Real Analysis

Sequences

Sequence is a function whose domain is the set of positive integers.

i.e., $\{a_n\}_{n=1}^{\infty}$

or $a_n = a(n)$,

$n = 1, 2, 3, \ldots\ldots$

e.g., (a) $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\right\}$

(b) Let P_n be the nth Prime number

$\{P_n\}_{n=1}^{\infty} = \{2, 3, 5, \ldots\}$

Convergence of Sequence—A sequence $\{a_n\}_{n=1}^{\infty}$ converges to a real number A iff for each $\in = 0$, there is a positive integer N, such that for all $n \geq N$, we have $|a_n - A| < \in$.

Neighbourhood—A set N_x of real numbers is a neighbourhood of real number x iff, N_x contains an interval of positive length centered at x, *i.e.*, iff there is $\varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \subset N_x$.

Convergent and divergent sequence— A sequence $\{a_n\}_{n=1}^{\infty}$ is convergent iff there is a real number. A such that $\{a_n\}_{n=1}^{\infty}$ converges to A. If $\{a_n\}_{n=1}^{\infty}$ is not convergent it is Divergent sequence.

Cauchy Sequence—A sequence $\{a_n\}_{n=1}^{\infty}$ is Cauchy iff for each $\varepsilon > 0$, there is a positive integer N : if $m, n \geq N$, then $|a_n - a_m| < \varepsilon$.

Limit of a Sequence—If a sequence is convergent the unique number to which it converges is the limit of the sequence.

Accumulation Point—For a set S of real numbers, a real number A is an accumulation point of S iff every neighbourhood of A contains infinitely many points of S.

Subsequence—Let $\{a_n\}_1^{\infty}$ be a sequence and $\{n_k\}_1^{\infty}$ be any sequence of positive integer such that $n_1 < n_2 < n_3 < \ldots$ the sequence $\{a_{nk}\}_{k=1}^{\infty}$ is called a subsequence of $\{a_n\}_{n=1}^{\infty}$.

Increasing sequence—Sequence $\{a_n\}_{n=1}^{\infty}$ is increasing, iff $b_n \geq b_{n+1}$ for all n.

Monotone Sequence—Sequence that is either increasing or decreasing.

Bounded above sequence—Sequence $\{a_n\}_{n=1}^{\infty}$ is bounded above, iff there exist a real number N : $a_n \leq N$ for all n.

Bounded below sequence—Sequence $\{a_n\}_{n=1}^{\infty}$ is bounded below iff there exist a real number M : $a_n \geq M$ for all n.

Bounded Sequence—Sequence $\{a_n\}_{n=1}^{\infty}$ is bounded, if it is bounded both from above and below $\Leftrightarrow$ the exist a real number S : $|a_n| \leq S$ for all n.

Series

Infinite series—An infinite series is a pair $\left\{\{a_n\}_{n=1}^{\infty}, \{S_n\}_{n=1}^{\infty}\right\}$, where $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers and $S_n = \sum_{k=1}^{\infty} a_k$ for all n, a_n is the nth term of the series and S_n is the nth partial sum of the series.

Convergence of Series—If $\sum_{n=1}^{\infty} a_n$ converges, then $\{S_n\}_{n=1}^{\infty}$ converges.

Converges absolutely—An infinite series $\sum_{n=1}^{\infty} a_n$ converges absolutely iff $\sum_{n=1}^{\infty} |a_n|$ converges.

Converges Conditionally—If $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Cauchy's Product—Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are two infinite series and for each n define $c_n =$

$\sum_{k=0}^{n} a_n b_{n-k}$. The infinite series $\sum_{n=0}^{\infty} c_n$ is called the Cauchy's product of two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$.

Rearrangement of Series—Let $\sum_{n=0}^{\infty} a_n$ be an infinite series. If T is one-one function from {0, 1, 2, ...} onto {0, 1, 2, ...), then the infinite series $\sum_{n=0}^{\infty} a\mathrm{T}_{(n)}$ is called a rearrangement of $\sum_{n=0}^{\infty} a_n$.

Interval of Convergence—If $\sum_{n=0}^{\infty} a_n x^n$ is a power series, then the set of points at which series converges is either.

(1) $\mathbb{R}$, a set of all real numbers $(-\infty, \infty)$, (Interval of infinite radius)

(2) {0}, (Interval of zero radius)

(3) An interval of positive finite length centered at zero which may contain all, none or one of its end points.

These intervals are called interval of convergence.

Radius of Convergence—If $\sum_{n=0}^{\infty} a_n x^n$ has an interval of convergence c which is different from R and {0}, then there is a unique real number r such that, $(-r, r) \subset \mathrm{C} \subset [-r, r]$. This number r is called the radius of convergence of the power series.

Uniform Convergence—A sequence $\{f_n\}_{n=1}^{\infty}$ of functions is said to converge uniformly on E if there is a function $f : \mathrm{E} \to \mathrm{R}$ such that for each $\in s_0$, there is N such that for each positive integer n, $n \geq \mathrm{N}$ implies that $|f_n(x) - f(x)| < \in$ for all $x \in \mathrm{E}$.

Power Series—Let $\{a_n\}_{n=0}^{\infty}$ be a sequene of real number. For each real number, x, a series $\sum_{n=0}^{\infty} a_n x^n$ is power series.

Maxima and Minima

(1) **Maximum value**—A continuous function $f(x)$ is said to have a maximum value for $x = a$, if $f(a)$ is greater than any other value of $f(x)$ lying in small neighbourhood of $x = a$.

In other words, $f(a)$ is a maximum value of $f(x)$, if $f(x)$ is increasing in $(a-h, a)$ and decreasing in $(a, a+h)$ where h is a small quantity.

(2) **Minimum value**—A continuous function $f(x)$ is said to have a minimum value at $x = a$, if $f(a)$ is smallest of all $f(x)$ lying in small neighbourhood of $x = a$.

In other words, $f(a)$ is minimum value of $f(x)$, if $f(x)$ is decreasing in $(a-h, a)$ and increasing in $(a, a+h)$, where h is a small quantity.

(3) **Conditions for Maxima**—The function $f(x)$ has a maximum value $f(a)$ if $f'(a) = 0$ and $f'(x)$ changes sign from positive to negative as x passes through a from left to right.

In general, for any even number n.

$f'(a) = f''(a) = f'''(a) = \ldots = f^{n-1}(a) = 0$ and $f''(a) < 0$, then $f(a)$ is a maximum value of $f(x)$.

In particular, if $n = 2$ then $f'(a) = 0$, $f''(a) < 0$, then the function is maximum at $x = a$.

(4) **Conditions for Minima**—The function $f(x)$ has a minimum value $f(a)$. If $f'(a) = 0$ and $f'(x)$ change sign from negative to positive as x passes through a from left to right.

In general, for any even number n.

$f'(a) = f''(a) = f'''(a) = \ldots\ldots\ldots = f^{n-1}(a) = 0$ and $f^n(a) \leq 0$, then $f(a)$ is a minimum value of the function $f(x)$.

In particular, If $n = 2$ and $f'(a) = 0$, $f''(a)$ so the function is minimum at $x = a$.

Extreme value—Either a maximum value or minimum value $f(a)$ of the function $f(x)$ is said to be extreme value.

> **Note :** The tangent at maximum or minimum point of the curve is parallel to x-axis.

Stationary value—If $f'(a) = 0$, then $f(a)$ is said to be stationary value of the function $f(x)$ at $x = a$.

> **Note :** Every extreme value is stationary but every stationary value need not be an extreme value.

Greatest Value—The greatest value of a function in an interval (a, b) is either a maximum value of $f(x)$ at a point inside the interval or end value (*i.e.*, at $x = a$, or $x = b$) of $f(x)$ which ever is greater.

Least Value—The least value of $f(x)$ in an interval (a, b) is either a minimum value of $f(x)$ at a point insdie the interval or an end value (*i.e.* at $x = a$ or $x = b$) of $f(x)$ which ever is less.

Maxima and Minima for the function of two independent variable—

(i) **Maximum value**—Let $f(x, y)$ be continuous function. The value $f(a, b)$ is said to be maximum value of $f(x, y)$ if there exist some neighbourhood of the points (a, b) such that $(a+h, b+k)$ of this neighbourhood, other than (a, b).

$$f(a, b) > f(a+h, b+k)$$

(ii) **Minimum value**—Let $f(x, y)$ be a continuous function the value $f(a, b)$ is said to be maximum value of (x, y) if there exist some neighbourhood of the point (a, b) such that $(a+h, b+k)$ of this neighbourhood, other than (a, b)

$$f(a, b) < f(a+h, b+k)$$

Sufficient Condition for Maximum value—If $f_x(a, b) = 0, f_y(a, b) = 0$ and $f_x^2(a, b) = A$, $f_{xy}(a, b) = B, f_y^2(a, b) = C$, then $f(a, b)$ is maximum value if $AC - B^2 > 0$ and $A < 0$.

Sufficient Condition for Minimum value—If $f_x(a, b) = 0, f_y(a, b) = 0$ and $f_x^2(a, b) = A, f_{xy}(a, b) = B, f_y^2(a, b) = C$, then $f(a, b)$ is minimum value if $AC - B^2 > 0$ and $A > 0$.

> **Note** : The value $f(a, b)$ is neither maximum nor minimum, if $AC - B^2 < 0$.

Extreme value—Either a maximum value or a minimum value $f(a, b)$ of the function $f(x, y)$ is said to be extreme value.

Sufficient Condition for Extreme value—If $f_x(a, b) = 0, f_y(a, b) = 0$ and $f_x^2(a, b) = A, f_{xy} = (a, b) = B, f_y^2(a, b) = C$, then $f(a, b)$ is an extreme value if $AC - B^2 > 0$.

Fourier Series

L(I)—Set of Lebesgue-integrable function on interval 1.

L^2(I)—Set of square integrable on I.

Orthogonal system and orthonormal on interval 1—Let $\delta = \{\phi_0, \phi_1, \phi_2, \ldots\}$ be a collection of function in $L^2(I)$.

If $(\phi_n, \phi_m) = 0$ whenever $m \neq n$, the collection S is orthogonal system on I.

If each ϕ_n has $\| \phi_n \| = 1$, then S is orthonormal on I.

Fourier Series and Coefficients—Let $S = \{\phi_1, \phi_2 \ldots\}$ be orthonormal on interval I and fE L^2 (I) then $f(x) \sim \sum_{n=0}^{\infty} c_n \phi_n(x)$ is called Fouries series of f relative to S and $c_n = (f_1\phi_n) = \int f(x)\, \phi_n (x)dx$ $(n = 0, 1, 2, \ldots)$ is called Fouries Coefficient of f relative to S.

Functions of Several Variables—If $D \subset R^n$ the function

$f(x) = f(x_1, x_2, \ldots x_n)$, where $(x \in \mathbb{R}^n$ and $f: D \to \mathbb{R}^n$ is called function of several variables.

Limit—A function $f(x), x \in D \subset R^n$ has a limit l, *i.e.*, $\lim_{x \to a} f(x) = l$, if for given $\in > 0$, there exist $\delta = 0$ such that $|f(x) - l| < \in$ for every $a \in D$, $\|x - a\| < \delta$.

Continuity—A function $f: D \to \mathbb{R}^n$ is continuous at $x = a$, if for each $\in > 0$ there exist $\delta > 0$ such that $|f(x) - f(a)| < \in$ whenever $\|x - a\| < \delta$ and $x \in D \subset \mathbb{R}^n$ or $f: D \to \mathbb{R}^n$ is continuous at a iff $\lim_{x \to a} f(x) = f(a)$.

Uniform Continuity—A function $f: D \to \mathbb{R}^n$ is uniformaly continuous on D if it is continuous at every $x \in D \subset \mathbb{R}^n$.

Some Important Results

1. $f: D \to \mathbb{R}^n, g: D \to \mathbb{R}^n$ then, $\lim (f \pm g)(x) = \lim f(x) \pm \lim g(x)$ $\lim (f . g)(x) = \lim f(x) . \lim g(x)$, $\lim (f/g)(x) = \lim f(x)/\lim g(x)$. If $\lim g(x) \neq 0$.
2. The range of a function continuous on a compact set is compact.
3. A real valued function continuous on a compact set is bounded and attains its bound.
4. A real valued function continuous on a closed rectangle $[a, b]$ is bounded and attains its bound.
5. A function continuous on a compact domain is uniformaly continuous.
6. Let f be a real valued function with domain $D \subset R^n$. Let D be such that

   $X, Y \in D \Rightarrow ex + (1 + \tau) Y \in D \ \forall \ \tau \in [0, 1]$,

   then f assumes every value between $f(x)$ and $f(y)$.
7. If $\lim_{x \to a} f(x) = b$

   and $b = (b_1, b_2, \ldots b_m)$

   $f = (f_1, f_2, \ldots f_m)$

   then $\lim_{x \to a} f_i(x) = b_i \ (1 \le i \le m)$ and conversely

Riemann (Stielties) Integration

Partition—A partition P of $[a, b]$ is a finite set $\{x_0, x_1, \ldots, x_n\}$ such that $a = x_0 < x_1 < \ldots < x_n = b$.

Refinement—If P and Q are partition of $[a, b]$ with $P \subset Q$, then Q is refinement of P.

Riemann Integration—Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function and P $\{x_1, x_2, \ldots, x_n\}$ is a partition of $[a, b]$. For each i, $(i = 1, 2, \ldots, n)$ define

$$M_i(f) = \sup\{f(x) : x \in [x_{i-1} - x_i]\}$$
$$m_i(f) = \inf\{f(x) : x \in [x_{i-1} - x_i]\}$$
$$U(P, f) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1})$$

(Upper darboux sum of f)

$$L(P, f) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1})$$

(lower darboux sum of f)

Then upper integration of f.

$$\int_a^{-b} f\,dx = \inf\{U(P, f) : P \text{ is a partition}\}$$

Lower integration of f.

$$\int_{-a}^{b} f\,dx = \text{Sup}\{L(P, f) : P \text{ is a Partition}\}$$

Riemann integrable on $[a, b]$—f is Riemann integrable if $\int_a^b f(x)\,dx$ exist.

i.e. $\int_a^{-b} f\,dx = \int_{-a}^{b} f\,dx = \int_a^b f\,dx$

Riemann—Stielties integration—Suppose $f : [a, b] \to R$ is bounded and $d : [a, b] \to R$ is an increasing function. For each partition

$$P = \{x_0, \ldots, x_n\} \text{ define}$$

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i(f)[x_i(\alpha) - x_{i-1}(\alpha)]$$

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i(f)[x_i(\alpha) - x_{i-1}(\alpha)]$$

$$\int_a^{-b} f\,d\alpha = \inf\{U(P, f, \alpha) : P \text{ is a partition}\}$$

$$\int_{-a}^{b} f\,d\alpha = \sup\{L(P, f, \alpha) : P \text{ is a partition}\}$$

f is Riemann-Stielties integrable with respect to α on $[a, b]$ if

$$\int_a^{-b} f\,d\alpha = \int_{-a}^{b} f\,d\alpha = \int_a^b f\,d\alpha$$

when $\alpha(x) = x$, the Riemann—Stielties integral with respect to α reduces to Riemann integration.

Some Important Theorems

1. Let $f : [a, b] \to \mathbb{R}$ be bounded and $\alpha : [a, b] \to R$ increasing function.
   Then if P and Q are any partions of $[a, b]$, we have
   (i) If $P \subset Q$, then $L(P, f, \alpha) \le L(Q, f, \alpha)$ and $U(Q, f, \alpha) \le U(P, f, \alpha)$
   (ii) $L(P, f, \alpha) \le U(Q, f, \alpha)$
   (iii) $\int_{-a}^{b} f\,d\alpha \le \int_a^b f\,d\alpha$
2. Let $f : [a, b] \to \mathbb{R}$ be bounded and $\alpha : [a, b] \to \mathbb{R}$ be increasing. Then f is Riemann integration on $[a, b]$ iff for each $\in$ so, there is a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) \le \in$.
3. If f'. $[a, b] \to R$ is monotone and $\alpha : [a, b] \to \mathbb{R}$ is increasing and continuous, then f is Riemann–Stielties integration on $[a, b]$.
4. If $f : [a, b] \to \mathbb{R}$ is continuous and $\alpha : [a, b] \to \mathbb{R}$ increasing, then f is Riemann–Stielties integration on $[a, b]$.
5. If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and f' is Riemann integration on $[a, b]$, then $\int_a^b f'\,dx = f(b) - f(a)$.
6. If $f_1, f_2 : [a, b] \to \mathbb{R}$ are bounded, $\alpha : [a, b] \to \mathbb{R}$ is increasing and f_1, f_2 are Riemann integration with respect to α on $[a, b]$, then—
   (a) For any real numbers c_1, c_2, $c_1f_1 + c_2f_2$ is also Riemann integration on $[a, b]$ and $\int_a^b (c_1f_1 + c_2f_2)\,d\alpha = c_1 \int_a^b f_1 d\alpha + c_2 \int_a^b f_2 d_\alpha$
   (b) If $f_1(x) \le f_2(x)$ for all $x \in [a, b]$, then $\int_a^b f_1(x)\,d\alpha \le \int_a^b f_2(x)\,d\alpha$
   (c) If $m \le f_1(x) \le M$ for all $x \in [a, b]$, then $m\{\alpha(b) - \alpha(a)\} \le \int_a^b f_1 d\alpha \le M\{a(b) - \alpha(a)\}$.
   (d) If $\beta : [a, b] \to R$ is increasing and f is Riemann integration with respect to β on $[a, b]$ and c_1 and c_2 are any non-negative real numbers, then f is Riemann integration with respect to $(c_1\alpha + c_2\beta)$ on $[a, b]$ and $\int_a^b fd\,(c_1\alpha + c_2\beta) = c_1 \int_a^b f d\alpha + c_2 \int_a^b f d\beta$.

7. Suppose $f : [a, b] \to R$ is bounded and $\alpha : [a, b] \to \mathbb{R}$ is increasing. If $a < c < b$, then f is Riemann–Stielties integration on $[a, b]$ iff f is Riemann–Stielties integration on $[a, c]$ and $[c, b]$ and $\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$.
8. Suppose $f : [a, b] \to [c, d]$, $\alpha : [a, b] \to \mathbb{R}$ is increasing. f is Riemann Stielties integrable on $[a, b]$ and $\phi : [c, d] \to \mathbb{R}$ is continuous. Then ϕ of is Riemann–Stielties integrable on $[a, b]$.
9. If $f, g : [a, b] \to \mathbb{R}$, $\alpha : [a, b] \to \mathbb{R}$ is increasing, f, g is Riemann–Stielties integrable on $[a, b]$ then
   (a) fg is Riemann Stielties integrable on $[a, b]$
   (b) $|f|$ is Riemann Stielties integrable on $[a, b]$
   (c) $\left|\int_a^b f d\alpha\right| \le \int_a^b |f|\, d\alpha$
10. If $f : [a, b] \to \mathbb{R}$ is continuous and $\alpha : [a, b] \to \mathbb{R}$ is increasing, then there is $c \in [a, b]$ such that $\int_a^b f\,d\alpha = f(c)\,[\alpha(b) - \alpha[a)]$

Line and Surface Integrals

Line Integral

Curve—A curve Γ, in the xy plane is the set of points (x, y) such that $\Gamma = \{(x, y) : x = \phi(\tau), y = \psi(\tau); a \le \tau \le b\}$.

Closed Curve—If $\phi(a) = \phi(b)$ and $\psi(a) = \psi(b)$ the curve Γ is closed.

Double Points—For $a < \tau_1, \tau_2 < b$ and $\tau_1 \ne \tau_2 \Rightarrow \phi(\tau_1) = \phi(\tau_2)$ and $\psi(\tau_1) = \psi(\tau_2)$.

Jordan Curve—If curve closed and have no double points.

Regular Curve—If curve Γ is regular if it has no double points and if the interval (a, b) can be divided into a finite subintervals in each of which $[\psi'(\tau)]^2 + [\psi'(\tau)]^2 > 0$ where $\phi(\tau), \psi(\tau) \in c'$.

Regular region—A region is regular if it is bounded and closed and if its boundary consists of a finite number of regular Jordan curves which have no points in common with each other.

Plane region— $R_x = R\,[a, b\,\phi(x), \psi(x)]$

$R_y = R\,[a, b, \phi(y), \psi(y)]$

Line Integral—Let $f(x, y)$ is a function defined at every point $(x, y) \in \Gamma$ and Δ is a sub division of interval $[a, b]$ by $a = \tau_0, \tau_1, \ldots, \tau_n = b$. The line integral of $f(x, y)$ are

$$\int_\Gamma f(x, y)\,dx = \int_{x_0 y_0}^{x_1 y_1} f(x, y)\,dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(\phi(\tau_i'), \psi(\tau_i'))\,[\phi(\tau_i) - \phi(\tau_{i-1})]$$

$$\int_\Gamma f(x, y)\,dy = \int_{x_0 y_0}^{x_1 y_1} f(x, y)\,dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(\phi(\tau_i'), \psi(\tau_i'))\,[\phi(\tau_i) - \phi(\tau_{i-1})]$$

where $x_0 = \phi(a)$, $y_0 = \psi(a)$

and $x_1 = \phi(b)$, $y_1 = \psi(b)$

and $\tau_{i-1} \le \tau_i' \le \tau_i$

Simply Connected domain—A domain D is simply connected if no Jordan curve in D contains in its interior a boundary point of D.

Some Important Theorems

1. If Γ is a regular curve and $f(x, y)$ on Γ exists, then $\int_\Gamma f(x, y)\,dx$ and $\int_\Gamma f(x, y)\,dy$ exists.
2. **Green theorem (Gauss's theorem) (First form)**—If R is region R_x and also R_y and Γ is bounded of R, where $P(x, y), Q(x, y) \in R$ then $\int_\Gamma P dx + Q dy = \iint_R Q_x(x, y) - P_y(x, y]\,dx$ the line integral taken in positive sense.
3. **Green's theorem (Second form)**—If R is a region Rx and regular region S, Γ is bounded of R and $P(x, y), Q(x, y) \in R$, then $\int_\Gamma P dx + Q dy = \iint_R [Q_x(x, y) - P_x(x, y)]ds$ the line integral taken in positive sense.
4. Area of the region R—
   $A = -\int_\Gamma y\,dx = \int_\Gamma x\,dy = \frac{1}{2}\int_\Gamma (-y)dx + x\,dy$
   the integral being in the positive sense.
5. If D is simply connected and $P(x, y)$, $Q(x, y) \in D$ and $Q_1(x, y) = P_2(x, y) \in D$ iff there exists $F(x, y) \in D$ such that $F_1 = P$, $F_2 = Q$.
6. If D is simply connected and $P(x, y)$, $Q(x, y) \in D$, $Q_1(x, y) = P_2(x, y) \in D$ and Γ is a regular curve in D joining (a, b) with
   $x_0 = \phi(a)$ and $y_0 = \psi(a)$

Then line integral extended over Γ is independent of Γ.

7. If $P(x, y)$, $Q(x, y) \in D$, a simply connected domain, then $Q_x = P_y \in D$ iff $\int_\Gamma Pdx + Qdy = 0$ for every regular closed curve Γ in D.
8. If R (x, y, z) a point in $Vxy = V(R, \phi(x, y), \psi(x, y)$ and function ϕ, ψ exists in R and y is the angle between the positive z-axis and the exterior normal to $\in$, the boundary of V_{xy} then

$$\iiint_{V_{xy}} R_3 dy = \iint_\in R \cos y d\in$$

Line Integrals in Space—If $\Gamma = (x, y, z) : x = \phi(\tau), y = \psi(\tau), z = w(\tau), a \leq \tau \leq 1$. The line integral is defined as

$$\int_\Gamma f(x, y, z)dx = \sum_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(\psi(\tau_i'),$$

$$\psi(\tau_i'), \psi(\tau_i')) [\phi(\tau_i) - \phi(\tau_{i-1})]\tau_{i-1} \leq \tau_i' \leq \tau_i$$

Stokes' theorem—If $\in$ is the surface $z = f(x, y)$ bounded by the regular closed curve Γ and P, Q, y belongs to $\in$ and α, β, γ are direction angles to a directed normal to $\in$. Then

$$\int_\Gamma Pdx + Qdy + Rdz = \iint_\in [(R_2 - Q_3) \cos \alpha$$

$$+ (P_3 - R_1) \cos \beta + (Q_1 - P_2) \cos \gamma]d\in$$

where the direction of integration is clockwise to an observer facing in the direction of the directed normal.

Surface Integral

Subdivision Δ of surface $\in$—It is a set of closed curves $\{e_k\}_1^n$ lying on surface $\in$.

Diameter of a region on $\in$—The diameter of a region on $\in$ is the length of the largest straight line segment whose ends lie in the region.

Norm of $\|\Delta\|$—It is the largest of the n-diameters of the subregions produced by the sub-division.

Surface integral over $\in$—Let $P(x, y, z)$ be a function defined at every point of $\in$ and let (ξ_K, n_K, ξ_K) be a point on $\in$ inside or on the boundary of the sub-region bounded by e_k. Then the surface integral of $P(x, y, z)$ over $\in$ is

$$\iint_E P(x, y, z)\, d\in = \lim_{\|\Delta\| \to 0} \sum_{k=1}^{n} P(\xi_K, n_K, \xi_K)\Delta\in_K$$

where $(\Delta\in_K)$ is the area of sub-region $\in_K$ and the limit exists.

Some Important Theorems

1. If $\in$ is the surface $z = f(x, y)$ over the region R and $\in$ lines in V, then

(a) $\iint_\in P(x, y, z)\, d\in$ exists

(b) $\iint_E P(x, y, z)\, d\in = \iint_R P(x, y, f(x, y))$

$$\sqrt{1 + f_1^2(x, y) + f_2^2(x, y)}\, ds$$

This reduces a surface integral to an ordinary double integral.

2. **Green's (Causs's) theorem**—If $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$ are the points on V and V is bounded by bounded region $\in^*$ and α, β, γ are the direction angle of the exterior normal to $\in^*$

$$\iiint_V [P_1(x, y, z) + Q_2(x, y, z) + R_3(x, y, z)]\, dv$$

$$= \iint_\in [P(x, y, z)\cos\alpha + Q(x, y, z)\cos\beta$$

$$+ R(x, y, z)\cos\gamma)]\, d\in$$

Metric Space

Metric Space—A metric space $<x, P>$ is a non-empty set X of elements (points) and $P : X \times X \to \mathbb{R}$ such that for $x, y, z \in X$.

1. $P(x, y) \geq 0$
2. $P(x, y) = 0$ iff $x = y$
3. $P(x, y) = P(y, x)$
4. $P(x, y) \leq P(x, y) + P(x, y)$

The function P is called metric.

Cartesian Product—If $<X, P_1>$ and $<Y, P_2>$ are two metric spaces, then $<X \times Y, \tau>$ is the cartesian product of $<X, P_1>$ and $<Y, P_2>$ defined as $X \times Y = \{(x, y) : x \in X, y \in Y\}$ and $\tau(<x_1, y_1>, <x_2, y_2>) = \sqrt{P_1(x_1, x_2)^2 + P_2(y_1, y_2)^2}$

Diameter—If (X, P) is a metric space, $E \neq \phi \subset X$, then diameter $E = \sup\{P(x, y); x, y \in E\}$.

Pseudometric—If $P(x, y) = 0$ for some $x \neq y$ then (X, P) is called Pseudometric space and P is a Pseudometric.

Extended Pseudometric—If $P(x, y) = \infty$ for some $x, y \in X$, then P is extended Pseudometric and (X, P) is extended Pseudometric space.

Ball—A set $\delta_{x, \delta}\ \{Y : P(x, y) < \delta\}$ is called ball centered at $x \in <X, P>$.

Convergence and Completeness

Convergence—A sequence $<x_n>$ from metric space $<X, P>$, converges to the point $x \in X$ (x is a limit), if given $\in > 0$, there is N such that $P(x, x_n) < \in$, $\forall_n \in N$.

Cluster Point—x is a cluster point of $<x_n>$ if given $\in > 0$ and given N there is $n \geq N : P(x, x_n) < \in$.

Cauchy's Sequence—A sequence $<x_n>$ from a metric space $<X, P>$ is called Cauchy sequence, if given $\in > 0$. There is $N : n, m > N$, we have $P(x_n, x_m) < \in$.

Complete metric space—A metric space $<X, P>$ is complete if every Cauchy sequence converges (to some point of X).

Some Important Theorems

1. If x is limit of $<x_n>$, then x is cluster point of $<x_n>$ (converge is not true).
2. If $<x_n>$ a Cauchy sequence converges to some $x \in <X, P>$, then sequence converges to $x \in <X, P>$ (converges is not true).
3. If $<X, P>$ is an incomplete metric space, it is possible to find a complete metric space X^* in which X is isometrically embeded as a dense subset. If X is contained in an arbitrary complete space Y, then X^* is isometric with the closure of X in Y.

Lebesgue Measure

Measure

Length of an interval—The length of an interval I is the difference of end points of the interval.

Measure of a set—Let M be a collection of sets of real numbers and $E \in M$. Then non-negative extended real number mE is called the measure of E. If m satisfies.

(a) mE is defined for each set E of real numbers *i.e.* $M = P(R)$, the power set of sets of real number.

(b) For an interval I, $mI = I(I)$

(c) If $<E_n>$ is a sequence of disjoint sets (for which m is defined) in M. $m\,(UE_n) = \in_m.E_n$.

(d) m is translation invariant, *i.e.* if E is the set on which m is defined, and $E + Y = \{x + y : x \in E)$, then $m\,(E + y) = mE$.

Countable Additive Measure—Let M be a σ-algebra of sets of real numbers and $E \in M$. Then non-negative extended real number mE is countable additive measure, if $m\,(UE_n) = \in mE_n$ for each sequence $<E_n>$ of disjoint sets in M.

Countable sub additive measure—Let M be a σ-algebra of sets of real numbers and $E \in M$. Then non-negative extended real number mE is countable subadditive measure, if $m(UE_n) \leq \in mE_n$, for each sequence $<E_n>$ of sets in M.

Counting measure—If

$$nE = \begin{cases} \infty, \text{ for infinite set E} \\ \text{The number of elements in E,} \\ \text{ for finite set E} \end{cases}$$

Some Important Results

Let m be a countable additive measure defined for all sets in a σ-algebra. M then

(a) Monotonicity : $A, B \in M, A \subset B \rightarrow mA \leq mB$

(b) If for some set $A \in M$, $mA < \infty$, then $m\phi = 0$

Lebesgue outer Measure

Lebesque outer Measure—Let A be a set of real number $\{I_n\}$ be the countable collection of open intervals that covers A, *i.e.* $A \subset U\,I_n$. Then Lebesgue outer measure m^*A of A is

$$m^*A = \underset{A \subset UI_n}{Inf} \in l(I_n)$$

Some Important Results

1. $m * \phi = 0$
2. $A \subset B \Rightarrow m^* A \leq m^* B$
3. For singleton set $\{x\}$, $m^* \{x\} = 0$
4. The Lebesgue outer measure of an interval is its length.
5. If $\{A_n\}$ is a countable collection of sets of real number. Then $m^* (UA_n) \leq m^* A_n$.
6. If A is countable, $m^* A = 0$
7. The set [0, 1] is not countable.
8. Given any set A and any $\in > 0$, there is an open set $O : A \subset O$ and $m * 0 \leq m^* A + \in$. There is a $G_\delta : A \subset G$ and $m^* A = m^* G$.

Lebesgue Measurable sets and Lebesgue Measure

Lebesgue Measurable sets—A set E is Lebesgue measurable its for each set A we have $m^*A = m^*(A \cap E) + m^*(A \cap E^C)$

Lebesgue Measure—If E is a Lebesgue measurable set, the Lebesgue measure mE is the Lebesgue outer measure of E.

Some Important Results

1. If $m^*E = 0$, then E is Lebesgue measurable.
2. If E_1 and E_2 are Lebesgue measurable, so $E_1 \cup E_2$.
3. The family M of Lebesgue measurable sets is in algebra of sets.
4. If A is any set and $E_1, E_2 \ldots, E_n$ a finite set sequence of disjoint Lebesgue measurable sets. Then
$$m^* \left(A \cap \left[\overset{n}{\underset{j=1}{\cup}} E_i\right]\right) = \sum_{i=1}^{n} m^*(A \cap E_i)$$
5. The collection M of Lebesgue measurable sets is a σ-algebra.
6. Every set with Lebesgue outer measure zero is Lebesgue measurable.
7. The interval (a, ∞) is Lebesgue measurable.
8. Every Borel set is Lebesgue measurable.
9. Each open set and closed set is Lebesgue measurable.
10. If $\langle E_i \rangle$ is a sequence of Lebesgue measurable set. Then for Lebesgue measure $m(UE_i) \le \in mE_i$.

    If the set E_i are pairwise disjoint then for Lebesgue measure $m(UE_i) = \in mE_i$.
11. Let $\langle E_i \rangle$ be an infinite decreasing sequence of Lebesgue measurable sets, *i.e.*, $E_{n+1} \subset E_n$ for each n. Let Lebesgue measure mE, is finite then $m\left(\overset{\infty}{\underset{n=1}{\cup}} E_i\right) = \lim_{n \to \infty} mE_n$.
12. For a given set following are equivalent :—
    (a) E is measurable
    (b) Given $\in > 0$, there is an open set $0 \supset E$ with $m^*(O \sim E) < \in$.
    (c) Given $\in > 0$, there is a closed set $F \subset E$ with $m^*(E \sim F) < \in$.
    (d) $G \in G_\delta$ with $E \subset G$, $m^*(G \sim E) = 0$
    (e) $F \in F_\delta$ with $F \subset E$, $m^*(E \sim F) = 0$ If $m^*E < \infty$, then these statements are equivalent to.
    (f) Given $\in > 0$, there is a finite union $\cup$ of open interval : $m^*(\cup \Delta E) < \in$.

Lebesgue Measurable Functions

Lebesgue Measurable Function—An extended real valued function f is Lebesgue measurable if its domain is measurable end if it satisfies one of the following five; for each real number α.

(a) $\{x : f(x) < \alpha\}$ is measurable
(b) $\{x : f(x) > \alpha\}$ is measurable
(c) $\{x : f(x) \le \alpha\}$ is measurable
(d) $\{x : f(x) \ge \alpha\}$ is measurable
(e) $\{x : f(x) = \alpha]$ is measurable

Almost everywhere Property—If a set of points where it fails to hold is a set of measure zero.

If $f = g$, almost everywhere if f and g have the same domain and $m\{x : f(x) \ne g(x)\} = 0$.

Characteristic function—χ_A : If A is any set, the characteristic function of set A is defined as
$$\chi_A(x) = \begin{cases} 1, \text{ if } x \in A \\ 0, \text{ if } x \notin A \end{cases}$$

Simple Function—A real valued function ϕ is simple function, if it is Lebesgue measurable and assume only a finite number of values.

Borel measurability—A function f is Borel measurable if for each α, the set $\{x : f(x) > \alpha\}$ is a Borel set.

Some Important Results

1. If f is an extended real valued function whose domain is measurable then the following statements are equivalent : For each real number α.
   (a) The set $\{x : f(x) > \alpha\}$ is Lebesgue measurable
   (b) The set $\{x : f(x) < \alpha\}$ is Lebesgue measurable
   (c) The set $\{x : f(x) \le \alpha\}$ is Lebesgue measurable
   (d) The set $\{x : f(x) \ge \alpha\}$ is Lebesgue measurable
2. If c is a constant and f and g two Lebesgue measurable real valued functions defined on the same domain. Then the functions $f + c$, cf, $f + g$, $g - f$ and fg are also Lebesgue measurable.
3. If $\langle f_n \rangle$ is a sequence of Lebesgue measurable functions (with the same domain of definition). Then the function syp $\{f_1, \ldots, f_n\}$, inf $\{f_1, \ldots, f_n\}$, $\sup_n f_n$, $\inf_n f_n$, $\lim f_n$ and $\lim f_n$ are all Lebesgue measurable.
4. If f is a measurable function and $f = g$ almost everywhere, then g is measurable.

5. If f is Lebesgue measurable function defined an interval $[a, b]$ and assume that f takes the values $\neq \infty$ only on a set of measure zero. Then given $\in > 0$, we can find a step function g and a continuous function h such that $|f - g| < \in$ and $|f - g| < \in$.

The Lebesgue Integration

The Riemann Integral—If f is a bounded real valued function defined on the interval $[a, b]$ and $a = x_0 < x_1 < \ldots < x_n = b$ is a subdivision of $[a, b]$.

$$S = \sum_{i=1}^{n} (x_i - x_{i-1}) M_i$$

$$S = \sum_{i=1}^{n} (x_i - x_{i-1}) m_i$$

where $$M_i = \sup_{x_{i-1}} < x < x_i f(x)$$

and $$m_i = \inf_{x_{i-1} < x < x_i} f(x)$$

The upper Riemann integral of f is

$$R\int_a^{-b} f(x)\,dx = \inf S$$

and the lower Riemann integral

$$R\int_a^{-b} f = R\int_{-a}^{b} f = R\int_a^b f(x)\,dx$$

Step Function—For the given subdivision $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ of the interval $[a, b]$, a function ϕ is a step function if,

$$\phi(x) = c_i,$$
$$x_{i-1} < x < x_i$$

Some Important Results

1. $\int_a^b \phi(x)\,dx = \sum_{i=1}^{n} e_i (x_i - x_{i-1})$

2. $R\int_a^{-b} f(x)\,dx = \inf \int_a^b \phi(x)\,dx$ for all step function $\phi(x) \geq f(x)$

3. $R\int_{-a}^{b} f(x)dx = \sup \int_a^b \phi(x)\,dx$ for all step function $\phi(x) \leq f(x)$

The Lebesgue Integral of a Bounded Function over a set of Finite Measure

Characteristic Function—A real valued function of set E.

$$\chi_E = \begin{Bmatrix} 1, & x \in E \\ 0, & x \notin E \end{Bmatrix}$$

Simple Function—A linear combination

$$\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$$

where sets E_i are measurable.

Canonical representation of simple function—If ϕ is a simple function and $\{a_1, \ldots a_n\}$ is the set of non-zero values of ϕ, then $\phi = \in a_i \chi_{A_i}$, where $A_i = \{x : \phi(x) = a_i\}$ is called canonical representation of simple function. Here A_i are disjoint and a_i are distinct and non-zero.

Integral of Simple function—If simple function ϕ vanishes outside a set of finite measure, the integral of ϕ is $\int \phi(x)\,dx = \sum_{i=1}^{n} a_i (mA_i)$ when ϕ has a canonical reprsentation $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$. and mA_i is the Lebesgue measure of A_i. If E is any measurable set, we have $\int_E \phi = \int^{-} \phi\chi_E$.

The Lebesgue Integral—If f is bounded measurable function defined on a measurable set E with mE finite, the Lebesgue integral of f over E is $\int_E f(x)\,dx = \inf \int_E \psi.(x)dx$ for all simple function $\psi \geq f$.

Some Important Theorems

1. Let $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$, with $E_i \cap E_j = \phi$ for $i \neq j$ suppose each set E_i is a measurable set of finite measure. Then $\int \phi = \sum_{i=1}^{n} a_i m_{E_i}$.

2. If ϕ and ψ is simple function which vanish out side a set of finite measure then $\int (a\phi + b\psi) = a\int \phi + b\int \psi$ and if $\phi \geq \psi$ almost everywhere then $\int \phi \geq \int \psi$.

3. If f is defined and bounded on a measurable set E with mE and $\inf_{f \leq \psi} \int_E \psi(x)dx = \sup_{f \geq \phi} \int_E \phi(x)\,dx$ for all simple functions ϕ and ψ. Then of is measurable function.

4. Let f be bounded function defined on $[a, b]$. If f is Riemann integrable on $[a, b]$, then it is measurable and

$$R\int_a^b f(x)dx = \int_a^b f(x)dx$$

5. If f and g are bounded measurable functions defined on set E of finite measure, then
   (a) $\int_E (af + bg) = a\int_E f + b\int_E g$
   (b) If $f = g$ almost everywhere, then $\int_E f = \int_E g$.
   (c) If $f \leq g$ almost everywhere, then $\int_E f \leq \int_E g$.
   (d) $\left|\int f\right| = \int |f|$
   (e) If $A \leq f(x) \leq B$, then $A(mE) \leq \int_E f \leq B(mE)$.
   (f) If A and B are dispoint measureable sets of finite measure, then
   $$\int_{A\cup B} f = \int_A f + \int_B f$$
6. Let $<f_n>$ is a sequence of measurable functions defined on a set E of finite measure and suppose that there is a real number M such that $|f_n(x)| \leq M$ for all n and all x. If $f(x) = \lim f_n(x)$ for each $x \in E$, then $\int_E f \lim \int_E f_n$.
7. A bounded function f on $[a, b]$ is Riemann integrable iff the set of points at which f is discontinuous has measure zero.

Fatou's Lemma—If $[f_n Y$ is a sequence of non-negative measurable function and $f_n(x) \to f(x)$, almost everywhere on a set E, then

$$\int_E f \leq \lim \int_E f_n$$

Dominated Convergence

Convergence in measure—A sequence $<f_n>$ of measurable functions is said to converge to f in measure if, given $\in > 0$, there is an N such that for all $n \geq N$, we have

$$m\{x : |f(x) - f_n(x)| \geq \in\} < \in$$

Cauchy sequence in measure—A sequence $<f_n>$ of measurable functions is Cauchy sequence in measure if given $\in > 0$ there is an N such that for all $m, n \geq N$, we have

$$m\{x : |f_n(x) - f_m(x)| \geq \in\} < \in$$

Some Important Theorem

1. If $<f_n>$ is a sequence of measurable functions that converges in measure to f. Then there is a subsequence $<f_{nk}>$ that converges to of almost everywhere.
2. If $<f_n>$ is a sequence of measurable functions defined on a measurable set E of finite measure. Then $<f_n>$ converges of f in measure iff every subsequence of $<f_n>$ has in turn a subsequence that converges almost everywhere to f.
3. Fatou's lemma and the monotone and Lebesgue convergence theorem remain valid if 'convergence almost everywhere' is replaced by 'convergence in measure'.

Weierstrass approximation theorem—A space X has Weierstrass property if every infinite sequence in X has atleast one limit point. Now-here dense—$A\subset X$ is nowhere dense, A the closure of A has no interior point.

First category (Merger)—A set $A\subset X$ is of first category in X, if it is the union of countabley many nowhere dense sets in X.

Second Category (Non-merger)—A set $A\subset X$ is of second category in X if it is not of first category.

Compactness

Compact metric space—(1) A metric space (X, P) is compact if every infinite subset of X has atleast one limit point.

(2) A metric space X is compact if every open covering μ of X a has finite subcovering.

Set $K\subset X$ is compact if (K, P) is compact.

Relatively Compact—If X is a metric space, $K\subset X$ and closure of K, $\bar{K}$ is compact, then K is relatively compact to X.

Total Boundedness—A metric space X is totally bounded if for every $\in > 0$, X contains a finite set, called an $\in$-net, such that the finite set of open spheres of radius $\in$ and centres in the $\in$-net covers X.

Some Solved Examples

Example 1. Test the convergence of the integral $\int_a^\infty \frac{dx}{x^n}$, where $a > 0$.

Solution :

Here $\int_a^\infty \frac{dx}{x^n} = \lim_{x\to\infty} \int_a^x \frac{dx}{x^n}$

$$= \lim_{x \to \infty} \frac{1}{1-n}[x^{1-n} - a^{1-n}]$$

where $n \neq 1$

Case I. When $1 - n < 0$

i.e., $n > 1$

$$\lim_{x \to \infty} \int_a^x \frac{dx}{x^n} = \frac{a^{1-n}}{n-1}$$

which is finite. The given integral is convergent when $n > 1$.

Case II. When $1 - n < 0$

i.e., $n < 1$

$$\lim_{x \to \infty} \int_a^x \frac{dx}{x^n} = \infty,$$

Therefore, the integral is divergent.

Case III. When $1 - n = 0$

i.e. $n = 1$

$$\lim_{x \to \infty} \int_a^x \frac{dx}{x} = \lim_{x \to \infty} [\log x - \log a]$$
$$= \infty$$

Therefore, divergent.

Hence, the given integral is divergent except when $n > 1$.

Example 2. If $f(x, y) = 2x^2 - xy + 2y^2$ find $\frac{\delta f}{\delta x}$, $\frac{\delta f}{\delta y}$ at (1, 2)

Solution :

$$\left(\frac{\delta f}{\delta x}\right)_{(1,2)} = \lim_{h \to 0} \frac{f(1+h, 2) - f(1, 2)}{h}$$
$$= \lim_{h \to 0} \frac{\{2(1+h)^2 - (1+h)^2 + 2 \cdot 2^2\} - \{2 \cdot 1^2 - 1 \cdot 2 + 2 \cdot 2^2\}}{h}$$
$$= \lim_{h \to 0} \frac{2h^2 + 2h}{h}$$
$$= \lim_{h \to 0} (2h + 2)$$
$$= 2$$

$$\left(\frac{\delta f}{\delta y}\right)_{(1,2)} = \lim_{k \to 0} \frac{f(1, 2+k) - f(1, 2)}{k}$$
$$= \lim_{k \to 0} \frac{\{2 - (2+k) + 2(2+k)^2\} - \{2 - 2 + 8\}}{k}$$
$$= \lim_{k \to 0} \frac{2k^2 + 7k}{k}$$
$$= \lim_{k \to 0} (2k + 7)$$
$$= 7$$

Example 3. Show that $\lim_{(x, y) \to (0, 0)} \frac{2xy}{x^2 + y^2}$ does not exist.

Solution : Let $f(x, y) = \frac{2xy}{x^2 + y^2}$

$(x, y) \neq (0, 0)$

Taking $y = \phi(x) = mx$

where m is a real number

We have
$$f(x, y) = \frac{2mx^2}{x^2 + m^2x^2}$$
$$= \frac{2m}{1 + m^2}$$

$$\therefore \quad \lim_{x \to 0} f(x, \phi(x)) = \lim_{x \to 0} \frac{2m}{1 + m^2}$$
$$= \frac{2m}{1 + m^2}$$

For $\phi_1(x) = m_1 x$,

$\phi_2(x) = m_2 x$

and $m_1 \neq m_2$

$$\lim_{x \to 0} f(x, \phi_1(x)) = \frac{2m_1}{1 + m_1^2} \quad \ldots\text{(i)}$$

and
$$\lim_{x \to 0} f(x, \phi_2(x)) = \frac{2m_2}{1 + m_2^2} \quad \ldots\text{(ii)}$$

By (i) and (ii) we have

$$\lim_{x \to 0} f(x, \phi_1(x)) \neq \lim_{x \to 0} f(x, \phi_2(x))$$

$\therefore \lim_{(x, y) \to (0, 0)} f(x, y)$ does not exist.

Also,
$$\lim_{y \to 0} \left(\lim_{x \to 0} \frac{2xy}{x^2 + y^2}\right) = \lim_{y \to 0} 0 = 0$$
$$\lim_{x \to 0} \left(\lim_{y \to 0} \frac{2xy}{x^2 + y^2}\right) = \lim_{x \to 0} 0 = 0$$

$\therefore$ The two repeated limits exist and are equal.

Example 4. If ϕ, ψ of $[1, 2] \to \mathbb{R}$ and $c = \{(x, y) : x = at^2, y = 2at\}$, then find the value of $\int_c \frac{dx}{x + y}$.

Solution : Here
$$f(x, y) = \frac{1}{x + y},$$
$$\phi(t) = at^2,$$
$$\psi(t) = 2at,$$
$$\psi'(t) = 2at$$

$\therefore$ The conditions are satisfied

$$\therefore \qquad \int_c \frac{1}{x+y}\,dx = \int_1^2 \frac{1}{at^2+2at}\,.\,2at\,dt$$

$$= \int_1^2 \frac{2}{t+2}\,dt$$

$$= [2\log(t+2)]^2$$

$$= 2\log(4/3)$$

Example 5. The integral $\int_0^{\pi/2} \sin^{m-1} x . \cos^{n-1} x\,dx$ converges if and only if , $m > 0, n > 0$.

Solution : The points $x = 0$ and $x = \frac{1}{2}\pi$ are the points of infinite discontinuous, when $0\subset m < 1$.

$$\sin^{m-1}x\cos^{n-1}x = \left(\frac{\sin x}{x}\right)^{m-1}\cos^{n-1}x\,x^{m-1}$$

$$\therefore \lim_{x\to 0} x^{\mu}\left(\frac{\sin x}{x}\right)^{m-1}\cos^{n-1}x\,x^{m-1}$$

$$= \lim_{x\to 0} x^{\mu}x^{m-1}$$

$$= 1$$

If $\qquad \mu = 1-m$

$\therefore$ The integral will be convergent if

$$\mu = 1-m < 1,$$

i.e., $\qquad m > 0$

and $\qquad 1-m > 0$

or, $\qquad m < 1$

Hence, the integral is convergent, if $0 < m < 1$ when $0 < n < 1$.

$$\sin^{m-1}\cos^{n-1}x = x$$

$$= \sin^{m-1}x\,.\,\frac{\cos^{n-1}x}{\left(\frac{1}{2}\pi - x\right)^{n-1}}\left(\frac{\pi}{2}-x\right)^{n-1}$$

and $$\lim_{x\to\pi/2}\left(\frac{\cos x}{\frac{1}{2}\pi - x}\right)^{n-1} = 1$$

$$\therefore \lim_{x\to\pi/2}\left(\frac{\pi}{2}-x\right)^{\mu}\sin^{m-1}x\left(\frac{\cos x}{\frac{1}{2}\pi - x}\right)^{n-1}\left(\frac{\pi}{2}-x\right)^{n-1}$$

$$= \lim_{x\to\pi/2}\left(\frac{\pi}{2}-x\right)^{\mu}.\left(\frac{\pi}{2}-x\right)^{n-1}$$

$$= 1$$

If $\qquad \mu = 1-n$

$\therefore$ The integral will be convergent if $0 < n < 1$ when $m > 1$ and $n > 1$.

When $m > 1$ and $n > 1$, the integral becomes a proper integral and hence converges.

$\therefore$ The given integral is convergent if and only if $m > 0, n > 0$.

Example 6. Show that (i) the area bounded by a simple closed curve c is given by $\frac{1}{2}\int_c x\,dy - y\,dx$.

(ii) Find the area of the ellipse $x = a\cos\theta, y = b\sin\theta, 0 \le \theta \le 2\pi$.

Solution : (i) By Green's theorem

$$\int_c P dx + Q dy = \iint_S \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dx\,dy$$

Put $\qquad P = -y$

and $\qquad Q = -x$

$$\because \qquad \frac{\delta P}{\delta y} = -1,$$

$$\frac{\delta Q}{\delta x} = +1$$

$$\therefore \quad \oint_c xdy - y\,dx = 2\int_S dx\,dy$$

$$= 2A$$

where A is the area of the surface S.

(ii) The area of surface S.

$$A = \frac{1}{2}\int_c xdy - ydx$$

Now for the ellipse

$$x = a\cos\theta,$$

$$y = b\sin\theta$$

$$\text{Area} = \frac{1}{2}\oint xdy - ydx$$

$$= \frac{1}{2}\int_0^{2\pi}(a\cos\theta)(b\cos\theta) - (b\sin\theta)(-a\sin\theta)\,d\theta$$

$$= \frac{1}{2}ab\int_0^{2\pi} d\theta$$

$$= \pi ab$$

Example 7. Evaluate by Green's theorem, $\oint_c (y - \sin x\,dx + \cos x dy$ where c is the triangle enclosed by the lines $x = 0, x = \pi/2, xy = 2x$, $P = y - \sin x$, $Q = \cos x$.

Solution : Here $\frac{\delta P}{\delta y} = 1,$

$$\frac{\delta Q}{\delta x} = -\sin x$$

Hence by Green's theorem.

$$\oint_c (y - \sin x)dx + \cos x dy$$

$$= \iint_S (-1 - \sin x)\, dx\, dy$$

$$= \int_{x=0}^{\pi/2} \int_{y=0}^{2x/\pi} (1 + \sin x)\, dx\, dy$$

$$= \int_0^{2/\pi} (1 + \sin x) \frac{2x}{\pi} dx$$

$$= -\frac{2}{\pi} \int_0^{2/\pi} (x + x \sin x)\, dx$$

$$= (\pi/4 + 2/\pi)$$

Example 8. If $S \subset R^n$, prove that S is compact iff set S is closed and bounded.

Solution : Suppose S is closed and bounded by Heine Borel theorem there is an open covering F of S such that a finite subcollection of F also covers S, so S is compact by definition of compactness. Suppose S is compact to prove S is bounded, choose $\bar{x} \in S$, then $\{B(\bar{x}, K), K = 1, 2, \ldots\}$ a collection of open covering of S. Since S is compact, a finite subcollection also cover S and hence S is bounded.

To prove S is closed. Suppose S is not closed, there is an accumulation point y of S such that $y \notin S$. If $x \in S$ and let $r_x = \|x - y\|/2$. Then the collection $\{B(x, r_x) : x \in S\}$ is an open covering of S. Since S is compact there is a finite sub collection which covers.

i.e. $$S \subseteq \bigcup_{k=1}^{P} B(x_k, r_k)$$

Choose $$r = \min\{r_1, r_2, \ldots, r_P\}$$

Then for open ball $B(y, r)$.

If $$x \in B(y, r)$$

$$\Rightarrow \quad \|x - y\| < r \leq r_k \quad (\text{for some } r_k)$$

Also we have

$$\|y - x_k\| \leq \|y - x\| + \|x - x_k\|$$

$$\Rightarrow \quad \|x - x_k\| \geq \|y - x_k\| - \|x - y\|$$

$$= 2r_k - \|x - y\|$$

Example 9. Verify Stokes' theorem for $F = -y^3 i + x^3 j$, where j is the circular disc

$$x^2 + y^2 \leq 1,$$
$$z = 0$$

Solution : Given $F = y^3 i + x^3 j$,

The boundary C of S is a circle in xy. Plane,

$$x^3 + y^3 = 1,$$
$$z = 0$$

In parametric form

$$x = \cos\theta,$$
$$y = \sin\theta,$$
$$z = 0 \qquad \text{where } 0 \leq \theta \leq 2\pi$$

$$\therefore \int_C F.dr = \int_C F_1 dx + F_2 dy + F_3 dz$$

$$= \int_C (-y^3 dx + x^3 dy)$$

$$= \int_C [-\sin^3\theta(-\sin\theta) + \cos^3\theta \cos\theta]\, d\theta$$

$$= \int_0^{2/\pi} (1 - 2\sin^2\theta\cos^2\theta)\, d\theta,$$

$$= \int_0^{2/\pi} d\theta - 2\int_0^{2/\pi} \sin^2\theta\cos^2\theta\, d\theta$$

$$= 2\pi - 2(u)\int_0^{\pi/2} \sin^2\theta \,.\, \cos^2\theta\, d\theta$$

$$= 2\pi - 8 \,.\, \frac{1}{4} \,.\, \frac{1}{2} \,.\, \frac{\pi}{2}$$

$$= \frac{3\pi}{2}$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ -y^3 & x^3 & 0 \end{vmatrix}$$

$$= K(3x^2 + 3y^2)$$

$$\therefore \int (\nabla \times F).N ds = 3\int_S (x^2 + y^2) K.N ds$$

$$= 3\iint_R (x^2 + y^2)\, dx.dy \quad \ldots(i)$$

Since $(K.N)\, ds = dx \,.\, dy$

and R is the region of xy-Plane.

Put $$x = r\cos\phi,$$
$$y = r\sin\phi$$

$$\therefore \quad dx \,.\, dy = r\, dr d\phi$$

and v varies from 0 to 1

and $$0 \leq \phi \leq 2\pi$$

$$\therefore \int (\nabla \times F) N.ds = 3\int_{\phi=0}^{2\pi} \int_{r=0}^{1} r^2$$

$$= r dr d\bar{\phi}$$

$$= \frac{3\pi}{2}$$

Hence, the verification of the theorem.

Example 10. Find the nth term of the sequence $\{0, 1, 0, 1, \ldots\}$.

Solution : The first term of the sequence.

$$a_1 = \frac{1-1}{2} = 0$$

and second term of the sequence.

$$a_2 = \frac{1+(-1)^2}{2} = 1$$

Third term of the sequence

$$a_3 = \frac{1+(-1)^3}{2} = 0$$

Fourth term of the sequence

$$a_4 = \frac{1+(-1)^4}{2} = 1$$

$\therefore$ nth term of the sequence

$$a_n = \frac{1+(-1)^n}{2}$$

Example 11. The sequence $\left\{\dfrac{\sin \frac{n\pi}{2}}{n}\right\}_{n=1}^{\infty}$ is a convergent sequence.

Solution :

The sequence $\left\{\sin \dfrac{n\pi}{2}\right\}_{n=1}^{\infty}$ is bounded sequence

$$\because \qquad \left|\sin \frac{n\pi}{2}\right| \leq 1$$

The sequence $\dfrac{1}{n}$ is a convergent sequence and it converges to 0.

By the theorem : If $\{a_n\}_{n=1}^{\infty}$ converges to 0 and $\{b_n\}_{n=1}^{\infty}$ is bounded then $\{a_nb_n\}_{n=0}^{\infty}$ converges to 0, we have $\left\{\sin \dfrac{n\pi}{2} . \dfrac{1}{n}\right\}$ converges to 0.

Example 12. The series $\sum\limits_{n=1}^{\infty}\left(1+\dfrac{1}{\sqrt{n}}\right)^{-n}$ is convergent.

Solution : We have $a_n = \left(1+\dfrac{1}{\sqrt{n}}\right)^{-n}$

$$\therefore \qquad \sqrt{a_n} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

By Root test

$$\therefore \qquad \lim_{n\to\infty} \sqrt{(a_n)} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

$$= \frac{1}{e} < 1$$

Hence the series is convergent.

Example 13. Every Cauchy sequence is bounded.

Solution : Let sequence $<x_n>$ is a Cauchy sequence.

For $\in > 0,$

$\exists N : n \geq N,$

$m \geq N$

$\Rightarrow \quad |x_n - x_m| < \in$

$\because \quad |x_n - x_m| \leq |x_n| - |x_m|$

$\Rightarrow \quad |x_n| > |x_m| < \in$

$\Rightarrow \quad |x_n| < |x_m| + \in \qquad \ldots$(i)

Choose $\quad M = \min (|x_m|) + \in$

$\therefore$ By (i), we have, $\in > 0,$

$\exists N : n \geq N,$

$|x_n| < M$

The sequence $<x_n>$ is bounded.

Example 14. The open interval is an open set.

Solution : Let (a, b) bean open interval,

If$x\in$ (a, b), then $a < x < b$

Choose $\quad \delta = \min \{x - a, b - x\}$

Thus $\forall$ $x\in$ (a, b), $\delta > 0$, whenever

$|x - y| < \delta,$

$y \in (a, b)$

i.e. (a, b) is open set.

Example 15. The closed interval is a closed set.

Solution : Let x be a point of closure of $[a, b]$, then for every $\delta > 0$ there is

$y \in [a, b],$

such that $\quad |x - y| < \delta$

A closed interval $[a, b]$ is closed set if it contains all its points of closure

i.e. $\quad x \in [a, b]$

On the contrary,

Let $\quad x \notin [a, b]$

Choose $\quad \delta = \min \{b - y, y - a\}$

For every $\quad y \in [a, b]$

Such that $|x - y| < \delta$,

$x \in [a, b]$,

which is the contradiction

Thus $x \in [a, b]$

$\Rightarrow [a, b]$ contains all its closure points

$\Rightarrow [a, b]$ is closed set.

Example 16. Prove that the continuous image of a compact set is compact.

Solution : Let f be a continuous function mapping the compact set K onto a space Y. If μ is an open covering for Y, then the collection of sets $f^{-1}[0]$ for all $0 \in \mu$ is an open covering of K. By the compactness of K, there are a finite number 0_1 ......0_n of sets of μ such that the sets $f^{-1}[0_1]$ cover K. Since f is onto, the sets $0_1 \ldots 0_n$ cover Y. Thus the continuous image of a compact set is compact.

Example 17. Prove that a metric space X has the Bolzano-Weierstrass property iff X is sequentially compact.

Solution : Since every limit of a subsequence of $<x_n>$, sequential compactness implies the Bolzano-Weierstrass property. Conversely, if $<x_n>$ has x for a cluster point, then for each K we can find an $n_k > n_{k-1}$ such that the ball of radius 1/K about x contains x_{nk}. Then $x_{nk} \to x$. Thus a metric space with the Bolzano-Weierstrass property is sequentially compact.

Example 18. Prove that if E is a labesgue measurable set, then each translate E + y is also measurable.

Solution : Let E + y is lebesgue measurable, then for some set A.

$$m^* A = m^* [A \cap (E + y)] + m^* [A \cap (E + y)^c] \quad \ldots(1)$$

Since E is measurable, we have

$$m^* [A \cap (E + y)]] = m^* (A \cap (E + y)^c \cap E] + m^* [A \cap (E + y) \cap E^c]$$

$$m^* [A \cap (E + y)^c] = m^* [A \cap (E + y)^c \cap E] + m^* [A \cap (E + y)^c \cap E^c]$$

Adding These, we have

$$m^* (A \cap (E + y)] + m^* [A \cap (E + y)^c]$$
$$= m^* [A \cap E \cap (E + y)E] + m^* [A \cap E \cap (E + y)^c] + m^* [A \cap E^c \cap (E + y)] + m^* [A \cap E^c \cap (E + y)]$$
$$= m^* (A \cap E) + m^* (A \cap E^c) = m^* A.$$

This proves (E + y) is lebesgue measurable

OBJECTIVE TYPE QUESTIONS

1. The nth term of the sequence $\left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots 0 \right\}$ is—
   (A) $\frac{1}{n-1}$
   (B) $\frac{1}{2n}$
   (C) $\frac{1}{2(n+1)}$
   (D) None of these

2. The following sequence {2, 3, 5, 7, …} is a sequence of—
   (A) Real number
   (B) Prime number
   (C) Even number
   (D) Odd number

3. The nth term of the sequence {1, 1/2, 1, 1/3, 1, 1/4, …} is—
   (A) For n even $\frac{1}{n - \frac{n}{2}}$ for n odd 1
   (B) For n even $\frac{1}{n + \frac{n}{2} + 1}$ for n odd 1
   (C) For n even $\frac{1}{n - \frac{n}{2} + 1}$ for n odd 1
   (D) For n even $\frac{1}{n + \frac{n}{2}}$ for n odd 1

4. The nth term of the sequence $\left\{ 2, \frac{-3}{2}, \frac{4}{3}, \frac{-5}{4}, \ldots \right\}$ is—
   (A) $1 + 1/n$
   (B) $(-1)^{n-1} (1 - 1/n)$
   (C) $(-1)^{n-1} (1 + 1/n)$
   (D) None of these

5. The nth term of the sequence $\left\{ 1, \frac{5}{2}, \frac{5}{3}, \frac{9}{4}, \frac{9}{5}, \ldots \right\}$ is—

(A) $\frac{2n+(-1)^n}{2n}$ (B) $\frac{2n+(-1)^n}{n}$

(C) $\frac{2n+(1)^n}{n}$ (D) None of these

6. The sequence $\{0, 1, 0, 1/2, 0, 1/3, \ldots\}$ has the nth term—

(A) $\frac{1+(-1)^n}{n}$ (B) $\frac{n-(1)^n}{n}$

(C) $\frac{n+(1)^n}{n}$ (D) None of these

7. A monotone sequence $\{a_n\}_{n=1}^{\infty}$ is convergent—

(A) It is bounded
(B) It is unbounded
(C) It is decreasing
(D) None of these

8. If sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are divergent sequence, then—

(A) $\{a_n+b_n\}_{n=1}^{\infty}$ is always divergent
(B) $\{a_n+b_n\}_{n=1}^{\infty}$ is always convergent
(C) $\{a_n+b_n\}_{n=1}^{\infty}$ sometimes convergent
(D) None of these

9. If sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergetn then—

(A) $\{a_n+b_n\}_{n=1}^{\infty}$ is always convergent
(B) $\{a_n+b_n\}_{n=1}^{\infty}$ is always divergent
(C) $\{a_n+b_n\}_{n=1}^{\infty}$ is sometimes divergent
(D) None of these

10. The sequence $\left\{\frac{\cos\frac{n\pi}{2}}{\mathrm{n}}\right\}_{n=1}^{\infty}$ is—

(A) Convergent to 0 (B) Divergent
(C) Convergent to 1 (D) None of these

11. (a) Every bounded sequence is convergent.
(b) Every convergent sequence is bounded

(A) (a) and (b) are true
(B) (a) is true, (b) is false
(C) (b) is true, (a) is false
(D) (a) and (b) are false

12. The sequence $\{1, 0, 1, 0, 1, 0, \ldots\ldots\}$ is—

(A) Increasing sequence
(B) Decreasing sequence
(C) Monotone sequence
(D) None of these

13. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequence such that $\{a_n\}_{n=0}^{\infty}$ and $\{a_n\, b_n\}_{n=0}^{\infty}$ converges respectively to A and AB, then $\{b_n\}_{n=0}^{\infty}$ converges iff—

(A) $\mathrm{A} \neq 0$ (B) $\mathrm{A} = 0$
(C) $\mathrm{B} = 0$ (D) None of these

14. If $\{b_n\}_{n=1}^{\infty}$ is an increasing bounded sequence, then for the sequence $\{b_n\}_{n=1}^{\infty}$ is following statement is false.

(A) $\{b_n\}_{n=1}^{\infty}$ is a convergent sequence
(B) $\{b_n\}_{n=1}^{\infty}$ is a divergent sequence
(C) $\{b_n\}_{n=1}^{\infty}$ is a monotone sequence
(D) $\{b_n\}_{n=1}^{\infty}$ is a Cauchy sequence

15. If $\{a_n\}_{n=0}^{\infty}$ converges to a, for all n, $a \geq 0$, then $\left\{\sqrt{a_n}\right\}_{n=0}^{\infty}$ is—

(A) Converges to $\sqrt{a}$
(B) Diverges to $\sqrt{a}$
(C) Converges to a
(D) Diverges to a

16. If $\{a_n\}_{n=0}^{\infty}$ converges to A, then—

(A) $\{|a_n|\}_{n=0}^{\infty}$ converges to A
(B) $\{|a_n|\}_{n=0}^{\infty}$ converges to $|\mathrm{A}|$
(C) $\{|a_n|\}_{n=0}^{\infty}$ divergent sequence
(D) None of these

17. A sequence $\{a_n\}_{n=1}^{\infty}$ is bounded iff there is a real number S such that—

(A) $|a_n| \leq \mathrm{S}$ for all n (B) $|a_n| \geq \mathrm{S}$ for all n
(C) $|a_n| = \mathrm{S}$ for all n (D) None of these

18. A sequence $\{a_n\}_{n=1}^{\infty}$ is bounded from below if for real number R—

(A) $a_n \leq \mathrm{R}$ for all n (B) $a_n \geq \mathrm{R}$ for all n
(C) $a_n = \mathrm{R}$ for all n (D) None of these

19. If $\{a_n\}_{n=1}^{\infty}$ converges to A and B both, then—

(A) $\mathrm{A} > \mathrm{B}$ (B) $\mathrm{A} = \mathrm{B}$
(C) $\mathrm{A} \leq \mathrm{B}$ (D) None of these

20. If a sequence $\{a_n\}_{n=1}^{\infty}$ converges to a real number A, then $\{a_n\}_{n=1}^{\infty}$ is—
(A) Unbounded sequence
(B) Bounded sequence
(C) Divergent sequence
(D) None of these

21. A sequence $\{a_n\}_{n=1}^{\infty}$ is bounded from above if for real number R—
(A) $a_n \geq$ R for all n
(B) $a_n \leq$ R for all n
(C) $a_n =$ R for all n
(D) None of these

22. If a sequence is not a Cauchy sequence y, then it is a—
(A) Divergent sequence
(B) Convergent sequence
(C) Bounded sequence
(D) None of these

23. (a) Every convergent sequence is a Cauchy sequence
(b) Every Cauchy sequence is a convergent sequence
(A) (a) and (b) both are false
(B) (a) is true
(C) (b) is true
(D) (a) and (b) both are true

24. Every Cauchy sequence is—
(A) Unbounded sequence
(B) Bounded sequence
(C) Divergent sequence
(D) None of these

25. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence converges to 0 and $\{b_n\}_{n=1}^{\infty}$ be a sequence that is bounded, then $\{a_n\, b_n\}_{n=1}^{\infty}$ is a sequence that—
(A) Converges to one
(B) Converges to zero
(C) Is divergent sequence
(D) None of these

26. Let sequence $\{a_n\}_{n=1}^{\infty}$ converges A and sequence $\{b_n\}_{n=1}^{\infty}$ converges to B, with $a_n \leq b_n$ for all n, then—
(A) A $\leq$ B (B) A = B
(C) A $\geq$ B (D) None of these

27. Let sequence $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ converges to A and B respectively, then $\{a_n/b_n\}_{n=1}^{\infty}$ converges to A/B if—
(A) $b_n \neq 0$ for all n and B = 0
(B) $b_n \neq$ for some n
(C) $b_n \neq 0$ for all n and B $\neq$ 0
(D) None of these

28. If $\{a_n\}_{n=1}^{\infty}$ is decreasing and bounded, then $\{a_n\}_{n=1}^{\infty}$ is—
(A) Convergent sequence
(B) Divergent sequence
(C) Non-Cauchy sequence
(D) None of these

29. Every Cauchy sequence is—
(A) Monotone sequence
(B) Divergent sequence
(C) Unbounded sequence
(D) None of these

30. A series $\sum_{n=1}^{\infty} a_n$ converges, absolutely iff—
(A) $\sum_{n=1}^{\infty} |a_n|$ converges
(B) $\left|\sum_{n=1}^{\infty} a_n\right|$ converges
(C) $\sum_{n=1}^{\infty} |a_n|$ diverges
(D) None of these

31. A series $\sum_{n=1}^{\infty} a_n$ converges, then sequence $\{a_n\}_{n=1}^{\infty}$ is—
(A) Diverges
(B) Converges to zero
(C) Converges to any number
(D) None of these

32. For infinite series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, $b_n \geq 0$ for n, and there is a real number N, such that for $n \geq$ N $\Rightarrow |a_n| \leq b_n$. If $\sum_{n=1}^{\infty} b_n$ converges, then—

(A) $\sum_{n=1}^{\infty} b_n$ is absolutely convergent

(B) $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

(C) $\sum_{n=1}^{\infty} a_n$ is absolutely divergent

(D) $\sum_{n=1}^{\infty} b_n$ is absolutely divergent

33. For real number α, if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges to A and B respectively, then $\sum_{n=1}^{\infty} (a_n + \alpha b_n)$—
(A) Converges to A + αB
(B) Diverges
(C) Converges to αA + B
(D) None of these

34. The series $\sum_{n=1}^{\infty} [(-1)^n/(2n-1)]$ is—
(A) Convergent
(B) Divergent
(C) Unbounded
(D) None of these

35. The series 2 + 4 + 6 + 8 + …… is—
(A) Divergent (B) Convergent
(C) Unbounded (D) None of these

36. The series $1 + \frac{1}{2^2} + \frac{1}{3^3} + \ldots$ is—
(A) Divergent (B) Convergent
(C) Bounded (D) None of these

37. The series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is—
(A) Convergent (B) Divergent
(C) Unbounded (D) None of these

38. The series $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ is—
(A) Divergent (B) Convergent
(C) Unbounded (D) None of these

39. The series 1 + 3 + 5 + 7 + … is—
(A) Divergent (B) Convergent
(C) Unbounded (D) None of these

40. The series $\sum_{n=1}^{\infty} (-1)^n$ is—
(A) Divergent (B) Convergent
(C) Unbounded (D) None of these

41. The series $1^3 + 2^3 + 3^3 + \ldots$ is—
(A) Divergent (B) Convergent
(C) Bounded (D) None of these

42. The series $\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \ldots$ is—
(A) Convergent (B) Divergent
(C) Decreasing (D) None of these

43. The sequence $\{(-1)^n\}_{n=1}^{\infty}$ is—
(A) Bounded and convergent
(B) Convergent and unbounded
(C) Bounded and divergent
(D) Divergent and unbounded

44. The sequence $\{1/n\}$ is—
(A) Unbounded and convergent
(B) Bounded and convergent
(C) Bounded and divergent
(D) Divergent and unbounded

45. The sequence $\{(-1)^n/n\}$ converges to—
(A) Zero (B) 1
(C) 2 (D) None of these

46. A monotone increasing is bounded, then—
(A) It is divergent (B) It is convergent
(C) It is constant (D) None of these

47. Every Cauchy sequence is—
(A) Bounded (B) Unbounded
(C) Divergent (D) None of these

48. Every Cauchy sequence is—
(A) Unbounded (B) Convergent
(C) Divergent (D) None of these

49. If sequence $\{a_n\}_{n=1}^{\infty}$ converges to A and $\{b_n\}$ converges to B, then for all $a_n \geq b_n$ we have—
(A) A $\geq$ B (B) A $\leq$ B
(C) A = B (D) None of these

50. The sequence $\{1, 2, 3, \dots\}$ is—
(A) Bounded below
(B) Bounded above
(C) Bounded
(D) None of these

51. The sequence $\{-1, -2, -3, \dots\}$ is—
(A) Bounded below (B) Bounded above
(C) Bounded (D) None of these

52. The sequence $\left\{\frac{n+1}{n}\right\}$ is—
(A) Bounded (B) Bounded above
(C) Unbounded (D) None of these

53. The sequence $\left\{\frac{n}{n+1}\right\}$ is—
(A) Decreasing sequence
(B) Bounded
(C) Unbounded
(D) None of these

54. The sequence $\left\{\frac{1}{3^n}\right\}$ is—
(A) Divergent sequence
(B) Bounded
(C) Unbounded
(D) None of these

55. The sequence $\left\{\frac{(-1)^n}{n}\right\}$ is—
(A) Divergent sequence
(B) Bounded
(C) Unbounded
(D) None of these

56. The sequence $\left\{\frac{n+1}{n}\right\}$ is—
(A) Increasing sequence
(B) Decreasing sequence
(C) Unbounded
(D) None of these

57. The sequence $\left\{\frac{n}{n+1}\right\}$ is—
(A) Increasing sequence
(B) Decreasing sequence
(C) Unbounded
(D) None of these

58. The sequence $\left\{\frac{(-1)^n}{n}\right\}$ is—
(A) Unbounded (B) Decreasing
(C) Increasing (D) None of these

59. The sequence $\{1, 0, 1, 0, 1, 0, \dots\}$ is—
(A) Increasing (B) Decreasing
(C) Bounded (D) None of these

60. The sequence $\left\{a + \frac{(-1)^n b}{n}\right\}$—
(A) Bounded (B) Unbounded
(C) Divergent (D) None of these

61. The series $\sum_{n=1}^{\infty} \frac{1}{n^n}$ is—
(A) Convergent (B) Divergent
(C) Oscillatory (D) None of these

62. The series $\sum_{n=1}^{\infty} \frac{1}{(1 + 1/n)^{n^2}}$ is—
(A) Convergent (B) Divergent
(C) Oscillatory (D) None of these

63. The series $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \dots$—
(A) Convergent (B) Divergent
(C) Oscillatory (D) None of these

64. The series $1 + x + \frac{x^2}{2!} + \dots\dots + \frac{x^n}{n!} + \dots$—
(A) Convergent (B) Divergent
(C) Oscillatory (D) None of these

65. The series $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$, for $|x| > 1$ is—
(A) Convergent (B) Divergent
(C) Oscillatory (D) None of these

66. A polynomial for real values of x is—
(A) Continuous (B) Discontinuous
(C) Convergent (D) None of these

67. A function $f(x) = \frac{x^2 - 1}{x - 1}$ is—
(A) Discontinuous at $x = 1$
(B) Discontinuous at $x = 2$
(C) Continuous at $x = 1$
(D) Continuous at $x = 2$

68. A function $f(x) = |x|$ is—
(A) Discontinuous
(B) Discontinuous at $x = 0$
(C) Continuous everywhere
(D) None of these

69. A function $f(x) = |x|$ is—
(A) Continuous and differentiable at $x = 0$
(B) Continuous but not differentiable at $x = 0$
(C) Discontinuous and differentiable at $x = 0$
(D) None of these

70. Let $f : E \to R$ and $x_0 \in E$ and x_0 an accumulation point of E if every sequence $\{x_n\}_{n=1}^{\infty}$ converses to x_0 with $x_n \in E$ for all n $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x_0)$, then—
(A) f is discontinuous at x_0
(B) f is discontinuous at x_0
(C) f is differentiable at x_0
(D) None of these

71. $f : E \to Q$ and $g : E \to R$ and both are continuous at x_0, then—
(A) f/g is continuous at x_0
(B) f/g is discontinuous at x_0
(C) f/g is continuous at x_0 if $g(x_0) \neq 0$
(D) None of these

72. Statement A : f is continuous
Statement B : f is differentiable
(A) If A is true, then B is true
(B) If A is false, then B is false
(C) If B is true, then A is true
(D) If B is true, then A is false

73. A set $E \subset R$ is compact—
(A) It is closed and bounded
(B) It is open and bounded
(C) It is open and unbounded
(D) It is closed and unbounded

74. The function $f(x)$ is continuous on closed interval $[a, b]$, then—
(A) f is bounded on $[a, b]$
(B) f is unbounded on $[a, b]$
(C) f is constant on $[a, b]$
(D) None of these

75. The function $f(x)$ is continuous on closed interval $[a, b]$, and $f(a) \cdot f(b) < 0$ if at some point c, $f(c) = 0$, then—
(A) $c \notin [a, b]$
(B) $c \in [a, b]$
(C) $c \in [a, b]$
(D) None of these

76. The function $f(x)$ is continuous on closed interval $[a, b]$ and $m = \min f(x)$ and $M = \max f(x)$. If for any A, $m \leq A \leq M$ there is $x_0 \in [a, b]$ for which—
(A) $f(x_0) = A$
(B) $f(x_0) = 0$
(C) $f(x_0) \neq A$
(D) None of these

77. A real number set E is compact, if—
(A) E is bounded and open
(B) E is unbounded and closed
(C) E is bounded and closed
(D) E is unbounded and open

78. If f_1 and f_2 are two real valued bounded functions defined on $[a, b]$, then for every partition P on $[a, b]$—
(A) $U(P, f_1 + f_2) = U(P, f_1) + U(P, f_2)$
(B) $U(P, f_1 + f_2) \leq U(P, f_1) + U(P, f_2)$
(C) $U(P, f_1 + f_2) \geq U(P, f_1) + U(P, f_2)$
(D) None of these

79. Let Lower Riemann Integral be $f(x)$ on $[a, b]$ and $L(P, f)$ is the Lower Riemann sum over all partitions on $[a, b]$ then—
(A) $\int_{-a}^{b} f(x)\, dx = i.u.b\ \{L(P, f)\}$
(B) $\int_{-a}^{b} f(x)\, dx = g.l.b\ \{L(P, f)\}$
(C) $\int_{-a}^{b} f(x)\, dx = L(P, f)$
(D) None of these

80. If f is a Riemann integrable function on $[a, b]$ and λ is any real number then following holds—
(A) $\lambda \int_{a}^{b} f(x)\, dx = \int_{a}^{b} \lambda f(x)\, dx$
(B) $\lambda \int_{a}^{b} f(x)\, dx \neq \int_{a}^{b} \lambda f(x)\, dx$
(C) $\lambda \int_{a}^{b} f(x)\, dx \geq \int_{a}^{b} \lambda f(x)\, dx$
(D) None of these

81. A real valued bounded function $f(x)$ is Riemann integrable on $[a, b]$, then—
(A) $\int_{-a}^{b} (x)\, dx$ and $\int_{a}^{-b} (x)\, dx$ exist
(B) $\int_{-a}^{b} (x)\, dx = \int_{a}^{-b} f(x)\, dx$
(C) $\int_{a}^{-b} f(x)\, dx \neq \int_{-a}^{b} f(x)\, dx$
(D) None of these

82. If $\int_{-a}^{b} f\, dx$ and $\int_{a}^{-b} dx$ are lower and upper Riemann integrable on $[a, b]$ then—
(A) $\int_{-a}^{b} f\, dx \geq \int_{a}^{-b} f\, dx$
(B) $\int_{-a}^{b} f dx = \int_{a}^{-b} f\, dx$
(C) $\int_{-a}^{b} f\, dx \leq \int_{a}^{-b} f dx$
(D) None of these

83. If f is real valued bounded function on $[a, b]$ and m, M are greatest lower bound and least upper bound respectively, then—
(A) $m(b-a) = \mathrm{M}(b-a)$
(B) $m(b-a) \geq \mathrm{M}(b-a)$
(C) $m(b-a) \leq \mathrm{M}(b-a)$
(D) None of these

84. The radius of convergence of the series
$$1 - \frac{x}{1} + \frac{x^2}{2!} - \frac{x^3}{3!}$$
is—
(A) ∞ (B) Zero
(C) 1 (D) None of these

85. The radius of convergence of the series
$$\frac{x+0{\cdot}2}{1} + \frac{(x+0{\cdot}2)^2}{2} + \ldots\ldots + \frac{(x+0{\cdot}2)^n}{n} + \ldots$$
(A) 1 (B) ∞
(C) Zero (D) None of these

86. The series $1{\cdot}x + 1{\cdot}2x^2 + 1{\cdot}2{\cdot}3x^3 + \ldots + \underline{|n}\ x^n + \ldots$ is—
(A) Divergent everywhere except $x = 0$
(B) Convergent everywhere except $x = 0$
(C) Divergent for $x = 0$
(D) None of these

87. The radius of convergence of the series $1 - x^2 + x^4 - x^6 + \ldots\ldots$ is—
(A) 1 (B) Zero
(C) 2 (D) None of these

88. The series $1 + x + x^2 + \ldots + x^n + \ldots$
(A) Converges for $|x| < 1$
(B) Diverges for $|x| > 1$
(C) Converges for $(x) = 1$
(D) None of these

89. The series $1 + \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \ldots\ldots$
(A) Diverges for $|x| > 1$
(B) Converges for $|x| < 1$
(C) Diverges for $|x| = 1$
(D) None of these

90. The domain of convergence for $1 + x + x^2 + \ldots$ is—
(A) $[-1, +1]$ (B) $[-1, +1]$
(C) $[-2, 2]$ (D) None of these

91. The domain of convergence for $1 + \frac{x}{1^2} + \frac{x^2}{2^2} + \ldots$ is—
(A) $[-1, +1]$ (B) $[-1, +1]$
(C) $[0, 1]$ (D) None of these

92. The domain of convergence for
$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$
is—
(A) $(-1, 1)$ (B) $(-1, 1)$
(C) $(-1, 2)$ (D) $(-1, 1)$

93. If $f : [a, b] \to \mathrm{R}$ is continuous and monotone function, then—
(A) f is Riemann integrable on $[a, b]$
(B) f is not Riemann integrable on $[a, b]$
(C) f is Riemann integrable on R
(D) None of these

94. If f_1 and f_2 are Riemann integral functions, then $\int_{a}^{b} f_1 dx + \int_{a}^{b} f_2 dx$ is equal to—
(A) $\int_{b}^{a} (f_1 + f_2)\, dx$
(B) Zero
(C) $\int_{b}^{a} (f_1 + f_2)\, dx$
(D) None of these

95. If f_1 is Riemann integrable, then

$$\int_b^a f_1\,dx + \int_b^c f_1\,dx$$

is—

(A) $\int_a^b f_1 dx + c$ (B) $\int_b^c f_1 dx + c$

(C) $\int_a^c f_1 dx$ (D) None of these

96. $f : [a, b] \to R$, f is Riemann integrable then—

(A) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

(B) $\int_a^b |f(x)|\,dx \le \left|\int_a^b f(x)dx\right|$

(C) $\int_a^b |f(x)|\,dx = \left|\int_a^b f(x)\,dx\right|$

(D) None of these

97. $f : [a, b] \to R$, P and Q are partitions of $[a, b]$ such $P \subset Q$, then—

(A) $L(P, f) \le (Q, f)$

(B) $L(P, f) \ge L(Q, f)$

(C) (A) is true (B) is false

(D) (B) is true (A) is false

98. A function f is Riemann integrable on $[a, b]$ if—

(A) Only $\int_a^{-b} f\,dx$ exist

(B) Only $\int_{-a}^{b} f\,dx$ exist

(C) $\int_a^{-b} f\,dx \neq \int_{-a}^{b} f\,dx$

(D) $\int_a^{-b} f\,dx = \int_{-a}^{b} f\,dx$

99. If f is Riemann integrable with respect to α on $[a, b]$, then—

(A) f is increasing and α is bounded function

(B) f is bounded and α is increasing function

(C) f and α are both bounded

(D) f and α are both increasing

100. $f : [0, 1] \to R$ such that $f(x) = \begin{cases} 0 & x \text{ is rational} \\ 1 & x \text{ is irrational} \end{cases}$

then—

(A) The upper and lower integral of f does not exist

(B) f is riemann integrable

(C) f is not riemann integrable

(D) None of these

101. $f : [a, b] \to R$ is continuous, then f is Riemann integrable on $[a, b]$—

(A) The statement is true

(B) The statement is false

(C) Neither true nor false

(D) Partially true

102. Given a function $f(x) = |x|$ for all $x \in R$. The function f is—

(A) Differentiable at zero

(B) Continuous at zero

(C) (A) is true (B) is false

(D) (A) and (B) are both true

103. A rational function (The quotient of two polynomial functions) is—

(A) Differentiable everywhere

(B) Not differentiable any where

(C) Differentiable at each point, where deno-minator is non zero

(D) None of these

104. Let $f : D \to R$. A point $x_0 \in D$ is a relative minimum of f, iff there is a neighbourhood Q of x_0 such that $x \in Q \cap D$, then—

(A) $f(x) \le f(x_0)$ (B) $f(x) \ge f(x_0)$

(C) $f(x) \not\le f(x_0)$ (D) $f(x) \not\ge f(x_0)$

105. Let $f : [a, b] \to R$ is continuous on $[a, b]$ and f is differentiable on (a, b), if $f(a) = f(b) = 0$ there exist c such that $f'(c) = 0$, then—

(A) $c \in [a, b]$ (B) $c \in (a, b)$

(C) $c \in [a, b)$ (D) $c \in (a, b]$

106. The set of positive integers is—

(A) Bounded above

(B) Unbounded above

(C) Unbounded below

(D) None of these

107. If $x \in R$, set of real numbers then—

(A) $\infty < x < -\infty$ (B) $-\infty \le x \le \infty$

(C) $-\infty < x < \infty$ (D) $-\infty \ge x \ge \infty$

108. The value of $(\infty - \infty)$ is—

(A) 0 (B) ∞

(C) $-\infty$ (D) Undefined

109. The set of real number is—
(A) Bounded above (B) Unbounded above
(C) Finite set (D) Countable set

110. If R* is an extended real number system, then the least upper bound of R* is—
(A) $+\infty$
(B) $-\infty$
(C) 0
(D) No least upper bound

111. If S is non-empty set of real numbers and S is unbounded below then—
(A) Inf S $= -\infty$ (B) Inf S $= +\infty$
(C) Sup S $= \infty$ (D) Sup S $= -\infty$

112. If R is a set of real numbers, $x, y, \in$ R, then—
(A) $x + y \neq y + x$ (B) $x + y = y + x$
(C) $xy \neq xy$ (D) $x + y \geq y + x$

113. If R is a set of real numbers, $x \in$ R, there exist $-x \in$ R—
(A) $x + (-x) = 0$ (B) $x + (-x) = 1$
(C) $x(-x) = 0$ (D) $x(-x) = 1$

114. If R is a set of real number $x \in$ R, then—
(A) $x + x^{-1} = 0$ (B) $x + x^{-1} = 1$
(C) $xx^{-1} = 1$ (D) $xx^{-1} = 0$

115. If R is a set of real numbers $x, y \in$ R, then—
(A) $x > 0$ and $y > 0 \Rightarrow xy > 0$
(B) $x > 0$ and $y > 0 \Rightarrow xy < 0$
(C) $x > 0$ and $y > 0 \Rightarrow xy = 0$
(D) $x > 0$ and $y > 0 \Rightarrow xy = 1$

116. If A and B are two non empty set of R if C = $\{x + y : x \in A, y \in B\}$, then—
(A) Inf C = inf A + inf B
(B) Inf C $\neq$ inf A + inf B
(C) Inf C < inf A + inf B
(D) Inf C > inf A + inf B

117. If $x, y, z \in$ R the set of real number, then following relation not holds—
(A) $x > 0$ and $y > 0$
(B) $x, y > 0 \Rightarrow x + y > 0$
(C) $x, y, z > 0 \Rightarrow x + y + z > 0$
(D) $xy > 0 \Rightarrow x - y > 0$

118. Given a real number a and b such that—
$a \leq b + \in, \ \forall \in > 0$
(A) $a = b$ (B) $a \leq b$
(C) $a \geq b$ (D) $ab = \in$

119. If $x, y, z \in$ R and $x < y$, then—
(A) $xz < yz, z > 0$
(B) $xz < yz, z < 0$
(C) $xz < yz, z = 0$
(D) $xz < yz$, z is positive, negative or zero

120. If $x, y, z \in$ R and $x < y$, then—
(A) $xz > yz, z < 0$
(B) $xz > yz, z > 0$
(C) $xz > yz, z = 0$
(D) $xz > yz$, z is positive, negative or zero

121. If $w, x, y, z \in$ R, the set of real numbers and $x > y, z > w$, then—
(A) $xz > yw$ if $y, w > 0$
(B) $xz > yw$ if $y > 0, w < 0$
(C) $xz > yw$ if $y < 0, w > 0$
(D) $xz > yw$ if $y, w < 0$

122. The set of integers is—
(A) Ordered set
(B) Non-ordered set
(C) Set of irrational numbers
(D) Does not satisfies principle of induction

123. Composite number n is—
(A) A prime number and $n > 1$
(B) Non prime number and $n < 1$
(C) Non prime number and $n > 1$
(D) A prime number and $n < 1$

124. An integer n is a prime number if—
(A) $n > 1$ and n is divisible by itself only
(B) $n < 1$ and n is divisible by itself only
(C) $n = 1$
(D) $n = 1$ and n/n

125. Every integer $n > 1$ is—
(A) Prime number
(B) Prime number or product of prime numbers
(C) Product of prime numbers
(D) Sum of prime numbers

126. For open interval (a, b), where $a < b, x \in (a, b)$ we have—
(A) $a \leq x \leq b$ (B) $a < x < b$
(C) $a \leq x < b$ (D) $a < x \leq b$

127. For closed interval $[a, b]$, where $a < b, x \in [a, b]$ we have—
(A) $a \leq x \leq b$ (B) $a < x \leq b$
(C) $a \leq x < b$ (D) $a < x < b$

128. If P is a prime number and P | ab, then—
(A) P|a of P|b (B) P + ab = 0
(C) P – ab = 0 (D) ab|P

129. If S is a set of real numbers, and c_1 and c_2 are two least upper bounds of S, then—
(A) $c_1 = c_2$ (B) $c_1 > c_2$
(C) $c_1 < c_2$ (D) $c_1 \neq c_2$

130. If S is a set of real numbers, c is a least upper bound and b is any upper bound of S, then—
(A) $c \leq b$ (B) $c \geq b$
(C) $c \neq b$ (D) $c > b$

131. The set $R^+ = (0, +\infty)$ has—
(A) No upper bound
(B) Upper bound
(C) Least upper bound
(D) None of these

132. The Set $R^+ = [0, 1]$ has—
(A) Upper bound only
(B) Lower bound only
(C) Upper and lower bound both
(D) No upper and lower bound

133. An element a is an minimal element of set S, then—
(A) $a \in S$
(B) $a \notin s$
(C) a is not lower bound of S
(D) a is not upper bound of S

134. The set of negative integers have the least upper bound—
(A) 1 (B) 2
(C) 0 (D) – 1

135. The non-empty set of real numbers which is bounded below has—
(A) Supremum (B) Infimum
(C) Upper bound (D) No lower bound

136. For every real number x, there is a positive integer n—
(A) $n > x$ (B) $n < x$
(C) $n = x$ (D) None of these

137. If S is a set of real numbers, c_1 and c_2 are two greatest lower bound of S, then—
(A) $c_1 > c_2$ (B) $c_1 < c_2$
(C) $c_1 = c_2$ (D) $c_1 \neq c_2$

138. If S is a set of real number, c is the greatest lower bound and b is any lower bound, then—
(A) $c < b$ (B) $c > b$
(C) $c \geq b$ (D) $c \leq b$

139. The set of positive integer have the greatest lower bound—
(A) 1 (B) 0
(C) ∞ (D) None of these

140. The set of positive real numbers is—
(A) Bounded above
(B) Bounded below
(C) Unbounded from below
(D) None of these

141. The non empty set S of real numbers which is bounded above has—
(A) Supremum (B) Infimum
(C) Lower bound (D) No upper bound

142. The set $R^+ = (0, +\infty)$ is—
(A) Bounded above
(B) Unbounded above
(C) Unbounded below
(D) None of these

143. The closed interval (0, 1) have—
(A) Maximal element only
(B) Minimal element only
(C) Both maximal and minimal element
(D) None of these

144. If an element is maximal element of Set S, then—
(A) $a \in S$
(B) $a \notin S$
(C) a is not upper bound of S
(D) a is lower bound of S

145. The set of natural numbers has—
(A) Upper bound
(B) Lower bound
(C) Maximal element
(D) None of these

146. The set $R^+ = (0, \infty)$ is—
(A) Bounded above
(B) Bounded below
(C) Unbounded below
(D) None of these

147. The set $R^+ = (0, \infty)$ has—
(A) Minimal element
(B) No minimal element
(C) Least upper bound
(D) Maximal element

148. The closed interval $S = [0, 1]$ is—
(A) Bounded above
(B) Unbounded below
(C) Unbounded above
(D) No maximal element

149. The half interval [0, 1] is—
(A) Bounded above only
(B) Bounded below only
(C) Bonded above and below both
(D) None of these

150. The half interval [0, 1] have—
(A) Maximal element only
(B) Minimal element only
(C) Maximal and minimal both elements
(D) No maximal and no minimal element

151. Let S = [0, 1], the maximal element S is—
(A) 0 (B) 1
(C) ϕ (D) 2

152. Let S = [0, 1], the least upper bound for S is—
(A) 0 (B) 1
(C) ϕ (D) 2

153. Let S = [0, 1], the maximal element of S is—
(A) 0
(B) 1
(C) No maximal element
(D) ϕ

154. If $S, T \subset R$ and $\forall s \in S$ and $t \in T$, $s \le t$, it S and T have supremum, then—
(A) Sup S ≤ Sup T
(B) Sup S ≥ Sup T
(C) Sup S = Sup T
(D) None of these

155. If $S, T \subset R$ and $\forall s \in S, t \in T, s \le t$ if S and T have infimum, then—
(A) Inf S ≥ Inf T (B) Inf S ≤ Inf T
(C) Inf S = Inf T (D) None of these

156. If S is a non empty set and S has no upper bound, then—
(A) Sup S = ∞ (B) Inf S = ∞
(C) Sup S = – ∞ (D) Inf S = – ∞

157. The set of real numbers is—
(A) Unbounded
(B) Bounded from below
(C) Bounded from above
(D) Bounded

158. The completeness aniam states—
(A) Every non-empty set S of real number which is bounded above has supremum
(B) Every non empty set S of real number which is bounded above has infimum
(C) Every non empty set S of real number which is bounded have no supremum
(D) Every non empty set S of real number which is bounded have no infimum

159. If $0 < x < y$ and $0 < z < w$, then—
(A) $xz < yw$ (B) $xz > yw$
(C) $xz < y$ (D) $xz < w$

160. If $x < y < 0$ and $0 < z < w$, then—
(A) $xz < yw$ (B) $xz > yw$
(C) $xz < y$ (D) $xz < w$

161. The value of (0, ∞) is—
(A) ∞ (B) 0
(C) Undefined (D) None of these

162. If $\mathbb{R}^*$ is an extended real number system then the inf $\mathbb{R}^*$ is—
(A) 0 (B) + ∞
(C) – ∞ (D) No infimum

163. If $S, T \subset \mathbb{R}$ and $S \le t \ \forall s \in S$ and $t \in T$, then—
(A) Inf T ≤ Inf S (B) Inf T ≥ Inf S
(C) Inf T = Inf S (D) None of these

164. If $S, T \subset \mathbb{R}$ and $A = \{x + y : x \in S$ and $y \in T\}$, then—
(A) Sup A = Sup S + Sup T
(B) Sup A > Sup S + Sup T
(C) Sup A < Sup S + Sup T
(D) Sup A ≠ Sup S + Sup T

165. "Every nonempty set S of real number which is bounded below, has infimum" is stated in—
(A) Field aniom

(B) Ordered axiom
(C) Completeness axiom
(D) None of these

166. If $x, y, z \in \mathbb{R}$, then—
(A) $x(y + z) = xy + yz$
(B) $x(y + z) = xy + yz$
(C) $x(y + z) = xyz$
(D) $xy = xz, y \neq z$

167. If the sequence, $\{x_n\}$ of real numbers have limits l_1 and l_2, then—
(A) $l_1 > l_2$ (B) $l_1 < l_2$
(C) $l_1 = l_2$ (D) $l_1 \neq l_2$

168. Every Cauchy sequence of real numbers is—
(A) Convergent
(B) Divergent
(C) Limit does not exist
(D) None of these

169. If a sequence of real number has a cluster points, then—
(A) It is convergent
(B) It s divergent
(C) Limit exist
(D) Existence of limit not definite

170. If a sequence of real numbers has a limit, then—
(A) Cluster point exist
(B) Cluster point does not exist
(C) Sequence is divergent
(D) None of these

171. (a)Sequence $<x_n>$ is convergent
(b)Sequence $<x_n>$ is bounded
(A) $A \Rightarrow B$ (B) $B \Rightarrow A$
(C) $A \Leftrightarrow B$ (D) None is true

172. (a)Cauchy sequence is convergent
(b)Cauchy sequence is bounded.
(A) (a) and (b) both true
(B) (a) is false
(C) (b) is false
(D) (a) and (b) both false

173. For the sequence $<(-1)^n>$, $\overline{\lim}\, x_n$ is equal to—
(A) -1 (B) $+1$
(C) $-\infty$ (D) $+\infty$

174. For the sequence $<(-1)^n n>$, $\overline{\lim}\, x_n$ is equal to—
(A) -1 (B) $+1$
(C) $-\infty$ (D) $+\infty$

175. For the sequence $<x_n>$ where $x_n = (-1)^n \left(1 + \frac{1}{n}\right)$, $\overline{\lim}\, x_n$ is equal to—
(A) -1 (B) $+1$
(C) $-\infty$ (D) $+\infty$

176. The set of real numbers $\mathbb{R}$ is—
(A) Countable (B) Uncountable
(C) Infinite (D) Bounded

177. Following inequality is false—
(A) $\underline{\lim}\, x_n + \overline{\lim}\, y_n \leq \overline{\lim}\, x_n + \overline{\lim}\, y_n$
(B) $\lim x_n + \overline{\lim}\, y_n \leq \overline{\lim}\, (x_n + y_n)$
(C) $\underline{\lim}\, x_n + \overline{\lim}\, y_n \leq \underline{\lim}\, x_n + \underline{\lim}\, y_n$
(D) $\underline{\lim}\, x_n + \overline{\lim}\, y_n \geq \lim x_n + \lim y_n$

178. Following statement is true—
(A) $\underline{\lim}\, x_n \leq \overline{\lim}\, x_n$ (B) $\overline{\lim}\, x_n \leq \underline{\lim}\, x_n$
(C) $\underline{\lim}\, x_n \geq \lim x_n$ (D) $\overline{\lim}\, x_n \leq \lim x_n$

179. If limit $x_n = l$ exist, then—
(A) $\lim x_n = \overline{\lim}\, x_n = l$
(B) $\underline{\lim}\, x_n \neq \overline{\lim}\, x_n = l$
(C) $l = \underline{\lim}\, x_n \neq \lim x_n$
(D) $\underline{\lim}\, x_n \leq \overline{\lim}\, x_n \leq l$

180. For the sequence $<x_n>$, where $x_n = (-1)^n n$, limit x_n is—
(A) $-\infty$ (B) $+\infty$
(C) 0 (D) 1

181. For the sequence $<(-1)^n>$, $\lim x_n$ is equal to—
(A) $-\infty$ (B) $+\infty$
(C) $+1$ (D) -1

182. For the sequence $<x_n>$ where $x_n = (-1)^n \left(1 + \frac{1}{n}\right)$, $\lim x_n$ is equal to—
(A) $-\infty$ (B) -1
(C) $+\infty$ (D) $+1$

183. The sequence diverges to $+\infty$ if—

(A) $\underline{\lim}\, x_n = \overline{\lim}\, x_n = +\infty$
(B) $\underline{\lim}\, x_n \le +\infty$
(C) $\overline{\lim}\, x_n \le +\infty$
(D) $\underline{\lim}\, x_n = \overline{\lim}\, x_n < +\infty$

184. If $\forall\, x_n, y_n \in \mathbb{R}, x_n < y_n$, then—

(A) $\underline{\lim}\, x_n \ge \underline{\lim}\, y_n$
(B) $\underline{\lim}\, x_n \le \underline{\lim}\, y_n$
(C) $\overline{\lim}\, x_n \le \overline{\lim}\, y_n$
(D) $\overline{\lim}\, x_n \le \underline{\lim}\, y_n$

185. If every $x_n < y_n$, then—

(A) $\overline{\lim}\, x_n \le \underline{\lim}\, y_n$ (B) $\overline{\lim}\, x_n \le \overline{\lim}\, y_n$
(C) $\underline{\lim}\, x_n \le \overline{\lim}\, y_n$ (D) $\lim x_n \le \lim y_n$

186. The open interval is—

(A) Open set (B) Closed set
(C) Empty set (D) Closure

187. The closed interval is—

(A) Open set (B) Closed set
(C) Empty set (D) Unbounded set

188. The half closed interval is $[a, \infty]$ is—

(A) Open set (B) Closed set
(C) Empty set (D) Unbounded set

189. The intersection of finite collection of open sets is—

(A) Open set (B) Closed set
(C) Empty set (D) None of these

190. The intersection of any collection of open sets is—

(A) Open set
(B) Closed set
(C) Can not defined
(D) Semi open

191. If S is a set of real numbers which is bounded above then Sup S is—

(A) A point of closure to S
(B) Not a point of closure to S
(C) Prime number
(D) None of these

192. If S is a set of real numbers which is bounded below the inf S is—

(A) A point of closure to S
(B) Not a point of closure to S
(C) Prime number
(D) None of these

193. If F is an open covering of a closed and bounded set A, then—

(A) There exist a infinite sub collection of F which covers A
(B) There exist a uncountable sub collection of F which covers A
(C) Both (A) and (B)
(D) None of these

194. Let F be an open covering of A, then—

(A) There exist countable sub collection of F which covers A
(B) There exist uncountable collection of F which covers A
(C) (A) and (B) both true
(D) (A) and (B) both false

195. The complement of open set is—

(A) Closed set (B) Open set
(C) Countable set (D) None of these

196. A finite set is—

(A) Open set (B) Closed set
(C) Uncountable set(D) None of these

197. Singleton set $\{x_0\}$ is—

(A) Open (B) Closed
(C) Uncountable (D) None of these

198. If f' exist and is monotonic on (a, b), then—

(A) f' is continuous on (a, b)
(B) f' is discontinuous on (a, b)
(C) f' is constant
(D) None of these

199. If f is continuous on $[a, b]$ and f' exist at each $x \in (a, b)$, then if $f' > 0$ in (a, b), then—

(A) f is strictly increasing
(B) f is strictly decreasing
(C) f is constant
(D) None of these

200. If f is continuous on $[a, b]$ and f' exist at each $x \in (a, b)$, then it $f' = 0$—

(A) f is increasing

(B) f is decreasing
(C) f is constant
(D) None of these

201. A function f is increasing function on $E \subset \mathbb{R}$ if $x, y \in E$—
(A) $x < y \Rightarrow f(x) \le f(y)$
(B) $x < y \Rightarrow f(y) \le f(x)$
(C) $x < y \Rightarrow f(x) = f(y)$
(D) None of these

202. A function f is strictly decreasing on $E \subset R$, if $x, y \in E$—
(A) $x < y \Rightarrow f(x) < f(y)$
(B) $x < y \Rightarrow f(y) < f(x)$
(C) $x < y \Rightarrow f(y) \le f(x)$
(D) $x < y \Rightarrow f(y) \ge f(x)$

203. A function is a monotonic function f if—
(A) f is only a decreasing function
(B) f is only an incresing function
(C) f is either increasing or decreasing function
(D) None of these

204. Singular monotonic function on $[a, b]$ have—
(A) $f(x) = 0, \forall x \in [a, b]$
(B) $f(x) = 0, \forall x \in [a, b]$
(C) $f(x) \neq 0, \forall x \in [a, b]$
(D) None of these

205. If f is an increasing function, then—
(A) $-f$ is decreasing function
(B) $-f$ is increasing function
(C) $-f$ is constant
(D) None of these

206. If f is an increasing function on closed interval $[a, b]$, then $c \in [a, b]$ we have—
(A) $f(c^-) \ge f(c) \ge f(c^+)$
(B) $f(c^-) \le f(c) \le f(c^+)$
(C) $f(c^-) = f(c) = f(c^+)$
(D) None of these

207. If f and g are of bounded variation, then following is false—
(A) $f + g$ is of bounded variation
(B) $f - g$ is of bounded variation
(C) fg is of bounded variation
(D) f/g is of bounded variation

208. If f is bounded variation, then f/g is of bounded variation if—
(A) f is of bounded above from zero
(B) g is of bounded above from zero
(C) g is of bounded variation
(D) None of these

209. If f is of bounded variation on $[a, b]$ and $c \in (a, b)$, then—
(A) $V_f(a, b) \le V_f(a, c) + V_f(c, b)$
(B) $V_f(a, b) \ge V_f(a, c) + V_f(a, b)$
(C) $V_f(a, b) = V_f(a, c) + V_f(c, b)$
(D) None of these

210. If f is of bounded variation on $[a, b]$ and $c \in (a, b)$, then—
(A) f is of bounded variation on $[a, c]$ and on $[c, b]$
(B) f is not of bounded variation on $[a, c]$ and on $[c, b]$
(C) f is constant on $[a, c]$ and $[c, b]$
(D) None of these

211. If f is of bounded variation on $[a, b]$ if—
(A) f is of difference of two monotonic real valued function on $[a, b]$
(B) f is the product of two monotonic real valued function on $[a, b]$
(C) f is the quotient of two monotone real valued function on $[a, b]$
(D) None of these

212. If f is continuous on $[a, b]$ then f is of bounded variation on $[a, b]$ if—
(A) f is the difference of two monotone continuous function on $[a, b]$
(B) f is the product of two monotone continuous function on $[a, b]$
(C) f is the quotient of two monotone continuous functions on $[a, b]$
(D) None of these

213. If $f : [a, b] \rightarrow R$ is monotonic, then—
(A) If is of bounded variation
(B) f is unbounded
(C) The set of discontinuous of f are uncountable
(D) None of these

214. If f is of bounded variation on $[a, b]$, then total variation of f on $[a, b]$ is—

(A) Finite number (B) Infinite
(C) Zero (D) None of these

Answers with Hints

1. (B) The first term $a_1 = \frac{1}{1\cdot 2} = \frac{1}{2}$

The second term $a_2 = \frac{1}{2\cdot 2} = \frac{1}{4}$

The third term $a_3 = \frac{1}{3\cdot 2} = \frac{1}{6}$

and the nth term $a_n = \frac{1}{n\,.\,2}$

2. (B) Prime number
3. (C) For n = odd $a_n = 1$

For n = even $a_n = \dfrac{1}{n - \frac{n}{2} + 1}$

4. (C)

$$a_1 = 2 = 1 + \frac{1}{1}$$
$$a_2 = \frac{-3}{2} = (-1)^{2-1}\left(1 + \frac{1}{2}\right)$$
$$a_3 = \frac{4}{3} = (-1)^{3-1}\left(1 + \frac{1}{3}\right)$$
$$a_4 = \frac{-5}{4} = (-1)^{4-1}\left(1 + \frac{1}{4}\right)$$
$$a_n = (-1)^{n-1}\left(1 + \frac{1}{n}\right)$$

5. (B)

$$a_1 = 1 = 2 - 1,$$
$$a_2 = \frac{5}{2} = 4 + 1,$$
$$a_3 = \frac{5}{3} = 6 - 1$$
$$\ldots\ldots a_n = \frac{2n + (-1)^n}{n}$$

6. (A) Given

$$a_1 = 0 = \frac{1-1}{1}$$
$$a_2 = 1 = \frac{1+1}{2}$$
$$a_3 = 0 = \frac{1-1}{3}$$
$$a_4 = \frac{1}{2} = \frac{1+1}{4}$$
$$a_n = \frac{1 + (-1)^n}{n}$$

7. (A)	8. (C)	9. (A)	10. (A)	11. (C)
12. (D)	13. (A)	14. (B)	15. (A)	16. (B)
17. (A)	18. (A)	19. (B)	20. (B)	21. (A)
22. (A)	23. (D)	24. (B)	25. (B)	26. (A)
27. (C)	28. (A)	29. (A)	30. (A)	31. (B)
32. (B)	33. (C)			

34. (A) The nth term $a_n = \dfrac{(-1)^n}{2n-1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left|\frac{(-1)^n}{2n-1}\right| = 0$$

$\Rightarrow (|a_n|)$ is a convergent sequence

$\Rightarrow \in |a_n|$ is convergent

$\Rightarrow \in a_n$ is convergent and converges to zero.

35. (A) The nth term $a_n = 2n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(2n)$$
$$= \infty \neq 0$$

$\because \{a_n\}$ is not converges to zero

$\therefore \in a_n$ is divergent.

36. (A) The nth term

$$a_n = 1/n^n$$
$$a_{n+1} = \frac{1}{(n+1)^{n+1}}$$

(By Ratio test)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1}{(n+1)^{n+1}}\cdot\frac{n^n}{1}\right|$$
$$= \left|\frac{n^n}{(n+1)^n}\cdot\frac{1}{n+1}\right|$$
$$= \left|\left(\frac{n^n}{n+1}\right)^n\left(\frac{1}{n+1}\right)\right|$$
$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1}{e}\,.\,\infty\right|$$
$$= \infty > 1$$
$$\left(\because \lim\left(\frac{n}{n+1}\right) = 1/e\right)$$

$\therefore$ Series diverges.

37. (A)

$$\frac{a_{n+1}}{a_n} = \frac{\lfloor\underline{n+1}}{(n+1)^{n+1}} = \frac{n^n}{\lfloor\underline{n}}$$
$$= \frac{n^n}{(1+n)^n} = \frac{1}{(1+1/n)^n}$$

$\therefore \quad \lim \left|\frac{a_{n+1}}{a_n}\right| = 1 \div \left\{\lim_{n\to\infty}(1+1/n)^n\right\}$

$= \frac{1}{e} < 1$

$\therefore$ The series is convergent.

38. (B) $\frac{a_{n+1}}{a_n} = \frac{1}{3}(1+1/n)^2$

$\Rightarrow \quad \lim \frac{a_{n+1}}{a_n} = \frac{1}{3} < 1$

$\Rightarrow$ Series is convergent.

39. (A) The partial sum

$S_n = \frac{1}{2} n\,[2(1) + (n-1).1]$

$= \frac{1}{2} n\,(n+1)\ \forall\ n \in N$

$\therefore \quad \forall n \in N,\ 1 \le S_n$ (bounded below)

and there is

$n\,\theta M : M \ge S_n$

$\therefore \{S_n\}$ is unbounded sequence

Also $S_{n+1} - S_n = \frac{1}{2}(n+1)(n+2)\frac{-1}{2}$

$n\,(n+1)$

$= \frac{1}{2}(n+1)(n+2-n)$

$= n+1$

$\Rightarrow \quad S_{n+1} - S_n \ge 0$

$\Rightarrow \{S_n\}$ is increasing sequence

$\because \{S_n\}$ is increasing and unbounded

$\therefore \{S_n\}$ is divergent

$\Rightarrow 1 + 3 + 5 + \ldots$ is divergent.

40. (A) $S_n = \sum_{n=0}^{n} (-1)^K = 1$

or 0 as n is odd or even

$\Rightarrow \quad \{S_n\} = \{1, 0, 1, 0, \ldots\}$

It is divergent sequence

$\Rightarrow \in (-1)^n$ is divergent

41. (A) The partial sum $S^n = \sum_{K=0}^{n} (n)^n$

$\{S_n\}$ is increasing sequence and unbounded from above

$\therefore$ It is divergent sequence

$\Rightarrow$ The series is divergent series.

42. (A) The nth term

$a_n = \frac{1}{n\,(n+2)}$

$= \frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+2}\right]$

$S_n = \sum_{K=1}^{n} a_K$

$= \frac{1}{2}\left[\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]$

$\because \forall n \in N \quad S_n \ge \frac{1}{6}$

and $\quad S_n \le \frac{3}{4}$

$\therefore \{S_n\}$ is bounded and also S_n is monotone sequence

$\Rightarrow \{S_n\}$ is convergent

$\therefore$ The series is convergent.

43. (C) $x_n = (-1)^n$,

$\forall n \in N, \forall n \in N, -1 \le x_n$

and $\quad \forall n \in N,\ 1 \ge x_n$

$\Rightarrow \{(-1)^n\}$ is bounded

For even number $\lim_{n\to\infty} (-1)^n = -1$

For odd number $\lim_{n\to\infty} (-1)^n = -1$

$\therefore$ Series is divergent.

44. (B) $x_n = \frac{1}{n}\ \forall n \in N, \forall n \in N,$

$1 \ge x_n, \forall n \in N,$

$0 \le 0$

$\Rightarrow \left\{\frac{1}{n}\right\}$ is bounded

$x_n = \frac{1}{n}$

and $\quad x_{n+1} = \frac{1}{n+1}$

$\Rightarrow \quad x_{n+1} - x_n = \frac{1}{n+1} - \frac{1}{n}$

$= \frac{n-n-1}{(n+1)n}$

$= \frac{-1}{(n+1)^n}$

$\Rightarrow \quad x_{n+1} - x_n \le 0$

$\Rightarrow \quad x_{n+1} \le x_n,$

$\forall n \in N$ decreasing sequence

$\because \left\{\frac{1}{n}\right\}$ is bounded and monotone sequence

$\therefore \left\{\frac{1}{n}\right\}$ is convergent.

45. (A) $x_n = a_n b_n$

$$= (-1)^n \cdot \left(\frac{1}{n}\right) \; \forall n \in \text{N},$$

$$a_n = (-1)^n$$

$\Rightarrow \quad \forall n \in \text{N} \; -1 \leq a_n$

and $\quad 1 \geq a_n$

$\Rightarrow \{a_n\}$ is a bounded sequence

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n} = 0$$

$\Rightarrow \{a_n\}$ is bounded sequence and $\{b_n\}$ converges to zero

$\therefore \{a_n b_n\}$ converges to zero.

46. (B) 47. (A) 48. (B) 49. (B)

50. (A) Let $x_n = n$,

$n \in \text{N}, \forall n \in \text{N}, 1 \leq x_n$ (bounded below)

But there exist no real number

$$\text{M} : x_n \leq \text{M}$$

$\therefore \{n\}$ is unbounded.

51. (B) The sequence $\{-n\}$ is bounded above since there exist a real number

$$-1 : -1 \geq -n, \forall n \in \text{N}.$$

52. (A) $x_n = \frac{n+1}{n} = 1 + 1/n$

$\forall n \in \text{N}, 1 \leq x_n$ (bounded below)

$\forall n \in \text{N}, 2 \geq x_n$ (bounded above)

$\therefore \left\{\frac{n+1}{n}\right\}$ is bounded sequence.

53. (B) $x_n = \frac{n}{n+1} = \frac{1}{1 + 1/n}$

$\forall n \in \text{N}, 1 \geq x_n$ (bounded above)

and $\forall n \in \text{N}, \frac{1}{2} \leq x_n$ (bounded below)

$\therefore \{x_n\}$ is bounded

54. (B) $x_n = 1/3^n$,

$\forall n \in \text{N}, 0 \leq x_n$ (bounded below)

$\forall n \in \text{N}, 1 \geq x_n$ (bounded above)

$\therefore \{x_n\}$ is bounded.

55. (B) $x_n = \frac{(-1)^n}{n}$

$\forall n \in \text{N}, -1 \leq x_n$

$\forall n \in \text{N}, 0 \geq x_n$

$\therefore \{x_n\}$ is bounded

56. (B)

$$x_n = \frac{n+1}{n}$$

$$= 1 + 1/n$$

$$x_{n+1} = \frac{(n+1)+1}{(n+1)}$$

$$= 1 + \frac{1}{n+1}$$

$$x_{n+1} - x_n = \frac{1}{n+1} \cdot \frac{-1}{n}$$

$$= \frac{-1}{n(n+1)}$$

$\Rightarrow \quad x_{n+1} - x_n < 0$

$\Rightarrow \quad x_{n+1} < x_n$

$\Rightarrow \{x_n\}$ is decreasing sequence.

57. (A)

$$x_n = \frac{n}{n+1},$$

$$x_{n+1} = \frac{n+1}{n+1+1}$$

$$= \frac{n+1}{n+2}$$

$$x_{n+1} - x_n = \frac{n+1}{n+2} - \frac{n}{n+1}$$

$$= \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)}$$

$$= \frac{1}{(n+2)(n+1)}$$

$\Rightarrow \quad x_{n+1} - x_n > 0$

$\Rightarrow \quad x_{n+1} > x_n$

$\Rightarrow \left[\frac{n}{n+1}\right]$ is increasing sequence.

58. (A)

$$x_n = \frac{(-1)^n}{n},$$

$$x_{n+1} = \frac{(-1)^{n+1}}{n+1}$$

$$x_{n+1} - x_n = \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^n}{n}$$

$$= \frac{n(-1)^{n+1} - (n+1)(-1)^n}{(n+1)^n}$$

$$= \frac{(-1)^n(-n-n+1)}{(n+1)\,n}$$
$$= \frac{(-1)^n}{n\,(n+1)}$$

Increasing n even, Decreasing n odd.

$\forall n \in \mathrm{N}, -1 \le x_n \le 0$, bounded,

59. (C) $\forall n \in \mathrm{N}, 0 \le x_n$,

$\forall n \in \mathrm{N}, 1 \ge x_n$

$\therefore$ {1, 0, 1, 0, …} is bounded.

60. (A) $\forall n \in \mathrm{N}, a \le x_n$,

$\forall n \in \mathrm{N}, a - b \ge x_n$

$\therefore \{x_n\}$ is bounded.

61. (A) The series $\Sigma \frac{1}{n^{\mathrm{P}}}$ is convergent iff P > 1

62. (A) Here $a_n = \frac{1}{(1+1/n)^{n^2}}$

By Root test

$$\lim_{n\to\infty} \sqrt{a_n} = \lim_{n\to\infty} \frac{1}{(1+1/n)^n}$$
$$= \frac{1}{e} < 1$$

$\therefore$ The series is convergent.

63. (A) The absolute series

$\frac{\sin x}{1^3} + \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} + \ldots$ is convergent.

$\therefore$ The series is convergent.

64. (A) By Ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^n}{\underline{|n}} \times \frac{\underline{|n-1}}{x^{n-1}}\right|$$
$$= \left|\frac{x}{n}\right|$$
$$\lim_{n\to\infty} \left|\frac{x}{n}\right| = 0 < 1,$$

The series is convergent.

65. (B) By Ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|(-1)^{n+1} \cdot \frac{x^{n+1}}{n+1} \times \frac{n}{(-1)^n \times x^n}\right|$$
$$= \left|\frac{x \cdot n}{n+1}\right|$$
$$= x \cdot \left|\frac{1}{1+1/n}\right|$$
$$x \lim_{n\to\infty} \left|\frac{1}{1+1/n}\right| = x > 1$$

when $|x| > 1$

$\therefore$ The series diverges for $|x| > 1$

66. (A)	67. (A)	68. (C)	69. (B)	70. (B)
71. (C)	72. (C)	73. (A)	74. (A)	75. (B)
76. (A)	77. (C)	78. (C)	79. (A)	80. (A)
81. (B)	82. (C)	83. (C)		

84. (A) Here $a_n = \frac{(-1)^n}{\underline{|n}}$

and
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{\underline{|n+1}} \times \frac{\underline{|n}}{1}$$
$$= \frac{1}{n}$$

$\therefore$
$$\mathrm{L} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$
$$= \lim_{n\to\infty} \frac{1}{n}$$
$$= 0$$

and the radius of convergence

$$\mathrm{R} = \frac{1}{\mathrm{L}} = \frac{1}{0}$$
$$= \infty$$

85. (A) Here $x_0 = 0{\cdot}2$

and $a_n = 1/n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{n+1} \cdot \frac{n}{1}$$
$$= \frac{n}{n+1}$$
$$= \frac{1}{1+1/n}$$
$$\Rightarrow \quad \mathrm{L} = \lim_{n\to\infty} \frac{1}{1+1/n}$$
$$= 1$$

and the radius of convergence.

$$\mathrm{R} = \frac{1}{\mathrm{L}} = 1$$

86. (A) Here $a_n = \underline{|n}$

and
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\underline{|n+1}}{\underline{|n}}$$
$$= n + 1$$

$$\Rightarrow \qquad L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \infty$$

The radius of convergence

$$R = 0$$

$\therefore$ The series converges only at

$$x = 0$$

87. (A)	88. (A)	89. (A)	90. (A)	91. (A)
92. (B)	93. (A)	94. (C)	95. (C)	96. (B)
97. (A)	98. (D)	99. (B)	100. (B)	101. (A)
102. (C)	103. (C)	104. (B)	105. (B)	106. (B)
107. (C)	108. (D)	109. (B)	110. (A)	111. (A)
112. (B)	113. (A)	114. (C)	115. (A)	116. (A)
117. (D)	118. (B)	119. (A)	120. (A)	121. (A)
122. (A)	123. (C)	124. (A)	125. (B)	126. (B)
127. (A)	128. (A)	129. (A)	130. (A)	131. (A)
132. (C)	133. (A)	134. (D)	135. (B)	136. (A)
137. (C)	138. (C)	139. (A)	140. (B)	141. (A)
142. (B)	143. (D)	144. (A)	145. (B)	146. (B)
147. (A)	148. (A)	149. (C)	150. (B)	151. (B)
152. (B)	153. (C)	154. (A)	155. (A)	156. (A)
157. (A)	158. (A)	159. (A)	160. (A)	161. (B)
162. (C)	163. (A)	164. (A)	165. (C)	166. (B)
167. (C)	168. (A)	169. (D)	170. (A)	171. (C)
172. (A)	173. (B)	174. (D)	175. (B)	176. (A)
177. (C)	178. (A)	179. (A)	180. (A)	181. (D)
182. (B)	183. (A)	184. (B)	185. (B)	186. (A)
187. (B)	188. (B)	189. (A)	190. (C)	191. (A)
192. (A)	193. (C)	194. (A)	195. (A)	196. (B)
197. (B)	198. (A)	199. (A)	200. (C)	201. (A)
202. (B)	203. (C)	204. (B)	205. (A)	206. (B)
207. (D)	208. (B)	209. (C)	210. (A)	211. (A)
212. (A)	213. (A)	214. (A)		