Calculus of Variations and Integral Equations

(1) Calculus of Variations

The First Variation

Admissible Arc—It is the arc of integration such that the integral $I = \int_a^b F\left(x, y, \frac{dy}{dx}\right) dx$, can be determined, where y is a function of x and $F = F(x, y, y')$ process a continuous partial derivatives.

Weak Variation—(1) Let $y = S(x)$ and $y = S(x) + \varepsilon t(x)$ be two admissible curves of integral, $I = \int_a^b F\left(x, y, \frac{dy}{dx}\right) dx$, where ε is an arbitrary constant independent of x and y, $t(x)$ is an arbitrary function of x, independent of ε. With this restriction on $t(x)$, the ordinate y is said to be a weak variation.

(2) The variation y is said to be of weak variation if $t(x)$ and $t'(x)$ are of same order of smallness.

First Variation—The integral $I_1 = \varepsilon \int_a^b \left(t \frac{\partial F}{\partial S} - \frac{dt}{dx}\frac{\partial F}{\partial S}\right) dx$, where $y = S(x)$, $t = t(x)$ and $F = F(x, y, y')$.

Second Variation—The integral $I_2 = \frac{\varepsilon}{2!} \int_a^b \left(t^2 \frac{\partial^2 F}{\partial S^2} - 2tt' \frac{\partial^2 F}{\partial S \partial S'} + t'^2 \frac{\partial^2 F}{\partial S'^2}\right) dx$ where $y = S(x)$, $t = t(x)$ and $F = F(x, y, y')$.

Euler's Characteristic Equation—Given the function $F = F(x, y, y')$, the differential equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$, is referred as Euler's characteristic equation. The Euler's characteristic equation is also known as Euler-Lagrange equation.

Extremals—It is solution to Euler's characteristic equation.

Discontinuous Solutions—Discontinuous solutions are obtained by joining several continuous arcs, each of which satisfies the Eulerian characteristic equation. The corners occur at the points of junctions.

Some Important Theorems

(1) The integral $I = \int_a^b F(x, y, y')\, dx$, whose end points are fixed, is stationary for weak variations, if y satisfies Euler's characteristic equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$.

(2) The integral $\int_a^b F(y, y')\, dx$, whose ends are fixed is stationary for weak variation if y satisfies the differential equation $F - y' \frac{\partial F}{\partial y'} = C$, where C is an arbitrary constant.

(3) The integral $\int_a^b F(x, y')\, dx$, whose ends are fixed, is stationary for weak variations if y satisfies the differential equation $\frac{\partial F}{\partial y'} = C$, where C is an arbitrary constant.

(4) **Legender's test**—If (i) Euler's characteristic equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ is satisfied, (ii) the range of integration (a, b) is sufficiently small and (iii) the sign of $\frac{\partial^2 F}{\partial y'^2}$ is constant throughout this range. Then stationary value $I_s = \int_a^b F(x, S, S')\, dx$ of integral $I = \int_a^b F(x, y, y')\, dx$, is a maximum if $\frac{\partial^2 F}{\partial y'^2} < 0$ and minimum if $\frac{\partial^2 F}{\partial y'^2} > 0$.

(5) If Euler's characteristic equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) = 0$ vanishes identically, then the indefinite integral $\int F(x, y, y')\, dx$ can be evaluated as a function of x and y. The integral has then a value which is independent of the path of integration and which is a function of the end positions only.

The Second Variation

Second Variation—The integral $I_2 = \frac{\varepsilon}{2!} \int_a^b (t^2 F_{00} + 2tt' F_{01} + t'^2 F_{11}) dx$.

Jacobi's Accessory Equation—The accessory equation $\left\{ F_{00} - \frac{d}{dx}(F_{01}) \right\} u - \frac{d}{dx} \left\{ F_{11} \frac{du}{dx} \right\} = 0$, where $\frac{\partial^2 F}{\partial S^2} = F_{00}$, $\frac{\partial^2 F}{\partial S \partial S'} = F_{01}$ and $\frac{\partial^2 F}{\partial S'^2} = F_{11}$.

Conjugate Points (Kinetic foci)—If (i) $u(x)$ is the solution of the accessory equation, (ii) a is the abscissa of the point A, and (iii) $u(a) = 0$, then the roots of $u(x) = 0$ are the abscissae of the point on the curve $y = S(x)$ conjugate to A.

Some Important Theorems

1. If $t(a) = t(b) = 0$

$$\int_a^b (t^2 F_{00} + 2tt' F_{01} + t'^2 F_{11}) dx$$
$$= \int_a^b \left\{ t^2 F_{00} - t^2 \frac{d}{dx}(F_{01}) - t \frac{d}{dx}(t' F_{11}) \right\} dx.$$

2. If $t(a) = t(b) = 0$ and u is a solution of $\left\{ F_{00} - \frac{d}{dx}(F_{01}) \right\} u - \frac{d}{dx} \left\{ F_{11} \frac{du}{dx} \right\} = 0$, then

$$I_2 = \frac{\varepsilon}{2!} \int_a^b F_{11} \left(t' - t \frac{u'}{u} \right)^2 dx.$$

3. Let $y = S(x)$ be the equation of the extremal through the fixed points A and B for which the integral $I = \int_a^b F(x, y, y') dx$ is stationary. Let A and A' be two adjacent conjugate points (kinetic foci) on the curve. If (i) B lies between A and A' and (ii) $\frac{\partial^2 F}{\partial S'^2}$ has constant sign for all points of the arc AB, then for weak variation I is a maximum when $\frac{\partial^2 F}{\partial S'^2}$ is negative and a minimum when it is positive.

- If (i) a is the abscissa of the point A, (ii) $u_1(x)$ and $u_2(x)$ are independent solutions of the accessory equation, then the equation for the abscissa of the points conjugate to A is, $\frac{u_1(x)}{u_2(x)} = \frac{u_1(a)}{u_2(a)}$.
- If $y = S(x, c_1, c_2)$ is the general solution of Euler's characteristic equation, then the equation for abscissa of points conjugate to A is $\frac{\partial y}{\partial c_1} / \frac{\partial y}{\partial c_2} = \left(\frac{\partial y}{\partial c_1} / \frac{\partial y}{\partial c_2} \right)_{x = a}$

4. If the extremals, which pass through the point A have an envelope E, then the points of contact of E and the extremal AB are the points conjugate to A.

5. If $u(x)$ a solution of the accessory equation, it cannot have double zero.

6. If $u_1(x)$ and $u_2(x)$ are independent solutions of the accessory equation, then the ratio $u_1(x)/u_2(x)$ steadily increases or steadily decreases as x increases.

The Generalized Results

1. Let the values of t_0 and t_1 and the functional form of F be given, then the integral $\int_{t_0}^{t_1} F(q_1, q_2, \ldots, q_n : \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n ; t) dt$, where the q_i, $i = 1, 2, \ldots, n$ are arbitrary functions of t, is stationary for weak variations when the q_i, $i = 1, 2, \ldots n$ satisfy the n equations $\frac{\partial F}{\partial q_m} - \frac{\partial}{dt} \left(\frac{\partial F}{\partial \dot{q}_m} \right) = 0$ $(m = 1, 2, \ldots, n)$.

2. Let $\int_{t_0}^{t_1} F(q_1, q_2 \ldots, q_n : \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n ; t) dt$, where the q_i, $i = 1, 2, \ldots n$ satisfy the n second order partial differential equations $\frac{\partial F}{\partial q_m} - \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{q}_m} \right) = 0$ $(m = 1, 2, \ldots n)$. If at every point of a sufficiently small range t_0 to t_1 (i) $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ $(r = 1, 2, \ldots, n)$ all have constant sign, (ii) $\left(\frac{\partial^2 F}{\partial \dot{q}^2_r} \right) \left(\frac{\partial^2 F}{\partial \dot{q}^2_s} \right) > \left(\frac{\partial^2 F}{\partial \dot{q}_r \partial \dot{q}_s} \right), \left(\begin{matrix} r = 1, 2, \ldots, n \\ s = 1, 2, \ldots, n \end{matrix}, r \neq s \right)$. Then I is a maximum if the sign of $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ are all negative and minimum if they are all positive [evidently (ii) ensures that they all have the same sign].

3. Let $\frac{d^m y}{dx^m}$ be denoted by y_m and let the values of $y, y_1, y_2, \ldots, y_{n-1}$ be given for both $x = a$ and $x = b$. Also let the functional form of F be given : then the integral $I = \int_a^b F(x, y, y_1, y_2, \ldots y_n) dx$ is stationary when y satisfies the equation $\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y_1} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y_2} \right) - \ldots + (-1)^n \frac{d^n}{dx^n} \left(\frac{\partial F}{\partial y_n} \right) = 0.$

4. Let F be a given functional form, let $p_r = \frac{\partial z}{\partial x_r}$, where z is a function of $x_1, x_2, \ldots, x_n$, and let the integral $I = \iiint \ldots \int F(z, x_1, x_2, \ldots, x_n : p_1, p_2, \ldots, p_n)\, dx_1\, dx_2 \ldots dx_n$ be taken through an n-dimensional region bounded by given fixed boundaries of dimension $n - 1$.

Then the integral I stationary when z is a solution of the second order partial differential equation $\frac{\partial F}{\partial z} - \frac{\partial}{dx_1}\left(\frac{\partial F}{\partial p_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial F}{\partial p_2}\right) - \frac{\partial}{\partial x_n}\left(\frac{\partial F}{\partial p_n}\right) = 0$.

Relative Maxima and Minima

1. By suitable choice of the constants λ, a solution of second order equation $\frac{\partial}{\partial y}(F - \lambda\phi) - \frac{d}{dx}\left\{\frac{d}{\partial y_1}(F - \lambda\phi)\right\} = 0$ can be found which renders the integral I of equation $I = \int_a^b F(x, y, y_1)\, dx$ stationary, which passes through the end points of the range of integration of $\int_a^b F(x, y, y_1)$ and $\int_a^b \phi(x, y, y_1)\, dx$.

Isoperimetrical Problem

1. By suitable choice of arbitrary constant and of the constant λ, it is possible to find solution of $\frac{\partial G}{\partial x} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{x}}\right) - \lambda\left\{\frac{\partial \psi}{\partial x} - \frac{d}{dt}\left(\frac{\partial \psi}{\partial \dot{x}}\right)\right\} = 0$ and $\frac{\partial G}{\partial y} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{y}}\right) - \lambda\left\{\frac{\partial \psi}{\partial y} - \frac{d}{dt}\left(\frac{\partial \psi}{\partial \dot{y}}\right)\right\} = 0$ which makes the integral of equations $I = \rho g \int_A^B y ds$ stationary and at same time satisfy condition $l = \int_A^B ds$.

2. Solution of the simultaneous equations $\frac{\partial G}{\partial x} - \frac{d}{dx}\left(\frac{\partial G}{\partial \dot{x}}\right) - \left\{\mu\frac{\partial S}{\partial x} - \frac{d}{dt}\left(\mu\frac{\partial S}{\partial \dot{x}}\right)\right\} = 0$ and $\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial \dot{y}}\right) - \left\{\mu\frac{\partial s}{\partial y} - \frac{d}{dt}\left(\mu\frac{\partial s}{\partial \dot{y}}\right)\right\} = 0$ can be found which satisfies equation $S(x, y, t, \dot{x}, \dot{y}) = 0$ and which make the integral $I = \int_{t_1}^{t_2} G(x, y, t, \dot{x}, \dot{y})\, dt$, stationary.

3. If the range of integration of the integral $I = \int_a^b F(x, y, y_1)\, dx$ is sufficient small, and if the expression $E = \frac{\partial^2 F}{\partial {y_1}^2} - \lambda\frac{\partial^2 \phi}{\partial {y_1}^2}$ has the same sign throughout this range, then the integral I has a maximum if E is negative and minimum if E is positive.

Integral with Variable End Points

1. If the end points A and B of the range of integration of the integral $I = \int_a^b F(x, y, y')\, dx$ can be displaced along prescribed curves, then I is stationary when the following conditions (necessary) satisfied :

(i) The ordinate of the external y, satisfies Eulerian equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$,

(ii) At $x = a$, $F + (g'_1 - y')\frac{\partial F}{\partial y'} = 0$, where a is the abscissa of the end point A, which can be displaced along the curve $y = g_1(x)$.

(iii) At $x = b$, $F + (g'_2 - y')\frac{\partial F}{\partial y'} = 0$, where b is the abscissa of the end point B, which can be displaced along the curve $y = g_2(x)$.

Strong Maximum and Minimum

1. **Necessary Condition**—The integral I has strong maximum if :

(i) The equation of $\lceil e$, the arc of integration, satisfy $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$.

(ii) The arc AB of $\lceil e$ contains no point conjugate to either A or B.

(iii) At all points of this arc $E = \frac{\partial^2 F}{\partial y'^2} < 0$.

(iv) At all points of this arc and for all finite values of p, $E(x, y, y', p) \le 0$.

For strong minimum, the inequality are reversed.

2. **Sufficient Conditions**—The integral I has strong maximum if :

(i) The equation of $\lceil e$, the arc of integration, satisfies $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$.

(ii) The arc AB of $\lceil e$, contains no point conjugate to either A or B.

(iii) At all points of the arc AB, of $\lceil e$, and for all finite values of p, $\frac{\partial^2 F(x, y, p)}{(\partial p^2)} < 0$.

For strong minimum the inequality sign is reversed.

3. The integral $I = \int_A^B G(x, y, \dot{x}, \dot{y})\, dt$, where the end points A and B are fixed and $G(x, y, \dot{x}, \dot{y})$ is homogeneous and of degree one in $\dot{x}$ and $\dot{y}$, have strong maximum iff (i) the variable x and y, which are functions of t, either satisfy two equations,

$$\left.\begin{aligned} \frac{\partial G}{\partial x} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{x}}\right) &= 0 \\ \frac{\partial G}{\partial y} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{y}}\right) &= 0 \end{aligned}\right\} \quad \ldots(1)$$

or the single equivalent equation $\frac{\partial^2 G}{\partial x \partial \dot{y}} - \frac{\partial^2 G}{\partial x \partial \dot{x}} - S(x, y, \dot{x}, \dot{y})(\ddot{x}\dot{y} - \ddot{y}\dot{x}) = 0$. Here $\lceil e$ is the curve, which passes through the fixed ends A and B satisfies the parametric equations (1).

(ii) The arc AB of $\lceil e$ contains no points conjugate either to A or to B.

(iii) At all points of this arc $S(x, y, \dot{x}, \dot{y}) > 0$, where

$$S(x, y, \dot{x}, \dot{y}) = \frac{1}{\dot{y}^2}\frac{\partial^2 G}{\partial \dot{x}^2} = \frac{-1}{\dot{x}\dot{y}}\frac{\partial^2 G}{\partial \dot{x}\partial \dot{y}} = \frac{1}{\dot{x}^2}\frac{\partial^2 G}{\partial \dot{y}^2}$$

(iv) At all points of this arc $E(x, y, \dot{x}, \dot{y}, p, q) < 0$ for every pair of finite values of p and q other, than $p = \dot{x}$ and $q = \dot{y}$ simultaneously.

4. The integral $I = \int_A^B G(x, y, \dot{x}, \dot{y})\, dt$, where A, B are fixed and where $G(x, y, \dot{x}, \dot{y})$ is homogeneous and of degree one in $\dot{x}$ and $\dot{y}$ iff

(i) The variables x and y which are functions of t, either satisfy

$$\left.\begin{aligned} \frac{\partial G}{\partial x} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{x}}\right) &= 0 \\ \frac{\partial G}{\partial y} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{y}}\right) &= 0 \end{aligned}\right\} \quad \ldots(1)$$

or $\frac{\partial^2 G}{\partial x \partial \dot{y}} - \frac{\partial^2 G}{\partial y \partial \dot{x}} - S(x, y, \dot{x}, \dot{y})(\ddot{x}(\dot{y}) - \ddot{y}(\dot{x})) = 0$

Here $\lceil e$ is the curve which passes through A and B and satisfies equation (1).

(ii) The arc AB of $\lceil e$ contains no points conjugate either to A or to B.

(iii) $S(x, y, p, q) < 0$ at every point of AB and for every pair of p and q.

- For strong minimum the sign of the in equality be reversed.

(2) Integral Equations

Integral Equations

Integral Equations—An unknown function appears under the integral sign in these equations.

Fredholm Integral Equations—Let $k(x, y)$ be a given real function defined for $0 \le x \le 1$, $0 \le y \le 1$ and $f(x)$ a real valued function defined for $0 \le x \le 1$ and λ an arbitrary complex number. Then

(i) Linear Fredholm integral equation of second kind (for a function $\phi(x)$) is $\phi(x) - \lambda \int_0^1 k(x, y)\, \phi(y)\, dy = f(x)$, $(0 \le x \le 1)$.

(ii) Linear Fredholm integral equation of first kind (for a function $\phi(x)$) is $\int_0^1 k(x, y)\, \phi(y)\, dy = f(x)$, $(0 \le x \le 1)$.

Note—For a sake of convenience the basic domain is assumed to be $0 \le x \le 1$, the basic domain can be $a \le x \le b$ or any bounded set on the x-axis.

Algebraic System of Equations—The problem of solving Fredholm integral equations of first and second kind is considered as a generalization of the problem of solving a set of n-linear algebraic equations in n-unknowns :

$$\sum_{S=1}^{n} a_{rs} x_s = b_r \ (r = 1, 2, \ldots, n).$$

Volterra Equations—Let $k(x, y)$ be a given real function defined for $0 \le x \le 1$, $0 \le y \le 1$, $f(x)$ a real valued function for $0 \le x \le 1$ and λ an arbitrary complex number, with $k(x, y) \equiv 0$ if $y > x$, then

(i) Voltera equation of first kind (for a function of $\phi(x)$ is $\int_0^x k(x, y)\, \phi(y)\, dy = f(x)$

(ii) Voltera integral equation of second kind (for a function $\phi(x)$ is $\phi(x) - \lambda \int_0^x k(x, y)\, \phi(y)\, dy = f(x)$.

Volterra Equations

L_2–Kernel—A kernel $k(x, y)$ which is quadratically integrable in the square $(0 \le x \le h, 0 \le y \le h)$. *i.e.*, $k(x, y)$ with norm $\|k\|$ such that $\|k\|^2 = \int_0^h \int_0^h k^2\, k^2(x, y)\, dxdy \le N^2$ exist, where N^2, a constant.

L_2–Function—A function (f), whose norm is given by $\|f\|^2 = \int_0^b f^2(x)\, dx$.

L^*_2–Function—A function f, which is L_2–function and bounded.

Solutions of Volterra Integral Equation of Second Kind

1. Volterra integral equation of second kind can be solved by Picard's process of successive approximations : $k_{n+1}(x, y) = \int_0^x k(x, z)\, k_n(z, y)\, dz$, $(h = 1, 2, \ldots)$.

2. The Volterra integral equation of the second kind $\phi(x) - \lambda \int_0^x k(x, y)\, \phi(y)\, dy = f(x)$, $(0 \le x \le h)$, where the kernel $k(x, y)$ and the function $f(x)$, belongs to the class L_2 has one and essentially only one solution in the same class L_2. The solution is given by $\phi(x) = f(x) - \lambda \int_0^x H(x, y, \lambda) f(y)\, dy$, where $H(x, y, \lambda)$ is the resolvent kernel given by the series of iterated kernels $H(x, y, \lambda) = -\sum_{n=0}^{\infty} \lambda^n k_{n+1}(x, y)$. This series converges almost every where. The resolvent kernel satisfies the integral equation $k(x, y) + H(x, y, \lambda) = \lambda \int_x^y k(x, z)\, H(x, y, \lambda) dz = \lambda \int_y^x H(x, z, \lambda)\, k(z, y)\, dz$.

3. Any solution of homogeneous Volterra integral equations of second kind $\phi(x) - \lambda \int_0^x k(x, y)\, \phi(x)\, dy = 0$ in L_2–space is necessarily a zero function.

4. By uniqueness theorem, the equations $k(x, y) + H(x, y, \lambda) = \lambda \int_y^x k(x, z)\, H(z, y, \lambda)\, dz = \lambda \int_y^x H(x, z, \lambda)\, k(z, y)\, dz$ are characteristic for the resolvent kernel $H(x, y, \lambda)$ in the space L_2. For only L_2–function satisfying above equation, it is the resolvent kernel corresponding to the kernel $k(x, y)$.

Solutions of Volterra Equation of First Kind

If in a Volterra equation of first kind $\int_0^x k(x, y)\, \phi(y)\, dy = f(x)$, $(0 \le x \le h)$. ...(1)

The diagonal $k(x, x)$ vanish now here in the basic interval $(0, h)$ and if the derivatives $\frac{df(x)}{dx} \equiv f'(x)$, $\frac{\partial k}{\partial x} \equiv k'_x(x, y)$ and $\frac{\partial k}{\partial x} \equiv k'_x(x, y)$ exist and continuous, the equation can be reduced to one of the two way :

1. Differentiate (1) with respect to x, one obtain

$$\phi(x) - \int_0^x \frac{k'_x(x, y)}{k(x, x)}\, \phi(y)\, dy = \frac{f'(x)}{k(x, x)}$$

2. Using $\int_0^x \phi(x)\, dy = \Phi(x)$, o ne g et $\Phi(x) = \frac{f(x)}{k(x, x)} - \int_0^x \frac{k'_y(x, y)}{k(x, x)}\, \Phi(y)\, dy = \frac{f(x)}{k(x, x)} - \int_0^x H^*(x, y, 1) \frac{f(y)}{k(y, y)}\, dy$, where $H^*(x, y, \lambda)$ is the resolvent kernel corresponding to $\frac{k'_y(x, y)}{k(x, x)}$ and $f(x)$ is differentiable.

Fredholm Equation of Third Kind

The equations $k(x, x)\, \phi(x) + \int_0^x k'_x(x, y)\, \phi(y)\, dy = f'(x)$ and $f(x) = [k(x, y)\, \Phi(y)]_{y=0}^{y=x} - \int_0^x k'_y(x, y)\, \Phi(y)\, dy$, where $k(x, x)$ vanishes at $x = 0$.

Linear Differential Equations and Volterra Integral Equations

Linear differential equation

$$a_0(x) \frac{d^n u}{dx^n} + a_1(x) \frac{d^{n-1} u}{dx^{n-1}} + \ldots + a_n(x)\, u = F(x) \quad \ldots(1)$$

with initial conditions,

$$u(0) = c_0,\ u'(0) = c_1,\ \ldots,\ u^{n-1}(0) = c_{n-1} \quad \ldots(2)$$

can be reduced to Volterra equation of second kind.

$$a_0(x)\, \phi(x) + \int_0^x k(x, y)\, \phi(y)\, dy = f(x) \quad \ldots(3)$$

$$\text{where } k(x, y) = \sum_{n=1}^{n} a_n(x) \frac{(x-y)^{n-1}}{(n-1)!} \quad \ldots(4)$$

$f(x) = \mathrm{F}(x) - c_{n-1}\, a_1(x) - (c_{n-1}\, x + c_{n-2})\, a_2(x) - \ldots - \left(c_{n-1}\frac{x^{n-1}}{(n-1)!} + \ldots + c_1 x + c_0\right) a_n(x)$ ...(5)

1. If the leading coefficient in (1) $a_0(x) = 1$ then the equation (1) reduced to, $\phi(x) + \int_0^x k(x, y)\, \phi(x)\, dy = f(x)$, where k and F are given by equations (4) and (5) respectively.

2. If $a_0(x) = 0$, the equation (1) reduces to Volterra equation of first kind.

Equations of Faltung Type (Closed Cycle/Convolution Type)

The equation $\phi(x) - \lambda \int_0^x k(x, y)\, \phi(y)\, dy = f(x)$ is called equation of Faltung type, which can be written as $y(x) = f(x) + k(x) * y(x)$ and solve by Laplace transformation.

Able Equation—The equation $\int_0^x \frac{\phi(y)}{(x-y)^{\alpha}})\, dy = f(x)$, $(0 < \alpha < 1)$ and its solution is given by

$$\phi(x) = \frac{\sin \alpha\pi}{\pi}\frac{d}{dx}\int_0^x \frac{f(y)}{(x-y)^{1-\alpha}}\, dy$$

$$= \frac{\sin(\alpha\pi)}{\pi}\left[\frac{f(0)}{x^{1-\alpha}} + \int_0^x \frac{f'(y)}{(x-y)^{1-\alpha}}\, dy\right]$$

Fredholm Equations

Fredholm Linear Operator—$\mathrm{F}_x[\phi(y)] \equiv \phi(x) - \lambda \int_0^1 \mathrm{K}(x, y)\, \phi(y)\, dy$.

Associated Fredholm Operator—$\overset{*}{\mathrm{F}}_x[\phi(y)] \equiv \phi(x) - \lambda \int_0^1 k(y, x)\, \phi(y)\, dy$.

Green's Formula—$\int \mathrm{F}_x[\phi(y)]\, \psi(x)\, dx = \int \overset{*}{\mathrm{F}}_x[\psi(y)]\, \phi(x)\, dx$.

Pincherie-Goursat Kernel—$k(x, y) = \sum_{k=1}^{n} \mathrm{X}_k(x)\, \mathrm{Y}_k(x)$, where $\{\mathrm{X}_i(x)\}$ and $\{\mathrm{Y}_i(x)\}$ are two sets of linearly independent (L_2) functions in the basic interval (0, 1).

1. To each quadratically integrable kernel $k(x, y)$, there corresponds a resolvent kernel H (x, y', λ) which is an analytic function of λ regular at least inside the circle $|\lambda| < \|k\|^{-1}$ and represented there by the power series

$-\mathrm{H}(x, y, \lambda) = k(x, y) + \lambda\, k_2(x, y) + \lambda^2\, k_3(x, y) + \ldots$

Let the domain of existence of the resolvent kernel in complex λ-plane be $\hbar$: Then if $f(x)$ belong to the class L_2, the unique, quadratically integrable solution of Fredholm equation of second kind.

$\phi(x) - \lambda \int k(x; y)\, \phi(y)\, dy = f(x)$, valid in $\hbar$ is given by the formula $\phi(x) = f(x) - \lambda \int \mathrm{H}(x, y; \lambda)\, f(y)\, dy$.

2. Fredholm integral equation of second kind, $\phi(x) - \lambda \int k(x, y)\, \phi(y)\, dy = f(x)$ has one and only one solution of the class L_2 given by formula $\phi(x) = f(x) - \lambda \int \mathrm{H}(x, y; \lambda)\, f(y)\, dy$. Here $\mathrm{H}(x, y; \lambda)$ the resolvent kernel is an analytic function of λ ...

If $|\lambda| \leq \|k\|^{-1}$, it is give by the Neumann series $-\mathrm{H}(x, y, \lambda) = k(x, y) + \lambda k_2(x, y) + \lambda^2 k_3(x, y) + \ldots$, where $k_2, k_3, \ldots$ are iterated kernels. The only exceptions are the singular points of $\mathrm{H}(x, y; \lambda)$ which coincide with the zeros (eigen values) of an analytic function $\mathrm{D}(\lambda)$ of λ. If kernel is the Pincheri-Goursat kernel then $\mathrm{D}(\lambda)$ is a polynomial.

3. If the homogeneous Fredholm integral equation has the only trivial solution, then the corresponding non-homogeneous equation always has one and only one solution. On the contrary if the homogeneous equation has non-trivial solutions, then the non-homogeneous integral equation has either no solution or infinity of solutions, depending on the given function $f(x)$.

4. A Volterra integral equation has no eigen values.

Symmetric Kernels and Orthogonal Systems of Functions

Orthogonal System of Functions—Given a system of functions $\{\phi_n\} = \phi_1(x), \phi_2(x), \ldots$ where $\phi_i, \phi_j\ (i \neq j)$ satisfies orthogonal condition $(\phi_i, \phi_j) = \int_a^b \phi_i(x)\, \phi_j(x)\, dx = 0\ (i \neq j)$ in basic interval (a, b).

Orthogonal System—An orthogonal system of functions $\{\phi_n\}$ satisfying normalizing condition $(\phi_i, \phi_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$.

Complete Orthonormal System—An orthogonal system is called complete in L_2 if Parseval's equation $\sum_{h=1}^{\infty} a^2{}_h = \int_a^b f^2(x)\, dx$ holds for any function $f(x)$ of L_2, where a_n is the Fourier coefficient of the function $f(x)$ with respect to the system $[\phi_h]$,

$$a_n = \int_a^b f(x)\,\phi_n(x)\,dx;\ n = 1, 2, \ldots$$

$k \in L^*_2$—if function $A(x)$ related to the kernel k is bounded, *i.e.* $\int k^2(x, y)\,dy = A^2(x) < N^2$.

Theorems

1. For a symmetric kernel $k(x, y) = k(y, x)$, the associated eigen function ψ_h coincide with the proper eigen functions $\{\phi_h\}$.

2. Any pair $\phi_h(x)$ and $\phi_k(x)$ of eigen functions of a symmetric kernel, corresponding to two different eigen values λ_h and λ_k satisfies the orthogonality condition

$$(\phi_h, \phi_k) \equiv \int_a^b \phi_h(x)\,\phi_k(x)\,dx = 0,\ (h \neq k)$$

3. **Process of Orthogonalization**—Given any (finite or denumerable) system of linearly independent L_2–functions $\psi_1(x)$, $\psi_2(x)$, … it is always possible to find constants hrs such that the functions

$$\phi_1(x) = \psi_1(x)$$
$$\phi_2(x) = h_{21}\psi_1(x) + \psi_2(x)$$
$$\phi_3(x) = h_{31}\psi_1(x) + h_{32}\psi_2(x) + h_{33}\psi_3(x)$$
$$\vdots$$
$$\vdots$$
$$\phi_n(x) = h_{n1}\psi_1(x) + h_{n2}\psi_2(x) + \ldots + h_{nn-1}\ \psi_{n-1} + \psi_n(x)$$

are orthogonal in the basic interval (a, b).

4. For a given system $\{\phi_h\}$, a given function $f(x)$ and a given number n of terms of the linear combination $c_1\phi_1(x) + c_2\phi_2(x) + \ldots$, the non-negative integral.

$$\begin{aligned} I_n &= \int_a^b f^2(x)\,dx - 2\sum_{h=1}^{n} a_h c_h + \sum_{n=1}^{n} c^2_h \\ &= \int_a^b f^2(x)\,dx - \sum_{h=1}^{n} a^2_h + \sum_{h=1}^{n} (a_h - c_h)^2 \end{aligned}$$

attains its minimal value iff the coefficients c_h coincide with the corresponding Fourier coefficients

$$a_h = \int_a^b f(x)\,\phi_h(x)\,dx,\ (h = 1, 2, \ldots n)$$

of the function $f(x)$.

And this minimal value is given by

$$\begin{aligned} I^*_n &= \int_a^b f^2(x)\,dx - \sum_{h=1}^{n} a^2_h \\ &= \left[\int_a^b f^2(x)\,dx - \sum_{n=1}^{\infty} a^2_h\right] + \sum_{h=n+1}^{\infty} a^2_h \end{aligned}$$

5. A necessary and sufficient condition for an L_2–function $f(x)$ to be approximated in the mean by a linear combination of $f(x)$ of a given orthonormal system is that Parseval's equation $\sum_{n=1}^{\infty} a^2_h = \int_a^b f^2(x)\,dx$ holds for given $f(x)$.

6. **Weyl Lemma**—A necessary and sufficient condition for the convergence in the mean over the interval (a, b) of a sequence. Of L_2–function $\{f_n\}$ to $f(x)$ is that for any $\varepsilon > 0$, $\exists$ an integer N such that for $m, n \geq N$, we have

$$\int_a^b [f_m(x) - f_n(x)]^2 dx < \varepsilon$$

7. **Rietz–Fischer Theorem**—Given sequence of L_2–functions $\{f_n\}$ over the interval (a, b) and for every $\varepsilon > 0$ there exist an integer N such that for $m, n \geq N$ we have $\int_a^b [f_m(x) - f_n(x)]^2\ dx \leq \varepsilon$, then there exist a function $f \in L_2$–function such that $\{f_n\}$ convergence to $f(x)$.

8. Approximation of general L_2–kernel by means of Pincherle–Goursat kernel–Any L_2–kernel $k(x, y)$ *i.e.*, any kernel for which both functions.

$$A(x) = \left[\int_0^1 k^2(x, y)\,dy\right]^{1/2}$$

and $$B(y) = \left[\int_0^1 k^2(x, y)\,dx\right]^{1/2}$$

exist almost everywhere in the basic interval (0, 1) and belong to the class L_2, can be decomposed into the sum of suitable Pincherle–Goursat kernel

$$S(x, y) = \sum_{k=1}^{n} X_k(x)\,Y_k(x)$$

and L_2–kernel $T(x, y)$ for $\varepsilon > 0$,

$$||T||^2 = \int_0^1\int_0^1 T^2(x, y)\,dxdy < \varepsilon^2$$

9. Every symmetric, non-zero L_2–kernel has at least one eigen value.

10. Any L_2–function $w(x)$ is orthogonal to all the eigen functions $\phi_h(x)$ of symmetric kernel $k(x, y)$ iff $\int k(x, y)\,w(y)\,dy = 0$.

11. **Hilbert-Schmidt Theorem**—If $f(x)$ can be written in the form

$$f(x) = \int k(x, y)\, g(y)\, dy \qquad \ldots(1)$$

where $k(x, y)$ is a symmetric L_2–kernel and $g(y)$ is an L_2–function, then $f(x)$ can also be represented by its Fourier series with respect to the orthonormal system h_n of eigen functions of $k(x, y)$ *i.e.*,

$$f(x) = \sum_{h=1}^{\infty} a_h \phi_h(x) \qquad \ldots(2)$$

where $a_n = \int f(x)\, \phi_h(x)\, dx$, $(h = 1, 2, \ldots)$ (2) moreover if $k \in L^*_2$ *i.e.*, $\int k^2(x, y)\, dy \le N^2$ (N = constant) then series (2) converges absolutely and uniformly for every $f(x)$ of type (1).

1. If the symmetric kernel $k(x, y)$ belong to class L_2, then all the corresponding iterated kernels $k_m(x, y)$, $(m \ge 2)$ can be represented by absolute and uniformly convergent series.

$$k_m(x, y) = \sum_{h=1}^{\infty} \lambda_n^{-m} \phi_h(x)\, \phi_h(y), \ (m = 2, 3, \ldots)$$

If $k \in L^*_2$ then each series converges uniformly.

2. The singular points of the resolvent kernel H corresponding to a symmetric L_2 kernel $K(x, y)$ are simple poles.

3. If the function $f(x)$ can be differentiated twice in the basic interval (0, 1) and if its second derivative $f''(x)$ belongs to the class L_2 and if $f(0) = f(1) = 0$, then $f(x)$ has the absolutely and uniformly convergent expansion

$$f(x) = \sum_{h=1}^{\infty} \alpha_h \sin(h\pi x), \ (0 \le x \le 1),$$

where $\alpha_h = 2\int_0^1 f(x) \sin(h\pi x)\, dx$, $(h = 1, 2, 3, \ldots)$

OBJECTIVE TYPE QUESTIONS

1. An arc of integration such that the integral $I = \int_a^b F\left(x, y, \frac{dy}{dx}\right) dx$, can be determined, where y is a function of x and $F = F(x, y, y')$ process a continuous partial derivatives is reffered as—
   (A) Admissible arc (B) Open arc
   (C) Closed arc (D) None of these

2. Let $y = S(x)$ be the equation of the extremal through the fixed point A and B for which the integral $I = \int_a^b F(x, y, y')\, dx$ is stationary. Let A and A' be two adjacent conjugate points (kinetic foci) on the curve. If (i) B lies between A and A' and (ii) $\frac{\partial^2 F}{\partial S'^2}$ has constant sign for all points of the arc AB, and $\frac{\partial^2 F}{\partial S'^2}$ is negative. Therefore for weak variation—
   (A) Integral I is a minimum
   (B) Integral I is a maximum
   (C) Integral I is zero
   (D) None of these

3. Let integral $I = \int_0^1 F(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t)\, dt$ where the q_i, $i = 1, 2, \ldots, n$ satisfy the n second order partial differential equations $\frac{\partial F}{\partial q_m} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_m}\right) = 0$ $(m = 1, 2, \ldots, n)$. If at every point of a sufficiently small range (t_0 to t_1) (i) $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ $(r = 1, 2, \ldots, n)$ all have constant sign, (ii) $\left(\frac{\partial^2 F}{\partial \dot{q}^2_r}\right)\left(\frac{\partial^2 F}{\partial \dot{q}^2_s}\right) > \left(\frac{\partial^2 F}{\partial \dot{q}_r \partial \dot{q}_s}\right)$, $\left(\begin{matrix} r = 1, 2, \ldots, n \\ s = 1, 2, \ldots, n \end{matrix}\ r \ne s\right)$. Then if the sign of $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ are all negative, then—
   (A) I is minimum (B) I is maximum
   (C) I is constant (D) None is true

4. The integral $I = \int_A^B G(x, y, \dot{x}, \dot{y})\, dt$, where the end points A and B are fixed and $G(x, y, \dot{x}, \dot{y})$ is homogeneous and of degree one is $\dot{x}$ and $\dot{y}$, have strong minimum, then—
   (A) $\frac{\partial^2 G}{\partial x \partial \dot{y}} - \frac{\partial^2 G}{\partial x \partial \dot{x}} - S(x, y, \dot{x}, \dot{y})(\ddot{x}\dot{y} - \ddot{y}\dot{x}) = 0$
   (B) $\frac{\partial^2 G}{\partial x \partial \dot{y}} - \frac{\partial^2 G}{\partial x \partial \dot{x}} = 0$
   (C) $\frac{\partial^2 G}{\partial x \partial \dot{y}} + \frac{\partial^2 G}{\partial x \partial \dot{x}} + S(x, y, \dot{x}, \dot{y})(\ddot{x}\dot{y} - \ddot{y}\dot{x}) = 0$
   (D) None of these

5. The integral $I = \int_A^B G(x, y, \dot{x}, \dot{y})\, dt$, where the end points A and B are fixed and $G(x, y, \dot{x}, \dot{y})$ is homogeneous and of degree one in $\dot{x}$ and $\dot{y}$ have strong maximum, then—
   (A) $\frac{\partial^2 G}{\partial x \partial \dot{y}} - \frac{\partial^2 G}{\partial x \partial \dot{x}} - S(x, y, \dot{x}, \dot{y})(\ddot{x}\dot{y} - \ddot{y}\dot{x}) = 0$
   (B) $\frac{\partial^2 G}{\partial x \partial \dot{y}} - \frac{\partial^2 G}{\partial x \partial \dot{x}} = 0$
   (C) $\frac{\partial^2 G}{\partial x \partial \dot{y}} + \frac{\partial^2 G}{\partial x \partial \dot{x}} + S(x, y, \dot{x}, \dot{y})(\ddot{x}\dot{y} - \ddot{y}\dot{x}) = 0$
   (D) None of these

6. The integral $I = \int_A^B G(x, y, \dot{x}, \dot{y})\, dt$, where the end points A and B are fixed and $G(x, y, \dot{x}, \dot{y})$ is homogeneous and of degree one in $\dot{x}$ and $\dot{y}$ have strong minimum, then—
   (A) $\left.\begin{array}{l} \frac{\partial G}{\partial x} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{x}}\right) = 0 \\ \frac{\partial G}{\partial y} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{y}}\right) = 0 \end{array}\right\}$
   (B) $\frac{\partial G}{\partial x} \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{x}}\right) = \frac{\partial G}{\partial y} \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{y}}\right) = 0$
   (C) $\frac{\partial G}{\partial x}\left(\frac{\partial G}{\partial \dot{x}}\right) = \frac{\partial G}{\partial y}\left(\frac{\partial G}{\partial \dot{y}}\right)$
   (D) None of these

7. The integral $I = \int_A^B G(x, y, \dot{x}, \dot{y})\, dt$, where the end points A and B are fixed and $G(x, y, \dot{x}, \dot{y})$ is homogeneous and of degree one in $\dot{x}$ and $\dot{y}$ have strong maximum, then—
   (A) $\left.\begin{array}{l} \frac{\partial G}{\partial x} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{x}}\right) = 0 \\ \frac{\partial G}{\partial y} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{y}}\right) = 0 \end{array}\right\}$
   (B) $\frac{\partial G}{\partial x} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{x}}\right) = \frac{\partial G}{\partial y} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{y}}\right) = 0$
   (C) $\frac{\partial G}{\partial x}\left(\frac{\partial G}{\partial \dot{x}}\right) = \frac{\partial G}{\partial y}\left(\frac{\partial G}{\partial \dot{y}}\right)$
   (D) None of these

8. The integral I has strong maximum if—
   (A) The arc AB of the arc of integration $\lceil e$, contains no point conjugate to either A or B
   (B) The arc AB of the arc of integration $\lceil e$, contains point conjugate to either A or B
   (C) The arc AB of the arc of integration $\lceil e$, contains point conjugate to both A or B
   (D) None of these

9. The integral $I = \int_A^B G(x, y, \dot{x}, \dot{y})\, dt$, where the end points A and B are fixed and $G(x, y, \dot{x}, \dot{y})$ is homogeneous and of degree one in $\dot{x}$ and $\dot{y}$ have strong maximum, then—
   (A) The variables x and y, which are functions of t
   (B) The variables x and y, which are not functions of t
   (C) The variables x and y, which are not functions
   (D) None of these

10. The integral I has strong maximum if—
   (A) The arc AB of the arc of integration $\lceil e$, contains no point conjugate to either A or B
   (B) The arc AB of the arc of integration $\lceil e$, contains point conjugate to either A or B
   (C) The arc AB of the arc of integration $\lceil e$, contains point conjugate to both A or B
   (D) None of these

11. The integral I has .... if—
   (1) The equation of $\lceil e$, the arc of integration, satisfy $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$,
   (2) The arc AB of $\lceil e$ contains no point conjugate to either A or B,
   (3) At all points of this arc $E = \frac{\partial^2 F}{\partial y'^2} < 0$. And at all points of this arc and for all finite values of p, $E(x, y, y', p) \le 0$
   (A) Strong maximum
   (B) Strong minimum
   (C) Constant
   (D) None of these

12. The integral I has strong maximum if—
   (A) At all points of the arc AB, of the arc of integration $\lceil e$, and for all finite values of p, $\frac{\partial^2 F(x, y, p)}{\partial p^2} < 0$

(B) At all points of the arc AB, of the arc of integration $\lceil e$, and for all finite values of p, $\dfrac{\partial^2 F(x, y, p)}{\partial p^2} > 0$

(C) At all points of the arc AB, of the arc of integration $\lceil e$, and for all finite values of p, $\dfrac{\partial^2 F(x, y, p)}{\partial p^2} = 0$

(D) None of these

13. The integral I has strong maximum if—

(A) The equation of $\lceil e$, the arc of integration, satisfies $\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$

(B) The equation of $\lceil e$, the arc of integration, satisfies $\dfrac{\partial F}{\partial y} - \left(\dfrac{\partial F}{\partial y'}\right) = 0$

(C) The equation of $\lceil e$, the arc of integration, satisfies $\dfrac{\partial F}{\partial y} = 0$

(D) The equation of $\lceil e$, the arc of integration, satisfies $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$

14. The integral I has .... if—

(1) The equation of $\lceil e$, the arc of integration, satisfy $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$,

(2) The arc of AB of $\lceil e$, contains no point conjugate to either A or B,

(3) At all points of this arc $E = \dfrac{\partial^2 F}{\partial y'^2} < 0$ and at all point of this arc and for all finite values of p, $E(x, y, y', p) \leq 0$

(A) Strong maximum

(B) Strong minimum

(C) Constant

(D) None of these

15. Let $y = S(x)$ be the equation of the extremal through the fixed point A and B for which the integral $I = \int_a^b F(x, y, y')\, dx$ is stationary. Let A and A' be two adjacent conjugate points (kinetic foci) on the curve. If (i) B lies between A and A' and (ii) $\dfrac{\partial^2 F}{\partial S'^2}$ has constant sign for all points of the arc AB, and $\dfrac{\partial^2 F}{\partial S'^2}$ is positive. Therefore for weak variation—

(A) Integral I is a maximum

(B) Integral I is zero

(C) Integral I is minimum

(D) None of these

16. Let integral $I = \int_{t_0}^{t_1} F(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, ; t)\, dt$ where the q_i, $i = 1, 2, \ldots, n$ satisfy the n second order partial differential equations $\dfrac{\partial F}{\partial q_m} - \dfrac{d}{dt}\left(\dfrac{\partial F}{\partial \dot{q}_m}\right) = 0$ $(m = 1, 2, \ldots, n)$. If at every point of a sufficiently small range t_0 to t_1 (i) $\dfrac{\partial^2 F}{\partial \dot{q}_r^2}$ $(r = 1, 2, \ldots, n)$ all have constant sign, (ii) $\left(\dfrac{\partial^2 F}{\partial \dot{q}_r^2}\right)\left(\dfrac{\partial^2 F}{\partial \dot{q}_s^2}\right) > \left(\dfrac{\partial^2 F}{\partial \dot{q}_r \partial \dot{q}_s}\right)$, $\left(\begin{matrix} r = 1, 2, \ldots, n \\ s = 1, 2, \ldots, n \end{matrix}\; r \neq s\right)$. Then if the sign of $\dfrac{\partial^2 F}{\partial \dot{q}_r^2}$ are all negative, then—

(A) I is minimum (B) I is maximum

(C) I is constant (D) None is true

17. The integral I has strong minimum if—

(A) The equation of $\lceil e$, the arc of integration, satisfies $\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$

(B) The equation of $\lceil e$, the arc of integration, satisfies $\dfrac{\partial F}{\partial y} - \left(\dfrac{\partial F}{\partial y'}\right) = 0$

(C) The equation of $\lceil e$, the arc of integration, satisfies $\dfrac{\partial F}{\partial y} = 0$

(D) The equation of $\lceil e$, the arc of integration, satisfies $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$

18. Let $y = S(x)$ be the equation of the extremal through the fixed points A and B for which the integral $I = \int_a^b F(x, y, y')\, dx$ is stationary. Let A and A' be two adjacent conjugate points

(kinetic foci) on the curve. If (i) B lies between A and A' and (ii) $\frac{\partial^2 F}{\partial S'^2}$ has constant sign for all points of the arc AB, then for weak variation I is a minimum—

(A) When $\frac{\partial^2 F}{\partial S'^2}$ is less than zero

(B) When $\frac{\partial^2 F}{\partial S'^2}$ is greater than zero

(C) When $\frac{\partial^2 F}{\partial S'^2}$ is equal to zero

(D) None of these

19. The integral I has strong minimum if—

(A) At all points of the arc AB, of the arc of integration $\lceil e$, and for all finite values of p, $\frac{\partial^2 F(x, y, p)}{\partial p^2} < 0$

(B) At all points of the arc AB, of the arc of integration $\lceil e$, and for all finite values of p, $\frac{\partial^2 F(x, y, p)}{\partial p^2} > 0$

(C) At all points of the arc AB, of the arc of integration $\lceil e$, and for all finite values of p, $\frac{\partial^2 F(x, y, p)}{\partial p^2} = 0$

(D) None of these

20. Let $I = \int_{t_0}^{t_1} F(q_1, q_2, \ldots, q_m, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t)\, dt$ where the q_i, $i = 1, 2, \ldots, n$ satisfy the n second order partial differential equations $\frac{\partial F}{\partial q_m} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_m}\right) = 0, (m = 1, 2, \ldots, n)$. If at every point of a sufficiently small range t_0 to t_1 (i) $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ $(r = 1, 2, \ldots, n)$ all have constant sign, (ii) $\left(\frac{\partial^2 F}{\partial \dot{q}^2_r}\right)\left(\frac{\partial^2 F}{\partial \dot{q}^2_s}\right) > \left(\frac{\partial^2 F}{\partial \dot{q}_r \partial \dot{q}_s}\right), \left(\begin{matrix} r = 1, 2, \ldots, n \\ s = 1, 2, \ldots, n \end{matrix}\ r \neq s\right)$. Then I is a minimum—

(A) If the signs of $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ are all negative

(B) If the signs of $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ are all positive

(C) If the signs of $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ varies from negative to positive

(D) None of these

21. Let $y = S(x)$ be the equation of the extremal through the fixed points A and B for which the integral $I = \int_a^b F(x, y, y')\, dx$ is stationary. Let A and A' be two adjacent conjugate points (kinetic foci) on the curve. If (i) B lies between A and A' and (ii) $\frac{\partial^2 F}{\partial S'^2}$ has constant sign for all points of the arc AB, then for weak variation I is a minimum—

(A) When $\frac{\partial^2 F}{\partial S'^2}$ is negative

(B) When $\frac{\partial^2 F}{\partial S'^2}$ is positive

(C) When $\frac{\partial^2 F}{\partial S'^2}$ is neither negative nor positive

(D) None of these

22. The integral I has strong minimum if—

(A) The arc AB of the arc of integration $\lceil e$, contains no point conjugate to either A or B

(B) The arc AB of the arc of integration $\lceil e$, contains point conjugate to either A or B

(C) The arc AB of the arc of integration $\lceil e$, contains point conjugate to both A or B

(D) None of these

23. The integral I has strong maximal if (1) the equation of $\lceil e$, the arc of integration, satisfy $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$, (2) The arc AB of $\lceil e$. contains no point conjugate to either A or B, (3) At all points of this arc $E = \frac{\partial^2 F}{\partial y'^2} < 0$ and at all points of this arc and for all finite values of p, E (x, y, y', p) is—

(A) Less than/equals to zero

(B) Less than zero

(C) Greater than/equals to zero

(D) Greater than zero

24. Let F be given functonal form, let $p_r = \frac{\partial z}{\partial x_r}$, where z is a function of $x_1, x_2, \ldots, x_n$ and let the integral $I = \iiint \ldots \int F(z, x_1, x_2, \ldots, x_n, p_1, p_2, \ldots, p_n)\, dx_1 dx_2 \ldots dx_n$ be taken through an n-dimensional region bounded by given fixed boundaries of dimension $n - 1$. Then the integral I is stationary when z is a solutions of—
   (A) The second-order partial differential equation
   $$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x_1}\left(\frac{\partial F}{\partial p_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial F}{\partial p_2}\right) - \cdots - \frac{\partial}{\partial x_n}\left(\frac{\partial F}{\partial p_n}\right) = 0$$
   (B) The second-order partial differential equation
   $$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x_1}\left(\frac{\partial F}{\partial p_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial F}{\partial p_2}\right) - \cdots - \frac{\partial}{\partial x_n}\left(\frac{\partial F}{\partial p_n}\right) = c$$
   (C) The first order partial differential equation
   $$\frac{\partial F}{\partial z} - \left(\frac{\partial F}{\partial p_1}\right) - \left(\frac{\partial F}{\partial p_2}\right) - \cdots - \left(\frac{\partial F}{\partial p_n}\right) = 0$$
   (D) None of these

25. Let the functional form of F be given, then the integral $I = \int_a^b F(x, y, y_1, y_2, \ldots, y_n)\, dx$ is stationary when y satisfies the equation—
   (A) $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial y_2}\right) - \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y_n}\right) = 0$
   (B) $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial y_2}\right) + \cdots + \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y_n}\right) = 0$
   (C) $(-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y_n}\right) = 0$
   (D) None of these

26. Let $y = S(x)$ be the equation of the extremal through the fixed points A and B for which the integral $I = \int_a^b F(x, y, y')\, dx$ is stationary. Let A and A' be two adjacent conjugate points (kinetic foci) on the curve. If (i) B lies between A and A' and (ii) $\frac{\partial^2 F}{\partial S'^2}$ has constant sign for all point of the arc AB, then for weak variation I is a minimum—
   (A) When $\frac{\partial^2 F}{\partial S'^2}$ is less than zero
   (B) When $\frac{\partial^2 F}{\partial S'^2}$ is greater than zero
   (C) When $\frac{\partial^2 F}{\partial S'^2}$ is equal to zero
   (D) None of these

27. The integral I has strong minimum if (1) the equation of $\lceil e$, the arc of integration, satisfy $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$, (2) The arc AB of $\lceil e$. contains no point conjugate to either A or B, (3) At all points of this arc $E = \frac{\partial^2 F}{\partial y'^2} < 0$ and at all points of this arc and for all finite values of p, $E(x, y, y', p)$ is—
   (A) Less than/equals to zero
   (B) Less than zero
   (C) Greater than/equals to zero
   (D) Greater than zero

28. Let $I = \int_{t_0}^{t_1} F(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t)\, dt$ where the q_i, $i = 1, 2, \ldots, n$ satisfy the n second order partial differential equations $\frac{\partial F}{\partial q_m} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_m}\right) = 0$ $(m = 1, 2, \ldots, n)$. If at every point of a sufficiently small range t_0 to t_1 (i) $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ $(r = 1, 2, \ldots, n)$ all have constant sign, (ii) $\left(\frac{\partial^2 F}{\partial \dot{q}^2_r}\right)\left(\frac{\partial^2 F}{\partial \dot{q}^2_s}\right) > \left(\frac{\partial^2 F}{\partial \dot{q}_r \partial \dot{q}_s}\right)$, $\left(\begin{matrix} r = 1, 2, \ldots, n \\ s = 1, 2, \ldots, n \end{matrix}\ r \neq s\right)$.
   Then I is a minimum—
   (A) If the signs of $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ are all negative
   (B) If the signs of $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ are all positive

(C) If the signs of $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ varies from negative to positive

(D) None of these

29. Given $y = S(x)$, $t = t(x)$ and $F = F(x, y, y')$ then second variation is defined as—

(A) $\varepsilon \int_a^b \left(t\frac{\partial F}{\partial S} - \frac{dt}{dx}\frac{\partial F}{\partial S'} \right) dx$

(B) $\int_a^b F(x, y, y')\, dx$

(C) $\frac{\varepsilon}{2!} \int_a^b (t^2 F_{00} + 2tt' F_{01} + t'^2 F_{11}) dx$

(D) None of these

30. In Legender's test the integral $I = \int_a^b F(x, y, y')\, dx$, is minimum of value of $\frac{\partial^2 F}{\partial y'^2}$ is—

(A) Positive

(B) Negative

(C) Neither positive nor negative

(D) None of these

31. Let the values of t_0 and t_1 and the functional form of F be given. Then the integral $\int_{t_0}^{t_1} F(q_1, q_2, \ldots, q_r, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n ; t)\, dt$ where the q_i, $i = 1, 2, \ldots, n$ are arbitrary functions of t, is stationary for weak variations when the q_i, $i = 1, 2, \ldots, n$ satisfy—

(A) n equations $\frac{\partial F}{\partial q_m} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}^2_m}\right) = 0$

$(m = 1, 2, \ldots, n)$

(B) $n + 1$ equations $\frac{\partial F}{\partial q_m} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_m}\right) = 0$

$(m = 1, 2, \ldots, n + 1)$

(C) $n - 1$ equation $\frac{\partial F}{\partial q_m} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_m}\right) = 0$

$(m = 1, 2, \ldots, n - 1)$

(D) None of these

32. Given $y = S(x)$, $t = t(x)$ and $F = F(x, y, y')$ then second variation is defined as—

(A) $\frac{\varepsilon}{2!}\int_a^b \left(t^2 \frac{\partial^2 F}{\partial S^2} - 2tt' \frac{\partial^2 F}{2S \partial S'} + t'^2 \frac{\partial^2 F}{\partial S'^2} \right) dx$

(B) $\frac{\varepsilon}{2!} \int_a^b (t^2 F_{00} + 2tt' F_{01} + t'^2 F_{11}) dx$

(C) (A) and (B) both

(D) None of these

33. Given $y = S(x)$, $t = t(x)$ and $F = F(x, y, y')$ then first variation is defined as—

(A) $\varepsilon \int_a^b \left(t\frac{\partial F}{\partial S} - \frac{dt}{dx}\frac{\partial F}{\partial S'} \right) dx$

(B) $\int_a^b F(x, y, y')\, dx$

(C) $\frac{\varepsilon}{2!} \int_a^b (t^2 F_{00} + 2tt' F_{01} + t'^2 F_{11}) dx$

(D) None of these

34. If $u_1(x)$ and $u_2(x)$ are independent solutions of the accessory equation, then … as x increases—

(A) The ratio $u_1(x)/u_2(x)$ steadily increases or steadily decreases

(B) The $u_1(x)/u_2(x)$ steadily increases only

(C) The ratio $u_1(x)/u_2(x)$ steadily decreases only

(D) None of these

35. If $u(x)$ is a solution of the accessory equation then—

(A) It cannot have double zero always

(B) It can have double zero always

(C) It may have double zero

(D) None of these

36. If $y = S(x, c_1, c_2)$ is the general solution of Euler's characteristic equation, then the equation for abscissa points conjugate to A is—

(A) $\frac{\partial y}{\partial c_1} = \left(\frac{\partial y}{\partial c_1}\right)_{x=a}$

(B) $\frac{\partial y}{\partial c_1} / \frac{\partial y}{\partial c_2} = \left(\frac{\partial y}{\partial c_1} / \frac{\partial y}{\partial c_2}\right)_{x=a}$

(C) $\frac{\partial y}{\partial c_2} = \left(\frac{\partial y}{\partial c_1}\right)_{x=a}$

(D) None of these

37. Let $y = S(x)$ be the equation of the extremal through the fixed point A and B for which the integral $I = \int_a^b F(x, y, y')\, dx$ is stationary. Let A and A' be two adjacent conjugate points (kinetic foci) on the curve. If (i) B lies

between A and A' and (ii) $\frac{\partial^2 F}{\partial S'^2}$ has constant sign for all points of the arc AB, then for weak variation I is a maximum—

(A) When $\frac{\partial^2 F}{\partial S'^2}$ is negative

(B) When $\frac{\partial^2 F}{\partial S'^2}$ is positive

(C) When $\frac{\partial^2 F}{\partial S'^2}$ is neither negative nor positive

(D) None of these

38. If (i) is the abscissa of the point A, (ii) $u_1(x)$ and $u_2(x)$ are independent solutions of the Jacobi's equation, then the equation for the abscissa of the points conjugate to A is—

(A) $\frac{1}{u_2(x)} = \frac{1}{u_2(a)}$ (B) $\frac{u_1(x)}{u_2(x)} = \frac{u_1(a)}{u_2(a)}$

(C) $u_1(x) = u_2(x)$ (D) None of these

39. If $t(a) = t(b) = 0$ and u is a solution of $\left\{F_{00} - \frac{d}{dx}(F_{01})\right\} u - \frac{d}{dx}\left\{F_{11}\frac{du}{dx}\right\} = 0$, then—

(A) $I_2 = \int_a^b F_{11}\left(t' - t\frac{u'}{u}\right)^2 dx$

(B) $I_2 = \frac{\varepsilon}{2!}\int_a^b F_{11}\left(t' - t\frac{u'}{u}\right)^2 dx$

(C) $I_2 = \frac{\varepsilon}{2!}\int_a^b F_{11}\left(\frac{u'}{u}\right)^2 dx$

(D) $I_2 = \frac{\varepsilon}{2!}$

40. The following result

$$\int_a^b (t^2F_{00} + 2tt'F_{01} + t'^2F_{11})dx =$$

$$\int_a^b \left\{t^2F_{00} - t^2\frac{d}{dx}(F_{01}) - t\frac{d}{dx}(t'F_{11})\right\}dx$$

holds if—

(A) $t(a) = t(b) = 0$

(B) $t(a) \neq t(b)$

(C) $t(a) = t(b)$

(D) None of these

41. The roots of $u(x) = 0$ are the abscissa of the points on the curve $y = S(x)$ conjugate to A if—

(A) $u(x)$ is a solution of the Jacobi's accessory equation

(B) a is the abscissa of the point A and $u(a) = 0$

(C) Both (A) and (B)

(D) None of these

42. If Euler's characteristic equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) = 0$ varishes identically, then the indefinite integral $\int F(x, y, y')\,dx$ can be evaluated as a function of—

(A) x and y (B) x only

(C) y only y (D) None of these

43. The accessory equation $\left\{F_{00} - \frac{d}{dx}(F_{01})\right\} u - \frac{d}{dx}\left\{F_{11}\frac{du}{dx}\right\} = 0$, where $\frac{\partial^2 F}{\partial S^2} = F_{00}$, $\frac{\partial^2 F}{\partial S \partial S'} = F_{01}$ and $\frac{\partial^2 F}{\partial S'^2} = F_{11}$ is referred as—

(A) Jacobi's equation

(B) Euler equation

(C) Volterra equation

(D) None of these

44. If Euler's characteristic equation $\left\{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) = 0\right\}$ vanishes identically, then the indefinite integral $\int F(x, y, y')\,dx$—

(A) Is a function of the end positions only

(B) Is not a function of the end positions

(C) Dependent on path

(D) None of these

45. If Euler's characteristic equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) = 0$ vanishes identically, then the indefinite integral $\int F(x, y, y')\,dx$—

(A) Has a value which is independent of the path of integration

(B) Has a value which is dependent of the path of integration

(C) Both (A) and (B) true

(D) None of these

46. The integral $\frac{\varepsilon}{2!}\int_a^b \left(t^2\frac{\partial^2 F}{\partial S^2} - 2tt'\frac{\partial^2 F}{\partial S \partial S'} + t'^2\frac{\partial^2 F}{\partial S'^2}\right) dx$, where $y = S(x)$, $t = t(x)$ and $F = F(x, y, y')$ is referred as—

(A) First variation (B) Second variation

(C) Third variation (D) Fourth variation

47. In Legendre's test the integral $I = \int_a^b F(x, y, y')\,dx$, is maximum if value of $\frac{\partial^2 F}{\partial y'^2}$ is—
(A) Positive
(B) Negative
(C) Neither positive nor negative
(D) None of these

48. For Legendre's test the sign of $\left(\frac{\partial^2 F}{\partial y'^2}\right)$ is—
(A) Constant throughout this range
(B) Increasing throughout the range
(C) Decreasing throughout the range
(D) None of these

49. For Legendre's test—
(A) Euler's characteristic equation is to be satisfied
(B) Euler's characteristic equation is not to be satisfied
(C) Jacobi's accessory equation is satisfied
(D) Jacobi's accessory equation is not to be satisfied

50. The integral $\int_a^b F(x, y')\,dx$, whose ends fixed, is stationary for weak variations if y satisfies the differential equation—
(A) $\frac{\partial F}{\partial y'} = c$, where c is an arbitrary constant
(B) $\frac{\partial F}{\partial y'} > c$, where c is an arbitrary constant
(C) $\frac{\partial F}{\partial y'} < c$, where c is an arbitrary constant
(D) None of these

51. The integral $\int_a^b F(y, y')\,dx$, whose ends are fixed, is stationary for weak variation if y satisfies the differential equation—
(A) $F - y'\frac{\partial F}{\partial y'} = c$, where c is an arbitrary constant
(B) $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$
(C) $F + y'\frac{\partial F}{\partial y'} = c$, where c is and arbitrary constant
(D) None of these

52. The integral $I = \int_a^b F(x, y, y')\,dx$, whose end points are fixed, is stationary for weak variations, if y satisfies—
(A) $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$
(B) $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = F$
(C) $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \geq 0$
(D) None of these

53. Given the function $F = F(x, y, y')$, the differential equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$, is referred as—
(A) Euler's characteristic equation
(B) Lagrange's equation
(C) Hamilton equation
(D) None of these

54. The integral $\frac{\varepsilon}{2!}\int_a^b\left(t^2\frac{\partial^2 F}{\partial S^2} - 2tt'\frac{\partial^2 F}{\partial S\partial S'} + t'^2\frac{\partial^2 F}{\partial S'^2}\right)dx$, where $y = S(x)$, $t = t(x)$ and $F = F(x, y, y')$ is referred as—
(A) First variation (B) Second variation
(C) Third variation (D) Fourth variation

55. The solution to Euler's characteristic equation is referred as—
(A) Extremals (B) Zeros
(C) Nulls (D) None of these

56. The integral $\varepsilon\int_a^b\left(t\frac{\partial F}{\partial S} - \frac{dt}{dx}\frac{\partial F}{\partial S'}\right)dx$, where $y = S(x)$, $t = t(x)$ and $F = F(x, y, y')$ is referred as—
(A) First variation (B) Second variation
(C) Third variation (D) Fourth variation

57. Let $y = S(x)$, and $y = S(x) + \in t(x)$ be two admissible curves of integral, $I = \int_a^b F\left(x, y, \frac{\partial y}{dx}\right)dx$ where $\in$ is an arbitrary constant independent of x and y, $t(x)$ is an arbitrary function of x, independent of $\in$, with this restriction on $t(x)$, the ordinate y is referred as—
(A) Strong variation (B) Weak variation
(C) Admissible arc (D) None of these

58. Let $y = S(x)$ and $y = S(x) + \in t(x)$ be two admissible curves of integral and $t(x)$ is an arbitrary function of x, independent of $\in$ then variation y is referred as—
(A) Strong variation (B) Weak variation
(C) Admissible arc (D) None of these

59. In integral equations—
(A) An unknown function appears under the integral sign
(B) A known function appears under the integral sign
(C) No function appears under the integral sign
(D) None of these

60. Let $K(x, y)$ be a given real function defined for $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $f(x)$ a real valued function defined for $0 \leq x \leq 1$ and λ an arbitrary complex number. Then $\phi(x) - \lambda \int_0^1 K(x, y)\, \phi(y)\, dy = f(x)$, $(0 \leq x \leq 1)$ is the—
(A) Linear Fredholm integral equation of second kind (for a function $\phi(x)$)
(B) Linear Fredholm integral equation of first kind (for a function $\phi(x)$)
(C) Volterra integral equation of first kind (for a function $\phi(x)$)
(D) Volterra integral equation of second kind (for a function $\phi(x)$)

61. Let $K(x, y)$ be a given real function defined for $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $f(x)$ a real valued function defined for $0 \leq x \leq 1$ and λ an arbitrary complex number. Then $\int_0^1 K(x, y)\, \phi(y)\, dy = f(x)$, $(0 \leq x \leq 1)$ is the—
(A) Linear Fredholm integral equation of second kind (for a function $\phi(x)$)
(B) Linear Fredholm integral equation of first kind (for a function $\phi(x)$)
(C) Volterra integral equation of first kind (for a function $\phi(x)$)
(D) Volterra integral equation of second kind (for a function $\phi(x)$)

62. Let $K(x, y)$ be a given real function defined for $0 \leq x \leq 1$, $0 \leq y \leq 1$, $f(x)$ a real valued function for $0 \leq x \leq 1$ and λ an arbitrary complex number, with $K(x, y) \equiv 0$ if $y > x$, the $\int_0^x K(x, y)\, \phi(y)\, dy = f(x)$ is the—
(A) Linear Fredholm integral equation of second kind (for a function $\phi(x)$)
(B) Linear Fredholm integral equation of first kind (for a function $\phi(x)$)
(C) Volterra integral equation of first kind (for a function $\phi(x)$)
(D) Volterra integral equation of second kind (for a function $\phi(x)$)

63. Let $K(x, y)$ be a given real function defined for $0 \leq x \leq 1$, $0 \leq y \leq 1$, $f(x)$ a real valued function for $0 \leq x \leq 1$ and λ an arbitrary complex number, with $K(x, y) \equiv 0$ if $y > x$, then $\phi(x) - \lambda \int_0^x K(x, y)\, \phi(y)\, dy = f(x)$ is the—
(A) Linear Fredholm integral equation of second kind (for a function $\phi(x)$)
(B) Linear Fredholm integral equation of first kind (for a function $\phi(x)$)
(C) Volterra integral equation of first kind (for a function $\phi(x)$)
(D) Volterra integral equation of second kind (for a function $\phi(x)$)

64. A kernel $K(x, y)$ which is quadratically integrable in the square $(0 \leq x \leq h, 0 \leq y \leq h)$ is referred as—
(A) L_1–kernel (B) L_2–kernel
(C) L_3–kernel (D) L_4–kernel

65. The equation $K(x, y)\, \phi(x) + \int_0^x K'_x(x, y)\, \phi(y)\, dy = f'(x)$ and $f(x) = [K(x, y)\, \Phi(y)]_{y=0}^{y=x} - \int_0^x K'_y(x, y)\, \Phi(y)\, dy$, where $K(x, y)$ vanishes at $x = 0$, are referred as—
(A) Fredholm equation of first kind
(B) Fredholm equation of second kind
(C) Fredholm equation of third kind
(D) Fredholm equation of fourth kind

66. A function f, whose norm is given by $\|f\|^2 = \int_0^h f^2(x)\, dx$ is—
(A) L–function (B) L_1–function
(C) L_2–function (D) L_3–function

67. A L_2–function is—
(A) A function f, which is L_2–function and bounded

(B) A function f, which is L_2–function only
(C) A function f, which is bounded only
(D) None of these

68. The Volterra integral equation of the second kind $\phi(x) - \lambda \int_0^x K(x, y)\,\phi(y)\,dy = f(x)$, $(0 \le x \le h)$, where the kernel $K(x, y)$ and the function (x), belongs to the class L_2 has—
(A) One and essentially only the solution in the same class L_2
(B) Two and essentially only two solutions in the same class L_2
(C) Three and essentially only three solutions in the same class L_2
(D) None of these

69. Any solution of homogeneous Volterra integral equations of second kind $\phi(x) - \lambda \int_0^x K(x, y)\,\phi(x)\,dy = 0$ in L_2–space is—
(A) Necessarily a zero function
(B) Necessarily a non zero function
(C) Absolute function
(D) None of these

70. A kernel $K(x, y)$ with norm $\|K\|$ such that $\|K\|^2 = \int_0^h \int_0^h K^2(x, y)\,dxdy \le N^2$ exist, where N^2, a constant is—
(A) L_1–kernel (B) L_2–kernel
(C) L_3–kernel (D) L_4–kernel

71. The equation $\int_0^x \frac{\phi(y)}{(x-y)^{\alpha}}\,dy = f(x), (0 \le \alpha < 1)$ is referred as—
(A) Fredholm equation
(B) Able equation
(C) Maxwell Equation
(D) Picard's equation

72. The solution to Able equation is given by—
(A) $\phi(x) = \frac{\sin \alpha\pi}{\pi} \frac{d}{dx} \int_0^x \frac{f(y)}{(x-y)^{1-\alpha}}\,dy$
(B) $\phi(x) = \lambda \int_0^1 K(x, y)\,\phi(y)\,dy + f(\alpha)$, $(0 \le x \le 1)$
(C) $\phi(x) = \lambda \int_0^1 K(x, y)\,\phi(y)\,dy$
(D) None of these

73. The solution of Able equation is given by—
(A) $\phi(x) = \frac{\sin \alpha\pi}{\pi}$
(B) $\phi(x) = \frac{\sin(\alpha\pi)}{\pi}\left[\frac{f(0)}{x^{1-\alpha}} + \int_0^x \frac{f'(y)}{(x-y)^{1-\alpha}}\,dy\right]$
(C) $\phi(x) = x$
(D) None of these

74. The following represents Linear Fredholm integral equation of second kind (for a function $\phi(x)$)—
(A) $\phi(x) - \lambda \int_0^1 K(x, y)\,\phi(y)\,dy = f(x)$, $(0 \le x \le 1)$
(B) $\int_0^1 K(x, y)\,\phi(y)\,dy = f(x)$
(C) $\int_0^x K(x, y)\,\phi(y)\,dy = f(x)$
(D) $\phi(x) - \lambda \int_0^x K(x, y)\,\phi(y)\,dy = f(x)$

75. The following represents Linear Fredholm integral equation of first kind (for a function $\phi(x)$)—
(A) $\phi(x) - \lambda \int_0^1 K(x, y)\,\phi(y)\,dy = f(x)$, $(0 \le x \le 1)$
(B) $\int_0^1 K(x, y)\,\phi(y)\,dy = f(x)$
(C) $\int_0^x K(x, y)\,\phi(y)\,dy = f(x)$
(D) $\phi(x) - \lambda \int_0^x K(x, y)\,\phi(y)\,dy = f(x)$

76. The following represents Volterra integral equation of second kind (for a function $\phi(x)$)—
(A) $\phi(x) - \lambda \int_0^1 K(x, y)\,\phi(y)\,dy = f(x)$, $(0 \le x \le 1)$
(B) $\int_0^1 K(x, y)\,\phi(y)\,dy = f(x)$
(C) $\int_0^x K(x, y)\,\phi(y)\,dy = f(x)$
(D) $\phi(x) - \lambda \int_0^x K(x, y)\,\phi(y)\,dy = f(x)$

77. A Volterra integral equation has—
(A) One eigen values
(B) Two eigen values
(C) Three eigen values
(D) No eigen values

78. The following represents Linear Fredholm integral equation of second kind (for a function $\phi(x)$)—
(A) $\phi(x) - \lambda \int_0^1 K(x, y)\, \phi(y)\, dy = f(x)$, $(0 \le x \le 1)$
(B) $\int_0^1 K(x, y)\, \phi(y)\, dy = f(x)$
(C) $\int_0^x K(x, y)\, \phi(y)\, dy = f(x)$
(D) $\phi(x) - \lambda \int_0^x K(x, y)\, \phi(y)\, dy = f(x)$

79. The Fredholm Linear operator is—
(A) $F_x[\phi(y)] \square\, \Phi(x) - \lambda \overset{1}{\underset{0}{\square}} K_Q(x, y)\, \phi(y)\, dy$
(B) $F^*_x[\phi(y)] \equiv \phi(x) - \lambda \int_0^1 K(y, x)\, \phi(y)\, dy$
(C) $\int F_x[\phi(y)]\, \psi(x)\, dx = \int F^*h[\psi(y)]\, \phi(x)\, dx$
(D) None of these

80. The associated Fredholm operator is—
(A) $F_x[\phi(y)] \square\, \Phi(x) - \lambda \overset{1}{\underset{0}{\square}} K_Q(x, y)\, \phi(y)\, dy$
(B) $F^*_x[\phi(y)] \equiv \phi(x) - \lambda \int_0^1 K(y, x)\, \phi(y)\, dy$
(C) $\int F_x[\phi(y)]\, \psi(x)\, dx = \int F^*_x[\psi(y)]\, \phi(x)\, dx$
(D) None of these

81. The Green's formula is—
(A) $F_x[\phi(y)] \square\, \Phi(x) - \lambda \overset{1}{\underset{0}{\square}} K_Q(x, y)\, \phi(y)\, dy$
(B) $F^*_x[\phi(y)] \equiv \phi(x) - \lambda \int_0^1 K(y, x)\, \phi(y)\, dy$
(C) $\int F_x[\phi(y)]\, \psi(x)\, dx = \int F^*_x[\psi(y)]\, \phi(x)\, dx$
(D) None of these

82. The Pincherie-Goursat kernel is given by—
(A) $K(x, y) = \sum_{k=1}^{n} Y_k(x)$, where $\{Y_i(x)\}$ is a set of linearly independent L_2–functions in the basic interval (0, 1)
(B) $K(x, y) = \sum_{k=1}^{n} X_k(x)$, where $\{X_i(x)\}$ is a set of linearly independent L_2–functions in the basic interval (0, 1)
(C) $K(x, y) = = \sum_{k=1}^{n} X_k(x)\, Y_k(x)$, where $\{X_i(x)\}$ and $\{Y_i(x)\}$ are two sets of linearly independent L_2–functions in the basic interval (0, 1)
(D) None of these

83. An orthogonal system of functions is a system of functions $\{\phi_n\} = \phi_1(x), \phi_2(x), \ldots$, where ϕ_i, ϕ_j $(i \neq j)$ satisfies orthogonal condition—
(A) $(\phi_i, \phi_j) \equiv \int_a^b \phi_i(x)\, \phi_j(x)\, dx = 0\ (i \neq j)$
(B) $(\phi_i, \phi_j) \equiv \int_a^b \phi_i(x)\, \phi_j(x)\, dx = 0$
(C) $(\phi_i, \phi_j) \equiv \int_a^b \phi_i(x)\, \phi_j(x)\, dx = 0\ (i = j)$
(D) None of these

84. An orthonormal system is—
(A) An orthogonal system of functions $\{\phi_n\}$ satisfying normalizing condition $(\phi_i, \phi_j) = \begin{cases} 0, & i = j \\ 1, & i \neq j \end{cases}$
(B) An orthogonal system of functions $\{\phi_n\}$ satisfying non normalizing condition
(C) An orthogonal system of functions $\{\phi_n\}$ satisfying normalizing condition $(\phi_i, \phi_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$
(D) None of these

85. An orthonormal system for which Parseval's equation $\sum_{h=1}^{\infty} a^2{}_h = \int_a^b f^2(x)\, dx$ holds for any function $f(x)$ of L_2, where a_n is the Fourier coefficient of the function $f(x)$ with respect to the system $\{\phi_h\}$, and $a_n = \int_a^b f(x)\, \phi_n(x)\, dx$, $n = 1, 2, \ldots$ is referrers as—
(A) Complete orthonormal system
(B) Complete normal system
(C) Symmetric system
(D) None of these

86. One states $K\, \varepsilon\, L^*{}_2$—
(A) If function $A(x)$ related to the kernel K is bounded

(B) If function A (x) related to the kernel K is not bounded
(C) If function A (x) related to the kernel K is linear function
(D) None of these

87. If function A (x) related to the kernel K is bounded then—
(A) $\int K^2 (x, y)\, dy = A^2 (x) > N^2$
(B) $\int K^2 (x, y)\, dy = A^2 (x) < N^2$
(C) $\int K^2 (x, y)\, dy = A^2 (x) = N^2$
(D) None of these

88. A necessary and sufficient condition for an L_2–function $f(x)$ to be approximated in the mean by a linear combination of $f(x)$ of a given orthonormal system is—
(A) That Parseval's equation $\sum_{h=1}^{\infty} a^2{}_h = \int_a^b f^2 (x)\, dx$ does not holds for given $f(x)$
(B) That Parseval's equation $\sum_{h=1}^{\infty} a^2{}_h = \int_a^b f^2 (x)\, dx$ holds for given $f(x)$
(C) Weyl lemma is satisfied
(D) None of these

89. The kernel is symmetric if—
(A) $K(x, y) = K(y, x)$
(B) $K(x, y) > K(y, x)$
(C) $K(x, y) < K(y, x)$
(D) None of these

90. For any pair $\phi_h (x)$ and $\phi_k (x)$ of eigen functions of a symmetric kernel corresponding to two different eigen values λ_h and λ_k satisfies orthogonality condition—
(A) $(\phi_h, \phi_k) \equiv \int_a^b \phi_h (x)\, \phi_k (x)\, dx = 0, (h \neq k)$
(B) $(\phi_h, \phi_k) \equiv \int_a^b \phi_h (x)\, \phi_k (x)\, dx = 1, (h = k)$
(C) $(\phi_h, \phi_k) \equiv \int_a^b \phi_h (x)\, \phi_k (x)\, dx = 0, (h = k)$
(D) None of these

91. A necessary and sufficient condition for the convergence in the mean over the interval (a, b) of a sequence of L_2–function $\{f_n\}$ to $f(x)$ is that for any $\varepsilon > 0$, $\exists$ an integer N such that for $m, n \geq N$, one have $\int_a^b [f_m (x) - f_n (x)]^2\, dx < \varepsilon$—
(A) Weyl lemma
(B) Parseval's lemma
(C) Reistz–Fischer theorem
(D) None of these

92. Given sequence of L_2–functions $\{f_n\}$ over the interval (a, b) and for every $\varepsilon > 0$ there exist an integer N such that for $m, n \geq N$ we have $\int_a^b [f_m (x) - f_n (x)]^2\, dx \leq \varepsilon$, then there exist a function $f \in L_2$–function such that $\{f_n\}$ converges to $f(x)$—
(A) Weyl lemma
(B) Parseval's lemma
(C) Riestc–Fischer theorem
(D) None of these

93. If the function $f(x)$ can be differentiated twice in the basic interval (0, 1) and if its …, $f''(x)$ belongs to the class L_2 and if $f(0) = f(1) = 0$, then $f(x)$ has the absolutely and uniformly convergent expansion $f(x) = \sum_{h=1}^{\infty} \alpha_h \sin (h\pi x)$, $(0 \leq x \leq 1)$, where $\alpha_h = 2 \int_0^1 f(x) \sin (h\pi x)\, dx$, $(h = 1, 2, 3, \ldots)$—
(A) First derivative (B) Second derivative
(C) Third derivative (D) Fourth derivative

94. If the function $f(x)$ can be differentiated … in the basic interval (0, 1) and if its second derivative $f''(x)$ belongs to the class L_2 and if $f(0) = f(1) = 0$, then $f(x)$ has the absolutely and uniformly convergent expansion $f(x) = \sum_{n=1}^{\infty} \alpha_h \sin (h\pi x)$, $(0 \leq x \leq 1)$, where $\alpha_h = 2 \int_0^1 f(x) \sin (h\pi x)\, dx$, $(h = 1, 2, 3, \ldots)$
(A) Once (B) Twice
(C) Thrice (D) None of these

95. Every symmetric, non-zero L_2–kernel has at least—
(A) One eigen value
(B) Two eigen value
(C) Three eigen value
(D) Four eigen value

96. Any L_2–function $w(x)$ is orthogonal to all the eigen functions $\phi_h (x)$ of symmetric kernel K (x, y) if—
(A) $\int K (x, y)\, w (y)\, dy > 0$

(B) $\int K(x, y)\, w(y)\, dy \neq 0$

(C) $\int K(x, y)\, w(y)\, dy = 0$

(D) None of these

97. If the symmetric kernel K (x, y) belongs to class L_2, then all the corresponding iterated kernels $K_m(x, y)$, $(m \geq 2)$ can be represented by … series—

(A) Absolute convergent

(B) Absolute and uniformly convergent

(C) Uniformly convergent

(D) None of these

98. The singular points of the resolvent kernel H corresponding to a symmetric L_2 kernel K (x, y) are—

(A) Simple poles

(B) Not simple pole

(C) Double poles

(D) None of these

99. If the function $f(x)$ can be differentiated twice in the basic interval (0, 1) and if its second derivative $f''(x)$ belongs to the class L_2 and …, then $f(x)$ has the absolutely and uniformly convergent expansion $f(x) = \sum_{h=1}^{\infty} \alpha_h \sin(h\pi x)$, $(0 \leq x \leq 1)$, where $\alpha_h = 2\int_0^1 f(x) \sin(h\pi x)\, dx$, $(h = 1, 2, 3, \ldots)$—

(A) If $f(0) > f(1)$

(B) If $f(0) \leq f(1)$

(C) If $f(0) = f(1) = 0$

(D) None of these

100. If $\int K(x, y)\, w(y)\, dy = 0$—

(A) Any L_2–function $w(x)$ orthogonal to all the eigen functions $\phi_h(x)$ of symmetric kernel K (x, y)

(B) Any L_2–function $w(x)$ is symmetric to all the eigen functions $\phi_h(x)$ of symmetric kernel K (x, y)

(C) Any L_2–function $w(x)$ if not orthogonal to all the eigen functions $\phi_h(x)$ of symmetric kernel K (x, y)

(D) None of these

Answers

1. (A)	2. (B)	3. (B)	4. (A)	5. (A)
6. (A)	7. (A)	8. (A)	9. (A)	10. (A)
11. (B)	12. (A)	13. (D)	14. (A)	15. (C)
16. (A)	17. (D)	18. (B)	19. (B)	20. (B)
21. (B)	22. (A)	23. (A)	24. (A)	25. (A)
26. (A)	27. (B)	28. (A)	29. (C)	30. (A)
31. (A)	32. (C)	33. (A)	34. (A)	35. (A)
36. (B)	37. (A)	38. (B)	39. (B)	40. (A)
41. (C)	42. (A)	43. (A)	44. (A)	45. (A)
46. (B)	47. (B)	48. (A)	49. (A)	50. (A)
51. (A)	52. (A)	53. (A)	54. (A)	55. (A)
56. (A)	57. (B)	58. (B)	59. (A)	60. (A)
61. (B)	62. (C)	63. (D)	64. (B)	65. (C)
66. (C)	67. (A)	68. (A)	69. (A)	70. (B)
71. (B)	72. (A)	73. (B)	74. (A)	75. (B)
76. (D)	77. (D)	78. (D)	79. (A)	80. (B)
81. (C)	82. (C)	83. (A)	84. (A)	85. (A)
86. (A)	87. (B)	88. (B)	89. (A)	90. (A)
91. (A)	92. (C)	93. (B)	94. (B)	95. (A)
96. (C)	97. (B)	98. (A)	99. (C)	100. (A)