Linear Programming

Convex-Set and n-Dimensional Space

Euclidean Space, E^n—An n-dimensional Euclidean space (Euclidean Vector Space) is a collection of all vector (points)

$$\vec{a} = [a_1, a_2, \ldots\ldots, a_n]$$

(1) Vector Multiplication (by scalar) and Vectors Addition—

$$\lambda\vec{a} = [\lambda a_1, \lambda a_2, \ldots\ldots, \lambda\, a_n]$$

$$\text{and } \vec{a} + \vec{b} = [a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n]$$

$$\text{were } \vec{a} = [a_1, a_2, \ldots, a_n]$$

$$\text{and } \vec{b} = [b_1, b_2, \ldots, b_n]$$

(2) Distance between Two Vectors—

$$|\vec{a} - \vec{b}| = \sqrt{\sum_{i=1}^{n} (a_1 - b_1)^2}$$

(3) Point Set—$X = \{\bar{x} : P(\vec{x})$, where $P(\vec{x})$ is a property defined on $\vec{x}\}$.

(4) Hypersphere—A hypersphere in E^n with centre at a and radius $\varepsilon > 0$ is the set of points

$$X = \left\{ \frac{\vec{x}}{|\vec{x} - \vec{a}|} = \in \right\}$$

(5) Inside—The inside of a hypersphere with centre at a and radius $\varepsilon > 0$, is the set of points

$$X = \left\{ \frac{\vec{x}}{|\vec{x} - \vec{a}|} < \in \right\}$$

(6) $\in$-Neighbourhood—An $\in$-neighbourhood of the point $\vec{a}$, is the set of points inside the hypersphere with centre at a and radius $\varepsilon > 0$,

$$X = \left\{ \frac{\vec{x}}{|\vec{x} - \vec{a}|} < \in \right\}$$

(7) Interior Point—A point $\vec{a}$ is an interior of set A if these exists an $\in$-neighbourhood about $\vec{a}$ which contains only points of set A.

(8) Boundary Points—A point is a boundary point of a set A every $\in$-neighbourhood about $\vec{a}$ contains points which are in the set and points which are not in the set.

(9) Open Set—A set is an open set if it contains only interior points.

(10) Closed Set—A set is closed if it contains all its boundary points.

(11) Complement—The complement of any set A in E^n, A^c is the set of all points in E^n but not in A.

(12) Strictly Bounded—A set A is strictly bounded if there exists a positive number r such that for every $\vec{a} \in A, |\vec{a}| < \vec{r}$.

(13) Bounded from Above—Set A is bounded from above if there exists $\vec{r}$, with each component finite such that for all $\vec{a} \in A, \vec{a} \leq \vec{r}$.

(14) Bounded from Below—Set A is bounded from below if there exists $\vec{r}$, with each component finite, such that for each

$\vec{a} \in A, \vec{a} \leq \vec{r}$.

(15) Line—The line passing through $\vec{x_1}$ and $\vec{x_2}$ $(\vec{x_1} \neq \vec{x_2})$ in E^n, is the set of the points

$$X = \{\vec{x} / \vec{x} = \lambda\vec{x_2} + (1 - \lambda)\vec{x_1},\ \lambda \text{ real number.}\}$$

(16) Polygonal Path—A path joining such that $x_1 = y_1, y_2, \ldots\ldots, y_R = x_2$ where y_1 joins y_2, y_2 joins y_3, …… and y_R joins x_2.

(17) Line Segment—The line segment joining points $\vec{x_1}$ and $\vec{x_2}$ in E^n, is the set of points

$$X = \{\vec{x} / \vec{x} = \lambda\vec{x_2} + (1 - \lambda)\vec{x_1},\ 0 \leq \lambda \leq 1\}$$

(18) Connected Set—An open set is connected if any two points in the set can be joined by a polygonal path lying entirely within the set.

(19) Region—A region in E^n is connected set of points in E^n.

(20) Normal—Given the hyper-plane $\overrightarrow{cx} = z$ in E^n ($\vec{c} \neq 0$), then $\vec{c}$ is a vector normal to the hyper-plane.

(21) Parallel Hyper Planes—Two hyper-planes are parallel if they have the same unit normal.

(22) Open Half-Spaces—The set $X_1 = (\vec{x}/\overrightarrow{cx} < z)$ and $X_2 = (\vec{x}/\overrightarrow{cx} > z)$ are open half-spaces.

(23) Closed Half-Spaces—The set

$$X_3 = (\vec{x}/\vec{c}\ \vec{x}) \leq z)$$

and $$X_4 = (\vec{x}/\vec{c}\ \vec{x} \leq z)$$

are closed half spaces.

(24) Convex Sets—A set is convex if for any points $\vec{x}_1$, $\vec{x}_2$ in the set, the line segment joining these points is also in the set.

(25) Convex Combination—A convex combination of a finite number of points $\vec{x}_1$, $\vec{x}_2$, ..., $\vec{x}_m$ is a point $\vec{x} = \sum_{i=1}^{m} \mu_i = 1$

(26) Extreme Points—A points $\vec{x} \in X$ is an extreme point of a convex set X iff there do not exist point $\vec{x}_1$, $\vec{x}_2$ ($\vec{x}_1 \neq \vec{x}_2$) in the set such that

$$\vec{x} = (1 - \lambda)\vec{x}_1 + \lambda\vec{x}_2, 0 < \lambda < 1.$$

(27) Convex Polyhedron—The convex hull of finite number of points is a convex polyhedron spanned by these points.

(28) Convex Hull—The convex hull of a set is the intersection of all convex sets which contain A.

(29) Simplex—The convex hull of any set of $n + 1$ points from E^n which do not lie on a hyper-plane in E^n is simplex.

(30) Supporting Hyper-Planes—Given a boundary points $\vec{w}$ of a convex set X : then $\overrightarrow{cx} = z$ is a supporting hyperplane at $\vec{w}$ if $\vec{c}\ \vec{w} = z$ iff of X lies in one closed half-space produced by the hyperplane, *i.e.*, $\overrightarrow{cu} \geq z$ for all $\vec{u} \in X$ or $\overrightarrow{cu} \leq z$ for all $\vec{u} \in X$.

(31) Edge—Let $\vec{x}_1$ and $\vec{x}_2$ be distinct extreme points of the convex set X. The line segment joining them in edge of convex set if it is the intersection of X with supporting hyperplane. If $\vec{x}_1$ is an extreme point of X, and if there exists another point $\vec{x} \in X$ such that $\vec{x} = \vec{x}_1 + \lambda(\vec{x}' - \vec{x}_1')$ is in X for every $\lambda \geq 0$ and if in addition, the set $L = \{\vec{x}/\vec{x}) = \vec{x}_i + \lambda(\vec{x}' - \vec{x}_1{}^*)$, $\lambda \geq 0\}$ is the Intersection of X with supporting hyper plane, then the set L is edge of X which extends to infinity.

(32) Adjacent extreme points—Two distinct extreme points $\vec{x}_i$ and $\vec{x}_2$ of the convex set X is adjecent if the line segment joining them is an edge of the convex set.

(33) Cone—A cone c is a set of points such that if $\vec{x}$ is in set then $\mu\vec{x}$, $\mu \geq 0$ is also in set *i.e.* the cone generated by the set $X = \{\vec{x}\}$ is the set $c = \{\vec{y}/\vec{y} = \mu\vec{x}, \mu \geq 0$ and $\vec{x} \in X\}$.

(34) Vertex—The point $\vec{O} = (0, 0, ..., 0)$ is an element of any cone and is called vertex of the cone.

(35) Negative—The negative C of the cone is $C = (\vec{y})$ is the set $C = (-\vec{y})$.

(36) Sum—The sum of two cones

$$C_1 = (\vec{y}), C_2 = (\vec{z}), \text{is}$$

$$C = \{\vec{y} + \vec{z}/\vec{y} \in C_1 \text{ and } \vec{z} \in C_2\}$$

(37) Polar Cone—If $c = \{\vec{y}\}$ is a cone, then C^+, the cone polar to c, is the set

$$C^+ = \{\vec{v}/\vec{v}, \vec{u} \geq 0, \vec{u} \in c\}$$

(38) Convex Cone—A cone is convex cone if it is a convex set.

(39) Half Line—Given a single point $\vec{a} \neq 0$, a half line (ray) is a set $L = \{\vec{y} : \vec{y} = u\vec{a}, u \geq 0\}$.

(40) Containing Half-Space—Given a cone C. Let $\vec{C} \neq 0$ be an element of C^+. Then containing half-space is the set

$$H_s = \{x : a' \vec{x} \geq 0\}$$

(41) Orthogonal Cone—Given a cone C in E^n. The orthogonal cone $C^\perp$ is the set of all vectors E^n such that $\vec{u}$ is orthogonal to every $D \in C$ *i.e.*

$$C^\perp = \{\vec{u} : u\vec{v} = 0, \vec{v} \in C\}.$$

(42) Dimension of a Cone—The dimension of a cone C is the maximum number of linearly independent vectors in C.

(43) Convex Polyhedral—A convex polyhedral cone C is the sum of a finite number of half lines, *i.e.* $C = \sum_{i=1}^{r} L_i$.

(44) Extreme Supporting Half-Spaces—The set of points in E^n such that $H_F = \{\vec{v} : c\vec{v} \geq 0\}$ is an extreme supporting half sapce for then-dimensional convex polyhedral cone C generated by the points $\vec{a_1} \ldots \vec{a_r}$ if C lies in the half space H_F and $(n - 1)$ linearly independent points from the set $\vec{a_1}, \vec{a_2}, \ldots \vec{a_n}$ lies on the hyperplane $c\vec{v} = 0$.

(45) Extreme Supporting Hyperplane—The hyperplane $c\vec{v} = 0$ which form the boundary of the extreme supporting half-space is called on extreme supporting hyperplane for the convex polyhedral cone C.

Some Important Theorems and Results

(I) The set E^n is both open and closed.

(II) Strictly bounded set lies inside a hyper-plane of radius r, with its centre at the origin.

(III) Extreme point is the boundary point of a set, converse is not true.

(IV) Hyper-plane is a convex set.

(V) Closed half space is a convex set.

(VI) Intersection of two convex sets is also a convex set.

(VII) Intersection of two closed sets is a closed set.

(VIII) Intersection of finite number of hyper-planes, half-spaces or both of is a convex set.

(IX) Intersection of finite number of hyper-planes or closed half spaces or of both is a closed convex set.

(X) The set of feasible solutions to a linear programming problem is a closed convex set.

(XI) The set of all convex combinations of a finite number of points $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m$ is a convex set.

(XII) The convex hull of a finite number of points $\vec{x}_1, \vec{x}_2, \ldots \vec{x}_m$ is the set of all convex combinations of $\vec{x}_1, \vec{x}_2, \ldots \vec{x}_m$.

(XIII) **Theorems on Separating Hyper-Planes**

(a) Given any closed convex set X, a point $\vec{y} \in X$ or there exist a hyper-plane which contains $\vec{y}$ such that all of X is contained in one open half space produced by that hyper-plane.

(b) If $\vec{w}$ is a boundary point of a closed convex set, then there is at least one supporting hyper-plane at $\vec{w}$.

(c) A closed convex set which is bounded from below has extreme points in every supporting hyper-plane.

(d) If a closed, strictly bounded convex set X has a finite number of extreme points, any point in the set can be written as a convex combination of the extreme points, *i.e.* set X is a convex hull of its extreme points.

(XIV) A set of points is a convex cone iff the sum $\vec{v}_1 + \vec{v}_2$ is in the set, when $\vec{v}_1$ and $\vec{v}_2$ are and if $u\vec{v}$ is in the set when $\vec{v}$ for any $u \geq 0$.

(XV) The sum $c_1 + c_2$ of two convex cones is also convex.

(XVI) The cone generated by a convex set is a convex cone.

(XVII) The cone C generated by a convex polyhedral is a convex polyhedral cone.

Linear Programming Problems (BASIC)

General Linear Programming Problem—Given a set of m-linear inequalities or equations in

r-variables, we wish to find non-negative values of these variables which satisfy the constraints and maximize or minimize some linear functions of the variables.

Optimize $z = c_1x_1 + c_2x_2 + \dots + c_rx_r$ ...(1)

Subject to

$a_{i1}X_1 + a_{i2}X_2 + \dots + a_{ir}X_r \{\leq, =, \geq\} b_i$, ...(2)

$i = 1, 2, \dots\dots, m$

with restrictions (non-negativity)

$x_j \geq 0, j = 1, 2, \dots\dots r$

Here a_{ij}, b_i, c_j are known constants and one of the sign $\leq$, $=$, $\geq$, holds for each constraints and vary from constraints to constraint.

Determine the value of x_j ($j = 1, 2, \dots r$) (Decision variables) ?

Decision Variables—x_j's are decision variables.

Objective Function—A linear function $z = \Sigma\, c_j\, x_j$, which is to be maximize or minimize.

Constraints—Inequalities $\sum_j a_{ij}\, x_j \{\leq, =, \geq\} b_i$ $(i = 1, 2, \dots\dots m)$

Non-negative Restrictions—$x_j \geq 0$ ($j = 1, 2, \dots r$).

Solution—The values of the variables x_j's ($j = 1, 2, \dots m$) satisfying constraints(2).

Feasible Soltuion—A solution satisfying non-negative restriction, $x_j \geq 0$ ($j = 1, 2, \dots r$)

Optimal Solution—A feasible solution which optimizes the objective function (1).

Conversion of Inequality into Equality

Slack variable—Given $\sum_{j=1}^{r} a_{kj}\, x_j \leq b_j$, then

$\sum_{j=1} a_{hj}\, x_j - x_{r+k} = b_j$, where $x_{r+k} = 0$ is slack variable

Surplus variable—Given $\sum_{j=1}^{r} a_{kj}\, x_j \geq b_j$, then

$\sum_{j=1}^{r} a_{kj}\, x_j - x_{r+k} = b_h$, where $x_{r+k} \geq 0$ is surplus variable.

Conversion of $-b_j$ to b_j—Given $\sum_{j=1}^{r} a_{hj}\, x_j$ $\{\leq, =, \geq\} - b_j$, multiply both side by (-1) gives $\sum_{j=1}^{r}$ $(-a_{nj})\, x_j \{\geq, =, \leq\}\, b_j$.

Conversion of minimization to maximization—Given min $z = \Sigma\, c_j\, x_j$ is equivalent to max $(-z) = \Sigma\, (c_j) x_j$ *i.e.* min (z) = max $(-z)$ and max (z) = min $(-z)$.

Cost (price) Associated with Slack and Surplus Variables

The cost associated with slack and surplus variables is assumed to be zero.

i.e. if
$$z = \sum_{j=1}^{r} c_j\, x_j + \sum_{h=1}^{u} c_n\, x_{r+h} + \sum_{k=1}^{v} c_k\, x_{k+r}$$

where c_h and c_k are cost associated with slack x_{r+h} and surplus x_{k+r} variables respectively and $c_h = c_k = 0$, which gives

$$z = \sum_{j=1}^{r} c_j\, x_j + \sum_{h=1}^{u} 0.x_{r+h} + \sum_{k=1}^{v} 0.\, x_{k+r} = \sum_{j=1}^{r} c_j\, x_j$$

Standard linear Programming Problem

$$\max z = \sum_{j=1}^{r} c_j\, x_j$$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, (i = 1, 2, \dots m)$$

with $\quad x_j \geq 0, (j = 1, 2, \dots\dots n)$

where $\quad b_i \geq 0\ (i = 1, \dots\dots m)$

Conversion of General Linear Programming Problem to Standard linear Programming Problem

By introducing slack and surplus veriables, making all $b_j \geq 0$ and maximize objective function z, we can convert general linear programming to standard linear programming problem, which gives

$$\max z = \sum_{j=1}^{r} c_j x_j$$

Subject to

$$\sum_{j=1}^{r} a_{hj}\, x_j + x_{r+n} = b_h, (h = 1, \dots\dots u)$$

$$\sum_{j=1}^{r} a_{kj}\, x_j - x_{r+h} = b_k, (k = u + 1, \dots v)$$

$$\sum_{j=1}^{r} a_{pj} x_j = b_p, (p = v+1, \ldots m)$$

with $x_j \geq 0,$

this gives max $z = \vec{cx}$

subject to $\vec{Ax} = \vec{b}$ with $\vec{x} \geq 0$

where $\vec{c} = \{c_1, c_2, \ldots, c_n\}$

$\vec{x} = \{x_1, x_2, \ldots x_n\}$

$\vec{A} = [a_{ij}]_{m \times m}$ matrix

and $\vec{b} = \{b_1, b_2, \ldots b_m\}$.

Simultaneous Linear Equations and Basic Solution

Given m-simultaneous linear equations is n known $(m < n)$

Given $\vec{Ax} = \vec{b}$

i.e. $\sum_{j=1}^{n} a_{ij} x_j = b_i, (i = 1, 2, \ldots m)$

where $\vec{A} = [a_{ij}]_{m \times n}$,

$\vec{b} = [b_i, \ldots\ldots b_m]$

and $\vec{x} = [x_1, x_2, \ldots\ldots x_n]$

Basic Solution—For a given system of m-simultaneous linear equations in n unknowns,

$$\vec{A}\,\vec{x} = \vec{b}\ (m < n)$$

and $\text{rank}(\vec{A}) = m$

If $\vec{B}$ is any $m \times m$ non singular matrix, chosen from $\vec{A}$ and if all the $n \times m$ variables not associated with the columns of this matrix $\vec{B}$ are set equal to zero, the solution of the resulting system of equation is called a basic solution. The m variables associated with matrix B are called basic variables *i.e.* $\vec{B}$ is a $m \times m$ non singular matrix from $\vec{A}$, then basic solution is $\vec{x}_B = B^{-1}b$, where all the variables not associated with columns from A in B are set equal to zero.

Degeneracy—A basic solution to $\vec{Ax} = \vec{b}$ is degenerate if one or more of the basic variables vanish.

Some Important Results

(1) A necessary and sufficient condition for the existence and non-degeneracy of all possible basic solutions of $\vec{Ax} = \vec{b}$ is the linear independence for every set of m columns from the argumented matrix $\vec{A}_b = (\vec{A}, \vec{b})$.

(2) A necessary and sufficient condition for any given basic solution $\vec{x}_B = \vec{B}^{-1}\vec{b}$ to be non-degenerated is the linear independence of $\vec{b}$ and every set of $(m-1)$ columns from $\vec{B}$.

Basic Feasible Solution—A basic solution which is feasible, *i.e.* $\vec{x}_B \geq \vec{0}$.

Associate Cost Vector—If $\vec{x}_B$ is a basic feasible solution to max $z = \vec{c}\,\vec{x}$, subject to $\vec{A}\,\vec{x} = \vec{b}$, $\vec{x} \geq \vec{o}$, then the vector $\vec{c}_B = (c_{B1}, \ldots c_{Bm})$, where c_B is the cost associated with basic varibale B, is called cost associated vector with basic feasible solution $\vec{x}_B$

Improved basic feasible solution—If $\vec{x}_B$ and $\vec{x}_B$ are two basic feasible solutions to the standard linear programming problem. Then $\vec{x}_B$ in the improved basic feasible solution in comparision to $\vec{x}_B$, if $\vec{c}_B\ \vec{x}_B \geq \vec{c}_B\ \vec{c}_B$, where $\vec{x}_B$ and $\vec{c}_B$ are associated cost vectors with basic feasible solution on $\vec{x}_B$ and $\vec{x}_B$.

Optimal Basic Feasible Solution

A basic feasible solution $\vec{x}_B$ is optimal basic feasible solution to max $z = \vec{c}\ \vec{x}$, subject to $\vec{A}\ \vec{x} = b, c = \vec{O}$ if $z^* = \vec{c}_B, \vec{x}_B$ for every basic feasible solution $\vec{x}_B$ and its associated cost vector $\vec{c}_B$.

Net Evaluation

If $\vec{c}_B$ is a basic feasible solution to max $z = \vec{c}\ \vec{x}$, subject to $\vec{A}\ \vec{x} = \vec{b}$, $\vec{x} \geq 0$ and $\vec{c}_B$ is the cost associated with $\vec{x}_B$. For each column vector $\vec{a_j}$ in $\vec{A}$, which is not column vector of $\vec{B}$, let $a_j = \sum_{i=1}^{m} y_{ij}\, b_i$.

Then $z_j = \sum_{i=1}^{m} \vec{c}_{bi}\, y_{ij}$ is called evaluation corresponding $\vec{a_j}$ and the number $z_j - c_j$ is called net evaluation corresponding to $\vec{a_j}$.

Some Important Theorems

(1) If a linear programming problem has a feasible solution then it also has a basic feasible solution.

(2) There exists finite number of basic feasible solutions to linear programming problem.

(3) If a linear programming problem has a basic feasible solution and we drop one of the basic vector and introduce a non-basic vector in the basis set, then the new solution obtained is also a basic feasible solution.

(4) If $\vec{x}_B$ is a basic feasible solution to, max $z = \vec{c}\ \vec{x}$, subject to $\vec{A}\ \vec{x} = \vec{b}$, $\vec{x} = 0$.

If $\vec{x}_B$ is another basic feasible solution, obtained by replacing one of the basic vector of B by non-basic vector a_j, for which net evaluation $z_j - c_j < 0$. Then $\vec{x}_B$ is an improve basic feasible solution *i.e.*

$$\vec{c}_B \vec{x}_B \geq \vec{c}_B \vec{x}_B$$

(5) If there exist a basic feasible solution to a given linear programming problem and if for at least one j, has which $z_j - c_j < 0$ and $y_{ij} \leq 0$ ($i = 1, \ldots\ldots m$). Then there does not exist any optimum solution to linear programming problem.

(6) A sufficient condition for a basic feasible solution to an linear programming problem to be an optimum (maximum) is that $z_j - c_j$ $\frac{\notin \vec{A}x = \vec{b}}{||\underline{n}|\underline{m}|\underline{n-m}}$ for all j for which the column vector $\vec{a_j} \geq \in \vec{A}$ is not in the basis B. *i.e.* $\vec{a}$ $\notin \vec{B}$

(7) A necessary and sufficient condition for a basic feasible solution to a linear programming problem to be an optimum (max) is that $z_j - c_j \geq$ for all j for which $\vec{a} \notin \vec{B}$.

(8) Any convex combination of k-different optimum solution to a linear programming problem is again on optimum solution to the problem.

Extreme Point and Basic Feasible Solution

(1) Set of feasible solutions to $\vec{A}x = \vec{b}$ is a convex set.

(2) Every basic feasible solution is an extreme point of the convex set of the feasible solution.

(3) Every extreme point is a basic feasible solution to the set of contraints.

(4) The convex set of feasible solutions has a finite number of the extreme point.

(5) The number of extreme points is equal to the number of different basic feasible solutions and the number of basic feasible solutions (extreme points) can not be greater, than $|\underline{n}|\ \underline{m}\,|\underline{n-m}$.

(6) In the absence of degeneracy, there is a one-to-one correspondence between the extreme points and the basic feasible solution.

Geometric Interpretation of L.P.P.

Whenever the feasible solution of linear programming problem exists, the region of feasible solution is a convex set and there also exist extreme proints. If the optimal solution exists one of the extreme point is optimal.

Whenever the optimal value of objective function z is finite, at least one extreme point of the region of feasible solution has an optimal solution.

If optimal solution is not unique, there are points other than extreme points that were optimal, but in any coase one extreme point is optimal.

Simplex Method and its Geometric Interpretation

Simplex method is an algebric iterative procedue for solving linear programming problem in finite number of steps (and states for unbounded solution, if exist).

If there is an optimal solution, one of the extreme point is optimal and there are finite number of extreme points. By simplex method we move step by step from a given extreme point to an optimal extreme point.

The simplex method moves along an 'edge' of the region of feasible solutions from one extreme point to adjacent one. Of all the adjacent extreme points, one chosen is that which gives greatest increase (or greatest decrease) in objective function z.

At which extreme point, by simplex method we know that, whether that extreme point is optimal if not, what the next extreme point will be.

Whenever by simplex method, on extreme point which, has on edge leading to infinite exist and the objective function can be increased (or decreased) by moving along that edge, the simplex method states for unbounded solution.

Imp—If origin is a feasible solution, it will be an extreme point because of all variables are non-negative.

Graphic Method of Solution

The graphic solution procedure is one method of solving two variables linear programming problems and involves the following steps :

(1) Formulate the problem in terms of a series of mathematical constraints and an objective function.

(2) Plot each of the constraints as follow. Each inequality in the constraints equation be written as equality. Give any arbitrary value to one variable and get the value of other variable by solving the equation. Similarly, give another arbitrary value to the variable and final the corresponding value of the other variable. Now plot these two sets of values. Connect these points by a straight line. This exercise is to be carried out for each of the constraints equations. Thus, there will be as many straight lines as there are equations, each straight lines representing one constraints.

(3) Identify the feasible region (or solution space), *i.e.* the area which satisfy all the contraints simultaneously. For 'greater than' constraints, the feasible region will be the area which lies above the constraints, lines. For 'greater than or equal to' or 'less than or equal to' constraints, the feasible region includes the points on the constraints lines also.

(4) **Corner point method**

   (a) Identify the each of the corner (or extreme points) of the feasible region either by visual inspection or the method of simultaneous equations.

   (b) Compute the profit/cost at each corner point by substituting the co-ordinates of that point into the objective function.

   (c) Identify the optimal solution at that corner point which shows highest profit (In a maximization problem) or lowest cost (In a minimization problem).

Linear Programming (Advanced)

The linear programming model includes

(a) Decision variables (continuous), we seek to determine.

(b) Objective (goal), we aim to optimize (linear function).

(c) Constraints, that need to be satisfied (linear function).

Non-negativity Restriction—$x_i \geq 0$.

Feasible Solution—Any solution that satisfies all the constraints.

Optimum Feasible Solution—Any feasible solution, that optimize the objective.

Graphical L.P. Solutions

Optimum solution is associated with a corner points of the solution space, where two lines intersects.

The Simplex Method

In graphical method, the optimum LP solution is always associated with corner points (extreme points) of the solution space.

The simplex method is a mathematical treatment to obtain and identify these extreme points algebraically.

To obtain optimal solution, we first convert the LP model to standard LP model by using slack and surplus variable, to convert inequality constraints into equality. After it we try to find basic solutions of the simultaneous linear equations. These (algebraic) basic solutions completly define all the (geometric) extreme points of the solution space. The simplex algorithm located the optimum among these basic solutions.

Standard Form and its Basic Solutions

Standard Form

(a) All the constraints with non-negativity restriction on variables are equations with non-negative right hand side.

(b) All variables are non-negative.

(c) Objective function is maximization, type.

Conversion of Inequality into Equality

(a) ($\leq$ type) – by slack variable.

(b) ($\geq$ type) – by surplus variable.

(c) The negative right hand side is converted to non-negative right hand side by multiplying – 1 to both sides.

(d) ($\leq$ type) inequality converts to ($\geq$ – type) inequality by multiplying – 1.

Conversion of Unrestricted Variables

Replace the unrestricted variable x_j by $x^*_j - x^-_j$ wher $x^+_j, x^-_j \geq 0$.

i.e. $$x_j = x^+_j - x^-_j$$

Convesion of Maximization to Minimization

$$\text{Max } f(x_1, \ldots x_n) = \min(-f(x_1, \ldots x_n)$$
$$\text{Min } f(x_1, \ldots x_n) = \max(-f(x_1, \ldots x_n)$$

Determination of Basic Solutions

The standard LP form includes m-simultaneous linear equations and n-unknowns (variables), $(m < n)$.

(a) Assign n–m variables. Zero value.

(b) Remaining m-variables are determined by solving resulting m-simultaneous equations. If m-equations yield a unique solution, then the associated m-variables are called basic variables and remaining n–m variables are referred to non-basic variables.

The unique solution (basic variables) and non-basic variables comprises a basic solution. If all variables are non-negative then basic solution is feasible solution otherwise it is infeasible.

The maximum number of possible basic solution for m equations and n unknown is $\binom{n}{m}$.

Unrestricted Variables and Basic Solutions

For unrestricted variables

$$x_j = x^+_j - x^-_j;\ x^+_j, x^-_j \geq 0.$$

Since x^+_j and x^-_j are dependent variables, only one can be the basic variable.

i.e., if x^+_j is basic variable, then x^-_j is non-basic variable.

If x^-_j is basic variable, then x^+_j is non-basic variable.

The Simplex Algorithm

The simplex algorithm is designed to locate the optimum by conectrating on a selected number of the basic feasible solutions of the problem.

The simplex method always starts with a basic feasible solution and then attempts to find other basic feasible solution that will impreve the objective value. This is possible only if an increase in a current zero (non-basic) feasibles can lead to an improvement in the objective value. However for a current zero variable to become positive, one of the current basic variables must be removed to guarantee that the new solution will include exactly m basic varaibles. The selected zero variable is the entering variable and the removing basic variable is the leaving variable.

Basic	z	$x_1 x_2 \ldots x_n$	Solution
z			z-row
x_1 x_2 . . . x_n			

Optimality Conditions

The entering variable in a maximization (minimization) problem is the non-basic variable having the most negative (positive) coefficient in the row.

Maximization—z-row coefficient-most negative.

Minimization—z-row coefficient-most positive.

The optimality reaches, at the iteration where all the z-row cofficients of non-basic variables are non-negative for maximization and non-positive for minimization.

Feasibility Condition

For both the maximization and minimization problem, the leaving variable is the basic variable associated with the smallest non-negative ratio.

Steps for Simplex Method

Step 1. Determine the starting basic variables.

Step 2. Select the entering variable using the optimality condition, stop if no entering variables.

Step 3. Select the leaving variable using feasibility conditions.

Step 4. Determine the new basic solution.

The simplex method computations are iterative, because next tableau is produced from the current tableau.

Artificial Starting Solution

For the LP's in which all the constraints are of (≤) type, with non-negative right hand side, the slack offers a convenient starting basic feasible solution. For (≥) type and (=) type constraints, new variables are introduced, known as artificial variable to produce initially basic feasible solution for simplex tableau.

Basic Solution

Given m-simultaneous linear equations in n-variable ($m < n$).

$$\vec{O}^{-1}\vec{x}_B = \vec{c}\ d_x^{\Delta}\ x$$

where $\vec{A} = [a_{ij}]_{m \times n}$

and rank A $= m$

If $\vec{B}$ is a sub matrix of $\vec{A}$, formed by m-linear independent columns of $\vec{A}$. Then solution obtained by setting n–m variables not associated with the columns of $\vec{B}$ equal to zero and solving resultant equation is called basic solution.

The m-variables associated of $\vec{B}$ are called basic variables and (n–m) remaining variables that are equal to zero are called non-basic variables.

The solution $\vec{x}_B = \vec{B}^{-1}\ \vec{b}$ is called basic solution.

Basic Feasible Solution—$\vec{x}_B \geq \vec{O}$

Associated Cost Vectors

If $\vec{x}_B$ is a basic feasible solution to standard L.P.P., then

$$\vec{c}_B = [c_{B_1}, \ldots\ldots c_{B_m}]$$

where c_{Bi} are components of $\vec{c}$ associated with the basic variables called cost vector associated with $\vec{x}_B$.

Improved Basic Feasible Solution

If $\vec{x}_B$ and $\overset{\Delta}{x}_B$ are two basic feasible solutions to standard L.P.P. $\overset{\Delta}{x}_B$ is improved basic feasible solution, compared to

$\vec{a}_j \in \overset{\Delta}{\vec{c}}_B\ \overset{\Delta}{\vec{x}}_B \geq \vec{B} \notin \to B$

if $\overset{\Delta}{\vec{c}}_B\ \overset{\Delta}{\vec{x}}_B > \vec{c}_B\ \vec{x}_B$ (maximization)

$\overset{\Delta}{\vec{c}}_B\ \overset{\Delta}{\vec{x}}_B < \vec{c}_B\ \vec{x}_B$ (minimization)

Fundamental Properties of Solutions

(1) If linear programming problem has a feasible solution, then it also has a basic feasible solution.

(2) There exist only finite number of basic feasible solutions to a linear programming problem.

(3) If linear programming problem have a basic feasible solution. Then if we drop one of the basic variable and introduce a non-basic variable to basic solution, then new solution is also a basic feasible solution.

(4) If $\vec{x}_B$ be a basic feasible solution, $\overset{\Delta}{\vec{x}}_B$ is a basic feasible solution obtained by admitting a non-basic column vector $\vec{a}_j$ in the basis for which net evaluation.
$z_j - c_j < 0$ (maximization),
$z_j - c_j > 0$ (minimization).

(5) If $z_j - c_j = 0$ for at least one j for which $y_{ij} > 0$, $i = 1, 2, \ldots\ldots, m$; then the another basic feasible solution is obtained which gives an unchanged value of the objective function.

(6) A sufficient condition for a basic feasible solution to be optimum is that $z_j - c_j \geq 0$ for all j for which the column vector $\vec{a}_j \in \vec{A}$ is not in the basis $\vec{B}$.

(7) A basis feasible solution is an optimum (maximum) is that $z_j - c_j \geq 0$ for all j for which $\overrightarrow{a_j} \notin \overrightarrow{B}$.

(8) Any convex combination of k different optimum solutions, is again an optimum solution.

The Simplex Tableau and Simplex Method

Given Max $Z = 6x_1 + 4x_2$

Subject to,

$$x_1 + 2x_2 \leq 720$$
$$2x_1 + x_2 \leq 780$$
$$x_1 \leq 320$$
$$x_1, x_2 \geq 0$$

Standard L.P. Problem :

$$\text{Max } z = 6x_1 + 4x_2 + Os_1 + Os_2 + Os_3$$

Subject to,

$$x_1 + 2x_2 + s_1 = 720$$
$$2x_1 + x_2 + s_2 = 780$$
$$x_1 + + s_3 = 320$$

Non-negativity restriction,

$x_1, x_2, s_1, s_2, s_3 \geq 0$

Here z-row is—$z - 6x_1 - 4x_2$ and slack variables s_1, s_2, s_3 have provided initial basic feasible solution.

Basic	x_1	x_2	s_1	s_2	s_3	Solution	Ratio
s_1	1	2	1	0	0	720	720
s_2	2	1	0	1	0	780	340
s_3	1	0	0	0	1	320	320
z	−6	−4	0	0	0		

↑ Entering variable (pivot column)

Here the entering variable is spotted as the non-basic variable having most negative cofficient in z-row.

The leaving variables is spotted as the minimum non-negative ratio.

Gauss-Jordan Computation (for new basic solution)

Pivot column—A column associated with entering variable.

Pivot row—A row associated with leaving variable.

Pivot element—An intersection of pivot row and pivot column provides pivot element.

Also, New pivot row = curmet pivot row + pivot element.

New row = current row – (its pivotcoefficient)

New Pivot Row

Basic	x_1	x_2	s_1	s_2	s_3	Solution	Ratio
s_1	0	2	1	0	–1	400	200
s_2	0	1	0	1	–2	140	140 →
s_3	1	0	0	0	1	320	:
z	0	– 4	0	0	6		

↑ Entering variable

Here s_3 is pivot row (current) and x_1 – row is the new pivot row (1 0 0 0 1/320) + 1 = (1 0 0 0 1/320)

s_2– row (new)

= (2 1 0 1 0/780) – 2 (1 0 0 0 1/320)

= (0 1 0 1 – 2/140)

s_1–row (new)

= (1 2 1 0 0/720) – 1 (1 0 0 0 1/320)

= (0 2 1 0 – 1/400)

z-row (new)

= (– 6 – 4 0 0 0/0) + 6(1 0 0 0 1/320)

= (0 – 4 0 0 6/320)

Entering variable (most negative coefficient in z-row)

= x_2

Leaving variable (minimum non-negative ratio) = s_2

Basic	x_1	x_2	s_1	s_2	s_3	Solution	Ratio
s_1	0	0	1	2	3^*	120	40 →
s_2	0	1	0	1	–2	140	:
s_3	1	0	0	0	1	320	320
z	0	0	0	4	–2	2480	

↑ Entering variable

s_2-row (pivot row) current and x_2-row (pivot row) new.

(0 1 0 1 – 2/140) + 1 = (0 1 0 1 – 2/140)

s_1-row (new)

= (0 2 1 0 – 1/400) – 2(0 1 0 1 – 2/140)

= (0 0 1 – 2 3/120)

x_1-row (new)

= (1 0 0 0 1/320) – 0(0 1 0 1 – 2/140)

= (1 0 0 0 1/320)

z-row (new)

$= (0 - 4\ 0\ 0\ 6/320) + 4(0\ 1\ 0\ 1\ -2/140)$

$= (0\ 0\ 0\ 4 - 2/2480)$

Basic	x_1	x_2	s_1	s_2	s_3	Solution	Ratio
s_1	0	0	$\frac{1}{3}$	$\frac{2}{3}$	1	40	
s_2	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	220	
s_3	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	280	
z	0	0	$\frac{2}{3}$	$\frac{8}{3}$	0	2560	

x_1-row pivot row (current) and x_2-row pivot row (new)

$$(0\ 0\ 1\ 2\ 3/120) + 3 = \left(0\ 0 - \frac{1}{3}\ \frac{2}{3}\ 1/140\right)$$

x_2-row (new)

$$= (0\ 1\ 0\ 1 - 2/140) + 2\left(0\ 0 - \frac{1}{3}\ \frac{2}{3}\ 1/140\right)$$

$$= \left(0\ 1\frac{2}{3}\ -\frac{1}{3}\ 0/220\right)$$

x_1-row (new)

$$= (1\ 0\ 0\ 0\ 1/320) - \left(0\ 0\ -\frac{1}{3}\ \frac{2}{3}\ 1/140\right)$$

$$= \left(1\ 0\ -\frac{1}{3}\ \frac{2}{3}\ 0/280\right)$$

z-row (new)

$$= (0\ 0\ 0\ 4 - 2/2480)$$

$$+ 2\left(0\ 0 - \frac{1}{3}\ \frac{2}{3}\ 1/140\right)$$

$$= \left(0\ 0\ \frac{2}{3}\ \frac{8}{3}\ 0/2560\right)$$

$$= (\geq)$$

$$\sum_{i=1}^{m} a_{ij}\, y_i$$

∴ **Decision variables**	**Optimum values**
x_1	220
x_2	280
z	2560

Some Important Consideration

Optimality Conditions (entering variable)

The entering variable in a maximization (minimization) problem is the non-basic variable having the most negative (positive) coefficient in z-row. The optimum is reached at the iteration where all the z-row coefficients of the non-basic variables are non-negative (non-positive).

Feasibility Condition (leaving variable)

For both the maximization and minimization problem, leaving variable is the basic variable associated with the smallest non-negative ratio.

For simplex method—Determine the starting basic feasible solution select entering variable using optimality condition select, leaving variable using feasibility condition determine new basic solution by Gauss-Jordan computation.

Artificial Starting Solution

For the linear programming problems in which all the constraints one of ($\leq$) type (with non-negative right hand side), the slack gives a starting basic feasible solution. When the model involves ($\geq$) and ($=$) constraints, then we introduce variable that are not the part of solution, known as artificial variables to get initial basic feasible solutions. The two known methods are

The Big-M method and Two phase method.

Big-M Method

It starts with linear programming problem in standard form. For any equation i that does not have a slack variable, we argument an artificial variable Ri, which become a part of initial basic solution and assign penalty in the objective function to force then to zero level at further iteration of the simplex algorithm. In objective function we use MRi in case of maximization an –MRi in case of minimization.

Here new z-row = old z-row M Σ(Ri – row) (maximization)

Now z-row = old z-row – M Σ(Ri – row) (minimization)

Standard L.P. Problem

(1) Minimize $z = 4x_1 + x_2$

(2) Subject to
$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 - x_3 = 6$$
$$x_1 + 2x_2 + x_4 = 4$$
$$x_1, x_2, x_3, x_4 \geq 0$$

(3) Using the stepping-stone path, determine the maximum number of items that can be allocated to the route selected in setp 2 and adjust the allocation appropriately.

(4) Return to Setp 1.

Modi Method

Use the current solution and operations (a) and (b) below to the marginal cost of sending material over each of the unoccupied routes.

(a) Set $u_1 = 0$. Find row indices $u_2,\ldots, u_m$ and column indices $v_1,\ldots,v_n$ such that $c_{ij} = u_i + v_i$ for every occupied cell.

(b) Let $ij = c_{ij} - (u_i + v_j)$ for every unused cell : ij is the mariginal cost of introducing cell i, j into the solution.

Setp 2 through 4 are the same and in the stepping-stone method.

Assignment Problems

It is a particular case of transportation problem in which a number of operations are to be assigned to an equal number of operators, where each operator performs only one operation. Where objective is to maximize overall profit or minimize overall cost for a given assignment schedule.

Let n-jobs be assigned to n-operators. Let c_{ij} be the cost in currend in assigning i-th job to j-th operator and let

$$x_{ij} \begin{cases} 1, \text{ if } i\text{-th job is assigned to } j\text{-th operator} \\ 0, \text{ otherwise} \end{cases}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}$$

W

$$\begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then the assignment problem is the linear programming problem,

$$\text{Min. } z = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij}\, c_{ij}$$

Subject to the Constraints

$$\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, \ldots\ldots, n$$

$$\sum_{j=1}^{m} x_{ij} = 1, i = 1, 2, \ldots\ldots, n$$

with $x_{ij} = 0$ or 1

(1) The optimum assignment schedule remains unaltered if a constant is add or substract to/from all the elements of the row or column of the assignment cost matrix.

(2) If for an assignment problem and $c_{ij} \geq 0$, then the assignment schedule (x_{ij}) which satisfies. $x_{ij}\, c_{ij} = 0$ is optimal.

(3) **Konig-Egervary theorem**—If P is a non-empty subset of the points of a matrix A = $[a_{ij}]$, then the maximum number of independent points that can be selected in P is equal the minimum number of lies covering all the elements of P.

Assignment Alogrithm (Hungarian Assignment Method)

(1) The cost matrix is square, if not make it square by adding suitable number of dummy rows (or columns) with zero cost element.

(2) Locate the smallest cost elements in each row of the cost matrix subtract this smallest element from each element of that row the resultant is that there is at lest one zero in each row of the reduced cost matrix.

(3) In the reduced cost matrix obtained, locate the smallest element in each column in it. Subtract the smallest value from every other entry in the column, the resulttant is that there is at least one zero in each of the rows and columns of the second reduced cost matrix.

(4) In second reduced cost matrix, search an optimum assignment :

(a) Examine the rows successively untiall a row with exactly one zero is found. Enterchanging this zero as an cross cut all O's in its column. Proceed in similar manner untill all the rows have been examined. If there are more than one zero in any row than do not touch that ow and pass on to the next row.

(b) Repeat the procedure for the columns of the reduced cost matrix. If there is no single zero in any row or column of the reduced matrix, then arbitrarily choose a row or column having the minimum number of O's. Arbitrarily select and enterchangel any one O in the row or column thus chosen and cross all other O's in its row and column. Repeat steps (a) and (b) until all the zero's have been either assigned or crossed.

(c) If each row and each column of the reduced matrix has one and only one assigned O, the optimum assignment is made in the cells of an-rectangled O's. Otherwise go to the next step.

(5) Draw the minimum number of horizontal and/or vertical lines through all the O's as follows:

(a) Mark (.) the rows in which assignment has not been made.

(b) Mark (.) columns which have zeros in the marked rows.

(c) Mark (.) rows (not already marked) which have assignments in marked columns.

(d) Repeat (b) and (c) untill the chain of marking is completed.

(e) Draw straight lines through all unmarked rows and marked columns.

(6) If the minimum number of lines passing through all the zero is equal to the number of rows or columns, the optimum solution is attained by an arbitrary allocation in the positions of the zeros not crosed in (3). Otherwise go to next.

(7) Revise the costs matrix os follows—

(a) Find the smallest element not covered by any of the lines of (4).

(b) Subtract this from all the uncrosed elements and add the same at the point of inter-section of the two lines.

(c) Other elements crossed by the lines remain unchanged.

(d) Go to 4 and repeat the procedure till an optimum solution is attained.

(8) **A basic Solution to the system is degenerated if :**

(A) Some basic variables are non-zero.

(B) Some basic variables are equal to zero.

(C) Some basic variable are positive.

(D) Some basic variable are negative.

OBJECTIVE TYPE QUESTIONS

1. If A = $[a_{ij}]$ is the pay off matrix, then saddle point exist when—

(A) $\min_j \max_i a_{ij} \le \max \min_j a_{ij}$

(B) $\min_j \max_i a_{ij} \ge \max_i \min_j a_{ij}$

(C) $\min_j \max_i a_{ij} = \max_i \min_j a_{ij}$

(D) None of these

2. In two person zero sum game—

(A) Gain by one player is equal to loss by the other player

(B) Two players gains

(C) Gain by one player is more than the loss by other player

(D) None of these

3. Game is a situation where—

(A) Players have same objectives

(B) Players have conflicting objectives

(C) Players have no objectives

(D) None of these

4. If a variable x_j is unrestricted in sign in the primal linear programming problem, then the corresponding (*i.e.* the jth) dual constraint is—

(A) With ≤ sign if the primal is a minimization problem.

(B) With ≥ sign if the primal in a maximization problem

(C) With equality sign

(D) None of these

5. If the dual of the problem has infeasible solution, then the value of objective function is—

(A) Unbounded (B) Bounded

(C) No solution (D) None of these

6. An assignment problem is—

(A) A special case of transportation problem

(B) Dual problem

(C) Non-linear programming problem

(D) None of these

7. A basic solution to the system is non-degenerated, if—

(A) One or more basic variables vanishes

(B) No basic variable vanishes

(C) Basic variables exist

(D) None of these

8. An hyper-plane is—

(A) Convex set

(B) Not a convex set

(C) Disk with hole

(D) None of these

9. Given a system of linear equations, if B is one of the basic solution for given basic B, then—
   (A) $\vec{x}_B = \neq$
   (B) $\vec{x}_B \, \vec{b} = \vec{B}^{-1}$
   (C) $\vec{x}_B = \vec{B}^{-1} b$
   (D) None of these

10. Set $A \subset E^n$ is bounded below if $J\vec{r}$ for every $\vec{a} \notin A$, is—
    (A) $\vec{r} \leq \vec{a}$ (B) $\vec{r} \geq \vec{a}$
    (C) $\vec{r} = \vec{a}$ (D) $\vec{r} > \vec{a}$

11. Set $A \subset E^n$ is bounded above if $J\,\vec{r}$ for every $\vec{a} \notin A$, is—
    (A) $\vec{r} \leq \vec{a}$ (B) $\vec{r} \geq \vec{a}$
    (C) $\vec{r} = \vec{a}$ (D) None of these

12. If either of the dual has a finite solution then, objective functions of two problems are—
    (A) Equal
    (B) Unequal
    (C) Equal or unequal
    (D) None of these

13. The neighbourhood about a point is the set of points—
    (A) $\left\{\dfrac{\vec{x}}{|\vec{x} - \vec{a}|} > \in\right\}$ (B) $\left\{\dfrac{\vec{x}}{|\vec{x} - \vec{a}|} < \in\right\}$
    (C) $\left\{\dfrac{\vec{x}}{|\vec{x} - \vec{a}|} = \in\right\}$ (D) None of these

14. The inside of a hypersphere with centre at and radius > 0, is the set of points—
    (A) $\left\{\dfrac{\vec{x}}{|\vec{x} - \vec{a}|} < \in\right\}$ (B) $\left\{\dfrac{\vec{x}}{|\vec{x} - \vec{a}|} > \in\right\}$
    (C) $\left\{\dfrac{\vec{x}}{|\vec{x} - \vec{a}|} = \in\right\}$ (D) None of these

15. Let E^n be an Euclidean space, a hypersphere in E^n with centre at and radius > 0 is a set of points—
    (A) $\left\{\dfrac{\vec{x}}{|\vec{x} - \vec{a}|} > \in\right\}$ (B) $\left\{\dfrac{\vec{x}}{|\vec{x} - \vec{a}|} < \in\right\}$
    (C) $\left\{\dfrac{\vec{x}}{|\vec{x} - \vec{a}|} = \in\right\}$ (D) None of these

16. The equation of line passing through points $\vec{x_1}$ and $(\vec{x_1} \neq \vec{x_2})$ in E^n is—
    (A) $\lambda\vec{x_2} + (1 - \lambda)\vec{x_1} = \vec{x} \in E^n.\lambda$ any real number
    (B) $\lambda\vec{x_2} + (1-\lambda)\vec{x_1} \in E^n, \lambda < 0$
    (C) $\lambda\vec{x_2} + (1-\lambda)\vec{x_1} = \vec{x_1}\ E^n, 1 < x < 0$
    (D) None of these

17. Set $\left\{\dfrac{(x_1, x_2)}{(x_2 - 1)^2 + (x_2 - 3)} \geq 2\right\}$ is—
    (A) Strictly bounded
    (B) Bounded above
    (C) Bounded below
    (D) None of these

18. The set $A \subset E^n$ is bounded below, if there exist—
    (A) Upper limit
    (B) Lower limit
    (C) Upper and lower both limits
    (D) None of these

19. Given $\begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ the one of the base is—
    (A) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix}$
    (C) $\begin{bmatrix} 1 & 6 \\ 3 & 4 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 6 \\ 3 & 4 \end{bmatrix}$

20. The basic to $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ is—
    (A) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$
    (C) $\begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 5 \end{bmatrix}$

21. The assignment problem—
    (A) Is a special case of transportation problem
    (B) Can be solved by simplex alogrithm
    (C) Both A and B are true
    (D) Both A and B are false

22. In cost matrix of assignment problem, from each row we select the … element, which can give atleast one zero elements in each row—
    (A) Smallest (B) Biggest
    (C) Any (D) None of these

23. The cost matrix in assignment problem is a—
    (A) Square matrix (B) Rectangle matrix
    (C) Diagonal matrix (D) None of these

24. Every basic feasible solution in convex set of solution is—
    (A) Extreme point
    (B) Boundary point
    (C) Non-extreme point
    (D) Non-boundary point

25. Set feasible solutions to $\vec{A}\ \vec{x} = \vec{b}$ is—
    (A) Convex sets
    (B) Non-convex set
    (C) Disconnected set
    (D) None of these

26. Two hyper-planes are parallel if they have—
    (A) Same unit normal
    (B) Distinct unit normal
    (C) Same normal
    (D) None of these

27. Simplex method is the—
    (A) Algebraic procedure for solving L.P.P.
    (B) Geometric procedure for solving L.P.P.
    (C) Both (A) and (B)
    (D) None of these

28. If a feasible solution of linear programming problem exist, the region of feasible solution is—
    (A) Closed convex set
    (B) Connected set
    (C) None convex set
    (D) None of these

29. If the dual of linear programming problem have a finite optimal solution, then primal process—
    (A) Finite optimal solution
    (B) Unbounded solution
    (C) No solution
    (D) None of these

30. The dual of a dual linear programming problem is—
    (A) Primal (B) Dual
    (C) Dual of a dual (D) None of these

31. The Euclidean space E^n is—
    (A) Open set
    (B) Closed set
    (C) Both open and closed set
    (D) Neither open nor closed

32. Let E^2 is Euclidean plane, a equation of circle with centre (0, 0) and radius $\in$ is—
    (A) $|\vec{x}| < \in$ (B) $|\vec{x}| > \in$
    (C) $|\vec{x}| = \in$ (D) None of these

33. The intersection of finite number of convex sets is—
    (A) Convex set (B) Open set
    (C) Closed set (D) None of these

34. The complement of a closed set is—
    (A) Open set
    (B) Closed set
    (C) Both open and closed
    (D) None of these

35. The complement of an open set is—
    (A) Open set
    (B) Closed set
    (C) Neither open nor closed
    (D) None of these

36. In a balanced transportation problem with m origins and n destinations the number of linearly independent constraints is—
    (A) $m + n$ (B) $m + n + 1$
    (C) $m + n - 1$ (D) None of these

37. A set A E^n is closed set, then—
    (A) It contains all its boundary points
    (B) It contains all its interior points
    (C) Both (A) and (B) are true
    (D) Both (A) and (B) are false

38. A set A E^n is open set, then—
   (A) It contains all its boundary points
   (B) It contains all its interior points
   (C) Both (A) and (B) are true
   (D) Both (A) and (B) are false

39. The optimal solution of the region are—
   (A) (0, 0) (0, 200) (0, 300) and (400, 0)
   (B) (0, 0) (0, 200) (300, 0) and (200, 100)
   (C) (0, 0) (0, 200) (0, 300) and (200, 100)
   (D) (0, 0) (0, 300) (400, 0) and (200, 100)

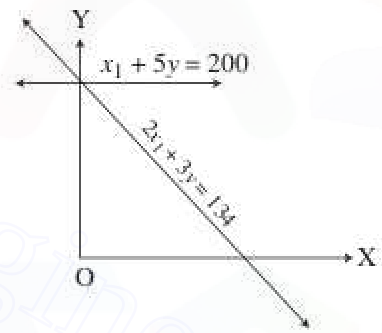
40. A company manufactures two types of telephone sets A and B. The A type telephone set requires 2 hrs and B type requires 4 hours to make. The company has 800 work hours per day 300 telephone can pack in a day. The selling prices of A and B types telephones are Rs. 300 and Rs. 400 respectively. For maximum profits company produces x telephones of A type and y telephones of B-types. Then except $x \geq 0$, and $y \geq 0$, linear constraints are—
   (A) $x + 2y \leq 400; x + y \geq 300$
   Max $z = 300x + 400y$
   (B) $2x + y \leq 400; x + y \geq 300$
   Max $z = 400x + 300y$
   (C) $2x + y \geq 400; x + y \geq 300$
   Max $z = 300x + 400y$
   (D) $x + 2y \leq 400; x + y \geq 300$
   Max $z = 300x + 400y$

41. The probable region of the linear programming problem in the above question is of the type—
   (A) Bounded (B) Unbounded
   (C) Parallelogram (D) Square

42. A whole sale merchant wants to start the business of cereal with Rs. 24,000. Wheat is Rs. 400 per quintal and rice is Rs. 600 per quintal. He has capacity for store 200 quintal cereal. He earns the profit Rs. 25 per quintal on wheat and Rs. 40 per quintal on rice. If he stores x quintal rice and y quintal wheat, then for max profit the objective function is—
   (A) $25x + 40y$ (B) $40x + 25y$
   (C) $400x + 600y$ (D) $\frac{400}{40}x + \frac{600}{25}y$

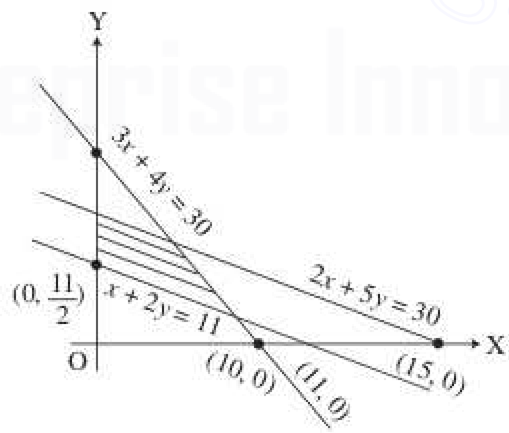
43. A firm makes pent and shirts. A shirt take 2 hours on machine and 3 hours of man labour while a pent takes 3 hours on machine and 2 hour of man labour. In a week there are 70 hours machine and 75 hours of man labour available. If the firm determine to make x shirts and y pents a week then for this linear constraints are—
   (A) $x \geq 0, y \geq 0,$ $2x + 3y \geq 70,$ $3x + 2y \geq 75$
   (B) $x \geq 0, y \geq 0,$ $2x + 3y \leq 70,$ $3x + 2y \geq 75$
   (C) $x \geq 0, y \geq 0,$ $2x + 3y \geq 70.$ $3x + 2y \leq 75$
   (D) $x \geq 0, y \geq 0,$ $2x + 3y \leq 70,$ $3x + 2y \leq 75$

44. The minimum value of objective function $c = 2x + 2y$ in the given feasible region, is—



   (A) 134 (B) 40
   (C) 38 (D) 80

45. The solution of set of constraints $x + 2y \geq 11$, $3x + 4y \leq 30$, $2x + 5y \leq 30$, $x \geq 0$, $y \geq 0$ includes the point



   (A) (2, 3) (B) (3, 2)
   (C) (3, 4) (D) (4, 3)

46. For the L.P. problem Min $z = 2x_1 + 3x_2$ such that $-x_1 + 2x_2 \le 4$, $x_1 + x_2 \le 6$, $x_1 + 3x_2 \ge 9$, and $x_1, x_2 \ge 0$ —
    (A) $x_1 = 1{\cdot}2, x_2 = 2{\cdot}4, z = 10{\cdot}4$
    (B) $x_1 = 2{\cdot}4, x_2 = 1{\cdot}2, z = 10{\cdot}2$
    (C) $x_1 = 1{\cdot}3, x_2 = 2{\cdot}5, z = 10{\cdot}6$
    (D) $x_1 = 1{\cdot}2, x_2 = 2{\cdot}6, z = 10{\cdot}2$

47. For the L.P. problem Min $z = x_1 + x_2$ such that $5x_1 + 10x_2 \le 0$, $x_1 + x_2 \ge 1$, $x_2 \le 4$ and $x_1, x_2 \ge 0$—
    (A) There is a bounded solution
    (B) There is a no solution
    (C) There is a infinite solution
    (D) None of these

48. For the L.P. problem Max $z = 3x_1 + 2x_2$ such that $2x_1 - x_2 \ge 2$, $x_1 + 2x_2 \le 8$ and $x_1, x_2 \ge 0$, $z =$ —
    (A) 12 (B) 24
    (C) 36 (D) 40

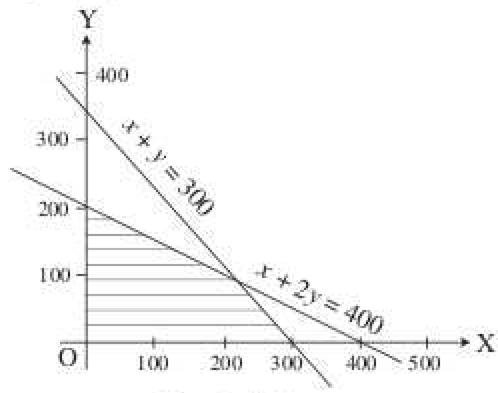
49. In Hungarian alogrithm the selected smallest elements of each row of cost matrix in assignment problem given atleast …… in each row of the reduced cost matrix—
    (A) One zero element
    (B) One positive element
    (C) One negative element
    (D) None of these

50. An assignment problem consists of—
    (A) A set of n-jobs
    (B) A set of n-operators
    (C) Cost associated with each jobs
    (D) All of these

Answer with Explanation

1. (C)	2. (A)	3. (B)	4. (C)	5. (A)
6. (A)	7. (A)	8. (A)	9. (C)	10. (A)
11. (B)	12. (A)	13. (B)	14. (A)	15. (C)
16. (A)	17. (C)	18. (B)	19. (A)	20. (A)
21. (A)	22. (A)	23. (A)	24. (A)	25. (A)
26. (A)	27. (A)	28. (A)	29. (A)	30. (A)
31. (C)	32. (C)	33. (A)	34. (A)	35. (B)
36. (C)	37. (C)	38. (B)	39. (D)	40. (B)
41. (A)	42. (B)	43. (D)	44. (D)	45. (C)
46. (D)	47. (C)	48. (B)	49. (A)	50. (D)

(Q. 39) to (41) The linear constraints are



$$x + 2y \le 400$$
$$x + y \le 300$$
and $$x, y \ge 0$$

Hence vertices of feasible region are (300, 0), (0, 200), (200, 100).

The optical solution in most at (200, 100)

42. Maximum profit
$$z = 40x + 25y$$

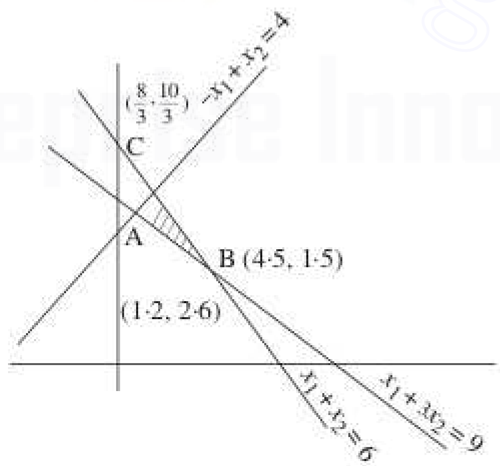
43.

	Machine	Man
Shirts : x	2 hrs	3 hrs
Pents : y	3 hrs	2 hrs

Linear constraints are
$$2x + 3y \le 70,$$
$$3x + 2y \le 75,$$
and $$x, y \ge 0$$

44. $$\text{Min } z = 2(0) + 2(40) = 80.$$

46. The graph of linear programming problem is as given below.



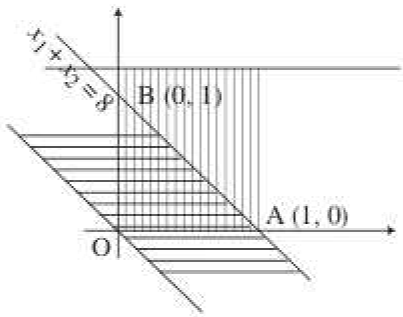
Hence the required feasible region given by the graph whose vertices are A(1·2, 2·6), B(4·5, 1·5) and C$\left(\frac{8}{3}, \frac{10}{3}\right)$.

Thus objective function is minimum at A(1·2, 2·6).

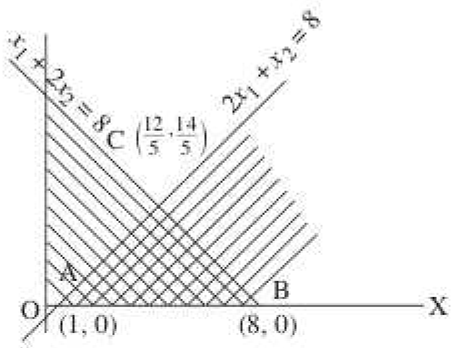
So $\quad x_1 = 1{\cdot}2, x_2 = 2{\cdot}6$

and $\quad z = 2 \times 1{\cdot}2 + 3 \times 2{\cdot}6 = 10{\cdot}2.$

47. As there may be infinite x_1 and x_2's on line $x_1 + x_2 = 1$.



48. Change the inequalities into equations and draw the graph of lines, thus we get the required feasible retion.



Hence it is bounded by the vertices A(1, 0), B(8, 0) and C$\left(\frac{12}{5}, \frac{14}{5}\right)$Now by evaluation of the objective function for the vertices of feasible region it is found to be maximum at (8, 0).

Hence solution is,

$$z = 3 \times 8 + 0 \times 2 = 24.$$

Common Data Questions

Directions (Q. 1 to 4)—A departmental head has four subordinates and four tasks to be performed. The subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. His estimate, of the time each man would take the perform each task, is given in the matrix below:

Tasks	**Men**			
	I	II	III	IV
A	18	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

1. The following is the—
   (A) Balanced problem
   (B) Unbalanced problem
   (C) Both (A) and (B) are true
   (D) None of these

2. The optimal man-hours is—
   (A) 59 (B) 49
   (C) 39 (D) 29

3. The optimal assignment is—
   (A) A — III, B — IV
   (B) A — III, C — II
   (C) B — I, C — II
   (D) A — IV, D — I

4. The following is a—
   (A) Transportation problem
   (B) Assignment problem
   (C) Game problem
   (D) None of these

Solution :

Step 1. Subtracting the smallest element of each row from every element of the corresponding row, we get the reduced matrix :

$$\begin{bmatrix} 7 & 15 & 6 & 0 \\ 0 & 15 & 1 & 13 \\ 23 & 4 & 3 & 0 \\ 9 & 16 & 14 & 0 \end{bmatrix}$$

Step 2. Subtracting the smallest element of each column of the reduced matrix from every element of the corresponding column, we get the following reduced matrix :

$$\begin{bmatrix} 7 & 11 & 5 & 0 \\ 0 & 11 & 0 & 13 \\ 23 & 0 & 2 & 0 \\ 9 & 12 & 13 & 0 \end{bmatrix}$$

Step 3. Starting with row 1, we entectangle — (*i.e.*, make assignment) a single zero, if any, and cross (×) all other zeros in the column so marked. Thus we get

$$\begin{bmatrix} 7 & 11 & 5 & \boxed{0} \\ \boxed{0} & 11 & 0 & 13 \\ 23 & \boxed{0} & 2 & 0 \\ 9 & 12 & 13 & 0 \end{bmatrix}$$

In the above matrix, we arbitrarily enrectangled a zero in column 1, because row 2 had two zeros.

Step 4.

$$\begin{bmatrix} 2 & 6 & 0 & 0 \\ 0 & 11 & 0 & 18 \\ 23 & 0 & 2 & 5 \\ 4 & 7 & 8 & 0 \end{bmatrix}$$

Step 5.

$$\begin{bmatrix} 2 & 6 & 0 & 0 \\ 0 & 11 & 0 & 18 \\ 23 & 0 & 2 & 5 \\ 4 & 7 & 8 & 0 \end{bmatrix}$$

Directions (Q. 5 to 9)—

Given Min $10x_1 + 5x_2 + 5x_3$,

Subject to

$5x_1 - 5x_2 - 3x_3 \longleftarrow 1$

$-x_1 + x_2 \longleftarrow -3$

$x_1 - x_3 \longleftarrow -7$

$-4x_1 + 4x_2 + x_3 \longleftarrow 5$

x_1 ? 0

5. In the dual of this problem we have—
   (A) Max $y_1 - 3y_2 - 7y_3 - 5y_4$
   (B) Max $10x_1 + 5x_2 + 5x_3$
   (C) Max $5x_1 - 5x_2 - 3x_3 - 1$
   (D) None of these
6. The problem have—
   (A) Possible solution
   (B) No-feasible solution
   (C) Bounded solution
   (D) None of these
7. The problem have—
   (A) Bounded solution
   (B) Unbounded solution
   (C) Feasible solution
   (D) None of these
8. The dual of the problem have—
   (A) Non-negative variables
   (B) Negative variables
   (C) Unrestricted variables
   (D) None of these
9. In the dual, we have—
   (A) Max $y_1 - 3y_2 - 7y_3 + 5y_4$
   and y_i $\square \longleftarrow \longleftarrow 0$, $i = 1, 2, 3, 4$
   (B) Min $y_1 - 3y_2 - 7y_3 + 5y_4$
   $y_i \longleftarrow 0$, $i = 1, 2, 3, 4$
   (C) Both (A) and (B) are true
   (D) None of these

Solution :

The dual is

Max $y_1 - 3y_2 - 7y_3 + 5y_4$

Subject to

$5y_1 - y_2 + y_3 - 4y_4 \longleftarrow 10$,

$-5y_1 + y_2 + 4y_4 \longleftarrow 5$,

$-3y_1 - y_3 + y_4 \longleftarrow 5$,

$y_i = 0$. (I = 1, 2, 3, 4)

The iterations are given as follows. The solution is unbounded.

Hence there is no feasible solution of the given L.P.

Table

Basic	y_1	y_2	y_3	y_4	s_1	s_2	s_3	**Sol.**
y_0	-1	3	7	-5	0	0	0	0
s_1	5	-1	1	-4	1	0	0	10
s_2	-5	1	0	4	0	1	0	5
s_3	-3	0	-1	1	0	0	1	5
	$\frac{-29}{4}$	$\frac{17}{4}$	7	0	0	$\frac{5}{4}$	0	$\frac{25}{4}$
s_1	0	0	1	0	1	1	0	15
y_4	$\frac{-5}{4}$	$\frac{1}{4}$	0	1	0	$\frac{1}{4}$	0	$\frac{5}{4}$
s_3	$\frac{-7}{4}$	$\frac{-1}{4}$	-1	0	0	$\frac{-1}{4}$	1	$\frac{15}{4}$

The optimal assignment

A — III, B — I, C — II and D — IV.

The minimum total time is 17 + 13 + 19 + 10 = 59 man hours.

10. Max $z = 5x_1 + 12x_2 + 4x_3$,
    Subject to $x_1 + 2x_2 + x_3 \longleftarrow 5$,

$$2x_1 - x_2 + 3x_3 = 2,$$
$$x_1, x_2, x_3 \geq 0.$$

We change the equality into inequalities and bring the LPP in canoncial form.

Solution :

Max $z = 5x_1 + 12x_2 + 4x_3$,

Subject to $x_1 + 2x_2 + x_3 \geq 5,\ y_1$

$$2x_1 - x_2 + 3x_3 \geq 2\ y_2^+$$
$$-2x_1 + x_2 - 3x_3 \geq -2,\ y_3^-$$
$$x_1, x_2, x_3 \geq 0$$

The dual of above is

Min $w = 5y_1 + 2(y_2^+ - y_2^-)$

Subject to

$$y_1 + 2(y_2^+ - y_2^-) \geq 5,$$
$$2y_1 - (y_2^+ - y_2^-) \geq 12,$$
$$y_1 + 3(y_2^+ - y_2^-) \geq 4,$$
$$y_1, y_2^+, y_2^- \geq 0$$

The above is same as

Min $w = 5y_1 + 2y_2$

Subject to $\quad y_1 + 2y_2 \geq 5$

$$2y_1 - y_2 \geq 12,$$
$$y_1 + 3y_3 \geq 4,$$

$y_1 \geq 0$ and y_2 unrestricted in sign.

●●