2, ...

D   THE DIFFERENCE METHOD FOR SEQUENCES
[2.12]

We have seen that a **linear** sequence is one in which each term differs from the previous term by the same constant. The general term will have the form $u_n = an + b$ where a and b are constants. You should notice how this form compares with that of a linear function.

In the same way, a **quadratic** sequence has general term $u_n = ax^2 + bx + c$ and a **cubic** sequence has general term $u_n = ax^3 + bx^2 + cx + d$.

In order to find the formula for one of these sequences, we use a technique called the **difference method**.

Discovery   *The difference method*

Part 1: Linear sequences

Consider the linear sequence $u_n = 3n + 2$ where $u_1 = 5$, $u_2 = 8$, $u_3 = 11$, $u_4 = 14$, and $u_5 = 17$.

We construct a **difference table** to display the sequence, and include a row for the **first difference** $\Delta 1$. This is the difference between successive terms of the sequence.

n	1	2	3	4	5
u_n	5	8	11	14	17
$\Delta 1$		3	3	3	3

What to do:

1 Construct a difference table for the sequence defined by:

    a $u_n = 4n + 3$ **b** $u_n = -3n + 7$

2 Copy and complete:

    For the linear sequence $u_n = an + b$, the values of $\Delta 1$ are ......

3 Copy and complete the difference table for the general linear sequence $u_n = an + b$:

n	1	2	3	4	5
u_n	$a+b$	$2a+b$	$3a+b$		
$\Delta 1$		a			

4 The circled elements of the difference table in **3** can be used to find the formula for u_n.

    For example, in the original example above, $a = 3$ and $a + b = 5$.

    $\therefore$ $a = 3$, $b = 2$, and hence $u_n = 3n + 2$.

    Use the difference method to find $u_n = an + b$ for the sequence:

    a 4, 11, 18, 25, 32, 39, ...   **b** 41, 37, 33, 29, 25, 21, ...

Part 2: Quadratic and cubic sequences

Now consider the quadratic sequence defined by $u_n = 2n^2 - n + 3$.

Its terms are: $u_1 = 4$ $u_2 = 9$ $u_3 = 18$ $u_4 = 31$ $u_5 = 48$ $u_6 = 69$

We again construct a difference table, and this time we include another row for the **second difference** $\Delta 2$. This is the difference between the terms of the first difference.

n	1	2	3	4	5	6
u_n	4	9	18	31	48	69
$\Delta 1$		5	9	13	17	21
$\Delta 2$			4	4	4	4

What to do:

5   Construct a difference table for the quadratic sequence defined by:

     a $u_n = n^2 + 2n + 3$ **b** $u_n = -n^2 + 5n + 4$

6   Copy and complete:

     For the quadratic sequence $u_n = an^2 + bn + c$, the values of $\Delta 2$ are ......

7   Copy and complete the difference table for the general quadratic sequence $u_n = an^2 + bn + c$:

n	1	2	3	4	5	6
u_n	$(a + b + c)$	$4a + 2b + c$	$9a + 3b + c$			
$\Delta 1$		$(3a + b)$	$5a + b$			
$\Delta 2$			$(2a)$			

8   Describe how the circled elements in **7** can be used to find the formula for u_n.

9   Use the difference method to find $u_n = an^2 + bn + c$ for the sequence:

     a   2, 0, 0, 2, 6, ...   **b** -5, 4, 19, 40, 67, ...

10   Consider any two cubic sequences of the form $u_n = an^3 + bn^2 + cn + d$.

     For each sequence, construct a difference table for $n = 1, 2, 3, 4, 5, 6$.

     Include rows for $\Delta 1$, $\Delta 2$ and $\Delta 3$. Record your observations.

11   Describe how the table in **10** can be used to find the formula for $u_n = an^3 + bn^2 + cn + d$.

You should have discovered that:

- For the **linear sequence** $u_n = an + b$, the **first differences** are constant and equal to a.
  The general difference table is:

n	1	2	3	4	5
u_n	$(a + b)$	$2a + b$	$3a + b$	$4a + b$	$5a + b$
$\Delta 1$		a	a	a	a

  We use $\Delta 1$ to find a, and the first term of u_n to find b.

- For the **quadratic sequence** $u_n = an^2 + bn + c$, the **second differences** are constant and equal to $2a$.
  The general difference table is:

n	1	2	3	4
u_n	$(a + b + c)$	$4a + 2b + c$	$9a + 3b + c$	$16a + 4b + c$
$\Delta 1$		$(3a + b)$	$5a + b$	$7a + b$
$\Delta 2$			$(2a)$	$2a$

  We use the circled terms to find a, b and c.

- For the **cubic sequence** $u_n = an^3 + bn^2 + cn + d$, the **third differences** are constant and equal to $6a$. The general difference table is:

n	1	2	3	4
u_n	$a + b + c + d$	$8a + 4b + 2c + d$	$27a + 9b + 3c + d$	$64a + 16b + 4c + d$
$\Delta 1$	$7a + 3b + c$	$19a + 5b + c$	$37a + 7b + c$	
$\Delta 2$	$12a + 2b$	$18a + 2b$		
$\Delta 3$	$6a$			

We use the circled terms to find a, b, c and d.

> You should not memorise these tables but learn to quickly generate them when you need.

Example 7   ◀) *Self Tutor*

Find a formula for the general term u_n of: 6, 13, 20, 27, 34, 41, ...

The difference table is:

n	1	2	3	4	5	6
u_n	6	13	20	27	34	41
$\Delta 1$	7	7	7	7	7	

The $\Delta 1$ values are constant, so the sequence is linear.

$$u_n = an + b \quad \text{with} \quad a = 7 \quad \text{and} \quad a + b = 6$$
$$\therefore \quad 7 + b = 6$$
$$\therefore \quad b = -1$$

$\therefore$ the general term is $u_n = 7n - 1$

Example 8   ◀) *Self Tutor*

Examine the dot sequence:

a How many dots are in the next *two* figures?

b Find a formula for u_n, the number of dots in the nth figure.

a

15 dots

21 dots
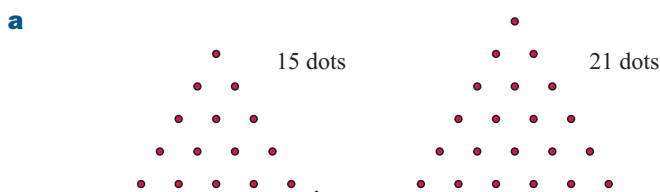
b $u_1 = 1$, $u_2 = 3$, $u_3 = 6$, $u_4 = 10$, $u_5 = 15$, $u_6 = 21$

The difference table is:

n	1	2	3	4	5	6
u_n	1	3	6	10	15	21
$\Delta 1$		2	3	4	5	6
$\Delta 2$			1	1	1	1

The $\Delta 2$ values are constant, so the sequence is quadratic with general term $u_n = an^2 + bn + c$.

$$2a = 1, \quad \text{so} \quad a = \tfrac{1}{2}$$

$$3a + b = 2, \quad \text{so} \quad \tfrac{3}{2} + b = 2 \quad \text{and} \quad \therefore \quad b = \tfrac{1}{2}$$

$$a + b + c = 1, \quad \text{so} \quad \tfrac{1}{2} + \tfrac{1}{2} + c = 1 \quad \text{and} \quad \therefore \quad c = 0$$

$\therefore$ the general term is $u_n = \tfrac{1}{2}n^2 + \tfrac{1}{2}n$

Example 9   ◀⟩ *Self Tutor*

Find a formula for the general term u_n of the sequence: -6, -4, 10, 42, 98, 184, ...

The difference table is:

n	1	2	3	4	5	6
u_n	-6	-4	10	42	98	184
$\Delta 1$		2	14	32	56	86
$\Delta 2$			12	18	24	30
$\Delta 3$			⑥	6	6	

The $\Delta 3$ values are constant, so the sequence is cubic with general term $u_n = an^3 + bn^2 + cn + d$.

$$6a = 6, \qquad\qquad \text{so} \quad a = 1$$

$$12a + 2b = 12, \qquad \text{so} \quad 12 + 2b = 12 \quad \text{and} \quad \therefore \quad b = 0$$

$$7a + 3b + c = 2, \qquad \text{so} \quad 7 + c = 2 \quad \text{and} \quad \therefore \quad c = -5$$

$$a + b + c + d = -6, \quad \text{so} \quad 1 - 5 + d = -6 \quad \text{and} \quad \therefore \quad d = -2$$

$\therefore$ the general term is $u_n = n^3 - 5n - 2$

For quadratic and cubic sequences, an alternative to writing the general difference tables down on the spot is to only use the difference table to identify the form of the sequence.

We can then use the **quadratic** or **cubic regression** functions on our graphics calculator to find the coefficients. Instructions for doing this are found on page **27**.

For **Examples 8** and **9** above, the results are:

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EXERCISE 26D

1 Use the method of differences to find the general term u_n of:

   a 1, 5, 9, 13, 17, 21, ...
             b 17, 14, 11, 8, 5, 2, ...

   c 2, 6, 12, 20, 30, 42, ...
             d 0, 6, 14, 24, 36, 50, ...

   e 6, 13, 32, 69, 130, 221, 348, ...
      f 2, 7, 18, 38, 70, 117, 182, ...

2 Consider the sequence: 2, 12, 30, 56, 90, 132, ...

   a Use the difference method to find the general term u_n.

   b Suggest an alternative formula for u_n by considering $u_1 = 1 \times 2$, $u_2 = \text{......} \times \text{......}$,
      $u_3 = \text{......} \times \text{......}$, and so on.

3 Consider the dot pattern:
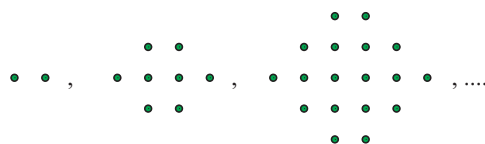
   a Find u_n for $n = 1, 2, 3, 4, 5, 6$ and 7.

   b Find a formula for the general term u_n.

   c How many dots are needed to make up the 30th figure in the pattern?

4 These diagrams represent the hand-shakes between two people (A and B), three people (A, B and C), four people (A, B, C and D), and so on.

   A —— B
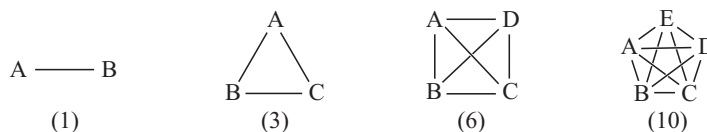
    (1)   (3)   (6)   (10)

   a Draw diagrams showing hand-shakes for 6, 7 and 8 people.

   b When you are sure that you have counted them correctly, find the formula for the general term u_n of the sequence 1, 3, 6, 10, ...

   c 179 delegates attend a conference. If every person shakes hands with every other person, how many hand-shakes take place?

5 Consider the sequence u_n where: $u_1 = 1 \times 3$

                               $u_2 = 1 \times 3 + 2 \times 5$

                               $u_3 = 1 \times 3 + 2 \times 5 + 3 \times 7$

                               $u_4 = 1 \times 3 + 2 \times 5 + 3 \times 7 + 4 \times 9$ and so on.

   a Find the values of $u_1, u_2, u_3, u_4, u_5, u_6$ and u_7.

   b Use the difference method to find a formula for u_n.

   c Hence, find the value of u_{50}.

6 Consider the **Opening Problem** on page **533**.

   Notice that: $u_1 = 1^2$

                  $u_2 = 1^2 + 2^2$

                  $u_3 = 1^2 + 2^2 + 3^2$

                  $u_4 = 1^2 + 2^2 + 3^2 + 4^2$

   a Find u_n for $n = 1, 2, 3, 4, 5, 6$ and 7.

   b Use the difference method to find a formula for u_n.

   c How many squares are contained in a 100 by 100 square?

7 Consider the pattern:
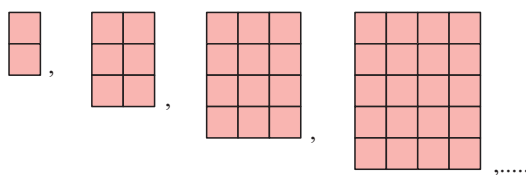
    a Suppose u_n is the number of squares contained in the nth figure, so $u_1 = 2$ and $u_2 = 6 + 2 = 8$.
       Find the values of u_3, u_4, u_5 and u_6.

    b Find a formula for u_n.

    c How many squares are contained in the 15th figure?

Review set 26A

1 Write down a rule for the sequence and find its next two terms:

    a 6, 10, 14, 18, 22, ...
    b 810, 270, 90, 30, ...

2 Draw the next *two* matchstick figures in the pattern and write down the number of matchsticks used as a number sequence:

    a

    b
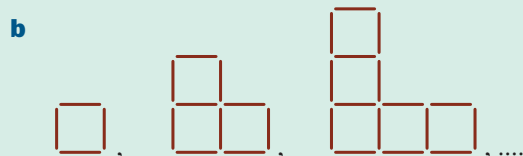
3 Find the first four terms of the sequence with nth term:

    a $u_n = 6n - 1$
    b $u_n = n^2 + 5n - 2$

4   **a** Find a formula for the general term u_n of the sequence: 4, 8, 12, 16, ...

    b *Hence* find u_n for:

       i 1, 5, 9, 13, ...
       ii $\frac{1}{3}$, $\frac{1}{7}$, $\frac{1}{11}$, $\frac{1}{15}$, ...

    c Find the 20th term for each of the sequences in **b**.

5 List the first four terms of the geometric sequence defined by:

    a $u_n = 27 \times \left(\frac{2}{3}\right)^n$
    b $u_n = 5 \times (-2)^{n-1}$

6 Find u_n for the sequence:

    a 3, 12, 48, 192, ...
    b 88, -44, 22, -11, ...

7 Use the method of differences to find the general term u_n of:

    a 5, 12, 19, 26, 33, ...
    b -1, 6, 15, 26, 39, 54, ...

8 Consider the figures:

    Suppose u_n is the number of triangles in the nth figure, so $u_1 = 1$ and $u_2 = 5$ (4 small triangles and 1 large triangle).

    a Find u_n for $n = 3, 4, 5$.

    b Use the method of differences to find a formula for u_n.

    c How many triangles are in the 50th figure?

9 Consider the sequence   6, 24, 60, 120, 210, 336, ...

   a Use the difference method to find the general term u_n.

   b Suggest an alternative formula for u_n by considering $u_1 = 1 \times 2 \times 3$, $u_2 = ... \times ...... \times ......$, $u_3 = ... \times ...... \times ......$, and so on.

10 Sarah has baked a cake, and wishes to divide it into pieces using straight line cuts.

1 cut   2 cuts   3 cuts
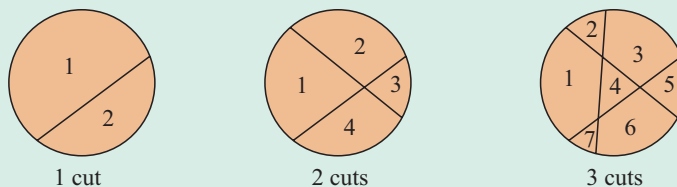
Suppose u_n is the maximum number of pieces which can be made from n cuts, so $u_1 = 2$, $u_2 = 4$, $u_3 = 7$.

   a Find u_n for $n = 4$ and 5.

   b Use the method of differences to find a formula for u_n.

   c If Sarah makes 10 cuts, what is the maximum number of pieces she can make?

Review set 26B

1 Write down a rule for the sequence and find its next two terms:

   a 17, 12, 7, 2, ...   **b** -2, 4, -8, 16, ...

2 Draw the next *two* figures and write down the number of dots used as a number sequence:

   a   **b**

3 Find the first four terms of the sequence with nth term:

   a $u_n = -4n + 5$ **b** $u_n = (n + 2)(n - 1)$

4 **a** Find a formula for the general term u_n of the sequence: 6, 12, 18, 24, ...

   b *Hence* find u_n for:

      i 10, 16, 22, 28, ...   **ii** $\frac{5}{7}$, $\frac{11}{13}$, $\frac{17}{19}$, $\frac{23}{25}$, ...

   c Find the 15th term for each of the sequences in **b**.

5 Find u_n for the sequence:

   a 2, 5, 10, 17, 26, ...   **b** $\frac{1}{1}$, $\frac{1}{8}$, $\frac{1}{27}$, $\frac{1}{64}$, ...

6 List the first *four* terms of the general sequence defined by:

   a $u_n = 5 \times 3^{n-1}$ **b** $u_n = 48 \times \left(-\frac{1}{4}\right)^n$

7 Find u_n for the sequence:

   a 4, -12, 36, -108, ...   **b** 224, 56, 14, $3\frac{1}{2}$, ...

8 Use the method of differences to find the general term u_n of:

     a   43, 34, 25, 16, 7, ...
                                    b   4, 12, 22, 34, 48, ...

9 Consider the sequence of figures:
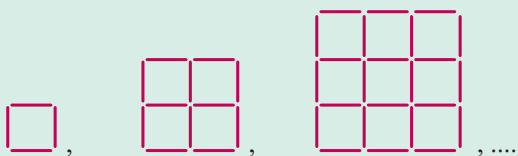
Suppose u_n is the number of matchsticks required to make the nth figure, so $u_1 = 4$ and $u_2 = 12$.

     a   Find u_n for $n = 3, 4, 5, 6$.

     b   Use the method of differences to find a formula for u_n.

     c   How many matches are in the 10th figure?

10 Consider the sequence of figures:
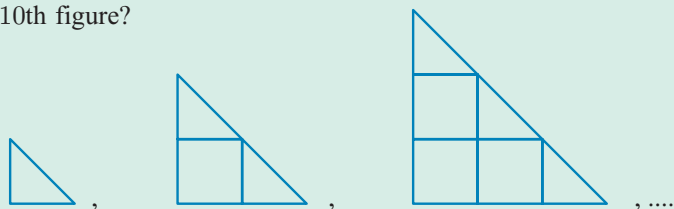
Suppose u_n is the number of rectangles present in the nth figure, so, $u_1 = 0$, $u_2 = 1$, and $u_3 = 5$.

     a   Show that $u_4 = 15$, $u_5 = 35$, and $u_6 = 70$.

     b   Show that neither a linear, quadratic or cubic model fit the data.

     c   Use technology to show that a quartic model fits the data. Give the quartic in the form
        $an^4 + bn^3 + cn^2 + dn + e$, where a, b, c, d and $e \in \mathbb{Q}$, written in fractional form.

Circle geometry

27

Contents:

Opening problem

The towns of Arden, Barne, Cowley and Dirnham are all the same distance from the town of Oakden. Both Barne and Dirnham are equidistant from Arden and Cowley.

Can you prove that Barne, Oakden and Dirnham are collinear?
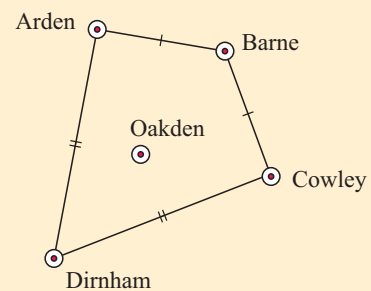
A   CIRCLE THEOREMS   [4.7]

Before we can talk about the properties and theorems of circles, we need to learn the appropriate language for describing them.

- A **circle** is the set of all points which are equidistant from a fixed point called the **centre**.

- The **circumference** is the distance around the entire circle boundary.
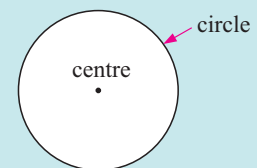
- An **arc** of a circle is any continuous part of the circle.

- A **chord** of a circle is a line segment joining any two points on the circle.

- A **semi-circle** is a half of a circle.

- A **diameter** of a circle is any chord passing through its centre.

- A **radius** of a circle is any line segment joining its centre to any point on the circle.

- A **tangent** to a circle is any line which touches the circle in exactly one point.
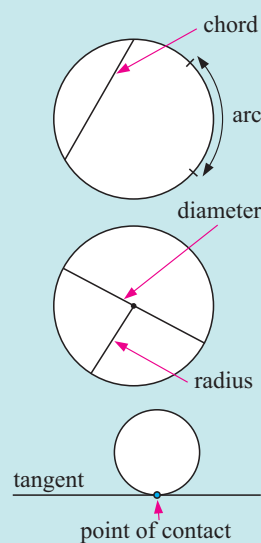
Discovery 1   *Properties of circles*

This discovery is best attempted using the **computer package** on the CD.

However, you can also use a compass, ruler and protractor.

Part 1:   The angle in a semi-circle

What to do:

1 Draw a circle and construct a diameter. Label it as shown.

2 Mark any point P not at A or B on the circle.
Draw AP and PB.

3 Measure angle APB.

4 Repeat for different positions of P and for different circles.
What do you notice?

5 Copy and complete: *The angle in a semi-circle is ......*
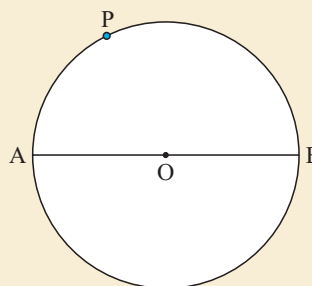
GEOMETRY PACKAGE

Part 2:   Chords of a circle theorem

What to do:

1 Draw a circle with centre C. Construct any chord AB.

2 Construct the perpendicular from C to AB which cuts the chord at M.

3 Measure the lengths of AM and BM. What do you notice?

4 Repeat the procedure above with another circle and chord.

5 Copy and complete:
The perpendicular from the centre of a circle to a chord ......
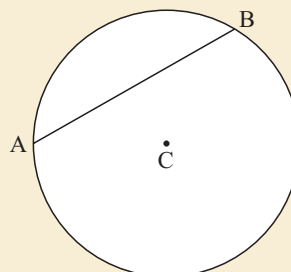
GEOMETRY PACKAGE

Part 3:   Radius-tangent theorem

What to do:

1   Use a compass to draw a circle with centre O, and mark on it a point A.

2   Draw at A, as accurately as possible, a tangent TA.

3   Draw the radius OA.

4   Measure the angle OAT with a protractor.

5   Repeat the procedure above with another circle and tangent.

6   Copy and complete:
The tangent to a circle is ...... to the radius at the point ......
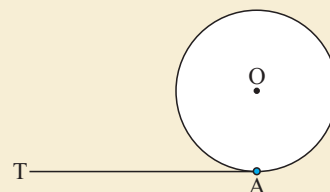
GEOMETRY PACKAGE

Part 4:   Tangents from an external point

What to do:

1   Use your compass to draw a circle, centre O.

2   From an external point P draw the two tangents to the circle to meet it at A and B.

3   Measure AP and BP.

4   Repeat with another circle of different size.

5   Copy and complete:
Tangents from an external point to a circle are ......
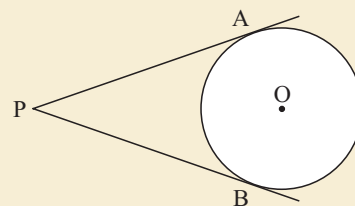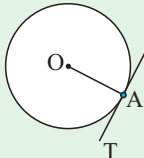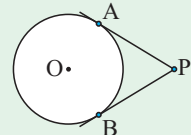
GEOMETRY PACKAGE

You should have discovered the following **circle theorems**:

Name of theorem	*Statement*	*Diagram*
Angle in a semi-circle	The angle in a semi-circle is a right angle.	$A\widehat{B}C = 90^o$
Chords of a circle	The perpendicular from the centre of a circle to a chord bisects the chord.	$AM = BM$

Name of theorem	*Statement*	*Diagram*
Radius-tangent	The tangent to a circle is perpendicular to the radius at the point of contact.	O$\widehat{A}$T $= 90^o$
Tangents from an external point	Tangents from an external point are equal in length.	AP $=$ BP

Two useful **converses** are:

- If line segment AB subtends a right angle at C then the circle through A, B and C has diameter AB.
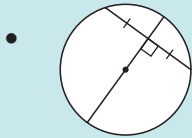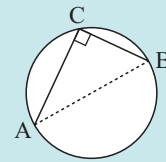
- The perpendicular bisector of a chord of a circle passes through its centre.

Example 1   ◀) **Self Tutor**

Find x, giving brief reasons for your answer.
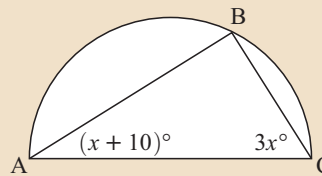
$$\begin{aligned}
&\text{A}\widehat{\text{B}}\text{C measures } 90^o && \{\text{angle in a semi-circle}\} \\
\therefore\quad &(x+10)+3x+90 = 180 && \{\text{angles in a triangle}\} \\
\therefore\quad &4x+100 = 180 \\
\therefore\quad &4x = 80 \\
\therefore\quad &x = 20
\end{aligned}$$

EXERCISE 27A.1

1 Find the value of any unknowns, giving brief reasons for your answers:

a

b

c

d

$x°$

O

$30°$

Y   X

e

$50°$

O

$a°$
$b°$

X   Y

f

$40°$
$a°$

O $30°$

X

$b°$

Y

g

X

$a°$
$70°$
Y

O•

$b°$

Z
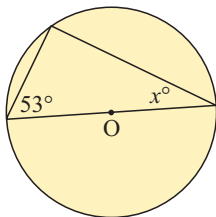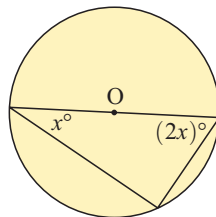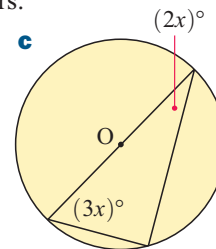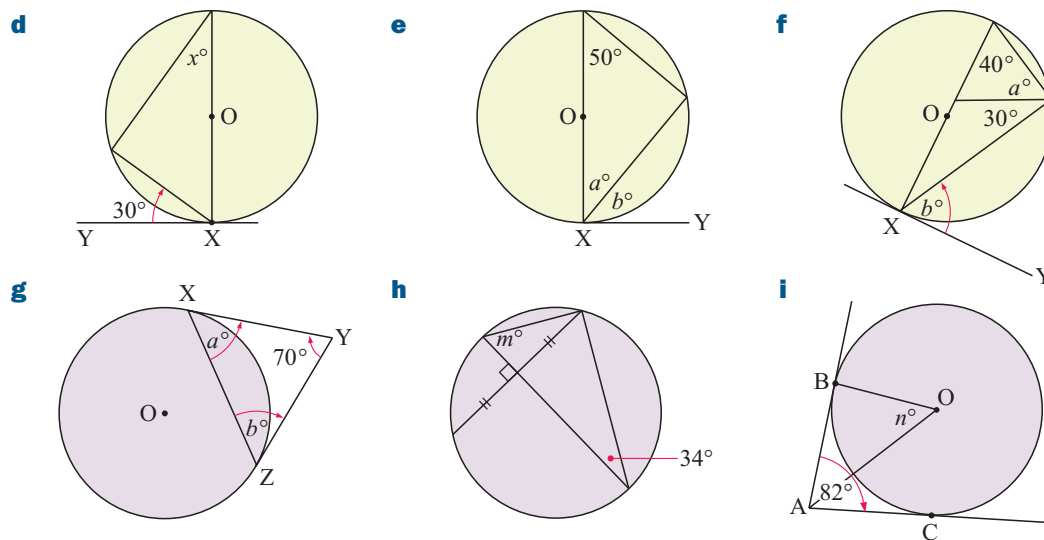
XY and YZ are tangents
from point Y

h

$m°$

$34°$

i

B   O

$n°$

A $82°$

C

2

A   P   B

S   Q

D

R   C
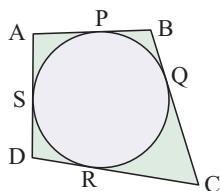
A circle is drawn and four tangents to it are constructed as
shown.
Deduce that   AB + CD = BC + AD.

3 Find the radius of the circle which touches the three sides
of the triangle as shown.

3 cm

O

4 cm

4

A   B

C
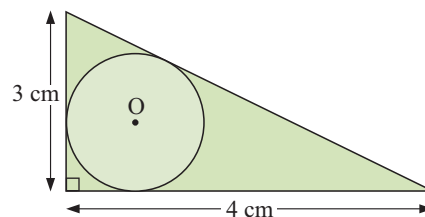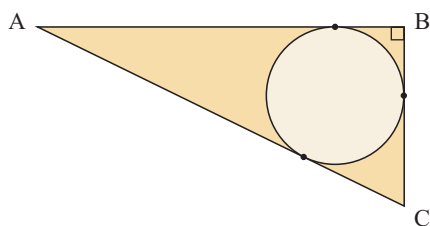
A circle is inscribed in a right angled triangle.
The radius of the circle is 3 cm, and BC has
length 8 cm.

Find the perimeter of the triangle ABC.

5 Consider this figure:

 a What can be deduced about:

   i A$\widehat{P}$O   **ii** B$\widehat{P}$O?

 b Using ΔAPB, explain why $2a + 2b = 180$ and
hence that $a + b = 90$.

 c What theorem have you proven?

P

$a°$ $b°$

A   O   B
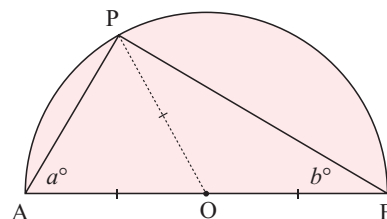
6 In this question we prove the *chords of a circle* theorem.

 a For the given figure join OA and OB and classify ΔOAB.

 b Apply the isosceles triangle theorem to triangle OAB. What
geometrical facts result?

A

O• X

B
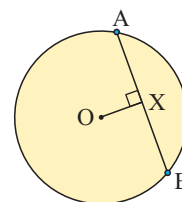
7 In this question we prove the *tangents from an external point* theorem.

  a Join OP, OA and OB.

  b Assuming the *tangent-radius* theorem, prove that Δs POA and POB are congruent.

  c What are the consequences of the congruence in **b**?
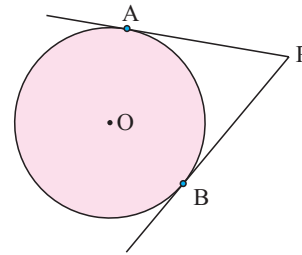
THEOREMS INVOLVING ARCS

Any continuous part of a circle is called an **arc**.
If the arc is less than half the circle it is called a **minor arc**.
If it is greater than half the circle it is called a **major arc**.

a minor arc BC   a major arc BC
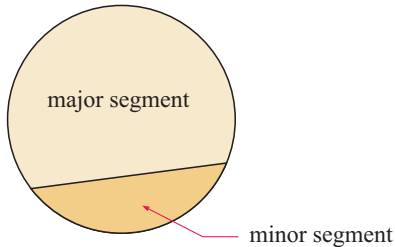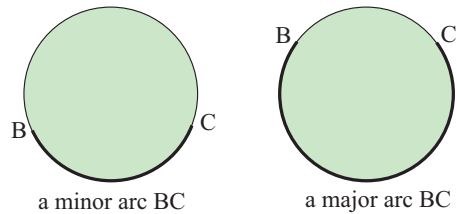
A chord divides the interior of a circle into two regions called **segments**. The larger region is called a **major segment** and the smaller region is called a **minor segment**.

major segment

minor segment

In the diagram opposite:

- the minor arc BC **subtends** the angle BAC, where A is on the circle
- the minor arc BC also subtends angle BOC at the centre of the circle.
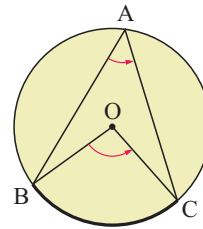
Discovery 2   *Theorems involving arcs*

The use of the **geometry package** on the CD is recommended, but the discovery can also be done using a ruler, compass and protractor.

Part 1:   Angle at the centre theorem

What to do:

 1 Use a compass to draw a large circle with centre O. Mark on it points A, B and P.

 2 Join AO, BO, AP and BP with a ruler.
    Measure angles AOB and APB.

 3 What do you notice about the measured angles?

 4 Repeat the above steps with another circle.

 5 Copy and complete:
    "The angle at the centre of a circle is ...... the angle on the circle subtended by the same arc."
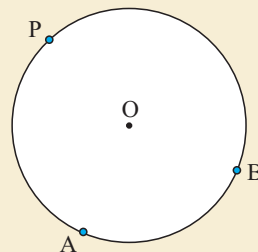
GEOMETRY PACKAGE
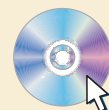
Part 2: Angles subtended by the same arc theorem

What to do:

1 Use a compass to draw a circle with centre O. Mark on it points A, B, C and D.

2 Draw angles ACB and ADB with a ruler.

3 Measure angles ACB and ADB. What do you notice?

4 Repeat the above steps with another circle.

5 Copy and complete:
"*Angles subtended by an arc on the circle are ...... in size.*"
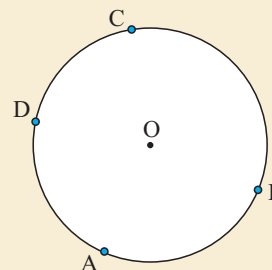
GEOMETRY
PACKAGE

You should have discovered the following theorems:

Name of theorem	Statement	Diagram	
Angle at the centre	The angle at the centre of a circle is twice the angle on the circle subtended by the same arc.		$A\widehat{O}B = 2 \times A\widehat{C}B$
Angles subtended by the same arc	Angles subtended by an arc on the circle are equal in size.		$A\widehat{D}B = A\widehat{C}B$

Note: • The following diagrams show other cases of **the angle at the centre theorem**. These cases can be easily shown using the **geometry package**.
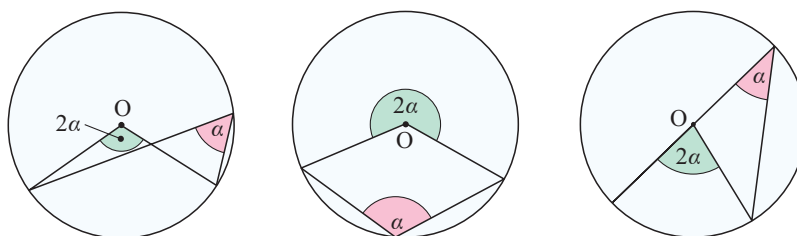
GEOMETRY
PACKAGE

• The **angle in a semi-circle theorem** is a special case of the angle at the centre theorem.
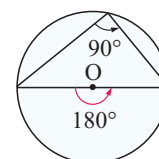
Example 2

◀)) **Self Tutor**

Solve for x:

a

b
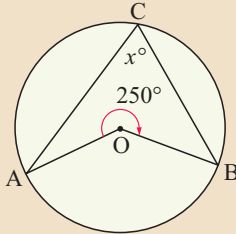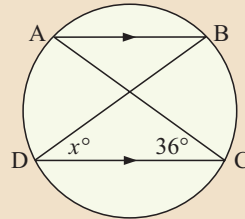
a Obtuse $\widehat{AOB} = 360^o - 250^o$
                        {angles at a point}

$\therefore \quad \widehat{AOB} = 110^o$

$\therefore \quad 2x = 110$ {angle at the centre}

$\therefore \quad x = 55$

b $\widehat{BDC} = \widehat{ABD}$ {alternate angles}

      $\widehat{ABD} = \widehat{ACD}$ {angles on same arc}

$\therefore \quad \widehat{BDC} = \widehat{ACD}$

$\therefore \quad x = 36$

EXERCISE 27A.2

1 Find, giving reasons, the value of x in each of the following:

a
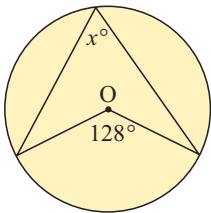
b
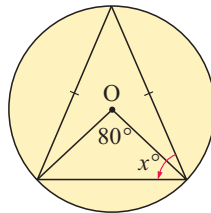
c
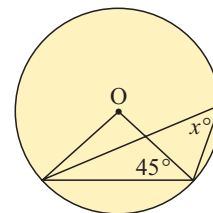
d
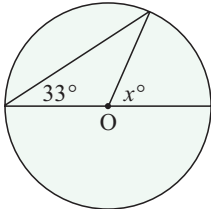
e
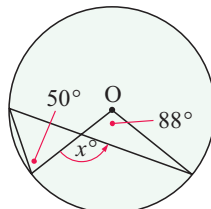
f
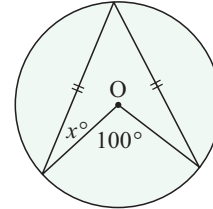
2 Find, giving reasons, the values of the unknowns in the following:

a

b

c

d

e

f

g

h

i
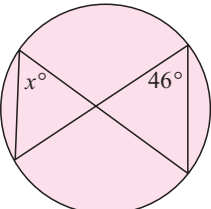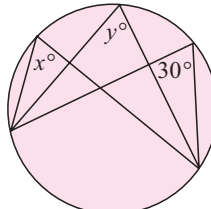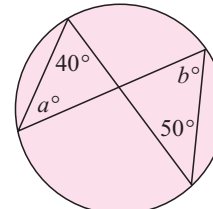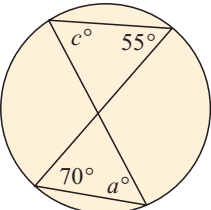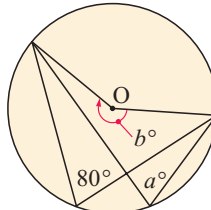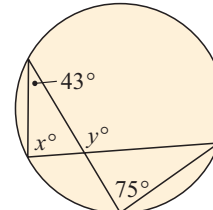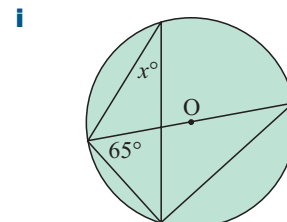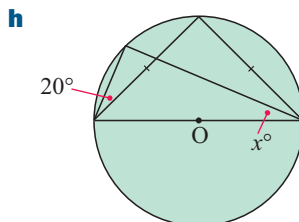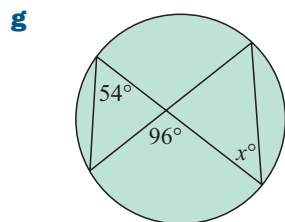
3 C is the point of contact of tangent CT.
Find x, giving reasons for your answer.
Hint: Draw diameter CD and join DB.
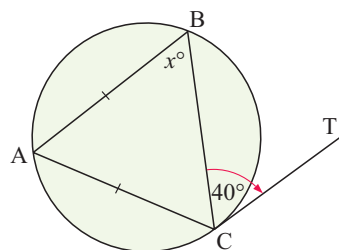
4 In this question we prove the *angle at the centre* theorem.

  a Explain why Δs OAP and OBP are isosceles.

  b Find the measure of the following angles in terms of a and b:

  i $A\hat{P}O$ **ii** $B\hat{P}O$ **iii** $A\hat{O}X$

  iv $B\hat{O}X$ **v** $A\hat{P}B$ **vi** $A\hat{O}B$

  c What can be deduced from **b v** and **b vi**?
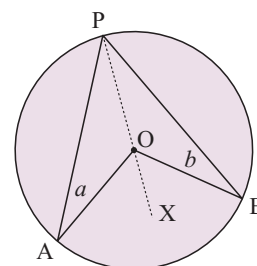
5 In this question we prove the *angles in the same segment* theorem.

  a Using the results of question **4**, find the size of $A\hat{O}B$ in terms of α.

  b Find the size of $A\hat{C}B$ in terms of α.

  c State the relationship between $A\hat{D}B$ and $A\hat{C}B$.

6
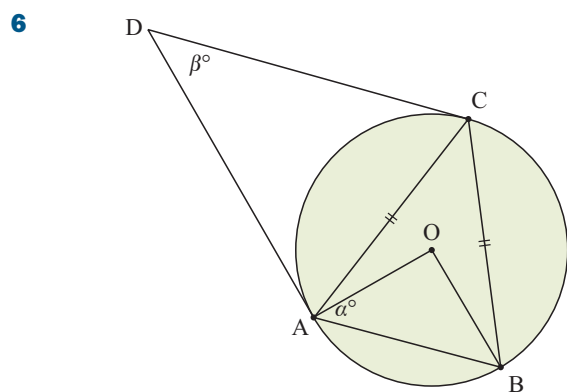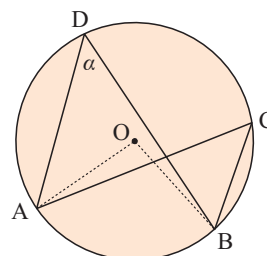
Challenge:
In the figure alongside, AC = BC, and the tangents at A and C meet at D.
Show that $\alpha + \beta = 90$.

| B | *CYCLIC QUADRILATERALS* | **[4.7]** |

A circle can always be drawn through any three points that are not collinear.

To find the circle's centre we draw the perpendicular bisectors of the lines joining two pairs of points.

Using the chords of a circle theorem, the centre is at the intersection of these two lines.

centre
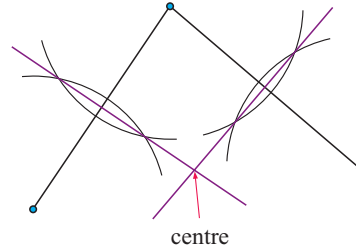
However, a circle may or may not be drawn through any four points in a plane.

For example, consider the sets of points opposite:
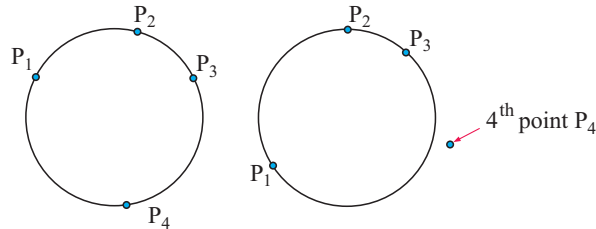
> If a circle can be drawn through four points we say that the points are **concyclic**.
>
> If any four points on a circle are joined to form a convex quadrilateral then the quadrilateral is said to be a **cyclic quadrilateral**.
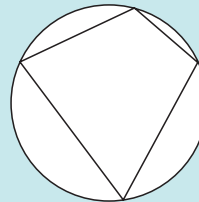
CYCLIC QUADRILATERAL THEOREMS

Name of theorem	*Statement*	*Diagram*
Opposite angles of a cyclic quadrilateral	The opposite angles of a cyclic quadrilateral are supplementary, or add to 180°.	$\alpha + \beta = 180$ $\theta + \phi = 180$
Exterior angle of a cyclic quadrilateral	The exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.	$\theta_1 = \theta_2$

Proof: Opposite angles of a cyclic quadrilateral

Join OA and OC.

Let $\widehat{ADC} = \alpha^o$ and $\widehat{ABC} = \beta^o$
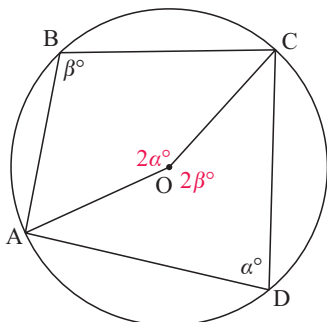
$\therefore$ $\widehat{AOC} = 2\alpha^o$ {angle at the centre}

and reflex $\widehat{AOC} = 2\beta^o$ {angle at the centre}

But $2\alpha + 2\beta = 360$ {angles at a point}

$\therefore$ $\alpha + \beta = 180$

$\therefore$ $\widehat{ABC} + \widehat{ADC} = 180^o$

and since the angles of any quadrilateral add to 360°,

$\widehat{BAD} + \widehat{BCD} = 180^o$

Proof: Exterior angle of a cyclic quadrilateral
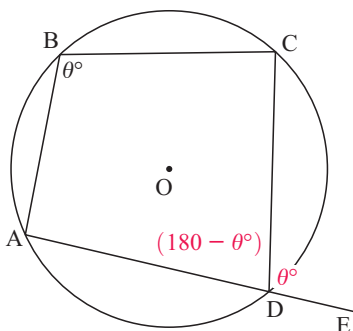
This theorem is an immediate consequence of the opposite angles of a cyclic quadrilateral being supplementary.

Let $\widehat{ABC} = \theta^o$

$\therefore$ $\widehat{ADC} = (180 - \theta)^o$ {opp. angles of cyclic quad.}

$\therefore$ $\widehat{CDE} = \theta^o$ {angles on a line}

$\therefore$ $\widehat{ABC} = \widehat{CDE}$

Example 3   ◀) *Self Tutor*

Solve for x:
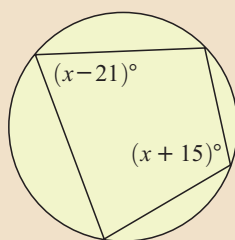
The angles given are opposite angles of a cyclic quadrilateral.

$\therefore$ $(x + 15) + (x - 21) = 180$

$\therefore$ $2x - 6 = 180$

$\therefore$ $2x = 186$

$\therefore$ $x = 93$

Example 4

🔊 *Self Tutor*

Find x and y, giving reasons for your answers:
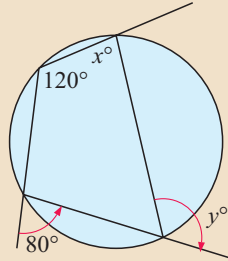
$x = 80$ and $y = 120$ {exterior angles of a cyclic quadrilateral}

EXERCISE 27B.1

1 Find x giving reasons:

a
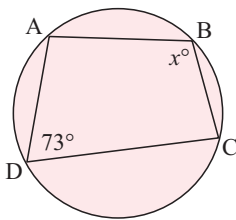
b
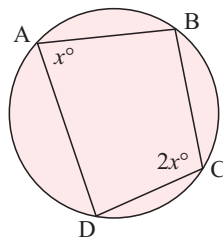
c
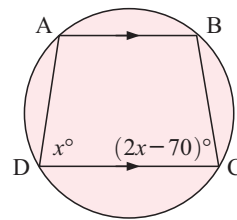
d
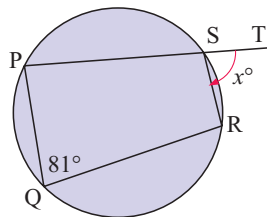
e
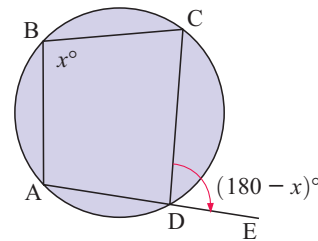
f
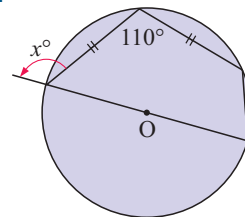
2 Find the values of the unknowns in the following:

a
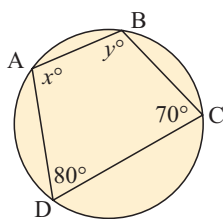
b
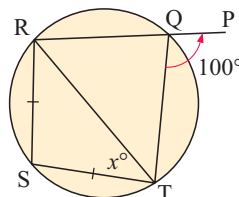
c
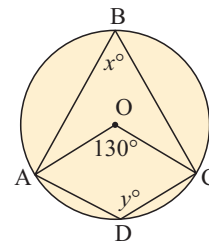
d
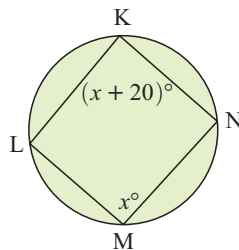
e
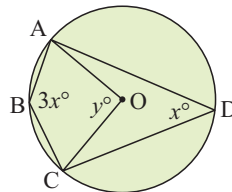
f
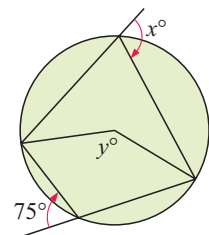
3 ABCD is a cyclic quadrilateral. Sides AB and DC are *produced* or extended to meet at E. Sides DA and CB are produced to F.

If angle AFB is 30° and angle BEC is 20°, find all angles of quadrilateral ABCD.

4 ABCDE is a pentagon inscribed in a circle with centre O. BD is a diameter of the circle. AE is parallel to BD, and is produced to F.

Angle BAC $= 45^o$ and angle CAE $= 70^o$.

  a Find the size of angles BCD and BDC.

  b Show that BC = CD

  c Calculate the size of angle DEF.
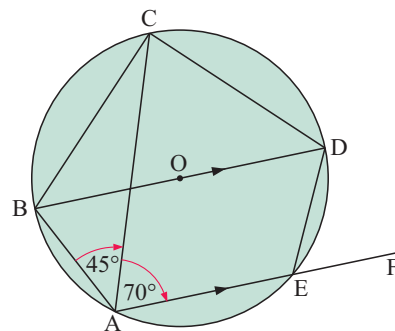
5 An alternative method for establishing the *opposite angles of a cyclic quadrilateral* theorem is to use the figure alongside. Show how this can be done.
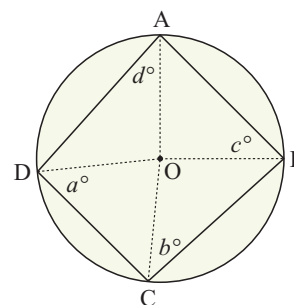
6 Answer the **Opening Problem** on page **547**.

TESTS FOR CYCLIC QUADRILATERALS

A quadrilateral is a **cyclic quadrilateral** if one of the following is true:

- one pair of opposite angles is supplementary

If $\alpha + \beta = 180^o$ then ABCD is a cyclic quadrilateral.

- one side subtends equal angles at the other two vertices
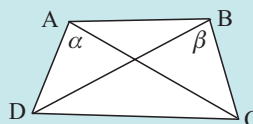
If $\alpha = \beta$ then ABCD is a cyclic quadrilateral.

- an exterior angle is equal to the opposite interior angle
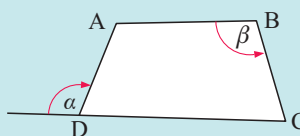
If $\alpha = \beta$ then ABCD is a cyclic quadrilateral.

Example 5   ◆) *Self Tutor*

Triangle ABC is isosceles with AB = AC. X and Y lie on AB and AC respectively such that XY is parallel to BC. Prove that XYCB is a cyclic quadrilateral.
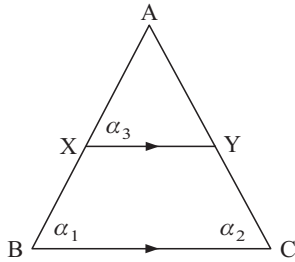
$\triangle$ABC is isosceles with AB = AC.

$$\therefore \quad \alpha_1 = \alpha_2 \quad \text{\{equal base angles\}}$$
Since   XY $\parallel$ BC, $\alpha_1 = \alpha_3$ {equal corresp. angles}
$$\text{so, } \alpha_2 = \alpha_3$$

$\therefore$ XYCB is a cyclic quadrilateral
      {exterior angle = opposite interior angle}

EXERCISE 27B.2

1   Is ABCD a cyclic quadrilateral? Give reasons for your answers.

a
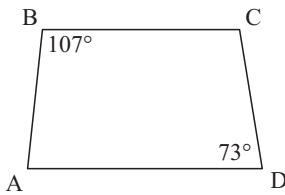
b
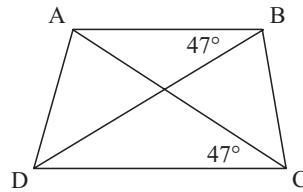
c
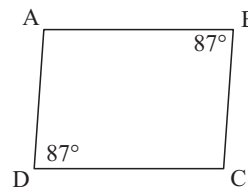
d
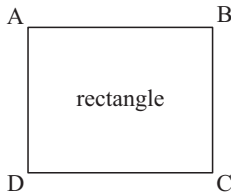
e
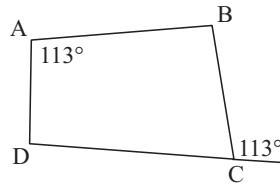
f
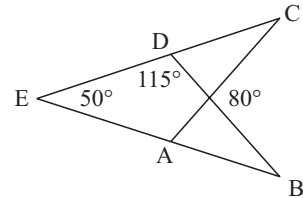
2   ABCD is a trapezium in which AB is parallel to DC and AD = BC.

     Show that ABCD is a cyclic quadrilateral.

     Hint: Draw BE parallel to AD, meeting DC at E.
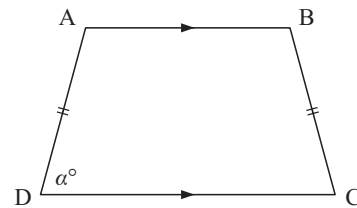
3
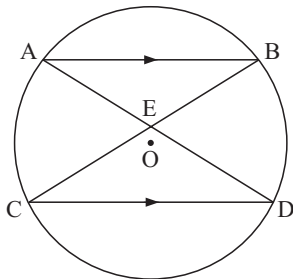
     AB and CD are parallel chords of a circle with centre O. BC and AD meet at E.

     Show that AEOC is a cyclic quadrilateral.

4   A circle has centre O. The tangents to the circle from an external point P meet the circle at points A and B. Show that PAOB is a cyclic quadrilateral.

5   Triangle ABC has its vertices on a circle. P, Q and R are any points on arcs AB, BC and AC respectively.
Prove that $A\widehat{R}C + B\widehat{Q}C + A\widehat{P}B = 360^o$.

6   Two circles intersect at X and Y. A line segment AB is drawn through X to cut one circle at A and the other at B. Another line segment CD is drawn through Y to cut one circle at C and the other at D, with A and C being on the same circle. Show that AC is parallel to BD.
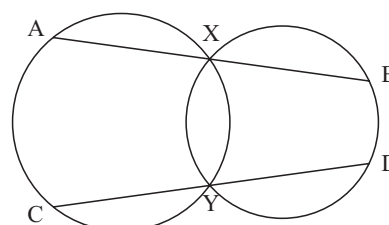
Review set 27A

1   Find the value of a, giving reasons:

a   **b**   **c**

d   **e**   **f**

g   **h**   **i**
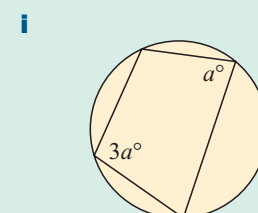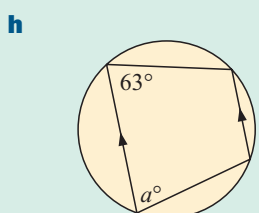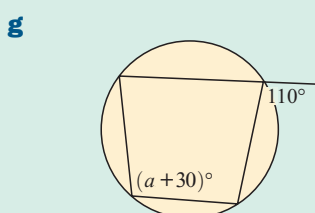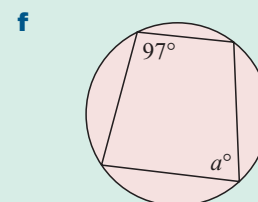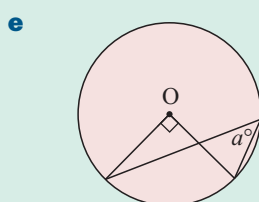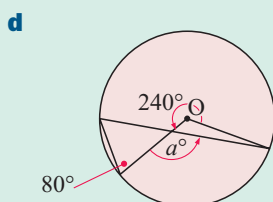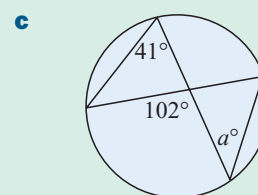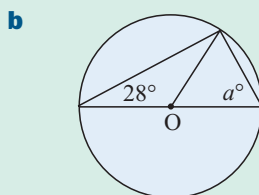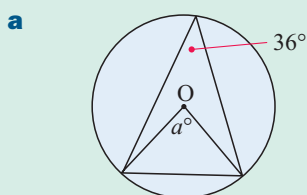
2   Copy and complete:

Triangle OAC is isosceles   {as AO = ...}

∴ $A\widehat{C}O = $ ......   {.......}

Likewise, triangle BOC is ......

∴ $B\widehat{C}O = $ ......   {......}

Thus the angles of triangle ABC measure a^o, b^o and ......

∴ $2a + 2b = $ ......   {......}

∴ $a + b = $ ......

and so $A\widehat{C}B = $ ......

3
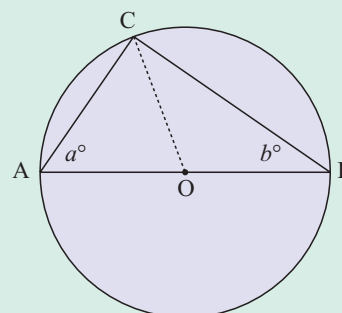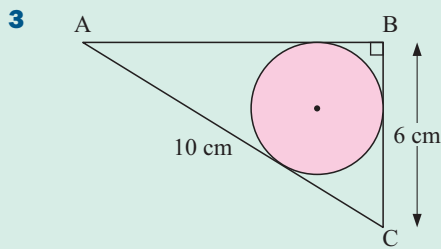
Find:

   a the length of side AB

   b the length of the radius of the circle.

4 AB and CM are common tangents to two circles.
Show that:

   a M is the midpoint of AB

   b $A\widehat{C}B$ is a right angle.
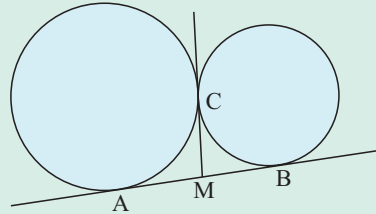
5 AB is the diameter of a circle with centre O.
AC and BD are any two chords.

If $B\widehat{D}O = \alpha$:

   a find $D\widehat{B}O$

   b show that $B\widehat{D}O = A\widehat{C}D$.
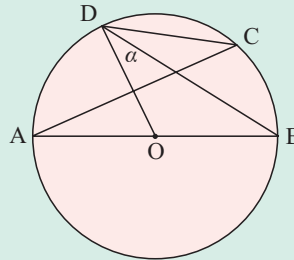
6 AB and AC are any two chords which are *not* diameters of a circle with centre O.
X and Y are the midpoints of AB and AC respectively.
Explain why quadrilateral OXAY is a cyclic quadrilateral.

Review set 27B

1 Find the value of x, giving reasons:

   a

   b

   c

   d

   e

   f
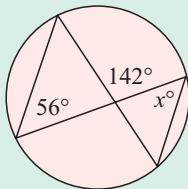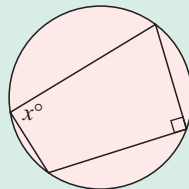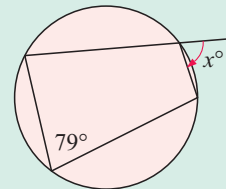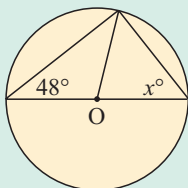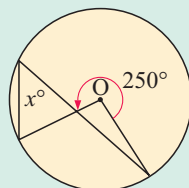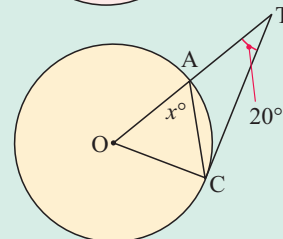
2 Copy and complete:

$$\text{Obtuse } D\widehat{O}B = \text{......} \{\text{......}\}$$

Likewise, reflex $D\widehat{O}B = \text{......} \{\text{.......}\}$

$\therefore \text{......} + \text{......} = 360^o \{\text{angles at a point}\}$

$\therefore \alpha + \beta = \text{......}$

Thus, the opposite angles of a ...... are ......
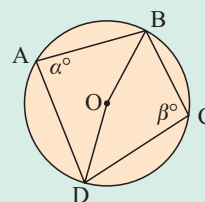
3
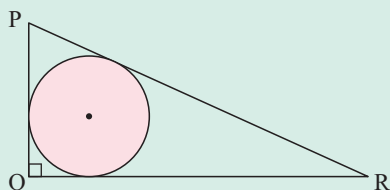
The circle inscribed in triangle PQR has radius of length 3 cm.

PQ has length 7 cm.

Find the perimeter of triangle PQR.

4 In triangle PQR, PQ = PR. A circle is drawn with diameter PQ, and the circle cuts QR at S.

Show that S is the midpoint of QR.
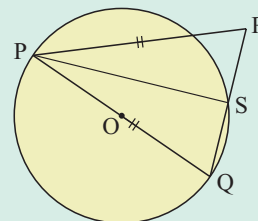
5 In this question we prove the *angle between a tangent and a chord* theorem.

  a We draw diameter AX and join CX.

  Find the size of: **i** $T\widehat{A}X$ **ii** $A\widehat{C}X$

  b Now let $T\widehat{A}C = \alpha$. Find, in terms of α:

  i $C\widehat{A}X$ **ii** $C\widehat{X}A$ **iii** $C\widehat{B}A$

  Give reasons for your answers.
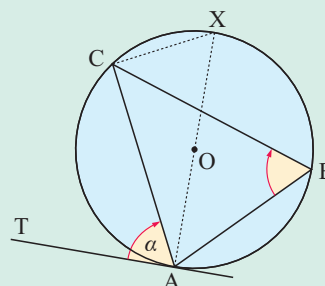
6 Using the result of **5**, find:

  a $B\widehat{C}X$ and $C\widehat{B}X$

  b $\alpha + \beta + \gamma$
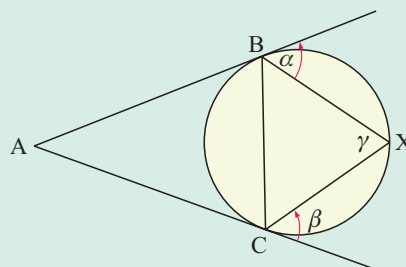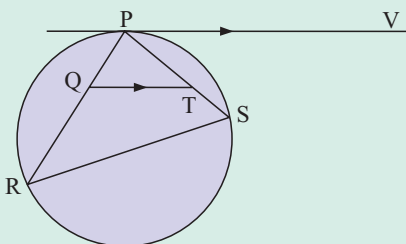
7 PV is a tangent to the circle and QT is parallel to PV. Use the result of **5** to prove that QRST is a cyclic quadrilateral.

Challenge

1
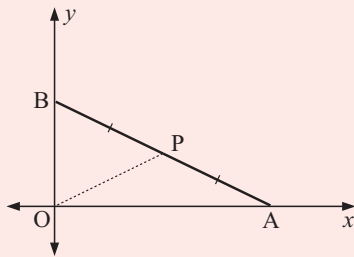
A solid bar AB moves so that A remains on the x-axis and B remains on the y-axis. At P, the midpoint of AB, is a small light.

Prove that as A and B move to all possible positions, the light traces out a path which forms a circle.

(Do not use coordinate geometry methods.)

2 PAB is a wooden set square in which $\widehat{APB}$ is a right angle. The set square is free to move so that A is always on the x-axis and B is always on the y-axis.

Show that the point P always lies on a straight line segment which passes through O.

(Do not use coordinate geometry methods.)
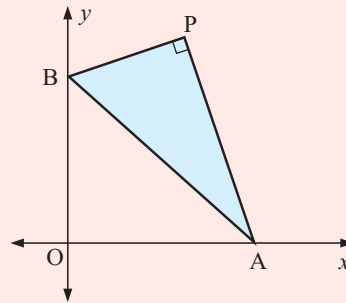
3 P is any point on the circumcircle of $\triangle ABC$ other than at A, B or C. Altitudes PX, PY and PZ are drawn to the sides of $\triangle ABC$ (or the sides produced).

Prove that X, Y and Z are collinear. XYZ is known as **Simson's line**.
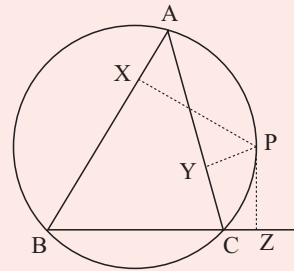
4
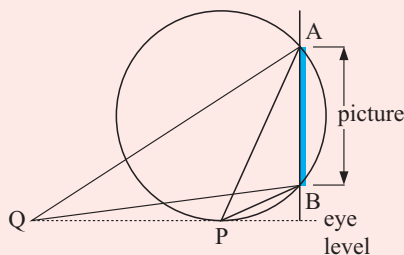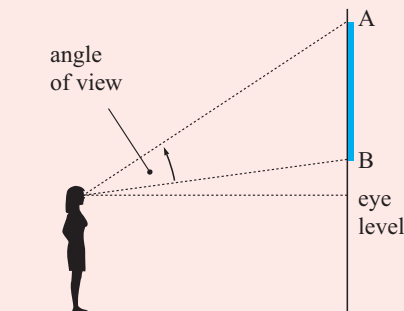
Britney notices that her angle of view of a picture on a wall depends on how far she is standing in front of the wall. When she is close to the wall the angle of view is small. When she moves backwards so that she is a long way from the wall the angle of view is also small. It becomes clear to Britney that there must be a point in the room where the angle of view is greatest. She is wondering whether this position can be found from a deductive geometry argument only. Kelly said that she thought this could be done by drawing an appropriate circle.

She said that the solution is to draw a circle through A and B which touches the 'eye level' line at P, then $\widehat{APB}$ is the largest angle of view. To prove this, choose any other point Q on the eye level line and show that this angle must be less than $\widehat{APB}$. Complete the full argument.

Exponential functions and equations

28

Contents:

Opening problem

Sergio is a land developer. He estimates that each year his business will increase by 30% over the previous year. Currently his assets are valued at 3.8 million pounds. If his prediction is accurate, the formula $F = 3.8 \times (1.3)^n$ can be used to find his future assets $£F$ million after n years.

Things to think about:

a What are his predicted assets in:

   i 2 years   **ii** 10 years?

b What would the graph of F against n look like?

c How long will it take for his assets to reach $£20$ million in value?

A   RATIONAL EXPONENTS   *[1.9, 2.4]*

In **Chapter 6** we saw the following **exponent laws** which are true for all positive bases a and b and all integer indices m and n.

- $a^m \times a^n = a^{m+n}$

  To **multiply** numbers with the same base, keep the base and **add** the indices.

- $\dfrac{a^m}{a^n} = a^{m-n}$

  To **divide** numbers with the same base, keep the base and **subtract** the indices.

- $(a^m)^n = a^{m \times n}$

  When raising a power to a power, keep the base and **multiply** the indices.

- $(ab)^n = a^n b^n$

  The power of a product is the product of the powers.

- $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

  The power of a quotient is the quotient of the powers.

- $a^0 = 1, \quad a \neq 0$

  Any non-zero number raised to the power of zero is 1.

- $a^{-n} = \dfrac{1}{a^n}$ and $\dfrac{1}{a^{-n}} = a^n$ and in particular $a^{-1} = \dfrac{1}{a}$.

These laws can also be applied to rational exponents, or exponents which are written as a fraction.

We have seen examples of rational indices already when we studied surds.

Notice that $(a^{\frac{1}{2}})^2 = a^{\frac{1}{2} \times 2} = a^1 = a$ and $(\sqrt{a})^2 = a$, so $\boxed{a^{\frac{1}{2}} = \sqrt{a}.}$

and $(a^{\frac{1}{3}})^3 = a^{\frac{1}{3} \times 3} = a^1 = a$ and $(\sqrt[3]{a})^3 = a$, so $\boxed{a^{\frac{1}{3}} = \sqrt[3]{a}.}$

In general, $\boxed{a^{\frac{1}{n}} = \sqrt[n]{a}}$ where $\sqrt[n]{a}$ is called the 'nth root of a'.

Example 1   🔊 *Self Tutor*

Simplify: **a** $49^{\frac{1}{2}}$ **b** $27^{\frac{1}{3}}$ **c** $49^{-\frac{1}{2}}$ **d** $27^{-\frac{1}{3}}$

a $\quad 49^{\frac{1}{2}}$
$= \sqrt{49}$
$= 7$

b $\quad 27^{\frac{1}{3}}$
$= \sqrt[3]{27}$
$= 3$

c $\quad 49^{-\frac{1}{2}}$
$= \dfrac{1}{49^{\frac{1}{2}}}$
$= \dfrac{1}{\sqrt{49}}$
$= \frac{1}{7}$

d $\quad 27^{-\frac{1}{3}}$
$= \dfrac{1}{27^{\frac{1}{3}}}$
$= \dfrac{1}{\sqrt[3]{27}}$
$= \frac{1}{3}$

Discovery 1   *Rational Exponents*

Our aim is to discover the meaning of numbers raised to rational exponents of the form $\frac{m}{n}$ where $m, n \in \mathbb{Z}$. For example, what does $8^{\frac{2}{3}}$ mean?

What to do:

1   Use the graphing package to draw the graphs of $y = 8^x$ and $x = \frac{2}{3}$ on the same set of axes.

GRAPHING PACKAGE

Find the value of $8^{\frac{2}{3}}$ by locating the intersection of the two graphs.

2   Use the rule $(a^m)^n$ to simplify $(8^2)^{\frac{1}{3}}$ and $\left(8^{\frac{1}{3}}\right)^2$.

3   Copy and complete: **i** $(8^2)^{\frac{1}{3}} = \sqrt[3]{8^2} = \ldots\ldots$ **ii** $\left(8^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{8}\right)^2 = \ldots\ldots$

4   Use **3** to write $a^{\frac{m}{n}}$ in two different forms.

You should have discovered that $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ or $\left(\sqrt[n]{a}\right)^m$.

In practice we seldom use these laws, but they do help to give meaning to rational exponents.

Example 2   ◀) *Self Tutor*

Simplify: **a** $27^{\frac{4}{3}}$ **b** $16^{-\frac{3}{4}}$

a $27^{\frac{4}{3}} = (3^3)^{\frac{4}{3}}$
$= 3^4$
$= 81$

b $16^{-\frac{3}{4}} = (2^4)^{-\frac{3}{4}}$
$= 2^{-3}$
$= \frac{1}{2^3}$
$= \frac{1}{8}$

EXERCISE 28A

1   Evaluate without using a calculator:

a $4^{\frac{1}{2}}$ **b** $4^{-\frac{1}{2}}$ **c** $16^{\frac{1}{2}}$ **d** $16^{-\frac{1}{2}}$ **e** $25^{\frac{1}{2}}$ **f** $25^{-\frac{1}{2}}$

g $8^{\frac{1}{3}}$ **h** $8^{-\frac{1}{3}}$ **i** $64^{\frac{1}{3}}$ **j** $64^{-\frac{1}{3}}$ **k** $32^{\frac{1}{5}}$ **l** $32^{-\frac{1}{5}}$

m $125^{\frac{1}{3}}$ **n** $(-125)^{\frac{1}{3}}$ **o** $(-1)^{\frac{1}{2}}$ **p** $(-1)^{\frac{1}{3}}$

2   Write the following in index form:

a $\sqrt{10}$ **b** $\frac{1}{\sqrt{10}}$ **c** $\sqrt[3]{15}$ **d** $\frac{1}{\sqrt[3]{15}}$

e $\sqrt[4]{19}$ **f** $\frac{1}{\sqrt[4]{19}}$ **g** $\sqrt[5]{13}$ **h** $\frac{1}{\sqrt[5]{13}}$

3   Use your calculator to evaluate, correct to 3 significant figures where necessary:

a $\sqrt[3]{64}$ **b** $\sqrt[4]{81}$ **c** $\sqrt[5]{1024}$ **d** $\sqrt[3]{200}$

e $\sqrt[4]{400}$ **f** $\sqrt[5]{1000}$ **g** $\sqrt[7]{128}$ **h** $\sqrt[3]{10.83}$

4   Without using a calculator, find the value of the following:

a $8^{\frac{4}{3}}$ **b** $8^{-\frac{2}{3}}$ **c** $4^{\frac{3}{2}}$ **d** $4^{-\frac{3}{2}}$ **e** $27^{\frac{2}{3}}$ **f** $27^{-\frac{2}{3}}$

g $32^{\frac{2}{5}}$ **h** $32^{-\frac{3}{5}}$ **i** $64^{\frac{5}{6}}$ **j** $125^{-\frac{2}{3}}$ **k** $81^{\frac{3}{4}}$ **l** $81^{-\frac{3}{4}}$

B   *EXPONENTIAL FUNCTIONS [3.2, 3.3, 3.5, 3.8]*

Consider a population of 100 mice which is growing under plague conditions.

If the mouse population doubles each week, we can construct a **table** to show the population number M after w weeks.

w (weeks)	0	1	2	3	4	......
M	100	200	400	800	1600	......

We can also represent this information on a graph as:
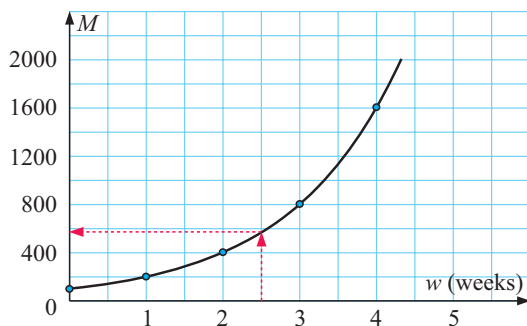
If we use a smooth curve to join the points, we can predict the mouse population when $w = 2.5$ weeks!

We can find a relationship between M and w using another table:

w	M values
0	$100 = 100 \times 2^0$
1	$200 = 100 \times 2^1$
2	$400 = 100 \times 2^2$
3	$800 = 100 \times 2^3$
4	$1600 = 100 \times 2^4$

So, we can write $M = 100 \times 2^w$.

This is an **exponential function** and the graph is an **exponential graph**.

We can use the function to find M for any value of $w \geqslant 0$.

For example, when $w = 2.5$, $M = 100 \times 2^{2.5}$
$$\approx 566 \text{ mice}$$

An **exponential function** is a function in which the variable occurs as part of the exponent or index.

The simplest exponential functions have the form $f(x) = a^x$ where a is a positive constant, $a \neq 1$.

For example, graphs of the exponential functions $f(x) = 2^x$ and $g(x) = (\frac{1}{2})^x = 2^{-x}$ are shown alongside.
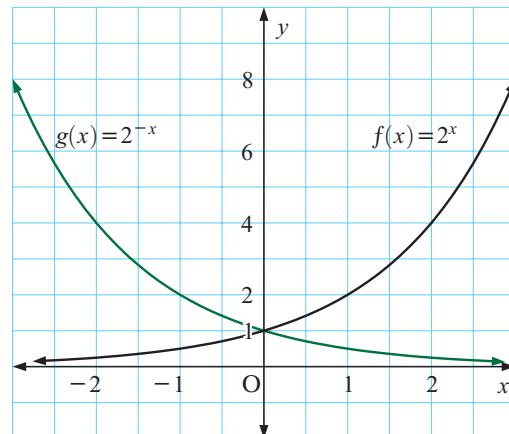
Discovery 2 — *Graphs of simple exponential functions*

This discovery is best done using a graphing package or graphics calculator.

GRAPHING PACKAGE

What to do:

1   **a**   On the same axes, graph $y = (1.2)^x$, $y = (1.5)^x$, $y = 2^x$, $y = 3^x$, $y = 7^x$.

    b   State the coordinates of the point which all of these graphs pass through.

    c   Explain why $y = a^x$ passes through this point for all $a \in \mathbb{R}$, $a > 0$.

    d   State the equation of the asymptote common to all these graphs.

    e   Comment on the shape of the family of curves $y = a^x$ as a increases in value.

2   On the same set of axes graph $y = (\frac{1}{3})^x$ and $y = 3^{-x}$.
    Explain your result.

3   **a**   On the same set of axes graph $y = 3 \times 2^x$, $y = 6 \times 2^x$ and $y = \frac{1}{2} \times 2^x$.

    b   State the y-intercept of $y = k \times 2^x$. Explain your answer.

4   **a**   On the same set of axes graph $y = 5 \times 2^x$ and $y = 5 \times 2^{-x}$.

    b   What is the significance of the factor 5 in each case?

    c   What is the difference in the shape of these curves, and what causes it?

All graphs of the form $f(x) = a^x$ where a is a positive constant not equal to 1:

- have a **horizontal asymptote** $y = 0$ (the x-axis)
- pass through $(0, 1)$ since $f(0) = a^0 = 1$.

An asymptote is a line which the graph approaches but never actually reaches.

Example 3 — ◀) *Self Tutor*

For the function $f(x) = 3 - 2^{-x}$, find: **a** $f(0)$ **b** $f(3)$ **c** $f(-2)$

a $f(0) = 3 - 2^0$
        $= 3 - 1$
        $= 2$

b $f(3) = 3 - 2^{-3}$
        $= 3 - \frac{1}{8}$
        $= 2\frac{7}{8}$

c $f(-2) = 3 - 2^{-(-2)}$
        $= 3 - 2^2$
        $= 3 - 4$
        $= -1$

EXERCISE 28B

1   If $f(x) = 3^x + 2$, find the value of: **a** $f(0)$ **b** $f(2)$ **c** $f(-1)$

2   If $f(x) = 5^{-x} - 3$, find the value of: **a** $f(0)$ **b** $f(1)$ **c** $f(-2)$

3   If $g(x) = 3^{x-2}$, find the value of: **a** $g(0)$ **b** $g(4)$ **c** $g(-1)$

4 a Complete the table of values shown for the function $f(x) = 3^x$.

x	-3	-2	-1	0	1	2	3
y							

b Use the table of values in **a** to graph $y = f(x)$.

c On the same set of axes and without a table of values, graph:

   i $y = -f(x)$ **ii** $y = f(-x)$ **iii** $y = 2f(x)$ **iv** $y = f(2x)$.

5 a Click on the icon to obtain a printable graph of $y = 2^x$.
Use the graph to estimate, to one decimal place, the value of:

   i $2^{0.7}$ **ii** $2^{1.8}$ **iii** $2^{-0.5}$.

Graph of $y = 2^x$

b Check your estimates in **a** using the $\boxed{\wedge}$ key on your calculator.

c Use the graph to estimate, correct to one decimal place, the solution of the equation:

   i $2^x = 5$ **ii** $2^x = 1.5$ **iii** $2^x = -1$.

6 Find the image of:

a $y = 2^x$ under the translation $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ **b** $y = 3^x$ under the translation $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

c $y = 2^{-x}$ under:

   i a reflection in the x-axis   **ii** a reflection in the y-axis   **iii** a reflection in the line $y = x$

d $y = 3^x$ under:

   i a stretch with invariant x-axis and scale factor 2

   ii a stretch with invariant y-axis and scale factor $\frac{1}{3}$.
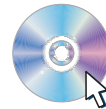
7 For the following functions:

   i sketch the graph   **ii** find the y-intercept   **iii** find the equation of any asymptote.

a $f(x) = (1.2)^x$ **b** $f(x) = 2^x - 1$ **c** $f(x) = 2^{x-1}$ **d** $f(x) = 2^{x+1}$

e $f(x) = 3 + 2^x$ **f** $f(x) = 2 - 2^x$ **g** $f(x) = \dfrac{2^{-x} + 1}{3}$ **h** $f(x) = 2(3^{-x}) + 1$

8 Explain why $(-2)^x$ is undefined for some real numbers x.

9 Find the exponential function corresponding to the graph:

a
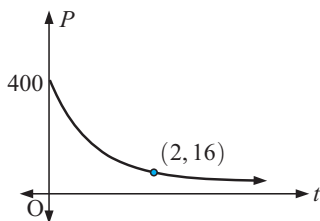
b
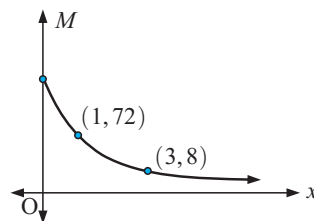
C *EXPONENTIAL EQUATIONS* *[2.11]*

An **exponential equation** is an equation in which the unknown occurs as part of the exponent or index.

For example: $2^x = 8$ and $30 \times 3^x = 7$ are both exponential equations.

If $2^x = 8$, then $2^x = 2^3$. Thus $x = 3$ is a solution, and it is in fact the only solution.

Since $f(x) = a^x$ is a one-one function, if $a^x = a^k$ then $x = k$.

If the base numbers are the same, we can **equate indices**.

Example 4   ◀) *Self Tutor*

Solve for x: **a** $2^x = 32$ **b** $3^{x-2} = \frac{1}{9}$

a $2^x = 32$
$\therefore$ $2^x = 2^5$
$\therefore$ $x = 5$

b $3^{x-2} = \frac{1}{9}$
$\therefore$ $3^{x-2} = 3^{-2}$
$\therefore$ $x - 2 = -2$
$\therefore$ $x = 0$

> If we can make the base numbers the same then we can equate the indices.

Example 5   ◀) *Self Tutor*

Solve for x: **a** $6 \times 3^x = 54$ **b** $4^{x-1} = \left(\frac{1}{2}\right)^{1-3x}$

a $6 \times 3^x = 54$
$\therefore$ $3^x = 9$
$\therefore$ $3^x = 3^2$
$\therefore$ $x = 2$

b $4^{x-1} = \left(\frac{1}{2}\right)^{1-3x}$
$\therefore$ $(2^2)^{x-1} = (2^{-1})^{1-3x}$
$\therefore$ $2(x - 1) = -1(1 - 3x)$
$\therefore$ $2x - 2 = -1 + 3x$
$\therefore$ $-2 + 1 = 3x - 2x$
$\therefore$ $x = -1$

EXERCISE 28C.1

1 Solve for x:

a $3^x = 3$ **b** $3^x = 9$ **c** $2^x = 8$ **d** $5^x = 1$

e $3^x = \frac{1}{3}$ **f** $5^x = \frac{1}{5}$ **g** $2^x = \frac{1}{16}$ **h** $5^{x+2} = 25$

i $2^{x+2} = \frac{1}{4}$ **j** $3^{x-1} = \frac{1}{27}$ **k** $2^{x-1} = 32$ **l** $3^{1-2x} = \frac{1}{27}$

m $4^{2x+1} = \frac{1}{2}$ **n** $9^{x-3} = 3$ **o** $\left(\frac{1}{2}\right)^{x-1} = 2$ **p** $\left(\frac{1}{3}\right)^{2-x} = 9$

2 Solve for x:

a $5 \times 2^x = 40$ **b** $6 \times 2^{x+2} = 24$ **c** $3 \times \left(\frac{1}{2}\right)^x = 12$

d $4 \times 5^x = 500$ **e** $8 \times \left(\frac{1}{2}\right)^x = 1$ **f** $7 \times \left(\frac{1}{3}\right)^x = 63$

g $2^{2-5x} = 4^x$ **h** $5^{x-1} = \left(\frac{1}{25}\right)^x$ **i** $9^{x-2} = \left(\frac{1}{3}\right)^{3x-1}$

j $2^x \times 4^{2-x} = 8$ **k** $3^{x+1} \times 9^{-x} = \left(\frac{1}{3}\right)^{x+1}$ **l** $2^{x^2-2x} = 8$

3 If $a \times 5^n = 150$ and $a \times 10^n = 600$, find a and n.   **Hint:** Consider $\dfrac{a \times 10^n}{a \times 5^n}$.

4 Find b and t given that $b \times 2^t = 8$ and $b \times 6^t = 1944$.

5 Suppose $2^{x+3y} = 32$ and $3^{2x-y} = \frac{1}{81}$. Find x and y.

SOLVING EXPONENTIAL EQUATIONS GRAPHICALLY

In many exponential equations we cannot easily make the base numbers on both sides the same. For example, if $3^x = 6$ we cannot easily write 6 with a base number of 3.

We can solve these types of exponential equations using a graphics calculator, using the methods learnt in **Chapter 23**.

Discovery 3   *Solving exponential equations graphically*

Consider the exponential equation $3^x = 6$.

Since $3^1 = 3$ and $3^2 = 9$, the solution for x must lie between 1 and 2.

A **graphics calculator** can be used to solve this equation by drawing the graphs of $y = 3^x$ and $y = 6$ and finding their **point of intersection**. To find out how to do this, consult the instructions on pages **23** to **24**.

Alternatively, click on the icon to obtain a graphing package.

GRAPHING PACKAGE
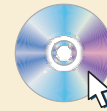
What to do:

1 Draw the graph of $y = 3^x$.

2 Draw the graph of $y = 6$ on the same set of axes.

3 Find the coordinates of the point of intersection of the graphs.

4 Solve for x, correct to 3 decimal places:

    a $3^x = 10$ **b** $3^x = 30$ **c** $3^x = 100$

    d $2^x = 12$ **e** $5^x = 40$ **f** $7^x = 42$

    If using a calculator you may have to change the viewing window scales.

Example 6   ◀◎ *Self Tutor*

Solve $2^x = 10$ correct to 3 decimal places.

$2^x = 10$ has the same solutions as
$2^x - 10 = 0$.

The x-intercept ≈ 3.322

$\therefore \quad x \approx 3.322$

≈ 3.322

$y = 2^x - 10$
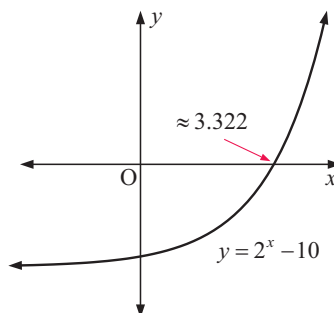
We could also plot $y = 2^x$ and $y = 10$ on the same set of axes.

EXERCISE 28C.2

1 Solve for x, giving answers correct to 3 decimal places:

a $2^x = 100$	**b** $2^x = 0.271$	**c** $2^x = -3$
d $5^x = 8$	**e** $7^{-x} = 23$	**f** $9^x = 10\,000$
g $3^x = 0.006\,51$	**h** $5 \times 2^{-x} = 18$	**i** $200 \times 2^x = 5800$
j $300 \times 2^{-3x} = 4.1$	**k** $25 \times 3^{-2x} = 0.035$	**l** $4 \times 2^{-0.02x} = 0.07$
m $3^{x+1} = 4$	**n** $2^{3x-2} = 5$	**o** $3(2^{x-2}) = 1$

D PROBLEM SOLVING WITH EXPONENTIAL FUNCTIONS [3.2]

Exponential functions model real life situations in many branches of science and commerce. Common applications include compound interest and biological modelling.

Example 7 ◀ᴐ) *Self Tutor*

During a locust plague, the area of land eaten is given by $A = 8000 \times 2^{0.5n}$ hectares where n is the number of weeks after the initial observation.

a Find the size of the area initially eaten.

b Find the size of the area eaten after: **i** 4 weeks **ii** 7 weeks.

c Graph A against n.

d How long would it take for the area eaten to reach $50\,000$ hectares?

a Initially, $n = 0$ $\therefore$ $A = 8000 \times 2^0$
$\therefore$ $A = 8000$ hectares

b **i** When $n = 4$,
$$A = 8000 \times 2^{0.5 \times 4}$$
$$= 8000 \times 2^2$$
$$= 32\,000 \text{ ha}$$

ii When $n = 7$,
$$A = 8000 \times 2^{0.5 \times 7}$$
$$= 8000 \times 2^{3.5}$$
$$\approx 90\,500 \text{ ha}$$

c

d We plot $Y_1 = 8000 \times 2^{0.5X}$ and $Y_2 = 50\,000$ on the same axes.
The graphs meet when $X \approx 5.29$
$\therefore$ it takes approximately 5.29 weeks for the area eaten to reach $50\,000$ hectares.

Example 8 ◀)) *Self Tutor*

The current I flowing through the electric circuit in a fan, t milliseconds after it is switched off, is given by $I = 320 \times 2^{-0.5t}$ milliamps.

a Find the initial current in the circuit.

b Find the current after: **i** 4 **ii** 10 milliseconds.

c Graph I against t.

d Use technology to find how long it takes for the current to drop to 50 milliamps.

a When $t = 0$, $I = 320 \times 2^{-0.5 \times 0}$
$= 320$ milliamps

b **i** When $t = 4$,
$I = 320 \times 2^{-0.5 \times 4}$
$= 320 \times 2^{-2}$
$= 320 \times \frac{1}{4}$
$= 80$ milliamps

ii When $t = 10$,
$I = 320 \times 2^{-0.5 \times 10}$
$= 320 \times 2^{-5}$
$= 320 \times \frac{1}{32}$
$= 10$ milliamps

c
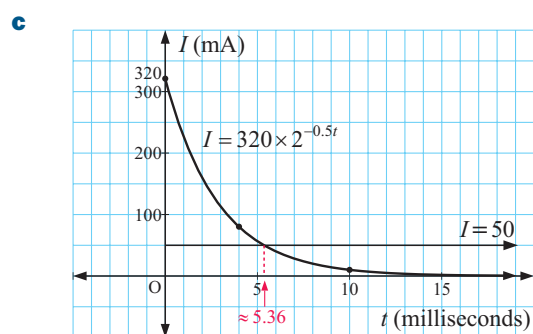
d We plot $Y_1 = 320 \times 2^{-0.5X}$ and $Y_2 = 50$ on the same axes.
The graphs meet when $X \approx 5.36$
$\therefore$ it takes approximately 5.36 milliseconds for the current to drop to 50 milliamps.

EXERCISE 28D

1 A local zoo starts a breeding program to ensure the survival of a species of mongoose. From a previous program, the expected population in n years' time is given by $P = 40 \times 2^{0.2n}$.

a What is the initial population purchased by the zoo?

b What is the expected population after:
i 3 years **ii** 10 years **iii** 30 years?

c Graph P against n.

d How long will it take for the population to reach 100?

2 In Tasmania a reserve is set aside for the breeding of echidnas. The expected population size after t years is given by $P = 50 \times 2^{\frac{t}{3}}$.

a What is the initial breeding colony size?

b Find the expected colony size after:
i 3 years **ii** 9 years **iii** 20 years.

c Graph P against t.

d How long will it take for the echidna population to reach 150?

3 In Uganda, the number of breeding females of an endangered population of gorillas is G_0. Biologists predict that the number of breeding females G in n years' time will, if left alone by man, grow according to $\quad G = G_0 \times 5^{0.07n}$.

   a If initially 28 breeding females are in the colony, find G_0.

   b Find G when: **i** $n = 3$ years   **ii** $n = 10$ years   **iii** $n = 30$ years.

   c Graph G against n.

   d Find the time it will take for the gorilla population to reach 200.

4 The weight of radioactive material in an ore sample after t years is given by
$W = 2.3 \times 2^{-0.06t}$ grams.

   a Find the initial weight.

   b Find the weight after: **i** 20 years   **ii** 200 years   **iii** 2000 years.

   c Graph W against t.

   d What is the percentage loss in weight from $t = 0$ to $t = 20$?

   e How long will it take for the weight to fall to 0.8 grams?

5 A cup of boiling water is placed in a refrigerator and after t minutes its temperature is given by
$T = 100 \times 2^{-\frac{t}{4}}$ °C.

   a Find the inital temperature of the water.

   b Find the water temperature after: **i** 2 minutes   **ii** 10 minutes   **iii** 1 hour.

   c Graph T against t.

   d There is no risk of the water causing scalding once its temperature falls to 49^oC. How long will this take?

6 The value of an investment in n years' time at 8.3% p.a. compound interest is given by
$F = 5800 \times (1.083)^n$ dollars.

   a What was the original investment?

   b Find the value of the investment after: **i** 3 years   **ii** 10 years.

   c Find the time taken for the value of the investment to double.

7 After m months, the value of a washing machine is given by $V = 500(0.981)^m$ dollars.

   a What was the washing machine's original value?

   b How much is it worth after: **i** 6 months   **ii** 4 years?

   c How long will it take for the washing machine's value to reduce to $150?

8 A bacteria population doubles every 1.5 days. Initially there are 20 bacteria.

   a Find a formula for the number of bacteria B after d days.

   b Sketch the graph of B against d for the first 15 days.

   c Find the number of bacteria after:

      i 2 days   **ii** 1 week   **iii** 2 weeks.

   d The presence of 5000 bacteria is considered dangerous. How long will it take the population to reach this level?

9 Answer the **Opening Problem** on page **565**.

E   *EXPONENTIAL MODELLING*   *[3.2]*

In **Chapter 22** we were given data for two related variables x and y, and we used technology to find a **line of best fit** $y = ax + b$ to connect the variables.

This process is called **linear regression**.

We will now use technology to connect variables x and y by an **exponential model** of the form $y = a \times b^x$.

For example, consider the table of values:

t	2	5	8	12
H	40	113	320	1280

We can perform **exponential regression** on this data using a **TI-84 Plus**. Instructions for doing this can be found on page **28**.

The **Casio fx-9860G** does not perform exponential regression in this form, but you can click on the icon to access an alternative exponential regression program.

EXPONENTIAL REGRESSION
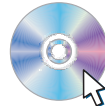
```
ExpReg
y=a*b^x
a=19.9887
b=1.4143
r²=1.0000
r=1.0000
```

For this data we find that $H \approx 20.0 \times (1.414)^t$.

Notice that the **coefficient of determination** $r^2 = 1$, which indicates that the data follows an exponential curve exactly.

To test whether two variables x and y are related by an exponential model of the form $y = a \times b^x$, we should do the following:

- Graph y against x. The graph should look like either

  or
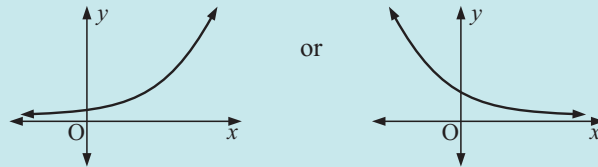
- Perform exponential regression using a graphics calculator. The closer r^2 is to 1, the better the exponential model fits the data.

Example 9   ◄》 *Self Tutor*

The weight W grams of bacteria in a culture was measured regularly. The results were:

Time, t days	1	2	3	4	5
Weight, W grams	11	20	35	62	118

EXPONENTIAL REGRESSION

a   Use technology to show that an exponential model fits the data well.

b   Find the exponential model.

c   Use the model to estimate the original weight of the culture.

a   Putting the data into lists and using exponential regression we get:
     As $r^2 \approx 1$, an exponential model fits the data very well.

b $W \approx 6.09 \times 1.80^t$ grams

c   When $t = 0$, $W \approx 6.09 \times 1.80^o$
             ≈ 6.09

   So, the original weight was about 6.09 grams.

```
ExpReg
y=a*b^x
a=6.0877
b=1.7998
r²=.9994
r=.9997
```

EXERCISE 28E

1 Show that an exponential model is appropriate for the following data, and state the equation of the exponential model.

a

x	2	5	10	20
P	44	100	401	6448

b

t	1	3	6	8	11
N	4.6	3.1	1.7	1.1	0.6

2 Over the last 8 years, Jane and Pierre have had their house valued four times. The valuations were:

Year	2000	2002	2005	2008
Value(€)	216 000	226 000	242 000	260 000

   a Does an exponential model fit this data? If so, what is the model? (Let year 2000 be $t = 0$.)

   b Estimate the value of their house in the years

      i 2004   **ii** 2009.

   c Which of the values in **b** is more reliable? Give reasons for your answer.

3 The table below shows the concentration of chemical X in the blood of an accident victim at various times after an injection was administered.

Time (t minutes)	1	2	3	4	5	6	7
Concentration of X (C micrograms/cm³)	104.6	36.5	12.7	4.43	1.55	0.539	0.188

   a Show that an exponential model fits the data well, and find the model.

   b The victim is considered safe to move when the concentration of X falls below 2×10^{-4} micrograms/cm³. Estimate how long it will take to reach this level.

Review set 28A

1 Write the following in exponent form:

   a $\sqrt{5}$ **b** $\dfrac{1}{\sqrt[3]{7}}$ **c** $\sqrt[4]{51}$ **d** $\dfrac{1}{\sqrt[5]{47}}$

2 Evaluate without using a calculator:

   a $16^{\frac{3}{4}}$ **b** $125^{\frac{2}{3}}$ **c** $9^{-\frac{3}{2}}$ **d** $32^{-\frac{3}{5}}$

3 Let $f(x) = 5^x$.

   a Find: **i** $f(-x)$ **ii** $-f(x)$ **iii** $2 \times f(x)$ **iv** $f(2x)$

   b Sketch all five graphs on the same set of axes.

4 If $f(x) = 3^x - 1$, find the value of:

   a $f(0)$ **b** $f(3)$ **c** $f(-1)$ **d** $f(-2)$

5 On the same set of axes, without using technology, draw the graphs of $y = 2^x$ and $y = 2^x + 2$.

   a State the y-intercepts and the equations of the horizontal asymptotes.

   b What transformation is needed to draw the graph of $y = 2^x + 2$ from the graph of $y = 2^x$?

6 Solve for x:

   a $3^x = \frac{1}{81}$ **b** $2^{x-2} = \frac{1}{8}$ **c** $25^{4-x} = 1$

   d $4 \times 5^x = 100$ **e** $5 \times (\frac{1}{2})^{x+1} = 80$ **f** $4^{x+1} \times 8^x = \frac{1}{4}$

7 Find k and n given that $k \times 2^n = 144$ and $k \times 8^n = 9$.

8 Solve for x, correct to 2 decimal places:

  a $5^x = 90$ **b** $3^{-2x} = 0.05$ **c** $20 \times 2^{0.2x} = 50$

9 The number of employees in a company after t years is given by $N = 30 \times 2^{\frac{t}{4}}$.

  a How many people were employed originally?

  b Find the number of employees after: **i** 4 years   **ii** 7 years.

  c Graph N against t.

  d How long will it take for the company to grow to 200 employees?

Review set 28B

1 Use your calculator to evaluate, correct to 3 decimal places:

  a $\sqrt[3]{25}$ **b** $\sqrt[5]{100}$ **c** $\sqrt[4]{15.1}$

2 Evaluate without using a calculator:

  a $81^{-\frac{1}{4}}$ **b** $4^{\frac{5}{2}}$ **c** $64^{\frac{2}{3}}$ **d** $81^{-\frac{5}{4}}$

3 If $P(x) = 2 \times 3^{-x}$, find the value of:

  a $P(0)$ **b** $P(1)$ **c** $P(2)$ **d** $P(-1)$ **e** $P(-2)$

4 On the same set of axes, without using technology, draw the graphs of $y = 2^x$ and $y = 3 \times 2^x$.

  a State the y-intercepts and the equations of the horizontal asymptotes.

  b What transformation is needed to draw the graph of $y = 3 \times 2^x$ from the graph of $y = 2^x$?

5 Suppose $f(x) = 2^x$.

  a Express in the form $k \times a^x$:

      i $f(-x)$ **ii** $f(2x)$ **iii** $f(x+1)$ **iv** $f(x-1)$

  b Sketch all five graphs on the same set of axes.

6 Solve for x:

  a $5^{x+1} = \frac{1}{25}$ **b** $27^{2x} = \frac{1}{3}$ **c** $8^x = 4^{5-x}$

  d $14 \times 9^{3-2x} = 14$ **e** $3 \times \left(\frac{1}{4}\right)^{x+1} = 96$ **f** $7^{x^2-x} = 49$

7 Suppose $3^{x-2y} = 27$ and $2^{2x+y} = \frac{1}{16}$. Find x and y.

8 Solve for x, correct to 2 decimal places:

  a $3^x = 20$ **b** $7^{2x} = -1$ **c** $12 \times 5^{-0.01x} = 10$

9 At the zoo, Terry the giant turtle is overweight. It is decided to put him on a diet. His weight t weeks after starting the diet is given by $W = 130 \times 3^{-0.01t}$ kilograms.

  a How much did Terry weigh before he was started on the diet?

  b How much weight will Terry have lost after:

      i 2 weeks   **ii** 6 weeks?

  c The keepers would like Terry to lose 20 kg. How long will it take to achieve this?

Further trigonometry

29

Contents:

Opening problem

A triangular property is bounded by two roads and a long, straight drain.

Can you find:

a the area of the property in m^2 and in hectares

b the length of the drain boundary

c the angle that the Johns Road boundary makes with the drain boundary?
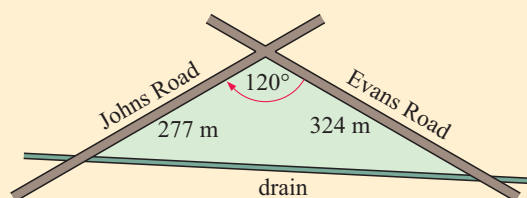
A   THE UNIT CIRCLE   [8.3]

In **Chapter 15** we introduced the **unit circle**, which is the circle with centre O(0, 0) and radius 1 unit. In that chapter we considered only the first quadrant of the circle, which corresponds to angles θ where $0^o \leqslant \theta \leqslant 90^o$. We now consider the complete unit circle including all four quadrants.

As P moves around the circle, the angle θ varies.

The coordinates of P are defined as $(\cos\theta, \sin\theta)$.
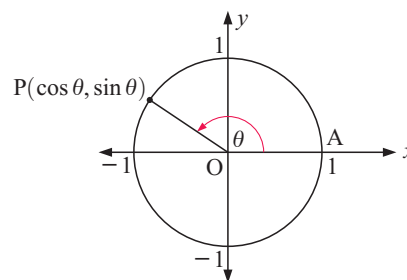
Below is a unit circle diagram from which we can estimate trigonometric ratios.

Example 1   ◀)) *Self Tutor*

a State the exact coordinates of:
  i A   **ii** B

b Find the coordinates of:
  i A   **ii** B
  correct to 3 decimal places.

a **i** A($\cos 152^o$, $\sin 152^o$)   **ii** B($\cos 297^o$, $\sin 297^o$)
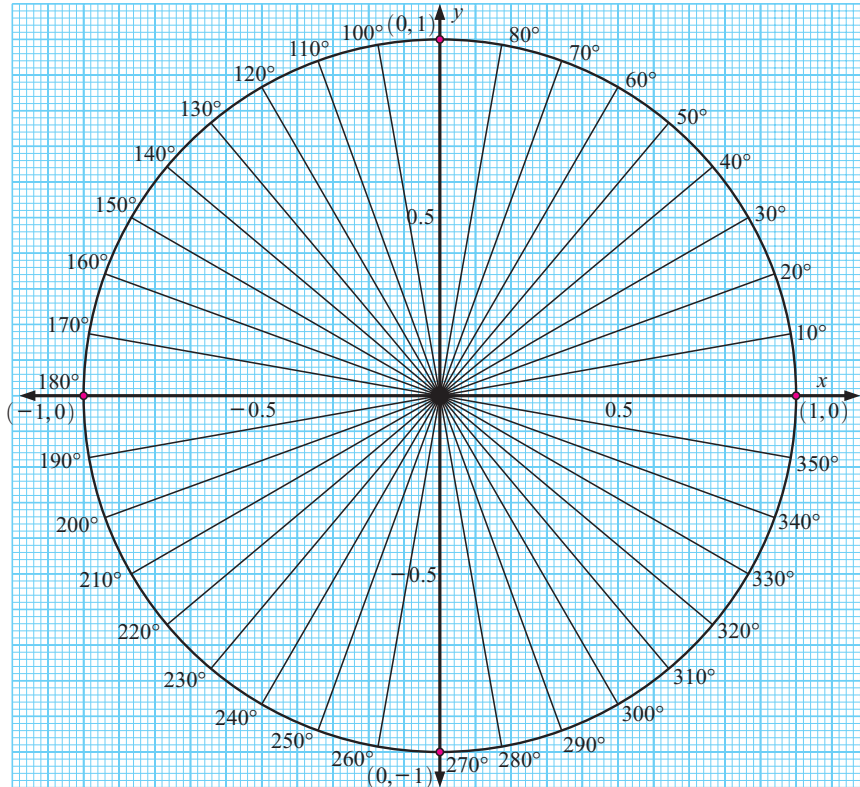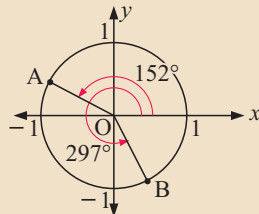
b **i** A(-0.883, 0.469)   **ii** B(0.454, -0.891)

Example 2   ◀)) *Self Tutor*
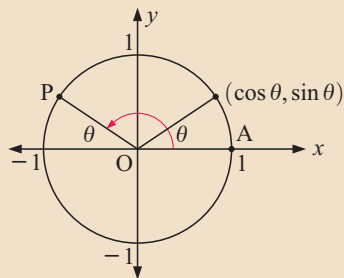
a Find the size of angle AOP marked with an arrow.

b Find the coordinates of P using:
  i the unit circle
  ii symmetry in the y-axis.

c What can be deduced from **b**?

d Use **c** to simplify $\tan(180^o - \theta)$.

a   angle AOP $= (180^o - \theta)$

b   **i**   P is $(\cos(180^o - \theta),\ \sin(180^o - \theta))$

    ii   P is $(-\cos\theta,\ \sin\theta)$

c $\cos(180^o - \theta) = -\cos\theta$ and $\sin(180^o - \theta) = \sin\theta$

d $\tan(180^o - \theta) = \dfrac{\sin(180^o - \theta)}{\cos(180^o - \theta)}$

$$= \dfrac{\sin\theta}{-\cos\theta} \qquad \{\text{using } \mathbf{c}\}$$

$$= -\tan\theta$$

EXERCISE 29A.1

1   **a**   State the exact coordinates of P.

    b   Find the coordinates of P correct to 3 decimal places.
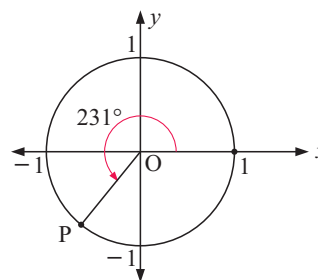
2   Use the unit circle diagram to find:

a $\sin 180^o$	**b** $\cos 180^o$	**c** $\sin 270^o$	**d** $\cos 270^o$
e $\cos 360^o$	**f** $\sin 360^o$	**g** $\cos 450^o$	**h** $\sin 450^o$

3   Use the unit circle diagram to estimate, to 2 decimal places:

a $\cos 50^o$	**b** $\sin 50^o$	**c** $\cos 110^o$	**d** $\sin 110^o$
e $\sin 170^o$	**f** $\cos 170^o$	**g** $\sin 230^o$	**h** $\cos 230^o$
i $\cos 320^o$	**j** $\sin 320^o$	**k** $\cos(-30^o)$	**l** $\sin(-30^o)$

4   Check your answers to **3** using your calculator.

5
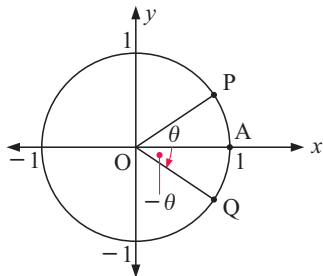
a   State the coordinates of point P.

b   Find the coordinates of Q using:

    i   the unit circle

    ii   symmetry in the x-axis.

c   What can be deduced from **b**?

d   Use **c** to simplify $\tan(-\theta)$.

$$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$$

6   By considering a unit circle diagram like that in **5**, show how to simplify
$\sin(180^o + \theta)$, $\cos(180^o + \theta)$, and $\tan(180^o + \theta)$.

Hint: Consider rotational symmetry.

IMPORTANT TRIGONOMETRIC RATIOS IN THE UNIT CIRCLE

In **Chapter 15** we found the trigonometric ratios for the angles 0°, 30°, 45°, 60° and 90°.

θ	$\cos\theta$	$\sin\theta$	$\tan\theta$
0°	1	0	0
30°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
90°	0	1	undefined

These angles correspond to the points shown on the first quadrant of the unit circle:
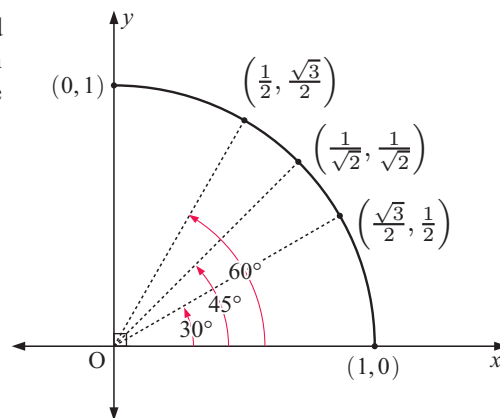
We can use the symmetry of the unit circle to find the coordinates of all points with angles that are multiples of 30° and 45°.

For example, the point Q corresponding to an angle of 120° is a reflection in the y-axis of point P with angle 60°.

Q has the negative x-coordinate and the same y-coordinate as P, so the coordinates of Q are $\left(-\frac{1}{2},\ \frac{\sqrt{3}}{2}\right)$.
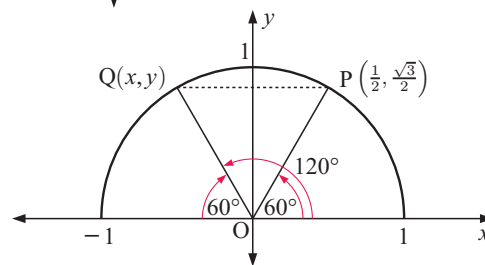
Multiples of 30°   **Multiples of 45°**
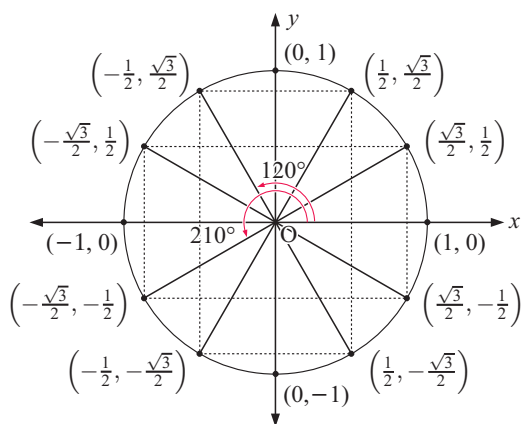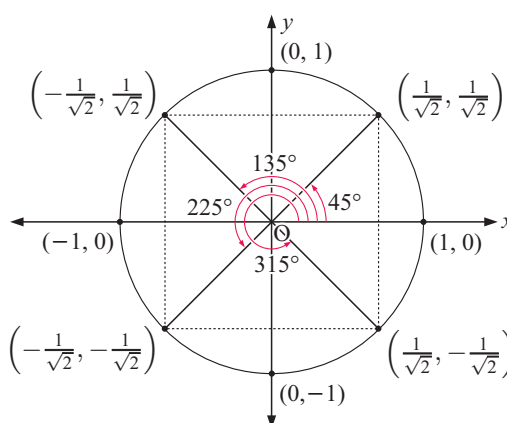
We can find the trigonometric ratios of these angles using the coordinates of the corresponding point on the unit circle.

Example 3   ◀) Self Tutor
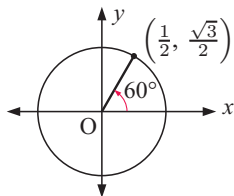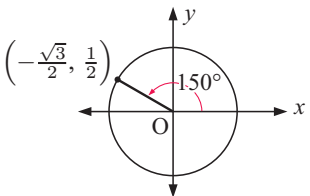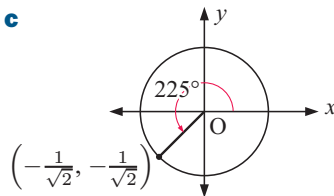
Use a unit circle diagram to find $\sin\theta$, $\cos\theta$ and $\tan\theta$ for:

   a $\theta = 60^o$ **b** $\theta = 150^o$ **c** $\theta = 225^o$

a

$\sin 60^o = \frac{\sqrt{3}}{2}$

$\cos 60^o = \frac{1}{2}$

$\tan 60^o = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

b

$\sin 150^o = \frac{1}{2}$

$\cos 150^o = -\frac{\sqrt{3}}{2}$

$\tan 150^o = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$

c

$\sin 225^o = -\frac{1}{\sqrt{2}}$

$\cos 225^o = -\frac{1}{\sqrt{2}}$

$\tan 225^o = 1$

EXERCISE 29A.2

1 Use a unit circle to find $\sin\theta$, $\cos\theta$ and $\tan\theta$ for:

    a $\theta = 30^o$ **b** $\theta = 180^o$ **c** $\theta = 135^o$ **d** $\theta = 210^o$

    e $\theta = 300^o$ **f** $\theta = 270^o$ **g** $\theta = 315^o$ **h** $\theta = 240^o$

2 Without using a calculator, find the exact values of:

    a $\sin^2 135^o$ **b** $\cos^2 120^o$ **c** $\tan^2 210^o$ **d** $\cos^3 330^o$

Check your answers using a calculator.

> $\sin^2\theta = (\sin\theta)^2$,
> $\cos^2\theta = (\cos\theta)^2$
> and so on.

3 Use a unit circle diagram to find all angles between 0° and 360° which have:

    a a sine of $\frac{1}{2}$ **b** a cosine of $\frac{\sqrt{3}}{2}$ **c** a sine of $\frac{1}{\sqrt{2}}$

    d a sine of $-\frac{1}{2}$ **e** a sine of -1 **f** a cosine of $-\frac{\sqrt{3}}{2}$.

B   AREA OF A TRIANGLE USING SINE   [8.6]

Consider the acute angled triangle alongside, in which the sides opposite angles A, B and C are labelled a, b and c respectively.

Area of triangle ABC $= \frac{1}{2} \times AB \times CN = \frac{1}{2}ch$

    But $\sin A = \frac{h}{b}$

    $\therefore$ $h = b\sin A$

    $\therefore$ area $= \frac{1}{2}c(b\sin A)$ or $\frac{1}{2}bc\sin A$
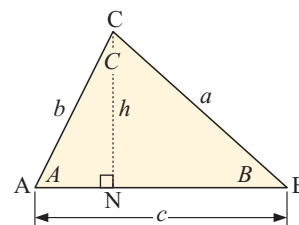
If the altitudes from A and B were drawn, we could also show that

    area $= \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C$.   **area $= \frac{1}{2}ab\sin C$**   is worth remembering.

For the obtuse angled triangle ABC alongside:

Area of triangle ABC $= \frac{1}{2} \times$ AB $\times$ CN $= \frac{1}{2}ch$

But $\sin(180^o - A) = \dfrac{h}{b}$

$\therefore \quad h = b\sin(180^o - A) = b\sin A$

$\therefore$ area of triangle ABC $= \frac{1}{2}cb\sin A$,
     which is the same result as when A was acute.
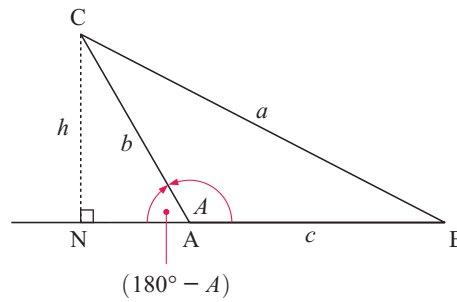
Summary:

> The area of a triangle is a half of the product of two sides and the sine of the included angle.
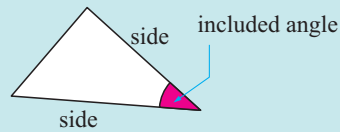
Example 4   ◀) Self Tutor

Find the area of triangle ABC.
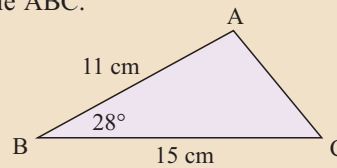
$$\text{Area} = \tfrac{1}{2}ac\sin B$$
$$= \tfrac{1}{2} \times 15 \times 11 \times \sin 28^o$$
$$\approx 38.7 \text{ cm}^2$$

EXERCISE 29B

1 Find the area of:

a
12 cm
45°
13 cm

b
28 km
82°
25 km

c
7.8 cm
112°
6.4 cm

d
32 m   84°   27 m

e
10.6 cm   125°
12.2 cm

f
1.65 m
78°
1.43 m
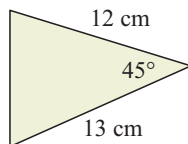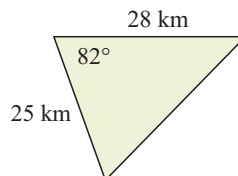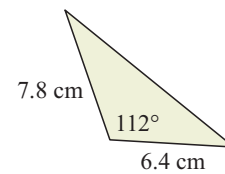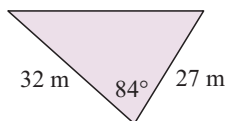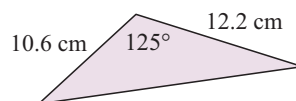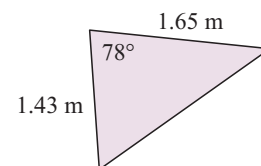
2 Find the area of a parallelogram with sides 6.4 cm and 8.7 cm and one interior angle 64°.

3 If triangle ABC has area 150 cm², find the value of x.

14 cm, A, B, 75°, x cm, C
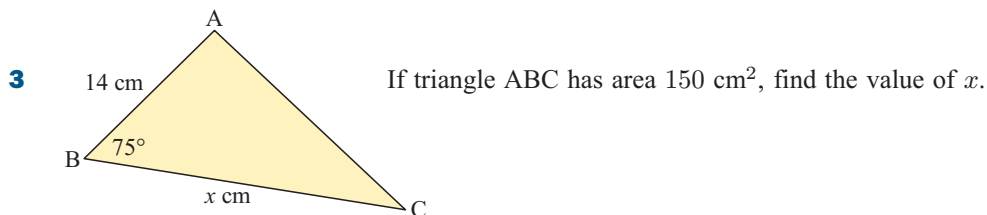
4 Triangle PQR has $P\widehat{Q}R = \theta$.   PQ = 10 m,   QR = 12 m,   and the area of the triangle is 30 m².
Find the possible values of θ.

5 Triangle ABC has   AB = 13 cm   and   BC = 17 cm,   and its area is 73.4 cm².   Find the measure
of $A\widehat{B}C$.

6

B, c, a, A, A, C, b, C

  a Find the area of triangle ABC using:

   i angle A **ii** angle C

  b Hence, show that $\dfrac{a}{c} = \dfrac{\sin A}{\sin C}$.
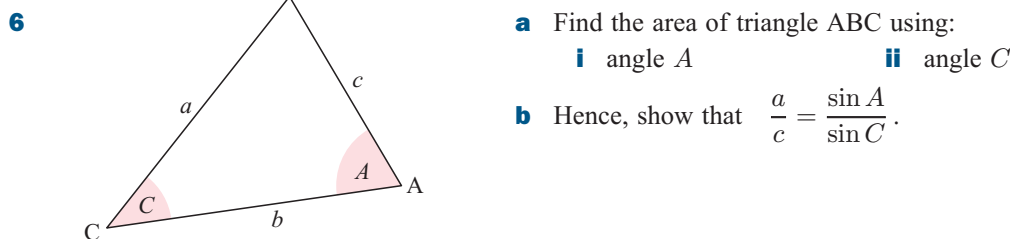
C   THE SINE RULE   [8.4]

The **sine rule** is a set of equations which connects the lengths of the sides of any triangle with the sines of
the opposite angles.

The triangle does not have to be right angled for the sine rule to be used.

THE SINE RULE

> In any triangle ABC with sides a, b and c units,
> and opposite angles A, B and C respectively,
>
> $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$
>
> C, b, a, A, c, B
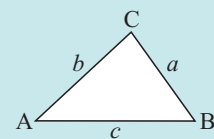
Proof: The area of any triangle ABC is given by $\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C$.

Dividing each expression by $\frac{1}{2}abc$ gives $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

The sine rule is used to solve problems involving triangles when angles and sides
opposite those angles are to be related.

**GEOMETRY
PACKAGE**

We use the **sine rule** when we are given:

- **two sides** and an **angle not included** between these sides, or
- **two angles** and a **side**.
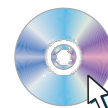
FINDING SIDES

> ### Example 5   ◀)) *Self Tutor*
>
> Find the length of side BC correct
> to 2 decimal places:
>
> Using the sine rule: $\dfrac{BC}{\sin 113^o} = \dfrac{18}{\sin 41^o}$
>
> $\therefore \quad BC = \dfrac{18 \times \sin 113^o}{\sin 41^o}$
>
> $\therefore \quad BC \approx 25.26$
>
> $\therefore$ BC is about 25.26 m long.
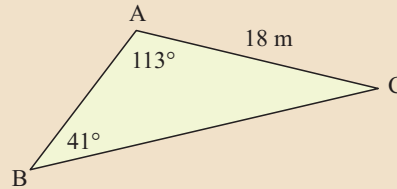
EXERCISE 29C.1

1 Find the value of x:
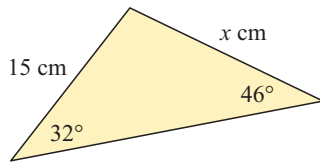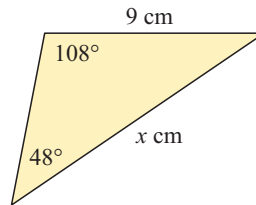
   a   **b**   **c**
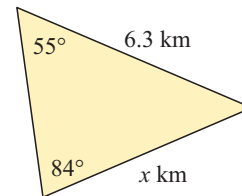
2 In triangle ABC find:

   a a if $A = 65^o$, $B = 35^o$, $b = 18$ cm   **b** b if $A = 72^o$, $C = 27^o$, $c = 24$ cm

   c c if $B = 25^o$, $C = 42^o$, $a = 7.2$ cm.

FINDING ANGLES

The problem of finding angles using the sine rule is more complicated because there may be two possible answers.

> This **ambiguous case** may occur when we are given two sides and one angle, where the angle is opposite the shorter side.
>
> It occurs because an equation of the form $\sin \theta = b$ produces answers of the form $\theta = \sin^{-1} b$ or $(180^o - \sin^{-1} b)$.
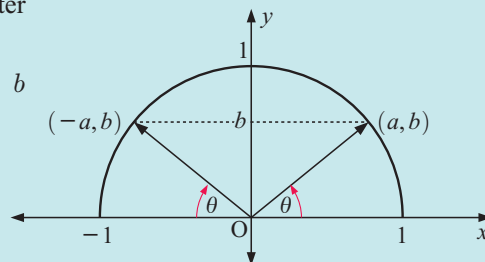
Example 6

Find, correct to 1 decimal place, the measure of angle C in triangle ABC if   AC = 8 cm,
AB = 12 cm,   and angle B measures 28°.
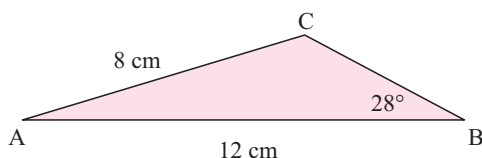
$$\frac{\sin C}{c} = \frac{\sin B}{b} \qquad \text{\{sine rule\}}$$

$$\therefore \quad \frac{\sin C}{12} = \frac{\sin 28^o}{8}$$

$$\therefore \quad \sin C = \frac{12 \times \sin 28^o}{8}$$

$$\text{Now} \quad \sin^{-1}\left(\frac{12 \times \sin 28^o}{8}\right) \approx 44.8^o$$

and since the angle at C could be acute or obtuse,

$$\therefore \quad C \approx 44.8^o \quad \text{or} \quad (180 - 44.8)^o$$

$\therefore$ C measures 44.8^o if it is acute, or 135.2^o if it is obtuse.

In this case there is insufficient information to determine the actual shape of the triangle.

The validity of the two answers in the above example can be demonstrated by a simple construction.

Step 1:   Draw AB of length 12 cm and construct an angle of 28° at B.
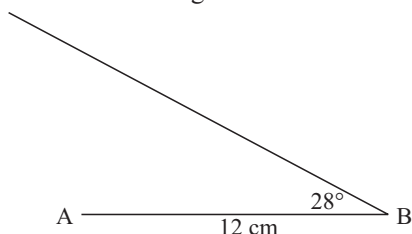
Step 2:   From A, draw an arc of radius 8 cm.
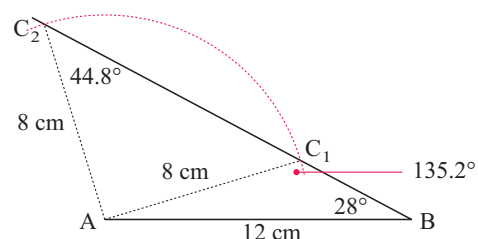
Sometimes there is information given in the question which enables us to **reject** one of the answers.

Example 7

Find the measure of angle L in triangle KLM given that $L\widehat{K}M$ measures 52°,
LM = 158 m, and KM = 128 m.
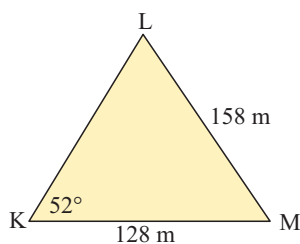
By the sine rule, $\quad \dfrac{\sin L}{128} = \dfrac{\sin 52^o}{158}$

$$\therefore \quad \sin L = \frac{128 \times \sin 52^o}{158}$$

$$\text{Now} \quad \sin^{-1}\left(\frac{128 \times \sin 52^o}{158}\right) \approx 39.7^o$$

$$\therefore \quad L \approx 39.7^o \quad \text{or} \quad (180 - 39.7)^o \approx 140.3^o$$

But KM < LM, so we know angle L < angle K. Hence $L \approx 39.7^o$.

EXERCISE 29C.2

1 Find the value of θ:

a

14.8 m   17.5 m   38° $\theta°$

b

$\theta°$ 54°   29 cm   35 cm

c

6.4 km   2.4 km $\theta°$ 15°

2 Solve for x:

a

8 m   6 m   20° $x°$

b

9 cm x cm   120°   25°

c

6 km   25°   30° x km

d

$x°$ 7 m   5 m   50°

e

110°   10 m x m

f

$x°$ 12 cm   10 cm   35°
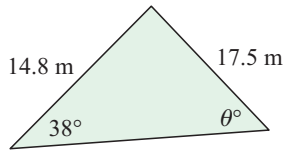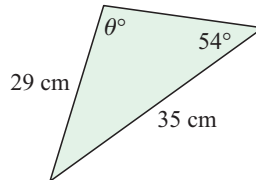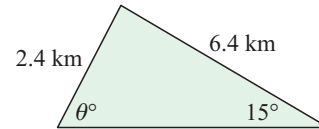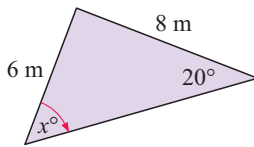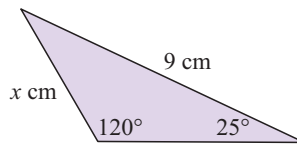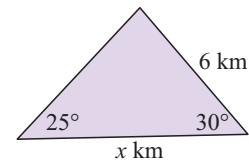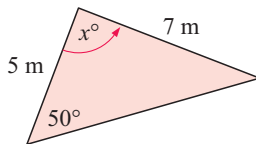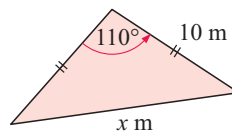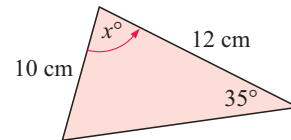
3 In triangle ABC, find the measure of:

    a angle A if $a = 12.6$ cm, $b = 15.1$ cm   and $A\widehat{B}C = 65^o$

    b angle B if $b = 38.4$ cm, $c = 27.6$ cm   and $A\widehat{C}B = 43^o$

    c angle C if $a = 5.5$ km, $c = 4.1$ km   and $B\widehat{A}C = 71^o$.

4 In triangle ABC, angle $A = 100^o$, angle $C = 21^o$, and AB $= 6.8$ cm. Find the length of side AC.

5 In triangle PQR, angle $Q = 98^o$, PR $= 22$ cm, and PQ $= 15$ cm. Find the size of angle R.

> There may be two possible solutions. A sketch may help to find them.

D   THE COSINE RULE   [8.5]

THE COSINE RULE

In any triangle ABC with sides a, b and c units and opposite angles A, B and C respectively,

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C.$$
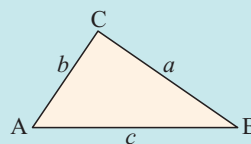
Proof (for a triangle with acute angle A):

Consider triangle ABC shown.

Using Pythagoras' theorem, we find

$$b^2 = h^2 + x^2, \quad \text{so} \quad h^2 = b^2 - x^2$$
$$\text{and} \quad a^2 = h^2 + (c - x)^2$$
$$\text{Thus,} \quad a^2 = (b^2 - x^2) + (c - x)^2$$
$$\therefore \quad a^2 = b^2 - x^2 + c^2 - 2cx + x^2$$
$$\therefore \quad a^2 = b^2 + c^2 - 2cx \quad \text{......} \ (1)$$
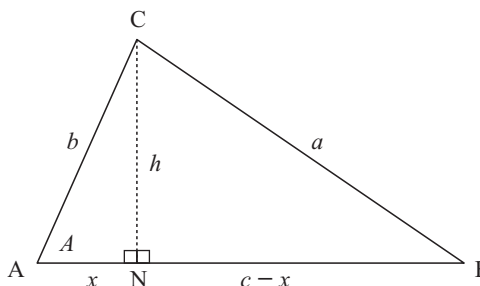
But in ΔACN, $\cos A = \dfrac{x}{b}$ and so $x = b \cos A$

So, in (1), $a^2 = b^2 + c^2 - 2bc \cos A$

Similarly, we can show the other two equations to be true.

Challenge: Prove the **Cosine Rule** $a^2 = b^2 + c^2 - 2bc \cos A$,   in the case where A is an obtuse angle.

You will need to use $\cos(180^o - \theta) = -\cos \theta$.

We use the **cosine rule** when we are given:

- **two sides** and the **included angle** between them,   or
- **three sides**.

GEOMETRY PACKAGE

Useful rearrangements of the cosine rule are:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

They can be used if we are given all three side lengths of a triangle.

Example 8   ◀)) *Self Tutor*

Find, correct to 2 decimal places, the length of BC.
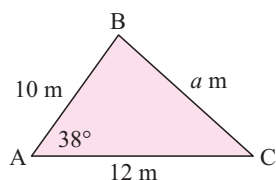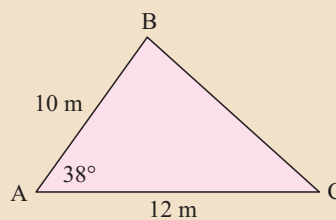
By the cosine rule:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$\therefore \quad a = \sqrt{12^2 + 10^2 - 2 \times 12 \times 10 \times \cos 38^o}$$
$$\therefore \quad a \approx 7.41$$

$\therefore$ BC is 7.41 m in length.

Example 9

🔊 *Self Tutor*

Find the size of $A\widehat{B}C$ in the given figure.

Give your answer correct to 1 decimal place.
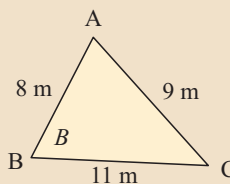
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\therefore \quad \cos B = \frac{11^2 + 8^2 - 9^2}{2 \times 11 \times 8}$$

$$\therefore \quad B = \cos^{-1}\left(\frac{11^2 + 8^2 - 9^2}{2 \times 11 \times 8}\right)$$

$$\therefore \quad B \approx 53.8^o$$

So, $A\widehat{B}C$ measures about 53.8^o.

EXERCISE 29D

1 Find the value of x in:

a
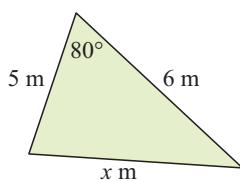
b
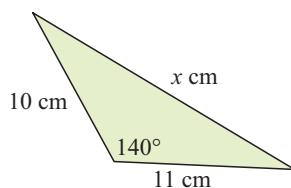
c
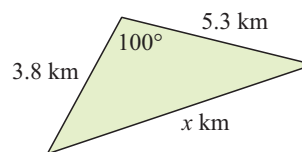
d
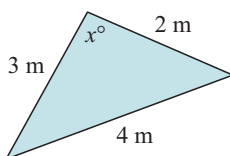
e
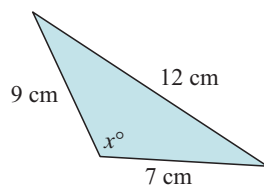
f
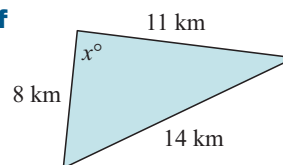
2 Find the length of the remaining side in the given triangle:

a
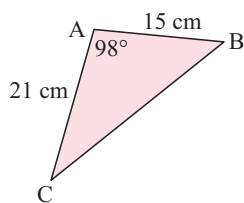
b
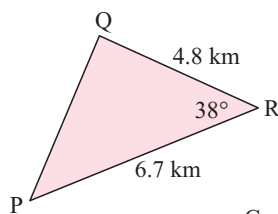
c
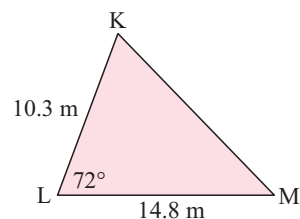
3 Find the measure of all angles of:
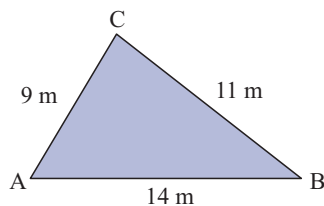
4 Find:

    a the smallest angle of a triangle with sides 9 cm, 11 cm and 13 cm

    b the largest angle of a triangle with sides 3 cm, 5 cm and 7 cm.

5
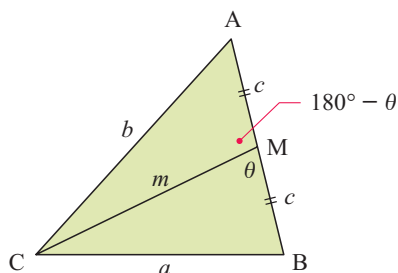
    a Use the cosine rule in triangle BCM to find $\cos\theta$ in terms of a, c and m.

    b Use the cosine rule in triangle ACM to find $\cos(180^o - \theta)$ in terms of b, c and m.

    c Use the fact that $\cos(180^o - \theta) = -\cos\theta$ to prove **Apollonius' median theorem**:
$$a^2 + b^2 = 2m^2 + 2c^2.$$

    d Hence find x in the following:

        i
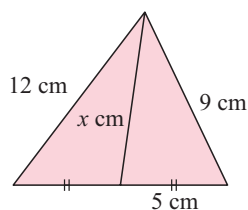
        ii
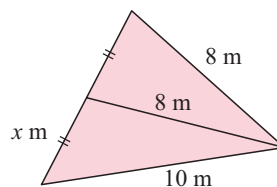
6 In triangle ABC, AB $= 10$ cm, AC $= 9$ cm and $\widehat{ABC} = 60^o$. Let BC $= x$ cm.

    a Use the cosine rule to show that x is a solution of $x^2 - 10x + 19 = 0$.

    b Solve the above equation for x.

    c Use a scale diagram and a compass to explain why there are two possible values of x.

7 Find, correct to 3 significant figures, the area of:

    a
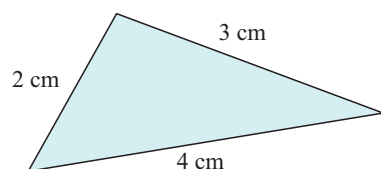
    b
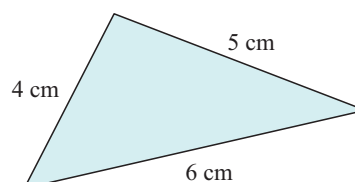
E   PROBLEM SOLVING WITH THE SINE AND COSINE RULES   [8.4, 8.5, 8.7]
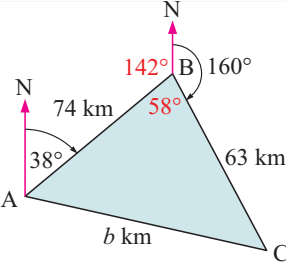
Whenever there is a choice between using the sine rule or the cosine rule, always use the **cosine rule** to avoid the ambiguous case.

Example 10   ◄⑴ Self Tutor

An aircraft flies 74 km on a bearing 038° and then 63 km on a bearing 160°.
Find the distance of the aircraft from its starting point.

By the cosine rule,

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$\therefore \quad b^2 = 63^2 + 74^2 - 2 \times 63 \times 74 \times \cos 58^o$$
$$\therefore \quad b^2 \approx 4504.03$$
$$\therefore \quad b \approx 67.1$$

$\therefore$ the aircraft is 67.1 km from its starting point.

EXERCISE 29E

1 Two farm houses A and B are 10.3 km apart. A third farm house C is located such that $\ B\widehat{A}C = 83^o$ and $\ A\widehat{B}C = 59^o$. How far is C from A?

2 A roadway is horizontal for 524 m from A to B, followed by a 23° incline 786 m long from B to C. How far is it directly from A to C?
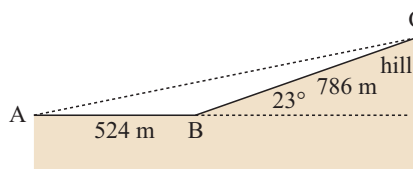
3 Towns A, B and C are located such that $\ B\widehat{A}C = 50^o$ and B is twice as far from C as A is from C. Find the measure of $B\widehat{C}A$.

4 Hazel's property is triangular with dimensions as shown in the figure.

     a Find the measure of the angle at A, correct to 2 decimal places.

     b Hence, find the area of her property correct to the nearest hectare.

5 An aeroplane flies from Geneva on a bearing of 031° for 200 km. It then changes course and flies for 140 km on a bearing of 075°. Find:

     a the distance of Geneva from the aeroplane

     b the bearing of Geneva from the aeroplane.

6 A ship sails northeast for 20 km and then changes direction, sailing on a bearing of 250° for 12 km. Find:

     a the distance of the ship from its starting position

     b the bearing it must take to return directly to its starting position.

7 An orienteer runs for 450 m then turns through an angle of 32° and runs for another 600 m. How far is she from her starting point?

8 A yacht sails 6 km on a bearing 127° and then 4 km on a bearing 053°. Find the distance and bearing of the yacht from its starting point.

9 Mount X is 9 km from Mount Y on a bearing 146°. Mount Z is 14 km away from Mount X and on a bearing 072° from Mount Y. Find the bearing of X from Z.

10 A parallelogram has sides of length 8 cm and 12 cm. Given that one of its angles is 65°, find the lengths of its diagonals.

11 Calculate the length of a side of a regular pentagon whose vertices lie on a circle with radius 12 cm.
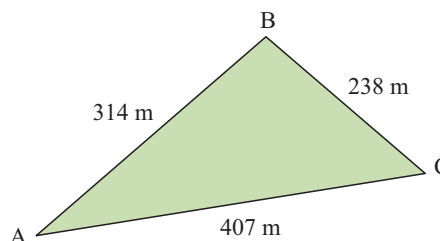
12   X is 20 km north of Y. A mobile telephone mast M is to be placed 15 km from Y so the bearing of M from X is 140°.

    a   Draw a sketch to show the two possible positions where the mast could be placed.

    b   Calculate the distance of each of these positions from X.

13   Bushwalkers leave point P and walk in the direction 238° for 11.3 km to point Q. At Q they change direction to 107° and walk for 18.9 km to point R. How far is R from the starting point P?

14
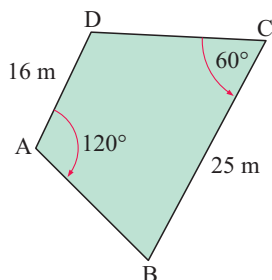
David's garden plot is in the shape of a quadrilateral. If the corner points are A, B, C and D then the angles at A and C are 120° and 60° respectively. AD = 16 m, BC = 25 m, and DC is 5 m longer than AB. A fence runs around the entire boundary of the plot. How long is the fence?

F   TRIGONOMETRY WITH COMPOUND SHAPES
[8.1, 8.4, 8.5, 8.7]

Example 11
◄)) Self Tutor

AD is a vertical mast and CE is a vertical flagpole.
Angle ABD is 30° and angle EBC is 50°.

Calculate:

    a   the length of DE

    b   the size of angle EDB.

a   angle DBE $= 180^o - 30^o - 50^o$
$= 100^o$
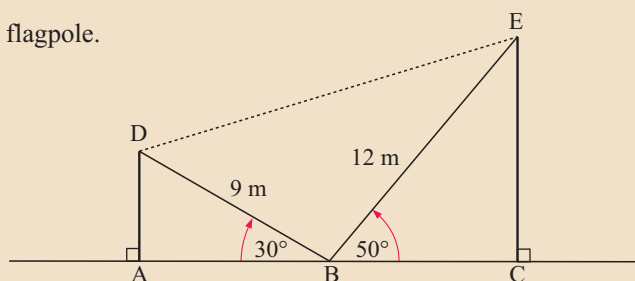
$\therefore$ $DE^2 = 9^2 + 12^2 - 2(9)(12)\cos 100^o$
{by the Cosine Rule}

$\therefore$ DE ≈ 16.202 m

$\therefore$ DE ≈ 16.2 m

b
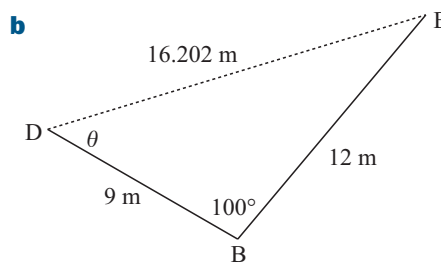
Using the sine rule: $\dfrac{\sin\theta}{12} \approx \dfrac{\sin 100^o}{16.202}$

$\therefore$ $\sin\theta \approx \dfrac{12 \times \sin 100^o}{16.202}$

$\therefore$ $\theta \approx 46.8^o$

$\therefore$ angle EDB is about 46.8^o.

EXERCISE 29F

1 A 40 m high tower is 8 m wide. Two students A and B are
on opposite sides of the top of the tower. They measure the
angles of depression to their friends at C and D to be 54° and
43° respectively. How far are C and D apart if A, B, C and
D are all in the same plane?
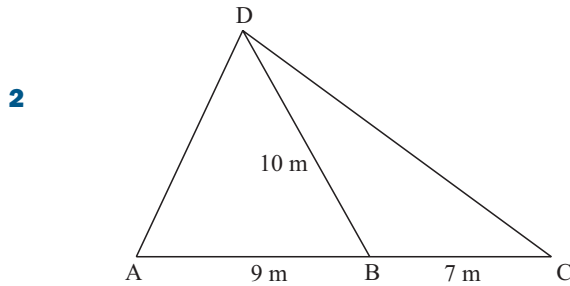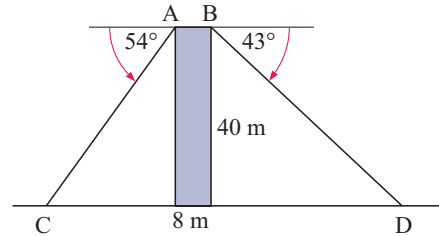
The area of triangle ABD is 33.6 m^2.
Find the length of CD.

2

3 Find x:
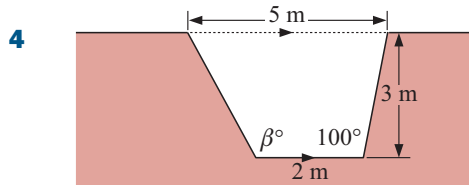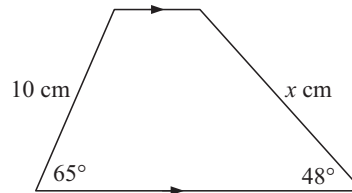
4 A stormwater drain is to have the shape shown. Determine
the angle the left hand side makes with the bottom of the
drain.

5 From points A and B at sea, the angles of elevation to
the top of the mountain T are 37° and 41° respectively.
A and B are 1200 m apart.
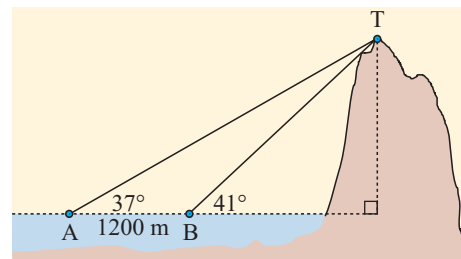
   a What is the size of $\widehat{ATB}$?

   b Find the distance from A to T.

   c Find the distance from B to T.

   d Find the height of the mountain.

   e Use the given figure to show that

$$d = h\left(\frac{1}{\tan\theta} - \frac{1}{\tan\phi}\right).$$

   f Use **e** to check your answer to **d**.
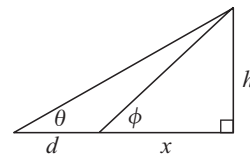
6 Find x and y in the given figure.
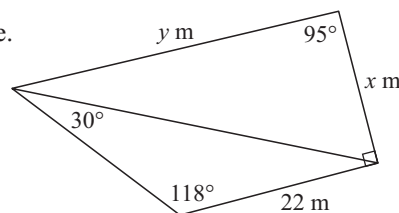
7 Jane and Peter are considering buying a block of land. The land agent supplies them with the given accurate sketch. Find the area of the property, giving your answer in:

   a m^2 **b** hectares.

8   **Plan of garden**
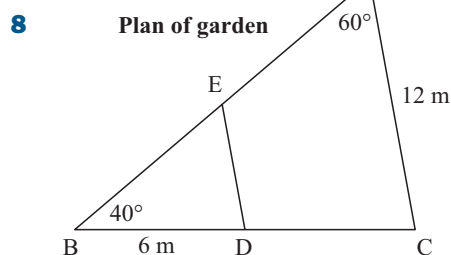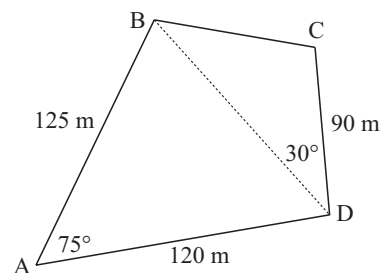
In the given plan view, AC = 12 m, angle BAC = 60°, and angle ABC = 40°. D is a post 6 m from corner B, E is another post, and BDE is a lawn of area 13.5 m^2.

   a Calculate the length of DC.

   b Calculate the length of BE.

   c Find the area of ACDE.

G | *TRIGONOMETRIC GRAPHS*   [3.2, 8.8]

GRAPHS FROM THE UNIT CIRCLE

The diagram alongside gives the y-coordinates for all points on the unit circle at intervals of 30°.
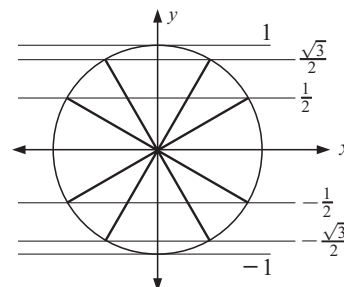
A table for $y = \sin \theta$ can be constructed from these values:

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = \sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0

Plotting $\sin \theta$ against θ gives:
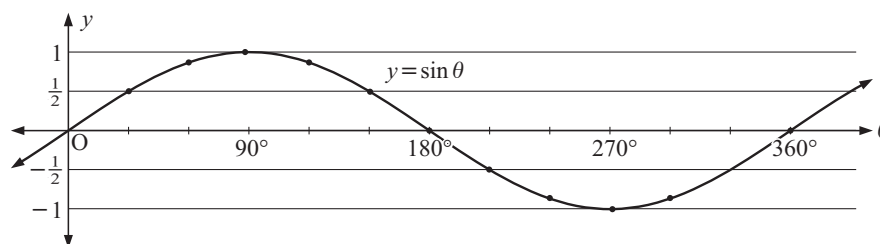
EXERCISE 29G.1

1   **a**   By finding x-coordinates of points on the unit circle, copy and complete:

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = \cos\theta$													

     b   Use **a** to graph $y = \cos\theta$ for $0^o \leqslant \theta \leqslant 360^o$,   making sure the graph is fully labelled.

     c   What is the maximum value of $\cos\theta$ and when does it occur?

     d   What is the minimum value of $\cos\theta$ and when does it occur?

2   **a**   By using $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$,   copy and complete:

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = \tan\theta$	0	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	und.									

     b   Find the equations of the vertical asymptotes of $y = \tan\theta$ for $0^o \leqslant \theta \leqslant 360^o$.

     c   Use **a** and **b** to graph $y = \tan\theta$ for $0^o \leqslant \theta \leqslant 360^o$.

PROPERTIES OF BASIC TRIGONOMETRIC GRAPHS

Click on the icon to see how the graphs of $y = \sin\theta$, $y = \cos\theta$ and $y = \tan\theta$ are generated from the unit circle.

DEMO
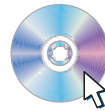
Before we consider these graphs in more detail, we need to learn appropriate language for describing them.

TERMINOLOGY
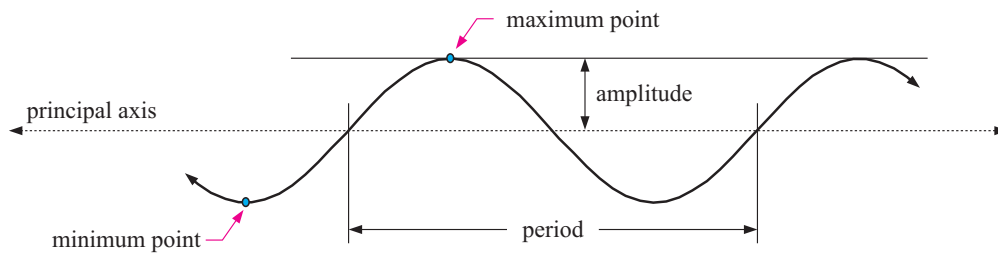
- A **periodic function** is one which repeats itself over and over in a horizontal direction.
- The **period** of a periodic function is the length of one repetition or cycle.
- The graph oscillates about a horizontal line called the **principal axis** or **mean line**.
- A **maximum point** occurs at the top of a crest.
- A **minimum point** occurs at the bottom of a trough.
- The **amplitude** is the vertical distance between a maximum or minimum point and the principal axis.

THE GRAPH OF $y = \sin x$

Instead of using θ, we now use x to represent the angle variable.
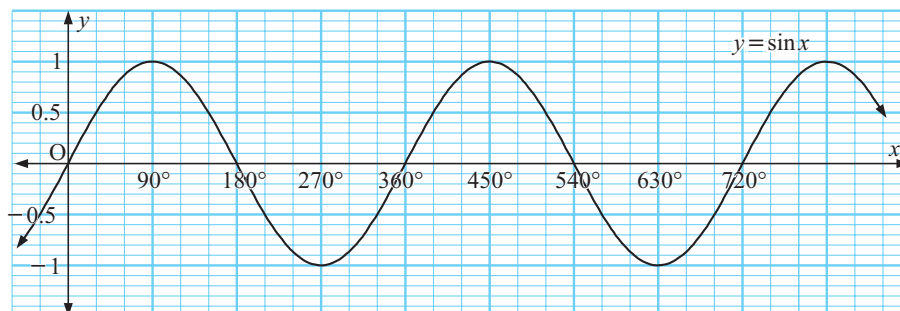
The **sine graph** has the following properties:

- it is continuous, which means it has no breaks
- its range is $\{y \mid -1 \leqslant y \leqslant 1, \ y \in \mathbb{R}\}$
- it passes through the origin and continues indefinitely in both directions
- its amplitude is 1
- its period is 360°.
- it has lines of symmetry $x = \pm 90^o, \ x = \pm 270^o, \ x = \pm 450^o, \ ......$
- it has points of rotational symmetry on the x-axis at $0^o, \pm 180^o, \pm 360^o, \pm 540^o, \pm 720^o, ...$

DEMO

THE GRAPH OF $y = \cos x$
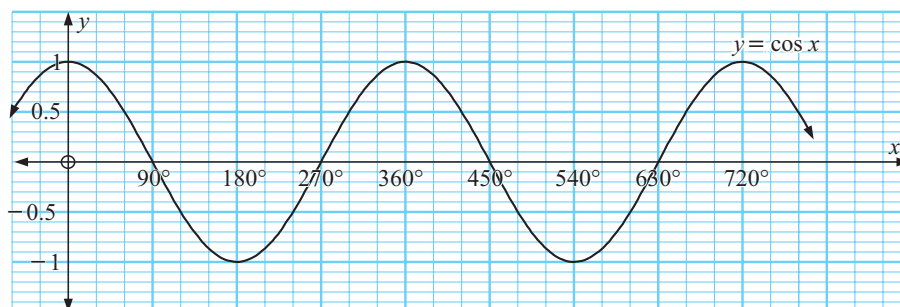
The **cosine graph** has the following properties:

- it is continuous
- its range is $\{y \mid -1 \leqslant y \leqslant 1, \ y \in \mathbb{R}\}$
- its y-intercept is 1
- its amplitude is 1
- its period is 360°
- it has exactly the same shape as the sine graph, but is translated 90° to the left, or with vector $\begin{pmatrix} -90^o \\ 0 \end{pmatrix}$
- it has vertical lines of symmetry $x = 0^o, \ x = \pm 180^o, \ x = \pm 360^o, \ x = \pm 540^o, \ ......$
- it has points of rotational symmetry on the x-axis at $\pm 90^o, \ \pm 270^o, \ \pm 450^o, \ \pm 630^o, \ ......$

DEMO

THE GRAPH OF $y = \tan x$

The vertical lines $x = \pm 90^o$, $x = \pm 270^o$, $x = \pm 450^o$, and so on, are all vertical asymptotes.
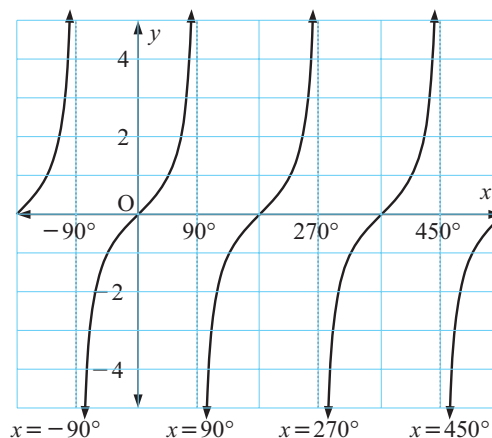
The tangent curve has the following properties:

- it is not continuous at $\pm 90^o$, $\pm 270^o$, $\pm 450^o$, and so on; these are the values of x where $\cos x = 0$
- its range is $\{y \mid y \in \mathbb{R}\}$
- it passes through the origin
- its period is 180°
- it has points of rotational symmetry on the x-axis at 0°, $\pm 180^o$, $\pm 360^o$, ...

DEMO

EXERCISE 29G.2

1 Use the graphs of $y = \sin x$ and $y = \cos x$ to find:

   a $\sin 150^o$ **b** $\sin 210^o$ **c** $\cos 120^o$ **d** $\cos 300^o$

2   **a** Use the graph of $y = \sin x$ to find all angles between 0° and 720° which have the same sine as:

       i 50° **ii** 45° **iii** 10°

   b Use the graph of $y = \cos x$ to find all angles between 0° and 720° which have the same cosine as:

       i 25° **ii** 90° **iii** 70°

   c Use the graph of $y = \tan x$ to find all angles between 0° and 600° which have the same tan as:

       i 20° **ii** 55° **iii** 80°

3 Use the graph of $y = \sin x$ and your calculator to solve these equations for $0^o \leqslant x \leqslant 720^o$. Give your answers correct to the nearest degree.

   a $\sin x = 0$ **b** $\sin x = 0.3$ **c** $\sin x = 0.8$ **d** $\sin x = -0.4$

4 Use the graph of $y = \cos x$ and your calculator to solve these equations for $0^o \leqslant x \leqslant 720^o$. Give your answers correct to the nearest degree.

   a $\cos x = 1$ **b** $\cos x = 0.7$ **c** $\cos x = 0.2$ **d** $\cos x = -0.5$

5 Use the graph of $y = \tan x$ and your calculator to solve these equations for $0^o \leqslant x \leqslant 600^o$. Give your answers correct to the nearest degree.

   a $\tan x = 3$ **b** $\tan x = -2$ **c** $\tan x = 10$.

6   **a** Draw the graphs of $y = \cos x$ and $y = \sin x$ on the same set of axes for $0^o \leqslant x \leqslant 720^o$.

   b Find all values of x on this domain such that $\cos x = \sin x$.

7 Plot on the same set of axes for $0 \leqslant x \leqslant 360^o$:

     a $y = \sin x$ **b** $y = \sin x + 2$ **c** $y = \sin(x + 90^o)$ **d** $y = \cos x$

     What do you notice?

8 Plot on the same set of axes for $0 \leqslant x \leqslant 360^o$:

     a $y = \cos x$ **b** $y = \cos x - 2$ **c** $y = \cos(x - 90^o)$ **d** $y = \sin x$

     What do you notice?

H   GRAPHS OF $y = a\sin(bx)$ AND $y = a\cos(bx)$ [3.2, 3.3, 8.8]

Discovery   The families $y = a\sin(bx)$ and $y = a\cos(bx)$

In this discovery we consider different transformations of the basic sine and cosine functions.

GRAPHING PACKAGE

What to do:

1 Use the graphing package or a graphics calculator to graph on the same set of axes:

     a $y = \sin x$ **b** $y = 2\sin x$ **c** $y = \frac{1}{2}\sin x$

     d $y = -\sin x$ **e** $y = -\frac{1}{3}\sin x$ **f** $y = -\frac{3}{2}\sin x$

2 All of the graphs in **1** have the form $y = a\sin x$.
Comment on the significance of:

     a the sign of a **b** the size of a, which is $|a|$.

3 Use the graphing package to graph on the same set of axes:

     a $y = \sin x$ **b** $y = \sin 2x$ **c** $y = \sin\left(\frac{1}{2}x\right)$ **d** $y = \sin 3x$

4 All of the graphs in **3** have the form $y = \sin(bx)$ where $b > 0$.

     a Does b affect the: **i** amplitude   **ii** period?

     b What is the period of $y = \sin(bx)$, $b > 0$?

5 Repeat **1** to **4** above, replacing sin by cos.

You should have observed that for $y = a\sin(bx)$ and $y = a\cos(bx)$:

- a affects the **amplitude** of the graph. It provides a stretch with invariant x-axis and scale factor a.
- b affects the **period** of the graph. It provides a stretch with invariant y-axis and scale factor b.
- the amplitude is $|a|$ and the period is $\dfrac{360^o}{b}$.

Example 12
🔊 *Self Tutor*

Draw free-hand sketches of $y = \sin x$, $y = 3\sin x$ and $y = 3\sin(2x)$ for $0^o \leqslant x \leqslant 720^o$.

EXERCISE 29H

1 Draw free-hand sketches of the following for $0^o \leqslant x \leqslant 720^o$:

  a $y = \sin x$ and $y = \sin(2x)$ **b** $y = \sin x$ and $y = 2\sin x$

  c $y = \sin x$ and $y = -\sin x$ **d** $y = \sin x$ and $y = \sin\left(\frac{1}{2}x\right)$

  e $y = \sin x$ and $y = 3\sin x$ **f** $y = \sin x$ and $y = 3\sin\left(\frac{1}{2}x\right)$

  g $y = \sin x$ and $y = 2\sin(3x)$ **h** $y = \sin x$ and $y = -\sin(2x)$

PRINTABLE WORKSHEET
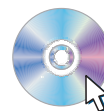
2 Draw free-hand sketches of the following for $0^o \leqslant x \leqslant 720^o$:

  a $y = \cos x$ and $y = 2\cos x$ **b** $y = \cos x$ and $y = \cos\left(\frac{1}{2}x\right)$

  c $y = \cos x$ and $y = -\cos x$ **d** $y = \cos x$ and $y = 3\cos(2x)$

  e $y = \cos x$ and $y = -2\cos x$ **f** $y = \cos x$ and $y = \frac{1}{2}\cos(3x)$

3 Use your calculator to solve correct to 1 decimal place, for $0^o \leqslant x \leqslant 360^o$:

  a $\sin x = 0.371$ **b** $\cos x = -0.673$ **c** $\sin(2x) = 0.4261$

  d $\tan x = 4$ **e** $\cos(3x) = \frac{1}{3}$ **f** $\sin\left(\frac{x}{2}\right) = 0.9384$

4 Find a and b if $y = a\sin(bx)$ has graph:

  a

  b

  c

  d

IGCSE01

e

f
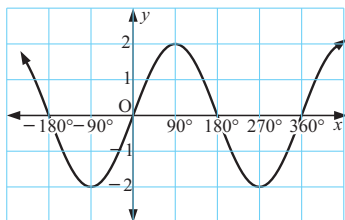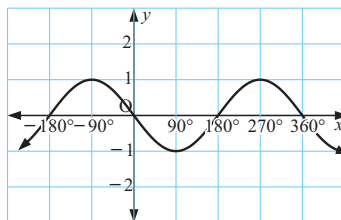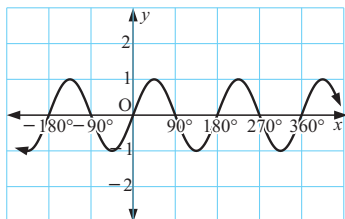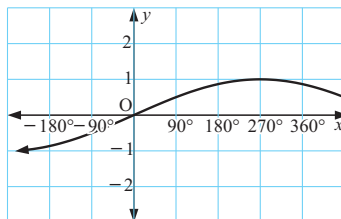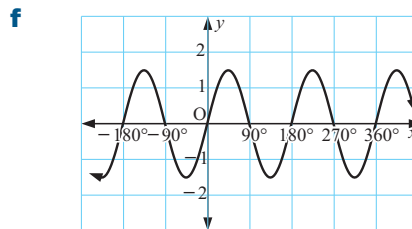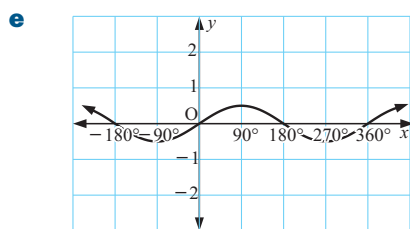
5 Find a and b if $y = a\cos(bx)$ has graph:

a

b

c

d

e

f
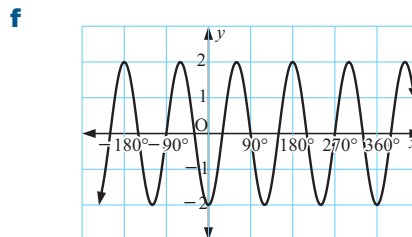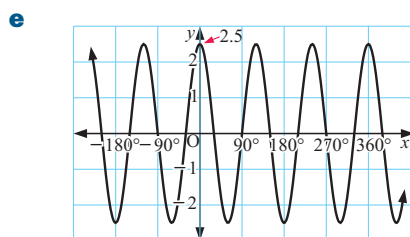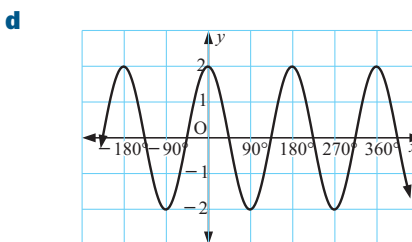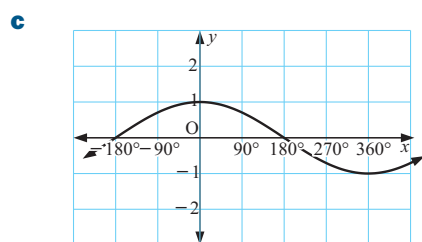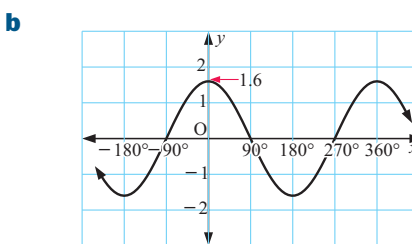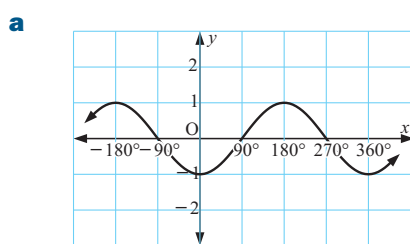
Review set 29A

1 Find:

   a the exact coordinates of point P

   b the coordinates of P correct to 3 decimal places.
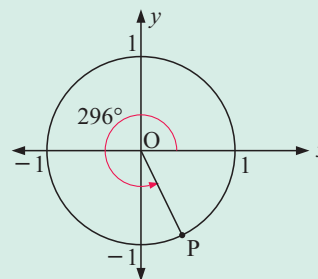
2 Use the unit circle to find the exact values of:

   a $\sin 120^o$

   b $\cos 360^o$

   c $\tan 330^o$
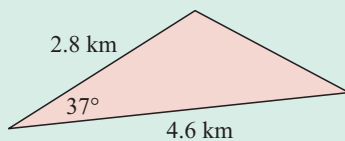
3 Find the area of:

4 The area of triangle ABC is 21 m^2. Find x.
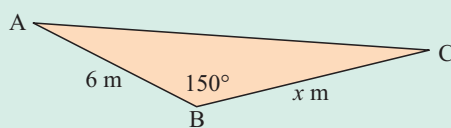
5 Use the diagram alongside to write, in terms of a and b, a value for:

    a $\cos\theta$ **b** $\sin\theta$

    c $\sin(180^o - \theta)$ **d** $\cos(180^o - \theta)$
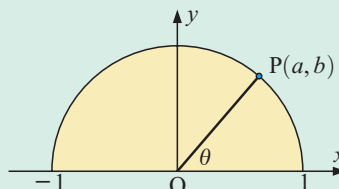
6 Two long straight roads intersect at P at an angle of 53°. Starting at P, cyclist A rides for 16.2 km along one of the roads, while cyclist B rides 18.9 km along the other road. How far apart are the cyclists now? Assume the angle between the paths of the cyclists is acute.

7 Find, correct to 3 significant figures, the values of the unknowns in:
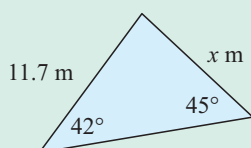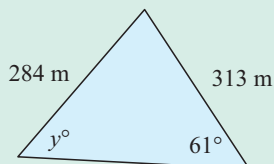
    a   **b**   **c**
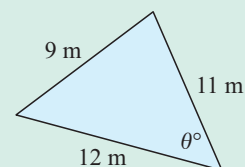
8 Triangle ABC has $\widehat{ABC} = 48^o$, $AB = 10$ cm, and $AC = 8$ cm. Show that $\widehat{ACB}$ has two possible sizes. Give each answer correct to 3 significant figures.
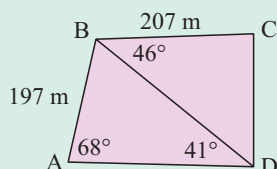
9   Find:

                                            a   the length of BD

                                            b   the total area of quadrilateral ABCD.

10 Draw sketch graphs of $y = \cos x$ and $y = -\cos(2x)$ on the same axes for $0^o \leqslant x \leqslant 720^o$.

11 Find a and b given the graph either has equation $y = a\cos(bx)$ or $y = a\sin(bx)$. State clearly which of these two functions is illustrated.
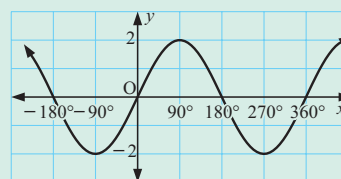
Review set 29B

1 Find:

    a   the exact coordinates of point Q

    b   the coordinates of Q correct to 4 decimal places.
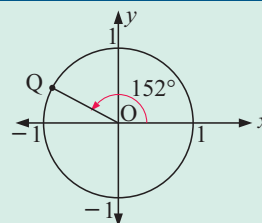
2
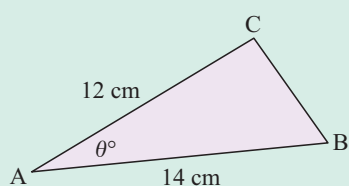
Triangle ABC has acute angle θ^o at vertex A. Find θ correct to 1 decimal place if the area of triangle ABC is 33.4 cm².

3 **a** State the coordinates of point Q in terms of a and b.

   b State the coordinates of point Q in terms of θ.

   c Explain why $\cos(180^o + \theta) = -\cos\theta$.
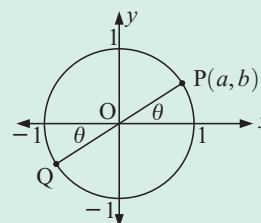
4 Find, correct to 3 significant figures, the value of the unknowns in:

   a   **b**   **c**
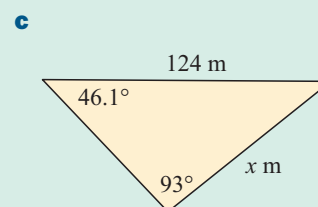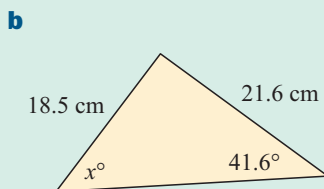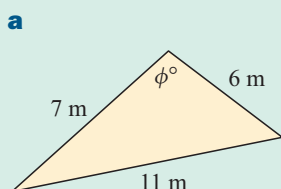
5 Jason's sketch of his father's triangular vegetable patch is shown alongside. Find:

   a the length of the fence AB

   b the area of the patch in hectares.
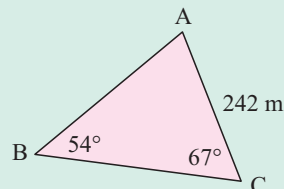
6 Triangle ABC has   AB = 12 m,   BC = 10 m,   AC = x m,   and $\widehat{BAC} = 40^o$.
    Show that there are two possible values for x.

7 A ship leaves port P and travels for 50 km in the direction 181°. It then sails 60 km in the direction 274° to an island port Q.

   a How far is Q from P?

   b To sail back directly from Q to P, in what direction must the ship sail?

8 Find x:
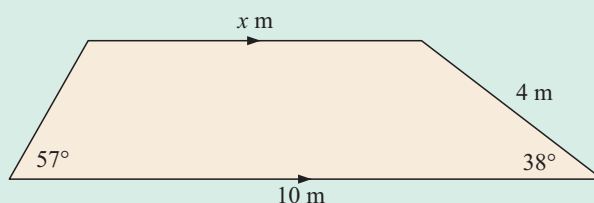
9 On the same set of axes for $0^o \leqslant x \leqslant 720^o$, sketch the graphs of $y = \cos x$, $y = \cos(2x)$ and $y = \frac{1}{2}\cos(2x)$.

10
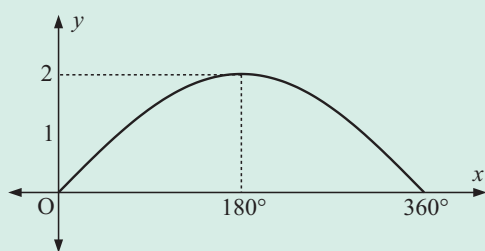
The illustrated graph has equation of the form
$y = a \sin(bx)$.
Find a and b.

11 Use your calculator to find all values of x for $0^o < x < 720^o$ for which $\cos\left(\frac{x}{2}\right) = 0.787$.
Give your answers correct to the nearest degree.

Challenge

1 A rocket is fired vertically above the North Pole.
How high must the rocket rise if the line of sight to the horizon makes an angle of 30° with the path of the rocket?

2 **a** Use a calculator to find the value of $\tan 60^o$ and $\tan 75^o$.
What do you notice?

    b Consider the triangle alongside:

        i Find all angle sizes within the figure.

        ii Find the lengths of BD, BC and AB.

        iii Use the figure to establish why your result of **a** is true.
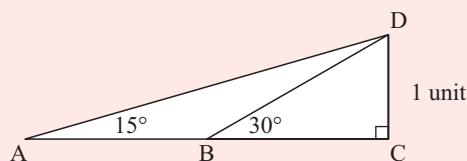
3
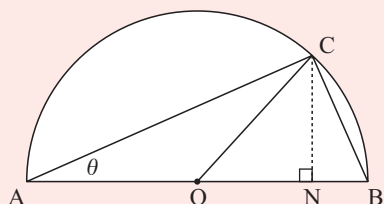
Consider the given figure.

    a Find, giving reasons, the size of $\widehat{\text{CON}}$ in terms of θ where $\widehat{\text{CAO}} = \theta$.

    b Let $AO = CO = BO = r$, $ON = a$, $CN = h$ and $AC = x$. Show that $\sin(2\theta) = \dfrac{h}{r}$.

    c Prove that $x^2 = 2r(a + r)$.

    d Find, in simplest form, $2\sin\theta\cos\theta$ in terms of the given variables.

    e What has been discovered from **b** and **d**?

    f By substituting $\theta = 23^o$, 48.6^o and 71.94^o, verify your discovery of **e**.

4 Triangle ABC has sides of length a, b and c units. A circle of radius r is drawn through the vertices of the triangle.

Show that the area of the triangle is given by the formula $A = \dfrac{abc}{4r}$.

Variation and power modelling

30

Contents:

Opening problem

A shop sells cans of soft drink for €2. Suppose we buy n cans and the total cost is €C.
To study the relationship between the two variables *number of cans* and *total cost*, we can use a table of values or a graph.

n	0	1	2	3	4	5
C	0	2	4	6	8	10

The graph of C against n consists of discrete points because we can only buy a whole number of cans. However, an imagined line passing through these points would also pass through the origin.
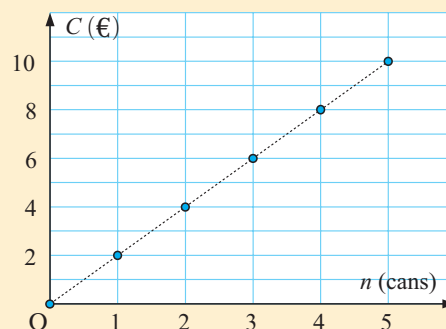
Things to think about:

- Which of the following are true:
  - ▸ doubling the number of cans doubles the total cost
  - ▸ halving the number of cans halves the total cost
  - ▸ increasing the number of cans by 30% increases the cost by 30%?

- How can we describe the relationship between n and C?

A   DIRECT VARIATION   [2.13]

> Two variables are **directly proportional** if multiplying one of them by a number
> results in the other one being multiplied by the same number.

In the **Opening Problem**, the variables C and n are *directly proportional*.

We say that C *is directly proportional* to n, and write $C \propto n$.

The variables are connected by the formula or law $C = 2n$ since 2 is the gradient of the line.

> If two quantities x and y are **directly proportional**, we write $y \propto x$.
>
> The symbol $\propto$ reads   *is directly proportional to*   or   *varies directly as*.
>
> If $y \propto x$ then $y = kx$ where k is a constant called the **proportionality constant**.
>
> When y is graphed against x then k is the **gradient** of the graph, and the line passes through the **origin**.

Example 1   ◀ᴼ) Self Tutor

Fruit buns cost 60 cents each. Suppose x buns are bought and the total price is \$$y$.

a   Show, by graphical means, that y is directly proportional to x.

b   Find: **i**   the proportionality constant   **ii**   the law connecting x and y.

a

x	0	1	2	3	4	5
y	0	0.60	1.20	1.80	2.40	3.00

Since the graph is a straight line passing through
$(0,\ 0)$, $y \propto x$.

b   **i**   Gradient $= \dfrac{0.60 - 0}{1 - 0} = 0.6$

So, the proportionality constant k is 0.6.

   ii   The law connecting x and y is $y = 0.6x$.

Example 2   ◀ᴼ) Self Tutor

Suppose $y \propto n$ and $y = 40$ when $n = 3$. Find n when $y = 137$.

Method 1:

Since $y \propto n$, $y = kn$ where k is the
proportionality constant.

Since $n = 3$ when $y = 40$, $40 = k \times 3$

$\qquad\qquad\qquad \therefore \quad \frac{40}{3} = k$

$\qquad\qquad\qquad \therefore \quad y = \frac{40}{3}n$

So, when $y = 137$, $137 = \frac{40}{3}n$

$\qquad\qquad \therefore \quad 137 \times \frac{3}{40} = n$

$\qquad\qquad\qquad \therefore \quad n \approx 10.3$

Method 2:

n	3	?
y	40	137

$\times \frac{137}{40}$

To change y from 40 to 137, we multiply by $\frac{137}{40}$.

Since $y \propto n$, we also multiply n by $\frac{137}{40}$

$\therefore \quad n = 3 \times \frac{137}{40} \approx 10.3$

EXERCISE 30A.1

1

h	0	1	2	3	4	5
W	0	12.50	25.00	37.50	50.00	62.50

The table alongside shows the total wages £W earned for working h hours.

    a Draw a graph of W against h.

    b Are W and h directly proportional? Explain your answer using the features of your graph.

    c If W and h are directly proportional, determine:

        i the proportionality constant   **ii** the law connecting W and h.

2 If y is directly proportional to x, state what happens to:

    a y if x is doubled   **b** y if x is trebled

    c x if y is doubled   **d** x if y is halved

    e y if x is increased by 20%   **f** x if y is decreased by 30%

    g y if 2 is added to x **h** y if 3 is subtracted from x.

3 Which of the following graphs indicate that y is directly proportional to x?

4 The law connecting the circumference C and radius r of a circle is $C = 2\pi r$.

    a Explain why $C \propto r$.

    b Find the proportionality constant.

    c What happens to: **i** C if r is doubled   **ii** r if C is increased by 50%?

5 If $y \propto x$ and $y = 123$ when $x = 7.1$, find:

    a y when $x = 13.2$ **b** x when $y = 391$.

6 The resistance R ohms to the flow of electricity in a wire varies in direct proportion to the length l cm of the wire. When the length is 10 cm, the resistance is 0.06 ohms. Find:

    a the law connecting R and l

    b the resistance when the wire is 50 cm long

    c the length of wire which has a resistance of 3 ohms.

7 A 4 litre can of paint will cover 18 m² of wall. If n is the number of litres and A is the area covered, write down a formula connecting A and n. Hence, find the number of litres required to paint a room with wall area 40.5 m².

8 The speed of a falling object is directly proportional to the time it falls. The speed after 5 seconds is 49 m/s.

    a What will be the speed of a falling object after 8 seconds?

    b How long will it take a falling object to reach a speed of 100 m/s?

OTHER DIRECT VARIATION

The formula for finding the area A of a circle of radius r, is $A = \pi r^2$.

r	0	1	2	3	4
A	0	3.14	12.57	28.27	50.27

If we graph A against r we get the graph alongside. The graph is not a straight line, but rather is part of a **parabola**.

However, if we graph A against r^2 we get this graph:

r	0	1	2	3
r^2	0	1	4	9
A	0	3.14	12.57	28.27

Since the graph of A against r^2 is a straight line through the origin O, A is directly proportional to r^2.

We write $A \propto r^2$, and so $A = kr^2$ where k is the proportionality constant. In this case we know $k = \pi$.

Notice from the table that as r is doubled from 1 to 2, r^2 is multiplied by 4. So, A is also increased by a factor of 4.

Example 3
🔊 **Self Tutor**

Consider the table of values:

x	2	4	6	8
y	6	24	54	a

a By finding $\dfrac{y}{x^2}$ for each point, establish that $y \propto x^2$.

b Write down the rule connecting y and x.

c Find the value of a.

a When $x = 2$,   When $x = 4$,   When $x = 6$,

$$\frac{y}{x^2} = \frac{6}{2^2} = 1.5 \qquad \frac{y}{x^2} = \frac{24}{4^2} = 1.5 \qquad \frac{y}{x^2} = \frac{54}{6^2} = 1.5$$

$\dfrac{y}{x^2} = 1.5$ in each case, and so $y = 1.5x^2$. Hence $y \propto x^2$.

b $y = 1.5x^2$

c When $x = 8$, $y = a$ $\therefore$ $a = 1.5 \times 8^2 = 96$.

Example 4 ◄» *Self Tutor*

Consider $y = \dfrac{x^4}{5}$. State which two variables are directly proportional and determine the proportionality constant k.

Since $y = \dfrac{x^4}{5}$, $y = \frac{1}{5}x^4$ $\therefore$ $y \propto x^4$ and $k = \frac{1}{5}$.

y is directly proportional to the fourth power of x, and the proportionality constant $k = \frac{1}{5}$.

Example 5 ◄» *Self Tutor*

Suppose T is directly proportional to d^2 and $T = 100$ when $d = 2$. Find:

a T when $d = 3$ **b** d when $T = 200$ if $d > 0$.

Method 1:

$T \propto d^2$
$\therefore$ $T = kd^2$ for some constant k.
When $d = 2$, $T = 100$
$\therefore$ $100 = k \times 2^2$
$\therefore$ $100 = 4k$
$\therefore$ $k = 25$
So, $T = 25d^2$.

a When $d = 3$, $T = 25 \times 3^2$
$\qquad = 25 \times 9$
$\qquad = 225$

b When $T = 200$,
$\qquad 200 = 25 \times d^2$
$\therefore$ $8 = d^2$
$\therefore$ $d = 2\sqrt{2}$ {as $d > 0$}

Method 2:

$T \propto d^2$

a

$\times \frac{3}{2}$

d	2	3
T	100	

d is multiplied by $\frac{3}{2}$
$\therefore$ d^2 is multiplied by $\left(\frac{3}{2}\right)^2$
$\therefore$ T is multiplied by $\left(\frac{3}{2}\right)^2$
$\therefore$ $T = 100 \times \left(\frac{3}{2}\right)^2 = 225$

b

d	2	
T	100	200

$\times 2$

T is multiplied by 2
$\therefore$ d^2 is multiplied by 2
$\therefore$ d is multiplied by $\sqrt{2}$ {$d > 0$}
$\therefore$ $d = 2 \times \sqrt{2} = 2\sqrt{2}$

Since T is directly proportional to d^2, whatever we multiply T by we must do the same to d^2.

Example 6 ◄» *Self Tutor*

The *period* or time for one complete swing of a pendulum is directly proportional to the square root of its length. When the length is 25 cm, the period is 1.00 seconds.

a If the length is 70 cm, find the period to 2 decimal places.

b What would the length be for a period of 2 seconds?

If T is the period and l is the length, then $T \propto \sqrt{l}$.

a

$\times \frac{70}{25}$

l	25	70
T	1.00	

l is multiplied by $\frac{70}{25}$

$\therefore$ $\sqrt{l}$ is multiplied by $\sqrt{\frac{70}{25}}$

$\therefore$ T is multiplied by $\sqrt{\frac{70}{25}}$ {as $T \propto \sqrt{l}$}

$\therefore$ $T = 1.00 \times \sqrt{\frac{70}{25}}$

      ≈ 1.67

So, the period would be 1.67 seconds.

b

l	25	
T	1.00	2

$\times 2$

T is multiplied by 2

$\therefore$ $\sqrt{l}$ is multiplied by 2   {as $T \propto \sqrt{l}$}

$\therefore$ l is multiplied by 2^2

$\therefore$ $l = 25 \times 2^2 = 100$

So, the length would be 100 cm.

Example 7
🔊 *Self Tutor*

The volume of a cylinder of fixed height varies in direct proportion to the square of the base radius. Find the change in volume when the base radius is increased by 18%.

Method 1:

If r is increased by 18% then

     $r_2 = 118\%$ of r_1

$\therefore$ $r_2 = 1.18r_1$

Now $V_1 = kr_1^2$ and $V_2 = kr_2^2$

$\therefore$ $V_2 = k(1.18r_1)^2$

$\therefore$ $V_2 = k \times 1.3924r_1^2$

$\therefore$ $V_2 = 1.3924V_1$

$\therefore$ $V_2 = 139.24\%$ of V_1

$\therefore$ V has increased by 39.24%.

Method 2:

If the volume is V and the radius is r, then $V \propto r^2$.

When the radius is increased by 18%,

     r is multiplied by 118% or 1.18

$\therefore$ r^2 is multiplied by $(1.18)^2$

$\therefore$ V is multiplied by $(1.18)^2 = 1.3924$

$\therefore$ V is multiplied by 139.24%

$\therefore$ V is increased by 39.24%.

EXERCISE 30A.2

1 By finding values of $\frac{y}{x^2}$ for each point in the following tables, establish that $y \propto x^2$ and state the rule connecting y and x. Hence find the value of a.

a

x	2	3	5	7
y	16	36	100	a

b

x	1	3	4	a
y	3	27	48	108

2 By finding values of $\frac{y}{x^3}$, establish that $y \propto x^3$ and find the value of b.

a

x	1	2	3	4
y	5	40	135	b

b

x	2	3	5	b
y	4	13.5	62.5	108

3 State which two variables are directly proportional and determine the proportionality constant k:

   a $A = 4.9t^2$ **b** $K = 2x^3$ **c** $T = \dfrac{\sqrt{l}}{5}$

   d $V = 500t^4$ **e** $P = 4\sqrt[3]{x}$ **f** $V = \frac{4}{3}\pi r^3$

4 P is directly proportional to a^2 and when $a = 5$, $P = 300$. Find:
   a the value of P when $a = 2$ **b** the value of a when $P = 2700$.

5 M is directly proportional to the cube of x and when $x = 2$, $M = 24$. Find:
   a the value of M when $x = 3$ **b** the value of x when $M = 120$.

6 D is directly proportional to $\sqrt{t}$ and when $t = 4$, $D = 16$. Find:
   a the value of D when $t = 9$ **b** the value of t when $D = 200$.

7 The value of a gem stone varies in direct proportion to the square of its weight. If a 4 carat stone is valued at \$200, find the value of a 5 carat stone.

8 The surface area of a sphere varies in direct proportion to the square of its radius. If a sphere of radius 6 cm has a surface area of 452 cm^2, find, to the nearest mm, the radius of a sphere with surface area 1000 cm^2.

9 When a stone falls freely, the time taken to hit the ground varies in direct proportion to the square root of the distance fallen. If it takes a stone 4 seconds to fall 78.4 m, find how long it would take for a stone to fall 500 m down a mine shaft.

10 At sea, the distance in km of the visible horizon is directly proportional to the square root of the height in metres of the observer's eye above sea level. So, $D \propto \sqrt{h}$.
If the horizon is 9 km when my eye is 5.4 m above sea level, how far can I see from a height of 10 m?

11 The volume of blood flowing through a blood vessel is directly proportional to the square of the internal diameter. If the diameter is reduced by 20%, what reduction in volume of blood flow would occur?

12 The volume V of a sphere varies in direct proportion to the cube of its radius.
   a What change in radius is necessary to double the volume?
   b What change in volume reduces the radius by 10%?

13 The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$ where r is the radius and h is the height.
   a If we consider cones of fixed radius but variable height, what proportionality exists between V and h?
   b If we consider cones of fixed height but variable radius, what proportionality exists between V and r?
   c What would happen to the volume of a cone if the base radius was increased by 20% and the height was decreased by 15%?

B	*INVERSE VARIATION*	*[2.13]*

If two painters can paint a house in 3 days, how long would it take four painters working at the same rate to paint the house?

The four painters will be able to do twice as much work in the same amount of time. So, it will take them a half of the time, or $1\frac{1}{2}$ days.

The two variables in this example are the *number of painters* and the *time taken*.

In a case like this where doubling one variable halves the other, we have an **inverse variation**.

> Two variables are **inversely proportional** or **vary inversely** if, when one is *multiplied* by a constant, the other is *divided* by the same constant.

Dividing by k is the same as multiplying by $\dfrac{1}{k}$, so we can see that if one of the variables is multiplied by 2, the other must be multiplied by $\frac{1}{2}$, or is *halved*.

Consider again the example of two painters completing a job in three days.

Suppose x is the number of painters and y is the number of days to complete the job. Clearly x must be an integer.

This table shows some of the possible combinations. They are plotted on the graph alongside.

x	1	2	3	4	6
y	6	3	2	$1\frac{1}{2}$	1

The shape of the graph is part of a **hyperbola**.

Now consider the graph of y against $\dfrac{1}{x}$.

x	1	2	3	4	6
$\dfrac{1}{x}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$
y	6	3	2	$1\frac{1}{2}$	1

Notice that the points on this graph form a straight line which passes through the origin.

So,   if y is **inversely proportional** to x, then y is **directly proportional** to $\dfrac{1}{x}$.

Consequently, $$y = k \times \frac{1}{x} \quad \text{or} \quad y = \frac{k}{x} \quad \text{or} \quad xy = k.$$

In the painters example, notice that $xy = 6$ for all points on the graph. So, in this case $k = 6$.

Example 8

◀) *Self Tutor*

Consider the table:

x	1	2	3	4	5
y	24	12	8	6	4.8

Explain why the variables x and y are inversely proportional.

x and y are inversely proportional if xy is constant for each ordered pair.

For all 5 ordered pairs in this table, $xy = 24$

$$\therefore \quad y = \frac{24}{x}$$

$\therefore \quad y$ is inversely proportional to x.

Example 9

◀) *Self Tutor*

If M is inversely proportional to t and $M = 200$ when $t = 3$, find:

    a M when $t = 5$ **b** t when $M = 746$ (to 2 d.p.)

$M \propto \dfrac{1}{t}$ **a**

$\times \frac{5}{3}$

t	3	5
M	200	

t is multiplied by $\frac{5}{3}$

$\therefore \quad \dfrac{1}{t}$ is multiplied by $\frac{3}{5}$

$\therefore \quad M$ is multiplied by $\frac{3}{5}$

$\therefore \quad M = 200 \times \frac{3}{5} = 120$

b

t	3	
M	200	746

$\times \frac{746}{200}$

M is multiplied by $\frac{746}{200}$

$\therefore \quad \dfrac{1}{t}$ is multiplied by $\frac{746}{200}$

$\therefore \quad t$ is multiplied by $\frac{200}{746}$

$\therefore \quad t = 3 \times \frac{200}{746} \approx 0.80$

Example 10

◀) *Self Tutor*

The velocity V of a body travelling a fixed distance is inversely proportional to the time taken t to complete the journey. When the velocity is 40 cm/s, the time taken is 280 seconds. Find the time when the velocity is 50 cm/s.

Method 1: $V \propto \dfrac{1}{t}$

$\therefore \quad V = k \left(\dfrac{1}{t} \right)$

When $V = 40$, $t = 280$

$\therefore \quad 40 = k \left(\frac{1}{280} \right)$

$\therefore \quad k = 40 \times 280$

$\therefore \quad k = 11\,200$

So, $V = \dfrac{11\,200}{t}$

Now when $V = 50$

$$50 = \frac{11\,200}{t}$$

$\therefore \quad 50t = 11\,200$

$\therefore \quad t = \dfrac{11\,200}{50} = 224$

$\therefore$ the time taken is 224 seconds.

Method 2: $V \propto \dfrac{1}{t}$ V is multiplied by $\frac{50}{40} = \frac{5}{4}$

t	280	
V	40	50

$\times \frac{50}{40}$

$\therefore$ $\dfrac{1}{t}$ is multiplied by $\frac{5}{4}$ {as $V \propto \dfrac{1}{t}$}

$\therefore$ t is multiplied by $\frac{4}{5}$

$\therefore$ $t = 280 \times \frac{4}{5} = 224$

$\therefore$ the time taken is 224 seconds.

EXERCISE 30B

1 Calculate the value of xy for each point in the following tables. Hence determine for each table whether x and y are inversely proportional. If an inverse proportionality exists, determine the law connecting the variables and draw the graph of y against x.

a

x	1	2	3	6
y	12	6	4	2

b

x	2	3	6	9
y	12	8	4	3

c

x	7	6	12	1
y	12	14	7	84

2 Which of the following graphs could indicate that y is inversely proportional to x?

a

b

> If y is inversely proportional to x then y is directly proportional to $\frac{1}{x}$.

c

d

3 If y is inversely proportional to x, and $y = 20$ when $x = 2$, find:

   a the value of y when $x = 5$ **b** the value of x when $y = 100$.

4 If y is inversely proportional to x, and $y = 24$ when $x = 6$, find:

   a the value of y when $x = 8$ **b** the value of x when $y = 12$.

5 The formula connecting average speed s, distance travelled d, and time taken t, is $s = \dfrac{d}{t}$. Complete the following statements by inserting 'directly' or 'inversely':

   a If d is fixed, s is ...... proportional to t.

   b If t is fixed, s is ...... proportional to d.

   c If s is fixed, d is ...... proportional to t.

6 M varies inversely to t^2 where $t > 0$. When $t = 2$, $M = 10$.

   a Find M when $t = 5$.   **b** Find t when $M = 250$.

7 P varies inversely to the square root of g. When $g = 9$, $P = 20$.

   a Find P when $g = 4$.   **b** Find g when $P = 50$.

8 Consider gas trapped inside an airtight cylinder. For a given mass of gas kept at a constant temperature, the volume is inversely proportional to the pressure. When $V = 10$ cm^3, the pressure is 40 units. Find:

   a the pressure when the volume is 20 cm^3

   b the volume when the pressure is 100 units.

9 The time taken to complete a certain job varies inversely to the number of workers doing the task. If 20 workers could do the job in 6 days, find how long it would take 15 workers to do the job.

10 Consider an object suspended from a spring. The object is dropped so it will bounce up and down. The time between successive bounces is called the *period* of the motion. This time varies inversely to the square root of the stiffness of the spring. The period of a spring of stiffness 100 units is 0.2 seconds. Find:

   a the period for a spring of stiffness 300 units

   b the stiffness required for a 0.1 second period.

11 The amount of heat received by a body varies inversely to the square of the distance of the body from the heat source. If the distance from the source is decreased by 60%, what effect does this have on the heat received?

12 The intensity of light on a screen varies inversely to the square of the distance between the screen and the light source.

If a screen is illuminated by a light source 20 m away, the intensity is one fifth of what is required. Where should the light be placed?

C   VARIATION MODELLING   [2.13]

The following families of graphs could be useful to help identify variation models:

DIRECT VARIATION

$y = kx^3$
$y = kx^2$
$y = kx$
$y = kx^{\frac{1}{2}}$
$y = kx^{\frac{1}{3}}$

The graph always passes through the origin (0, 0).

INVERSE VARIATION

$y = \dfrac{k}{\sqrt{x}}$
$y = \dfrac{k}{x}$
$y = \dfrac{k}{x^2}$

The graph is always asymptotic to both the x and y-axes.

Discussion   *Possible models*

Why do the following graphs not represent direct or inverse variation models?

a   **b**   **c**

In some cases, we know what type of variation exists. We use the data values given to find the exact equation for the model.

Example 11   ◀》 *Self Tutor*

The area A of a sector of given angle is directly proportional to the square of its radius r.

Find the equation of the variation model given the data on the graph.

Since $A \propto r^2$, $A = kr^2$ where k is a constant.

From the graph we see that when $r = 2$, $A = 3$

$\therefore$ $3 = k \times 4$ and so $k = \frac{3}{4}$

The model is $A = \frac{3}{4}r^2$.

> Check the other data points on the graph to make sure they obey this model.

In the previous sections we observed that:

> If $y \propto x^n$ then:
>
> - $y = kx^n$ where k is the proportionality constant
> - the graph of y against x^n is a straight line with gradient k.

Example 12   ◀) *Self Tutor*

A small glass ball is rolled down a sloping sheet of hard board. At time t seconds it has rolled a distance d cm. The results are:

t	0	1	2	3	4	5
d	0	0.2	0.8	1.8	3.2	5.0

Show that the model is quadratic and find its equation.

To show the model is quadratic, we graph d against t^2:

Since this graph is linear, the model is indeed quadratic.

$d = kt^2$ where k is the gradient of the line.

$k = \dfrac{5 - 0}{25 - 0} = \frac{1}{5} = 0.2$

$\therefore \quad d = 0.2t^2$

Example 13   ◀) *Self Tutor*

Copies of the Eiffel Tower are sold in different sizes all over Paris. Of those made of brass, measurement of the height and mass of several samples was made:

height (h cm)	8	12	16
mass (m g)	61.44	207.36	491.52

Assuming that $m \propto h^n$ for some integer n, find an equation connecting m and h.

We are given that $m \propto h^n$, so $m = kh^n$ where k and n are constants.

When $h = 8$, $m = 61.44$, so $61.44 = k \times 8^n$ ...... (1)
When $k = 16$, $m = 491.52$, so $491.52 = k \times 16^n$ ...... (2)

Dividing (2) by (1) gives $\dfrac{k \times 16^n}{k \times 8^n} = \dfrac{491.52}{61.44}$ Using (1), $61.44 = k \times 8^3$

$\therefore \quad \left(\frac{16}{8}\right)^n = 8$ $\therefore \quad k = \dfrac{61.44}{512}$

$\therefore \quad 2^n = 2^3$ $\therefore \quad k = 0.12$

$\therefore \quad n = 3$ $\therefore$ the equation is $m = 0.12h^3$

We check this model using the third data point: $0.12 \times 12^3 = 207.36$ ✓

EXERCISE 30C

1 Find the variation model for these data sets:

a

x	1	2	3	4	5
y	4	32	108	256	500

b

x	1	$2\frac{1}{4}$	4	9
y	3	$4\frac{1}{2}$	6	9

c

x	1	2	3	4
y	16	8	$5\frac{1}{3}$	4

d

x	1	2	4	5
y	100	25	$6\frac{1}{4}$	4

2 The distance to the horizon (d km) is proportional to the square root of the height (h m) of a person above sea level.

The graph of d against h is shown alongside.

Find a model connecting d and h.

3

Scale models of a car are made in different sizes.

The mass of a car (m kg) is directly proportional to the cube of its length (l m).

The graph of m against l is shown.

Find a model connecting m and l.

4 It is suspected that for two variables x and y, y varies inversely to x.

Find the equation of the model connecting y and x using data from the graph.

5 The table opposite contains data from an experiment.

Show that the model relating x to y is of the form $y = \dfrac{k}{x^2}$ and find the value of k.

x	0.25	0.5	1	2
y	80	20	5	1.25

6 A student wants to find the relationship between the length (l m) of a pendulum and the time period (T seconds) that it takes for one complete swing. In an experiment she collected the following results:

l	0.25	0.36	0.49	0.64	0.81	1.00
T	1	1.2	1.4	1.6	1.8	2.0

    a Show that the model relating T and l is of the form $T = k\sqrt{l}$.

    b Find the value of k.

    c If a pendulum has length 2 m, what will its period be?

7 The designer of a car windscreen needs to find the relationship between the air resistance R and the velocity v km/h of the car. He carries out an experiment and the results are given in the table alongside:

v	10	20	30	40
R	0.5	4	13.5	32

From previous experiences, the designer expects that $R \propto v^n$.

a Plot the graphs of R against v, R against v^2, and R against v^3.

b Hence deduce the approximate model for R in terms of v.

D POWER MODELLING [2.13, 11.2]

The direct and inverse variations we have studied are all examples of **power models**. These are equations of the form $y = ax^b$.

If $b > 0$, we have **direct variation**.

If $b < 0$, we have **inverse variation**.

So far we have only considered data which a power model fits *exactly*. We now consider data for which the 'best' model must be fitted. In such a case we use technology to help find the model.

If we suspect two variables are related by a power model, we should:

- Graph y against x. We look for either:
  - a direct variation curve which passes through the origin.
  - an inverse variation curve for which the x and y-axes are both asymptotes.

- Use the **power regression** function on your calculator. Instructions for this can be found on page **28**.

For example, consider the data in **Example 13**:

height (h cm)	8	12	16
mass (m g)	61.44	207.36	491.52

A graph of the data suggests a power model is reasonable.

Using the power regression function on a graphics calculator, we obtain the screen dump opposite.

So, $m = 0.12h^3$.

The coefficient of determination $r^2 = 1$ indicates the model is a perfect fit.


```
PwrReg
 y=a*x^b
 a=.12
 b=3
 r²=1
 r=1
```

Example 14   ◄⑴ *Self Tutor*

Gravity is the force of attraction which exists between any
two objects in the universe. The force of attraction F
Newtons between two masses d cm apart is given in the
following table:

d (cm)	24	26	28	30	32	34	36	38	40
F (N)	237	202	174	152	133	118	105	95	85

a Graph F against d and show that a power model is reasonable.
b Obtain the power model which best fits the data.
c What factor confirms that the fit is excellent?
d Estimate the force of attraction between two spheres that are:
  i 20 cm apart   **ii** 29 cm apart   **iii** 50 m apart.

a

The graph has both the x and y-axes as asymptotes
and it looks like an example of inverse variation.
A power model is therefore reasonable.

```
PwrReg
y=a*x^b
a=137564.8648
b=-2.00236699
r²=.9999420984
r=-.9999710488
```

b $F \approx 138\,000 \times d^{-2.00}$,   so $F \approx \dfrac{138\,000}{d^2}$.

c $r^2 \approx 0.9999$ is very close to 1, so the fit is excellent.

d **i** When $d = 20$

$F \approx \dfrac{138\,000}{20^2}$

$\therefore \quad F \approx 345$ N

ii When $d = 29$

$F \approx \dfrac{138\,000}{29^2}$

$\therefore \quad F \approx 164$ N

iii 50 m = 5000 cm
when $d = 5000$

$F = \dfrac{138\,000}{5000^2}$

$\therefore \quad F \approx 0.005\,52$ N

EXERCISE 30D

1 In **Example 12** with the small glass ball rolling down the
incline, we found that the model was $d = kt^2$ with $k = 0.2$.
Use technology to check this result.

t	0	1	2	3	4	5
d	0	0.2	0.8	1.8	3.2	5.0

2 Use your graphics calculator to find the power model which best fits the following data:

a

x	2	3	5	8
y	3.25	0.96	0.20	0.05

b

x	2	5	12	25
M	35.4	22.3	14.4	10.1

3 Warm blooded animals maintain a remarkably constant body temperature by using most of the food
they consume to produce heat. The larger the animal, the more heat it needs to produce. The following
table gives the heat required (h Joules) for animals of various masses (m kg).

Mass (m kg)	0.02	0.5	4	20	50	100	500
Heat required (h Joules)	25	170	630	1670	2500	4200	16 750

a Obtain a power model for this data.

b Is the power model appropriate? Explain your answer.

c What would be the heat required by an elephant of mass 5 tonnes?

4 Johann Kepler is a very famous man in the history of astronomy and mathematics. He used data from observations of planetary orbits to show that these motions are not random but rather obey certain mathematical laws which can be written in algebraic form. He took the Earth as his "base unit", so the orbital periods are given as multiples of one Earth year, and orbital radii as multiples of one Earth orbit. Some of his observational data are given in this table.

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn
Orbital period (T)	0.241	0.615	1.000	1.881	11.862	29.457
Orbital radius (R)	0.387	0.723	1.000	1.542	5.202	9.539

a Obtain a power model for this data.

b Explain whether the power model is appropriate.

c What is the resulting equation when the equation of **a** is cubed?

5 If you compress a gas quickly, it gets hot. The more you compress it, the hotter it gets. Just think how hot a bike pump gets when you pump up a tyre quickly. In such cases, Boyle's law which connects volume (V ml) and pressure (P hectopascals) no longer applies. To find a new model for this case, data has been obtained. It is:

P	1015	2000	3000	4000	5000	6000
V	1000	615	460	375	320	281

a Find the power model of best fit.

b Estimate the volume when: **i** $P = 2400$ hPa **ii** $P = 7000$ hPa

---

Discovery *The pendulum*

A **pendulum** can be made by tying an object such as a mobile phone to a shoe lace and allowing the object to swing back and forth when supported by the string.

The object of this discovery is to determine any rule which may connect the period of the pendulum (T secs) with its length (l cm).

We will find the **period**, which is the time taken for one complete oscillation, back and forth, for various lengths of the pendulum.

The maximum angle θ of the pendulum should be 15°. The length of the pendulum is the distance from the point of support to the object's centre of mass.

What to do:

1 Which of the following ideas have merit when finding the period of the pendulum for a particular length?

- Several students should time one period using their stopwatches.
- Timing 8 complete swings and averaging is better than timing one complete swing.
- If several students do the timing, the highest and lowest scores should be removed and the remaining scores should be averaged.

2 List possible factors which could lead to inaccurate results.

3 After deciding on a method for determining the period, measure the period for pendulum lengths of 20 cm, 30 cm, 40 cm, ..., 100 cm and record your results in a table like the one alongside.

length (l cm)	period (T s)
20	
30	
40	
⋮	

4 Use technology to determine the law connecting T and l.

Review set 30A

1 The variables x and y in the table alongside are inversely proportional. Find a and b.

x	1	3	6	a
y	24	8	b	60

2 If 7 litres of petrol are needed to drive 100 km, how far could you travel on 16 litres of petrol?

3 If 3 men could paint a grain silo in 18 days, how long would it take 8 men to paint the silo working at the same rate?

4 Draw the graph of y against x^2 for the points in the table alongside. From your graph, verify that a law of the form $y = kx^2$ applies. Hence find:

x	1	2	3
y	4	16	36

  a the value of k **b** y when $x = 6$ **c** x when $y = 64$.

5 P is directly proportional to the square root of Q, and $P = 12$ when $Q = 9$.   Find:

  a the law connecting P and Q **b** the value of P when $Q = 121$
  c the value of Q when $P = 13$.

6 The period of a pendulum varies in direct proportion to the square root of its length. If a 45 cm pendulum has period 1.34 seconds, find the period of a 64 cm pendulum.

7 The volume of a cylinder is directly proportional to the square of its radius. Find:

  a the change in volume produced by doubling the radius

  b the change in radius needed to produce a 60% increase in volume.

8 It is suspected that two variables D and p are related by a law of the form $D = \dfrac{k}{\sqrt{p}}$ where k is a constant. An experiment to find D for various values of p was conducted and the results alongside were obtained.

p	1	4	9	16	25
D	36	18	12	9	7.2

  a Graph D against $\sqrt{p}$ for these data values.

   b What features of the graph suggest that $D = \dfrac{k}{\sqrt{p}}$ is an appropriate model for the data?

   c Determine k.

   d Find D, correct to 1 decimal place, when $p = 13$.

9 Water flows into a channel at a rate of R m³/s. The depth of the channel is d metres. The following table gives corresponding values of R and d.

d	0.5	1.0	1.4	1.7	2.0	2.5
R	0.7	4.0	9.3	15.0	22.6	39.7

   a Find the power model which best fits the data.

   b Explain whether the power model is appropriate.

   c Use the model from **a** to estimate the rate of flow when the depth is 2.2 m.

   d How much water flows into the channel each minute when the depth is 2.2 m?

Review set 30B

1 The variables p and q in the table alongside are directly proportional. Find a and b.

p	2	4	7	a
q	6	12	b	42

2 If y is inversely proportional to x, what happens to:

   a y if x is trebled   **b** x if y is increased by 50%?

3 If x is inversely proportional to y, and $y = 65$ when $x = 10$, find:

   a the law connecting x and y **b** y when $x = 12$

   c x when $y = 150$.

4 If 5 bags of fertiliser are required to fertilise a lawn 26 m by 40 m, how many bags would be required to fertilise a lawn 39 m by 100 m?

5 The resistance, R ohms, to the flow of electricity in a wire varies inversely to the area of the cross-section of the wire. When the area is 0.15 cm², the resistance is 0.24 ohms. Find:

   a the resistance when the area is 0.07 cm²

   b the area when the resistance is 0.45 ohms.

6 The intensity of light on a screen varies inversely to the square of the distance between the screen and the light source. If the screen has 24 units of illumination when the source is 4 m away, determine the illumination when the screen is 6 m away.

7 The surface area of a sphere is directly proportonal to the square of its radius. Find:

   a the change in surface area if the radius is decreased by 19%

   b the change in radius if the surface area is trebled.

8 Kelly makes small glass pyramids of height h cm. She suspects that the volume of glass V cm³ is directly proportional to a power of h, so $V \propto h^n$. The following table of volumes for various heights was obtained:

h	2	4	6
V	3.2	25.6	86.4

   a Use the *first two* pieces of data to find k and n for which $V = kh^n$.

   b Check that the model you found in **a** is satisfied by the final data point.

   c Can you suggest why this value of n could be expected?

   d Find: **i** V when $h = 8$ **ii** h when $V = 50$.

9 The diameter of a scallop is measured at regular intervals of 3 months. The results were:

Age (n quarters)	1	2	4	5	6	8	9	10	12
Diameter (D mm)	32	42	56	62	67	75	79	82	89

    a Find the *power* model which best fits the data.

    b Use the model from **a** to estimate the diameter of the scallop when:

        i $n = 3$ **ii** $n = 7$ **iii** $n = 11$ **iv** $n = 14$.

    c Which of your estimates in **b** is least likely to be reliable?

Challenge

1 The square of the velocity of a vehicle varies inversely as the distance travelled from its starting point. If the distance is increased by 20%, find the approximate percentage decrease in the velocity.

2 **a** If y varies directly as x^n, $y = y_1$ when $x = x_1$ and $y = y_2$ when $x = x_2$, show that

$$\frac{y_2}{y_1} = \left(\frac{x_2}{x_1}\right)^n.$$

    b If y varies inversely as x^n, show that $\dfrac{y_2}{y_1} = \left(\dfrac{x_1}{x_2}\right)^n.$

3 **a** The sum of the reciprocals of two variables is constant. Show that the sum of the variables is directly proportional to their product.

    b Suppose that when the two variables in **a** are equal, each has a value of 4. If one variable has a value of 8, find the value of the other variable.

4 **a** One variable is directly proportional to another.
       Show that their sum is directly proportional to their difference.

    b For different models of a tricycle, the front wheel radius is directly proportional to the back wheel radius.
       Show that the difference between the areas of the front and back wheels is directly proportional to the square of the difference between their radii.

5 The income made by a train service varies directly as the excess speed above 40 km/h. Expenses however, vary directly as the square of the excess speed above 40 km/h. Given that a speed of 60 km/h just covers expenses, find the speed which will maximise the profits.

Logarithms

31

Contents:

Opening problem

Tony invests \$8500 for n years at 7.8% p.a. compounding annually. The interest rate is fixed for the duration of the investment. The value of the investment after n years is given by $V = 8500 \times (1.078)^n$ dollars.

Things to think about:

a   How long will it take for Tony's investment to amount to \$12 000?
b   How long will it take for his investment to double in value?

In **Chapter 28** we answered problems like the one above by graphing the exponential function and using technology to find when the investment is worth a particular amount.

However, we can also solve these problems without a graph using **logarithms**.

A   LOGARITHMS IN BASE a [3.10]

We have seen previously that $y = x^2$ and $y = \sqrt{x}$ are *inverse functions*.

For example, $5^2 = 25$ and $\sqrt{25} = 5$.

Logarithms were created to be the **inverse of exponential functions**.

> If $y = a^x$ then we say "x is the logarithm of y in base a", and write this as $x = \log_a y$.

For example, since $8 = 2^3$ we can write $3 = \log_2 8$. The two statements '2 to the power 3 equals 8' and 'the logarithm of 8 in base 2 equals 3' are **equivalent**, and we write:

$$2^3 = 8 \Leftrightarrow \log_2 8 = 3$$

Further examples are: $10^3 = 1000 \Leftrightarrow \log_{10} 1000 = 3$

$$3^2 = 9 \Leftrightarrow \log_3 9 = 2$$

$$4^{\frac{1}{2}} = 2 \Leftrightarrow \log_4 2 = \tfrac{1}{2}$$

The symbol $\Leftrightarrow$ "is equivalent to"

In general, $y = a^x$ and $x = \log_a y$ are **equivalent** statements

and we write $y = a^x \Leftrightarrow x = \log_a y.$

Example 1   ◀) *Self Tutor*

Write an equivalent:

a logarithmic statement for $2^5 = 32$ **b** exponential statement for $\log_4 64 = 3$.

a $2^5 = 32$ is equivalent to $\log_2 32 = 5$.
So, $2^5 = 32 \Leftrightarrow \log_2 32 = 5$.

b $\log_4 64 = 3$ is equivalent to $4^3 = 64$.
So, $\log_4 64 = 3 \Leftrightarrow 4^3 = 64$.

Example 2   ◀) *Self Tutor*

Find the value of $\log_3 81$.

Let $\log_3 81 = x$
$\therefore \quad 3^x = 81$
$\therefore \quad 3^x = 3^4$
$\therefore \quad x = 4$
$\therefore \quad \log_3 81 = 4$

The logarithm of 81 in base 3 is the exponent or power of 3 which gives 81.

EXERCISE 31A

1 Write an equivalent logarithmic statement for:

a $2^2 = 4$ **b** $4^2 = 16$ **c** $3^2 = 9$ **d** $5^3 = 125$

e $10^4 = 10\,000$ **f** $7^{-1} = \tfrac{1}{7}$ **g** $3^{-3} = \tfrac{1}{27}$ **h** $27^{\frac{1}{3}} = 3$

i $5^{-2} = \tfrac{1}{25}$ **j** $2^{-\frac{1}{2}} = \tfrac{1}{\sqrt{2}}$ **k** $4\sqrt{2} = 2^{2.5}$ **l** $0.001 = 10^{-3}$

2 Write an equivalent exponential statement for:

a $\log_2 8 = 3$ **b** $\log_2 1 = 0$ **c** $\log_2 \left(\tfrac{1}{2}\right) = -1$ **d** $\log_2 \sqrt{2} = \tfrac{1}{2}$

e $\log_2 \left(\tfrac{1}{\sqrt{2}}\right) = -\tfrac{1}{2}$ **f** $\log_{\sqrt{2}} 2 = 2$ **g** $\log_{\sqrt{3}} 9 = 4$ **h** $\log_9 3 = \tfrac{1}{2}$

3 Without using a calculator, find the value of:

a $\log_{10} 100$ **b** $\log_2 8$ **c** $\log_3 3$ **d** $\log_4 1$

e $\log_5 125$ **f** $\log_5 (0.2)$ **g** $\log_{10} 0.001$ **h** $\log_2 128$

i $\log_2 \left(\tfrac{1}{2}\right)$ **j** $\log_3 \left(\tfrac{1}{9}\right)$ **k** $\log_2 (\sqrt{2})$ **l** $\log_2 \left(\sqrt{8}\right)$

   m $\log_7\left(\sqrt[3]{7}\right)$ **n** $\log_2(4\sqrt{2})$ **o** $\log_{\sqrt{2}}2$ **p** $\log_2\left(\frac{1}{4\sqrt{2}}\right)$

   q $\log_{10}(0.01)$ **r** $\log_{\sqrt{2}}4$ **s** $\log_{\sqrt{3}}\left(\frac{1}{3}\right)$ **t** $\log_3\left(\frac{1}{9\sqrt{3}}\right)$

4   Rewrite as logarithmic equations:

   a $y = 4^x$ **b** $y = 9^x$ **c** $y = a^x$ **d** $y = (\sqrt{3})^x$

   e $y = 2^{x+1}$ **f** $y = 3^{2n}$ **g** $y = 2^{-x}$ **h** $y = 2 \times 3^a$

5   Rewrite as exponential equations:

   a $y = \log_2 x$ **b** $y = \log_3 x$ **c** $y = \log_a x$ **d** $y = \log_b n$

   e $y = \log_m b$ **f** $T = \log_5\left(\frac{a}{2}\right)$ **g** $M = \frac{1}{2}\log_3 p$ **h** $G = 5\log_b m$

   i $P = \log_{\sqrt{b}} n$

6   Rewrite the following, making x the subject:

   a $y = \log_7 x$ **b** $y = 3^x$ **c** $y = (0.5)^x$ **d** $z = 5^x$

   e $t = \log_2 x$ **f** $y = 2^{3x}$ **g** $y = 5^{\frac{x}{2}}$ **h** $w = \log_3(2x)$

   i $z = \frac{1}{2} \times 3^x$ **j** $y = \frac{1}{5} \times 4^x$ **k** $D = \frac{1}{10} \times 2^{-x}$ **l** $G = 3^{x+1}$

7   Explain why, for all $a > 0$, $a \neq 1$:   **a** $\log_a 1 = 0$ **b** $\log_a a = 1$

B   THE LOGARITHMIC FUNCTION   [3.10]

> The **logarithmic function** is $f(x) = \log_a x$ where $a > 0$, $a \neq 1$.

Consider $f(x) = \log_2 x$ which has graph $y = \log_2 x$.

Since $y = \log_2 x \Leftrightarrow x = 2^y$,   we can obtain the table of values:

y	-3	-2	-1	0	1	2	3
x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Notice that:

- the graph of $y = \log_2 x$ is asymptotic to the y-axis
- the domain of $y = \log_2 x$ is $\{x \mid x > 0\}$
- the range of $y = \log_2 x$ is $\{y \mid y \in \mathbb{R}\}$

THE INVERSE FUNCTION OF $f(x) = \log_a x$

Given the function $y = \log_a x$, the inverse is $x = \log_a y$ {interchanging x and y}

$$\therefore \quad y = a^x$$

So, $f(x) = \log_a x \Leftrightarrow f^{-1}(x) = a^x$

For example,   if $f(x) = \log_2 x$ then $f^{-1}(x) = 2^x$.

The inverse function $y = \log_2 x$ is the reflection of $y = 2^x$ in the line $y = x$.

Example 3   ◀)) *Self Tutor*

Find the inverse function $f^{-1}(x)$ for: **a** $f(x) = 5^x$ **b** $f(x) = 2\log_3 x$

a $y = 5^x$ has inverse function $x = 5^y$

$\therefore \quad y = \log_5 x$

So, $f^{-1}(x) = \log_5 x$

b $y = 2\log_3 x$ has inverse function

$x = 2\log_3 y$

$\therefore \quad \dfrac{x}{2} = \log_3 y$

$\therefore \quad y = 3^{\frac{x}{2}}$

So, $f^{-1}(x) = 3^{\frac{x}{2}}$

EXERCISE 31B

1   Find the inverse function $f^{-1}(x)$ for:

    a $f(x) = 4^x$ **b** $f(x) = 10^x$ **c** $f(x) = 3^{-x}$ **d** $f(x) = 2 \times 3^x$

    e $f(x) = \log_7 x$ **f** $f(x) = \frac{1}{2}(5^x)$ **g** $f(x) = 3\log_2 x$ **h** $f(x) = 5\log_3 x$

    i $f(x) = \log_{\sqrt{2}} x$

2   **a**   On the same set of axes graph $y = 3^x$ and $y = \log_3 x$.

    b   State the domain and range of $y = 3^x$.

    c   State the domain and range of $y = \log_3 x$.

3   Prove using algebra that if $f(x) = a^x$ then $f^{-1}(x) = \log_a x$.

> If $f(x) = g(x)$,
> graph $y = f(x)$
> and $y = g(x)$
> on the same set
> of axes.

4   Use the logarithmic function   log   on your graphics calculator to solve the following equations correct to 3 significant figures. You may need to use the instructions on page **15**.

    a $\log_{10} x = 3 - x$ **b** $\log_{10}(x - 2) = 2^{-x}$

    c $\log_{10}\left(\frac{x}{4}\right) = x^2 - 2$ **d** $\log_{10} x = x - 1$

    e $\log_{10} x = 5^{-x}$ **f** $\log_{10} x = 3^x - 3$

C   *RULES FOR LOGARITHMS*   [3.10]

Consider two positive numbers x and y. We can write them both with base a: $x = a^p$ and $y = a^q$,   for some p and q.

$\therefore \quad p = \log_a x$ and $q = \log_a y$ ...... (*)

Using exponent laws, we notice that: $xy = a^p a^q = a^{p+q}$

$$\frac{x}{y} = \frac{a^p}{a^q} = a^{p-q}$$

$$x^n = (a^p)^n = a^{np}$$

$\therefore \quad \log_a(xy) = p + q = \log_a x + \log_a y$ {from *}

$$\log_a\left(\frac{x}{y}\right) = p - q = \log_a x - \log_a y$$

$$\log_a(x^n) = np = n \log_a x$$

$$\log_a(xy) = \log_a x + \log_a y$$
$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$
$$\log_a(x^n) = n \log_a x$$

Example 4   ◀⑴ *Self Tutor*

Simplify: **a** $\log_2 7 - \frac{1}{2}\log_2 3 + \log_2 5$ **b** $3 - \log_2 5$

a $\log_2 7 - \frac{1}{2}\log_2 3 + \log_2 5$

$= \log_2 7 + \log_2 5 - \log_2 3^{\frac{1}{2}}$

$= \log_2(7 \times 5) - \log_2 \sqrt{3}$

$= \log_2\left(\frac{35}{\sqrt{3}}\right)$

b $3 - \log_2 5$

$= \log_2 2^3 - \log_2 5$

$= \log_2\left(\frac{8}{5}\right)$

$= \log_2(1.6)$

Example 5   ◀⑴ *Self Tutor*

If $\log_3 5 = p$ and $\log_3 8 = q$,   write in terms of p and q:

a $\log_3 40$ **b** $\log_3 25$ **c** $\log_3\left(\frac{64}{125}\right)$

a $\log_3 40$

$= \log_3(5 \times 8)$

$= \log_3 5 + \log_3 8$

$= p + q$

b $\log_3 25$

$= \log_3 5^2$

$= 2\log_3 5$

$= 2p$

c $\log_3\left(\frac{64}{125}\right)$

$= \log_3\left(\frac{8^2}{5^3}\right)$

$= \log_3 8^2 - \log_3 5^3$

$= 2\log_3 8 - 3\log_3 5$

$= 2q - 3p$

EXERCISE 31C

1   Write as a single logarithm:

   a $\log_3 2 + \log_3 8$ **b** $\log_2 9 - \log_2 3$ **c** $3 \log_5 2 + 2 \log_5 3$

   d $\log_3 8 + \log_3 7 - \log_3 4$ **e** $1 + \log_3 4$ **f** $2 + \log_3 5$

   g $1 + \log_7 3$ **h** $1 + 2 \log_4 3 - 3 \log_4 5$ **i** $2 \log_3 m + 7 \log_3 n$

   j $5 \log_2 k - 3 \log_2 n$

2   If $\log_2 7 = p$ and $\log_2 3 = q$, write in terms of p and q:

   a $\log_2 21$ **b** $\log_2 \left(\frac{3}{7}\right)$ **c** $\log_2 49$ **d** $\log_2 27$

   e $\log_2 \left(\frac{7}{9}\right)$ **f** $\log_2 (63)$ **g** $\log_2 \left(\frac{56}{9}\right)$ **h** $\log_2 (5.25)$

3   Write y in terms of u and v if:

   a $\log_2 y = 3 \log_2 u$ **b** $\log_3 y = 3 \log_3 u - \log_3 v$

   c $\log_5 y = 2 \log_5 u + 3 \log_5 v$ **d** $\log_2 y = u + v$

   e $\log_2 y = u - \log_2 v$ **f** $\log_5 y = -\log_5 u$

   g $\log_7 y = 1 + 2 \log_7 v$ **h** $\log_2 y = \frac{1}{2} \log_2 v - 2 \log_2 u$

   i $\log_6 y = 2 - \frac{1}{3} \log_6 u$ **j** $\log_3 y = \frac{1}{2} \log_3 u + \log_3 v + 1$

4   Without using a calculator, simplify:

   a $\dfrac{\log_2 16}{\log_2 4}$ **b** $\dfrac{\log_p 16}{\log_p 4}$ **c** $\dfrac{\log_5 25}{\log_5 \left(\frac{1}{5}\right)}$ **d** $\dfrac{\log_m 25}{\log_m \left(\frac{1}{5}\right)}$

D   LOGARITHMS IN BASE 10   [3.10]

Logarithms in base 10 are called **common logarithms**.

$y = \log_{10} x$ is often written as just $y = \log x$, and we *assume* the logarithm has base 10.

Your calculator has a $\boxed{\text{log}}$ key which is for base 10 logarithms.

Discovery   Logarithms

The logarithm of any positive number can be evaluated using the $\boxed{\text{log}}$ key on your calculator. You will need to do this to evaluate the logarithms in this discovery.

What to do:

1   Copy and complete:

Number	Number as a power of 10	log of number
10		
100		
1000		
100 000	10^5	$\log(100\,000) = 5$
0.1		
0.001		

2 Copy and complete:

Number	Number as a power of 10	log of number
$\sqrt{10}$		
$\sqrt[3]{10}$		
$\sqrt{1000}$		
$\dfrac{1}{\sqrt{10}}$		

3 Can you draw any conclusion from your table? For example, you may wish to comment on when a logarithm is positive or negative.

Example 6   ◆) *Self Tutor*

Use the property $a = 10^{\log a}$ to write the following numbers as powers of 10:

a 2   **b** 20

a $\log 2 \approx 0.301$ **b** $\log 20 \approx 1.301$
$\therefore \quad 2 \approx 10^{0.301}$ $\therefore \quad 20 \approx 10^{1.301}$

> If the base for a logarithm is not given then we assume it is 10.

RULES FOR BASE 10 LOGARITHMS

The rules for base 10 logarithms are clearly the same rules for general logarithms:

$$\log(xy) = \log x + \log y$$
$$\log\left(\frac{x}{y}\right) = \log x - \log y$$
$$\log(x^n) = n \log x$$

> These rules correspond closely to the **exponent laws**.

Example 7   ◆) *Self Tutor*

Write as a single logarithm:

a $\log 2 + \log 7$ **b** $\log 6 - \log 3$ **c** $2 + \log 9$ **d** $\dfrac{\log 49}{\log\left(\frac{1}{7}\right)}$

a $\quad \log 2 + \log 7$
$= \log(2 \times 7)$
$= \log 14$

b $\quad \log 6 - \log 3$
$= \log\left(\frac{6}{3}\right)$
$= \log 2$

c $\quad 2 + \log 9$
$= \log 10^2 + \log 9$
$= \log(100 \times 9)$
$= \log 900$

d $\quad \dfrac{\log 49}{\log\left(\frac{1}{7}\right)}$
$= \dfrac{\log 7^2}{\log 7^{-1}}$
$= \dfrac{2 \log 7}{-1 \log 7}$
$= -2$

EXERCISE 31D.1

1 Write as powers of 10 using $a = 10^{\log a}$:

    a   8   **b**   80   **c**   800   **d**   0.8   **e**   0.008

    f   0.3   **g**   0.03   **h**   0.000 03   **i**   50   **j**   0.0005

2 Write as a single logarithm in the form $\log k$:

    a $\log 6 + \log 5$ **b** $\log 10 - \log 2$ **c** $2 \log 2 + \log 3$

    d $\log 5 - 2 \log 2$ **e** $\frac{1}{2} \log 4 - \log 2$ **f** $\log 2 + \log 3 + \log 5$

    g $\log 20 + \log(0.2)$ **h** $- \log 2 - \log 3$ **i** $3 \log \left(\frac{1}{8} \right)$

    j $4 \log 2 + 3 \log 5$ **k** $6 \log 2 - 3 \log 5$ **l** $1 + \log 2$

    m $1 - \log 2$ **n** $2 - \log 5$ **o** $3 + \log 2 + \log 7$

3 Explain why $\log 30 = \log 3 + 1$ and $\log(0.3) = \log 3 - 1$

4 Without using a calculator, simplify:

    a $\dfrac{\log 8}{\log 2}$ **b** $\dfrac{\log 9}{\log 3}$ **c** $\dfrac{\log 4}{\log 8}$ **d** $\dfrac{\log 5}{\log \left(\frac{1}{5} \right)}$

    e $\dfrac{\log(0.5)}{\log 2}$ **f** $\dfrac{\log 8}{\log(0.25)}$ **g** $\dfrac{\log 2^b}{\log 8}$ **h** $\dfrac{\log 4}{\log 2^a}$

5 Without using a calculator, show that:

    a $\log 8 = 3 \log 2$ **b** $\log 32 = 5 \log 2$ **c** $\log \left(\frac{1}{7} \right) = - \log 7$

    d $\log \left(\frac{1}{4} \right) = -2 \log 2$ **e** $\log \sqrt{5} = \frac{1}{2} \log 5$ **f** $\log \sqrt[3]{2} = \frac{1}{3} \log 2$

    g $\log \left(\frac{1}{\sqrt{3}} \right) = -\frac{1}{2} \log 3$ **h** $\log 5 = 1 - \log 2$ **i** $\log 500 = 3 - \log 2$

6 $7^4 = 2401 \approx 2400$

    Show that $\log 7 \approx \frac{3}{4} \log 2 + \frac{1}{4} \log 3 + \frac{1}{2}$.

LOGARITHMIC EQUATIONS

The logarithm laws can be used to help rearrange equations. They are particularly useful when dealing with exponential equations.

Example 8   ◀) *Self Tutor*

Write the following as logarithmic equations in base 10:

    a $y = a^3 b^2$ **b** $y = \dfrac{m}{\sqrt{n}}$

    a $y = a^3 b^2$ **b** $y = \dfrac{m}{\sqrt{n}}$

      $\therefore$ $\log y = \log(a^3 b^2)$

      $\therefore$ $\log y = \log a^3 + \log b^2$ $\therefore$ $\log y = \log \left(\dfrac{m}{n^{\frac{1}{2}}} \right)$

      $\therefore$ $\log y = 3 \log a + 2 \log b$

                                          $\therefore$ $\log y = \log m - \log n^{\frac{1}{2}}$

                                          $\therefore$ $\log y = \log m - \frac{1}{2} \log n$

> ### Example 9
> ◀ ⟩ **Self Tutor**
>
> Write these equations without logarithms:
>
> **a** $\log D = 2x + 1$ **b** $\log N \approx 1.301 - 2x$
>
> **a** $\log D = 2x + 1$
>
> $\therefore \quad D = 10^{2x+1}$
>
> or $D = (100)^x \times 10$
>
> **b** $\log N \approx 1.301 - 2x$
>
> $\therefore \quad N \approx 10^{1.301-2x}$
>
> $\therefore \quad N \approx \dfrac{10^{1.301}}{10^{2x}} \approx \dfrac{20}{10^{2x}}$

> ### Example 10
> ◀ ⟩ **Self Tutor**
>
> Write these equations without logarithms:
>
> **a** $\log C = \log a + 3 \log b$ **b** $\log G = 2 \log d - 1$
>
> **a** $\log C = \log a + 3 \log b$
>
> $\qquad\quad = \log a + \log b^3$
>
> $\qquad\quad = \log(ab^3)$
>
> $\therefore \quad C = ab^3$
>
> **b** $\log G = 2 \log d - 1$
>
> $\qquad\quad = \log d^2 - \log 10^1$
>
> $\qquad\quad = \log \left(\dfrac{d^2}{10} \right)$
>
> $\therefore \quad G = \dfrac{d^2}{10}$

EXERCISE 31D.2

1 Write the following as logarithmic equations in base 10:

   a $y = ab^2$ **b** $y = \dfrac{a^2}{b}$ **c** $y = d\sqrt{p}$

   d $M = a^2 b^5$ **e** $P = \sqrt{ab}$ **f** $Q = \dfrac{\sqrt{m}}{n}$

   g $R = abc^2$ **h** $T = 5\sqrt{\dfrac{d}{c}}$ **i** $M = \dfrac{ab^3}{\sqrt{c}}$

2 Write these equations without logarithms:

   a $\log Q = x + 2$ **b** $\log J = 2x - 1$ **c** $\log M = 2 - x$

   d $\log P \approx 0.301 + x$ **e** $\log R \approx x + 1.477$ **f** $\log K = \frac{1}{2}x + 1$

3 Write these equations without logarithms:

   a $\log M = \log a + \log b$ **b** $\log N = \log d - \log e$

   c $\log F = 2 \log x$ **d** $\log T = \frac{1}{2} \log p$

   e $\log D = -\log g$ **f** $\log S = -2 \log b$

   g $\log A = \log B - 2 \log C$ **h** $2 \log p + \log q = \log s$

   i $-\log d + 3 \log m = \log n - 2 \log p$ **j** $\log m - \frac{1}{2} \log n = 2 \log P$

   k $\log N = 1 + \log t$ **l** $\log P = 2 - \log x$

E   EXPONENTIAL AND LOGARITHMIC EQUATIONS   [3.10]

We have already seen how to solve equations such as $2^x = 5$ using technology. We now consider an algebraic method.

By definition, the exact solution is $x = \log_2 5$, but we need to know how to evaluate this number.

We therefore consider taking the logarithm of both sides of the original equation:

$$\log(2^x) = \log 5$$
$$\therefore \quad x \log 2 = \log 5 \qquad \{\text{logarithm law}\}$$
$$\therefore \quad x = \frac{\log 5}{\log 2}$$

We conclude that $\log_2 5 = \dfrac{\log 5}{\log 2}$.

In general: the solution to $a^x = b$ where $a > 0$, $b > 0$ is $x = \log_a b = \dfrac{\log b}{\log a}$.

Example 11   ◀) Self Tutor

Use logarithms to solve for x, giving answers correct to 3 significant figures:

   a $2^x = 30$ **b** $(1.02)^x = 2.79$ **c** $3^x = 0.05$

a $\quad 2^x = 30$	**b** $\quad (1.02)^x = 2.79$	**c** $\quad 3^x = 0.05$
$\therefore \quad x = \dfrac{\log 30}{\log 2}$	$\therefore \quad x = \dfrac{\log(2.79)}{\log(1.02)}$	$\therefore \quad x = \dfrac{\log(0.05)}{\log 3}$
$\therefore \quad x \approx 4.91$	$\therefore \quad x \approx 51.8$	$\therefore \quad x \approx -2.73$

Example 12   ◀) Self Tutor

Show that $\log_2 11 = \dfrac{\log 11}{\log 2}$.   Hence find $\log_2 11$.

Let $\log_2 11 = x$

$$\therefore \quad 2^x = 11$$
$$\therefore \quad \log(2^x) = \log 11$$
$$\therefore \quad x \log 2 = \log 11$$
$$\therefore \quad x = \frac{\log 11}{\log 2}$$
$$\therefore \quad \log_2 11 = \frac{\log 11}{\log 2} \approx 3.46$$

To solve logarithmic equations, we can sometimes write each side as a power of 10.

Example 13   ◀) Self Tutor

Solve for x: $\log_3 x = -1$

$$\log_3 x = -1$$
$$\therefore \quad \frac{\log x}{\log 3} = -1 \qquad \{\log_a b = \frac{\log b}{\log a}\}$$
$$\therefore \quad \log x = -1\log 3$$
$$\therefore \quad \log x = \log(3^{-1})$$
$$\therefore \quad \log x = \log\left(\tfrac{1}{3}\right)$$
$$\therefore \quad x = \tfrac{1}{3}$$

EXERCISE 31E

1 Solve for x using logarithms, giving answers to 4 significant figures:

  a $10^x = 80$ **b** $10^x = 8000$ **c** $10^x = 0.025$

  d $10^x = 456.3$ **e** $10^x = 0.8764$ **f** $10^x = 0.0001792$

2 Solve for x using logarithms, giving answers to 4 significant figures:

  a $2^x = 3$ **b** $2^x = 10$ **c** $2^x = 400$

  d $2^x = 0.0075$ **e** $5^x = 1000$ **f** $6^x = 0.836$

  g $(1.1)^x = 1.86$ **h** $(1.25)^x = 3$ **i** $(0.87)^x = 0.001$

  j $(0.7)^x = 0.21$ **k** $(1.085)^x = 2$ **l** $(0.997)^x = 0.5$

3 The weight of bacteria in a culture t hours after it has been established is given by $W = 2.5 \times 2^{0.04t}$ grams.
After what time will the weight reach:

  a 4 grams   **b** 15 grams?

4 The population of bees in a hive t hours after it has been discovered is given by $P = 5000 \times 2^{0.09t}$.
After what time will the population reach:

  a 15 000   **b** 50 000?

5 Answer the **Opening Problem** on page **625**.

6 Show that $\log_5 13 = \dfrac{\log 13}{\log 5}$.   Hence find $\log_5 13$.

7 Find, correct to 3 significant figures:

  a $\log_2 12$ **b** $\log_3 100$ **c** $\log_7 51$ **d** $\log_2(0.063)$

8 Solve for x:

  a $\log_2 x = 2$ **b** $\log_5 x = -2$ **c** $\log_2(x+2) = 2$ **d** $\log_5(2x) = -1$

Review set 31A

1   **a**   On the same set of axes, sketch the graphs of $y = 2^x$ and $y = \log_2 x$.

      b   What transformation would map $y = 2^x$ onto $y = \log_2 x$?

      c   State the domain and range of $y = \log_2 x$.

2   Copy and complete:

      a $\log_a a^x = $ **b**   if $y = b^x$ then $x = $, and vice versa.

3   Find the value of:

      a $\log_2 16$ **b** $\log_3 \left(\frac{1}{3} \right)$ **c** $\log_2 \sqrt{32}$ **d** $\log_4 8$

4   Write the following in terms of logarithms:

      a $y = 5^x$ **b** $y = 7^{-x}$

5   Write the following as exponential equations:

      a $y = \log_3 x$ **b** $T = \frac{1}{3} \log_4 n$

6   Make x the subject of:

      a $y = \log_5 x$ **b** $w = \log(3x)$ **c** $q = \dfrac{7^{2x}}{3}$

7   Find the inverse function, $f^{-1}(x)$ of:

      a $f(x) = 4 \times 5^x$ **b** $f(x) = 2 \log_3 x$

8   Solve for x, giving your answers correct to 5 significant figures:

      a $4^x = 100$ **b** $4^x = 0.001$ **c** $(0.96)^x = 0.013\,74$

9   The population of a colony of wasps t days after discovery is given by $P = 400 \times 2^{0.03t}$.

      a   How big will the population be after 10 days?

      b   How long will it take for the population to reach 1200 wasps?

10   Write as a single logarithm:

      a $\log 12 - \log 2$ **b** $2 \log 3 + \log 4$ **c** $2 \log_2 3 + 3 \log_2 5$

11   Write as a logarithmic equation in base 10:

      a $y = \dfrac{a^3}{b^2}$ **b** $M = 3\sqrt{\dfrac{a}{b}}$

12   Write as an equation without logarithms:

      a $\log T = -x + 3$ **b** $\log N = 2 \log c - \log d$

13   If $\log_2 3 = a$ and $\log_2 5 = b$, find in terms of a and b:

      a $\log_2 15$ **b** $\log_2 \left(1\frac{2}{3} \right)$ **c** $\log_2 10$

14   Find y in terms of u and v if:

      a $\log_2 y = 4 \log_2 u$ **b** $\log_5 y = -2 \log_5 v$ **c** $\log_3 y = \frac{1}{2} \log_3 u + \log_3 v$

15   Find $\log_3 15$ correct to 4 decimal places.

16   Use a graphics calculator to solve, correct to 4 significant figures:

      a $2^x = 4 - 3x$ **b** $\log x = 3^{-x}$

Review set 31B

1 **a** On the same set of axes, sketch the graphs of $y = 3^x$ and $y = \log_3 x$.

    b State the domain and range of each function.

2 Find the value of:

    a $\log_2 \sqrt{2}$ **b** $\log_2 \frac{1}{\sqrt{8}}$ **c** $\log_{\sqrt{3}} 27$ **d** $\log_9 27$

3 Write the following in terms of logarithms:

    a $y = 4^x$ **b** $y = a^{-n}$

4 Write the following as exponential equations:

    a $y = \log_2 d$ **b** $M = \frac{1}{2} \log_a k$

5 Make x the subject of:

    a $y = \log_3 x$ **b** $T = \log_b(3x)$ **c** $3t = 5 \times 2^{x+1}$

6 Find the inverse function $f^{-1}(x)$ of:

    a $f(x) = 6^x$ **b** $f(x) = \frac{1}{2} \log_5 x$

7 Solve for x, giving your answers correct to 4 significant figures:

    a $3^x = 3000$ **b** $(1.13)^x = 2$ **c** $2^{(2^x)} = 10$

8 The value of a rare banknote has been modelled by $V = 400 \times 2^{0.15t}$ US dollars, where t is the time in years since 1970.

    a What was the value of the banknote in 1970?

    b What was the value of the banknote in 2005?

    c When is the banknote expected to have a value of $100 000?

9 Write as a single logarithm:

    a $\log_2 5 + \log_2 3$ **b** $\log_3 8 - \log_3 2$ **c** $2 \log 5 - 1$ **d** $2 \log_2 5 - 1$

10 Write as a logarithmic equation in base 10:

    a $D = \dfrac{100}{n^2}$ **b** $G^2 = c^3 d$

11 Write as an equation without logarithms:

    a $\log M = 2x + 1$ **b** $\log G = \frac{1}{2} \log d - 1$

12 If $\log_3 7 = a$ and $\log_3 4 = b$, find in terms of a and b:

    a $\log_3 \left(\frac{4}{7}\right)$ **b** $\log_3 28$ **c** $\log_3 \left(\frac{7}{3}\right)$

13 Find y in terms of c and d if:

    a $\log_2 y = 2 \log_2 c$ **b** $\log_3 y = \frac{1}{3} \log_3 c - 2 \log_3 d$

14 Find $\log_7 200$ correct to 3 decimal places.

15 Use a graphics calculator to solve, correct to 4 significant figures:

    a $3^x = 0.6x + 2$ **b** $\log(2x) = (x - 1)(x - 4)$

Challenge

1   Where is the error in the following argument?

$$\tfrac{1}{2} > \tfrac{1}{4}$$

$\therefore \quad \log(\tfrac{1}{2}) > \log(\tfrac{1}{4})$

$\therefore \quad \log(\tfrac{1}{2}) > \log(\tfrac{1}{2})^2$

$\therefore \quad \log(\tfrac{1}{2}) > 2\log(\tfrac{1}{2})$

$\qquad \therefore \quad 1 > 2 \quad$ {dividing both sides by $\log(\tfrac{1}{2})$}

2   Solve for x:

   a $4^x - 2^{x+3} + 15 = 0$ **Hint:**   Let $2^x = m,$ say.

   b $\log x = 5\log 2 - \log(x + 4)$.

3   **a**   Find the solution of $2^x = 3$ to the full extent of your calculator's display.

   b   The solution of this equation is not a rational number, so it is irrational.

Consequently its decimal expansion is infinitely long and neither terminates nor recurs. Copy and complete the following argument which proves that the solution of $2^x = 3$ is irrational without looking at the decimal expansion.

Proof:

Assume that the solution of $2^x = 3$ is rational.

(The opposite of what we are trying to prove.)

$\therefore$ there exist positive integers p and q such that $x = \dfrac{p}{q}, \quad q \neq 0$

Thus $2^{\frac{p}{q}} = 3$

$\therefore \quad 2^p = \ldots\ldots$

and this is impossible as the LHS is $\ldots\ldots$ and the RHS is $\ldots\ldots$ no matter what values p and q may take.

Clearly, we have a contradiction and so the original assumption is incorrect.

Consequently, the solution of $2^x = 3$ is $\ldots\ldots$

4   Prove that:

   a   the solution of $3^x = 4$ is irrational

   b   the exact value of $\log_2 5$ is irrational.

Inequalities

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Contents:

Opening problem

Jason and Kate have set up a stall at the school fete. Their plan is to provide barbecued chops and sausages for visitors at lunch time. Their butcher will sell them sausages for 40 cents each and chops for $1.00 each. Initially they think of what they could purchase for $20. Examine the following questions:

a Could they spend all the money buying chops?
If so, how many could they purchase?

b Could they spend all the money buying sausages?
If so, how many could they purchase?

c If they purchase 10 chops, how many sausages could they purchase?

d Clearly these are *three* solutions to their problem, but are there more? If so, how many more solutions are there and how would we find them?

A   SOLVING ONE VARIABLE INEQUALITIES WITH TECHNOLOGY   [2.2]

Some inequalities can be easily solved by examining a graph. For example:

- if we want to find x such that $f(x) > 0$, we graph $y = f(x)$ and find values of x where the function is **above the x-axis**

- if we want to find x such that $f(x) < 0$, we graph $y = f(x)$ and find values of x where the function is **below the x-axis**
- if we want to find x such that $f(x) > g(x)$, we graph $y = f(x)$ and $y = g(x)$ on the same axes and find values of x for which the graph of $y = f(x)$ is **above** $y = g(x)$.

Example 1
◀) **Self Tutor**

Solve the inequality: $5 - 2x - x^2 < 0$.

Using a graphics calculator we plot $Y = 5 - 2X - X\char`\^2$.

The graph cuts the x-axis when $x \approx -3.45$ and 1.45.

So, the solution is $x < -3.45$ or $x > 1.45$
(to 3 significant figures).

$y = 5 - 2x - x^2$
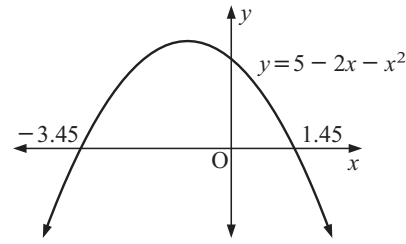
Example 2
◀) **Self Tutor**

Solve the inequality: $3 + x > 2^x$.

Using a graphics calculator we plot $Y = 2\char`\^X$ and
$Y = 3 + X$ on the same set of axes.

The x-coordinates of the points of intersection are:
$x \approx -2.86$ and 2.44.

The line $y = 3 + x$ is above the exponential $y = 2^x$
when $-2.86 < x < 2.44$.

$y = 2^x$
$y = 3 + x$
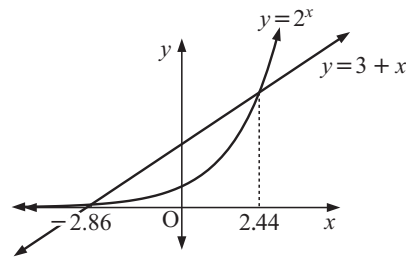
EXERCISE 32A

1 Solve for x using technology:

   a $1 - x < 3$ **b** $3 + x \geqslant 2$ **c** $4x - 1 > 5$

   d $2 - 3x \leqslant 1$ **e** $\frac{1}{2}x - 1 > 0$ **f** $3 - 5x \leqslant 3x + 1$

2 Solve for x:

   a $x^2 - x - 6 \geqslant 0$ **b** $x^2 + x - 12 \leqslant 0$ **c** $10 - x^2 - 3x \geqslant 0$

   d $x^2 - 2x - 5 < 0$ **e** $2x^3 - 5x + 6 > 0$ **f** $x^3 - x^2 - 2x + 1 < 0$

3 Solve for x:

   a $x^2 \geqslant x + 3$ **b** $3^x > 2$ **c** $2^x \geqslant 4 - x^2$

   d $x^2 > 2^x$ **e** $1 - x - x^2 > \dfrac{3}{x}$ **f** $\dfrac{1}{x} \leqslant x^2 + 1$

4 Solve for x: $\sin(2x) > 0.461$ for $0^o < x < 360^o$

5 Solve for x: $|x| + |1 - 2x| < 3$

B   LINEAR INEQUALITY REGIONS   [7.7]

Linear inequalities define regions of the Cartesian plane.

Consider the region $\mathcal{R}$ which is on or to the right of the line $x = 3$. This region is specified by the linear inequality $x \geqslant 3$, since all points within $\mathcal{R}$ have x-coordinates which are more than 3.
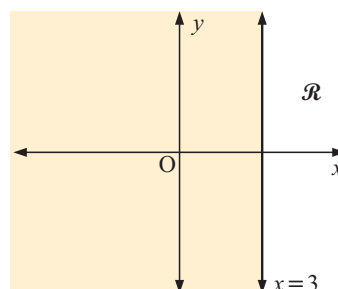
To illustrate this region we shade out all **unwanted** points. This makes it easier to identify the required region $\mathcal{R}$ when several inequalities define a region, as $\mathcal{R}$ is the region left unshaded.

We consider the boundary separately.

We use a solid boundary line to indicate that points on the boundary are wanted.

If the boundary is unwanted, we use a dashed boundary line.
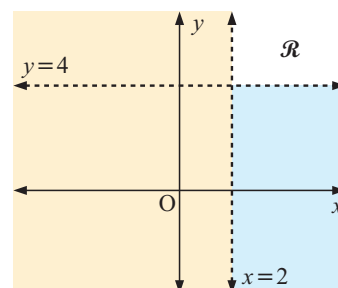
For example, to illustrate the region specified by $x > 2$ and $y > 4$, we shade the region on and to the left of the line $x = 2$, and the region on and below the line $y = 4$. The region $\mathcal{R}$ left completely unshaded is the region specified by $x > 2$ and $y > 4$. The lines $x = 2$ and $y = 4$ are dashed, which indicates the boundaries are not included in the region.

Example 3   ◀� Self Tutor

Write inequalities to represent the following unshaded regions:

a
b
c
d
e
f
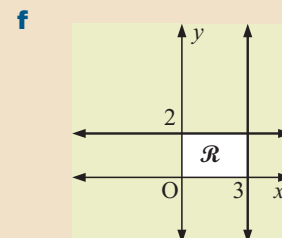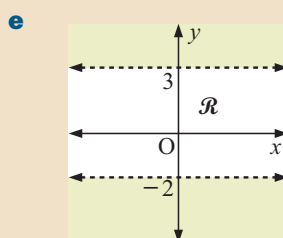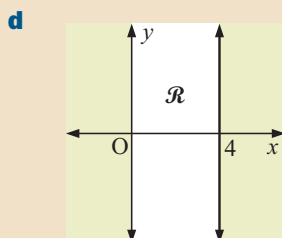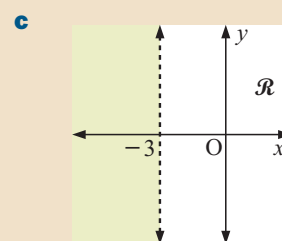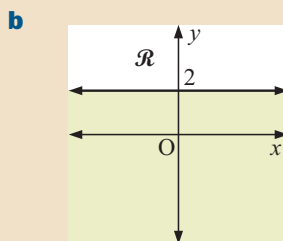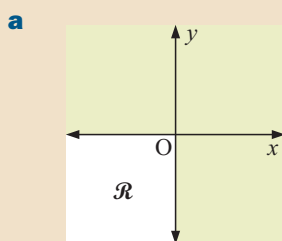
| a | $x \leqslant 0$ and $y \leqslant 0$ | b | $y \geqslant 2$ | c | $x > -3$ |
| d | $0 \leqslant x \leqslant 4$ | e | $-2 < y < 3$ | f | $0 \leqslant x \leqslant 3$ and $0 \leqslant y \leqslant 2$ |

Discovery   *Linear inequalities of the form* $ax + by < d$ *or* $ax + by > d$

Consider the regions on either side of the line with equation
$3x + 2y = 6$.

What to do:

a Copy and complete:

Point	$3x + 2y$	Point	$3x + 2y$
A(1, 3)	$3(1) + 2(3) = 9$	H(-2, 4)	$3(-2) + 2(4) = 2$
B(,)		I	
C(,)		J	
D(,)		K	
E(,)		L	
F(,)		M	
G(,)		O(0, 0)	
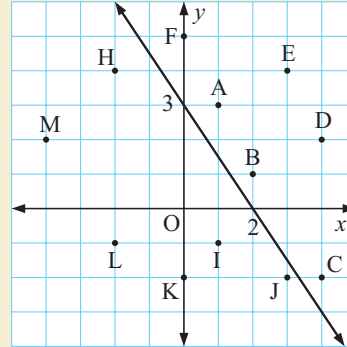
b Using **a**, what inequality defines the region under the line $3x + 2y = 6$?
What inequality defines the region above the line $3x + 2y = 6$?

You should have discovered that:

> All points satisfying $ax + by < d$ lie on one side of the line $ax + by = d$ and all points
> satisfying $ax + by > d$ lie on the other side.

To find the region which corresponds to an inequality, we substitute into the inequality a point not on the boundary line, usually O(0, 0).

If a true statement results then this point lies in the region we want. If not, then the required region is the other side of the line.

Example 4

🔊 *Self Tutor*

Graph $3x - 4y > 12$.

The boundary line is $3x - 4y = 12$.
It is not included in the region.

When $x = 0$, $-4y = 12$
$\therefore$ $y = -3$

When $y = 0$, $3x = 12$
$\therefore$ $x = 4$

So, $(0, -3)$ and $(4, 0)$ lie on the boundary.

If we substitute $(0, 0)$ into $3x - 4y > 12$
we obtain $0 > 12$ which is false.

$\therefore$ $(0, 0)$ does not lie in the region.

So, in this case we want the region below the line $3x - 4y = 12$.
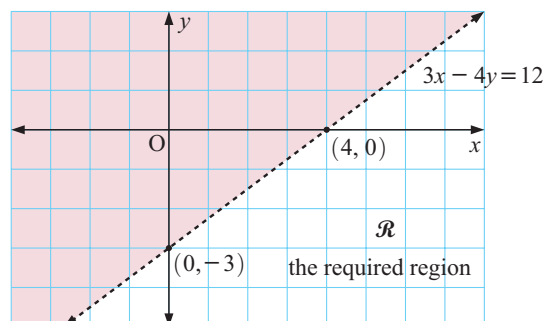
EXERCISE 32B

1 Write inequalities to represent the following unshaded regions, $\mathcal{R}$:

a

b

c

d

e

f

g

h

i
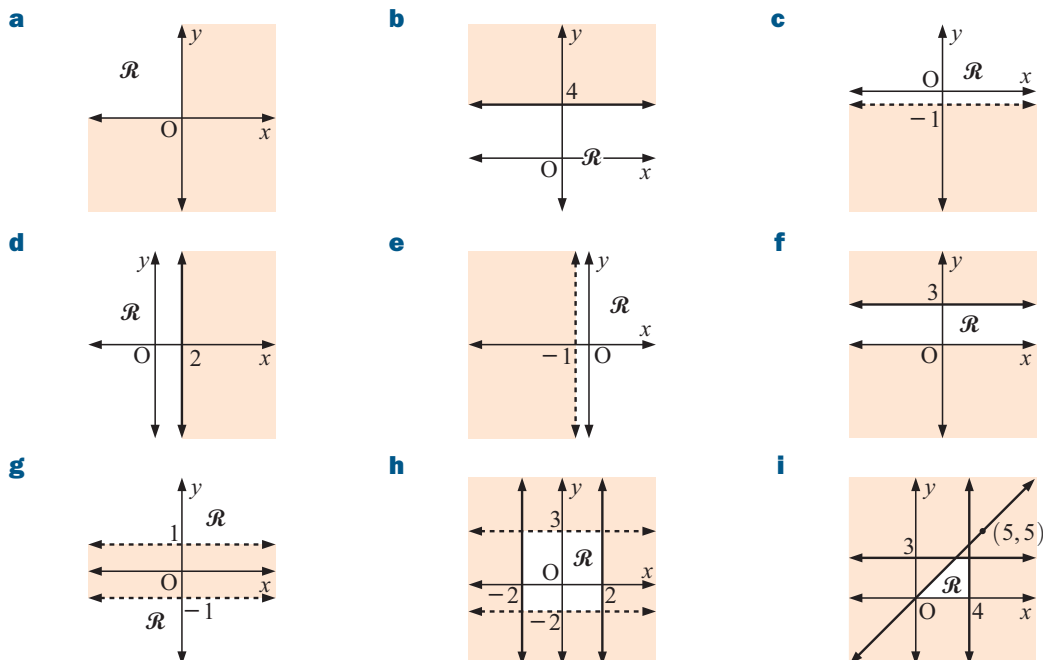
2 Graph the regions defined by:

   a $x < 4$
   b $x \geqslant -1$
   c $y \geqslant 2$

   d $y < -1$
   e $0 < x < 3$
   f $-1 \leqslant y \leqslant 2$

   g $-2 < x \leqslant 3$
   h $-1 \leqslant y < 4$
   i $0 \leqslant x \leqslant 3$ and $1 \leqslant y \leqslant 4$

3 Graph the regions defined by:

   a $x + 2y \leqslant 4$
   b $2x + y > 5$
   c $3x + 2y < 6$

   d $2x + 3y \geqslant 6$
   e $2x - 3y < 12$
   f $5x + 2y > 10$

   g $2x - 5y < -10$
   h $4x + 3y < 6$
   i $4x - 3y \geqslant 12$

   j $y \geqslant x$
   k $y < -x$
   l $2x + 5y \geqslant 0$

4 Graph the regions defined by:

   a $x \geqslant 0$ and $y \geqslant 2$
        b $x \leqslant -2$ and $y \geqslant 4$

   c $x \geqslant 0$, $y \geqslant 0$ and $x + y \leqslant 4$
    d $x \geqslant 0$, $y \geqslant 0$ and $2x + y < 6$

   e $x \geqslant 2$, $y \geqslant 0$ and $x + y \geqslant 6$
    f $x \geqslant 0$, $y \geqslant 3$ and $2x + 3y \geqslant 12$

5 Write down inequalities which represent the unshaded region $\mathcal{R}$ of:

a

b
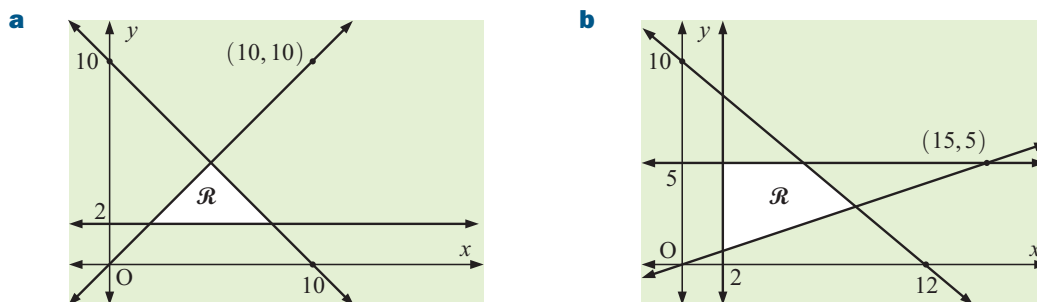
C   *INTEGER POINTS IN REGIONS*   [7.7]

In many problems involving inequalities, the only points which are possible are those with integer coordinates. We therefore consider only these points within the region $\mathcal{R}$.

We are often asked to find the minimum or maximum value of a function in the region $\mathcal{R}$. If all of the constraints are linear, the minimum and maximum **must** occur at a vertex or corner point of the region $\mathcal{R}$. If all of the vertices have integer coordinates, we need only consider these points when finding the maximum or minimum. However, if we are considering only integer points and the vertices are not all integer points, we need to be more careful.

Example 5   ◀ଽ *Self Tutor*

A region $\mathcal{R}$ is defined by $x \geqslant 2$, $y \geqslant 3$, $x + y \leqslant 10$, $x + 2y \leqslant 14$.

  a On grid paper, graph the region $\mathcal{R}$.

  b How many points in $\mathcal{R}$ have integer coordinates?

  c Find all points in $\mathcal{R}$ with integer coordinates that lie on the line $y = \frac{1}{2}x + 3$.

  d Find the maximum value of $5x + 4y$ for the 15 feasible points in $\mathcal{R}$.

a
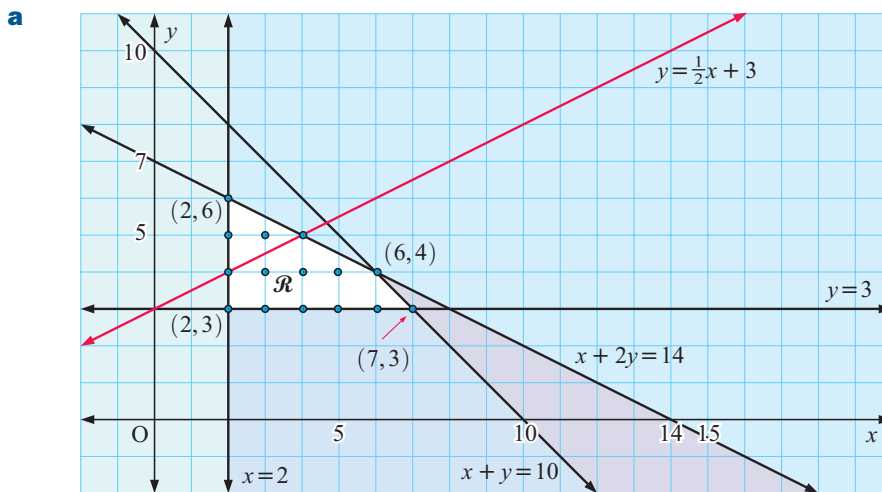
  b We look for all points in $\mathcal{R}$ where the grid lines meet.
         There are 15 points with integer coordinates, as shown.

  c The line $y = \frac{1}{2}x + 3$ passes through $(2, 4)$ and $(4, 5)$ in $\mathcal{R}$.

  d The vertices all have integer coordinates, so we need only consider these points:

Points	$5x + 4y$
$(2, 3)$	$10 + 12 = 22$
$(2, 6)$	$10 + 24 = 34$
$(6, 4)$	$30 + 16 = 46$
$(7, 3)$	$35 + 12 = 47$

$5x + 4y$ has maximum value 47 when $x = 7$, $y = 3$.

EXERCISE 32C

1 Consider the region $\mathcal{R}$ defined by: $x \geqslant 0$, $x + 2y \geqslant 8$ and $x + y \leqslant 6$.

    a Graph the region $\mathcal{R}$ on grid paper.

    b How many points in $\mathcal{R}$ have integer coordinates?

    c How many of these points obey the rule $y - x = 4$?

    d Find the greatest and least values of $2x + y$ for all $(x, y) \in \mathcal{R}$ with integer coordinates.

2 Consider the region $\mathcal{R}$ defined by: $x \geqslant 0$, $y \geqslant 0$, $x + y \leqslant 8$ and $x + 3y \leqslant 12$.

    a Graph the region $\mathcal{R}$ on grid paper.

    b Find the largest value of the following and the corresponding values of integers x and y:

        i $2x + 3y$ **ii** $x + 4y$ **iii** $3x + 3y$.

3 **a** Graph the region $\mathcal{R}$ in which $x \geqslant 0$, $y \geqslant 0$, $x + 2y \geqslant 12$ and $3x + 2y \geqslant 24$.

    b Find all integer values of x and y in $\mathcal{R}$ such that $2x + 3y = 24$.

    c Find the minimum value of $2x + 5y$ for all $(x, y) \in \mathcal{R}$ with integer coordinates.

    d Find the minimum value of $12x + 8y$ for all $(x, y) \in \mathcal{R}$ with integer coordinates.
State the coordinates of any point where this minimum value occurs.

D   PROBLEM SOLVING (EXTENSION)

This section is an extension to the syllabus item **7.7** and is a good example of **mathematical modelling**.

In some problems we need to construct our own inequalities. These inequalities are called **constraints**. A table may be used to help sort out the given information.

Example 6   ◆ϑ *Self Tutor*

Consider the production of 2-drawer and 5-drawer filing cabinets. We only have 34 drawers, 8 locks, and 42 square metres of sheet metal available.

Each 2-drawer cabinet requires 1 lock and 2 square metres of sheet metal, while each 5-drawer cabinet requires 1 lock and 4 m^2 of sheet metal. Let x be the number of 2-drawer filing cabinets made and y be the number of 5-drawer filing cabinets made. List the constraints connecting x and y.

We put the data in table form:

Type	Number of cabinets	Locks per cabinet	Metal per cabinet
2-drawer	x	1	2 m^2
5-drawer	y	1	4 m^2

$x \geqslant 0$, $y \geqslant 0$ since we cannot make negative cabinets.

The total number of drawers $= 2x + 5y$,   so $2x + 5y \leqslant 34$.

The total number of locks $= x + y$,   so $x + y \leqslant 8$.

The total number of m^2 of metal $= 2x + 4y$,   so $2x + 4y \leqslant 42$.

EXERCISE 32D

1 4 litre cans of base white paint are converted into lime green or pine green by adding yellow tint and blue tint in different proportions. For lime green we add 5 units of yellow tint to one unit of blue tint. For pine green we add 1 unit of yellow tint to 4 units of blue tint.

If 15 units of yellow tint and 12 units of blue tint are available, state inequalities connecting x, the number of cans of lime green paint that can be made, and y, the number of cans of pine green paint that can be made.

2 An importer purchases two types of baseball helmet: standard helmets cost \$80 each and deluxe helmets cost \$120 each. The importer wants to spend a maximum of \$4800, and because of government protection to local industry, can import no more than 50 helmets. Suppose the importer purchases x standard helmets and y deluxe helmets. List the constraints on the variables x and y.

3 A farmer has a week in which to plant lettuces and cauliflowers. Lettuces can be planted at a rate of 8 ha per day and cauliflowers at a rate of 6 ha per day. 50 ha are available for planting.

Suppose the farmer plants lettuces for x days and cauliflowers for y days. List, with reasons, the constraints involving x and y.

4 Two varieties of special food are used by athletes. Variety A contains 30 units of carbohydrate, 30 units of protein, and 100 units of vitamins. Variety B contains 10 units of carbohydrate, 30 units of protein, and 200 units of vitamins. Each week an athlete must consume at least 170 units of carbohydrate, at least 1400 units of vitamins, but no more than 330 units of protein.
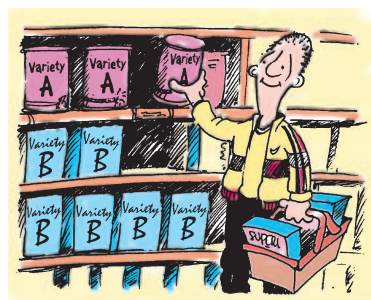
   a Let the number of tins of variety A be x and the number of tins of variety B be y. Explain each of the constraints: $x \geqslant 0$, $y \geqslant 0$, $3x + y \geqslant 17$, $x + y \leqslant 11$ and $x + 2y \geqslant 14$.

   b Graph the region $\mathcal{R}$ defined by these five inequalities.

   c If variety A costs £5 per tin and variety B costs £3 per tin, find the combination which minimises the cost.

   d If the prices of tins change to £4 per tin for variety A and £9 for variety B, what combination will now minimise the cost?

5 A librarian has space for 20 new books. He needs to spend at least €240 to use his annual budget. Hardback books cost €30 each and softback books cost €10 each.
If he buys x hardbacks and y softbacks:

   a explain why $x \geqslant 0$, $y \geqslant 0$, $x + y \leqslant 20$, and $3x + y \geqslant 24$.

   b graph the region defined by the constraints.

   c Use your region to find:

      i the smallest number of books the librarian can buy

      ii the largest amount of money the librarian can spend.

6 A manufacturer produces two kinds of table-tennis sets:

        Set A contains 2 bats and 3 balls,   Set B contains 2 bats, 5 balls and 1 net.

In one hour the factory can produce at most 56 bats, 108 balls and 18 nets. Set A earns a profit of \$3 and Set B earns a profit of \$5.

   a Summarise the information in a table assuming x sets of A and y sets of B are made each hour.

   b Write an expression for the total profit made \$P.

   c List the constraints.

   d Determine how many of each set the manufacturer should make each hour to maximise the profit.

   e Are any components under-utilised when the maximum profit is achieved?

7 A manufacturer of wheelbarrows makes two models, Deluxe
   and Standard.
   For the Deluxe model he needs machine A for 2 minutes
   and machine B for 2 minutes.
   For the Standard model he needs machine A for 3 minutes
   and machine B for 1 minute.
   Machine A is available for at most 48 minutes and machine
   B for 20 minutes every hour.
   He knows from past experience that he will sell at least twice as many of the Standard model as the
   Deluxe model. The Deluxe model earns him £25 profit and the Standard model earns £20 profit.

   a Construct a set of constraints if x Deluxe models and y Standard models are made.

   b Write an expression for the profit $£P$ in terms of x and y.

   c Graph the region defined by the constraint inequalities.

   d How many of each model should be made per hour in order to maximise the profits?
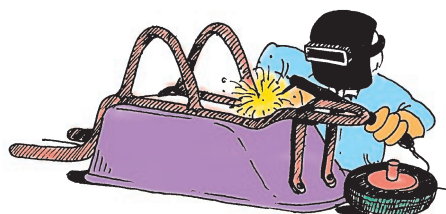
Review set 32A

1 Use a graphics calculator to solve these inequalities:

   a $x^2 + 4x - 1 > 0$ **b** $x^3 + 11x < 6x^2 + 5$

2 Write inequalities which completely specify these unshaded regions $\mathcal{R}$:

   a

   b
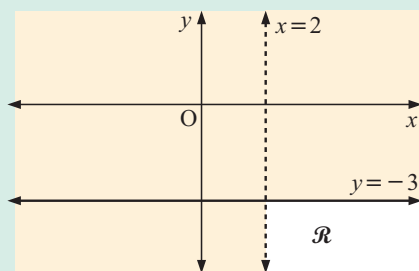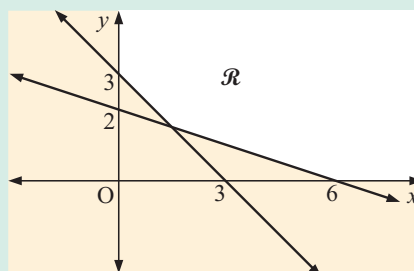
3 **a** Find all points with integer coordinates which lie in the region defined by:
          $x \geqslant 2$, $y \geqslant 1$, $x + y \leqslant 7$, $x + y \geqslant 5$, $2x + y \leqslant 10$.

   b If the constraint $x \geqslant 2$ changes to $x > 2$, what effect does this have on your answers in **a**?

4 **a** Graph the region $\mathcal{R}$ defined by the inequalities: $x \geqslant 0$, $y \geqslant 0$, $x + y \geqslant 12$, $x + 2y \geqslant 16$.

   b Find all points (x, y) in $\mathcal{R}$ with integer coordinates such that $x + y = 14$.

   c Find the minimum value of $3x + 2y$ for all points in $\mathcal{R}$ where x and y are integers.

5 A factory makes gas meters and water meters. Gas meters need 4 gears, 1 dial, and 8 minutes of
   assembly time for a profit of $20. Water meters need 12 gears, 1 dial, and 4 minutes of assembly
   time for a profit of $31. There are 60 gears, 9 dials, and 64 minutes of assembly time available for
   use in this production.

a Copy and complete the table:

Meter	Gears	Dials	Assembly time (min)	Profit	Number
Gas					x
Water					y

b Construct five constraints involving x and y.

c Graph the region $\mathcal{R}$ defined by the inequalities in **b**.

d Write an expression for the profit in terms of x and y.

e How many of each meter need to be made to maximise the profit?

Review set 32B

1 Use a graphics calculator to solve these inequalities:

a $2^x > \dfrac{1}{x}$

b $x^4 + x \leqslant x^3 + x^2 + 1$

2 Write inequalities which completely specify these unshaded regions $\mathcal{R}$:

a

b
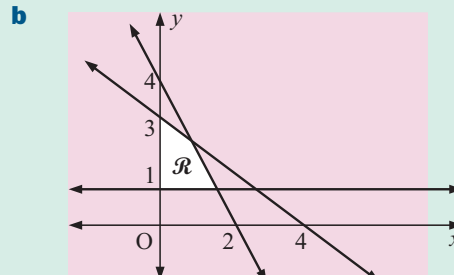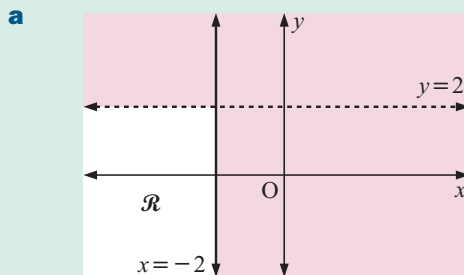
3 a Find all points with integer coordinates which lie in the region defined by:
$x \geqslant 0, \quad y \geqslant 2, \quad x + y \leqslant 5, \quad x + 2y \geqslant 6, \quad 3x + 2y \leqslant 12.$

b List the points in **a** for which $y = x + 3$.

c Which of the integer solutions in **a** would maximise the expression $5x + 4y$?

4 A manufacturer makes 2-drawer filing cabinets, and desks with a single drawer. The 2-drawer cabinet uses 1 lock and 3 square metres of metal, and yields €34 profit. The desk uses 1 lock and 9 m² of metal, and yields €47 profit. There are 14 drawers, 8 locks, and 54 square metres of metal available. Find how many of each should be produced to earn the highest possible profit.

The following questions are classified as multi-topic as they consist of questions from at least two different parts of the syllabus.

For example, consider:

Adapted from May 2008, Paper 4

6 The pentagon OABCD is shown on the grid.

  a Write as column vectors:

    i $\overrightarrow{OD}$ **ii** $\overrightarrow{BC}$

  b Describe fully the single transformation which maps side BC onto side OD.

  c One of the inequalities which defines the shaded region inside the pentagon is $y \leqslant \frac{1}{2}x + 4$. What are the other four inequalities?
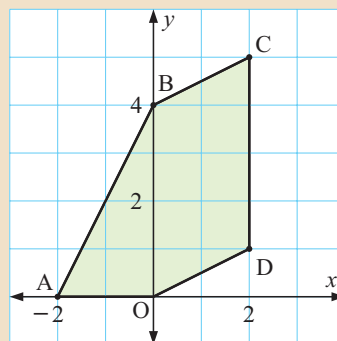
Notice that this is a multi-topic question as **vectors**, **transformations** and **inequality regions** are being considered within it.

Many of the questions in **Chapters 33** and **34** are adapted from past examination papers for IGCSE Mathematics 0580 by permission of the University of Cambridge Local Examinations Syndicate. The 0580 course is a different syllabus from that followed by students of the 0607 course, but has many features in common. These questions are certainly appropriate for practising mathematical techniques and applications relevant to the 0607 curriculum, but do not necessarily represent the style of question that will be encountered on the 0607 examination papers. Teachers are referred to the specimen papers of the 0607 syllabus for a more representative group of questions. The University of Cambridge Local Examinations Syndicate bears no responsibility for the example answers to questions taken from its past question papers which are contained in this publication.

MULTI-TOPIC QUESTIONS

1 The pentagon OABCD is shown on the grid.

   a Write as column vectors:

     i $\overrightarrow{OD}$ **ii** $\overrightarrow{BC}$

   b Describe fully the single transformation which maps side BC onto side OD.

   c One of the inequalities which defines the shaded region inside the pentagon is $y \leqslant \frac{1}{2}x + 4$. What are the other four inequalities?
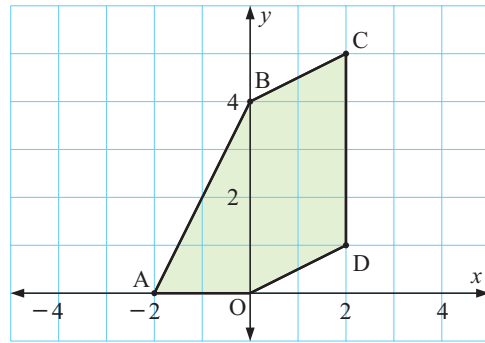
2 **May 2008, Paper 4**

A circle, centre O, touches all the sides of the regular octagon ABCDEFGH shaded in the diagram.

The sides of the octagon are of length 12 cm.

BA and GH are extended to meet at P. HG and EF are extended to meet at Q.

   a   **i** Show that angle BAH is 135°.

      ii Show that angle APH is 90°.

   b Calculate

      i the length of PH

      ii the length of PQ

      iii the area of triangle APH

      iv the area of the octagon.

   c Calculate

      i the radius of the circle

      ii the area of the circle as a percentage of the area of the octagon.
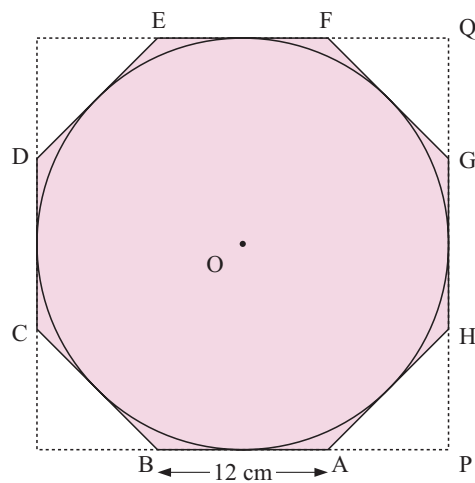
3 **Adapted from May 2008, Paper 4**

Vreni took part in a charity walk. She walked a distance of 20 kilometres.

   a She raised money at a rate of $12.50 for each kilometre.

      i How much money did she raise by walking the 20 kilometres?

      ii The money she raised in **a i** was $\frac{5}{52}$ of the total money raised.
         Work out the total money raised.

      iii In the previous year the total money raised was $2450.
         Calculate the percentage increase on the previous year's total.

   b Part of the 20 kilometres was on a road and the rest was on a footpath.
     The ratio   road distance : footpath distance   was 3 : 2.

      i Work out the road distance.

      ii Vreni walked along the road at 3 km/h and along the footpath at 2.5 km/h. How long, in hours and minutes, did Vreni take to walk the 20 kilometres?

      iii Work out Vreni's average speed.

      iv Vreni started at 08:55. At what time did she finish?

   c On a map, the distance of 20 kilometres was represented by a length of 80 centimetres.
     The scale of the map was 1 : n.
     Calculate the value of n.

4 **Nov 2006, Paper 4**

Maria, Carolina and Pedro receive $800 from their grandmother in the ratio

Maria : Carolina : Pedro $= 7 : 5 : 4$.

a Calculate how much money each receives.

b Maria spends $\frac{2}{7}$ of her money and then invests the rest for two years at 5% per year simple interest. How much money does Maria have at the end of the two years?

c Carolina spends all of her money on a hi-fi set and two years later sells it at a loss of 20%. How much money does Carolina have at the end of the two years?

d Pedro spends some of his money and at the end of the two years he has $100. Write down and simplify the ratio of the amounts of money Maria, Carolina and Pedro have at the end of the two years.

e Pedro invests his $100 for two years at a rate of 5% per year *compound interest*. Calculate how much money he has at the end of these two years.

5 **Adapted from Nov 2007, Paper 4**

The table shows some terms of several sequences.

Term	1	2	3	4	......	8	......
Sequence **P**	7	5	3	1	......	p	......
Sequence **Q**	1	8	27	64	......	q	......
Sequence **R**	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	......	r	......
Sequence **S**	4	9	16	25	......	s	......
Sequence **T**	1	3	9	27	......	t	......
Sequence **U**	3	6	7	-2	......	u	......

a Find the values of p, q, r, s, t and u.

b Find the nth term of sequence

   i **P**   **ii** **Q**   **iii** **R**   **iv** **S**   **v** **T**   **vi** **U**

c Which term in sequence **P** is equal to -777?

d Which term in sequence **T** is equal to $177\,147$?

6 **Adapted from May 2007, Paper 4**

OBCD is a rhombus with sides of 25 cm. The length of the diagonal OC is 14 cm.

a Show, *by calculation*, that the length of the diagonal BD is 48 cm.

b Calculate, correct to the nearest degree,

   i angle BCD   **ii** angle OBC.

c $\overrightarrow{DB} = 2\mathbf{p}$ and $\overrightarrow{OC} = 2\mathbf{q}$.

Find, in terms of $\mathbf{p}$ and $\mathbf{q}$, **i** $\overrightarrow{OB}$ **ii** $\overrightarrow{OD}$

d BE is parallel to OC and DCE is a straight line.

Find, in its simplest form, $\overrightarrow{OE}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

e M is the midpoint of CE.

Find, in its simplest form, $\overrightarrow{OM}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

f O is the origin of a coordinate grid. OC lies along the x-axis and $\mathbf{q} = \binom{7}{0}$.

($\overrightarrow{DB}$ is vertical and $|\overrightarrow{DB}| = 48$). Write down as column vectors **i** $\mathbf{p}$ **ii** $\overrightarrow{BC}$

g Write down the value of $|\overrightarrow{DE}|$.
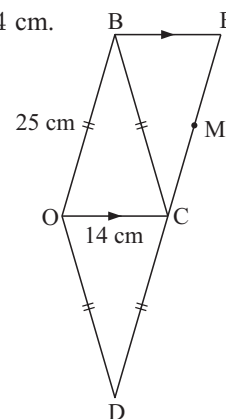
7 May 2006, Paper 4

The length, y, of a solid is inversely proportional to the square of its height, x.

  a Write down a general equation for x and y.
      Show that when $x = 5$ and $y = 4.8$ the equation becomes $x^2y = 120$.
  b Find y when $x = 2$.
  c Find x when $y = 10$.
  d Find x when $y = x$.
  e Describe exactly what happens to y when x is doubled.
  f Describe exactly what happens to x when y is decreased by 36%.
  g Make x the subject of the formula $x^2y = 120$.

8 Adapted from May 2007, Paper 4

diagram 1   diagram 2   diagram 3
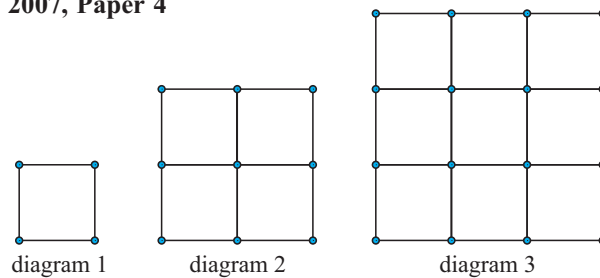
The first three diagrams in a sequence are shown above.
The diagrams are made up of dots and lines. Each line is one centimetre long.

  a Make a sketch of the next diagram in the sequence.
  b The table below shows some information about the diagrams.

Diagram	1	2	3	4	......	n
Area	1	4	9	16	......	x
Number of dots	4	9	16	p	......	y
Number of one centimetre lines	4	12	24	q	......	z

  i Write down the values of p and q.
  ii Write down each of x, y and z in terms of n.

  c The *total* number of one centimetre lines in the first n diagrams is given by the expression $en^3 + fn^2 + gn$.
      i Find the values of e, f and g.
      ii Find the total number of one centimetre lines in the first 10 diagrams.

9 Nov 2005, Paper 4

A Spanish family went to Scotland for a holiday.

  a The family bought 800 pounds (£) at a rate of £1 = 1.52 euros (€). How much did this cost in euros?
  b The family returned home with £118 and changed this back into euros. They received €173.46. Calculate how many euros they received for each pound.
  c A toy which costs €11.50 in Spain costs only €9.75 in Scotland. Calculate, as a percentage of the cost in Spain, how much *less* it costs in Scotland.
  d The total cost of the holiday was €4347.00. In the family there were 2 adults and 3 children. The cost for one adult was double the cost for one child. Calculate the cost for one child.

    e The *original* cost of the holiday was *reduced* by 10% to €4347.00. Calculate the original cost.

    f The plane took 3 hours 15 minutes to return to Spain. The length of this journey was 2350 km. Calculate the average speed of the plane in

        i kilometres per hour   **ii** metres per second.

10 **Adapted from May 2006, Paper 4**

The diagram shows a pyramid on a horizontal rectangular base ABCD. The diagonals of ABCD meet at E. P is vertically above E. AB = 8 cm, BC = 6 cm and PC = 13 cm.
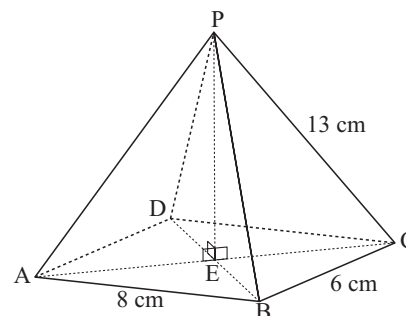
    a Calculate PE, the height of the pyramid.

    b Calculate the volume of the pyramid.

        (The volume of a pyramid is given by $\frac{1}{3} \times$ area of base $\times$ height.)

    c Calculate angle PCA.

    d M is the midpoint of AD and N is the midpoint of BC. Calculate angle MPN.

    e **i** Calculate angle PBC.

        ii K lies on PB so that BK = 4 cm. Calculate the length of KC.

11 **Adapted from May 2006, Paper 4**

    a The numbers 0, 1, 1, 1, 2, k, m, 6, 9, 9 are in order $(k \neq m)$.

        Their median is 2.5 and their mean is 3.6.

        i Write down the mode.

        ii Find the value of k.

        iii Find the value of m.

        iv Maria chooses a number at random from the list.

            The probability of choosing this number is $\frac{1}{5}$. Which number does she choose?

    b 100 students are given a question to answer.

        The time taken (t seconds) by each student is recorded and the results are shown in the table.

t	$0 < t \leqslant 20$	$20 < t \leqslant 30$	$30 < t \leqslant 35$	$35 < t \leqslant 40$	$40 < t \leqslant 50$	$50 < t \leqslant 60$	$60 < t \leqslant 80$
Frequency	10	10	15	28	22	7	8

        i Calculate an estimate of the mean time taken.

        ii Two students are picked at random. What is the probability that they both took more than 50 seconds? Give your answer as a fraction in its lowest terms.

    c **Answer this part on a sheet of graph paper.**

        The data in part **b** is re-grouped to give the following table.

t	$0 < t \leqslant 30$	$30 < t \leqslant 60$	$60 < t \leqslant 80$
Frequency	p	q	8

        i Write down the values of p and q.

        ii Draw an accurate histogram to show these results. Use a scale of 1 cm to represent 5 seconds on the horizontal time axis. Use a scale of 1 cm to 0.2 units of frequency density (so that 1 cm² on your histogram represents 1 student).

12 **May 2007, Paper 4**
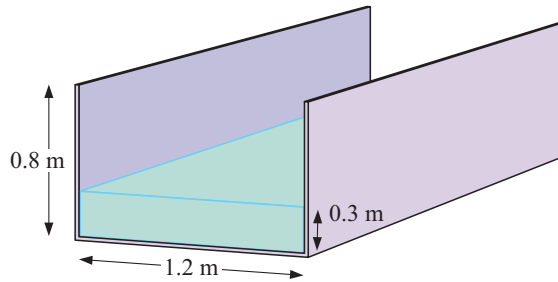
The diagram shows water in a channel.
This channel has a rectangular cross-section, 1.2 metres by 0.8 metres.

a When the depth of water is 0.3 metres, the water flows along the channel at 3 metres/minute.
Calculate the number of cubic metres which flow along the channel in one hour.

b When the depth of water in the channel increases to 0.8 metres, the water flows at 15 metres/minute.
Calculate the percentage increase in the number of cubic metres which flow along the channel in one hour.

c The water comes from a cylindrical tank.
When 2 cubic metres of water leave the tank, the level of water in the tank goes down by 1.3 millimetres. Calculate the radius of the tank, in *metres*, correct to one decimal place.

d When the channel is empty, its *interior* surface is repaired. This costs $0.12 per square metre.
The total cost is $50.40. Calculate the length, in metres, of the channel.

13 **Adapted from Nov 2004, Paper 4**

Quadrilaterals P and Q have diagonals which ● are unequal ● intersect at right angles.
P has two lines of symmetry, Q has one line of symmetry.

a **i** Sketch quadrilateral P. Write down its geometrical name.
ii Sketch quadrilateral Q. Write down its geometrical name.

b In quadrilateral P, an angle between one diagonal and a side is x^o.
Write down, in terms of x, the four angles of quadrilateral P.

c The diagonals of quadrilateral Q have lengths 20 cm and 12 cm.
Calculate the area of quadrilateral Q.

d Quadrilateral P has the same area as quadrilateral Q in **c**.
The lengths of the diagonals and sides of quadrilateral P are all integer values.
Find the length of a side of quadrilateral P.

14 **May 2005, Paper 4**

OABCDE is a regular hexagon. With O as origin the position vector of C is **c** and the position vector of D is **d**.

a Find, in terms of **c** and **d**,
i $\overrightarrow{DC}$
ii $\overrightarrow{OE}$
iii the position vector of B.

b The sides of the hexagon are each of length 8 cm. Calculate
i the size of angle ABC
ii the area of triangle ABC
iii the length of the straight line AC
iv the area of the hexagon.
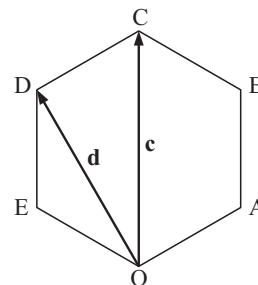
15 **Nov 2004, Paper 4**

Water flows through a pipe into an empty cylindrical tank. The tank has a radius of 40 cm and a height of 110 cm.

a Calculate the volume of the tank.

b The pipe has a cross-sectional area of 1.6 cm^2. The water comes out of the pipe at a speed of 14 cm/s. How long does it take to fill the tank? Give your answer in hours and minutes, correct to the nearest minute.

c All the water from the tank is added to a pond which has a surface area of 70 m^2. Work out the increase in depth of water in the pond. Give your answer in millimetres, correct to the nearest millimetre.

16 **Adapted from Nov 2004, Paper 4**

a During a soccer match a player runs from A to B and then from B to C as shown in the diagram.
AB = 40 m, BC = 45 m and AC = 70 m.

i Show by calculation that angle   BAC = 37°, correct to the nearest degree.

ii The bearing of C from A is   051°. Find the bearing of B from A.

iii Calculate the area of triangle ABC.

b x- and y-axes are shown in the diagram.

$\overrightarrow{AC} = \begin{pmatrix} p \\ q \end{pmatrix}$, where p and q are measured in metres.

i Show that $p = 54.4$.   **ii** Find the value of q.

c Another player is standing at D. BC = 45 m, angle BCD = 54° and angle DBC = 32°. Calculate the length of BD.
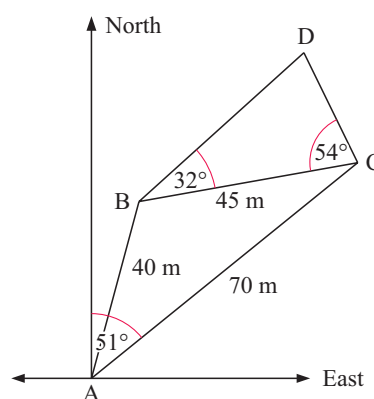
17 **May 2005, Paper 4**
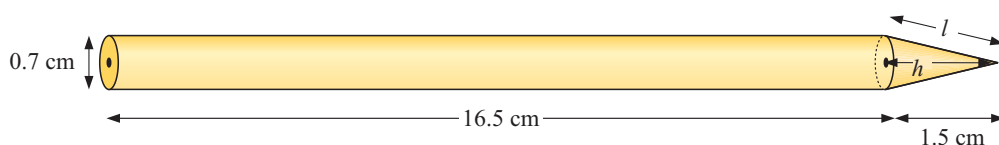
The diagram shows a pencil of length 18 cm. It is made from a cylinder and a cone. The cylinder has *diameter* 0.7 cm and length 16.5 cm. The cone has *diameter* 0.7 cm and length 1.5 cm.

a Calculate the volume of the pencil.
(The volume, V, of a cone of radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.)

b Twelve of these pencils just fit into a rectangular box of length 18 cm, width w cm and height x cm. The pencils are in 2 rows of 6 as shown in the diagram.

i Write down the values of w and x.   **ii** Calculate the volume of the box.

iii Calculate the percentage of the volume of the box occupied by the pencils.

c Showing all your working, calculate

i the slant height, l, of the cone,
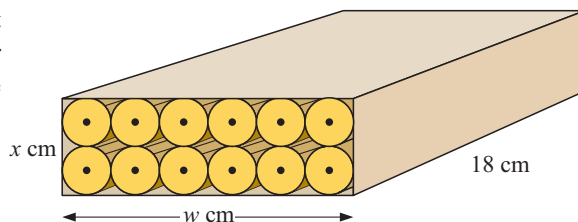
    ii the *total* surface area of *one* pencil, giving your answer correct to 3 significant figures.
        (The curved surface area, A, of a cone of radius r and *slant* height l is given by $A = \pi r l$.)

18 **Adapted from May 2002, Paper 4**

An equilateral 16-sided figure $APA'QB.....$ is formed when the square ABCD is rotated 45° clockwise about its centre to position $A'B'C'D'$. $AB = 12$ cm and $AP = x$ cm.

  a   **i** Use triangle $PA'Q$ to explain why
         $2x^2 = (12 - 2x)^2$.

     ii Show that this simplifies to
         $x^2 - 24x + 72 = 0$.

    iii Solve $x^2 - 24x + 72 = 0$. Give your answers correct to 2 decimal places.

  b   **i** Calculate the perimeter of the 16-sided figure.

     ii Calculate the area of the 16-sided figure.
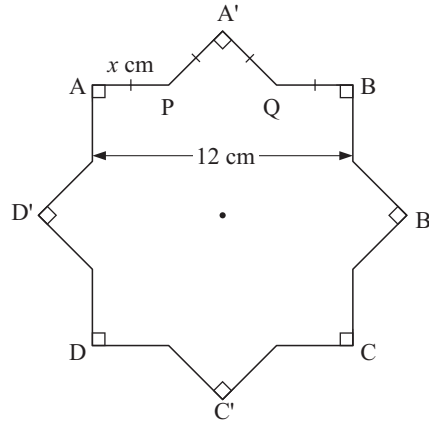
19 **Adapted from May 2004, Paper 4**

A cone, ATB, and a section of a sphere, ASB, share the same circular base, centre C, radius r. The height, TC, of the cone is h and STOC is a straight line. The radius, OB, of the sphere is R and the height, CS, of the section of the sphere is H.

  a $r = 6$ cm, $h = 14$ cm and $R = 10$ cm.

     i Calculate the volume of the cone ABT.
       (The volume of a cone with base radius r
       and height h is $\frac{1}{3}\pi r^2 h$.)

     ii Show that the height, SC, of the section of the sphere is 18 cm.

    iii Calculate the volume of the section of the sphere ASB.
       (The volume of a section of a sphere, radius R, height H is $\frac{1}{3}\pi H^2(3R - H)$.)

     iv Find the percentage of the volume of the section of the sphere **not** occupied by the cone.

  b In a different sphere section, $R = 3$ cm, $h = 2r$ cm and $TS = 1$ cm.

     i Write down the height, SC, in terms of r and show that $OC = (2r - 2)$ cm.

     ii Use Pythagoras' theorem in triangle OCB to find OC^2 in terms of r.

    iii Use your answers to parts **b i** and **b ii** to show that $5r^2 - 8r - 5 = 0$.

     iv Solve the equation $5r^2 - 8r - 5 = 0$.
       Give your answers correct to 2 decimal places.

     v Write down the height of the cone.
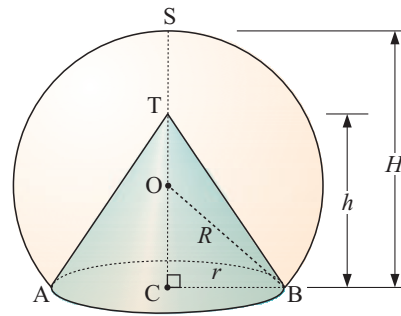
20   **Adapted from Nov 2000, Paper 4**

On television a weather forecaster uses a cloud
symbol shown in the diagram.

Its perimeter consists of a straight line AE, two
semicircular arcs APB and DQE and the major
arc BRD of a circle, centre C.

AE = 7.5 cm,   AB = DE = 3 cm   and
BC = CD = 2.8 cm.

Angle BAE = angle DEA = 70° and X is the
midpoint of BD.

a   **i**   Show that   BX = 2.724 cm.

ii   Calculate the angle BCX.

b   Calculate

i   the area of triangle BCD

ii   the area of the trapezium ABDE

iii   the area of the major sector BCD

iv   the total area of the cloud symbol.
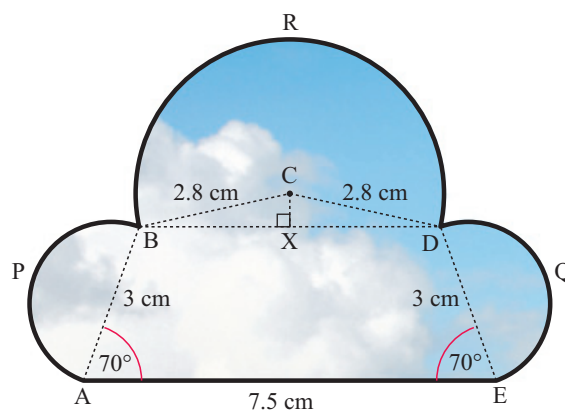
21   **May/June 2000, Paper 4**

The points P, Q and R lie on the circumference of a
circle, centre O. PQ = 5 cm, PR = 8 cm and angle
QPR = 70°.

a   Calculate the area of triangle PQR.

b   Calculate the length of the chord QR.

c   Find the size of the obtuse angle QOR.

d   Show that the radius of the circle is 4.18 cm,
correct to three significant figures.

e   Taking the radius of the circle as 4.18 cm, calculate the length of the minor arc QR.

f   Find the size of the reflex angle QOR.
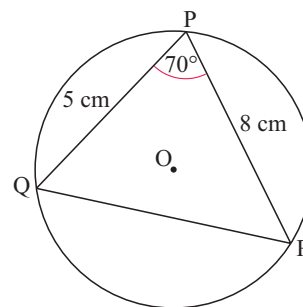
22   **Adapted from May/June 2000, Paper 4**

a   In the pyramid ABCD, ABC is the base and D is
the vertex.

Angle BCA = angle DAC = angle DAB = 90°.
AD = h cm,   AC = b cm   and   BC = a cm.
(The formula for the volume of a pyramid is
$\frac{1}{3}$ base area × perpendicular height.)

i   Write down a formula for the volume of the
pyramid ABCD in terms of a, b and h.

ii   Calculate the volume of pyramid ABCD when
$a = 6$, $b = 5$ and $h = 8$.

b   The pyramid PQRST has a rectangular base with
ST = x cm   and   RS = $(x + 3)$ cm.

The height of the pyramid, OP, is 12 cm, where
O is the centre of the rectangle.
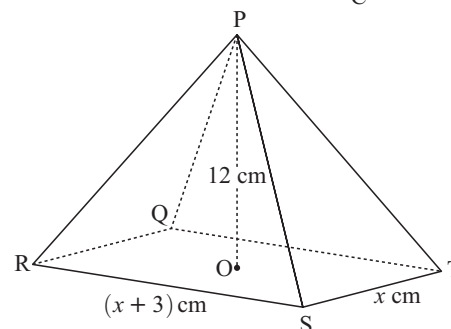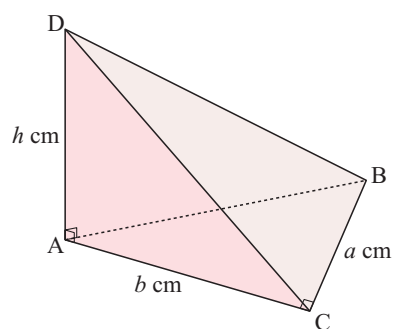
   i Write down a formula for the volume of this pyramid in terms of x.
   ii When the volume is numerically equal to the perimeter of the rectangular base, show that
      $2x^2 + 4x - 3 = 0$.
   iii Solve the equation $2x^2 + 4x - 3 = 0$, giving your answers correct to 2 decimal places.
   iv Use your answer to **iii** to write down the length of RS.
   v M is the midpoint of ST. Calculate angle PMO.

23 **May 2002, Paper 4**
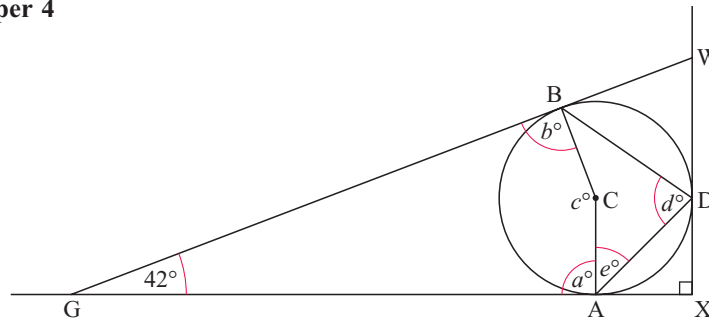
A sphere, centre C, rests on horizontal ground at A and touches a vertical wall at D. A straight plank
of wood, GBW, touches the sphere at B, rests on the ground at G and against the wall at W. The wall
and the ground meet at X. Angle WGX $= 42^o$.

a Find the values of a, b, c, d and e marked on the diagram.

b Write down one word which completes the following sentence.
   'Angle CGA is 21° because triangle GBC and triangle GAC are ....'

c The radius of the sphere is 54 cm.
   i Calculate the distance GA. Show all your working.
   ii Show that GX $= 195$ cm correct to the nearest centimetre.
   iii Calculate the length of the plank GW.
   iv Find the distance BW.

24 **Adapted from May/June 2000, Paper 4**
   The diagram shows a window ABCDE. ABDE is a rectangle.
   BCD is an arc of a circle with centre O and radius x cm.
   The total height of the window is 90 cm. AB $=$ ED $= 80$ cm
   and AE $=$ BD $= 40$ cm. The line OC is perpendicular to
   BD, and BF $=$ FD.

   a **i** Write down, in terms of x, the length of OC and the
         length of OF.
      ii Use Pythagoras' Theorem in triangle OFD to write
         down an equation in x.
      iii By solving the equation, show that $x = 25$.
   b Using a scale of 1 cm to represent 10 cm, construct an
      accurate drawing of the window.
   c Find the area of the window.
   d The window is made of glass 2 mm thick. The mass of 1 cm^3 of the glass is 6.5 grams.
      Calculate the mass of glass in the window, giving your answer in kilograms.
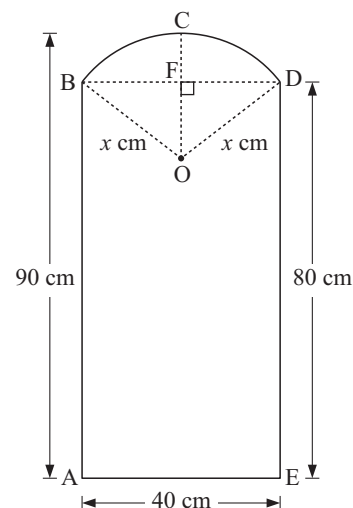
25 **Adapted from Nov 1999, Paper 4**

On December 21st, the sun rises in Buenos Aires at 0542 and sets at 2013.

   a Find the length of time between sunrise and sunset in hours and minutes.

   b
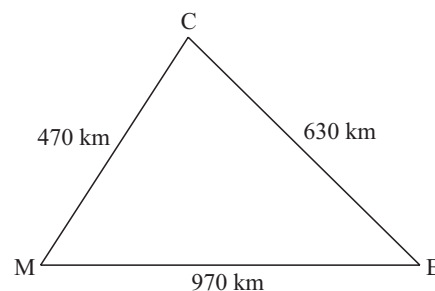
A plane flies from Buenos Aires (B) to Cordoba (C). It continues to Mendoza (M) before returning to Buenos Aires. The flight distances are shown on the diagram above.

     i Showing all your working, calculate angle MCB to the nearest degree.

    ii The bearing of Buenos Aires from Cordoba is 124°. Write down the bearing of Mendoza from Cordoba.

   c The average speed of the plane was 500 kilometres per hour. The times spent at Cordoba and at Mendoza were 1 hour 30 minutes and 2 hours respectively.

     i Calculate the total time from leaving Buenos Aires until landing there again.
      Give your answer in hours and minutes to the nearest minute.

    ii The plane left Buenos Aires on December 21st at 1240. Will it land in Buenos Aires before sunset?

26 **Nov 1999, Paper 4**

A large circular window is shown in the diagram. The unshaded part is glass and is made up of a small circle and 12 identical shapes. The shaded part is stone.

   a The diagram shows one of the 12 identical shapes.
     ABC is an isosceles triangle and BCD is a semicircle.
     BC = 1.4 m and angle BAC = 30°.
     Calculate

     i the area of the semicircle BCD

    ii the length of AC, showing that it rounds off to 2.705 m

   iii the area of triangle ABC

   iv the area of the shape ABDC.
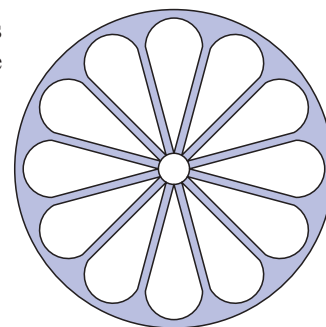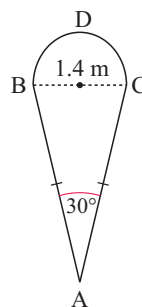
   b The radius of the small circle is 0.3 m. Calculate the total area of glass, including the small circle.

   c The radius of the large circular window is 4 m. Calculate the percentage of the window's area which is stone.

27 **Nov 1995, Paper 4**

The end, A, of a pendulum, OA, moves along the arc AB as shown in the diagram. The length of the pendulum is h metres and the time, t seconds, taken to move from A to B is

given by $t = \pi\sqrt{\dfrac{h}{9.81}}$.

   a Find t when $h = 1.6$.
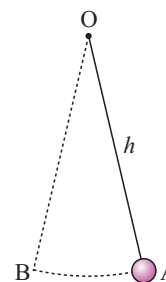
b Find the length of a pendulum which takes 1 second to move from A to B.

c Write h in terms of π and t.

d If $h = 1$ and the arc length, AB, is 1 m, calculate

    i angle AOB   **ii** the area of the sector AOB.

28 **Adapted from Nov 1995, Paper 4**

In the triangle ABC, AB $= x$ cm. The side AC is 3 cm shorter than AB and the side BC is 5 cm shorter than AB.

  a   **i** Show that the perimeter of the triangle, p cm, is given by $p = 3x - 8$.

      ii The perimeter is $2\frac{1}{2}$ times the length of AB. Find the length of AB.

    iii Given that angle ACB $= 83.2^o$ and case **ii** applies, calculate the smallest angle of the triangle, giving your answer correct to the nearest degree.

  b   **i** If, instead, the triangle ABC is right angled, show that $x^2 - 16x + 34 = 0$.

      ii Solve the equation $x^2 - 16x + 34 = 0$ giving your answers correct to 2 decimal places.

    iii Hence find the lengths of the sides of the right-angled triangle.

29 **Nov 1995, Paper 4**

O is the centre of the circle. Angle BOD $= 132^o$.

The chords AD and BC meet at P.

  a   **i** Calculate angles BAD and BCD.

      ii Explain why triangles ABP and CDP are similar.

    iii AP $= 6$ cm, PD $= 8$ cm, CP $= 3$ cm and
AB $= 17.5$ cm.
Calculate the lengths of PB and CD.

    iv If the area of triangle ABP is n cm^2, write down, in terms of n, the area of triangle CPD.

  b   **i** The tangents at B and D meet at T. Calculate angle BTD.

      ii Use OB $= 9.5$ cm to calculate the diameter of the circle which passes through O, B, T and D, giving your answer to the nearest centimetre.
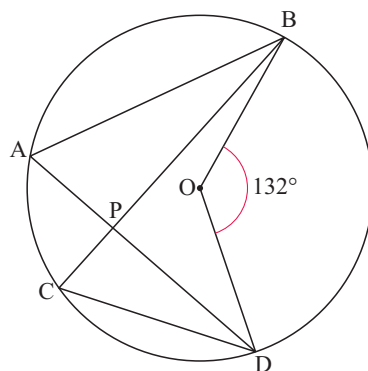
30 **Adapted from Nov 1998, Paper 4**

  a ABCDE is a semicircle of diameter 10 centimetres.
AC $=$ CE and angle ACE $= 90^o$. Calculate

    i the area of the semicircle

    ii the area of triangle ACE

    iii the area of the shaded segment ABC.

  b PQ and QR are tangents to a semicircle with centre O and diameter 10 centimetres. POR is a straight line, PQ $=$ QR and angle PQR $= 90^o$. Calculate the area of triangle PQR.
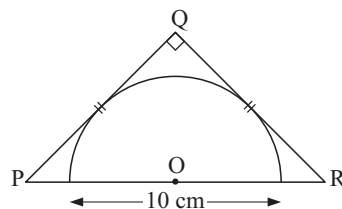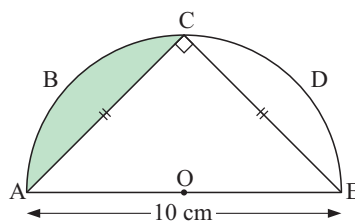
PRINTABLE QUESTIONS
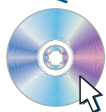
Click on the icon to obtain **24 more multi-topic questions**.

31 **Adapted from June 1998, Paper 4**

Six cylindrical bales of hay, each with radius 0.8 m and length 1.5 m, are stacked as shown in the diagram. The centres of three of the bales are marked A, B and C.

a **i** What type of triangle is ABC?
  ii What is the length of AB?
  iii Calculate h, the vertical height of the stack in metres (see diagram alongside).

b Calculate the total volume of hay in the stack.

c Ten bales of hay are stacked as shown in the diagram. Calculate the vertical height of this stack.
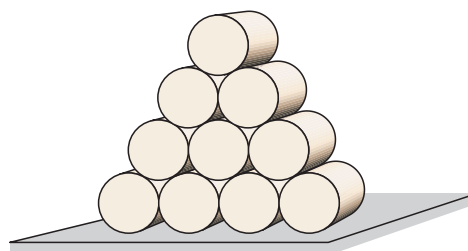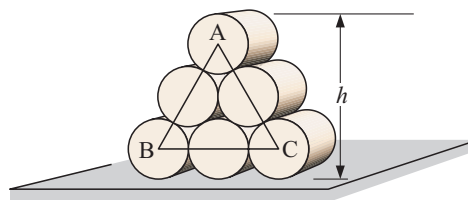
32 **Adapted from June 1997, Paper 4**

A cinema has 200 seats. Ticket prices are \$5 for an adult and \$2.50 for a child.

a One evening, 80% of the seats in the cinema are occupied. Twenty of the people present are children. Calculate the total money taken from the sale of tickets.

b Another evening, x children are present and *all* the seats are occupied. The money taken for the tickets is \$905.
  i Write down an equation in x.   **ii** Calculate the value of x.

c The money taken for tickets for a week is \$10 800. This sum is divided between costs, wages and profit in the ratio $2 : 3 : 7$. Calculate
  i the profit for the week
  ii the simple interest earned if this profit is invested at a rate of 5% per annum for 4 months.

33 **June 1997, Paper 4**

A child's toy consists of a cone inside a sphere. The radius of the sphere, OA, is 6 cm and the radius of the base of the cone, AC, is 3.6 cm.

(The volume of a sphere of radius R is $\frac{4}{3}\pi R^3$. The volume of a cone of base radius r and height h is $\frac{1}{3}\pi r^2 h$.)

a Show that VOC, the height of the cone, is 10.8 cm.

b Calculate
  i the volume of the sphere
  ii the volume of the cone
  iii the percentage of the volume of the sphere *not* occupied by the cone.

c The sphere rolls 3 metres across the floor in a straight line. Calculate:
  i the circumference of the sphere
  ii the number of *complete* revolutions made by the sphere
  iii the number of degrees through which the sphere must still turn in order to complete another revolution.
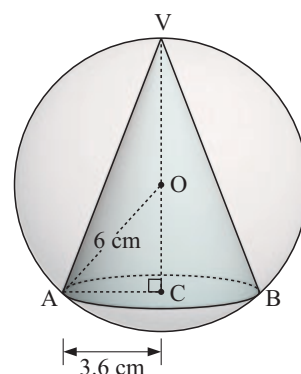
34 **Nov 1996, Paper 4**

Diagram 1 shows a regular tetrahedron WXYZ with all sides of length 6 cm. Diagram 2 shows the base XYZ of the tetrahedron. O is the centre of the base, M is the midpoint of XZ, and N is the midpoint of XY.
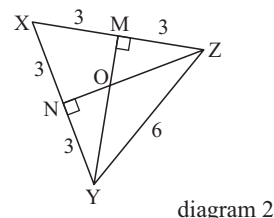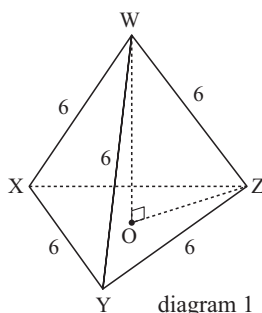
a Write down the size of angle OZM.

b Show that, correct to 4 significant figures, the length of OZ is 3.464 cm.

c Calculate the height, OW, of the tetrahedron.

d Calculate the volume of the tetrahedron. (Volume of a tetrahedron $= \frac{1}{3}$ base area $\times$ height.)

e Calculate the angle between the edge WZ and the base XYZ.

35 **Adapted from June 1997, Paper 4**

A circular board is divided into twelve equal sectors numbered from 1 to 12. A dart is thrown and lands on the board. Assume that it is equally likely to land anywhere in the circle. The score is the number in the sector where the dart lands.
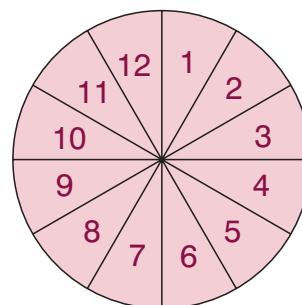
a When one dart is thrown, find the probability that the score is

i a square number

ii a prime number or less than 6 or both

b When two darts are thrown, the *sum* of the two scores is calculated.

i List *all* the possible ways of scoring 21.

ii Find the probability of scoring 21 with two darts. Give your answer as a fraction in its lowest terms.

c The table shows the scores when a player throws one dart 30 times.

Score	1	2	3	4	5	6	7	8	9	10	11	12
Frequency	1	1	1	0	1	1	2	3	4	5	6	5

For these scores, find   **i** the mode   **ii** the median   **iii** the mean.

d To prevent damage to the wall, the board, which has a radius of 10 cm, is placed on a wooden square of side 30 cm. One dart is thrown by a beginner who is equally likely to hit anywhere within the *square*.

i Calculate the area of the sector numbered 2.

ii Calculate the probability that the beginner scores 2, giving your answer as a decimal.
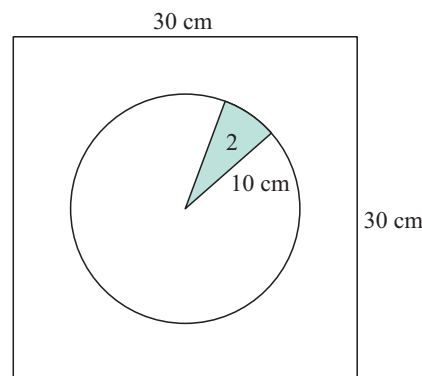
36 **June 1995, Paper 4**

a In the right angled triangle PQR, $PR = x$ cm and $QR = (x + 1)$ cm. The area of the triangle PQR is 5 cm^2.

   i Show that $x^2 + x - 10 = 0$.

   ii Solve the equation $x^2 + x - 10 = 0$, giving your answers correct to 1 decimal place. Hence write down the length of PR.
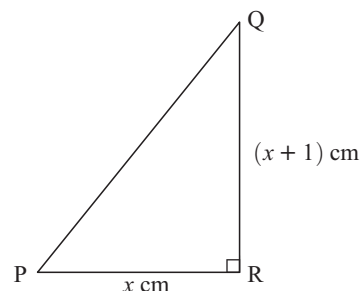
b In triangle ABC, angle $ACB = 120^o$, $AC = y$ cm and $BC = (y + 2)$ cm.

   i Use the cosine rule to find an expression for AB^2 in terms of y.

   ii When $AB = 7$ cm, show that $y^2 + 2y - 15 = 0$.

   iii Factorise $y^2 + 2y - 15$.

   iv Solve the equation $y^2 + 2y - 15 = 0$. Hence write down the lengths of AC and CB.
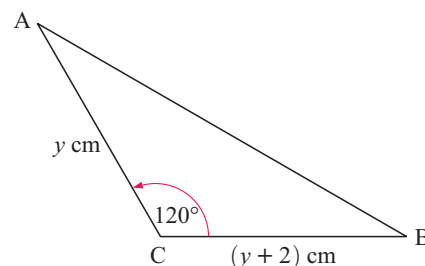
37 **Adapted from June 1995, Paper 4**

a BCA is the diameter of a circle, centre C, radius r. DAE is a tangent to the circle at A. $DE = 3r$ and angle $DCA = 30^o$.

   i Draw the diagram accurately when $r = 3$ cm.

   ii Measure and write down the length of BE in your diagram.

   iii Calculate the length of the semi-circular arc BA when $r = 3$ cm.

b In the case when $r = 10$ cm, *calculate*, to 2 decimal places,

   i the length of DA   **ii** the length of AE   **iii** the length of BE

   iv the length of the semi-circular arc BA.

c Comment on the relationship between the length of BE and the length of the semi-circular arc BA.
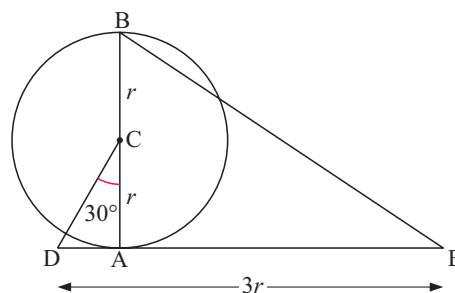
38 **June 1995, Paper 4**

Ahmed earns \$20 000 each year.

a In 1991, he paid no tax on the first \$3000 of his earnings. He paid 25% of the rest as tax. Show that he paid \$4250 as tax.

b In 1992, he paid no tax on the first \$4000 of his earnings. He paid 30% of the rest as tax. Calculate how much he paid as tax.

c In 1993, he paid no tax on the first \$$x$ of his earnings. He paid 30% of the rest as tax.

   i Find an expression in terms of x for the amount of tax he paid.

   ii Calculate the value of x if he paid \$4950 as tax.

39 **Adapted from June 1995, Paper 4**

    a **i** The diagram shows a hollow cone with base radius
            $AC = 3$ cm and edge $OA = 18$ cm. Calculate
            a the height OC
            b angle AOC
            c the circumference of base.
      ii The cone is cut along the line OA and opened out
         to form the sector AOA'. Calculate
         a the circumference of a circle of radius 18 cm
         b angle AOA'.

    b The top part of a *solid* cone is removed.
      The height of the remaining solid is half the height of
      the original cone.

      i Write down, in the form $1 : n$, the ratios
         a the base radius of the cone removed : the base radius of the original cone
         b the curved surface area of the cone removed : the curved surface area of the original
            cone
         c the volume of the cone removed : the volume of the original cone.
     ii The curved surface area of the original cone was 24π cm^2.
        Calculate, in terms of π, the curved surface area of the remaining solid.
    iii The volume of the original cone was V cm^3.
        Give the volume of the remaining solid in terms of V.

40 **Nov 1994, Paper 4**

Mahmoud enjoys flying his kite. On any given day, the probability that there is a good wind is $\frac{3}{4}$.
If there is a good wind, the probability that the kite will fly is $\frac{5}{8}$. If there is not a good wind, the
probability that the kite will fly is $\frac{1}{16}$.

    a **i** Copy the given tree diagram. Write
         on your diagram the probability for
         each branch.
      ii What is the probability of a good wind
         and the kite flying?
    iii Find the probability that, whatever the
        wind, the kite does *not* fly.

    b If the kite flies, the probability that it sticks in a tree is $\frac{1}{2}$. Calculate the probability that, whatever
      the wind, the kite sticks in a tree.

c

Wind strength	1	2	3	4	5	6	7	8	9
Number of days	3	5	6	8	6	7	9	5	1

The table shows the wind strength measured on each of 50 days.

  i   State the mode and find the median wind strength.

  ii   Calculate the mean wind strength.

  iii   A 'good wind' has strength x such that $3 \leqslant x \leqslant 7$.
        Estimate the probability of a good wind from this data.

41   Nov 1994, Paper 4

Alexis, Biatriz and Carlos are business partners.

  a   60% of each week's income is used for the business. The rest is divided between Alexis, Biatriz and Carlos in the ratio $5 : 3 : 1$.

   i   Calculate how much they each receive in a week when the income is \$9000.

   ii   Calculate the income in a week when Carlos receives \$420.

  b   Alexis buys Carlos' share of the business for \$16 000, which he borrows from the bank at a rate of 12% simple interest per year. How much interest will he have to repay in 6 months?

42   Adapted from Nov 1994, Paper 4

A spherical ball, radius r, diameter AB, is floating in water with its centre O at a depth h below the surface. CD is a diameter of the circular cross-section formed at the surface. If $r = 13$ cm and $h = 5$ cm, calculate

  a   **i**   the length of CD

   ii   the angle COD

   iii   the length of the arc CBD

   iv   the distance from C to D round the semi-circle, diameter CD.

  b   **i**   The area of surface *above* the water level is given by the formula $2\pi r(r - h)$.
        Find the area above the water level.

   ii   The total surface area of a sphere is $4\pi r^2$.
        Find the area above the water level as a percentage of the total surface area of the sphere.

43   Adapted from June 1992, Paper 4

Before the Euro was introduced, France had a French franc.

  a   **i**   If £1 = 9.80 French francs, calculate how much 100 francs are worth in pounds (£), giving your answer correct to two decimal places.

   ii   If £1 = x francs, write down an expression, in terms of x, for the value in pounds of 100 francs.

  b   A French holidaymaker toured Britain in 1989 and in 1990. In 1990, the exchange rate was £1 = x francs. In 1989, it was £1 = $(x + 1)$ francs.

The holidaymaker found that, for 100 francs, she received £1 more in 1990 than in 1989.

   i   Write down an equation in x and show that it reduces to $x^2 + x - 100 = 0$.

   ii   Use the above equation to calculate the value of x, giving your answer correct to two decimal places.

   iii   Use your answer to **b ii** to find the value, in pounds, of 100 francs in the year 1990. Give your answer correct to two decimal places.

44   **Adapted from June 1994, Paper 4**

cross-section

The diagram shows a prism of cross-sectional area 0.42 cm^2 and volume 7.56 cm^3.

   a   Calculate the length of the prism.

   b   The prism is made of wood and 1 cm^3 of this wood has a mass of 0.88 g. Calculate the mass of the prism.

   c   The prisms are made from a block of wood of volume 0.5 m^3. It is known that 25% of the wood is wasted. Calculate the number of prisms which can be made, giving your answer to the nearest thousand.

   d   **i**   State the area of triangle OAB.

       ii   What special type of triangle is triangle OAB?

      iii   Given that the length of AB is x cm, find an expression for the area of triangle OAB in terms of x. Hence find the length of AB correct to the nearest millimetre.

45   **Adapted from June 1993, Paper 4**

In the diagram, SR is parallel to PQ. $SR = 4$ cm, $SX = 2$ cm, $RX = 3$ cm and $PQ = 7$ cm.

   a   Explain why the triangles RSX and PQX are similar.

   b   Calculate the length of PX and the length of QX.

   c   It is also given that the area of triangle RSX is 2.90 cm^2. Calculate the area of triangle PQX, correct to two significant figures.

   d   Use trigonometry to calculate the size of angle SRX, to the nearest degree.

46   **Adapted from Nov 1991, Paper 4**

The main road, from A to B, through Newmarket, is straight for 15 kilometres. The ring road, around Newmarket, is an arc AB of a circle, centre O, of radius 12 kilometres.

   a   **i**   Calculate the size of the angle marked θ in the diagram.

      ii   Use your answer to **a i** to show that, correct to three significant figures, the length of the ring road between A and B is 16.2 kilometres.

   b   Mr Carson can drive at a steady 100 km/h along the ring road. If he drives from A to B through Newmarket, he can average 80 km/h for 12 kilometres but averages only 40 km/h for the 3 kilometres through the town. Calculate, to the nearest minute, the amount of time that he saves by driving round the ring road.

47 Nov 1991, Paper 4

The volume of a regular hexagonal prism is given by the formula $V = 2.6s^3$, where s is the length of each edge of the prism.

a Find V if $s = 3.3$ cm.

b Make s the subject of the formula.

c   **i** Use trigonometry to obtain an expression for the area of the (shaded) hexagon, in terms of s.

   ii Hence show that the original formula is approximately correct.

48 Nov 1989, Paper 4

In the triangle XYZ, angle $XZY = 60^o$, $XY = 9.5$ cm, $XZ = 8$ cm and $YZ = x$ cm.

a   **i** Use the cosine rule to show that x satisfies the equation $4x^2 - 32x - 105 = 0$.

   ii By solving this quadratic equation, find the length of YZ.

b Calculate angle XYZ.

49 Adapted from Nov 1990, Paper 4

The shape ABCDEF consists of a trapezium ACDF and a minor segment ABC of a circle centre O. The lines FA and DC are tangents to the circle at A and C respectively. The radius of the circle is 2 m. $AC = 3.6$ m and $AF = CD = 5$ m.

a Show that angle AOC is 128.3^o, correct to one decimal place.

b Calculate the area of the sector OABC.

c Calculate the area of the triangle OAC and hence the area of the minor segment ABC.

d Show that the perpendicular distance between AC and FD is 4.5 m.

e Find the area of the trapezium ACDF and hence the area of the whole shape ABCDEF.

50 June 1994, Paper 4

In a fitness exercise, students run across a field from A to B, then from B to C and then from C to A.

a A student runs from A to B in 10 seconds. Calculate his speed in

   i metres/second   **ii** kilometres/hour

b Another student runs from A to B in 10.5 seconds, from B to C in 13 seconds and from C to A at a *speed* of 8.5 m/s. Calculate her overall average speed in metres/second.

c Showing all your working, calculate angle BAC.

d The bearing of B from A is 062°. Calculate

   i the bearing of C from A   **ii** the bearing of A from C

51 **Adapted from June 1990, Paper 4**

A(-3, 4), B(5, -2) and C(2, -6) are three vertices of a parallelogram ABCD.

a Write down the vector $\overrightarrow{BA}$ in the form $\begin{pmatrix} p \\ q \end{pmatrix}$.

b Find the coordinates of the vertex D.

c Calculate the lengths of the line segments AB, BC and AC.

d Use your answers in **c** to show that the parallelogram ABCD is a rectangle.

e Calculate the area of ABCD.

f The equation of the line through A and B is $y = -\frac{3}{4}x + \frac{7}{4}$.

    i What is the gradient of this line?

    ii Write down the coordinates of the point at which this line cuts the y-axis.

52 **Adapted from June 1988, Paper 4**

Aristotle Jones wants to sail his yacht from P to Q. In order to reach T from P, he has to sail along PS (vector **a**) and then along ST (vector **b**). This is called a "tack".

a Write the vector $\overrightarrow{PT}$ in terms of **a** and **b**.

b $\begin{pmatrix} x \\ y \end{pmatrix}$ is a vector with components x km east and y km north.

If $\mathbf{a} = \begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} \frac{3}{4} \\ -\frac{1}{4} \end{pmatrix}$, find

    i the components of $\overrightarrow{PT}$ **ii** the length of $\overrightarrow{PT}$.

c The distance from P to Q is 15 km.

    i How many tacks does he need to take to reach Q?

    ii Write the vector $\overrightarrow{PQ}$ in terms of **a** and **b**.

d Find the bearing of Q from P.

53 **Adapted from June 1988, Paper 4**

Note: 1 hectare $= 10\,000$ m^2

On a still day, a helicopter hovers at a height of 200 m and sprays the ground with fertiliser. The shaded part of the diagram shows the circular area sprayed.

a If the "angle of spray" is 32°, calculate the sprayed area in square metres. Give your answer correct to three significant figures.

b The farmer wants to spray a circular area of 3 hectares from the same height. What "angle of spray" should he use?

54 **1988, Specimen paper 6**

Mark	10	20	30	40	50	60	70	80	90	100
Number of girls scoring this mark or less	1	4	8	16	24	29	32	34	35	36

36 girls sat an examination in which the maximum mark was 100. The table above shows the number of girls who scored a particular mark, or less, in the examination.

a Calculate how many girls scored a mark between 61 and 70 inclusive.

b Using a vertical scale of 2 cm to represent 5 girls and a horizontal scale of 1 cm to represent 10 marks, plot these values on graph paper and draw a smooth curve through your points.

c Showing your method clearly, use your graph to estimate the median mark.

d State, as a fraction in its lowest terms, the probability that a girl chosen at random will have a mark

    i less than or equal to 50   **ii** greater than 70.

CHAPTER 33 (CD question answers)

31 a i equilateral **ii** 3.2 m **iii** $h \approx 4.37$ m
b ≈ 18.1 m^3 **c** $h \approx 5.76$ m

32 a $750 **b i** $2.5x + 5(200 - x) = 905$ **ii** $x = 38$
c i $6300 **ii** $105

33 a VC $= 6 + \sqrt{6^2 - 3.6^2} = 10.8$ cm
b i ≈ 905 cm^3 **ii** ≈ 147 cm^3 **iii** $\approx 83.8\%$
c i ≈ 37.7 cm **ii** 7 times **iii** 15.2^o

34 a 30° **b** OZ $= \dfrac{3}{\cos 30^o} \approx 3.464$ cm
c OW ≈ 4.899 cm **d** $V \approx 25.5$ cm^3 **e** $\approx 54.7^o$

35 a i $\frac{1}{4}$ **ii** $\frac{7}{12}$
b i 12 and 9, 11 and 10, 10 and 11, 9 and 12 **ii** $\frac{1}{36}$
c i 11 **ii** 10 **iii** 8.9 **d i** 26.2 cm^2 **ii** 0.0291

36 a i $\frac{1}{2}x(x+1) = 5$ $\therefore$ $x^2 + x = 10$, etc.
ii $x \approx 2.7$ or -3.7, PR ≈ 2.7 cm $\{$PR $> 0\}$
b i AB$^2 = 3y^2 + 6y + 4$
ii $3y^2 + 6y + 4 = 49$ simplifies to $y^2 + 2y - 15 = 0$
iii $(y + 5)(y - 3)$
iv $y = -5$ or 3, but we reject -5
$\therefore$ AC $= 3$ cm, CB $= 5$ cm

37 a i

Half size

ii BE ≈ 9.42 cm **iii** ≈ 9.42 cm
b i DA ≈ 5.77 cm **ii** AE ≈ 24.2 cm
iii BE ≈ 31.4 cm **iv** ≈ 31.4 cm
c BE $\approx$ length of semi-circular arc BA

38 a Tax $= \$3000 \times 0\% + \$17\,000 \times 25\%$ **b** $4800
$= \$4250$
c i $T = (20\,000 - x) \times 0.3$ **ii** $x = 3500$

39 a i *a* OC ≈ 17.7 cm *b* $\approx 9.59^o$ *c* ≈ 18.8 cm
ii *a* ≈ 113 cm *b* 60°
b i *a* $1:2$ *b* $1:4$ *c* $1:8$
ii 18π cm^2 **iii** $\frac{7}{8}V$ cm^3

40 a i

ii $\frac{15}{32}$
iii $\frac{33}{64}$
b $\frac{31}{128}$
c i mode $= 7$, median $= 5$ **ii** mean $= 4.94$ **iii** 0.72

41 a i Alexis $2000, Biatriz $1200, Carlos $400 **ii** $9450
b $960

42 a i 24 cm **ii** $\approx 135^o$ **iii** ≈ 30.6 cm **iv** ≈ 37.7 cm
b i ≈ 653 cm^2 **ii** $\approx 30.8\%$

43 a i £10.20 **ii** $\dfrac{100}{x}$ pounds
b i $\dfrac{100}{x+1} + 1 = \dfrac{100}{x}$ and on multiplying both sides by
$x(x+1)$ reduces to $x^2 + x - 100 = 0$.
ii $x \approx 9.51$ $\{$the other solution is $< 0\}$
iii £10.51

44 a 18 cm **b** ≈ 6.65 g **c** $\approx 50\,000$
d i 0.07 cm^2 **ii** equilateral
iii $A = \dfrac{\sqrt{3}x^2}{4}$, AB ≈ 4 mm

45 a • $\widehat{RSX} = \widehat{PQX}$ $\{$equal alternate angles$\}$
• $\widehat{SXR} = \widehat{QXP}$ $\{$vertically opposite$\}$
$\therefore$ Δs are equiangular and so are similar.
b PX $= 5.25$ cm, QX $= 3.5$ cm **c** ≈ 8.9 cm^2
d $\approx 29^o$

46 a i $\theta \approx 38.7^o$ **ii** $l = \left(\frac{77.364}{360}\right) \times 2\pi(12) \approx 16.2$ km
b Time saved $= 3.778....$ min ≈ 4 min

47 a $V \approx 93.4$ cm^3 **b** $s = \sqrt[3]{\dfrac{V}{2.6}}$
c i $A = \frac{3\sqrt{3}}{2}s^2$ **ii** $\approx 2.598\,076s^2 \approx 2.60s^2$

48 a i $9.5^2 = x^2 + 8^2 - 2x(8)\cos 60^o$ $\{$cosine rule$\}$
and this simplifies to $4x^2 - 32x - 105 = 0$
ii $x = 10\frac{1}{2}$ or $-2\frac{1}{2}$ but x cannot be < 0
$\therefore$ $x = 10\frac{1}{2}$
So, YZ $= 10.5$ cm
b $\approx 46.8^o$

49 a $\widehat{AOC} = 2\sin^{-1}(0.9) \approx 128.3^o$ **b** 4.48 m^2
c OAC ≈ 1.57 m^2, ABC ≈ 2.91 m^2
d $\widehat{AOC} \approx 128.3^o$ $\therefore$ $\widehat{OAC} \approx 25.85^o$ $\therefore$ $\widehat{AFE} \approx 64.15^o$
So, $\sin 64.15^o \approx \dfrac{d}{5}$ $\therefore$ $d \approx 4.5$
e ≈ 26.0 m^2, ≈ 28.9 m^2

50 a i 8 m/s **ii** 28.8 km/h **b** ≈ 7.97 m/s **c** $\approx 55.8^o$
d i $\approx 118^o$ **ii** $\approx 298^o$

51 a $\begin{pmatrix} -8 \\ 6 \end{pmatrix}$ **b** D$(-6, 0)$ **c** AB $= 10$, BC $= 5$, AC $= 5\sqrt{5}$
d AC$^2 = 125 = 100 + 25 = $ AB$^2 + $ BC^2
$\therefore$ $\widehat{ABC}$ is a right angle
$\{$converse of Pythagoras' theorem$\}$
So, the parallelogram is a rectangle.
e 50 units^2 **f i** $-\frac{3}{4}$ **ii** $(0, 1\frac{3}{4})$

52 a $\mathbf{a} + \mathbf{b}$ **b i** $\overrightarrow{PT} = \begin{pmatrix} 1 \\ \frac{3}{4} \end{pmatrix}$ **ii** $1\frac{1}{4}$ km
c i 12 tacks **ii** $\overrightarrow{PQ} = 12(\mathbf{a} + \mathbf{b})$ **d** 053.1^o

53 a $\approx 49\,100$ m^2 (≈ 4.91 ha) **b** $\approx 26.0^o$

54 a 3 girls

Y:\HAESE\IGCSE01\IG01_33\popup\ans\001IB_IGC1_33puan.CDR Friday, 21 November 2008 11:01:25 AM PETER

b

Examination marks

c median ≈ 42.5 **d**   **i** $\frac{2}{3}$ **ii** $\frac{1}{9}$

Investigation and modelling questions

34

Contents:

- **A** Investigation questions
- **B** Modelling questions

Many of the questions in **Chapters 33** and **34** are adapted from past examination papers for IGCSE Mathematics 0580 by permission of the University of Cambridge Local Examinations Syndicate. The 0580 course is a different syllabus from that followed by students of the 0607 course, but has many features in common. These questions are certainly appropriate for practising mathematical techniques and applications relevant to the 0607 curriculum, but do not necessarily represent the style of question that will be encountered on the 0607 examination papers. Teachers are referred to the specimen papers of the 0607 syllabus for a more representative group of questions. The University of Cambridge Local Examinations Syndicate bears no responsibility for the example answers to questions taken from its past question papers which are contained in this publication.

A INVESTIGATION QUESTIONS

1 It is given that $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{k}$ where $k \in \mathbb{Z}$.

a If $n = 1$, the LHS is 1^2.

If $n = 2$, the LHS is $1^2 + 2^2$.

Use $n = 1$ to find the value of k, and check that you get the same value of k for $n = 2$ and $n = 3$.

b Use the given formula to find the value of $1^2 + 2^2 + 3^2 + 4^2 + ... + 100^2$.

c Notice that $2^2 + 4^2 + 6^2 = (2 \times 1)^2 + (2 \times 2)^2 + (2 \times 3)^2$
$$= 2^2 1^2 + 2^2 2^2 + 2^2 3^2$$
$$= 2^2(1^2 + 2^2 + 3^2)$$

Hence,

    i find m if $2^2 + 4^2 + 6^2 + 8^2 + ... + 100^2 = 2^2(1^2 + 2^2 + 3^2 + ... + m^2)$

    ii find the value of $2^2 + 4^2 + 6^2 + 8^2 + ... + 100^2$.

d Use **b** and **c ii** to find the value of $1^2 + 3^2 + 5^2 + 7^2 + ... + 99^2$.

e Use some of the previous answers to find the value of:
$$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + ... + 99^2 - 100^2.$$

2 A farmer makes a sheep pen in the shape of a quadrilateral from four pieces of fencing. Each side of the quadrilateral is 5 metres long and one of the angles is 60°.

    a Using a scale of 1 to 100, make an accurate drawing of the quadrilateral.

    b Mark in its axes of symmetry with broken lines and describe how they cut each other.

    c What is the special geometrical name of this shape?

    d Calculate the area enclosed by the sheep pen, giving your answer in square metres.

    e By changing the angles (but leaving the lengths of the sides unchanged), the area enclosed by the sheep pen can be varied. What is the greatest possible area that can be enclosed? Justify your answer.

3 Throughout this question, remember that 1 is *not* a prime number.

    a Find a prime number which can be written as the sum of two prime numbers.

    b Consider the statement "All even numbers greater than 15 can be written as the sum of two different prime numbers in at least two different ways." For example, $20 = 3 + 17 = 7 + 13$.

       i Show that the above statement is true for 16.
      ii Find a number between 30 and 50 which shows that the statement is false.

    c Show that 16 can be written as the sum of three different prime numbers.

    d Consider the statement "All odd numbers greater than 3 can be written as the sum of two prime numbers". Is this statement true or false? Justify your answer.

4 **Adapted from June 1989, Paper 4**

A firm which manufactures golf balls is experimenting with the packaging of its product. 3 golf balls, each of radius 2.15 centimetres, are packaged in a rectangular box, a cross-section of which is shown in the diagram alongside. The box is 12.9 centimetres long, 4.3 centimetres wide and 4.3 centimetres high.

    a Given that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$, calculate the amount of space within the box which is unfilled.

    b The marketing department suggests that an equilateral triangular box of side 11.75 centimetres and height 4.3 centimetres might be more attractive. The diagrams show a plan and side view of the new box.

    Calculate the amount of space within this new box which is unfilled.

    c Give your answer to **a** and **b** as percentages of the capacity of each container.

    d Design a box of your own which gives a smaller percentage of unfilled space.

5 **Adapted from June 1989, Paper 6**

Consider the figures P to T:

a All five figures have something important in common. What is it?

b Calculate the area of a regular hexagon (H) of side 4 centimetres.

c Using the letters P, Q, R, S, T and H, list the areas in order of size, starting with the smallest.

d Explain any conclusions you arrive at.

6 **June 1988, Specimen Paper 6**

The diagram shows 3 squares, the sides of which are 1 cm, 2 cm and 3 cm respectively. Each of the small squares on the diagram has a side of length 1 cm and alternate squares are coloured black and white.

a The number of small squares of each colour used is shown in the following table. Copy and complete the table.

Length of side of given square	L	1	2	3	4	5
Number of black squares	B	1	2	5		
Number of white squares	W	0	2	4		
Total number of squares	T	1	4	9		

b **i** How many small white squares will there be when a square of side 11 cm is drawn?

ii Find the length of the side of a square when 1681 small black and white squares are needed to cover it.

c Write down a formula connecting T and L.

d Write down a formula connecting T and B when

i B is an even number **ii** B is an odd number.

7 **Adapted from June 1989, Paper 6**

  a Copy and complete the following two sets of calculations.

1	$=$	1^3	$=$
$1 + 2$	$=$	$1^3 + 2^3$	$=$
$1 + 2 + 3$	$=$	$1^3 + 2^3 + 3^3$	$=$
$1 + 2 + 3 + 4$	$=$	$1^3 + 2^3 + 3^3 + 4^3$	$=$

  b How are the two sets of results related?

  c Find the value of $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3$.

  d Given that the sum of the first 25 numbers, $1 + 2 + 3 + ... + 25$, is 325, find the value of $1^3 + 2^3 + 3^3 + ... + 25^3$.

  e $1 + 2 + 3 + 4 + 5 + ... + n = an^2 + bn$. Find the values of a and b, and test your answers.

  f Find the value of $1^3 + 2^3 + 3^3 + 4^3 + ... + 250^3$.

8 **June 1988, Paper 6**

The diagram shows the first eight rows of a continuing pattern of black and white triangles.

  a Find a formula for each of the following:

    i the number of triangles in the nth row

    ii the total number of triangles in the first n rows

    iii the total number of white triangles in the first n rows

    iv the total number of black triangles in the first n rows.

  b Show algebraically that your answer to **a ii** is the sum of your answers to **a iii** and **iv**.

9 **Nov 2002, Paper 4**

Sarah investigates cylindrical plant pots. The standard pot has base radius r cm and height h cm. Pot A has radius $3r$ and height h. Pot B has radius r and height $3h$. Pot C has radius $3r$ and height $3h$.

  a **i** Write down the volumes of pots A, B and C in terms of π, r and h.

    ii Find in its lowest terms the ratio of the volumes of A : B : C.

    iii Which one of the pots A, B or C is mathematically similar to the standard pot? Explain your answer.

    iv The surface area of the standard pot is S cm^2. Write down in terms of S the surface area of the similar pot.

  b Sarah buys a cylindrical plant pot with radius 15 cm and height 20 cm. She wants to paint its outside surface (base and curved surface area).

    i Calculate the area she wants to paint.

    ii Sarah buys a tin of paint which will cover 30 m^2. How many plant pots of this size could be *completely* painted on their outside surfaces using this tin of paint?

10 **Nov 2002, Paper 4**

  a Write down the 10th term and the nth term of the following sequences.

   i 1, 2, 3, 4, 5, ..., ...,   **ii** 7, 8, 9, 10, 11, ..., ...,   **iii** 8, 10, 12, 14, 16, ..., ...

  b Consider the sequence $1(8 - 7)$, $2(10 - 8)$, $3(12 - 9)$, $4(14 - 10)$, ..., ...

   i Write down the next term and the 10th term of this sequence in the form $a(b - c)$ where a, b and c are integers.

   ii Write down the nth term in the form $a(b - c)$ and then simplify your answer.

11 **Nov 2000, Paper 4**

  A teacher asks four students to write down an expression using each of the integers 1, 2, 3 and n exactly once. Ahmed's expression was $(3n + 1)^2$. Bumni's expression was $(2n + 1)^3$. Cesar's expression was $(2n)^{3+1}$. Dan's expression was $(3 + 1)^{2n}$. The value of each expression has been worked out for $n = 1$ and put in the table below.

  a Copy and complete this table, giving the values for each student's expression for $n = 2$, 0, -1 and -2.

	$n = 2$	$n = 1$	$n = 0$	$n = -1$	$n = -2$
Ahmed		16			
Bumni		27			
Cesar		16			
Dan		16			

  b Whose expression will always give the greatest value   **i** if $n < -2$ **ii** if $n > 2$?

  c Cesar's expression $(2n)^{3+1}$ can be written as an^b and Dan's expression $(3 + 1)^{2n}$ can be written as c^n. Find the values of a, b and c.

  d Find any expression, using 1, 2, 3 and n exactly once, which will always be greater than 1 for any value of n.

12 **Adapted from Nov 1997, Paper 4**

  a A tin of soup is 11 centimetres high and has a diameter of 8 centimetres (Diagram 1). Calculate the volume of the tin.

  b The tins are packed tightly in boxes of 12, seen from above in Diagram 2. The height of each box is 11 centimetres.

   i Write down the length and the width of the box.

11 cm

8 cm

Diagram 1   Diagram 2

   ii Calculate the percentage of the volume of the box which is not occupied by the tins.

  c A shopkeeper sells the tins of soup for $0.60 each. By doing this he makes a profit of 25% on the cost price. Calculate the cost price of

   i one tin of soup   **ii** a box of 12 tins.

  d The shopkeeper tries to increase sales by offering a box of 12 tins for $6.49. At this price:

   i how much does a customer save by buying a box of 12 tins

   ii what percentage profit does the shopkeeper make on each box of 12 tins?

13 **Nov 1997, Paper 4**

A "Pythagorean triple" is a set of three **whole** numbers that could be the lengths of the three sides of a right-angled triangle.

    a   Show that $\{5, 12, 13\}$ is a Pythagorean triple.

    b   Two of the numbers in a Pythagorean triple are 24 and 25. Find the third number.

    c   The largest number in a Pythagorean triple is x and one of the other numbers is $x - 2$.

        i   If the third number is y, show that $y = \sqrt{4x - 4}$.

        ii   If $x = 50$, find the other two numbers in the triple.

        iii   If $x = 101$, find the other two numbers in the triple.

        iv   Find two other Pythagorean triples in the form $\{y, x - 2, x\}$, where $x < 40$.
             Remember that all three numbers must be whole numbers.

14 **Adapted from June 1991, Paper 4**

    a   Show that

        i $2\frac{1}{2} \times 1\frac{2}{3} = 2\frac{1}{2} + 1\frac{2}{3}$ **ii** $1\frac{3}{4} \times 2\frac{1}{3} = 1\frac{3}{4} + 2\frac{1}{3}$ **iii** $2\frac{1}{5} \times 1\frac{5}{6} = 2\frac{1}{5} + 1\frac{5}{6}$

    b   Write $2\frac{1}{2}$ and $1\frac{2}{3}$ in the form $1 + \frac{x}{y}$ and repeat for $1\frac{3}{4}$ and $2\frac{1}{3}$.

    c   From your observations in **b**, find another statement like those in **a** which is true.

    d   Write down a generalisation of what you have discovered and prove it algebraically.

15 **Adapted from June 1997, Paper 4**

Maria thinks of 3 possible savings schemes for her baby son.

Scheme A: save $10 on his 1st birthday, $20 on his 2nd birthday, $30 on his 3rd birthday, $40 on his 4th birthday, ...

Scheme B: save $1 on his 1st birthday, $2 on his 2nd birthday, $4 on his 3rd birthday, $8 on his 4th birthday, ...

Scheme C: save $1 on his 1st birthday, $4 on his 2nd birthday, $9 on his 3rd birthday, $16 on his 4th birthday, ...

She puts these ideas in a table.

Scheme/Birthday	1st	2nd	3rd	4th
A	$10	$20	$30	$40
B	$1	$2	$4	$8
C	$1	$4	$9	$16

    a   Write down, for each of the Schemes A, B and C, the amount to be saved on

        i   his 7th birthday   **ii**   his nth birthday.

    b   The formulae for the *total* amount saved up to and including his nth birthday are as follows.

$$\text{Scheme A:} \quad \text{total} = \$5n(n + 1)$$
$$\text{Scheme B:} \quad \text{total} = \$(2^n - 1)$$
$$\text{Scheme C:} \quad \text{total} = \$\frac{n(n + 1)(2n + 1)}{6}$$

        i   For *each* of the schemes A, B and C, find the total amount saved up to and including his 10th birthday.

        ii   Which scheme gives the smallest total amount of savings up to and including his 18th birthday?

        iii   Find the birthday when the scheme you have selected in **b ii** first gives the smallest total amount of savings.

16 **Adapted from June 1990, Paper 4**

   a Work out:

      i 26×93 and 62×39 **ii** 36×42 and 63×24

   b Find *one* other pair of multiplications with the same property.

   c Explain why every two digit number can be written in the form $10a + b$ where $a, b \in \mathbb{Z}^+$.

   d What can be deduced from the equation $(10m + n)(10r + s) = (10n + m)(10s + r)$?

17 **Adapted from Nov 1992, Paper 4**

n	1	2	3	4
n^2	1		9	
n^4	1		81	

   a Copy and complete the table of values above.

   b In the table below, $p = 1^2 + 2^2$

$$q = 1^2 + 2^2 + 3^2 + 4^2$$
$$r = 3(2^2) + 3(2) - 1$$
$$s = 3(3^2) + 3(3) - 1$$
$$t = 1^4 + 2^4 + 3^4$$
$$u = 1^4 + 2^4 + 3^4 + 4^4$$

Calculate the values of p, q, r, s, t and u.

n	1	2	3	4	......	20
Row X	1	p	14	q	......	
Row Y	5	r	s	59	......	
Row Z	1	17	t	u	......	

   c For the first four values of n in the table, consider the (Row X value) × (Row Y value) and the Row Z value. Find the formula which connects Row X and Row Y with Row Z.

   d **i** The value in Row X for $n = 20$ can be found by putting $n = 20$ into the formula $X = \dfrac{n(n+1)(2n+1)}{6}$. Find this value of X.

      ii The value in Row Y for $n = 20$ can be found by putting $n = 20$ into the formula $Y = 3n^2 + 3n - 1$. Find this value of Y exactly.

   e Use your answers to **c** and **d** to find the exact value of $1^4 + 2^4 + 3^4 + ... + 19^4 + 20^4$.

18 **Adapted from Nov 1992, Paper 4**

One central circle, of radius 3 cm and centre O, is completely surrounded by other circles which touch it and touch each other, as shown in the diagram. These outer circles are identical to each other.

   a If the radius of each outer circle is x cm, write down the following lengths in terms of x:

      i OA   **ii** OB   **iii** AB.

   b On one occasion there are 6 circles completely surrounding the central circle.

      i Calculate angle AOB   **ii** What special type of triangle is AOB in this case?

      iii Use your previous answers to find x.

c On another occasion there are 20 small circles completely surrounding the central circle.

    i Calculate angle AOB.

    ii M is the midpoint of AB. Consider the triangle MAO and write down the equation involving x and a trigonometric ratio.

    iii Solve this equation to find x correct to 2 decimal places.

d Extend the result to n small circles and test your result when $n = 20$.

19 **Adapted from Nov 1996, Paper 4**

  a As the product of its prime factors, $1080 = 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 5$.

    Write 135, 210 and 1120 as the product of their prime factors.

  b Copy this grid.

$a = 1$	$b =$	$c =$
$d =$	$e =$	$f =$
$g =$	$h =$	$i = 8$

    The nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are to be placed in your grid in such a way that the following four statements are *all* true.

$$a \times b \times d \times e = 135$$
$$b \times c \times e \times f = 1080$$
$$d \times e \times g \times h = 210$$
$$e \times f \times h \times i = 1120$$

    The digits 1 and 8 have already been placed for you.

    Use your answers to **a** to answer the following questions.

    i Which is the only digit, other than 1, that is a factor of 135, 1080, 210 and 1120?

    ii Which is the only letter to appear in all four statements above?

    iii 7 is a factor of only two of the numbers 135, 1080, 210 and 1120. Which two?

  c Now complete your grid.

20 **June 1994, Paper 4**

  a Calculate the gradient of the straight line joining the points (3, 18) and (3.5, 24.5).

  b The diagram shows part of the curve $y = 2x^2$.

    i P is the point (c, d). Write down d in terms of c.

    ii Q is the point $(c + h, e)$. Write down e in terms of c and h.

    iii Write down the length of PR. Find an expression for the length of QR in terms of c and h, and simplify your answer.

    iv Show that the gradient of the line PQ is $4c + 2h$.

    v If P is the point (3, 18) and Q is the point (3.5, 24.5), state the value of c and the value of h, and use these values to show that **b iv** gives the same answer as **a**.

    vi If P is the point (3, 18) and Q is the point (3.1, 19.22), state the value of c and the value of h, and use **b iv** to find the gradient of the line PQ.

    vii If P is the point (3, 18) and Q gets closer and closer to P, what happens

        a to the value of h

        b to the value of the gradient of the line PQ?

Click on the icon to obtain 4 more investigation questions.

PRINTABLE QUESTIONS

B   MODELLING QUESTIONS

1   June 1990, Paper 4

A gardener has 357 tulip bulbs to plant.

a If she planted a rectangle of 15 rows, with 23 bulbs in each row, how many bulbs would be left over?

b How many bulbs would there be in the largest square that she could plant?

c   **i** If she plants x rows, with y bulbs in each row, write down a formula for the number of bulbs left over.

   ii If $10 < x < 20$ and $y > 20$, find the value of x and the value of y such that no bulbs are left over.

2 The depth of water (d metres) in a harbour is given by the formula $d = a + b\sin(ct)^o$ where a, b and c are constants, and t is the time in hours after midnight. It is known that both b and c are non-zero and $20 < c < 35$.

The following table gives depths at particular times:

t	midnight	noon	1300	1400
d	5	5	7	8.46

a Using the first three pieces of information in the table, deduce the values of a, b and c.

b Check that the formula is correct by substituting the fourth piece of information.

c Find the depth of water at 10 am.

d What is the greatest depth of water in the harbour?

e At what times of day is the depth of water greatest?

f What is the least depth of water in the harbour?

3   Adapted from Nov 1994, Paper 4

a In a chemical reaction, the mass M grams of a chemical is given by the formula $M = 160a^{-t}$ where a is a constant integer and t is the time (in minutes) after the start. A table of values for t and M is given below.

t	0	1	2	3	4	5	6	7
M	160	80	40	20	q	5	r	1.25

   i Find the value of a.

   ii Find the values of q and r.

   iii Sketch the graph of M against t.

   iv Draw an accurate graph and add to it a tangent at $t = 2$. Estimate the rate of change in the mass after 2 minutes.

b The other chemical in the same reaction has mass m grams where $m = 160 - M$.

   i On the same graph as in **a iii**, sketch the graph of m against t.

   ii For what value of t do the chemicals have equal mass?

   iii State a single transformation which would give the graph for m from the graph for M.

4   The surge model has form $y = \dfrac{at}{2^{bt}}$ where a and b are constants and t is the time, $t \geqslant 0$.

This model has extensive use in the study of medical doses where there is an initial rapid increase to a maximum and then a slow decay to zero.

a   Use a graphics calculator to graph the model (on the same set of axes) for:

     i $a = 10$, $b = 2$ **ii** $a = 15$, $b = 3$

b   The effect of a pain killing injection t hours after it has been given is shown in the following table: The effect E follows a surge model of the form $E = \dfrac{at}{2^{bt}}$.

Time (t hours)	0	2	4	6	12
Effect (E units)	0	25	25	r	s

     i   By using *two* of the points of this table, find the values of a and b.

     ii   Hence, find the values of r and s in the table.

     iii   Use your calculator to find the maximum effect of the injection and when it occurs.

     iv   It is known that surgical operations can only take place when the effectiveness is more than 15 units. Between what two times can an operation take place?

5   The logistic model has form $y = \dfrac{a \times 2^{bt}}{2^{bt} + c}$ where t is the time, $t \geqslant 0$. The logistic model is useful in describing limited growth problems, i.e., when the y variable cannot grow beyond a particular value for some reason.

a   Use technology to help graph the logistic model for $a = 3$, $b = \frac{1}{2}$ and $c = 2$.

(Use the window $-1 \leqslant x \leqslant 40$, $-1 \leqslant y \leqslant 5$.)

b   What feature of the graph indicates a limiting value?

c   What is the limiting value?

d   Bacteria is present in a carton of milk and after t hours the bacteria (B units) was recorded as follows:

t	0	1	2
$B(t)$	10	12.70	15.03

It is known that $c = 1$.

     i   Use the first two sets of data to find a and b, and hence determine the logistic model.

     ii   Use the model found in **i** to check the third data set.

     iii   What is the limiting quantity of bacteria for this model?

     iv   In the general model $y = \dfrac{a \times 2^{bt}}{2^{bt} + c}$,   explain why the limiting quantity has value a.

6   **June 1991, Paper 4**

A farmer keeps x goats and y cows. Each goat costs \$2 a day to feed and each cow costs \$4 a day to feed. The farmer can only afford to spend \$32 a day on animal food.

a   Show that $x + 2y \leqslant 16$.

b   The farmer has room for no more than 12 animals. He wants to keep at least 6 goats and at least 3 cows. Write down three more inequalities.

c   Using a scale of 1 cm to represent 1 unit on each axis, represent the four inequalities on a graph.

d   One possible combination which satisfies all the inequalities is 6 goats and 4 cows. Write down all the other possible combinations.

e   If he makes a profit of \$50 on each goat and \$80 on each cow, which combination will give him the greatest profit? Calculate the profit in this case.

7 **May 2001, Paper 4**

a Write down which one of the sketch graphs above labelled **A** to **H** shows each of the following:

   i a *speed-time* graph for a car which starts from the rest and has constant acceleration

   ii $y = x^3 + 1$

   iii y is inversely proportional to x^2

   iv the sum of x and y is constant

   v $y = \cos x$ for $0^o \leqslant x \leqslant 90^o$

   vi a *distance-time* graph when the speed is decreasing.

b Write down an equation for sketch graph **D** if it passes through the points $(1, 1)$ and $(2, 4)$ and, when extended to the left, has line symmetry about the vertical axis.

8 **June 1994, Paper 4**

In a school gardening project, teachers and students carry earth to a vegetable plot. A teacher can carry 24 kg and a student can carry 20 kg. Each person makes one trip. Altogether at least 240 kg of earth must be carried. There are x teachers and y students.

a Show that $6x + 5y \geqslant 60$.

b There must not be more than 13 people carrying earth, and there must be at least 4 teachers and at least 3 students. Write down three more inequalities in x and/or y.

c   **i** Draw x and y axes from 0 to 14, using 1 cm to represent 1 unit of x and y.

   ii On your grid, represent the information in parts **a** and **b**. Shade the *unwanted* regions.

d From your graph, find

   i the least number of people required

   ii the greatest amount of earth which can be carried.

9 **Adapted from Nov 1993, Paper 4**

Anna throws a ball from a point A, one metre above the ground, towards a wall. The ball travels along the arrowed path from A to B, given by the equation $y = a + bx - x^2$ where the x-axis represents the horizontal ground and the y-axis represents the wall. The ball passes through the point T$(2, 4)$ and hits the wall 4 m above O.

a Find the values of a and b.

b Show that the x-coordinate of A satisfies the equation $x^2 - 2x - 3 = 0$.

c Find the x-coordinate of A.

d The ball rebounds from the wall at B to the ground at G. The equation of the path B to G is $y = c - 2x - x^2$.

    i Find the value of c.   **ii** Find the x-coordinate of G correct to 2 decimal places.

e How far from the wall is the ball when it is **i** 0.5 m **ii** 3 m above the ground?

f Find the greatest height of the ball during its motion.

10 **May 2005, Paper 4**

A taxi company has "SUPER" taxis and "MINI" taxis. One morning a group of 45 people needs taxis. For this group the taxi company uses x "SUPER" taxis and y "MINI" taxis. A "SUPER" taxi can carry 5 passengers and a "MINI" taxi can carry 3 passengers. So $5x + 3y \geqslant 45$.

a The taxi company has 12 taxis. Write down *another* inequality in x and y to show this information.

b The taxi company always uses at least 4 "MINI" taxis. Write down an inequality in y to show this.

c Draw x and y axes from 0 to 15 using 1 cm to represent 1 unit on each axis.

d Draw three lines on your graph to show the inequality $5x + 3y \geqslant 45$ *and* the inequalities from parts **a** and **b**. Shade the *unwanted* regions.

e The cost to the taxi company of using a "SUPER" taxi is \$20 and the cost of using a "MINI" taxi is \$10. The taxi company wants to find the cheapest way of providing "SUPER" and "MINI" taxis for this group of people. Find the *two* ways in which this can be done.

f The taxi company decides to use 11 taxis for this group.

    i The taxi company charges \$30 for the use of each "SUPER" taxi and \$16 for the use of each "MINI" taxi. Find the two possible *total* charges.

    ii Find the largest possible *profit* the company can make, using 11 taxis.

21 **Adapted from Nov 1993, Paper 4**

   a Show that $\dfrac{1}{n} - \dfrac{1}{n+1} = \dfrac{1}{n(n+1)}$.

   b Copy the following table, completing the rows for $n = 2$, 3, 4, 99 and 100.

   c Use **a** and your table to find another expression for

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots\ldots + \frac{1}{100 \times 101}.$$

     Write your answer as a single fraction.

n	$\dfrac{1}{n} - \dfrac{1}{n+1}$	$\dfrac{1}{n(n+1)}$
1	$\dfrac{1}{1} - \dfrac{1}{2}$	$\dfrac{1}{1 \times 2}$
2	$\ldots\ldots - \ldots\ldots$	$\ldots\ldots$
3	$\ldots\ldots - \ldots\ldots$	$\ldots\ldots$
4	$\ldots\ldots - \ldots\ldots$	$\ldots\ldots$
$\downarrow$	$\downarrow \quad \downarrow$	$\downarrow$
99	$\ldots\ldots - \ldots\ldots$	$\ldots\ldots$
100	$\ldots\ldots - \ldots\ldots$	$\dfrac{1}{100 \times 101}$

22 **Adapted from Nov 1999, Paper 4**

The height of a cylinder is 10 cm and its radius is 2.5 cm.

   a
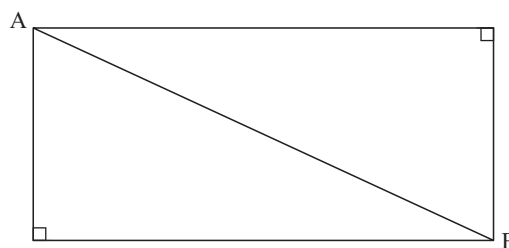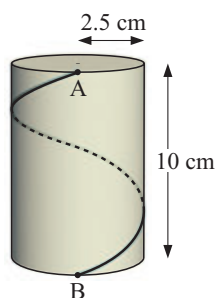
     A piece of string is wound evenly once around the curved surface of the cylinder, starting at a point A on the circumference of the top circular face and finishing at B, vertically below A.

     A sketch of the net of the curved surface of the cylinder, together with the string AB, is shown above in the diagram on the right. Calculate the length of the string, AB.

   b
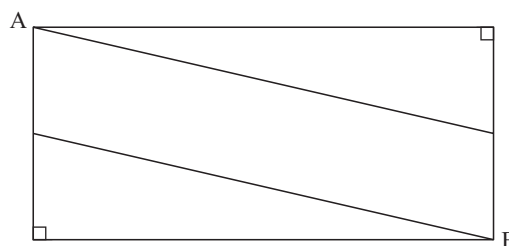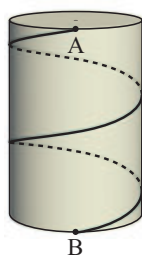
     Another string, starting at A, is wound evenly *twice* around the cylinder, finishing again at B. Calculate the length of this string.

   c Sketch the net when a string, starting at A, is wound evenly three times around the cylinder, finishing at B.

   d A string is wound evenly n times around the cylinder, from A to B. Find a formula, in terms of n, for the length of the string.

23 **Adapted from Nov 1998, Paper 4**

Look at this table of numbers.

Row 1	1			
Row 2	3	5		
Row 3	7	9	11	
Row 4	13	15	17	19

....

....

a   **i** Write down the sum of each of the first four rows.

     ii What is the sum of the numbers in the hundredth row? Explain your answer.

b   **i** Write down the sum of all the numbers   in the first two rows

                                          in the first three rows

                                          in the first four rows

                                          in the first n rows.

     ii What is the sum of all the numbers in the first 100 rows?

c What is the last number in the nth row?

24 In our **decimal system** the number 2358 represents 2 thousands, three hundreds, five tens and 8 ones.

That is, $2358 = 2000 + 300 + 50 + 8$ or $2 \times 10^3 + 3 \times 10^2 + 5 \times 10^1 + 8 \times 1$

All positive integers can be written in this form where the **base** is 10.

The **binary system** uses **base 2**. So, place values of digits correspond to powers of 2 (instead of 10). Each number contains only the digits 0 and 1. Binary numbers are used in a computer. Numbers are stored in binary code.

For example, $101\,011_2 = 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1$

So, $101\,011_2$ has a decimal value of $32 + 8 + 2 + 1 = 43$.

To convert a decimal number to a binary number, observe the following division method:

```
2 | 43
2 | 21   + 1  ↑
2 | 10   + 1
2 | 5    + 0
2 | 2    + 1
2 | 1    + 0
    0    + 1
```

So, $43 = 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1$

a Convert these binary numbers to decimal numbers:

    i 101_2 **ii** 1001_2 **iii** $11\,111_2$ **iv** $1\,001\,001_2$ **v** $11\,011\,011_2$

b Convert these decimal numbers into binary numbers:

    i 7   **ii** 17   **iii** 39   **iv** 81   **v** 138   **vi** 432

c The **hexadecimal system** uses base 16 and contains the digits 0 to 9 as well as the letters A to F. This is also used in computers as hexadecimal numbers are very easily converted to binary ones.

    Notice that $BED_{16} = 11 \times 16^2 + 14 \times 16^1 + 13 \times 1 = 3053_{10}$

    i Convert the following hexadecimal numbers into decimal numbers:

        a 98_{16} **b** 213_{16} **c** DAD_{16} **d** $1BC4_{16}$

    ii Convert these decimal numbers into hexadecimal numbers:

        a 123   **b** 389   **c** 2164   **d** 18 341

    iii Convert these hexadecimal numbers into binary numbers:

        a 25_{16} **b** BE_{16} **c** $1A4_{16}$ **d** ABE_{16}

CHAPTER 34 (CD question answers)

21 a $\dfrac{1}{n} - \dfrac{1}{n+1} = \dfrac{n+1-n}{n(n+1)} = \dfrac{1}{n(n+1)}$

b

n	$\dfrac{1}{n} - \dfrac{1}{n+1}$	$\dfrac{1}{n(n+1)}$
1	$\dfrac{1}{1} - \dfrac{1}{2}$	$\dfrac{1}{1\times 2}$
2	$\dfrac{1}{2} - \dfrac{1}{3}$	$\dfrac{1}{2\times 3}$
3	$\dfrac{1}{3} - \dfrac{1}{4}$	$\dfrac{1}{3\times 4}$
4	$\dfrac{1}{4} - \dfrac{1}{5}$	$\dfrac{1}{4\times 5}$
$\vdots$	$\vdots$	$\vdots$
99	$\dfrac{1}{99} - \dfrac{1}{100}$	$\dfrac{1}{99\times 100}$
100	$\dfrac{1}{100} - \dfrac{1}{101}$	$\dfrac{1}{100\times 101}$

c $\quad \dfrac{1}{1\times 2} + \dfrac{1}{2\times 3} + \dfrac{1}{3\times 4} + \ldots\ldots + \dfrac{1}{100\times 101}$

$= \dfrac{1}{1} - \dfrac{1}{101}$

$= 1 - \dfrac{1}{101}$

$= \dfrac{100}{101}$

22 a $\sqrt{10^2 + (5\pi)^2} \approx 18.6$ cm

b $2\sqrt{(\frac{10}{2})^2 + (5\pi)^2} \approx 33.0$ cm

c
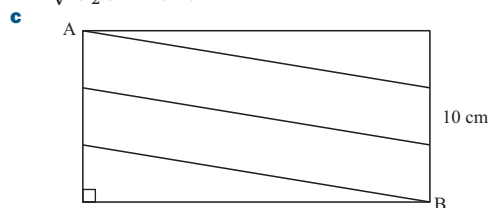
d $L = n\sqrt{(\frac{10}{n})^2 + (5\pi)^2}$ cm

23 a **i** 1, 8, 27, 64

ii $100^3 = 1\,000\,000$ $\quad \{1 = 1^3,\ 8 = 2^3,\ 27 = 3^3,$
$64 = 4^3\}$

b **i** 9, 36, 100, $(1+2+3+4+\ldots+n)^2$ or $\dfrac{n^2(n+1)^2}{4}$

ii $25\,502\,500$

c $n^2 + n - 1$

24 a **i** 5 **ii** 9 **iii** 31 **iv** 73 **v** 219

b **i** 111_2 **ii** $10\,001_2$ **iii** $100\,111_2$ **iv** $1\,010\,001_2$
v $10\,001\,010_2$ **vi** $110\,110\,000_2$

c **i** *a* 152 *b* 531 *c* 3501 *d* 7108

ii *a* $7B_{16}$ *b* 185_{16} *c* 874_{16} *d* $47A5_{16}$

iii *a* $100\,101_2$ *b* $10\,111\,110_2$
c $110\,100\,100_2$ *d* $101\,010\,111\,110_2$

EXERCISE 1A

1 a $3x + 3$ **b** $10 - 2x$ **c** $-x - 2$ **d** $x - 3$
e $4a + 8b$ **f** $6x + 3y$ **g** $5x - 5y$ **h** $-6x^2 + 6y^2$
i $-2x - 8$ **j** $-6x + 3$ **k** $x^2 + 3x$ **l** $2x^2 - 10x$
m $-3x - 6$ **n** $-4x + 12$ **o** $2x - 7$ **p** $-2x + 2y$
q $a^2 + ab$ **r** $ab - a^2$ **s** $2x^2 - x$ **t** $2x^3 - 2x^2 - 4x$

2 a $2x + 5$ **b** $1 - 4x$ **c** $3x - 1$ **d** $2 - 4x$ **e** x^2
f $6x - x^2$ **g** $2ab + a^2$ **h** $7x - 3x^2$ **i** $2x^2 - 10x$

3 a $5x - 2$ **b** $3a - 2b$ **c** $a + 2b$ **d** $15 - 3y$ **e** $-6y - 10$
f $15x - 8$ **g** $a + 5b$ **h** $x^2 + 6x - 6$ **i** $x^2 + 2x + 6$
j $2x^2 - x$ **k** $-2x^2 + 2x$ **l** $x^2 - y^2$ **m** $5 - 3x$
n $8x - 8$ **o** $6x^2 - 22x$

EXERCISE 1B

1 a $A_2 = ac$ **b** $A_2 = ad$ **c** $A_3 = bc$ **d** $A_4 = bd$
e $A = (a + b)(c + d)$
$(a + b)(c + d) = ac + ad + bc + bd$

2 a $x^2 + 10x + 21$ **b** $x^2 + x - 20$ **c** $x^2 + 3x - 18$
d $x^2 - 4$ **e** $x^2 - 5x - 24$ **f** $6x^2 + 11x + 4$
g $1 + 2x - 8x^2$ **h** $12 + 5x - 2x^2$ **i** $6x^2 - x - 2$
j $25 - 10x - 3x^2$ **k** $7 + 27x - 4x^2$ **l** $25x^2 + 20x + 4$

3 a $x^2 - 4$ **b** $a^2 - 25$ **c** $16 - x^2$ **d** $4x^2 - 1$
e $25a^2 - 9$ **f** $16 - 9a^2$

4 a $x^2 + 6x + 9$ **b** $x^2 - 4x + 4$ **c** $9x^2 - 12x + 4$
d $1 - 6x + 9x^2$ **e** $9 - 24x + 16x^2$ **f** $25x^2 - 10xy + y^2$

5 $A = (x + 6)(x + 4) = x^2 + 10x + 24$

EXERCISE 1C

1 a $x^2 - 4$ **b** $x^2 - 4$ **c** $4 - x^2$ **d** $4 - x^2$
e $x^2 - 1$ **f** $1 - x^2$ **g** $x^2 - 49$ **h** $c^2 - 64$
i $d^2 - 25$ **j** $x^2 - y^2$ **k** $16 - d^2$ **l** $25 - e^2$

2 a $4x^2 - 1$ **b** $9x^2 - 4$ **c** $16y^2 - 25$
d $4y^2 - 25$ **e** $9x^2 - 1$ **f** $1 - 9x^2$
g $4 - 25y^2$ **h** $9 - 16a^2$ **i** $16 - 9a^2$

3 a $4a^2 - b^2$ **b** $a^2 - 4b^2$ **c** $16x^2 - y^2$
d $16x^2 - 25y^2$ **e** $4x^2 - 9y^2$ **f** $49x^2 - 4y^2$

4 a **i** 43×37 **ii** 24×26
$= (40 + 3)(40 - 3)$ $= (25 - 1)(25 + 1)$
$= 40^2 - 3^2$ $= 25^2 - 1^2$

b **i** 396 **ii** 2499 **iii** 9991

EXERCISE 1D

1 a $A_1 = a^2$ **b** $A_2 = ab$ **c** $A_3 = ab$ **d** $A_4 = b^2$
e $A = (a + b)^2$, $(a + b)^2 = a^2 + 2ab + b^2$

2 a $x^2 + 10x + 25$ **b** $x^2 + 8x + 16$ **c** $x^2 + 14x + 49$
d $a^2 + 4a + 4$ **e** $9 + 6c + c^2$ **f** $25 + 10x + x^2$

3 a $x^2 - 6x + 9$ **b** $x^2 - 4x + 4$ **c** $y^2 - 16y + 64$
d $a^2 - 14a + 49$ **e** $25 - 10x + x^2$ **f** $16 - 8y + y^2$

4 a $9x^2 + 24x + 16$ **b** $4a^2 - 12a + 9$ **c** $9y^2 + 6y + 1$
d $4x^2 - 20x + 25$ **e** $9y^2 - 30y + 25$ **f** $49 + 28a + 4a^2$
g $1 + 10x + 25x^2$ **h** $49 - 42y + 9y^2$ **i** $9 + 24a + 16a^2$

5 a $x^4 + 4x^2 + 4$ **b** $y^4 - 6y^2 + 9$ **c** $9a^4 + 24a^2 + 16$
d $1 - 4x^2 + 4x^4$ **e** $x^4 + 2x^2y^2 + y^4$ **f** $x^4 - 2a^2x^2 + a^4$

6 a $-x^2 - 3x - 8$ **b** $x^2 + x + 2$ **c** $2x^2 + 6x + 5$
d $-6x - 13$ **e** $11 - 13x + 3x^2$
f $10x^2 - 7x - 5$ **g** $3x^2 - 2x - 10$ **h** $3x^2 - x - 10$
i $2x^2 + 2x + 5$ **j** $-6x - 3$

EXERCISE 1E

1 a $x^3 + 3x^2 + 6x + 8$ **b** $x^3 + 5x^2 + 3x - 9$
c $x^3 + 5x^2 + 7x + 3$ **d** $2x^3 + x^2 - 6x - 5$
e $2x^3 + 7x^2 + 8x + 3$ **f** $2x^3 - 9x^2 + 4x + 15$
g $3x^3 + 14x^2 - x + 20$ **h** $8x^3 - 14x^2 + 7x - 1$

2 a $x^3 + 3x^2 + 3x + 1$ **b** $x^3 + 9x^2 + 27x + 27$
c $x^3 - 12x^2 + 48x - 64$ **d** $x^3 - 9x^2 + 27x - 27$
e $27x^3 + 27x^2 + 9x + 1$ **f** $8x^3 - 36x^2 + 54x - 27$

3 a $x^3 + 6x^2 + 8x$ **b** $x^3 - x^2 - 6x$ **c** $x^3 - 9x^2 + 20x$
d $2x^3 + 14x^2 + 20x$ **e** $-3x^3 + 15x^2 - 18x$
f $x^3 - 4x^2 - 12x$ **g** $-9x^3 - 33x^2 + 12x$
h $-10x^3 - 13x^2 + 3x$ **i** $x^3 - 3x^2 - 4x + 12$

4 a $x^3 + 9x^2 + 26x + 24$ **b** $x^3 - x^2 - 14x + 24$
c $x^3 - 10x^2 + 31x - 30$ **d** $2x^3 + x^2 - 12x + 9$
e $3x^3 + 14x^2 + 21x + 10$ **f** $12x^3 + 11x^2 - 2x - 1$
g $-3x^3 + 26x^2 - 33x - 14$ **h** $-3x^3 + 16x^2 - 12x - 16$

5 a 4 **b** 6 **c** 6 **d** 9 **e** 8 **f** 12 **g** 8 **h** 12

EXERCISE 1F

1 a $2a$ **b** $5b$ **c** $4xy$ **d** $4x$ **e** x **f** $-2x$
g $-b$ **h** $4a$ **i** $-3xy$

2 a 2 **b** c **c** 1 **d** k **e** $3a$ **f** $5x$
g $5x$ **h** $8y$ **i** 18

3 a ab **b** abc **c** $12a$ **d** a **e** r **f** q
g $3b$ **h** dp **i** $4r$ **j** $3pq$ **k** $2ab$ **l** $6xy$
m 5 **n** $12wz$ **o** $12pqr$

4 a $(x + 2)$ **b** $2(x + 5)$ **c** x **d** $2(x + 1)$
e $2(x + 3)$ **f** $2x(x - 3)$

EXERCISE 1G

1 a $2(x + 2)$ **b** $3(a - 4)$ **c** $5(3 - p)$
d $6(3x + 2)$ **e** $4x(x - 2)$ **f** $2m(1 + 4m)$

2 a $4(x + 4)$ **b** $5(2 + d)$ **c** $5(c - 1)$ **d** $d(c + e)$
e $2a(3 + 4b)$ **f** $2x(3 - x)$ **g** $7a(b - 1)$ **h** $2b(2a - 3c)$

3 a $3(a + b)$ **b** $8(x - 2)$ **c** $3(p + 6)$ **d** $14(2 - x)$
e $7(x - 2)$ **f** $6(2 + x)$ **g** $c(a + b)$ **h** $6(2y - a)$
i $a(5 + b)$ **j** $c(b - 6d)$ **k** $x(7 - y)$ **l** $y(x + 1)$
m $a(1 + b)$ **n** $y(x - z)$ **o** $p(3q + r)$ **p** $c(d - 1)$

4 a $x(x + 2)$ **b** $x(5 - 2x)$ **c** $4x(x + 2)$
d $7x(2 - x)$ **e** $6x(x + 2)$ **f** $x^2(x + 9)$
g $xy(x + y)$ **h** $2x^2(2x - 3)$ **i** $9x(x^2 - 2y)$
j $a(a^2 + a + 1)$ **k** $2(a^2 + 2a + 4)$ **l** $3a(a^2 - 2a + 3)$

5 a $9(b - a)$ **b** $3(2b - 1)$ **c** $4(b - 2a)$
d $c(d - 7)$ **e** $a(b - 1)$ **f** $6x(2 - x)$
g $5x(3x - 1)$ **h** $2b(2a - b)$ **i** $a(a - 1)$

6 a $-6(a + b)$ **b** $-4(1 + 2x)$ **c** $-3(y + 2z)$
d $-c(9 + d)$ **e** $-x(1 + y)$ **f** $-5x(x + 4)$
g $-3y(4 + y)$ **h** $-9a(2a + b)$ **i** $-8x(2x + 3)$

7 a $(x - 7)(2 + x)$ **b** $(x + 3)(a + b)$ **c** $(x + 2)(4 - x)$
d $(x + 9)(x + 1)$ **e** $(b + 4)(a - 1)$ **f** $(b + c)(a + d)$
g $(m + n)(a - b)$ **h** $(x + 3)(x - 1)$

8 a $(x + 3)(x - 1)$ **b** $(x - 7)(x + 7)$ **c** $(x + 6)(x - 4)$
d $(x - 2)(x - 8)$ **e** $x + 2$ **f** $(a + b)(4 - a)$
g $3(a - 2)(a - 4)$ **h** $(x + 4)(4x + 1)$
i $5(x - 1)(6 - x)$ **j** $-(x + 5)(4x + 17)$

EXERCISE 1H

1
a $(x+2)(x-2)$ **b** $(2+x)(2-x)$
c $(x+9)(x-9)$ **d** $(5+x)(5-x)$
e $(2x+1)(2x-1)$ **f** $(3x+4)(3x-4)$
g $(2x+3)(2x-3)$ **h** $(6+7x)(6-7x)$

2
a $3(x+3)(x-3)$ **b** $-2(x+2)(x-2)$
c $3(x+5)(x-5)$ **d** $-5(x+1)(x-1)$
e $2(2x+3)(2x-3)$ **f** $-3(3x+5)(3x-5)$

3
a $(x+\sqrt{3})(x-\sqrt{3})$ **b** no linear factors
c $(x+\sqrt{15})(x-\sqrt{15})$ **d** $3(x+\sqrt{5})(x-\sqrt{5})$
e $(x+1+\sqrt{6})(x+1-\sqrt{6})$ **f** no linear factors
g $(x-2+\sqrt{7})(x-2-\sqrt{7})$
h $(x+3+\sqrt{17})(x+3-\sqrt{17})$ **i** no linear factors

4
a $(x+3)(x-1)$ **b** $4(x+2)(x-1)$
c $(x-5)(x+3)$ **d** $3(x+1)(3-x)$
e $(3x+2)(x-2)$ **f** $(2x+3)(4x-3)$
g $(3x-1)(x+3)$ **h** $8x(x-1)$ **i** $-3(4x+3)$

5
a $(x-1)$ m
b $A = x^2 - (x-1)^2$
$ = (x+[x-1])(x-[x-1])$
$ = (2x-1)(1)$
$ = 2x - 1$ square metres

EXERCISE 1I

1
a $(x+3)^2$ **b** $(x+4)^2$ **c** $(x-3)^2$
d $(x-4)^2$ **e** $(x+1)^2$ **f** $(x-5)^2$
g $(y+9)^2$ **h** $(m-10)^2$ **i** $(t+6)^2$

2
a $(3x+1)^2$ **b** $(2x-1)^2$ **c** $(3x+2)^2$
d $(5x-1)^2$ **e** $(4x+3)^2$ **f** $(5x-2)^2$
g $-(x-1)^2$ **h** $-2(x+2)^2$ **i** $-3(x+5)^2$

3
a $x^2 + 12x + 36 = (x+6)^2$ and no perfect square is ever negative for real x.
b Consider $x^2 - 4x + 4$
$ x^2 - 4x + 4 = (x-2)^2$
where $(x-2)^2 \geqslant 0$ for all real x
$\therefore \quad x^2 - 4x + 4 \geqslant 0$
$\therefore \quad x^2 + 4 \geqslant 4x$ for all real x

EXERCISE 1J

1
a $(b+2)(a+1)$ **b** $(a+4)(c+d)$ **c** $(a+2)(b+3)$
d $(m+p)(n+3)$ **e** $(x+3)(x+7)$ **f** $(x+4)(x+5)$
g $(2x+1)(x+3)$ **h** $(3x+2)(x+4)$ **i** $(5x+3)(4x+1)$

2
a $(x+5)(x-4)$ **b** $(x+2)(x-7)$ **c** $(x-3)(x-2)$
d $(x-5)(x-3)$ **e** $(x+7)(x-8)$ **f** $(2x+1)(x-3)$
g $(3x+2)(x-4)$ **h** $(4x-3)(x-2)$ **i** $(9x+2)(x-1)$

EXERCISE 1K

1
a $3, 4$ **b** $3, 5$ **c** $2, 8$ **d** $2, 9$ **e** $-3, 7$
f $3, -7$ **g** $-6, 2$ **h** $-2, 15$

2
a $(x+1)(x+3)$ **b** $(x+12)(x+2)$ **c** $(x+3)(x+7)$
d $(x+6)(x+9)$ **e** $(x+4)(x+5)$ **f** $(x+3)(x+5)$
g $(x+4)(x+6)$ **h** $(x+2)(x+7)$ **i** $(x+2)(x+4)$

3
a $(x-1)(x-2)$ **b** $(x-1)(x-3)$ **c** $(x-2)(x-3)$
d $(x-3)(x-11)$ **e** $(x-3)(x-13)$ **f** $(x-3)(x-16)$
g $(x-4)(x-7)$ **h** $(x-2)(x-12)$ **i** $(x-2)(x-18)$

4
a $(x-8)(x+1)$ **b** $(x+7)(x-3)$ **c** $(x-2)(x+1)$
d $(x-4)(x+2)$ **e** $(x+8)(x-3)$ **f** $(x-5)(x+2)$
g $(x+9)(x-6)$ **h** $(x+9)(x-8)$ **i** $(x-7)(x+3)$
j $(x-3)(x+2)$ **k** $(x-12)(x+5)$ **l** $(x+12)(x-5)$
m $(x+6)(x-3)$ **n** $(x+2)(x-9)$ **o** $(x-5)(x-7)$

5
a $(x+6)(x+1)$ **b** $(x-9)(x+7)$ **c** $(x-2)(x-9)$
d $(x+8)(x-2)$ **e** $(x-1)(x-4)$ **f** $(x+7)(x+5)$
g $(x-5)(x+4)$ **h** $(x-11)(x+2)$ **i** $(x+12)(x-4)$
j $(x-7)(x+4)$ **k** $x(x+13)$ **l** $(x-7)^2$

6
a $2(x+1)(x+4)$ **b** $3(x-1)(x-6)$ **c** $2(x+3)(x+4)$
d $2(x-10)(x-12)$ **e** $4(x-3)(x+1)$ **f** $3(x-3)(x-11)$
g $2(x-10)(x+9)$ **h** $3(x-4)(x+2)$ **i** $2(x+4)(x+5)$
j $x(x-8)(x+1)$ **k** $4(x-3)^2$ **l** $7(x+5)(x-2)$
m $5(x-8)(x+2)$ **n** $x(x-7)(x+4)$ **o** $x^2(x+1)^2$

7
a $-(x+9)(x-6)$ **b** $-(x+2)(x+5)$ **c** $-(x+3)(x+7)$
d $-(x-3)(x-1)$ **e** $-(x-2)^2$ **f** $-(x+3)(x-1)$
g $-(x-8)(x+6)$ **h** $-(x-3)^2$ **i** $-(x-3)(x-7)$
j $-2(x-9)(x+7)$ **k** $-2(x-5)^2$ **l** $-x(x-2)(x+1)$

EXERCISE 1L

1
a $(2x+3)(x+1)$ **b** $(2x+5)(x+1)$ **c** $(7x+2)(x+1)$
d $(3x+4)(x+1)$ **e** $(3x+1)(x+4)$ **f** $(3x+2)(x+2)$
g $(4x+1)(2x+3)$ **h** $(7x+1)(3x+2)$ **i** $(3x+1)(2x+1)$
j $(6x+1)(x+3)$ **k** $(5x+1)(2x+3)$ **l** $(7x+1)(2x+5)$

2
a $(2x+1)(x-5)$ **b** $(3x-1)(x+2)$ **c** $(3x+1)(x-2)$
d $(2x-1)(x+2)$ **e** $(2x+5)(x-1)$ **f** $(5x+1)(x-3)$
g $(5x-3)(x-1)$ **h** $(11x+2)(x-1)$ **i** $(3x+2)(x-3)$
j $(2x+3)(x-3)$ **k** $(3x-2)(x-5)$ **l** $(5x+2)(x-3)$
m $(3x-2)(x+4)$ **n** $(2x-1)(x+9)$ **o** $(2x-3)(x+6)$
p $(2x-3)(x+7)$ **q** $(5x+2)(3x-1)$ **r** $(21x+1)(x-3)$

3
a $(3x+2)(5x+3)$ **b** $(3x+2)(5x-3)$ **c** $(3x-2)(5x+3)$
d $2(3x-2)(5x-3)$ **e** $2(3x-1)^2$ **f** $3(4x+3)^2$
g $2(4x+1)(2x+1)$ **h** $2(4x-1)(2x+1)$
i $5(4x+1)(2x-1)$ **j** $4(4x-1)(2x-1)$
k $(5x+3)(5x+2)$ **l** $(5x-3)(5x-2)$ **m** $(5x-4)(5x+2)$
n $(25x+1)(x-6)$ **o** $(6x+5)(6x-1)$ **p** $(9x+5)(4x-1)$
q $(12x-5)(3x+2)$ **r** $(18x-1)(2x+3)$

EXERCISE 1M

1
a $x(3x+2)$ **b** $(x+9)(x-9)$ **c** $2(p^2+4)$
d $3(b+5)(b-5)$ **e** $2(x+4)(x-4)$
f $n^2(n+2)(n-2)$ **g** $(x-9)(x+1)$ **h** $(d+7)(d-1)$
i $(x+9)(x-1)$ **j** $4t(1+2t)$ **k** $3(x+6)(x-6)$
l $2(g-11)(g+5)$ **m** $(2a+3d)(2a-3d)$
n $5(a-2)(a+1)$ **o** $2(c-3)(c-1)$
p $x^2(x+1)(x-1)$ **q** $d^2(d+3)(d-1)$ **r** $x(x+2)^2$

2
a $(x-3)^2$ **b** $(x+11)(x-11)$ **c** $(x-1)^2$
d $(y+5)^2$ **e** $(x+11)^2$ **f** $(x-y)^2$ **g** $(1+x)(1-x)$
h $(5y+1)(5y-1)$ **i** $(7y+6z)(7y-6z)$ **j** $(2d+7)^2$
k $a(2b+c)(2b-c)$ **l** $2\pi(R+r)(R-r)$

3
a $a(b+c-2)$ **b** $ab(ab-2)$ **c** $2x(3+x)(3-x)$
d $(x+7)^2$ **e** $4a(a+b)(a-b)$ **f** $xy(x+2)(x-2)$
g $4x^2(x+1)(x-1)$ **h** $(x-2)(y-z)$ **i** $(a+b)(x+1)$
j $(x-y)(a+1)$ **k** $(x+2)(x+3)$ **l** $(x^2+1)(x+1)$

4
a $7(x-5y)$ **b** $2(g+2)(g-2)$ **c** $-5x(x+2)$
d $m(m+3p)$ **e** $(a+3)(a+5)$ **f** $(m-3)^2$
g $5x(x+y-xy)$ **h** $(x+2)(y+2)$ **i** $(y+5)(y-9)$
j $(2x+1)(x+5)$ **k** $3(y+7)(y-7)$ **l** $3(p+q)(p-q)$
m $(2c+1)(2c-1)$ **n** $3(x+4)(x-3)$ **o** $2(b+5)(x-3)$

5 **a** $-(x-1)(x+12)$ **b** $-2(x-1)(x-3)$
  c $-(x+7)(x-2)$ **d** $-2x(x-1)^2$
  e $(a+b+3)(a+b-3)$ **f** $x(x+4)$

6 **a** $(2x+3)(x+7)$ **b** $(2x+5)(x+3)$
  c $(2x+1)(2x+5)$ **d** $(4x+3)(3x+1)$
  e $(x-5)(6x+1)$ **f** $(4x+1)^2$
  g $(5x+4)(5x-4)$ **h** $(12x+1)(x-6)$
  i $2(6x-1)(x-3)$ **j** $3(3x+4)(x-1)$
  k $(3x-5)(4x-3)$ **l** $(3x+2)(12x-7)$

REVIEW SET 1A

1 **a** $3x^2-6x$ **b** $15x-3x^2$ **c** $x^2-5x-24$
  d x^2+6x+9 **e** $-x^2+4x-4$ **f** $16x^2-1$
  g $12x^2-5x-2$ **h** $2x^2+3x-15$

2 **a** x^4+6x^2+9 **b** $4-9d^2$ **c** $x^3-15x^2+75x-125$
  d x^3+3x^2-2x+8 **e** $13x-20-2x^2$ **f** $16x^2-y^2$

3 **a** $3c$ **b** $4p$ **c** $3rs$

4 **a** $3x(x-4)$ **b** $3x(5-2x)$ **c** $2(x+7)(x-7)$
  d $(x-3)^2$ **e** $(a+b)^2$ **f** $(x+2)(x-1)$

5 **a** $(x-1)(5+y)$ **b** $(3x+7)(1+2b)$

6 **a** $(x+3)(x+7)$ **b** $(x-3)(x+7)$ **c** $(x-7)(x+3)$
  d $(x-2)(x-3)$ **e** $4(x-3)(x+1)$ **f** $-(x+4)(x+9)$

7 **a** $(4x+5)(2x+3)$ **b** $(6x-1)(2x-3)$ **c** $(4x-5)(3x+2)$

8 **a** $(x+\sqrt{10})(x-\sqrt{10})$ **b** cannot be done
  c $(x-4+\sqrt{13})(x-4-\sqrt{13})$

REVIEW SET 1B

1 **a** $9x^2-6xy+y^2$ **b** $2a^2-2ab$ **c** $-12x^2+x+1$
  d $4x^2+28x+49$ **e** $-25+10x-x^2$ **f** $1-49x^2$
  g $20x^2-11x-4$ **h** $-x^2+7x+18$

2 **a** $10x-11$ **b** $9x^2+12x+4$ **c** $64-q^2$
  d $4x^2+21x-18$ **e** $8x^3-12x^2+6x-1$
  f $x^3-7x^2+14x-6$

3 **a** $5b(a+2b)$ **b** $3(x+2)(x-2)$ **c** $(x+4)^2$
  d $2(a-b)^2$ **e** $3x(x+3)(x-1)$ **f** $(x-3)(x-6)$

4 $(y-z)(2x+1)$

5 **a** $(x+5)(x+7)$ **b** $(x+7)(x-5)$ **c** $(x-5)(x-7)$
  d $2(x-7)(x+5)$ **e** $(x-5)(x-6)$ **f** $-(x-2)(x-10)$

6 **a** $(x+9)(x-9)$ **b** $2(x+\sqrt{19})(x-\sqrt{19})$
  c cannot be done

7 **a** $(3x+2)(4x-1)$ **b** $(3x-2)(4x+3)$
  c $4(3x-1)(2x+3)$

8 **a** $(c+3)(d+3)$ **b** $(4-x)(x-1)$ **c** $(3x-4)(2x-3)$

9 **b** $(5x+7)(x-3)$ **c** $(5x+7)(x-3)$
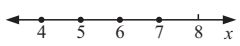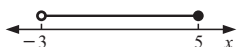   The same result as **b**.

EXERCISE 2A
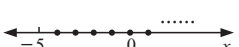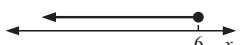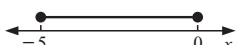
1 **a** **i** 12 **ii** 5 **b** $S'=\{1,3,5,6,8,10,12\}$
      iii 2 $T'=\{1,2,3,5,6,7,8,9,10,12\}$
  c **i** true **ii** false **iii** true **d** **i** false **ii** true
  e $x=2$ **f** finite as it contains a finite number of elements

2 **a** $\varnothing$, $\{a\}$
  b $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{b,c\}$, $\{a,c\}$, $\{a,b,c\}$
  c 16 (For a set with 1 element, 2 subsets;
      for a set with 2 elements, 4 subsets;
      for a set with 3 elements, 8 subsets.)

3 $\varnothing$, $\{2\}$, $\{4\}$, $\{7\}$, $\{9\}$, $\{2,4\}$, $\{2,7\}$, $\{2,9\}$, $\{4,7\}$, $\{4,9\}$,
  $\{7,9\}$, $\{2,4,7\}$, $\{2,4,9\}$, $\{2,7,9\}$, $\{4,7,9\}$ (15 of them)

4 **a** $S=\{1,2,3,6\}$ **b** $S=\{6,12,18,24,......\}$
  c $S=\{1,17\}$ **d** $S=\{17,34,51,68,......\}$
  e $S=\{2,3,5,7,11,13,17,19\}$
  f $S=\{12,14,15,16,18,20,21,22,24,25,26,27,28\}$

5 **a** 4 **b** is infinite **c** 2 **d** is infinite **e** 8 **f** 13

6 **a** $A=\{23,29\}$, $B=\{22,24,26,28\}$,
      $C=\{21,22,24,25,26,27,28\}$, $D=\varnothing$
  b **i** 2 **ii** 0 **c** **i** A and D **ii** B and D
  d **i** false **ii** true **iii** false

7 **a** **i** 7 **ii** 4 **iii** 3 **iv** 2 **v** 5
  b $n(S)+n(S')=n(U)$
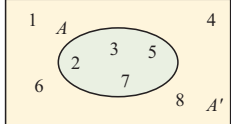
EXERCISE 2B
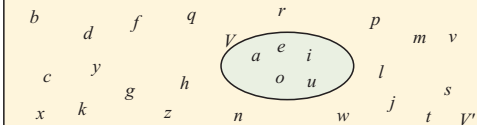
1 **a** true **b** true **c** true **d** true
  e false **f** false **g** true **h** true

2 **a, b, c, d, f, g, h** are rational; **e** is irrational

3 **a** $0.\overline{7}=\frac{7}{9}$ **b** $0.\overline{41}=\frac{41}{99}$ **c** $0.\overline{324}=\frac{12}{37}$

4 **a** 0.527 can be written as $\frac{527}{1000}$, and 527, 1000 are integers
  b $0.\overline{9}=1$

5 **a** e.g., $\sqrt{2}+(-\sqrt{2})=0$ which is rational
  b e.g., $\sqrt{2}\times\sqrt{50}=\sqrt{100}=10$ which is rational

6 **a** true **b** false **c** true

EXERCISE 2C

1 **a** The set of all real x such that x is greater than 4.
  b The set of all real x such that x is less than or equal to 5.
  c The set of all real y such that y lies between 0 and 8.
  d The set of all real x such that x lies between 1 and 4 or is
      equal to 1 or 4.
  e The set of all real t such that t lies between 2 and 7.
  f The set of all real n such that n is less than or equal to 3 or
      n is greater than 6.

2 **a** $\{2,3,4,5,6\}$ **b** $\{5,6,7,8,9,......\}$
  c $\{......,-4,-3,-2,-1,0,1,2\}$ **d** $\{0,1,2,3,4,5\}$
  e $\{-4,-3,-2,-1,0,1,2,3,......\}$ **f** $\{1,2,3,4,5,6\}$

3 **a** $\{x \mid -5 \leqslant x \leqslant -1, \ x \in \mathbb{Z}\}$
      There are other correct answers.
  b $\{x \mid x \leqslant 5, \ x \in \mathbb{N}\}$ There are other correct answers.
  c $\{x \mid x \geqslant 4, \ x \in \mathbb{Z}\}$ There are other correct answers.
  d $\{x \mid x \leqslant 1, \ x \in \mathbb{Z}\}$ There are other correct answers.
  e $\{x \mid -5 \leqslant x \leqslant 1, \ x \in \mathbb{Z}\}$
      There are other correct answers.
  f $\{x \mid x \leqslant 44, \ x \in \mathbb{Z}\}$ There are other correct answers.

4 **a** $\{x \mid x > 3\}$ **b** $\{x \mid 2 < x \leqslant 5\}$
  c $\{x \mid x \leqslant -1 \text{ or } x \geqslant 2\}$ **d** $\{x \mid -1 \leqslant x \leqslant 4, \ x \in \mathbb{Z}\}$
  e $\{x \mid 0 \leqslant x \leqslant 6, \ x \in \mathbb{N}\}$ **f** $\{x \mid x < 0\}$

5 **a**   **b**
  c   **d**
  e   **f**

6 **a** finite **b** infinite **c** infinite **d** finite
  e finite **f** infinite
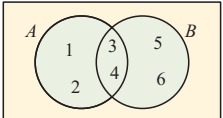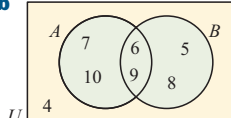
EXERCISE 2D
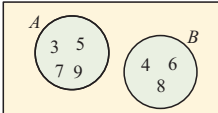
1 a

$A = \{2, 3, 5, 7\}$

b $A' = \{1, 4, 6, 8\}$

c 4, 4
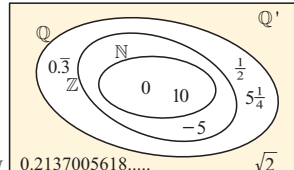
2 a
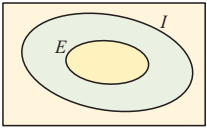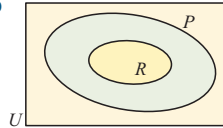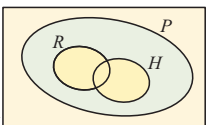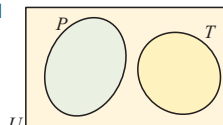
b $V' = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$
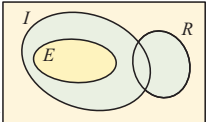
3 a i $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

ii $N = \{3, 8\}$ **iii** $M = \{1, 3, 4, 7, 8\}$

b $n(N) = 2$, $n(M) = 5$ **c** No, $N \subseteq M$.
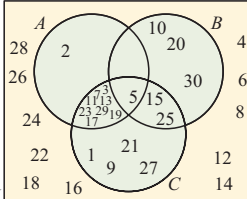
4 a

b

c

5 a/b

c i true

ii true

iii true

d (shaded section of diagram alongside)

6 a

b

c

d
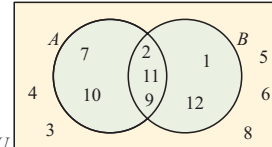
e

7

EXERCISE 2E.1
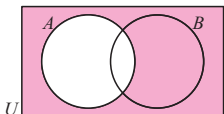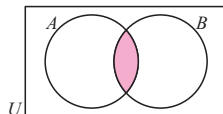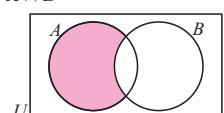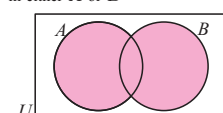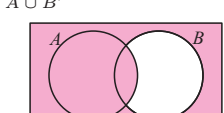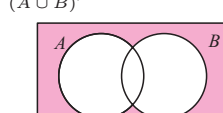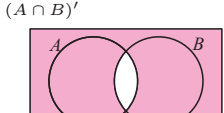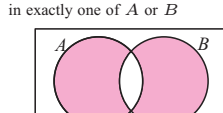
1 a i $C = \{1, 3, 7, 9\}$ **ii** $D = \{1, 2, 5\}$

iii $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ **iv** $C \cap D = \{1\}$

v $C \cup D = \{1, 2, 3, 5, 7, 9\}$

b i $n(C) = 4$ **ii** $n(D) = 3$ **iii** $n(U) = 9$

iv $n(C \cap D) = 1$ **v** $n(C \cup D) = 6$

2 a i $A = \{2, 7\}$ **ii** $B = \{1, 2, 4, 6, 7\}$

iii $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ **iv** $A \cap B = \{2, 7\}$

v $A \cup B = \{1, 2, 4, 6, 7\}$

b i $n(A) = 2$ **ii** $n(B) = 5$ **iii** $n(U) = 8$

iv $n(A \cap B) = 2$ **v** $n(A \cup B) = 5$

3 a

b i $A \cap B = \{2, 9, 11\}$

ii $A \cup B = \{1, 2, 7, 9, 10, 11, 12\}$

iii $B' = \{3, 4, 5, 6, 7, 8, 10\}$

c i $n(A) = 5$ **ii** $n(B') = 7$

iii $n(A \cap B) = 3$ **iv** $n(A \cup B) = 7$

4 a $A \cap B = \{1, 3, 9\}$

b $A \cup B = \{1, 2, 3, 4, 6, 7, 9, 12, 18, 21, 36, 63\}$

5 a $X \cap Y = \{B, M, T, Z\}$

b $X \cup Y = \{A, B, C, D, M, N, P, R, T, W, Z\}$

6 a i $n(A) = 8$ **ii** $n(B) = 10$

iii $n(A \cap B) = 3$ **iv** $n(A \cup B) = 15$

b $n(A) + n(B) - n(A \cap B) = 8 + 10 - 3 = 15 = n(A \cup B)$

7 a $n(A) + n(B) - n(A \cap B) = (a + b) + (b + c) - b$
$$= a + b + c = n(A \cup B)$$

b $A \cap B = \varnothing$, $\therefore$ $n(A \cap B) = 0$

8 a $\varnothing$ **b** U **c** $\varnothing$

EXERCISE 2E.2

1 a not in A - shaded pink

b in both A and B

c $A \cap B'$

d in either A or B

e $A \cup B'$

f $(A \cup B)'$

g $(A \cap B)'$

h in exactly one of A or B

2 **a** in X but not in Y
   b the complement of 'in exactly one of X and Y'
   c in at least 2 of X, Y and Z

3 **a** A'

b $A' \cap B$

c $A' \cup B$

d $A' \cap B'$

4 **a** A

b B'

c $B \cap C$

d $A \cup C$

e $A \cap B \cap C$

f $(A \cup B) \cap C$

g $(A \cup C) \cap B$

h $(A \cap C) \cup B$

i $(A \cup B)' \cap C$

5 **a**

represents $A \cap B$
represents $(A \cap B)'$

represents A'
represents B'
whole shaded region is $A' \cup B'$

b

represents $B \cap C$
represents A
whole shaded region represents
$A \cup (B \cap C)$

represents $A \cup B$
represents $A \cup C$
represents
$(A \cup B) \cap (A \cup C)$

c

represents A
represents $B \cup C$
represents $A \cap (B \cup C)$

represents $A \cap B$
represents $A \cap C$
whole shaded region represents
$(A \cap B) \cup (A \cap C)$
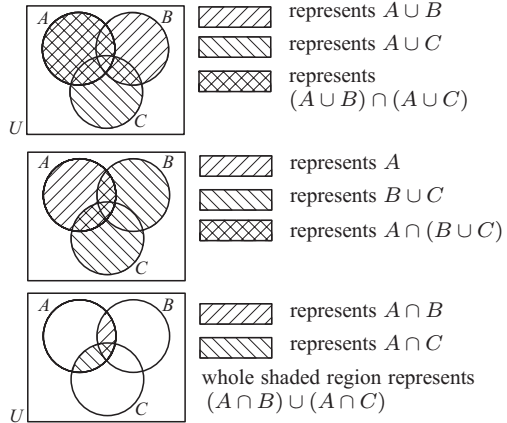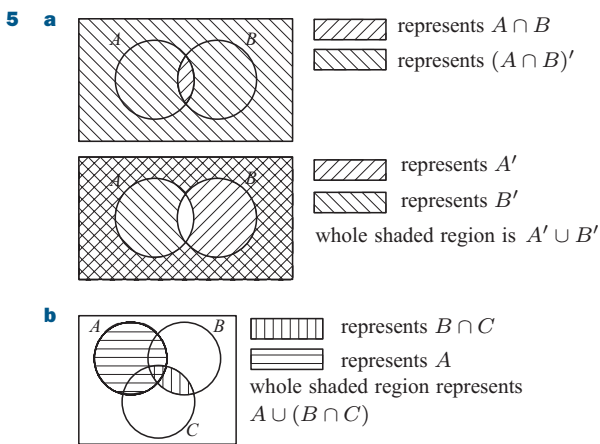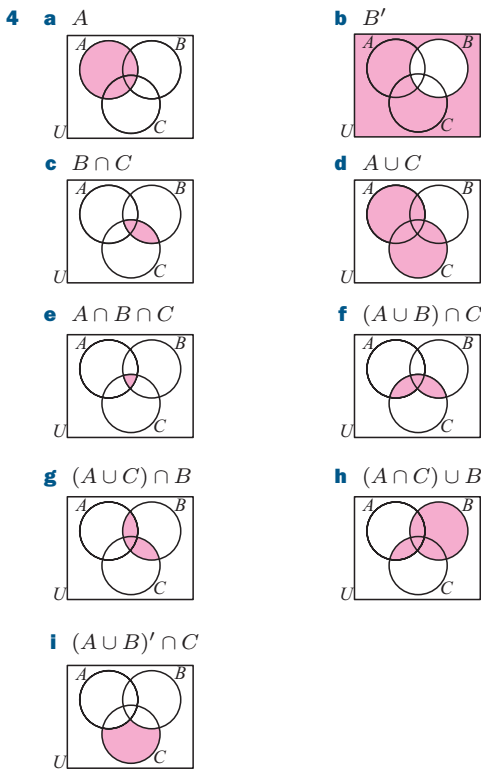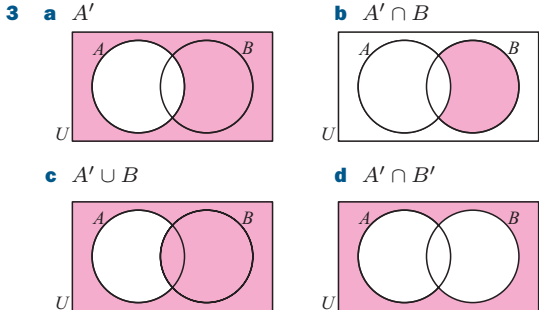
EXERCISE 2F

1 **a** 26 **b** 20 **c** 25 **d** 5 **e** 7
2 **a** 48 **b** 27 **c** 23 **d** 16 **e** 30

3

4
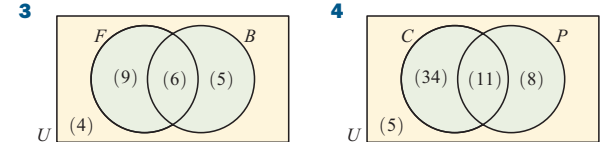
   a 5 **b** 20 **c** 9   **a** 13 **b** 53 **c** 42
   d 4 **e** 14   **d** 8 **e** 34

5 6 girls **6** **a** 12 books **b** 14 books
7 **a** 150 people **b** 150 people
8 **a** 7 shops **b** 12 shops **c** 18 shops
9 14 cars **10** 39 people attended **11** 7% of them
12 1 worker uses both
13 **a** 9% **b** 91% **c** 58% **d** 72% **e** 19%
14 7 women **15** **a** 11 of them **b** 65 of them
16 5 students

REVIEW SET 2A

1 **a** 1.3 can be written as $\frac{13}{10}$, and 13, 10 are integers
   b false **c** $\{23, 29, 31, 37\}$
   d The set of all real t such that t lies between -1 and 3, including -1.
   e $\{x \mid 0 < x \leqslant 5\}$ **f**

2 **a**

   b $A' = \{1, 2, 4, 5, 7, 8, 10, 11\}$ **c** $n(A') = 8$ **d** false
3 $\varnothing$, $\{1\}$, $\{3\}$, $\{6\}$, $\{8\}$, $\{1, 3\}$, $\{1, 6\}$, $\{1, 8\}$, $\{3, 6\}$, $\{3, 8\}$, $\{6, 8\}$, $\{1, 3, 6\}$, $\{1, 3, 8\}$, $\{1, 6, 8\}$, $\{3, 6, 8\}$ (15 of them)
4 **a** false **b** false
5 **a** **i** $A = \{1, 2, 3, 4, 5\}$ **ii** $B = \{1, 2, 7\}$
   iii $U = \{1, 2, 3, 4, 5, 6, 7\}$
   iv $A \cup B = \{1, 2, 3, 4, 5, 7\}$ **v** $A \cap B = \{1, 2\}$
   b **i** $n(A) = 5$ **ii** $n(B) = 3$ **iii** $n(A \cup B) = 6$
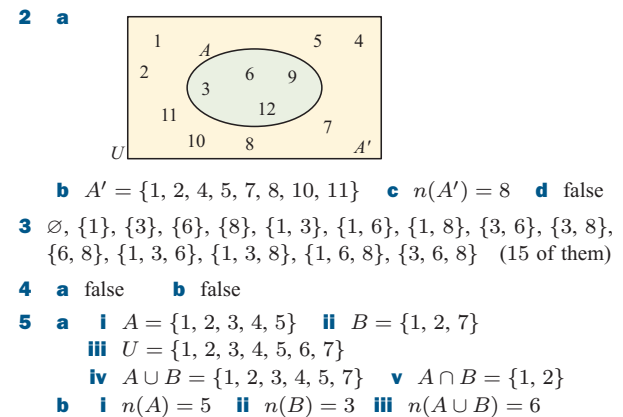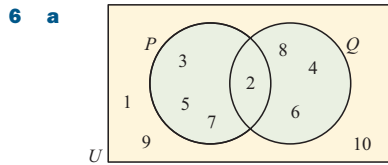
6 **a**

b **i** $P \cap Q = \{2\}$ **ii** $P \cup Q = \{2, 3, 4, 5, 6, 7, 8\}$
iii $Q' = \{1, 3, 5, 7, 9, 10\}$
c **i** $n(P') = 6$ **ii** $n(P \cap Q) = 1$ **iii** $n(P \cup Q) = 7$
d true

7 **a** The shaded region is the complement of X, i.e., everything
not in X.
b The shaded region represents 'in exactly one of X or Y but
not both'.
c The shaded region represents everything in X or in neither
set.

8
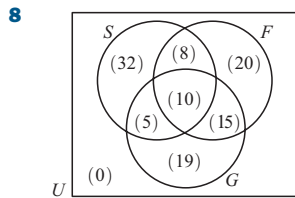
109 took part

REVIEW SET 2B

1 **a** **i** false **ii** false **b** $0.\overline{51} = \frac{51}{99}$, and 51, 99 are integers
c $\{t \mid t \leqslant -3 \text{ or } t > 4\}$ **d**

2

3 **a**

b

c
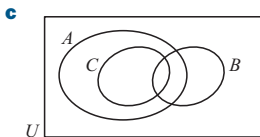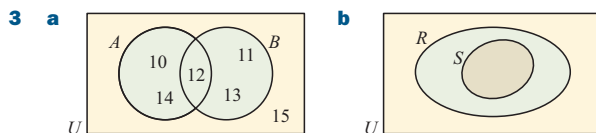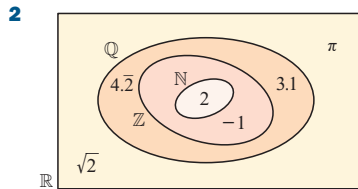
4 **a** $A \cap B = \{1, 2, 3, 6\}$
b $A \cup B = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$
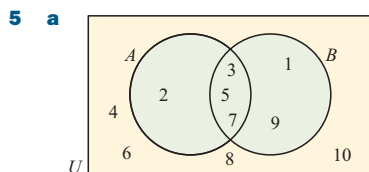
5 **a**

b **i** $A' = \{1, 4, 6, 8, 9, 10\}$ **ii** $A \cap B = \{3, 5, 7\}$
c **i** false **ii** true **d** **i** 4 **ii** 5 **iii** 6
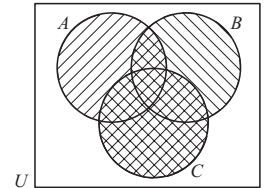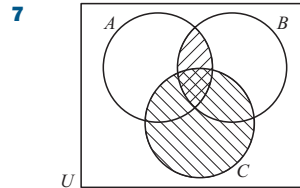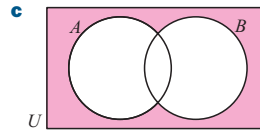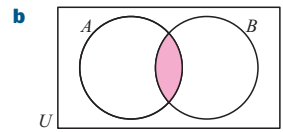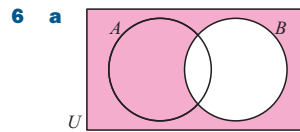
6 **a**

b

c

7

represents $A \cap B$ represents $A \cup C$
represents C represents $B \cup C$
whole shaded region   represents
represents $(A \cap B) \cup C$ $(A \cup C) \cap (B \cup C)$

Area shaded is the same in each case.

8
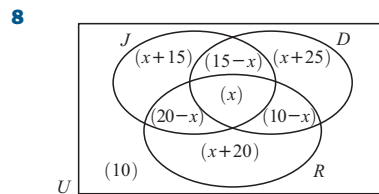
$x = 5$
$\therefore$ 5 were members of
all 3 clubs

EXERCISE 3A.1

1 **a** $x = -11$ **b** $x = -3$ **c** $x = -7$ **d** $x = -3$
e $x = 5$ **f** $x = 9$ **g** $x = 1$ **h** $x = -5$
i $x = -2$ **j** $x = 3$ **k** $x = -1\frac{1}{2}$ **l** $x = -6$

2 **a** $x = 11$ **b** $x = -5\frac{1}{2}$ **c** $x = -4$ **d** $x = 3\frac{1}{2}$
e $x = 1$ **f** $x = 11$ **g** $x = -6$ **h** $x = 11$
i $x = -\frac{1}{2}$ **j** $x = -2$ **k** $x = 4$ **l** $x = -9$

3 **a** $x = 28$ **b** $x = -15$ **c** $x = -16$ **d** $x = -12$
e $x = 19$ **f** $x = -11$ **g** $x = 10$ **h** $x = 24$

4 **a** $x = -5\frac{1}{2}$ **b** $x = -3$ **c** $x = 17$ **d** $x = -7$
e $x = 3$ **f** $x = 8\frac{1}{2}$

EXERCISE 3A.2

1 **a** $x = 9$ **b** $x = -12$ **c** $x = 1$ **d** $x = -2$
e $x = 2\frac{2}{3}$ **f** $x = -3$

2 **a** $x = -3$ **b** $x = 6$ **c** $x = 2$ **d** $x = 3$
e $x = 2$ **f** $x = 1$

3 **a** $x = 6$ **b** $x = -3$ **c** $x = 1\frac{1}{5}$ **d** $x = -3$
e $x = -3\frac{1}{2}$ **f** $x = -4$

4 **a** $x = 3$ **b** $x = 2$ **c** $x = 2$ **d** $x = 6\frac{1}{2}$
e $x = 1$ **f** $x = 6$

5 **a** $x = 0$ **b** $x = 2$ **c** $x = 3$ **d** $x = 3$
e $x = -1$ **f** $x = -\frac{7}{9}$ **g** $x = -5$ **h** $x = 6$
i $x = 3\frac{1}{2}$ **j** $x = \frac{2}{3}$ **k** no solution
l infinite number of solutions (true for all x)

EXERCISE 3B

1 a $x = -\frac{1}{4}$ **b** $x = -18$ **c** $x = -5$ **d** $x = 18$
  e $x = -1$ **f** $x = -\frac{1}{5}$ **g** $x = -1$ **h** $x = -5$
  i no solution

2 a $x = 15$ **b** $x = 4\frac{1}{2}$ **c** $x = 17\frac{1}{2}$ **d** $x = 2\frac{1}{4}$
  e $x = \frac{3}{8}$ **f** $x = -\frac{7}{12}$ **g** $x = \frac{4}{15}$ **h** $x = -\frac{2}{15}$

3 a $x = 1$ **b** $x = -1$ **c** $x = 8\frac{1}{2}$ **d** $x = -12$
  e $x = \frac{1}{5}$ **f** $x = -1$

4 a $x = 12$ **b** $x = -\frac{36}{5}$ **c** $x = -\frac{16}{5}$ **d** $x = \frac{13}{7}$
  e $x = 16$ **f** $x = \frac{40}{7}$ **g** $x = \frac{16}{3}$ **h** $x = \frac{11}{5}$
  i $x = \frac{55}{28}$ **j** $x = \frac{4}{5}$ **k** $x = -\frac{6}{5}$ **l** $x = \frac{17}{24}$

EXERCISE 3C

1 a $x + 6 = 13$ **b** $x - 5 = -4$ **c** $2x + 7 = 1$
  d $\frac{x-1}{2} = 45$ **e** $3x = 17 - x$ **f** $5x = x + 2$

2 a $x + (x+1) = 33$ **b** $x + (x+1) + (x+2) = 102$
  c $x + (x+2) = 52$ **d** $x + (x+2) + (x+4) = 69$

3 a $40t + 25(10 - t) = 280$ **b** $40p + 70(p - 3) = 340$
  c $250t + 720t = 2910$
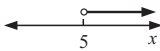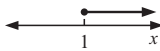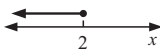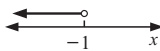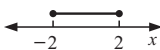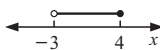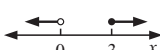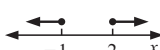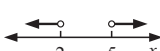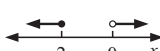
EXERCISE 3D

1 7 **2** 37 and 38 **3** 19 **4** 13 **5** 50
6 84 **7** 5 roses **8** 17 five cent and 23 ten cent
9 16 of 6-pack, 9 of 10-pack
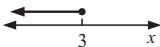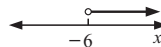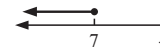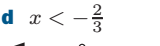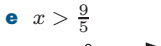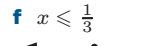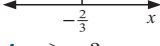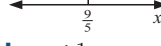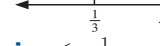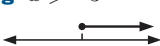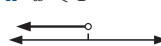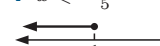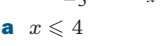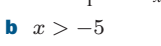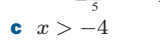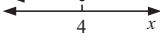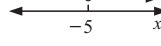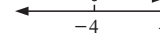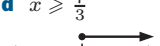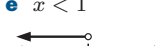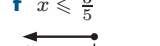
EXERCISE 3E

1 a $x \approx \pm 3.32$ **b** no solutions exist **c** $x \approx \pm 8.43$
  d $x \approx 9.43$ **e** $x = 2$ **f** $x \approx 2.22$ **g** $x \approx -2.22$
  h $x = \pm 3$ **i** no solutions exist **j** $x \approx 1.87$
  k $x \approx -2.57$ **l** $x \approx 1.84$

2 a $x = \pm\sqrt{12}$ (or $\pm 2\sqrt{3}$) **b** $x = \pm 6$ **c** $x = \pm\sqrt{3}$
  d $x = \pm 7$ **e** $x = \pm\sqrt{10}$ **f** $x = \pm\sqrt{35}$
  g $x = \pm 4$ **h** no solutions exist

EXERCISE 3F

1 a number line: open circle at 5, arrow right
  b number line: closed circle at 1, arrow right
  c number line: closed circle at 2, arrow left
  d number line: open circle at -1, arrow left
  e number line: closed circles at -2 and 2, segment between
  f number line: open circle at -3, closed circle at 4
  g number line: closed circle at 1, open circle at 6
  h number line: open circles at -1 and 0
  i number line: open circle at 0 arrow left, closed circle at 3 arrow right
  j number line: closed circle at -1 arrow left, closed circle at 2 arrow right
  k number line: open circle at 2 arrow left, open circle at 5 arrow right
  l number line: closed circle at -2 arrow left, open circle at 0 arrow right

2 a $x > 0$ **b** $x \leqslant -2$ **c** $x \geqslant 100$
  d $x < 0.2$ **e** $-1 \leqslant x \leqslant 4$ **f** $3 \leqslant x < 10$
  g $-5 < x < 5$ **h** $7 < x \leqslant 20$
  i $x < 2$ or $x \geqslant 7$ **j** $x < -3$ or $x > 0$
  k $x \leqslant 4$ or $x \geqslant 17$ **l** $x \leqslant 21$ or $x > 33$
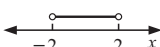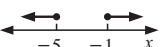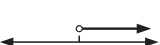
EXERCISE 3G

1 a $x \leqslant 3$ **b** $x > -6$ **c** $x \leqslant 7$
  d $x < -\frac{2}{3}$ **e** $x > \frac{9}{5}$ **f** $x \leqslant \frac{1}{3}$
  g $x \geqslant -3$ **h** $x < 1$ **i** $x \leqslant -\frac{1}{5}$

2 a $x \leqslant 4$ **b** $x > -5$ **c** $x > -4$
  d $x \geqslant \frac{7}{3}$ **e** $x < 1$ **f** $x \leqslant \frac{3}{5}$

3 a $x > -\frac{7}{2}$ **b** $x > \frac{4}{3}$ **c** $x \leqslant \frac{1}{3}$
  d $x \geqslant 1$ **e** $x < 0$ **f** $x \leqslant -\frac{5}{3}$

4 a No solutions as 1 6. **b** True for all real x, i.e., $x \in \mathbb{R}$.
  c True for all real x, i.e., $x \in \mathbb{R}$.
  d No solutions in **a**, an infinite number of solutions for **b**, and
    an infinite number of solutions in **c** when $2x - 4 = 2(x - 2)$.

REVIEW SET 3A

1 a $x = -10$ **b** $x = 19$ **2 a** $x = 2\frac{6}{7}$ **b** $x = -2$
3 a $x = \frac{1}{15}$ **b** $x = 9$ **4 a** $x = 7$ **b** $x = -\frac{1}{7}$
5 a number line: closed circle at -1, open circle at 3
  b number line: open circles at 0 arrow left and 3 arrow right
6 a $x < 3$ **b** $x \geqslant -7$
7 a $2(x + 11) = 48$ **b** $x + (x+1) + (x+2) = 63$
8 The number is 7. **9** twelve 5-cent coins
10 a $x = \pm 2$ **b** $x = \pm\sqrt{21}$

REVIEW SET 3B

1 a $x = 8$ **b** $x = 9$ **2 a** $x = \frac{3}{4}$ **b** $x = 1$
3 a $x = -4\frac{2}{3}$ **b** $x = 4\frac{1}{4}$ **4 a** $x = 1\frac{1}{9}$ **b** $x = -2$
5 a number line: open circles at -2 and 2, segment between
  b number line: closed circles at -5 and -1 arrows
6 a $x \leqslant 9$ **b** $x > \frac{1}{4}$
7 a $4x = x + 15$ **b** $x + (x+2) = 36$ **8** The number is 7.
9 a $x = 4$ **b** no solutions exist **10** 332 votes

EXERCISE 4A

1 a $y = 44$ **b** $x = 122$ **c** $x = 141$ **d** $y = 180$
  e $y = 99$ **f** $x = y = 90$

2 a $y = 55$ {angles in right angle}
  b $f = 117$ {co-interior angles}
  c $c = 102$ {co-interior angles}
  d $y = 65$ {alternate angles}

e $x = 100$ {corresponding angles}

f $x = 49$ {co-interior angles}

g $x = 112$ {vertically opposite angles}

h $x = 45$ {angles at a point}

i $d = 333$ {angles at a point}

j $a = 116$ {corresponding angles}

k $x = 138$ {angles on a line}

l $b = 108$ {alternate angles}

m $a = 65$ {angles on a line}, $b = 65$ {corresponding angles}

n $a = 57$ {vert. opp. angles},
$b = 57$ {corresponding angles}

o $a = 102$ {co-interior angles}, $b = 102$ {co-interior angles}

p $c = 90$ {angles on a line}

q $a = 59$ {angles on a line}, $b = 59$ {corresponding angles}

r $i = 218$ {angles at a point}

s $a = 122$ {vert. opp. angles}, $b = 58$ {co-interior angles}
$c = 58$ {corresponding angles}, $d = 122$ {angles on a line}

t $b = 137$ {angles at a point}

u $r = 81$, $s = 81$ {corresponding angles}

3 **a** $d = 125$ {angles at a point}

b $e = 120$ {angles at a point}

c $f = 45$ {angles at a point}

d $h = 36$ {angles on a line}

e $g = 60$ {corresponding angles/angles on a line}

f $x = 85$ {co-interior angles}

g $x = 45$ {angles on a line}

h $x = 15$ {angles in right angle}

i $x = 42$ {angles at a point}

4 **a** KL ∥ MN {alternate angles equal}

b KL ∦ MN {co-interior angles not supplementary}

c KL ∥ MN {corresponding angles equal}

5 **a** $x = 60$, $y = 60$ **b** $a = 90$, $b = 35$

c $p = 90$, $q = 25$, $r = 25$

EXERCISE 4B

1 **a** $a = 62$ {angles of a triangle}

b $b = 91$ {angles of a triangle}

c $c = 109$ {angles of a triangle}

d $d = 128$ {exterior angle of a triangle}

e $e = 136$ {exterior angle of a triangle}

f $f = 58$ {exterior angle of a triangle}

2 **a** AB   **b** AC   **c** BC   **d** AC and BC   **e** BC

f BC   **g** BC   **h** BC   **i** AB

3 **a** true   **b** false   **c** false   **d** false   **e** true

4 **a** $a = 20$ {angles of a triangle}

b $b = 60$ {angles of a triangle}

c $c = 56$ {corresponding angles/angles of a triangle}
$d = 76$ {angles of a triangle}

d $a = 84$ {vert. opp. angles}, $b = 48$ {angles of a triangle}

e $a = 60$ {angles on a line}
$b = 100$ {ext. angle of a triangle}

f $a = 72$, $b = 65$ {vertically opposite angles}
$c = 137$ {ext. angle of triangle}
$d = 43$ {angles on a line}

5 46.5^o, 34.5^o and 99°

EXERCISE 4C

1 **a** $x = 36$ {isosceles triangle theorem/angles of a triangle}

b $x = 55$ {isosceles triangle theorem/angles of a triangle}

c $x = 36$ {isosceles triangle theorem/angles of a triangle}

d $x = 73$ {isosceles triangle theorem}

e $x = 60$ {angles on a line/isos. Δ theorem/angles of a Δ}

f $x = 32.5$ {isos. Δ theorem/angles on a line/angles of a Δ}

2 **a** $x = 16$ {isosceles triangle theorem}

b $x = 9$ {isosceles triangle theorem}

c $x = 90$ {isosceles triangle theorem/
line from apex to midpoint of base}

3 **a** equilateral   **b** isosceles   **c** equilateral   **d** isosceles

e equilateral   **f** isosceles

4 **a** $x = 52$ **b** ΔABC is isosceles (BA = BC)

5 **a** $\theta = 72$ **b** $\phi = 54$ **c** $\widehat{ABC} = 108^o$

6 **a** $\theta = 36$ **b** $\phi = 72$ **c** $\widehat{ABC} = 144^o$

EXERCISE 4D

1 **a** 360° **b** 540° **c** 720° **d** 1080°

2 **a** $x = 87$ **b** $x = 43$ **c** $x = 52.5$ **d** $a = 108$

e $a = 120$ **f** $a = 65$

3 **a** $x = 60$ {angles of a pentagon}

b $x = 72$ {angles of a hexagon}

c $x = 120$ {angles of a hexagon}

d $x = 60$ {angles of a hexagon}

e $x = 125$ {angles of a heptagon}

f $x = 135$ {angles of an octagon}

4 135° **5** **a** 108° **b** 120° **c** 135° **d** 144°

6 12 angles   **7** No such polygon exists.

8

Regular Polygon	No. of sides	No. of angles	Size of each angle
equilateral triangle	3	3	60°
square	4	4	90°
pentagon	5	5	108°
hexagon	6	6	120°
octagon	8	8	135°
decagon	10	10	144°

9 • $180(n - 2)^o$ • $\theta = \dfrac{180(n - 2)^o}{n}$

10 **a** Yes, these are all 30°.   **b** Yes, the two inner ones.

11 **a** $128\frac{4}{7}^o$ **b** $\alpha = 25\frac{5}{7}$, $\beta = 102\frac{6}{7}$, $\gamma = 77\frac{1}{7}$, $\delta = 51\frac{3}{7}$

12 $\alpha = 60$, $\beta = 80$

13 We must be able to find integers k such that

$$k\left[\frac{(n - 2) \times 180}{n}\right] = 360 \quad \text{{angles at a point}}$$

$$\therefore \quad k = \frac{2n}{n - 2} \quad \text{where} \quad n = 3, 4, 5, 6, ...$$

The only possibilities are:

$k = 6$, $n = 3$; $k = 4$, $n = 4$; $k = 3$, $n = 6$

So, only equilateral triangles, squares and regular hexagons tessellate.

14 **a**   **b**

Y:\HAESE\IGCSE01\IG01_an\681IB_IGC1_an.CDR Tuesday, 18 November 2008 2:35:57 PM PETER
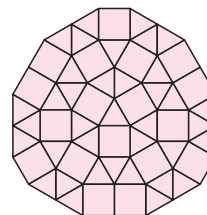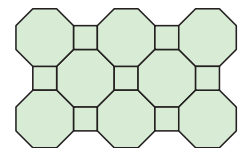
EXERCISE 4E

1 a $x = 120$ **b** $x = 95$ **c** $x = 60$

2 a 108° **b** 135° **c** 144° **d** 162° **e** 176.4^o
    f $\left(180 - \frac{360}{n}\right)^o$

3 a 8 sides   **b** 24 sides   **c** 180 sides   **d** 720 sides

4 a 6 sides   **b** 12 sides   **c** 72 sides   **d** 360 sides

REVIEW SET 4A

1 a ...... are equal in size.
    b ...... are supplementary (add to 180°).

2 a $x = 72$ {corr. angles}   **b** $x = 20$ {angles on a line}
    c $x = 242$ {angles at a point}
    d $x = 65$ {angles on a line/alternate angles}
    e $x = 40$ {vertically opposite/co-interior angles}

3 yes {angles on a line/alternate angles equal}

4 a $x = 62$ **b** $x = 152$ **c** $x = 68$

5 $\triangle$ABC is isosceles since $A\widehat{C}B = A\widehat{B}C = 62^o$
    {angles on a line/angles in a triangle/isosceles triangle theorem}

6 a $x = 122$ {angles in a quadrilateral}
    b $x = 60$ {opposite angles of parallelogram}
    $y = 5$ {opposite sides of parallelogram}
    c Must be a parallelogram and so a rectangle.
    {equal opposite sides and one angle a right angle}
    $x = 35$, $y = 55$ {alternate angles}

7 a $x = 110$ {angles in a pentagon}
    b $x = 120$ {angles in a hexagon}
    c $x = 100$ {exterior angles of a polygon}
    d $x = 75$ {exterior angles of a polygon}

REVIEW SET 4B

1 a $a = 73$ {angles on a line}
    b $b = 90$ {angles on a line}   **c** $c = 56$ {vert. opp. angles}
    d $a = 110$ {alternate angles}, $b = 110$ {vert. opp. angles}
    e $x = 40$ {angles on a line}

2 a $x = 60$ {angles in a $\triangle$}   **b** $a = 131$ {ext. angle of a $\triangle$}
    c $x = 2$ {equal sides of equilateral triangle}

3 $x = 96$ {corr. angles/isosceles $\triangle$ theorem/angles in a $\triangle$},
    $y = 96$ {corr. angles}

4 a right angled scalene   **b** obtuse angled scalene
    c obtuse angled isosceles

5 a $x = 56$ {angles on a line/angles in a quadrilateral}
    b $a = 44$ {angles in a triangle}, $b = 46$ {alternate angles},
    $c = 88$ {isosceles triangle theorem/angles in a triangle}

6

Polygon	Number of sides	Sum of interior angles
pentagon	5	540°
hexagon	6	720°
octagon	8	1080°

7 a 150° **b** 160°

8 a $x = 100$ {vert. opp. angles/angles of a regular polygon}
    b $x = 140$ {angles of a regular polygon}

CHALLENGE

1 a sum = 180°
    b Sum always seems to be 180° (or close to it).
    c The sum of the angles of all '5-point stars' is 180°.
    d **Hint:** $B\widehat{E}I + E\widehat{B}I = J\widehat{I}D$
    e The angle sum of all '7-point stars' is 540°.

EXERCISE 5A

1 a

Subjects taught by teachers in a school
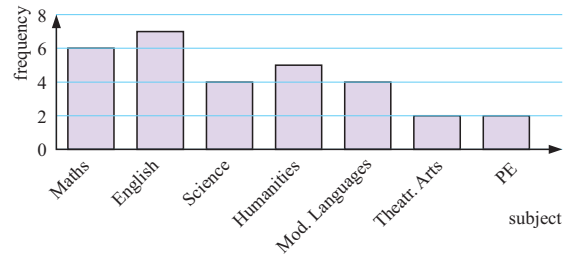
b 33.3%

2 a Aston 195°,   **b** Toy car makes
    Bentley 90°,
    Corvette 75°
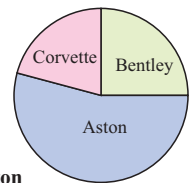
3 a

Midday temperature in Kingston
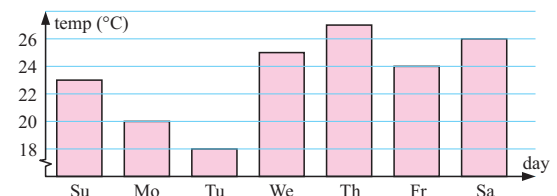
b

Rainfall in Kingston
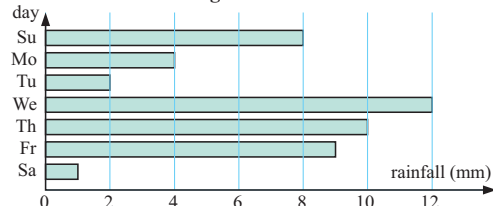
4 a

Stem	Leaf
2	9 7 4 1 7 5
3	3 0 5 4 6 4
4	6 0 2 8
5	8 7 1 0

b

Stem	Leaf
2	1 4 5 7 7 9
3	0 3 4 4 5 6
4	0 2 6 8
5	0 1 7 8

    Key: 5 | 8 means 58

5 a Science Test results   **b** 28%

2	9
3	8 9
4	3 4 7
5	1 3 5 6 6 6 8 9 9
6	1 2 2 3 4 4 4 6 7 7 8 8 9
7	0 1 2 3 4 5 6 8
8	0 0 0 1 7 8 8 9 9 9
9	0 2 2 5

    c 12%

    Key: 2 | 9 means 29
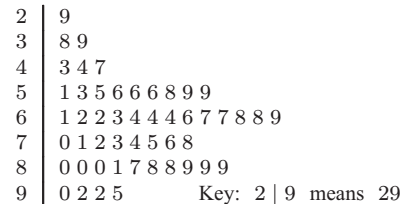
6 a 188 students   **b** 9.5^o **c** 1440 students

7 a The data is continuous. Therefore we use a line graph.
    b

Shadow length over time data
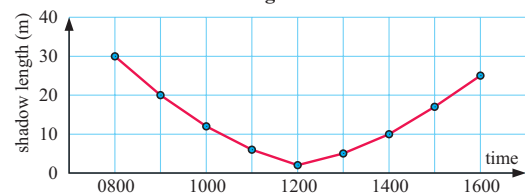
8 a 1   **b** 43   **c** 10   **d** 1   **e** 21.4%

9 a A scatterplot as the data is discrete.

b

Golf score data for Guy
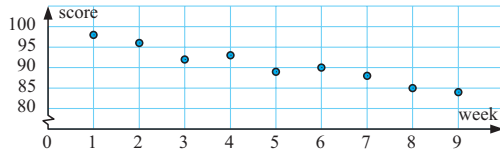
10 a

Stem	Leaf
0	
1	8
2	9 9 7
3	4 7 9 3 7 5 9 1 4 7 4
4	4 0 2 3 3 7 1 3 8 4 4 4 5 9 1 2 2 3 3 5
5	1 3 8 0 5 2 4 9 7 1 Key: 5	1 means 51

b

Stem	Leaf
0	
1	8
2	7 9 9
3	1 3 4 4 4 5 7 7 7 9 9
4	0 1 1 2 2 2 3 3 3 3 3 4 4 4 4 5 5 7 8 9
5	0 1 1 2 3 4 5 7 8 9

c The stem-and-leaf plot shows all the actual data values.

d i 59 **ii** 18   **e** 22.2%   **f** 8.9%

EXERCISE 5B

1 a A and B are very similar, with minor variations.
Both are positively skewed.

b The values of A are generally higher than the values of B.
A is negatively skewed and B is positively skewed.

c A is positively skewed, and B is almost symmetrical.
The values of B are generally higher than the values of A.

d A and B are very similar, and both are negatively skewed.

e A and B are very similar, and both are positively skewed.

f A and B are very similar, and both are symmetrical.

2 'Outside USA' service makes Pedro happier than 'Inside USA' service. This is because there are generally fewer breakages with the private company.

3 a In 2008 as the scores are more closely grouped.

b In 2008, 842 compared with 559 in 2007.

4 a

Real estate sales data 07-08
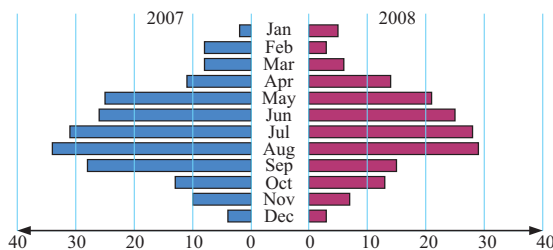
b The sales each month are quite similar for both years although more houses were sold in total in 2007. (This is Northern hemisphere data as more houses sell in warmer weather.)

5 a

Travel times to university

Alex (train)		Stan (bus)
7 5	1	8 9
7 5 3 2	2	2 4 5 8 9
6 4 4 0	3	0 1 1 2 4 6 8
8 5 1	4	2
5	5	
0	6	Key: 4	2 means 42 min

b As the bus data is less spread out, travelling by bus is more reliable.

6 a

Golf driving distances

Amon		Maddie
8 7	14	
9 2	15	0 2
5 5 0	16	7 8 9
7 4 4 2	17	4 5 7 8 9
8 8 3 1 0	18	0 0 2 2 8 9
4 2 1 1	19	4 5 6 7 Key: 17	4 means 174 m

b Yes, shots with Maddie's driver are less spread out and therefore more consistent. They are also generally longer hits.

EXERCISE 5C

1 a

Colour	Sector angle
Green	88°
Blue	78°
Red	67°
Purple	48°
Other	78°

b
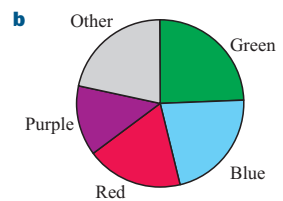
2 a

Comparison of weekly incomes

b
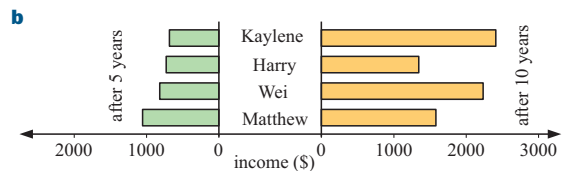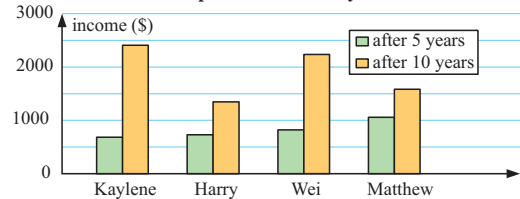
c

Graduate	% increase
Kaylene	252
Henry	84.8
Wei	173
Matthew	49.8

d Kaylene and Wei

3 a i

ii
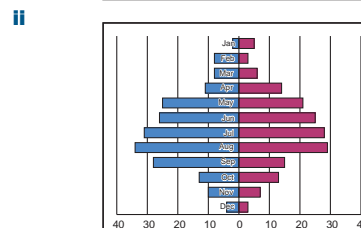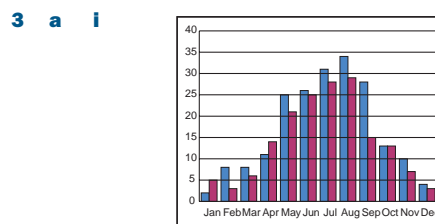
b Either; a personal preference.

REVIEW SET 5A

1 a £720 000

   b Admin. 72°, Production 100°, Marketing 90°,
      Distribution 60°, Advert. 38°

   c **Business expenses**

Marketing 25%, Production 27.8%, Distribution 16.7%, Admin 20%, Advertising 10.6%
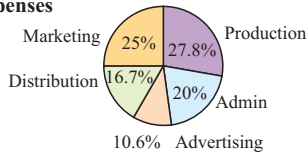
2 a **Animal numbers data**

   b 10%

   c Cows, Goats, Sheep
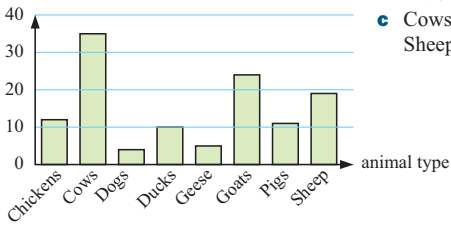
3 a **School drama participants**

   males   females

   b 2006
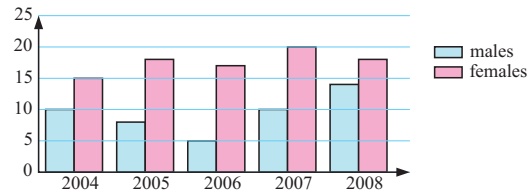
4 a 5 | 1 means 51 kg   **b** 28 kg   **c** 65 kg   **d** 54 kg

5 a **Cost**   **Sales**

Cost: Lilies 48°, Roses 102°, Azaleas 30°, Orchids 42°, Tulips 48°, Carnations 90°

Sales: Lilies 42°, Roses 105°, Azaleas 45°, Orchids 45°, Tulips 48°, Carnations 75°
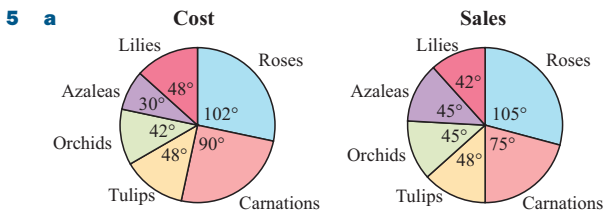
   b Roses and Carnations   **c** Roses and Carnations

   d Lilies as he makes least profit (only £6000).

6 a **Weights of men data**

Before		After
	6	8 8
9	7	4 8 9 9
6 3 2	8	3 3 4 5 9
9 8 7 5 3	9	2 3 5 6 9 9
4 4 1 0	10	2 6 6 7
9 7 4 1 1	11	0 3
5 3 2 1 0	12	2 5
5 2	13	Key: 10	2 means 102 kg

   b Yes, the 'After' weights are significantly less than the
      'Before' weights as a group.

REVIEW SET 5B

1 **Speeding driver data**
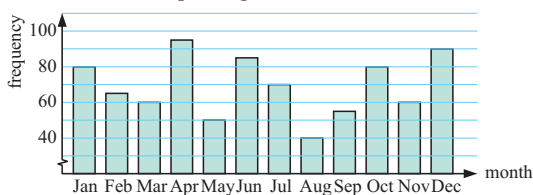
2 a A line graph.

   b **IBM share price data**
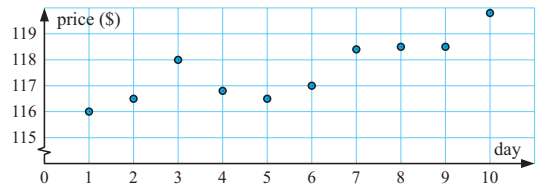
   c **i** Day 10   **ii** Day 1

3 **Weights of new-born babies**

Stem	Leaf
2	3 8
3	1 3 4 5 6 8 9 9
4	0 0 1 2 4
5	1 Key: 5	1 means 5.1 kg

4 a **Rebounds**

Rhys   Evan
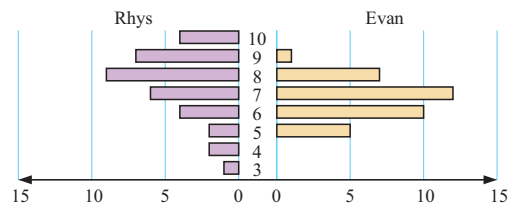
   b Evan, as his data is more closely grouped.   **c** Rhys

5 a **Pet owner data**

   men   women

   b Women own more pets.

6

Travel 15.5%, Work 36.3%, University 29.0%, Apprenticeship 19.2%
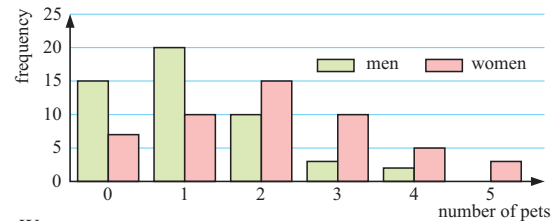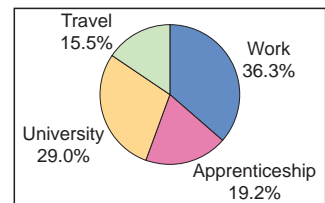
EXERCISE 6A.1

1 a 8   **b** 27   **c** 32   **d** 125   **e** 540

   f 1176   **g** 2925   **h** 4400

2 a $50 = 2 \times 5^2$ **b** $98 = 2 \times 7^2$ **c** $108 = 2^2 \times 3^3$

   d $360 = 2^3 \times 3^2 \times 5$ **e** $1128 = 2^3 \times 3 \times 47$

   f $784 = 2^4 \times 7^2$ **g** $952 = 2^3 \times 7 \times 17$

   h $6500 = 2^2 \times 5^3 \times 13$

3 a $n = 5$ **b** $n = 8$ **c** $n = 12$

4 a $n = 3$ **b** $n = 6$ **c** $n = 10$ **5** 3   **6** 7

7 a $n = 625$ **b** $n = 7$ **c** $n = 5$ **d** $n = 6$

EXERCISE 6A.2

1 a 1   **b** -1 **c** 1   **d** -1 **e** 1   **f** -1

   g -1 **h** -27 **i** -27 **j** 27   **k** -36 **l** 64

2　**a** -72　**b** 108　**c** -648

3　**a** 256　**b** 625　**c** -243　**d** 2401　**e** 512
　f $117\,649$　**g** $-117\,649$　**h** $1.795\,856\,326$
　i $-0.005\,487\,423\,936$　**j** $-325\,687.9871$

EXERCISE 6B

1　**a** $2^4 = 16$　**b** $2^4 = 16$　**c** $3^9 = 19\,683$
　d $5^5 = 3125$　**e** x^6　**f** a^4　**g** n^{10}　**h** b^8

2　**a** $2^1 = 2$　**b** $3^3 = 27$　**c** $5^4 = 625$
　d $4^4 = 256$　**e** x^3　**f** y^3　**g** a　**h** b^4

3　**a** $2^6 = 64$　**b** $3^{12} = 531\,441$　**c** $2^{18} = 262\,144$
　d $10^{10} = 10\,000\,000\,000$　**e** x^6　**f** x^{15}　**g** a^{20}
　h b^{24}

4　**a** a^7　**b** n^8　**c** a^4　**d** a^6　**e** b^5　**f** a^{18}
　g a^{n+5}　**h** b^8　**i** b^3　**j** m^{14}　**k** a^{10}　**l** g^{11}

5　**a** $a^3 b^3$　**b** $a^4 c^4$　**c** $b^5 c^5$　**d** $a^3 b^3 c^3$
　e $16a^4$　**f** $25b^2$　**g** $81n^4$　**h** $8b^3 c^3$
　i $\dfrac{8}{p^3}$　**j** $\dfrac{a^3}{b^3}$　**k** $\dfrac{m^4}{n^4}$　**l** $\dfrac{32c^5}{d^5}$

6　**a** $8b^{12}$　**b** $\dfrac{9}{x^4 y^2}$　**c** $25a^8 b^2$　**d** $\dfrac{m^{12}}{16n^8}$
　e $\dfrac{27a^9}{b^{15}}$　**f** $32m^{15} n^{10}$　**g** $\dfrac{16a^8}{b^4}$　**h** $125x^6 y^9$

EXERCISE 6C

1　**a** 1　**b** $\frac{1}{6}$　**c** $\frac{1}{4}$　**d** 1　**e** 9　**f** $\frac{1}{9}$
　g 125　**h** $\frac{1}{125}$　**i** 49　**j** $\frac{1}{49}$　**k** 1000　**l** $\frac{1}{1000}$

2　**a** 1　**b** 1　**c** 2　**d** 1　**e** 1　**f** 3　**g** $\frac{1}{25}$
　h $\frac{1}{16}$　**i** 4　**j** $\frac{8}{3}$　**k** $\frac{3}{2}$　**l** 5　**m** 3　**n** $\frac{4}{5}$
　o $\frac{11}{3}$　**p** 9　**q** $\frac{27}{8}$　**r** $\frac{8}{27}$　**s** $\frac{25}{16}$　**t** $\frac{4}{25}$

3　**a** $\dfrac{1}{3b}$　**b** $\dfrac{3}{b}$　**c** $\dfrac{7}{a}$　**d** $\dfrac{1}{7a}$　**e** t^2　**f** $\dfrac{1}{16t^2}$
　g $\dfrac{1}{25t^2}$　**h** $\dfrac{t^2}{5}$　**i** $\dfrac{x}{y}$　**j** $\dfrac{1}{xy}$　**k** $\dfrac{x}{y^3}$　**l** $\dfrac{1}{x^3 y^3}$
　m $\dfrac{1}{3pq}$　**n** $\dfrac{3}{pq}$　**o** $\dfrac{3p}{q}$　**p** $x^3 y^5$

4　**a** 5^2　**b** 5^{-2}　**c** 3^3　**d** 3^{-3}
　e 2^4　**f** 2^{-4}　**g** $2^1 3^{-1}$　**h** $3^1 5^{-1}$
　i $3^2 5^{-3}$　**j** $2^5 3^{-4}$　**k** $2^2 3^1 5^{-1}$　**l** $3^1 5^2 2^{-3}$

5　**a** 10^3　**b** 10^{-2}　**c** 10^5　**d** 10^{-6}

EXERCISE 6D.1

1　**a** 10^2　**b** 10^3　**c** 10^1　**d** 10^5　**e** 10^{-1}
　f 10^{-2}　**g** 10^{-4}　**h** 10^8

2　**a** 3.87×10^2　**b** 3.87×10^4　**c** 3.87×10^0
　d 3.87×10^{-2}　**e** 3.87×10^{-3}　**f** 2.05×10^1
　g 2.05×10^2　**h** 2.05×10^{-1}　**i** 2.05×10^4
　j 2.05×10^7　**k** 2.05×10^{-4}

3　**a** 4.0075×10^4 km　**b** 1.495×10^{11} m
　c 4×10^{-4} mm　**d** 4×10^7 bacteria
　e 1.4162×10^{-7}　**f** 1×10^{-2} mm

4　**a** 300　**b** 2000　**c** $36\,000$　**d** $920\,000$
　e $5\,600\,000$　**f** 34　**g** $7\,850\,000$　**h** $900\,000\,000$

5　**a** 0.03　**b** 0.002　**c** $0.000\,47$　**d** $0.000\,063$
　e 1.7　**f** $0.000\,95$　**g** 0.349　**h** $0.000\,007$

6　**a** $0.000\,000\,9$ m　**b** $6\,606\,000\,000$　**c** $100\,000$ light years
　d $0.000\,01$ mm　**e** $0.\underbrace{000......00}_{26 \text{ zeros}}1\,66$ kg

7　**a** 1.817×10^7　**b** 9.34×10^{10}　**c** 4.1×10^{-4}

8　**a** 1.6×10^8　**b** 3.2×10^9　**c** 1.5×10^{10}　**d** 8×10^9
　e 3.6×10^7　**f** 4.9×10^{-3}　**g** 3×10^1　**h** 2×10^{-1}

EXERCISE 6D.2

1　**a** $\boxed{4.65^{06}}$　**b** $\boxed{5.12^{-05}}$　**c** $\boxed{5.99^{-04}}$
　d $\boxed{3.761^{10}}$　**e** $\boxed{4.95^{07}}$　**f** $\boxed{8.44^{-06}}$

2　**a** 3×10^{-8}　**b** 1.36×10^{10}　**c** 4.64×10^{10}
　d 9.87×10^9　**e** 3.57×10^{-8}　**f** 8.74×10^{-6}

3　**a** 2.55×10^8　**b** 7.56×10^{-6}　**c** 2.75×10^{-10}
　d 3×10^1　**e** 2.44×10^{-5}　**f** 1.02×10^7

4　**a** 4.80×10^8 mm　**b** 3.15×10^7 seconds
　c 3.15×10^{10} seconds　**d** 5×10^{-7} kg

5　**a** 8.64×10^4 km　**b** 6.05×10^5 km　**c** 6.31×10^7 km

6　**a** 1.8×10^{10} m　**b** 2.59×10^{13} m　**c** 9.46×10^{15} m

EXERCISE 6E

1　**a** 7　**b** 13　**c** 15　**d** 24　**e** $\frac{1}{3}$　**f** $\frac{1}{11}$　**g** $\frac{1}{17}$
　h $\frac{1}{23}$

2　**a** 2　**b** -5　**c** $\frac{1}{5}$

3　**a** 24　**b** -30　**c** -30　**d** 12　**e** 18
　f $54\sqrt{2}$　**g** 12　**h** $24\sqrt{3}$　**i** 64

4　**a** $2\sqrt{2}$　**b** 0　**c** $\sqrt{2}$　**d** $\sqrt{3}$　**e** $7\sqrt{7}$
　f $-3\sqrt{5}$　**g** $6\sqrt{2}$　**h** $-3\sqrt{2}$　**i** $8\sqrt{5}$　**j** $-3\sqrt{2}$
　k $4\sqrt{3}$　**l** $10\sqrt{5} - 10$

5　**a** $2\sqrt{2} + 7\sqrt{3}$　**b** $5\sqrt{2} - \sqrt{3}$　**c** $-7\sqrt{2} + 4\sqrt{3}$
　d $11\sqrt{5} - 5\sqrt{2}$　**e** $2\sqrt{2} - 10\sqrt{7}$　**f** $2\sqrt{2} + 3\sqrt{11} + 3$
　g $\sqrt{6} - 3\sqrt{2} + 4$　**h** $9\sqrt{3} - 5\sqrt{7} - 13$

EXERCISE 6F

1　**a** $\sqrt{10}$　**b** $\sqrt{21}$　**c** $\sqrt{33}$　**d** 7　**e** 6　**f** $2\sqrt{10}$

2　**a** $6\sqrt{6}$　**b** $6\sqrt{15}$　**c** $\sqrt{30}$　**d** $4\sqrt{3}$　**e** -12
　f $162\sqrt{6}$

3　**a** 2　**b** $\frac{1}{2}$　**c** 3　**d** $\frac{1}{3}$　**e** 2　**f** $\frac{1}{2}$
　g 3　**h** $\sqrt{6}$　**i** $\frac{1}{\sqrt{10}}$　**j** 5　**k** 1　**l** 25

4　**a** $2\sqrt{2}$　**b** $3\sqrt{2}$　**c** $5\sqrt{2}$　**d** $7\sqrt{2}$
　e $10\sqrt{2}$　**f** $12\sqrt{2}$　**g** $100\sqrt{2}$　**h** $\frac{1}{2}\sqrt{2}$

5　**a** $2\sqrt{3}$　**b** $3\sqrt{3}$　**c** $5\sqrt{3}$　**d** $\frac{1}{3}\sqrt{3}$

6　**a** $2\sqrt{5}$　**b** $3\sqrt{5}$　**c** $5\sqrt{5}$　**d** $\frac{1}{5}\sqrt{5}$

7　**a** **i** 7　**ii** 5　**iii** 2　**iv** 4
　b $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$　and　$\sqrt{a-b} \neq \sqrt{a} - \sqrt{b}$

8　**a** $2\sqrt{6}$　**b** $5\sqrt{2}$　**c** $3\sqrt{6}$　**d** $2\sqrt{10}$　**e** $2\sqrt{14}$
　f $3\sqrt{7}$　**g** $2\sqrt{13}$　**h** $2\sqrt{11}$　**i** $2\sqrt{15}$　**j** $3\sqrt{10}$
　k $4\sqrt{6}$　**l** $2\sqrt{17}$　**m** $5\sqrt{7}$　**n** $9\sqrt{2}$　**o** $8\sqrt{2}$
　p $10\sqrt{7}$

9　**a** $\frac{\sqrt{5}}{3}$　**b** $\frac{3\sqrt{2}}{2}$　**c** $\frac{\sqrt{3}}{2}$　**d** $\frac{5\sqrt{3}}{6}$

EXERCISE 6G

1 a $\sqrt{10}+2$ **b** $3\sqrt{2}-2$ **c** $3+\sqrt{3}$ **d** $\sqrt{3}-3$
e $7\sqrt{7}-7$ **f** $2\sqrt{5}-5$ **g** $22-\sqrt{11}$ **h** $\sqrt{6}-12$
i $3+\sqrt{6}-\sqrt{3}$ **j** $6-2\sqrt{15}$ **k** $6\sqrt{5}-10$ **l** $30+3\sqrt{10}$

2 a $2-3\sqrt{2}$ **b** $-2-\sqrt{6}$ **c** $2-4\sqrt{2}$ **d** $-3-\sqrt{3}$
e $-3-2\sqrt{3}$ **f** $-5-2\sqrt{5}$ **g** $-3-\sqrt{2}$ **h** $-5+4\sqrt{5}$
i $\sqrt{7}-3$ **j** $11-2\sqrt{11}$ **k** $\sqrt{7}-\sqrt{3}$ **l** $4-2\sqrt{2}$
m $9-15\sqrt{3}$ **n** $-14-14\sqrt{3}$ **o** $4-6\sqrt{2}$

3 a $4+3\sqrt{2}$ **b** $7+4\sqrt{3}$ **c** $1+\sqrt{3}$ **d** $10+\sqrt{2}$ **e** -2
f $3-3\sqrt{7}$ **g** $-1-\sqrt{5}$ **h** 4 **i** 5 **j** $14-7\sqrt{2}$

4 a $3+2\sqrt{2}$ **b** $7-4\sqrt{3}$ **c** $7+4\sqrt{3}$
d $6+2\sqrt{5}$ **e** $5-2\sqrt{6}$ **f** $27-10\sqrt{2}$
g $9+2\sqrt{14}$ **h** $22-8\sqrt{6}$ **i** $8-4\sqrt{3}$
j $13+4\sqrt{10}$ **k** $13-4\sqrt{10}$ **l** $44+24\sqrt{2}$
m $51-10\sqrt{2}$ **n** $17-12\sqrt{2}$ **o** $19+6\sqrt{2}$

5 a 13 **b** 23 **c** 1 **d** -9 **e** 14
f 19 **g** -2 **h** -28 **i** -174

6 a 1 **b** -4 **c** $x-y$

EXERCISE 6H

1 a $\frac{\sqrt{2}}{2}$ **b** $\sqrt{2}$ **c** $2\sqrt{2}$ **d** $5\sqrt{2}$ **e** $\frac{\sqrt{14}}{2}$
f $\frac{\sqrt{3}}{3}$ **g** $\sqrt{3}$ **h** $\frac{4\sqrt{3}}{3}$ **i** $6\sqrt{3}$ **j** $\frac{\sqrt{33}}{3}$
k $\frac{\sqrt{5}}{5}$ **l** $\frac{3\sqrt{5}}{5}$ **m** $\frac{\sqrt{15}}{5}$ **n** $3\sqrt{5}$ **o** $25\sqrt{5}$
p $\sqrt{5}$ **q** $\frac{\sqrt{3}}{6}$ **r** $\frac{2\sqrt{6}}{3}$ **s** $\frac{3\sqrt{5}}{2}$ **t** $\frac{\sqrt{2}}{4}$

2 a $\frac{3+\sqrt{5}}{4}$ **b** $2-\sqrt{3}$ **c** $\frac{4+\sqrt{11}}{5}$
d $\frac{5\sqrt{2}-2}{23}$ **e** $\frac{\sqrt{3}-1}{2}$ **f** $\frac{10+15\sqrt{2}}{-14}$
g $\frac{3\sqrt{5}-10}{11}$ **h** $\frac{5\sqrt{7}-13}{3}$

3 a $4+2\sqrt{2}$ **b** $5-5\sqrt{2}$ **c** $-3+2\sqrt{2}$
d $-\frac{4}{7}+\frac{1}{7}\sqrt{2}$ **e** $1+\sqrt{2}$ **f** $3+2\sqrt{2}$
g $\frac{9}{7}+\frac{3}{7}\sqrt{2}$ **h** $\frac{10}{7}+\frac{6}{7}\sqrt{2}$

REVIEW SET 6A

1 a 81 **b** 40 **2 a** $36=2^2\times3^2$ **b** $242=2\times11^2$
3 a -8 **b** -1 **c** 108 **4 a** $\frac{1}{27}$ **b** $\frac{9}{16}$ **c** -2
5 2^{-4} **6 a** $5^7=78\,125$ **b** b^5 **c** x^{12}
7 a $16c^6$ **b** $8a^6b^3$ **c** $\frac{s^4}{81t^8}$
8 a 9×10^0 **b** 3.49×10^4 **c** 7.5×10^{-3}
9 a $2\,810\,000$ **b** 2.81 **c** $0.002\,81$
10 a 4.26×10^8 **b** 6×10^3
11 a 2.57×10^6 km **b** 1.80×10^7 km **c** 9.37×10^8 km
12 a 18 **b** -24 **c** $\sqrt{2}$ **d** $4\sqrt{3}$
13 a $8\sqrt{3}-6$ **b** $16-6\sqrt{7}$ **c** 1 **d** $\sqrt{5}-4$ **e** $8+5\sqrt{2}$
14 a $4\sqrt{2}$ **b** $5\sqrt{3}$ **c** $\frac{\sqrt{6}}{2}$ **d** $\frac{30+5\sqrt{3}}{33}$
15 $\frac{1}{7}\sqrt{7}$ **16 a** $7+4\sqrt{3}$ **b** 13

REVIEW SET 6B

1 a 343 **b** 225
2 a $42=2\times3\times7$ **b** $144=2^4\times3^2$
3 a 1 **b** -64 **c** -288 **4 a** $\frac{1}{36}$ **b** $\frac{2}{3}$ **c** $2\frac{7}{9}$
5 a $3^8=6561$ **b** 1 **c** y^{15}
6 a 2^45^{-2} **b** $2^35^13^{-4}$ **c** $2^23^25^1$ **d** $2^25^23^{-2}$
7 a $\frac{1}{5c}$ **b** $\frac{7}{k^2}$ **c** $\frac{1}{64d^6}$
8 a 2.6357×10^2 **b** 5.11×10^{-4} **c** 8.634×10^8
9 a 2.78 **b** $39\,900\,000$ **c** $0.002\,081$
10 a 6.4×10^7 **b** 6×10^6 **11** 2.1×10^{-7} km
12 a $6\sqrt{15}$ **b** $40\sqrt{5}$ **c** $-2\sqrt{2}$ **d** $2-2\sqrt{2}$
e 9 **f** 15
13 a $5\sqrt{3}$ **b** $\frac{2\sqrt{5}}{3}$
14 a 22 **b** $4\sqrt{5}-9$ **c** $6-4\sqrt{3}$ **d** $7\sqrt{2}-9$
15 a $7\sqrt{2}$ **b** $\frac{\sqrt{6}}{3}$ **c** $\frac{3\sqrt{2}-2}{7}$ **d** $\frac{-20-5\sqrt{3}}{13}$
16 a $-7-3\sqrt{5}$ **b** $\frac{1}{2}-\frac{5}{2}\sqrt{5}$

CHALLENGE

1 $3+2\sqrt{2}$
2 $x^2=8-2\sqrt{15}$, $x^4=124-32\sqrt{15}$, $\therefore$ $x^4-16x^2=-4$,
$x=\sqrt{5}-\sqrt{3}$ is one of the solutions of the equation
$x^4-16x^2+4=0$
3 a Hint: $(\sqrt{a+b})^2=(\sqrt{a}+\sqrt{b})^2$ **b** $b=0$ or $a=b$
$\therefore$ $a+b=......$, etc.
4 a $\sqrt{5}$, $\sqrt{5}$, $2\sqrt{5}$, $3\sqrt{5}$
b The answer always has form $k\sqrt{5}$ where $k\in\mathbb{Z}^+$.
(In fact the successive values of k are 1, 1, 2, 3, 5, 8, ...
which are the numbers of the Fibonacci sequence.)

EXERCISE 7A

1 a 26.4 cm **b** 17.8 cm **c** 127.3 m
2 a 19.6 m **b** 112.9 m
3 a 71.4 km/h **b** 220 km **c** 8 h 19 min
4 a 128.7 cm^2 **b** 7.14 m
5 a 4263 cm^3 **b** 1.06 cm **c** 4.99 mm
6 a 706.9 cm^2 **b** 39.9 cm **7 a** 55.8 m^3 **b** 8.42 cm
8 a 15.9 km **b** 49.3 m **9 a** 1.34 sec **b** 81 cm

EXERCISE 7B.1

1 a $y=2-\frac{2}{5}x$ **b** $y=5-\frac{3}{4}x$ **c** $y=2x-8$
d $y=2-\frac{2}{7}x$ **e** $y=10-\frac{5}{2}x$ **f** $y=\frac{2}{3}x+4$
2 a $x=r-p$ **b** $x=\frac{z}{y}$ **c** $x=\frac{d-a}{3}$
d $x=\frac{d-2y}{5}$ **e** $x=\frac{p-by}{a}$ **f** $x=\frac{y-c}{m}$
g $x=\frac{s-2}{t}$ **h** $x=\frac{m-p}{q}$ **i** $x=\frac{6-a}{b}$
3 a $y=mx-c$ **b** $y=\frac{c-p}{2}$ **c** $y=\frac{a-t}{3}$
d $y=\frac{n-5}{k}$ **e** $y=\frac{a-n}{b}$ **f** $y=\frac{a-p}{n}$

4 **a** $z = \dfrac{b}{ac}$ **b** $z = \dfrac{a}{d}$ **c** $z = \dfrac{2d}{3}$

 d $z = \pm\sqrt{2a}$ **e** $z = \pm\sqrt{bn}$ **f** $z = \pm\sqrt{m(a-b)}$

5 **a** $a = \dfrac{F}{m}$ **b** $r = \dfrac{C}{2\pi}$ **c** $d = \dfrac{V}{lh}$ **d** $K = \dfrac{b}{A}$

 e $h = \dfrac{2A}{b}$ **f** $T = \dfrac{100I}{PR}$ **g** $M = \dfrac{E}{C^2}$ **h** $a = 2M - b$

EXERCISE 7B.2

1 $y = -\frac{5}{3}x + 6$ **a** $-\frac{5}{3}$ **b** 6

2 **a** $a = \dfrac{d^2}{2bK}$ **b** **i** 1.29 **ii** 16.2

3 **a** $d = st$ **i** 180 km **ii** 120 km **iii** 126.7 km

 b $t = \dfrac{d}{s}$ **i** 3 hours **ii** 4 hours **iii** 2 hours 12 mins

4 **a** $n = \dfrac{I}{Pr}$ **b** **i** 2.05 years **ii** 10 years

EXERCISE 7C

1 **a** $A = 200 \times 17$ **b** $A = 200m$ **c** $A = Dm$

2 **a** $A = 2000 + 150 \times 8$ **b** $A = 2000 + 150w$

 c $A = 2000 + dw$ **d** $A = P + dw$

3 **a** $C = 40 + 60 \times 5$ **b** $C = 40 + 60t$

 c $C = 40 + xt$ **d** $C = F + xt$

4 **a** $A = 200 - 8 \times 5$ **b** $A = 200 - 5x$

 c $A = 200 - bx$ **d** $A = P - bx$

5 **a** $C = 5000 - 10 \times 200$ **b** $C = 5000 - 200r$

 c $C = 5000 - mr$ **d** $C = L - mr$

6 **a** **i** $P = (2 \times 5 + 4)$ m   **ii** $P = (2a + 4)$ m

   iii $P = (2a + b)$ m

 b **i** $P = (2 \times 3 + 4 + 5)$ m **ii** $P = (2x + 4 + 5)$ m

   iii $P = (2x + y + 5)$ m   **iv** $P = (2x + y + z)$ m

EXERCISE 7D

1 **a** $x = \dfrac{c - a}{3 - b}$ **b** $x = \dfrac{c}{a + b}$ **c** $x = \dfrac{a + 2}{n - m}$

 d $x = \dfrac{-a}{b + 8}$ **e** $x = \dfrac{b - a}{c - 1}$ **f** $x = \dfrac{e - d}{r + s}$

2 **a** $r = \sqrt{\dfrac{A}{\pi}}$ **b** $x = \sqrt[5]{aN}$ **c** $r = \sqrt[3]{\dfrac{3V}{4\pi}}$

 d $x = \sqrt[3]{\dfrac{n}{D}}$ **e** $x = \pm\sqrt{\dfrac{y + 7}{4}} = \pm\dfrac{\sqrt{y + 7}}{2}$

 f $Q = \pm\sqrt{P^2 - R^2}$

3 **a** $a = d^2 n^2$ **b** $l = 25T^2$ **c** $a = \pm\sqrt{b^2 + c^2}$

 d $d = \dfrac{25a^2}{k^2}$ **e** $l = \dfrac{gT^2}{4\pi^2}$ **f** $b = \dfrac{16a}{A^2}$

4 **a** $a = \dfrac{P}{2} - b$ **b** $h = \dfrac{A - \pi r^2}{2\pi r}$ **c** $r = \dfrac{E}{I} - R$

 d $q = p - \dfrac{B}{A}$ **e** $x = \dfrac{3 - Ay}{2A}$ **f** $y = \sqrt{\dfrac{4}{M} - x^2}$

5 **a** $x = \dfrac{y}{1 - y}$ **b** $x = \dfrac{2y + 3}{1 - y}$ **c** $x = \dfrac{3y + 1}{3 - y}$

 d $x = \dfrac{y - 2}{y - 5}$ **e** $x = \dfrac{2y + 1}{y + 4}$ **f** $x = \dfrac{3y - 7}{2y + 3}$

 g $x = \dfrac{3y - 1}{y - 1}$ **h** $x = -\dfrac{4y + 3}{y + 2}$ **i** $x = \dfrac{2y}{y + 3}$

6 **a** $r = \sqrt[3]{\dfrac{3V}{4\pi}}$ **b** **i** 2.12 cm **ii** 62.0 cm

7 **a** $d = \sqrt{\dfrac{Gm_1 m_2}{F}}$

 b **i** 2.22×10^8 m   **ii** 3.02×10^{10} m

8 **a** $v = \sqrt{c^2\left(1 - \dfrac{m_0{}^2}{m^2}\right)} = \dfrac{c}{m}\sqrt{m^2 - m_0{}^2}$

 b $v = \dfrac{\sqrt{8}}{3}c$ **c** 2.998×10^8 m/s

EXERCISE 7E.1

1 **a** $x = -4,\ y = -6$ **b** $x = 5,\ y = 7$ **c** $x = 2,\ y = 6$

 d $x = -4,\ y = -7$ **e** $x = -2,\ y = -8$ **f** no solution

 g $x = 1,\ y = 5$ **h** no solution **i** $x = \frac{2}{5},\ y = 0$

2 **a** $x = \frac{1}{2},\ y = 4\frac{1}{2}$ **b** $x = 3,\ y = 4$ **c** $x = 2,\ y = -1$

 d $x = 0,\ y = -4$ **e** $x = -1,\ y = -1$ **f** $x = 0,\ y = 6$

EXERCISE 7E.2

1 **a** $x = 2,\ y = 5$ **b** $x = 3,\ y = 1$ **c** $x = 0,\ y = 5$

 d $x = 5,\ y = 9$ **e** $x = 0,\ y = 4$ **f** $x = 2,\ y = 1$

2 **a** $x = 5,\ y = 3$ **b** $x = 1,\ y = \frac{2}{5}$

 c $x = 2,\ y = 1\frac{1}{3}$ **d** $x = 5,\ y = -2$

 e $x = -3,\ y = -\frac{1}{2}$ **f** $x = -4\frac{1}{2},\ y = -3\frac{1}{2}$

3 **a** reduces to $5 = 7$ which is never true

 b no solution - the two lines they represent are parallel (same gradient)
  $\therefore$ they never intersect

4 **a** reduces to $8x + 6 = 8x + 6$ which is always true

 b infinite number of solutions - the lines they represent are coincident

EXERCISE 7E.3

1 **a** $11x = 11$ **b** $4y = 12$ **c** $9x = 9$ **d** $9x = 6$

 e $-y = 11$ **f** $-11y = -11$

2 **a** $x = 2,\ y = 6$ **b** $x = 1,\ y = -2$

 c $x = -1,\ y = -3$ **d** $x = 1,\ y = -5$

 e $x = 2,\ y = -2$ **f** $x = -3,\ y = 1$

3 **a** $10x + 25y = 5$ **b** $-3x + y = -4$

 c $3x - 21y = 24$ **d** $-10x - 8y = -18$

 e $-18x - 12y = 12$ **f** $-16x + 8y = -12$

4 **a** $x = 5,\ y = -2$ **b** $x = 1,\ y = 2$

 c $x = 1,\ y = -6$ **d** $x = 5,\ y = 1$

 e $x = 1,\ y = 1$ **f** $x = -3,\ y = 4$

 g $x = 2,\ y = -3$ **h** $x = 1,\ y = -1$

 i $x = -1,\ y = 3$ **j** $x = 5,\ y = -2$

 k $x = \frac{16}{29},\ y = \frac{1}{29}$ **l** $x = -3,\ y = -3$

 m $x = -\frac{34}{13},\ y = -\frac{85}{26}$ **n** $x = -2,\ y = -3$

5 **a** infinite number of solutions, lines are coincident

 b no solution, lines are parallel

EXERCISE 7F

1 34 and 15 **2** 51 and 35 **3** 16 and 24

4 nectarines 56 pence, peaches 22 pence

5 adults €12, children €8

6 15 small trucks and 10 large trucks

7 24 twenty cent coins and 13 fifty cent coins

8 9 male and 4 female costumes   **9** $x = 5$, $y = 2$

10 $45 call out and $60 an hour

11 **a** $a = \frac{5}{9}$, $b = -\frac{160}{9}$ **b** 25°C

REVIEW SET 7A

1 **a** 11.4 g/cm^3 **b** 37.9 g   **c** 9230.8 cm^3

2 **a** $t = \dfrac{s+13}{3}$ **b** $t = \pm 2\sqrt{r}$ **3** $x = \dfrac{2y-3}{y-2}$

4 **a** **i** $V = 6 \times 8$ **ii** $V = 8n$ **iii** $V = ln$
   b $V = 25 + ln$

5 **a** $c = -3$ **b** $c = 6$ **c** $c = -10$

6 **a** $x = -1$, $y = 3$ **b** $x = 2$, $y = -1$ **7** $x = 2$, $y = 7$

8 **a** $x = -2$, $y = 3$ **b** $x = 5$, $y = 6$

9 A sausage costs $0.80 and a chop $2.50.   **10** 9 drivers

REVIEW SET 7B

1 **a** 204 800 units   **b** 7.75 cm

2 **a** $y = \dfrac{20-6x}{5}$ or $y = 4 - \frac{6}{5}x$ **b** $r = \dfrac{C}{2\pi}$

3 4.46 cm   **4** **a** $x = \dfrac{y}{2} - \dfrac{3}{K}$ **b** $x = \dfrac{25c^2}{2M^2}$

5 **a** $C = 2 + 1.20$ **b** $C = 2 + 1.20 \times 5$
   c $C = p + 1.20b$ **d** $C = p + bx$

6 **a** $x = y = -2$ **b** $x = -4$, $y = -16$

7 $x = \frac{24}{13}$, $y = \frac{33}{13}$ **8** $x = 1$, $y = 0$ **9** 5 and 17   **10** $1.50

CHALLENGE

1 3, 8 and 14   **2** $x = 5$, $y = 1$

3

a	0	1	-1	-1	1
b	0	1	-1	1	-1
c	0	1	1	-1	-1

4 only 3961

5 $a = b = c$

6 **a** $x = 3$, $y = 5$, $z = -2$ **b** $x = 4$, $y = 2\frac{1}{2}$, $z = -1\frac{1}{2}$

7 $ab = -1$

EXERCISE 8A.1

1 **a** 8.06 cm   **b** 7.07 cm   **c** 15.3 km

2 **a** 9.22 cm   **b** 2.06 km   **c** 6.72 cm

3 **a** $x = \sqrt{11}$ **b** $x = \sqrt{2}$ **c** $x = \sqrt{5}$

4 **a** $x = \sqrt{\frac{5}{4}}$ or $\frac{1}{2}\sqrt{5}$ **b** $x = \sqrt{\frac{10}{4}}$ or $\frac{1}{2}\sqrt{10}$ **c** $x = \frac{1}{2}$

5 **a** $x \approx 5.20$ **b** $x \approx 7.21$ **c** $x = 2$

6 **a** $x = \sqrt{17}$, $y = 2\sqrt{2}$ **b** $x = \sqrt{29}$, $y = 3\sqrt{5}$
   c $x = \sqrt{5}$, $y = \sqrt{6}$

7 **a** $x \approx 1.41$ **b** $x = 14$ **8** AC ≈ 6.24 m

9 **a** AB $= \sqrt{17}$ cm   **b** AB $= \sqrt{29}$ m   **c** AB $= \sqrt{41}$ m

10 Area of trapezium $= \left(\dfrac{a+b}{2}\right)(a+b)$

Sum of areas of triangles $= \frac{1}{2}ab + \frac{1}{2}c^2 + \frac{1}{2}ab$

Equate these, etc.

11 $3\sqrt{2}$ cm (≈ 4.24 cm)

EXERCISE 8A.2

1 **a**, **b**, **d**, **f** are Pythagorean triples.

2 **a** $k = 17$ **b** $k = 10$ **c** $k = 48$ **d** $k = 25$
   e $k = 24$ **f** $k = 60$

3 $(3k)^2 + (4k)^2 = 9k^2 + 16k^2 = 25k^2 = (5k)^2$
   $\therefore$ there are infinitely many Pythagorean triples as k takes infinitely many values.

EXERCISE 8B

1 **b**, **e**, **f** are right angled.   **2** **a** $\widehat{\text{BAC}}$ **b** $\widehat{\text{ABC}}$ **c** $\widehat{\text{ACB}}$

3 Yes, as $6800^2 + 3500^2 \approx 7648^2$.

4 AB $= \sqrt{117}$ cm, BC $= \sqrt{52}$ cm
   AB^2 + BC$^2 = 169 = 13^2 = $ AC^2
   $\therefore$ $\triangle$ABC is right angled at B.
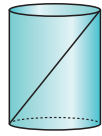
EXERCISE 8C

1 8.54 cm   **2** 3.16 cm $\times$ 9.49 cm

3 **a** 53.7 cm   **b** 160 cm^2 **4** 6.63 cm   **5** 7.07 cm

6 25.6 cm

7 **a** 15.8 km   **b** 22.4 km   **c** 22.4 km   **d** 25.5 km
   e 29.2 km   **f** 31.6 km

8 9.43 km   **9** 5 km

10 **a** 21.5 m   **b** 8 m   **c** 21.5 m   **d** 40.8 m

11 **a** $x = \sqrt{2}$, $y = 45$ **b** $x = 6$, $h = \sqrt{13}$
   c $x = \frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$

12 10.4 cm   **13** **a** 22.2 cm^2 **b** 8 cm   **14** 1.80 m

15 19.2 km   **16** $\sqrt{5} \approx 2.24$ km   **17** 13.4 km

EXERCISE 8D

1 10.9 cm   **2** 4.66 cm   **3** $4\sqrt{3}$ cm or 6.93 cm

4 $\frac{1}{2}\sqrt{157} \approx 6.26$ cm   **5** 6 cm   **6** ≈ 8.49 cm

7 3.71 cm   **8** 4.24 cm   **9** 12.5 cm   **10** $6\sqrt{2}$ cm

11 10.6 cm   **12** 1.3 m   **13** 10.05 m   **14** ≈ 1.42 cm

15 **a** O$\widehat{\text{B}}$A is a right angle and so BA is a tangent to the circle at B.
   b As PQ$^2 = $ PR$^2 + $ RQ^2, $\triangle$PRQ is right angled at R and so PQ is a diameter.

17 AD $= 16$ cm

EXERCISE 8E

1 15 cm

2

When placed like this the greatest length a stirrer could be and lie within the glass is 11.6 cm. So a 12 cm stirrer would go outside the glass.

3 3.16 cm   **4** $2\sqrt{3}$ cm ≈ 3.46 cm   **5** 6.80 m   **6** 61.6 cm

7 **a** 10 m   **b** $5\sqrt{5}$ m ≈ 11.2 m   **8** 12.65 m   **9** ≈ 66.8 m

10 32.2 m

REVIEW SET 8A

1 **a** $\sqrt{29} \approx 5.39$ cm   **b** $\sqrt{33} \approx 5.74$ cm
   c $3\sqrt{3} \approx 5.20$ cm, $6\sqrt{3} \approx 10.4$ cm

2 Right angled at A as AB^2 + AC$^2 = 25 + 11$
   $= 36$
   $= $ BC^2

3 $5^2 + 11^2 = 146 \neq 13^2$.

4 **a** $4\sqrt{10} \approx 12.6$ units   **b** $4\sqrt{13} \approx 14.4$ units
   c $4\sqrt{17} \approx 16.5$ units

5 $3\sqrt{13} \approx 10.8$ cm   **6** radius is $5\sqrt{2}$ cm ≈ 7.07 cm

7 5.77 m

8 If R is the radius of the larger circle and r is the radius of the smaller, then PQ $= $ PR $= \sqrt{R^2 - r^2}$.

9 **a** $x = 2\sqrt{5} \approx 4.47$ **b** $x = \sqrt{171} \approx 13.1$ **10** 6.55 m

REVIEW SET 8B

1 a $x = \sqrt{18}$ **b** $x = \sqrt{61}$ **c** $x = \sqrt{2}$

2 $2^2 + 5^2 = (\sqrt{29})^2$, $\widehat{ABC}$ is a right angle.

3 No, $AB = \sqrt{41}$ m, $BC = \sqrt{61}$ m
   and $AB^2 + BC^2 = 102 \neq AC^2$

4 a $6\sqrt{5} \approx 13.4$ km   **b** $3\sqrt{29} \approx 16.2$ km
   c $3\sqrt{13} \approx 10.8$ km

5 6 cm   **6** $12\sqrt{2} \approx 17.0$ m   **7** $6\sqrt{5} \approx 13.4$ cm

8 $5\sqrt{3} \approx 8.66$ cm

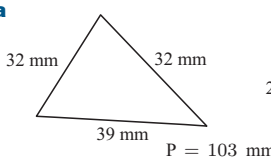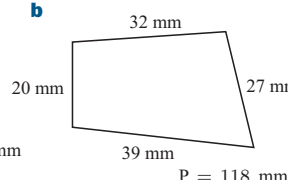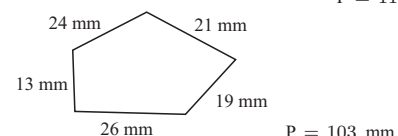**9 a** $y = 10\sqrt{2} \approx 14.1$ **b** $y = 15$ **10** ≈ 17.7 m

CHALLENGE

1 D is 8 km from C.   **2 a** $10\sqrt{17} \approx 41.2$ cm   **b** 50 cm
3 ≈ 80.3 m   **4** $3:8$ **5** 2.5 cm

EXERCISE 9A

1 a cm   **b** mm   **c** km   **d** m   **e** mm   **f** cm

3 a 52 000 m   **b** 1150 mm   **c** 165 cm   **d** 6300 mm
   e 62 500 cm   **f** 8 100 000 mm

4 a 4.8 m   **b** 5.4 cm   **c** 5.28 km   **d** 2 m
   e 5.8 km   **f** 7 km

5 a 42 100 m   **b** 2.1 m   **c** 7.5 cm   **d** 1.5 km
   e 185 cm   **f** 425 mm   **g** 280 000 cm
   h 16.5 m   **i** 250 000 mm

6 18 m   **7** 105 laps   **8** 2125 steps

9 a 0.375 mm   **b** 1.17 m   **c** 320 cards

EXERCISE 9B

1 a

32 mm   32 mm
39 mm
P = 103 mm

b

32 mm
20 mm   27 mm
39 mm
P = 118 mm

c

24 mm   21 mm
13 mm
19 mm
26 mm   P = 103 mm

2 a 38.8 cm   **b** 17.8 cm   **c** 11.9 km   **d** 29 m
   e 11.6 cm   **f** 30 m   **g** 17.6 m   **h** 99.6 km
   i 19.1 m

3 a $P = 2a + 2b$ or $P = 2(a+b)$
   b $P = k + l + m$ **c** $P = 12a$

4 11.1 cm   **5** 8.7 m   **6** 18.8 cm   **7** 38.4 m

8 $10 + 8\sqrt{2} \approx 21.3$ m   **9** 113 cm

EXERCISE 9C.1

1 a cm^2 **b** m^2 **c** ha   **d** m^2 **e** km^2 **f** mm^2

2 a 0.23 cm^2 **b** 36 000 m^2 **c** 0.0726 m^2
   d 7 600 000 mm^2 **e** 0.853 ha   **f** 35 400 000 cm^2
   g 1354 mm^2 **h** 4 320 000 cm^2 **i** 4820 mm^2
   j 3 000 000 m^2 **k** 70 ha   **l** 6 600 000 m^2
   m 6.6 km^2 **n** 500 cm^2 **o** 0.0025 m^2
   p 0.052 cm^2 **q** 720 000 000 000 mm^2

3 a 1000 cm^2 **b** 240 cm^2 **c** 1.2 cm^2 **d** 80 000 m^2

4 625 m^2 **5** 625 coasters

EXERCISE 9C.2

1 a 40 cm^2 **b** 625 m^2 **c** 42 m^2 **d** 39 cm^2
   e 15 cm^2 **f** 45 cm^2 **g** 180 m^2 **h** 24 cm^2
   i 25 cm^2

2 a $h = 6$ **b** $h = 3.6$ **c** $h = 5$

3 a 112 m^2 **b** 39 m^2 **c** 74 m^2 **d** 84 cm^2
   e 31.5 cm^2 **f** 189 cm^2 **g** 38 m^2 **h** 34 cm^2
   i 34 cm^2

4 96 cm^2 **5** \$280.80   **6** 5.09 m^2 **7** 375 tiles

8 a 48 cm^2 **b** $4\sqrt{2} \approx 5.66$ cm   **9** 144 cm^2

10 a 80 cm^2 **b** $\frac{1}{2}ab$ cm^2 **11** $5\frac{1}{3}$ cm   **12** 204 m^2

EXERCISE 9D

1 a 50.3 cm   **b** 3.39 m   **c** 34.6 cm

2 a 314 cm^2 **b** 468 m^2 **c** 73.9 cm^2

3 a 13.1 cm   **b** 17.6 m   **4 a** 22.1 m^2 **b** 139 cm^2

5 a 28.3 m^2 **b** 38.5 cm^2 **c** 56.5 m^2
   d 12.6 cm^2 **e** 26.2 m^2 **f** 30.2 cm^2

6 a $r = \frac{10}{\pi} \approx 3.18$ cm   **b** 2.52 cm   **c** 3 m

7 a $\frac{4}{3}\pi$ cm   **b** $16 + \frac{4}{3}\pi$ cm   **c** $\frac{16}{3}\pi$ cm^2

8 a 84.2 cm   **b** 16.8 cm   **c** 30.6 cm

9 a 451 m, 13 000 m^2 **b** 28.0 cm, 33.0 cm^2
   c 357 m, 6960 m^2 **d** 34.6 cm, 77.9 cm^2
   e 61.7 cm, 211 cm^2 **f** 94.2 cm, 571 cm^2
   g 13.7 cm, 9.53 cm^2 **h** 37.7 cm, 37.7 cm^2
   i 62.8 cm, 157 cm^2

10 15.9 cm   **11** 6.23 m   **12** 154 m^2 **13** 20.9 cm

14 a 7.13 cm^2 **b** 32.1 cm^2 **15** 23.9^o **16** 76.4^o

17 a $A = \frac{1}{2}\pi r^2$ **b** $A = \pi(R+r)(R-r)$
   c $A = (x + 2a)^2 - \pi a^2$
   d $A = 2ab + \pi a^2$ or $A = a(2b + \pi a)$
   e $A = (5 + \pi)a^2$ **f** $A = (\sqrt{3} + \frac{\pi}{2})x^2$

18 a 263 m   **b** ≈ 3480 m^2

19 a 1080 mm = 1.08 m   **b** 32.7 mm = 3.27 cm
   c i 1.13 cm^2 **ii** 6.91 cm^2 **iii** 39.3 cm^2
      iv 43.6 cm^2 **v** 46.1 cm^2

REVIEW SET 9A

1 a i 3280 m   **ii** 75.5 cm   **iii** 0.32 m   **b** 40 000 staples

2 a 9.2 m   **b** 42 m   **c** 41.7 cm

3 a 20 cm^2 **b** 19.6 m^2 **c** 22.5 m^2

4 a 157 cm   **b** 1273 revolutions   **5** 262°

6 a i $P = a + h + \sqrt{h^2 - a^2}$ cm
      ii $A = \frac{1}{2}a\sqrt{h^2 - a^2}$ cm^2
   b i $P = (10 + \pi)x$ m   **ii** $A = (8 + \frac{\pi}{2})x^2$ m^2

7 a 1.60 m   **b** 10.0 m

8 Area $= \frac{1}{2} \times 5 \times 12$ or $\frac{1}{2} \times 13 \times d$,   etc.

REVIEW SET 9B

1 a i 30 cm^2 **ii** 4 m^2 **iii** 60 ha **b** 62 rectangles

2 6 ha **3 a** 30 cm **b** 22.7 cm **c** 85.1 cm

4 a 489 cm^2 **b** 121 m^2 **c** $25\sqrt{3} \approx 43.3$ cm^2

5 a $\frac{3}{8}$ **b** 52.3 cm **c** 170 cm^2

6 a 6.49 m **b** 132 m^2

7 a $A = \left(\dfrac{a+b}{2}\right)h + bc$ **b** $A = \left(4 + \dfrac{\pi}{2}\right)x^2$

 c $A = \left(2 - \dfrac{\pi}{4}\right)x^2$

8 π m, $\frac{\pi}{2} - 1$ m^2

CHALLENGE

1 $\approx 9.31\%$ **2** $3:2$ **6** 8 cm^3

8 $r = \dfrac{25}{\pi + 6}$ cm ≈ 2.73 cm

EXERCISE 10A

1 a 70% **b** 26% **c** 12% **d** 5.5% **e** 200%
 f 20% **g** 5% **h** 100% **i** 98% **j** 0.3%

2 a $\frac{3}{4} = 0.75$ **b** $\frac{7}{100} = 0.07$ **c** $\frac{3}{200} = 0.015$
 d $1\frac{3}{5} = 1.6$ **e** $1\frac{13}{100} = 1.13$

3 a 70% **b** 15% **c** $33\frac{1}{3}\%$ **d** $26\frac{2}{3}\%$

4 7.85% **5 a** €2541 **b** €254 100

6 a 600 g **b** $1092 **c** 84 kg **d** 25 litres
 e 18.7 tonnes **f** £16 469.78

7 108 marks **8 a** 69.8 kg **b** 2.05% loss

9 a £345 **b** £4355 **c** 7.34%

10 87.5% **11** 30.4% **12** 44% **13** 21%

EXERCISE 10B.1

1 €1.50 **2** €28 loss **3** £1862 profit **4** $120 profit

5 a i $2 **ii** $22 **b i** $200 **ii** $450
 c i £7.50 **ii** £57.50 **d i** €53 550 **ii** €308 550

6 a i €16.50 **ii** €38.50 **b i** £61.25 **ii** £113.75
 c i $26.25 **ii** $78.75 **d i** €1344 **ii** €8256

7 $110.40 **8 a** $44.80 **b** $115.20 **c** $129.60

EXERCISE 10B.2

1 38.9% **2 a** $812.50 **b** $377.50 **c** 86.8%

3 a $1.60 **b** $1.25 **c** 78.1%

4 a €470 **b** 32.4% loss **5 a** £4600 **b** 25.3%

6 a $15 **b** 27.3%

7 a £140 300 **b** £32 800 **c** 23.4%

EXERCISE 10C.1

1 a $1734 **b** $1617.19 **2 a** $2891.25 **b** €5549.01

3 a 6.07% p.a. **b** 2.80% p.a.

4 a ≈ 1.87 years $= 1$ year 45 weeks
 b ≈ 3.01 years $= 3$ years 5 days

EXERCISE 10C.2

1 a $480 **b** £992.80 **c** $10 880 **d** €434.14

2 The first one, £7000 compared with £7593.75 for the second.

3 a 2.67% p.a. **b** 6.82% p.a. **4** 7.5% p.a.

5 10.3% p.a. **6 a** 3 years 9 months **b** 6 years

7 ≈ 2 years 9 months

EXERCISE 10D

1 75 marks **2** 620 students **3** 40 kg **4** 2250 m

5 a £4700 **b** £235 **6** 140 000 people **7** ≈ 1.247 m

8 $500 **9** €8400

EXERCISE 10E.1

1 a 1.1 **b** 0.9 **c** 1.33 **d** 0.79 **e** 1.072 **f** 0.911

2 a $84.80 **b** £81.60 **c** 57 kg **d** €22.95
 e £655.20 **f** 31.5 m

3 a €26 per hour **b** 31.86 m **c** 2.61 cm

4 a 50% increase **b** 20% decrease **c** 15.8% decrease
 d 31.1% increase **e** 25% increase **f** 18.8% decrease

5 46.7% increase **6** 10.6% decrease **7** 7.06% increase

8 2.11% decrease **9** 8% increase

EXERCISE 10E.2

1 a $2880 **b** £2805 **c** €3239.60 **d** $5170

2 False, e.g., $1000 increased by 10% is $1100,
 $1100 decreased by 10% is $990, not $1000.

3 $86.24 (including tax) **4** £180.79 **5** €78

6 £4428.74 **7** €40 575 **8** $40 279.90 **9** $55 163.86

10 a 31.8% increase **b** 16.1% decrease **c** 21.6% increase

11 16.3% **12** 20.0%

EXERCISE 10F

1 a $2977.54 **b** £5243.18 **c** €11 433.33

2 a €105.47 **b** $782.73 **c** £4569.19

3 a 13 738.80 Yuan **b** 1738.80 Yuan

4 a $5887.92 **b** $887.92

5 1st plan earns 3200 pesos, 2nd plan earns 3485 pesos
 $\therefore$ plan 2.

6 a $1514.44 **b** $15 886.59

7 a 12% p.a. **b** 5.5% p.a.

8 a Molly's, as it earns $44.89 more.
 b Max's, as it earns $7471.91 more.

9 $\approx 20.6\%$ p.a. **10** 12% p.a.

EXERCISE 10G

1 a 9.95 m/s **b** 35.8 km/h **2** $151\frac{2}{3} \approx 152$ km

3 3.39 seconds **4** 13.8 km/h

5 a 2 h 56 min 51 s **b** 54 min 33 s

6 a 7.875 km **b** 5530 km

7 a 9 km **b** 3 km/h **8** 7.94 km/h **9** 92.3 km/h

10 a 54 km/h **b** $33\frac{1}{3}$ m/s

11 a 28.8 sec **b** 248 m

EXERCISE 10H

1 a 1.8 km **b** 28 min **c i** 1 km **ii** 1.6 km
 d i 6 min **ii** 16 min **e** 3.86 km/h

2 a 12 min **b** 80 km **c** 20 km
 d A: 1 h 48 min, B: 2 h 12 min **e** 24 min
 f A: 44.4 km/h, B: 36.4 km/h

3 a the Smith family **b** the Reynolds family
 c the Reynolds family **d** 2 hours **e** $1\frac{1}{2}$ hours
 f 100 km

4 a after 2 min **b** 1 min **c** 3 min
 d 2 km **e** between the 3 and 4 minute marks

REVIEW SET 10A

1 a 0.87 **b** 1.109 **2 a** $2900 **b** 58.5 kg **3** 6.05 m
4 a ¥8845 **b** ¥2745 **5 a** £500 **b** 11.1%
6 a $12 **b** $68 **7** 180% increase **8** $38.50
9 $53.30 **10** $15 000 **11** 5.88% p.a.
12 ≈ 2 years 4 months **13** $1530 **14** $24 845.94
15 *Plan 1:* $2700, *Plan 2:* $2703.67
∴ *Plan 2 earns more interest.*
16 4 h 21 min 26 sec **17 a** 105 km **b** 70 km/h

REVIEW SET 10B

1 a 1.1 **b** 0.883 **2 a** £3915 **b** ≈ 380 km
3 a €35 **b** 21.2% **4 a** €513 **b** €436.05
5 $70 400 **6** €52.46 **7** $365 911 **8** $1768 **9** $1200
10 4.2 years, i.e., 4 years 73 days
11 a $34 264.87 **b** $9264.87
12 a €17 271.40 **b** €9271.40 **c** 11.6% p.a. **13** 1300 km
14 a 325 km **b** 20 min
c Johnson: 87.5 km/h, Maple: 100 km/h **d** 88.6 km/h
15 £2465.75 **16** 6.52% p.a. **17** 75 km/h

CHALLENGE

1 172 pages **2** 10 hours **3** ≈ 2.21 billion beats
4 The manager breaks even. **5** add ≈ 28.6 grams of salt

6
2	4	9
5	8	3
7	1	6

7
14	1	8	11
15	5	4	10
2	16	9	7
3	12	13	6

8
2	1	8
2	9	4
8	8	1

EXERCISE 11A.1

1 a 54 cm^2 **b** 121.5 cm^2 **c** 576.24 mm^2
2 a 276 cm^2 **b** 6880 mm^2 **c** 8802 m^2
3 a ≈ 198 m^2 **b** ≈ 496 cm^2 **c** ≈ 148 cm^2
4 a 360 cm^2 **b** 340 m^2 **c** ≈ 9840 cm^2
5 a 576 cm^2 **b** 384 m^2 **c** ≈ 823 m^2 **6** 1160 cm^2
7 a 364 cm^2 **b** 72.8 ml **8** 34 m^2 **9** 167.4 m^2

EXERCISE 11A.2

1 a ≈ 207 cm^2 **b** ≈ 339 cm^2 **c** ≈ 196 cm^2
d ≈ 56.7 m^2 **e** ≈ 124 m^2 **f** ≈ 79.5 cm^2
2 a 64π cm^2 **b** 60π m^2 **c** $5\pi(5 + \sqrt{106})$ cm^2
3 a ≈ 5030 cm^2 **b** ≈ 145 km^2 **c** ≈ 84.8 cm^2
4 a ≈ 1640 cm^2 **b** ≈ 758 cm^2 **c** ≈ 452 cm^2
d ≈ 942 m^2 **e** 603 cm^2
5 ≈ 18.1 cm^2 **6** ≈ 207 cm^2 **7** 35 spheres **8** ≈ 33.8 cm^2
9 a ≈ 5.64 m **b** ≈ 21.8 cm **c** ≈ 25.8 m
10 a $A = 18x^2$ **b** $A = 6x^2 + 12x$
c $A = 5x^2 + 6x + x\sqrt{5}(x + 2)$ **d** $A = 10\pi x^2$
e $A = 7\pi x^2 + 2\pi x$ **f** $A = (39 + 5\sqrt{10})x^2$
11 a $A = 2(ab + bc + ac)$ **b** $A = ab + bc + ac + c\sqrt{a^2 + b^2}$
12 $A = \frac{1}{2}\pi r^2 + rh + \frac{1}{2}\pi r\sqrt{h^2 + r^2}$

EXERCISE 11B.1

1 a 8650 mm^3 **b** 86 cm^3 **c** 0.3 m^3
d 124 000 mm^3 **e** 0.3 cm^3 **f** 3 700 000 cm^3
2 540 540 coins **3** 27 200 cm^3

EXERCISE 11B.2

1 a 385 m^3 **b** ≈ 339 cm^3 **c** 320 cm^3
d 45 cm^3 **e** 704 cm^3 **f** 432 cm^3
g ≈ 670 cm^3 **h** 32 cm^3 **i** ≈ 288 cm^3
j ≈ 125 m^3 **k** ≈ 1440 cm^3 **l** ≈ 262 cm^3
2 a $V = Al$ **b** $V = (a^2 - b^2)c$ **3** ≈ 0.905 m^3
4 0.6 m^3 **5** 4500 m^3 **6** 0.9 m^3 **7** 1145 sinkers
8 1.79 m high **9** Area of end $= 2h \times h - \frac{1}{2}\pi r^2$, etc.

EXERCISE 11C.1

1 a ml **b** ml or cl **c** kl **d** litres
2 a 680 ml **b** 376 cl **c** 37.5 cl **d** 47.32 kl
e 3500 litres **f** 423 ml **g** 54 litres **h** 0.5834 kl
3 15 jugs **4** 1 092 000 kl

EXERCISE 11C.2

1 a 2300 cc **b** 800 cc **c** 1800 cc **d** 3500 cc
2 a 25 ml **b** 3200 m^3 **c** 7320 litres
3 a 22.05 kl **b** ≈ 23.6 kl **c** ≈ 186 kl
4 ≈ 524 ml **5** 368 bottles **6** 12.7 cm **7** 10 hours
8 1.06 cm **9** 3.07 cm **10** 3 h 27 min **11** every 24 min

EXERCISE 11D

1 a 3.2 kg **b** 1870 kg **c** 0.047 835 kg
d 4.653 g **e** 2 830 000 g **f** 63 200 g
g 0.074 682 t **h** 1 700 000 000 mg **i** 91.275 kg
2 54 kg **3** 7500 cubes **4 a** 1.5 g/cm^3 **b** 0.973 g/cm^3
5 No, as there is 1887 cm^3 of flour and the capacity of the canister is 1800 cm^3.
6 1.05 kg **7** 72 kg

EXERCISE 11E

1 a i 226 m^2 **ii** 385 m^3
b i $V = \frac{8}{3}\pi r^3$ **ii** $r = 9$ **iii** $r = 8$
2 ≈ 1410 kl **3 a** 2378 cm^3 **b** 6 kg
4 a i ≈ 183 cm^2 **ii** ≈ 195 cm^3
b $h \approx 8.96$ cm **c** $r \approx 2.80$ cm **d** ≈ 13.6 cm
5 a ≈ 682 cm^2 **b** ≈ 1410 cm^3 **c** ≈ 9.47 kg
6 a i $A = (4 + 3\pi)a^2 + 8ab$ **ii** $V = a^2(4b + \pi a)$
iii $M = a^2 d(4b + \pi a)$
b i $A = 5\pi x^2$ **ii** $V = \frac{5}{3}\pi x^3$ **iii** $M = \frac{5}{3}\pi dx^3$
7 a capacity $= \frac{4}{3}\pi x^3$ kl **b** $x \approx 4.924$
c $A = \pi(3 + \sqrt{2})x^2$ m^2 **d** ≈ 255°
8 a $V = \left(1 + \frac{\pi}{4}\right)a^3$ m^3
b base is 6.05 m by 12.1 m with $\frac{a}{2} \approx 3.03$ m
c Each 4 m^2 of floor space has a 'double bunk' bed.
9 $\frac{91}{108}$ cm

REVIEW SET 11A

1 a 2.6 cm^3 **b** 8 m^3 **c** 1 200 000 cm^3
d 5600 ml **e** 0.25 kl **f** 56 ml
2 a 34 cm^2 **b** 36 cm^2 **c** ≈ 51.8 m^2 **3** 1.24 cm
4 a ≈ 729.17 cm^3 **b** ≈ 302.58 cm^3 **c** ≈ 37.70 m^3
5 175 grams **6 a** 1.44 g/cm^3 **b** 0.317 g/cm^3
7 a ≈ 3850 cm^3 **b** ≈ 3.85 litres **c** ≈ 2280 cm^3
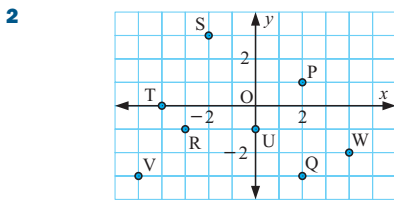
8 ≈ 1.24 m   **9** **a** $V = \left(\dfrac{\theta}{360}\right) \times \pi r^2 h$

**b** $A = \left(\dfrac{\theta}{360}\right) \times 2\pi r(r + h) + 2hr$

REVIEW SET 11B

1 **a** 0.35 g   **b** 0.25 t   **c** 16 800 g
d 0.15 litres   **e** 26 000 cl   **f** 800 ml
2 **a** $A = 16x^2$ **b** $A = 6\pi x^2$ **c** $A = 7x^2$
3 **a** 3.99 cm   **b** 3 m   **4** 220 ml
5 **a** ≈ 339.29 cm^3 **b** 1272.35 cm^3 **c** $\approx 14\,137.17$ cm^3
6 ≈ 33.5 kg   **7** ≈ 170 litres   **8** ≈ 872 cm^3
9 **a** 4 m   **b** 768 m^3 **c** ≈ 527 m^2

EXERCISE 12A

1 J(4, 3), K$(-2, -3)$, L$(-4, 2)$, M$(3, -1)$, N(0, 3)

2

3 **a** 1st   **b** 4th   **c** 3rd   **d** 2nd
e None, it is on the negative x-axis.
f None, it is on the negative y-axis.   **g** 3rd   **h** 4th

4 **a**

b

c

d

e

f

g

h

5 **a**

b

i yes
ii D

i yes   **ii** E
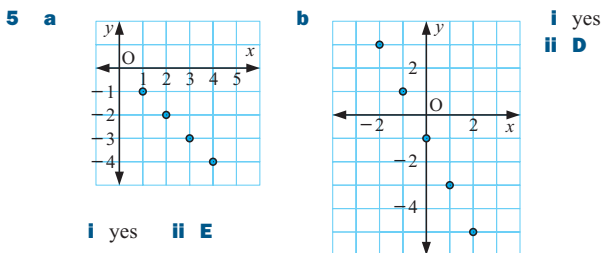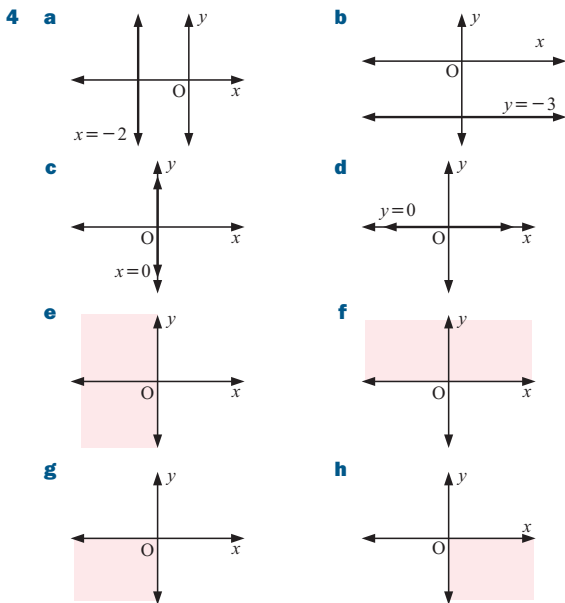
EXERCISE 12B.1

1 **a** 2 units   **b** $\sqrt{37}$ units   **c** $\sqrt{13}$ units   **d** 3 units
e $2\sqrt{2}$ units   **f** $\sqrt{29}$ units   **g** 5 units   **h** 7 units
i $5\sqrt{2}$ units
2 **a** 1.41 units   **b** 5.39 units   **c** 6.71 units
d 5.10 units   **e** 6.40 units   **f** 6.32 units
g 7.07 units   **h** 4.12 units   **i** 2.24 units

EXERCISE 12B.2

1 **a** $2\sqrt{2}$ units   **b** 7 units   **c** $2\sqrt{5}$ units   **d** 6 units
e 7 units   **f** $\sqrt{5}$ units   **g** $\sqrt{10}$ units   **h** $3\sqrt{5}$ units
2 **a** isosceles (AB = AC)   **b** scalene
c isosceles (AB = BC)   **d** isosceles (BC = AC)
e equilateral   **f** isosceles (AC = BC)
3 **a** $A\widehat{B}C$ **b** $A\widehat{B}C$ **c** $B\widehat{A}C$ **d** $B\widehat{A}C$
4 **a** $a = 2$ **b** $a = 3$ or -5 **c** $a = \pm 2$ **d** $a = -1$

EXERCISE 12C

1 **a** $\left(-1\frac{1}{2}, 3\frac{1}{2}\right)$ **b** $(-1, -2)$ **c** $\left(1, 1\frac{1}{2}\right)$ **d** $(2, 1)$
e $\left(1, -1\frac{1}{2}\right)$ **f** $\left(-4, 1\frac{1}{2}\right)$ **g** $\left(-4\frac{1}{2}, \frac{1}{2}\right)$ **h** $\left(-1\frac{1}{2}, \frac{1}{2}\right)$
2 **a** $(5, 3)$ **b** $(1, -1)$ **c** $\left(1\frac{1}{2}, 3\right)$ **d** $(0, 4)$
e $\left(2, -1\frac{1}{2}\right)$ **f** $(1, 1)$ **g** $\left(1, 2\frac{1}{2}\right)$ **h** $\left(2, -3\frac{1}{2}\right)$
3 **a** B$(0, -6)$ **b** B$(5, -2)$ **c** B$(0, 6)$ **d** B$(0, 7)$
e B$(-7, 3)$ **f** B$(-3, 0)$
4 **a** P$(-9, 10)$ **b** P$(6, 3)$ **5** C$(1, -3)$ **6** P$(7, -3)$
7 S$(-2, 0)$ **8** $\dfrac{\sqrt{89}}{2}$ units   **9** $a = 2\frac{1}{3}$, $b = 5\frac{1}{2}$

EXERCISE 12D.1

1 **a** $\frac{1}{3}$ **b** 0   **c** -3 **d** $\frac{2}{3}$ **e** $-\frac{3}{4}$ **f** undefined
g -4 **h** $-\frac{2}{5}$

2 **a**

b

c

d

e

f
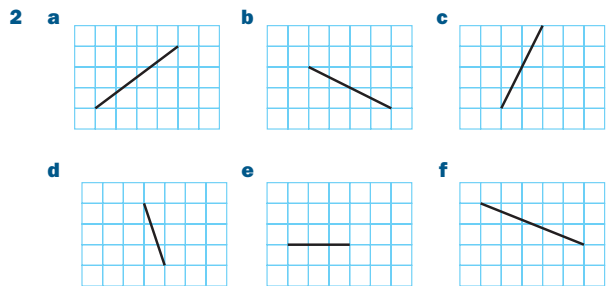
3 **a** $\frac{1}{5}$ **b** $\frac{1}{4}$ **c** 4   **d** 0   **e** undefined   **f** $\frac{2}{7}$ **g** $-\frac{2}{7}$
h 1

4

5
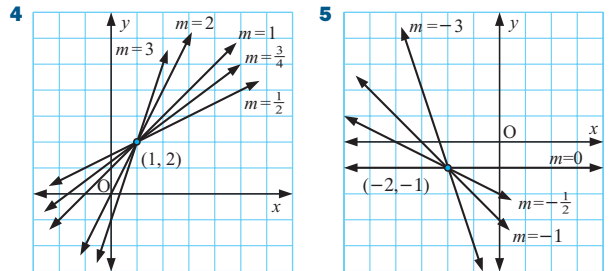
EXERCISE 12D.2

1 a $1\frac{1}{4}$ **b** Tan walks at $1\frac{1}{4}$ metres per second.
c Constant as the gradient is constant.

2 a 105 km/h **b i** 115 km/h **ii** 126.7 km/h
c From time 2 hours until time 5 hours.

3 a The y-intercept $(0, 40)$ indicates that taxi drivers earn a base rate of €40 before doing any work.
b The gradient of 18 means that a taxi driver earns €18 per hour of work.
c i €148 **ii** €310 **d** €23 per hour

4 a A has gradient $\frac{350}{29} \approx 12.1$; B has gradient $\frac{320}{33} \approx 9.70$
b A travels 12.1 km per litre of fuel; B travels 9.70 km per litre of fuel.
c $28.38

5 a $3 base charge
b AB has gradient $1\frac{1}{2}$; BC has gradient 1. These gradients indicate the charge per kilometre travelled.
c AC has gradient $1\frac{1}{5}$ which means that the average charge is $1.20 per kilometre travelled.
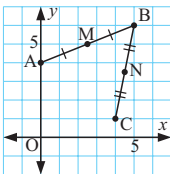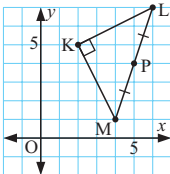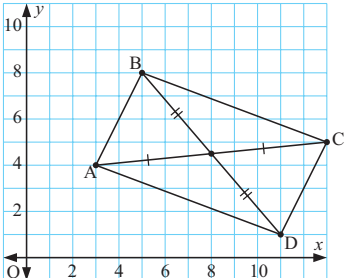
EXERCISE 12E.1

1 a -2 **b** $-\frac{5}{2}$ **c** $-\frac{1}{3}$ **d** $-\frac{1}{7}$ **e** $\frac{5}{2}$ **f** $\frac{3}{7}$ **g** $\frac{1}{5}$
h 1

2 **c**, **d**, **f** and **h** are perpendicular.

3 a $a = 9$ **b** $a = 1$ **c** $a = 6\frac{1}{3}$

4 a $t = \frac{1}{5}$ **b** $t = 5$ **c** $t = 3\frac{3}{5}$

5 a $t = 4$ **b** $t = 4$ **c** $t = 14$ **d** $t = 3\frac{1}{7}$

EXERCISE 12E.2

1 a not collinear **b** collinear **c** not collinear **d** collinear

2 a $c = 3$ **b** $c = -5$

EXERCISE 12F

1 a   **b** gradient of MN $= -\frac{3}{4}$
gradient of AC $= -\frac{3}{4}$
c MN $= 2\frac{1}{2}$ units and AC $= 5$ units

2 a   **b** KL $=$ KM $= 2\sqrt{5}$ units $\therefore$ ΔKLM is isosceles
c P$(5, 4)$
d gradient of KP $= -\frac{1}{3}$ gradient of LM $= 3$, etc.

3 a

b i gradient of AB $= 2$
gradient of DC $= 2$, $\therefore$ AB $\parallel$ DC
ii gradient of BC $= -\frac{3}{8}$
gradient of AD $= -\frac{3}{8}$, $\therefore$ BC $\parallel$ AD
c a parallelogram
d AB $=$ DC $= 2\sqrt{5}$ units
BC $=$ AD $= \sqrt{73}$ units
e i midpoint of AC is $(8, 4\frac{1}{2})$
ii midpoint of BD is $(8, 4\frac{1}{2})$
f Diagonals bisect each other.
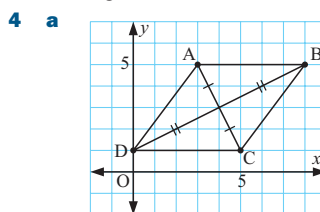
4 a
b AB $=$ BC $=$ DC $=$ DA $= 5$ units $\therefore$ ABCD is a rhombus.
c midpoint of AC is $(4, 3)$; midpoint of DB is $(4, 3)$
d gradient of AC $= -2$; gradient of DB $= \frac{1}{2}$, etc.
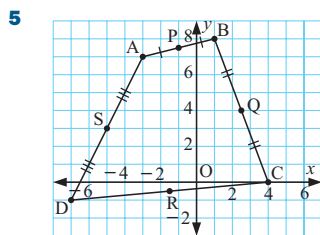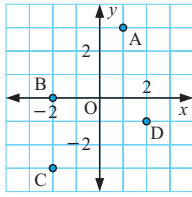
5
a i P$(-1, 7\frac{1}{2})$
ii Q$(2\frac{1}{2}, 4)$
iii R$(-1\frac{1}{2}, -\frac{1}{2})$
iv S$(-5, 3)$
b i -1 **ii** $\frac{9}{8}$ **iii** -1 **iv** $\frac{9}{8}$
c Opposite sides are parallel and so it is a parallelogram.

6 a OB $=$ radius $= 10$, $\therefore$ OC $= 10$ also.
Using the distance formula, $\sqrt{6^2 + a^2} = 10$
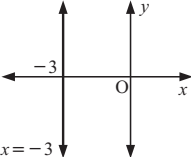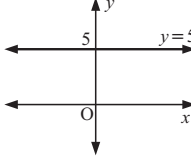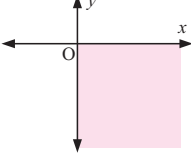$\therefore$ $a = 8$ {as $a > 0$}
b i $\frac{1}{2}$ **ii** -2
c AC and CB are perpendicular $\therefore$ $\widehat{ACB}$ is a right angle.

REVIEW SET 12A

1
2 a 5 units **b** $\sqrt{13}$ units
3 $(5, -1)$
4 a $\frac{2}{5}$ **b** $-3\frac{3}{4}$

5 a i 40 km/h **ii** 90 km/h **iii** 20 km/h
b i 40 **ii** 90 **iii** 20
The gradients are the same as the average speeds in **a**.
c 60 km/h
6 AB $=$ BC $= \sqrt{65}$ units, AC $= \sqrt{26}$ units $\therefore$ ΔABC is isosceles
7 a $k = -4\frac{1}{2}$ **b** $k = -15$
8 gradient of AB $=$ gradient of BC $= 2$ and B is common

9 $b = -3$

10 **a** AB $=$ BC $= 5$ units   **b** X$(\frac{1}{2}, \frac{1}{2})$

   c gradient BX $\times$ gradient AC $= 7 \times -\frac{1}{7} = -1$

REVIEW SET 12B

1 **a** $(-3, 3)$ **b** $\sqrt{58}$ units   **c** $-\frac{3}{2}$

2 **a**   **b**

c

3 $m = 2 \pm 3\sqrt{5}$ **4** B is $(5, -4)$ **5** **a** -1 **b** undefined

6 **a** The value of 60 at A indicates that the plumber charges a call-out fee of $60.

   b AB has gradient 50;   BC has gradient 40
   These gradients indicate the average hourly charge for the time intervals $0 \leqslant h \leqslant 3$ (AB) or $h > 3$ (BC).

   c AC has gradient 46 which means that the plumber charges an average of $46 per hour for a job lasting 5 hours.

7 **a** D$(4, -3)$ **b** 5 units   **8** $c = 20$ **9** $a = -8$

10 **a** gradient AB $=$ gradient DC $= \frac{1}{5}$

   b gradient AD $=$ gradient BC $= -2$

   c AB $\parallel$ DC   and   AD $\parallel$ BC $\therefore$ ABCD is a parallelogram.

   d $(\frac{1}{2}, \frac{1}{2})$;   diagonals bisect each other

CHALLENGE
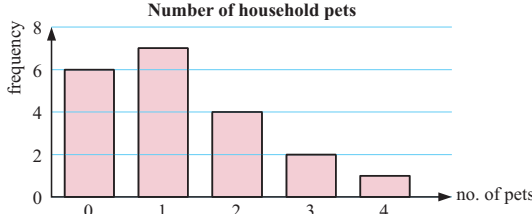
1 **a** $\sqrt{(b+a)^2 + c^2}$ units   **b** $\sqrt{(b-a)^2 + c^2}$ units

   d The perpendicular bisector of the base of an isosceles triangle passes through its apex.

2 **a** B is $(a + b, c)$ **b** midpoint of AC is $\left(\dfrac{a+b}{2}, \dfrac{c}{2}\right)$

   midpoint of OB is $\left(\dfrac{a+b}{2}, \dfrac{c}{2}\right)$

   c The diagonals of a parallelogram bisect each other.

3 **a** $ax + cy = a^2 + c^2$ and $(b-a)x - cy = b^2 - a^2 - c^2$

   b x-coord. of S is b **c** RS is a vertical line.

   d The perpendicular bisectors of the sides of a triangle are concurrent.

EXERCISE 13A

1 **a** categorical   **b** numerical   **c** categorical

   d numerical   **e** numerical   **f** categorical

   g categorical   **h** numerical

2 **a** male, female

   b soccer, gridiron, AFL, rugby league, rugby union, Gaelic

   c black, blond, brown, grey, red

3 **a** sample   **b** census   **c** sample   **d** census   **e** sample

   f sample

5 **a** quantitative discrete   **b** quantitative continuous

   c quantitative discrete   **d** quantitative discrete

   e quantitative discrete   **f** quantitative continuous

g quantitative discrete   **h** quantitative continuous

i quantitative continuous   **j** quantitative discrete

EXERCISE 13B.1

1 **a** number of pets in a household

   b Discrete, since you can't have part of a pet.

   c

   Number of household pets

   d positively skewed   **e** 30%   **f** 15%

2 **a** 45 shoppers   **b** 18 shoppers   **c** 15.6%

   d positively skewed

3 **a** number of toothpicks in a box   **b** discrete

   c

No. of toothpicks	Tally	Freq.
47	|	1
48	卌	5
49	卌 卌	10
50	卌 卌 卌 卌 |||	23
51	卌 卌	10
52	卌 ||||	9
53				2

   d

   Toothpicks in a box
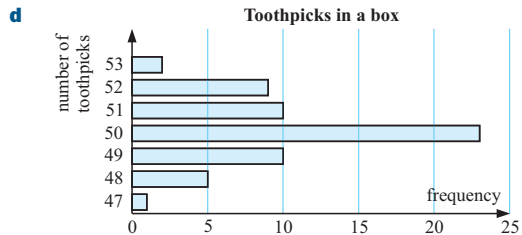
   e approximately symmetrical   **f** 38.3%

4 **a**

No. of accidents/month	Tally	Freq.
3				2
4						4
5					3
6	卌		6
7	卌 卌	10
8	卌 |||	8
9					3

   b Yes, negatively skewed.

   c

   Comparison of number of accidents
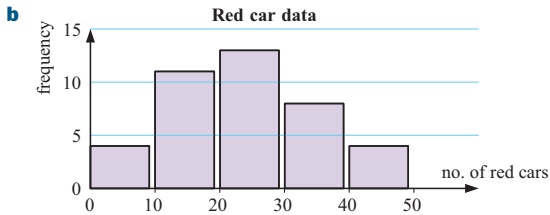
   before upgrade
   after upgrade
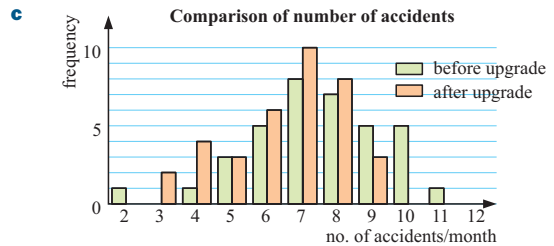
   d The distribution seems to have shifted towards the lower end. Before there were 11 times when 9 or more accidents/month occurred, now only 3.

EXERCISE 13B.2
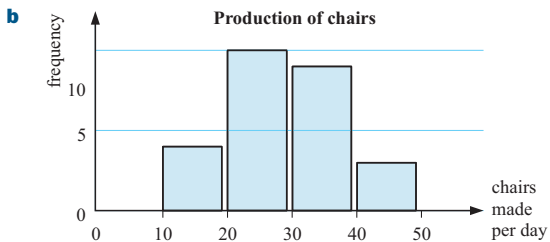
1 a

Number of red cars	Tally	Frequency
0 - 9	||||	4
10 - 19	ⅢⅢ Ⅲ |	11
20 - 29	ⅢⅢ ⅢⅢ Ⅲ	13
30 - 39	ⅢⅢ Ⅲ	8
40 - 49	||||	4
	Total	40

b

Red car data

c 15 students   **d** 30%   **e** 20 - 29 red cars

2 a

Chairs made per day	Tally	Frequency
10 - 19	||||	4
20 - 29	ⅢⅢ ⅢⅢ	10
30 - 39	ⅢⅢ ||||	9
40 - 49	Ⅲ	3
	Total	26

b

Production of chairs

c 88.5%   **d** 12 days   **e** 20 - 29 chairs/day

3 a

Visitors each day	Tally	Frequency
0 - 99		0
100 - 199	Ⅲ	3
200 - 299	||||	4
300 - 399	ⅢⅢ |	6
400 - 499	ⅢⅢ Ⅲ	8
500 - 599	ⅢⅢ ⅢⅢ Ⅲ	12
600 - 699	ⅢⅢ ||||	9
	Total	42

b

Museum visitors
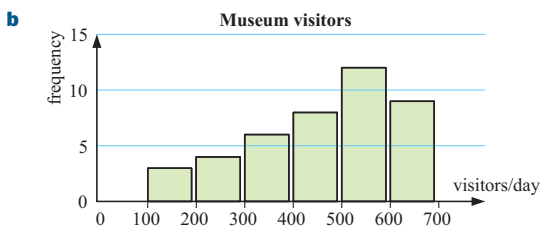
c 21 days   **d** 500 - 599 visitors   **e** negatively skewed

4 a

Year 12 students	Tally	Frequency
50 - 59	||	2
60 - 69	ⅢⅢ ||	7
70 - 79	ⅢⅢ ⅢⅢ	10
80 - 89	ⅢⅢ Ⅲ	8
90 - 99	Ⅲ	3
	Total	30

b

Year 12 student enrolments
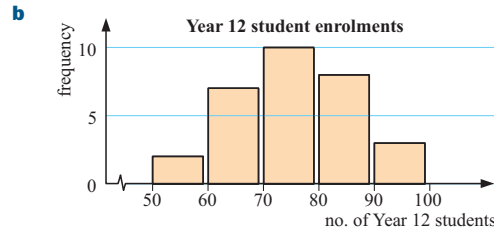
c 11 schools   **d** 30%   **e** 70 - 79 students
f a symmetrical distribution

EXERCISE 13C

1 a i 24   **ii** 24   **iii** no mode
   b i 13.3   **ii** 11.5   **iii** 8
   c i 10.3   **ii** 10.0   **iii** 11.2
   d i 428.6   **ii** 428   **iii** 415 and 427

2 a A: 7.73, B: 8.45   **b** A: 7, B: 7
   c The data sets are the same except for the last value, and the last value of A is less than the last value of B, so the mean of A is less than the mean of B.
   d The middle value of the data sets is the same, so the median is the same.

3 a mean: \$582 000, median: \$420 000, mode: \$290 000
   b The mode is the second lowest value, so does not take the higher values into account.
   c No, since the data is unevenly distributed, the median is not in the centre.

4 a mean: 3.11, median: 0, mode: 0
   b The data is very positively skewed so the median is not in the centre.
   c The mode is the lowest value so does not take the higher values into account.

5 a 44   **b** 44   **c** 40.6   **d** increase mean to 40.75

6 105.6   **7** 1712 km   **8** \$2 592 000   **9** $x = 12$
10 $a = 8$ **11** 27   **12** 7.875
13 $A = \dfrac{30S + 31O + 30N}{91}$ **14** 15

EXERCISE 13D

1 a i 9   **ii** $Q_1 = 7$, $Q_3 = 10$ **iii** 7   **iv** 3
   b i 18.5   **ii** $Q_1 = 16$, $Q_3 = 20$ **iii** 14   **iv** 4
   c i 26.9   **ii** $Q_1 = 25.5$, $Q_3 = 28.1$ **iii** 7.7   **iv** 2.6

2 a median $= 2.35$, $Q_1 = 1.4$, $Q_3 = 3.7$
   b range $= 5.1$,   IQR $= 2.3$
   c i ...... greater than 2.35 minutes
      ii ...... less than 3.7 minutes
      iii The minimum waiting time was 0.1 minutes and the maximum waiting time was 5.2 minutes. The waiting times were spread over 5.1 minutes.

3 a 20   **b** 58   **c** 40   **d** 30   **e** 49   **f** 38   **g** 19

EXERCISE 13E

1 a 1 **b** 2 **c** ≈ 1.90 **d** 3

2 a i 5.74 **b**
 ii 7
 iii 8
 iv 10
 c bimodal

Number of books read
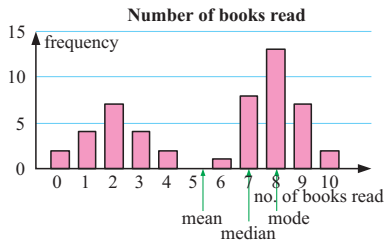
d The mean takes into account the full range of numbers of books read and is affected by extreme values. Also the values which are lower than the median are well below it.

e median

3 a i 4.25 **ii** 5 **iii** 5 **b** 6 **c** Yes, negatively skewed.

d The mean is less than the mode and median.

4 a

Foreign countries visited	Frequency
0	6
1	8
2	5
3	4
4	2
5	1
6	2

b 28 people
c i ≈ 1.96
 ii 1.5
 iii 1
 iv 6

EXERCISE 13F

1 a 71.225 **c**

Chem. exam mark	f	x	fx
30 - 39	1	34.5	34.5
40 - 49	2	44.5	89.0
50 - 59	7	54.5	381.5
60 - 69	6	64.5	387.0
70 - 79	10	74.5	745.0
80 - 89	10	84.5	845.0
90 - 99	4	94.5	378.0
	40		2860

b 73.5
d i 71.5
 ii 74
e very close

2 a 8.41 **b** 6 - 10 lunches **c** ≈ 8

3 a ≈ 55.2 **b** ≈ 58.3

c For this test the boys performed better than the girls.

EXERCISE 13G

1 a Helen: mean ≈ 18.8, median $= 22$
 Jessica: mean ≈ 18.1, median $= 18$
b Helen: range $= 38$, IQR $= 15.5$
 Jessica: range $= 29$, IQR $= 6.5$
c Helen, but not by much.
d Jessica: smaller range and IQR.

5 a *Boys:* mean ≈ 171, median $= 170.5$
  range $= 28$, IQR $= 7$
 Girls: mean ≈ 166, median $= 166$
  range $= 25$, IQR $= 7.5$
b The distributions show that in general, the boys are taller than the girls but there is little variation between their height distribution.

REVIEW SET 13A

1 a discrete **b** continuous **c** discrete

2 a 49 **b** 15 **c** 26.5% **d** positively skewed
 e ≈ 3.51 employees

3 a

No. call-outs/day	Tally	Frequency
0 - 9	||	2
10 - 19	ⵀ |	6
20 - 29	ⵀ ⵀ	10
30 - 39	ⵀ ||	7
Total		25

b

Fire department call-outs
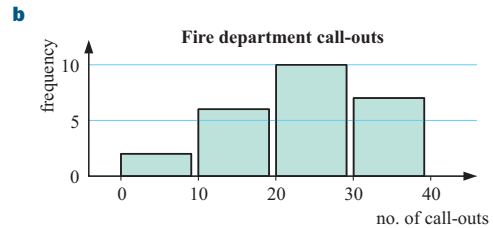
c 17 days **d** 8% **e** 20 - 29 call-outs/day

4 a $\overline{x} \approx 29.6$ **b** 16 and 28 **c** 29 **d** 45
 e $Q_1 = 22$, $Q_3 = 41.5$ **f** 19.5

5 a

Test scores	Tally	Frequency
0 - 9	|	1
10 - 19	|||	3
20 - 29	ⵀ ⵀ ||	12
30 - 39	ⵀ ⵀ ||||	14
Total		30

b

Test scores
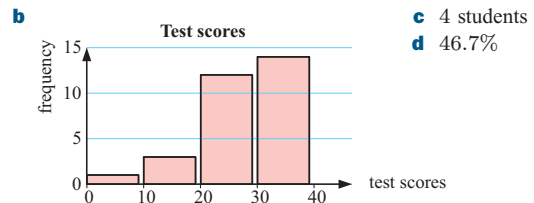
c 4 students
d 46.7%

6 $x = 7$

7 a 12 **b** 43 **c** 29 **d** 20 **e** 34.5 **f** 31 **g** 14.5

8 a

Goals/game	Frequency
0	8
1	7
2	4
3	1
Total	20

b i 20 games
 ii 18 goals
c i 0.9
 ii 1
 iii 0
 iv 3

9 8.53 potatoes per plant

10 a $\overline{x} \approx 159$ **b** 157 **c** 143 **d** 174

REVIEW SET 13B

1 a The number of children per household. **b** discrete

c

Number of children	Tally	Frequency
0		0
1	ⵀ	5
2	ⵀ ||	7
3	||||	4
4	||	2
5	|	1
6		0
7		0
8	|	1
Total		20

d

Number of children in a household
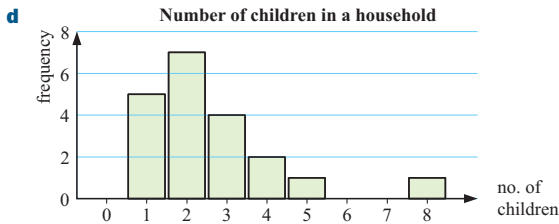
e positively skewed   **f** mode $= 2$

2 a

Emails sent last week	Tally	Frequency
0 - 9	~~IIII~~ I	6
10 - 19	~~IIII~~ IIII	9
20 - 29	~~IIII~~ II	7
30 - 39	~~IIII~~	5
40 - 49	II	2
50 - 59	I	1
	Total	30

b
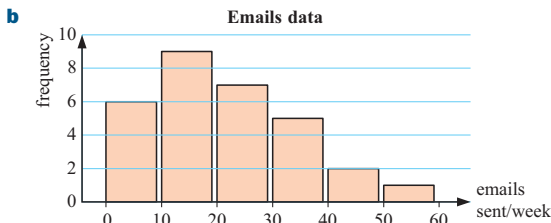
Emails data

c 10 - 19 emails   **d** 26.7%   **e** positively skewed

3 a 10 days   **b** 27.5%   **c** 170 - 179 vehicles   **d** $\overline{x} \approx 169$

4 ≈ 13.6 **5** 6, $x = 2$

6 a median $= 101.5$, $Q_1 = 98$, $Q_3 = 105.5$
   b IQR $= 7.5$.   The middle 50% of the data lies in an interval
   of length 7.5 scores.

7 a $\overline{x} \approx 52.3$ **b** 51   **c** $Q_1 = 42$
   d $Q_3 = 61$ **e** 32   **f** IQR $= 19$

8 a   i $\overline{x} \approx 8.47$ **ii** 9   **iii** 9   **iv** 4
   b well above the average   **c** negative

9 $\overline{x} \approx 7.58$,   median ≈ 6.90

10 a   i $\overline{x} \approx 371$ **ii** 372   **iii** 228.5   **iv** 469.5
   b range $= 682$,   IQR $= 241$

EXERCISE 14A
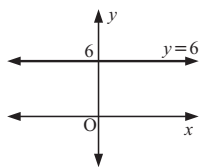
1 a horizontal line   **b** vertical line
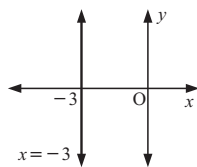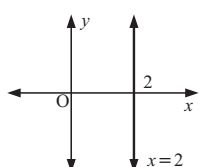
c vertical line   **d** horizontal line
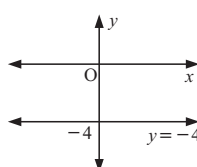
2 a horizontal line   **b** vertical line

3 a $y = 0$ **b** $x = 0$ **c** $y = -3$ **d** $x = 4$
4 a $y = 3$ **b** $x = 4$

EXERCISE 14B

1 a i

x	-3	-2	-1	0	1	2	3
y	-3	-2	-1	0	1	2	3

ii
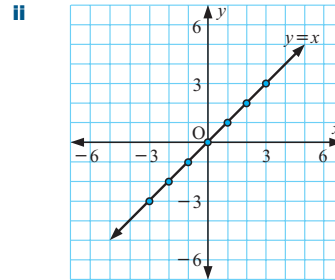
iii gradient $= 1$, x- and y-intercepts are both 0

b i

x	-3	-2	-1	0	1	2	3
y	-9	-6	-3	0	3	6	9

ii
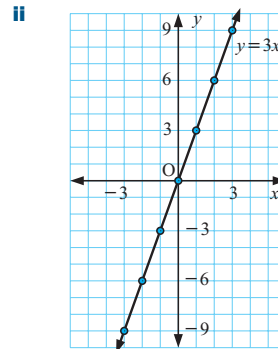
iii gradient $= 3$,
x- and y-intercepts
are both 0

c i

x	-3	-2	-1	0	1	2	3
y	-1	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1

ii
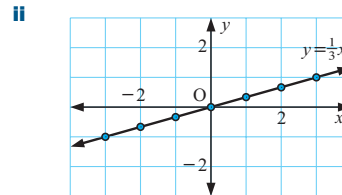
iii gradient $= \frac{1}{3}$, x- and y-intercepts are both 0

d i

x	-3	-2	-1	0	1	2	3
y	9	6	3	0	-3	-6	-9

ii
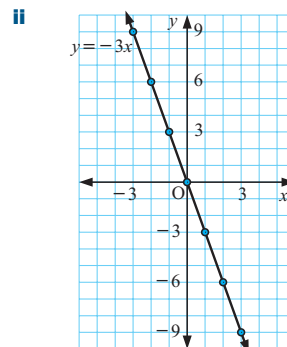
iii gradient $= -3$,
x- and y-intercepts
are both 0

e **i**

x	-3	-2	-1	0	1	2	3
y	-5	-3	-1	1	3	5	7

ii
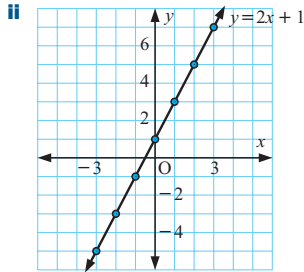
iii gradient $= 2$, x-intercept is $-\frac{1}{2}$, y-intercept is 1

f **i**

x	-3	-2	-1	0	1	2	3
y	7	5	3	1	-1	-3	-5

ii
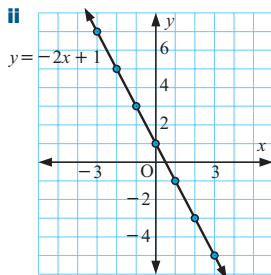
iii gradient -2, x-intercept is $\frac{1}{2}$, y-intercept is 1

g **i**

x	-3	-2	-1	0	1	2	3
y	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$

ii
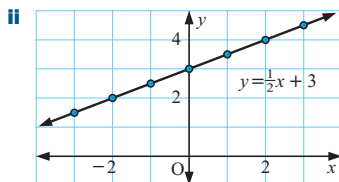
iii gradient $= \frac{1}{2}$, x-intercept is -6, y-intercept is 3

h **i**

x	-3	-2	-1	0	1	2	3
y	$4\frac{1}{2}$	4	$3\frac{1}{2}$	3	$2\frac{1}{2}$	2	$1\frac{1}{2}$

ii
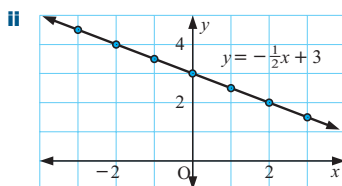
iii gradient $= -\frac{1}{2}$, x-intercept is 6, y-intercept is 3

2 **b**, **a**, **c**; coefficient of x **3** the sign of the coefficient of x

4 the constant term of the equation

EXERCISE 14C

1 **a** $y = 3x - 2$ **b** $y = -4x + 8$ **c** $y = \frac{1}{2}x + \frac{2}{3}$

   d $y = -\frac{2}{3}x + \frac{3}{4}$

2 **a** $m = 3, c = 11$ **b** $m = -2, c = 6$ **c** $m = \frac{1}{2}, c = 0$

   d $m = -\frac{1}{3}, c = -2$ **e** $m = 0, c = 3$

   f m is undefined, no y-intercept **g** $m = -2, c = 3$

   h $m = \frac{1}{2}, c = -1$ **i** $m = \frac{3}{2}, c = \frac{1}{2}$ **j** $m = \frac{2}{3}, c = -\frac{1}{3}$

   k $m = -\frac{1}{4}, c = \frac{1}{4}$ **l** $m = -\frac{2}{5}, c = \frac{3}{5}$

3 **a** $y = x - 1$ **b** $y = \frac{1}{2}x + 1$ **c** $y = -\frac{3}{4}x + 3$

   d $y = -\frac{1}{3}x + 3$ **e** $y = -2x - 4$ **f** $y = 3$

4 **a** $y = 2x + 2$ **b** $y = -3x + 10$ **c** $y = \frac{2}{3}x - 2$

   d $y = -\frac{1}{4}x - 2\frac{1}{2}$ **e** $y = \frac{4}{5}x - 12$ **f** $y = -\frac{1}{6}x - 7$

5 **a** $y = 2x + 3$ **b** $y = \frac{1}{2}x - 1$ **c** $y = -\frac{1}{3}x + 2$

   d $y = -3x + 4$ **e** $y = 2$ **f** $x = 3$

   g $y = -\frac{5}{3}x + 4$ **h** $y = \frac{5}{4}x + 3\frac{1}{2}$ **i** $y = \frac{1}{5}x - 1\frac{4}{5}$

   j $y = \frac{7}{4}x - 1\frac{1}{4}$ **k** $y = -\frac{6}{7}x - 1\frac{4}{7}$ **l** $y = 5x + 8$

6 **a** $M = \frac{1}{3}t + 4$ **b** $N = \frac{2}{3}x - 2$ **c** $G = -\frac{3}{4}s + 3$

   d $H = -g + 2$ **e** $F = \frac{3}{10}x + 5$ **f** $P = -\frac{1}{3}t - 2$

7 **a** $y = \frac{1}{2}x + 3$ **b** $y = 2x + 6$ **c** $y = \frac{2}{5}x - 2$

   d $y = -2x - 2$ **e** $y = -\frac{4}{3}x + 6\frac{2}{3}$ **f** $y = \frac{1}{2}x + 4$

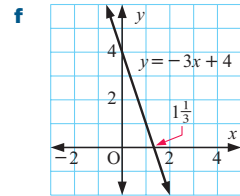**8** **a** $a = 2$ **b** $a = 13$ **c** $a = 5$ **d** $a = 4$
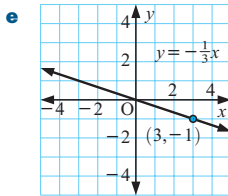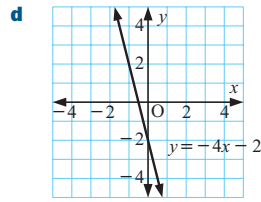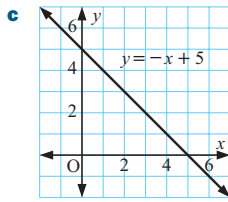
EXERCISE 14D.1

1 **a** $x - 2y = -6$ **b** $2x + y = 7$ **c** $x + 3y = -6$

   d $2x - 3y = -1$

2 **a** $y = -\frac{1}{2}x + 4$ **b** $y = -\frac{2}{5}x + 2$ **c** $y = 3x - 11$

   d $y = -\frac{1}{4}x + 1\frac{1}{2}$ **e** $y = \frac{6}{5}x - 3$ **f** $y = -\frac{4}{3}x + 4$

   g $y = \frac{9}{2}x - 9$ **h** $y = -\frac{5}{6}x + 5$

3 **a** $m = -\frac{1}{3}, c = 2$ **b** $m = \frac{1}{4}, c = 2$ **c** $m = -\frac{3}{5}, c = 1$

   d $m = 4, c = -8$ **e** $m = -\frac{1}{5}, c = \frac{4}{5}$ **f** $m = \frac{7}{5}, c = 3$

   g $m = -\frac{4}{3}, c = 8$ **h** $m = \frac{a}{b}, c = -\frac{d}{b}$

4 **a** $2x - y = -4$ **b** $3x + y = -2$ **c** $2x - 4y = -1$

   d $x + 3y = 0$ **e** $3x + 4y = 8$ **f** $2x - 5y = 5$

5 **a** $2x - 3y = -6$ **b** $5x - 4y = 8$ **c** $3x + 5y = 15$

   d $x - y = -5$ **e** $5x + 3y = -10$ **f** $5x + 7y = -15$

6 **a** $k = 3$ **b** $k = -2$ **c** $k = 2$ **d** $k = -2$
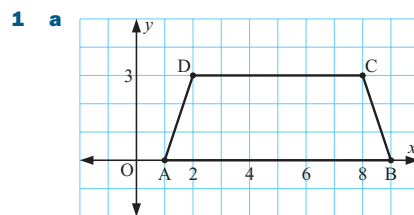
EXERCISE 14D.2

1 **a** $x - 2y = 2$ **b** $2x - 3y = -19$ **c** $3x - 4y = 15$

   d $3x - y = 11$ **e** $x + 3y = 13$ **f** $3x + 4y = -6$

   g $2x + y = 4$ **h** $3x + y = 4$

2 **a** $-\frac{2}{3}$ **b** $\frac{3}{7}$ **c** $\frac{6}{11}$ **d** $-\frac{5}{6}$ **e** $-\frac{1}{2}$ **f** 3

3 **a** parallel lines have the same gradient of $-\frac{3}{5}$

   b $-\frac{3}{5} \times \frac{5}{3} = -1$ for perpendicular lines, $\therefore$ $m = \frac{5}{3}$

4 **a** $3x + 4y = 10$ **b** $2x - 5y = 3$ **c** $3x + y = -12$

   d $x - 3y = 0$

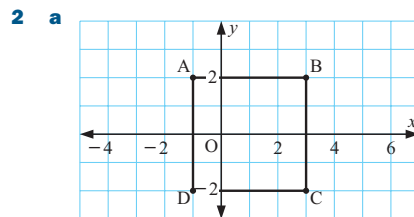**5** **a** $\frac{2}{3}, -\frac{6}{k}$ **b** $k = -9$ **c** $k = 4$

EXERCISE 14E
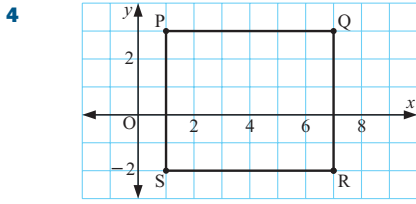
1 **a**   **b**

c

$y = -x + 5$

d

$y = -4x - 2$

e

$y = -\frac{1}{3}x$
$(3, -1)$

f

$y = -3x + 4$
$1\frac{1}{3}$

g

$(4, 3)$
$y = \frac{3}{4}x$

h

$y = \frac{1}{3}x - 1$

i

$y = -\frac{3}{2}x + 2$
$1\frac{1}{3}$

2 a

$y = 2x - 1$
$\frac{1}{2}$
$y = -2x + 1$

b

$y = \frac{1}{2}x + 2$
$y = -\frac{1}{2}x + 2$

3 a

$2x + y = 4$

b

$3x + y = 6$

c

$3x - 2y = 12$

d

$3x + 4y = 12$

e

$x - y = 2$

f

$x + y = -2$

g

$2x - 3y = -9$
$-4\frac{1}{2}$

h

$4x + 5y = 20$

i

$5x - 2y = -10$

4 a

$(-1, 2)$
new
$\frac{1}{2}$
$\frac{1}{3}$
$3x + 2y = 1$
old

b $2x - 3y = -8$

5

$3x - 5y = -15$
new
$3x - 5y = 15$
old

EXERCISE 14F

1 a

b a trapezium
c $x = 5$

2 a
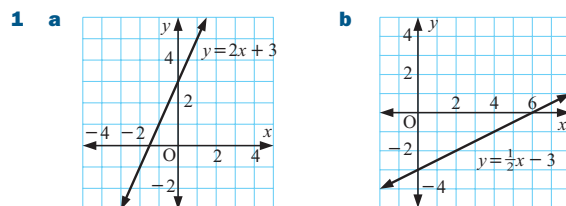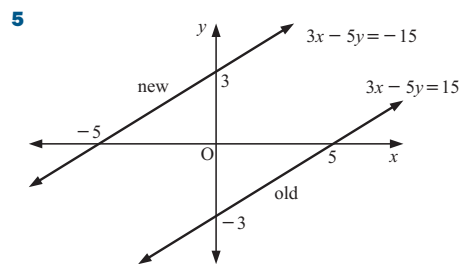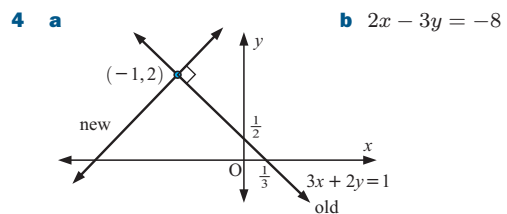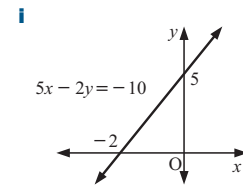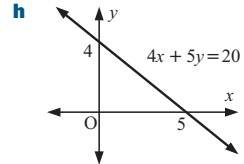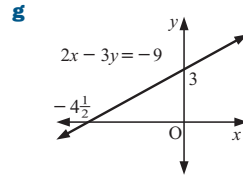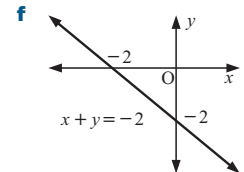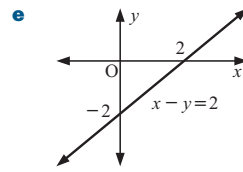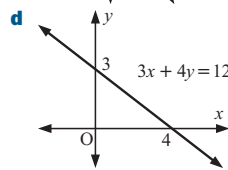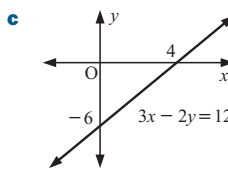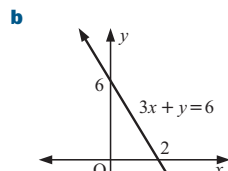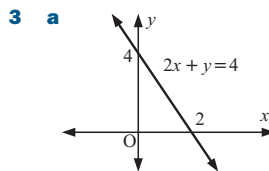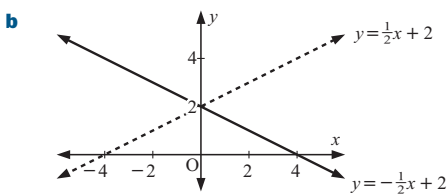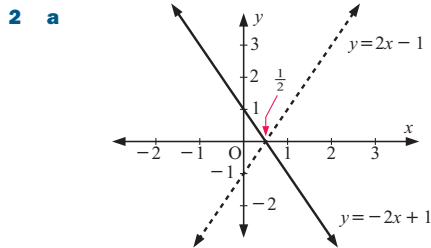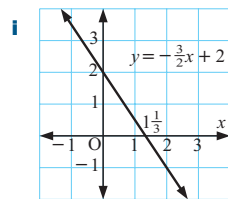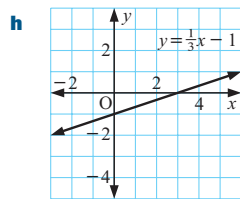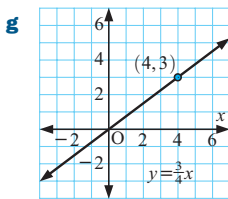
b a square
c $x = 1$, $y = 0$,
$y = x - 1$,
$y = -x + 1$

3 a R is at $(7, -2)$ **b** $(4, \frac{1}{2})$ and $(4, \frac{1}{2})$ **c** $x = 4$, $y = \frac{1}{2}$
d The diagonals of a rectangle bisect each other.

4

a gradient of OA = 4, gradient of BC = 4 $\therefore$ OA $\parallel$ BC
OA = BC = $\sqrt{17}$ units
b gradient of OA = 4
gradient of AB = $-\frac{1}{4}$ and $4 \times -\frac{1}{4} = -1$
$\therefore$ OA and AB are perpendicular.
c From **a**, a pair of opposite sides ara parallel and equal in
length, $\therefore$ OABC is a parallelogram.
From **b**, angle OAB is a right angle
$\therefore$ the parallelogram is a rectangle.
d $(\frac{1}{2}, 2)$ and $(5, 3)$ **e** $2x + 8y = 17$ and $4x - y = 17$

5 a
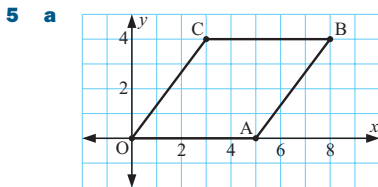
b $y = 4$ **c** OA = AB = BC = CO = 5 units
$\therefore$ OABC is a rhombus
d $(4, 2)$ and $(4, 2)$
e Diagonals of a rhombus bisect each other
f $y = \frac{1}{2}x$ and $2x + y = 10$

6 a gradient of AC = $\frac{k}{3}$, gradient of BC = $-\frac{k}{3}$
b As the right angle must be at C, $\frac{k}{3} = \frac{3}{k}$
$\therefore$ $k^2 = 9$ and so $k = 3$ {as $k > 0$}
c isosceles   **d** $x = 0$
7 a C is at $(0, 2\sqrt{3})$ **b** M is at $(1, \sqrt{3})$ and N$(-1, \sqrt{3})$
c $x = 0$, $\sqrt{3}x - 3y = -2\sqrt{3}$, $\sqrt{3}x + 3y = 2\sqrt{3}$

REVIEW SET 14A

1 a $x = -1$ **b** $-\frac{4}{5}$ **c** x-int = 3, y-int = 2, $m = -\frac{2}{3}$
d $8x - 3y = -7$
2 $y = -\frac{1}{2}x + 4$ (or $x + 2y = 8$)   **3** $y = 3x - 5$
4 $y = -2x + 3$ **5** $2x - 3y = -18$ (or $y = \frac{2}{3}x + 6$)
6

7 $k = -11$
8 $y = -4$
9 $2x - 3y = 18$

10

11
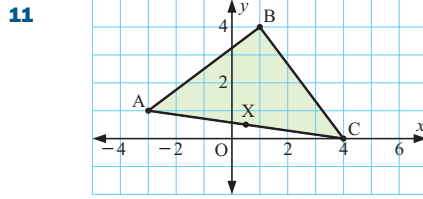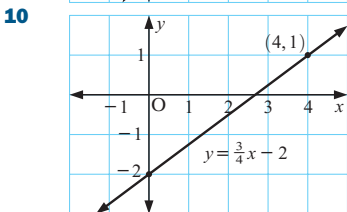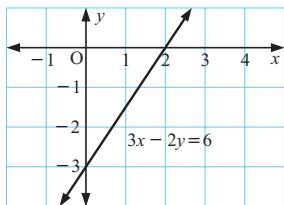
a AB = BC = 5 units $\therefore$ $\triangle$ABC is isosceles
b X is at $(\frac{1}{2}, \frac{1}{2})$
c gradient of BX = 7, gradient of AC = $-\frac{1}{7}$
$\therefore$ BX and AC are perpendicular.
d BX is a line of symmetry with equation $y = 7x - 3$.

12
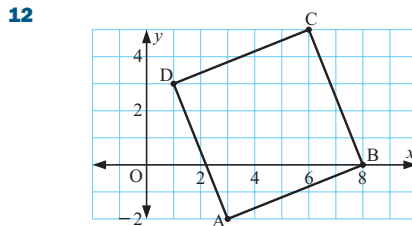
a gradient of AB = $\frac{2}{5}$, gradient of DC = $\frac{2}{5}$, $\therefore$ AB $\parallel$ DC.
Also AB = CD = $\sqrt{29}$ units
b gradient of AB = $\frac{2}{5}$, gradient of BC = $-\frac{5}{2}$, and as these
are negative reciprocals, AB is perpendicular to BC.
Also AB = BC = $\sqrt{29}$ units.
c ABCD is a square
d $7x - 3y = 27$, $3x + 7y = 24$, $4x - 10y = 3$,
$10x + 4y = 51$

REVIEW SET 14B

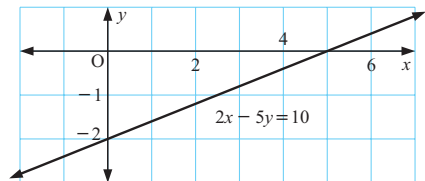**1 a** $y = 0$ **b** $m = -2, c = 5$ **c** $4x + 7y = 6$
2 $y = 2x + 2$
3 a $y = -2x + 7$ **b** $y = -\frac{2}{3}x + 2\frac{1}{3}$ **c** $y = \frac{3}{2}x - 7\frac{1}{2}$
4 $k = -1$
5 $P = -\frac{2}{3}t + 3\frac{1}{3}$
6 x-intercept is 5,
y-intercept is -2

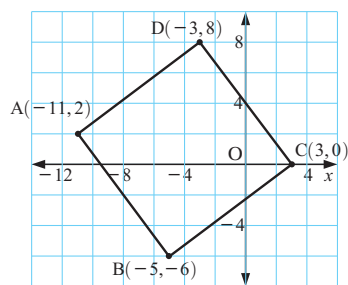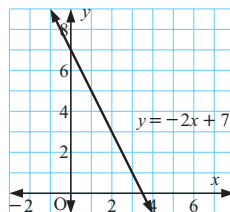**7 a** $y = 3$ **b** $x = -1$ **c** $y = 1.6x + 20$
8 $m = -\frac{4}{3}$ **9 a** $m = 2$ **b** $y = -\frac{1}{2}x + 3$
10

x	0	1	2	3
y	7	5	3	1

11 $5x + 3y = 19$
12 a

b AB = BC = CD = DA = 10 units

c gradient of AB $= -\frac{4}{3}$ gradient of BC $= \frac{3}{4}$

∴ AB and BC are perpendicular.

From **b**, ABCD is a rhombus and from **c**, one angle is a right
angle. ∴ ABCD is a square.

d $x + 7y = 3$, $7x - y = -29$, $4x + 3y = -13$,
$3x - 4y = -16$

e midpoint is $(-4, 1)$ **f** $x + 7y = -4 + 7 = 3$ ✓
$7x - y = -28 - 1 = -29$ ✓
$4x + 3y = -16 + 3 = -13$ ✓
$3x - 4y = -12 - 4 = -16$ ✓

EXERCISE 15A

1 **a** **i** BC **ii** AC **iii** AB **b** **i** KM **ii** KL **iii** LM
c **i** PR **ii** QR **iii** PQ **d** **i** XZ **ii** XY **iii** YZ
e **i** CE **ii** DE **iii** CD **f** **i** ST **ii** RT **iii** RS

2 **a** **i** PR **ii** QR **iii** PQ **iv** PQ **v** QR
b **i** AC **ii** AB **iii** BC **iv** BC **v** AB

EXERCISE 15B.1

1 **a** **i** $\frac{p}{r}$ **ii** $\frac{q}{r}$ **iii** $\frac{p}{q}$ **iv** $\frac{q}{r}$ **v** $\frac{p}{r}$ **vi** $\frac{q}{p}$

b **i** $\frac{y}{x}$ **ii** $\frac{z}{x}$ **iii** $\frac{y}{z}$ **iv** $\frac{z}{x}$ **v** $\frac{y}{x}$ **vi** $\frac{z}{y}$

c **i** $\frac{4}{5}$ **ii** $\frac{3}{5}$ **iii** $\frac{4}{3}$ **iv** $\frac{3}{5}$ **v** $\frac{4}{5}$ **vi** $\frac{3}{4}$

d **i** $\frac{4}{7}$ **ii** $\frac{\sqrt{33}}{7}$ **iii** $\frac{4}{\sqrt{33}}$ **iv** $\frac{\sqrt{33}}{7}$ **v** $\frac{4}{7}$ **vi** $\frac{\sqrt{33}}{4}$

e **i** $\frac{5}{\sqrt{34}}$ **ii** $\frac{3}{\sqrt{34}}$ **iii** $\frac{5}{3}$ **iv** $\frac{3}{\sqrt{34}}$ **v** $\frac{5}{\sqrt{34}}$ **vi** $\frac{3}{5}$

f **i** $\frac{7}{\sqrt{65}}$ **ii** $\frac{4}{\sqrt{65}}$ **iii** $\frac{7}{4}$ **iv** $\frac{4}{\sqrt{65}}$ **v** $\frac{7}{\sqrt{65}}$ **vi** $\frac{4}{7}$

2 **a** $\sin 70^o = \frac{x}{a}$ **b** $\sin 35^o = \frac{x}{b}$ **c** $\tan 64^o = \frac{x}{c}$

d $\cos 40^o = \frac{d}{x}$ **e** $\cos 49^o = \frac{x}{e}$ **f** $\tan 73^o = \frac{f}{x}$

g $\cos 54^o = \frac{g}{x}$ **h** $\tan 30^o = \frac{h}{x}$ **i** $\sin 68^o = \frac{i}{x}$

3 **a** $x \approx 15.52$ **b** $x \approx 12.99$ **c** $x \approx 9.84$
d $x \approx 6.73$ **e** $x \approx 11.86$ **f** $x \approx 22.94$
g $x \approx 24.41$ **h** $x \approx 16.86$ **i** $x \approx 5.60$
j $x \approx 16.37$ **k** $x \approx 22.66$ **l** $x \approx 10.43$

4 **a** $\theta = 62^o$, $a \approx 10.6$, $b \approx 5.63$
b $\theta = 27^o$, $a \approx 16.8$, $b \approx 7.64$
c $\theta = 65^o$, $a \approx 49.65$, $b \approx 21.0$

EXERCISE 15B.2

1 **a** 56.3^o **b** 34.8^o **c** 48.2^o **d** 34.8^o **e** 41.1^o
f 48.6^o **g** 25.3^o **h** 37.1^o **i** 35.5^o

2 **a** $x \approx 6.24$, $\theta \approx 38.7$, $\phi \approx 51.3^o$ **3** All check ✓
b $x \approx 5.94$, $\alpha \approx 53.9^o$, $\beta \approx 36.1^o$
c $x \approx 7.49$, $a \approx 38.4^o$, $b \approx 51.6^o$

4 **a** $x \approx 2.65$ **b** $\theta \approx 41.4^o$ **c** $x \approx 10.1$
d $x \approx 4.21$, $\theta = 59^o$ **e** $\theta \approx 56.3^o$, $\phi \approx 33.7^o$
f $x \approx 12.2$, $y \approx 16.4$ **g** $x = 12$, $\theta \approx 33.7^o$
h $x \approx 13.1$, $\theta \approx 66.6^o$ **i** $x \approx 14.3$, $\theta \approx 38.9^o$

5 The 3 triangles do not exist.

EXERCISE 15C

1 110 m **2** 32.9^o **3** 238 m **4** 2.65 m
5 54.6 m **6** 761 m **7** 280 m **8** $\theta \approx 44.4^o$
9 1.91^o **10** 23.5 m **11** 73.2 m **12** 67.4^o

13 **a** 10.8 cm **b** 21.8^o **14** 106° **15** 35.3 cm^2
16 15.8 cm **17** **a** ≈ 5.03 m **b** AB ≈ 7.71 m **18** 252 m
19 14.3^o **20** $\approx 118^o$ **21** 53.2^o **22** **a** 248 m **b** 128 m
23 7.48 m **24** 163 m **25** 729 m **26** 1.66 units
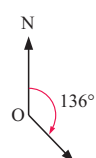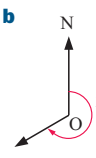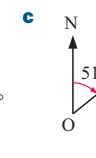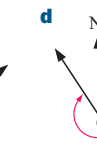**27** AB ≈ 8.66 m, BC ≈ 9.85 m, AC ≈ 6.43 m

EXERCISE 15D.1

1 **a** P$(\cos 28^o, \sin 28^o)$,   Q$(\cos 68^o, \sin 68^o)$
b P$(0.883, 0.469)$,   Q$(0.375, 0.927)$

2 **a** 0.2588 **b** 0.9659 **c** ≈ 3.73

3 **a** $\sqrt{2}$ units **b** **i** $\frac{1}{\sqrt{2}}$ **ii** $\frac{1}{\sqrt{2}}$ **iii** 1

4 **a** right angled isosceles **b** ON = NP $= \frac{1}{\sqrt{2}}$ {Pythagoras}
c P$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ **d** **i** $\frac{1}{\sqrt{2}}$ **ii** $\frac{1}{\sqrt{2}}$ **iii** 1

5 **a** A$(1, 0)$, B$(0, 1)$ **b** **i** 0 **ii** 1 **iii** undefined
c **i** 1 **ii** 0 **iii** 0

6 **a** $\triangle$ABC is equilateral, ∴ $\theta = 60^o$
$\phi = 180^o - 90^o - 60^o = 30^o$ {in $\triangle$BMC}
b **i** MC = 1 **ii** MB $= \sqrt{3}$ {Pythagoras}
c **i** $\frac{1}{2}$ **ii** $\frac{\sqrt{3}}{2}$ **iii** $\sqrt{3}$ **d** **i** $\frac{\sqrt{3}}{2}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{\sqrt{3}}$

7 **a** OP = OA = 1 {equal radii} ∴ $\triangle$OPA is isosceles.
But the included angle is 60° ∴ $\triangle$OPA is equilateral.
b **i** ON $= \frac{1}{2}$ **ii** PN $= \frac{\sqrt{3}}{2}$ {Pythagoras}
c P$(\frac{1}{2}, \frac{\sqrt{3}}{2})$ **d** **i** $\frac{1}{2}$ **ii** $\frac{\sqrt{3}}{2}$ **iii** $\sqrt{3}$
e 30° **f** **i** $\frac{\sqrt{3}}{2}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{\sqrt{3}}$

8 **a** **i** $\cos\theta$ **ii** $\sin\theta$ **iii** $\tan\theta$
b $\tan\theta$ is the length of the tangent at A to the point where the
extended radius for the angle θ meets the tangent.

EXERCISE 15D.2

1 **Hint:** Replace $\sin 30^o$ by $\frac{1}{2}$, $\cos 30^o$ by $\frac{\sqrt{3}}{2}$, $\sin 60^o$ by $\frac{\sqrt{3}}{2}$
and $\cos 60^o$ by $\frac{1}{2}$.

2 **a** $\frac{3}{4}$ **b** $\frac{1}{\sqrt{3}}$ **c** 3 **d** 2 **e** $\frac{3}{4}$ **f** $\frac{2}{3}$ **g** $\sqrt{3}$
h $\frac{1}{2}$ **i** $2\frac{1}{2}$

3 **a** $a = 6\sqrt{3}$ **b** $h = \frac{20\sqrt{3}}{3}$ **c** $c = 8\sqrt{3}$
d $d = 15\sqrt{3}$ **e** $x = 2\sqrt{3}$ **f** $y = 2$

EXERCISE 15E

1 **a** **b** **c** **d**

2 **a** 234° **b** 293° **c** 083° **d** 124°

3 **a** **i** 041° **ii** 142° **iii** 322° **iv** 099° **v** 221°
vi 279°
b **i** 027° **ii** 151° **iii** 331° **iv** 066° **v** 207°
vi 246°

4 123° **5** 7.81 km, 130° **6** 22.4 km **7** 38.6 km

8 58.3 m, 239° **9** 221 km

10 **a** 36.1 km, 057.7^o **b** 238°

EXERCISE 15F.1

1 **a** ≈ 21.2 cm   **b** $\approx 35.3^o$

2 **a** ≈ 9.43 cm   **b** $\approx 32.5^o$ **c** ≈ 10.8 cm   **d** $\approx 29.1^o$

3 **a** ≈ 8.94 cm   **b** $\approx 18.5^o$ **4** 69.2 cm   **5 a** 45°

EXERCISE 15F.2

1 **a** **i** GF   **ii** HG   **iii** HF   **iv** GM

   b **i** MA   **ii** MN

   c **i** CD   **ii** DE   **iii** DF   **iv** DX

2 **a** **i** $D\widehat{E}H$ **ii** $C\widehat{E}G$ **iii** $A\widehat{G}E$ **iv** $B\widehat{X}F$

   b **i** $P\widehat{Y}S$ **ii** $Q\widehat{W}R$ **iii** $Q\widehat{X}R$ **iv** $Q\widehat{Y}R$

   c **i** $A\widehat{Q}X$ **ii** $A\widehat{Y}X$

3 **a** **i** 45° **ii** $\approx 35.3^o$ **iii** $\approx 63.4^o$ **iv** $\approx 41.8^o$

   b **i** $\approx 21.8^o$ **ii** $\approx 18.9^o$ **iii** $\approx 21.0^o$

   c **i** $\approx 36.9^o$ **ii** $\approx 33.9^o$ **iii** $\approx 33.9^o$

   d **i** $\approx 58.6^o$ **ii** $\approx 64.9^o$
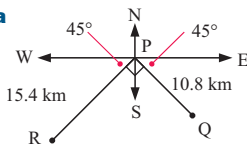
EXERCISE 15F.3

1 $\approx 54.7^o$ **2** $\approx 29.5^o$ **3** $\approx 71.7^o$ **4** $\approx 51.6^o$

REVIEW SET 15A

1 $\sin\theta = \frac{5}{13}$, $\cos\theta = \frac{12}{13}$, $\tan\theta = \frac{5}{12}$

2 **a** $x \approx 14.0$ **b** $x \approx 35.2^o$

3 $\theta = 36^o$, $x \approx 12.4$, $y \approx 21.0$ **4** 80.9 m   **5** $1\frac{3}{4}$

6 **a** 0.934   **b** 2.61   **c** $P(\frac{\sqrt{3}}{2}, \frac{1}{2})$ **7** ≈ 9.38 m

8 ≈ 22.4 km   **9 a** 56.3^o **b** $\approx 33.9^o$

10 ≈ 589 km on bearing 264°

REVIEW SET 15B

1 **a** **i** $\frac{y}{z}$ **ii** $\frac{y}{z}$ **iii** $\frac{y}{x}$ **b** **i** 0.8 **ii** 0.8 **iii** $\frac{4}{3}$

2 **a** $x \approx 38.7^o$ **b** $x \approx 37.1^o$

3 $x \approx 25.7$, $\alpha \approx 36.4^o$, $\theta \approx 53.6^o$ **4** ≈ 638 m   **5** $\approx 32.2^o$

6 **a** $\frac{1}{3}$ **b** $\frac{3}{4}$

7 **a** 84 km   **b** ≈ 48.2 km   **c** ≈ 24.1 km/h

8 **a**

               N
     $45°$ $45°$
          P
  W   E
  15.4 km   10.8 km
          S
   R   Q

   b 18.8 km   **c** 080.0^o

9 $(30 - 6\sqrt{3})$ cm ≈ 19.6 cm   **10 a** 45° **b** $\approx 54.7^o$

EXERCISE 16A.1

1 **a** $2a$ **b** $2b$ **c** $\frac{1}{2x}$ **d** 8   **e** cannot be simplified

   f $2a$ **g** $\frac{1}{2b}$ **h** 2   **i** $\frac{1}{3a^2}$ **j** $\frac{x}{2}$

   k cannot be simplified   **l** $\frac{a}{b}$ **m** cannot be simplified

   n $\frac{5x}{y}$ **o** $2ac$ **p** $4a$ **q** 1   **r** $3a^3$ **s** a^2 **t** $\frac{a}{2}$

2 **a** $\frac{x}{3} + 1$ **b** $2a + \frac{1}{2}$ **c** $\frac{a}{c} + \frac{b}{c}$ **d** $\frac{a}{b} + 2$

   e $a + 2$ **f** $a + 2b$ **g** $m + 2n$ **h** $2 + \frac{4n}{m}$

3 **e**, **f** and **g** produce simplified answers. They have common factors in the numerator and denominator.

4 **a** $x + 2$ **b** $2(x - 1)$ **c** $\frac{b + 2}{2}$ **d** $\frac{n + 5}{6}$ **e** $\frac{2}{x + 2}$

   f $\frac{3}{3 - a}$ **g** 6   **h** $\frac{1}{2}$ **i** $\frac{2}{x}$ **j** $\frac{x(x - 5)}{3}$

   k $\frac{x + 3}{2(x + 2)}$ **l** $\frac{x + 2}{5}$ **m** $\frac{x + 2}{x + 3}$ **n** $\frac{x + 5}{3}$

   o $\frac{x + 6}{3}$ **p** $\frac{x}{x - 1}$ **q** $\frac{(x + 2)(x + 1)}{4}$ **r** $\frac{(x + 2)^2}{x^2}$

EXERCISE 16A.2

1 **a** $\frac{3}{x + 2}$ **b** $x + 3$ **c** $x + 4$ **d** $\frac{x + 2}{2}$

   e $\frac{x + 4}{2}$ **f** $\frac{a + 4}{3}$ **g** $y + z$ **h** x

   i $\frac{a + c}{a - c}$ **j** $\frac{1}{2(x - 4)}$ **k** $\frac{x + 3}{6}$ **l** $\frac{x - y}{3}$

2 **a** 2   **b** $\frac{m + n}{2}$ **c** x **d** $\frac{1}{m}$ **e** $x + 2$ **f** $x + 2$

   g x **h** $\frac{1}{b + c}$ **i** $3x$ **j** $x + 3$ **k** $2x$ **l** x

3 **a** -2 **b** $-\frac{1}{2}$ **c** -1 **d** cannot be simplified

   e $-\frac{1}{2}$ **f** -3 **g** $-\frac{3}{x}$ **h** $-\frac{xy}{3}$ **i** $-3x$ **j** $\frac{2x + 3}{2}$

   k cannot be simplified   **l** $x + 2$ **m** $x - 2$ **n** $-(x + 2)$

   o $\frac{1}{x - 3}$ **p** $m - n$ **q** $-(m + n)$ **r** $\frac{3}{2 - x}$

   s $-\frac{x + 4}{x}$ **t** -1 **u** $\frac{x + y}{2y}$ **v** $-\frac{2(a + d)}{a}$

   w $\frac{4x}{x + 2}$ **x** $\frac{-3x}{x + 2}$

4 **a** $\frac{x}{x + 1}$ **b** $\frac{x + 1}{x + 2}$ **c** $\frac{x - 2}{2x}$ **d** $\frac{x + 3}{x + 4}$

   e $\frac{x - 2}{x - 5}$ **f** $\frac{x + 4}{2x}$ **g** $\frac{x - 1}{2(x - 2)}$ **h** $\frac{x + 3}{x}$

   i $\frac{2x + 1}{x + 2}$ **j** $\frac{x + 2}{x - 1}$ **k** $\frac{2x + 5}{x + 3}$ **l** $\frac{4x + 1}{x - 2}$

EXERCISE 16B

1 **a** $\frac{ab}{6}$ **b** $\frac{1}{2}$ **c** $\frac{1}{2}$ **d** $\frac{a^2}{6}$ **e** $\frac{ax}{by}$ **f** 1

   g $\frac{x^2}{3}$ **h** $\frac{2x}{y}$ **i** $\frac{1}{2n}$ **j** 3   **k** $\frac{m}{n}$ **l** 2

   m $5t$ **n** $\frac{x^2}{y^2}$ **o** $\frac{16}{d^2}$ **p** 1

2 **a** $\frac{2}{3}$ **b** $\frac{1}{2}$ **c** $3x$ **d** $3y$ **e** 3   **f** $\frac{c}{n^2}$

   g $\frac{d^2}{5}$ **h** 3   **i** $\frac{b}{a}$ **j** $\frac{3}{2d}$ **k** $\frac{8}{x^3}$ **l** $\frac{x}{2}$

   m $\frac{3}{a}$ **n** $\frac{1}{x}$ **o** $\frac{10}{a^2}$ **p** $\frac{3a}{5}$

EXERCISE 16C

1 **a** $\frac{7x}{10}$ **b** $\frac{x}{6}$ **c** $\frac{17x}{20}$ **d** $\frac{3x}{10}$ **e** $\frac{t}{12}$ **f** $\frac{8n}{21}$

   g $\frac{5a}{6}$ **h** $\frac{2a + b}{4}$ **i** $\frac{7n}{15}$ **j** $\frac{g}{2}$ **k** $\frac{2s}{15}$

   l $\frac{2a}{5}$ **m** x **n** $\frac{5y}{12}$ **o** $\frac{z}{12}$ **p** $\frac{41q}{21}$

2 a $\dfrac{3b+2a}{ab}$ **b** $\dfrac{4d+3a}{ad}$ **c** $\dfrac{5b-3a}{ab}$ **d** $\dfrac{a}{m}$

e $\dfrac{3a+b}{3y}$ **f** $\dfrac{3}{2a}$ **g** $\dfrac{3b-2}{ab}$ **h** $\dfrac{dc+ab}{ad}$

i $\dfrac{4+a}{b}$ **j** $\dfrac{2d-ac}{ad}$ **k** $\dfrac{15+x^2}{3x}$ **l** $\dfrac{pd-12}{6d}$

m $\dfrac{m^2+3n}{3m}$ **n** $\dfrac{2mn-mp}{np}$ **o** $\dfrac{17b}{20}$ **p** $\dfrac{16b}{15}$

3 a $\dfrac{x+6}{3}$ **b** $\dfrac{m-2}{2}$ **c** $\dfrac{4a}{3}$ **d** $\dfrac{b-10}{5}$

e $\dfrac{x-18}{6}$ **f** $\dfrac{12+x}{4}$ **g** $\dfrac{30-x}{6}$ **h** $\dfrac{2x+3}{x}$

i $\dfrac{6x-3}{x}$ **j** $\dfrac{b^2+3}{b}$ **k** $\dfrac{5+x^2}{x}$ **l** $-\dfrac{11y}{6}$

4 a $\dfrac{14x}{15}$ **b** $\dfrac{11x}{35}$ **c** $\dfrac{11}{2a}$ **d** $\dfrac{21}{4y}$

e $\dfrac{3c+4b}{bc}$ **f** $\dfrac{5b-24a}{4ab}$ **g** $\dfrac{x+30}{10}$ **h** $\dfrac{12-x}{3}$

EXERCISE 16D.1

1 a $\dfrac{9x-4}{20}$ **b** $\dfrac{5x+10}{6}$ **c** $\dfrac{20x-7}{42}$ **d** $\dfrac{a+5b}{6}$

e $\dfrac{13x-9}{20}$ **f** $\dfrac{5x+11}{14}$ **g** $\dfrac{x+15}{30}$ **h** $\dfrac{x-7}{42}$

i $\dfrac{2-3x}{10}$ **j** $\dfrac{3x-1}{12}$ **k** $\dfrac{2x+1}{15}$ **l** $\dfrac{25x+5}{24}$

2 a $\dfrac{5x-1}{(x+1)(x-2)}$ **b** $\dfrac{12x+17}{(x+1)(x+2)}$

c $\dfrac{x+14}{(x-1)(x+2)}$ **d** $\dfrac{-6}{(x+2)(2x+1)}$

e $\dfrac{7x+8}{(x-1)(x+4)}$ **f** $\dfrac{3(5x+2)}{(1-x)(x+2)}$

g $\dfrac{4x+3}{x(x+1)}$ **h** $\dfrac{3(x+5)}{x(x+3)}$ **i** $\dfrac{x^2-x+6}{(x+2)(x-4)}$

j $\dfrac{2(x-1)}{x-3}$ **k** $\dfrac{2(x-1)}{x+2}$ **l** $\dfrac{2x^2+4x-3}{(x+3)(x+2)}$

m $\dfrac{7}{(2x-1)(x+3)}$ **n** $\dfrac{17x-7}{x(3x-1)}$

o $\dfrac{-5x-2}{(x+2)(x-2)}$ **p** $\dfrac{x+2}{x(x+1)}$

q $\dfrac{x^2+1}{x(x-1)(x+1)}$ **r** $\dfrac{4x^2-x-9}{(x+1)(x-1)(x+2)}$

s $\dfrac{2x^3-x^2+1}{x(x+1)(x-1)}$

3 a $\dfrac{6}{x+3}$ **b** $1\frac{1}{4}$ **c** $\dfrac{4}{x-4}$ **d** 5

e $\dfrac{x-1}{x+2}$ **f** $\dfrac{x-2}{x}$ **g** $\dfrac{x-1}{3x+5}$ **h** $\dfrac{2x+3}{x-1}$

4 a $\cos\theta=\dfrac{x+2}{x^2}$, $\sin\theta=\dfrac{x-1}{x^2}$, $\tan\theta=\dfrac{x-1}{x+2}$

b $\sin\theta\div\cos\theta=\dfrac{x-1}{x^2}\div\dfrac{x+2}{x^2}=\dfrac{x-1}{x^2}\times\dfrac{x^2}{x+2}=\dfrac{x-1}{x+2}$

EXERCISE 16D.2

1 a $\dfrac{2+x}{x(x+1)}$ **b** $\dfrac{2+x^2}{x(x+1)}$ **c** $\dfrac{2(x^2+2x+2)}{(x+2)(x-3)}$

d $\dfrac{2(x+5)}{x+2}$ **e** $\dfrac{x^2-2x+3}{(x-2)(x+3)}$ **f** $\dfrac{x-5}{x-2}$

---

g $\dfrac{2(x-5)}{x-1}$ **h** $\dfrac{x+14}{x+7}$

2 a $\dfrac{2(x+1)^2}{(x+2)(x-3)}$ **b i** $x=-2$ or 3 **ii** $x=-1$

3 a $\dfrac{-2}{x-2}$ **b** $\dfrac{2}{x+4}$ **c** $\dfrac{x-2}{x+2}$

d $\dfrac{x-6}{2-x}$ **e** $\dfrac{-(x+2)}{4x^2}$ **f** $\dfrac{12-x}{16x^2}$

4 a $\dfrac{x+3}{3(x+1)}$ **b i** $x=-1$ or 2 **ii** $x=-3$

REVIEW SET 16A

1 a $3x$ **b** $3n$ **c** $\dfrac{x}{6}$ **d** $\dfrac{2}{x}$

2 a $\dfrac{2}{c+3}$ **b** cannot be simplified **c** $x+2$ **d** $\dfrac{x}{3(x+2)}$

3 a $\dfrac{19x}{15}$ **b** $\dfrac{2x^2}{5}$ **c** $\dfrac{10}{9}$ **d** $\dfrac{x}{15}$

4 a 4 **b** -5 **c** $2x$

5 a $\dfrac{11x+1}{12}$ **b** $\dfrac{16x-9}{14}$ **c** $\dfrac{3x+2}{x(x+2)}$

6 a $\dfrac{-2}{x+4}$ **b** $\dfrac{x+3}{x}$ **c** $\dfrac{2x+1}{3x+2}$

7 a $\dfrac{2(x-2)}{x+1}$ **b i** $x=-3$ or -1 **ii** $x=2$

REVIEW SET 16B

1 a $\frac{2}{3}$ **b** $2x$ **c** $3n$ **d** $2x$

2 a cannot be simplified **b** $x+5$ **c** $\dfrac{2}{a+4}$ **d** $\dfrac{b}{2(b-a)}$

3 a $\dfrac{11x}{4}$ **b** $\dfrac{-5x}{4}$ **c** $\dfrac{3x^2}{2}$ **d** $\frac{3}{8}$

4 a -1 **b** $\frac{5}{2}$ **c** $\dfrac{3x}{a}$

5 a $\dfrac{13x-5}{15}$ **b** $-\dfrac{5x+3}{6}$ **c** $\dfrac{x+6}{2x(x+2)}$

6 a $2(x-2)$ **b** $\dfrac{x-7}{x-2}$ **c** $\dfrac{3x+1}{4x+1}$

7 a $\dfrac{-3(x-4)}{x-2}$ **b i** $x=\pm2$ **ii** $x=4$

8 a $\dfrac{-10}{x-1}$ **b** $\dfrac{x+3}{x}$

EXERCISE 17A

1 a The variable can take any value in the continuous range 75 to 105.

b $85\leqslant w<90$. This class has the highest frequency.

c symmetrical **d** ≈ 89.5 kg

2 a $40\leqslant h<60$ **b** ≈ 69.6 mm **c** 46 of them **d** 30%

e i ≈ 754 **ii** ≈ 686

3 a $1\leqslant d<2$ **b** ≈ 1.66 km **c** 33.5% **d** 9 students

4 a 32.5% **b** ≈ 22.9 min **c** ≈ 1490 people

EXERCISE 17B

1 a

b $8 \leqslant t < 12$
c ≈ 7.26 min
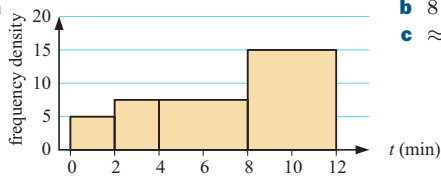
2 a

b $65 \leqslant m < 70$
c ≈ 61.4 kg
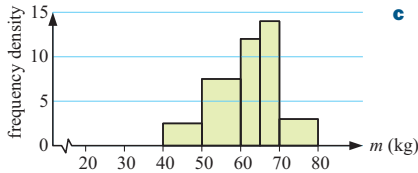
3 a

b $35 \leqslant d < 45$
c ≈ 41.1 m
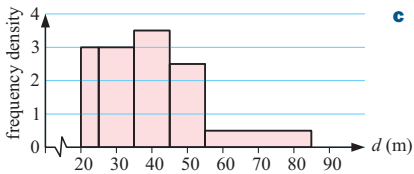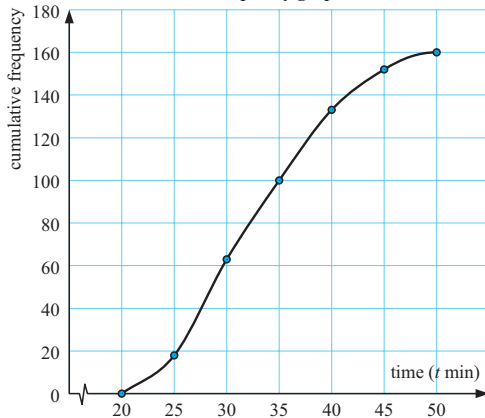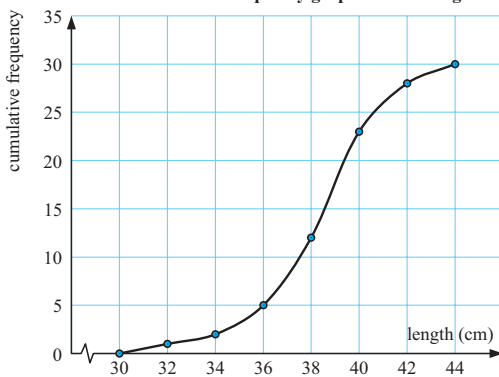
4 430 employees

EXERCISE 17C

1

Cumulative frequency graph of race data
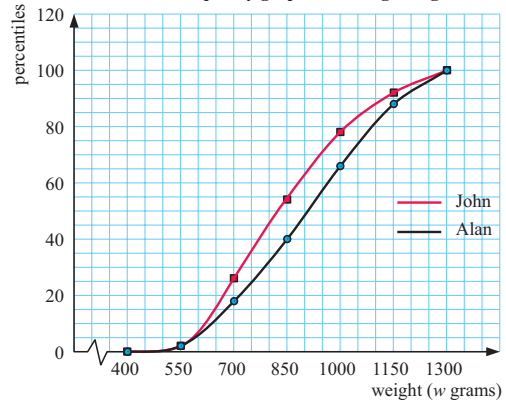
a 32 min **b** 80 **c** 28 min **d** IQR $= 10$ min
2 a 120 **b** 29 kg **c** 48 **d** 17.5 kg
**3 a**

Cumulative frequency graph of trout length

b 57%   **c** 38.6 cm   **d** 3.0 cm
e 37.6 cm. 35% of the lengths are less than or equal to this value.
f ≈ 38.3 cm   **g** negatively skewed

4 a

Cumulative frequency graph of cabbage weight data

b **i** Alan, ≈ 910; John, ≈ 830
    ii Alan, ≈ 310; John, ≈ 290
c Alan, ≈ 970; John, ≈ 890
d Alan's cabbages are generally heavier than John's. The spread of each data set is about the same.

5 a

Cumulative frequency graph of Ben Nevis climb data
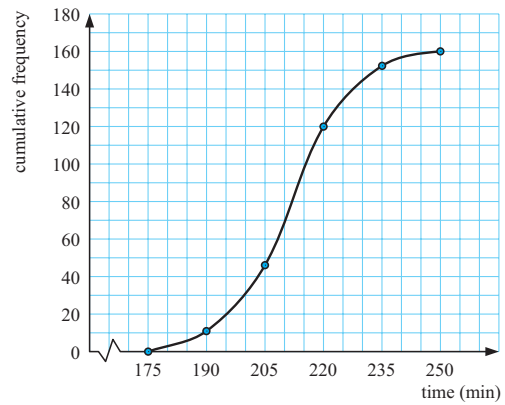
b ≈ 210 min
c $17\frac{1}{2}$ min. This is the length of time in which the middle 50% of the data lies.
d ≈ 20 **e** symmetrical

6 a **i** 4 kg
    ii 2.0 kg

c ≈ 4.25 kg

b

Weight (w grams)	Freq.	Cum. Freq.
$0 \leqslant w < 1$	1	1
$1 \leqslant w < 2$	2	3
$2 \leqslant w < 3$	5	8
$3 \leqslant w < 4$	12	20
$4 \leqslant w < 5$	8	28
$5 \leqslant w < 6$	6	34
$6 \leqslant w < 7$	3	37
$7 \leqslant w < 8$	2	39
$8 \leqslant w < 9$	1	40

REVIEW SET 17A

1 a The variable can take any value in the continuous range $48 \leqslant m < 53$ grams.

   b $50 \leqslant m < 51$ **c** ≈ 50.8 grams

   d slightly negatively skewed

2 a ≈ 54.9 km/h   **b** $55 \leqslant v < 60$ **c** $\approx 24.3\%$

   d symmetrical

3 a   **b** $0.5 \leqslant C < 1$
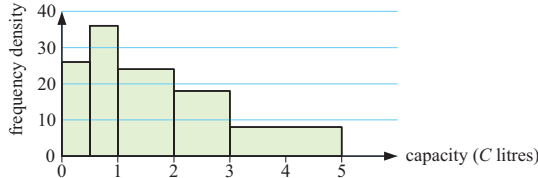
4 a   **b** $20 \leqslant h < 30$

   c ≈ 34.0 cm

5 a
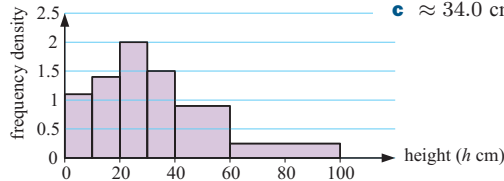
Cumulative frequency graph of wage data

   b i $\approx £950$ **ii** $\approx £1200$

6 a
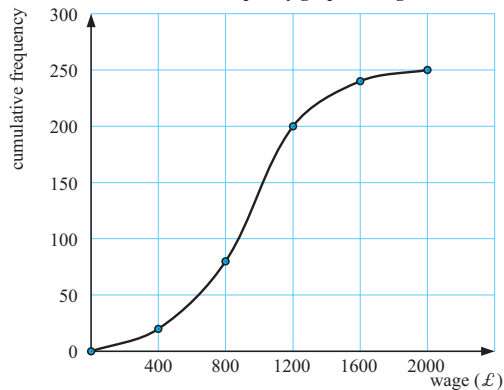
Time (t min)	Freq.	Cum. Freq.
$0 \leqslant t < 5$	2	2
$5 \leqslant t < 10$	3	5
$10 \leqslant t < 15$	5	10
$15 \leqslant t < 20$	10	20
$20 \leqslant t < 25$	20	40
$25 \leqslant t < 30$	15	55
$30 \leqslant t < 35$	5	60
$35 \leqslant t < 40$	10	70
$40 \leqslant t < 45$	6	76
$45 \leqslant t < 50$	4	80

   b i 25 min
   ii 15 min
   iii 37 min

   c i 27
   ii 18

REVIEW SET 17B

1 a 2 kg   **b** $2 \leqslant m < 4$ **c** ≈ 4.76

d

Histogram of masses of cats   **e** 377 of them
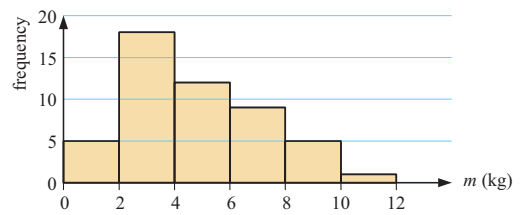
2 a ≈ 39.1 sec   **b** $35 \leqslant t < 40$

3 a

Histogram of long jump data

   b $5 \leqslant d < 5.5$

4 a
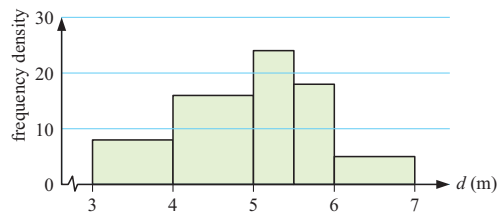
Area (A m²)	Frequency
$300 \leqslant A < 500$	20
$500 \leqslant A < 600$	20
$600 \leqslant A < 700$	35
$700 \leqslant A < 800$	25
$800 \leqslant A < 1100$	45

   b ≈ 712 m²

5 a

UE point	CF (boys)	CF (girls)	Percentiles B	Percentiles G
10	5	0	3.1	0.0
20	3	4	8.1	3.3
30	25	12	15.6	10.0
40	35	22	21.9	18.3
50	75	37	46.9	30.8
60	125	62	78.1	51.7
70	145	102	90.6	85
80	155	112	96.9	93.3
90	160	117	100.0	97.5
100	160	120	100.0	100.0

Cumulative frequency graph of test scores
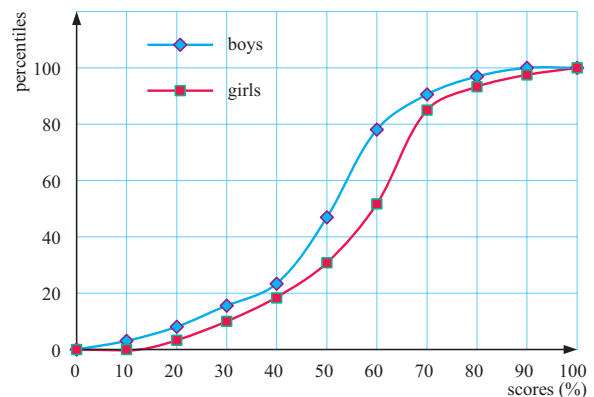
   b For boys: medium ≈ 52,   IQR ≈ 19
   For girls: medium ≈ 59,   IQR ≈ 22

   c As the girls graph is further to the right of the boys graph, the girls are outperforming the boys. Both distributions are negatively skewed.

6 a

Histogram of lengths of newborn babies

b 27 babies
c 70% of them

d
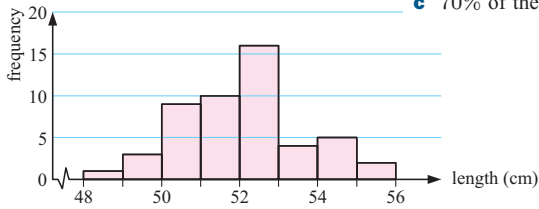
Upper end point	Cumulative frequency
49	1
50	4
51	13
52	23
53	39
54	43
55	48
56	50

e

Cumulative frequency graph of lengths of newborns
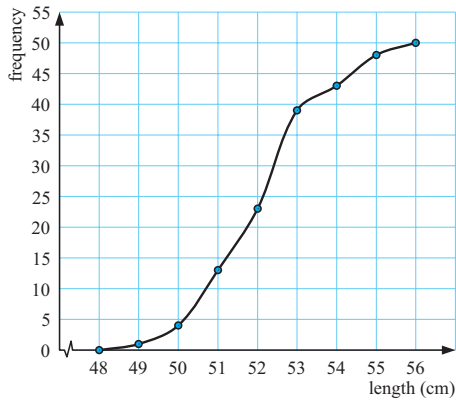
f **i** ≈ 52.1 cm   **ii** 18 babies

EXERCISE 18A

1 a $x = 0.8$ **b** $x \approx 1.13$ **c** $x \approx 0.706$
2 a $x = 8$ **b** $x = 8.75$ **c** $x = 4.8$ **d** $x \approx 3.18$
3 a 6 cm   **b** $k = 1.5$
4 a true   **b** false, e.g.,

   c true   **d** false, e.g.,

5 They are **not** similar.
   Comparing lengths; $k = \frac{16}{12} = \frac{4}{3}$
   Comparing widths; $k = \frac{12}{8} = \frac{3}{2}$
   and $\frac{4}{3} \neq \frac{3}{2}$.
6 FG $= 2.4$ m
7 a

   b

8 No, any two equiangular triangles are similar.

EXERCISE 18B.1

1 All of these figures have triangles which can be shown to be equiangular and therefore are similar.
   For example, in **a**, $\triangle CBD$ is similar to $\triangle CAE$ as they share an equal angle at C and $\widehat{CBD} = \widehat{CAE} = 90^o$.

EXERCISE 18B.2

1 a $x = 2.4$ **b** $x \approx 2.8$ **c** $x \approx 3.27$ **d** $x = 9.6$
   e $x = 11.2$ **f** $x = 5$ **g** $x \approx 6.67$ **h** $x = 7$
   i $x = 7.2$

EXERCISE 18C

1 a 7 m   **b** 7.5 m   **2** 1.8 m   **3** 2.67 m   **4** 1.52 m
5 1.44 m   **6** 9 seconds   **7** ≈ 117 m   **8** 1013 m
9 a SU $= 5.5$ m,   BC $= 8.2$ m
   b No, the ball's centre is ≈ 11 cm on the D side of C.

EXERCISE 18D

1 a $x = 18$ **b** $x = 6$ **c** $x = 5$ **d** $x \approx 4.38$
2 a $k = 4$ **b** 20 cm and 24 cm   **c** area A : area B $= 1 : 16$
3 a $k = 2.5$ **b** 100 cm^2 **c** 84 cm^2 **4** ED ≈ 1.45
5 a $V = 80$ **b** $V = 40.5$ **c** $x \approx 5.26$ **d** $x = 8$
6 6750 cm^3 **7 a** 4 cm   **b** 648 cm^3 **8** 6 cm^2
9 a $k = \frac{2}{3}$ **b** 14 850 cm^3 **c** 280 cm^2
10 No.   Comparing capacities, $k \approx 1.37$
        Comparing lengths, $k = 1.6$
   These values should be the same if the containers are similar.
11 a 2.5 m   **b** 16 : 25   **c** 64 : 125

REVIEW SET 18A

1

2 a $x \approx 1.71$ **b** $x \approx 1.83$ **3** $x = 2.8$
4 **Hint:** Carefully show that triangles are equiangular, giving reasons.
5 a $x \approx 6.47$ **b** $x = 2\sqrt{6} \approx 4.90$
6 a $A = 7$ **b** $x \approx 8.14$ **7 a** $x = 15$ **b** $y = 32$
8 ≈ 66.7 m wide   **9 a** $k = 4$ **b** 0.99 m   **c** 0.5 m^3

REVIEW SET 18B

1

2 a $k = \frac{7}{5}$ **b** 49 : 25
3 a $x = 3$ **b** $x = 4$ **c** $x = 12$
4 a $\widehat{BAC} = \widehat{NMC} = 90^o$ {given}
      $\angle C$ is common to both $\therefore$ $\triangle$s ABC and MNC are equiangular, i.e., similar.
   b $\frac{x}{8} = \frac{6}{15}$ $\therefore$ $x = \frac{48}{15} = 3.2$ **c** 6.4 cm
5 a $x = 4$ **b** $x \approx 42.7$ **6 a** $x = 3.6$ **b** $y = 6.4$
7 a **Hint:** Explain carefully, with reasons, why they are equiangular.
   b CD $= 7.2$ cm   **c** 22.4 cm^2
8 $2\sqrt{13} \approx 7.21$ cm by $3\sqrt{13} \approx 10.8$ cm   **9** 648 cm^3

CHALLENGE

1 ≈ 17.1 m   **2** 3.75 m

EXERCISE 19A

1 **a**

$$\begin{array}{ccc} -2 & \to & -9 \\ 0 & \to & -5 \\ 7 & \to & 9 \\ x & & y \end{array}$$

one-one

b

$$\begin{array}{ccc} 0 & & 0 \\ 1 & & 1 \\ 2 & & 2 \\ 3 & & 3 \\ x & & y \end{array}$$

many-many

c

$$\begin{array}{ccc} 0 & & 0 \\ 1 & & 1 \\ 4 & & -1 \\ & & 2 \\ & & -2 \\ x & & y \end{array}$$

one-many

d

$$\begin{array}{ccc} 0 & & 0 \\ 1 & & 1 \\ 2 & & 2 \\ 3 & & 3 \\ x & & y \end{array}$$

many-many

e

$$\begin{array}{ccccccccccc} -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ & & & & & & & & & \\ -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

one-one
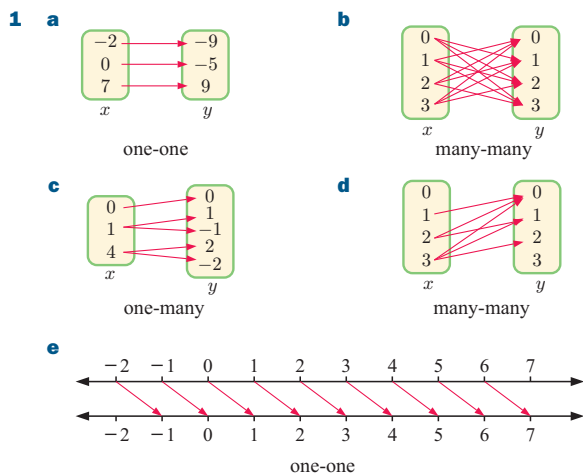
2 **a** {real numbers}
   b {multiples of 2 which are not multiples of 4}
   c {positive real numbers}   **d** {real numbers $\geqslant 10$}
   e {all integers}

EXERCISE 19B.1

1 **a** Domain $= \{-3, -2, -1, 0, 6\}$ Range $= \{-1, 3, 4, 5, 8\}$.
   b Domain $= \{-3, -1, 0, 2, 4, 5, 7\}$ Range $= \{3, 4\}$.

2 **a** Domain is $\{x \mid x > -4\}$.   Range is $\{y \mid y > -2\}$.
      Is a function.
   b Domain is $\{x \mid x = 2\}$.   Range is $\{y \mid y \text{ is in } \mathbb{R}\}$.
      Is not a function.
   c Domain is $\{x \mid -3 \leqslant x \leqslant 3\}$. Range is $\{y \mid -3 \leqslant y \leqslant 3\}$.
      Is not a function.
   d Domain is $\{x \mid x \text{ is in } \mathbb{R}\}$.   Range is $\{y \mid y \leqslant 0\}$.
      Is a function.
   e Domain is $\{x \mid x \text{ is in } \mathbb{R}\}$.   Range is $\{y \mid y = -5\}$.
      Is a function.
   f Domain is $\{x \mid x \text{ is in } \mathbb{R}\}$.   Range is $\{y \mid y \geqslant 1\}$.
      Is a function.
   g Domain is $\{x \mid x \text{ is in } \mathbb{R}\}$.   Range is $\{y \mid y \leqslant 4\}$.
      Is a function.
   h Domain is $\{x \mid x \geqslant -5\}$.   Range is $\{y \mid y \text{ is in } \mathbb{R}\}$.
      Is not a function.
   i Domain is $\{x \mid x \neq 1, \ x \text{ is in } \mathbb{R}\}$.
      Range is $\{y \mid y \neq 0, \ y \text{ is in } \mathbb{R}\}$.
      Is a function.

3 **a** $\{2, 3, 5, 10, 12\}$ **b** $\{0, \frac{1}{2}, 2\}$
   c $\{y \mid -3 < y < 5\}$ **d** $\{y \mid -27 \leqslant y \leqslant 64\}$

4 **a** **i**

   ii Range is
      $\{y \mid -5 \leqslant y \leqslant 7\}$.

   b **i**

   ii Range is
      $\{y \mid 0 \leqslant y \leqslant 16\}$.

c **i**

   ii Range is
      $\{y \mid -9 \leqslant y \leqslant 11\}$.

d **i**

   ii Range is
      $\{y \mid y \leqslant -1 \ \text{ or } \ y \geqslant \frac{1}{2}\}$.

e **i**

   ii Range is
      $\{y \mid y \leqslant -2 \ \text{ or } \ y \geqslant 2\}$.

f **i**

   ii Range is
      $\{y \mid y \geqslant 1\}$.

g **i**

   ii Range is
      $\{y \mid y \in \mathbb{R}\}$
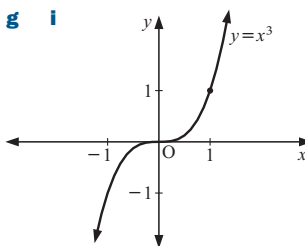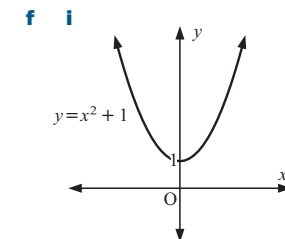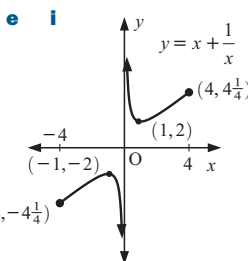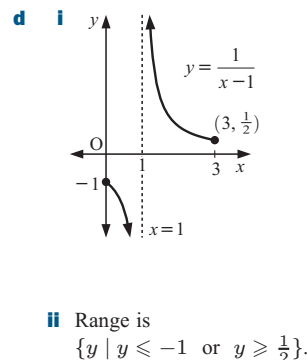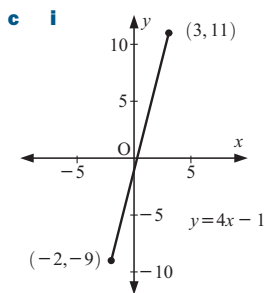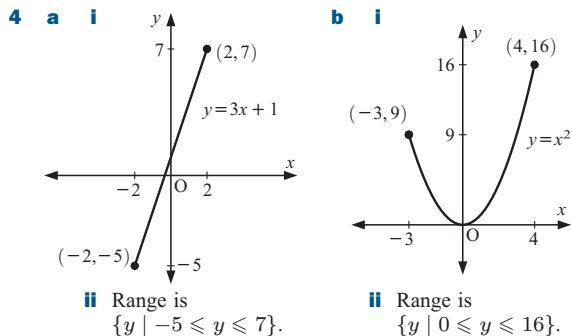
EXERCISE 19B.2

1 **a**, **b**, **e** are functions as no two ordered pairs have the same x-coordinate.

2 **a**, **b**, **d**, **e**, **g**, **h**, **i** are functions.

3 No, a vertical line is not a funtion as it does not satisfy the vertical line test.

EXERCISE 19C

1 **a** $f(5) = 8$ which means that 5 is mapped onto 8 and $(5, 8)$ lies on the graph of the function f.
   b $g(3) = -6$ which means that 3 is mapped onto -6 and $(3, -6)$ lies on the graph of the function g.
   c $H(4) = 4\frac{1}{3}$ which means that 4 is mapped onto $4\frac{1}{3}$ and $(4, 4\frac{1}{3})$ lies on the graph of the function H.

2 **a** **i** 5   **ii** -7 **iii** 21   **iv** -395
   b **i** 6   **ii** 4   **iii** -4 **iv** 10
   c **i** -2 **ii** $2\frac{1}{2}$ **iii** $1\frac{3}{7}$ **iv** $\frac{7}{10}$

3 **a** **i** $f(4) = -11$ **ii** $x = \pm 2$ **b** **i** $\frac{1}{2}$ **ii** $a = 19$
   c **i** $x = \pm\sqrt{3}$ **ii** $x = \pm 2$ **d** $x = -8$

4 **a** $V(4) = \$12\,000$, the value of the car after 4 years.
   b $t = 5$;   the car is worth $\$8000$ after 5 years.   **c** $\$28\,000$
   d No, as when $t > 7$, $V(t) < 0$ which is not valid.

5

Range is
$\{y \mid -7 \leqslant y \leqslant 1\}$.

6
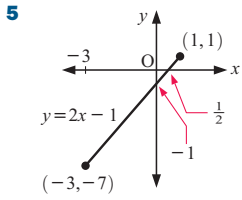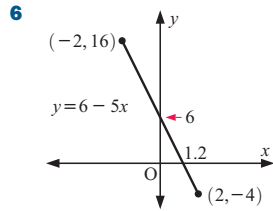
Range is
$\{y \mid -4 \leqslant y \leqslant 16\}$.

7 a $f(2) = -3$ **b** $x \approx -1.4$, 2 or 3.4

8 a $5 - 2a$ **b** $5 + 2a$ **c** $3 - 2a$ **d** $11 - 2x$ **e** $5 - 4x$

9 a $x^2 + 8x + 9$ **b** $x^2 - 6x + 2$ **c** $x^2 - 4x - 3$
  d $x^4 + 4x^2 - 3$ **e** $x^4 + 6x^2 + 2$

EXERCISE 19D

1 a 10 **b** 3 **c** 10 **d** 17 **e** 101 **f** 3

2 a $21, -8$ **b** **i** $21 - 10x$ **ii** $2 - 10x$

3 a $0, x^2 - 2x$ **b** $6, x^2 + 2x - 2$

4 a $2 - 3x$ **b** $6 - 3x$ **c** $9x - 16$ **d** x

5 a $x^2 + 5x$ **b** **i** 6 **ii** 36 **iii** -6

6 a $\sqrt{4x - 3}$ **b** $16x - 15$

7 a $f(x) = \sqrt{x}, \ g(x) = x - 3$ **b** $f(x) = x^3, \ g(x) = x + 5$
  c $f(x) = \dfrac{5}{x}, \ g(x) = x + 7$
  d $f(x) = 3 - 4x, \ g(x) = \dfrac{1}{\sqrt{x}}$
  e $f(x) = 2x + 1, \ g(x) = 3^x$ **f** $f(x) = x - 1, \ g(x) = \dfrac{4}{x^2}$

8 $x = 3$ **9** $g(x) = x^2 + 2$ **10** $f(x) = x^2 - 3$

11 $a = 2, \ b = -5$ or $a = -2, \ b = 15$

EXERCISE 19E

1 a when x or $y = 0$, $xy = 0 \neq 5$
  b vertical asymptote $x = 0$, horizontal asymptote $y = 0$
  c **i** $y = 0.01$ **ii** $y = -0.01$
  d **i** $x = 0.01$ **ii** $x = -0.01$
  e $y = \dfrac{5}{x}$ **f** $y = -\dfrac{5}{x}$
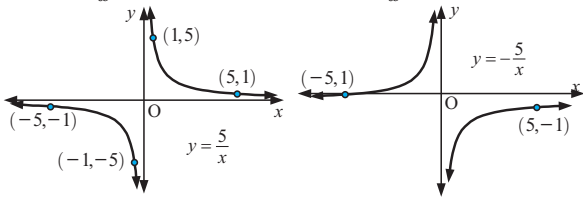
2 a

n	4	8	12	16	20	24	28
t	10	5	3.3	2.5	2	1.7	1.4

b
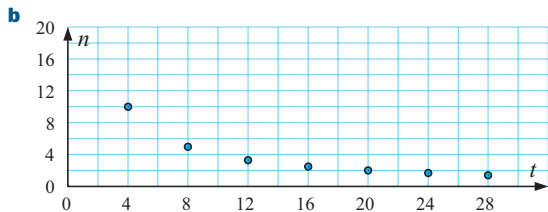
c Yes, one part of a hyperbola. **d** $t = \dfrac{40}{n}$ or $nt = 40$

3 a $y = \dfrac{8}{x}$ **b** $y = \dfrac{3}{x}$ **c** $y = -\dfrac{12}{x}$

4 a i

  b i

  ii $x = 2, \ y = 0$ **ii** $x = -1, \ y = 0$
  c i

  ii $x = 3, \ y = 1$
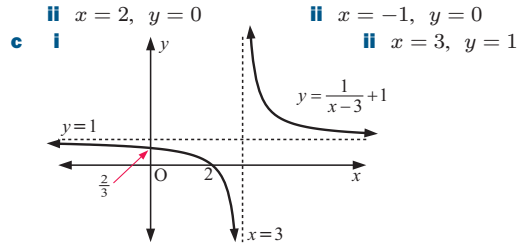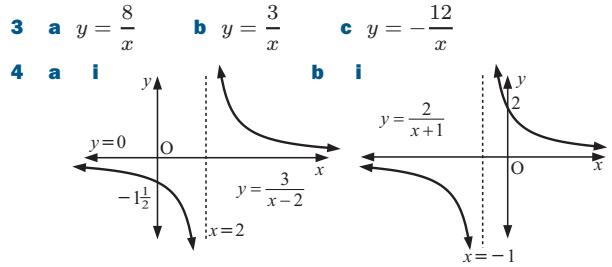
EXERCISE 19F.1

1 a 7 **b** 7 **c** 0.93 **d** $2\frac{1}{4}$ **e** 0.0932

2 a 2 **b** 10 **c** 5 **d** 11 **e** 11 **f** 40 **g** 40 **h** -2

3 a **i** both 9 **ii** both 0 **iii** both 4 **iv** both 81
    v both 81 **vi** both 400
  b $|a|^2 = a^2$

4 a $x = \pm 4$ **b** $x = \pm 1.4$
  c no solution as $|x| > 0$ for all x **d** $x = \pm 6$
  e $x = 2$ or -4 **f** $x = 7$ or -3 **g** $x = \pm 4$
  h $x = 4$ or 6

5 a

a	b	$|ab|$	$|a|\,|b|$	$\left\|\dfrac{a}{b}\right\|$	$\dfrac{|a|}{|b|}$
12	3	36	36	4	4
12	-3	36	36	4	4
-12	3	36	36	4	4
-12	-3	36	36	4	4

  b It is likely that $|ab| = |a|\,|b|$ and $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}, \ b \neq 0$.

6 a $y = \begin{cases} -x & \text{if } x \geqslant 0 \\ x & \text{if } x < 0 \end{cases}$ **b** $y = \begin{cases} 2x & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases}$
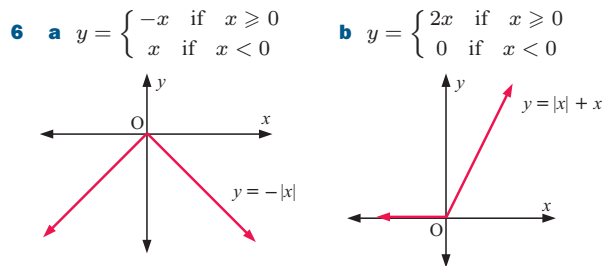
  c $y = \begin{cases} 1 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$
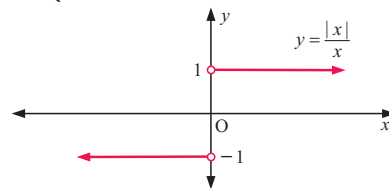
d $y = \begin{cases} -x & \text{if } x \geqslant 0 \\ 3x & \text{if } x < 0 \end{cases}$

$y = x - 2|x|$
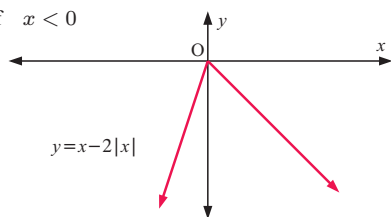
7 a

a	b	$|a+b|$	$|a|+|b|$	$|a-b|$	$|a|-|b|$
2	5	7	7	3	−3
2	−5	3	7	7	−3
−2	5	3	7	7	−3
−2	−5	7	7	3	−3

b $|a+b| \neq |a|+|b|$, $|a-b| \neq |a|-|b|$

EXERCISE 19F.2

1 a $y=|x+1|$ **b** $y=|x-1|$

c $y=|2x-1|$ **d** $y=|4-x|$

e $y=|2-3x|$ **f** $y=-|3x+2|$

2 $x = -\dfrac{b}{a}$

3 a $f(x) = |x-2|$ **b** $f(x) = |2x+2|$

c $f(x) = \left|\tfrac{1}{2}x + 1\right|$
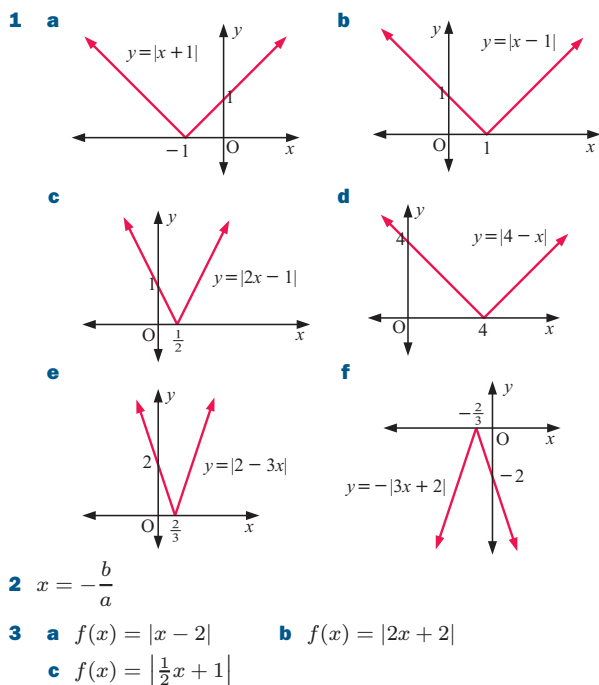
REVIEW SET 19A

1 a Domain is $\{x \mid -4 < x \leqslant 3\}$. Range is $\{y \mid -2 \leqslant y < 4\}$.
b Domain is $\{x \mid x \geqslant -5\}$. Range is $\{y \mid y \geqslant -145\}$.
c Domain is $\{x \mid x \neq 0\}$. Range is $\{y \mid y \neq 0\}$.

2 a −13 **b** **i** 0 **ii** −21 **iii** $3 - x^2 - 2x$

3 a Domain is $\{x \mid x \in \mathbb{R}\}$. Range is $\{y \mid y = 2\}$.
Is a function.
b Domain is $\{x \mid x \geqslant -3,\ x \in \mathbb{R}\}$. Range is $\{y \mid y \in \mathbb{R}\}$.
Is not a function.
c Domain is $\{x \mid -2 \leqslant x \leqslant 1\}$.
Range is $\{y \mid -3 \leqslant y \leqslant 2\}$.   Is a function.

4 a $5 - 2x^2 - 4x$ **b** $4x^2 - 24x + 35$

5 a −19 **b** $5x - 9$ **c** $25x - 24$

6 a $y = \dfrac{12}{x}$ **b** $y = -\dfrac{9}{x}$

7 a $y = \dfrac{4}{x+3} - 1$ **b** $x = -3,\ y = -1$

8 a $f(x) = |x-4|$ **b**
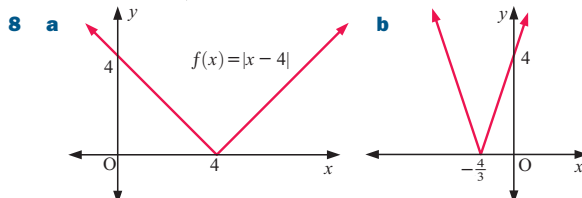
REVIEW SET 19B

1 A many-one function.

2 a **i** Range is $\{y \mid y > -2\}$. Domain is $\{x \mid x > -6\}$.
ii function
b **i** Range is $\{y \mid y \neq 3\}$. Domain is $\{x \mid x \neq -2\}$.
ii function
c **i** Range is $\{y \mid y \in \mathbb{R}\}$. Domain is $\{x \mid x = -3\}$.
ii not a function
d **i** Range is $\{y \mid y \in \mathbb{R}\}$. Domain is $\{x \mid x \leqslant 1\}$.
ii not a function

3 a −24 **b** $-5x - x^2$ **c** $-x^2 + 3x + 4$

4 $y = 3 - 2x$ Range is $\{y \mid -3 \leqslant y \leqslant 9\}$

5 a $4x^2 + 6x$ **b** $4x + 3$ **c** 4

6 a $y = \dfrac{6}{x}$ **b** $y = -\dfrac{12}{x}$

7 $f(x) = |3x - 3|$ or $f(x) = |3 - 3x|$

8 If $x \geqslant 0$, $y = x + 4$
If $x < 0$, $y = -x + 4$
$y = -x + 4$ $y = x + 4$ is $y = |x| + 4$

EXERCISE 20A

1 a $(5, 3)$ **b** $(4, 6)$ **2** $\binom{0}{3}$ **3** $(0, 1)$

4 a $\binom{8}{0}$ **b** $\binom{-8}{0}$ **c** $\binom{0}{-4}$ **d** $\binom{0}{4}$ **e** $\binom{4}{1}$
f $\binom{4}{6}$ **g** $\binom{-8}{-4}$ **h** $\binom{-4}{-6}$ **i** $\binom{0}{5}$ **j** $\binom{4}{-6}$

5 **a/b**
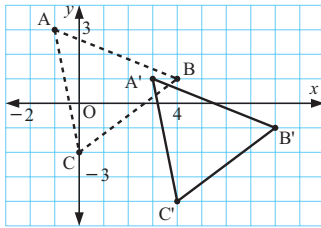
c A′(3, 1), B′(8, −1), C′(4, −4)   **d** $2\sqrt{5}$ units

6 a translation of $\binom{5}{5}$

7 **a** $y = 2x + 7$ **b** $y = \frac{1}{3}x + 1$ **c** $y = -x + 7$

 d $y = -\frac{1}{2}x - 6$

EXERCISE 20B

1 **a** (3, 2)   **b** (−3, −2)   **c** (2, −3)

2 **a** (3, 6)   **b** (−4, 5)   **c** (−3, −2)

3 **a**

b A′(−1, 5),
 B′(2, 7),
 C′(4, 4)

4 **a**

b P′(−5, 4),
 Q′(−3, −2),
 R′(−1, 1)

5 **a** $y = -\frac{1}{2}x$ **b** $x = 3$

6 a rotation about O(0, 0) through $(\theta + \phi)^o$ **7** $2x - 3y = -3$

EXERCISE 20C

1
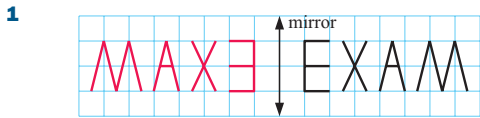
2 **a** (4, 1)   **b** (−4, −1)   **c** (−1, 4)   **d** (1, −4)

3 **a** (1, −3)   **b** (3, 1)   **c** (5, −3)   **d** (−1, 1)

 e (−1, 3)   **f** (−5, −3)   **g** (−3, −1)   **h** (−1, 7)

4
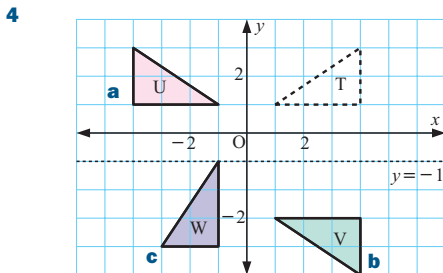
5 **a** (1, −1)   **b** (5, −1)   **c** (3, −9)   **d** (1, 7)   **e** (5, 1)

6 $y = x - 5$

7 **a** on the graph   **b** on the graph

 c **i/ii** on the graph

 iii A clockwise rotation of 90° about O.

 d **i/ii** on the graph   **iii** A 180° rotation about O.
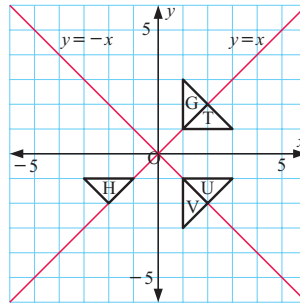
8 **a** (3, 2)   **b** (−8, 3)   **c** (1, −4)

9 **a** (1) is $y = 2x + 3$ (2) is its translation   and
 (3) is the reflection in $y = -1$
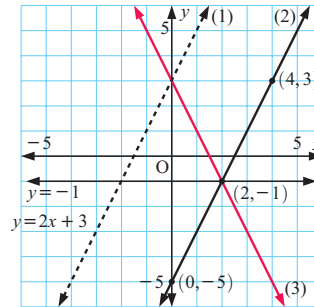
 b $m = -2$, $c = 3$ ∴ $y = -2x + 3$

10 **a** $y = -2x$ **b** $y = \frac{1}{2}x$ **c** $y = -2x + 4$

 d $y = -2x + 6$

EXERCISE 20D

1 **a**

 b

 c

2 **a**

 b
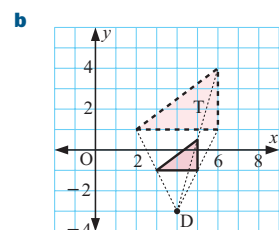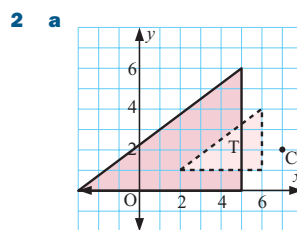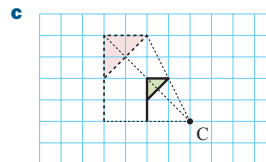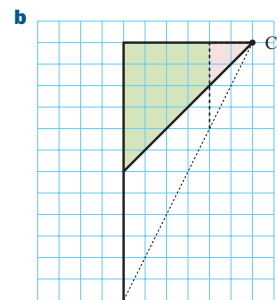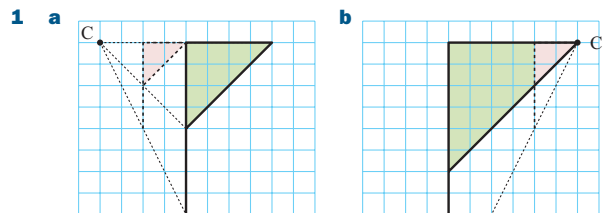
3 **a** $(4\frac{1}{2}, 6)$ **4** **a** **i** $y = 2x$ **ii** $y = 2x$

b $(0, 2)$ **b** **i** $y = -x + 8$ **ii** $y = -x + \frac{4}{3}$

c **i** $y = 2x + 9$ **ii** $y = 2x$

6 **a** $y = 6x$ **b** $y = \frac{9}{4}x$ **c** $y = \frac{1}{4}x + 2\frac{1}{4}$

EXERCISE 20F

1 **a**
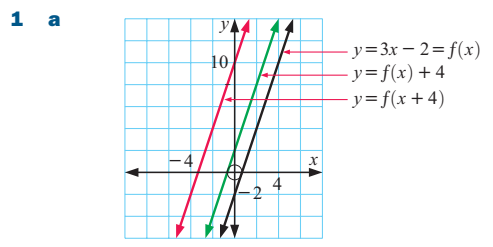
b $y = f(x) + 4$ is a vertical translation of $y = f(x)$ through $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$.

$y = f(x + 4)$ is a horizontal translation of $y = f(x)$ through $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$.

2 **a**
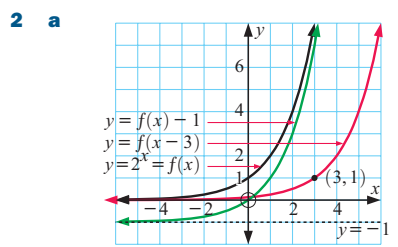
b **i** a horizontal translation of $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ **ii** a translation of $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

3 **a**

b $y = -1$

c

$y = -1$

4 **a**
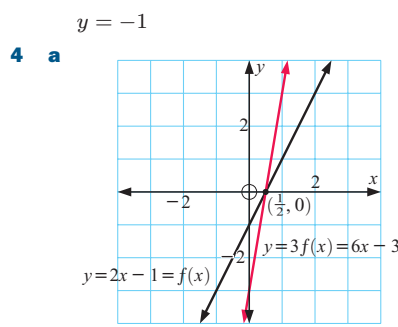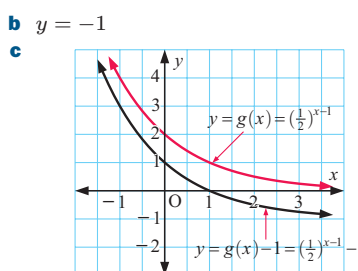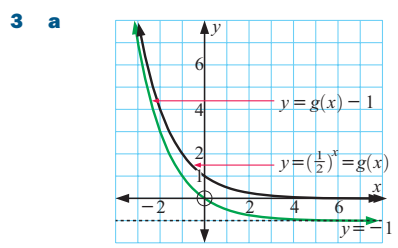
b points on the x-axis,   i.e., $(\frac{1}{2}, 0)$

EXERCISE 20E

1 **a**

b

c

d

2 **a**

b

3 **a** $(3, -4)$ **b** $(4, 10)$ **c** $(-1, 1)$ **d** $(4\frac{1}{2}, -4)$

e $(\frac{3}{4}, 1\frac{3}{4})$

4 **a**

b

A′ is $(1, 3)$, B′ is $(4, 1)$

C′ is $(2, 9)$

A′ is $(0, 2)$, B′ is $(1\frac{1}{2}, 1)$

C′ is $(\frac{1}{2}, 5)$

5 **a** Vertical stretch of factor $k = \frac{3}{4}$ with invariant x-axis.

b Horizontal stretch of factor $k = 2$, with invariant y-axis.

c Horizontal stretch of factor $k = 4$, with invariant y-axis.

d Horizontal stretch of factor $k = \frac{1}{4}$, with $x = -1$ the invariant line.

5 **a**
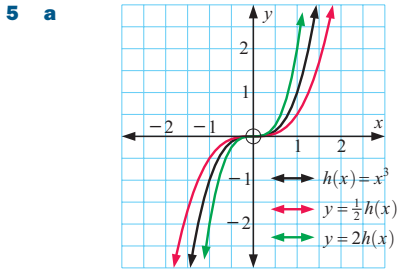
  b a stretch with invariant x-axis, factor $\frac{1}{4}$

6 **a/b** x-intercepts are ± 1, y-intercept is -1
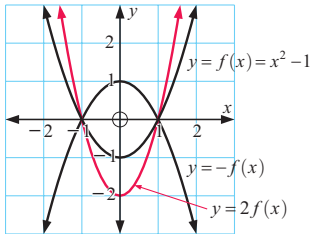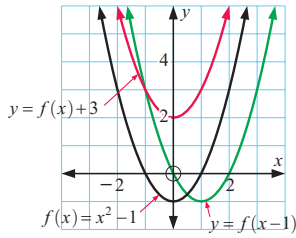
  c **i** a vertical translation of $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

    ii a horizontal translation of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

    iii a stretch with invariant x-axis and scale factor 2
    iv a reflection in the x-axis

  d
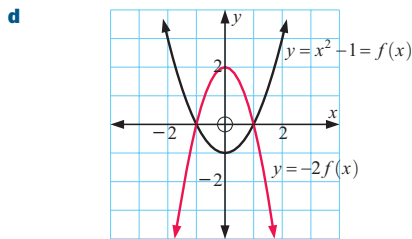
    A stretch with invariant x-axis and scale factor $k = -2$.
  e $(-1, 0)$ and $(1, 0)$

7 **a** **i** A stretch with invariant x-axis and scale factor $k = 3$.
    ii $g(x) = 3f(x)$

  b **i** A vertical translation of $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$.   **ii** $g(x) = f(x) - 2$

  c **i** A stretch with invariant x-axis and scale factor $k = \frac{1}{2}$.
    ii $g(x) = \frac{1}{2}f(x)$

8 **a**

  b

c

d

e

f

9
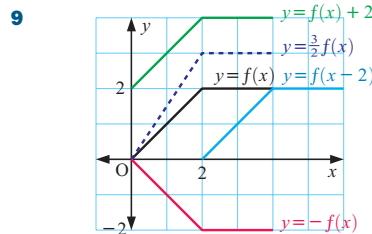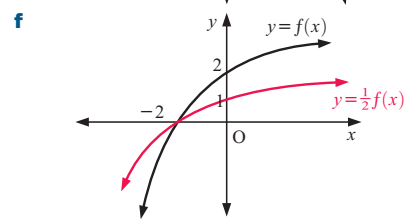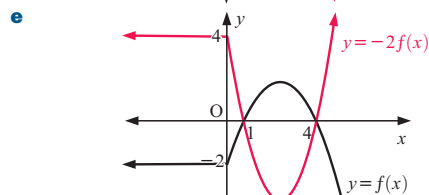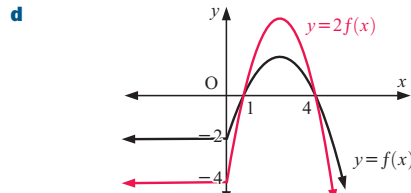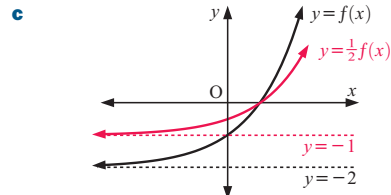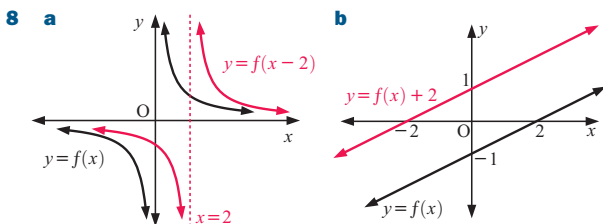
10 **a** $g(x) = 2x^2 - 8$, $h(x) = 4x^2 - 4$
  b
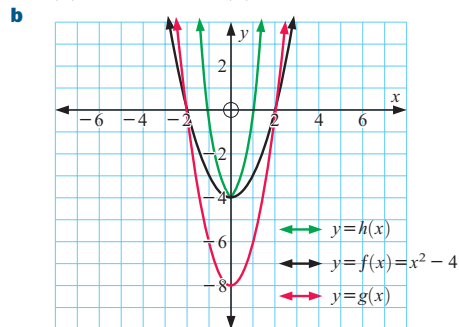
  c A stretch with invariant x-axis and $k = 2$.
  d $(-2, 0)$ and $(2, 0)$ **e** zeros are ± 1
  f A stretch with invariant y-axis and $k = \frac{1}{2}$.

EXERCISE 20G

1 **a** a reflection in the y-axis
  b a rotation about $O(0, 0)$ through 180°

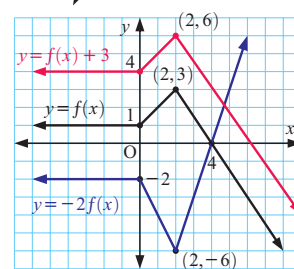**c** a translation of $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ **d** a translation of $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

e a translation of $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

f a 90° anticlockwise rotation about O(0, 0)

g a reduction, centre O(0, 0), scale factor $\frac{1}{4}$

h an enlargement, centre O(0, 0), scale factor 3

i a reflection in $y = -x$

j a stretch with invariant x-axis and scale factor $\frac{2}{3}$

k a reflection in $y = -2$

l a stretch with invariant y-axis and scale factor 2

m a rotation about P, anticlockwise through 43°

EXERCISE 20H

1 a i **ii**
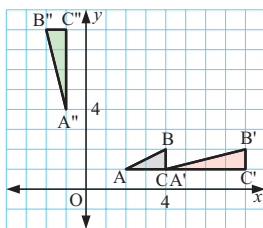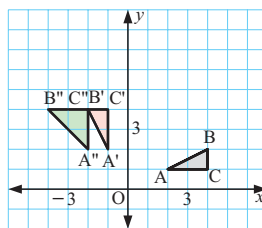
b i **ii**
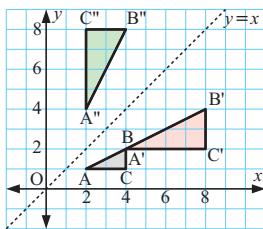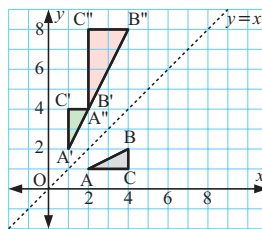
c i a 90° anticlockwise rotation about O
  ii a 90° clockwise rotation about O

d i a reflection in the x-axis **ii** a reflection in the x-axis

2 a i T_3 **ii** T_3 **iii** T_6 **iv** T_0 **v** T_1

b

first transformation \ second transformation	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7
T_0	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7
T_1	T_1	T_0	T_3	T_2	T_5	T_4	T_7	T_6
T_2	T_2	T_7	T_4	T_1	T_6	T_3	T_0	T_5
T_3	T_3	T_6	T_5	T_0	T_7	T_2	T_1	T_4
T_4	T_4	T_5	T_6	T_7	T_0	T_1	T_2	T_3
T_5	T_5	T_4	T_7	T_6	T_1	T_0	T_3	T_2
T_6	T_6	T_3	T_0	T_5	T_2	T_7	T_4	T_1
T_7	T_7	T_2	T_1	T_4	T_3	T_6	T_5	T_0

REVIEW SET 20A

1 a i (2, 5) **ii** (−5, 2) **iii** (4, −5)
  b i (4, 1) **ii** (5, 4) **c** (2, 2) **d** (6, −2)
  e $(3, -2\frac{1}{2})$ **f** $(2\frac{1}{2}, -1)$

2 (−8, 1)

3 a $y = 2x - 9$ **b** $y = -2x + 1$ **c** $y = -\frac{1}{2}x - \frac{1}{2}$
  d $y = x - 1$

4 Stretch with invariant line the x-axis and scale factor $k = \frac{5}{3}$.
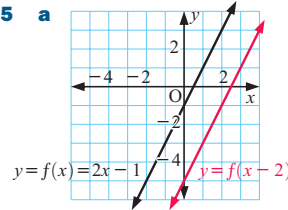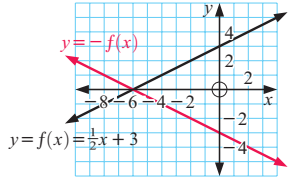
5 a **b**
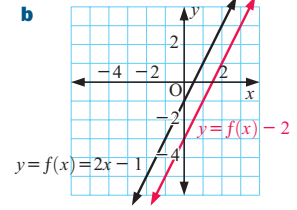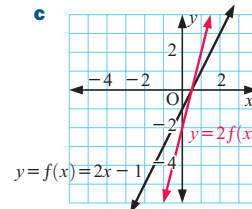
c **d**
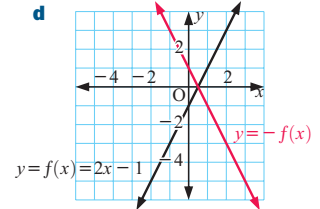
6

7 a a translation of $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ **b** a reflection in the line $y = -x$
  c a stretch with invariant y-axis and scale factor $\frac{1}{2}$

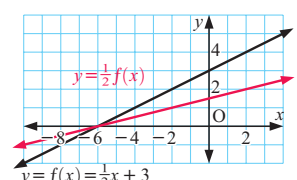**8 a** a reflection in the y-axis **b** a reflection in the x-axis

REVIEW SET 20B

1 a (3, 2) **b** (2, −3) **c** (3, 10)

2 a (−5, −11) **b** (−7, −3)

3 a (9, 15) **b** (−4, 3) **c** $(-5, -1\frac{1}{2})$

4 a $y = -2x + 6$ **b** $x + 2y = 1$ **c** $x - 2y = -1$
  d $2x + 2y = 1$

5 A stretch with invariant line the y-axis with scale factor $k = 3$.

6 a **b**
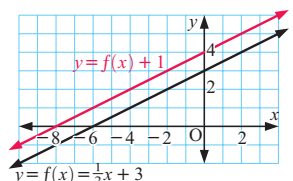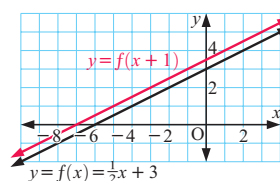
c **d**

7 a A stretch with invariant x-axis and scale factor $k = 2$.
  b $g(x) = 2f(x)$

8 a a reflection in the x-axis
  b a rotation of 180° about O(0, 0)
  c A reduction about O(0, 0) with scale factor $k = \frac{1}{3}$.

9 a a reflection in $y = x$ **b** a reflection in $y = -x$

EXERCISE 21A

1   **a** $x = \pm 10$ **b** $x = \pm 5$ **c** $x = \pm 2$
   d $x = \pm 3$ **e**   no solution   **f** $x = 0$
   g $x = \pm 3$ **h**   no solution   **i** $x = \pm\sqrt{2}$

2   **a** $x = 4$ or -2 **b** $x = 0$ or -8 **c**   no solution
   d $x = 4 \pm \sqrt{5}$ **e**   no solution   **f** $x = -2$
   g $x = 2\frac{1}{2}$ **h** $x = 0$ or $-\frac{4}{3}$ **i** $x = \dfrac{\pm\sqrt{6} - 3}{2}$

EXERCISE 21B.1

1   **a** $x = 0$ **b** $a = 0$ **c** $y = 0$ **d** $a = 0$ or $b = 0$
   e $x = 0$ or $y = 0$ **f** $a = 0$ or $b = 0$ or $c = 0$ **g** $a = 0$
   h $p = 0$ or $q = 0$ or $r = 0$ or $s = 0$ **i** $a = 0$ or $b = 0$

2   **a** $x = 0$ or 5 **b** $x = 0$ or -3 **c** $x = -1$ or 3
   d $x = 0$ or 7 **e** $x = 0$ or -1 **f** $x = -6$ or $\frac{3}{2}$
   g $x = \pm\frac{1}{2}$ **h** $x = -2$ or 7 **i** $x = 5$ or $-\frac{2}{3}$
   j $x = 0$ **k** $x = 5$ **l** $x = \frac{1}{3}$

EXERCISE 21B.2

1   **a** $x = 0$ or 7 **b** $x = 0$ or 5 **c** $x = 0$ or 8
   d $x = 0$ or 4 **e** $x = 0$ or -2 **f** $x = 0$ or $-\frac{5}{2}$
   g $x = 0$ or $\frac{3}{4}$ **h** $x = 0$ or $\frac{5}{4}$ **i** $x = 0$ or 3

2   **a** $x = 1$ or -1 **b** $x = 3$ or -3 **c** $x = 5$
   d $x = -2$ **e** $x = -1$ or -2 **f** $x = 1$ or 2
   g $x = -2$ or -3 **h** $x = 2$ or 3 **i** $x = -1$ or -6
   j $x = -2$ or -7 **k** $x = -5$ or -6 **l** $x = -5$ or 3
   m $x = -6$ or 2 **n** $x = 3$ or 8 **o** $x = 7$

3   **a** $x = -4$ or -5 **b** $x = -4$ or -7 **c** $x = -4$ or 2
   d $x = -4$ or 3 **e** $x = 3$ or 2 **f** $x = 2$
   g $x = 3$ or -2 **h** $x = 12$ or -5 **i** $x = 10$ or -7
   j $x = -5$ or 2 **k** $x = 3$ or 4 **l** $x = 12$ or -3

4   **a** $x = \frac{1}{2}$ or 2 **b** $x = -3$ or $\frac{1}{3}$ **c** $x = -4$ or $-\frac{5}{3}$
   d $x = \frac{1}{2}$ or -3 **e** $x = \frac{1}{2}$ or 5 **f** $x = -1$ or $-\frac{5}{2}$
   g $x = -\frac{1}{3}$ or -4 **h** $x = -\frac{2}{5}$ or 3 **i** $x = \frac{1}{2}$ or -9
   j $x = 1$ or $-\frac{5}{2}$ **k** $x = \frac{4}{3}$ or -2 **l** $x = \frac{3}{2}$ or -6

5   **a** $x = \frac{1}{3}$ or $-\frac{5}{2}$ **b** $x = \frac{2}{3}$ or $-\frac{1}{2}$ **c** $x = -\frac{1}{2}$ or $-\frac{1}{3}$
   d $x = -\frac{1}{21}$ or 3 **e** $x = \frac{2}{5}$ or $-\frac{1}{2}$ **f** $x = -\frac{3}{10}$ or 1

6   **a** $x = -4$ or -3 **b** $x = -3$ or 1 **c** $x = \pm 3$
   d $x = -1$ or $\frac{2}{3}$ **e** $x = -\frac{1}{2}$ **f** $x = \frac{5}{2}$ or 4

7   **a** $x = \pm\sqrt{6}$ **b** $x = \pm\sqrt{8}$ **c** $x = \pm\sqrt{10}$
   d $x = 4$ or -3 **e** $x = -1$ or -5 **f** $x = 2$ or -1
   g $x = \frac{1}{2}$ or -1 **h** $x = 1$ or $-\frac{1}{3}$ **i** $x = -1$ or 4

8   **a** $x = -\frac{1}{2}$ or 2 **b** $x = 1$ or -3 **c** $x = -\frac{1}{3}$ or -1
   d $x = 1$ or $1\frac{1}{7}$

9   **a** $x = \pm 1$ or ± 2 **b** $x = \pm\sqrt{3}$ or ± 2 **c** $x = \pm\sqrt{5}$

EXERCISE 21C.1

1   **a** $x = -2 \pm \sqrt{7}$ **b** $x = -3 \pm 2\sqrt{2}$ **c** $x = -2 \pm \sqrt{11}$
   d $x = -1 \pm \sqrt{3}$ **e** $x = 3 \pm \sqrt{7}$ **f** $x = 2 \pm \sqrt{5}$
   g $x = \dfrac{3 \pm \sqrt{5}}{2}$ **h** $x = -4 \pm \sqrt{11}$ **i** $x = \dfrac{1 \pm \sqrt{3}}{2}$
   j $x = \dfrac{1 \pm \sqrt{2}}{3}$ **k** $x = \dfrac{2 \pm \sqrt{3}}{5}$ **l** $x = \dfrac{-3 \pm \sqrt{7}}{2}$

2   **a** $x \approx 5.24$ or 0.76 **b** $x \approx 0.22$ or -2.22
   c $x \approx 0.72$ or -1.12 **d** $x \approx -1.22$ or 0.55
   e $x \approx 2.62$ or 0.38 **f** $x \approx 2.30$ or -1.30

3   **a** $x = \dfrac{-1 \pm \sqrt{29}}{2}$ **b** $x = \dfrac{-1 \pm \sqrt{5}}{2}$ **c** $x = 1 \pm \sqrt{3}$
   d $x = 1 \pm 2\sqrt{2}$ **e** $x = \dfrac{7 \pm \sqrt{217}}{6}$ **f** $x = \dfrac{3 \pm \sqrt{13}}{2}$
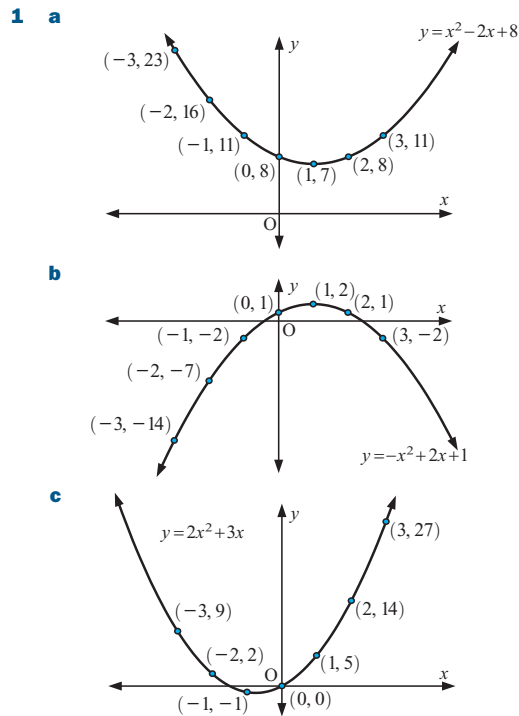
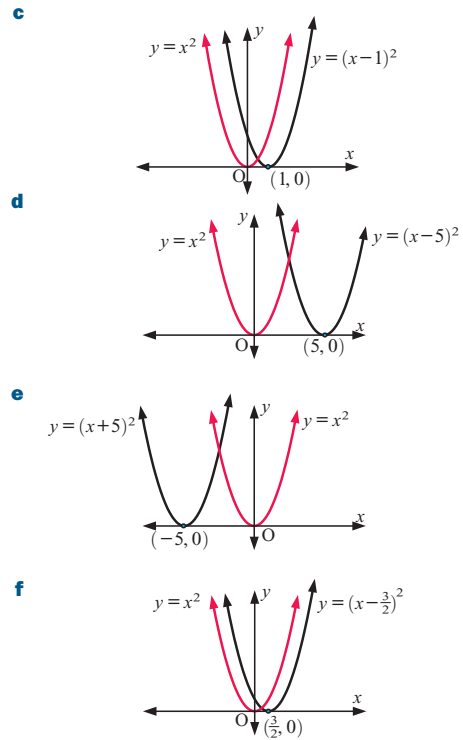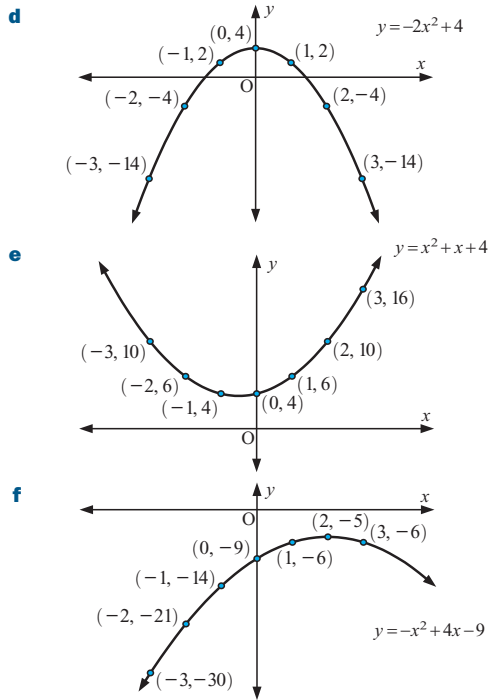## EXERCISE 21C.2

1   **a** $x = \dfrac{3 \pm \sqrt{-39}}{2}$ $\therefore$ no real solutions exist
   b $x = -1 \pm \sqrt{-3}$ $\therefore$ no real solutions exist
   c $x = \dfrac{1 \pm \sqrt{-7}}{4}$ $\therefore$ no real solutions exist

2   **a** $x = \pm 5$ **b**   no real solutions exist   **c** $x = \pm\sqrt{7}$
   d   no real solutions exist   **e** $x = \pm\frac{3}{2}$
   f   no real solutions exist   **g**   no real solutions exist
   h $x = 5$ or -1 **i**   no real solutions exist
   j   no real solutions exist   **k** $x = \dfrac{3 \pm \sqrt{19}}{2}$
   l $x = \dfrac{-1 \pm \sqrt{17}}{4}$
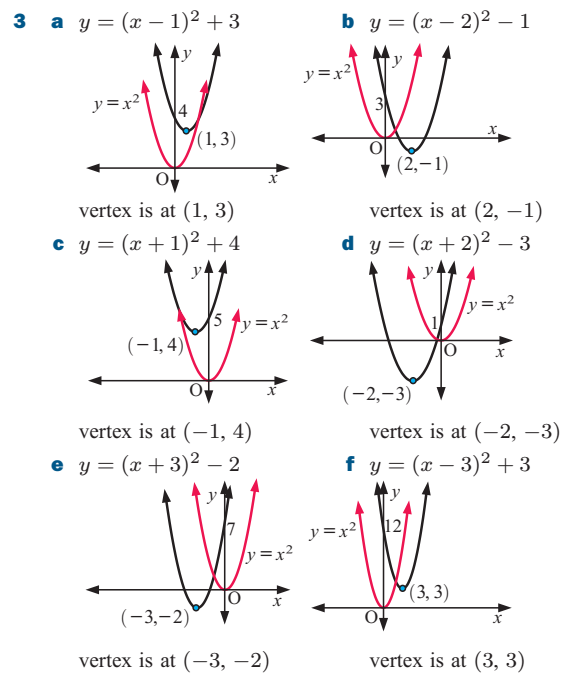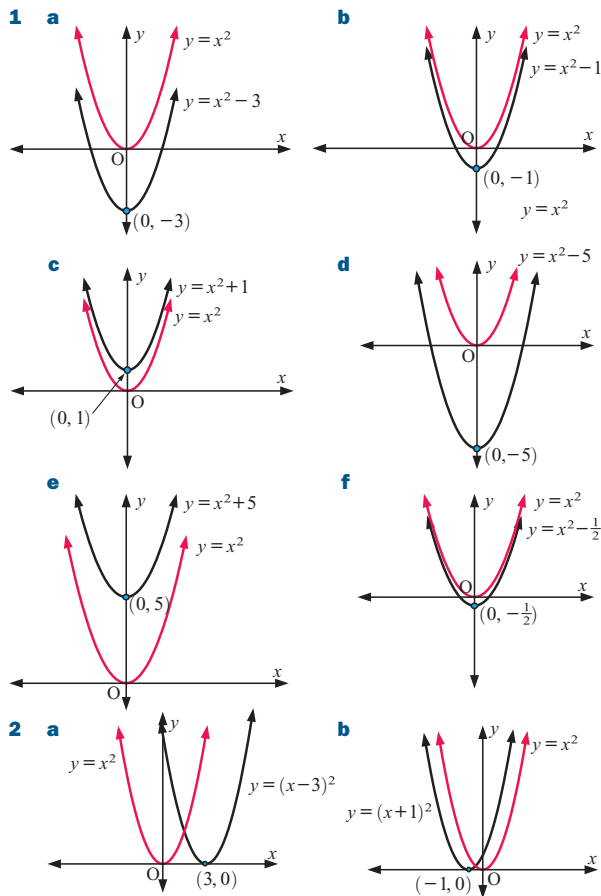
EXERCISE 21D

1   **a**, **c**, **d** and **e** are quadratic functions.

2   **a** $y = 0$ **b** $y = 5$ **c** $y = -15$ **d** $y = 12$

3   **a**   No   **b**   Yes   **c**   Yes   **d**   No   **e**   No   **f**   No

4   **a** $x = -3$ **b** $x = -3$ or -2 **c** $x = 1$ or 4
   d   no solution

5   **a** $x = 0$ or 1 **b** $x = 3$ or -1 **c** $x = \frac{1}{2}$ or -7
   d $x = 2$ or 3

EXERCISE 21E.1
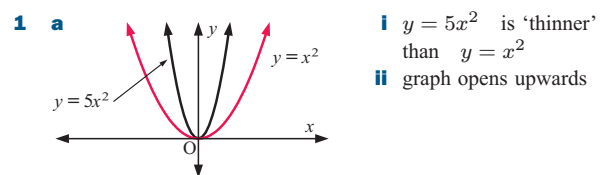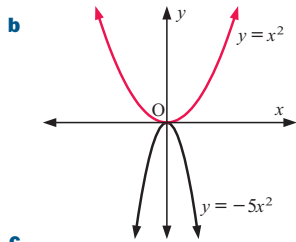
1   **a**

   b

   c

d

$y = -2x^2 + 4$

$(0, 4)$
$(-1, 2)$ $(1, 2)$
$(-2, -4)$ $(2, -4)$
$(-3, -14)$ $(3, -14)$

e

$y = x^2 + x + 4$
$(3, 16)$
$(-3, 10)$ $(2, 10)$
$(-2, 6)$ $(1, 6)$
$(-1, 4)$ $(0, 4)$

f

$(2, -5)$ $(3, -6)$
$(0, -9)$ $(1, -6)$
$(-1, -14)$
$(-2, -21)$
$y = -x^2 + 4x - 9$
$(-3, -30)$

EXERCISE 21E.2

1 a

$y = x^2$
$y = x^2 - 3$
$(0, -3)$

b

$y = x^2$
$y = x^2 - 1$
$(0, -1)$
$y = x^2$

c

$y = x^2 + 1$
$y = x^2$
$(0, 1)$

d

$y = x^2 - 5$
$(0, -5)$

e

$y = x^2 + 5$
$y = x^2$
$(0, 5)$

f

$y = x^2$
$y = x^2 - \frac{1}{2}$
$(0, -\frac{1}{2})$

2 a

$y = x^2$
$y = (x-3)^2$
$(3, 0)$

b

$y = x^2$
$y = (x+1)^2$
$(-1, 0)$

c

$y = x^2$
$y = (x-1)^2$
$(1, 0)$

d

$y = x^2$
$y = (x-5)^2$
$(5, 0)$

e

$y = (x+5)^2$
$y = x^2$
$(-5, 0)$

f

$y = x^2$
$y = (x - \frac{3}{2})^2$
$(\frac{3}{2}, 0)$

3 a $y = (x-1)^2 + 3$

$y = x^2$
4
$(1, 3)$

vertex is at $(1, 3)$

b $y = (x-2)^2 - 1$

$y = x^2$
3
$(2, -1)$

vertex is at $(2, -1)$

c $y = (x+1)^2 + 4$

5 $y = x^2$
$(-1, 4)$

vertex is at $(-1, 4)$

d $y = (x+2)^2 - 3$

$y = x^2$
1
$(-2, -3)$

vertex is at $(-2, -3)$

e $y = (x+3)^2 - 2$

7
$y = x^2$
$(-3, -2)$

vertex is at $(-3, -2)$

f $y = (x-3)^2 + 3$

$y = x^2$
12
$(3, 3)$

vertex is at $(3, 3)$

EXERCISE 21E.3

1 a

$y = x^2$
$y = 5x^2$

i $y = 5x^2$ is 'thinner' than $y = x^2$
ii graph opens upwards

b

i $y = -5x^2$ is 'thinner' than $y = x^2$
ii graph opens downwards

c

i $y = \frac{1}{3}x^2$ is 'wider' than $y = x^2$
ii graph opens upwards

d
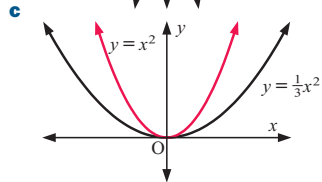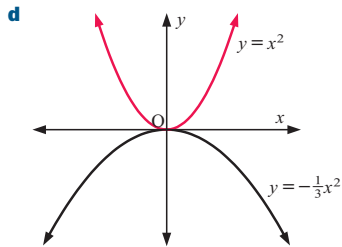
i $y = -\frac{1}{3}x^2$ is 'wider' than $y = x^2$
ii graphs opens downwards

e
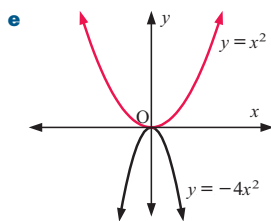
i $y = -4x^2$ is 'thinner' than $y = x^2$
ii graph opens downwards

f

i $y = \frac{1}{4}x^2$ is 'wider' than $y = x^2$
ii graph opens upwards

2 a

$y = -(x-1)^2 + 3$, $(1, 3)$

b

$y = 2x^2 + 4$, $(0, 4)$

c

$(2, 4)$, $y = -(x-2)^2 + 4$

d

$y = 3(x+1)^2 - 4$, -1, $(-1, -4)$

e

$y = \frac{1}{2}(x+3)^2$, $4\frac{1}{2}$, $(-3, 0)$

f

$(-3, 1)$, $-3\frac{1}{2}$, $y = -\frac{1}{2}(x+3)^2 + 1$

g

$(-4, 3)$, $y = -2(x+4)^2 + 3$

h

$y = 2(x-3)^2 + 5$, $(3, 5)$

i

$y = \frac{1}{2}(x-2)^2 - 1$, 1, $(2, -1)$
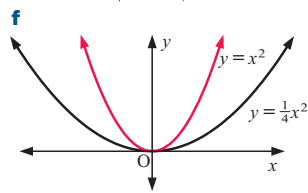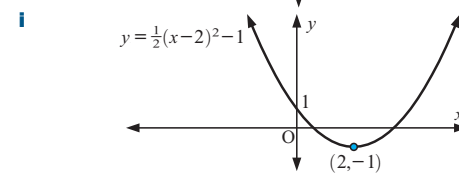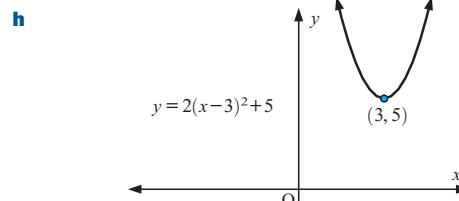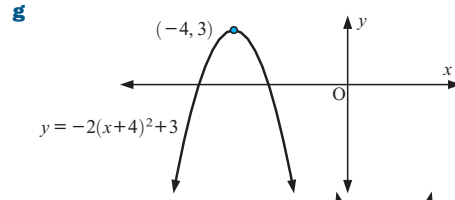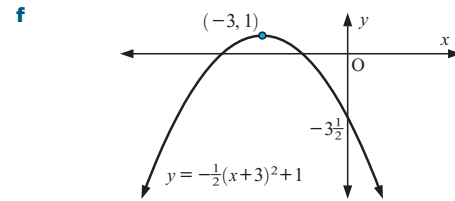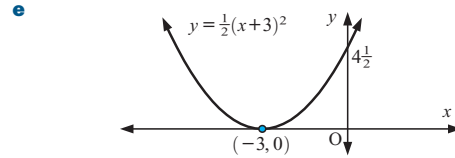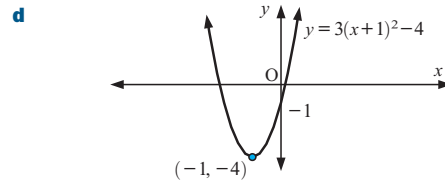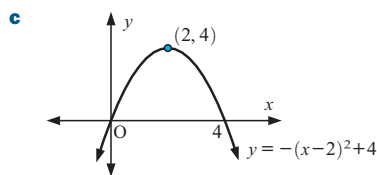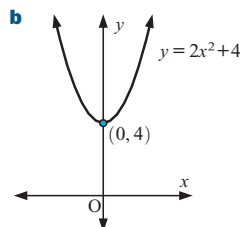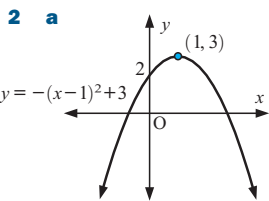
3 a G   **b** A   **c** E   **d** B   **e** I   **f** C
g D   **h** F   **i** H

4 a i

$y = 2x^2$, 1, $y = 2x^2 - 3x + 1$, $\frac{1}{2}$, 1, $\left(\frac{3}{4}, -\frac{1}{8}\right)$

ii same shape

b i

$(-3, 13)$, $y = -x^2 - 6x + 4$, 4, $y = -x^2$

ii same shape

c i

$y = 3x^2$

$y = 3x^2 - 5x$

$\left(\frac{5}{6}, -\frac{25}{12}\right)$

ii same shape

d i

$(0, 5)$

$y = -2x^2 + 5$

$y = -2x^2$
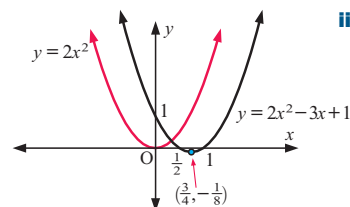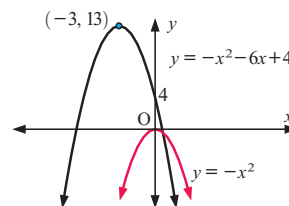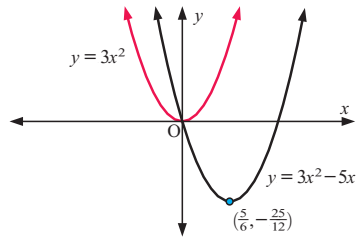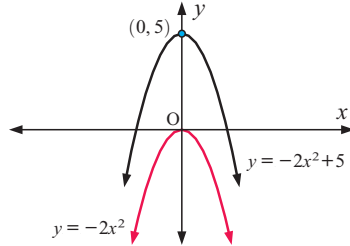
ii same shape

EXERCISE 21F.1

1 a 3   **b** 2   **c** -8 **d** 1   **e** 6   **f** 5
   g 6   **h** 8   **i** -2

2 a 3 and -1 **b** 2 and 4   **c** -3 and -2
   d 4 and 5   **e** -3 (touching)   **f** 1 (touching)

3 a ± 3 **b** $\pm\sqrt{3}$ **c** -5 and -2 **d** 3 and -4
   e 0 and 4   **f** -4 and -2 **g** -1 (touching)
   h 3 (touching)   **i** $2 \pm \sqrt{3}$ **j** $-2 \pm \sqrt{7}$ **k** $3 \pm \sqrt{11}$
   l $-4 \pm \sqrt{5}$

EXERCISE 21F.2

1 a

b

2

-3 1

c

2 5

-4

d

4

2
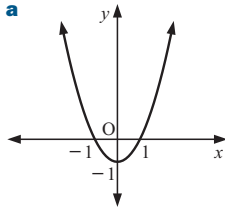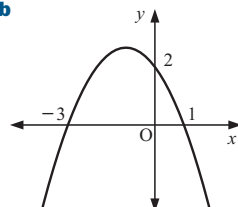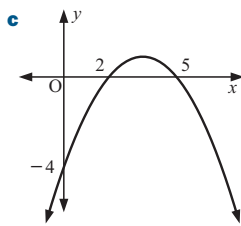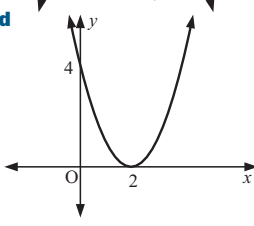
2 a i $a = 1$
   ii 4
   iii $x = 2$ (repeated)

$y = x^2 - 4x + 4$

4

2

b i $a = 1$
   ii -3
   iii $x = -3$ or 1

$y = (x-1)(x+3)$

-3 1

-3
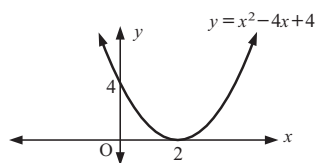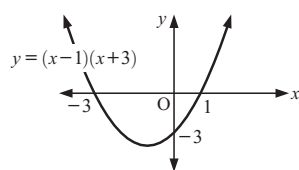
c i $a = 2$
   ii 8
   iii $x = -2$ (repeated)
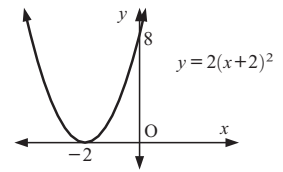
8

$y = 2(x+2)^2$

-2

d i $a = -1$
   ii 2
   iii $x = -1$ or 2

2

-1 2

$y = -(x-2)(x+1)$

e i $a = -3$
   ii -3
   iii $x = -1$ (repeated)

-1

-3

$y = -3(x+1)^2$

f i $a = -3$
   ii -12
   iii $x = 1$ or 4

4

1

$y = -3(x-4)(x-1)$

-12

g i $a = 2$
   ii 6
   iii $x = -3$ or -1

6

-1

-3

$y = 2(x+3)(x+1)$
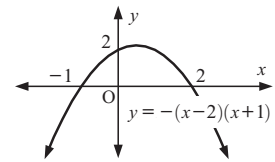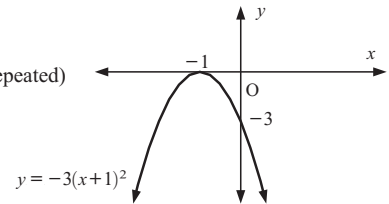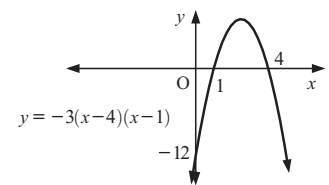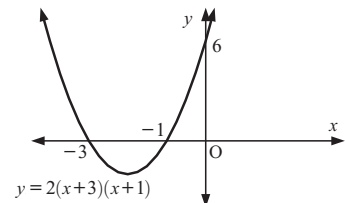
h i $a = 2$
   ii 2
   iii no x-intercepts

$y = 2x^2 + 3x + 2$

2
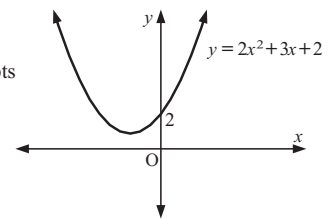
i i $a = -2$
   ii 5
   iii $x = -\frac{5}{2}$ or 1

$y = -2x^2 - 3x + 5$

5

$-\frac{5}{2}$ 1
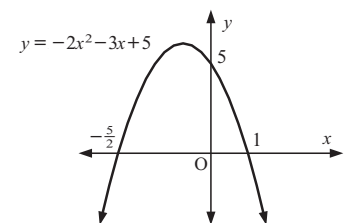
EXERCISE 21G

1 a $x = -2$ **b** $x = \frac{3}{2}$ **c** $x = -\frac{2}{3}$
   d $x = -2$ **e** $x = \frac{5}{4}$ **f** $x = 10$
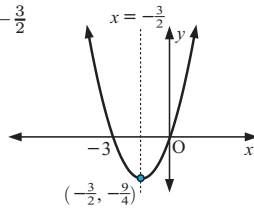
g $x = -6$ **h** $x = \frac{25}{2}$ **i** $x = 150$

2 a $(2, -2)$ **b** $(-1, -4)$ **c** $(0, 4)$

   d $(0, 1)$ **e** $(-2, -15)$ **f** $(-2, -5)$

   g $(-\frac{3}{2}, -\frac{11}{2})$ **h** $(\frac{5}{2}, -\frac{19}{2})$ **i** $(1, -\frac{9}{2})$
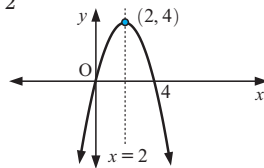
3 a   **i** x-intercepts $4, -2$, y-intercept -8

      ii axis of symmetry $x = 1$

      iii vertex $(1, -9)$

      iv $y = x^2 - 2x - 8$
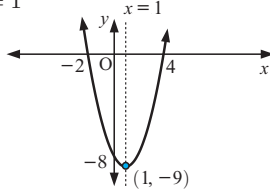
   b   **i** x-intercepts $0, -3$, y-intercept 0

      ii axis of symmetry $x = -\frac{3}{2}$

      iii vertex $(-\frac{3}{2}, -\frac{9}{4})$

      iv $y = x^2 + 3x$

   c   **i** x-intercepts $0, 4$, y-intercept 0

      ii axis of symmetry $x = 2$

      iii vertex $(2, 4)$

      iv $y = 4x - x^2$

   d   **i** x-intercept -2, y-intercept 4

      ii axis of symmetry $x = -2$

      iii vertex $(-2, 0)$

      iv $y = x^2 + 4x + 4$
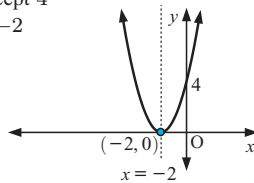
   e   **i** x-intercepts $-4, 1$, y-intercept -4

      ii axis of symmetry $x = -\frac{3}{2}$

      iii vertex $(-\frac{3}{2}, -\frac{25}{4})$

      iv $y = x^2 + 3x - 4$
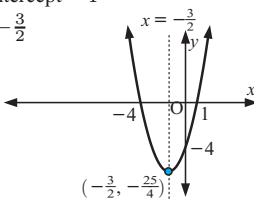
   f   **i** x-intercept 1, y-intercept -1

      ii axis of symmetry $x = 1$

      iii vertex $(1, 0)$

      iv $y = -x^2 + 2x - 1$
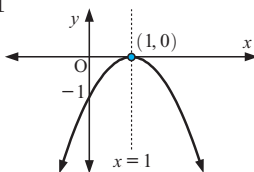
g   **i** x-intercepts $-2, -4$, y-intercept -8

      ii axis of symmetry $x = -3$

      iii vertex $(-3, 1)$

      iv $y = -x^2 - 6x - 8$
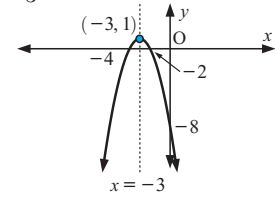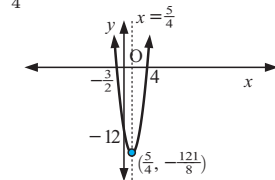

   h   **i** x-intercepts $1, 2$, y-intercept -2

      ii axis of symmetry $x = \frac{3}{2}$

      iii vertex $(\frac{3}{2}, \frac{1}{4})$

      iv $y = -x^2 + 3x - 2$
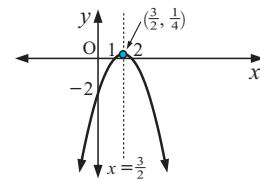
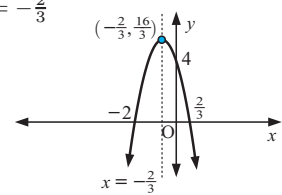   **i**   **i** x-intercepts $\frac{1}{2}, -3$, y-intercept -3

      ii axis of symmetry $x = -\frac{5}{4}$

      iii vertex $(-\frac{5}{4}, -\frac{49}{8})$

      iv $y = 2x^2 + 5x - 3$
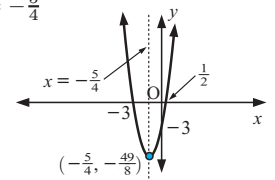
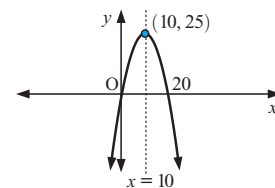   **j**   **i** x-intercepts $-\frac{3}{2}, 4$, y-intercept -12

      ii axis of symmetry $x = \frac{5}{4}$

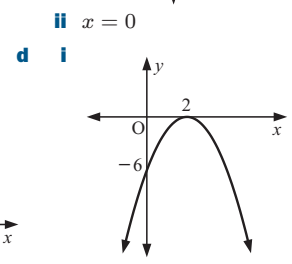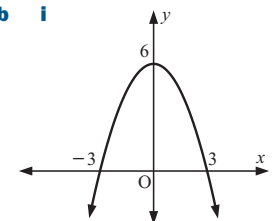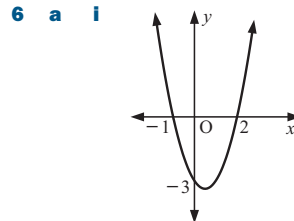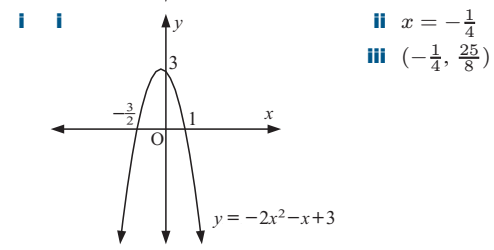   **iii** vertex $(\frac{5}{4}, -\frac{121}{8})$

      iv $y = 2x^2 - 5x - 12$

   k   **i** x-intercepts $\frac{2}{3}, -2$, y-intercept 4

      ii axis of symmetry $x = -\frac{2}{3}$

      iii vertex $(-\frac{2}{3}, \frac{16}{3})$

      iv $y = -3x^2 - 4x + 4$

   l   **i** x-intercepts $0, 20$, y-intercept 0

      ii axis of symmetry $x = 10$

      iii vertex $(10, 25)$

      iv $y = -\frac{1}{4}x^2 + 5x$

4 a $x = 2$ **b** $x = 1$ **c** $x = -4$

5 a i

$y = x^2 + 4x + 4$

ii $x = -2$
iii $(-2, 0)$

b i

$y = x(x-2)$

ii $x = 1$
iii $(1, -1)$

c i

$y = 2(x-2)^2$

ii $x = 2$
iii $(2, 0)$

d i

$y = -(x-1)(x+3)$

ii $x = -1$
iii $(-1, 4)$

e i

$y = -2(x-1)^2$

ii $x = 1$
iii $(1, 0)$

f i

$y = -5(x+2)(x-2)$

ii $x = 0$
iii $(0, 20)$

g i

$y = 2(x+1)(x+4)$

ii $x = -2\frac{1}{2}$
iii $(-2\frac{1}{2}, -4\frac{1}{2})$

h i

$y = 2x^2 - 3x - 2$

ii $x = \frac{3}{4}$
iii $(\frac{3}{4}, -\frac{25}{8})$

i i

$y = -2x^2 - x + 3$

ii $x = -\frac{1}{4}$
iii $(-\frac{1}{4}, \frac{25}{8})$

6 a i

ii $x = \frac{1}{2}$

b i

ii $x = 0$

c i

ii $x = -2$

d i

ii $x = 2$
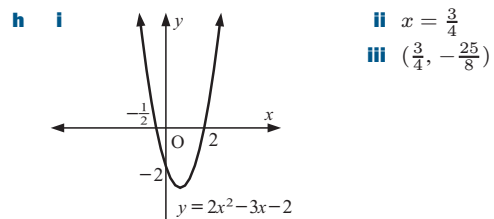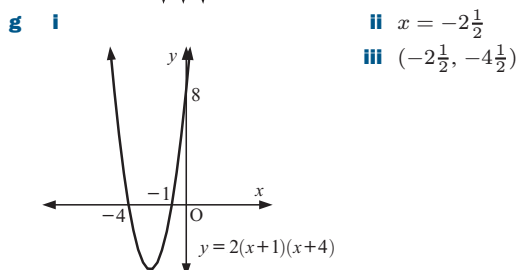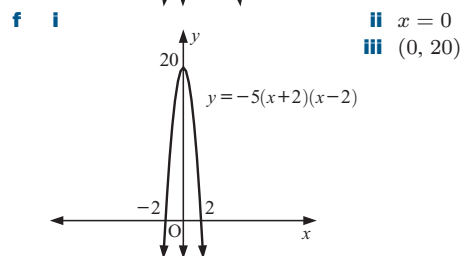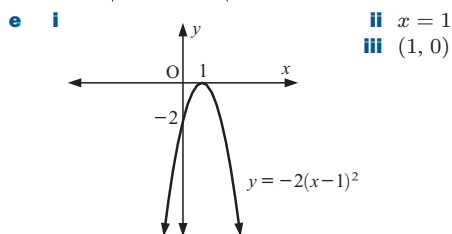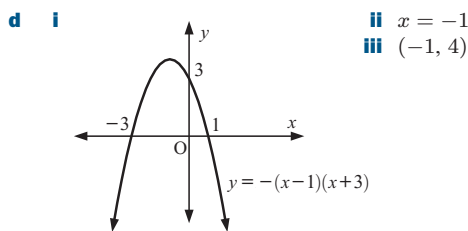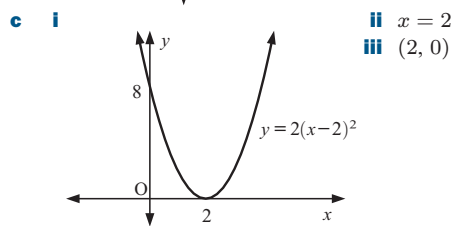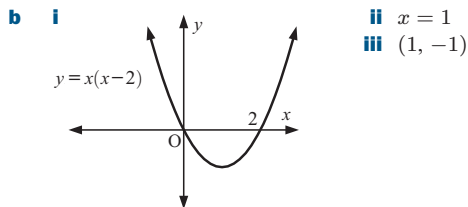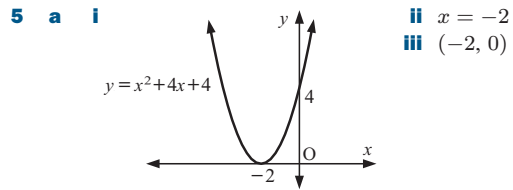
7 a 1 and 3   **b** -1 and -2 **c** 2

EXERCISE 21H

1 a $f(x) = x^2 - 2x + 4$ **b** $f(x) = x^2 + 2$
   c $f(x) = x^2 - 6x + 9$ **d** $f(x) = x^2 + 6x + 7$
   e $f(x) = x^2 - x + 1\frac{1}{4}$

2 a $f(x) = x^2 - 2x$ **b** $f(x) = x^2 + 3x - 4$
   c $f(x) = x^2 + 3x - 10$ **d** $f(x) = x^2 + 7x$

3 a $y = 2(x-1)^2 + 3$ **b** $y = -(x+2)^2 + 3$
   c $y = -2(x+1)^2 - 2$

4 a $f(x) = 2(x-2)^2 - 5$ **b** $f(x) = -(x+4)^2 + 19$
   c $f(x) = -(x-1)^2 + 8$ **d** $f(x) = -2(x+2)^2 + 11$

5 a $f(x) = -3x^2 + 6x - 7$ **b** $f(x) = 3x^2 + 12x + 15$
   c $f(x) = \frac{4}{3}x^2 + 8x + 7$ **d** $f(x) = -2x^2 + 12x - 10$

6 a $f(x) = 2(x+2)^2 - 5$ **b** $f(x) = 2(x-3)^2 - 19$

7 a $f(x) = -x^2 - 3x - 5$ **b** $f(x) = -2x^2 - 32x + 7$

8 a $f(x) = -2x^2 + 8$ **b** $f(x) = -3x^2 + 15x - 12$
   c $f(x) = 3x^2 - 3x - 18$ **d** $f(x) = -2x^2 + 2x + 40$
   **e** $f(x) = 2x^2 - 9x + 9$ **f** $f(x) = 16x^2 - 8x - 15$

EXERCISE 21I

1 **a** $x \approx -3.414$ or -0.586 **b** $x \approx 0.317$ or -6.317
   c $x \approx 2.766$ or -1.266 **d** $x \approx 3.409$ or -1.076
   e $x \approx 3.642$ or -0.892 **f** $x \approx 1.339$ or -2.539

2 **a** $x \approx 1.115$ or -0.448 **b** $x \approx 1.164$ or -2.578
   c $x \approx 0.451$ or 1.282

EXERCISE 21J

1 -11 or 10 **2** 6 or -2 **3** 9 and -3 **4** 1 **5** 11 metres

6 2 m   **7** $12\frac{1}{2}$ m by 10 m   **8** $\dfrac{5+\sqrt{5}}{2}$ and $\dfrac{5-\sqrt{5}}{2}$

9 **a** $x = \sqrt{31} - 1$ **b** $x = \dfrac{3+\sqrt{5}}{2}$

10 $BC = 5$ cm or 16 cm   **11** 10 cm, 24 cm, 26 cm

12 ≈ 51.6 m   **14** **a** $t = 20$ **b** 6 hours   **c** 30 km/h

15 $\frac{1}{6}$ or $\frac{-10}{-5}$ **16** 4 m   **17** $\frac{4}{3}$ or $\frac{3}{2}$

18 **a** **i** 75 m   **ii** 195 m   **iii** 275 m
   b **i** 2 sec and 14 sec   **ii** 0 sec and 16 sec
   c Each height occurs once on the way up and once on the way down.

19 **a** **i** a $40 loss   **ii** $480   **b** 10 or 62 cakes

20 Max takes $4\frac{1}{2}$ h, Sam takes 6 h   **21** 2 cm by 2 cm
22 **b** ≈ 8.8 cm   **23** 8 cm, 15 cm, 17 cm

REVIEW SET 21A

1 **a** $x = \pm\sqrt{2}$ **b** no real solutions   **c** $x = 0$ or 3
   d $x = 3$ or 8 **e** $x = -\frac{2}{5}$ or $\frac{3}{2}$ **f** $x = -\frac{7}{3}$ or 3

2 **a** $x = 0$ or 1 **b** no real solutions   **c** $x = 2 \pm \sqrt{5}$
   d no real solutions   **e** $x = 1$ or $-3\frac{1}{3}$ **f** $x = 5$ or $-1\frac{1}{2}$

3 **a** $x = \dfrac{-1 \pm \sqrt{3}}{2}$ **b** $x = \dfrac{2 \pm \sqrt{2}}{2}$

4 **a** $x \approx 0.349$ or -1.635 **b** $x \approx \pm 2.12$
   c $x \approx 1.45$ or -3.45

5 **a** -15 **b** -17 **c** $x = 6$ or -3

6 **a**

   b

   c
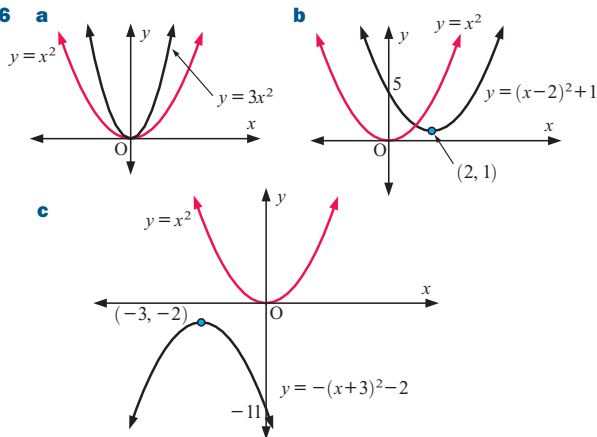
7 **a** **i** opens downwards
     ii 6
     iii 1 and -3
     iv $x = -1$
   b
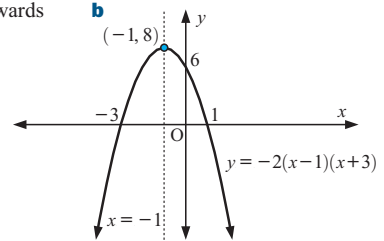
8 **a** **i** -15
     ii 5 and -3
     iii $x = 1$
     iv $(1, -16)$
   b
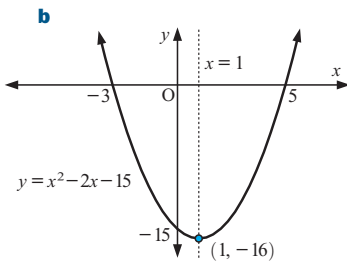
9 $f(x) = x^2 + 6x - 2$

10 **a** $V(2, -1)$ **c**
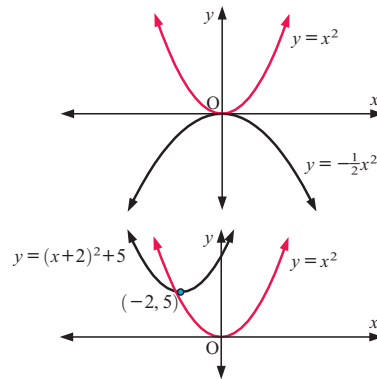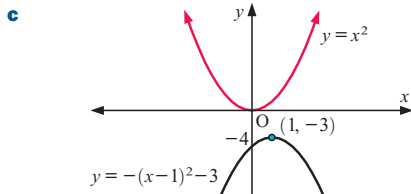   **b** 11
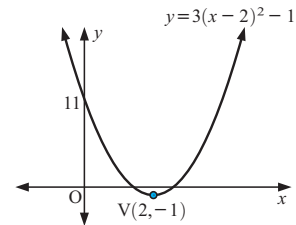
11 $f(x) = 2(x+1)^2 - 5$ **12** $f(x) = 9x^2 + 39x + 12$
13 $f(x) = \frac{9}{2}x^2 - 54x + 160$ **14** $3\sqrt{3}$ cm by $\sqrt{3}$ cm

15 $\dfrac{15 + 30\sqrt{2}}{7}$ cm   **16** **a** $x = 2$ **b** $x = 5$ **c** $x = 6$

17 **a** 2 seconds   **b** 80 m   **c** 6 seconds

REVIEW SET 21B

1 **a** $x = 3$ **b** $x = -5$ or 4 **c** no real solutions
   d $x = 8$ or -3 **e** $x = \pm 2$ **f** $x = -\frac{1}{2}$ or $\frac{2}{3}$

2 **a** $x = 2 \pm \sqrt{14}$ **b** $x = \dfrac{-1 \pm \sqrt{37}}{2}$

3 **a** $x = -5$ or 3 **b** $x = 2\frac{1}{2}$ or 4 **c** $x = -\frac{1}{3}$ or 3

4 **a** 1 **b** 13 **c** $x = -2$ or $\frac{3}{2}$

5 **a**

   b

c

$y = x^2$

O $(1, -3)$

-4

$y = -(x-1)^2 - 3$

6 **a** **i** opens upwards **ii** 12 **iii** 2 (touching) **iv** $x = 2$

b

$y = 3(x-2)^2$ $x = 2$

12

O $(2, 0)$
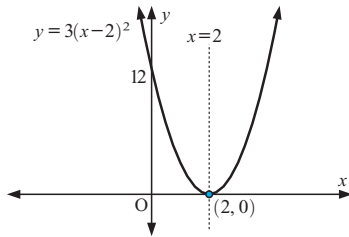
7 **a** **i** -10 **ii** 2 and 5 **iii** $x = \frac{7}{2}$ **iv** $(\frac{7}{2}, \frac{9}{4})$

b

$(\frac{7}{2}, \frac{9}{4})$

2   5

O

$y = -x^2 + 7x - 10$

$x = \frac{7}{2}$

8 **a** $h = 2$ **b** $a = -\frac{3}{4}$, $k = 3\frac{1}{2}$

9 **a**

-1 O 3

-6

$f(x) = 2(x-3)(x+1)$

b

$g(x) = -x(x+4)$

-4 O
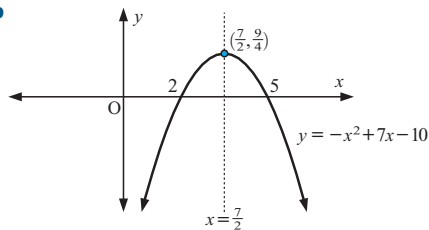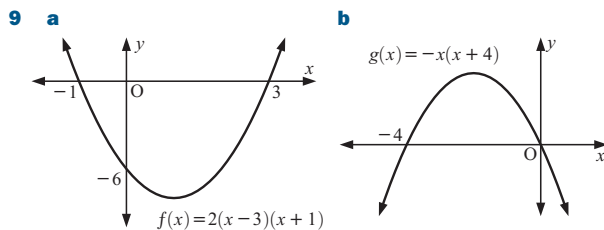
10 $f(x) = x^2 + 2x - 15$ **11** $f(x) = -(x-2)^2 + 2$

12 $f(x) = -2x^2 - 16x - 17$ **13** $\frac{4}{5}$ or -1

14 $\sqrt{3} - \frac{3}{2}$ and $\sqrt{3} + \frac{3}{2}$ **15** 12 grandchildren
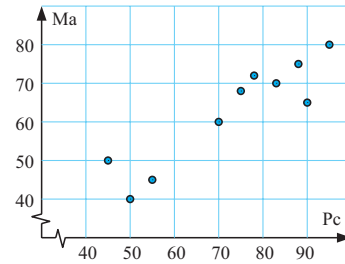
16 $CX = 60$ m, $AY \approx 34.377$ m

CHALLENGE

1 **b** $3\frac{1}{8}$ when $x = \frac{1}{4}$ **2** 150 of them

3 44.4 m by 50 m for each

EXERCISE 22A

1 **a** **i** negative association   **ii** linear   **iii** strong

  b **i** no association

  c **i** positive association   **ii** linear   **iii** moderate

  d **i** positive association   **ii** linear   **iii** weak

  e **i** positive association   **ii** not linear   **iii** moderate

  f **i** negative association   **ii** not linear   **iii** strong

2 **a** "...... as x increases, y increases"

  b "...... as T increases, d decreases"

  c "...... randomly scattered"

3 **a** **i**

Ma

80

70

60

50

40

40 50 60 70 80 90   Pc

  ii A moderate, positive, linear association.

  b

Art

90

80

70

60

40 50 60 70 80 90   Pc
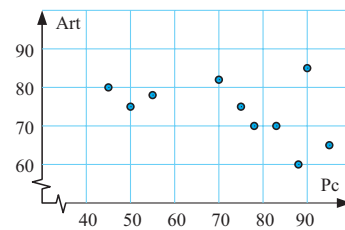
A weak, negative, linear association (virtually zero association).

4

Sales (£ '000)

25

20

15

10

5

0

0   5   10   15   20   25   Temp (°C)
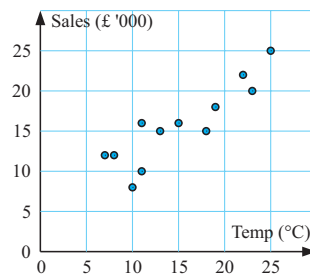
A moderate, positive, linear association exists between *Temperature* and *Sales*.

5

7   Shoe size
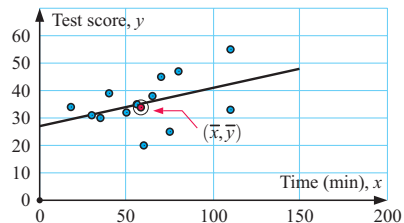
6

5

4

140 145 150 155 160 165   Height
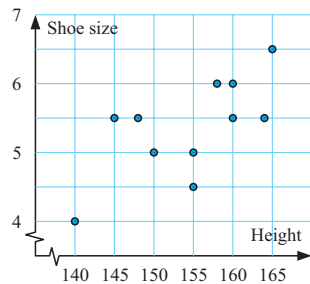
A weak, positive, linear association exists between *Shoe size* and *Height*.

EXERCISE 22B

1 **a** *Minutes spent preparing* (x)   **b** $\overline{x} \approx 61.5$, $\overline{y} \approx 35.7$

  c

Test score, y

60

50

40

30

20

10

0

0   50   100   150   200

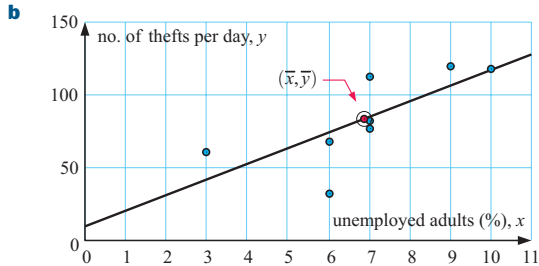$(\overline{x}, \overline{y})$

Time (min), x

  d There is a weak, positive correlation between the variables.

e There appears to be a weak, positive correlation between *minutes spent preparing* and *test score*. This means that as the time preparing increases, the score increases.
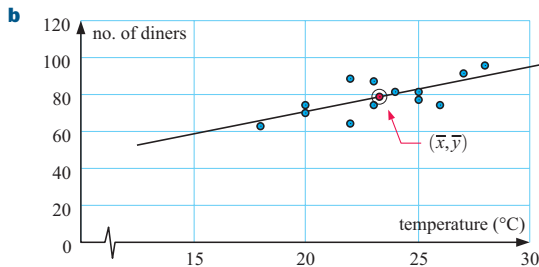
f/g On the graph above. **h** ≈ 31 marks

2 a $\overline{x} \approx 6.88$, $\overline{y} = 83.5$

b

c A moderate, positive correlation between the variables.

d/e On the graph above. **f** About 95 thefts per day.

3 a $\overline{x} \approx 23.3$, $\overline{y} \approx 78.8$
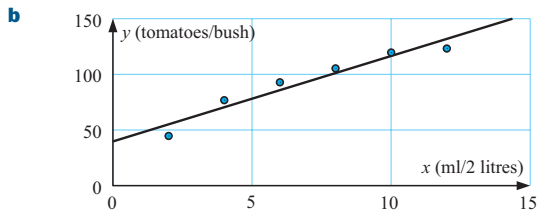
b

c There is a moderate, positive correlation between *number of diners* and *noon temperature*.

d/e On the graph above. **f i** 68 diners **ii** 93 diners

EXERCISE 22C

1 a $w \approx 0.903h - 81.0$ **b** 99.7 kg

2 a spray concentration

b

c $y \approx 7.66x + 40.1$

d The slope indicates that an increase in concentration of 1 ml/2 litres will increase the yield by approximately 7.66 tomatoes per bush. The vertical intercept indicates that a bush can be expected to produce 40.1 tomatoes when no spray is applied.

e 94 tomatoes/bush. Even though this is an interpolation this seems low compared with the yield at 6 and 8. Looking at the graph it would appear that the relationship is not linear.

3 a number of lawn beetles

b
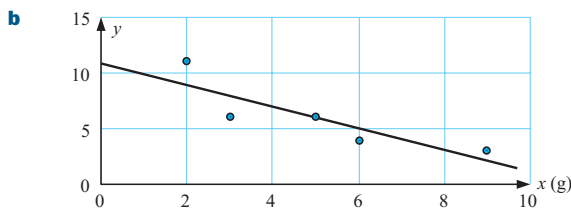
c $y \approx -0.967x + 10.8$

d The slope indicates that an increase in chemical of 1 gram will decrease the number of lawn beetles by 0.967 per square metre of lawn. The vertical intercept indicates that there are expected to be 10.8 beetles per square metre of lawn when no chemical is applied.
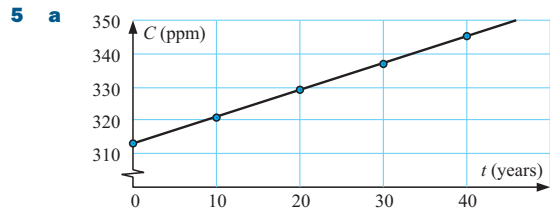
e 4 beetles. This is an interpolation and since the correlation is strong, the prediction is reasonable.

4 a
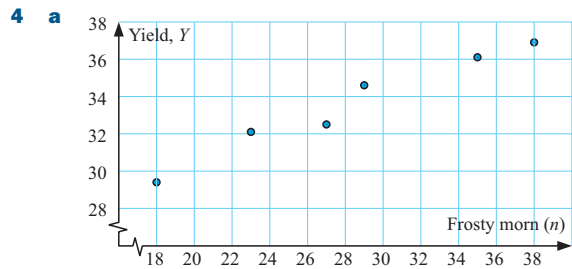
b $Y \approx 0.371n + 23.1$. A very strong, positive association.

c i ≈ 34.6 **ii** ≈ 38.7
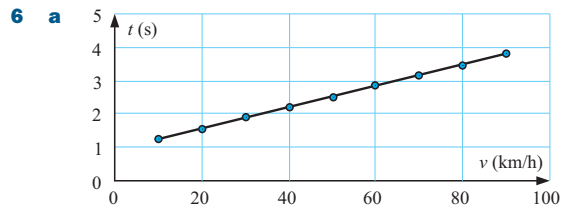
d "......, the *greater* the yield of cherries."

5 a

The linear model seems to fit exactly.

b $C = 0.8t + 313$ **c** 335 parts per million

d 361 parts per million

6 a

The most likely model is linear.

b $t \approx 0.0322v + 0.906$ **c i** 2.68 secs **ii** 4.44 secs

d 55 lies within the poles, ∴ **c i** should be reasonably reliable. 110 lies outside the poles, ∴ **c ii** could be unreliable.

e The vertical intercept is the reaction time, i.e., the time taken for the driver to respond to the red light when the car is stationary.

f A three second margin should allow sufficient time to avoid a collision with the car in front, taking into consideration that the car is slowing down as it comes to a stop.
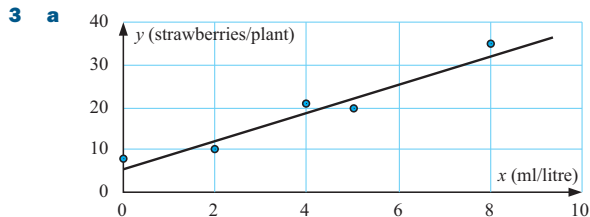
REVIEW SET 22A

1 a The independent variable is *Weeks of experience*. The dependent variable is *Defective items*.

b i weak **ii** negative

2 a
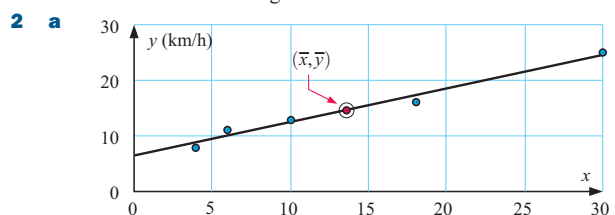
b $\overline{x} = 13.6, \quad \overline{y} = 14.6$ **c/d** On the graph.

e i 21 km/h **ii** 31 km/h

3 a

y (strawberries/plant)

x (ml/litre)

b There is a strong, positive correlation between *spray concentration* and *yield of strawberries*.

c This suggests that the higher the spray concentration, the higher the yield of strawberries.

d $y \approx 3.45x + 5.71$

e i 16 strawberries/plant **ii** 40 strawberries/plant

f As 10 lies outside the data range, this involves extrapolation and therefore may not be a reliable prediction.

4 a The width of the whorl is the dependent variable, the position of the whorl is the independent variable.
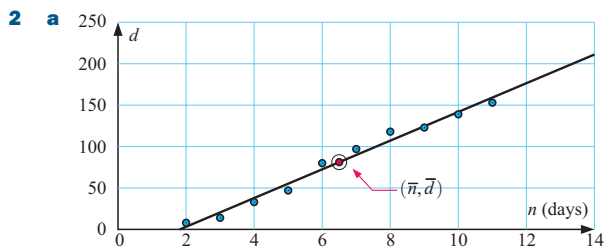
b

w (cm)

p
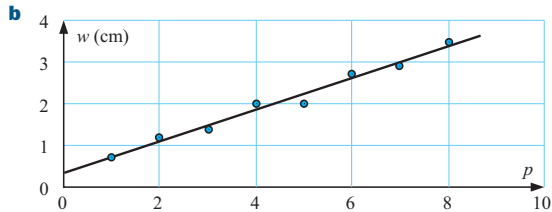
c There is a very strong, positive association between the variables.

d $w \approx 0.381p + 0.336$

e 5.67 cm As $p = 14$ is outside the poles, this prediction could be unreliable.

REVIEW SET 22B

1 a The independent variable is *age*. **b** No association exists.

c It is not sensible to find it as the variables are not linearly related.

2 a

d

$(\overline{n}, \overline{d})$

n (days)

A linear model does seem appropriate.

b $\overline{n} = 6.5, \quad \overline{d} = 81.2$ **c/d** On the graph.

e About 210 diagnosed cases.
Very unreliable as is outside the poles. The medical team have probably isolated those infected at this stage and there could be a downturn which may be very significant.
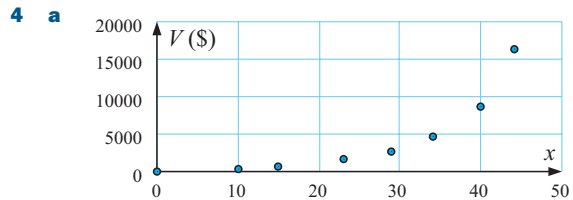
3 a

R

I (€ ’000)
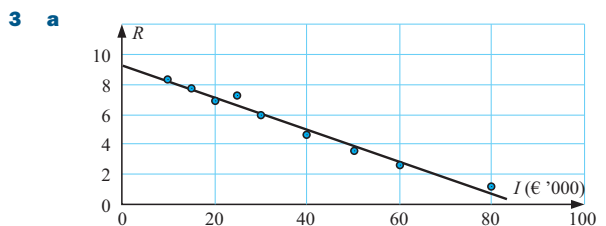
b $R \approx -0.106I + 9.25$ **c** 4.48 per 1000 people

d €100 000 gives a rate of -1.35 which is meaningless, i.e., €100 000 is outside the data range of this model.

e i $I = 25, \quad R = 7.3$ This goes against the trend of decrease in R for increase in I.

ii b $R \approx -0.104I + 9.08$ **c** 4.4 per 1000 people

4 a

V ($)

x

b A moderate, positive association exists between x and V.

c $V \approx 305x - 3050$ dollars

d The points clearly lie on a curve and not on a straight line. Also, when $x = 0$, $V \approx -3050$ dollars.

e As we cannot use the model in **c**, we need to draw a smooth curve through the points and extend it to $x = 50$.
A reasonable estimate is $V \approx \$28\,000$ (if the trend continues).

V ($)

x
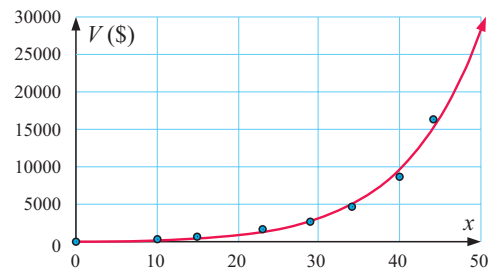
EXERCISE 23A.1

1 a $f(x) = x^3 - 7x + 6$ **b** $f(x) = 2x^3 + 9x^2 + x - 12$

c $f(x) = 2x^3 + 3x^2 - 12x - 20$

d $f(x) = x^3 + 3x^2 + 3x + 3$

2 a

$y = (x + 1)(x - 2)(x - 3)$

b

$y = -2(x + 1)(x - 2)(x - \frac{1}{2})$

c

$y = \frac{1}{2}x(x - 4)(x + 3)$

d

$y = 2x^2(x-3)$

e

$y = -\frac{1}{4}(x-2)^2(x+1)$

f

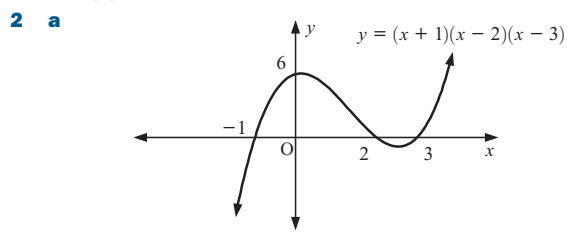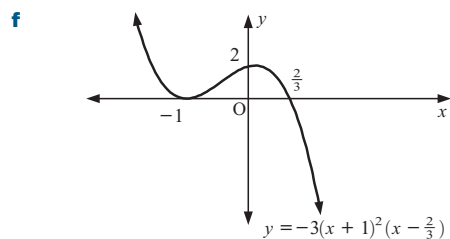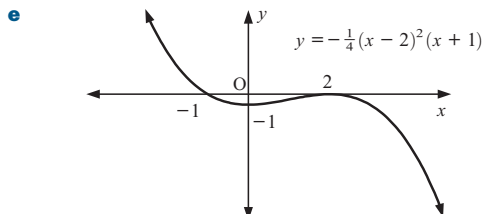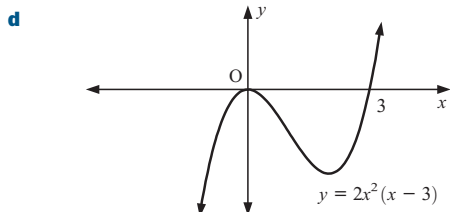$y = -3(x+1)^2\left(x-\frac{2}{3}\right)$
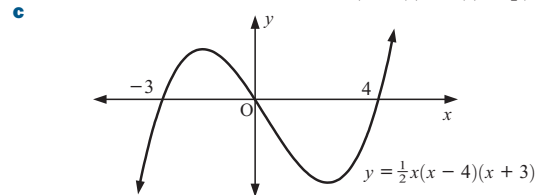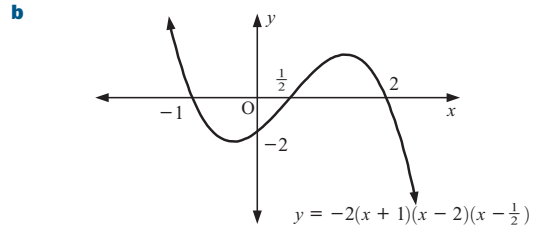
EXERCISE 23A.2

1 **a** $f(x) = 2(x+1)(x-2)(x-3)$
  b $f(x) = -2(x+3)(x+2)(2x+1)$
  c $f(x) = \frac{1}{4}(x+4)^2(x-3)$

2 **a** $f(x) = 2x^3 - 5x^2 - 6x + 9$
  b $f(x) = 3x^3 - 16x^2 + 15x + 18$

3 $b = 4, \ c = -10$ **4** $b = 2, \ d = -8$

EXERCISE 23B

1 **a** $f^{-1}(x) = x + 7$ **b** $f^{-1}(x) = \dfrac{x-2}{3}$

  c $f^{-1}(x) = \dfrac{3-4x}{2}$ **d** $f^{-1}(x) = \sqrt[3]{x}$

  e $f^{-1}(x) = \sqrt[3]{\dfrac{x-1}{2}}$ **f** $f^{-1}(x) = \dfrac{3x+1}{4}$

  g $f^{-1}(x) = x^2 - 1, \ x \geqslant 0$ **h** $f^{-1}(x) = \dfrac{x^2+5}{3}, \ x \geqslant 0$

  i $f^{-1}(x) = \dfrac{1}{x} + 2$

2 **a** **i** $f^{-1}(x) = 8 - x$ **ii** $f^{-1}(x) = \dfrac{9}{x}$

  b Both functions are their own inverses.

3 **a** For $f(x) = mx + c, \ m \neq 0$,

  $f^{-1}(x) = \dfrac{1}{m}x - \dfrac{c}{m}$ which is linear as $m \neq 0$

  b The gradients are reciprocals.

  c If (a, b) lies on $y = f(x)$ then $b = ma + c$

  and $f^{-1}(b) = \dfrac{1}{m}b - \dfrac{c}{m}$

  $= \dfrac{b-c}{m}$

  $= \dfrac{ma}{m}$

  $= a$ as $m \neq 0$

d **i** $f^{-1}(x) = 4x + 2$ **ii** $f^{-1}(x) = -\frac{3}{2}x - 1$

4 **b** **i** does   **ii** does not   **iii** does   **iv** does not

5 **a**

This horizontal line cuts the graph more than once.
$\therefore \ f^{-1}(x)$ does not exist.

  b

Likewise, $f^{-1}(x)$ does not exist.

  c

Likewise, $f^{-1}(x)$ does not exist.

6 **a** **i**

  ii

  iii
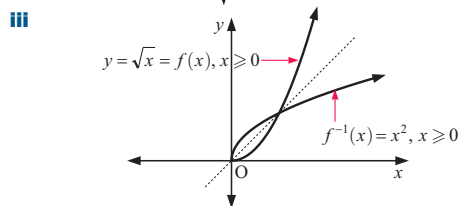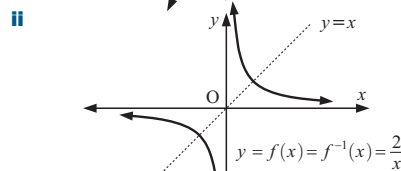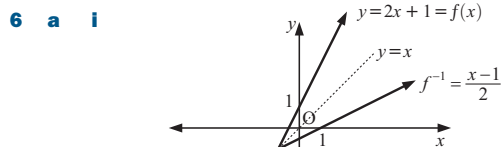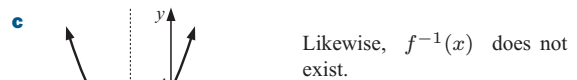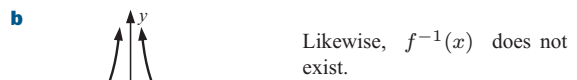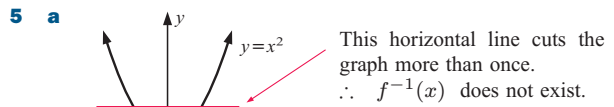
  b ...... a reflection of $y = f(x)$ in the line $y = x$.

7 From **6**, $y = f^{-1}(x)$ is a reflection of $y = f(x)$ in the line $y = x$. So, when $x = k$ is reflected in $y = x$ to become $y = k$, $y = k$ must be a horizontal asymptote of $y = f^{-1}(x)$.

8 **a** **i** $f^{-1}(x) = \dfrac{2}{x} + 3$

  ii

b i $f^{-1}(x) = -1 - \dfrac{3}{x}$

ii

$y = f^{-1}(x)$

$y = -1$

$f(x) = -\dfrac{3}{x+1}$

$x = -1$

c i $f^{-1}(x) = \dfrac{2x}{x-1}$

ii

$y = f^{-1}(x)$

$y = 2$

$y = 1$

$f(x) = \dfrac{x}{x-2}$

$x = 2$

$x = 1$

d i $f^{-1}(x) = \dfrac{x+1}{x-1} = f(x)$

ii

$y = 1$

$f(x) = f^{-1}(x) = \dfrac{x+1}{x-1}$

$x = 1$

e i $f^{-1}(x) = \sqrt[3]{\dfrac{x+1}{x}}$

ii

$y = 1$

$f(x) = \dfrac{1}{x^3 - 1}$

$y = f^{-1}(x)$

$x = 1$

f i $f^{-1}(x) = \dfrac{3x+1}{x-2}$

ii

$y = f^{-1}(x)$

$y = 3$

$y = 2$

$f(x) = \dfrac{2x+1}{x-3}$

$x = 2$ $x = 3$
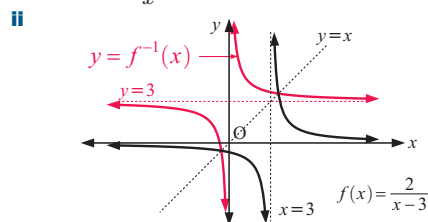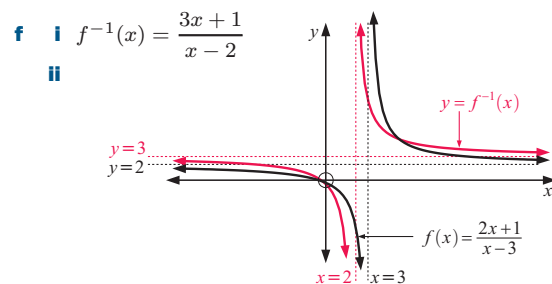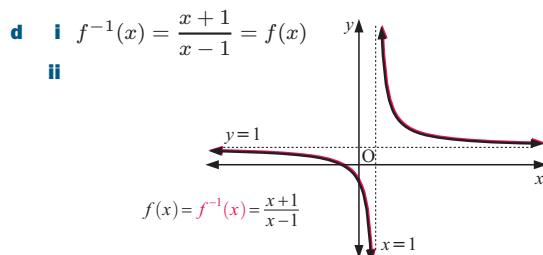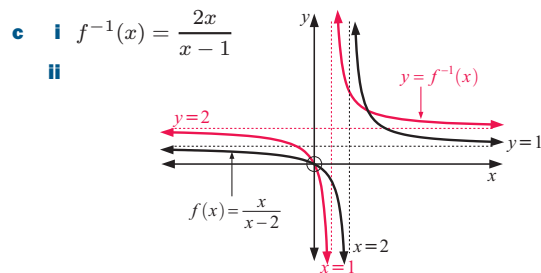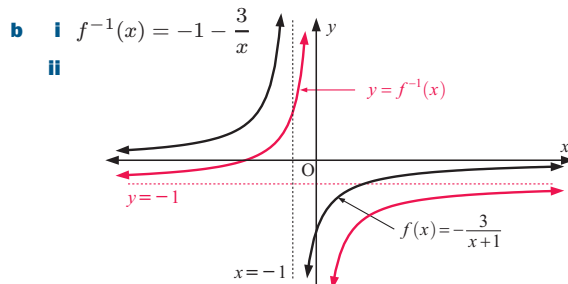
They are reflections of one another in the line $y = x$.
The domain of $y = f(x)$ is the range of $y = f^{-1}(x)$ and the range of $y = f(x)$ is the domain of $y = f^{-1}(x)$.

EXERCISE 23C.1

1 a i $(0, -3)$ **ii** -3 **iii** ± 1.73
b i $(\frac{1}{2}, -\frac{3}{2})$ **ii** -1 **iii** $-0.366, 1.366$
c i $(-\frac{1}{3}, -5)$ **ii** -4 **iii** $-1.079, 0.412$

2 a $y = |2x - 1| + 2$

$(\frac{1}{2}, 2)$

LS: $x = \frac{1}{2}$

b $y = |x(x-3)|$

LS: $x = 1\frac{1}{2}$

c $y = |(x-2)(x-4)|$

LS: $x = 3$

d $y = |x| + |x-2|$

$(2, 2)$

LS: $x = 1$

e $y = |x| - |x+2|$

$(-2, 2)$

$(0, -2)$

f $y = \left|9 - x^2\right|$

LS: $x = 0$

3 a $f(x) = x^3 - 4x^2 + 5x - 3, \quad -1 \leqslant x \leqslant 4$

$(4, 17)$

$f(x) = x^3 - 4x^2 + 5x - 3$

≈ 2.47

$(-1, -13)$
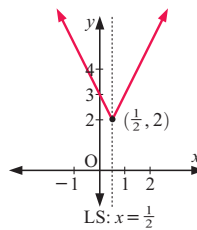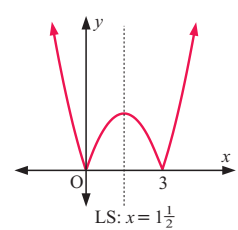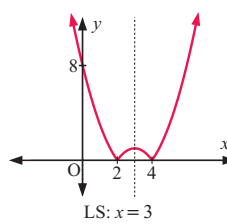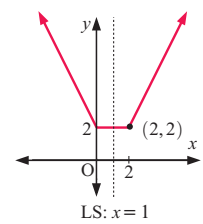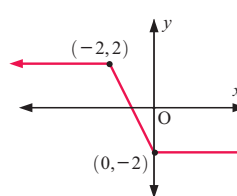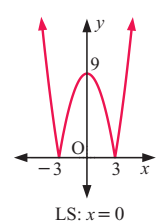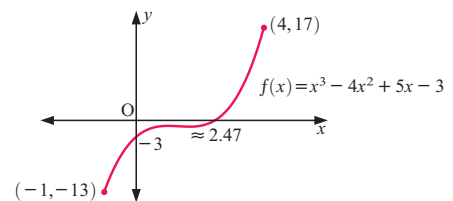
b x-intercept ≈ 2.47, y-intercept -3
c local maximum at $(1, -1)$
local minimum at $\approx (1.67, -1.15)$
d Range is $\{y \mid -13 \leqslant y \leqslant 17\}$

e

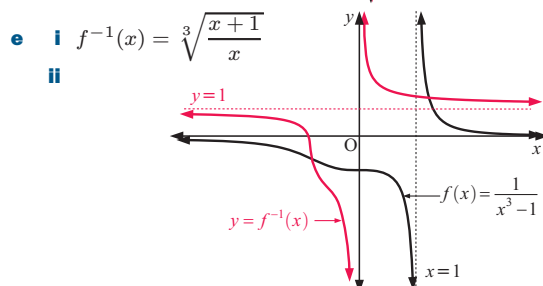| x	y	x	y
-1	-13	2	-1
-0.5	-6.63	2.5	0.125
0	-3	3	3
0.5	-1.38	3.5	8.38
1	-1	4	17
1.5	-1.13		

4 a $f(x) = x^4 - 3x^3 - 10x^2 - 7x + 3, \quad -4 \leqslant x \leqslant 6$

$(-4, 319)$

$(6, 249)$

3

b ≈ 5.17 **c** $\approx (3.72, -124)$

d

$(-2, 17)$ 3 $(1, -16)$
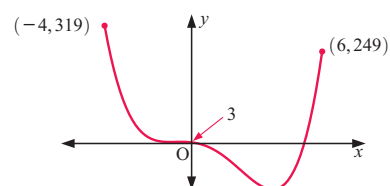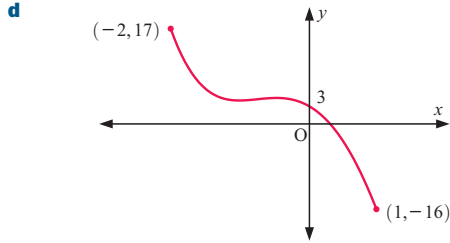
e local maximum at $\approx (-0.470, 4.44)$
local minimum at $(-1, 4)$

f

x	y	x	y
0	3	0.6	-5.32
0.1	2.20	0.7	-7.59
0.2	1.18	0.8	-10.1
0.3	-0.07	0.9	-12.9
0.4	-1.57	1.0	-16
0.5	-3.31		

5 a i $f(x) = \dfrac{4}{x-2}$
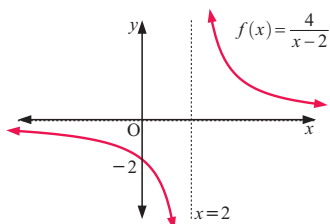
   ii $x = 2$, $y = 0$
   iii no x-intercept
     y-intercept -2
   iv no turning points
     exist

b i $f(x) = 2 - \dfrac{3}{x+1}$
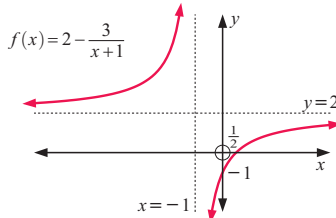
   ii $x = -1$, $y = 2$
   iii x-intercept $\frac{1}{2}$
     y-intercept -1
   iv no turning points
     exist

c i $f(x) = 2^x - 3$
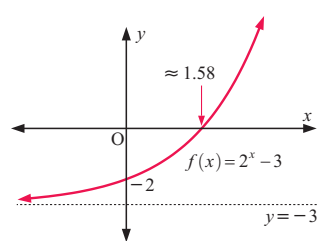
   ii $y = -3$
   iii x-intercept ≈ 1.58
     y-intercept -2
   iv no turning points
     exist

d i $f(x) = 2x + \dfrac{1}{x}$
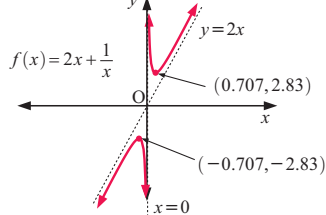
   ii $x = 0$, $y = 2x$
   iii no intercepts exist
   iv local maximum
     at $\approx (-0.707, -2.83)$
     local minimum
     at $\approx (0.707, 2.83)$

e i $f(x) = \dfrac{4x}{x^2 - 4x - 5}$
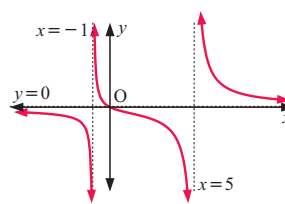
   ii $x = -1$, $x = 5$,
     $y = 0$
   iii x- and y-intercepts
     are both 0
   iv no turning points
     exist

f i $f(x) = 3^{-x} + 2$
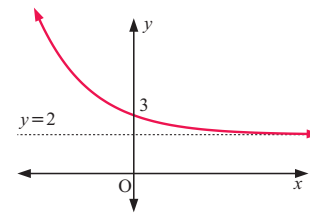
   ii $y = 2$
   iii no x-intercepts,
     y-intercept is 3
   iv no turning points
     exist

g i $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$
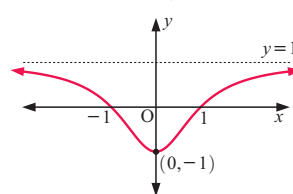
   ii $y = 1$
   iii x-intercepts ± 1
     y-intercept -1
   iv minimum turning
     point at $(0, -1)$

h i $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$
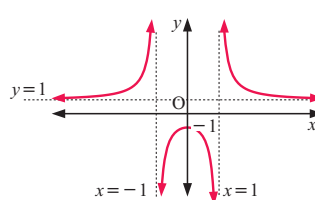
   ii $x = -1$, $x = 1$
     $y = 1$
   iii no x-intercepts
     y-intercept -1
   iv maximum turning
     point at $(0, -1)$

i i $f(x) = \dfrac{2^x + 3}{2^x + 1}$
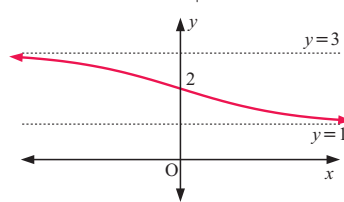
   ii $y = 1$, $y = 3$
   iii no x-intercepts
     y-intercept 2
   iv no turning points
     exist

6 a

x	$f(x)$
-2	-3.75
-1	-0.5
0	1
1	1
2	0
3	-1
4	0
5	7

b

$(5, 7)$ $(0.485, 1.16)$ ≈ -0.767 $(-2, -3\frac{3}{4})$ $f(x) = 2^x - x^2$
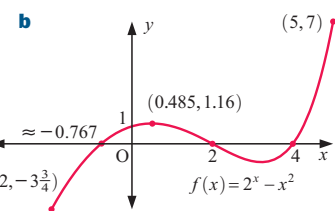
c ≈ -0.767, 2, 4

d maximum turning point at $\approx (0.485, 1.16)$
minimum turning point at $\approx (3.21, -1.05)$

7

x	$f(x)$	$g(x)$	$h(x)$
-2	0.25	3	1
-1.5	0.354	1.25	1.25
-0.5	0.707	-0.75	undefined
0	1	-1	-1
0.5	1.414	-0.75	-0.25
1	2	0	0
2	4	3	0.2
2.7	6.498	6.290	0.266
3.61	12.210	12.032	0.318

8 a

b $x = 0, \ y = x$

c $\approx (-0.725, 0.653), \ \approx (0.808, -0.429)$

9 a

b Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid 1 < y \leqslant 4\}$

c $y = 1$ **d** It does not meet $y = 5$.   **e** $1 < k < 4$

10 a

b $(2, 4)$ and $\approx (-2.595, 0.1655)$

c For $x > 2$ and for $-2.595 < x < 1$.
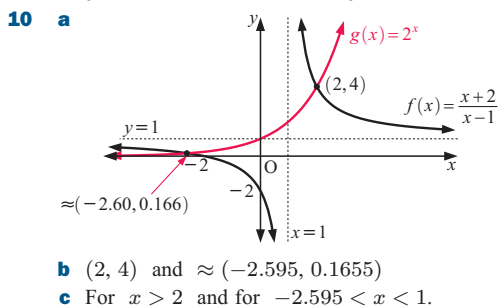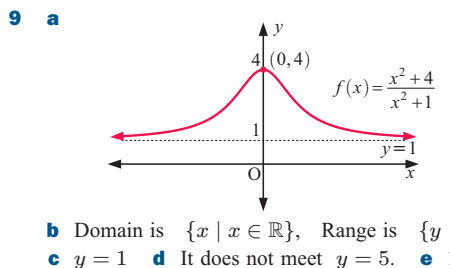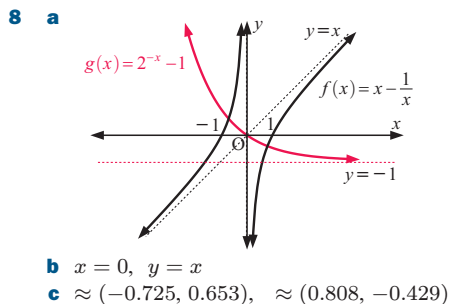
EXERCISE 23C.2

1 a $x = -1.5$ or 1 **b** $x \approx -1.24$ or 3.24

  c $x \approx -1.30$ or 2.30 **d** $x \approx -4.83$ or 0.828

  e $x \approx -1.57$ or 0.319 **f** $x \approx 0.697$ or 4.30

2 a $x \approx 0.458$ or 3.31 **b** $x \approx 1.70$

  c $x = -1$ or ≈ 1.35 **d** $x \approx 1.46$

  e $x \approx -0.686$ **f** $x \approx -2.512$

3 a $x \approx -1.67$ **b** $x \approx 3.21$ **c** $x \approx -0.846, 0, 2$ or 3.22

  d $x \approx -1.37$ or 2.07 **e** $x \approx 4.22$ **f** no solutions exist

4 a $(-2.56, -1.56), \ (-1, -4)$ and $(1.56, 2.56)$

  b no points of intersection   **c** $(3.21, 1.56)$

  d $(-1.21, -0.57)$ and $(1, 1)$

EXERCISE 23D

1 a ≈ -2 **b** ≈ 4 **c** 0 **d** ≈ -2

  e ≈ -1.5 **f** ≈ 0.7

2 a ...... point is *zero*   **b** ...... can *intersect* the curve. ......

REVIEW SET 23A

1 a

b
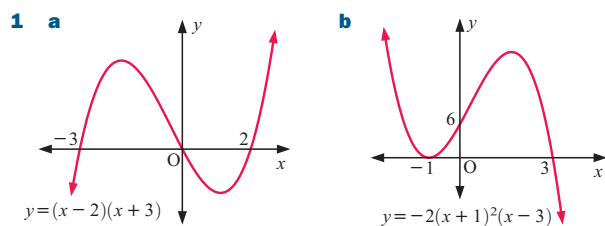
2 a $f(x) = -3x(x + 1)(x - 2)$ **b** $f(x) = (x - 1)^2(x - 4)$

3 $f(x) = 2(x + 1)^2(x - 3)$

  or $f(x) = 2x^3 - 2x^2 - 10x - 6$

4 a $f^{-1}(x) = \dfrac{x + 1}{4}$ **b** $g^{-1}(x) = \dfrac{1}{x} - 3$

5 a $f^{-1}(x) = \dfrac{3x + 5}{2}$ **b**
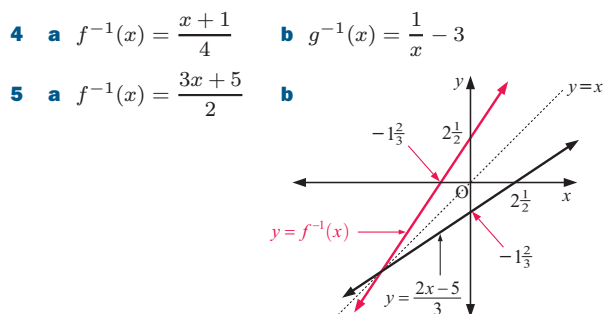
6 a $x \approx 2.01$ **b** $x \approx 1.38$ **c** $x \approx -7.26, \ -1.65$ or 1.34

7 a $\approx (-2.17, 2.41)$ and $(1.93, 0.457)$

  b $\approx (0.0773, 4.94), \ (0.594, 4.49)$ and $(2.705, -0.521)$

8 a
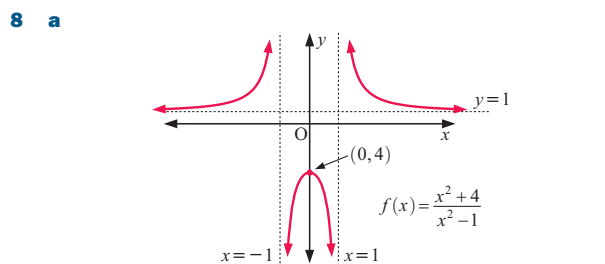
b $x = -1, \ x = 1, \ y = 1$

c Domain is $\{x \mid x \neq \pm 1, \ x \in \mathbb{R}\}$

  Range is $\{y \mid y \leqslant -4 \text{ or } y > 1\}$

d $k < -4$ or $k > 1$

9 ≈ -1

REVIEW SET 23B

1 a

b
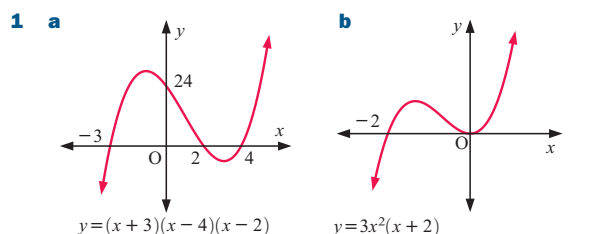
2 $c = 3, \ d = -10$

3 a $f^{-1}(x) = \dfrac{x}{8}$ **b** $f^{-1}(x) = 1 + \dfrac{2}{x}$

  c $f^{-1}(x) = x^2 - 3, \ x \geqslant 0$

4 a yes   **b** no   **c** yes

5 **a**

b $x = -4$, $x = 1$, $y = 0$

c x-intercept is 3, y-intercept is $\frac{3}{4}$

d minimum turning point at $\approx (-0.742, 0.659)$ maximum turning point at $\approx (6.74, 0.0607)$
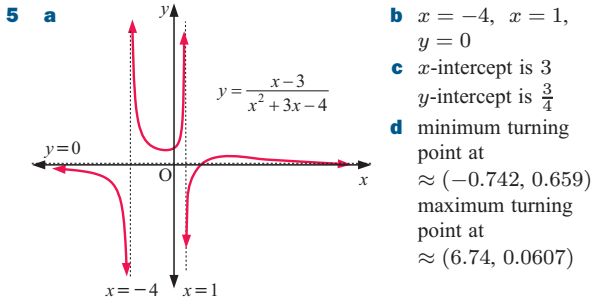
6 **a** $x \approx 2.18$ **b** $x \approx 3.29$ **c** $x \approx 0.505$

7 **a** $\approx (-1.70, -4.94)$ and $(1.26, 1.98)$
    b $\approx (-0.63, 2.50)$ and $(0.52, 3.76)$

8 **a**

x	$f(x)$	$g(x)$
-10	0.833	1.25
-5	0.913	1.118
0	1	1
5	1.095	0.894
10	1.2	0.8
15	1.315	0.716
20	1.44	0.64

b for $f(x)$: largest is 1.44 smallest is 0.833
for $g(x)$: largest is 1.25 smallest is 0.64

c
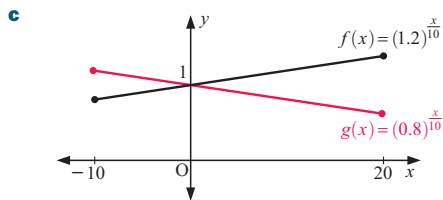
d $(0, 1)$ **e** $y = -0.01x + 1$

9 ≈ 3

EXERCISE 24A

1 **a** *Scale:* 1 cm ≡ 10 km/h

b *Scale:* 1 cm ≡ 10 m/s

c *Scale:* 1 cm ≡ 10 m

d

Scale: 1 cm ≡ 10 m/s

2 **a** *Scale:* 1 cm ≡ 15 Newtons

b *Scale:* 1 cm ≡ 15 Newtons

3 **a** *Scale:* 1 cm ≡ 20 km/h

b *Scale:* 1 cm ≡ 15 kg m/s

c *Scale:* 1 cm ≡ 10 km

d *Scale:* 1 cm ≡ 30 km/h
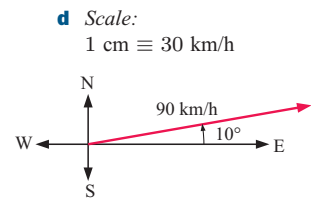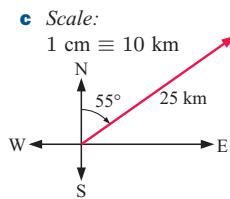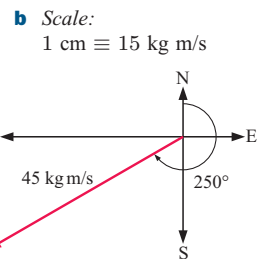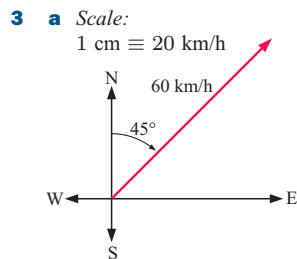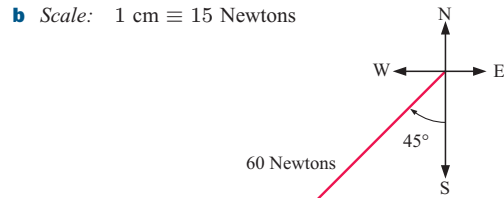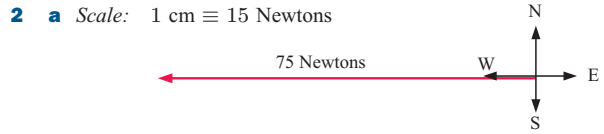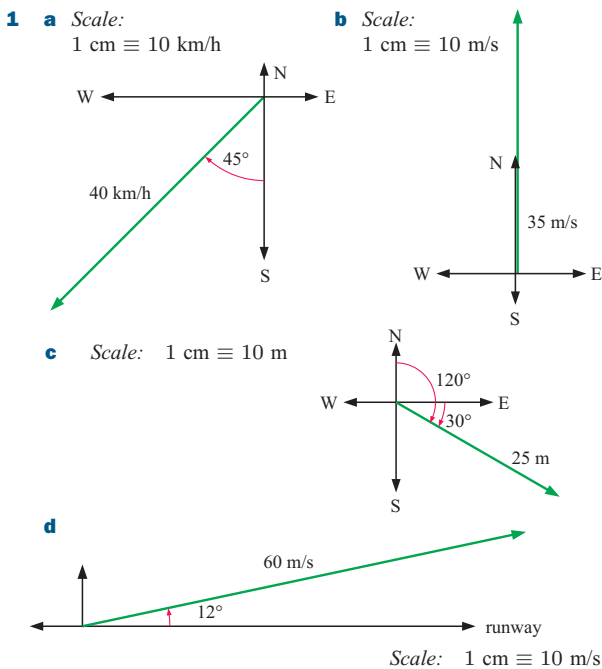
EXERCISE 24B

1 **a** **a**, **c** and **e**; **b** and **d**   **b** **a**, **b**, **c** and **d**
    c **a** and **b**; **c** and **d**   **d** none are equal
    e **a** and **c**; **b** and **d**

2 **a** **p**   **b** $-\mathbf{p}$ **c** **q**   **d** $-\mathbf{q}$ **e** **r**   **f** $-\mathbf{r}$

3 **a** false   **b** true   **c** false   **d** true   **e** true

EXERCISE 24C

1 **a**

b

c

d

e

f
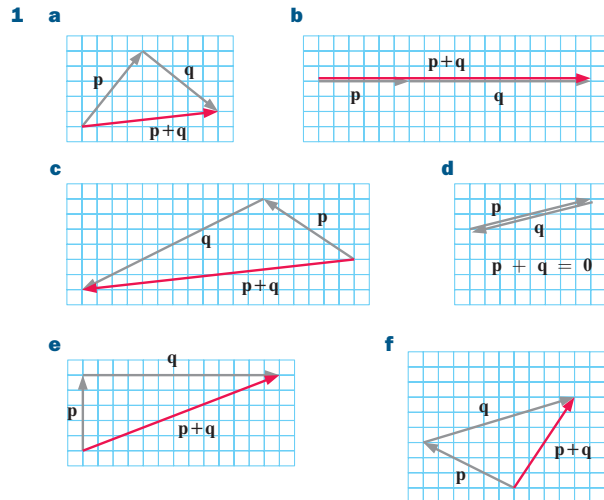
2 **a** $\overrightarrow{QS}$ **b** $\overrightarrow{PR}$ **c** $\overrightarrow{PQ}$ **d** $\overrightarrow{PS}$

3 Paola is 25 km from his starting point at a bearing of 347°.

4 **a** 600 km/h due north **b** 400 km/h due north
  c 510 km/h at a bearing of 011.3^o

5 The ship is travelling at a speed of 23.3 knots at a bearing of 134°.

EXERCISE 24D

1 **a**

  b

  c

  d

  e

  f
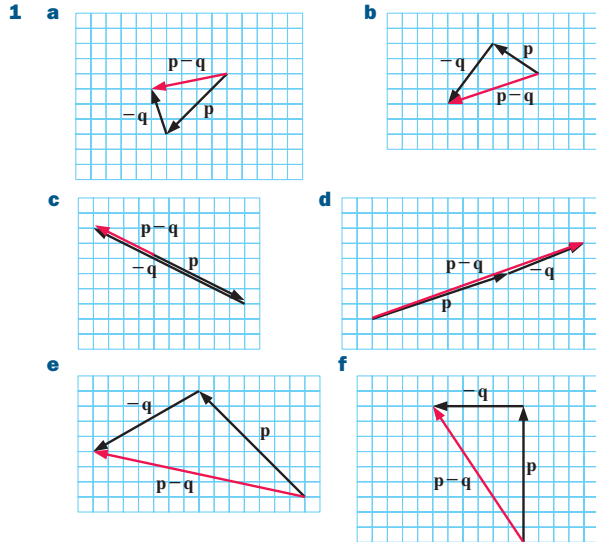
2 **a** $\overrightarrow{QS}$ **b** $\overrightarrow{PR}$ **c** $\mathbf{0}$ (zero vector) **d** $\overrightarrow{RQ}$ **e** $\overrightarrow{QS}$
  f $\overrightarrow{RQ}$

3 The plane must fly 4.57^o west of north at 501.6 km/h.

4 **a** The boat must head 26.6^o west of north. **b** 28.3 km/h

EXERCISE 24E.1

1 **a**

  b

  c

  d

2 **a** $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$ **c** $\begin{pmatrix} -3 \\ -4 \end{pmatrix}$ **d** $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$ **e** $\begin{pmatrix} -6 \\ 4 \end{pmatrix}$
  f $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

3 **a** $\begin{pmatrix} 9 \\ 3 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ **c** $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ **d** $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ **e** $\begin{pmatrix} 11 \\ -1 \end{pmatrix}$
  f $\begin{pmatrix} -4 \\ -1 \end{pmatrix}$ **g** $\begin{pmatrix} -9 \\ -3 \end{pmatrix}$ **h** $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$ **i** $\begin{pmatrix} -11 \\ 1 \end{pmatrix}$

4 **a** $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$ **c** $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ **d** $\begin{pmatrix} 3 \\ -3 \end{pmatrix}$ **e** $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$

EXERCISE 24E.2

1 **a** $\begin{pmatrix} 5 \\ -4 \end{pmatrix}$ **b** $\begin{pmatrix} 5 \\ -4 \end{pmatrix}$ **c** $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ **d** $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ **e** $\begin{pmatrix} 0 \\ -6 \end{pmatrix}$
  f $\begin{pmatrix} 0 \\ -6 \end{pmatrix}$ **g** $\begin{pmatrix} 4 \\ -6 \end{pmatrix}$ **h** $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$

2 **a** $\begin{pmatrix} 1 \\ 6 \end{pmatrix}$ **b** $\begin{pmatrix} -5 \\ 1 \end{pmatrix}$ **c** $\begin{pmatrix} -6 \\ 4 \end{pmatrix}$ **d** $\sqrt{10}$ units
  e $\sqrt{26}$ units **f** $5\sqrt{2}$ units

3 **a** $\begin{pmatrix} -3 \\ -3 \end{pmatrix}$ **b** $\begin{pmatrix} -4 \\ 9 \end{pmatrix}$

4 **a** $\sqrt{17}$ units **b** 6 units **c** $\sqrt{13}$ units
  d $\sqrt{26}$ units **e** $2\sqrt{5}$ units **f** $13|a|$ units

5 **a** **i** $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$ **ii** $\sqrt{13}$ units **b** **i** $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ **ii** $\sqrt{29}$ units
  c **i** $\begin{pmatrix} -3 \\ -4 \end{pmatrix}$ **ii** 5 units **d** **i** $\begin{pmatrix} -12 \\ 5 \end{pmatrix}$ **ii** 13 units

6 **a** $\begin{pmatrix} 160 \\ 240 \end{pmatrix}$ **b** $\begin{pmatrix} -80 \\ 120 \end{pmatrix}$ **c** $\begin{pmatrix} 80 \\ 40 \end{pmatrix}$
  d **i** ≈ 431 m **ii** ≈ 288 m **iii** ≈ 144 m **iv** ≈ 89 m
  e $\begin{pmatrix} 160 \\ 400 \end{pmatrix}$ This vector describes the position of the hole from the tee.
  Its length gives the length of the hole, i.e., ≈ 431 m.

7 **a** $\overrightarrow{SA} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$, $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, $\overrightarrow{CD} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$,
  $\overrightarrow{DE} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$, $\overrightarrow{EF} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}$, $\overrightarrow{FG} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$, $\overrightarrow{GS} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$
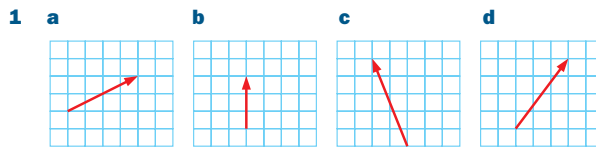 **b** $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
  c The finishing point is the same as the starting point, i.e., we are back where we started.

8 $k = \pm 4$ **9** **a** $k = 5$ **b** $|\mathbf{u}| = |\mathbf{v}| = 5\sqrt{2}$ units

EXERCISE 24F

1 **a**

  b

  c

  d

  e

  f

  g

  h
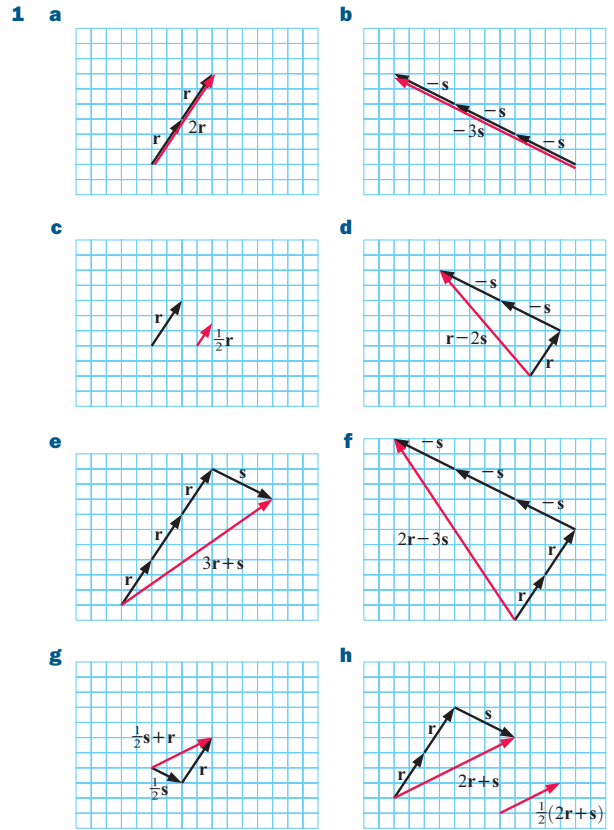
So,

  a $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$ **b** $\begin{pmatrix} -12 \\ 6 \end{pmatrix}$ **c** $\begin{pmatrix} 1 \\ 1\frac{1}{2} \end{pmatrix}$ **d** $\begin{pmatrix} -6 \\ 7 \end{pmatrix}$
  e $\begin{pmatrix} 10 \\ 7 \end{pmatrix}$ **f** $\begin{pmatrix} -8 \\ 12 \end{pmatrix}$ **g** $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ **h** $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

3 **a**

  b

  c
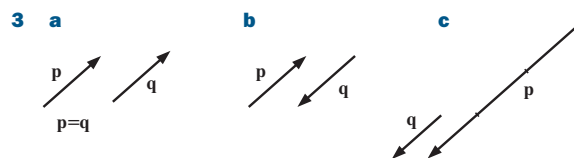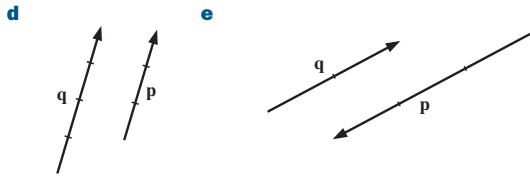
4 **a** $\begin{pmatrix} 9 \\ -8 \end{pmatrix}$ **b** $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$ **c** $\begin{pmatrix} -2\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$ **d** $\begin{pmatrix} -2\frac{1}{2} \\ 1 \end{pmatrix}$

5 $|k\mathbf{a}| = \left| k \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \right|$

$= \left| \begin{matrix} ka_1 \\ ka_2 \end{matrix} \right|$

$= \sqrt{k^2 a_1{}^2 + k^2 a_2{}^2}$

$= \sqrt{k^2(a_1{}^2 + a_2{}^2)}$

$= \sqrt{k^2}\sqrt{a_1{}^2 + a_2{}^2}$

$= |k|\,|\mathbf{a}|$

EXERCISE 24G

1 **a** $\mathbf{p} \parallel \mathbf{q}$ and $|\mathbf{p}| = 2\,|\mathbf{q}|$ **b** $\mathbf{p} \parallel \mathbf{q}$ and $|\mathbf{p}| = \frac{1}{2}\,|\mathbf{q}|$

c $\mathbf{p} \parallel \mathbf{q}$ and $|\mathbf{p}| = 3\,|\mathbf{q}|$ **d** $\mathbf{p} \parallel \mathbf{q}$ and $|\mathbf{p}| = \frac{1}{3}\,|\mathbf{q}|$

2 $k = -10$

3 $\overrightarrow{PQ} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$, $\overrightarrow{SR} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$ $\Rightarrow$ $\overrightarrow{PQ} \parallel \overrightarrow{SR}$ and $\left|\overrightarrow{PQ}\right| = \left|\overrightarrow{SR}\right|$

which is sufficient to deduce that PQRS is a parallelogram.

4 **a** C(2, 5) **b** D(9, 0)

5 **a** $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, $\overrightarrow{CD} = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$

b $\overrightarrow{CD} = 2\overrightarrow{AB}$, which implies that $\overrightarrow{CD}$ and $\overrightarrow{AB}$ have the same direction and $\therefore$ are parallel.

c $k = -7$

6 **a** yes **b** M is (3, 4) and N is (6, 2)

$\overrightarrow{MN} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 6 \\ -4 \end{pmatrix} = 2\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$\therefore$ $\overrightarrow{BC} \parallel \overrightarrow{MN}$ {same direction}

7 **a** $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} -6 \\ -12 \end{pmatrix}$ **b** $\overrightarrow{BC} = -3\overrightarrow{AB}$, i.e., $k = -3$

c B, C and A are collinear and CA : AB = 2 : 1

8 **a** P(2, 8), Q(5, 7), R(0, 4), S(−3, 5)

b $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\overrightarrow{SR} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ **c** $\overrightarrow{SP} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, $\overrightarrow{RQ} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

d PQ ∥ SR and PQ = SR, lengthwise
SP ∥ RQ and SP = RQ, lengthwise
$\therefore$ PQRS is a parallelogram.

EXERCISE 24H

1 **a** $\mathbf{r} + \mathbf{s}$ **b** $-\mathbf{t} - \mathbf{s}$ **c** $\mathbf{r} + \mathbf{s} + \mathbf{t}$

2 **a** $\mathbf{p} + \mathbf{q}$ **b** $\mathbf{q} + \mathbf{r}$ **c** $\mathbf{p} + \mathbf{q} + \mathbf{r}$

3 **a** $\frac{1}{2}\mathbf{q}$ **b** $\mathbf{p} + \mathbf{q}$ **c** $-\frac{1}{2}\mathbf{q}$ **d** $\mathbf{p} + \frac{1}{2}\mathbf{q}$

4 **a** $-\mathbf{s} - \mathbf{r} + \mathbf{t}$ **b** $\frac{1}{2}(-\mathbf{s} - \mathbf{r} + \mathbf{t})$

c $\frac{1}{2}\mathbf{s} + \frac{1}{2}\mathbf{t} - \frac{1}{2}\mathbf{r}$ **d** $\frac{1}{2}\mathbf{r} + \frac{1}{2}\mathbf{s} + \frac{1}{2}\mathbf{t}$

5 $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$ **6** $\overrightarrow{OP} = \frac{3}{7}\mathbf{a} + \frac{4}{7}\mathbf{b}$

$= \mathbf{a} + \frac{1}{2}\overrightarrow{AB}$, etc.

7 **a** $\mathbf{q}$ **b** $2\mathbf{q}$ **c** $\mathbf{p} + \mathbf{q}$ **d** $\mathbf{p} + 2\mathbf{q}$

e $-\mathbf{p} - 2\mathbf{q}$ **f** $\mathbf{p} + \mathbf{q}$

8 **a** **i** $\mathbf{a} - \mathbf{b}$ **ii** $\overrightarrow{MN} = \overrightarrow{MB} + \overrightarrow{BN}$

$= \frac{1}{2}\mathbf{b} + \frac{1}{2}(\mathbf{a} - \mathbf{b})$

$= \frac{1}{2}\mathbf{a}$

b $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{OA}$ i.e., MN is parallel to OA and has half its length.

c **i** $\overrightarrow{OM} = \frac{1}{2}\mathbf{b}$ **ii** $\overrightarrow{ON} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$ **iii** $\frac{1}{2}\mathbf{b} + \frac{1}{4}\mathbf{a}$

9 **a** **i** $\frac{1}{2}\mathbf{p}$ **ii** $\frac{1}{3}\mathbf{p} + \frac{2}{3}\mathbf{q}$ **iii** $\mathbf{p} - \mathbf{q}$ **iv** $\frac{1}{3}\mathbf{p} - \frac{1}{3}\mathbf{q}$

b As $\overrightarrow{DP} = \frac{1}{3}\overrightarrow{DB}$, DP is parallel to DB and $\frac{1}{3}$ its length. Thus, D, P and B are collinear and DP : PB = 1 : 2.

REVIEW SET 24A

1 **a** **b** **c**
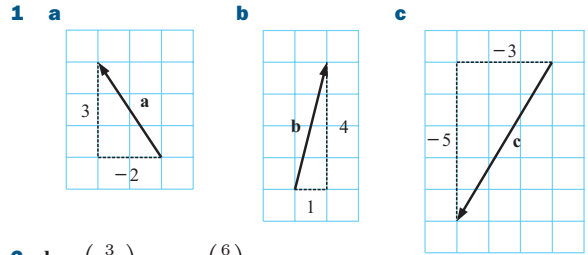
2 $\mathbf{d} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\mathbf{e} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$

3

Scale:
1 mm ≡ 2 km/h

4 **a**

b 093.8^o

c 300.7 km/h

5 **a** **b** **i** $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$ **ii** $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$ **iii** $\begin{pmatrix} 2 \\ 12 \end{pmatrix}$
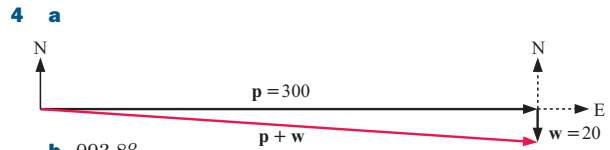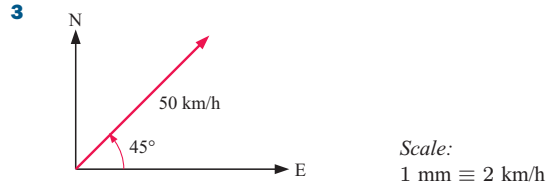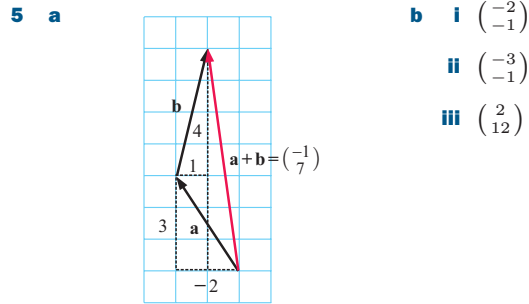
6 **a** **b** **i** $\begin{pmatrix} -6 \\ -10 \end{pmatrix}$ **ii** $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ **iii** $\begin{pmatrix} 9 \\ 12 \end{pmatrix}$
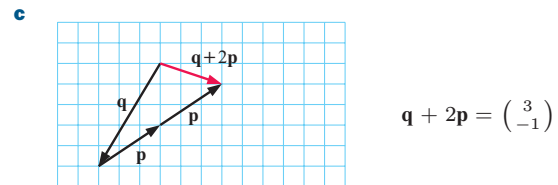
c $\mathbf{q} + 2\mathbf{p} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

7 a
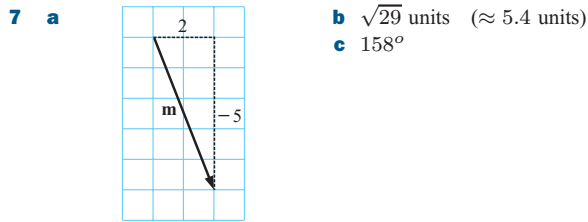
b $\sqrt{29}$ units   (≈ 5.4 units)

c 158°

8 a i $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ **ii** $\begin{pmatrix} -3 \\ -3 \end{pmatrix}$ **b** $\overrightarrow{AC} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$ **c** $\sqrt{34}$ units

9 $k = -10$ **10 a** $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ **b** 5 units

11 a $\overrightarrow{CA} = -\mathbf{c}$ **b** $\overrightarrow{AB} = -\mathbf{a} + \mathbf{b}$ **c** $\overrightarrow{OC} = \mathbf{a} + \mathbf{c}$

d $\overrightarrow{BC} = -\mathbf{b} + \mathbf{a} + \mathbf{c}$

12 a $\overrightarrow{DC} = 3\mathbf{q}$ **b** $\overrightarrow{CB} = -\mathbf{p} - \mathbf{q}$ **c** $\overrightarrow{BT} = \frac{2}{3}(\mathbf{p} + \mathbf{q})$

REVIEW SET 24B

1

2 a $\mathbf{m} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$ **b** $\mathbf{n} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$

3

Scale:
1 cm $\equiv$ 50 km

4 a
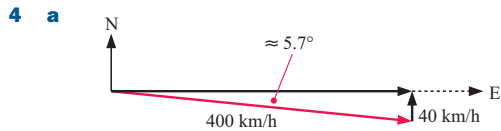
b ≈ 398 km/h in direction 096°

5 a $\begin{pmatrix} 1 \\ -6 \end{pmatrix}$ **b** $\sqrt{37}$ units **6** $n = 6$

7 a

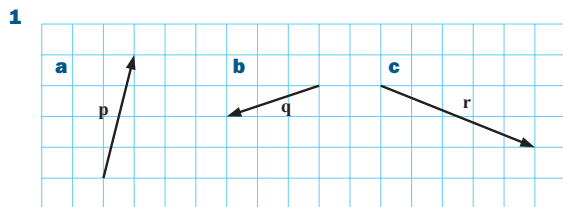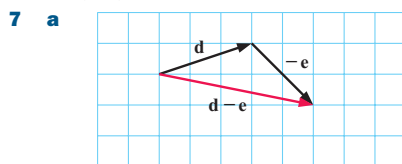**b** $\mathbf{d} - \mathbf{e} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$ **c i** $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ **ii** $\begin{pmatrix} 18 \\ -2 \end{pmatrix}$

8 We get the zero vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$

9 a $-\mathbf{p}$ **b** $\frac{1}{2}\mathbf{q}$ **c** $\frac{1}{2}\mathbf{q} - \mathbf{p}$ **d** $\mathbf{q}$

10 *Scale:*
1 cm $\equiv$ 2 km/h
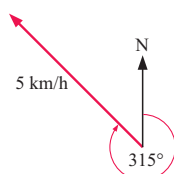
11 a i $\mathbf{p}$ **ii** $\mathbf{q}$ **iii** $\mathbf{p} + \mathbf{q}$ **iv** $2\mathbf{p} + 2\mathbf{q}$

b $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AC}$ which means that PQ and AC are parallel and PQ is half as long as AC.

c ...... triangle is *parallel* to the third side and *half* its length.

12 A$(-2, -4)$

13 a i $\overrightarrow{UX} = \frac{4}{5}\mathbf{v}$ **ii** $\overrightarrow{OX} = \mathbf{u} + \frac{4}{5}\mathbf{v}$

b i $\overrightarrow{VY} = \frac{5}{4}\mathbf{u}$

ii $\overrightarrow{VW} = \mathbf{u}$ and $\overrightarrow{VY} = \frac{5}{4}\mathbf{u}$ $\therefore$ $\overrightarrow{VY} = \frac{5}{4}\overrightarrow{VW}$

Thus VY is parallel to VW and $1\frac{1}{4}$ times its length.
i.e.,   V, Y and W are collinear and   VW : WY = 4 : 1.

CHALLENGE

1 a The quadrilateral formed is a parallelogram.

b The quadrilateral formed by joining the midpoints of the sides of any quadrilateral is a parallelogram.

2 a $r\mathbf{a} + s\mathbf{b} = m\mathbf{a} + n\mathbf{b}$
$\Rightarrow$ $(r - m)\mathbf{a} = (n - s)\mathbf{b}$
If $r - m \neq 0$ and $n - s \neq 0$
then $\mathbf{a} \parallel \mathbf{b}$
But $\mathbf{a} \nparallel \mathbf{b}$, $\therefore$ $r - m = 0$ and $n - s = 0$
$\therefore$ $r = m$ and $n = s$

b i $\overrightarrow{OP} = k\overrightarrow{OC}$ **ii** $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$
$= k(\mathbf{b} + \frac{1}{2}\mathbf{a})$ $= \mathbf{a} + t\overrightarrow{AB}$
    $= \mathbf{a} + t(\mathbf{b} - \mathbf{a})$

iii $k = \frac{2}{3}$, $\overrightarrow{OP} = \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$

EXERCISE 25A

1 a impossible **b** equally likely
c extremely unlikely **d** very unlikely **e** certain
f extremely unlikely **g** very likely
h extremely likely **i** likely **j** very unlikely
k extremely unlikely **l** very likely

2 a Event B **b** Event A

3 a unlikely **b** extremely unlikely **c** likely
d impossible **e** extremely likely

EXERCISE 25B

1 0.55 **2** 0.84 **3** ≈ 0.0894 **4** ≈ 0.256 **5** ≈ 0.331

6 a ≈ 0.243 **b** ≈ 0.486

7 a P(winning) ≈ 0.548 **b** P(2 child family) ≈ 0.359

8 a 407 people **b**

c i ≈ 0.229
ii ≈ 0.201
iii ≈ 0.307

Brand	Freq	Rel Freq
Silktouch	125	0.307
Super	107	0.263
Just Soap	93	0.229
Indulgence	82	0.201
Total	407	1

9 a

Outcome	Freq	Rel Freq
0 heads	121	0.247
1 head	259	0.530
2 heads	109	0.223
Total	489	1

b i ≈ 0.247
ii ≈ 0.530
iii ≈ 0.223

10 a 1083 people **b**

c i 0.25
ii 0.75

Colour	Freq	Rel Freq
Green	361	0.25
Red	1083	0.75
Total	1444	1

11 a 5235 tickets **b**

c ≈ 0.207

Ticket Type	Freq	Rel Freq
Adult	3762	0.719
Concession	1084	0.207
Child	389	0.074
Total	5235	1

12 a

Councillor	Freq	Rel Freq
Mr Tony Trimboli	2167	0.361
Mrs Andrea Sims	724	0.121
Mrs Sara Chong	2389	0.398
Mr John Henry	720	0.12
Total	6000	1

b ≈ 0.519

EXERCISE 25C

1 **a** ≈ 0.501 **b** ≈ 0.519 **c** ≈ 0.272 **d** ≈ 0.563
   e ≈ 0.416

2 **a**

	Like	Dislike	Σ
Junior students	87	38	125
Senior students	129	56	185
Σ	216	94	310

b **i** ≈ 0.281 **ii** ≈ 0.416 **iii** ≈ 0.697 **iv** ≈ 0.597

3 **a** ≈ 0.863 **b** ≈ 0.137 **c** ≈ 0.333 **d** ≈ 0.0344
   e ≈ 0.436 **f** ≈ 0.0633 **g** ≈ 0.691

EXERCISE 25D

1 65 days   **2** 13 of them   **3** **a** 0.36   **b** 26 backs
4 125 of them   **5** **a** 100   **b** 200   **c** 400   **d** 500

6 **a** $\frac{1}{4}$ **b** 75 times

7 **a** 0.11   **b** **i** 3140 points   **ii** 31.4 points

EXERCISE 25E

1 **a** {A, B, C, D}   **b** {BB, BG, GB, GG}
   c {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD,
   BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB,
   CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC,
   DBCA, DCAB, DCBA}
   d {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}
   e **i** {HH, HT, TH, TT}
      ii {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}
      iii {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT,
         HTTH, THHT, THTH, TTHH, HTTT, THTT, TTHT,
         TTTH, TTTT}

2 **a** [coin vs die scatter graph]
   b [die 2 vs die 1 scatter graph]
   c [spinner vs die scatter graph]
   d [spinner 2 vs spinner 1 scatter graph]

3 **a** [tree diagram 5-cent / 10-cent]
   b [tree diagram coin / spinner]

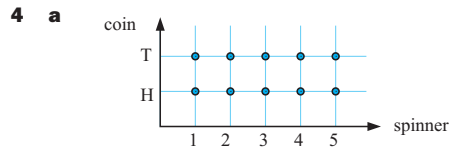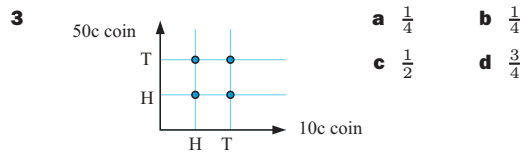**c** [tree diagram spinner 1 / spinner 2]

d [tree diagram ticket 1 / ticket 2]

e [tree diagram bead 1 / bead 2]

4 [Venn diagram with sets H and G; regions 3, 7, 8, 2, universe U]
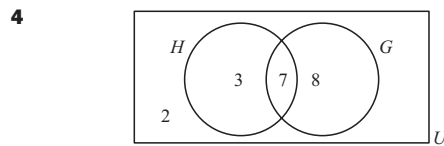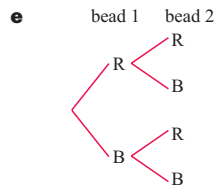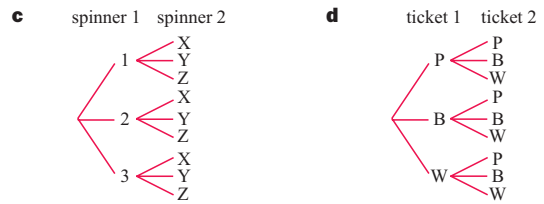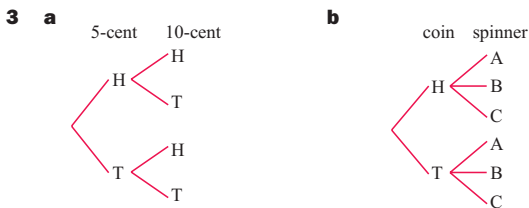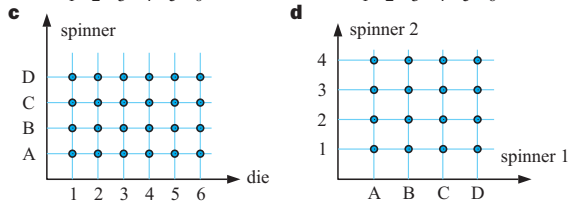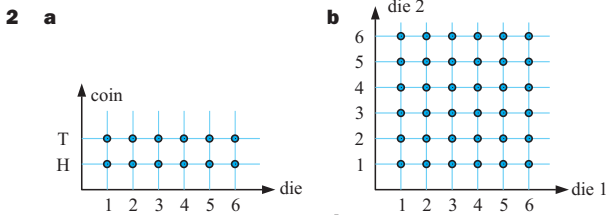
EXERCISE 25F

1 **a** {ODG, OGD, DOG, DGO, GOD, GDO}
   b **i** $\frac{1}{6}$ **ii** $\frac{1}{3}$ **iii** $\frac{2}{3}$ **iv** $\frac{1}{3}$

2 **a** $\frac{17}{43}$ **b** $\frac{5}{43}$ **c** $\frac{24}{43}$ **d** $\frac{19}{43}$ **e** $\frac{19}{43}$ **f** $\frac{17}{24}$ **g** $\frac{5}{17}$

3 [50c coin vs 10c coin scatter graph]   **a** $\frac{1}{4}$ **b** $\frac{1}{4}$
   c $\frac{1}{2}$ **d** $\frac{3}{4}$

4 **a** [coin vs spinner scatter graph]
   b 10 outcomes   **c** **i** $\frac{1}{10}$ **ii** $\frac{3}{10}$ **iii** $\frac{2}{5}$ **iv** $\frac{3}{5}$

5 {ABK, AKB, BAK, BKA, KAB, KBA}
   a $\frac{1}{3}$ **b** $\frac{1}{3}$ **c** $\frac{1}{3}$ **d** $\frac{2}{3}$

6 **a** {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}
   b **i** $\frac{1}{8}$ **ii** $\frac{1}{8}$ **iii** $\frac{1}{8}$ **iv** $\frac{3}{8}$ **v** $\frac{1}{2}$ **vi** $\frac{7}{8}$

7 **a** $\frac{3}{8}$ **b** $\frac{19}{24}$

8 **a** {PQRS, PQSR, PRQS, PRSQ, PSQR, PSRQ, QPRS, QPSR,
   QRPS, QRSP, QSPR, QSRP, RPQS, RPSQ, RQPS, RQSP,
   RSPQ, RSQP, SPQR, SPRQ, SQPR, SQRP, SRPQ, SRQP}
   b **i** $\frac{1}{2}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{2}$ **iv** $\frac{1}{2}$

9 **a** [die 2 vs die 1 scatter graph]   **b** **i** $\frac{1}{18}$ **ii** $\frac{1}{6}$
      iii $\frac{11}{36}$ **iv** $\frac{5}{9}$
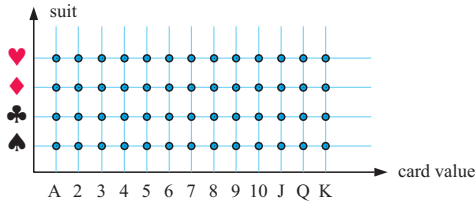   **v** $\frac{1}{4}$ **vi** $\frac{1}{6}$

10 **a** $\frac{4}{15}$ **b** $\frac{3}{10}$ **c** $\frac{7}{10}$

11 {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, HTTH, THTH, TTHH, THHT, TTTH, TTHT, THTT, HTTT, TTTT}

  a $\frac{1}{16}$ **b** $\frac{3}{8}$ **c** $\frac{5}{16}$ **d** $\frac{15}{16}$ **e** $\frac{1}{4}$

12 **a**

  b **i** $\frac{1}{13}$ **ii** $\frac{1}{52}$ **iii** $\frac{1}{4}$ **iv** $\frac{3}{13}$ **v** $\frac{1}{26}$ **vi** $\frac{1}{2}$

  vii $\frac{4}{13}$ **viii** 0

13 **a** $\frac{11}{14}$ **b** $\frac{5}{28}$ **c** $\frac{1}{28}$

EXERCISE 25G.1

1 **a** $\frac{1}{10}$ **b** $\frac{1}{5}$ **2** **a** $\frac{3}{14}$ **b** $\frac{4}{21}$

3 **a** $\frac{8}{21}$ **b** $\frac{1}{7}$ **c** $\frac{2}{7}$ **4** **a** 0.0441 **b** 0.6241

5 **a** $\frac{2}{15}$ **b** $\frac{2}{5}$ **c** $\frac{1}{5}$ **d** $\frac{4}{15}$

6 **a** **i** 0.405 **ii** 0.595 **b** 0.164 **c** 0.354

EXERCISE 25G.2

1 **a** **i** $\frac{5}{14}$ **ii** $\frac{15}{56}$ **iii** $\frac{15}{56}$ **iv** $\frac{3}{28}$

  b Because these 4 events are the only possible outcomes. One of them must occur.

2 **a** **i** $\frac{1}{3}$ **ii** $\frac{4}{15}$ **iii** $\frac{8}{15}$

  b The possibilities are: WW, WY, YW, YY.
  The 3 events do not cover all these possibilities.
  So, the probability sum should not be 1.

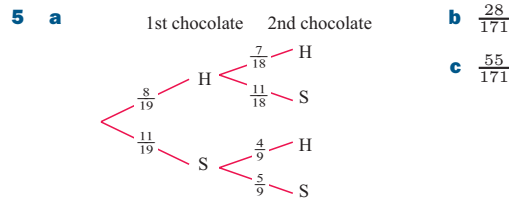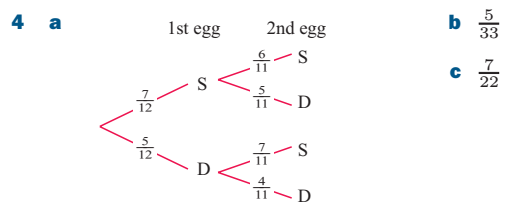**3** **a** $\frac{1}{14}$ **b** $\frac{1}{56}$ **c** $\frac{1}{28}$

EXERCISE 25H

1 **a**

  b $\frac{4}{25}$

  c $\frac{1}{25}$

  d $\frac{16}{25}$

  e $\frac{16}{25}$

2 **a** **i** $\frac{5}{9}$ **ii** $\frac{4}{9}$

  b

  c **i** $\frac{25}{81}$

  ii $\frac{16}{81}$

  iii $\frac{20}{81}$

  iv $\frac{40}{81}$

3 **a**

  b $\frac{11}{80}$
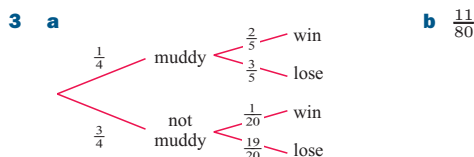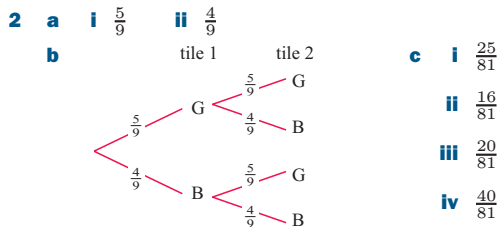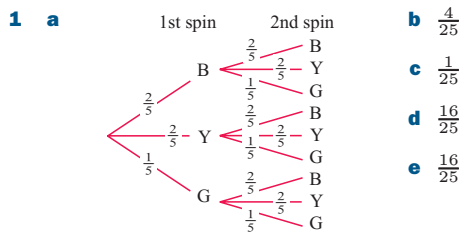
4 0.034 **5** $\frac{23}{60}$ **6** **a** $\frac{23}{42}$ **b** $\frac{19}{42}$

EXERCISE 25I

1

  a $\frac{4}{9}$ **b** $\frac{1}{9}$ **c** $\frac{2}{9}$ **d** $\frac{4}{9}$
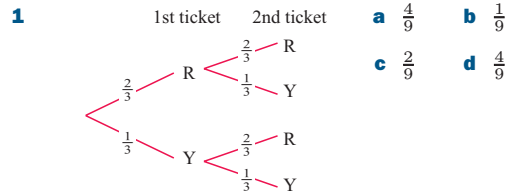
2 **a** $\frac{2}{7}$ **b** $\frac{1}{7}$ **c** $\frac{2}{7}$ **d** $\frac{4}{7}$

3 **a** $\frac{1}{6}$ **b** $\frac{5}{18}$ **c** $\frac{5}{18}$ **d** $\frac{5}{18}$

  These cases cover all possibilities, so their probabilities must add up to 1.
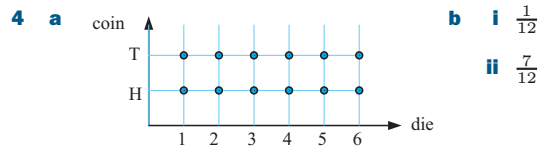
4 **a**

  b $\frac{5}{33}$ **c** $\frac{7}{22}$

5 **a**

  b $\frac{28}{171}$ **c** $\frac{55}{171}$

6 **a** $\frac{1}{25}$ **b** $\frac{24}{25}$ **c** $\frac{7}{4975} \approx 0.001\,41$
  d $\frac{4584}{4975} \approx 0.921$ **e** $\frac{8}{199}$

EXERCISE 25J

1 **a** A and B, A and D, A and E, A and F, B and D, B and F, C and D

  b **i** $\frac{2}{3}$ **ii** $\frac{5}{6}$ **iii** $\frac{2}{3}$ **iv** $\frac{1}{2}$ **v** 1 **vi** $\frac{5}{6}$

2 **a** $\frac{5}{17}$ **b** $\frac{8}{17}$ **c** $\frac{6}{17}$ **3** $\frac{4}{15}$

4 **a**

  b **i** $\frac{1}{12}$ **ii** $\frac{7}{12}$

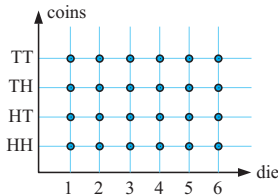 **c** $P(H) + P(5) - P(H \text{ and } 5) = \frac{6}{12} + \frac{2}{12} - \frac{1}{12}$
  $= \frac{7}{12} = P(H \text{ or } 5)$

5 **a**

  b **i** $\frac{1}{18}$ **ii** $\frac{5}{9}$
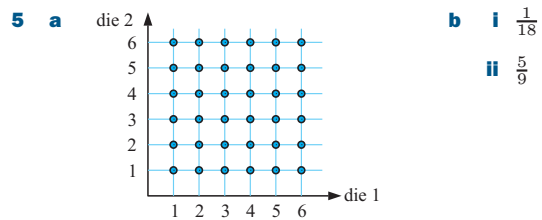
  c $P(3) + P(4) - P(3 \text{ and } 4) = \frac{11}{36} + \frac{11}{36} - \frac{2}{36}$
  $= \frac{5}{9} = P(3 \text{ or } 4)$

EXERCISE 25K

1 a 11 **b i** $\frac{29}{50}$ **ii** $\frac{11}{40}$ **2 a** $\frac{7}{30}$ **b** $\frac{7}{15}$

3 a $\frac{12}{25}$ **b** $\frac{3}{25}$ **c** $\frac{7}{25}$ **d** $\frac{12}{19}$ **4 a** $\frac{1}{25}$ **b** 8 days

5 a $\frac{5}{7}$ **b** $\frac{13}{14}$ **c** $\frac{1}{14}$ **d** $\frac{5}{23}$ **e** $\frac{9}{10}$

6 a $\frac{24}{25}$ **b** $\frac{1}{25}$ **7 a** $\frac{1}{8}$ **b** $\frac{3}{8}$ **8 a** $\frac{3}{4}$ **b** $\frac{5}{9}$

9 a

b i $\frac{1}{24}$ **ii** $\frac{1}{8}$ **iii** $\frac{7}{12}$

10 a $\frac{1}{20}$ **b** $\frac{19}{20}$ **c** $\frac{31}{50}$ **d** $\frac{39}{50}$ **e** $\frac{12}{19}$ **f** $\frac{17}{78}$

11 $\frac{35}{47}$

REVIEW SET 25A

1 a 39 days **b i** $\frac{1}{39}$ **ii** $\frac{11}{39}$ **iii** $\frac{19}{39}$

2

3 a ≈ 0.885
b ≈ 0.424
c ≈ 0.376
d ≈ 0.421

4 The occurrence of either event does not affect the occurrence of the other event.

5 a $\frac{1}{16}$ **b** $\frac{3}{8}$ **c** $\frac{5}{16}$

6 a 0.72 **b** 0.02 **c** 0.98 **d** 0.18

7 $\frac{1}{6}$ **8 a** $\frac{2}{7}$ **b** $\frac{4}{7}$

9 a

b i $\frac{1}{10}$
ii $\frac{3}{5}$

10

a $\frac{13}{20}$ **b** $\frac{1}{5}$
c $\frac{4}{5}$ **d**
e $\frac{9}{13}$

11
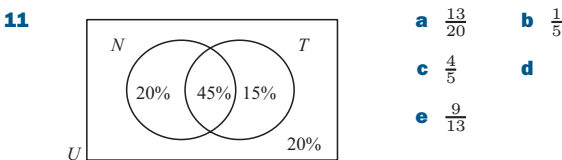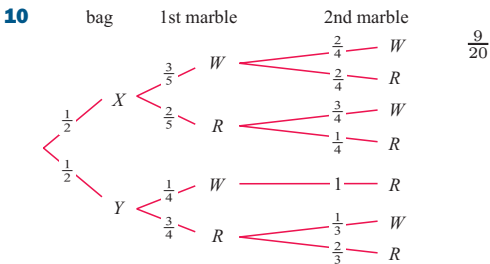
REVIEW SET 25B

1 a 0.364 **b** 0.551 **c** 0.814
2 a {AH, BH, CH, DH, AT, BT, CT, DT}

b i

ii

3 a 0.60
b 0.40

4 a $\frac{3}{14}$
b

c i $\frac{33}{91}$
ii $\frac{36}{91}$
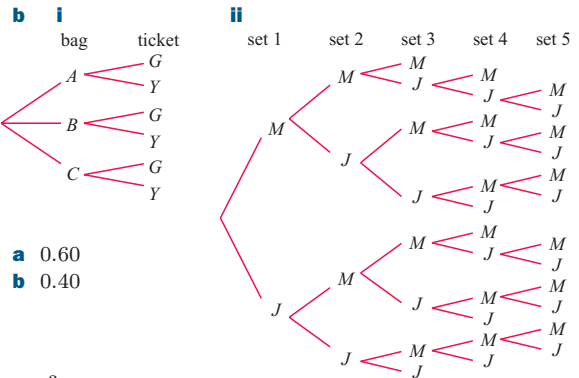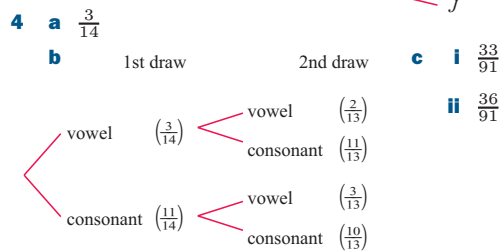
5 a $\frac{1}{9}$ **b** $\frac{4}{9}$ **c** $\frac{4}{9}$ **6 a** $\frac{1}{20}$ **b** $\frac{3}{10}$

7

8

a $\frac{4}{25}$ **b** $\frac{12}{25}$
c $\frac{1}{5}$ **d** $\frac{2}{5}$
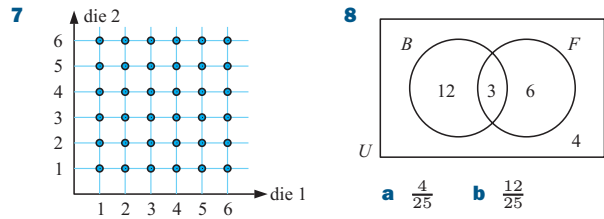
a $\frac{1}{36}$ **b** $\frac{11}{36}$ **c** $\frac{1}{6}$ **d** $\frac{2}{9}$

9 a ≈ 0.707 **b** ≈ 0.293 **c** ≈ 0.0467 **d** ≈ 0.340
e ≈ 0.540

10 a ≈ 0.045 **b** ≈ 0.124 **c** ≈ 0.876
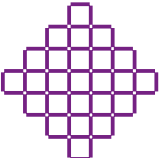**d** These probabilities add to 1 as the events are complementary.

EXERCISE 26A

1 a Start with 5 and then add 3 successively to get further terms: 20, 23.
b Start with 2 and then add 7 successively to get further terms: 37, 44.
c Start with 8 and then add 11 successively to get further terms: 63, 74.
d Start with 38 and then subtract 4 successively to get further terms: 18, 14.
e Start with 3 and then subtract 5 successively to get further terms: -22, -27.
f Start with $\frac{1}{2}$ and then add $1\frac{1}{2}$ successively to get further terms: 8, $9\frac{1}{2}$.

2 a Start with 3 and then multiply each successive term by 2: 96, 192.
b Start with 1 and then multiply each successive term by 2: 32, 64.
c Start with 2 and then multiply each successive term by 5: 1250, 6250.
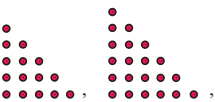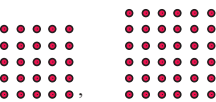**d** Start with 36 and then multiply each successive term by $\frac{1}{2}$: $2\frac{1}{4}$, $1\frac{1}{8}$.

e Start with 162 and then multiply each successive term by $\frac{1}{3}$:
$2, \frac{2}{3}$.

f Start with 405 and then multiply each successive term by $\frac{1}{3}$:
$5, 1\frac{2}{3}$.

3 **a** 25, 36   **b** 36, 49   **c** 125, 216   **d** $\frac{1}{36}, \frac{1}{49}$

e 16, 22   **f** 42, 56

4 **a** The first two terms are 1 and 1 and from then on the next term is the sum of the previous two terms: 13, 21.

b The first two terms are 1 and 3 and from then on the next term is the sum of the previous two terms: 18, 29.

c Each term is made up by adding two successive prime numbers starting $2 + 3, 3 + 5, 5 + 7,$ etc.: 36, 42.
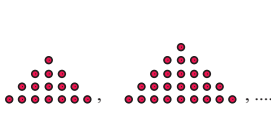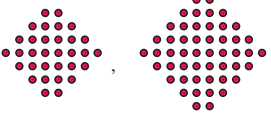
5 **a**

3, 6, 9, 12, 15, ...

b

7, 10, 13, 16, 19, ...

c

4, 13, 26, 43, 64, ...

d

5, 12, 19, 26, 33, ...

e

6, 11, 16, 21, 26, ...

f

4, 16, 36, 64, 100, ...

6 **a**

1, 3, 6, 10, 15, 21, ...

b

1, 4, 9, 16, 25, 36, ...

c

1, 4, 9, 16, 25, ...

d

2, 8, 18, 32, 50, ...

EXERCISE 26B

1 **a** 5, 7, 9, 11, ...   **b** 7, 9, 11, 13, ...

c 5, 8, 11, 14, ...   **d** 8, 11, 14, 17, ...

e $-1, -3, -5, -7,$ ......   **f** $1, -1, -3, -5,$ ......

g $3, 0, -3, -6,$ ......   **h** 13, 9, 5, 1, ...

i 69, 62, 55, 48, ...

2 **a** 2, 5, 10, 17, ...   **b** 0, 3, 8, 15, ...

c 2, 6, 12, 20, ...   **d** 3, 8, 15, 24, ...

e 2, 9, 28, 65, ...   **f** 2, 15, 44, 95, ...

3 **a** $u_n = 2n$ **b** **i** $u_n = 2n + 2$ **ii** $u_n = 2n - 1$

4 **a** $u_n = 5n$ **b** **i** $u_n = 5n + 1$ **ii** $u_n = 5n - 2$

5 **a** $u_n = 2^n$ **b** $u_n = 3 \times 2^n$

6 **a** $u_n = n$

b **i** $u_n = n + 1$ **ii** $u_n = n + 2$ **iii** $u_n = \dfrac{1}{n}$

iv $u_n = \dfrac{1}{n+1}$ **v** $u_n = \dfrac{n}{n+1}$ **vi** $u_n = \dfrac{n+2}{n}$

vii $u_n = n(n + 1)$ **viii** $u_n = (n+1)(n+2)$

ix $u_n = n(n + 2)$ **x** $u_n = \dfrac{3n - 2}{3n}$

7 **a** $u_n = n^2$

b **i** $u_n = (n + 1)^2$ **ii** $u_n = n^2 - 1$

iii $u_n = \dfrac{1}{n^2}$ **iv** $u_n = \dfrac{n}{(n+1)^2}$

8 **a** $u_n = n^3$ **b** $u_n = n^3 - 1$

c $u_n = 3 \times 2^{n-1}$ **d** $u_n = 24 \times (\frac{1}{2})^{n-1}$

EXERCISE 26C

1 **a** 6, 12, 24, 48, 96, ...   **b** 3, 6, 12, 24, 48, ...

c 12, 24, 48, 96, 192, ...   **d** 5, 10, 20, 40, 80, ...

e $24, 12, 6, 3, 1\frac{1}{2}$, ...   **f** $36, 12, 4, 1\frac{1}{3}, \frac{4}{9}$, ...

g $-48, 96, -192, 384, -786$, ...

h $8, -8, 8, -8, 8$, ...   **i** $-4, 2, -1, \frac{1}{2}, -\frac{1}{4}$, ...

2 **a** $-1, 1$; $u_n = (-1)^{n-1}$ or $u_n = (-1)^{n+1}$

b $1, -1$; $u_n = (-1)^n$ **c** 64, 128; $u_n = 2^n$

d $-64, 128$; $u_n = 2 \times (-2)^{n-1}$

e 486, 1458; $u_n = 6 \times 3^{n-1}$

f $486, -1458$; $u_n = 6 \times (-3)^{n-1}$

g 324, 972; $u_n = 4 \times 3^{n-1}$

h $4802, -33\,614$; $u_b = 2 \times (-7)^{n-1}$

i $48, -96$; $u_n = 3 \times (-2)^{n-1}$

j $-1, \frac{1}{2}$; $u_n = -16 \times (-\frac{1}{2})^{n-1}$
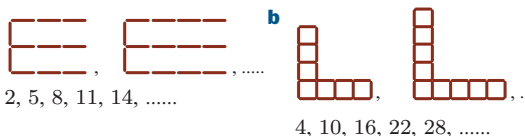
EXERCISE 26D

1 **a** $u_n = 4n - 3$ **b** $u_n = 20 - 3n$ **c** $u_n = n^2 + n$

d $u_n = n^2 + 3n - 4$ **e** $u_n = n^3 + 5$

f $u_n = \frac{1}{2}n^3 + \frac{3}{2}n$ or $u_n = \dfrac{n(n^2 + 3)}{2}$

2 **a** $u_n = 4n^2 - 2n$

b $u_1 = 1 \times 2$, $u_2 = 3 \times 4$, $u_3 = 5 \times 6$, $u_4 = 7 \times 8$, etc

$\therefore$ $u_n = (2n - 1) \times 2n$

3 **a** $u_1 = 2$, $u_2 = 8$, $u_3 = 18$, $u_4 = 32$, $u_5 = 50$,
$u_6 = 72$, $u_7 = 98$
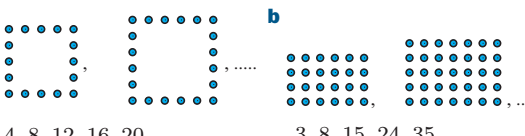
b $u_n = 2n^2$ **c** $u_{30} = 1800$ dots

4 **a**
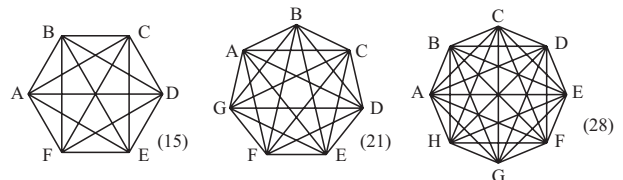
b $u_n = \frac{1}{2}n^2 + \frac{1}{2}n$ or $u_n = \dfrac{n(n+1)}{2}$

c 15 931 handshakes

5 a $u_1 = 3$, $u_2 = 13$, $u_3 = 34$, $u_4 = 70$, $u_5 = 125$, $u_6 = 203$, $u_7 = 308$

  b $u_n = \frac{2}{3}n^3 + \frac{3}{2}n^2 + \frac{5}{6}n$ or $u_n = \dfrac{n(n+1)(4n+5)}{6}$

  c $u_{50} = 87\,125$

6 a $u_1 = 1$, $u_2 = 5$, $u_3 = 14$, $u_4 = 30$, $u_5 = 55$, $u_6 = 91$, $u_7 = 140$

  b $u_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$ or $u_n = \dfrac{n(n+1)(2n+1)}{6}$

  c $338\,350$

7 a $u_3 = 20$, $u_4 = 40$, $u_5 = 70$, $u_6 = 112$

  b $u_n = \frac{1}{3}n^3 + n^2 + \frac{2}{3}n$ or $u_n = \dfrac{n(n+1)(n+2)}{3}$

  c 1360

REVIEW SET 26A

1 a Start with 6 and then add 4 successively to get further terms: $u_6 = 26$, $u_7 = 30$.

  b Start with 810 and then multiply each successive term by $\frac{1}{3}$: $u_5 = 10$, $u_6 = 3\frac{1}{3}$.

2 a

2, 5, 8, 11, 14, ...

  b

4, 10, 16, 22, 28, ...

3 a $u_1 = 5$, $u_2 = 11$, $u_3 = 17$, $u_4 = 23$

  b $u_1 = 4$, $u_2 = 12$, $u_3 = 22$, $u_4 = 34$

4 a $u_n = 4n$ **b** **i** $u_n = 4n - 3$ **ii** $u_n = \dfrac{1}{4n - 1}$

  c **i** $u_{20} = 77$ **ii** $u_{20} = \frac{1}{79}$

5 a $u_1 = 18$, $u_2 = 12$, $u_3 = 8$, $u_4 = 5\frac{1}{3}$

  b $u_1 = 5$, $u_2 = -10$, $u_3 = 20$, $u_4 = -40$

6 a $u_n = 3 \times 4^{n-1}$ **b** $u_n = 88 \times \left(-\frac{1}{2}\right)^{n-1}$

7 a $u_n = 7n - 2$ **b** $u_n = n^2 + 4n - 6$

8 a $u_3 = 9$, $u_4 = 13$, $u_5 = 17$ **b** $u_n = 4n - 3$

  c $u_{50} = 197$

9 a $u_n = n^3 + 3n^2 + 2n$

  b $u_1 = 6 = 1 \times 2 \times 3$
  $u_2 = 24 = 2 \times 3 \times 4$
  $u_3 = 60 = 3 \times 4 \times 5$
  $u_4 = 120 = 4 \times 5 \times 6$
  suggesting that $u_n = n(n+1)(n+2)$

10 a $u_4 = 11$, $u_5 = 16$

  b $u_n = \frac{1}{2}n^2 + \frac{1}{2}n + 1$ or $u_n = \dfrac{n^2 + n + 2}{2}$

  c $u_{10} = 56$ pieces

REVIEW SET 26B

1 a Start with 17 and then subtract 5 successively to get further terms: $u_5 = -3$, $u_6 = -8$.

  b Start with -2 and then multiply each successive term by -2: $u_5 = -32$, $u_6 = 64$.

2 a

4, 8, 12, 16, 20, ...

  b

3, 8, 15, 24, 35, ...

3 a $u_1 = 1$, $u_2 = -3$, $u_3 = -7$, $u_4 = -11$

  b $u_1 = 0$, $u_2 = 4$, $u_3 = 10$, $u_4 = 18$

4 a $u_n = 6n$ **b** **i** $u_n = 6n + 4$ **ii** $u_n = \dfrac{6n - 1}{6n + 1}$

  c **i** $u_{15} = 94$ **ii** $u_{15} = \frac{89}{91}$

5 a $u_n = n^2 + 1$ **b** $u_n = \dfrac{1}{n^3}$

6 a 5, 15, 45, 135, ... **b** -12, 3, $-\frac{3}{4}$, $\frac{3}{16}$, ...

7 a $u_n = 4 \times (-3)^{n-1}$ **b** $u_n = 224 \times (\frac{1}{4})^{n-1}$

8 a $u_n = 52 - 9n$ **b** $u_n = n^2 + 5n - 2$

9 a $u_3 = 24$, $u_4 = 40$, $u_5 = 60$, $u_6 = 84$

  b $u_n = 2n^2 + 2n$ or $u_n = 2n(n+1)$ **c** $u_{10} = 220$

10 c $u_n = \frac{1}{24}n^4 + \frac{1}{12}n^3 - \frac{1}{24}n^2 - \frac{1}{12}n$

EXERCISE 27A.1

1 a $x = 27$ **b** $x = 30$ **c** $x = 18$ **d** $x = 30$

  e $a = 40$, $b = 50$ **f** $a = 60$, $b = 40$

  g $a = 55$, $b = 55$ **h** $m = 56$ **i** $n = 49$

3 1 cm **4** 40 cm

5 a **i** $\widehat{\text{APO}} = a^o$ **ii** $\widehat{\text{BPO}} = b^o$

  b $\widehat{\text{APO}} = a + b$

  $\therefore$ $a + a + b + b = 180^o$ {angle sum of a Δ} etc.

  c The 'angle in a semi-circle' theorem.

6 a ΔOAB is isosceles as OA = OB {equal radii}

  b X is the midpoint of chord AB, $\widehat{\text{OAX}} = \widehat{\text{OBX}}$, $\widehat{\text{AOX}} = \widehat{\text{BOX}}$.

7 b The triangle are congruent {RHS} as:

  • OA = AB {equal radii}
  • $\widehat{\text{OAP}} = \widehat{\text{OBP}} = 90^o$ {tangent property}
  • OP is common

  c Consequences are: • AP = BP
  • OP bisects $\widehat{\text{APB}}$ and $\widehat{\text{AOB}}$

EXERCISE 27A.2

1 a $x = 64$ **b** $x = 70$ **c** $x = 45$ **d** $x = 66$

  e $x = 94$ **f** $x = 25$

2 a $x = 46$ **b** $x = 30$, $y = 30$ **c** $a = 50$, $b = 40$

  d $a = 55$, $c = 70$ **e** $a = 80$, $b = 200$

  f $x = 75$, $y = 118$ **g** $x = 42$ **h** $x = 25$ **i** $x = 25$

3 $x = 70$

4 a equal radii

  b **i** a **ii** b **iii** $2a$ **iv** $2b$ **v** $a+b$ **vi** $2a+2b$

  c The angle at the centre of a circle is twice the angle at the circle subtended by the same arc.

5 a 2α **b** α **c** $\widehat{\text{ADB}} = \widehat{\text{ACB}}$

EXERCISE 27B.1

1 a $x = 107$ {opp. angles of cyclic quad.}

  b $x = 60$ {opp. angles of cyclic quad.}

  c $x = 70$ {opp. angles of cyclic quad.}

  d $x = 81$ {exterior angles of cyclic quad.}

  e $x = 90$ {exterior angles of cyclic quad.}

  f $x = 125$ {exterior angles of cyclic quad.}

2 a $x = 110$, $y = 100$ **b** $x = 40$

  c $x = 65$, $y = 115$ **d** $x = 80$

  e $x = 45$, $y = 90$ **f** $x = 105$, $y = 150$

3 $\widehat{\text{BAD}} = 95^o$, $\widehat{\text{ADC}} = 65^o$, $\widehat{\text{DCB}} = 85^o$, $\widehat{\text{ABC}} = 115^o$

4 **a** $\widehat{BCD} = 90^o$, $\widehat{BDC} = 45^o$
   b From **a**, $\widehat{CBD} = 45^o$ {angles in a triangle}
       $\therefore$ ΔBCD is isosceles.
   c $\widehat{DEF} = 65^o$
6 **Hint:** A, B, C and D lie on a circle with centre O.

EXERCISE 27B.2

1 **a** Yes, one pair of opposite angles are supplementary.
   b Yes, AD subtends equal angles at B and C.   **c** No.
   d Yes, opposite angles are supplementary.
   e Yes, one pair of opposite angles are supplementary.
   f Yes, AD subtends equal angles at B and C.

REVIEW SET 27A

1 **a** $a = 72$ **b** $a = 62$ **c** $a = 61$ **d** $a = 140$
   e $a = 45$ **f** $a = 83$ **g** $a = 80$ **h** $a = 63$
   i $a = 45$
3 **a** 8 cm **b** 2 cm **5** **a** $\widehat{DBO} = \alpha$

REVIEW SET 27B

1 **a** $x = 86$ **b** $x = 90$ **c** $x = 79$ **d** $x = 42$
   e $x = 55$ **f** $x = 55$
3 56 cm

5 **a** **i** 90° **ii** 90°
   b **i** $90^o - \alpha$ {radius tangent theorem}
       ii α {angle sum of triangle}
       iii α {angles on same arc}

6 **a** $\widehat{BCX} = \alpha$, $\widehat{CBX} = \beta$ **b** $\alpha + \beta + \gamma = 180^o$

EXERCISE 28A

1 **a** 2   **b** $\frac{1}{2}$ **c** 4   **d** $\frac{1}{4}$ **e** 5   **f** $\frac{1}{5}$
   g 2   **h** $\frac{1}{2}$ **i** 4   **j** $\frac{1}{4}$ **k** 2   **l** $\frac{1}{2}$
   m 5   **n** -5 **o** no real solution   **p** -1

2 **a** $10^{\frac{1}{2}}$ **b** $10^{-\frac{1}{2}}$ **c** $15^{\frac{1}{3}}$ **d** $15^{-\frac{1}{3}}$ **e** $19^{\frac{1}{4}}$
   f $19^{-\frac{1}{4}}$ **g** $13^{\frac{1}{5}}$ **h** $13^{-\frac{1}{5}}$

3 **a** 4   **b** 3   **c** 4   **d** ≈ 5.85
   e ≈ 4.47 **f** ≈ 3.98 **g** 2   **h** ≈ 2.21

4 **a** 16   **b** $\frac{1}{4}$ **c** 8   **d** $\frac{1}{8}$ **e** 9   **f** $\frac{1}{9}$
   g 4   **h** $\frac{1}{8}$ **i** 32   **j** $\frac{1}{25}$ **k** 27   **l** $\frac{1}{27}$

EXERCISE 28B

1 **a** 3   **b** 11   **c** $2\frac{1}{3}$ **2** **a** -2 **b** $-2\frac{4}{5}$ **c** 22

3 **a** $\frac{1}{9}$ **b** 9   **c** $\frac{1}{27}$

4 **a**

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27

   b, c

$y = f(x)$
$y = -f(x)$
$y = f(-x)$
$y = 2f(x)$
$y = f(2x)$
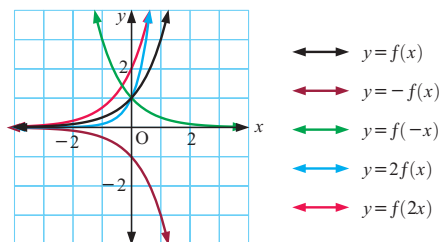
5 **a** **i** ≈ 1.6 **ii** ≈ 3.5 **iii** ≈ 0.7
   b **i** ≈ 1.62 **ii** ≈ 3.48 **iii** ≈ 0.707

c **i** $x \approx 2.3$ **ii** $x \approx 0.6$ **iii** no solution

6 **a** $y = 2^{x+1} + 3$ **b** $y = 3^{x-2} - 4$
   c **i** $y = -2^{-x}$ **ii** $y = 2^x$ **iii** $x = 2^{-y}$
   d **i** $y = 2(3^x)$ **ii** $y = 3^{3x}$

7 **a** **i**

$f(x) = (1.2)^x$

   ii 1
   iii $y = 0$

   b **i**

$f(x) = 2^x - 1$

   ii 0
   iii $y = -1$

   c **i**

$f(x) = 2^{x-1}$

   ii $\frac{1}{2}$
   iii $y = 0$

   d **i**

$f(x) = 2^{x+1}$

   ii 2
   iii $y = 0$

   e **i**

$f(x) = 3 + 2^x$

   ii 4
   iii $y = 3$

   f **i**

$f(x) = 2 - 2^x$

   ii 1
   iii $y = 2$

   g **i**

$f(x) = \dfrac{2^{-x} + 1}{3}$

   ii $\frac{2}{3}$
   iii $y = \frac{1}{3}$

h i

$f(x) = 2(3^{-x}) + 1$

ii 3

iii $y = 1$
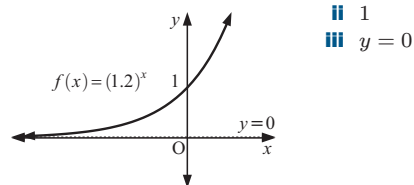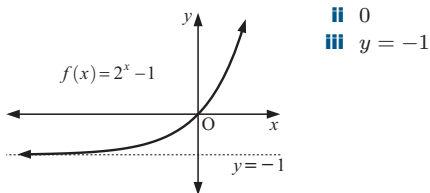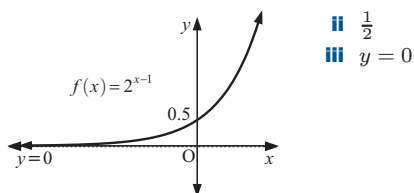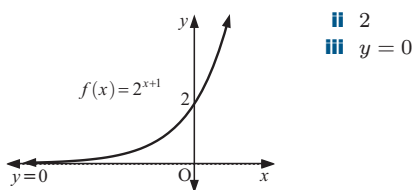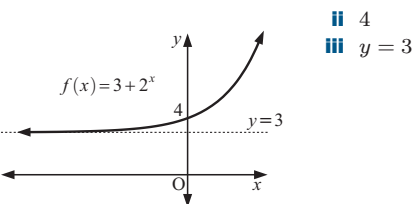
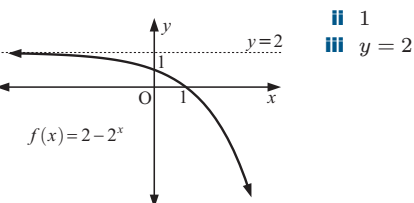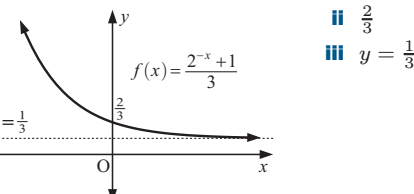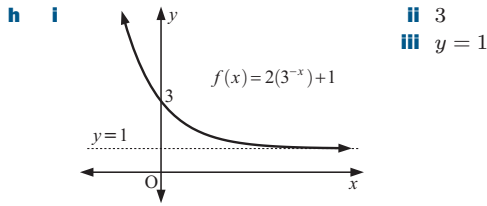**8** If $x = \frac{1}{2}$, $(-2)^{\frac{1}{2}} = \sqrt{-2}$ which is undefined.

9 a $P = \dfrac{400}{5^t}$ **b** $M = \dfrac{216}{3^x}$

## EXERCISE 28C.1

1 a $x = 1$ **b** $x = 2$ **c** $x = 3$ **d** $x = 0$
  e $x = -1$ **f** $x = -1$ **g** $x = -4$ **h** $x = 0$
  i $x = -4$ **j** $x = -2$ **k** $x = 6$ **l** $x = 2$
  m $x = -\frac{3}{4}$ **n** $x = \frac{7}{2}$ **o** $x = 0$ **p** $x = 4$

2 a $x = 3$ **b** $x = 0$ **c** $x = -2$ **d** $x = 3$
  e $x = 3$ **f** $x = -2$ **g** $x = \frac{2}{7}$ **h** $x = \frac{1}{3}$
  i $x = 1$ **j** $x = 1$ **k** no solution   **l** $x = 3$ or -1

3 $a = 6$, $n = 2$ **4** $b = \frac{1}{4}$, $t = 5$ **5** $x = -1$, $y = 2$

EXERCISE 28C.2

1 a $x \approx 6.644$ **b** $x \approx -1.884$ **c** no solution exists
  d $x \approx 1.292$ **e** $x \approx -1.611$ **f** $x \approx 4.192$
  g $x \approx -4.583$ **h** $x \approx -1.848$ **i** $x \approx -4.858$
  j $x \approx 2.064$ **k** $x \approx 2.991$ **l** $x \approx 291.825$
  m $x \approx 0.262$ **n** $x \approx 1.441$ **o** $x \approx 0.415$

EXERCISE 28D

1 a 40 mongooses   **c**
  b i ≈ 61
    ii ≈ 160
    iii 2560
  d ≈ 6.61 years

$P = 40 \times 2^{0.2n}$
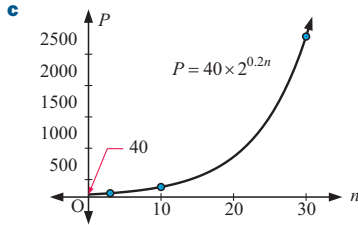
2 a 50 echidnas   **c**
  b i 100
    ii 400
    iii ≈ 5080
  d ≈ 4.75 years

$P = 50 \times 2^{\frac{t}{3}}$
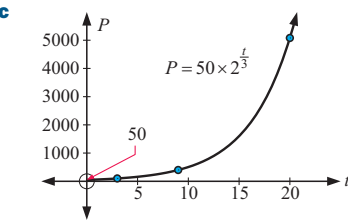
3 a $G_0 = 28$ **c**
  b i ≈ 39
    ii ≈ 86
    iii ≈ 822
  d ≈ 17.5 years

$G = 28 \times 5^{0.07n}$
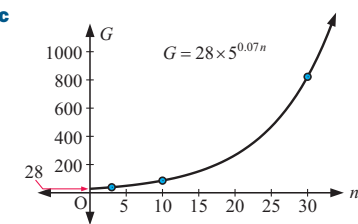
4 a 2.3 grams
  b i 1.00 grams   **ii** 0.000 562 grams
    iii 1.73×10^{-36} grams

c

$W = 2.3 \times 2^{-0.06t}$
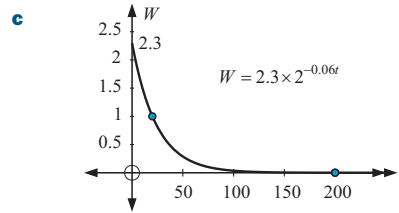
d 56.5%   **e** ≈ 25.4 years

5 a 100°C   **c**
  b i 70.7^{o}C
    ii 17.7^{o}C
    iii $0.003\,05^{o}$C
  d ≈ 4.12 minutes

$T = 100 \times 2^{-\frac{t}{4}}$
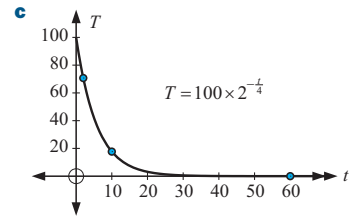
6 a $5800
  b i $7367.38   **ii** $12\,873.97   **iii** ≈ 8.69 years

7 a $500   **b i** $445.64   **ii** $199.11
  c ≈ 62.8 months ≈ 5 years 3 months

8 a $B \approx 20 \times 2^{\frac{2d}{3}}$ **b**
  c i ≈ 50
    ii ≈ 508
    iii $\approx 12\,900$
  d ≈ 11.9 days

$B = 20 \times 2^{\frac{2d}{3}}$
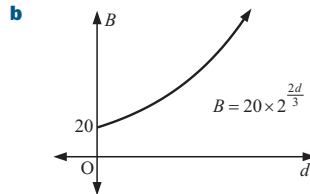
9 a i £6.42 million   **b**
    ii £52.4 million
  c ≈ 6.33 years
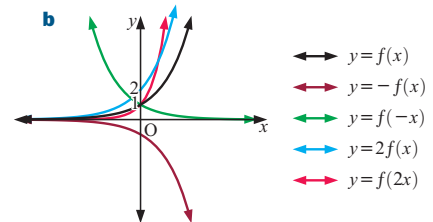

$F = 3.8 \times (1.3)^n$
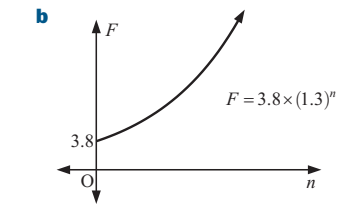
EXERCISE 28E

1 a $P \approx 25.1(1.32)^x$ **b** $N \approx 5.70(0.815)^t$

2 a Yes, very well. $V \approx 216\,000(1.023)^t$
  b i $\approx$ €237 000   **ii** $\approx$ €266 000
  c The 2004 figure is reliable as it is between the poles $t = 0$ and $t = 8$. For this reason, the 2009 figure may not be reliable.

3 a The fit is excellent as $r^2 \approx 1$. $C \approx 300(0.3487)^t$
  b $\approx 13\frac{1}{2}$ minutes
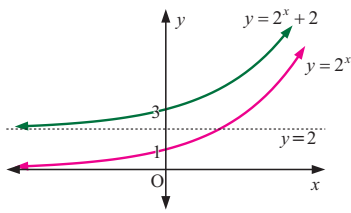
REVIEW SET 28A

1 a $5^{\frac{1}{2}}$ **b** $7^{-\frac{1}{3}}$ **c** $51^{\frac{1}{4}}$ **d** $47^{-\frac{1}{5}}$

2 a 8   **b** 25   **c** $\frac{1}{27}$ **d** $\frac{1}{8}$

3 a i 5^{-x} **b**
    ii -5^x
    iii 2×5^x
    iv 5^{2x}

$y = f(x)$
$y = -f(x)$
$y = f(-x)$
$y = 2f(x)$
$y = f(2x)$

4 a 0 **b** 26 **c** $-\frac{2}{3}$ **d** $-\frac{8}{9}$

5

a For $y = 2^x$: y-intercept 1, horizontal asymptote $y = 0$
For $y = 2^x + 2$: y-intercept 3, horizontal asymptote $y = 2$

b a vertical translation of $\binom{0}{2}$

6 a $x = -4$ **b** $x = -1$ **c** $x = 4$ **d** $x = 2$
e $x = -5$ **f** $x = -\frac{4}{5}$

7 $k = 576$, $n = -2$

8 a $x \approx 2.80$ **b** $x \approx 1.36$ **c** $x \approx 6.61$

9 a 30 people **c**
b **i** 60 people
ii 101 people
d ≈ 10.9 years
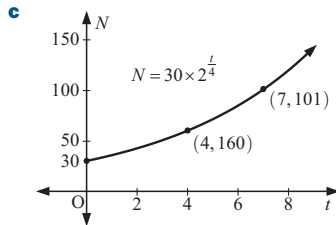
REVIEW SET 28B

1 a ≈ 2.924 **b** ≈ 2.512 **c** ≈ 1.971

2 a $\frac{1}{3}$ **b** 32 **c** 16 **d** $\frac{1}{243}$

3 a 2 **b** $\frac{2}{3}$ **c** $\frac{2}{9}$ **d** 6 **e** 18

4
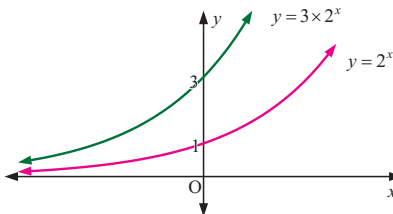
a For $y = 2^x$: y-intercept 1; horizontal asymptote $y = 0$
For $y = 3 \times 2^x$: y-intercept 3; horizontal asymptote $y = 0$
b stretch with invariant x-axis and scale factor 3

5 a **i** $(\frac{1}{2})^x$ **b**
ii 4^x
iii 2×2^x
iv $\frac{1}{2} \times 2^x$

$y = f(x)$
$y = -f(x)$ $y = f(x+1)$
$y = f(2x)$ $y = f(x-1)$
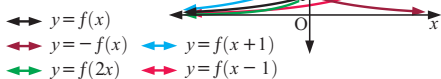
6 a $x = -3$ **b** $x = -\frac{1}{6}$ **c** $x = 2$ **d** $x = \frac{3}{2}$
e $x = -\frac{7}{2}$ **f** $x = 2$ or -1

7 $x = -1$, $y = -2$

8 a $x \approx 2.73$ **b** no solutions exists as LHS is always > 0
c $x \approx 11.33$

9 a 130 kg **b** **i** ≈ 2.83 kg **ii** ≈ 8.29 kg
iii ≈ 15.2 weeks or 15 weeks 1 day

EXERCISE 29A.1

1 a $(\cos 231^o, \sin 231^o)$ **b** $\approx (-0.629, -0.777)$

2 a 0 **b** -1 **c** -1 **d** 0 **e** 1 **f** 0 **g** 0 **h** 1

3 a 0.64 **b** 0.77 **c** -0.34 **d** 0.94
e 0.17 **f** -0.98 **g** -0.77 **h** -0.64
i 0.77 **j** -0.64 **k** 0.87 **l** -0.5

5 a $P(\cos \theta, \sin \theta)$
b **i** $Q(\cos(-\theta), \sin(-\theta))$ **ii** $Q(\cos \theta, -\sin \theta)$
c $\cos(-\theta) = \cos \theta$, $\sin(-\theta) = -\sin \theta$
d $\tan(-\theta) = -\tan \theta$

6

$\sin(180^o + \theta) = -\sin \theta$
$\cos(180^o + \theta) = -\cos \theta$
$\tan(180^o + \theta) = \tan \theta$

EXERCISE 29A.2

1 a $\sin 30^o = \frac{1}{2}$, $\cos 30^o = \frac{\sqrt{3}}{2}$, $\tan 30^o = \frac{1}{\sqrt{3}}$
b $\sin 180^o = 0$, $\cos 180^o = -1$, $\tan 180^o = 0$
c $\sin 135^o = \frac{1}{\sqrt{2}}$, $\cos 135^o = -\frac{1}{\sqrt{2}}$, $\tan 135^o = -1$
d $\sin 210^o = -\frac{1}{2}$, $\cos 210^o = -\frac{\sqrt{3}}{2}$, $\tan 210^o = \frac{1}{\sqrt{3}}$
e $\sin 300^o = -\frac{\sqrt{3}}{2}$, $\cos 300^o = \frac{1}{2}$, $\tan 300^o = -\sqrt{3}$
f $\sin 270^o = -1$, $\cos 270^o = 0$, $\tan 270^o$ is undefined
g $\sin 315^o = -\frac{1}{\sqrt{2}}$, $\cos 315^o = \frac{1}{\sqrt{2}}$, $\tan 315^o = -1$
h $\sin 240^o = -\frac{\sqrt{3}}{2}$, $\cos 240^o = -\frac{1}{2}$, $\tan 240^o = \sqrt{3}$

2 a $\frac{1}{2}$ **b** $\frac{1}{4}$ **c** $\frac{1}{3}$ **d** $\frac{3\sqrt{3}}{8}$

3 a 30° and 150° **b** 30° and 330° **c** 45° and 135°
d 210° and 330° **e** 270° **f** 150° and 210°

EXERCISE 29B

1 a 55.2 cm^2 **b** 347 km^2 **c** 23.1 cm^2 **d** 430 m^2
e 53.0 cm^2 **f** 1.15 m^2

2 50.0 cm^2 **3** $x \approx 22.2$ **4** $\theta = 30^o$ or 150°

5 $\widehat{ABC} \approx 41.6^o$ or 138.4^o

6 a **i** area $= \frac{1}{2}bc \sin A$ **ii** area $= \frac{1}{2}ab \sin C$
b Equate areas.

EXERCISE 29C.1

1 a $x \approx 11.1$ **b** $x \approx 11.5$ **c** $x \approx 5.19$

2 a $a = 28.4$ cm **b** $b \approx 52.2$ cm **c** $c \approx 5.23$ cm

EXERCISE 29C.2

1 a $\theta \approx 31.4$ **b** $\theta \approx 77.5$ or 102.5
c $\theta \approx 43.6$ or 136.4

2 a $x \approx 27.1$ or 152.9 **b** $x \approx 4.39$ **c** $x \approx 11.6$
d $x \approx 96.8$ **e** $x \approx 16.4$ **f** $x \approx 101.5$ or 8.50

3 a $\widehat{A} \approx 49.1^o$ **b** $\widehat{B} \approx 71.6^o$ or 108.4^o **c** $\widehat{C} \approx 44.8^o$

4 $AC \approx 16.3$ cm **5** $R \approx 42.5^o$

EXERCISE 29D

1 **a** $x \approx 7.11$ **b** $x \approx 19.7$ **c** $x \approx 7.04$
 d $x \approx 104.5$ **e** $x \approx 96.4$ **f** $x \approx 93.6$

2 **a** 27.5 cm **b** 4.15 km **c** 15.2 m

3 $\widehat{A} \approx 51.8^o$, $\widehat{B} \approx 40.0^o$, $\widehat{C} \approx 88.3^o$ **4** **a** 43.0^o **b** 120°

5 **a** $\cos \theta = \dfrac{m^2 + c^2 - a^2}{2cm}$

 b $\cos(180^o - \theta) = \dfrac{m^2 + c^2 - b^2}{2cm}$

 d **i** $x \approx 9.35$ **ii** $x \approx 4.24$

6 **b** $x = 5 \pm \sqrt{6}$
 c
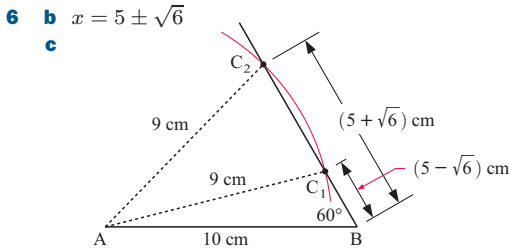
7 **a** ≈ 2.90 cm^2 **b** 9.92 cm^2

EXERCISE 29E

1 $AC \approx 14.3$ km **2** $AC \approx 1280$ m **3** $\widehat{BCA} \approx 107.5^o$
4 **a** 35.69^o **b** 4 ha **5** **a** 316 km **b** 228.9^o
6 **a** 10.4 km **b** 195.9^o **7** ≈ 1010 m
8 8.08 km, 099° **9** $\approx 214^o$
10 11.3 cm and 17.0 cm **11** 14.1 cm
12 **a**

 b 7.59 km or 23.05 km
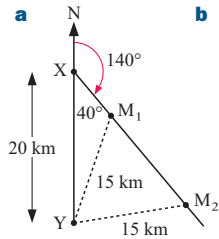
 13 ≈ 14.3 km **14** ≈ 63.4 m

EXERCISE 29F

1 $CD \approx 80.0$ m **2** $CD \approx 15.6$ m **3** $x \approx 12.2$ **4** $\approx 129^o$
5 **a** 4° **b** $\approx 11\,300$ m **c** $\approx 10\,400$ m
 d ≈ 6792 m **e** ≈ 6792 m
6 $x \approx 17.7$, $y \approx 33.1$ **7** **a** $\approx 10\,600$ m^2 **b** ≈ 1.06 ha
8 **a** $DC \approx 10.2$ m **b** $BE \approx 7.00$ m **c** ≈ 82.0 m^2

EXERCISE 29G.1

1 **a**

θ	0°	30°	60°	90°	120°	150°	180°
y	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1

θ	210°	240°	270°	300°	330°	360°
y	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

 b

 c 1 when $\theta = 0^o$ or 360° **d** -1 when $\theta = 180^o$

2 **a**

θ	0°	30°	60°	90°	120°	150°	180°
y	0	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	undef.	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	0

θ	210°	240°	270°	300°	330°	360°
y	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	undef.	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	0

 b $\theta = 90^o$, $\theta = 270^o$
 c

EXERCISE 29G.2

1 **a** 0.5 **b** -0.5 **c** -0.5 **d** 0.5
2 **a** **i** $x = 50^o, 130^o, 410^o, 490^o$
  ii $x = 45^o, 135^o, 405^o, 495^o$
  iii $x = 10^o, 170^o, 370^o, 530^o$
 b **i** $x = 25^o, 335^o, 385^o, 695^o$
  ii $x = 90^o, 270^o, 450^o, 630^o$
  iii $x = 70^o, 290^o, 430^o, 650^o$
 c **i** $x = 20^o, 200^o, 380^o, 560^o$
  ii $x = 55^o, 235^o, 415^o, 595^o$
  iii $x = 80^o, 260^o, 440^o$
3 **a** $x = 0^o, 180^o, 360^o, 540^o, 720^o$
 b $x \approx 17^o, 163^o, 377^o, 523^o$
 c $x \approx 53^o, 127^o, 413^o, 487^o$
 d $x \approx 204^o, 336^o, 564^o, 696^o$
4 **a** $x = 0^o, 360^o, 720^o$ **b** $x \approx 46^o, 314^o, 406^o, 674^o$
 c $x \approx 78^o, 282^o, 438^o, 642^o$
 d $x = 120^o, 240^o, 480^o, 600^o$
5 **a** $x \approx 72^o, 252^o, 432^o$ **b** $x \approx 117^o, 297^o, 477^o$
 c $x \approx 84^o, 264^o, 444^o$
6 **a**

 b $x = 45^o, 225^o, 405^o, 585^o$

7

We notice that:

- $y = \sin x + 2$ is $y = \sin x$ translated $\binom{0}{2}$
- $\sin(x + 90^o) = \cos x$ {as graphs coincide}

8

We notice that:
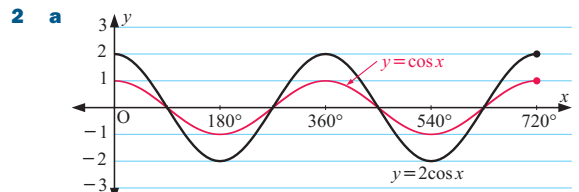
- $\cos(x - 90°) = \sin x$ {as graphs coincide}
- $y = \cos x - 2$ is $y = \cos x$ translated $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

EXERCISE 29H
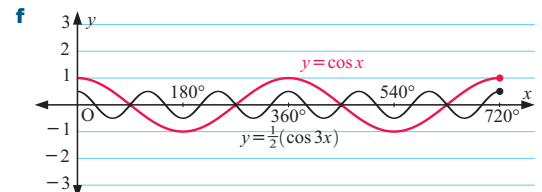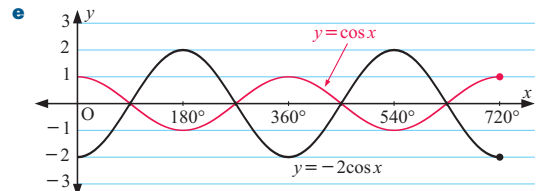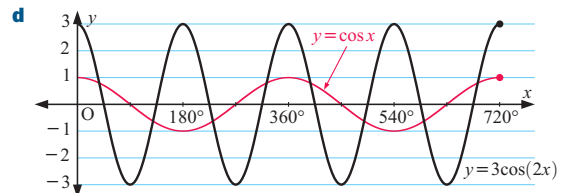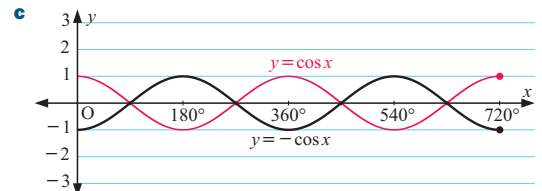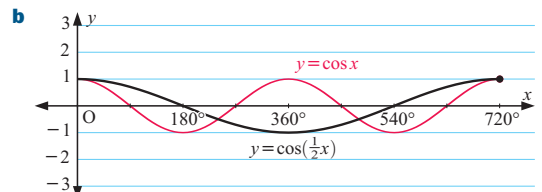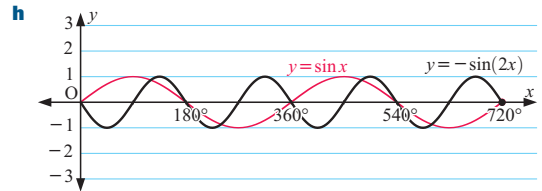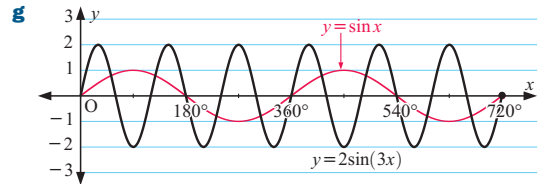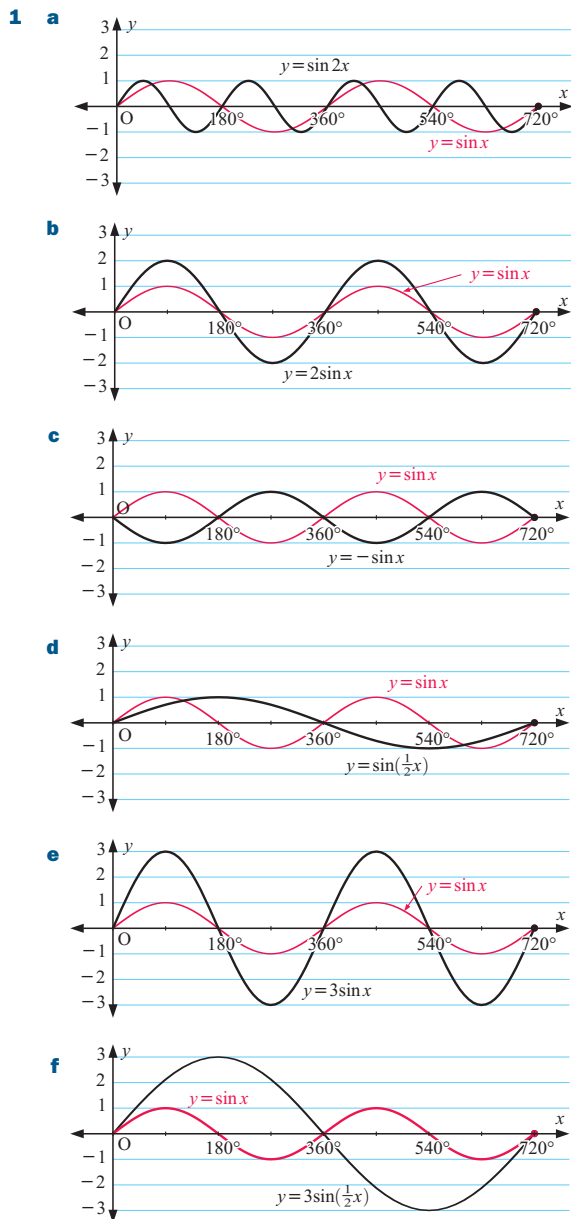
1 a

b

c

d

e

f

g

h

2 a

b

c

d

e

f

3 **a** $x \approx 21.8^o$ or 158.2^o **b** $x \approx 132.3^o$ or 227.7^o
    c $x \approx 12.6^o, 77.4^o, 192.6^o, 257.4^o$ **d** $x \approx 76.0^o, 256^o$
    e $x \approx 23.5^o, 96.5^o, 143.5^o, 216.5^o, 263.5^o, 336.5^o$
    f $x \approx 139.6^o, 220.4^o$

4 **a** $a = 2, b = 1$ **b** $a = -1, b = 1$ **c** $a = 1, b = 2$
    d $a = 1, b = \frac{1}{3}$ **e** $a = \frac{1}{2}, b = 1$ **f** $a = \frac{3}{2}, b = 2$

5 **a** $a = -1, b = 1$ **b** $a = 1.6, b = 1$ **c** $a = 1, b = \frac{1}{2}$
    d $a = 2, b = 2$ **e** $a = 2.5, b = 3$ **f** $a = -2, b = 3$
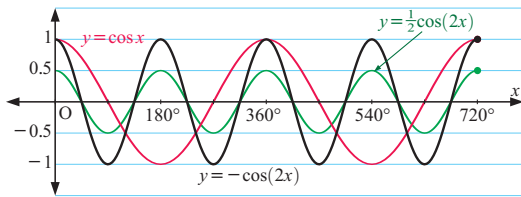
REVIEW SET 29A

1 **a** $(\cos 296^o, \sin 296^o)$ **b** $\approx (0.438, -0.899)$

2 **a** $\frac{\sqrt{3}}{2}$ **b** 1 **c** $-\frac{1}{\sqrt{3}}$ **3** ≈ 3.88 km^2 **4** $x = 14$

5 **a** a **b** b **c** b **d** $-a$ **6** ≈ 15.8 km

7 **a** $x \approx 11.1$ **b** $y \approx 74.6$ or 105.4 **c** $\theta \approx 45.8$

8 $A\widehat{C}B \approx 68.3^o$ or 111.7^o

9 **a** BD ≈ 278 m   **b** $46\,700$ m^2

10
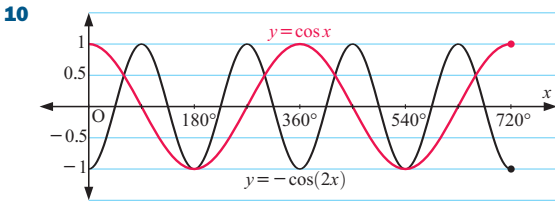
11 $a = 2, b = 1$; $y = 2 \sin x$

REVIEW SET 29B

1 **a** Q$(\cos 152^o, \sin 152^o)$ **b** $\approx$Q$(-0.8829, 0.4695)$

2 $\theta \approx 23.4$

3 **a** Q$(-a, -b)$ **b** Q$(-\cos \theta, -\sin \theta)$
    c The angle measured from the positive x-axis is $180^o + \theta$
    $\therefore$ Q is $(\cos(180^o + \theta), \sin(180^o + \theta))$
    $\therefore$ $\cos(180^o + \theta) = -\cos \theta$

4 **a** $\phi \approx 115$ **b** $x \approx 50.8$ or 129
    c $x \approx 89.5$

5 **a** AB ≈ 275 m   **b** 2.86 ha   **6** $x \approx 2.83$ or 15.6

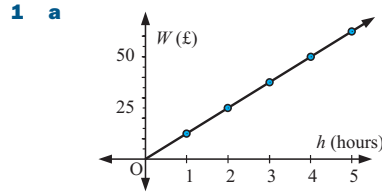**7** **a** ≈ 76.1 km   **b** 053.0^o **8** $x \approx 5.25$

9

10 $a = 2$, $b = \frac{1}{2}$ **11** $x \approx 76^o, 644^o$

CHALLENGE

1 The rocket must rise the same height as the earth's radius.

2 **a** $\tan 75^o = \tan 60^o + 2$

    b **i**

    ii BD $= 2$ units
    BC $= \sqrt{3}$ units
    AB $= 2$ units

    iii $\tan 75^o = \frac{2 + \sqrt{3}}{1} = 2 + \sqrt{3}$, $\tan 60^o = \frac{\sqrt{3}}{1} = \sqrt{3}$

3 **a** $C\widehat{O}N = 180^o - 2\theta$ **d** $2 \sin \theta \cos \theta = \frac{h}{r}$
    e $\sin(2\theta) = 2 \sin \theta \cos \theta$

EXERCISE 30A.1

1 **a**

    b $W \propto h$, as the graph is a straight line passing through the origin.
    c **i** $k = 12.5$ **ii** $W = 12.5h$

2 **a** y is doubled   **b** y is trebled   **c** x is doubled
    d x is halved   **e** y is increased by 20%
    f x is decreased by 30% **g** cannot be determined
    h cannot be determined

3 **a** No, does not pass through the origin.
    b Yes, linear and passes through the origin.   **c** No, not linear.
    d No, does not pass through the origin.

4 **a** The law is of the form $C = kr$, k a constant.
    b $k = 2\pi$ **c** **i** C is doubled   **ii** r is increased by 50%

5 **a** $y \approx 229$ **b** $x \approx 22.6$

6 **a** $R = 0.006l$ **b** $R = 0.3$ ohms   **c** $l = 500$ cm

7 **a** $A = 4.5n$, 9 litres   **8** **a** 78.4 m/s   **b** ≈ 10.2 sec

EXERCISE 30A.2

1 **a** $\frac{y}{x^2} = 4$; $y = 4x^2$; $a = 196$
    b $\frac{y}{x^2} = 3$; $y = 3x^2$; $a = 6$

2 **a** $\frac{y}{x^3} = 5$; $y = 5x^3$; $b = 320$
    b $\frac{y}{x^3} = \frac{1}{2}$; $y = 0.5x^3$; $b = 6$

3 **a** $A \propto t^2$, $k = 4.9$ **b** $K \propto x^3$, $k = 2$
    c $T \propto \sqrt{l}$, $k = \frac{1}{5}$ **d** $V \propto t^4$, $k = 500$
    e $P \propto \sqrt[3]{x}$, $k = 4$ **f** $V \propto r^3$, $k = \frac{4}{3}\pi$

4 **a** $P = 48$ **b** $a = 15$

5 **a** $M = 81$ **b** $x = \sqrt[3]{40} \approx 3.42$

6 **a** $D = 24$ **b** $t = 625$ **7** \$312.50   **8** 89 mm

9 10.1 seconds   **10** 12.2 km   **11** 36% reduction

12 **a** r is multiplied by $\sqrt[3]{2}$ (a 26% increase)
    b volume is decreased by 27.1%

13 **a** $V \propto h$ **b** $V \propto r^2$ **c** volume increases 22.4%

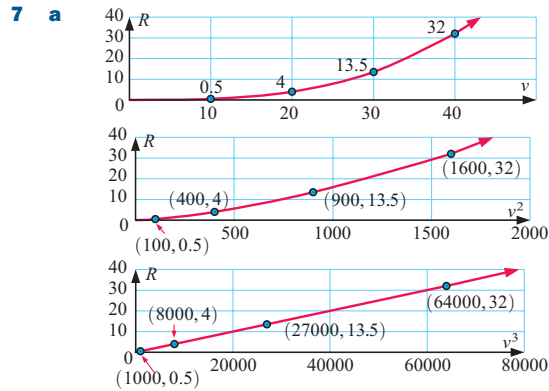### EXERCISE 30B

1 **a** $xy = 12$ **c** $xy = 84$

    b not inversely proportional

2 b and **c**   **3**   **a** $y = 8$ **b** $x = 0.4$

4   **a** $y = 18$ **b** $x = 12$

5   **a** inversely   **b** directly   **c** directly

6   **a** $M = 1.6$ **b** $t = 0.4$ **7**   **a** $P = 30$ **b** $g = 1.44$

8   **a** 20 units   **b** 4 cm^3 **9**   8 days

10   **a** 0.115 seconds   **b** 400 units

11 heat is increased by 525%   **12** ≈ 8.94 m away

EXERCISE 30C

1   **a** $y = 4x^3$ **b** $y = 3\sqrt{x}$ **c** $y = \dfrac{16}{x}$ **d** $y = \dfrac{100}{x^2}$

2 $d = 4\sqrt{h}$ **3** $m = \dfrac{l^3}{2}$ **4** $y = \dfrac{10}{x}$

5 Hint: Show that $x^2 y$ is a constant for all data points

$$y = \frac{5}{x^2} \quad \therefore \quad k = 5$$

6   **a** Show that $\dfrac{T}{\sqrt{l}}$ is always constant   **b** $k = 2$

    c ≈ 2.83 seconds

7   **a**

    b $R = kv^3$ {as 3rd graph is linear}

    Show $\dfrac{R}{v^3}$ is a constant to get $R = 0.0005x^3$

EXERCISE 30D

1 Notice that $(0,0)$ should not be entered.

2   **a** $y \approx 26.3x^{-3.02}$ **b** $M \approx 49.8x^{-0.497}$

3   **a** $h \approx 265\,m^{0.624}$ joules

    b It seems appropriate for the given data as r^2 is very close to 1.

    c $\approx 53\,900$ Joules. This could be unreliable as $m = 5000$ lies well outside the poles $(0.02 \leqslant m \leqslant 500)$.

4   **a** $R \approx 1.00T^{0.6667}$

    b The model seems very appropriate as r^2 is very close to 1.

    c $R^3 \approx T^2$

5   **a** $V \approx 140\,000P^{-0.714}$ $(r^2 \approx 1)$

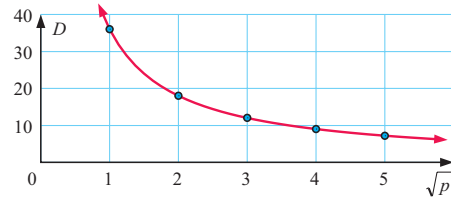   **b**   **i** ≈ 542 ml   **ii** ≈ 252 ml

REVIEW SET 30A

1   **a** $a = \frac{2}{5},\ b = 4$ **2** 229 km   **3** 6.75 days

4

    a $k = 4$
    b $y = 144$
    c $x = 4$

5   **a** $P = 4\sqrt{Q}$ **b** $P = 44$ **c** $Q \approx 10.6$

6 1.60 seconds

7   **a** V is multiplied by 4   **b** 26.5% increase in r

8   **a**

$\sqrt{p}$	1	2	3	4	5
D	36	18	12	9	7.2

    b The graph seems to be asymptotic to the axes.

    c $k = 36$ **d** $D \approx 10.0$

9   **a** $R \approx 3.99 \times d^{2.51}$

    b r^2 is very close to 1,

    $\therefore$ the power model seems appropriate

    c $R \approx 28.8$ m^3/s   **d** ≈ 1730 m^3 each minute

REVIEW SET 30B

1 $a = 14,\ b = 21$

2   **a** y is divided by 3   **b** x is decreased by $33\frac{1}{3}\%$

3   **a** $xy = 650$ **b** $y \approx 54.2$ **c** $x \approx 4.33$

4 18.75 bags   **5**   **a** 0.514 ohms   **b** 0.08 cm^2

6 10.7 units   **7**   **a** reduced by 34.4%   **b** increased by 73.2%

8   **a** $k = 0.4,\ n = 3 \ \therefore \ V = 0.4h^3$

    b when $h = 6$, $V = 86.4$ ✓

    c Volumes in general depend on 3 lengths multiplied together

    d   **i** ≈ 205 cm^3 **ii** $h = 5$ cm

9   **a** $D \approx 31.8n^{0.413}$ mm

    b   **i** ≈ 50.1 mm   **ii** ≈ 71.0 mm   **iii** ≈ 85.6 mm

     iv ≈ 94.6 mm

    c b iv is least likely to be reliable.

CHALLENGE

1 velocity is decreased by $\approx 8.71\%$

2   **a** $y = kx^n \ \therefore \ y_1 = kx_1{}^n$ and $y_2 = kx_2{}^n$ etc.

    b similar argument

3   **b** $2\frac{2}{3}$ **5** 50 km/h

EXERCISE 31A

1   **a** $\log_2 4 = 2$ **b** $\log_4 16 = 2$

    c $\log_3 9 = 2$ **d** $\log_5 125 = 3$

    e $\log_{10} 10\,000 = 4$ **f** $\log_7(\frac{1}{7}) = -1$

    g $\log_3(\frac{1}{27}) = -3$ **h** $\log_{27}(3) = \frac{1}{3}$

    i $\log_5(\frac{1}{25}) = -2$ **j** $\log_2\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$

    k $\log_2(4\sqrt{2}) = 2.5$ **l** $\log_{10}(0.001) = -3$

2   **a** $2^3 = 8$ **b** $2^0 = 1$ **c** $2^{-1} = \frac{1}{2}$

    d $2^{\frac{1}{2}} = \sqrt{2}$ **e** $2^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}$ **f** $(\sqrt{2})^2 = 2$

    g $(\sqrt{3})^4 = 9$ **h** $9^{\frac{1}{2}} = 3$

3   **a** 2   **b** 3   **c** 1   **d** 0   **e** 3

    f -1 **g** -3 **h** 7   **i** -1 **j** -2

k $\frac{1}{2}$ **l** $\frac{3}{2}$ **m** $\frac{1}{3}$ **n** $\frac{5}{2}$ **o** 2

p $-\frac{5}{2}$ **q** -2 **r** 4 **s** -2 **t** $-\frac{5}{2}$

4 **a** $\log_4 y = x$ **b** $\log_9 y = x$ **c** $\log_a y = x$

   d $\log_{\sqrt{3}} y = x$ **e** $\log_2 y = x + 1$ **f** $\log_3 y = 2n$

   g $\log_2 y = -x$ **h** $\log_3 \left(\frac{y}{2}\right) = a$

5 **a** $2^y = x$ **b** $3^y = x$ **c** $a^y = x$

   d $b^y = n$ **e** $m^y = b$ **f** $5^T = \frac{a}{2}$

   g $3^{2M} = p$ **h** $b^{\frac{G}{5}} = m$ **i** $(\sqrt{b})^P = n$

6 **a** $7^y = x$ **b** $x = \log_3 y$ **c** $x = \log_{0.5} y$

   d $x = \log_5 z$ **e** $x = 2^t$ **f** $\frac{\log_2 y}{3} = x$

   g $x = 2\log_5 y$ **h** $x = \frac{3^w}{2}$ **i** $x = \log_3(2z)$

   j $x = \log_4(5y)$ **k** $x = -\log_2(10D)$

   l $x = -1 + \log_3 G$

7 **a** As $a^0 = 1$ for all $a > 0$, then $\log_a 1 = 0$.

   b As $a^1 = a$ for all $a > 0$, then $\log_a a = 1$.

EXERCISE 31B

1 **a** $f^{-1}(x) = \log_4 x$ **b** $f^{-1}(x) = \log_{10} x$

   c $f^{-1}(x) = -\log_3 x$ **d** $f^{-1}(x) = \log_3\left(\frac{x}{2}\right)$

   e $f^{-1}(x) = 7^x$ **f** $f^{-1}(x) = \log_5(2x)$

   g $f^{-1}(x) = 2^{\frac{x}{3}}$ **h** $f^{-1}(x) = 3^{\frac{x}{5}}$

   i $f^{-1}(x) = (\sqrt{2})^x$ or $2^{\frac{x}{2}}$

2 **a**
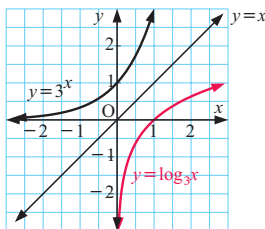
   b Domain is $\{x \mid x \in \mathbb{R}\}$ Range is $\{y \mid y > 0,\ y \in \mathbb{R}\}$

   c Domain is $\{x \mid x > 0,\ x \in \mathbb{R}\}$ Range is $\{y \mid y \in \mathbb{R}\}$

3 The inverse of $y = a^x$ is $x = a^y$

   $\therefore\ y = \log_a x$ i.e., $f^{-1}(x) = \log_a x$

4 **a** $x \approx 2.59$ **b** $x \approx 3.27$ **c** $x \approx 0.0401$ or 1.22

   d $x \approx 0.137$ or 1 **e** $x \approx 1.32$ **f** $x \approx 0.0103$ or 1

EXERCISE 31C

1 **a** $\log_3 16$ **b** $\log_2 3$ **c** $\log_5 72$

   d $\log_3 14$ **e** $\log_3 12$ **f** $\log_3 45$

   g $\log_7 21$ **h** $\log_4\left(\frac{36}{125}\right)$ **i** $\log_3(m^2 n^7)$

   j $\log_2\left(\frac{k^5}{n^3}\right)$

2 **a** $p + q$ **b** $q - p$ **c** $2p$ **d** $3q$

   e $p - 2q$ **f** $p + 2q$ **g** $p + 3 - 2q$ **h** $q + p - 2$

3 **a** $y = u^3$ **b** $y = \frac{u^3}{v}$ **c** $y = u^2 v^3$ **d** $y = 2^{u+v}$

   e $y = \frac{2^u}{v}$ **f** $y = \frac{1}{u}$ **g** $y = 7v^2$ **h** $y = \frac{\sqrt{v}}{u^2}$

   i $y = \frac{36}{\sqrt[3]{u}}$ **j** $y = 3\sqrt{uv}$

4 **a** 2 **b** 2 **c** -2 **d** -2

EXERCISE 31D.1

1 **a** $\approx 10^{0.903}$ **b** $\approx 10^{1.903}$ **c** $\approx 10^{2.903}$

   d $\approx 10^{-0.0969}$ **e** $\approx 10^{-2.0969}$ **f** $\approx 10^{-0.523}$

   g $\approx 10^{-1.523}$ **h** $\approx 10^{-4.523}$ **i** $\approx 10^{1.699}$

   j $\approx 10^{-3.301}$

2 **a** $\log 30$ **b** $\log 5$ **c** $\log 12$ **d** $\log(\frac{5}{4})$ **e** $\log 1 = 0$

   f $\log 30$ **g** $\log 4$ **h** $-\log 6$ or $\log(\frac{1}{6})$ **i** $\log(\frac{1}{512})$

   j $\log 2000$ **k** $\log(\frac{64}{125})$ **l** $\log 20$ **m** $\log 5$ **n** $\log 20$

   o $\log(14\,000)$

3 $\log 30 = \log(3 \times 10) = \log 3 + \log 10$, etc.

4 **a** 3 **b** 2 **c** $\frac{2}{3}$ **d** -1 **e** -1 **f** $-1\frac{1}{2}$

   g $\frac{b}{3}$ **h** $\frac{2}{a}$

6 **Hint:** $2400 = 2^3 \times 3 \times 5^2$

EXERCISE 31D.2

1 **a** $\log y = \log a + 2\log b$ **b** $\log y = 2\log a - \log b$

   c $\log y = \log d + \frac{1}{2}\log p$ **d** $\log M = 2\log a + 5\log b$

   e $\log P = \frac{1}{2}(\log a + \log b)$ **f** $\log Q = \frac{1}{2}\log m - \log n$

   g $\log R = \log a + \log b + 2\log c$

   h $\log T = \log 5 + \frac{1}{2}(\log d - \log c)$

   i $\log M = \log a + 3\log b - \frac{1}{2}\log c$

2 **a** $Q = 10^{x+2}$ or $Q = 100(10^x)$

   b $J = 10^{2x-1}$ or $J = \frac{1}{10}(100^x)$

   c $M = 10^{2-x}$ or $M = \frac{100}{10^x}$

   d $P \approx 10^{0.301+x}$ or $P \approx 2(10^x)$

   e $R \approx 10^{x+1.477}$ or $R \approx 30(10^x)$

   f $K \approx 10^{\frac{1}{2}x+1}$ or $K \approx 10(10^{\frac{x}{2}})$

3 **a** $M = ab$ **b** $N = \frac{d}{e}$ **c** $F = x^2$ **d** $T = \sqrt{p}$

   e $D = \frac{1}{g}$ **f** $S = \frac{1}{b^2}$ **g** $A = \frac{B}{C^2}$ **h** $p^2 q = s$

   i $\frac{m^3}{d} = \frac{n}{p^2}$ **j** $\frac{m}{\sqrt{n}} = p^2$ **k** $N = 10t$ **l** $P = \frac{100}{x}$

EXERCISE 31E

1 **a** $x \approx 1.903$ **b** $x \approx 3.903$ **c** $x \approx -1.602$

   d $x \approx 2.659$ **e** $x \approx -0.057\,30$ **f** $x \approx -3.747$

2 **a** $x = 1.585$ **b** $x = 3.322$ **c** $x = 8.644$

   d $x = -7.059$ **e** $x = 4.292$ **f** $x = -0.099\,97$

   g $x = 6.511$ **h** $x = 4.923$ **i** $x = 49.60$

   j $x = 4.376$ **k** $x = 8.497$ **l** $x = 230.7$

3 **a** 17.0 hours   **b** 64.6 hours

4 **a** 17.6 hours   **b** 36.9 hours

5 **a** 4.59 years ≈ 4 years 7 months

   b 9.23 years ≈ 9 years 3 months

6 ≈ 1.59

7 **a** ≈ 3.58 **b** ≈ 4.19 **c** ≈ 2.02 **d** ≈ -3.99

8 **a** $x = 4$ **b** $x = \frac{1}{25}$ **c** $x = 2$ **d** $x = \frac{1}{10}$

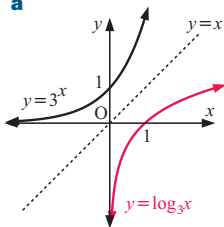**REVIEW SET 31A**

1 **a**
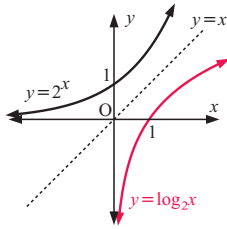
b A reflection in the line $y = x$

c Domain is $\{x \mid x > 0, \ x \in \mathbb{R}\}$
Range is $\{y \mid y \in \mathbb{R}\}$

2 **a** $\log_a a^x = x$ **b** $x = \log_b y$

3 **a** 4 **b** -1 **c** $\frac{5}{2}$ **d** $\frac{3}{2}$

4 **a** $\log_5 y = x$ **b** $\log_7 y = -x$

5 **a** $3^y = x$ **b** $n = 4^{3T}$

6 **a** $x = 5^y$ **b** $x = \dfrac{10^w}{3}$ **c** $x = \frac{1}{2}\log_7(3q)$

7 **a** $f^{-1}(x) = \log_5\left(\dfrac{x}{4}\right)$ **b** $f^{-1}(x) = 3^{\frac{x}{2}}$

8 **a** $x \approx 3.3219$ **b** $x \approx -4.9829$ **c** $x \approx 105.03$

9 **a** ≈ 492 wasps   **b** ≈ 52.8 days

10 **a** $\log 6$ **b** $\log 36$ **c** $\log_2 1125$

11 **a** $\log y = 3\log a - 2\log b$
b $\log M = \log 3 + \frac{1}{2}\log a - \frac{1}{2}\log b$

12 **a** $T = \dfrac{1000}{10^x}$ **b** $N = \dfrac{c^2}{d}$

13 **a** $a + b$ **b** $b - a$ **c** $1 + b$

14 **a** $y = u^4$ **b** $y = \dfrac{1}{v^2}$ **c** $y = \sqrt{uv}$

15 ≈ 2.4650 **16** **a** $x \approx 0.07663$ **b** $x \approx 1.533$

REVIEW SET 31B

1 **a**

b For $y = 3^x$:
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y > 0, y \in \mathbb{R}\}$
For $y = \log_3 x$:
Domain is $\{x \mid x > 0, x \in \mathbb{R}\}$
Range is $\{y \mid y \in \mathbb{R}\}$

2 **a** $\frac{1}{2}$ **b** $-\frac{3}{2}$ **c** 6 **d** $\frac{3}{2}$

3 **a** $x = \log_4 y$ **b** $n = -\log_a y$

4 **a** $d = 2^y$ **b** $k = a^{2M}$

5 **a** $x = 3^y$ **b** $x = \dfrac{b^T}{3}$ **c** $x = -1 + \log_2\left(\dfrac{3t}{5}\right)$

6 **a** $f^{-1}(x) = \log_6 x$ **b** $f^{-1}(x) = 5^{2x}$

7 **a** $x \approx 7.288$ **b** $x \approx 5.671$ **c** $x \approx 1.732$

8 **a** \$400   **b** $\approx \$15\,200$ **c** Year 2024

9 **a** $\log_2 15$ **b** $\log_3 4$ **c** $\log(2.5)$ **d** $\log_2\left(\frac{25}{2}\right)$

10 **a** $\log D = 2 - 2\log n$ **b** $2\log G = 3\log c + \log d$

11 **a** $M = 10^{2x+1}$ or $M = 10 \times 100^x$ **b** $G = \dfrac{\sqrt{d}}{10}$

12 **a** $b - a$ **b** $a + b$ **c** $a - 1$

13 **a** $y = c^2$ **b** $y = \dfrac{\sqrt[3]{c}}{d^2}$ **14** ≈ 2.723

15 **a** $x \approx -3.288$ or 0.8342 **b** $x \approx 0.9149$ or 4.284

CHALLENGE

1 The last step {dividing by $\log(\frac{1}{2})$ which is negative reverses the inequality sign}.

2 **a** $x \approx 1.58$ or 2.32 **b** $x = 4$ as x must be > 0

3 **a** $x \approx 1.584\,962\,501$ **b** $\therefore \ 2^p = 3^q$
LHS is even and RHS is odd
...... is irrational

EXERCISE 32A

1 **a** $x > -2$ **b** $x \geqslant -1$ **c** $x > 1\frac{1}{2}$ **d** $x \geqslant \frac{1}{3}$
e $x > 2$ **f** $x \geqslant \frac{1}{4}$

2 **a** $x \leqslant -2$ or $x \geqslant 3$ **b** $-4 \leqslant x \leqslant 3$
c $-5 \leqslant x \leqslant 2$ **d** $-1.45 < x < 3.45$
e $x > -2$ **f** $x < -1.25$ or $0.445 < x < 1.80$

3 **a** $x \leqslant -1.30$ or $x \geqslant 2.30$ **b** $x > 0.631$
c $x \leqslant -1.93$ or $x \geqslant 1.26$
d $x < -0.767$ or $2 < x < 4$
e $-2.13 < x < 0$ **f** $x < 0$ or $x \geqslant 0.682$

4 $13.7^o < x < 76.3^o$ or $193.7^o < x < 256.3^o$

5 $-\frac{2}{3} < x < 1\frac{1}{3}$

EXERCISE 32B

1 **a** $x \leqslant 0$ and $y \geqslant 0$ **b** $y \leqslant 4$ **c** $y > -1$
d $x \leqslant 2$ **e** $x > -1$ **f** $0 \leqslant y \leqslant 3$
g $y < -1$ or $y > 1$ **h** $-2 \leqslant x \leqslant 2$ and $-2 < y < 3$
i $y \leqslant x$, $0 \leqslant x \leqslant 4$, $0 \leqslant y \leqslant 3$

2 **a**

b

c

d

e

f

g

h

i

3 a

$x + 2y = 4$

b

$2x + y = 5$

c

$3x + 2y = 6$

d

$2x + 3y = 6$

e

$2x - 3y = 12$

f

$5x + 2y = 10$

g

$2x + 5y = -10$

h

$4x + 3y = 6$

i

$4x - 3y = 12$

j

$y = x$

k

$y = -x$

l

$2x + 5y = 0$

4 a

$y = 2$

b

$y = 4$, $x = -2$

c

$x + y = 4$

d

$2x + y = 6$

e

$(2, 4)$, $(6, 0)$, $x = 2$, $x + y = 6$

f

$2x + 3y = 12$, $(0, 4)$, $y = 3$, $(1\frac{1}{2}, 3)$
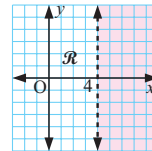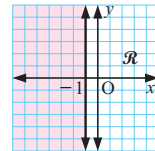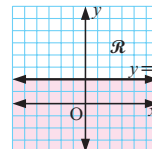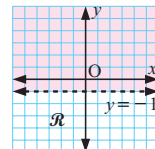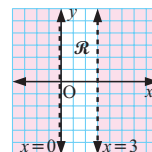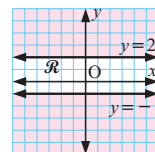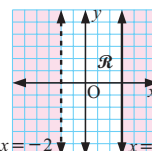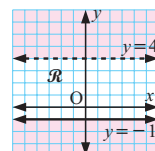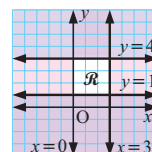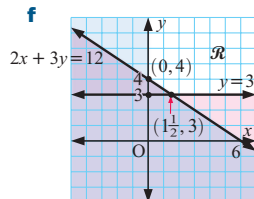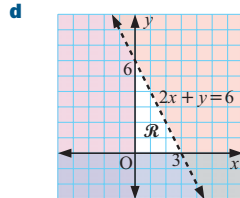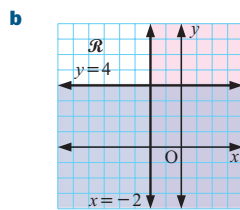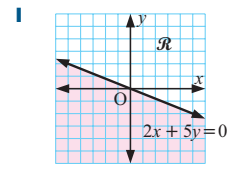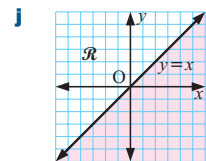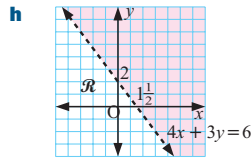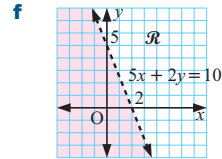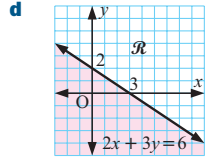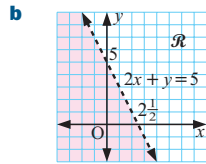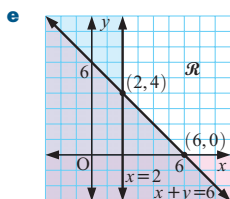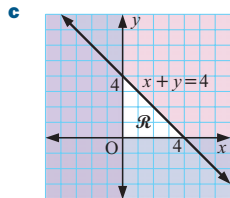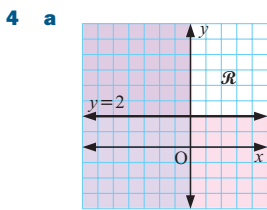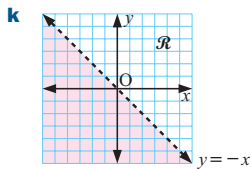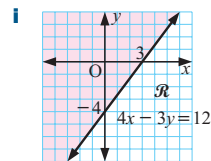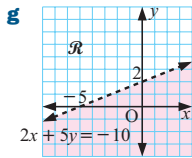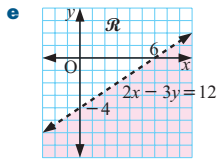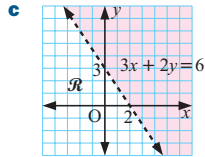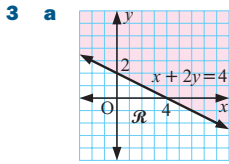
5 a $y \geqslant 2$, $y \leqslant x$, $x + y \leqslant 10$
  b $x \geqslant 2$, $y \leqslant 5$, $y \geqslant \frac{1}{3}x$, $5x + 6y \leqslant 60$

EXERCISE 32C

1 a

$(4, 2)$, $x + 2y = 8$, $x + y = 6$
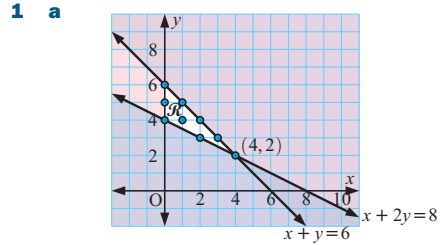
b 9 points   **c** 2 points, $(0, 4)$ and $(1, 5)$
d 8, greatest: 10; least: 4

2 a

$(6, 2)$, $x + 3y = 12$, $x + y = 8$
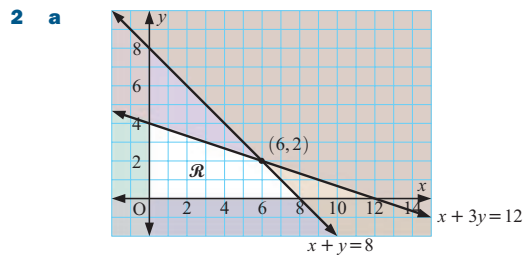
b   **i** 18   when $x = 6$, $y = 2$
    ii 16   when $x = 0$, $y = 4$
    iii 24   when $x = 6$, $y = 2$ or $x = 7$, $y = 1$ or $x = 8$, $y = 0$

3 a

$(0, 12)$, $(6, 3)$, $(12, 0)$, $3x + 2y = 24$, $x + 2y = 12$
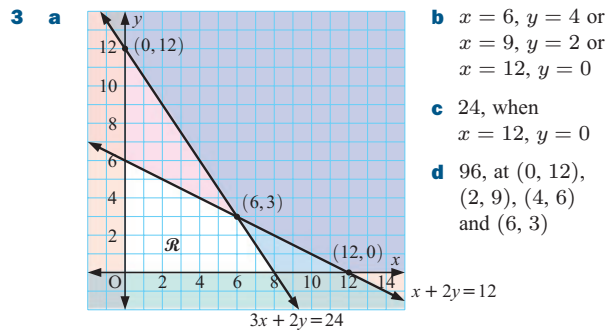
b $x = 6$, $y = 4$ or $x = 9$, $y = 2$ or $x = 12$, $y = 0$

c 24, when $x = 12$, $y = 0$

d 96, at $(0, 12)$, $(2, 9)$, $(4, 6)$ and $(6, 3)$

EXERCISE 32D

1 $x \geqslant 0$, $y \geqslant 0$, $5x + y \leqslant 15$, $x + 4y \leqslant 12$
2 $x \geqslant 0$, $y \geqslant 0$, $x + y \leqslant 50$, $2x + 3y \leqslant 120$
3 $x \geqslant 0$, $y \geqslant 0$, $x + y \leqslant 7$ as there are 7 days in a week
   $8x + 6y \leqslant 50$ as there are no more than 50 ha available

4 a x and y cannot be negative
$\therefore x \geqslant 0$, $y \geqslant 0$,
$3x + y \geqslant 17$ as at least 170 units of carbohydrate must be consumed.
$x + y \leqslant 11$ as at most 330 units of protein are consumed. $x + 2y \geqslant 14$ as at least 1400 units of vitamins are needed.
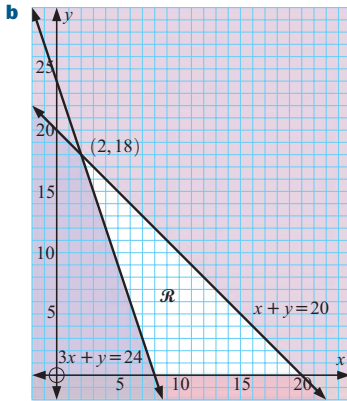
b

$3x + y = 17$, $(3, 8)$, $x + y = 11$, $(4, 5)$, $(8, 3)$, $x + 2y = 14$
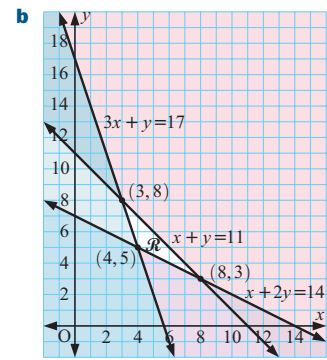
c 4 of A and 5 of B   **d** 8 of A and 3 of B

5 a

	No.	Cost
Hard	x	€20
Soft	y	€8

b

cannot have a negative number of books purchased
$\therefore$ $x \geqslant 0$, $y \geqslant 0$ $x + y \leqslant 20$, as can purchase at most 20 books
$30x + 10y \geqslant 240$, as needs to spend at least €240.
$\therefore$ $3x + y \geqslant 24$.

c **i** 8 when 8 hard, 0 soft   **ii** €600 (20 hard, 0 soft)

6 a

	Bats	Balls	Nets	Profit	Made/h
Set A	2	3	0	$3	x
Set B	2	5	1	$5	y

b $P = 3x + 5y$
c $x \geqslant 0$, $y \geqslant 0$, $x + y \leqslant 28$, $3x + 5y \leqslant 108$, $y \leqslant 18$
d (1) 6 of A and 18 of B   or   (2) 11 of A and 15 of B   or
(3) 16 of A and 12 of B
e For (1), the making of bats is under-utilised.
For (2), the making of bats and nets is under-utilised.
For (3), the making of nets is under-utilised.

7 a

	A	B	Profit	No.
Deluxe	2 min	2 min	£25	x
Standard	3 min	1 min	£20	y

$x \geqslant 0$, $y \geqslant 0$, $2x + 3y \leqslant 48$, $2x + y \leqslant 20$, $y \geqslant 2x$

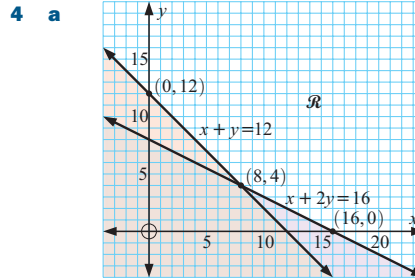**b** $P = 25x + 20y$
c
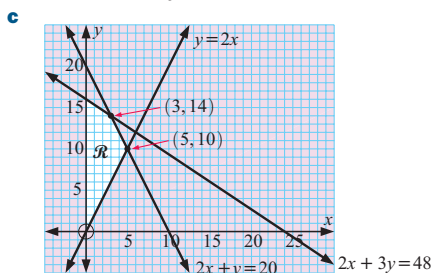
d Max. profit £355 when 3 Deluxe and 14 Standard are made.

REVIEW SET 32A

1 a $x < -4.236$ or $x > 0.236$ **b** $x < 0.675$
2 a $x > 2$, $y \leqslant -3$
b $x \geqslant 0$, $y \geqslant 0$, $x + y \geqslant 3$, $x + 3y \geqslant 6$
3 a (2, 3), (2, 4), (2, 5), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2)
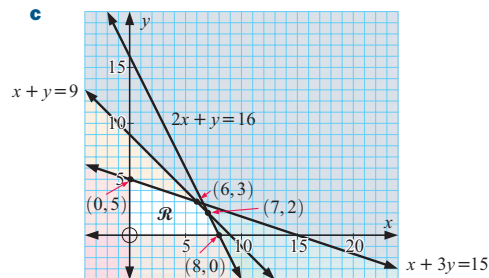**b** 3 solutions are removed. Only (3, 2), (3, 3), (3, 4), (4, 1) and (4, 2) remain.

4 a

b (0, 14), (1, 13), (2, 12), (3, 11), (4, 10), (5, 9), (6, 8), (7, 7), (8, 6), (9, 5), (10, 4), (11, 3), (12, 2)
c 24 when $x = 0$, $y = 12$

5 a

Meter	Gears	Dials	Time	Profit	No.
Gas	4	1	8	$20	x
Water	12	1	4	$31	y

b $x \geqslant 0$, $y \geqslant 0$, $x + 3y \leqslant 15$, $x + y \leqslant 9$, $2x + y \leqslant 16$
c

d $P = 20x + 31y$ **e** $213 when 6 gas and 3 water

REVIEW SET 32B

1 a $x < 0$ or $x > 0.641$ **b** $-1.18 \leqslant x \leqslant 1.51$
2 a $x \leqslant -2$, $y < 2$
b $x \geqslant 0$, $y \geqslant 1$, $2x + y \leqslant 4$, $3x + 4y \leqslant 12$
3 a (0, 3), (0, 4), (0, 5), (1, 3), (1, 4), (2, 2), (2, 3)
b (0, 3), (1, 4)   **c** $x = 2$, $y = 3$ with maximum value 22
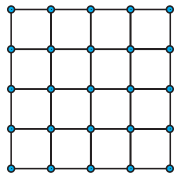**4** 3 filing cabinets and 5 desks for a €337 profit

CHAPTER 33

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1 a **i** $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **ii** $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **b** A translation of $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$.
c $x \leqslant 2$, $y \geqslant \frac{1}{2}x$, $y \geqslant 0$, $y \leqslant 2x + 4$

2 a **i** $\widehat{BAH} = \dfrac{(8-2) \times 180^o}{8} = 135^o$
ii   So, $\widehat{HAP} = 45^o$
Likewise $\widehat{AHP} = 45^o$
$\therefore$ $\widehat{APH} = 90^o$ {angles of a triangle}

b **i** $6\sqrt{2}$ cm   **ii** $12(1 + \sqrt{2})$ cm   **iii** 36 cm^2
iv $288(1 + \sqrt{2})$ cm^2

c **i** $6(1 + \sqrt{2})$ cm   **ii** 94.8%

3 a **i** $250.00   **ii** $2600.00   **iii** $\approx 6.12\%$
b **i** 12 km **ii** 7 h 12 min **iii** ≈ 2.78 km/h **iv** 16:07
c $n = 25\,000$

4 a Maria $350, Carolina $250, Pedro $200
b $275   **c** $200   **d** $11 : 8 : 4$ **e** $110.25

b   **i** $u_n = 9 - 2n$ **ii** $u_n = n^3$ **iii** $u_n = \dfrac{n}{n+1}$

     iv $u_n = (n+1)^2$ **v** $u_n = 3^{n-1}$

     vi $u_n = (n+1)^2 - 3^{n-1}$

c u_{393} **d** u_{11}

6   **b**   **i** $\approx 147^o$ **ii** $\approx 33^o$ **c**   **i** $\mathbf{q} + \mathbf{p}$ **ii** $\mathbf{q} - \mathbf{p}$

   d $\mathbf{p} + 3\mathbf{q}$ **e** $\frac{1}{2}\mathbf{p} + \frac{5}{2}\mathbf{q}$ **f**   **i** $\mathbf{p} = \begin{pmatrix} 0 \\ 24 \end{pmatrix}$ **ii** $\begin{pmatrix} 7 \\ -24 \end{pmatrix}$

   g   50 units

7   **a** $y = \dfrac{k}{x^2}$ $k = 5^2 \times 4.8 = 120$ **b** $y = 30$

          $\therefore \quad x^2 y = 120$

   c $x = 2\sqrt{3}$ {as $x > 0$}   **d** $x \approx 4.93$ **e** y is quartered

   f x is increased by 25%   **g** $x = \sqrt{\dfrac{120}{y}}$ {as $x > 0$}

8   **a**

                          b   **i** $p = 25$

                                ii $q = 40$

                                iii $x = n^2$, $y = (n+1)^2$,

                                   $z = 2n(n+1)$

c

Diagram	1	2	3	4	5
Total lines	4	16	40	80	140

     i $e = \frac{2}{3}$, $f = 2$, $g = \frac{4}{3}$ **ii**   880 one cm lines

9   **a**   €1216   **b**   €1.47   **c** $\approx 15.2\%$ **d**   €621   **e**   €4830

   f   **i** ≈ 723 km/h   **ii** ≈ 201 m/s

10   **a**   12 cm   **b**   192 cm^3 **c** $\approx 67.4^o$ **d** $\approx 36.9^o$

   e   **i** $\approx 76.7^o$ **ii** ≈ 6.40 cm

11   **a**   **i** mode $= 1$ **ii** $k = 3$ **iii** $m = 4$ **iv** 9

   b   **i** $\overline{x} \approx 38.2$ **ii** $\frac{7}{330}$

   c   **i** $p = 20$, $q = 72$

     ii
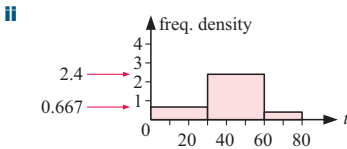
12   **a**   64.8 m^3 **b**   864 m^3, $\approx 1230\%$ increase

   c   22.1 m   **d**   150 m

13   **a**   **i** a rhombus   **ii** a kite

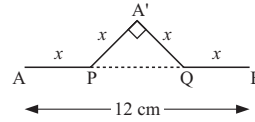   **b** $(2x)^o$ and $(180 - 2x)^o$ **c**   120 cm^2 **d**   13 cm

14   **a**   **i** $\mathbf{c} - \mathbf{d}$ **ii** $\mathbf{d} - \frac{1}{2}\mathbf{c}$ **iii** $\frac{3}{2}\mathbf{c} - \mathbf{d}$

   b   **i** 120° **ii** $16\sqrt{3}$ cm^2 **iii** $8\sqrt{3}$ cm   **iv** $96\sqrt{3}$ cm^2

15   **a** $\approx 553\,000$ cm^3 (or ≈ 0.553 m^3)

   b ≈ 6.86 hours (≈ 6 hours 51 min)   **c** ≈ 8 mm

16   **a**   **i** $\widehat{BAC} = 36.9550.... \approx 37^o$ **ii** 014.0^o

     iii ≈ 842 m^2

   b   **i** $p = 70 \cos 39^o \approx 54.4$ **ii** $q = 70 \sin 39^o \approx 44.1$

c $BD \approx 36.5$ m

17   **a** ≈ 6.54 cm^3

   b   **i** $w = 4.2$, $x = 1.4$ **ii** ≈ 106 cm^3 **iii** $\approx 74.2\%$

   c   **i** $l = \sqrt{(1.5)^2 + (0.35)^2} \approx 1.54$ cm   **ii** ≈ 38.4 cm^2

18   **a**   **i**

             $PQ = (12 - 2x)$ cm

         $\therefore \quad x^2 + x^2 = (12 - 2x)^2$ {Pythagoras}

         i.e., $2x^2 = (12 - 2x)^2$

     iii $x \approx 3.51$

   b   **i** ≈ 56.2 cm   **ii** ≈ 169 cm^2

19   **a**   **i** ≈ 528 cm^3

     ii $OC = \sqrt{R^2 - r^2} = \sqrt{10^2 - 6^2} = 8$ cm

           $OS = R = 10$ cm

         $\therefore \quad SC = OC + OS = 18$ cm

     iii ≈ 4070 cm^3 **iv** $\approx 87.0\%$

   b   **i** $SC = CT + TS$ **ii** $OC^2 = 9 - r^2$

              $= 2r + 1$

       $\therefore \quad OC = SC - OS$

              $= (2r + 1) - 3$

              $= (2r - 2)$ cm

     iii simplify $(2r - 2)^2 = 9 - r^2$ **iv** $r \approx 2.08$

     v $CT = h \approx 4.16$ cm

20   **a**   **i** $BX = \frac{1}{2}BD$ **ii** $\widehat{BCX} \approx 76.6^o$

         $= \frac{1}{2}(7.5 - 2 \times 3 \cos 70^o)$

         ≈ 2.724 cm

   b   **i** ≈ 1.77 cm^2 **ii** ≈ 18.3 cm^2 **iii** ≈ 14.1 cm^2

     iv ≈ 41.2 cm^2

21   **a** ≈ 18.8 cm^2 **b** ≈ 7.85 cm   **c** 140° {angle at centre}

   d $r = \dfrac{\frac{1}{2}(QR)}{\sin 70^o} \approx 4.18$ cm   **e** ≈ 10.2 cm   **f** 220°

22   **a**   **i** $V = \frac{1}{6}abh$ cm^3 **ii** 40 cm^3

   b   **i** $V = 4x(x + 3)$ cm^3

     ii simplify $4x(x + 3) = 2(2x + 3)$

     iii $x \approx 0.58$ {as $x > 0$}   **iv** $RS \approx 3.58$ cm

     v $\widehat{PMO} \approx 81.5^o$

23   **a** $a = 90$, $b = 90$, $c = 138$, $d = 69$, $e = 45$

   b   '...... GAC are *congruent*.'

   c   **i** $GA \approx 141$ cm   **ii** $GX = GA + AX$

                                $\approx 141 + 54$

     iii ≈ 262 cm   **iv** ≈ 121 cm

24   **a**   **i** $OC = x$ cm, $OF = (x - 10)$ cm

     ii $x^2 = 20^2 + (x - 10)^2$

     iii Equation in **ii** simplifies to $20x = 500$ etc.

   c ≈ 3480 cm^2 **d** ≈ 4.52 kg

25   **a**   14 hours 31 min   **b**   **i** $\widehat{MCB} \approx 123^o$ **ii** 247°

   c   **i** 7 hours 38 min   **ii** arrives at 20:18, after sunset

26   **a**   **i** ≈ 0.770 m^2 **ii** $= 2.704\,592...... \approx 2.705$ m

     iii ≈ 1.83 m^2 **iv** ≈ 2.60 m^2

26 **a** **i** ≈ 0.770 m^2 **ii** $= 2.704\,592...... \approx 2.705$ m
  iii ≈ 1.83 m^2 **iv** ≈ 2.60 m^2
  b ≈ 31.5 m^2 **c** $\approx 37.4\%$

27 **a** $t \approx 1.27$ s   **b** $h \approx 0.994$ m   **c** $h = \dfrac{9.81t^2}{\pi^2}$

  d **i** $\widehat{AOB} \approx 57.3^o$ **ii** 0.500 m^2

28 **a** **i** sides are x cm, $(x-3)$ cm, $(x-5)$ cm
    $\therefore$ $P = 3x - 8$
  ii 16 cm   **iii** $\widehat{CBA} \approx 43^o$
  b **i** $x^2 = (x-3)^2 + (x-5)^2$ {Pythagoras}
    gives $x^2 - 16x + 34 = 0$ on simplification
  ii $x \approx 13.48$ or 2.52
  iii AB ≈ 13.5 m,   AC ≈ 10.5 m,   BC ≈ 8.5 m
    (case $x = 2.52$ is impossible)

29 **a** **i** $\widehat{BAD} = \widehat{BCD} = 66^o$

  ii • $\widehat{BAD} = \widehat{BCD}$ {from **i**}
    • $\widehat{APB} = \widehat{CPD}$ {vertically opposite}
    $\therefore$ Δs are similar {as they are equiangular}
  iii PB $= 16$ cm,   CD $= 8.75$ cm
  iv area ΔCPD $= \dfrac{n}{4}$ cm^2

  b **i** $\widehat{BTD} = 48^o$ **ii** diameter is OT and OT ≈ 23 cm
30 **a** **i** ≈ 39.3 cm^2 **ii** 25 cm^2 **iii** 7.13 cm^2 **b** 50 cm^2

CHAPTER 34

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A INVESTIGATION QUESTIONS

1 **a** $k = 6$

  $n = 2$,   LHS $= 5$,   RHS $= \dfrac{2 \times 3 \times 5}{6} = 5$ ✓

  $n = 3$,   LHS $= 14$,   RHS $= \dfrac{3 \times 4 \times 7}{6} = 14$ ✓

  b $338\,350$ **c** **i** $m = 50$ **ii** $171\,700$ **d** $166\,650$
  e -5050

2 **a/b**
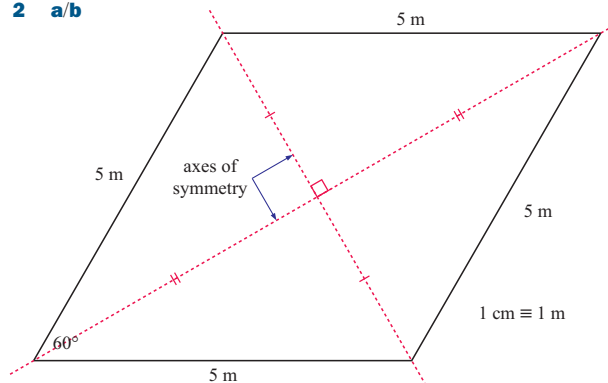
The axes of symmetry bisect each other at right angles.

  c a rhombus   **d** $\dfrac{25}{2}\sqrt{3}$ m$^2 \approx 21.7$ m^2
  e Area is a maximum of 25 m^2 when the angle is 90°,
    i.e., when the rhombus is a square. This is because when 60°
    is replaced by θ,   area $= \left(\frac{1}{2} \times 5^2 \times \sin \theta\right) \times 2 = 25 \sin \theta$
    which is a maximum of 25 m^2 when $\sin \theta = 1$ (a maximum).

3 **a** $5 = 2 + 3$ is one example
  b **i** $16 = 3 + 13 = 5 + 11$
    ii 38 can only be written as $7 + 31$
  c $16 = 2 + 3 + 11$
  d False, as 11 cannot be written as the sum of two prime numbers.

4 **a** ≈ 114 cm^3
  b ≈ 132 cm^3
  c $\approx 47.6\%$, $\approx 51.4\%$
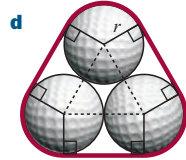
  d   box shape (from above)

  Area from above $= \pi r^2 + 3(2r^2) + \sqrt{3}r^2$
    $= (\pi + 6 + \sqrt{3}) \times 2.15^2$
    ≈ 50.263 cm^2
  $\therefore$ volume ≈ 216.13 cm^3
  $\therefore$ unfilled space ≈ 91.2 cm^3
  and $\%$ unfilled $\approx 42.2\%$

5 **a** They all have perimeter 24 units.
  b $24\sqrt{3}$ cm$^2 \approx 41.6$ cm^2
  c R: 24 cm^2, S: 30 cm^2, P: 32 cm^2, Q: 36 cm^2,
    H: 41.6 cm^2, T: ≈ 45.8 cm^2
  d For figures with the same perimeter, as the figures get more regular the area gets larger with the circle providing max. area.

6 **a**

L	1	2	3	4	5
B	1	2	5	8	13
W	0	2	4	8	12
T	1	4	9	16	25

  b **i** 60 **ii** 41 cm
  c $T = L^2$
  d **i** $T = 2B$
    ii $T = 2B - 1$

7 **a**
  $1 \qquad\qquad = 1$ $\qquad 1^3 \qquad\qquad\qquad = 1$
  $1 + 2 \qquad\quad = 3$ $\qquad 1^3 + 2^3 \qquad\qquad = 9$
  $1 + 2 + 3 \quad\; = 6$ $\qquad 1^3 + 2^3 + 3^3 \qquad = 36$
  $1 + 2 + 3 + 4 = 10$ $\qquad 1^3 + 2^3 + 3^3 + 4^3 = 100$

  b
    $1^3 = 1^2$
    $1^3 + 2^3 = 9 = (1 + 2)^2$
    $1^3 + 2^3 + 3^3 = 36 = (1 + 2 + 3)^2$
    $1^3 + 2^3 + 3^3 + 4^3 = 100 = (1 + 2 + 3 + 4)^2$

  c $1^3 + 2^3 + 3^3 + 4^3 + ... + 9^3$
    $= (1 + 2 + 3 + 4 + ... + 9)^2$
    $= 45^2$
    $= 2025$

  d $325^2 = 105\,625$
  e If $n = 1$,   LHS $= 1$,   RHS $= a + b$
      $\therefore$ $a + b = 1$
    If $n = 2$,   LHS $= 3$,   RHS $= 4a + 2b$
      $\therefore$ $4a + 2b = 3$
      $\therefore$ $a = \frac{1}{2}$, $b = \frac{1}{2}$
  f $984\,390\,625$

8 **a** **i** $2n - 1$ **ii** n^2 **iii** $\dfrac{n(n+1)}{2}$ **iv** $\dfrac{n(n-1)}{2}$

  b $\dfrac{n(n+1)}{2} + \dfrac{n(n-1)}{2} = \dfrac{n}{2}(n + 1 + n - 1)$

    $= \dfrac{n}{2} \times 2n$

    $= n^2$

9 **a** **i** $V_A = 9\pi r^2 h$, $V_B = 3\pi r^2 h$, $V_C = 27\pi r^2 h$
    ii $3 : 1 : 9$

b   **i** $5(16-11)$, $10(26-16)$
    ii $n([2n+6]-[n+6])=n^2$

11 a

n	2	1	0	-1	-2
A	49	16	1	4	25
B	125	27	1	-1	-27
C	256	16	0	16	256
D	256	16	1	$\frac{1}{16}$	$\frac{1}{256}$

  b   **i** Cesar's   **ii** Dan's   **c** $a=16$, $b=4$, $c=16$
  d n^2+13 is one of them

12 a ≈ 553 cm^3 **b**   **i** 32 cm by 24 cm   **ii** $\approx 21.5\%$
  c   **i** \$0.48   **ii** \$5.76   **d**   **i** \$0.71   **ii** $\approx 12.7\%$

13 a $5^2+12^2=25+144=169=13^2$ **b** 7
  c   **i** $y^2+(x-2)^2=x^2$ gives $y=\sqrt{4x-4}$
      $\{y$ must be $>0\}$
    ii 48 and 14   **iii** 99 and 20
    iv 3, 4, 5   or   6, 8, 10   or   8, 15, 17   or   10, 24, 26
      or   12, 35, 37   {any 2 of these}

14 a   **i** LHS = RHS = $\frac{25}{6}$ **ii** LHS = RHS = $\frac{49}{12}$
    iii LHS = RHS = $\frac{121}{30}$
  b $1+\frac{3}{2}$, $1+\frac{2}{3}$; $1+\frac{3}{4}$, $1+\frac{4}{3}$
  c e.g., $1\frac{3}{5}\times 2\frac{2}{3}=1\frac{3}{5}+2\frac{2}{3}$
  d $\left(1+\dfrac{x}{y}\right)\left(1+\dfrac{y}{x}\right)=\left(1+\dfrac{x}{y}\right)+\left(1+\dfrac{y}{x}\right)$

$$\text{LHS = RHS} = \frac{(x+y)^2}{xy}$$

15 a   **i** 7th;   A - \$70,   B - \$64,   C - \$49
    ii nth;   A - \$$10n$,   B - \$$2^{n-1}$,   C - \$$n^2$
  b   **i** A - \$550,   B - \$1023,   C - \$385
    ii scheme A   (\$1710)   **iii** 15th birthday

16 a   **i** both $=2418$ **ii** both equal 1512
  b $43\times 68=34\times 86$
  c In the digit form 'ab' the a represents the 10s and the b the
    ones. So   'ab' $=10a+b$
  d $12\times 63=21\times 36$, $12\times 84=21\times 48$, $23\times 64=32\times 46$,
    $23\times 96=32\times 69$, $31\times 26=13\times 62$, $34\times 86=43\times 68$,
    $48\times 42=84\times 24$, $46\times 96=64\times 69$, $93\times 13=39\times 31$,
    $14\times 82=41\times 28$, $36\times 84=63\times 48$, $21\times 24=12\times 42$

17 a

n	1	2	3	4
n^2	1	4	9	16
n^4	1	16	81	256

  b $p=5$, $q=30$, $r=17$, $s=35$, $t=98$, $u=354$
  c Row X $\times$ Row Y $=5\times$ Row Z
  d   **i** X $=2870$ **ii** Y $=1259$ **e** $722\,666$

18 a   **i** $(x+3)$ cm   **ii** $(x+3)$ cm   **iii** $2x$ cm
  b   **i** 60° **ii** equilateral   **iii** $x=3$
  c   **i** 18° **ii** $\sin 9^o=\dfrac{x}{x+3}$ **iii** $x\approx 0.56$ cm
  d $\widehat{AOB}=\dfrac{360^o}{n}$, $x=\dfrac{3\sin\left(\frac{180}{n}\right)^o}{1-\sin\left(\frac{180}{n}\right)^o}$

19 a $135=3^3\times 5$, $210=2\times 3\times 5\times 7$, $1120=2^5\times 5\times 7$
  b   **i** 5
    ii e
    iii 210 and 1120

  c

1	9	6
3	5	4
2	7	8

20 a 13

---

b   **i** $d=2c^2$ **ii** $e=2(c+h)^2$
    iii PR $=h$,   QR $=e-d=2(c+h)^2-2c^2$
      $\therefore$ QR $=4ch+2h^2$
    iv $m=\dfrac{4ch+2h^2}{h}=4c+2h$
    v $c=3$, $h=0.5$ and $4c+2h=13$ ✓
    vi $c=3$, $h=0.1$, $m=12.2$
    vii   **a** h approaches 0   **b** gradient approaches 12

B   MODELLING QUESTIONS

1 a 12   **b** 324
  c   **i** $357-xy$ bulbs   **ii** $xy=357$
    As $357=3\times 7\times 17$ then $x=17$, $y=21$

2 a $a=5$, $b=4$ and $c=30$
  b $d=5+4\sin(420^o)\approx 8.46$ **c** ≈ 1.54 m
  d 9 m   **e** at 0300 and 1500
  f 1 m when $\sin(30t)^o=-1$
    i.e.,   at   0900 and every 12 hours thereafter.

3 a   **i** $a=2$ **ii** $q=10$, $r=2.5$
    iii $M=\dfrac{160}{2^t}$

    iv

    Rate of change is $-\dfrac{95}{3.4}$ {tangent slope} ≈ -28
    $\therefore$ decreasing at about 28 g/min.
  b   **i** On graph **3 a iii**.   **ii** $t=1$ min
    iii A reflection in $M=80$.

4 a

  b   **i** $a=25$, $b=\frac{1}{2}$ **ii** $r=18.75$, $s\approx 4.69$
    iii Max. effect ≈ 26.5 units at $t\approx 2.89$ hours
    iv between 48 min and 7 h 9 min

5 a

$$y = \frac{3 \times 2^{\frac{t}{2}}}{2^{\frac{t}{2}} + 2}$$
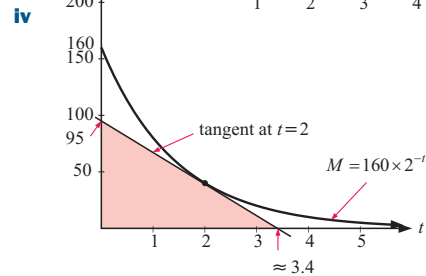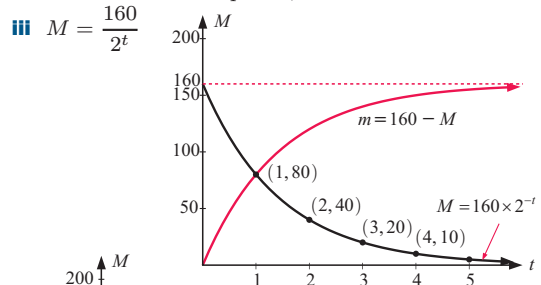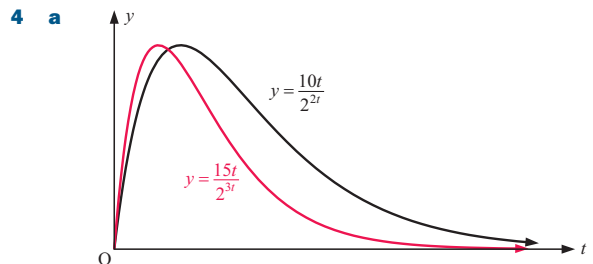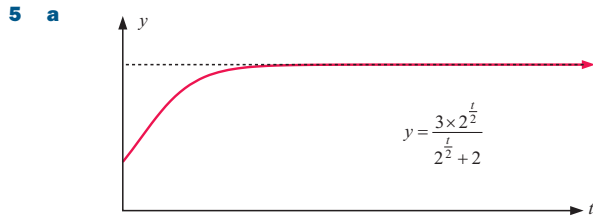
b the horizontal asymptote   **c** 3

d   **i** $B(t) = \dfrac{20 \times 2^{0.799t}}{2^{0.799t} + 1}$ **ii** $B(3) \approx 15.03$ ✓

   iii 20 units   **iv** If t is very large, 2^{bt} is huge
   and $2^{bt} \approx 2^{bt} + c$
   $\therefore \quad y \approx a$

6 a goats cost $\$2x$,   cows cost $\$4y$
   $\therefore \quad 2x + 4y \leqslant 32$ and hence $x + 2y \leqslant 16$
b $x + y \leqslant 12$, $x \geqslant 6$, $y \geqslant 3$
c
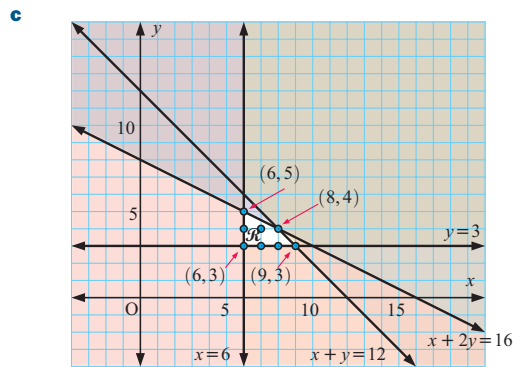
d 6g, 3c and 6g, 5c and 7g, 3c and 7g, 4c and
   8g, 3c and 8g, 4c and 9g, 3c
e 8 goats and 4 cows for $\$720$ profit

7 a   **i** B   **ii** G   **iii** F   **iv** E   **v** A   **vi** C
b $y = x^2$

8 a The x teachers carry $24x$ kg.
   The y students carry $20y$ kg.
   $\therefore \quad 24x + 20y \geqslant 240$
   $\therefore \quad 6x + 5y \geqslant 60$ $\{\div$ by 4$\}$
b $x + y \leqslant 13$, $x \geqslant 4$, $y \geqslant 3$
c
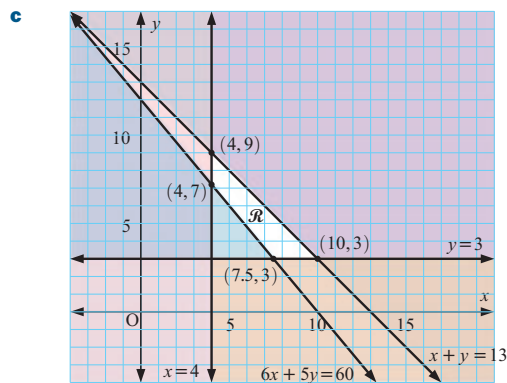
d   **i** 11 people   **ii** 300 kg

9 a $a = 4$, $b = 2$
b As $4 + 2x - x^2 = 1$, $x^2 - 2x - 3 = 0$ **c** 3
d   **i** $c = 4$ **ii** $\sqrt{5} - 1 \approx 1.24$
e   **i** ≈ 1.12 m   **ii** ≈ 0.414 m, 2.41 m
f 5 m above the court

10 a $x + y \leqslant 12$ **b** $y \geqslant 4$
c/d
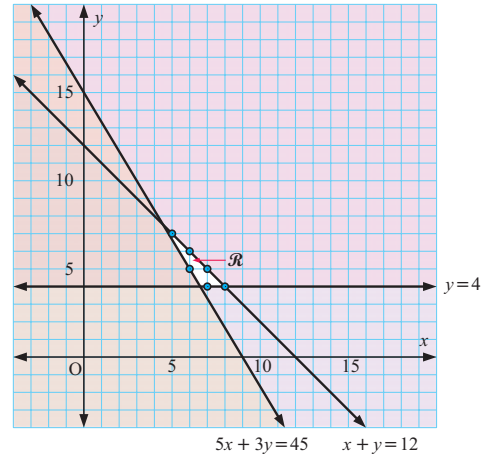
e Cheapest is $\$170$ using 5 SUPER taxis and 7 MINI taxis
   or 6 SUPER taxis and 5 MINI taxis
f   **i** $\$260$ and $\$274$
   ii $\$94$ {using 7 SUPER taxis and 4 MINI taxis}

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