Numerical Analysis

Error

Error—Error in numerical calculation occurs due to—

(i) Error in input data/experimental error.

(ii) **Round off error**—Occurs due to rounding of digits (use of finite number of digits).

(iii) **Truncation error**—Occurs due to approximation.

(iv) Errors that occur due to mistakes in numerical computation.

Absolute error—$\in = a - \tilde{a}$, where $\tilde{a}$ is an approximation of exact value a.

$$\text{True Value} = \text{Approximation} + \text{Error}$$

Relative Error—

$$\begin{aligned} \in_r &= \frac{\in}{a} \\ &= \frac{a - \tilde{a}}{a} \\ &= \frac{\text{Error}}{\text{True Value}} \end{aligned}$$

Error bound β—For $|\in| \leq \beta$, $|a - \tilde{a}| \leq \beta$

Error Propagation

1. In addition and subtraction, an error bound for the results is given by the sum of error bounds for the terms.
2. In multiplication and division, a bound for the relative error of the results is given (approximately) by the sum of the bounds for the relative errors of the given numbers.

Finite Difference Operators

(a) **Forward Operator—**

Let h be the finite difference.

then

$$\begin{aligned} \Delta f(x) &= f(x+h) - f(x) \\ \Delta^2 f(x) &= f(x+2h) - 2f(x+h) + f(x) \\ \Delta^3 f(x) &= f(x+3h) - 3f(x+2h) + 2f(x+h) - f(x) \end{aligned}$$

$$\Delta^n f(x) \sum_{r=0}^{n} (-1)^{n-r}\ {}^nC_r\, f(x+rh)$$

(b) **Shift Operator—**

Let h be the finite difference.

Then $$\begin{aligned} E\, f(x) &= f(x+h) \\ E^n\, f(x) &= f(x+nh) \end{aligned}$$

(c) **Backward differences—**

Let h be the finite difference.

Then $$\begin{aligned} \nabla f(x) &= f(x) - f(x-h) \\ \nabla^2 f(x) &= f(x-2h) - 2f(x-h) + f(x) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \nabla^n f(x) &= \sum_{r=0}^{n} (-1)^{n-r}\ {}^nC_r\, f(x-rh) \end{aligned}$$

(d) **Factorial Notation—**

Let h be the finite difference.

Then $$\begin{aligned} x^{(n)} &= x(x-h)(x-2h)\ldots\left(x-\overline{n-1}\right)h \\ x^{(n)} &= \frac{x\,!}{(x-n)\,!},\ (n < x) \\ \Delta x^{(n)} &= nhx^{(n-1)} \\ \Delta^n x^n &= n\,!\,h^n \end{aligned}$$

(e) **Central difference operator δ—**

$$\begin{aligned} \delta f(x) &= f\left(x + \frac{1}{2}h\right) - f\left(x - \frac{1}{2}h\right) \\ \delta^n f(x) &= \Delta^n f\left(x - \frac{1}{2}nh\right) \end{aligned}$$

(f) **Averaging operator μ—**

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{1}{2}h\right) + f\left(x-\frac{1}{2}h\right)\right]$$

Relation Between Different Finite Operators

1. Relation between Δ and E

$$\begin{aligned} E &\equiv 1 + \Delta \text{ and } \Delta \equiv E - 1 \\ E^n &\equiv (1+\Delta)^n \text{ and } \Delta^n \equiv (E-1)^n \end{aligned}$$

$$E^n f(x) = f(x + nh)$$
$$= \sum_{r=0}^{n} {}^nC_r \, \Delta^r f(x)$$

Theorem—1. If $f(x)$ is a rational integral function (polynomial) of degree n in x the nth difference of this polynomial is constant and $(n + 1)$th and higher difference are zero.

2. Relation between ∇, Δ and E

- $\nabla E \equiv E \nabla \equiv \Delta$
- $E^{-1} \equiv 1 - \nabla$
- $E \equiv (1 - \nabla)^{-1}$

3. Central difference

- $\delta = E^{1/2} - E^{-1/2}$
- $\delta \equiv \Delta E^{-1/2} \equiv E^{1/2} \Delta$
- $\delta \equiv \Delta E^{1/2} \equiv E^{1/2} \nabla$
- $\delta^n f(x) = \Delta^n f\left(x - \frac{1}{2}nh\right)$
- $\delta^n f(x) = \nabla^n f\left(x + \frac{1}{2}nh\right)$
- $\Delta\nabla = \nabla\Delta = \Delta - \nabla = \delta^2$
- $\mu \equiv \frac{1}{2}[E^{1/2} + E^{-1/2}]$

Interpolation

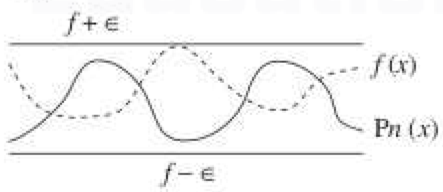
Polynomial of degree n (Algebraic polynomial)—

$$p_n f(x) = a_n x^n + a_n x^{n-1} + \ldots\ldots a_1 x + a_n$$

where n is non-negative integer and a_n, a_{n-1} …… a_1, a_0 are real constants.

Weierstrass approximation theorem—

For any continuous function $f(x)$ on an interval J : $a \leq x \leq b$ and error bound $\beta > 0$, there is a polynomial $p_n(x)$ of sufficiently high degree n such that $|f(x) - p_n(x)| < \beta$ for all $x \in$ J. *i.e.*, $p_n(x) \approx (x)$



Taylor's polynomial approximation—In Taylor's polynomial all the information ued in the aproximation is concentrated at the signle point x_0. This limits Taylor polynomial approximation to the situation in which approximations are needed only at points close to x_0.

Interpolation—It means to find (approximate) value of function $f(x)$ for an x between different x values $x_0, x, \ldots\ldots, x_n$ at which the value of $f(x)$ are given *i.e.*, $f(x_i) = f_i$ $(i = 0, 1 \ldots\ldots n)$.

If $x_0, x_1, \ldots\ldots x_n$ are $(n + 1)$ distinct values of real valued function $f(x)$. One has a polynomial $p_n(x_i) \approx f(x)$ of degree n or less *i.e.*, there is at most one polynomial of degree $\leq n$ which interpolates $f(x)$ at $(n + 1)$ distinct points $x_0, x_1, \ldots\ldots x_n$.

Lagrange Interpolation polynomial—

If $x_0, x_1, \ldots\ldots x_n$ are $(n + 1)$ distinct numbers and f is a function whose values are given at these numbers, then there exists a unique polynomial $p(x)$ of degree at most n with property

$$f(x_k) \approx p(x_k),\ k = 0, 1 \ldots\ldots n.$$

This polynomial (Lagrange interpolation polynomial of degree n) is given by

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} L_k(x) f_k$$
$$= \sum_{k=0}^{n} \frac{L_k(x)}{L_{k(xk)}} f_k$$

where
$$L_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{(x - x_j)}{(x_k - x_j)}$$

and
$$L_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^{n} (x_k - x_j)$$

Remainder term (error)—

$$\text{Remainder} = f(x) - p_n(x)$$
$$= \left(\prod_{j=1}^{n} (x - x_j)\right) \frac{f^{n+1}(\xi)}{(n+1)!},$$

where $\xi \in (a, b)$ if $x_1 \ldots\ldots x_n \in [a, b]$

Interpolation with unequal interval first divided difference—

$$\underset{x_1}{\Delta} y_0 = \frac{y_1 - y_0}{x_1 - x_0}$$
$$= \frac{y_0 - y_1}{x_0 - x_1}$$
$$= \underset{x_0}{\Delta} y_1$$

$$\underset{x_{n-1}}{\Delta} y_n = \frac{y_{n-1} - y_n}{x_{n-1} - x_n}$$
$$= \frac{y_n - y_{n-1}}{x_n - x_{n-1}}$$
$$= \underset{x_n}{\Delta} y_{n-1}$$

The divided difference are symmetrical in all their arguments *i.e.*, the value of any difference is independent of the other of their arguments.

i.e., $\underset{x_{n-1}}{\Delta} y_n = \underset{x_n}{\Delta} y_{n-1}$

n–th divided difference

$$\underset{x_1 \ldots x_n}{\Delta^n} y_0 = \sum_{i=0}^{n} \frac{y_1}{\prod_{\substack{j=0 \\ j \neq i}}^{n} (x_i - x_j)}$$

$$= f(x_0, x_1 \ldots x_n)$$

Then-divided difference of a polynomial of the nth degree are constant.

Newton's divided difference formula—

$$f(x) = y_0 + (x - x_0) f(x_0, x_1)$$

$$+ (x - x_0)(x - x_1) f(x_0, x_1, x_2) + \ldots\ldots + (x - x_0) \ldots\ldots (x - x_{n-1}) f(x_0, x_1 \ldots\ldots x_n) + R_n$$

Relation between divided difference and ordinary difference—

$$\underset{x_1}{\Delta} y_0 = \frac{y_0 - y_1}{x_0 - x_1}$$

$$= \frac{1}{h} \Delta f(0)$$

$$= \frac{\Delta y_0}{h}$$

$$\underset{x_1 \ldots x_n}{\Delta^n} y_0 = f(x_0, x_1 \ldots x_n)$$

$$= \frac{\Delta y_0}{n!\, h^n}$$

Interpolation with Equal Interval

Newton's Binomial expansion method (missing terms)—

Suppose $y = f(x)$ has values $y_0, y_1, \ldots\ldots\ldots y_n$ $(n + 1)$ values corresponding to the arguments

$$a, a + h, a + 2h \ldots\ldots a + nh$$

1. **If one value y_1 is missing—**

Then $\Delta^n y_x = 0$

or, $(E - 1)^n y_0 = y_n - {}^n c_1\, y_{n-1} + {}^n c_2\, y_{n-2} \ldots\ldots (-1)^n y_0 = 0$

$f(x)$ can be represented by a polynomial of degree $(n - 1)$ since n–values of $f(x)$ is known.

2. **If two values are missing—**

$f(x)$ can be represented, as a polynomial of degree $n - 2$ since $n - 1$ value of $f(x)$ is known.

Newton-Gregory forward difference interpolation formula—

$$f(x + nh) = \sum_{r=0}^{n} {}^n c_r\, \Delta^r f(x)$$

Newton–Gregory advance difference formula—

$$f(x) = \sum_{r=0}^{n} \frac{x^{(r)}}{r!} \Delta^r f(0)$$

$$= f(0) + x\, \Delta f(0) + \frac{x(x-1)}{2!} \Delta^2 f(0) + \frac{x(x-1)(x-2)}{3!} \Delta^3 f(0) \ldots\ldots + \frac{x^{(n)}}{n!} \Delta^n f(0)$$

Newton backward difference formula—

$$f(x) = \sum_{r=0}^{n} \frac{x^{(r)}}{r!} \Delta^r f(0)$$

$$= f(0) + x\, \nabla f(0) + \frac{x(x+1)}{2!} \nabla^2 f(0) + \ldots\ldots + \frac{x(x+1) \ldots\ldots (x+n-1)}{n!} \nabla^n f(0)$$

Central Difference Interpolation Formula

Gauss's forward formula for equal intervals.

$$y_u = y_0 + {}^u c_1\, \Delta y_0 + {}^u c_2\, \Delta^2 y - 1 + {}^{u+1} c_3\, \Delta^3 y_{-1} + {}^{u+1} c_4\, \Delta^4 y_{-2} + \ldots\ldots$$

Gauss's backward formula for equal intervals

$$y_u = y_0 + {}^u c_1\, \Delta y_{-1} + {}^{u+1} c_2\, \Delta^2 y_{-1} + {}^{u+1} c_3\, \Delta^3 y_{-2} + {}^{u+1} c_4\, \Delta^4 y_{-2} + \ldots\ldots$$

Sterling's formula—The arithmetic mean of Gauss's forward and backward formula is known as Sterling's formula.

$$y_u = y_0 + u\left(\frac{\Delta y_0 + \Delta_{y-1}}{2}\right) + \frac{u^2}{2!} \Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) \frac{u^2(u^2-1)}{4!} \Delta^4 y_{-2} + \ldots\ldots$$

where, $u = \dfrac{x - x_0}{h}$

Bessel's formula—Shifting the origin in Gauss's backward formula one have Bessel's formula (Gauss's third formula)

$$y_u = y_0 + u\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{\Delta^2{}_{y-1} + \Delta^2 y_0}{2}\right) + \frac{p(p-1)\left(p-\frac{1}{2}\right)}{3!}\Delta^3 y - 1 + \frac{(p+1)\,p\,(p-1)\,(p-2)}{4!}\left(\frac{\Delta^4{}_{y-2} + \Delta^4{}_{y-1}}{2}\right) + \ldots\ldots$$

Everett's formula—

$$y_u = vy_0 + \frac{v(v^2-1^2)}{3!}\Delta^2{}_{y-1} + \frac{v(v^2-1^2)(v^2-2^2)}{5!}\Delta^4 y - 2 + \ldots + uy_1 + \frac{u(u^2-1^2)}{3!}\Delta^2 y_0 + \frac{u(u^2-1^2)(u^2-2^2)}{5!}\Delta^4 y - 1 + \ldots\ldots$$

where $v = 1-u$.

Everett's formula truncated after second difference is equivalent to Bessel's formula truncated after third differences.

Numerical Soulution of Algebraic Equations

Given an equation $f(x) = 0$

A solution of $f(x) = 0$,

is a number $x = s : f(s) = 0$.

Iteration method—To solve $f(x) = 0$, when there is no formula for the exact solution one can use approximation method, an iteration method, in it one start from an initial guess x_0 (which may be poor) and compute step by step (*i.e.*, searching better) approximation $x_0, x_1, x_2, \ldots\ldots$ of an unknown solution of $f(x) = 0$.

(A) Fixed point iteration—

$$f(x) = 0$$

$$\Leftrightarrow \quad x = g(x)$$

choose x_0 and compute $(x_{n+1}) = g(x_n)$

The solution of $x = g(x)$ is called a fixed point of g.

Convergence of fixed-point iteration—

Let $x = s$ be a solution of $x = g(x)$ and suppose that g has a continuous derivative in some interval J containing s. Then if $|g'(x)| \le k < 1$ in J, the iteration process defined by $x_{n+1} = g(x_n)$ converges for any x_0 in J.

Existence of solution—

If g is continuous in closed interval J, its range lies in J then $x = g(x)$ has at least one solution in J.

Fixed point—

A fixed point for a given function g is a number p for which $g(p) = p$.

Theorems—

(1) If g is continuous on interval $[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$. then g has a fixed point in $[a, b]$.

(2) If in addition $g'(x)$ exists on (a, b) and a positive constant $k < 1$ exist with $|g'(x)| \le k$ or $g'(x) \le k$ for all $x \in (a, b)$, then the fixed point in $[a, b]$ is unique.

(3) **Fixed-point theorem**—Let g is a continuous function on $[a, b]$ such that $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose, in addition that g' exist on (a, b) and positive constant $k < 1$ exist with $|g'(x)| \le k$, for all $x \in (a, b)$. Then for any number P in $[a, b]$, the sequence defined by $p_n = g(p_{n-1})$, $n \ge 1$, converges to the unique fixed point P in $[a, b]$.

Corollary—If g satisfies the hypothesis of fixed point therorem, bounds for the error invalved in using p_n to approximate p are given by—

$$|p_n - p| \le k^n \max.\{p_0 - a, b - p_0\} \text{ and } |p_n - p| \le \frac{k_n}{1-k}|p_1 - p_0| \text{ for all } n \ge 1$$

(B) **Newton-Rapson Method—**

$$f(x) = 0$$

where f is assumed to have a continuous derivative

$$f'x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The convergence is at least quardratic for simple root if $f''(x)$ is continuous.

Second order quadratic convergence of Newton's method—

If $f(x)$ is three times differentiable and f', f'' are not zero at a solution s of $f(x) = 0$, then for x_0 sufficiently close to s. Newton's method is of second order.

Theorem—Let f is continuous in $[a, b]$ in XY plane. If $p \in [a, b]$ is such that $f(p) = 0$ and $f'(p) \neq 0$, then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any intial approximation $p_0 \in [p-\delta, p+\delta]$.

(C) Second Method (Regula Falsi)—Replacing the derivative $f'(x_n)$ by the difference quotient in Newton's method.

$$f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

$$x_{n+1} = x_n - f(x_n)\frac{x_n\ x_{n-1}}{f(x_n) - f(x_{n-1})}$$

(D) Bisection (Binary–Search) Method—Let f is continuous function defined on the interval $[a, b]$, with $f(a)$ and $f(b)$ of opposite sign. By the intermediate value theorem, there exists a number p in (a, b) with $f(p) = 0$.

[**Note**—This procedure will work for the case when $f(a)$ and $f(b)$ have opposite signs and there is more than one root in the interval (a, b)].

'The bisection method calls for a repeated halving of sub-intervals of $[a, b]$ and locating the half-containing p.

To find a solution to $f(x) = 0$ given the continuous function f on the interval $[a, b]$ where $f(a)$ and $f(b)$ have opposite signs.

Set $a_1 = a$

and $b_1 = b$

and let $p_1 = \frac{1}{2}(a_1 + b_1)$

If $f(p_1) = 0$,

then $p = p_1$

If not, then $(f_{p_1})f(a_1) < 0$

or $f(p_1)f(b_1) > 0$

opposite sign

If $f(p_1)f(a_1) < 0$

then $p \in (a_1, p_1)$

and set $a_2 = a_1$,

$b_2 = p_1$

Some sign

If $f(p_1)f(a_1) > 0$

Then $p \in (p_1, b_1)$

and set $a_2 = p_1$,

$b_2 = b_1$

Repeat the procedure.

Theorem—Let f is continuous function on $[a, b]$ and $f(a)\,f(b) < 0$, then bisection method generater a sequence $\{p_n\}$ approximating a zero p of f with—

$$|p_n - p| \le \frac{b-a}{2^n},$$

$$n \ge 1$$

Solution of linear system of equations :

(A) **Gauss elimination method**—

Given system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2\,y + c_2z = d_2$$
$$a_3x + b_3\,y + c_3z = d_3 \quad \ldots(1)$$

Step 1. Eliminate x from second and third equation, here $r_1 = \frac{a_2}{a_1}$, multiply first equation by r_1 and then subtract it by second equations.

Here $r_2 = \frac{a_3}{a_1}$,

multiply first equation by r_2 and then subtract it by third equation.

The resulting system is,

$$a_1x + b_1y + c_1z = d_1$$
$$b'_2y + c_2'\,z = d'_2$$
$$b'_3\,y + c_3'\,z = d_3' \quad \ldots(2)$$

The first equation is pivotal equation and a_1 is the first pivot.

Step 2. Eliminate y from third equation in (2).

Here $r_2 = \frac{b_3'}{b_2'}$

multiply second equation in (2) by r_3 and then subtract it by third equation.

The resulting equation is,

$$a_1x + b_1y + c_1z = d_1$$
$$b_2'y + c_2'\,z = d_2'$$
$$c''_3z = d''_3$$

The second equation is pivotal equation and b'_2 is the new pivot.

Step 3. The value of x, y, z can be found from (3) by back substitution.

(B) Cruot's Triangularization Method—

Given the system of equations :

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

which is equivalent to

$$AX = B \quad \ldots(1)$$

where $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and $$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let $$A = LU \qquad \ldots(2)$$

where, the lower triangular matrix

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

and the upper triangular matrix

$$U = \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ 0 & \mu_{22} & \mu_{23} \\ 0 & 0 & \mu_{33} \end{bmatrix}$$

Then (1) becomes

writting $$LUX = B, \qquad \ldots(3)$$
$$UX = V \qquad \ldots(4)$$

Equation (3) becomes LV = B which is equivalent to the equations :

$$v_1 = b_1$$
$$l_{21}\, v_1 + v_2 = b_2$$
$$l_{31}\, v_1 + l_{32}\, v_2 + v_3 = b_3$$

Solving these for v_1, v_2, v_3 we get V. Then (4) becomes

$$u_{11}\, x_1 + u_{12}\, x_2 + u_{13}\, x_3 = v_1$$
$$u_{22}\, x_2 + u_{23}\, x_3 = v_2$$

and $$u_{33}\, x_3 = v_3$$

from which x_3, x_2 and x_i can be found by back-substitution. To compute the matrices L and U write (2) as,

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 0 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiplying the matrices on the left and equating corresponding elements from both sides, we obtain.

(i) $$u_{11} = a_{11},$$
$$u_{12} = a_{12}$$

and $$u_{13} = a_{13}$$

(ii) $$l_{21}\, u_{11} = a_{21}$$

or, $$l_{21} = a_{21} / a_{11}$$

and $$l_{31}\, u_{11} = a_{31}$$

or, $$l_{31} = a_{31} / a_{11}$$

(iii) $$l_{21}\, u_{12} + u_{22} = a_{22}$$

or, $$u_{22} = a_{22} - \frac{a_{21}}{a_{11}} a_{13}$$

and $$l_{31}\, u_{13} + u_{23} = a_{23}$$

or, $$u_{23} = a_{23} - \frac{a_{21}}{a_{11}} a_{13}$$

(iv) $$l_{31}\, u_{12} + l_{32}\, u_{22} = a_{32}$$

or $$l_{32} = \frac{1}{\mu_{22}} \left[a_{32} - \frac{a_{31}}{a_{11}} a_{12} \right]$$

(v) $$l_{31}\, u_{13} + l_{33}\, u_{23} + u_{33} = a_{33}$$ which gives u_{33}.

And we compute the elements of L and U in the following set order—

(i) First row of U,

(ii) First Column of L

(iii) Second row of U

(iv) Second column of L

(v) Third row of U

(C) Matrix Inversion Method

Given
$$\left.\begin{aligned} a_1x + b_1y + c_1\, z &= d_1 \\ a_2x + b_2y + c_2\, z &= d_2 \\ a_3x + b_3y + c_3\, z &= d_3 \end{aligned}\right\} \qquad \ldots(1)$$

Which can be written as—

where $$AX = D \qquad \ldots(2)$$

where $$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix},$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and $$D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Multiplying both sides of (2) by the inverse matrix A^{-1}, we get $A^{-1}\, AX = A^{-1}\, D$

$$\text{IX} = \text{A}^{-1}\text{D}$$
$$\text{X} = \text{A}^{-1}\text{D}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|\text{A}|}\begin{bmatrix} \text{A}_1 & \text{A}_2 & \text{A}_3 \\ \text{B}_1 & \text{B}_2 & \text{B}_3 \\ \text{C}_1 & \text{C}_2 & \text{C}_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \ldots(3)$$

Where A_1, B_1, …… are the cofactors of a_1 b_1, …… in the determinant $|A|$.

Numerical Integration—It is the numerical evaluation of integrals $J = \int_a^b f(x)\,dx$, where a and b are given and f is a function given analytically by a formula or empirically by table of values *i.e.*,

$$J = \int_a^b f(x)\,dx \cong \int_a^b \phi(x)\,dx,$$

where $\phi(x)$ is a polynomial which is equivalent to $f(x)$.

Rectangular rule—Subdivide the interval of integration $a \le x \le b$ into n-subintervals of equal length $h = \frac{b-a}{n}$ and in each subinterval approximate f by $f(x^*_j)$ = value of f at the midpoint x_j of the jth subinterval.

$$J = \int_a^a f(x)\,dx = h\,[f(x^*_1) + f(x^*_2) + \ldots\ldots + f(x^*_n)]$$

where $h = \frac{b-a}{n}$

Trapezoidal rule—Subdivide the interval $a \le x \le b$ in to n-subintervals of equal length

$$h = \frac{b-a}{n},$$

where $x_0 = a$

and $x_0 + nh = b$

$$J = \int_a^h f(x)\,dx$$
$$= \int_{x_0}^{x_0+nh} f(x)dx$$
$$= \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \ldots\ldots + y_{n-1})]$$

Given y_0 and y_1 for corresponding values of x_0 and x_1 respectively for formula for function $f(x)$. Let $f(x)$ is a polynomial of degree 1. Then Trapezoidal rule is—

$$J = \frac{1}{2}h(y_0 + y_1)$$

Simpson's one-third rule—Sub-divide the interval $a \le x \le b$ into even number of equal interval $n = 2m$ of length

$$h = \frac{b-a}{2m}$$
$$J = \int_a^b f(x)dx,$$
$$= \int_{x_0}^{x_0+nh} f(x)dx$$
$$= \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \ldots\ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots\ldots + y_{n-2})]$$

1. Given y_0, y_1, y_2 corresponding to values x_0, x_1, x_2 for function $y = f(x)$ and let $f(x)$ is polynomial of degree 2, then Simpson's one-third rule is.

$$\boxed{J = \frac{1}{3}h\,[y_0 + 4y_1 + y_2]}$$

Simpson's three-eight rule—Sub-divide the interval $a \le x \le b$ into multiple of 3 of equal intervals $n = 3m$ of length $h = \frac{b-a}{3m}$

$$J = \int_a^b f(x)dx,$$
$$= \int_{x_0}^{x_0+nh} f(x)dx$$
$$= \frac{3h}{8}[y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{n-1}) + 2(y_3 + y_6 + \ldots\ldots + y_{n-3})]$$

2. Given y_0, y_1, y_2, y_3 corresponding to values x_0, x_1, x_2, x_3 for function $y = f(x)$.

Let $f(x)$ is a polynomial of degree 3, then Simpson's three eight rule is—

$$\boxed{J = \frac{3}{8}h\,[y_0 + 3y_1 + 3y_2 + y_3]}$$

Weddle's Rule :

Subdivide the interval $a \le x \le b$ into multiple of 6 of equal intervals $n = 6m$ length $h = \frac{b-a}{6\,m}$

$$J = \int_a^b f(x)dx,$$
$$= \int_{x_0}^{x_0+nh} f(x)dx$$

$$= \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \ldots)$$

Cote's formula :

$$1 = nh\,(c_0{}^n y_0 + c_1{}^n y_1 + c_2{}^n y_2 + \ldots\ldots + c_k{}^n y_n)$$
$$= \int_a^b f(x)dx,$$

Where $c_0{}^n$ is a cote number.

Closed Cote's formula—a and b are nodes $(x_0 = a, x_n = b)$

Newton-cote's formula and different rules of integration—The Trapezoidal and Simpson's rule are special Newton cotes formula.

(i) Trapezoidal rule :

$f(x)$ is interpolated at equally spaced nodes by a polynomial of degree $n = 1$.

(ii) Simpson rule—$f(x)$ is interpolated at equally spaced nodes by a polynomial of degree $n = 2$.

(iii) Three-eight rule—$f(x)$ is interpolated at equally spaced nodes by a polynomial of degree $n = 3$.

(iv) Booles rule—$f(x)$ is interpolated at equally spaced nodes by polynomial of degree $n = 4$.

(v) Weddle's rule—$f(x)$ is interpolated at equally spaced nodes by polynomial of degree, $n = 6$.

Numerical Solution of Ordinary Differential Equation—

(A) Taylor's Series Method :

Given, initial value problem

$$y' = \frac{dy}{dx} = f(x, y),$$

with $\quad y(x_0) = y_0 \qquad \ldots(1)$

Then

$$y'' = \frac{d^2y}{dx^2} = f_x + f_y y' = f_x + f_y f$$

$$y''' = \frac{d^3y}{dx^3} = f_{xx} + 2f_{xy}f + f_{yy}f^2 + f_x f_y + f_y{}^2 f \quad \ldots(2)$$

By Taylor's theorem, the series about a point $x = x_0$, $y = y_0 + (x - x_0)(y')_0 + \frac{(x + x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \ldots\ldots$ (3) From (3) one can find the value y_1 of y for $x = x_1$ and $y', y'', y''' \ldots$ can be found at $x = x_1$ with (1) and (2) and so on.

(B) Picard's Method :

Given initial value problem

$$y' = \frac{dy}{dx} = f(x, y),$$

with $\quad y(x_0) = y_0$

Integrating both sides, we have

$$y = y_0 + \int_{x_0}^{x} f(x, y)\,dx$$

First approximation :

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

Second approximation :

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

nth approximation :

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx$$

Theorem :

If the function $f(x, y)$ is bounded in same region about the point (x_0, y_0) and if $f(x, y)$ satisfies Lipshitz condition $|f(x, y) - f(x, \bar{y})| \le k|y - \bar{y}|$ (k being a constant), then the sequence y_1, $y_2 \ldots\ldots$ converges to the solution of initital value problem $y' = \frac{dy}{dx} = f(x, y)$, with $y(x_0) = y_0$.

(C) Euler's method :

Given initial value problem

$$y' = \frac{dy}{dx} = f(x, y),$$

with $\quad y(x_0) = y_0$

By Taylor series, for $h \to 0$, $y(x + h) \approx y(x) + hy'(x) = y(x) + hf(x, y)$

This gives, $\quad y_{n+1} = y_n + hf(x_n, y_n)$

where $\quad h = \frac{x_n - x_0}{n}$

$(i.e., x_n = x_0 + nh)$

(D) Modified Euler's Method :

Given initial value problem

$$y' = \frac{dy}{dx} = f(x, y),$$

with $y(x_0) = y_0$

Integrating both sides,

$$y = y_0 + \int_{x_0}^{x} f(x, y)\, dx$$

Approximating by means of one have

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

And the iteration formula

$$y_1^{(n+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1)\, y_1^{(n)}]$$

where $y_1^{(n)}$ is the nth approximation to y_1.

And $y_1^{(0)} = y_0 + hf(x_0, y_0)$,

Where $h = \dfrac{x_n - x_0}{n}$ $(i.e.,\ x_n = x_0 + nh)$

(D) Runge-Kutta Method :

Given initial value problem

$$y' = \frac{dy}{dx} = f(x, y),$$

where $y(x_0) = y_0$

Calculate : $k_1 = hf(x_n, y_n)$

$$k_2 = hf\left(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}k_2\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

and $y_{n+1} = y_n + \dfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

where $h = \dfrac{x_n - x_0}{n}$ $(i.e.,\ x_n = x_0 + nh)$

(E) Predictor–Corrector method :

Given initial value problem :

$$y' = \frac{dy}{dx} = f(x, y)$$

where $y(x_0) = y_0$

Predicted Value—

$$y^*_{n+1} = y_n + hf(x_n, y_n)$$

Corrected Value—

$$y_{n+1} = y_n + h[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})]$$

where $h = \dfrac{x_n - x_0}{n}$ $(i.e.,\ x_n = x_0 + nh)$

Existence and uniquenece of solutions for initial value problem :

Given $y' = f(x, y)$

and $y(x_0) = y_0$

1. Existence Theorem :

If $f(x, y)$ continuous at all points (x, y) is some rectangle R : $|\ x - x_0\ | < a$, $|\ y - y_0\ | < b$ and bounded in R, $|\ f(x, y)\ | \le k$, for all $(x, y) \in$ R, then the initial value problem has at least one solution $y(x)$.

2. Uniqueness Theorem :

If $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous for all (x, y) in the rectangle R and bounded $|\ f\ | \le k$ $\left|\dfrac{\partial f}{\partial y}\right| \le m$ for all $(x, y) \in$ R, then the initial value problem has at least one solution $y(x)$.

Some Solved Examples

Example 1. Prove that—

$$\mu_4 = \mu_3 + \Delta\mu_2 + \Delta^2\mu_1 + \Delta^3\mu_1$$

Solution : We know that

$$\Delta\mu_x = \mu_{x+h} - \mu_x$$

$$\therefore\quad \mu_4 - \mu_3 = \Delta\mu_3,$$

$$\mu_3 - \mu_2 = \Delta\mu_2,$$

$$\mu_2 - \mu_1 = \Delta\mu_1$$

Hence $\mu_4 = \mu_3 + \Delta\mu_3$

$= \mu_3 + \Delta(\mu_2 - \Delta\mu_2)$ $[\because\ \mu_3 - \mu_2 = \Delta\mu_2]$

$= \mu_3 + \Delta\mu_2 + \Delta^2\mu_2$

$= \mu_3 + \Delta\mu_2 + \Delta^2(\mu_1 + \Delta\mu_1)$ $[\because\ \Delta\mu_1 = \mu_2 - \mu_1]$

$= \mu_3 + \Delta\mu_2 + \Delta^2\mu_1 + \Delta^3\mu_1$

Example 2. Let $x = s$ be a solution of $x = g(x)$ and suppose that g has a continuous derivative in some interval J containing s. Then if $|\ g'(x)\ | \le k < 1$ in J, the iteration process defined by $x_n = g(x_{n-1})$ converges any $x_0 \in$ J.

Solution : Since g has a continuous derivative in interval I by mean value theorem.

$$g'(t) = \frac{g(x)-g(s)}{x-s} \; (s \in \mathrm{I})$$

$$\Rightarrow \quad g(x) - g(s) = g'(t)\,(x - \delta)\;(\delta \in \mathrm{I})$$

Here $g(s) = s$

and $x_1 = g(x_0)$,

$x_2 = g(x_1)$, ...

we have
$$\begin{aligned} |x_n - s| &= |g(x_{n-1}) - g(s)| \\ &= |g'(t)|\,|x_{n-1} - s| \\ &\le k\,|x_{n-1} - s| \\ &= k\,|g(x_{n-2}) - g(s)| \\ &= k\,|g'(t)|\,|x_{n-2} - s| \\ &\le k^2\,|x_{n-2} - s| \\ &\;\vdots \\ &\le k^n\,|x_0 - s| \end{aligned}$$

$\because$ $k < 1$ we have $k^n \to 0 \Rightarrow |x_n - s| \to 0$ as $n \to \infty$

$\Rightarrow$ The iteration process defined by $x_n = g(x_{n-1})$ converges to $x_0 \in \mathrm{I}$.

Example 3. Find the square root of 18, correct to four decimal places by fixed point iteration formula.

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{18}{x_n}\right]$$

Solution : Given the iteration formula

$$x_1 = \frac{1}{2}\left[x_n + \frac{18}{x_n}\right] \qquad ......(1)$$

Put $n = 0$,

$$x_1 = \frac{1}{2}\left[x_0 + \frac{18}{x_0}\right] \qquad ......(2)$$

Let $x_0 = \sqrt{18}$

$= 4{\cdot}2$ (Approximately)

Then from (2), we get

$$\begin{aligned} x_1 &= \frac{1}{2}\left[4{\cdot}2 + \frac{18}{4{\cdot}2}\right] \\ &= \frac{1}{2}[4{\cdot}2 + 4{\cdot}286] \\ &= 4{\cdot}243 \qquad ......(3) \end{aligned}$$

Putting $n = 1$ in (1), we get

$$\begin{aligned} x_2 &= \frac{1}{2}\left[x_1 + \frac{18}{x_1}\right] \\ &= \frac{1}{2}\left[4{\cdot}243 + \frac{18}{4{\cdot}243}\right], \text{ from (3)} \\ &= \frac{1}{2}[4{\cdot}243 + 4{\cdot}2423] \\ &= 4{\cdot}24265 \qquad ......(4) \end{aligned}$$

Putting $n = 2$ in (1), we get

$$\begin{aligned} x_3 &= \frac{1}{2}\left[x_2 + \frac{18}{x_2}\right] \\ &= \frac{1}{2}\left[4{\cdot}24265 + \frac{18}{4{\cdot}2465}\right] \\ &= \frac{1}{2}[4{\cdot}24265 + 4{\cdot}24263] \\ &= 4{\cdot}24264 \qquad ...(5) \end{aligned}$$

$\therefore$ From (4) and (5), the square root of 18, correct to four decimal places is $4{\cdot}2426$.

Example 4. Evaluate the following—

(a) $\Delta\left(\frac{x^2}{\cos 2x}\right)$ (b) $\Delta^2 \cos 2x$.

Solution : (a) $\Delta\left(\frac{x^2}{\cos 2x}\right)$

$$\begin{aligned} &= \frac{(x+h)^2}{\cos 2(x+h)} - \frac{x^2}{\cos 2x} \\ &= \frac{(x+h)^2 \cos 2x - x^2 \cos 2(x+h)}{\cos 2(x+h)\cos 2x} \\ &= \frac{[(x+h)^2 - x^2]\cos 2x + x^2[\cos 2x - \cos 2(x+h)]}{\cos 2(x+h)\cos 2x} \\ &= \frac{(2hx + h^2)\cos 2x + 2x^2 \sin h \sin(2x+h)}{\cos 2(x+h)\cos 2x} \end{aligned}$$

(b) $\Delta^2 \cos 2x$

$$\begin{aligned} &= \Delta(\cos 2(x+h) - \cos 2x] \\ &= \Delta \cos 2(x+h) - \Delta \cos 2x \\ &= [\cos 2(x+2h) - \cos 2(x+h)] - [\cos 2(x+h) - \cos 2x] \\ &= -2\sin(2x+3h)\sin h + 2\sin(2x+h)\sin h \\ &= -2\sin h[\sin(2x+3h) - \sin(2x+h)] \\ &= -2\sin h[2\cos(2x+2h)\sin h] \\ &= -4\sin^2 h\cos(2x+2h) \end{aligned}$$

Example 5. Evaluate $\Delta^2\left(\frac{5x+12}{x^2+5x+6}\right)$, where the interval of differencing is unity.

Solution : $\Delta^2\left(\frac{5x+12}{x^2+5x+6}\right)$

$$= \Delta^2 \left\{ \frac{5x+12}{(x+2)(x+3)} \right\}$$

$$= \Delta^2 \left\{ \frac{2}{x+2} + \frac{3}{x+3} \right\}$$

$$= \Delta \left\{ \Delta\left(\frac{2}{x+2}\right) + \Delta\left(\frac{3}{x+3}\right) \right\}$$

$$= \Delta \left\{ 2\left(\frac{1}{x+3} - \frac{1}{x+2}\right) + 3\left(\frac{1}{x+4} - \frac{1}{x+3}\right) \right\}$$

$$= -2\Delta \left\{ \frac{1}{(x+2)(x+3)} \right\} - 3\Delta \left\{ \frac{1}{(x+3)(x+4)} \right\}$$

$$= -2 \left\{ \frac{1}{(x+3)(x+4)} - \frac{1}{(x+2)(x+3)} \right\}$$

$$= -3 \left\{ \frac{1}{(x+4)(x+5)} - \frac{1}{(x+3)(x+4)} \right\}$$

$$= \frac{4}{(x+2)(x+3)(x+4)} + \frac{6}{(x+3)(x+4)(x+5)}$$

$$= \frac{2(5x+16)}{(x+2)(x+3)(x+4)(x+5)}$$

Example 6. Find the real root of the equation $x^4 - x - 10 = 0$. By Newton—Raphson method upto 3 decimal places.

Solution : Here,

$$f(x) = x^4 - x - 10 = 0 \qquad \ldots(1)$$

and $$f'(x) = 4x^3 - 1$$

By Newton-Raphson formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{x_i^4 - x_i - 10}{4x_i^3 - 1}$$

$$\Rightarrow \quad x_{i+1} = \frac{4x_i^4 - x_i - x_i^4 + x_i + 10}{4x_1^3 - 1}$$

$$= \frac{3x_i^4 + 10}{4x_i^3 - 1} \qquad \ldots(2)$$

Also from (1), we have

$$f(1) = 1 - 1 - 10$$
$$= -10$$
$$= \text{Negative}$$

$$f(2) = 2^4 - 2 - 10$$
$$= 16 - 2 - 10$$
$$= 4$$
$$= \text{Positive}$$

$\therefore$ The root of the given equation lies between 1 and 2.

Also $$f(1{\cdot}5) = (1{\cdot}5)^4 - 1{\cdot}5 - 10$$
$$= -\text{ve}$$

$$f(1{\cdot}8) = (1{\cdot}8)^4 - 1{\cdot}8 - 10$$
$$= 10{\cdot}5 - 1{\cdot}8 - 10$$
$$= -\text{ve}$$

$$f(1{\cdot}9) = (1{\cdot}9)^4 - 1{\cdot}9 - 10$$
$$= 13{\cdot}032 - 1{\cdot}9 - 10$$
$$= +\text{ve}$$

$\therefore$ The root of (1) lies between 1·8 and 1·9.

First approximation—Let $i = 0$,

So $$x_i = x_0$$
$$= 1{\cdot}8 \text{ (say)}$$

Then from (2)

$$x_1 = \frac{3x_0^4 + 10}{4x_0^3 - 1}$$

$$= \frac{3(1{\cdot}8)^4 + 10}{4(1{\cdot}8)^3 - 1}$$

$$= \frac{3(10{\cdot}498) + 10}{4(5{\cdot}832) - 1}$$

$$= \frac{41{\cdot}494}{22{\cdot}328}$$

$$= 1{\cdot}858$$

Second approximation—Let $i = 1$,

So $$x_i = x_1$$
$$= 1{\cdot}858$$

Then from (2),

$$x_2 = \frac{3(x_1^4) + 10}{4x_1^3 - 1}$$

$$= \frac{3(1{\cdot}858)^4 + 10}{4(1{\cdot}858)^3 - 1}$$

$$\Rightarrow \quad x_2 = \frac{3(11{\cdot}9174) + 10}{4(6{\cdot}414) - 1}$$

$$= \frac{45{\cdot}7522}{24{\cdot}656}$$

$$= 1{\cdot}8556$$

Third approximation—Let

$$i = 2,$$

So $$x_i = x_2$$
$$= 1{\cdot}8556$$

Then from (2),

$$x_3 = \frac{3(x_2)^4 + 10}{4x_2^3 - 1}$$

$$= \frac{(1{\cdot}8556)^4 + 10}{4\,(1{\cdot}8556)^3 - 1}$$

$$\Rightarrow \quad x_3 = \frac{3\,(11{\cdot}856) + 10}{4\,(6{\cdot}3893) - 1}$$

$$= \frac{45{\cdot}568}{24{\cdot}557}$$

$$= 1{\cdot}855602$$

The root of the equation is 1·855.

Example 7. By using Newton–Raphson's method (a) prove that the recurrence formula for solution of $x^n = a$ is x_{i+1}

$$= \frac{(n-1)\,x_i^n + a}{nx_i^{n-1}}$$

and (b) evaluate $x^n = 12$.

Solution : (a) Given

$$x^n = a$$

Let $\quad f(x) \equiv x^n - a = 0;$

$$f'(x) = n \cdot x^{n-1}$$

By Newton–Raphson formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{x_i^x - a}{nx_i^{n-1}}$$

$$= \frac{nx_i^n - x_i^n + a}{nx_i^{n-1}}$$

$$= \frac{(n-1)\,x_i^n + a}{nx_i^{n-1}} \quad \ldots(1)$$

(b) To evaluate $\quad x^n = 12$

We know $\quad (8)^{1/3} = (2)^{1/3} = 2$

$\therefore$ Taking $\quad a = 12$

and $\quad n = 3$ in (1)

$$x_{i+1} = \frac{2x_i^3 + 12}{3x_i^2} \quad \ldots(2)$$

For first approximation—Let

$$x_0 = 2{\cdot}2$$

$$x_1 = \frac{2x_0^3 + 12}{3x_0^2}$$

$$= \frac{2(2{\cdot}2)^3 + 12}{3(2{\cdot}2)^2}$$

$$= \frac{2(10{\cdot}648) + 12}{3(4{\cdot}84)}$$

$$= \frac{33{\cdot}296}{14{\cdot}52}$$

$$= 2{\cdot}293$$

For second approximation—Let $i = 1$,

$$x_i = x_1$$

$$= 2{\cdot}293,$$

then

$$x_2 = \frac{2x_1^3 + 12}{3x_1^2}$$

$$= \frac{2\,(2{\cdot}293)^3 + 12}{3(2{\cdot}293)^2}$$

$$= \frac{2(12{\cdot}0562) + 12}{3(5{\cdot}2578)}$$

$$= \frac{36{\cdot}1125}{15{\cdot}7785}$$

$$= 2{\cdot}2894$$

$\therefore$ The required value of x correct to three places of decimal, is 2·289.

Example 8. Solve $\dfrac{dy}{dx} = \dfrac{1}{x+y}$ for $x = 0{\cdot}5$ by using Runge-Kutta method with $x_0 = 0$, $y(x_0) = 1$, taking $h = 0{\cdot}5$.

Solution : Given $x_0 = 0$, $y_2 = 1$, $h = 0{\cdot}5$ …(1)

and

$$f(x, y) = \frac{1}{x+y} \quad \ldots(2)$$

$$\therefore \quad f(x_0, y_0) = \frac{1}{(x_0 + y_0)}$$

$$= \frac{1}{(0+1)}$$

$$= 1$$

Here

$$k_1 = hf(x_0, y_0)$$

$$= (0{\cdot}5)\,(1)$$

$$= 0{\cdot}5 \quad \ldots(3)$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1\right)$$

$$= (0{\cdot}5)\,f\,(0 + 2{\cdot}5,\ 1 + 0{\cdot}25)$$

$$= (0{\cdot}5)\,f\,(0{\cdot}25,\ 1{\cdot}25)$$

$$= (0{\cdot}5)\left[\frac{1}{(1{\cdot}50)}\right]$$

$$= 0{\cdot}333 \quad \ldots(4)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2\right)$$

$$= (0{\cdot}5)\,f\,(0{\cdot}25,\ 1{\cdot}167)$$

$$= (0{\cdot}5)\left[\frac{1}{(0{\cdot}25 + 1{\cdot}167)}\right]$$

$$= (0{\cdot}5)\left[\frac{1}{1{\cdot}417}\right]$$

$$= 0{\cdot}353 \qquad \ldots(5)$$

and $$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= (0{\cdot}5)\, f(0{\cdot}5, 1{\cdot}353)$$

$$= (0{\cdot}5)\left[\frac{1}{(0{\cdot}5 + 1{\cdot}353)}\right]$$

$$= (0{\cdot}5)\left[\frac{1}{(1{\cdot}853)}\right]$$

$$= 0{\cdot}270 \qquad \ldots(6)$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0{\cdot}5 + 2(0{\cdot}333) + 2(0{\cdot}353) + 0{\cdot}27]$$

From (3), (5), (4), (6)

$$= \frac{1}{6}[0{\cdot}5 + 0{\cdot}666 + 0{\cdot}706 + 0{\cdot}27]$$

$$= \frac{1}{6}[2{\cdot}142]$$

$$= 0{\cdot}357$$

$$\therefore \quad y_{n+1} = y_{n+k}$$

$$\Rightarrow \quad y_1 = y_0 + k$$

$$\Rightarrow \quad 1 + 0{\cdot}357 = 1{\cdot}357$$

Thus the required value of y when

$$x = 0{\cdot}5 \text{ is } 1{\cdot}357.$$

Example 9. Evaluate.

(a) $\frac{\Delta^2}{E}x^2$

(b) Δ (sin x cos $3x$).

Solution : (a) $\frac{\Delta^2}{E}x^2$

$$= \Delta^2 E^{-1} x^2$$

$$= \Delta^2 (x-h)^2 \qquad [\because \; E^n y_x = y_{x+nh}]$$

$$= \Delta[\Delta(x-h)^2]$$

$$= \Delta[\{(x+h)-h\}^2 - \{x-h\}^2]$$

$$= \Delta[x^2 - (x-h)^2]$$

$$= \Delta[x^2 - (x^2 - 2xh - h^2)]$$

$$= \Delta(2xh - h^2)$$

$$= [\{2(x+h)h - h^2\} - \{2xh - h^2\}] \qquad [\because \Delta y_n = y_{x+h} - y_x]$$

$$= \{2xh + 2h^2 - h^3\} - (2xh - h^2\}$$

$$= 2h^2$$

(b) Δ (sin x cos $3x$)

$$= \Delta\left[\frac{1}{2}(2\sin x\cos 3x)\right]$$

$$= \Delta\left[\frac{1}{2}(\sin 4x - \sin 2x)\right]$$

$$= \frac{1}{2}[\Delta(\sin 4x) - \Delta(\sin 2x)]$$

$$= \frac{1}{2}[\{\sin 4(x+h) - \sin 4x\} - \{\sin 2(x+h) - \sin 2x\}]$$

$$= \frac{1}{2}[2\cos(4x+2h)\sin 2h - 2\cos(2x+h)\sin h]$$

$$= \cos(4x+2h)\sin 2h - \cos(2x+h)\sin h$$

$$= (\sin h)[2\cos(4x+2h)\cos h - \cos(2x+h)]$$

Example 10. Using Picard's method solve $\frac{dy}{dx} = 1 - 2xy$, given $y = 0$ at $x = 0$, upto third approximation.

Solution : Here, $y_0 = 0$

Integrating the given equation between the limits given

$$[dy]_{y=0}^{y} = \int_{x=0}^{x}(1-2xy)\,dx$$

or $$y = \int_0^x (1-2xy)\,dx \qquad \ldots(i)$$

and $$y_n = y_0 + \int_0^x f(x, y_{n-1})\,dx$$

First approximation y_1, replace y by 0 in $(1 - 2xy)$, from (i),

$$y_1 = \int_0^\infty (1)\,dx = x \qquad \ldots(ii)$$

Second approximation y_2, replace y by 0 in $(1 - 2xy)$ from (i),

$$y_2 = \int_0^x (1 - 2xy_1)\,dx$$

$$= \int_0^x (1-2x^2)\,dx, \qquad \text{from (ii)}$$

$$= \left(x - \frac{2}{3}x^3\right)_0^x$$

$$= x - \frac{2}{3}x^3 \qquad \ldots(iii)$$

Third approximation y_3, replace y by y_2 in $(1 - 2xy)$ from (i),

$$y_3 = \int_0^x (1 - 2xy_2)\, dx$$

$$= \int_0^x \left[1 - 2x\left(x - \frac{2}{3}x^3\right)\right] dx,$$

from (iii)

or $$y_3 = \int_0^x \left(1 - 2x^2 + \frac{4}{3}x^4\right) dx$$

$$= x - \frac{2}{3}x^3 - \frac{4}{15}x^5$$

Thus the solution of the given equation upto third approximation are

$$y_1 = x$$

$$y_2 = x - \frac{2}{3}x^3,$$

$$y^3 = x - \frac{2}{3}x^3 - \frac{4}{15}x^5$$

Example 11. Solve

$$10x - 7y + 3z + 5u = 6$$
$$-6x + 8y - z - 4u = 5$$
$$3x + y + 4z + 11u = 2$$
$$5x - 9y - 2z + 4u = 7$$

by Crout's triangularization method.

Solution : Here,

$$\begin{bmatrix} 10 & -7 & 3 & 5 \\ -6 & 8 & -1 & -4 \\ 3 & 1 & 4 & 11 \\ 5 & -9 & -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 7 \end{bmatrix}$$

A = LU

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -7 & 3 & 5 \\ -6 & 8 & -1 & -4 \\ 3 & 1 & 4 & 11 \\ 5 & -9 & -2 & 4 \end{bmatrix}$$

So that

(i) R_1 of U : $u_{11} = 10$, $u_{12} = -7$, $u_{13} = 3$, $u_{14} = 5$

(ii) C_1 of L : $l_{21} = -0{\cdot}6$, $l_{31} = 0{\cdot}3$, $l_{41} = 0{\cdot}5$

(iii) R_2 of U : $u_{22} = 3{\cdot}8$, $u_{23} = 0{\cdot}8$, $u_{24} = -1$

(iv) C_2 of L : $l_{32} = 0{\cdot}815$, $l_{42} = 1{\cdot}447$

(v) R_3 of U : $u_{33} = 2{\cdot}447$, $u_{34} = 10{\cdot}315$

(vi) C_3 of L : $l_{43} = -0{\cdot}956$

(vii) R_4 of U : $u_{44} = 9{\cdot}924$

Thus

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0{\cdot}6 & 1 & 0 & 0 \\ 0{\cdot}3 & 0{\cdot}815 & 1 & 0 \\ 0{\cdot}5 & -1{\cdot}447 & -0{\cdot}956 & 1 \end{bmatrix} \begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3{\cdot}8 & 0{\cdot}8 & -1 \\ 0 & 0 & 2{\cdot}447 & 10{\cdot}315 \\ 0 & 0 & 0 & 9{\cdot}924 \end{bmatrix}$$

Writing UX = V, the given system becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -0{\cdot}6 & 1 & 0 & 0 \\ -0{\cdot}3 & 0{\cdot}815 & 1 & 0 \\ 0{\cdot}5 & -1{\cdot}447 & -0{\cdot}956 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 7 \end{bmatrix}$$

Solving this system, we have

$v_1 = 6$, $v_2 = 8{\cdot}6$, $v_3 = -6{\cdot}815$, $v_4 = 9{\cdot}924$

Hence the original system becomes.

$$\begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3{\cdot}8 & 0{\cdot}8 & -1 \\ 0 & 0 & 2{\cdot}447 & 10{\cdot}315 \\ 0 & 0 & 0 & 9{\cdot}924 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 6 \\ 8{\cdot}6 \\ -6{\cdot}815 \\ 9{\cdot}924 \end{bmatrix}$$

$$10x - 7y + 3z + 5u = 6,$$
$$3{\cdot}8y + 0{\cdot}8z - u = 8{\cdot}6$$
$$2{\cdot}447z + 10{\cdot}315u = -6{\cdot}815u = 1$$

So by back substitution, we get

$u = 1$, $z = -7$, $y = 4$ and $x = 5$.

Example 12. By using Runge-Kutta method find the value of y when $x = 1{\cdot}1$ given.

$$y(x_0) = 1{\cdot}2,\ x_0 = 1 \text{ and } \frac{dy}{dx} = 3x + y.$$

Solution : Given

$$x_0 = 1,\ y_0 = 1{\cdot}2,\ h = 0{\cdot}1 \qquad \ldots\text{(i)}$$

and $$f(x, y) = 3x + y^2 \qquad \ldots\text{(ii)}$$

So $$f(x_0, y_0) = 3x_0 + y_0^2$$
$$= 3(1) + (1{\cdot}2)^2$$
$$= 3 + 1{\cdot}44$$
$$= 4{\cdot}44$$

Here $$k_1 = hf(x_0, y_0)$$
$$= (0{\cdot}1)(4{\cdot}44)$$
$$= 0{\cdot}444 \qquad \ldots\text{(iii)}$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h \cdot y_0 + \frac{1}{2}k_1\right)$$

$$= (0{\cdot}1)\, f(1{\cdot}05,\ 1{\cdot}422)$$
$$= (0{\cdot}1)\,(3{\cdot}15 + 2{\cdot}022)$$
$$= (0{\cdot}1)\,(5{\cdot}172)$$
$$= 0{\cdot}517 \quad \ldots\text{(iv)}$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2\right)$$
$$= (0{\cdot}1)\, f(1{\cdot}05,\ 1{\cdot}459)$$
$$= (0{\cdot}1)\,(3{\cdot}15 + 2{\cdot}129)$$
$$= (0{\cdot}1)\,(5{\cdot}279)$$
$$= 0{\cdot}528 \quad \ldots\text{(v)}$$

$$k_4 = hf\,(x_0 + h,\ y_0 + k_3)$$
$$= (0{\cdot}1)\, f(1{\cdot}1,\ 1{\cdot}728)$$
$$= (0{\cdot}1)\,(3{\cdot}3 + 2{\cdot}986)$$
$$= (0{\cdot}1)\,(6{\cdot}286)$$
$$= 0{\cdot}629 \quad \ldots\text{(vi)}$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0{\cdot}444 + 2\,(0{\cdot}517) + 2\,(0{\cdot}528) + 0{\cdot}629]$$
$$= \frac{1}{6}[3{\cdot}163] = 0{\cdot}527$$

$$\therefore \quad y_{n+1} = y_n + k$$
$$\Rightarrow \quad y_1 = y_0 + k$$
$$= 1{\cdot}2 + 0{\cdot}527 = 1{\cdot}727$$

Thus, the required value of y when $x = 1{\cdot}1$ is $1{\cdot}727$.

Example 13. By Gauss elimination method solve the system of equations,

$$x + 4y - z = -5$$
$$x + y - 6z = -12$$
$$3x - y - z = 4.$$

Solution : We have

$$x + 4y - z = -5 \quad \ldots\text{(i)}$$
$$x + y - 6z = -12 \quad \ldots\text{(ii)}$$
$$3x - y - z = 4 \quad \ldots\text{(iii)}$$

Step I. Here $r_1 = 1$ operate (ii)—(i) and $r_2 = 3$. (iii)—3 (i) to eliminate x,

$$-3y - 5z = -7 \quad \ldots\text{(iv)}$$
$$-13y + 2z = 19 \quad \ldots\text{(v)}$$

Step II. Here $r_3 = \frac{13}{3}$ operate (v) $- \frac{13}{3}$ (iv) to eliminate y,

$$\frac{71}{3}z = \frac{148}{3} \quad \ldots\text{(vi)}$$

Step III. By back-substitution, we get

From (vi), $z = \dfrac{148}{71}$

From (iv), $y = \dfrac{7}{3} - \dfrac{5}{3}\left(\dfrac{148}{71}\right) = -\dfrac{81}{71}$

From (i), $x = -5 - 4\left(-\dfrac{81}{71}\right) + \dfrac{148}{71} = \dfrac{117}{71}$

$\therefore$ The solution is $x = \dfrac{117}{71}$, $y = -\dfrac{81}{71}$ and $z = \dfrac{148}{71}$

Example 14. Given

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3

Find the value of $\int_0^{80} y\,dx$ by Trapezoidal rule.

Solution : The trapezoidal rule

$$\int_a^b y\,dx = h\left[\frac{1}{2}y_0 + y_1 + y_2 + \ldots + y_{n-1} + \frac{1}{2}y_n\right]$$

Taking $h = 10$, we have

$$\therefore \int_0^{80} y\,dx$$
$$= h\left[\frac{1}{2}y_0 + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + \frac{1}{2}y_8\right]$$
$$= 10\left[\frac{1}{2}(0) + 4 + 7 + 9 + 12 + 15 + 14 + 8 + \frac{1}{2}(3)\right] \quad \text{(From table)}$$
$$= 10\,[69 + 1{\cdot}5]$$
$$= 10\,(70{\cdot}5)$$
$$= 705$$

The value of $\int_0^{80} y\,dx = 705$.

Example 15. Evaluate $\int_1^2 \log x\,dx$ by Trapezoidal rule.

Solution : Dividing the interval (1, 2) into five equal parts each of width $0{\cdot}2$.

The values of the function $y = \log_e x$ for each point of sub-division is given by

x	$y = \log x$	x	$y = \log x$
1·0	$0{\cdot}00000 = y_0$	1·6	$0{\cdot}20412 = y_3$
1·2	$0{\cdot}07918 = y_1$	1·8	$0{\cdot}25527 = y_4$
1·4	$0{\cdot}14613 = y_2$	2·0	$0{\cdot}30103 = y_5$

By Trapezoidal rule, we have

$$\int_1^2 \log x\, dx = \int_1^{1+5h} y\, dx$$

$$= h\left[\frac{1}{2}y_0 + y_1 + y_2 + y_3 + y_4 + \frac{1}{2}y_5\right]$$

$$= (0{\cdot}2)\left[\frac{1}{2}(0) + 0{\cdot}07918 + 0{\cdot}14613 \quad 0{\cdot}20412 + 0{\cdot}25527 + \frac{1}{2}(0{\cdot}30103)\right]$$

$$= (0{\cdot}2)\,[0{\cdot}07918 + 0{\cdot}14613 + 0{\cdot}20412 + 0{\cdot}25527 + 0{\cdot}15052]$$

$$= (0{\cdot}2)\,[0{\cdot}83522]$$

$$= 0{\cdot}16704$$

Example 16. Using Taylor's series method, find y to five places of decimals when $x = 1{\cdot}02$, given that

$$\frac{dy}{dx} = xy - 1 \text{ and } y = 2, \text{ when } x = 1.$$

Solution : Given

$$(y)_0 = 2 \qquad \ldots\text{(i)}$$

and $\quad y' = xy - 1 \qquad \ldots$(ii)

Differentiating (ii) successively,

$$y'' = xy' + y;$$

$$y''' = xy'' + 2y',$$

$$y^{(iv)} = xy''' + 3y'' \qquad \ldots\text{(iii)}$$

Put $x = 1$ and using (i), from (ii) and (iii), we have

$$(y')_0 = (y')_{x=1} = 1\,(2) - 1 = 1$$

$$(y'')_0 = (y'')_{x=1} = 1\,(1) + 2 = 3$$

$$(y''')_0 = (y''')_{x=1} = (1)\,(3) + 2\,(1) = 5$$

$$(y^{iv})_0 = (y^{iv})_{x=1} = (1)\,(5) + 3\,(3) = 14$$

By Taylor's series about $x = x_0$

$$y = (y_0) + (x - x_0)\,(y')_0 + \frac{(x-x_0)^2}{2\,!}(y'')_0 + \frac{(x-x_0)^3}{3\,!}(y''')_0 + \ldots\ldots$$

$$\Rightarrow y = 2 + (x-1)\,(1) + \frac{(x-1)^2}{2\,!}(3) + \frac{(x-1)^3}{3\,!}(5) + \frac{(x-1)^4}{4\,!}(14) + \ldots\ldots,$$

Since $x_0 = 1$ (given)

Put $x = 1{\cdot}02$ the required value of y is

$$y = 2 + (0{\cdot}02) + \frac{(0{\cdot}02)^2}{2}(3) + \frac{(0{\cdot}02)^3}{6}(5) + \frac{(0{\cdot}02)^4}{24}(14) + \ldots$$

$$= 2 + 0{\cdot}02 + 0{\cdot}0006 + 0{\cdot}0000067 + \ldots$$

$$= 2{\cdot}020606$$

$$= 2{\cdot}02061$$

Example 17. By Euler's method to initial value problem.

$\frac{dy}{dx} = x + y$, $y = 0$, when $x = 0$ in the range $x = 0$ to $x = 0{\cdot}04$

Taking $h = 0{\cdot}2$, find the value at y_1, y_2, y_3.

Solution : Taking $h = 0{\cdot}2$ split up the interval (0, 1) into sub-intervals of width 0·2 each with the help of $x_n = 0 + nh$ so that the points of division are given by

$x_0 = 0$, $x_1 = 0{\cdot}2$, $x_2 = 0{\cdot}4$, $x_3 = 0{\cdot}6$, $x_4 = 0{\cdot}8$, $x_5 = 1{\cdot}0$

Let us write

$$f(x, y) = x + y \qquad \ldots\text{(i)}$$

Here $x_0 = 0$, $y = 0$ and

$$h = 0{\cdot}2 \qquad \ldots\text{(ii)}$$

and we have to calculate y_5

$$\therefore \quad f(x_0, y_0) = x_0 + y_0 \text{ from (i)}$$

$$= 0 + 0 = 0$$

$$[\because x_0 = 0 = y_0]$$

Now $\quad y_1 = y_0 + hf(x_0, y_0)$

i.e., $\quad y_1 = 0 + (0{\cdot}2)\,(0) = 0$

$$y_1 = 0 \qquad \ldots\text{(iii)}$$

$$\therefore \quad f(x_1, y_1) = x_1 + y_1 \text{ from (i)}$$

$$= 0{\cdot}2 + 0 = 0{\cdot}2 \qquad \ldots\text{(iv)}$$

Now $\quad y_2 = y_1 + hf(x_1, y_1)$

$$= 0 + (0{\cdot}2)\,(0{\cdot}2), \text{ from (iv)}$$

or $\quad y_2 = 0{\cdot}04$

$$\therefore \quad f(x_2, y_2) = x_2 + y_2 \text{ from (i)}$$

$$= 0{\cdot}4 + 0{\cdot}04$$

$$= 0{\cdot}44$$

Now $$\begin{aligned} y_3 &= y_2 + hf(x_2, y_2) \\ &= (0{\cdot}04) + (0{\cdot}02)(0{\cdot}44) \\ &= 0{\cdot}128 \end{aligned}$$

Thus we have $y_1 = 0$, $y_2 = 0{\cdot}04$ and $y_3 = 0{\cdot}128$.

Example 18. Using Taylor's series, find the solution of the differential equation

$xy' = x - y$ (y (2) = 2 at $x = 2{\cdot}1$ correct to five decimal places.

Solution : The first few derivatives and their values at $x = 2$, $y = 2$ are

$$y' = 1 - y/x \qquad y'_0 = 0$$

$$y'' = \frac{-y'}{x} + \frac{y}{x^2} \qquad y''_0 = \frac{1}{2}$$

$$y''' = \frac{-y''}{x} + \frac{2y'}{x^2} - \frac{2y}{x^3} \qquad y'''_0 = \frac{-3}{4}$$

$$y^{iv} = \frac{-y'''}{x} + \frac{3y''}{x^2} - \frac{6y'}{x^3} + \frac{6y}{x^4} \qquad y^{iv}{}_0 = \frac{3}{2}$$

The Taylor series expansion about $x_0 = 2$ is

$$\begin{aligned} y(x) &= y_0 + (x-2)\,y'_0 + \frac{1}{2}(x-2)^2\,y''_0 \\ &\quad + \frac{1}{6}(x-2)^3\,y'''_0 + \frac{1}{24}(x-2)\,y^{iv}{}_0 + \ldots\ldots \\ &= 2 + (x-2)\,0 + \frac{1}{4}(x-2)^2 \\ &\quad - \frac{1}{8}(x-2)^3 + \frac{1}{16}(x-2)^4 + \ldots\ldots \end{aligned}$$

At $x = 2{\cdot}1$, we obtain

$$\begin{aligned} y(2{\cdot}1) &= 2 + 0{\cdot}0025 - 0{\cdot}000125 \\ &\quad + 0{\cdot}0000062\ldots\ldots \\ &= 2{\cdot}00238 \end{aligned}$$

Example 19. Given

$$\begin{aligned} 3x + y + 2z &= 3 \\ 2x - 3y - z &= -3 \\ x - 2y + z &= 4 \end{aligned}$$

Solve it by matrix inversion method.

Solution : Here

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$\Leftrightarrow \quad AX = D$$

$$\therefore \quad X = \frac{1}{|A|}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix},$$

where $A_1, B_1, \ldots\ldots$ are cofactors of $a_1, b_1, \ldots\ldots$ in the determinant $|A|$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & 11 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Thus $x = 1$, $y = 2$ and $z = -1$.

Example 20. $\Delta \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$.

Solution : $\Delta \log f(x) = \log f(x+h) - \log f(x)$, by definition of Δ

$$\begin{aligned} &= \log\left[\frac{f(x+h)}{f(x)}\right] \\ &= \log\left[\frac{E\,f(x)}{f(x)}\right] \qquad [\because E\,u_x = u_{x+h}] \\ &= \log\left[\frac{(\Delta+1)\,f(x)}{f(x)}\right] \qquad [\because E = \Delta + 1] \\ &= \log\left[\frac{\Delta f(x) + f(x)}{f(x)}\right] \\ &= \log\left[\frac{f(x)}{f(x)} + \frac{\Delta f(x)}{f(x)}\right] \\ &= \log\left[1 + \frac{\Delta f(x)}{f(x)}\right] \end{aligned}$$

Example 21. Prove that

$$\Delta\sqrt{u_x} = \frac{\Delta u_x}{\sqrt{u_x} + \sqrt{u_{x+h}}}.$$

Solution : $\quad \Delta\sqrt{u_x} = \sqrt{u_{x+h}} - \sqrt{u_x}$

By definition of Δ

$$\begin{aligned} &= \frac{(\sqrt{u_{x+h}} - \sqrt{u_x})(\sqrt{u_{x+h}} + \sqrt{u_x})}{(\sqrt{u_{x+h}} + \sqrt{u_x})} \\ &= \frac{u_{x+h} - u_x}{\sqrt{u_x} + \sqrt{u_{x+h}}} \\ &= \frac{\Delta u_x}{\sqrt{u_x} + \sqrt{u_{x+h}}} \end{aligned}$$

OBJECTIVE TYPE QUESTIONS

1. Given initial value problem $y' = \frac{dy}{dx} = f(x, y)$, where $y(x_0) = y_0$. In Runge-Kutta method—
   (A) $y_{n+1} = y_n + \frac{1}{6}(k_1 + k_2 + k_3 + k_4)$
   (B) $y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
   (C) $y_{n+1} = y_n + \frac{1}{4}(k_1 + 2k_2 + 3k_3 + 4k_4)$
   (D) None of these

2. Given initial value problem $y' = \frac{dy}{dx} = f(x, y)$, where $y(x_0) = y_0$. In Runge-Kutta method—
   (A) $k_3 = hf(x_n + h, y_n + k_2)$
   (B) $k_4 = hf(x_n + h, y_n + k_3)$
   (C) $k_3 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right)$
   (D) None of these

3. Given initial value problem $y' = \frac{dy}{dx} = f(x, y)$. where $y(x_0) = y_0$. In Runge-Kutta method—
   (A) $k_3 = hf(x_n + h, y_n + k_2)$
   (B) $k_3 = hf(x_n, y_n)$
   (C) $k_3 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right)$
   (D) None of these

4. By Taylor's theorem, the series about a point $x = x_0$ is given by—
   (A) $y = y_0 + x_0 (y')_0 + \frac{x_0^2}{2!}(y'')_0 + \frac{x_0^3}{3!}(y'')_0 + \ldots\ldots$
   (B) $y = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \ldots\ldots$
   (C) $y = y_0 + (x + x_0)(y')_0 + \frac{(x + x_0)^2}{2!}(y'')_0 + \frac{(x + x_0)^3}{3!}(y''')_0 + \ldots\ldots$
   (D) None of these

5. Given initial value problem $y' = \frac{dy}{dx} = f(x, y)$, where $y(x_0) = y_0$. In Runge-Kutta method—
   (A) $k_1 = hf(x_n)$ (B) $k_1 = hf(x_n, y_n)$
   (C) $k_1 = f(y_n)$ (D) None of these

6. In Newton-Cote's formula, if $f(x)$ is interpolated at equally spaced nodes by a polynomial of degree six, then it represents—
   (A) Trapezoidal rule
   (B) Simpson rule
   (C) Three-eight rule
   (D) Weddles rule

7. In Newton-Cotes formula, if (x) is interpolated at equally spaced nodes by a polynomial of degree one, then it represents—
   (A) Trapezoidal rule
   (B) Simpson rule
   (C) Three-eight rule
   (D) Booles rule

8. Given initial value problem $y' = \frac{dy}{dx} = f(x, y)$, where $y(x_0) = y_0$. In Runge-Kutta method—
   (A) $k_2 = hf(x_n + h, y_n + k_1)$
   (B) $k_2 = hf(x_n, y_n)$
   (C) $k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$
   (D) None of these

9. In Newton-Cotes formula, if $f(x)$ is interpolated at equally spaced nodes by a polynomial of degree four, then it represents—
   (A) Trapezoidal rule
   (B) Simpson rule
   (C) Three-eight rule
   (D) Booles rule

10. In Newton-Cotes formula, if $f(x)$ is interpolated at equally spaced nodes by a polynomial of degree three, then it represents—
    (A) Trapezoidal rule (B) Simpson rule
    (C) Three-eight rule (D) Booles rule

11. In Newton-Cotes formula, if $f(x)$ is interpolated at equally spaced nodes by a polynomial of degree two, then it represents—
    (A) Trapezoidal rule (B) Simpson rule
    (C) Three-eight rule (D) Booles rule

12. The n-divided difference of a polynomial of the nth degree are—
    (A) Constant (B) Zero
    (C) Variable (D) None of these

13. The n-th divided difference is defined as—
    (A) $\underset{x_1 \ldots x_n}{\Delta^n} y_0 = \sum_{i=0}^{n} \dfrac{y_i}{\prod_{\substack{j=0 \\ j \neq 1}}^{n} (x_i - x_j)}$
    (B) $\underset{x_1 \ldots x_n}{\Delta^n} y_0 = \sum_{i=0}^{n} \dfrac{y_i}{(x_i - x_j)}$
    (C) $\underset{x_1 \ldots x_n}{\Delta^n} y_0 = \sum_{i=0}^{n} \dfrac{y_1}{\prod_{j=0}^{n} (x_i - x_j)}$
    (D) None of these

14. Let $\tilde{a}$ is an approximation of exact value a and absolute error is $\in$, then exact value is given by—
    (A) $a = \varepsilon\tilde{a}$ (B) $a = \varepsilon + \tilde{a}$
    (C) $a = \dfrac{\varepsilon}{\tilde{a}}$ (D) None of these

15. Let $\tilde{a}$ is an approximation of axact value a, then absolute error ε, is defined as—
    (A) $\varepsilon = a\tilde{a}$ (B) $\varepsilon = \dfrac{a}{\tilde{a}}$
    (C) $\varepsilon = a - \tilde{a}$ (D) None of these

16. If $f(x, y)$ continuous at all points (x, y) is some rectangle R : $|x - x_0| < a$, $|y - y_0| < b$ and bounded in R, if $(x, y)| \leq k$, for all $(x, y) \in$ R, then the initial value problem has at least one solution $y(x)$. This is—
    (A) Uniqueness theorem for initial value problem
    (B) Existence theorem for initial value problem
    (C) Green's theorem
    (D) None of these

17. The bisection (Binary-search) method : Let f is continuous function defined on the interval $[a, b]$. with $f(a)$ and $f(b)$ of opposite sign—
    (A) By the intermidiate value theorem, there exists a number p in (a, b) with $f(p)$ equals to zero
    (B) By the intermidiate value theorem, there exists anumber p in (a, b) with $f(p)$ not equals to zero
    (C) By the intermidiate value theorem, there exists a number p in (a, b) with $f(p)$ equals to positive number
    (D) None of these

18. Subdivide the interval $a \leq x \leq b$ into mutiple of 6 of equal intervals $n = 6$m length $h = \dfrac{b-a}{6m}$ the integral $J = \int_a^b f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx$ by Weddles rule is given by—
    (A) $\dfrac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \ldots\ldots)$
    (B) $\dfrac{3h}{10}(y_0 + 2y_1 + 3y_2 + 4y_3 + 5y_4 + 6y_5 + 7y_6 + 8y_7 + 9y_8 + \ldots\ldots)$
    (C) $(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \ldots\ldots)$
    (D) None of these

19. If $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous for all (x, y) in the rectangle R and bounded, $|f| \leq k \left|\dfrac{\delta f}{\partial y}\right| \leq$ M for all $(x, y) \in$ R, then the initial value problem has at least one solution $y(x)$. This is—
    (A) Uniquencess theorem for initial value problem
    (B) Existence theorem for initial value problem
    (C) Greens theorem
    (D) None of these

20. Let f is ...... on $[a, b]$ and $f(a) f(b) < 0$. Then bisection method generates a sequence $\{P_n\}$ approximating a zero p of f with $|P_n - P| \leq \dfrac{b-a}{2^n}$, $n \geq 1$—

(A) Continuous function
(B) Discontinuous function
(C) Constant function
(D) None of these

21. Relation between divided difference and ordinary difference is—
(A) $\underset{x_1}{\Delta} y_0 = \frac{y_0 - y_1}{x_0 - x_1} = \frac{\Delta y_0}{h}$
(B) $\underset{x_1}{\Delta} y_0 = \frac{y_0 - y_1}{x_0 - x_1} = \Delta y_0$
(C) $\underset{x_1}{\Delta} y_0 = \frac{y_0 - y_1}{x_0 - x_1} = h\,\Delta y_0$
(D) None of these

22. Newton's divided difference formula is—
(A) $f(x) = y_0 + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + \ldots\ldots + (x - x_0) \ldots\ldots (x - x_{n-1}) f(x_0, x_1 \ldots x_n) + R_n$
(B) $f(x) = y_0 + (x - x_0) + (x - x_0)(x - x_1) + \ldots + (x - x_0) \ldots (x - x_{n-1}) + R_n$
(C) $f(x) = y_0 + f(x_0, x_1) + f(x_0, x_1, x_2) + \ldots + f(x_0, x_1, \ldots x_n) + R_n$
(D) None of these

23. Relation between divided difference and ordinary difference is—
(A) $\underset{x_1 \ldots x_n}{\Delta^n} y_0 = \Delta y_0$
(B) $\underset{x_1 \ldots x_n}{\Delta^n} y_0 = \frac{\Delta y_0}{n\,!}$
(C) $\underset{x_1 \ldots x_n}{\Delta^n} y_0 = \frac{\Delta y_0}{n\,!\,h^n}$
(D) None of these

24. Given y_0, y_1, y_2, y_3 corresponding to values x_0, x_1, x_2, x_3 for function $y = f(x)$. Let $f(x)$ is a polynomial of degree 3, then by Simpson's three-eight rule, the integral $J = \int_a^b f(x)dx$ is equivalent to—
(A) $J = \frac{3}{8} h\,[y_0 + 3y_1 + 3y_2 + y_3]$
(B) $J = \frac{1}{3} h\,[y_0 + 4y_1 + y_2]$
(C) $J = \frac{1}{3} h\,[2y_0 + 4y_1 + 2y_2]$
(D) None of these

25. If the function $f(x, y)$ is bounded in some region about the point (x_0, y_0) and if $f(x, y)$ satisfies Lipschitz condition $|f(x, y) - f(x, \bar{y})| < k\,|y - \bar{y}|$ (k being a constant), then the sequence $y_1, y_2, \ldots$ the solution of initial value problem $y' = \frac{dy}{dx} = f(x, y)$, with $y(x_0) = y_0$—
(A) Converges (B) Diverges
(C) Oscillate (D) None of these

26. In Picard's method, given initial value problem $y' = \frac{dy}{dx} = f(x, y)$, with $y(x_0) = y_0$, n-th approximation is—
(A) $y_n = y_0 + \int_{x_0}^{x} f(x, y_n)dx$
(B) $y_n = y_0 + \int_{x_0}^{x} f(x, y)dx$
(C) $y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx$
(D) None of these

27. In Predictor-Corrector method, given initial value problem $y' = \frac{dy}{dx} = f(x, y)$, where $y(x_0) = y_0$. Predicated and corrected values are respectively given by—
(A) $y^*_{n+1} = y_n + hf(x_n, y_n)$ and $y_{n+1} = y_n + h\,[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})]$
(B) $y_{n+1} = y_n + h\,[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})]$ and $y^*_{n+1} = y_n + hf(x_n, y_n)$
(C) $y^*_{n+1} = y_n + h$, $y_{n+1} = y_n + h\,[f(x_{n+1}, y^*_{n+1})]$
(D) None of these

28. Uniqueness theorem for initial value problem states—
(A) If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous for all (x, y) in the rectangle R and bounded, $|f| \leq k \frac{|\partial f|}{\partial y} \leq M$ for all $(x, y) \in R$, then the initial value problem has at least one solution $y(x)$
(B) If $f(x, y)$ continuous at all points (x, y) is some rectangle $R : |x - x_0| < a, |y - y_0| < b$ and bounded in R. If $(x, y)| \leq k$, for all $(x, y) \in R$, then the initial value problem has at least one solution $y(x)$

(C) Both (A) and (B) above

(D) None of these

29. Newton backward difference formula is—

(A) $f(x) = \sum_{r=0}^{n} \frac{x^{(r)}}{r!} \nabla f(0)$

(B) $f(x) = \sum_{r=0}^{n} \nabla^r f\ (0)$

(C) $f(x) = \sum_{r=0}^{n} \frac{x^{(r)}}{r!} \nabla^r f(0)$

(D) None of these

30. In Euler's method—Given initial value problem $y' = \frac{dy}{dx} = f(x, y)$, with $y(x_0) = y_0$, then approximation is given by—

(A) $y_{n+1} = y_n + hf(x_{n-1}, y_{n-1})$, where $h = \frac{x_n - x_0}{n}$

(B) $y_{n+1} = y_n + hf(x_n, y_n)$, where $h = \frac{x_n - x_0}{n}$

(C) $y_{n+1} = y_n$

(D) None of these

31. Fixed-point theorem states that—

(A) Let g is a continuous function on $[a, b]$ such that $g(x) \in (a, b)$ for all $x \in [a, b]$ suppose in addition that g' exist on (a, b) and positive contant $k < 1$ exist with $|g'(x) \leq k$ for all $x \in (a, b)$, then for any number p in $[a, b]$, the sequence defined by $p_n = g(p_{n-1})$ $n \geq 1$, converges to the unique fixed point p in $[a, b]$

(B) If $f(x, y)$ an $\frac{\partial f}{\partial y}$ are continuous for all (x, y) in the rectangle R and bounded $|f| \leq k \left|\frac{\partial f}{\partial y}\right| \leq M$ for all $(x, y) \in R$, then the initial value problem has at least one solution $y(x)$

(C) If $f(x, y)$ continuous at all points (x, y) is some rectangle $R : |x - x_0| < a,\ |y - y_0| < b$ and bounded in R, $/f(x, y) \leq k$, for all $(x, y) \in R$, then the initial value problem has at least one solution $y(x)$

(D) None of these

32. Newton-Gregory forward difference interpolation formula is—

(A) $f(x + nh) = \sum_{r=0}^{n} {}^nC_r \Delta^r f(x)$

(B) $f(x + nh) \sum_{r=0}^{n} \Delta^r f(x)$

(C) $f(x + nh) = \sum_{r=0}^{n} {}^nC_r f(x)$

(D) None of these

33. Newton-Gregory advance difference formula is—

(A) $f(x) = \sum_{r=0}^{n} \Delta^r f(0)$

(B) $f(x) = \sum_{r=0}^{n} \frac{x^{(r)}}{r!} \Delta f(0)$

(C) $f(x) = \sum_{r=0}^{n} \frac{x^{(r)}}{r!} \Delta^r f(0)$

(D) None of these

34. Existence theorem for initial value problem states that—

(A) If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous for all (x, y) in the rectangle R and bounded $|f| \leq k \left|\frac{\partial f}{\partial y}\right| \leq M$ for all $(x, y) \in R$, then the initial value problem has at least one solution $y(x)$

(B) If $f(x, y)$ continuous at all points (x, y) is some rectangle $R : |x - x_0| < a, |y - y_0| < b$ and bounded in R, $|f(x, y)| \leq k$, for all $(x, y) \in$ R, then the initial value problem has at least one solution $y(x)$

(C) Both (A) and (B) above

(D) None of these

35. If $f(x)$ is three times differentiable and f', f'' are not zero at a solution of $f(x) = 0$, then for x_0 sufficiently close to s—

(A) Newton's method is of first order

(B) Newton's method is of second order*

(C) Newton's method is of third order

(D) None of these

36. Newton-Raphson method states that—

(A) If $f(x) = 0$, where f is assumed to have a continuous derivative

$$f', x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} *$$

(B) If $f(x) = 0$, where f is assumed to have a continuous derivative

$$f', x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)}$$

(C) If $f(x) = 0$, where f is assumed to have a continuous derivative $f', x_{n+1} = \frac{x(x_n)}{f'(x^n)}$

(D) None of these

37. The divided difference are……in all their arguments—

(A) Asymmetrical (B) Symmetrical
(C) Inverse (D) None of these

38. If g is continuous on interval $[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has a fixed point in $[a, b]$. If $g'(x)$ exists on (a, b) and a positive constant $k < 1$ exist with $|g'(x)| \le k$ or $g'(x) \le k$ for all $x \in (a, b)$, then—

(A) The fixed point in $[a, b]$ is unique
(B) The fixed point in $[a, b]$ is not unique
(C) There is no fixed point in $[a, b]$
(D) None of these

39. If g is continuous on interval $[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then—

(A) g has a fixed point in $[a, b]$
(B) g has not a fixed point in $[a, b]$
(C) g has a fixed point in $[a, b]$
(D) None of these

40. A fixed point for a given function g is a number p for which—

(A) $g(p) = p$ (B) $g(p) = 0$
(C) $g(p) \ne p$ (D) None of these

41. Convergence of fixed point iteration states—

(A) Let $x = s$ be a solution of $x = g(x)$ and suppose that g has a continuous derivative in some interval J containing s. Then if $|g'(x)| \le k < 1$ in J, the iteration process defined by $x_{n+1} = g(x_n)$ converges for any x_0 in J

(B) If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous for all (x, y) in the rectangle R and bounded $|f| \le k \left|\frac{\partial f}{\partial y}\right| \le M$ for all $(x, y) \in R$, then the initial value problem has at least one solution $y(x)$

(C) If $f(x, y)$ continuous at all points (x, y) is some rectangle. $R : |x - x_0| < a, |y - y_0| < b$ and bounded in R, $|f(x, y)| \le k$, for all $(x, y) \in R$, then the initial value problem has at least one solution $y(x)$

(D) None of these

42. Fixed point theorem states : Let g is a continuous function on $[a, b]$ such that $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose, in addition, that g' exist on (a, b) and positive constant $k < 1$ exist with $|g'(x)| \le k$ for all $x < (a, b)$. Then for any number p in $[a, b]$, the sequence defined by $p_n = g(p_{n-1})\ n \ge 1$—

(A) Converges to the unique fixed point p in $[a, b]$
(B) Diverges to the unique fixed point p in $[a, b]$
(C) Converges to the different fixed point p in $[a, b]$
(D) Diverges to the different fixed point p in $[a, b]$

43. Everett's formula truncated after second differences is equivalent to……truncated after third differences—

(A) String's formula
(B) Bessel's formula
(C) Everett's formula
(D) None of these

44. $$y_u = v_{y_0} + \frac{v(v^2 - 1^2)}{3!}\Delta^2 y_{-1} + \frac{v(v^2 - 1^2)(v^2 - 2^2)}{5!}\Delta^4 y_{-2} + \ldots\ldots + uy_1 + \frac{u(u^2 - 1^2)}{3!}\Delta^2 y_0 + \frac{u[u^2 - 1^2(u^2 - 2^2)]}{5!}\Delta^4 y_{-1} + \ldots\ldots,$$

where $v = 1 - u$, represents—

(A) String's formula
(B) Bessel's formula
(C) Everett's formula
(D) None of these

45. Shifting the origin in Gauss's backward formula one have—

(A) String formula
(B) Bessel's formula
(C) Everett's formula
(D) None of these

46. $y_u = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2\,!}\Delta^2 y_{-1} + \frac{u\,(u^2-1)}{3\,!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{u^2\,(u^2-1)}{4\,!}\Delta^4 y_{-2} + \ldots\ldots,$

where $u = \frac{x - x_0}{h}$ represents—

(A) String's formula
(B) Bessel's formula
(C) Everett's formula
(D) None of these

47. The.........of Gauss's forward and backward and formula is known as Sterling's formula—

(A) Arithmetic mean
(B) Geometric mean
(C) Harmonic mean
(D) None of these

48. Gauss's backward formula for equal intervals is—

(A) $y_u = y_0 + {}^uC_1\,\Delta y_0 + {}^uC_2\,\Delta^2 y_{-1} + {}^{u+1}C_3\,\Delta^3 y_{-1} + {}^{u+1}c_4\,\Delta^4 y_{-2} + \ldots\ldots$
(B) $y_u = y_0 + {}^uC_1\,\Delta y_{-1} + {}^{u+1}c_2\,\Delta^2 y_{-1} + {}^{u+1}C_3\,\Delta^3 y_{-2} + {}^{u+1}C_4\,\Delta^4 y_{-2} + \ldots\ldots$
(C) $y_u = y_0 + \Delta y_{-1} + \Delta^2 y_{-1} + \Delta^3 y_{-2} + \Delta^4 y_{-2} + \ldots\ldots$
(D) None of these

49. Gauss's forward formula for equal intervals—

(A) $y_u = y_0 + {}^uC_1\,\Delta y_0 + {}^uC_2\,\Delta^2 y_{-1} + {}^{u+1}C_3\,\Delta^3 y_{-1} + {}^{u+1}c_4\,\Delta^4 y_{-2} + \ldots\ldots$
(B) $y_u = y_0 + {}^uC_1\,\Delta y_{-1} + {}^{u+1}c_2\,\Delta^2 y_{-1} + {}^{u+1}C_3\,\Delta^3 y_{-2} + {}^{u+1}C_4\,\Delta^4 y_{-2} + \ldots\ldots$
(C) $y_u = y_0 + \Delta y_{-1} + \Delta^2 y_{-1} + \Delta^3 y_{-2} + \Delta^4 y_{-2} + \ldots\ldots$
(D) None of these

50. Let $\tilde{a}$ an approximation of exact value a, then error bound β is defined as—

(A) $|a - \tilde{a}| = \beta$
(B) $|a - \bar{a}| \geq \beta$
(C) $|a - \tilde{a}| \leq \beta$
(D) None of these

51. $f(x) = f(0) + x\ \nabla f(0) + \frac{x(x+1)}{2!}\nabla^2 f(0) + \ldots\ldots + \frac{x(x+1)\ldots\ldots(x+n-1)}{n\,!}\nabla^n f\ (0)$ represents—

(A) Newton backward difference formula
(B) Newton forward difference formula
(C) Gauss's forward formula
(D) None of these

52. There is at most one polynomial of degree $\leq n$—

(A) Which interpolates, $f(x)$ at $(n+1)$ distinct points $x_0, x_1 \ldots\ldots, x_n$
(B) Which interpolates, $f(x)$ at n distinct points, $x_1 \ldots\ldots x_n$
(C) Which interpolates, $f(x)$ at $(n-1)$ distinct points $x_0,\ x_1, \ldots\ldots, x_{n-2}$
(D) None of these

53. Let h be the finite difference, factorial notation is defined as—

(A) $x^{(n)} = x\,(x-h), (x-2h)\ldots(x-\overline{n-1}\,)\,h$
(B) $x^{(n)} = \frac{x\,!}{(x-n)\,!}$, $(n < x)$
(C) Both (A) and (B)
(D) None of these

54. In divided difference, the value of any difference is......... of the order of their arguments.

(A) Independent
(B) Dependent
(C) Inverse
(D) None of these

55. Which of the following is true for backward difference operator ?

(A) $\nabla^n f(x) = \sum_{r=0}^{n} {}^nC_r\, f(x - rh)$
(B) $\nabla^n f(x) = \sum_{r=0}^{n} (-1)^{n-r} f(x - rh)$
(C) $\nabla^n f(x) = \sum_{r=0}^{n} (-1)^{n-r}\, {}^nC_r\, f(x - rh)$
(D) None of these

56. Which of the following is true for backward difference operator ?

(A) $\nabla^2 f(x) = f(x - 2h) - 2f(x - h) + f(x)$

(B) $\nabla^2 f(x) = f(x-2h) + 2f(x-h) + f(x)$
(C) $\nabla^2 f(x) = f(x-2h) - 2f(x-h) - f(x)$
(D) None of these

57. In interpolation if $x_0, x_1, \ldots\ldots x_n$ are $(n+1)$ distinct value of real valued function $f(x)$, then—
(A) One has a polynomial $p_n(x_i) \approx f(x)$ of degree n or more
(B) One has a polynomial $p_n(x_i) \approx f(x)$ of degree n exactly
(C) One has a polynomial $p_n(x_i) \approx f(x)$ of degree n or less
(D) None of these

58. Interpolation means—
(A) To find exact value of function $f(x)$ for an x between different x values $x_0, x_1, \ldots\ldots x_n$ at which the value $f(x)$ are given
(B) To find approximate value of function $f(x)$ for an x between different x values $x_0, x_1, \ldots\ldots x_n$ at which the value of $f(x)$ are given
(C) To find approximate value of function $f(x)$ for an x outside different x values $x_0, x_1, \ldots\ldots x_n$ at which the value of $f(x)$ are given
(D) To find exact value of function $f(x)$ for an x outside different x values $x_0, x_1, \ldots\ldots x_n$ at which the value of $f(x)$ are given

59. Which of the following is true ?
(A) $\mu = \frac{1}{2}[E^{1/2} + E^{1/2}]$
(B) $\mu = [E^{1/2} + E^{-1/2}]$
(C) $\mu = [E^{1/2} - E^{-1/2}]$
(D) None of these

60. Which of the following is true ?
(A) $\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$
(B) $\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$
(C) $\mu = \frac{1}{2}[E^{1/2}\, E^{-1/2}]$
(D) None of these

61. The central difference operator δ^2 is equal to—
(A) $\Delta\nabla$ (B) $\nabla\Delta$
(C) $\Delta - \nabla$ (D) All of these

62. Which of the following relation is true ?
(A) $\delta^n f(x) = \nabla f\left(x + \frac{1}{2}nh\right)$
(B) $\delta^n f(x) = \nabla^n f\left(x - \frac{1}{2}nh\right)$
(C) $\delta^n f(x) = \nabla^n f\left(x + \frac{1}{2}nh\right)$
(D) None of these

63. Which of the following relation is true ?
(A) $\delta^n f(x) = \Delta^n f\left(x + \frac{1}{2}nh\right)$
(B) $\delta^n f(x) = \Delta f\left(x + \frac{1}{2}nh\right)$
(C) $\delta^n f(x) = \Delta^n f\left(x - \frac{1}{2}nh\right)$
(D) None of these

64. Relation between central forward and shift operator is—
(A) $\delta \equiv \nabla E^{1/2} \equiv E^{1/2}\,\nabla$
(B) $\delta \equiv \nabla - E^{1/2} \equiv E^{1/2} - \nabla$
(C) $\delta \equiv \nabla + E^{1/2} \equiv E^{1/2} + \nabla$
(D) None of these

65. Relation between central forward and shift operator is—
(A) $\delta \equiv \Delta\, E^{-1/2} \equiv E^{-1/2}\,\Delta$
(B) $\delta \equiv \Delta - E^{-1/2} \equiv E^{-1/2} - \Delta$
(C) $\delta \equiv \Delta + E^{-1/2} = \equiv E^{-1/2} + \Delta$
(D) None of these

66. Central difference equivalent to shift operator—
(A) $\delta \equiv E^{1/2} - E^{-1/2}$ (B) $\delta \equiv E^{1/2} + E^{-1/2}$
(C) $\delta \equiv E^{1/2}\, E^{-1/2}$ (D) None of these

67. Which of the following relation is true ?
(A) $E \equiv \nabla^{-1}$ (B) $E \equiv (1+\nabla)^{-1}$
(C) $E \equiv (1-\nabla)^{-1}$ (D) None of these

68. Weierstrass approximation theorem states : for any continuous function $f(x)$ on an interval $J : a \le x \le b$ and error bound $\beta > 0$, there is a polynomial $p_n(x)$ (of sufficiently high degree) such that—

(A) $|f(x) - p_n(x)| = \beta$ for all $x \in J$
(B) $|f(x) - p_n(x)| > \beta$ for all $x \in J$
(C) $|f(x) - p_n(x)| < \beta$ for all $x \in J$
(D) None of these

69. Which of the following relation is true ?
(A) $E^{-1} \equiv 1 + \nabla$ (B) $E^{-1} \equiv \nabla$
(C) $E^{-1} \equiv 1 - \nabla$ (D) None of these

70. If $f(x)$ is a polynomial of degree n in x, then $(n + 1)$ and higher difference, this polynomial are—
(A) Constant (B) Variable
(C) Zero (D) None of these

71. If $f(x)$ is a polynomial of degree n in x, then nth difference of this polynomial is—
(A) Constant (B) Variable
(C) Zero (D) None of these

72. Which of the following relation is true ?
(A) $E^n f(x) = f(x + nh) \equiv \sum_{r=0}^{n} {}^{n-r}C_r \Delta^r f(x)$
(B) $E^n f(x) = f(x + nh) \equiv \sum_{r=0}^{n} {}^{n}C_r \Delta f(x)$
(C) $E^n f(x) = f(x + nh) \equiv \sum_{r=0}^{n} {}^{n}C_r \Delta^r f(x)$
(D) None of these

73. Which of the following relation is true ?
(A) $\Delta \equiv E - 1$
(B) $E^n \equiv (1 + \Delta)^n$
(C) Both (A) and (B)
(D) None of these

74. Relation between ∇, Δ and E is—
(A) $\nabla + E \equiv E + \nabla \equiv \Delta + E$
(B) $\nabla/E \equiv E/\nabla \equiv E/\Delta$
(C) $\nabla E \equiv E\nabla \equiv \Delta$
(D) None of these

75. Which of the following relation is true ?
(A) $E = 1 + \Delta$
(B) $\Delta^n \equiv (E - 1)^n$
(C) Both (A) and (B)
(D) None of these

76. The Averaging operator μ is—
(A) $\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{1}{2}h\right)\right]$
(B) $\mu f(x) = \frac{1}{2}\left[f\left(x - \frac{1}{2}h\right)\right]$
(C) $\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{1}{2}h\right) + f\left(x - \frac{1}{2}h\right)\right]$
(D) None of these

77. The central difference operator's δ is equivalent to—
(A) $\delta^n f(x) = \Delta^n f(xh)$
(B) $\delta^n f(x) = \Delta^n f\left(x + \frac{1}{2}nh\right)$
(C) $\delta^n f(x) = \Delta^n f\left(x - \frac{1}{2}nh\right)$
(D) None of these

78. The central difference operator δ is defined as—
(A) $\delta f(x) = f\left(x + \frac{1}{2}h\right) - f\left(x - \frac{1}{2}h\right)$
(B) $\delta f(x) = f\left(x + \frac{1}{2}h\right)$
(C) $\delta f(x) = f\left(x - \frac{1}{2}h\right)$
(D) None of these

79. Backward differences is defined as—
(A) $\nabla f(x) = -f(x - h)$
(B) $\nabla f(x) = f(x) + f(x - h)$
(C) $\nabla f(x) = f(x) - f(x - h)$
(D) None of these

80. Let h be the finite difference, then which of the following is true for shift operator ?
(A) $E^n f(x) = f(x)$
(B) $E^n f(x) = f(x + nh)$
(C) $E^n f(x) = f(x - nh)$
(D) None of these

81. Let h be the finite difference, then sift operator is defined as—
(A) $E f(x) = f(x)$
(B) $E f(x) = f(x - h)$
(C) $E f(x) = f(x + h)$
(D) None of these

82. Let h be the finite difference, then which of the following is true for forward difference operator ?
(A) $\Delta^n f(x) = \sum_{r=0}^{n} (-1)^{n-r} \, {}^{n}C_r f(x + rh)$

(B) $\Delta^n f(x) = \sum_{r=0}^{n} {}^nC_r f(x+rh)$

(C) $\Delta^n f(x) = \sum_{r=0}^{n} (-1)^{n-r} f(x+rh)$

(D) None of these

83. Let h be the finite difference, then which of the following is true for forward difference operator ?

(A) $\Delta^3 f(x) = f(x+3h) + 3f(x+2h) + 2f(x+h) + f(x)$

(B) $\Delta^3 f(x) = f(x+3h) - 3f(x+2h) + 2f(x+h) - f(x)$

(C) $\Delta^3 f(x) = f(x+3h) - 3f(x+2h) - 2f(x+h) - f(x)$

(D) None of these

84. Let h be the finite difference, then which of the following is true for forward difference operator ?

(A) $\Delta^2 f(x) = f(x+2h) + f(x)$

(B) $\Delta^2 f(x) = f(x+2h) + 2f(x+h) + f(x)$

(C) $\Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$

(D) None of these

85. Let h be the finite difference, then f forward difference operator is defined as—

(A) $\Delta f(x) = f(x+h)$

(B) $\Delta f(x) = f(x+h) + f(x)$

(C) $\Delta f(x) = f(x+h) - f(x)$

(D) None of these

Answers

1. (B)	2. (B)	3. (C)	4. (B)	5. (B)
6. (D)	7. (A)	8. (B)	9. (D)	10. (C)
11. (B)	12. (A)	13. (A)	14. (B)	15. (C)
16. (B)	17. (A)	18. (A)	19. (A)	20. (A)
21. (A)	22. (A)	23. (C)	24. (A)	25. (A)
26. (C)	27. (A)	28. (A)	29. (C)	30. (B)
31. (B)	32. (A)	33. (C)	34. (A)	35. (B)
36. (A)	37. (B)	38. (A)	39. (A)	40. (A)
41. (A)	42. (A)	43. (B)	44. (C)	45. (B)
46. (A)	47. (A)	48. (B)	49. (A)	50. (C)
51. (A)	52. (A)	53. (C)	54. (A)	55. (C)
56. (A)	57. (C)	58. (B)	59. (A)	60. (A)
61. (D)	62. (C)	63. (C)	64. (A)	65. (A)
66. (A)	67. (C)	68. (C)	69. (C)	70. (C)
71. (A)	72. (C)	73. (C)	74. (C)	75. (C)
76. (C)	77. (C)	78. (A)	79. (C)	80. (C)
81. (C)	82. (A)	83. (B)	84. (C)	85. (C)