UNIT –V

LAPLACE TRANSFORM

Def. Exponential order

A function $f(t)$ is said to be of exponential order if

$$\underset{t \to \infty}{\text{Lt}}\ e^{-st} f(t) = 0$$

Example 1 **Show that x^n is of exponential order as $x \to \infty$, $n > 0$.**

Solution :

$$\underset{x \to \infty}{\text{Lt}}\ e^{-ax} x^n = \underset{x \to \infty}{\text{Lt}}\ \frac{x^n}{e^{ax}} \left[\frac{\infty}{\infty} \text{ i.e., Indeterminant form}\right]$$

$$= \underset{x \to \infty}{\text{Lt}}\ \frac{n x^{n-1}}{a e^{ax}} \left[\frac{\infty}{\infty} \text{ i.e., Indeterminant form}\right]$$

[Apply L' Hospital Rule]

$$= \underset{x \to \infty}{\text{Lt}}\ \frac{n(n-1)\ldots 1}{a^n e^{ax}} \text{ [Repeating this process we get]}$$

$$= \underset{x \to \infty}{\text{Lt}}\ \frac{n!}{a^n e^{ax}} \text{ [Applying L'Hospital's rule]}$$

$$= \frac{n!}{\infty} = 0$$

Hence x^n is of exponential order.

Example **Show that t^2 is of exponential order.**

Solution : $$\underset{t \to \infty}{\text{Lt}}\ e^{-st} t^2 = \underset{t \to \infty}{\text{Lt}}\ \frac{t^2}{e^{st}} \left[\frac{\infty}{\infty} \text{ i.e., Indeterminant form}\right]$$

[Apply L'Hospital's rule]

$$= \underset{t \to \infty}{\text{Lt}}\ \frac{2t}{se^{st}} \left[\frac{\infty}{\infty} \text{ form}\right] \text{[Apply L'Hospital's Rule]}$$

$$= \underset{t \to \infty}{\text{Lt}}\ \frac{2}{s^2 e^{st}} = \frac{2}{\infty}$$

$$= 0$$

Hence t^2 is of exponential order.

Example **Show that the function**

$f(t) = e^{t^2}$ is not of exponential order.

Solution : $$\underset{t \to \infty}{\text{Lt}}\ e^{-st} e^{t^2} = \underset{t \to \infty}{\text{Lt}}\ e^{-st + t^2}$$

$$= e^{\infty} = \infty$$

So $f(t) = e^{t^2}$ is not of exponential order.

Define function of class A.

Solution : A function which is sectionally continuous over any finite interval and is of exponential order is known as a function of class A.

◆ **Important Result**

(1) $L[1] = \frac{1}{s}$ where $s > 0$

(2) $L[t^n] = \frac{n!}{s^{n+1}}$ where $n = 0, 1, 2, ...$

(3) $L[t^n] = \frac{\Gamma n+1}{s^{n+1}}$ where n is not a integer.

(4) $L[e^{at}] = \frac{1}{s-a}$ where $s > a$ or $s - a > 0$

(5) $L[e^{-at}] = \frac{1}{s+a}$ where $s + a > 0$

(6) $L[\sin at] = \frac{a}{s^2+a^2}$ where $s > 0$

(7) $L[\cos at] = \frac{s}{s^2+a^2}$ where $s > 0$

(8) $L[\sinh at] = \frac{a}{s^2-a^2}$ where $s > |a|$ or $s^2 > a^2$

(9) $L[\cosh at] = \frac{s}{s^2-a^2}$ where $s^2 > a^2$

(10) $L[af(t) \pm bg(t)] = a\, L[f(t)] \pm b\, L[g(t)]$ [Linearity property]

Note : (1) $e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + ...$

$e^\infty = 1 + \frac{\infty}{\lfloor 1} + \frac{\infty^2}{\lfloor 2} + ...$

(2) $e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$

(3) $\Gamma_{n+1} = n!$

(4) $\Gamma_{n+1} = \int_0^\infty x^n e^{-x}\, dx$

(5) $\Gamma_{n+1} = n\,\Gamma_n$

(6) $\Gamma_{1/2} = \sqrt{\pi}$

(7) $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx]$

(8) $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx]$

(9) $\sin^3 \theta = \frac{1}{4}[3 \sin \theta - \sin 3\theta]$

(10) $\cos^3 \theta = \frac{1}{4}[\cos 3\theta + 3 \cos \theta]$

(11) $\sin A \cos B = \frac{1}{2} [\sin (A + B) + \sin (A - B)]$

(12) $\cos A \sin B = \frac{1}{2} [\sin (A + B) - \sin (A - B)]$

(13) $\cos A \cos B = \frac{1}{2} [\cos (A + B) + \cos (A - B)]$

(14) $\sin A \sin B = -\frac{1}{2} [\cos (A + B) - \cos (A - B)]$

5.2 TRANSFORMS OF ELEMENTARY FUNCTIONS - BASIC PROPERTIES

Result (1) : Prove that $L[1] = \frac{1}{s}$ where $s > 0$

Proof : We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

Here $f(t) = 1$

$$\therefore L[1] = \int_0^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s}\left[e^{-st}\right]_0^\infty = -\frac{1}{s}\left[e^{-\infty} - e^{-0}\right]$$

$$= -\frac{1}{s}[0 - 1] \text{ by note (2)}$$

$$= \frac{1}{s},\ s > 0$$

Result (2) : Prove that $L[t^n] = \frac{n!}{s^{n+1}}$ [n = 0, 1, 2, ...]

Proof : We know that

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[t^n] = \int_0^\infty e^{-st} t^n\,dt = \int_0^\infty t^n\, d\left[\frac{e^{-st}}{-s}\right]$$

$$= t^n \left(\frac{e^{-st}}{-s}\right)\Bigg]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, n\, t^{n-1}\,dt$$

$$= (0-0) + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1}\,dt$$

i.e., $$L[t^n] = \frac{n}{s} L[t^{n-1}]$$

Similarly $$L[t^{n-1}] = \frac{n-1}{s} L[t^{n-2}]$$

$$L[t^{n-2}] = \frac{n-2}{s} L[t^{n-3}]$$

....

....

$$L[t^{n-(n-1)}] = \frac{n-(n-1)}{s} L\left[t^{[n-(n-1)]-1}\right]$$

$$= \frac{1}{s} L\left[t^{o}\right] = \frac{1}{s} L[1] = \frac{1}{s}\frac{1}{s}$$

$$\therefore L[t^n] = \frac{n}{s}\frac{n-1}{s}\cdots\frac{2}{s}\frac{1}{s}\frac{1}{s} = \frac{n!}{s^n}\frac{1}{s}$$

$$= \frac{n!}{s^{n+1}} \text{ where } [n = 0, 1, 2, \ldots]$$

Result (3) Prove that $L[t^n] = \frac{\Gamma_{n+1}}{s^{n+1}}$ where n is not a integer.

Proof : We know that $$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[t^n] = \int_0^\infty e^{-st} t^n\,dt$$

Put $st = x$ as $t \to 0 \Rightarrow x \to 0$

$s\,dt = dx$ as $t \to \infty \Rightarrow x \to \infty$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \int_0^\infty e^{-x} \frac{x^n}{s^{n+1}}\, dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty x^n e^{-x}\, dx$$

i.e., $L[t^n] = \dfrac{\Gamma_{n+1}}{s^{n+1}}$ $\left[\because \int_0^\infty x^n e^{-x}\, dx = \Gamma_{n+1}\right]$

when n is a positive integer.

we get $\Gamma_{n+1} = n\,!$

$$L\left[t^n\right] = \frac{n\,!}{s^{n+1}}$$

II. PROBLEMS BASED ON TRANSFORMS OF ELEMENTARY FUNCTIONS - BASIC PROPERTIES

Example 1 Find L[t]

Solution : $L[t^n] = \dfrac{n\,!}{s^{n+1}}$ [we know that]

$$L[t] = \frac{1\,!}{s^{1+1}} = \frac{1}{s^2}$$

Example 2 Find $L[t^3]$

Solution : We know that $L[t^n] = \dfrac{n\,!}{s^{n+1}}$

$$L[t^3] = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$$

Example 3 Find $L[\sqrt{t}]$

Solution : We know that $L[t^n] = \dfrac{\Gamma_{n+1}}{s^{n+1}}$

$$L[\sqrt{t}] = L[t^{1/2}] = \frac{\Gamma_{1/2+1}}{s^{1/2+1}}$$

$$= \frac{\frac{1}{2}\Gamma_{1/2}}{s^{3/2}} \qquad [\because \Gamma_{n+1} = n\,\Gamma_n \;;\; \Gamma_{1/2} = \sqrt{\pi}]$$

$$= \frac{\Gamma_{1/2}}{2s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

Example 4. Find L $[t^{3/2}]$

Solution :

We know that $L[t^n] = \frac{\Gamma_{n+1}}{s^{n+1}}$

$$L\left[t^{3/2}\right] = \frac{\Gamma_{3/2+1}}{s^{3/2}+1} = \frac{\frac{3}{2}\Gamma_{3/2}}{s^{5/2}}$$

$$= \frac{\frac{3}{2}\Gamma_{1/2+1}}{s^{5/2}} = \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma_{1/2}}{s^{5/2}}$$

$$= \frac{\left(\frac{3}{4}\right)\sqrt{\pi}}{s^{5/2}} \qquad [\because \Gamma_{1/2} = \sqrt{\pi}]$$

$$= \frac{3\sqrt{\pi}}{4s^{5/2}}$$

Example 5.2.5. Find L $\left[\frac{1}{\sqrt{t}}\right]$

Solution : We know that $L[t^n] = \frac{\Gamma_{n+1}}{s^{n+1}}$

$$L\left[\frac{1}{\sqrt{t}}\right] = L\left[t^{-1/2}\right] = \frac{\Gamma_{-1/2+1}}{s^{-1/2+1}}$$

$$= \frac{\Gamma_{1/2}}{s^{1/2}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}} \qquad [\because \Gamma_{1/2} = \sqrt{\pi}]$$

Result 4. Prove that L $[e^{at}] = \frac{1}{s-a}$ where s > a.

Proof : We know that

$$L[f(t)] = \int_C^{\infty} e^{-st} f(t)\,dt$$

$$L\left[e^{at}\right] = \int_0^\infty e^{-st}\ e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = -\frac{1}{s-a}\left[e^{-(s-a)t}\right]_0^\infty$$

$$= \frac{-1}{s-a}[0 - 1] = \frac{1}{s-a} \text{ where } s-a>0$$

Example 6. Find the value $L\left[e^{3t}\right]$

Solution : We know that

$$L[e^{at}] = \frac{1}{s-a}$$

$$L[e^{3t}] = \frac{1}{s-3}$$

Example 7 Find $L\,[e^{3t+5}]$

Solution :

W.K.T $\quad L[e^{at}] = \dfrac{1}{s-a}$

$$L[e^{3t+5}] = L[e^{3t}\ e^{5}]$$

$$= e^5\ L\ [e^{3t}] = e^5\left[\frac{1}{s-3}\right] = \frac{e^5}{s-3}$$

Example 8 Find $L\left[\dfrac{e^{at}}{a}\right]$

Solution : W.K.T $\quad L[e^{at}] = \dfrac{1}{s-a}$

$$L\left[\frac{e^{at}}{a}\right] = \frac{1}{a}L\,[e^{at}] = \frac{1}{a}\left[\frac{1}{s-a}\right]$$

Example 9 Find $L\,[2^t]$

Solution W.K.T. $\quad L[e^{at}] = \dfrac{1}{s-a}$

$$L[2^t] = L\left[e^{\log 2^t}\right]$$

$$= L\left[e^{t\log 2}\right]$$

$$= L\left[e^{(\log 2)\,t}\right]$$

$$= \frac{1}{s-\log 2}$$

Result 5. Prove that $\mathbf{L\,[e^{-at}] = \dfrac{1}{s+a}}$, $(s+a) > 0$

Proof : W.K.T. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[e^{-at}] = \int_0^\infty e^{-st} e^{-at}\, dt$$

$$= \int_0^\infty e^{-(s+a)t}\, dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = -\frac{1}{s+a}\left[e^{-(s+a)t}\right]_0^\infty$$

$$= -\frac{1}{s+a}[0 - 1]$$

$$= \frac{1}{s+a} \text{ where } (s+a) > 0$$

Example 10. Find $\mathbf{L\,[e^{-bt}]}$

Solution : W.K.T $L[e^{-at}] = \dfrac{1}{s+a}$

$$L[e^{-bt}] = \frac{1}{s+b}$$

Example 11. Find $\mathbf{L\,[2\,e^{-3t}]}$

Solution : W.K.T. $L[e^{-at}] = \dfrac{1}{s+a}$

$$L[2\,e^{-3t}] = 2\,L\,[e^{-3t}]$$

$$= 2\left[\frac{1}{s+3}\right] = \left[\frac{2}{s+3}\right]$$

Result 6. Prove that $\mathbf{L\,[\sin at] = \dfrac{a}{s^2+a^2}}$ $(s > 0)$

Proof : W.K.T. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$\int e^{ax}\, b\,(x)\, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]$$

$$L\,[\sin at] = \int_0^\infty e^{-st} \sin at\, dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}[-s \sin at - a \cos at]\right]_0^\infty \quad \text{by Note 7.}$$

$$= 0 - \left[\frac{(-a)}{s^2 + a^2}\right] = \frac{a}{s^2 + a^2} \text{ where } s > 0.$$

Example 5.2.12. Find L [sin 2t]

Solution : W.K.T $L[\sin at] = \dfrac{a}{a^2 + a^2}$

$$L[\sin 2t] = \frac{2}{s^2 + 2^2}$$

$$= \frac{2}{s^2 + 4}$$

Example 5.2.13. Find L [sin π t]

Solution : W.K.T $L[\sin at] = \dfrac{a}{s^2 + a^2}$

$$L[\sin \pi t] = \frac{\pi}{s^2 + \pi^2}$$

Result : 7. Prove that $L[\cos at] = \dfrac{s}{s^2 + a^2}$ $(s > 0)$

Proof : W.K.T. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[\cos at] = \int_0^\infty e^{-st} \cos at\, dt$$

$$= \left[\frac{e^{-st}}{s^2 + a^2}[-s \cos at + a \sin at]\right]_0^\infty$$

$$= 0 - \left[\frac{1}{s^2 + a^2}(-s)\right]$$

$$= \frac{s}{s^2 + a^2} \quad (s > 0)$$

Example 5.2.14. Find L [cos 2t]

Solution : W.K.T. $L[\cos at] = \dfrac{s}{s^2 + a^2}$

$$L[\cos 2t] = \frac{s}{s^2 + 4}$$

Example 15 Prove that L [cos at] = $\frac{s}{s^2+a^2}$ and L [sin at] = $\frac{a}{s^2+a^2}$

Solution : By Euler's theorem

$$e^{ix} = \cos x + i\sin x$$

$$e^{iat} = \cos at + i\sin at$$

$$L[e^{iat}] = L[\cos at + i\sin at]$$

$$= L[\cos at] + i\,L[\sin at]$$

$$L[\cos at] + i\,L[\sin at] = L[e^{iat}]$$

$$= \frac{1}{s-ia}$$

$$= \left[\frac{1}{s-ia}\right]\left[\frac{s+ia}{s+ia}\right]$$

$$= \frac{s+ia}{s^2+a^2}$$

Equating real & Imaginary parts we get

$$L[\cos at] = \frac{s}{s^2+a^2}$$

$$L[\sin at] = \frac{a}{s^2+a^2}$$

Example 16 Find L [cos (at + b)]

Solution : $L[\cos(at+b)]$

$$= L[\cos at\cos b - \sin at\sin b]$$

$$= \cos b\, L[\cos at] - \sin b\, L[\sin at]$$

$$= \cos b\left[\frac{s}{s^2+a^2}\right] - \sin b\left[\frac{a}{s^2+a^2}\right]$$

$$= \frac{s\cos b - a\sin b}{s^2+a^2}$$

Example 17 Find L [sin² 2t]

Solution : $L[\sin^2 2t] = L\left[\frac{1-\cos 4t}{2}\right] = \frac{1}{2}L[1-\cos 4t]$

$$= \frac{1}{2}\left[L[1] - L[\cos 4t]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$$

Example 18 Find L [sin 5t cos 2t]

Solution : $L[\sin 5t \cos 2t] = \frac{1}{2} L[\sin 7t + \sin 3t]$ by Note 11.

$$= \frac{1}{2}\left[\frac{7}{s^2+49} + \frac{3}{s^2+9}\right]$$

Example 19 Find L[(sin t − cos t)²]

Solution : $L[(\sin t - \cos t)^2] = L[\sin^2 t + \cos^2 t - 2\sin t \cos t]$

$$= L[1 - \sin 2t] = L[1] - L[\sin 2t]$$

$$= \frac{1}{s} - \frac{2}{s^2+4}$$

Result 8. Prove that $L[\sinh at] = \frac{a}{s^2 - a^2}$ **where** $s > |a|$

Proof : $\sinh at = \frac{e^{at} - e^{-at}}{2}$

$$L[\sinh at] = L\left[\frac{e^{at} - e^{-at}}{2}\right]$$

$$= \frac{1}{2} L[e^{at} - e^{-at}] = \frac{1}{2}\left[L[e^{at}] - L[e^{-at}]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right] = \frac{a}{s^2-a^2}, \; s > |a|$$

Result 9. Prove that $L[\cosh at] = \frac{s}{s^2 - a^2}$, $s > |a|$

Proof : $\cosh at = \frac{e^{at} + e^{-at}}{2}$

$$L[\cosh at] = L\left[\frac{e^{at} + e^{-at}}{2}\right]$$

$$= \frac{1}{2} L[e^{at} + e^{-at}] = \frac{1}{2}\left[L[e^{at}] + L[e^{-at}]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right] = \frac{s}{s^2-a^2}, \; s > |a|$$

Result 10. Linearity property.

Prove that $L[a f(t) \pm b g(t)] = a L[f(t)] \pm b L[g(t)]$

Proof : W.K.T. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[a f(t) \pm b g(t)] = \int_0^\infty e^{-st} [a f(t) \pm b g(t)]\, dt$$

$$= \int_0^\infty e^{-st} a f(t)\, dt \pm \int_0^\infty e^{-st} b g(t)\, dt$$

$$= a \int_0^\infty e^{-st} f(t)\, dt \pm b \int_0^\infty e^{-st} g(t)\, dt$$

$$= a L[f(t)] \pm b L[g(t)]$$

Example $L[e^{4t} + t^4 + 7]$

Solution : $L[e^{4t} + t^4 + 7]$

$$= L[e^{4t}] + L[t^4] + L[7]$$

$$= \frac{1}{s-4} + \frac{4!}{s^5} + 7 L[1]$$

$$\frac{1}{s-4} + \frac{24}{s^5} + 7\left[\frac{1}{s}\right]$$

Example 5.2.26. Find $L[f(t)]$ **if** $f(t) = \begin{cases} e^{-t}, & 0 < t < 4 \\ 0, & t > 4 \end{cases}$

Solution : W.K.T. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$= \int_0^4 e^{-st} e^{-t}\, dt + \int_4^\infty e^{-st}\, 0\, dt$$

$$= \int_0^4 e^{-(s+1)t}\, dt + 0$$

$$= \left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_0^4 = \frac{-1}{(s+1)} \left[e^{-(s+1)t}\right]_0^4$$

$$= \frac{-1}{s+1}\left[e^{-4(s+1)} - 1\right] = \frac{1}{s+1}[1 - e^{-4(s+1)}]$$

Result 11. Prove that L [f′ (t)] = s L [f (t)] − f (0)

Proof : W.K.T. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt$$

$$= \int_0^\infty e^{-st}\, d[f(t)]$$

$$= e^{-st} f(t)\Big]_0^\infty - \int_0^\infty f(t)(-s)\, e^{-st}\, dt$$

$$= [0 - f(0)] + s \int_0^\infty e^{-st} f(t)\, dt$$

$$= -f(0) + s\, L[f(t)]$$

$$= s\, L[f(t)] - f(0)$$

Result 12. Prove that L [f′′ (t)] = s^2 L [f (t)] − s f (0) − f ′ (0)

Proof : W.K.T. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[f''(t)] = \int_0^\infty e^{-st} f''(t)\, dt$$

$$= \int_0^\infty e^{-st}\, d[f'(t)]$$

$$= e^{-st} f'(t)\Big]_0^\infty - \int_0^\infty f'(t)(-s)\, e^{-st}\, dt$$

$$= [0 - f'(0)] + s\int_0^{\infty} e^{-st} f'(t)\, dt$$

$$= -f'(0) + s\, L[f'(t)]$$

$$= -f'(0) + s\,[sL[f(t)] - f(0)] \text{ by result } (11)$$

$$= s^2 L[f(t)] - s f(0) - f'(0)$$

Note : (15)

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{n-1}(0)$$

Result : 13. FIRST SHIFTING THEOREM

- If $L[f(t)] = \varphi(s)$ then $L[e^{at} f(t)] = \varphi(s-a)$
- If $L[f(t)] = \varphi(s)$ then $L[e^{-at} f(t)] = \varphi(s+a)$

Proof : W.K.T $L[f(t)] = \varphi(s) = \int_0^{\infty} e^{-st} f(t)\, dt$

$$L[e^{at} f(t)] = \int_0^{\infty} e^{-st} e^{at} f(t)\, dt$$

$$= \int_0^{\infty} e^{-(s-a)t} f(t)\, dt$$

$$= \varphi(s-a)$$

$$L[e^{-at} f(t)] = \int_0^{\infty} e^{-st} e^{-at} f(t)\, dt$$

$$= \int_0^{\infty} e^{-(s+a)t} f(t)\, dt$$

$$= \int_0^{\infty} e^{-(s+a)t} f(t)\, dt$$

$$= \varphi(s+a)$$

III. PROBLEMS BASED ON FIRST SHIFTING THEOREM AND SECOND SHIFTING THEOREM

Example **Find $L\,[t^n e^{-at}]$**

Solution : $$L[t^n e^{-at}] = \left[L\,(t^n)\right]_{s \to (s+a)}$$

$$= \left[\frac{n\,!}{s^{n+1}}\right]_{s \to (s+a)}$$

$$= \frac{n\,!}{(s+a)^{n+1}}$$

Example **Find $L\,[e^{-at} \cos bt]$**

Solution : $$L[e^{-at} \cos bt] = \left[L\,[\cos bt]\right]_{s \to (s+a)}$$

$$= \left[\frac{s}{s^2 + b^2}\right]_{s \to (s+a)}$$

$$= \frac{s+a}{(s+a)^2 + b^2}$$

Example **Find $L\,[e^{at} \sinh bt]$**

Solution : $$L[e^{at} \sinh bt] = \left[L\,[\sinh bt]\right]_{s \to (s-a)}$$

$$= \left[\frac{b}{s^2 - b^2}\right]_{s \to (s-a)} = \frac{b}{(s-a)^2 - b^2}$$

Example **Find $L\left[e^t\, t^{-1/2}\right]$**

Solution : $$L\left[e^t\, t^{-1/2}\right] = \left[L\,[t^{-1/2}]\right]_{s \to (s-1)}$$

$$= \left[\frac{\Gamma_{-1/2+1}}{s^{-1/2+1}}\right]_{s \to (s-1)} = \left[\frac{\Gamma_{1/2}}{s^{1/2}}\right]_{s \to (s-1)}$$

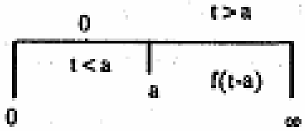
$$= \left[\frac{\sqrt{\pi}}{\sqrt{s}}\right]_{s \to (s-1)} = \left[\sqrt{\frac{\pi}{s}}\right]_{s \to (s-1)}$$

$$\sqrt{\pi}$$

Result 14. Second shifting theorem.

- If $L[f(t)] = \varphi(s)$ and $G(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

  then $L[G(t)] = e^{-as}\,\varphi(s)$



Proof : $L[G(t)] = \int_0^\infty e^{-st}\, G(t)\, dt$

$$= \int_0^a e^{-st}\, 0\, dt + \int_a^\infty e^{-st} f(t-a)\, dt$$

$$= \int_a^\infty e^{-st} f(t-a)\, dt$$

Put $t - a = u \qquad t \to a \Rightarrow u \to 0$

$\quad dt = du \qquad t \to \infty \Rightarrow u \to \infty$

$$= \int_0^\infty e^{-s(u+a)} f(u)\, du$$

$$= e^{-sa} \int_0^\infty e^{-su} f(u)\, du$$

$$= e^{-sa} \int_0^\infty e^{-st} f(t)\, dt \quad [\because u \text{ is a dummy variable}]$$

$$= e^{-sa}\, L[f(t)]$$

$$= e^{-sa}\, \varphi(s)$$

Result : 15. If $L[F(t)] = \varphi(s)$ and $C > 0$ then

$L[F(t-c)\, H(t-c)] = e^{-cs}\,\varphi(s)$ where $H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$

Proof : $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[F(t-c)\, H(t-c) = \int_0^\infty e^{-st}\, F(t-c)\, H(t-c)\, dt$$

DERIVATIVES AND INTEGRALS OF TRANSFORMS - TRANSFORMS OF DERIVATIVES AND INTEGRALS

Result : 17. Transforms of Derivatives

If $L[f(t)] = \varphi(s)$ **then** $L[t f(t)] = -\frac{d}{ds}\varphi(s) = -\varphi'(s)$

Proof :

$$\varphi(s) = L[f(t)]$$

$$\frac{d}{ds}\varphi(s) = \frac{d}{ds}L[f(t)]$$

$$\varphi'(s) = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)\,dt\right] = \int_0^\infty \frac{\partial}{\partial s}(e^{-st}) f(t)\,dt$$

$$= \int_0^\infty e^{-st}(-t) f(t)\,dt = -\int_0^\infty e^{-st}\, t f(t)\,dt$$

$$= -L[t f(t)]$$

Put $t - c = u$ $\qquad$ $t \to 0 \Rightarrow u \to -c$

$dt = du$ $\qquad$ $t \to \infty \Rightarrow u \to \infty$

$$= \int_{-c}^{\infty} e^{-s(u+c)}\, F(u)\, H(u)\, du$$

$$= e^{-sc}\int_{-c}^{\infty} e^{-su}\, F(u)\, H(u)\, du$$

$$= e^{-sc}\left[\int_{-c}^{0} e^{-su}\, F(u)\, 0\, du + \int_0^\infty e^{-su}\, F(u)\, du\right]$$

$$= e^{-sc}\int_0^\infty e^{-su}\, F(u)\, du$$

$$= e^{-sc}\int_0^\infty e^{-st}\, F(t)\, dt \quad [\because u \text{ is a dummy variable}]$$

$$= e^{-sc} L[F(t)] = e^{-sc}\varphi(s)$$

$$L[t f(t)] = -\varphi'(s)$$

Corollary :- If $L[f(t)] = \varphi(s)$ then $L[t^n f(t)] = (-1)^n \varphi^n(s)$.

Proof : W.K.T. $L[t f(t)] = -\varphi'(s)$

$$L[t^2 f(t)] = L[t \,.\, t f(t)]$$

$$= -\frac{d}{ds} L[t f(t)]$$

$$= -\frac{d}{ds}\left[\frac{-d}{ds} L f(t)\right]$$

$$= (-1)^2 \frac{d^2}{ds^2}[L f(t)]$$

$$= (-1)^2 \frac{d^2}{ds^2} \varphi(s)$$

....

....

....

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} \varphi(s) = (-1)^n \varphi^n(s)$$

PROBLEMS BASED ON TRANSFORMS OF DERIVATIVES

Example 1. Find L [t sin 2t]

Solution : W.K.T. $L[t^n f(t)] = (-1)^n \varphi^n(s)$

$$L(t \sin 2t) = -\frac{d}{ds}[L(\sin 2t)] = -\frac{d}{ds}\left[\frac{2}{s^2+4}\right]$$

$$= -\left[\frac{-4s}{(s^2+4)^2}\right] = \frac{4s}{(s^2+4)^2}$$

Example 2. Find L $[t^2 e^{-3t}]$

Solution : W.K.T $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\varphi(s)]$

$$L[t^2 e^{-3t}] = (-1)^2 \frac{d^2}{ds^2} L[e^{-3t}] = \frac{d^2}{ds^2}\left[\frac{1}{s+3}\right]$$

$$= \frac{d}{ds}\left[\frac{-1}{(s+3)^2}\right] = \frac{2}{(s+3)^3}$$

Example 3. Find $L[te^{-2t} \sin t]$

Solution : $L[t e^{-2t} \sin t] = -\frac{d}{ds}[L(e^{-2t} \sin t)]$

$$= -\frac{d}{ds}\left[[L[\sin t]]_{s \to (s+2)}\right] = -\frac{d}{ds}\left[\left[\frac{1}{s^2+1}\right]_{s \to (s+2)}\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{(s+2)^2+1}\right] = \frac{2(s+2)}{[(s+2)^2+1]^2}$$

Example Find $L[t \sin 3t \cos 2t]$

Solution : $L[t \sin 3t \cos 2t] = -\frac{d}{ds}[L(\sin 3t \cos 2t)]$

$$= -\frac{d}{ds}\left[\frac{1}{2}[L(\sin 5t) + L(\sin t)]\right] = -\frac{1}{2}\frac{d}{ds}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$$

$$= \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}$$

Example 5. Given that $L[\sin \sqrt{t}] = \frac{1}{2s}\sqrt{\frac{\pi}{s}}\, e^{-1/4s}$ find L.T. of $\frac{1}{\sqrt{t}} \cos \sqrt{t}$

Solution : Let $f(t) = \sin \sqrt{t}$

$$f'(t) = \frac{1}{2\sqrt{t}} \cos \sqrt{t}$$

$$L[f'(t)] = s\,L[f(t)] - f(0)$$

$$L\left[\frac{1}{2\sqrt{t}} \cos \sqrt{t}\right] = L[f'(t)]$$

$$= s\,\frac{1}{2s}\sqrt{\frac{\pi}{s}}\; e^{-1/4s} - 0 \quad [\because f(0) = 0]$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{s}}\; e^{-1/4s}$$

$$\frac{1}{2} L\left[\frac{1}{\sqrt{t}} \cos \sqrt{t}\right] = \frac{1}{2}\sqrt{\frac{\pi}{s}}\; e^{-1/4s}$$

$$L\left[\frac{1}{\sqrt{t}} \cos \sqrt{t}\right] = \sqrt{\frac{\pi}{s}}\; e^{-1/4s}$$

Example 6 show that $\int_0^{\infty} e^{-t}\, t\cos t\, dt = 0$

Solution : $\int_0 \ldots t\, dt = \left[L\,[t\cos t]\right]_{s=1}$

$$= \left[-\frac{d}{ds}L(\cos t)\right]_{s=1} = \left[-\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]_{s=1} = \left[-\left[\frac{(s^2+1)(1)-s(2s)}{(s^2+1)^2}\right]\right]_{s=1}$$

$$= \left[-\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]\right]_{s=1} = \left[-\left[\frac{1-s^2}{(s^2+1)^2}\right]\right]_{s=1} = [-(0)] = 0$$

Example 7 Find $L\,[t\, e^{-t}\cosh t]$

Solution : $L\,[t\, e^{-t}\cosh t] = -\frac{d}{ds}L\,[e^{-t}\cosh t]$

$$= -\frac{d}{ds}\left[\frac{s+1}{(s+1)^2-1}\right] = -\left[\frac{[(s+1)^2-1]-(s+1)\,2\,(s+1)]}{[(s+1)^2-1]^2}\right]$$

$$= -\left[\frac{(s+1)^2-1-2\,(s+1)^2}{[(s+1)^2-1]^2}\right] = \frac{(s+1)^2+1}{(s^2+2s)^2} = \frac{s^2+2s+2}{s^4+4s^2+4s^3}$$

Result 18. Integrals of transform

If $L\,[f(t)] = \varphi(s)$ and $\frac{1}{t}f(t)$ has a limit as $t \to 0$ then

$$L\left[\frac{1}{t}f(t)\right] = \int_s^{\infty} \varphi(s)\, ds$$

Proof : $\varphi(s) = L\,[f(t)]$

$$\int_s^{\infty} \varphi(s)\, ds = \int_s^{\infty} L\,[f(t)]\, ds$$

$$= \int_s^{\infty}\int_0^{\infty} e^{-st} f(t)\, dt\, ds = \int_0^{\infty}\int_s^{\infty} e^{-st} f(t)\, ds\, dt$$

[since s and t are independent variables and hence the order of integration in the double integral can be interchanged]

$$= \int_0^{\infty} f(t)\left[\int_s^{\infty} e^{-st}\, ds\right] dt = \int_0^{\infty} f(t)\left[\frac{e^{-st}}{-t}\right]_s^{\infty} dt$$

$$= \int_0^{\infty} f(t)\left[0+\frac{e^{-st}}{t}\right] dt = \int_0^{\infty} f(t)\frac{e^{-st}}{t}\, dt$$

$$= \int_0^{\infty} e^{-st}\frac{f(t)}{t}\, dt = L\left[\frac{1}{t}f(t)\right]$$

i.e., $L\left[\frac{1}{t}f(t)\right] = \int_s^{\infty} \varphi(s)\,ds$

PROBLEMS BASED ON INTEGRALS OF TRANSFORM

Example 8 . **Find** $L\left[\frac{1-e^t}{t}\right]$

Solution : $L\left[\frac{1}{t}f(t)\right] = \int_s^{\infty} \varphi(s)\,ds = \int_s^{\infty} L[f(t)]\,ds$

$$L\left[\frac{1-e^t}{t}\right] = \int_s^{\infty} L[1-e^t]\,ds = \int_s^{\infty}\left[\frac{1}{s}-\frac{1}{s-1}\right]ds$$

$$= \Big[\log s - \log(s-1)\Big]_s^{\infty} = \left[\log\frac{s}{s-1}\right]_s^{\infty}$$

$$= \left[\log\frac{s}{s(1-1/s)}\right]_s^{\infty} = \left[\log\frac{1}{1-1/s}\right]_s^{\infty}$$

$$= 0 - \log\frac{s}{s-1} = \log\left(\frac{s-1}{s}\right)$$

Example 9 **Find** $L\left[\frac{\sin at}{t}\right]$ **[A.U., March 1996]**

Solution : $L\left[\frac{1}{t}f(t)\right] = \int_s^{\infty} L[f(t)]\,ds$

$$L\left[\frac{\sin at}{t}\right] = \int_s^{\infty} L[\sin at]\,ds = \int_s^{\infty}\frac{a}{s^2+a^2}\,ds$$

$$= a\left[\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)\right]_s^{\infty} = \left[\tan^{-1}\frac{s}{a}\right]_s^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left[\frac{s}{a}\right] = \tan^{-1}\left[\frac{a}{s}\right]$$

Note : $\cot^{-1}\left(\frac{s}{a}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$

$$\frac{s}{a} = \cot\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{s}{a}\right)\right] = \frac{s}{a}$$

INITIAL AND FINAL VALUE THEOREMS

♦ INITIAL VALUE THEOREM

If $L[f(t)] = F(s)$, then $\underset{t \to 0}{Lt}\, f(t) = \underset{s \to \infty}{Lt}\, sF(s)$

Proof : W.K.T.

$$L[f'(t)] = s\,L[f(t)] - f(0)$$

$$= s\,F(s) - f(0)$$

$$sF(s) - f(0) = L[f'(t)]$$

$$= \int_0^\infty e^{-st} f'(t)\,dt$$

$$\underset{s \to \infty}{Lt}\left[s\,F(s) - f(0)\right] = \underset{s \to \infty}{Lt} \int_0^\infty e^{-st} f'(t)\,dt$$

$$\underset{s \to \infty}{Lt}\, s\,F(s) - f(0) = 0 \qquad [\because e^{-\infty} = 0]$$

i.e., $\underset{s \to \infty}{Lt}\, s\,F(s) = f(0) = \underset{t \to 0}{Lt}\, f(t)$

Hence $\underset{t \to 0}{Lt}\, f(t) = \underset{s \to \infty}{Lt}\, s\,F(s)$

♦ FINAL VALUE THEOREM

If $L\,f(t) = F(s)$, then $\underset{t \to \infty}{Lt}\, f(t) = \underset{s \to 0}{Lt}\, sF(s)$

Proof : W.K.T. $L[f'(t)] = s\,L[f(t)] - f(0)$

$$s\,L[f(t)] - f(0) = L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

$$\underset{s \to 0}{Lt}\,[s\,L[f(t)] - f(0)] = \underset{s \to 0}{Lt} \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \int_0^\infty f'(t)\,dt = \int_0^\infty d[f(t)] = f(t)\Big]_0^\infty$$

$$\underset{s \to 0}{Lt}\, s\,F(s) - f(0) = f(\infty) - f(0)$$

$$\underset{s \to 0}{Lt}\, s\,F(s) = f(\infty) = \underset{t \to \infty}{Lt}\, f(t)$$

Hence $\underset{t \to \infty}{Lt}\, f(t) = \underset{s \to 0}{Lt}\, s\,F(s)$

PROBLEMS BASED ON INITIAL VALUE AND FINAL VALUE THEOREM

Example 5.4.1. If $L[f(t)] = \dfrac{1}{s(s+a)}$, **find** $\underset{t \to \infty}{\text{Lt}} f(t)$ **and** $\underset{t \to 0}{\text{Lt}} f(t)$

Solution : $\underset{t \to 0}{\text{Lt}}\, f(t) = \underset{s \to \infty}{\text{Lt}}\, s \cdot F(s)$

$$= \underset{s \to \infty}{\text{Lt}}\, s \frac{1}{s(s+a)} = \underset{s \to \infty}{\text{Lt}} \frac{1}{s+a} = \frac{1}{\infty} = 0$$

$$\underset{t \to \infty}{\text{Lt}}\, f(t) = \underset{s \to 0}{\text{Lt}}\, s F(s) = \underset{s \to 0}{\text{Lt}}\, s \frac{1}{s(s+a)}$$

$$= \underset{s \to 0}{\text{Lt}} \frac{1}{s+a} = \frac{1}{a}$$

Example 2. Verify the initial and final value theorem for the function $f(t) = 1 + e^{-t}(\sin t + \cos t)$

Solution : Initial value theorem states that

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s F(s)$$

$$L[f(t)] = F(s) = \frac{1}{s} + L[\sin t + \cos t]_{s \to s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2 + 1} + \frac{s+1}{(s+1)^2 + 1}$$

$$= \frac{1}{s} + \frac{s+2}{(s+1)^2 + 1}$$

$$\text{L.H.S} = \lim_{t \to 0} f(t) = 1 + 1 = 2$$

$$\text{R.H.S} = \lim_{s \to \infty} s \left[\frac{1}{s} + \frac{s+2}{(s+1)^2 + 1}\right]$$

$$= \lim_{s \to \infty} \left[1 + \frac{s(s+2)}{(s+1)^2 + 1}\right] = \lim_{s \to \infty} \left[1 + \frac{s^2\left(1 + \frac{2}{s}\right)}{s^2\left[1 + \frac{2}{s} + \frac{2}{s^2}\right]}\right]$$

$$= \lim_{s \to \infty} \left[1 + \frac{1 + \frac{2}{s}}{1 + \frac{2}{s} + \frac{2}{s^2}}\right] = 1 + 1 = 2$$

$$\text{L.H.S.} = \text{R.H.S.}$$

Initial value theorem verified.

Final value theorem states that

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,F(s)$$

$$\text{L.H.S.} = \lim_{t \to \infty} [1 + e^{-t}(\sin t + \cos t)]$$

$$= 1 + 0 = 1$$

$$\text{R.H.S.} = \lim_{s \to 0} \left[1 + \frac{s(s+2)}{(s+1)^2 + 1}\right]$$

$$= 1 + 0 = 1$$

$$\text{L.H.S.} = \text{R.H.S.}$$

Final value theorem verified.

Example 3. Verify the initial and final value theorems for $f(t) = 3e^{-2t}$

Solution : $f(t) = 3e^{-2t}$

$$F(s) = L[f(t)] = L[3e^{-2t}] = \frac{3}{s+2}$$

Initial value theorem : $\underset{t \to 0}{\mathrm{Lt}}\, f(t) = \underset{s \to \infty}{\mathrm{Lt}}\, s\,F(s)$

$$\text{L.H.S.} = \underset{t \to 0}{\mathrm{Lt}}\, f(t) = \underset{t \to 0}{\mathrm{Lt}}\, 3e^{-2t} = 3$$

$$\text{R.H.S} = \underset{s \to \infty}{\mathrm{Lt}}\, s\,F(s) = \underset{s \to \infty}{\mathrm{Lt}}\, s\left(\frac{3}{s+2}\right) = \underset{s \to \infty}{\mathrm{Lt}}\, \frac{3s}{s+2}$$

$$= \underset{s \to \infty}{\mathrm{Lt}}\, \frac{3s}{s\left(1 + \frac{2}{s}\right)}$$

$$= \underset{s \to \infty}{\mathrm{Lt}}\, \frac{3}{1 + \left(\frac{2}{s}\right)} = 3$$

$$\text{L.H.S} = \text{R.H.S.}$$

Hence Initial value theorem verified.

Final value theorem $\underset{t \to \infty}{\text{Lt}} \; f(t) = \underset{s \to 0}{\text{Lt}} \; s\,F(s)$

$$\text{L.H.S.} = \underset{t \to \infty}{\text{Lt}} \; f(t) = \underset{t \to \infty}{\text{Lt}} \; 3e^{-2t} = 0 \qquad [\because e^{-\infty} = 0]$$

$$\text{R.H.S.} = \underset{s \to 0}{\text{Lt}} \; s\,F(s) = \underset{s \to 0}{\text{Lt}} \; s\left(\frac{3}{s+2}\right) = 0$$

$$\text{L.H.S.} = \text{R.H.S.}$$

Hence Final value theorem verified.

TRANSFORMS OF UNIT STEP FUNCTION AND IMPULSE FUNCTION

◆ UNIT STEP FUNCTION (OR) HEAVISIDE'S UNIT STEP FUNCTION

PROBLEMS BASED ON UNIT STEP FUNCTION

Example 1. Define the unit step function.

Solution :

The unit step function, also called Heavi side's unit function is defined as

$$U\,(t-a) = \begin{cases} 0 \text{ for } t < a \\ 1 \text{ for } t > a \end{cases}$$

This is the unit step functions at $t = a$

It can also be denoted by $H\,(t-a)$.

Example 2. Give the L.T. of the unit step function. **[M.U. Oct., 96]**

Solution :

The L.T. of the unit step function is given by

$$L\,[U\,(t-a)] = \int_0^\infty e^{-st}\,U\,(t-a)\,dt$$

[0 =

$$= \int_0^a e^{-st}\,0\,dt + \int_a^\infty e^{-st}\,(1)\,dt$$

$$= \int_a^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= 0 - \left(\frac{e^{-sa}}{-s}\right) = \frac{e^{-as}}{s}$$

TRANSFORM OF PERIODIC FUNCTIONS

Define periodic function and state its Laplace transform formula.

♦ Def. Periodic

A function $f(x)$ is said to be "periodic" if and only if $f(x+p) = f(x)$ is true for some value of p and every value of x. The smallest positive value of p for which this equation is true for every value of x will be called the period of the function.

The Laplace Transformation of a periodic function $f(t)$ with period p given by $\dfrac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt$

Proof : $\quad L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$= \int_0^p e^{-st} f(t)\, dt + \int_p^\infty e^{-st} f(t)\, dt$$

Put $t = u + p$ in the second integral

i.e., $u = t - p \qquad t \to p \Rightarrow u \to 0$

i.e., $du = dt \qquad t \to \infty \Rightarrow u \to \infty$

$$= \int_0^p e^{-st} f(t)\, dt + \int_0^\infty e^{-(u+p)s} f(u+p)\, du$$

$$= \int_0^p e^{-st} f(t)\, dt + e^{-sp} \int_0^\infty e^{-su} f(u)\, du \quad [\because f(u+p) = f(u)]$$

$$= \int_0^p e^{-st} f(t)\, dt + e^{-sp} \int_0^\infty e^{-st} f(t)\, dt \quad [\because u \text{ is a dummy variable}]$$

$$L[f(t)] = \int_0^p e^{-st} f(t)\, dt + e^{-sp}\, L[f(t)]$$

$$[1 - e^{-sp}]\, L[f(t)] = \int_0^p e^{-st} f(t)\, dt$$

$$L[f(t)] = \frac{1}{1-e^{-sp}} \int_0^p e^{-st} f(t)\, dt$$

Example 1 Find the Laplace transform of the Half wave rectifier function

$$f(t) = \begin{cases} \sin \omega t, & 0 < t < \dfrac{\pi}{\omega} \\ 0 \quad , & \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega} \end{cases}$$

Solution : This function is a periodic function with period $\frac{2\pi}{\omega}$ in the interval $\left(0, \frac{2\pi}{\omega}\right)$

$$L[f(t)] = \frac{1}{1 - e^{\frac{-2\pi s}{\omega}}} \int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\, dt$$

$$= \frac{1}{1 - e^{\frac{-2\pi s}{\omega}}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt + 0 \right]$$

$$= \frac{1}{1 - e^{\frac{-2\pi s}{\omega}}} \left[\frac{e^{-st}}{s^2 + \omega^2} \left[-s \sin \omega t - \omega \cos \omega t\right] \right]_0^{\pi/\omega}$$

$$= \frac{1}{1 - e^{\frac{-2\pi s}{\omega}}} \left[\frac{e^{-s\pi/\omega}\, \omega + \omega}{s^2 + \omega^2} \right]$$

$$= \frac{\omega\left[1 + e^{\frac{-s\pi}{\omega}}\right]}{\left[1 - e^{-s\pi/\omega}\right]\left[1 + e^{-s\pi/\omega}\right](s^2 + \omega^2)}$$

$$= \frac{\omega}{(s^2 + \omega^2)\left(1 - e^{-s\pi/\omega}\right)}$$

Example 2 **Find the Laplace Transform of**

$$f(t) = \begin{cases} t, & 0 < t < a \\ 2a-t, & a < t < 2a \end{cases} \text{ with } f(t+2a) = f(t)$$

Solution : $$L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st}\, t\, dt + \int_a^{2a} e^{-st}\, (2a-t)\, dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right) \right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right) \right]_a^{2a} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left[-t\,\frac{e^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^a + \left[-(2a-t)\frac{e^{-st}}{s} + \frac{e^{-st}}{s^2} \right]_a^{2a} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left(-a\frac{e^{-as}}{s} - \frac{e^{-as}}{s^2} \right) - \left(-\frac{1}{s^2} \right) \right] + \left[\left(\frac{e^{-2as}}{s^2} \right) - \left(-\frac{ae^{-as}}{s} + \frac{e^{-as}}{s^2} \right) \right]$$

$$= \frac{1}{1-e^{-2as}} \left[-\frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + a\frac{e^{-as}}{s} - \frac{e^{-as}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{1 + e^{-2as} - 2e^{-as}}{s^2} \right]$$

$$= \frac{[1-e^{-as}]^2}{s^2(1-e^{-as})(1+e^{-as})} = \frac{1-e^{-as}}{s^2(1+e^{-as})} = \frac{1}{s^2}\tanh\left[\frac{as}{2}\right]$$

INVERSE LAPLACE TRANSFORM

Now we obtain $f(t)$ when $\phi(s)$ is given, then we say that inverse Laplace transform of $\phi(s)$ is $f(t)$.

(1) If $L[f(t)] = \phi(s)$, then $L^{-1}[\phi(s)] = f(t)$

where L^{-1} is called the inverse Laplace transform operator.

(2) If $\varphi_1(s)$ and $\varphi_2(s)$ are L.T. of $f(t)$ and $g(t)$ respectively then

$$L^{-1}[C_1\varphi_1(s) + C_2\varphi_2(s)] = C_1L^{-1}[\varphi_1(s)] + C_2L^{-1}[\varphi_2(s)]$$

Proof : Given : $L[f(t)] = \varphi_1(s)$

$$f(t) = L^{-1}[\varphi_1(s)]$$

$$L[g(t)] = \varphi_2(s)$$

$$g(t) = L^{-1}[\varphi_2(s)]$$

W.K.T. $L[C_1f(t) + C_2g(t)] = C_1L[f(t)] + C_2L[g(t)]$

$$= C_1\varphi_1(s) + C_2\varphi_2(s)$$

$$C_1f(t) + C_2g(t) = L^{-1}[C_1\varphi_1(s) + C_2\varphi_2(s)]$$

i.e., $L^{-1}[C_1\varphi_1(s) + C_2\varphi_2(s)] = C_1f(t) + C_2g(t)$

$$= C_1L^{-1}\varphi(s) + C_2L^{-1}\varphi_2(s)$$

Note : (1) If $L[f(t)] = \varphi(s)$ then $L[e^{at}f(t)] = \varphi(s-a)$

i.e., If $L^{-1}[\varphi(s)] = f(t)$ then

$$L^{-1}[\varphi(s-a)] = e^{at}f(t) = e^{at}L^{-1}[\varphi(s)]$$

Note : (2) If $L[f(t)] = \varphi(s)$ then $L[e^{-at}f(t) = \varphi(s+a)$

i.e., If $L^{-1}[\varphi(s)] = f(t)$ then

$$L^{-1}[\varphi(s+a)] = e^{-at}f(t) = e^{-at}[L^{-1}\varphi(s)]$$

♦ IMPORTANCE FORMULA

1. $L^{-1}\left[\frac{1}{s}\right] = 1$

2. $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{\underline{|n-1}}$

3. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

4. $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

5. $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$

6. $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

7. $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

8. $L^{-1}[F(s-a)] = e^{at}f(t)$

9. $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b}e^{at}\sin b\,t$

10. $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at}\cos bt$

11. $L^{-1}\left[\frac{1}{(s-a)^2-b^2}\right] = \frac{1}{b}e^{at}\sinh bt$

12. $L^{-1}\left[\frac{s-a}{(s-a)^2-b^2}\right] = e^{at}\cosh bt$

13. $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}(\sin at - at\cos at)$

14. $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}t\sin at$

15. $L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right] = t\cos at$

16. $L^{-1}[1] = \delta(t)$

17. $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2a}[\sin at + at\cos at]$

Example 1 Find $L^{-1}\left[\frac{2s+1}{s^2+4s+13}\right]$

Solution : $L^{-1}\left[\frac{2s+1}{s^2+4s+13}\right] = L^{-1}\left[\frac{2s+1}{(s+2)^2+3^2}\right]$

$$= L^{-1}\left[\frac{2s+4-3}{(s+2)^2+3^2}\right]$$

$$= L^{-1}\left[\frac{2(s+2)}{(s+2)^2+3^2}\right] - L^{-1}\left[\frac{3}{(s+2)^2+3^2}\right]$$

$$= 2e^{-2t}L^{-1}\left[\frac{s}{s^2+3^2}\right] - e^{-2t}L^{-1}\left[\frac{3}{s^2+3^2}\right]$$

$$= 2e^{-2t}\cos 3t - e^{-2t}\sin 3t$$

Example 2 Find $L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right]$

Solution : $L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right]$

$$= \frac{1}{b^2-a^2}L^{-1}\left[\frac{b^2-a^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= \frac{1}{b^2-a^2}L^{-1}\left[\frac{1}{s^2+a^2} - \frac{1}{s^2+b^2}\right]$$

$$= \frac{1}{b^2-a^2}\frac{1}{a}L^{-1}\left[\frac{a}{s^2+a^2}\right] - \frac{1}{b^2-a^2}\frac{1}{b}L^{-1}\left[\frac{b}{s^2+b^2}\right]$$

$$= \frac{1}{a(b^2-a^2)}\sin at - \frac{1}{b(b^2-a^2)}\sin bt$$

$$= \frac{1}{b^2-a^2}\left[\frac{1}{a}\sin at - \frac{1}{b}\sin bt\right]$$

♦ MULTIPLICATION BY s

If $L[f(t)] = \varphi(s)$ then

$$L[f'(t)] = SL[f(t)] - f(0)$$

i.e., If $L^{-1}[\varphi(s)] = f(t)$

then $L^{-1}[s\varphi(s)] = f'(t)$

$$= \frac{d}{dt}f(t) = \frac{d}{dt}[L^{-1}\varphi(s)]$$

Provided $f(0) = 0$, $L^{-1}[f(0)] = 0$

when $t \to 0$

$$L^{-1}[s\,F(s)] = \frac{d}{dt}f(t) + f(0)\,\delta(t)$$

PROBLEMS BASED ON INVERSE L.T. MULTIPLICATION BY s

Example 1 **Find** $L^{-1}\left[\dfrac{s}{s^2+1}\right]$

Solution : $L^{-1}\left[\dfrac{1}{s^2+1}\right] = \sin t$

$$L^{-1}\left[\frac{s}{s^2+1}\right] = \frac{d}{dt}(\sin t) + \sin(0)\,\delta(t) = \cos t$$

Example 2 **Find** $L^{-1}\left[\dfrac{s}{4s^2-25}\right]$

Solution : $L^{-1}\left[\dfrac{1}{4s^2-25}\right] = \dfrac{1}{4}L^{-1}\left[\dfrac{1}{s^2-\frac{25}{4}}\right]$

$$= \frac{1}{4}\,\frac{2}{5}L^{-1}\left[\frac{5/2}{s^2-(5/2)^2}\right] = \frac{1}{10}\sinh\frac{5}{2}t$$

$$L^{-1}\left[\frac{s}{4s^2-2t}\right] = \frac{d}{dt}\left[\frac{1}{10}\sinh\frac{5}{2}t\right] + \frac{1}{10}\sinh\frac{5}{2}(0)f(t)$$

$$= \frac{1}{10}\cosh t\frac{5}{2}\left(\frac{5}{2}\right) + 0 = \frac{1}{4}\cosh\frac{5}{2}t$$

♦ MULTIPLICATION BY 1/S.

PROBLEMS BASED ON INVERSE L.T. [Multiplication by 1/s]

Example 3 If $L[f(t)] = \varphi(s)$, then $L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}\varphi(s)$

i.e., $L^{-1}\left[\frac{1}{s}\varphi(s)\right] = \int_0^t f(t)\,dt = \int_0^t L^{-1}[\varphi(s)]\,dt$

Proof : $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$L\left[\int_0^t f(t)\,dt\right] = \int_0^\infty e^{-st}\left[\int_0^t f(t)\,dt\right] dt$$

$$= \int_0^\infty \left[\int_0^t f(t)\,dt\right] d\left[\frac{e^{-st}}{-s}\right]$$

$$= \left(\int_0^t f(t)\,dt\right)\left(\frac{e^{-st}}{-s}\right)\Bigg]_0^\infty - \int_0^\infty \left(\frac{e^{-st}}{-s}\right) f(t)\,dt$$

$$[\because \text{diff} \int f(t)\,dt = f(t)]$$

$$= (0-0) + \frac{1}{s}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \frac{1}{s}L[f(t)] = \frac{1}{s}\varphi(s)$$

Example 4 **Find** $L^{-1}\left[\frac{1}{s(s+3)}\right]$

Solution : $L^{-1}\left[\frac{1}{s(s+3)}\right] = \int_0^t L^{-1}\left[\frac{1}{s+3}\right] dt$

$$= \int_0^t e^{-3t}\,dt = \left[\frac{e^{-3t}}{-3}\right]_0^t = -\frac{1}{3}\left[e^{-3t}\right]_0^t$$

$$= -\frac{1}{3}\left[e^{-3t} - 1\right] = \frac{1}{3}\left[1 - e^{-3t}\right]$$

♦ INVERSE LAPLACE TRANSFORMS OF DERIVATIVES.

W.K.T. If $L[f(t)] = \varphi(s)$ then $L[t f(t)] = -\varphi'(s)$

i.e., If $L^{-1}[\varphi(s)] = f(t)$ then $L^{-1}[\varphi'(s)] = -t f(t)$

$$= -t L^{-1}[\varphi(s)]$$

Example 5. Find $L^{-1}\left[\dfrac{s}{(s^2-a^2)^2}\right]$

Solution : Let $\varphi'(s) = \dfrac{s}{(s^2-a^2)^2}$

Put $s^2 - a^2 = t$

$2s\,ds = dt$

$s\,ds = \dfrac{dt}{2}$

$$\int \varphi'(s)\,ds = \int \frac{s}{(s^2-a^2)^2}\,ds$$

$$\varphi(s) = \int \frac{s}{(s^2-a^2)^2}\,ds$$

$$\therefore \varphi(s) = \int \frac{1}{t^2}\frac{dt}{2} = \frac{1}{2}\left[\frac{-1}{t}\right] = -\frac{1}{2t} = -\frac{1}{2(s^2-a^2)}$$

W.K.T. $L^{-1}(\varphi'(s)] = -t L^{-1}[\varphi(s)]$

$$= -t L^{-1}\left[\frac{-1}{2(s^2-a^2)}\right] = \frac{t}{2} L^{-1}\left[\frac{1}{(s^2-a^2)}\right]$$

$$= \frac{t}{2}\frac{1}{a} L^{-1}\left[\frac{a}{s^2-a^2}\right] = \frac{t}{2a}\sinh at$$

Example 6 Find $L^{-1}\left[\tan^{-1}\left(\dfrac{1}{s}\right)\right]$

Solution : W.K.T. $L^{-1}[\varphi'(s)] = -t L^{-1}[\varphi(s)]$

$$L^{-1}[\varphi(s)] = -\frac{1}{t} L^{-1}[\varphi'(s)]$$

$$L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\tan^{-1}\left(\frac{1}{s}\right)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{1+\frac{1}{s^2}}\left(\frac{-1}{s^2}\right)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{-1}{s^2+1}\right] = \frac{1}{t} L^{-1}\left[\frac{1}{1+s^2}\right] = \frac{1}{t}\sin t$$

Example .7 **Find** $L^{-1}\left[\log\left(\frac{s^2-1}{s^2}\right)\right]$

Solution : W.K.T. $L^{-1}[\varphi'(s)] = -t\,L^{-1}[\varphi(s)]$

(or) $L^{-1}[\varphi(s)] = -\frac{1}{t}L^{-1}[\varphi'(s)]$

$$L^{-1}\left[\log\left(\frac{s^2-1}{s^2}\right)\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left(\log\frac{s^2-1}{s^2}\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}[\log(s^2-1)-\log s^2]\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{2s}{s^2-1}-\frac{2s}{s^2}\right] = -\frac{1}{t}L^{-1}\left[\frac{2s}{s^2-1}-\frac{2}{s}\right]$$

$$= -\frac{2}{t}L^{-1}\left[\frac{s}{s^2-1}-\frac{1}{s}\right] = -\frac{2}{t}[\cosh t-1]$$

$$= \frac{2}{t}[1-\cosh t]$$

Example 8 **Find** $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$

Solution : W.K.T. $L^{-1}[\varphi(s)] = -\frac{1}{t}L^{-1}[\varphi'(s)]$

$$L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left[\log\left(\frac{s+1}{s-1}\right)\right]\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}[\log(s+1)-\log(s-1)]\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{1}{s+1}-\frac{1}{s-1}\right] = -\frac{1}{t}[e^{-t}-e^{t}]$$

$$= \frac{1}{t}\left[e^{t}-e^{-t}\right] = \frac{2}{t}\left[\frac{e^{t}-e^{-t}}{2}\right] = \frac{2}{t}\sinh t$$

♦ INVERSE LAPLACE TRANSFORM OF INTEGRALS.

$$L^{-1}\left[\int_s^\infty \varphi(s)\,ds\right] = \frac{1}{t}f(t) = \frac{1}{t}L^{-1}[\varphi(s)]$$

(or) $$L^{-1}[\varphi(s)] = t\,L^{-1}\left[\int_s^\infty \varphi(s)\,ds\right]$$

PROBLEMS BASED ON INVERSE LAPLACE TRANSFORM OF INTEGRALS

Example 9 Obtain $L^{-1}\left[\frac{2s}{(s^2+1)^2}\right]$

Solution : W.K.T. $L^{-1}[\varphi(s)] = tL^{-1}\left[\int_s^\infty \varphi(s)\,ds\right]$

$$L^{-1}\left[\frac{2s}{(s^2+1)^2}\right] = t\,L^{-1}\left[\int_s^\infty \frac{2s}{(s^2+1)^2}\,ds\right]$$

$$= t\,L^{-1}\left[\left(\frac{-1}{s^2+1}\right)_s^\infty\right] = t\,L^{-1}\left[0+\frac{1}{s^2+1}\right]$$

$$= t\,L^{-1}\left[\frac{1}{s^2+1}\right] = t\sin t$$

PROBLEMS BASED ON PARTIAL FRACTION

Example 9 Find $L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right]$

Solution : $\frac{1-s}{(s+1)(s^2+4s+13)} = \frac{A}{s+1}+\frac{Bs+C}{s^2+4s+13}$

$$1-s = A(s^2+4s+13)+(Bs+c)+(s+1)$$

Put $s = -1$ we get $1+1 = A(1-4+13)$

$$2 = 10A$$

$$A = 1/5$$

Put $s = 0$ we get $1 = 13A + C$

$$1 = \frac{13}{5}+C$$

$$C = 1-\frac{13}{5} \qquad \therefore C = -\frac{8}{5}$$

Equating the coefficients of s^2 on both sides

$$0 = A + B$$

$$B = -A$$

$$B = -\frac{1}{5}$$

$$\frac{1-s}{(s+1)(s^2+4s+13)} = \frac{1/5}{s+1} + \frac{-1/5\,s - 8/5}{s^2+4s+13}$$

$$= \frac{1}{5}\left[\frac{1}{s+1}\right] - \frac{1}{5}\left[\frac{s+8}{s^2+4s+13}\right]$$

$$= \frac{1}{5}\left[\frac{1}{s+1}\right] - \frac{1}{5}\left[\frac{s+8}{(s+2)^2+13-4}\right]$$

$$= \frac{1}{5}\left[\frac{1}{s+1}\right] - \frac{1}{5}\left[\frac{s+2}{(s+2)^2+3^2} + \frac{6}{(s+2)^2+3^2}\right]$$

$$L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right]$$

$$= \frac{1}{5}L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{5}L^{-1}\left[\frac{s+2}{(s+2)^2+3^2}\right] - \frac{2}{5}L^{-1}\left[\frac{3}{(s+2)^2+3^2}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}L^{-1}\left[\frac{s}{s^2+3^2}\right] - \frac{2}{5}e^{-2t}L^{-1}\left[\frac{3}{s^2+3^2}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\cos 3t - \frac{2}{5}e^{-2t}\sin 3t$$

Example 10 **Find** $L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]$

Solution : $\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3}$

$5s^2 - 15s - 11 = A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1)$

Put $s = -1$ we get

$$5 + 15 - 11 = A(-1-2)^3$$

$$9 = -27A$$

$$A = -\frac{1}{3}$$

equating the coefficients of s^3 on both sides

$$0 = A + B$$

$$B = -A$$

$$B = \frac{1}{3}$$

Put $s = 2$ we get

$$-21 = 3D$$

$$D = -7$$

Put $s = 0$ we get

$$-11 = -8A + 4B - 2C + D$$

$$= -8\left(-\frac{1}{3}\right) + 4\left(\frac{1}{3}\right) - 2C - 7$$

$$-4 = \frac{8}{3} + \frac{4}{3} - 2C$$

$$= 4 - 2C$$

$$-8 = -2C$$

$$C = 4$$

$$\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} = \frac{-\frac{1}{3}}{s+1} + \frac{\frac{1}{3}}{s-2} + \frac{4}{(s-2)^2} - \frac{7}{(s-2)^3}$$

$$L^{-1}\left[\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3}\right] = -\frac{1}{3} L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{3} L^{-1}\left[\frac{1}{s-2}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2}\right] - 7L^{-1}\left[\frac{1}{(s-2)^3}\right]$$

$$= -\frac{1}{3} e^{-t} + \frac{1}{3} e^{2t} + 4e^{2t} L^{-1}\left[\frac{1}{s^2}\right] - 7e^{2t} L^{-1}\left[\frac{1}{s^3}\right]$$

♦ SECOND SHIFTING PROPERTY

$$L^{-1}\,[e^{-as} F(s)] = f(t-a)\,U(t-a)$$

PROBLEMS BASED ON INVERSE LAPLACE TRANSFORM [SECOND SHIFTING PROPERTY]

Example 5.7.54. Find $L^{-1}\left[\frac{e^{-\pi s}}{s+3}\right]$

Solution : $L^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}$

$$L^{-1}\left[\frac{e^{-\pi s}}{s+3}\right] = e^{-3(t-\pi)}\,U(t-\pi)$$

Example 12 Find $L^{-1}\left[\frac{1}{\sqrt{1+s^2}}\right]$

Solution : $$\frac{1}{\sqrt{1+s^2}} = \frac{1}{\sqrt{s^2\left[1+\frac{1}{s^2}\right]}}$$

$$= \frac{1}{s\sqrt{1+\frac{1}{s^2}}} = \frac{1}{s}\left[1+\frac{1}{s^2}\right]^{-1/2}$$

$$= \frac{1}{s}\left[1-\frac{1}{2}\frac{1}{s^2}+\frac{1.3}{2^2}\frac{1}{2!}\frac{1}{s^4}-\frac{1.3.5}{2^3}\frac{1}{3!}\frac{1}{s^6}+\dots\right]$$

$$= \frac{1}{s}-\frac{1}{2}\frac{1}{s^3}+\frac{1.3}{2^2}\frac{1}{2!}\frac{1}{s^5}-\frac{1.3.5}{2^3}\frac{1}{3!}\frac{1}{s^7}+\dots$$

$$L^{-1}\left[\frac{1}{\sqrt{1+s^2}}\right] = L^{-1}\left[\frac{1}{s}\right]-\frac{1}{2}L^{-1}\left[\frac{1}{s^3}\right]+\frac{1.3}{2^2}\frac{1}{2!}L^{-1}\left[\frac{1}{s^5}\right]-\dots$$

$$= 1-\frac{1}{2}\frac{1}{2!}t^2+\frac{1.3}{2^2}\frac{1}{2!}\frac{1}{4!}t^4-\frac{1.3.5}{2^3}\frac{1}{3!}\frac{1}{6!}t^6+\dots$$

$$= 1-\frac{t^2}{4}+\frac{t^4}{64}-\frac{t^6}{384}+\dots$$

♦ **CHANGE OF SCALE PROPERTY.**

STATE AND PROVE THE SCALLING PROPERTY OF L.T.

PROBLEMS BASED ON INVERSE LAPLACE TRANSFORM
CHANGE OF SCALE PROPERTY

Example 13 If $L f(t) = F(s)$ then $L[f(at)] = \frac{1}{a} F\left[\frac{s}{a}\right]$

If $f(t) = L^{-1}[F(s)]$ then $L^{-1}[F(cs)] = \frac{1}{c} f\left(\frac{t}{c}\right)$

Solution : $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[f(at)] = \int_0^\infty e^{-st} f(at)\, dt$$

Put $at = x$ $\quad t \to 0 \quad \Rightarrow x \to 0$

$adt = dx$ $\quad t \to \infty \quad \Rightarrow x \to \infty$

$$= \int_0^\infty e^{-sx/a} f(x) \frac{dx}{a} = \frac{1}{a} \int_0^\infty e^{-(s/a)x} f(x)\, dx$$

$$= \frac{1}{a} \int_0^\infty e^{-(s/a)t} f(t)\, dt \qquad [\because x \text{ is dummy variable}]$$

$$= \frac{1}{a} F\left[\frac{s}{a}\right]$$

$$L^{-1}\left[\frac{1}{a} F\left(\frac{s}{a}\right)\right] = f(at)$$

$$\frac{1}{a} L^{-1}\left[F\left(\frac{s}{a}\right)\right] = f(at)$$

$$L^{-1}\left[F\left(\frac{s}{a}\right)\right] = a f(at)$$

Put $a = \frac{1}{c}$ we get $L^{-1}[F(cs)] = \frac{1}{c} f\left(\frac{t}{c}\right)$

Example 14 If $L[f(t)] = F(s)$ find $L\left[f\left(\frac{t}{a}\right)\right]$

Solution : $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[f(t/a)] = \int_0^\infty e^{-st} f\left(\frac{t}{a}\right) dt$$

Put $u = \frac{t}{a}$ $\quad t \to 0 \Rightarrow u \to 0$

$du = \frac{dt}{a}$ $\quad t \to \infty \Rightarrow u \to \infty$

$$L[f(t/a)] = \int_0^\infty e^{-s(au)} f(u)\, a du$$

$$= a \int_0^\infty e^{-asu} f(u)\, du$$

$$= a \int_0^\infty e^{-ast} f(t)\, dt \qquad [\because u \text{ is a dummy variable}]$$

$$= a\, F[as]$$

CONVOLUTION THEOREM

Define convolution.

Solution .

The convolution of two functions $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

Example 14 . Prove that f(t) * g(t) = g(t) * f(t).

Solution : $f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$

$$\text{W.K.T. } \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$$

$$= \int_0^t f(t-u)\, g\, [t-(t-u)]\, du = \int_0^t f(t-u)\, g(u)\, du$$

$$= \int_0^t g(u) f(t-u)\, du = g(t) * f(t)$$

State and prove convolution theorem.

Solution : If $f(t)$ and $g(t)$ are functions defined for $t \geq 0$

then $L[f(t) * g(t)] = L[f(t)] \,.\, L[g(t)]$

Proof : We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[f(t) * g(t)] = \int_0^\infty e^{-st} [f(t) * g(t)]\, dt$$

$$= \int_0^\infty e^{-st} \left[\int_0^t f(u)\, g(t-u)\, du\right] dt$$

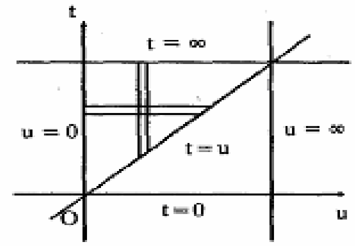
by def. of convolution

$$= \int_0^\infty \int_0^t e^{-st} f(u)\, g(t-u)\, du\, dt$$

Change the order of the integration

Given $t = 0$ to $t = \infty$

$u = 0$ to $u = t$



$$= \int_0^\infty \int_u^\infty e^{-st} f(u)\, g(t-u)\, dt\, du$$

$$= \int_0^\infty f(u) \int_u^\infty e^{-st} g(t-u)\, dt\, du$$

put $t - u = v$, $dt = dv$ | $t \to u \Rightarrow v \to 0$, $t \to \infty \Rightarrow v \to \infty$

$$= \int_0^\infty f(u) \int_0^\infty e^{-(u+v)s} g(v)\, dv\, du$$

$$= \int_0^\infty f(u)\, e^{-us} \int_0^\infty e^{-vs} g(v)\, dv\, du$$

$$= \int_0^\infty f(u)\, e^{-us}\, du \int_0^\infty e^{-vs} g(v)\, dv \quad [\because u \text{ and } v \text{ are dummy variable}]$$

$$= \int_0^\infty e^{-st} f(t)\, dt \int_0^\infty e^{-st} g(t)\, dt$$

$$= L[f(t) * g(t)] = L[f(t)] \,.\, L[(g(t)] = F(s) \,.\, G(s)$$

where $L[f(t)] = F(s)$

$L[g(t)] = G(s)$

$$L^{-1}[F(s)\; G(s)] = f(t) * g(t)$$

$$= L^{-1}[F(s)] * L^{-1}[G(s)]$$

PROBLEMS BASED ON CONVOLUTION THEOREM

Example 15 **Using convolution theorem find** $L^{-1}\left[\dfrac{1}{(s+a)(s+b)}\right]$

Solution :

$$L^{-1}[F(s).\, G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

$$L^{-1}\left[\frac{1}{s+a} \cdot \frac{1}{s+b}\right] = L^{-1}\left[\frac{1}{s+a}\right] * L^{-1}\left[\frac{1}{s+b}\right]$$

$$= e^{-at} * e^{-bt}$$

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t e^{-au} e^{-b(t-u)} du = \int_0^t e^{-au} e^{-bt} e^{bu} du$$

$$= e^{-bt} \int_0^t e^{-(a-b)u}\, du = e^{-bt} \left[\frac{e^{-(a-b)u}}{-(a-b)}\right]_0^t$$

$$= e^{-bt} \left[\frac{e^{-(a-b)t}}{-(a-b)} - \frac{1}{-(a-b)}\right] = \frac{e^{-bt}}{a-b} [1 - e^{-at} e^{bt}]$$

$$= \frac{1}{a-b} [e^{-bt} - e^{-at}]$$

Example 16 **Using convolution theorem find** $L^{-1} \left[\frac{s}{(s^2+a^2)^2}\right]$

Solution :

$$L^{-1} [F(s)\, G(s)] = L^{-1} [F(s)] * L^{-1} [G(s)]$$

$$L^{-1} \left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1} \left[\frac{s}{s^2+a^2}\right] * L^{-1} \left[\frac{1}{s^2+a^2}\right]$$

$$= L^{-1} \left[\frac{s}{s^2+a^2}\right] * \frac{1}{a} L^{-1} \left[\frac{a}{s^2+a^2}\right]$$

$$= \cos at * \frac{1}{a} \sin at = \frac{1}{a} [\cos at * \sin at]$$

$$= \frac{1}{a} \int_0^t \cos au \sin a(t-u)\, du$$

$$= \frac{1}{a} \int_0^t \sin(at - au) \cos au\, du$$

$$= \frac{1}{a} \int_0^t \frac{\sin(at - au + au) + \sin(at - au - au)}{2}\, du$$

$$= \frac{1}{2a} \int_0^t [\sin at + \sin a(t-2u)]\, du$$

$$= \frac{1}{2a} \left[(\sin at)\, u + \left(\frac{-\cos a(t-2u)}{-2a}\right)\right]_0^t$$

$$= \frac{1}{2a} \left[u (\sin at) + \frac{\cos a(t-2u)}{2a}\right]_0^t$$

$$= \frac{1}{2a} \left[\left(t \sin at + \left(\frac{\cos at}{2a}\right) - \left(0 + \frac{\cos at}{2a}\right)\right)\right]$$

$$= \frac{1}{2a} \left[t \sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a}\right] = \frac{1}{2a}\, t \sin at$$

Example 17 Find $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ using convolution theorem.

Solution :

$$L^{-1}[F(s)\,.\,G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

$$L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$= \cos at * \cos bt$$

$$= \int_0^t \cos au \cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t [\cos(au+bt-bu) + \cos(au-bt+bu)\,du$$

$$= \frac{1}{2}\int_0^t \left[\cos[(a-b)u+bt] + \cos[(a+b)u-bt]\right] du$$

$$= \frac{1}{2}\left[\frac{\sin(bt+(a-b)u)}{a-b} + \frac{\sin[(a+b)u-bt]}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\left(\frac{\sin(bt+at-bt)}{a-b} + \frac{\sin(at+bt-bt)}{a+b}\right) - \left(\frac{\sin bt}{a-b} - \frac{\sin bt}{a+b}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at}{a^2-b^2} - \frac{2b\sin bt}{a^2-b^2}\right] = \frac{1}{2}\left[\frac{2a\sin at - 2b\sin bt}{a^2-b^2}\right]$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2}$$

♦ Solving of Integral equations of convolution type.

Definition : An integral equation of the form

$$y(t) = f(t) + \int_0^t F(t-u)\, G(u)\, du$$

is called integral equation of convolution type.

This equation can also be expressed as

$$y(t) = f(t) + F(t) * G(t).$$

PROBLEMS BASED ON INTEGRAL EQUATIONS OF CONVOLUTION TYPE

Example 17 **Solve the integral eqn.**

$$\mathbf{y(t) = 1 + \int_0 y(u) \sin(t-u)\, du}$$

Solution : The given eqn. can be written as

$$y(t) = 1 + y(t) * \sin t$$

$$L[y(t)] = L[1] + L[y(t) * \sin t]$$

$$= \frac{1}{s} + L[y(t)]\, L[\sin t]$$

$$= \frac{1}{s} + L[y(t)] \left[\frac{1}{s^2+1}\right]$$

$$\left[1 - \frac{1}{s^2+1}\right] L[y(t)] = \frac{1}{s}$$

$$\left[\frac{s^2}{s^2+1}\right] L[y(t)] = \frac{1}{s}$$

$$L[y(t)] = \frac{s^2+1}{s^3} = \frac{1}{s} + \frac{1}{s^3}$$

$$y(t) = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s^3}\right] = 1 + \frac{1}{2}t^2$$

SOLUTION OF LINEAR ODE OF SECOND ORDER WITH CONSTANT COEFFICIENTS AND FIRST ORDER SIMULTANEOUS EQUATIONS WITH CONSTANT COEFFICIENTS USING LAPLACE TRANSFORMATION.

PROBLEMS BASED ON SOLUTION OF LINEAR ODE OF SECOND ORDER WITH CONSTANT COEFFICIENTS

Example 18 Solve by using L.T. $(D^2+9)y=\cos 2t$, given that if $y(0)=1,\ y\left(\frac{\pi}{2}\right)=-1$

Solution : Given $(D^2+9)\,y = \cos 2t$ i.e., $y''(t)+9y(t) = \cos 2t$

Taking Laplace transforms on both sides,

$$L[y''(t)]+9L[y(t)] = L[\cos 2t]$$

$$s^2 L[y(t)] - sy(0) - y'(0) + 9L[y(t)] = \frac{s}{s^2+4}$$

Using the initial conditions

$$y(0) = 1, \text{ and taking } y'(0) = k$$

we have

$$s^2 L[y(t)] - (s)(1) - k + 9L[y(t)] = \frac{s}{s^2+4}$$

$$(s^2+9)L[y(t)] = \frac{s}{s^2+4} + s + k$$

$$L[y(t)] = \frac{s}{(s^2+4)(s^2+9)} + \frac{s+k}{s^2+9} \qquad \ldots (1)$$

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9} \qquad \ldots (A)$$

$$s = (As+B)(s^2+9) + (Cs+D)(s^2+4)$$

equating s^3 on bothsides we get $0 = A + C$ ... (2)

equating s^2 on bothsides we get $0 = B + D$ ... (3)

Put $s = 0$ on bothsides we get $0 = 9B + 4D$ ... (4)

equating s on bothsides we get $1 = 9A + 4C$ ... (5)

$(2) \Rightarrow C = -A$

$\therefore \quad (5) \Rightarrow 9A + 4(-A) = 1$

$$9A - 4A = 1$$

$$5A = 1$$

$$A = \frac{1}{5}, \ C = -\frac{1}{5}$$

$$(3) \Rightarrow \quad D = -B$$

$$(4) \Rightarrow 9B + 4(-B) = 0$$

$$9B - 4B = 0$$

$$5B = 0$$

$$B = 0, \ D = 0$$

$$(A) \Rightarrow \frac{s}{(s^2+4)(s^2+9)} = \frac{\frac{1}{5}s + 0}{s^2+4} + \frac{-\frac{1}{5}s + 0}{s^2+9}$$

$$= \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)}$$

$$\therefore (1) \Rightarrow L[y(t)] = \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)} + \frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$y(t) = \frac{1}{5}L^{-1}\left[\frac{s}{s^2+4}\right] - \frac{1}{5}L^{-1}\left[\frac{s}{s^2+9}\right] + L^{-1}\left[\frac{s}{s^2+9}\right] + k\,L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$= \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t + \frac{k}{3}\sin 3t$$

Put $t = \frac{\pi}{2}$ we get $y\left(\frac{\pi}{2}\right) = \frac{1}{5}(-1) - \frac{1}{5}(0) + 0 + \frac{k}{3}(-1) = -\frac{1}{5} - \frac{k}{3}$

But given $y\left(\frac{\pi}{2}\right) = -1 \qquad \therefore -1 = -\frac{1}{5} - \frac{k}{3}$

$$1 = \frac{1}{5} + \frac{k}{3}$$

$$1 - \frac{1}{5} = \frac{k}{3}$$

$$\frac{4}{5} = \frac{k}{3} \Rightarrow k = \frac{12}{5}$$

$$\therefore \quad y(t) = \cos 3t + \frac{4}{5}\sin 3t + \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t$$

$$y(t) = \frac{4}{5}[\cos 3t + \sin 3t] + \frac{1}{5}\cos 2t$$

Example 19 Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ given that $y = \frac{dy}{dx} = 1$ at $x = 0$ using L.T. method.

Solution : Given $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ and $y(0) = 1,\ y'(0) = 1$

i.e., $y''(x) - 2y'(x) + 2y(x) = 0$

$$L[y''(x)] - 2L(y'(x)] + 2L[y(x)] = 0$$

$$s^2 L[y(x) - s\,y(0) - y'(0)] - 2[s L[y(x)] - y(0)] + 2L[y(x)] = 0$$

$$s^2 L[y(x) - s - 1] - 2[s L[y(x)] - 1] + 2L[y(x)] = 0$$

$$[s^2 - 2s + 2]\,L[y(x)] - s - 1 + 2 = 0$$

$$[s^2 - 2s + 2]\,L[y(x)] = s - 1$$

$$L[y(x)] = \frac{s-1}{s^2 - 2s + 2} = \frac{s-1}{(s-1)^2 + 1}$$

$$y(x) = L^{-1}\left[\frac{s-1}{(s-1)^2+1}\right] = e^t L^{-1}\left[\frac{s}{s^2+1}\right] = e^t \cos t$$

Example 20 Using L.T solve $y'' - 3y' + 2y = e^{-t}$ given $y(0) = 1$, $y'(0) = 0$.

Solution : $y'' - 3y' + 2y = e^{-t}$ and $y(0) = 1,\ y'(0) = 0$

$$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L[e^{-t}]$$

$$s^2 L[y(t)] - s\,y(0) - y'(0) - 3[s L[y(t)] - y(0)] + 2L[y(t)] = \frac{1}{s+1}$$

$$s^2 L[y(t)] - s - 0 - 3s L[y(t)] + 3 + 2L[y(t)] = \frac{1}{s+1}$$

$$(s^2 - 3s + 2)\,L[y(t)] = \frac{1}{s+1} + s - 3$$

$$(s-1)(s-2)\,L[y(t)] = \frac{s^2-2s-2}{s+1}$$

$$L[y(t)] = \frac{s^2-2s-2}{(s+1)(s-1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s-2}$$

$$s^2-2s-2 = A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1)$$

Put $s = 1$ we get $1-2-2 = -2B$

$$-3 = -2B$$

$$B = 3/2$$

Put $s = 2$ we get $4-4-2 = 3C$

$$C = -2/3$$

Put $s = 1$ we get $1+2-2 = 6A$

$$A = 1/6$$

$$L[y(t)] = \frac{1/6}{s+1} + \frac{3/2}{s-1} - \frac{2/3}{s-2}$$

$$= \frac{1}{6}\frac{1}{s+1} + \frac{3}{2}\frac{1}{s-1} - \frac{2}{3}\frac{1}{s-2}$$

$$y(t) = \frac{1}{6}L^{-1}\left[\frac{1}{s+1}\right] + \frac{3}{2}L^{-1}\left[\frac{1}{s-1}\right] - \frac{2}{3}L^{-1}\left[\frac{1}{s-2}\right]$$

$$= \frac{1}{6}e^{-t} + \frac{3}{2}e^{t} - \frac{2}{3}e^{2t}$$

Example 21 Using L.T. Solve $\frac{dx}{dt} + 3x - 2y = 1;\ \frac{dy}{dt} - 2x + 3y = e^t$ given that $x = 0 = y$ when $t = 0$.

Solution : The given differential equations can be written as

$$x'(t) + 3x(t) - 2y(t) = 1$$

$$y'(t) - 2x(t) + 3y(t) = e^t$$

Taking L.T. on both sides

$$L[x'(t)] + 3L[x(t)] - 2L[y(t)] = L[1]$$

$$L[y'(t)] - 2L[x(t)] + 3L[y(t)] = L[e^t]$$

$$s\,L[x(t)] - x(0) + 3L[x(t)] - 2L[y(t)] = \frac{1}{s}$$

$$s\,L[y(t)] - y(0) - 2L[x(t)] + 3L[y(t)] = \frac{1}{s-1}$$

Given $x(0) = 0$, $y(0) = 0$

$$s\,L[x(t)] + 3\,L[x(t)] - 2L[y(t)] = \frac{1}{s} \quad \ldots (1)$$

$$s\,L[y(t)] - 2\,L[x(t)] + 3L[y(t)] = \frac{1}{s-1} \quad \ldots (2)$$

$$(1) \Rightarrow \quad (s+3)\,L[x(t)] - 2\,L[y(t)] = \frac{1}{s} \qquad \dots (3)$$

$$-2\,L[x(t)] + (s+3)\,L[y(t)] = \frac{1}{s-1} \qquad \dots (4)$$

Solving (3) & (4) we get

$$L[x(t)] = \frac{\begin{vmatrix} 1/s & -2 \\ 1/(s-1) & s+3 \end{vmatrix}}{\begin{vmatrix} s+3 & -2 \\ -2 & s+3 \end{vmatrix}} = \frac{s^2+4s-3}{s(s-1)(s+1)(s+5)}$$

$$\frac{s^2+4s-3}{s(s-1)(s+1)(s+5)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} + \frac{D}{s+5}$$

$$s^2+4s-3 = A(s-1)(s+1)(s+5) + B(s)(s+1)(s+5) + C(s)(s-1)(s+5) + D(s)(s-1)(s+1)$$

Put $s = 0$ we get $-3 = A(-1)(1)(5)$

$$-3 = -5A$$

$$A = \frac{3}{5}$$

Put $s = 1$ we get

$$1+4-3 = 0 + B(1)(2)(6)$$

$$2 = 12\,B$$

$$B = 1/6$$

Put $s = -1$ we get

$$1-4+3 = 0+0+C(-1)(-2)(4)+0$$

$$-6 = 8C$$

$$C = -\frac{3}{4}$$

Put $s = -5$ we get

$$25-20-3 = 0+0+0+D(-5)(-6)(-4)$$

$$2 = -120D$$

$$D = -\frac{1}{60}$$

$$L[x(t)] = \frac{s^2+4s-3}{s(s-1)(s+1)(s+5)} = \frac{3/5}{s} + \frac{1/6}{s-1} + \frac{-3/4}{s+1} + \frac{-1/60}{s+5}$$

$$x(t) = \frac{3}{5}L^{-1}\left[\frac{1}{s}\right] + \frac{1}{6}L^{-1}\left[\frac{1}{s-1}\right] - \frac{3}{4}L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{60}L^{-1}\left[\frac{1}{s+5}\right]$$

$$= \frac{3}{5}(1) + \frac{1}{6}e^{t} - \frac{3}{4}e^{-t} - \frac{1}{60}e^{-5t}$$

Example 22 Solve $\dot{x} + y = \sin t$; $x + \dot{y} = \cos t$ with $x = 2$ and $y = 0$, when $t = ..$

Solution : Given $x'(t) + y(t) = \sin t$

$x(t) + y'(t) = \cos t$ and $x(0) = 2$, $y(0) = 0$

$$L[x'(t)] + L[y(t)] = L[\sin t]$$

$$L[x(t)] + L[y'(t)] = L[\cos t]$$

$$s\,L[x(t)] - x(0) + L[y(t)] = \frac{1}{s^2+1}$$

$$L[x(t)] + s\,L[y(t)] - y(0)] = \frac{s}{s^2+1}$$

$$s\,L[x(t)] + L[y(t)] = 2 + \frac{1}{s^2+1} \quad \text{... (1)}$$

$$L[x(t)] + s\,L[y(t)] = \frac{s}{s^2+1} \quad \text{... (2)}$$

Solving (1) and (2) we get

$$(1-s^2)\,L[y(t)] = 2 + \frac{1-s^2}{s^2+1}$$

$$(1-s^2)\,L[y(t)] = \frac{2s^2+2+1-s^2}{s^2+1}$$

$$L[y(t)] = \frac{s^2+3}{(s^2+1)(1-s^2)}$$

$$\frac{s^2+3}{(s^2+1)(1-s^2)} = \frac{As+B}{s^2+1}+\frac{Cs+D}{1-s^2}$$

$$s^2+3 = (As+B)(1-s^2)+(Cs+D)(s^2+1)$$

Equating s^3 on bothsides

$$0 = -A+C$$
$$A = C$$

Equating s^2 on bothsides

$$1 = -B+D$$

Equating s on both sides

$$0 = A+C$$

Put $s = 0$

$$3 = B+D$$
$$\therefore\ A = 0$$
$$C = 0$$

$$D = 2$$
$$B = 1$$

$$\Rightarrow y(t) = L^{-1}\left[\frac{1}{s^2+1}\right]-2L^{-1}\left[\frac{1}{s^2-1}\right]$$
$$= \sin t - 2\sinh t$$

To find $x(t)$, we have

$$x(t)+y'(t) = \cos t$$
$$x(t) = \cos t - y'(t)$$
$$y(t) = \sin t - 2\sinh t$$
$$\frac{dy}{dt} = \cos t - 2\cosh t$$
$$\therefore\ x(t) = \cos t - \cos t + 2\cosh t$$
$$= 2\cosh t$$

Hence $x(t) = 2\cosh t,\ y(t) = \sin t - 2\sinh t$