Ordinary Differential Equations

Ordinary Differential Equation—Ordinary differential equation is a differential equation in which unknown functions depends on single independent variables.

Order (of differential equation) is the order of the highest derivative in the equation.

Degree (of differential equation) is the degree of highest of derivative occurring in differential equation, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.

Solution (of differential equation)/ Integral—

(a) A relation between variables, which satisfies the given differential equation.

(b) Any function which reduces the differential equation to an identity when substituted for dependent variable.

e.g., $y = \sin x$ is a solution (integral) of $y'' + y = 0$ because substituting $y = \sin x$ in $y'' + y = 0$ $\Rightarrow (\sin x)'' + \sin x = 0$, which is an identity.

General (complete) solution is a solution in which the number of arbitrary constants is equal to the order of the differential equation.

Particular solution is a solution obtained from general solution, by giving particular value to arbitrary constant.

Singular solution is an additional solution, other than general solution. It is a solution that can not be obtained by assigning particular values to arbitrary constants of general solution.

e.g. $$\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} - y = 0$$

General solution

$$y = cx - c^2$$

Particular solution—Giving arbitrary values to c in general solution.

Singular solution

$$y = \frac{x^2}{4}$$

which can not be obtained from general solution by substituting value to c.

First Order Differential Equation

The general form

$$F(x, y, y') = 0, \; y' = \frac{dy}{dx} \quad \text{(Implicit from)}$$

$$y' = f(x, y) \quad \text{(Explicit form)}$$

Initial Value Problem

$$y' = f(x, y), \; y(x_0) = y_0$$

where x_0 and y_0 are given values

Solution—A function $y = f(x)$ such that $y' = f(x, y)$ becomes identity if we replace y by h and y' by h'.

General Solution—A solution involving an arbitrary constant.

Particular Solution—A solution obtained by given particular value to an arbitrary constant of general solution.

(A) Solution of different differential equations—

$$y' = f(x)$$

$$\frac{dy}{dx} = f(x)$$

$$\Rightarrow \quad dy = f(x)\, dx$$

Integration gives

$$y = \int f(x)\, dx + c$$

(B) Separable differential equation—

$$y' = \frac{f(x)}{g(y)}$$

$$\Rightarrow \quad g(y)\, dy = f(x)\, dx$$

Integration gives

$$\int g(y)\, dy = \int f(x)\, dx + c$$

Equation reducible to separable form,

(a) $\quad y' = g(y/x)$

let $\quad y = ux$

$$\Rightarrow \quad y' = u'x + u \quad \ldots(a)$$

Substitution gives

$$u'x + u = g(u) \Rightarrow x\frac{du}{dx} + u = g(u)$$

$$\Rightarrow \quad \frac{du}{g(u) - u} = \frac{dx}{x} \quad \ldots (b)$$

(b) $y' = f(ax + by)$ substitution $z = ax + by$, $z' = a + by'$ gives equation (a).

(c) $y' = f\left(\frac{ax + by + c}{px + gx + r}\right)$

(i) $c = r$ equation (a)

(ii) $\frac{a}{p} = \frac{b}{q}$ equation (b)

(iii) Substitute $x = u + \alpha,\ y = v + \beta$

$\therefore \ \frac{dy}{dx} = \frac{dv}{du}$, where α, β solution of

$$\begin{cases} a\alpha + b\beta + c = 0 \\ p\alpha + q\beta + c = 0 \end{cases}$$

gives an equation (a) in u and v.

Existence and Uniqueness Theorem for $\frac{dy}{dx} = f(x, y)$

Lipschitz Condition—If $f(x, y)$ is a function defined for (x, y) in a domain D, then the function $f(x, y)$ is said to satisfy the Lipschitz condition on D. If there exists a positive constant K such that $|f(x, y) - f(x_1, y_1)| \leq K\,|(y_2 - y_1)|$ for every pair of points (x, y_2) and (x, y_1) in D, the constant K is called Lipschitz constant.

Existence Theorem—The initial value problem $\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$ has atleast one solution $y(x)$ provided the function $f(x, y)$ is continuous and bounded for all values of x in a domain D and there exists positive constants M and K such that $|f(x_1, y)| \leq M$ and satisfies the Lipschitz condition $|f(x, y_2) - f(x, y_1) \leq K\,|y_2 - y_1|$ for all points in D.

Uniqueness Theorem—The initial value problem $\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$ has a unique solution provided the function $f(x, y)$ is continuous and bounded for all values of x in domain D and there exists positive constants M and K such that $|f(x, y)| \leq M$ and satisfies the Lipschitz condition $|f(x, y_2) - f(x, y_1)| \leq K|y_2 - y_1)$ for all points in domain D.

Existence and Uniqueness Theorem—The initial value problem $\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$, has a unique solution for all values of x in the range $|x - x_0| \leq a$ provided the function $f(x, y)$ is continuous and satisfy the conditions.

(i) $|f(x, y)| \leq M$

(ii) $|f(x, y_2) - f(x, y_1)| \leq K\,|y_2 - y_1|$ for all values of x and y. M, K are positive constants in the rectangle R defined by $|x - x_0| \leq a$ and $|y - y_0| \leq Ma$.

Theorem—If δ is either a rectangle $|x - x_0| \leq a$, $|y - y_0| \leq b$ $(a, b > 0)$ or a strip $|x - x_0| \leq a$, $|y| \leq \infty$, $(a > 0)$ and if f is real valued function defined on δ such that $\delta f/\delta y$ exists, is continuous on S and

$$\left|\frac{\delta}{\delta y} f(x, y)\right| \leq K,\ (x, y)\ \Sigma\delta, \text{ for } K \geq 0, \text{ then } f$$

satisfies a Lipschitz condition on δ with Lipschitz constant K.

Systems of Linear First Order Differential Equations

An equation of the form $\frac{dy}{dx} + Py = Q$.

where P and Q are functions of x (or constants) is called a linear equation of the first order.

Suppose R is an integrating factor, then the left hand side of $R\frac{dy}{dx} + RPy = RQ$ is the differential coefficient of some product. Since the term $R\frac{dy}{dx}$ can only be derived by differentiating Ry, we put.

$$R\frac{dy}{dx} + R \cdot Py = \frac{d}{dx}(Ry) = R\frac{dy}{dx} + y\frac{dR}{dx}$$

i.e., $$RP = \frac{dR}{dx}$$

or $$\log R = \int P\,dx, \text{ when } R = e^{\int P dx}$$

Hence, we have determined the integrating factor, R and so we have the rule.

To solve $\frac{dy}{dx} + Py = Q$, multiply both sides by $e^{\int P dx}$ which is an integrating factor and then we get the integral as

$$y \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} \cdot dx$$

Linear Ordinary Differential Equations of Higher Order with Coefficients

An ordinary linear differential equation of n^{th} order has the form

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_{n-1}\frac{dy}{dx} + P_n y = 0 \qquad \ldots(1)$$

where $P_1, P_2, \ldots P_n$ are functions of x or contains and do not contain y or derivatives of y.

If $P_1, P_2, \ldots P_n$ are constants and Q is any function of x, the equation of the form given by (1) are called ordinary differential equation with constant coefficients.

Theorem—If $y = f(x)$ is the general solution of

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_n y = 0 \qquad \ldots(2)$$

and $y = \phi(x)$ is the solution of equation (1), then

$$y = f(x) + \phi(x)$$

is the general solution of

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_{n-1}\frac{dy}{dx} + P_n y = Q \qquad \ldots(3)$$

Substituting the values of y from (3) in (1), we get the left hand side of (1),

$$= \left(\frac{d^n f}{dx^n} + P_1 \frac{d^{n-1}f}{dx^{n-1}} + \ldots + P_n f\right) + \left(\frac{d^n \phi}{dx^n} + P_1 \frac{d^{n-1}\phi}{dx^{n-1}} + \ldots + P_n \phi\right)$$

If the coefficient of $\frac{d^n y}{dx^n}$ is not unity, is can always be made unity by dividing the equation by its, coefficient. Thus (1) is the most general form.

Now, since $y = f(x)$ is a solution of (2), the first group of terms within the brackets is zero. Similarly the second group of terms is equal to Q, since $y = \phi(x)$ is a solution of (1).

Hence (3) is a solution of (1).

This theorem enables us to divide the method of solving a linear equation into two parts.

First we find the general solution of the equation (2) which we will denote, say, by

$$y = f_1(c_1, c_2, \ldots\ldots c_n, x)$$

Next, we find a solution of (1) which does not contain an arbitrary constant. Let us denote this solution, say, by

$$y = f_2(x)$$

$$\therefore \qquad y = f_1(c_1, c_2, \ldots\ldots cn_x) + f_2(x)$$

gives evidently, the general solution of equation (1)

Linear Second Order Ordinary Differential Equations with Variable Coefficients—Linear equations of second order of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad \ldots(1)$$

where P, Q and R are functions of x.

If an integral included in the complementary function of a second order linear differential equation be known its order can be depressed and the complete solution can be found.

Let $y = u$ be an integral in the complementary function of (1). Put $y = uv$, so that $y_1 = u_1 v + v_1 u$ and $y_2 = u_2 v + 2u_1 v_1 + uv_1$.

Substituting the values of y, y_1 and y_2 in (1), we get

$$uv_2 + (2u_1 + Pu)\,v_1 + (u_2 + Pu_1 + Qu)\,v = R \qquad \ldots(2)$$

The coefficient of v varnishes since $y = u$ is a solution of

$$y_2 + Py_1 + Qy = 0$$

Thus, (2) becomes

$$v_2 + \left(2\frac{u_1}{u} + P\right)v_1 = \frac{R}{u}$$

This is a linear equation in v_1. Hence v_1 can be determined; and then

$y = u\int v_1 dx + c_1 u$ is the integral of equation (1), the other constant of integration will occur in expression for v_1. Thus we get the complete primitive.

Change of Independent Variable—Let the linear equation of second order be

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

where P, Q and R are functions of x and let the independent variable be changed from x to z, z being a given function of x.

Since $\displaystyle \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$

and $\displaystyle \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right)$

$$= \frac{d}{dz}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right)\frac{dz}{dx}$$

$$= \frac{d^2y}{dx^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\cdot\frac{d^2z}{dx^2}$$

the original equation becomes.

$$\left(\frac{dz}{dx}\right)^2\frac{d^2y}{dx^2} + \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)\frac{dy}{dz} + Qy = R$$

or $$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \qquad \ldots(1)$$

where $$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

and $$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

P_1, Q_1, R_1 are functions of x as shown above but can be readily expressed as functions of z by the given relation between z and x.

If by equating $\frac{Q}{\left(\frac{dz}{dx}\right)^2}$ to a constant quantity we find that P, also becomes constant then equation (2) is at once integrable. Since z is quite arbitrary. It may therefore be chosen to satisfy an assignable condition. Thus, we may choose z to make the coefficient of $\frac{dy}{dz}$ vanish hence if we put

$P_1 = 0$ or $\frac{d^2z}{dx^2} + P\frac{dz}{dx} = 0$ *i.e.*, $z = \int e^{-\int P dx}\,dx$.

Above is the required relation between z and x which will make $P_1 = 0$ and the equation reduce to the form

$$\frac{d^2y}{dz^2} + Q_1 y = R_1$$

If the value of Q_1, comes out to be a constant or a constant divided by z^2, then equation (1) becomes integrable.

If we choose z such that $Q_1 = a^2$, then

$$a\frac{dz}{dx} = \sqrt{Q}$$

$$\therefore \qquad az = \int\sqrt{Q}\,dx$$

Now the equation will be reduced to

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + a^2y = R$$

If P_1 comes out to be a constant, then the above equation can be easily integrated.

Differential Equation with Variable Coefficients

Consider the differential equation of the type

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \ldots(1)$$

where R, S, T are continuous function of x and y possessing continuous partial derivatives and

$$r = \frac{\delta^2 z}{\delta x^2},\ s = \frac{\delta^2 z}{\delta y\delta x},\ t = \frac{\delta^2 z}{\delta y^2}$$

Sometimes r may be written as u_{zz}, s as u_{yx}, t as u_{yy}.

Classification

If $S^2 - 4RT > 0$, the differential equation of this type is hyperbolic.

If $S^2 - 4RT = 0$, the differential equation of this types is parabolic .

If $S^2 - 4RT < 0$, the differential equation of this type is elliptic.

Power Series Method—The power series method is the method for solving linear differential equations with variable coefficients.

Power Series—$\sum_{m=0}^{\infty} a_m\,(x - x_0)^m = a_0 + a_1 (x - x_0) + a_2\,(x - x_0)^2 + \ldots$

where a_0, a_1 … are constants (coefficients) and x_0 a constant called centre of the series.

Power Series in Power of x—$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1x + a_2x^2 + \ldots\ldots$

Real Analytic Function—A real analytic function $f(x)$ is called analytic at a point $x = x_0$ if it can be represented by a power series of $x - x_0$ with radius of convergence.

$$R = \frac{1}{\lim_{m\to\infty}\sqrt[m]{|a_m|}} > 0$$

Existence of Power Series Solution—If p, q, r in differential equation $y'' + p(x)\,y' + q(x)\,y = r(x)$ are analytic at $x = x_0$, then every solution of differential equation is analytic at $x = x_0$ and can be represented by power series in power of $x - x_0$ with radius of convergence $R > 0$.

Legendre's Equation and Polynomials

Legendreis equation—

$$(1 - x^2)\, y'' - 2xy' + n\,(n + 1)\, y = 0$$

where n is a parameter and a real number.

The solution of Legendre's equation is called Legendre function.

Legendre's polynomial (of degree n)—

$$\mathrm{P}_n(x) = \sum_{m=0}^{\mathrm{M}} (-1)^m \frac{\lfloor 2n - 2m}{2^n \lfloor m \, \lfloor n - m \, \lfloor m - 2m} x^{n-2m}$$

$$= \frac{\lfloor 2n}{2^n \left(\lfloor n\right)^2} - \frac{\lfloor 2n - 2}{2^n \lfloor 1 \, \lfloor n - 1 \, \lfloor n - 2} x^{n-2} + \dots$$

where $\mathrm{M} = \frac{n}{2}$ (n is even) or $\mathrm{M} = \frac{n+1}{2}$ (n is odd)

Rodriques formula—

$$\mathrm{P}_n(x) = \frac{1}{2^n \lfloor n} \frac{dx}{dx^n} [(x^2 - 1)^n]$$

Recurrence relations for Legendre polynomials—

$$(2n + 1)\, x\, \mathrm{P}_n(x) = (n + 1)\, \mathrm{P}_{n+1}(x) + n\mathrm{P}_{n-1}(x)$$

$$n\mathrm{P}_n = x\mathrm{P}'_n - \mathrm{P}'_{n-1}$$

$$(2n + 1)\, \mathrm{P}_n = \mathrm{P}'_{n+1} - \mathrm{P}'_{n-1}$$

$$(n + 1)\, \mathrm{P}_n = \mathrm{P}'_{n+1} - x\mathrm{P}'_n$$

$$(1 - x^2)\, \mathrm{P}'_n = n\,(\mathrm{P}_{n-1} - x\mathrm{P}_n)$$

$$(1 - x^2)\, \mathrm{P}'_n = (x + 1)\,(x\mathrm{P}_n - \mathrm{P}_{n+1})$$

Laplace first integral for $\mathrm{P}_n(x)$—

$$\mathrm{P}_n(x) = \frac{1}{\pi} \int_0^{\pi} \{x \pm \sqrt{(x^2 - 1)} \cos \phi\}^n \, d\phi$$

Laplace second integral for $\mathrm{P}_n(x)$—

$$\mathrm{P}_n(x) = \frac{1}{\pi} \int_0^{\pi} \frac{d\phi}{\{x \pm \sqrt{(x^2 - 1)} \cos \phi\}^{n+1}}$$

Generating function for $\mathrm{P}_n(x)$—

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n\, \mathrm{P}_n(x).$$

Bessel's Equations and Bessel's Functions

Bessel's differential equation—

$$x^2 y'' + xy' + (x^2 - n^2)\, y = 0$$

where n is a parameter and real number.

Bessel functions of first kind (of order n)—

$$\mathrm{J}_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} \lfloor m \, \lfloor m + n}$$

General Solution of Bessel's Equation—If n is not an integer, the general solution of Bessel's equation for all $x \neq 0$ is $y(x)$

$$= c_1\, \mathrm{J}_n(x) + c_2\, \mathrm{J}_{-n}(x)$$

Linear Dependence of Bessel Function—For integer n, the Bessel function $\mathrm{J}_n(x)$ and $\mathrm{J}_{-n}(x)$ are linearly dependent because.

$$\mathrm{J}_{-n}(x) = (-1)^n\, \mathrm{J}_n(x).$$

Recurrence relations for $\mathrm{J}_n(x)$—

$$\frac{d}{dx}\{x^n\, \mathrm{J}_n(x)\} = x^n\, \mathrm{J}_{n-1}(x)$$

$$\frac{d}{dx}\{x^{-n}\, \mathrm{J}_n(x)\} = -x^{-n}\, \mathrm{J}_{n+1}(x)$$

$$\mathrm{J}_{n-1}(x) + \mathrm{J}_{n+1}(x) = \frac{2n}{x} \mathrm{J}_n(x)$$

$$\mathrm{J}_{n-1}(x) - \mathrm{J}_{n+1}(x) = 2\mathrm{J}'_n(x)$$

$\mathrm{J}_n(x)$ for $n = \pm\frac{1}{2}$—

$$\mathrm{J}_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

$$\mathrm{J}_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

Elementary Bessel's Functions—Bessel functions $\mathrm{J}_n(x)$, $n = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \dots$ are elementary functions and can be expressed by finitely many sine and cosines and power of x.

Generating Functions of $\mathrm{J}_n(x)$—

$$e^{\frac{1}{2}x\left(t - \frac{1}{t}\right)} = \sum_{-\infty}^{\infty} t^n\, \mathrm{J}_n(x)$$

Some Solved Examples

Example 1. Show that $x\mathrm{L}_n''(x) + (1 - x)\, \mathrm{L}_n'(x) + n\mathrm{L}_n(x) = 0$.

Solution : Since $\mathrm{L}_n(x)$ satisfies Laquerre's equation

$$xy'' + (1 - x)\, y' + ny = 0$$

$$\Rightarrow x\mathrm{L}_n''(x) + (1 - x)\, \mathrm{L}_n'(x) + n\mathrm{L}_n(x) = 0$$

Example 2. Show that $L_n'(0) = -n\lfloor n$

Solution : Since

$$\sum_{n=0}^{\infty} \frac{L_n(x)}{\lfloor n} t^n = \frac{1}{1-t} e^{-\frac{xt}{1-t}}$$

Differentiating with respect to x, we have

$$\sum_{n=0}^{\infty} \frac{L_n(x)}{\lfloor n} t^n = -\frac{1}{(1-t)^2} e^{-\frac{xt}{1-t}}$$

For $x = 0$,

$$\sum_{n=0}^{\infty} \frac{L_n'(0)}{\lfloor n} t^n = -\frac{1}{(1-t)^2} = -t(1-t)^{-2}$$

$$= -t(1 + 2t + 3t^2 + \ldots nt^{n-1} + \ldots)$$

$$= -\sum_{n=1}^{\infty} nt^n$$

Equating the coefficient of t^n,

$$\frac{L_n'(0)}{\lfloor n} = -n$$

$$\Rightarrow \quad Ln'(0) = -n\lfloor n$$

Example 3. Classify and solve the following :

$u_{xx} - 2\sin x u_{xy} - \cos^2 x u_{yy} - \cos x u_y = 0.$

Solution : Comparing with the general second order differential equation, we have

$A = 1$, $B = -2\sin x$, $C = -\cos^2 x$, $D = 0$, $E = -\cos x$, $F = 0$, $G = 0$. The discriminant $B^2 - 4AC = 4(\sin^2 x + \cos^2 x) = 4 > 0$

Hence, the given partial differential equation is hyperbolic.

The characteristic equation are

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -\sin x - 1$$

and

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = 1 - \sin x$$

$$\Rightarrow \quad y = \cos x - x + c_1$$

and

$$\mu = \cos x + x + c_2$$

$$\Rightarrow \quad \zeta = x + y - \cos x = c_1$$

$$\eta = -x + y - \cos x = c_2$$

Integrating with respect to ζ, we obtain $\mu_\eta = f(\eta)$

Where f is arbitrary. Integrating once again with respect to η, we have

$$\mu = \int f(\eta)\, d\eta + g(\zeta)$$

$$\Rightarrow \quad \mu = \psi(\eta) + g(\zeta)$$

where $g(\zeta)$ is and arbitrary function. Returning to the old variables x, y, the solution of the given partial differential equation is

$$\mu(x, y) = \psi(y - x - \cos x) + g(y + x - \cos x)$$

Example 4. Show that

$$\int_0^x x^{-n} J_{n+1}(x)\, dx = \frac{1}{2^n \lceil (x+1)} - x^n J_n(x),\ n > 1$$

Solution : Since

$$\frac{d}{dx}\{x^{-n} J_n(x)\} = -x^{-n} J_{n+1}(x)$$

Integrating, we get

$$[x^{-n} J_n(x)]_0^x = \int_0^x x^{-n} J_{n+1}(x)\, dx$$

$$\Rightarrow \int_0^x x^{-n} J_{n+1}(x)\, dx$$

$$= -x^{-n} J_n(x) - \lim_{x \to 0} \left[\frac{J_n(x)}{x^n}\right]$$

$$= -x^n J_n(x) - \lim_{x \to 0} \frac{1}{x^n} \left[\frac{x^n}{2^n \lceil n+1} \left\{1 - \frac{x^2}{2 \cdot 2(n+1)} + \ldots\right\}\right]$$

$$= -x^{-n} J_n(x) - \frac{1}{2^n \lceil (n+1)}$$

Example 5. Prove that : $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$.

Solution : Since

$$2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$$

$$\Rightarrow \quad J_{n+1}(x) = \frac{1}{x}[2n J_n(x) - x J_{n-1}(x)] \quad \ldots(1)$$

For $n = 1, 2, 3$, we have

$$J_2(x) = \frac{1}{x}[2J_1(x) - xJ_0(x)] \quad \ldots(2)$$

$$J_3(x) = \frac{1}{x}[4J_2(x) - xJ_1(x)] \quad \ldots(3)$$

$$J_4(x) = \frac{1}{x}[6J_3(x) - x J_2(x)] \quad \ldots(4)$$

By (2) and (3)

$$J_3(x) = \frac{8}{x^2} J_1(x) - \frac{4}{x} J_0(x) - J_1(x)$$

$$= \frac{8-x^2}{x^2} J_1(x) - \frac{4}{x} J_0(x) \quad \ldots(5)$$

By (4) and (5), we have $J_4(x) = \frac{48-6x^2}{x^3} J_1(x)$ $- \frac{24}{x^2} J_0(x) - \frac{2}{x} J_1(x) + J_0(x)$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$$

Example 6. Show that

(i) $(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)$

(ii) $P_n(x) = x P_n'(x) - P'_{n-1}(x)$

(iii) $(2n+1) P_n(x) = P'_{n+1}(x) - P'_{n+1}(x)$.

Solution :

(i) Since $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$.

Differentiating partially with respect to t, we get,

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum n P_n(x) t^{n-1}$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-1/2} = (1-2xt+t^2) \sum n P_n(x) t^{n-1}$$

Equating coefficients of t^n from both sides we get,

$$x P_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2nx P_n(x) + (n-1) P_{n-1}(x)$$

$$\Rightarrow \quad (n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)$$

(ii) Since $(1-2xt+t^2)^{-1/2}$

$$= \sum_{n=0}^{\infty} P_n(x) t^n \quad \ldots(1)$$

Differentiating partially with respect to x

$$\Rightarrow \quad -\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2t) = \sum P_n'(x) t^n$$

$$\Rightarrow t(1-2xt+t^2)^{-3/2} = \sum P_n'(x) t^n \quad \ldots(2)$$

Again differentiating (1) partially with respect to t, we have

$$\Rightarrow \quad (x-t)(1-2xt+t^2)^{-3/2} = \sum n P_n(x) t^{n-1} \quad \ldots(3)$$

Dividing (3) by (2), we get

$$\frac{x-t}{t} = \frac{\sum n P_n(x) t^{n-1}}{\sum n P_n'(x) t^n}$$

$$\Rightarrow \quad \sum n P_n(x) t^n = (x-t) \sum P_n'(x) t^n$$

Equating coefficients of t^n from both sides, we get

$$n P_n(x) = x P_n'(x) - P'_{n-1}(x)$$

(iii) Since $(1-2xt+t^2)^{-1/2}$

$$= \sum_{n=0}^{\infty} P_n(x) t^n \quad \ldots(1)$$

Differentiating with respect to x, we get

$$(n+1) P'_{n+1}(x) = (2n+1) P_n(x) + (2n+1) x P'_n(x) - n P'_{n-1}(x) \quad \ldots(2)$$

From (1) and (2), we get

$$(n+1) P'_{n+1}(x) = (2n+1) P_n(x) + (2n+1)[n P'_n(x) + P'_{n-1}(x)] - n P'_{n-1}(x)$$

or $(2n+1) P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

Example 7. Prove that $\int x J_0^2(x)\, dx = \frac{1}{2} x^2 [J_0^2(x) + J_1^2(x)]$.

Solution : Here

$\int x J_0^2(x)\, dx = \int J_0^2(x)\, dx \cdot x$ [integrate by parts]

$$= J_0^2(x) \cdot \frac{1}{2} x^2 - \int 2 J_0(x) J_0'(x) \cdot \frac{1}{2} x^2\, dx$$

$$= \frac{1}{2} x^2 J_0^2(x) + \int x^2 J_0(x) J_1(x)\, dx$$

$$= \frac{1}{2} x^2 J_0^2(x) + \int x J_1(x)\, d/dx\, [x J_1(x)]\, dx$$

$$\left[\because \frac{d}{dx}[x J_1(x)] = x J_0(x)\right]$$

$$= \frac{1}{2} x^2 J_0^2(x) + \frac{1}{2}[x J_1(x)]^2$$

$$= \frac{1}{2} x^2 [J_0^2(x) + J_1^2(x)]$$

Example 8. Show that $\int J_3(x)\, dx = c - J_2(x) - \frac{2}{x} J_1(x)$.

Solution : Since $\frac{d}{dx}[x^{-n} J_n(x)]$

$$= -x^{-n} J_{n+1}(x) \quad \ldots(1)$$

$$\Rightarrow \quad \int x^{-n} J_{n+1}(x)\, dx = -x^{-n} J_n(x) \quad \ldots(2)$$

$$\therefore \int J_3(x)\, dx = \int x^2 \cdot x^{-2} J_3(x)\, dx + c$$

$$= x^2 \int x^{-2} J_3(x)\, dx - \int 2x \left[\int x^{-2} J_3(x)\, dx\right] dx + c$$

$$= x^2 [-x^{-2} J_2(x)] - \int 2x [-x^{-2} J_2(x)]\, dx + c$$

[by (2) when $n = 2$]

$$= c - J_2(x) + \int \frac{2}{x} J_2(x)\, dx$$

$$= c - J_2(x) - \frac{2}{x} J_1(x) \text{ [by (2) when } n = 1]$$

Example 9. Are the following linear independent sin x, cos x, sin $2x$?

Solution : The Wronskion of the given variable

$$w(x) = \begin{vmatrix} \sin x & \cos x & \sin 2x \\ \cos x & -\sin x & 2\cos 2x \\ -\sin x & -\cos x & -4\sin 2x \end{vmatrix}$$

$$= \sin x \begin{vmatrix} -\sin x & 2\cos 2x \\ -\cos x & -4\sin 2x \end{vmatrix}$$

$$- \cos x \begin{vmatrix} \cos x & \sin 2x \\ -\cos x & -4\sin 2x \end{vmatrix}$$

$$- \sin x \begin{vmatrix} \cos x & \sin 2x \\ -\sin x & -2\cos 2x \end{vmatrix}$$

$$= \sin x\,(4\sin 2x \sin x + 2\cos 2x \cos x)$$
$$- \cos x\,(-4 \sin 2x \cos x + \sin 2x \cos x)$$
$$- \sin x\,(2\cos x \cos 2x + \sin x \sin 2x)$$
$$= 3\sin^2 x \sin 2x + 3\cos^2 x \sin 2x$$
$$+ 2\sin x \cos x \cos 2x - 2\sin 2x \cos x \cos 2x$$
$$= 3\,(\sin^2 x + \cos^2 x) \sin 2x$$
$$+ \sin 2x \cos 2x - \sin 2x \cos 2x$$
$$= 3\sin 2x \neq 0$$

Since the Wronskion is not zero; they are linear independent.

Example 10. Show that $(2n + 1)\,(x^2 - 1)\, P_n'(x)$

$$= n\,(n + 1)\,\{P_{n+1}(x) - P_{n-1}(x)].$$

Solution : Since

$$(x^2 - 1)\, P_n'(x) = n\,(x\, P_n(x) - P_{n-1}(x)) \quad \ldots(1)$$

$$\text{and } (x^2 - 1)\, P_n'(x) = (n + 1)\,\{P_{n+1}(x) - x\, P_n(x)\} \quad \ldots(2)$$

Eliminating $xP_n(x)$ from (1) and (2), we get

$$(x^2 - 1)\, P'_n(x) =$$

$$n \left\{ P_{n+1}(x) - \frac{x^2 - 1}{n + 1} P'_n(x) - P_{n-1}(x) \right\}$$

$$\text{or } \left\{ x^2 - 1 + n\,\frac{x^2 - 1}{n + 1} \right\} P_n'(x)$$

$$= n\,\{P_{n+1}(x) - P_{n-1}(x)\}$$

$$\text{or } (2n + 1)\,(x^2 - 1)\, P_n'(x)$$

$$= n\,(n + 1)\,\{P_{n+1}(x) - P_{n-1}(x)\}$$

Example 11. Show that $\int_{-1}^{1} (x^2 - 1)\, P_{n+1}(x)\, P_n'(x)\, dx = \dfrac{2n\,(n + 1)}{(2n + 1)\,(2n + 3)}$.

Solution : Since

$$(2n + 1)\, xP_n(x) = (n + 1)\, P_{n+1}(x) + nP_{n-1}(x) \quad \ldots(1)$$

$$\text{and } (x^2 - 1)\, P_n'(x) = (n + 1)\,[P_{n+1}(x) - xP_n] \quad \ldots(2)$$

Multiplying (2) by $P_{n+1}(x)$ and integrating with respect to x, between the limit -1 and to $+1$, we get

$$\int_{-1}^{1} (x^2 - 1)P_{n+1}(x)\, P'_n(x)\, dx$$

$$= (n + 1) \int_{-1}^{1} P_{n+1}(x)\,[P_{n+1}(x) - xP_n(x)]dx$$

$$= (n + 1) \int_{-1}^{1} P^2_{n+1}(x)\, dx - (n + 1) \int_{-1}^{1} P_{n+1}(x)$$

$$\left\{ \frac{(n + 1)\, P_{n+1}(x) - nP_{n-1}(x)}{2n + 1} \right\} dx \text{ (from (1))}$$

$$= (n + 1)\frac{2}{2n + 3} - \frac{(n + 1)^2}{2n + 1} \cdot \frac{2}{2n + 3}$$

$$= \frac{2n\,(n + 1)}{(2n + 1)\,(2n + 3)}$$

Example 12. Show that $f(x, y) = xy^2$ satisfies the Lipschitz condition on the rectangle $|x| \leq 1$, $|y| \leq 1$ but does not satisfy a Lipschitz condition on the strip $|x| \leq 1$, $|y| < \infty$.

Solution : $f(x, y) = xy^2$ and $\left| \dfrac{\delta}{\delta y} f(x, y) \right| = |2xy| \leq 2$ for (x, y) on rectangle $|x| \leq 1$, $|y| \leq 1$.

$\therefore \dfrac{\delta f}{\delta y}$ exists and is continuous and bounded in the rectangle $|x| \leq 1$, $|y| \leq 1$.

Thus Lipschitz condition is satisfied on $|x| \leq 1$, $|y| \leq 1$ on strip $|x| \leq 1$, $|y| \leq \infty$, there does not exists any positive constant $K > 0$, such that

$$\left| \frac{\delta}{\delta y} f(x, y) \right| = |2xy| \leq K$$

Hence, Lipschitz condition not satisfied.

Examples 13. Prove that continuous function $f(x, y) = y^{2/3}$ on rectangle $|x| \leq 1$, $|y| \leq 1$ does not satisfies Lipschitz condition on a rectangle.

Solution :

Here $f(x, y) = x^{2/3}$ is continuous but $\left|\frac{\delta}{\delta y} f(x, y)\right|$ $= \left|\frac{2}{3y^{1/3}}\right| \neq K > 0$

for $y = 0$ which is a point of the rectangle.

Hence, Lipschitz condition is not satisfied on the rectangle.

Example 14. Show that

$$\int_0^a r J_0(xr)\, dr = \frac{a}{x} J_1(ax)$$

Solution : $\int_0^a r J_0(xr)\, dr$

$$= \int_0^a r\left[1 - \frac{(xr)^2}{2^2} + \frac{(xr)^4}{2^2 4^2} - \frac{(xr)^6}{2^2\cdot 4^2\cdot 6^2} + \ldots\right] dr$$

$$= \int_0^a \left[r + \frac{x^2}{2^2} r^3 + \frac{x^4}{2^2\cdot 4^2} r^5 + \frac{x^6}{2^2\cdot 4^2\cdot 6^2} r^7 + \ldots\right] dr$$

$$= \left[\frac{x^2}{2} - \frac{x^2 r^4}{2^2\cdot 4} + \frac{x^4 r^6}{2^2\cdot 4^2\cdot 6} - \frac{x^6 r^8}{2^2\cdot 4^2\cdot 6^2\cdot 8} + \ldots\right]_0^a$$

$$= \frac{a}{x}\left[\frac{ax}{2} - \frac{(ax)^3}{2^2\cdot 4} + \frac{(ax)^5}{2^2\cdot 4^2\cdot 6} + \ldots\right]$$

$$= \frac{a}{x} J_1(ax)$$

Example 15. Prove that $4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$.

Solution : Since

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

Differentiating it with respect to x, we get

$$2J_n''(x) = J'_{n-1}(x) - J'_{n+1}(x)$$

or

$$\begin{aligned} 4J''_n(x) &= 2J'_{n-1}(x) - 2J'_{n+1}(x) \\ &= [J_{n-2}(x) - J_n(x)] \\ &\quad - [J_n(x) - J_{n+2}(x)] \\ &= J_{n-2}(x) - 2J_n(x) + J_{n+2}(x) \end{aligned}$$

Example 16. Show that if δ is either a rectangle $|x - x_0| \le a$, $|y - y_0| \le b$ $(a, b > 0)$ or strip $|x - x_0| \le a$, $|y| < \infty$ $(a > 0)$ and if f is real valued continonus function defined on δ and $\frac{\delta f}{\delta y}$ exists and also.

Also,

$\left|\frac{\delta}{\delta y} f(x, y)\right| \le K$, $(x, y) \sum \delta$ for a positive constant K, then f satisfies Lipschitz condition on δ with Lipschitz constant K.

Solution : By fundamental theorem of calculus

$$f(x, y_2) - f(x, y) = \int_{y_1}^{y_2} \frac{\delta}{\delta y} f(x, y)\, dy$$

$$\Rightarrow \quad |f(x, y_2) - f(x, y) =$$

$$\left|\int_{y_1}^{y_2} \frac{\delta}{\delta y} f(x, y)\, dy\right| \le \int_{y_1}^{y_2} \left|\frac{\delta}{\delta y} f(x, y)\right| |dy|$$

$$\le K \int_{y_1}^{y_2} |dy|$$

$$\Rightarrow \quad |f(x, y_2) - f(x, y_1) \le K\, |y_2 - y_1)$$

for all $(x, y_1), (x, y_2) \sum \delta$

$\Rightarrow f(x, y)$ satisfy Lipschitz condition on δ, where K is Lipschitz constant.

Example 17. Show that : $\int_{-1}^{1} (1 - x^2)[P'_n(x)]^2\, dx = \frac{2n(n+1)}{2n+1}$.

Solution : $\int_{-1}^{1} (1 - x^2)[P'_n(x)]^2\, dx$

$$= \int_{-1}^{1} [(1 - x^2) P'n(x)]\, P'_n(x)\, dx$$

$$= [(1 - x^2) P'_n(x)\, P_n(x)]_1^1 - \int_{-1}^{1} \frac{d}{dx}[(1 - x^2) P'_n(x)]\, P_n(x)\, dx$$

$$= -\int_{-1}^{1} P_n(x) \frac{d}{dx}[(1 - x^2) P'_n(x)]\, dx$$

$$= -\int_{-1}^{1} P_n(x)\, [-n(n+1) P_n(x)]\, dx$$

Since $\frac{d}{dx}\ [(1 - x^2) P'_n(x)] + n(n+1) P_n(x) = 0$

$$= n(n+1) \int_{-1}^{1} P^2_n(x)\, dx$$

$$= n(n+1) \cdot \frac{2}{2n+1}$$

$$= \frac{2n(n+1)}{2n+1}$$

Example 18. Prove that

$$(1 - x^2) P'_n(x) = \frac{n(n+1)}{2n+1} [P_{n-1}(x) - P_{n+1}(x)].$$

Solution : Since

$$(1 - x^2) P'n(x) = n[P_{n-1}(x) - x\cdot P_n(x)] \qquad \ldots(a)$$

$$x\,\mathrm{P}_n(x) = \frac{1}{2n+1}\,[(n+1)\,\mathrm{P}_{n+1}(x) + n\mathrm{P}_{n-1}(x)] \quad \ldots\text{(b)}$$

By (a) and (b), eliminating $\mathrm{P}_n(x)$, we get

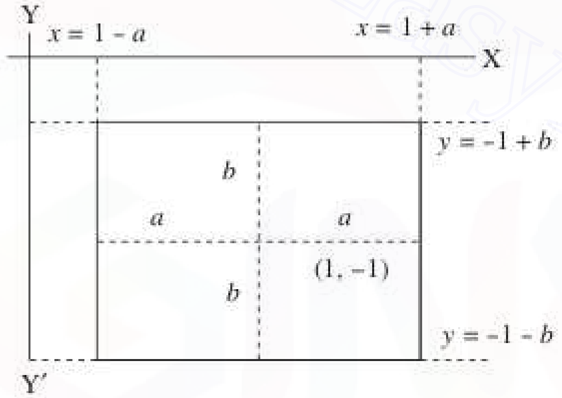
$$(1-x^2)\cdot \mathrm{P}'_n(x) = n\cdot(\mathrm{P}_{n-1}\left\{x) - \frac{1}{2n+1}\right.$$
$$\left.[(n+1)\,\mathrm{P}_{n+1}(x) + n\,\mathrm{P}_{n-1}(x)]\right\}$$
$$= \frac{n(n+1)}{2n+1}\,[\mathrm{P}_{n-1}(x) - \mathrm{P}_{n+1}(x)\}$$

Example 19. Show the existence and a uniqueness of solution for the initial value problem $\frac{dy}{dx} = y^2,\ y(1) = -1$.

Solution : The existence—Given,

$$f(x, y) = \frac{dy}{dx} = y^2 \text{ and } y(1) = 1.$$

Here $\frac{\delta f}{\delta y} = 2y$, so f, and $\frac{\delta f}{\delta y}$ are both continuous for all (x, y) consider the rectangle R : $|x-1| < a$, $|y+1| \le b$, about $(1, -1)$



Then,

$|f(x, y_2) - f(x, y_1)| = |y_2^2 - y_1^2| = |y_2 + y_1|\,|y_2 - y_1| \le (2+2b)\,|y_2 - y_1|$ and Lipschitz condition is satisfied in the rectangle R.

Thus by existence theorem, there exist atleast one solution.

To uniqueness—Now let

M = *max* $|f(x, y)|$ for $x, y \in$ R and $m = \min(a, b/\mathrm{M})$

M = *max* $|f(x, y)| = \max |y^2| = |(-1-b^2)| = (1+b^2)$

and $m = \min(a, b/\mathrm{M}) = \min\left(a, \frac{b}{(1+b)^2}\right)$

For maximum or minimum of

$$\frac{b}{(1+b)^2} = \phi(b)$$

$$\phi'(b) = \frac{1-b}{(1+b)^3}$$

and $$\phi''(b) = \frac{2b-4}{(1+b)^4}$$

$\Rightarrow$ when $b = 1$, $\phi'(1) = 0$ and $\phi''(1) = \frac{-2}{2^4} < 0$

$\therefore\ \phi(b)$ is max. at $a + b = 1$ and $\phi(1) = \frac{1}{4}$

$\therefore$ If $a < \frac{1}{4}$, $m < \frac{1}{4}$ if $a \ge \frac{1}{4}$, $m = \frac{1}{4}$

and thus $m = \min\left\{a, \frac{b}{\mathrm{M}}\right\} = \min\left\{a, \frac{1}{4}\right\}$
$$= \frac{1}{4} \text{ for } b = 1$$

Hence, the given problem possesses a unique solution when $|x-1| < \frac{1}{4}$.

Example 20. Show that :

$$\mathrm{J}_n(x) = (-2)^n x^n \left[\frac{d}{d(x)^2}\right]^n \mathrm{J}_0(x).$$

Solution : Since

$$\frac{d}{dx}[x^{-n}\mathrm{J}_n(x)] = x^{-n}\,\mathrm{J}_{n+1}(x) \quad \ldots(1)$$

When $n = 0$

$$\frac{d}{dx}[\mathrm{J}_0(x) = -\mathrm{J}_1(x)$$

or $$\frac{1}{x}\frac{d}{dx}[\mathrm{J}_0(x)] = -x^{-1}\mathrm{J}_1(x) \quad \ldots(2)$$

Differentiating with respect to x, we have

$$\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}\mathrm{J}_0(x)\right) = -\frac{d}{dx}[x^{-1}\,\mathrm{J}_1(x)]$$

or $$\frac{1}{x}\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}\mathrm{J}_0(x)\right) = (-1)^2\,x^{-2}\mathrm{J}_2(x)$$

or $$\left(\frac{1}{x}\frac{d}{dx}\right)^2 \mathrm{J}_0(x) = (-1)^2\,x^{-2}\,\mathrm{J}_2(x)$$

and so $$\left(\frac{1}{x}\frac{d}{dx}\right)^n \mathrm{J}_0(x) = (-1)^n\,x^{-n}\,\mathrm{J}_n(x)$$

$$\Rightarrow \quad \mathrm{J}_n(x) = (-1)^n\cdot x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \mathrm{J}_0(x)$$
$$= (-1)^n\,x^n\left(x\cdot\frac{d}{d(x^2)}\right)^n \mathrm{J}_0(x)$$
$$= (-2)^2\,x^n\left(\frac{d}{d(x^2)}\right)^n \mathrm{J}_0(x)$$

i.e., $$\mathrm{J}_n(x) = (-2)^n\,x^n\left(\frac{d}{d(x^2)}\right)^n \mathrm{J}_0(x)$$

Example 21. Find the solution of the one dimensional diffusion satisfying the following boundary conditions.

(i) T is bounded at $t \to \infty$

(ii) $\left.\frac{\delta T}{\delta x}\right|_{x=0} = 0$ for all t

(iii) $\left.\frac{\delta T}{\delta x}\right|_{x=0} = 0$ for all t

(iv) $T(x, 0) = x(a - x), 0 < x < a$

Solution : The possible solution is $T(x, t) = \exp(-a\lambda^2 t)(A\cos\lambda x + B\sin\lambda x)$ and $\frac{\delta T}{\delta x}$
$= \exp(-a\lambda^2 t)(-A\lambda\sin\lambda x + B\lambda\cos\lambda x)$ ... (1)

Using boundary conditions (ii), equation (1) gives B = 0 and boundary conditions (iii) into equation (1) gives.

$\sin\lambda a = 0$ implying $\lambda a = n\pi, \pi = 0\ 1, 2, \ldots$

By the principle of superposition, we have

$$T(x, t) = \in A_n \exp(-a\lambda^2 t)\cos\lambda x$$

$$= \sum_{n=0}^{\infty} A_n \exp\left[-a\left(\frac{n\pi}{a}\right)^2 t\right] \cos\left(\frac{n\pi}{a}\right)x$$

The boundary conditions (iv) gives

$$T(x, 0) = x(a - x) = A_0 + \sum_{n=1}^{\infty} A_n \exp.\left[-a\left(\frac{n\pi}{a}\right)^2 t\right]\cos\left(\frac{n\pi}{a}\right)x$$

where

$$A_0 = \frac{2}{a}\int_0^a (ax - x^2)\,dx = \frac{a^2}{6}$$

$$A_n = \frac{2}{a}\int_0^a (ax - x^2)\cos\left(\frac{n\pi}{a}x\right)dx$$

$$= \frac{2a^2}{n^2\pi^2}(1 + \cos n\pi)$$

$$= \frac{2a^2}{n^2\pi^2}(1 + (-1)^n]$$

$$\therefore \quad A_n = \begin{cases} \frac{-4a^2}{n^2\pi^2}, & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd} \end{cases}$$

The required solutions is

$$T(x, t) = \frac{a^2}{6} - \frac{4a^2}{\pi^2}\sum_{n=2,4,\ldots\text{even}}^{\infty}\frac{1}{n^2}\cos\left(\frac{n\pi}{a}\right)x\exp.\left[-\alpha\left(\frac{n\pi}{a}\right)^2 t\right]$$

Example 22. Show that :

(i) $P'_n(1) = \frac{1}{2}n(n + 1)$

(ii) $P'_n(-1) = (-1)^{n-1}\frac{1}{2}n(n + 1)$

Solution : We know that $P_n(x)$ is Legendre's function of first kind of the Legender's equation.

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

$\therefore (1 - x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0$ ... (1)

(i) Putting $x = 1$ in (1), we get, $0 - 2P'_n(1) + n(n+1)P_n(1) = 0$

or $P'n(1) = \frac{1}{2}n(n+1)P_n(1) = \frac{1}{2}n(n+1)$

[$\because P_n(1) = 1$]

(ii) Putting $x = -1$ in (1), we get

$0 + 2P'_n(-1) + n(n+1)P_n(-1) = 0$

or $\quad P'_n(-1) = -\frac{1}{2}n(n+1)P_n(-1)$

$$= -\frac{1}{2}n(n+1)(-1)^n P_n(1)$$

$$= (-1)^{n-1}\frac{1}{2}n(n+1)$$

[$\because P_n(-1) = (-1)^2 P_n$ and $P_n(1) = 1$]

Example 23. Find the solution of the wave equation $\mu_{tt} = c^2\mu_{xx}$

Given : (i) $\mu(0, t) = \mu(2, t) = 0$

(ii) $\mu(x, 0) = \sin^3 x\pi/2$

(iii) $\mu_t(x, 0) = 0$

Solution : The possible solution is $\mu(x, t) = c_1\cos\lambda x + c_2\sin\lambda x)[c_3\cos(c\lambda t) + c_4\sin(c\lambda t)]$

Using the condition $\mu(0, t) = 0$, we obtain $c_1 = 0$. Also, condition (iii) $\Rightarrow c_4 = 0$.

The condition $\mu(2, t) = 0$ gives, $\sin 2\lambda = 0$, implying that $\lambda = \frac{n\pi}{2}$, $n = 1, 2, \ldots$

Thus, the solution is, $\mu(x, t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{2}\cos\frac{n\pi ct}{2}$.

Using condition (ii), we have,

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{2} = \sin^3\frac{\pi x}{2} = \frac{3}{4}\sin\frac{\pi x}{2} - \frac{1}{4}\sin\frac{3\pi x}{2}$$

which gives $A_1 = 3/4$, $A_3 = -1/4$, while all other $A'_n s$ are zero. Hence the required solution is

$$\mu(x, t) = \frac{3}{4}\sin\frac{\pi x}{2}\cos\frac{\pi ct}{2} - \frac{1}{4}\sin\frac{3\pi x}{2}\cos\frac{3\pi ct}{2}$$

Example 24. Prove that

(i) $P_n(1) = 1$ (ii) $P_n(-x) = (-1)^n P_n(x)$.

Solution : (i) Since

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1 - 2xh + h^2)^{-1/2}$$

Substituting $x = 1$, we have,

$$\begin{aligned}\sum_{n=0}^{\infty} h^n P_n(1) &= (1 - 2h + h^2)^{-1/2}\\ &= (1 - h)^{-1}\\ &= 1 + h + h^2 + \ldots\\ &= \sum_{n=0}^{\infty} h^n\end{aligned}$$

Equating the coefficients of h^n, we have $P_n(1) = 1$

(ii) Since $\sum_{n=0}^{\infty} h^n P_n(x) = (1 - 2xh + h^2)^{-1/2}$

Substituting $x = -x$, we have, $\sum_{n=0}^{\infty} h^n P_n(-x)$

$$= (1 + 2xh + h^2)^{-1/2} \quad \ldots(a)$$

Also replacing h by $-h$, we have

$$\sum_{n=0}^{\infty} (-h)^n P_n(x) = (1 - 2xh + h^2)^{-1/2} \quad \ldots(b)$$

From (a) and (b), we have

$$\sum_{n=0}^{\infty} h^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n h^n P_n(x)$$

Equating the coefficients of h^n, we get

$$P_n(-x) = (-1)^n P_n(x)$$

Example 25. Show that :

$$\frac{d}{dx}[x J_n(x) J_{n+1}(x)] = x[J^2_n(x) - J^2_{n+1}(x)]$$

Solution : Since $\frac{d}{dx}[x J_n(x) J_{n+1}(x)]$

$$\begin{aligned}&= J_n(x) J_{n+1}(x) + x\,[J_n(x) J'_{n+1}(x)\\ &\qquad + J'_n(x) J_{n+1}(x)] \quad \ldots(1)\end{aligned}$$

and since $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

$$\Rightarrow x^{-n} J'_n(x) - nx^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x)$$

$$\Rightarrow \quad J'_n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x) \quad \ldots(2)$$

Also since

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

$$\Rightarrow x^n J'_n(x) + nx^{n-1} J_n(x) = x^n J_{n-1}(x)$$

Replace n by $n + 1$, we have

$$x^{n+1} J'_{n+1}(x) + (n+1) x^n J_{n+1}(x) = x^{n+1} J'_n(x)$$

$$\Rightarrow \quad J'_{n+1}(x) = J_n(x) - \frac{n+1}{x} J_{n+1}(x) \quad \ldots(3)$$

By (2), (3) and (1) we have

$$\begin{aligned}&\frac{d}{dx}[x J_n(x) J_{n+1}(x)]\\ &= J_n(x) J_{n+1}(x)\\ &\quad + x\left\{J_n(x)\left[J_n(x) - \frac{n+1}{x} J_{n+1}(x)\right]\right.\\ &\quad \left. + J_{n+1}\left[\frac{n}{x} J_n(x) - J_{n+1}(x)\right]\right\}\\ &= x \cdot [J^2_n(x) - J^2_{n+1}(x)]\end{aligned}$$

Example 26. Prove that :

(a) $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

(b) $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

Solution : (a) Since,

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\lfloor r \,\lfloor n+r+1}$$

Multiplying by x^n, we have

$$\begin{aligned}x^n J_n(x) &= x^n \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r! \,\lfloor (n+r+1)}\\ &= \sum_{r=0}^{\infty} \frac{(-1)^r x^{2(n+r)}}{2^{n+2r} r! \,\lfloor (n+r+1)}\end{aligned}$$

$$\begin{aligned}\therefore\ &\frac{d}{dx}[x^n J_n(x)]\\ &= \sum_{r=0}^{\infty} \frac{(-1)^r 2(n+r) x^{2(n+r)-1}}{2^{n+2r} r! \,\lfloor (n+r+1)}\\ &= x^n \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n-1+2r}}{r+ \,\lfloor (n-1+r+1)}\\ &= x^n J_{n-1}(x)\end{aligned}$$

(b) Since, $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\lfloor r \,\lfloor n+r+1}$

Multiplying by x^{-n}, we have

$$x^{-n} J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2^{n+2r} r! \,\lfloor (n+r+1)}$$

$$\therefore \frac{d}{dx}[x^{-n}J_n(x)]$$
$$= \sum_{r=0}^{\infty} \frac{(-1)^r 2rx^{2r-1}}{2^{n+2r} r! \, \lceil (n+r+1)}$$
$$= x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^{r-1} x^{n+1+2(r-1)}}{2^{n+1+2(r-1)}(r-1)! \lceil (n+r+1)}$$
$$= -x^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+1+2k}}{k! \lceil (n+1+k+1)}$$
$= -x^{-n} J_{n+1}(x)$, where $k = r - 1$
$$\Rightarrow \frac{d}{dx}[x^{-n}J_n(x)] = -x^n J_{n+1}(x)$$

Example 27. Show that $J_n(x) = \frac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$.

Solution : Since, $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$
$$\Rightarrow \quad x^n J_n'(x) + nx^{n-1} J_n(x) = x^n J_{n-1}(x)$$
or dividing by x^n, $J_n'(x) + (n/x) J_n(x)$
$$= J_{n-1}(x) \quad \ldots (1)$$
Since, $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n}J_{n+1}(x)$
$$\Rightarrow x^{-n} J_n'(x) - nx^{-n+1} J_n(x)$$
$$= x^{-n} J_{n+1}(x) - J_n'(x) + \frac{n}{x} J_n(x)$$
$$= J_{n+1}(x) \quad \ldots (2)$$
Adding (1) and (2), we obtain
$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$
$$\Rightarrow \quad J_n(x) = \frac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$$

Example 28. $\int_{-1}^{1} (x^2 - 1) P_{n+1}(x) P'_n(x)\, dx$
$$= \frac{2n(n+1)}{(2n+1)(2n+3)}$$

Solution : Since, $(x^2 - 1) P'_n(x) = (n+1)[P_{n+1}(x) - x \cdot P_n(x)]$

Multiplying it by $P_{n+1}(x)$ and integrating, we have,
$$\int_{-1}^{1} (x^2 - 1) P_{n+1}(x) P'_n(x)\, dx$$
$$= (n+1) \int_{-1}^{1} [P_{n+1}(x) - x P_n(x)] P_{n+1}(x)\, dx$$
$$= (n+1) \int_{-1}^{1} P^2_{n+1}(x) - (n+1) \int_{-1}^{1} P_{n+1}(x)$$
$$\left\{ \frac{(n+1) P_{n+1}(x) - nP_{n-1}(x)}{(2n+1)} \right\} dx$$
$$[\because (2n+1)\, x\, P_n(x) = (n+1) P_{n+1}(x) + nP_{n-1}(x)]$$
$$= (n+1) \frac{2}{2n+3} - \frac{(n+1)^2}{2n+1} \cdot \frac{2}{2n+3}$$
$$= \frac{2n(n+1)}{(2n+1)(2n+3)}$$

Example 29. Prove that :
$$\int P_n(x)\, dx + c = \frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1}$$

Solution : Since
$$(2n+1) P_n = P'_{n+1}(x) - P'_{n+1}(x)$$
or
$$P_n(x) = \frac{P'_{n+1}(x) - P'_{n-1}(x)}{2n+1}$$
Integrating, we have
$$\int P_n(x)\, dx + c = \frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1}$$

Example 30. Given the equation $x = A \cos (Pt - \alpha)$, eliminate A, P and α.

Solution :
$$\frac{dx}{dt} = -PA \sin (Pt - \alpha)$$
$$\frac{d^2x}{dt^2} = -P^2 A \cos (Pt - \alpha)$$
$$= -P^2 x$$
$$\frac{d^3x}{dt^2} = -P^2 \frac{dx}{dt}$$
$$\Rightarrow \quad \frac{\frac{d^3x}{dt^3}}{\frac{dx}{dt}} = -P^2 = d^2x/dt^2/x$$
$$\Rightarrow \quad x \cdot \frac{d^3x}{dt^3} = \frac{dx}{dt} \cdot \frac{d^2x}{dt^2}$$

Example 31. Solve $\frac{dx}{x} = \tan y dy$.

Solution : $\log x = -\log \cos y + c$
$$\Rightarrow \quad \log (x \cos y) = c$$
$$\Rightarrow \quad x \cos y = e^c = a \text{ (say)}$$

Example 32. Solve $\frac{dy}{dx} = 2xy$.

Solution :
$$\frac{dy}{y} = 2x \cdot dx$$
$$\Rightarrow \quad \log y = x^2 + \log a$$
$$\Rightarrow \quad y = ae^{x^2}$$

Example 33. Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$.

Solution : Auxiliary equation

$$m^2 - 6m + 13 = 0$$

$$\Rightarrow \quad m = 3 \pm 2i$$

$\therefore$ General solution is

$$y = Ae^{(3+2i)x} + Be^{(3-2i)x}$$

Example 34. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 50\,e^{2x}$.

Solution : $(D^2 - 2D + 4)\,y = 50e^{2x}$

$$\Rightarrow \quad (D-2)^2\,y = 50e^{2x}$$

$$\text{C.F. } (m-2)^2 = 0$$

$$\Rightarrow \quad m = +2, +2$$

$$\Rightarrow \quad y = (A + Bx)\,e^{2x}$$

$$\text{P.I. } \frac{1}{(D-2)^2}\cdot 50e^{2x} = 50e^{2x}\cdot\frac{1}{D^2}$$

$$= 50e^{2x}\cdot\frac{1}{2}x^2$$

$$= 25x^2e^{2x}$$

$\therefore$ General solution $y = 25x^2e^{2x} + (A + Bx)\,e^{2x}$

Example 35. Find the complete integral of the following equation

$$2xz - \left(\frac{\delta z}{\delta x}\right)x^2 - 2\left(\frac{\delta z}{\delta y}\right)xy + \left(\frac{\delta z}{\delta x}\right)\left(\frac{\delta z}{\delta y}\right) = 0$$

Solution : By Charpit's method the subsidiary equation of F $(x, y, z, p, q) = 0$

$$\frac{dx}{-\frac{\delta F}{\delta P}} = \frac{dy}{-\frac{\delta F}{\delta q}} = \frac{dz}{-P\frac{\delta F}{\delta P} - q\frac{\delta F}{\delta q}}$$

$$= \frac{dP}{\frac{\delta F}{\delta x} + P\frac{\delta F}{\delta z}}$$

$$= \frac{dz}{\frac{\delta F}{\delta y} + q\frac{\delta F}{\delta x}} = \frac{\delta f}{0}$$

$$\Rightarrow \frac{dx}{x^2 - q} = \frac{dy}{2xy - P} = \frac{dz}{Px^2 + 2xyq - 2Pq}$$

$$= \frac{dP}{2z - 2qy} = \frac{dq}{0} = \frac{df}{0}$$

Integrating gives $q = a$ ...(2)

and (1) and (2) gives

$$P = \frac{2x\,(z - ay)}{x^2 - a}$$

Hence, $dz = Pdx + 2dy$

$$= \frac{2x\,(z - ay)\,dx}{x^2 - a} + ady$$

i.e., $\frac{dz - ady}{z - ay} = \frac{2xdx}{x^2 - a^2}$

$\Rightarrow \; z = ay + b\,(x^2 - a^2)$

Example 36. Solve $\frac{dy}{dx} + 3y = e^{2x}$.

Solution : $\int Pdx = 3x$

$\therefore$ Integrating factor e^{3x}

$$\therefore \quad e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{5x}$$

$$\Rightarrow \quad \frac{d}{dx}\,(ye^{3x}) = e^{5x}$$

$$\Rightarrow \quad ye^{3x} = \frac{1}{5}e^{5x} + c$$

$$\Rightarrow \quad y = \frac{1}{5}e^{2x} + ce^{-3x}$$

Example 37. Show that $V = \frac{A}{r} + B$ is a solution of $\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0$

Solution : Differentiating $V = \frac{A}{r} + B$ twice with respect to r,

$$\frac{dV}{dr} = -\frac{A}{r^2} \text{ and } \frac{d^2V}{dr^2} = \frac{2A}{r^3}$$

$$\therefore \quad \left(\frac{d^2V}{dr^2}\right)\Big/\left(\frac{dV}{dr}\right) = \frac{2A}{r^3}\times\frac{r^2}{-A}$$

$$= \frac{-2}{r}$$

$$\Rightarrow \quad \frac{d^2V}{dr^2} = \frac{-2}{r}\frac{dV}{dr}$$

$$\Rightarrow \quad \frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0$$

Example 38. The differential equation satisfied by the system of parabolas,

$$y^2 = 4a\,(x + y)$$

Solution : Differentiating

$$y^2 = 4a\,(x + y) \quad \text{...(1)}$$

with respect to x,

$$2y\frac{dy}{dx} = 4a \quad \text{...(2)}$$

Dividing (1) by (2), we get

$$\frac{y^2}{2\left(\frac{dy}{dx}\right)} = x + a$$

$$\Rightarrow \quad a = \frac{y^2}{2\left(\frac{dy}{dx}\right)} - x$$

Substituting this value of 'a' in (2) a is eliminated and we get

$$y\left(\frac{dy}{dx}\right) = 2\left(\frac{y}{2\left(\frac{dy}{dx}\right)} - x\right)$$

$$\text{or} \quad y\left(\frac{dy}{dx}\right)^2 - y + 2x\frac{dy}{dx} = 0$$

$$\text{or} \quad y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$$

Example 39. Eliminate the arbitrary constants a, b and c from $z = a(x + y) + b(x - y) + abt + c$

Solution :

$$\frac{\delta z}{\delta x} = a + b$$

$$\frac{\delta z}{\delta y} = a - b$$

$$\frac{\delta z}{\delta t} = ab$$

$$\because \quad (a + b)^2 - (a - b)^2 = 4ab$$

$$\Rightarrow \quad \left(\frac{\delta z}{\delta x}\right)^2 - \left(\frac{\delta z}{\delta y}\right)^2 = 4\frac{\delta z}{\delta t}$$

Example 40. Given $cy = c^2x + 1$ find the c–discriminant.

Solution : $f(x, y, c) = 0$

$$\Rightarrow \quad cy = c^2x + 1 = 0$$

$$\Rightarrow \quad y - cx - \frac{1}{c} = 0 \qquad \ldots(1)$$

$$\text{and} \quad \frac{\delta f}{\delta c} = 0$$

$$\Rightarrow \quad -x + \frac{1}{c^2} = 0$$

$$\Rightarrow \quad c = \pm\frac{1}{\sqrt{x}} \qquad \ldots(2)$$

Substituting c in (1) given

$$y = \pm 2\sqrt{x}$$

$$\text{or} \quad y^2 = 4x$$

Example 41. Find the P-discriminant of $(x + c)^2 + y^2 = r^2$.

Solution : $(x + c)^2 + y^2 = r^2$

$$\Rightarrow \quad x + c = \pm\sqrt{r^2 - y^2}$$

$$\text{or} \quad 1 = \pm yp/\sqrt{r^2 - p^2}$$

$$\Rightarrow \quad y^2p^2 + y^2 - r^2 = 0$$

The P-discriminant of $y^2p^2 + y^2 - r^2 = 0$ is $\frac{\delta f}{\delta p} = 0$

$$\Rightarrow \quad y^2(y^2 - r^2) = 0$$

Example 42. Given $y = px \pm \sqrt{(p^2 + m^2)}$, find its singular solution.

Solution : The Chairaut's form,

$$y = px + f(p)$$

$$\Rightarrow \quad y = cx \pm \sqrt{(c^2 + m^2)}$$

$$\text{or} \quad (y^2 - cx)^2 = (c^2 + m^2) = 0$$

$$\text{or} \quad c^2(x^2 - 1) - 2xyc + (y^2 - m^2) = 0$$

The x-discriminant is

$$4x^2y^2 - 4(x^2 - 1)(y^2 - m^2) = 0$$

$$\text{or} \quad y^2 - m^2x^2 = m^2$$

which is the singular solution.

OBJECTIVE TYPE QUESTIONS

1. The equation $e^x dx + e^y dy = 0$ is of order—
   (A) Zero (B) One
   (C) Two (D) None of these

2. $y = x\frac{dy}{dx} + x/dy/dx$ is of degree—
   (A) Zero (B) Two
   (C) Three (D) One

3. The order of the equation $d^2y/dx^2 + n^2x = 0$ is—
   (A) Zero (B) One
   (C) Two (D) Three

4. The degree of the equation $d^2y/dx^2 + n^2x = 0$ is—
   (A) Zero (B) One
   (C) Two (D) Three

5. The degree of the equation $y\,dy./dx = x\left(\frac{dy}{dx}\right)^2 + x$ is—
(A) Zero (B) One
(C) Two (D) Three

6. The number of arbitrary constant. A general solution of first order equation contains—
(A) Zero (B) One
(C) Two (D) Three

7. The general solution of first order equation $\frac{dy}{dx} = \cos x$ is given by—
(A) $y = \cos x$ (B) $\sin x$
(C) $y = \cos x + c$ (D) $y = \sin x + c$

8. The general solution of the equation $\frac{dy}{dx} = -\frac{x}{y}$ is—
(A) $x^2 + y^2 = a^2$ (B) $x^2 - y^2 = a^2$
(C) $x = -y$ (D) $x + y = a$

9. The particulars solution of the initial value problem $xdx + ydy = 0, x_0 = 4, y_0 = -3$—
(A) $y = \pm\sqrt{25 - x^2}$ (B) $y = \pm\sqrt{x^2}$
(C) $y = -4x$ (D) $y^2 = 3x$

10. The general solution of the equation $\sin x\, dx + \frac{dy}{\sqrt{y}} = 0$ is—
(A) $2\sqrt{y} + \cos x = c$ (B) $2\sqrt{y} - \cos x = c$
(C) $2\sqrt{y} = \sin x$ (D) $\sin x + \cos x = \sqrt{y}$

11. The general solution of the equation of $y\,dx - xdy = 0$ is—
(A) $\frac{x}{y} = c$ (B) $x + y = c$
(C) $xy = c$ (D) $x - y = c$

12. Particular solution is a solution, that can be obtained from general solution by giving particular values to arbitrary constants—
(A) True
(B) False
(C) Neither true nor false
(D) All are true

13. The complete solution of differential equation contains arbitrary constants—
(A) More than the order of equation
(B) Can't say
(C) Equal to the order of equation
(D) Less than the order of equation

14. Singular solution of differential equation contains—
(1) Arbitrary constant
(2) Can be obtained from general
(3) Do not contains arbitrary constant
(4) Can not be obtained from general solution
(A) 1, 2 are true (B) 3, 4 are true
(C) 1, 4 are true (D) 2, 4 are true

15. A first order differential equation M $(x, y)\, dx$ + N $(x, y)\, dy = 0$ is exact if—
(A) $\frac{\delta M}{\delta x} = \frac{\delta N}{\delta y}$ (B) $\frac{\delta M}{\delta x} \neq \frac{\delta N}{\delta y}$
(C) $\frac{\delta M}{\delta y} \neq \frac{\delta M}{\delta x}$ (D) $\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$

16. The integrating factor for the differential equation $(x + 1)\frac{dy}{dx} - y = e^{3x}(x + 1)^2$ is—
(A) $\frac{1}{x+1}$ (B) $x + 1$
(C) $\frac{1}{x^2+1}$ (D) $x^2 + 1$

17. The integrating factor for the differential equation $2ydx + xdy = 0$ is—
(A) x (B) y
(C) xy (D) y^2

18. The integration factor for the Leibnitz linear equation $\frac{dy}{dx} + Py = Q$ is—
(A) $\int P dx$ (B) $\int Q dx$
(C) $\exp(\int Q dx)$ (D) $\exp(\int P dx)$

19. In linear ordinary differential equation, the dependent variable and its differential coefficients are not multiplied together and occurs only in—
(A) First degree (B) Second degree
(C) Third degree (D) Fourth degree

20. If M $(x, y)\, dx$ + N $(x, y)\, dy = 0$ and $\frac{\delta M}{\delta y} = \frac{\delta M}{\delta x}$, then—
(A) The equation is exact
(B) Not exact
(C) Linear
(D) Non-linear

21. If an equation Mdx + Ndy = 0 is not exact. Let F (x, y) be such that FMdx + FNdy = 0 is exact, then the function F is called—
(A) Differential function
(B) Arbitrary function
(C) Integrating factor
(D) None of these

22. For the equation $Y' + P(x)y = r(x)$ to be homogeneous—
(A) $r(x) \neq 0$ (B) $r(x) = 0$
(C) $r(x) = P(x)$ (D) $P(x) = 0$

23. For non homogeneous equation $y' + P(x) y = r(x)$, if y_1 and y_2 are its solutions, then the solution of homogeneous equation $y' + P(x) y = 0$ is—
(A) $y = y_1 - y_2$ (B) $y = \frac{y_1}{y_2}$
(C) $y = \frac{y_2}{y_1}$ (D) None of these

24. The homogeneous differential equation $M(x, y)\, dx + N(x, y)\, dy = 0$ can be reduced to a differential equation, in which the variable are separated, by the substitution—
(A) $y = vx$ (B) $xy = v$
(C) $x + y = v$ (D) $x - y = v$

25. Which of the following is not a solution of $y'' + y = 0$?
(A) $y = \sin x$ (B) $y = \cos x$
(C) $y = 3 \cos x$ (D) $y = \sin x + \frac{1}{2}$

26. Which of the following is not a solution of $xy' + y = 0$?
(A) $xy = \sqrt{3}$ (B) $xy = -2$
(C) $x = \sqrt{3}y$ (D) $xy = \frac{\pi}{2}$

27. What is the order and degree of a differential equation

$$\frac{d^2y}{dx^2} + \sqrt{1 + \left(\frac{dy}{dx}\right)^3} = 0\ ?$$

(A) First order, second degree
(B) First order, first degree
(C) Second degree, first order
(D) Second order, second degree

28. What is the order and degree of a differential equation $[1 + (y')^2]^{3/2} - Py'' = 0$—
(A) First order, second degree
(B) Second order, first degree
(C) Second order, second degree
(D) Third order, second degree

29. The solution of $\frac{dy}{dx} + \frac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0$ is—
(A) $y \sin x + (\sin y + y) x = c$
(B) $y \sin x + (\sin x + x) = c$
(C) $y = \sin x + y \sin y + c$
(D) None of these

30. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$, has the solution—
(A) $y = c_1 e^{-2x} + c_2 e^{-x}$
(B) $y = ce^{-2x}$
(C) $y = c_1 e^{-2x} + c_2 e^{-x} + c_3$
(D) None of these

31. $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4 = 0$, has the solution—
(A) $y = c_1 e^{-x} + x c_2 e^{-x} + x^2 c_3 e^{-x}$
(B) $y = c_1 \cos 2x + c_2 \sin 2x$
(C) $y = c_1 e^{-x} + c_2 \cos x + c_3 \sin 2x$
(D) None of these

32. The particular solution of $\frac{d^3y}{dx^3} + y = \cos(2x - 1)$ is—
(A) $\frac{1}{65}[\cos(2x - 1) - 8]$
(B) $\frac{1}{65}[\cos(2x - 1) - 8\sin(2x - 1)]$
(C) $\frac{1}{65}[\cos(2x + 1) + 8]$
(D) None of these

33. The nth order ordinary linear homogenous differential equation have—
(A) n-singular solution
(B) No singular solution
(C) One singular solution
(D) None of these

34. General solution of nth order ordinary linear differential equation contains—
(A) Every solution (B) Some solutions
(C) No solution (D) None of these

35. The n-order differential linear equation with constant coefficient, $y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$, $a_1 \ldots, a_n$ are constant and characteristic equation are—
(A) All distinct
(B) All same
(C) Partially distinct
(D) None of these

36. The difference of two solutions of non homogeneous n-order ordinary differential equation is a solution of corresponding homogeneous equation. The statement is—
(A) Always true (B) Always false
(C) Partially true (D) Partially false

37. If y_1 is the solution of non homogeneous nth order ordinary differential equation and y_2 is the solution of corresponding homogeneous equation. Then the solution $y_1 + y_2$ is also a solution of—
(A) Homogeneous equation
(B) Non-homogeneous equation
(C) Both (A) and (B)
(D) None of these

38. The linearity principle for ordinary differential equation holds for—
(A) Non-homogeneous equation
(B) Non-linear equation
(C) Linear differential equation
(D) None of these

39. For homogeneous linear ordinary differential equation, the linear combination of two solutions is again a solution of the equation. The statement is—
(A) True
(B) False
(C) Neither true nor false
(D) Can't say

40. If y_1 and y_2 are two solutions of $y'' + p(x) y' + g(x) y = 0$, then for general solution of this given equation. y_1 and y_2 are—
(A) Linearly dependent
(B) Linearly independent
(C) Proportional
(D) None of these

41. If y_1 and y_2 are two solutions of initial value problem $y'' + p(x) y' + g(x) y = 0$, $y(x_2) = y_0$, $y'(x_0) = y'_0$ and the wronskion $w(y, y_2) = 0$, then y_1 and y_2 are—
(A) Linearly dependent
(B) Linearly independent
(C) Proportional
(D) None of these

42. The nth order homogeneous linear differential equation $y^{(n)} + P_0^{(n-1)} + \ldots + P_n y = 0$ has general solution if the coefficievts $P_0(x), \ldots, P_n(x)$ on some internal I are—
(A) Continuous
(B) Discontinuous
(C) Discontinuous and differentiable
(D) None of these

43. The n-order ordinary linear differential equation have no other solution other than general solution—
(A) Always true (B) Always false
(C) Partially true (D) None of these

44. For $\frac{d^2y}{dx^2} + 4y = \tan 2x$ solving by variation of parameters. The value of wronskion w is—
(A) 1 (B) 2
(C) 3 (D) 4

45. Solving by variation of parameter $y'' - 2y' + y = e^x \log x$, the value of wronskion w is—
(A) e^{2x} (B) 2
(C) e^{-2x} (D) None of these

46. The value of wronskion $w(x, x^2, x^3)$ is—
(A) $2x^4$ (B) $2x^2$
(C) $2x^3$ (D) None of these

47. Solving by variation of parameter for the equation $y'' + y = \sec x$, the value of wronskion is—
(A) 1 (B) 2
(C) 3 (D) 4

48. The solution of $\frac{\delta^2 z}{\delta x^2} - 5\frac{\delta^2 z}{\delta x \delta y} + 4\frac{\delta^2 z}{\delta y^2} = \sin(4x + y)$ is—
(A) $z = \frac{1}{3} x \cos(4x + y)$
(B) $z = f_1(y + x) + f_2(y + 4x)$

(C) $z = f(x+y) - \frac{1}{3} x \cos(4x+y)$

(D) $z = f_1(y+x) + f_2(y+4x) - \frac{1}{3} x \cos(4x+y)$

49. $\frac{\delta^3 z}{\delta x^3} - 4\frac{\delta^3 z}{\delta x^2 \delta y} + 4\frac{\delta^3 z}{\delta x \delta y^2} = 0$ has the solution—

(A) $z = \phi(y) + f_1(y+2x) + f_2(y-2x)$

(B) $z = \phi(y) + 2f_1(y+2x)$

(C) $z = \phi(y) + f_1(y+2x) + xf_2(y+2x)$

(D) None of these

50. $\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2} = 12(x+y)$ has the solution—

(A) $z = f_1(y+ix) + f_2(y-ix)$

(B) $z = f_1(y+ix) + f_2(y-ix) + (x+y)^3$

(C) $z = (x+y)^3$

(D) None of these

51. $\frac{\delta^2 z}{\delta x^2} - \frac{\delta^2 z}{\delta y^2} + \frac{\delta z}{\delta x} - \frac{\delta z}{\delta y} = 0$ has the solution—

(A) $z = f_1(y-x) + e^x f_2(y-x)$

(B) $z = f_1(y+x) + f_2(y-x)$

(C) $z = e^{-x} f(y-x)$

(D) $z = f_1(y+x) + e^{-x} f_2(y-x)$

52. The solution of the equation $\frac{\delta^2 z}{\delta x \delta y}\left(\frac{\delta z}{\delta x} - 2\frac{\delta z}{\delta y} - 3\right) = 0$ is—

(A) $z = f_1(y) + f_2(x)$

(B) $z = e^{2x} f_1(y+3x)$

(C) $z = f_1(y) + e^{2x} f_2(y+3x)$

(D) $z = f_1(y) + f_2(x) + e^{3x} f_3(y+3x)$

53. $\frac{\delta^2 z}{\delta x^2} - \frac{\delta^2 z}{\delta x^2} + \frac{\delta z}{\delta x} = 1$ has the solution—

(A) $z = f_1(y) + e^{-x} f_2(y+x) + x$

(B) $z = f_1(y) + e^{-x} f_2(y+x)$

(C) $z = x$

(D) None of these

54. $\frac{\delta^2 z}{\delta x^2} + 2\frac{\delta^2 z}{\delta x \delta y} + \frac{\delta^2 z}{\delta y^2} + 2\left(\frac{\delta z}{\delta x} + \frac{\delta z}{\delta y}\right) + z = 0$—

(A) $z = 2e^{-x} f_1(y+x)$

(B) $z = e^{-x} f_1(y-x) + xe^{-x} f_2(y-x)$

(C) $z = f_1(y-x) + xf_2(y-x)$

(D) None of these

55. The solution of $\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2} = \frac{z}{a}$ is—

(A) $z = e^{y/a} f(x-y)$ (B) $z = e^{y/a} f(x)$

(C) $z = e^{y} f(y)$ (D) $z = e^{a} f(x+y)$

56. The general integral of $yzp + zxp = xy$ is—

(A) $f(x+y, y+z) = 0$

(B) $f(x^2+y^2, x^2+z^2) = 0$

(C) $f(x^2-y^2, x^2-z^2) = 0$

(D) None of these

57. The general integral of $\frac{\delta z}{\delta y} = 3\left(\frac{\delta z}{\delta x}\right)^2$ is—

(A) $z = ax + 3ay + c$

(B) $z = ax + 3a^2 y + c$

(C) $z = ax^2 + 3ay + c$

(D) None of these

58. The solution of $\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2} = x + y$ is—

(A) $z = \frac{2}{3}[(x+a)^{3/2} + (y-a)^{3/2}] + b$

(B) $z = \frac{2}{3}[(x+a) + (y-a)] + b$

(C) $z = \frac{2}{3}[(x+a)^2 + (y-a)^2] + b$

(D) $z = \frac{2}{3}[(x+a)^3 + (y-a)^3 + b$

59. The equation $\frac{\delta z}{\delta x} e^y = \frac{\delta z}{\delta y} e^x$ gives the general solution—

(A) $z = ae^x + be^x$

(B) $z = e^x + e^y$

(C) $z = a(e^x + e^y) + b$

(D) None of these

60. The complete solution of $z = px + qy + p^2 + q^2$ is—

(A) $z = ax + by + a^2 + b^2$

(B) $z = ax + by$

(C) $z = a^2x^2 + z^2 + by^2$

(D) $z = ax^2 + by^2 + z^2$

61. The solution of the differential equation

$$\left(x \sin\left(\frac{y}{x}\right)\right) dx - \left(y \sin\left(\frac{y}{x}\right) - x\right) dx = 0$$

(A) $\cos(y/x) = 0$

(B) $\sin(y/x) = 0$

(C) $\cos(y/x) - \log x = \text{constant}$

(D) $\sin(y/x) - \log x = \text{constant}$

62. The solution of the differential equation

$$\frac{dy}{dx} = \frac{3x^2y^4 + 2xy}{x^2 - 2x^3y^3} \text{ —}$$

(A) $x^3y^2 + \frac{x^2}{y} = \frac{x}{y}$ (B) $x^3y^2 + \frac{x^2}{y^2} = x$

(C) $x^3y^2 + \frac{x^2}{y} = c$ (D) None of these

63. The equation $y = Ae^{3x} + Be^{5x}$ can be represented as—

(A) $y'' - 8y' + 15y = 0$

(B) $y'' + 8y' = 0$

(C) $y' + 8y' = 0$

(D) $y'' + 8y' + 15y = 0$

64. The equation of the curve where sub-normal is equal to a constant a is—

(A) $y = ax + b$ (B) $y^2 = 2ax + 2b$

(C) $ay^2 - x^3 = a$ (D) None of these

65. The differential $\frac{dy}{dx}$ + Py = Q, P and Q are functions of x only, have the integrating factor—

(A) $e^{\int Pdx}$ (B) $e^{\int Qdy}$

(C) $e^{\int Pdy}$ (D) $e^{\int Qdx}$

66. A solution of a differential equation which contains no arbitrary constants is—

(A) Particular solution

(B) General solution

(C) Primitive

(D) None of these

67. A general solution of linear differential equation with constant coefficients is—

(A) Sum of particular solution and complementary function

(B) Product of particular solution and complementary function

(C) Quotient of particular solution and complementary function

(D) None of these

68. (a) The solution of ordinary differential equation of order n have n arbitrary constants.

(b) The solution of partial differential equation of order n have n arbitrary functions.

(A) (a) is true, (b) is false

(B) (a) is false (b) is true

(C) (a) and (b) are both true

(D) (a) and (b) both are false

69. The solution of ordinary differential equation of n order contains—

(A) n arbitrary constants

(B) More than n-arbitrary constants

(C) No arbitrary constants

(D) None of these

70. The solution of equation of $dx + xdy = 0$ is—

(A) $xy = c$ (B) $x = y + c$

(C) $x - y = c$ (D) None of these

71. Given a differential equation of order n, then its completes primitive contains—

(A) n-arbitrary constants

(B) More than n-arbitrary

(C) Less than n-arbitrary constants

(D) No arbitrary constant

72. The necessary condition for the equation M (x, y) dx + N (x, y) dy = 0 to be exact is—

(A) $\frac{\delta N}{\delta y} = \frac{\delta M}{\delta x}$ (B) $\frac{\delta N}{\delta y} = -\frac{\delta M}{\delta x}$

(C) $\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$ (D) $\frac{\delta M}{\delta y} = -\frac{\delta N}{\delta x}$

73. The equation $ydx + xdy = 0$ is—

(A) Exact differential equation

(B) Not exact differential equation

(C) Partial differential equation

(D) None of these

74. The solution of differential equation contains as may as arbitrary constants as the order of a differential equation then the solution is said—

(A) Particulars solution

(B) Complete primitive

(C) Singular solution

(D) None of these

75. The solution derived from complete primitive by giving particular values to arbitrary constants is—

(A) Singular solution

(B) Particular integral

(C) Complete solution

(D) None of these

76. The number of arbitrary constants complete primitive of differential equation

$$\frac{d^5y}{dx^5} + 2\frac{d^4y}{dx^4} = 0 \text{ contains—}$$

(A) 5 (B) 4
(C) 1 (D) None of these

77. Any equation contains n-arbitrary constants, then the order of differential equation derived from it, is—

(A) n (B) $n - 1$
(C) 2 (D) $n + 1$

78. The differential equation, derived from $y = Ae^{2x} + Be^{-2x}$ have the order, where A, B are constants—

(A) 3 (B) 2
(C) 1 (D) None of these

79. A differential equation of first order and first degree is homogeneous is—

(A) $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ (B) $\frac{dy}{dx}$ = constant
(C) $\frac{dy}{dx} = f(x)$ (D) None of these

80. The differential equation $\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$ is—

(A) Homogeneous equation
(B) Non homogenous equation
(C) Non-exact equation
(D) None of these

81. The differential equation $\frac{dy}{dx} = \frac{y^3}{x^2}$ is—

(A) Homogeneous (B) Non-homogenous
(C) Exact equation (D) None of these

82. The equation $\frac{dy}{dx} + Py = Q$ is linear differential equation of first order if—

(A) P, Q are function of x only
(B) P, Q are function of y only
(C) P, Q are function of x and y
(D) None of these

83. The equation $x = A\cos(pt - \alpha)$ can be expressed as—

(A) $\frac{d^2x}{dt^2} = x$ (B) $\frac{d^2x}{dt^2} = -p^2x$
(C) $\frac{d^2x}{dt^2} = 0$ (D) None of these

84. Given equation $\frac{d^3y}{dx^3} = \frac{dy}{dx}$ and a solution of it is $y = a_0 + a_1 \sinh/x + a_2 \cosh/x$, where a_0, a_1, a_2 are arbitrary constants, then this solution is—

(A) Particular solution
(B) Complete primitive
(C) Singular solution
(D) None of these

85. General solution of $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$ is—

(A) $y = Ae^x + Be^{2x} + Ce^{3x}$
(B) $y = 3e^x$
(C) $y = A + Be^{2x}$
(D) None of these

86. The differential equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 50e^{2x}$ have particular integral—

(A) $\frac{2e^{2x}}{3}$ (B) $2e^{2x}$
(C) e^{2x} (D) None of these

87. The general solution of $\frac{dy}{dx} + \frac{1}{x}y = x^2$ is—

(A) $xy = \frac{1}{4}x^4 + c$ (B) $xy = c$
(C) $\frac{x}{y} = c$ (D) None of these

88. Solution of $\frac{dy}{dx} + \frac{1}{x}y = x^3$ is—

(A) $e^{\log x} = x$ (B) $e = x$
(C) $\log x = y$ (D) None of these

89. General solution of $\frac{dy}{dx} + 2xy = 2e^{-x^2}$ is—

(A) $y = (2x + c)e^{-x^2}$ (B) $y = 2xe^{-x}$
(C) $y = e^{-x}$ (D) None of these

90. General solution of $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 2y = 0$ is—

(A) $y = Ae^{-2x}$
(B) $y = Be^{1/2x}$
(C) $y = Ae^{-2x} + Be^{-1/2x}$
(D) None of these

91. For a given differential equation—
(a) An envelope gives a singular solution
(b) Node locus gives a solution
(c) Cusp-locus gives a solution
(A) (a) is true, (b) and (c) are false
(B) (a), (b), (c) are true
(C) (a) is false, (b) and (c) are true
(D) (a), (b), (c) are false

92. The p-discriminant does not contains one of the following—
(A) The envelop
(B) The tac-locus
(C) The cusp-locus
(D) The node-locus

93. The c-discriminant when equated to zero includes nodal locus—
(A) Once (B) Twice
(C) Thrice (D) None of these

94. The c-discriminant and p-discriminant both contains—
(A) Envelope (B) Tac-locus
(C) Node-locus (D) None of these

95. The c-discriminant of the equation $(y - c)^2 = x(x - a)^2$ is—
(A) $x(x - a^2) = 0$ (B) $x - a^2 = 0$
(C) $x = a$ (D) None of these

96. The p-discriminant of the equation $y = px + \frac{1}{p}$ is—
(A) $y^2 = x$ (B) $y = x$
(C) $y^2 = 4x$ (D) None of these

97. The p-discriminant of equation $4xp^2 = (3x - a)^2$ is—
(A) $x = 0$ (B) $3x = b^2$
(C) $(3x - a)^2 = 0$ (D) None of these

98. The p-discriminant of equation $y^2p^2 + y^2 - r^2 = 0$ is—
(A) $y^2 = 0$ (B) $y^2 = r^2$
(C) $y^2(y^2 - r^2) = 0$ (D) $y^4 = 0$

99. Following is the solution of differential equation—
(A) Envelope (B) Cusp locus
(C) Node-locus (D) Tac-locus

100. If $\phi(x, y) = 0$ is a singular solution, then $\phi(x, y)$ is a factor of—
(A) p-discriminant only
(B) c-discriminant only
(C) p-and c-discriminant both
(D) None of these

101. The discriminant of a quadratic equation $ax^2 + bx + c = 0$ is—
(A) $b^2 - 4ac$ (B) b^2
(C) $4ac$ (D) None of these

102. The nodes have a double point—
(A) With distinct tangent
(B) With coincident tangent
(C) Same tangent
(D) None of these

103. (a) Singular solution contains no arbitrary constants.
(b) Singular solution can be obtained from complete primitive.
(A) (a) is true, (b) is false
(B) (a) and (b) both true
(C) (a) and (b) both false
(D) None of these

104. Which of the following is an ordinary differential equation ?
(A) $y + 2x = \frac{d}{dx}(\sin x)$
(B) $\frac{d}{dx}(y + x) = x^2 + \sin x$
(C) $\frac{d}{dx}(\cos x + \sin x) = x + y$
(D) $y^2 + x^2 = \frac{d}{dx}(x\alpha + 1)$

105. The degree of the differential equation $\left[y + x\left(\frac{d^2y}{dx^2}\right)^2\right]^{1/4} = \frac{d^3y}{dx^3}$ is given by—
(A) 2 (B) 3
(C) 4 (D) 1

106. The order of the differential equation $\left[1 + \left(\frac{d^3y}{dx^3}\right)^2\right]^{4/3} = \frac{d^2y}{dx^2}$ is given by—
(A) 1 (B) 2
(C) 3 (D) 4

107. Which of the following is false ?
(A) The degree of the differential equation is the highest power of the highest order differential coefficient occurring it
(B) The order of the differential equation is the highest order derivative occurring in the equation
(C) The order of a differential equation is unique
(D) The order of a differential equation is any integer

108. If m and n are the order and degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^3 + 4\left(\frac{dy}{dx}\right)^2 + 3y^2 + y = 0$, then—
(A) $m = n$ (B) $m < n$
(C) $m > n$ (D) $m \geq n$

109. The general solution of the differential equation $ydx - xdy = xydx$ is—
(A) $y = \text{A}xe^x$ (B) $\text{A}x = ye^x$
(C) $y = \text{A}e^x$ (D) $y = \text{A}e^{-x}$

110. The solution of the equation $\frac{dy}{dx} + \frac{1+y}{1+x} = 0$, is—
(A) $1 + x = c\,(1 + y)$
(B) $y + \frac{y^2}{2} = x + \frac{x^2}{2} + c$
(C) $1 + x + y + xy = c$
(D) None of these

111. The differential equation which is linear is—
(A) $\frac{dy}{dx} + x^2y = \sin y$
(B) $\frac{dy}{dx} - x^2y = \sin x$
(C) $(1 + y)\frac{dy}{dx} + \sin x = 0$
(D) $\frac{dy}{dx} + y\,(y + x) = x^2$

112. Solution of the equation $\frac{dy}{dx} = 1 + x + y + xy$ is —
(A) $1 + y = x + \frac{x^2}{2} + c$
(B) $\log (1 + x) = y + \frac{y^2}{2} + c$
(C) $\log (1 + y) = x + \frac{x^2}{2} + c$
(D) $\log (x + y) = y + \frac{y^2}{2} + c$

113. If $(x + y)\,dx + (x - y)\,dy = 0$, then—
(A) $x^2 + 2xy - y^2 = c$
(B) $x^2 - 2xy - y^2 = c$
(C) $x^2 + 2xy + y^2 = c$
(D) $x^2 - 2xy + y^2 = c$

114. The solution of the equation $(a^2 - 2xy - y^2)\,dx - (x + y)^2\,dy = 0$ is—
(A) $a^2x - x^2y + \frac{x^3}{3} - \frac{y^3}{3} = c$
(B) $a^2x - xy^2 + \frac{x^3}{3} - \frac{y^3}{3} = 0$
(C) $a^2x - x^2y - y^2x - \frac{y^3}{3} = c$
(D) $a^2x + x^2y + y^2 + \frac{y^3}{3} = c$

115. An integrating factor of the differential equation $(1 - x^2)\frac{dy}{dx} - xy = 1$ is—
(A) $-x$ (B) $\frac{x}{1 - x^2}$
(C) $\sqrt{1 - x^2}$ (D) $\frac{1}{2}\log_e (1 - x^2)$

116. The solution of the differential equation
$$\frac{dy}{dx} = e^{x-y} + x^2e^{-y} \text{ is—}$$
(A) $e^y = e^x + \frac{1}{3}x^3 + c$
(B) $e^y = e^{-x} + \frac{1}{3}x^3 + c$
(C) $e^y = e^x + x^3 + c$
(D) $e^{-y} = \frac{1}{3}x^3 + e^x + c$

117. If $\frac{1}{\text{M}}\left(\frac{\delta \text{N}}{\delta x} - \frac{\delta \text{M}}{\delta y}\right) = f\,(y)$, a function of y alone, then the integrating factor of $\text{M}dx + \text{N}dy = 0$ is—
(A) $e^{-\int f(y)\,dy}$
(B) $e^{\int f(y)\,dy}$
(C) $f(y)\int e^{f(y)\,dy}$
(D) $\int e^{f(y)} f(y)\,dy$

118. The solution of the differential equation $(x+y)^2 \frac{dy}{dx} = a^2$ is given by—

(A) $(y+x) = a \tan\left(\frac{y-c}{a}\right)$

(B) $(y-x) = a \tan(y-c)$

(C) $(y-x) = \tan\left(\frac{y-c}{a}\right)$

(D) $a(y-x) = \tan\left(\frac{y-c}{a}\right)$

119. The solution of $\frac{dy}{dx} = \frac{1}{x+y+1}$ is—

(A) $y = \log(x+y+2) + 1 + c$

(B) $y + 1 = \log(x+y+2) + c$

(C) $y + \log(x+y+2) = c$

(D) $y + 1 + \log(x+y+2) = c$

120. The differential equation $ydx - 2xdy = 0$ represents—

(A) A family of straight lines

(B) A family of parabolas

(C) A family of circles

(D) A family of catenaries

121. The integrating factor of the differential equation $(1+y^2)\,dx = (\tan^{-1} y - x)\,dy$ is—

(A) $e^{\tan^{-1} y}$ (B) $e^{\cot^{-1} y}$

(C) $\log(1+y^2)$ (D) None of these

122. If P, Q are function of x, then solution of differential equation $\frac{dy}{dx} + Py = Q$ is—

(A) $y \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx}\, dx + c$

(B) $y = e^{\int P dx} \int Q \cdot e^{\int P dx} + c$

(C) $y^P = \int Q \cdot e^{\int P dx}\, dx + c$

(D) None of these

123. The differential equation of the form $\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$ is called—

(A) Auxiliary equation

(B) Clairaut's equation

(C) Bessel's equation

(D) Bernoulli's equation

124. The general solution of the equation $\frac{dy}{dx} = \frac{x+y}{x-y}$ is—

(A) $\tan^{-1}\frac{y}{x} = \log(x^2+y^2) + c$

(B) $2\tan\frac{y}{x} = \log(x^2+y^2) + c$

(C) $2\tan^{-1}\frac{y}{x} = \log(x^2+y^2) + c$

(D) $\log\frac{y}{x} = 2\log(x^2+y^2) + c$

125. $Pdx + x \sin y\, dy = 0$ is exact, then P can be—

(A) $\sin y + \cos y$ (B) $-\sin y$

(C) $x^2 - \cos y$ (D) $\cos y$

126. The integrating factor for the differential equation $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy$ is given by—

(A) $\frac{1}{xy}$ (B) xy

(C) x^2y^2 (D) $\frac{1}{x^2y^2}$

127. To solve the equation $(x-y-2)\,dx - (2x-2y-3)\,dy = 0$ we shall put—

(A) $y = vx$ (B) $x - y = v$

(C) $y = c \sin x$ (D) $y^2 = x^2 + v$

128. Solution of the differential equation

$\frac{xdx + ydy}{xdy - ydx} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2 + y^2}}$ is—

(A) $\sqrt{x^2+y^2} = a \sin(\tan^{-1} y + cx)$

(B) $\sqrt{x^2+y^2} = a \sin(\tan^{-1} x/y + c)$

(C) $\sqrt{x^2+y^2} = a \sin(\tan^{-1} y/x + c)$

(D) $x^2 + y^2 = a \sin(\tan^{-1} y/x + c)^2$

129. Solution of the differential equation $(x-y)^2\, dx + 2xy\, dy = 0$ is—

(A) $ye^{y^2/x} = A$ (B) $xe^{y^2/x} = A$

(C) $ye^{x/y^2} = A$ (D) $xe^{x/y^2} = A$

130. Solution of the differential equation

$(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$ is—

(A) $3x^4y^2 + 6xy^2 + 2y^6 = c$

(B) $3x^2y^4 + 6xy^2 + 2y^6 = c$

(C) $3x^2y^4 + 6xy^2 + y^6 = c$

(D) $3x^2y^4 + 6x^2y^2 + 2y^6 = c$

131. Solution of the differential equation $x^2ydx - (x^3 + y^3)\,dy = 0$ is—

(A) $y^2 = ce^{y^2/x^2}$ (B) $y^3 = ce^{x^3/y^3}$
(C) $y^3 = ce^{y^3/x^3}$ (D) $y^3 = ce^{x^2/y^2}$

132. Solution of the equation $(y - xy^2)dx - (x + x^2y)\,dy = 0$ is—

(A) $x = cye^{1/xy}$ (B) $y = cxe^{1/xy}$
(C) $xy = ce^{1/xy}$ (D) $c = xye^{1/xy}$

133. Solution of the equation $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$ is—

(A) $y + xe^{x/y} = c$ (B) $x + ye^{x/y} = c$
(C) $y + xe^{y/x} = c$ (D) $x + ye^{y/x} = c$

134. The general solution of the differential equation $(x^2 + y^2)\,dx - 2xydy = 0$ is—

(A) $x^2 - cx - y^2 = 0$
(B) $(x - y)^2 = cx$
(C) $x + y + 2xy = c$
(D) $y = x^2 - 2x + c$

135. The solution of the equation $\frac{dy}{dx} + 2xy = 2xy^2$ is—

(A) $y = \frac{cx}{1 + e^{x^2}}$ (B) $y = \frac{1}{1 - ce^{x}}$
(C) $y = \frac{1}{1 + ce^{x^2}}$ (D) $y = \frac{cx}{1 + e^{x^2}}$

136. If $f'(x) = \tan^{-1} x$ and $f(0) = 0$, then $f(1)$ is—

(A) $\frac{\pi}{4} + \log 2$ (B) $\frac{\pi}{2} - \log 2$
(C) $\frac{1}{4}(\pi - \log 4)$ (D) $\frac{1}{4}(\pi + \log 4)$

137. Solution of differential equation $3e^x \tan ydx + (1 - e^x)\sec^2 ydy = 0$ is—

(A) $\tan y\,(1 - e^{3x}) = c$
(B) $\tan y\,(1 - e^{x})^{-3} = c$
(C) $\tan y\,(1 - e^{-3x}) = c$
(D) $\text{tam}\, y\,(1 - e^{-x})^3 = c$

138. Integrating factor of $(3x - 10y^3)\frac{dy}{dx} + y = 0$ is—

(A) e^{y^2} (B) e^{x^2}
(C) y^3 (D) x^2

139. Integrating factor of $\left\{x(x-1)\frac{dy}{dx} - (x-2)y\right\} = x^3(2x - 1)$ is given by—

(A) $\frac{x-1}{x^3}$ (B) $\frac{x^2}{x-1}$
(C) $\frac{x-1}{x^2}$ (D) $\frac{x^3}{2x-1}$

140. The transformation which will transform the differential equation $\frac{dz}{dx} + \frac{2zx}{x^2+1}\log z = \frac{xz}{x^2+1}(\log z)^3$ to the form $\frac{dy}{dx} + P(x)\,y = Q(x)$ is—

(A) $y = \log z$ (B) $y = \frac{1}{(\log z)^2}$
(C) $y = \frac{z}{(\log z)^2}$ (D) $y = (\log z)^2$

141. The solution of the equation $xP + 2y = Pxy$ is—

(A) $xy^2 = Ae^{y}$ (B) $xy^2 = Ae^{x}$
(C) $x^2y = Ae^{y}$ (D) $xy = Ae^{y}$

142. The primitive of the differential equation $y = Px + f(P), P = \frac{dy}{dx}$ is—

(A) $y = x + f(c)$
(B) $y = cx = f(c)$
(C) $y = cx + f(c) = 0$
(D) $y = cx + f(c)$

143. The differential equation whose primitive is $y^2 = 2cx + c^2$ is—

(A) $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} = y$
(B) $\frac{dy}{dx} + yx = x$
(C) $\frac{dy}{dx} = yx + x$
(D) $\left(\frac{dy}{dx}\right)^2 + y^2x = x$

144. The general solution of the differential equation $P^2 + 2Py\cot x - y^2 = 0$ is—

(A) $y(1 \pm \cos x) = c$
(B) $y = c \pm \cos x$
(C) $y\cos x = c$
(D) $cy \pm \cos x = 0$

145. The general solution of the differential equation $xy\,(P^2 + 1) = (x^2 + y^2)\,P$ is—
(A) $(y - cx)(x^2 - y^2 - c) = 0$
(B) $(y - c - x)(xy - c) = 0$
(C) $(y - x - c)(x^2 + y^2 - c) = 0$
(D) $(x^2y - c)(y - cx^2) = 0$

146. The differential equation $x^2P^2 + yP\,(2x + y) + y^2 = 0$ can be reduced to the Clairaut's form using the transformation—
(A) $u = y,\ v = xy$ (B) $u = y,\ v = x/y$
(C) $u = x,\ v = y^2$ (D) $u = y,\ v = x^2$

147. The solution of the differential equation $P^3y^2 - 2Px + y = 0$ is—
(A) $2cx = c^3 + y$ (B) $2cx = c^3 + y^2$
(C) $2cy = c^3 + x^3$ (D) $2cy = c^3 + y^3$

148. The general solution of the differential equation $y = 2Px + y^{x-1}\,P^n$ is—
(A) $y = 2cx + c^{n-1}$
(B) $y = 2cx + c^n$
(C) $y^2 = 2cx + c^n$
(D) $y^2 = 2cx + c^{n-1}$

149. The general solution of the differential equation $y = x\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2$ is—
(A) $y = cx - c^2$ (B) $y = (x + c)$
(C) $y = cx - c$ (D) $y = cx + c^2$

150. The solution of $\frac{dy}{dx} = \log\left[x\frac{dy}{dx} - y\right]$ is—
(A) $y = cx - e^c$
(B) $c = \log(-cx + y)$
(C) $c = \log(cx + y)$
(D) None of these

151. The general solution of the differential equation $(x + a)\,P^2 + (x + y)\,P - y = 0$ is—
(A) $y = cx + \frac{ac^2}{c + 1}$
(B) $y = cx + \frac{ac}{c + 1}$
(C) $y = -cx + \frac{ac^2}{c + 1}$
(D) $y = -cx + \frac{ac}{c + 1}$

152. Solution of the equation $y - 2px = f\,(xP^2)$, where $p = \frac{dy}{dx}$ is—
(A) $y = 2cx + f(c^2)$
(B) $y = cx + f(c^2x)$
(C) $y = 2c\sqrt{x} + f(cx^2)$
(D) $y = 2c\sqrt{x} + f(c^2)$

153. Solution of $x^2P^2 + xyP - 6y^2 = 0$ is—
(A) $(yx - c)(y - c) = 0$
(B) $(y^3x - c)(y - cx^2) = 0$
(C) $(yx^3 - c)(y - cx^2) = 0$
(D) $(yx^2 - c)(y - cx^2) = 0$

154. The general solution of $P^3 - 4xyP + 8y^2 = 0$ is—
(A) $x = c^2\,(y - c)^2$
(B) $4x^2 = 27c^2y + c^2$
(C) $27x = 4y^2c^2 + c^2$
(D) $64y = c\,(4x - c)^2$

155. The general solution of the differential equation $(x^2 + y^2)(1 + P)^2 - 2\,(x + y)(1 + P)(x + Py) + (x + Py)^2 = 0$ can be obtained by substituting—
(A) $u = x^2,\ v = y^2$
(B) $u = xy,\ v = x + y$
(C) $u = x + y,\ v = x^2 + y^2$
(D) $u = x^2 + y^2,\ v = xy$

156. The differential equation $e^{3x}\,(P - 1) + P^3e^{3y} = 0$ can be reduced to Clairaut's equation by substituting—
(A) $u = e^x,\ v = e^y$ (B) $u = e^y,\ v = e^x$
(C) $u = x^2$ (D) $v = y^2$

157. The solution of the differential equation $y = Px + \log P$, is—
(A) $y = cx + \log c$
(B) $y = cx^2 + \log c$
(C) $y^2 = cx + \log c$
(D) $y^2 = cx^2 + \log c$

158. The general solution of the differential equation $P^2 - 5P + 6 = 0$ is—
(A) $(y - 3x - c_1)(y - 2x - c_2) = 0$
(B) $(3x + y + c_1)(2y + x - c_2) = 0$
(C) $(y + 3x - c_1)(y + 2x - c_2) = 0$
(D) $(y - x - c_1)(y + x - c_2) = 0$

159. If f (D) $y = 0$ be a linear differential equation with constant coefficients, then its auxiliary equation is—
(A) $f(m^3) = 0$ (B) $f(m) = 0$
(C) $f(D - m) = 0$ (D) $f(x) = 0$

160. A differential equation in which the dependent variable and its derivatives occur only in first degree and are not multiplied together is called a .... differential equation—
(A) Homogeneous
(B) Non-homogeneous
(C) Linear
(D) Linear with constant coefficient

161. If the general solution of the linear differential equation with constant coefficients $f(0) = 0$ be $c_1e^{a_1x} + c_2e^{a_2x} + \ldots\ldots + c_ne^{a_nx} = 0$, then the roots of the auxiliary equation are—
(A) All real
(B) All imaginary
(C) Some real and some imaginary
(D) All real and distinct

162. The auxiliary equation $aD^2 + bD + c = 0$ is obtained from the linear differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, where a, b, c are constants, by substituting—
(A) $y = x^m$ (B) $y = b \sin mx$
(C) $y = m^x$ (D) $y = e^{mx}$

163. The roots of the auxiliary equation of the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$ are—
(A) 1, 4 (B) –1, 4
(C) –1, –4 (D) 1, –4

164. The general solution of the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$ is—
(A) $y = C_1e^{-x}$
(B) $y = C_2e^{4x}$
(C) $y = C_1e^{-x} + C_2e^{2x}$
(D) $y = C_1e^{-x} + C_2e^{4x}$

165. The general solution of the linear differential equation $(D^4 - 81)\, y = 0$ is given by—
(A) $(C_1 + C_2x)\, e^{3x} + (C_3 + C_4x) \sin 3x$
(B) $(C_1 + C_2x)\, e^{3x} + (C_3 + C_4x)\, e^{-3x} + (C_5 + C_6x) \cos 3x + (C_1 + C_8x) \sin 3x$
(C) $C_1e^{3x} + C_2e^{-3x} + C_3 \cos 3x + C_4 \sin 3x$
(D) $C_1e^{3x} + C_2e^{-3x} + e^{3x}(C_3 \cos x + C_4 \sin x)$

166. $\frac{1}{D - a}$ Q (x) is equal to—
(A) $e^{ax}\int Q(x)\, dx$ (B) $e^{-ax}\int e^{ax}\, Q(x)\, dx$
(C) $e^{-ax}\int Q(x)\, dx$ (D) $e^{ax}\int e^{-ax}\, Q(x)\, dx$

167. The general solution of the differential equation $D^2 (D + 1)^2\, y = e^x$ is—
(A) $y = C_1 + C_2x + (C_3 + C_4x)\, e^x$
(B) $y = C_1 + C_2x + (C_3 + C_4x)\, e^{-x} + \frac{1}{4}e^x$
(C) $y = (C_1 + C_2e^{-x}) + (C_3 + C_4)\, e^{-x} + \frac{1}{4}e^x$
(D) None of these

168. The general solution of the differential equation $\frac{d^2y}{dx^2} - y = e^x$ is—
(A) $y = A \cos (x + B) + e^x$
(B) $y = A \cosh (x + B) + \frac{1}{2}xe^x$
(C) $y = A \cosh (x + B) + xe^x$
(D) $y = A \cos (x + B) + xe^x$

169. The general solution of the differential equations $(D^2 + D - 2)\, y = e^x$ is given by—
(A) $y = C_1e^x + C_2e^{-2x} + \frac{1}{3}xe^x$
(B) $y = C_1e^x + C_2e^{-2x}$
(C) $y = C_1e^x + C_2e^{-2x} + \frac{1}{6}x^2e^x$
(D) $y = \frac{1}{3}xe^x + (C_1 + C_2x)\, e^{-2x}$

170. A P. I. of the differential equation $(D^2 + 4)\, y = x$ is—
(A) xe^{-2x} (B) $x \cos 2x$
(C) $x \sin 2x$ (D) $x/4$

171. The solution of the differential equation $\frac{d^2y}{dx^2} + \omega^2y = 10\omega^2$ is—
(A) $y = A \cos \omega x \sin \omega x + B + 10$
(B) $y = A \sin (\omega x + B) + 10\omega^2$
(C) $y = Ax + Bx \cos \omega x + 10\omega^2$
(D) $y = A \cos (\omega x + B) + 10$

172. The general solution of the differential equation $\frac{d^2y}{dx^2} + a^2y = \sec ax$ is—

(A) $y = C_1 \cos ax + C_2 \sin ax + x \sin ax \log (\cos ax)$

(B) $y = C_1 \cos ax + C_2 \sin ax + \frac{1}{a} \{x \sin ax + \log (\cos ax)\}$

(C) $y = C_1 \cos ax + C_2 \sin ax + \frac{1}{a}\left\{x \sin ax + \frac{1}{a} \log (\cos ax)\right\}$

(D) $y = C_1 \cos ax + C_2 \sin ax + \frac{1}{a}\left\{x \sin ax + \frac{1}{a} \log (\cos ax) \cos ax\right\}$

173. For the differential equation $t (t-2)^2 y'' + ty' + y = 0$, $t = 0$ is—

(A) An ordinary point

(B) A branch point

(C) An irregular point

(D) A regular singular point

174. Suppose that y_1 and y_2 form a fundamental set at solutions of a second order ordinary differential equation on the interval $-\infty < t < +\infty$, then—

(A) There is only one zero of y_1 between consecutive zeros of y_2

(B) There are two zeros of y_1 between consecutive zeros of y_2

(C) There are finite number of zeros of y_1 between consecutive zeros of y_2

(D) None of the above

175. The general solution of the system of differential equations $\frac{dX}{dt} = MX + b$, where $X = \begin{bmatrix} x_1 (t) \\ x_2 (t) \end{bmatrix}$; M, a 2×2 matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and b, a 2×1 constant vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is given by—

(A) $e^{m_1} C + b$ (B) $e^{m_1} C + bt$

(C) $e^{m_1} C - b$ (D) $e^{m_1} C - bt$

Where C is any 2×1 constant vector.

176. The general solution of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is of the form—

(A) $u = f(x + iy) + g(x - iy)$

(B) $u = f(x + y) + g(x - y)$

(C) $u = Cf(x - iy)$

(D) $u = g(x + iy)$

177. If $y' - x \neq 0$, a solution of the differential equation $y'(y' + y) = x(x + y)$ is given by y—

(A) $1 - x - e^{-x}$ (B) $1 - x + e^{x}$

(C) $1 + x + e^{-x}$ (D) $1 + x + e^{x}$

178. For the differential equation $4x^3y'' + 6x^2y' + y = 0$, the point at infinity is—

(A) An ordinary point

(B) A regular singular point

(C) An irregular singular point

(D) A critical point

179. For the differential equation $xy' - y = 0$ which of the following function is not an integrating factor ?

(A) $\frac{1}{x^2}$ (B) $\frac{1}{y^2}$

(C) $\frac{1}{xy}$ (D) $\frac{1}{x + y}$

180. The differential equation $(3a^2x^2 + by \cos x) dx + (2 \sin x - 4ay^3) dy = 0$ is exact for—

(A) $a = 3, b = 2$ (B) $a = 2, b = 3$

(C) $a = 3, b = 4$ (D) $a = 2, b = 5$

181. Let $u(x, t)$ be the solution of $u_{11} = u_{xx}$; $0 < x < 1$, $t > 0$, $u(x, 0) = x(1 - x)$, $u_1(x, 0) = 0$, then $u\left(\frac{1}{2}, \frac{1}{4}\right)$ is—

(A) $\frac{3}{16}$ (B) $\frac{1}{4}$

(C) $\frac{3}{4}$ (D) $\frac{1}{16}$

182. The differential equation $\frac{dy}{dx} = k(a - y)(b - y)$, when solved with the condition $y(0) = 0$, yields the result—

(A) $\frac{b(a - y)}{a(b - y)} = e^{(a-b)kx}$

(B) $\frac{b(a - x)}{a(b - x)} = e^{(b-a)ky}$

(C) $\frac{a(b - y)}{b(a - y)} = e^{(a-b)kx}$

(D) $xy = ke$

183. The equation $x^2 (y - 1) Z_{xx} - x (y^2 - 1) Z_{xy} + y (y^2 - 1) Z_{yy} + Z_x = 0$ is hyperbolic in the entire xy –plane except along—
(A) x-axis
(B) y-axis
(C) A line parallel to y-axis
(D) A line parallel to x-axis

184. Determine the type of the following differential equation $\frac{d^2y}{dx^2} + \sin(x + y) = \sin x$—
(A) Linear, homogeneous
(B) Non linear, homogeneous
(C) Linear, non-homogenous
(D) Non-linear, non-homogeneous

185. Which of the following is not an integrating factor of $xdy - ydx = 0$?
(A) $\frac{1}{x^2}$ (B) $\frac{1}{x^2 + y^2}$
(C) $\frac{1}{xy}$ (D) $\frac{x}{y}$

186. The general solution of the differential equation $\frac{dy}{dx} + \tan y \tan x = \cos x \sec y$ is—
(A) $2\sin y = (x + c - \sin x \cos x) \sec c$
(B) $\sin y = (x + c) \cos x$
(C) $\cos y = (x + c) \sin x$
(D) $\sec y = (x + c) \cos x$

187. The differential equation whose linearly independent solutions are $\cos 2x$, $\sin 2x$ and e^{-x} is—
where $D = \frac{d}{dx}$
(A) $(D^3 + D^2 + 4D + 4) y = 0$
(B) $(D^3 - D^2 + 4D - 4) y = 0$
(C) $(D^3 + D^2 - 4D - 4) y = 0$
(D) $(D^3 - D^2 - 4D + 4) y = 0$

188. The order of the numerical differentiation formula $f''(x_0) = \frac{1}{12h^2} [- \{f(x_0 - 2h) + f(x_0 + 2h)\} + 16 \{f(x_0 - h) + f(x_0 + h)\} - 30 f(x_0)]$ is—
(A) 2 (B) 3
(C) 4 (D) 1

189. The Bessel's function $\{J_0 (a_k x)\}_{k=1}^{\infty}$ with a_k denoting the k-th zero of $J_0(x)$ form on orthogonal system on [0, 1] with respect to weight function—
(A) 1 (B) x^2
(C) x (D) $\sqrt{x}$

190. Linear combinations of solutions of an ordinary differential equation are also solutions if the differential equation is—
(A) Linear non-homogenous
(B) Linear homogeneous
(C) Non-linear homogeneous
(D) Non-linear non-homogeneous

191. Which of the following, concerning the solution of the Neumann problem for Laplace's equation, on a smooth bounded domain, is true ?
(A) Solution is unique
(B) Solution is unique upto an additive constant
(C) Solution is unique upto a multiplicative constant
(D) No conclusion can be drawn about uniqueness

192. Which of the following satisfies the heat equation (without sources terms and with diffusion constant 1) in one space dimension ?
(A) $\sin\left[\frac{x^2}{4t}\right]$ (B) $e^t \sin x$
(C) $x^2 - t$ (D) $\frac{e^{-x^2/4t}}{\sqrt{t}}$

193. Which of the following is elliptic ?
(A) Laplace equation
(B) Wave equation
(C) Heat equation
(D) $u_{xx} + 2u_{xy} - 4u_{yy} = 0$

194. Let $X = (0,1) \cup (2, 3)$ be an open set in R. Let f be a continuous function on X such that the derivative $f'(x) = 0$ for all x. the range of f has—
(A) Uncountable number of points
(B) Countably infinite number of points
(C) At most 2 points
(D) At most 1 point

195. The orthogonal trajectory to the family of circles $x^2 + y^2 = 2\,cx$ (c arbitrary) is described by the differential equation—
   (A) $(x^2 + y^2)\,y' = 2xy$
   (B) $(x^2 - y^2)\,y' = 2xy$
   (C) $(y^2 - x^2)\,y' = xy$
   (D) $(y^2 - x^2)\,y' = 2xy$

196. Pick the region in which the following differential equation is hyperbolic $y^{u_{xx}} + 2xyu_{xy} + xu_{xy} = u_x + u_y$—
   (A) $xy \neq 1$ (B) $xy \neq 0$
   (C) $xy > 1$ (D) $xy > 0$

197. Extremals $y = y\,(x)$ for the variational problem $u\,[y\,(x)] = \int_0^1 (y + y')^2\,dx$ satisfy the differential equation—
   (A) $y'' + y = 0$ (B) $y'' - y = 0$
   (C) $y'' + y' = 0$ (D) $y' + y = 0$

198. In a sufficiently small neighbourhood around $x = 2$, the differential equation $\frac{dy}{dx} = \frac{y}{\sqrt{x}}$, $y\,(2) = 4$ has—
   (A) No solution
   (B) A unique solution
   (C) Exactly two solutions
   (D) Infinitely many solutions

199. The set of linearly independent solutions of the differential equation $\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = 0$ is—
   (A) $\{1, x, e^x, e^{-x}\}$ (B) $\{1, x, e^{-x}, xe^{-x}\}$
   (C) $\{1, x, e^x, xe^x\}$ (D) $\{1, x, e^x, xe^{-x}\}$

200. For the differential equation
$$x^2\,(1 - x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0\text{—}$$
   (A) $x = 1$ is an ordinary point
   (B) $x = 1$ is a regular singular point
   (C) $x = 0$ is an irregular singular point
   (D) $x = 0$ is an ordinary point

201. The general solution of the system of differential equations
$$y + \frac{dz}{dx} = 0,$$
$$\frac{dy}{dx} - z = 0$$
is given by—
   (A) $y = \alpha e^x + \beta e^{-x}$
   $z = \alpha e^x - \beta^{-x}$
   (B) $y = \alpha \cos x + \beta \sin x$
   $z = \alpha \sin x - \beta \cos x$
   (C) $y = \alpha \sin x - \beta \cos x$
   $z = \alpha \cos x + \beta \sin x$
   (D) $y = \alpha e^x - \beta e^{-x}$
   $z = \alpha e^x + \beta e^{-x}$

202. It is required to find the solution to the differential equation $2x\,(2 + x)\frac{d^2x}{dx^2} + 2(3 + x)\frac{dy}{dx} - xy = 0$ around the point $x = 0$. The roots of the indicial equation are—
   (A) $0, \frac{1}{2}$ (B) $0, 2$
   (C) $\frac{1}{2}, \frac{1}{2}$ (D) $0, \frac{1}{2}$

203. A function $u\,(x,\ t)$ satisfies the wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 1$, $t > 0$. If $u\left[\frac{1}{2}, 0\right] = \frac{1}{4}$, $u\left[1, \frac{1}{2}\right] = 1$ and $u\left[0, \frac{1}{2}\right] = \frac{1}{2}$, then $u\left[\frac{1}{2}, 1\right]$ is—
   (A) $\frac{1}{4}$ (B) $\frac{5}{4}$
   (C) $\frac{4}{5}$ (D) $\frac{4}{7}$

204. The Fourier transform $f\,(\omega)$ of $f\,(x)$ $-\infty < x < \infty$ is defined by $f\,(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f\,(x)\,e^{-iwx}\,dx$ the Fourier transform with respect to x of the solution $u\,(x,\ y)$ of the boundary value problem $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $-\infty < x < \infty$, $y > 0$, $u\,(x,\ 0) = f\,(x)$, $-\infty < x < \infty$ which remains bounded for large y is given by $U\,(\omega,\ y) = f\,(\omega)\,e^{-|w|y}$ then, the solution $u\,(x,\ y)$ is given by—
   (A) $u\,(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f\,(x - z)}{y^2 + z^2}\,dz$
   (B) $u\,(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f\,(x + z)}{y^2 + z^2}\,dz$

(C) $u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x-z)}{y^2+z^2} dz$

(D) $u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x+z)}{y^2+z^2} dz$

205. It is required to solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $0 < x < a$, $0 < y < b$, satisfying the boundary conditions $u(x, 0) = 0$, $u(x, b) = 0$, $u(0, y) = 0$ and $u(a, y) = f(y)$. If C_n's are constants, then the equation and the homogeneous boundary conditions determine the fundamental set of solutions of the form—

(A) $u(x, y) = \sum_{n=1}^{\infty} C_n \sinh\frac{n\pi x}{b} \sin\frac{n\pi x}{b}$

(B) $u(x, y) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{b} \sin\frac{n\pi y}{b}$

(C) $u(x, y) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{b} \sinh\frac{n\pi y}{b}$

(D) $u(x, y) = \sum_{n=1}^{\infty} C_n \sinh\frac{n\pi x}{b} \sinh\frac{n\pi y}{b}$

Answers with Explanation

1. (B)	2. (B)	3. (C)	4. (B)	5. (C)
6. (B)	7. (D)	8. (A)	9. (A)	10. (B)
11. (A)	12. (A)	13. (C)	14. (B)	15. (D)
16. (A)	17. (A)	18. (D)	19. (A)	20. (A)
21. (C)	22. (B)	23. (A)	24. (A)	25. (D)
26. (C)	27. (D)	28. (C)		

29. (A) $(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0$

$$\frac{\delta M}{\delta y} = \cos x + \cos y + 1 = \frac{\delta N}{\delta x}$$

The equation is exact and solution is
$\int (y\cos x + \sin y + y)\,dx + f(0)\,dx = c$

30. (A) Characteristic equation

$$m^2 + m - 2 = 0$$
$$\Rightarrow \quad (m+2)(m-1) = 0$$
$$\Rightarrow \quad m = -2, 1$$
$$\Rightarrow \quad y = c_1 e^{-2x} + c_2 e^{-x}$$

31. (C) Charactristic equation

$$m^3 + m^2 + um + u = 0$$
$$\Rightarrow \quad (m^2 + u)(m+1) = 0,$$
$$m = -1 \pm 2i$$
$$\therefore \quad y = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x$$

32. (B) $y_p = \frac{1}{D^3+1} \cos(2x-1)$

(Put $D^2 = -2^2 = -4$)

$$= \frac{1}{D(-4)+1} \cos(2x-1)$$
$$= \frac{1+4D}{1-16D^2} \cos(2x-1)$$

(Put $D^2 = -2^2 = -4$)

$$= \frac{1}{65}[\cos(2x-1) + 4D\cos(2x-1)]$$

or $y_p = \frac{1}{65}[\cos(2x-1) - 8\sin(2x-1)]$

33. (B)	34. (A)	35. (A)	36. (A)	37. (B)
38. (C)	39. (A)	40. (B)	41. (B)	42. (A)
43. (A)				

44. (B) $\frac{d^2y}{dx^2} + 4y = \tan x$

Corresponding homogeneous equation

$$\frac{d^2y}{dx^2} + 4y = 0$$
$$\Rightarrow \quad m^2 + 4 = 0$$
$$\therefore \quad m = \pm 2i$$
$$\therefore \quad y_1 = c_1 \cos 2x + c_2 \sin 2x$$
$$\therefore \quad w = \begin{vmatrix} y_1 & y_2 \\ y^1{}_1 & y^1{}_2 \end{vmatrix}$$
$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

45. (A) The corresponding homogeneous equation is

$$y'' - 2y' + y = 0$$
$$\Rightarrow \quad m^2 - 2m + 1 = 0$$
$$\Rightarrow \quad m = 1, 1$$
$$\therefore \quad y_n = (c_1 + c_2 x)e^x$$

Then $w = \begin{vmatrix} y_1 & y_2 \\ y^1{}_1 & y^1{}_2 \end{vmatrix}$

$$= \begin{vmatrix} e^x & xe^x \\ e^x & (1+x)e^x \end{vmatrix}$$
$$= e^{2x}$$

46. (C) $w(x, x^2, x^3) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3$

47. (A) The solution of corresponding homogeneous equation is

$$y = c_1 \cos x + c_2 \sin x$$

$$\therefore \quad w(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

48. (D) $(D^2 - 5DD' + 4D'^2)z = \sin(4x + y)$

Alternate equation

$$m^2 - 5m + 4 = 0$$

$$\Rightarrow (m-1)(m-4) = 0$$

$$\Rightarrow m = 1, 4$$

$$z_n = f_1(y + x) + f_2(y + 4x)$$

$$z_p = \frac{1}{D^2 - 5DD' + 4D'^2}\sin(4x + y)$$

$$= \frac{1}{(D - 4D')(D - D')}\sin(4x + y)$$

$$= \frac{1}{(D - 4D')}\cdot\frac{1}{4 - 1}\{-\cos(4x + y)\}$$

$$= \frac{1}{(D - 4D')}\left\{-\frac{1}{3}\cos(4x + y)\right\}$$

$$= -\frac{x}{3}\cos(4x + y)$$

$$\therefore \ z = z_n + z_p$$

$$= f_1(y + x) + f_2(y + 4x) - \frac{1}{3}x\cos(4x + y)$$

49. (C) $(D^3 - 4D^2D' + 4DD'^2) = 0$

Alternate equation

$$m^3 - 4m^2 + 4m = 0$$

$$\Rightarrow m(m-2)^2 = 0$$

$$\Rightarrow m = 0, 2, 2$$

$$z = \phi(y) + f_1(y + 2x') + xf_2(y + 2x)$$

50. (B) $(D^2 + D'^2)z = 12(x + y)$

Alternate equation

$$m^2 + 1 = 0, m \pm i$$

$$z_n = f_1(y + ix) + f_2(y - ix)$$

$$z_p = \frac{1}{(D^2 + D'^2)}12(x + y)$$

$$= \frac{1}{1^2 + 1^2}12\frac{(x + y)^3}{6}$$

$$= (x + y)^3$$

$$\therefore \ z = z_n + z_p = f_1(y + ix) + f_2(y - ix) + (x + y)^3$$

51. (D) $(D^2 - D'^2 + D - D')z = 0$

$$\Rightarrow (D - D')(D + D' + 1) = 0$$

$$z = f_1(y + x) + e^{-x}f_2(y - x)$$

52. (D) $DD'(D - 2D' - 3)z = 0$

$$z = e^{0.x}f_1(y) + e^{0.x}f_2(0 - x) + e^{3x}f_3(y + 3x)$$

$$= f_1(y) + f_2(-x) + e^{3x}f_3(y + 3x)$$

or $z = f_1(y) + f_2(x) + e^{3x}f_3(y + 3x)$

53. (A) $(D^2 - DD' + D)z = 1$

$$\Rightarrow D(D - D' + 1)z = 1$$

$$z_n = e^{0.x}f_1(y) + e^{-x}f_2(y + x)$$

$$= f_1(y) + e^{-x}f_2(y + x)$$

$$z_p = \frac{1}{D(D - D' + 1)}\cdot 1$$

$$= \frac{1}{D}(1 + D - D')^{-1}\cdot 1$$

$$= \frac{1}{D}\cdot 1 = x$$

$$\therefore \ z = z_n + z_p = f_1(y) + e^{-x}f_2(y + x) + x$$

54. (B) $(D^2 + 2DD' + D'^2 + 2D + 2D' + 1)z = 0$

$$\Rightarrow (D + D' + 1)z = 0$$

$$z = e^{-x}f_1(y - x) + xe^{-x}f_2(y - x)$$

55. (A) Lagrange's subsidiary equation

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{z/a}$$

$$\Rightarrow x - y = c_1$$

and $z = c_2 e^{y/a}$

$$\therefore \ z = e^{y/a}f(x - y)$$

56. (C) By Lagrange's subsidiary equation

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

$\Rightarrow$ First and second $x^2 - y^2 = c_1$

First and last $x^2 - y^2 = c_2$

$$\therefore \ f(x^2 - y^2, x^2 - z^2) = 0$$

57. (B) The equation is of the form $f(p, q) = 0$

General integral is

$$z = ax + by + c$$

where $b = 3a^2 \ \therefore \ z = ax + 3a^2y + c$

58. (A) Let $p^2 - x = y - q^2 = a$

$$\therefore \ p = \sqrt{(x + a)}$$

$$q = \sqrt{(y - a)}$$

Substituting in $dz = pdx + qdy$ and integrating

59. (C) $pe^{-x} = qe^{-y} = a$ (say)

and $dz = pdx + qdy$

$\Rightarrow dz = ae^x dx + ae^y dy$

60. (A) The Charpit's equations are

$$\frac{dp}{-p+p} = \frac{dq}{-q+q} = \frac{dz}{-p\,(-x-2p)-q\,(-y-2p)} = \frac{dx}{+(x+2p)} = \frac{dy}{-y-2q} = \frac{dF}{0}$$

First and second $dp = 0$, $dq = 0$, gives $p = a$, $q = b$

and $z = ax + by + a^2 + b^2$

61. (C) Rewriting the equation

$$\frac{dy}{dx} = \frac{y\sin(y/x) - x}{x\sin(y/x)}$$

Substitute $\quad y/x = 0$

We have, $v + x\dfrac{dv}{dx} = \dfrac{v\sin v - 1}{\sin v} = v - \operatorname{cosec} v$

$$\Rightarrow \quad -\sin v\,dv = \frac{dx}{x}$$

$$\Rightarrow \quad \cos y/x = \log x + c$$

$$\Rightarrow \quad \cos\left(\frac{y}{x}\right) - \log x = c$$

62. (C) The given equation can be rewritten as

$$(3x^2y^4 + 2xy) + (2x^3y^3 - x^2)\frac{dy}{dx} = 0$$

Dividing by y^2, we get

$$\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0$$

$$\Rightarrow \quad d\left(x^3y^2 + \frac{x^2}{y}\right) = 0$$

$$\Rightarrow \quad x^3y^2 + \frac{x^2}{y} = \text{constant}$$

63. (A) $y = Ae^{3x} + Be^{5x}$

$$\frac{dy}{dx} = 3Ae^{3x} + 5Be^{5x} = 3y + 2Be^{5x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 3\frac{dy}{dx} + 10Be^{5x} = 3\frac{dy}{dx} + 5\left(\frac{dy}{dx} - 3y\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$$

64. (B) $y\dfrac{dy}{dx} = a$

$$\Rightarrow \quad ydy = adx$$

$$\Rightarrow \quad \frac{y^2}{2} = ax + b$$

$$\Rightarrow \quad y^2 = 2ax + 2b$$

65. (A)	66. (A)	67. (A)	68. (A)	69. (A)
70. (A)	71. (A)	72. (C)	73. (A)	74. (B)
75. (B)	76. (A)	77. (A)	78. (B)	79. (A)
80. (A)	81. (B)	82. (A)	83. (B)	84. (B)
85. (A)	86. (B)	87. (A)	88. (A)	89. (A)
90. (C)	91. (B)	92. (D)	93. (B)	94. (A)
95. (A)	96. (C)	97. (C)	98. (C)	99. (A)
100. (C)	101. (A)	102. (A)	103. (A)	

104. (B) Only the equation in (B) possesses $\dfrac{dy}{dx}$, hence (B) is ordinary differential equation.

105. (C) Rationalizing the degree of differential equation (d.e.), we see that $y + x)\left(\dfrac{d^2y}{dx^2}\right)^2 = \left(\dfrac{d^3y}{dx^3}\right)^4$. Hence the degree of given d.e. is 4.

106. (C)

107. (D) The order of a differential equation can not be negative integer.

108. (B) Here order = 2 and degree = 3

$$\Rightarrow \quad m = 2$$

and $\quad n = 3$

$$\Rightarrow \quad m < n$$

109. (B) The given equation can be written as

$$\frac{dx}{x} - \frac{dy}{y} = dx$$

$$\Rightarrow \quad \left(\frac{1}{x} - 1\right)dx = \frac{dy}{y}$$

$$\Rightarrow \quad (\log x - x) = \log y - \log A$$

$$\Rightarrow \quad x = \log(Ax/y)$$

$$\Rightarrow \quad \frac{Ax}{y} = e^x$$

$$\Rightarrow \quad Ax = ye^x$$

110. (C) $\displaystyle\int\frac{dy}{1+y} + \int\frac{dx}{1+x} = \log c$

$$\Rightarrow \quad \log(1+y) + \log(1+x) = \log c$$

$$\Rightarrow \quad \log\{(1+x)(1+y)\} = \log c$$

$$\Rightarrow \quad (1+x)(1+y) = c$$

111. (B) The differential equation (d. e.) in (B) is linear as the shape of linear differential equation is

$$\frac{dy}{dx} + py = Q$$

Where P and Q are functions of x.

112. (C)

113. (A) Given equation can be written as

$$\frac{dy}{dx} = \frac{y+x}{y-x}$$

Put $\quad y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{vx+x}{vx-x}$$

$$x\frac{dv}{dx} = \frac{-v^2+2v+1}{v-1}$$

$$\Rightarrow \quad \left(\frac{v-1}{1+2v-v^2}\right) dv = \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{2}\log(1+2v-v^2) = \log x + \log c$$

$$\Rightarrow \quad \log x\sqrt{1+2v-v^2} = \log c$$

$$\Rightarrow \quad x^2 + 2xy - y^2 = c$$

114. (C) Here $M = a^2 - 2xy - y^2$

$$N = -(x+y)^2$$

$$\therefore \quad \frac{\delta M}{\delta y} = -2x - 2y,$$

$$\frac{\delta N}{\delta x} = -2(x+y)$$

$$\therefore \quad \frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$$

Hence given differential equation is exact.

$\therefore$ Required solution is

$$\int(a^2 - 2xy - y^2)\,dx + \int -y^2 dy = c$$

$$\Rightarrow \quad a^2x - x^2y - y^2x - \frac{1}{3}y^3 = c$$

115. (C) $\quad$ I. F. $= e^{\int \frac{-x}{1-x^2}dx}$

$$= e^{\frac{1}{2}\log(1-x^2)}$$

$$= \sqrt{1-x^2}$$

116. (A) The given equation can be written as

$$e^y dy = (e^x + x^2)\,dx$$

Integrating we get

$$e^y = e^x + \frac{1}{3}x^3 + c$$

which is the required solution.

117. (B)

118. (A) Let $\quad x + y = v$

$$\Rightarrow \quad 1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad 1 + \frac{a^2}{v^2} = \frac{dv}{dx}$$

$$\therefore \quad \int \frac{v^2}{a^2+v^2}\,dv = x + c$$

$$\text{or} \quad \int \left(1 - \frac{a^2}{a^2+v^2}\right) dv = x + c$$

$$\text{or} \quad v - \frac{a^2}{a}\tan^{-1}\frac{v}{a} = x + c$$

$$\text{or} \quad x + y - a\tan^{-1}\frac{x+y}{a} = x + c$$

$$\text{or} \quad y - a\tan^{-1}\frac{x-y}{a} = c$$

$$\text{or} \quad y + x = a\tan\left(\frac{y-c}{a}\right)$$

119. (B) Put $x + y + 1 = v$

$$\Rightarrow \quad 1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad 1 + \frac{1}{v} = \frac{dv}{dx}$$

$$\therefore \quad \int \frac{v}{v+1}\,dv = \int dx + c$$

$$\Rightarrow \quad \int \left(1 - \frac{1}{v+1}\right) dv = x + c$$

$$\text{or} \quad v - \log(v+1) = x + c$$

$$\text{or} \quad \log(v+1) = v - x - c$$

$$\therefore \quad \log(x+y+2) = y + 1 - c$$

120. (B) The given equation is

$$\frac{dx}{x} - 2\frac{dy}{y} = 0$$

$$\Rightarrow \quad \log x - 2\log y = \log c$$

$$\Rightarrow \quad \log\frac{x}{y^2} = \log c$$

$$\Rightarrow \quad x = cy^2$$

which represent a family of parabolas.

121. (A) The given equation is

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

$$\therefore \quad \text{I.F.} = e^{\int \frac{1}{1+y^2}dx}$$

$$= e^{\tan^{-1}y}$$

122. (A) $\quad$ 123. (D)

124. (C) Given that : $\frac{dy}{dx} = \frac{x+y}{x-y}$

This is homogeneous equation putting $y = vx$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{x+vx}{x-vx}$$

$$\text{or} \quad \left(\frac{1}{1+v^2} - \frac{v}{1+v^2}\right) dv = \frac{dx}{x}$$

Integrating, we get

$$\tan^{-1} v = \frac{1}{2}\log(1+v^2)$$
$$= \log x + k$$

$$\text{or} \quad 2\tan^{-1}\frac{y}{x} = \log(y^2+x^2) + 2k\,(=c)$$

125. (C) $pdx + x \sin y \cdot dy = 0$

Here $M = P$, $N = x \sin y$

For exact, $\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$

$$\Rightarrow \quad \frac{\delta P}{\delta y} = \sin y$$

$$\Rightarrow \quad P = -\cos y + f(x)$$

Let $f(x) = x^2$

then $P = -\cos y + x^2$

126. (D) $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

This is homogeneous and

$$Mx + Ny = (x^3y - 2x^2y^2) - (x^3y - 3x^2y^2)$$
$$= x^2y^2 \neq 0$$

$$\therefore \quad \text{I.F.} = \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$$

127. (B)

128. (C) $\frac{xdx+ydy}{xdy-ydx} = \sqrt{\frac{a^2-x^2-y^2}{x^2+y^2}}$

Put $x = r\cos\theta$

$y = r\sin\theta$

$$\Rightarrow \quad r^2 = x^2+y^2$$

$$\frac{y}{x} = \tan\theta$$

Differentiating these relation, we get

$$rdr = xdx + ydy$$

$$\text{and} \quad \frac{-ydx+xdy}{x^2} = \sec^2\theta\, d\theta$$

$$\Rightarrow \quad -ydx + xdy = r^2\cos^2\theta \sec^2\theta d\theta$$
$$= r^2 d\theta$$

$\therefore$ Given problem takes the form

$$\frac{rdr}{r^2d\theta} = \sqrt{\frac{a^2-r^2}{r^2}}$$

$$\int \frac{1}{\sqrt{a^2-r^2}}dr = \int d\theta + c$$

$$\therefore \quad r = a\sin(\theta + c)$$

$$\text{or} \quad \sqrt{x^2+y^2} = a\sin\left(\tan^{-1}\frac{y}{x} + c\right)$$

129. (B) The given equation can be written as

$$(x-y^2) + 2xy\frac{dy}{dx} = 0$$

Let $x - y^2 = v$

$$\Rightarrow \quad 1 - 2y\frac{dv}{dx} = \frac{dv}{dx}$$

$$\therefore \quad v + x\left(1 - \frac{dv}{dx}\right) = 0$$

$$\text{or} \quad \frac{dv}{dx} - \frac{v}{x} = 1$$

which is linear equation

$$\therefore \quad \text{I. F.} = e^{\int -1/x\, dx}$$
$$= e^{-\log x}$$
$$= 1/x$$

$$\text{Solution is} \quad v \cdot \frac{1}{x} = \int 1 \cdot \frac{1}{x}dx + \log c$$

$$\text{or} \quad v \cdot \frac{1}{x} = \log x + \log c$$

$$\text{or} \quad cx = e^{v/x}$$

$$cx = e^{(x-y^2)/x}$$

$$\text{or} \quad cx = e^{1-y^2/x}$$

$$\text{or} \quad cx = e \cdot e^{-y^2/x}$$

$$\text{or} \quad xe^{y^2/x} = e/c = A \text{ (say)}$$

130. (C) The given equation is

$(xy^3 + y)\,dx + 2\,(x^2y^2 + x + y^4)\,dy = 0$

$M = xy^3 + y$, $N = 2\,(x^2y^2 + x + y^4)$

$$\frac{\delta M}{\delta y} = 3xy^2 + 1, \frac{\delta N}{\delta x} = 4xy^2 + 2$$

$$\frac{1}{M}\left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) = \frac{1}{(xy^2+1)\,y}(xy^2+1)$$
$$= \frac{1}{y}$$

$\therefore$ I. F. $= e^{\int 1/y dy} = e^{\log y} = y$

Multiplying by y in the given equation

$(xy^4 + y^2)\,dx + 2\,(x^2y^3 + xy + y^5)\,dy = 0$

$\therefore$ Solution is $\int (xy^4 + y^2)\,dx + \int y^5 \cdot dy = c$

$$\text{or} \quad \frac{x^2}{2}y^4 + xy^2 + \frac{1}{6}y^6 = c$$

$$\text{or} \quad 3x^2y^4 + 6xy^2 + y^6 = c$$

131. (B) The given equation is $x^2y dx - (x^3 + y^3) dy = 0$

Here
$$M = x^2y$$
$$N = -(x^3 + y^3)$$
The given equation is homogeneous
$$\therefore \quad Mx + Ny = x^3y - (x^3y + y^4) = -y^4 \neq 0$$
$$\therefore \quad \text{I. F.} = \frac{1}{Mx + Ny} = -\frac{1}{y^4}$$
Multiplying by I. F. in the given equation
$$\frac{x^2}{y^3} dx - \left(\frac{x^3}{y^4} + \frac{1}{y}\right) dy = 0 \quad \ldots(1)$$
Solution is
$$\int \frac{x^2}{y^3} dx - \int \frac{1}{y} dx = \log k$$
or $\frac{x^3}{3y^3} - \log y = \log k$

or $y \cdot k = e^{x3/3y3}$

or $y^3k^3 = e^{x3/y3}$

or $y^3 = \frac{1}{k^3} e^{x3/y3}$

or $y^3 = c \cdot e^{x3/y3}$

132. (D) The given equation can be written as
$$(1 - xy)\, ydx - (1 + xy)\, xdy = 0$$
This is the form
$$f_1(x, y)\, ydx + f_2(x, y)\, xdy = 0$$
$$\therefore \quad Mx - Ny = (xy - x^2y^2) - (xy + x^2y^2) = -2x^2y^2$$
$$\therefore \quad \text{I. F.} = \frac{1}{Mx - Ny} = \frac{1}{-2x^2y^2} \text{ or } \frac{1}{-x^2y^2}$$
Multiplying by $\frac{1}{-x^2y^2}$ in the given equation we get
$$\left(\frac{-1}{x^2y} + \frac{1}{x}\right) dx - \left(\frac{1}{-xy^2} - \frac{1}{y}\right) dy = 0 \ldots(1)$$
Solution is
$$\int \left(\frac{1}{x^2y} - \frac{1}{x}\right) dx - \int \frac{1}{y} dy = \log k$$
or $-\frac{1}{xy} - \log x = \log k$

or $xyk = e^{-1/xy}$

or $xye^{1/xy} = \frac{1}{k}$

or $xye^{1/xy} = c$

133. (B) The given equation is
$$(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$$
$$M = 1 + e^{x/y}$$
$$N = e^{x/y}\left(1 - \frac{x}{y}\right)$$
$$\frac{\delta M}{\delta y} = e^{x/y}\left(-\frac{x}{y^2}\right)$$
$$\frac{\delta N}{\delta x} = e^{x/y}\left(-\frac{1}{y}\right) + \left(1 - \frac{x}{y}\right) e^{x/y}\left(\frac{1}{y}\right) = e^{x/y}\left(-\frac{x}{y^2}\right)$$
$$\therefore \quad \frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$$
Hence the given equation is exact.

$\therefore$ Solution is $\int (1 + e^{x/y})\, dx + \int 0 \cdot dy = c$

or $x + e^{x/y} \cdot y = c$

or $x + ye^{x/y} = c$

134. (A) The given equation is
$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \quad \ldots(1)$$
Put $y = vx$
$$\Rightarrow \quad \frac{dy}{dx} = v + x\left(\frac{dv}{dx}\right)$$
$$\Rightarrow \quad v + x\frac{dv}{dx} = 1 + \frac{v^2}{2v}$$
$$\Rightarrow \quad x\frac{dy}{dx} = \frac{1 - v^2}{2v}$$
$$\Rightarrow \quad \left(\frac{2v}{1 - v^2}\right) dv = \frac{dx}{x}$$
$$\Rightarrow \quad -\log(1 - v^2) = \log x - \log c$$
$$\Rightarrow \quad \log x(1 - v^2) = \log c$$
$$\Rightarrow \quad x^2 - y^2 = cx$$

135. (C) The given equation can be written as
$$y^{-2}\frac{dy}{dx} + 2xy^{-1} = 2x$$
Let $y^{-1} = v$
$$\Rightarrow \quad -y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$$
$$\therefore \quad -\frac{dv}{dx} + 2xv = 2x$$
or
$$\frac{dv}{dx} - 2xv = -2x$$
which is linear equation
$$\text{I. F.} = e^{\int -2x\, dx} = e^{-x^2}$$

Solution is $v \cdot e^{-x^2} = \int -2xe^{-x^2}\,dx + c$

or $v \cdot e^{-x^2} = e^{-x^2} + c$

or $\frac{1}{y} = 1 + ce^{x^2}$

or $y = \frac{1}{1+ce^{x^2}}$

136. (C) $f'(x) = \tan^{-1}(x)$

$$f(x) = \tan^{-1} x\,dx$$

$$= x\tan^{-1} x - \int \frac{1}{1+x^2}\,dx$$

$$f(x) = x\tan^{-1} x - \frac{1}{2}\log(1+x^2) + c$$

Put $x = 0$

Here $f(0) = 0 + c$

$\Rightarrow$ $c = 0$

$\therefore$ $f(x) = x\tan^{-1} x - \frac{1}{2}\log(1+x^2)$

$$f(1) = \frac{\pi}{4} - \frac{1}{2}\log 2$$

$$= \frac{\pi}{4} - \frac{1}{4}\log 4$$

$$= \frac{1}{4}(\pi - \log 4)$$

137. (B) Variable separable

$$3\frac{e^x}{1-e^x}\,dx = -\int \frac{\sec^2 y}{\tan y}\,dy$$

$$-3\log(1-e^x) = -\log\tan y + k$$

$$\log\tan y\,(1-e^x)^{-3} = k = \log c$$

$\therefore$ $\tan y\,(1-e^x)^{-3} = c$

138. (C) $\frac{dx}{dy} = -\frac{3x - 10y^3}{y}$

$$\frac{dx}{dy} + \frac{3}{y}x = 10y^2$$

$$\text{I. F.} = e^{3\int 1/y\,dy}$$

$$= e^{3\log y} = y^3$$

139. (C) $\frac{dy}{dx} - \frac{x-2}{x(x-1)}y = \frac{x^2(2x-1)}{x-1}$

Here $P = -\frac{x-2}{x(x-1)}$

$$= -\frac{2}{x} + \frac{1}{x-1}$$

$$e^{\int P dx} = e^{-2\log x + \log(x-1)}$$

$$= e^{\log\frac{x-1}{x-2}}$$

$$= \frac{x-1}{x^2}$$

140. (B) Divide the differential equation by $z(\log z)^3$, we get

$$\frac{1}{z(\log z)^3}\frac{dz}{dx} + \frac{2x}{x^2+1}\cdot\frac{1}{(\log z)^2} = \frac{x}{x^2+1}$$

Put $y = \frac{1}{(\log z)^2}\frac{dy}{dx} = -\frac{2}{(\log z)^3}\cdot\frac{1}{z}\cdot\frac{dz}{dx}$

$\therefore$ $-\frac{1}{z(\log z)^3}\frac{dz}{dx} = \frac{1}{2}\cdot\frac{dy}{dx}$

$\therefore$ Differential equation becomes

$$\frac{1}{2}\frac{dy}{dx} - \frac{2x}{x^2+1}\cdot y = -\frac{x}{x^2+1}$$

or $\frac{dy}{dx} - \frac{4x}{x^2+1}\cdot y = -\frac{2x}{x^2+1}$

which is a linear equation

$\therefore$ Transform required is

$$y = \frac{1}{(\log z)^2}$$

141. (C) The given equation is

$$Px + 2y = Pxy$$

or $Px = \frac{2y}{y-1}$

or $\frac{dy}{dx}x = \frac{2y}{y-1}$

or $2\frac{dx}{x} = \left(\frac{y-1}{y}\right)dy$

Integrating, we get

$$2\log x = (y - \log y) + \log A$$

or $\log\frac{x^2y}{A} = y$

or $\log\frac{x^2y}{A} = y$

or $x^2y = Ae^y$

142. (D) The given equation is

$$y = Px + f(P)$$

This is Clairaut's equation

$\therefore$ Its solution is

$$y = cx + f(c)$$

143. (A) Given equation is

$$y^2 = 2cx + c^2 \quad \ldots(1)$$

$\Rightarrow$ $2y\frac{dy}{dx} = 2c$

$\Rightarrow$ $y\frac{dy}{dx} = c$

Substituting the value of c in (1), we get

$$y\left(\frac{dy}{dx}\right)^2 + 2x\left(\frac{dy}{dx}\right) - y = 0$$

which the required differential equation

144. (A) Equation is $P^2 + 2Py \cot x - y^2 = 0$

$$\therefore \quad P = \frac{-2y \cot x \pm \sqrt{4y^2 \cot^2 x + 4y^2}}{2}$$

or $\quad P = -y \cot x \pm y \operatorname{cosec} x$

$\therefore \quad P = y \operatorname{cosec} x - y \cot x \quad \ldots(1)$

and $\quad P = -y (\operatorname{cosec} x + \cot x) \quad \ldots(2)$

From (1), $\frac{dy}{y} = (\operatorname{cosec} x - \cot x)\, dx$

Integrating, we get

$$\log y = \log \tan \frac{x}{2} - \log \sin x + \log c$$
$$= -\log (1 + \cos x) + \log c$$

$\therefore \quad y(1 + \cos x) = c$

Similarly the solution of (2) is

$y(1 - \cos x) = c$

Hence the required solution of the given equation

$y(1 \pm \cos x) = c$

145. (A) The given equations can be written as

$(xP - y)(yP - x) = 0$

i.e. $\quad xP = y$

and $\quad yP = x$

or $\quad \frac{dy}{y} = \frac{dx}{x}$

and $\quad ydy = x{\cdot}dx$

Integrating these equation we get

$\log y = \log x + \log c$

and $\quad \frac{y^2}{2} = \frac{x^2}{2} + c$

or $\quad y = cx$

and $\quad x^2 - y^2 = c$

$\therefore$ Hence the required solution is

$(y - cx)(x^2 - y^2 - c) = 0$

146. (A) Equation is $x^2P^2 + xP(2x + y) + y^2 = 0$

Let $y = u$, and $xy = v$, or $y = u$ and $x = \frac{v}{y} = \frac{v}{u}$ $\quad \ldots(1)$

$\therefore \quad dy = du$

and $\quad dx = \frac{(udv - vdu)}{u^2} \quad \ldots(2)$

$\therefore \quad P = \frac{dy}{dx} = \frac{u^2du}{udv - vdu}$

Substituting the values of x, y and P from (1) and (2) in the original equation

$$\frac{v^2}{u^2}\left(\frac{u^2du}{udv - vdu}\right)^2 + u\left(\frac{u^2du}{udv - vdu}\right)\left(\frac{2v}{u} + u\right) + u^2 = 0$$

or $u^2v^2 (du)^2 + u^2 (2v + u^2)\, du\, (udv - vdu) + u^2 (udv - vdu)^2 = 0$

or $[u^2v^2 - vu^2 (2v + u^2) + u^2v^2] (du)^2 + [u^2 (2v + u^2) - 2u^2v]\, du \cdot dv + u^4 (dv)^2 = 0$

or $\quad -v (du)^2 + ududv + (dv)^2 = 0$

or $\quad v = u \frac{dv}{du} + \left(\frac{dv}{du}\right)^2$

or $\quad v = uP + P^2$

where $\left(P = \frac{dv}{du}\right)$. This equation is Clairaut's form

147. (B) The given equation is solvable for x and can be written as

$$2x = P^2y^2 + \frac{y}{P}$$

Differentiating with respect to y, we get

$$2\frac{dx}{dy} = P^2{\cdot}2y + y^2{\cdot}2P\frac{dP}{dy} + \frac{P{\cdot}1 - y\frac{dP}{dy}}{P^2}$$

or $\quad \frac{2}{P} = 2yP^2 + \left(2Py^2 - \frac{y}{P^2}\right)\frac{dP}{dy} + \frac{1}{P}$

or $\quad \left(\frac{1}{P} - 2yP^2\right) - \left(2Py^2 - \frac{y}{P^2}\right)\frac{dP}{dy} = 0$

$$\frac{(1 - 2yP^3)}{P} + y\frac{(1 - 2yP^3)}{P^2}\frac{dP}{dy} = 0$$

$\Rightarrow \quad 1 + \frac{y}{P}\frac{dP}{dy} = 0$

$\therefore \quad \frac{dP}{P} + \frac{dy}{y} = 0$

Integrating, we get

$\log P + \log y = \log c$

or $\quad Py = c$

$\Rightarrow \quad P = c/y$

Substituting the value of P in the original equation, we get

$$\frac{c^3}{y} - 2\frac{c}{y}x + y = 0$$

or $\quad 2cx = c^3 + y^2$

148. (C) Equation $y = 2Px = y^{n-1}P^n$ is solvable for x and can be written as $2x = y/P - y^{n-1} P^{n-1}$

Differentiating with respect to y, we get

$$2\frac{dx}{dy} = \frac{P\cdot - y\cdot\frac{dP}{dy}}{P^2} - (y^{n-1}(n-1)\cdot P^{n-2}\, dP/dy + P^{n-1}(n-1)\, y^{n-2})$$

or $\dfrac{2}{P} = \dfrac{1}{P} - (n-1)P^{n-1}y^{n-2}$

$$-\left(\frac{y}{P^2} + (n-1)\,y^{n-1}P^{n-2}\right)\frac{dP}{dy}$$

or $\left(\dfrac{1+(n-1)P^n y^{n-2}}{P}\right)$

$$+\,y\left(\frac{1+(n-1)\,y^{n-2}P^n}{P^2}\right)\frac{dP}{dy} = 0$$

$\Rightarrow \quad 1 + \dfrac{y}{P}\dfrac{dP}{dy} = 0$

or $\quad \dfrac{dP}{P} + \dfrac{dy}{y} = 0$

Integrating, $\log P + \log y = \log c$

or $\quad P = \dfrac{c}{y}$

Substituting the value of P in the original equation

$$y = 2\frac{c}{y}x + y^{n-1}\frac{c^n}{y^n}$$

or $\quad y^2 = 2cx + c^n$

149. (D) The given equation is $y = Px + P^2$. which is the Clairaut's form

$\therefore$ Required solution is

$$y = cx + c^2$$

150. (A) The given equation is $P = \log(Px - y)$

$\Rightarrow \quad e^P = Px - y$

$\Rightarrow \quad y = Px - e^P$

which is the Clairaut's form

$\therefore$ Solution is $\quad y = cx - e^c$

151. (A) The given equation can be written as

$$(1+P)\,y = (P+P^2)\,x + aP^2$$

$\Rightarrow \quad y = Px + \dfrac{aP^2}{P+1}$

which is a Clairaut's form

$\therefore$ Solution is $y = cx + \dfrac{ac^2}{c+1}$

152. (D) Differentiating w.r. to x, we get

$$dy/dx - 2P - 2x\,dP/dx = f'(xP^2)\left\{2Px\frac{dP}{dx} + P^2\right\}$$

$$\Rightarrow \quad -\left(P + 2x\frac{dP}{dx}\right) = Pf'(xP^2)\left\{2x\frac{dP}{dx} + P\right\}$$

$\Rightarrow \quad \left(2x\dfrac{dP}{dx} + P\right)[Pf'(xP^2) + 1] = 0$

$\Rightarrow \quad 2x\dfrac{dP}{dx} + P = 0$

$\Rightarrow \quad 2\dfrac{dP}{dx} + \dfrac{dx}{x} = 0$

$\Rightarrow \quad 2\log P + \log x = \log c^2$

$\Rightarrow \quad P^2x = c^2$

Substituting in the given equation, we get

$$y - 2\sqrt{c^2x} = f(c^2)$$

$\Rightarrow \quad y = 2c\sqrt{x} + f(c^2)$

153. (C) Equation is $x^2P^2 + xyP - 6y^2 = 0$

or $\quad x^2P^2 + 3xyP - 2xyP - 6y^2 = 0$

or $\quad (xP + 3y)(xP - 2y) = 0$

or $\quad x\dfrac{dy}{dx} + 3y = 0$

and $\quad x\dfrac{dy}{dx} - 2y = 0$

or $\quad \dfrac{dy}{y} + 3\dfrac{dx}{x} = 0$

and $\quad \dfrac{dy}{y} - \dfrac{2dx}{x} = 0$

Integrating these gives

$$\log y + 3\log x = \log c$$

$\Rightarrow \quad x^3y = c$

and $\quad \log y - 2\log x = \log c$

$\Rightarrow \quad y = cx^2$

$\therefore$ The complete solution is

$$(x^3y - c)(y - cx^2) = 0$$

154. (D) The equation $P^3 - 4xyP + 8y^2 = 0$ is solvable for x and can be written as

$$4x = \frac{P^2}{y} + \frac{8y}{P}$$

Differentiating with respect to y, we get

$$4\frac{dx}{dy} = \frac{y\cdot 2P\dfrac{dP}{dy} - P^2\cdot 1}{y^2} + 8\,\frac{\left(P\cdot 1 - y\dfrac{dP}{dy}\right)}{P^2}$$

or $\quad \dfrac{y}{P} = \left(\dfrac{2P}{y} - \dfrac{8y}{P^2}\right)\dfrac{dP}{dy} - \dfrac{P^2}{y^2} + \dfrac{8}{P}$

or $\quad 2\left(\dfrac{P^3 - 4y^2}{yP^2}\right)\dfrac{dP}{dy} = \left(\dfrac{P^3 - 4y^2}{y^2P}\right)$

or $$\frac{2}{P}\frac{dP}{dy} = \frac{1}{y}$$

or $$2\frac{dP}{P} = \frac{dy}{y}$$

Integrating gives

$$2\log P = \log y + \log c$$

or $$P^2 = yc$$

or $$P = \sqrt{yc}$$

Substituting the value of P in the original equation, we get

$$4x = c + \frac{8y}{\sqrt{cy}}$$

or $$c\,(4x - c)^2 = 64y$$

155. (C) The given equation can be written as

$(x^2 + y^2)\,(dx + dy)^2 - 2\,(x + y)\,(dx + dy)\,(x\,dx + y\,dy) + (xdx + ydy)^2 = 0$

Now let $u = x + y$ and $v = x^2 + y^2$

$\therefore$ $$du = dx + dy$$

and $$dv = 2\,(xdx + ydy)$$

From equation (1)

$$v\,(du)^2 - u\,(du)\,(dv) + \left(\frac{1}{2}dv\right)^2 = 0$$

or $$4v\,(du)^2 - 4vdudv + (dv)^2 = 0$$

or $$\left(\frac{dv}{du}\right)^2 - 4u\left(\frac{dv}{du}\right) + 4v = 0$$

or $$P^2 - 4uP + 4v = 0 \quad \ldots(2)$$

where $$P = \frac{dv}{du}$$

It is a solvable for v

$$4v = 4uP - P^2$$

Differentiating with respect to u, we get

$$4\frac{dv}{du} = 4\left(u\frac{dP}{du} + P\cdot 1\right) - 2P\frac{dP}{du}$$

or $$4P = 4\left(u\frac{dP}{du} + P\right) - 2P\frac{dP}{du}$$

or $$(4u - 2P)\frac{dP}{du} = 0$$

or $$dP = 0$$

Integrating, we get $P = c$

Substituting the value of P in equation (2)

$$c^2 - 4uc + 4v = 0$$

or $c^2 - 4\,(x + y)\,c + 4\,(x^2 + y^2) = 0$

156. (B) The given equation is

$$e^{3x}\,(P - 1) + P^3\,e^{3y} = 0 \quad \ldots(1)$$

Let $$e^x = u$$

and $$e^y = v$$

$\therefore$ $$e^x dx = du$$

and $$e^y dy = dv$$

$\therefore$ $$P = \frac{dy}{dx} = \frac{e^x du}{e^y du} = \frac{udv}{vdu}$$

$\therefore$ From (1)

$$u^3\left(\frac{udv}{vdu} - 1\right) + \left(\frac{udv}{vdu}\right)v^3 = 0$$

or $$u^3\left(u\frac{dv}{du} - v\right) + u^3\left(\frac{dv}{du}\right)^3 = 0$$

or $$u\frac{dv}{du} - v + \left(\frac{dv}{du}\right)^3 = 0 \quad \ldots(1)$$

or $v = uP + P^3\left(\text{as } P = \frac{dv}{du}\right)$ which is Clairaut's form

157. (A) $y = Px + \log P$ is the Clairaut's form

$\therefore$ The solution is

$$y = cx + \log c$$

158. (A) $$P^2 - 5P + 6 = 0$$

$\therefore$ $$(P - 3)\,(P - 2) = 0$$

$\therefore$ $$P = 3$$

$\Rightarrow$ $$\frac{dy}{dx} = 3$$

$\Rightarrow$ $$dy = 3dx$$

$\Rightarrow$ $$y = 3x + c$$

Similarly $P = 2 \Rightarrow y = 2x + c_2$

$\therefore$ The solution of the given equation is

$$(y - 3x - c_1)\,(y - 2x - c_2) = 0$$

159. (B) 160. (D) 161. (D) 162. (D)

163. (B) The auxiliary equation of the given differential equation is

$$m^3 - 3m - 4 = 0$$

$\Rightarrow$ $$(m + 1)\,(m - 4) = 0$$

$\Rightarrow$ $$m = -1, 4$$

164. (D) The given equation is $(D^2 - 3D - 4)\,y = 0$

As above roots of the auxiliary equation are -1 and 4

$\therefore$ The solution is $y = C_1e^{-x} + C_2e^{4x}$.

165. (C) $(D^4 - 81)\,y = 0$ auxiliary equation $m^4 - 81 = 0$

$\Rightarrow$ $(m + 3)\,(m - 3)\,(m + 3i)\,(m - 3i) = 0$

$\Rightarrow \quad m = \pm 3, \quad m = \pm 3i$

$\therefore$ Solution is

$y = C_1e^{3x} + C_2e^{-3x} + C_3\cos 3x + C_4 \sin 3x$

166. (D)

167. (B) The A. E. of the given equation is

$$m^2(m+1)^2 = 0$$

$$\Rightarrow \quad m = 0, 0, -1, -1$$

$\therefore$ C. F. is $y = (C_1 + C_2x) + (C_3 + C_4x)e^{-x}$

$$\text{P. I.} = \frac{1}{D^2(D+1)^2}\cdot e^x = \frac{1}{1^2(1+1)^2}e^x = \frac{1}{4}e^x$$

Hence general solution is

$y = (C_1 + C_2x) + (C_3 + C_4x)e^{-x} + \frac{1}{4}e^x$

168. (B) The A. E. of the given differential equation is

$$m^2 - 1 = 0, \quad m = \pm 1$$

$$\therefore \quad \text{C. F. is } y = c_1e^x + c_2e^{-x} = A\cosh(x+B)$$

$$\text{P. I.} = \frac{1}{D^2-1}e^x = x\cdot\frac{1}{2D}e^x = \frac{x}{2}e^x$$

$\therefore$ Required solution is

$$y = A\cosh(x+B) + \frac{1}{2}xe^x$$

169. (A) The A. E. of the given differential equation is

$$m^2 - m - 2 = 0$$

$$\Rightarrow \quad m = 1, m = -2$$

$\therefore$ C. F. is $y = C_1e^x + C_2e^{-2x}$

and
$$\text{P. I.} = \frac{1}{D^2+D^{-2}}e^x = x\cdot\frac{1}{2D+1}e^x = x\cdot\frac{1}{2+1}e^x = \frac{1}{3}xe^x$$

$\therefore$ Required solution is

$y = C_1e^x + C_2e^{-2x} + \frac{1}{3}xe^x$

170. (D) $\text{P. I.} = \frac{1}{D^2+4}\cdot x$

$$= \frac{1}{4}\left(1 - \frac{D^2}{4}\right)^{-1}\cdot x = \frac{1}{4}\left(1 - \frac{D^2}{4} + \ldots\right)x = \frac{x}{4}$$

171. (D) A. E. is $m^2 + \omega^2 = 0$

$$\Rightarrow \quad m = \pm i\omega$$

$\therefore$ C. F. is $y = A\cos(\omega x + B)$

$$\text{P. I.} = \frac{1}{D^2+\omega^2}\cdot 10\omega^2 = \frac{1}{\omega^2}\left(1 + \frac{D^2}{\omega^2}\right)^{-1}\cdot 10\omega^2$$

$$\frac{1}{\omega^2}\left(1 - \frac{D^2}{\omega^2} + \ldots\right)\cdot 10\omega^2 = 10$$

$\therefore$ Solution is $y = A\cos(wx + B) + 10$

172. (D) The given equation is $(D^2 + a^2)\,y = \sec ax$

A. E. is $m^2 + a^2 = 0$

$$\Rightarrow \quad m = \pm ai$$

$\therefore$ C. F. is $y = C_1\cos ax + C_2\sin ax$

and
$$\text{P. I.} = \frac{1}{D^2+a^2}\sec ax = \frac{1}{2ai}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right]\sec ax$$

But $\frac{1}{D-ai}\sec ax$

$$= e^{iax}\int e^{-iax}\sec ax\,dx = e^{iax}\int\frac{\cos ax - i\sin ax}{\cos ax}\,dx = e^{iax}\left(x + \frac{i}{a}\log\cos ax\right)$$

Similarly $\frac{1}{D+ai}\sec ax$

$$= e^{iax}\left(x - \frac{i}{a}\log\cos ax\right)$$

$$\therefore \quad \text{P. I.} = \frac{1}{2ai}\left[e^{iax}\left(x + \frac{i}{a}\log\cos ax\right) - e^{-iax}\left(x - \frac{i}{a}\log\cos ax\right)\right]$$

$$= \frac{1}{a}\left[x\sin ax + \frac{1}{a}(\log\cos ax)\cos ax\right]$$

Hence the general solution of the given equation is

$$y = C_1 \cos ax + C_2 \sin ax + \frac{1}{a}\left[x \sin ax + \frac{1}{a}(\log \cos ax)\cos ax\right]$$

173. (D) 174. (D)

175. (A) $\frac{dX}{dt} = MX + b$

or $\frac{dX}{dt} - MX = b$

where, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

and $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$ I. F. $= e^{-\int M dt} = e^{-Mt}$

$\therefore$ $Xe^{-Mt} = \int be^{-Mt}\,dt + c$

$= -b \cdot \frac{e^{-Mt}}{M} + c$

$\Rightarrow$ $x = \frac{-b}{M} + ce^{Mt}$

176. (A) Given $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$

A. E. is $D^2 + D'^2 = 0$

or $D^2 + 1 = 0$

or $D = \pm i$

$\therefore$ $u = f(x + iy) + g(x - iy)$

177. (A) $y'(y' + y) = x(x + y)$

$\therefore$ $y^2 + yy' - x^2 - xy = 0$

$y'(y' + x + y) - x(y' + x + y) = 0$

$(y' - x)(y' + x + y) = 0$

Since $y' \neq x$

$\therefore$ $y' + x + y = 0$

or $\frac{dy}{dx} + x + y = 0$

or $\frac{dy}{dx} + y = -x$

Now I. F. $= e^{\int P dx}$

$= e^x$

$y \times e^x = \int -xe^x dx + c$

$= -[xe^x - \int 1 \cdot e^x\, dx] + c$

or $ye^x = -xe^x + e^x + c$

or $y = -x + 1 + ce^{-x}$

where c is constant

Choosing $c = -1$, we get

$y = 1 - x - e^{-x}$

178. (B) Given, $4x^3y'' + 6x^2y' + y = 0$

We transform the independent variable x to t by the relation

$t = \frac{1}{x}$

or $x = \frac{1}{t}$ ...(1)

Now $y' = -\frac{1}{x^2}\frac{dy}{dt}$

or $y'' = \frac{1}{x^4}\frac{d^2y}{dt} + \frac{2}{x^3}\frac{dy}{dt}$

and the given differential equation transform to

$4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$ ...(2)

The point at ∞ is transformed to the origin. From equation (2), we not that the origin is regular singular point.

Hence, the point at ∞ is a regular singular point of the given equation.

179. (D)

180. (A) $M = 3a^2x^2 + by \cos x$

$N = 2\sin x - 4ay^3$

For the equation to be exact,

$\frac{\delta N}{\delta x} = \frac{\delta M}{\delta y}$

or $2 \cos x = b \cos x$

or $b = 2$

181. (A) For the initial value problem

PDE : $u_{tt} - c^2u_{xx} = 0, -\infty < x < \infty, t \geq 0$

ICS : $u(x, 0) = \eta(x)$, $u_t(X, 0) = v(x)$

D' Alember solution is given by,

$$u(x, t) = \frac{1}{2}[\eta(x + ct) + \eta(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} v(x)\,dx$$

For the given problem $v(x) = 0$

$\eta(x) = x(1 - x), c = 1$

$\therefore$ The solution is of the form,

$u(x, t) = \frac{1}{2}[\eta(x + t) + \eta(x - t)]$

$= \frac{1}{2}[Cx + t)(1 - x - t) + (x - t)(1 - x + t)]$

$\therefore$ $u\left(\frac{1}{2}, \frac{1}{4}\right) = \frac{1}{2}\left[\frac{3}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{3}{4}\right]$

$= \frac{3}{16}$

182. (A) 183. (B) 184. (B) 185. (D) 186. (B)
187. (A) 188. (A)
189. (C) Because if λ_i and λ_j are roots of equation $J_n(\lambda a) = 0$

then $\int_0^a x J_n(\lambda_i x) J_n(\lambda_j x)\, dx$

$$= \begin{cases} 0 & \text{If } i \neq J \text{ (diff. roots)} \\ \frac{a^2}{2} J^2_{n+1}(\lambda_i\, a) & \text{If } i = j \text{ (equal roots)} \end{cases}$$

For orthogonality of system on [0, 1], the weight function is x.

190. (B) A linear homogeneous equation can always be reduced to a linear differential equation with constant coefficients of the form

$$\frac{d^n y}{dx^n} + a_n \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_1 y = x \quad \ldots(1)$$

$$f(D) = x$$

We will now show that if $y_1, y_2, \ldots y_n$ are linearly independent solution of equation (1) then $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ is also a solution of equation (1),

Where $c_1, c_2, \ldots c_n$ being arbitrary constants.

Since $y_1, y_2, \ldots y_n$ are solution of (1), we get

$$f(D)\, y_1 = 0$$
$$f(D)\, y_2 = 0$$
$$\ldots\ldots\ldots \quad \ldots$$
$$\ldots\ldots\ldots \quad \ldots$$
$$f(D)\, y_n = 0$$

$f(D)\{c_1 y_1 + c_2 y_2 + \ldots + c_n y_n\}$

$= f(D)(c_1 y_1) + f(D)(c_2 y_2) + \ldots + f(D)(c_n y_n)$

$= c_1 f(D)\, y_1 + c_2 f(D)\, y_2 + \ldots + c_n f(D)\, y_n$

$= c_1 \cdot 0 + c_2 \cdot 0 + \ldots + c_n \cdot 0 = 0$ (using (2))

This proves that statement

191. (B)
192. (D) Heat equation is given by

$$\frac{\delta U}{\delta t} = c^2 \frac{\delta^2 U}{\delta x^2},\ c^2 = 1$$

Let $$U = \frac{e^{-x^2}}{\sqrt{t}}$$

$$\therefore \quad \frac{\delta U}{\delta t} = \frac{\sqrt{t} \cdot e^{-x^2/4t}\left(\frac{x^2}{4t^2}\right) - e^{\frac{-x^2}{4t}} \cdot \frac{1}{2\sqrt{t}}}{t}$$

$$= \frac{e^{-x^2/4t}}{\sqrt{t}} \left[\frac{x^2}{4t^2} - \frac{1}{2t}\right]$$

$$= u\left(\frac{x^2}{4t^2} - \frac{1}{2t}\right)$$

$$\frac{\delta U}{\delta x} = \frac{1}{\sqrt{t}} \cdot e^{-x^2/4t}\left(\frac{2x}{4t}\right) = U - \left(\frac{x}{2t}\right)$$

$$\frac{\delta^2 U}{\delta x^2} = \frac{\delta U}{\delta x}\left(\frac{-x}{2t}\right) - U\left(\frac{1}{2t}\right) = U\left\{\frac{x^2}{4t^2} - \frac{1}{2t}\right\}$$

$$\therefore \quad \frac{\delta U}{\delta t} = \frac{\delta^2 U}{\delta x^2}$$

193. (A) Laplace equation $\Delta^2 \psi = 0$ is the elliptic equation occurring most frequently in physical problems.
194. (D)
195. (B) Given, $x^2 + y^2 = 2cx$

$$\Rightarrow \quad c = \frac{x^2 + y^2}{2x}$$

Differentiating with respect to x, we get

$$2x + 2y\frac{dy}{dx} = 2c$$

$$\Rightarrow \quad x + y\frac{dy}{dx} = c$$

$$= \frac{x^2 + y^2}{2x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{x^2 + y^2}{2x} - x}{y}$$

$$= \frac{y^2 - x^2}{2xy}$$

Its orthogonal trajectory is given by $-\frac{dx}{dy}$

$$-\frac{dx}{dy} = \frac{y^2 - x^2}{2xy}$$

$$\Rightarrow \quad \frac{dy}{dx}(y^2 - x^2) = -2xy$$

$$\Rightarrow \quad y'(x^2 - y^2) = 2xy$$

196. (C) 197. (B)
198. (B) $$\frac{dy}{dx} = \frac{y}{\sqrt{x}}$$

$$\Rightarrow \quad \frac{1}{y}\, dy = x^{-1/2}\, dx$$

$$\Rightarrow \quad y = e^{2\sqrt{x}} \cdot e^c$$

At $x = 2, y = 4$

$$\therefore \quad e^c = 4e^{-2\sqrt{2}}$$

$\therefore \quad y = e^{2\sqrt{x}} \cdot 4e^{-2\sqrt{2}}$

Hence differential equation has unique solution.

199. (A) Given : $D^4 - D^2 = 0$

$\Rightarrow \quad D^2 (D-1)(D+1) = 0$

$\Rightarrow \quad D = 0, 0, 1, -1$

$\therefore \quad y = 1 + c_1 x + c_2 e^x + c_3 e^{-x}$

Hence, the set of linearly independent solution is $\{1, x, e^{-x}, e^x\}$

200. (B) Given : $x^2 (1-x)\dfrac{d^2y}{dx} + x\dfrac{dy}{dx} + y = 0$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + \frac{1}{x(1-x)}\frac{dy}{dx} + \frac{1}{x^2(1-x)}y = 0$$

$$\therefore \quad P = \frac{1}{x(1-x)}$$

$$Q = \frac{1}{x^2(1-x)}$$

$\Rightarrow \quad P - xQ = 0$

Hence, solution of C. F. is $u = x$. So, the $x = 1$ is a regular singular point.

201. (C) $y = \alpha \sin x + \beta \cos x \quad \ldots(1)$

$z = \alpha \cos x + \beta \sin x \quad \ldots(2)$

Differentiate equation (1) with respect to x, we get

$$\frac{dy}{dx} = \alpha \cos x + \beta \sin x = z$$

$$\frac{dy}{dx} - z = 0 \qquad \ldots(3)$$

Differentiate equation (2) with respect to x, we get

$$\frac{dy}{dx} = -\alpha \sin x + \beta \cos x = -y$$

$$\Rightarrow \quad y + \frac{dy}{dx} = 0$$

Hence, equation (1) and (2) are general solution of differential equation (3) and (4).

202. (B) $2x(2+x)\dfrac{d^2y}{dx^2} + 2(3+x)\dfrac{dy}{dx} - xy = 0$

Let $\quad 2x(2+x) = v^2$

$\Rightarrow \quad 2x^2 + 4x - v^2 = 0$

$$\Rightarrow \quad x = \frac{-4 \pm \sqrt{16 + 8v^2}}{4}$$

$$x = \frac{-2 \pm \sqrt{4 + 2v^2}}{2}$$

203. (C) 204. (B) 205. (C)

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