Linear Algebra

Finite Dimensional Vector Space—The vector space V(F) is said to be finite dimensional or finitely generated, if there exists a finite subset S of V such that

$$V = L(S)$$

Note—A vector space which is not finitely generated may be referred as an infinite dimensional space.

Some Important Theorems

1. There exists a basis for each finite dimensional vector space.
2. If V(F) is a finite dimensional vector space, then any two basis of V have the same number of elements.
3. Every linearly independent subset of a finitely generated vector space V(F) is either a basis of V or can be extended to form a basis of V.
4. Each subspace ω of a finite dimensional vector space V(F) of dimension n is a finite dimensional space with dim $m \leq n$ also $V = \omega$, iff dim V = dim ω.
5. If ω_1 and ω_2 are two subspaces of a finite dimensional vector space V(F), then
$$\dim(\omega_1 + \omega_2) = \dim \omega_1 + \dim \omega_2 - \dim(\omega_1 \cap \omega_2)$$
6. Each set of $(n + 1)$ or more vectors of a finite dimensional vector space V(F) of dimension n is linearly dependent.
7. If V(F) is a finite dimensional vector space of dimension n, then any set of n linearly independent vector in V forms a basis of V.
8. If a sets of n vectors of a finite dimensional vector space V(F) of dimension n generates V(F), then S is a basis of V.
9. If ω is a subspace of finite dimensional vector space V(F), every linearly independent subset of ω is finite and is part of a finite basis for ω.

Linear Transformation—Let U(F) and V(F) be to vector spaces over the same field F.

Then, a function T from U into V such that

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta),$$

$\forall\ \alpha, \beta \in$ U and $a, b \in$ F is called linear transformation from U into V.

Properties of linear Transformations—Let T be a linear Transformation from U(F) into V(F), then

1. T (0) = 0, where 0 on the left hand side is a zero vector of U and 0 on the right hand side is zero vector of V.

2. $$T(-\alpha) = -T(\alpha),\ \forall\ \alpha \in U$$

3. $$T(a_1\alpha_1 + a_2\alpha_2 + \ldots\ldots + a_n\alpha_n) = a_1T(\alpha_1) + a_2T(\alpha_2) + \ldots\ldots + a_nT(\alpha_n)$$

where $\alpha_1, \ldots\ldots \alpha_n \in$ U and $a_1 \ldots, a_n \in$ F.

Rank of a Matrix—A number r is defined as the rank of a $m \times n$ matrix A if,

1. A has at least one minor of order r which is not equal to zero.
2. There is no minor of order $(r + 1)$ which is not equal to zero.

The rank of the matrix A is denoted by $\rho(A)$.

Note (1) : The rank of a null matrix is defined as zero *i.e.*, $\rho(0) = 0$.

Note (2) : If I_n is a unit matrix of order n, then its rank $I = n$. *i.e.* $\rho(I_n) = n$

Note (3) : From the definition of the rank of a matrix, we concluded that

(a) If a matrix A does not possesses any minor of order $(r + 1)$, then $\rho(A) \leq r$.

(b) If at least one minor of order r of the matrix is not equal to zero, then $\rho(A) \geq r$.

Note (4) : If every $(r + 1)^{th}$ order minor of A is zero, then any higher order minor will also be zero.

Note (5) : If A is $n \times n$ non-singular matrix, then $\rho(A) = n$.

System of linear equation—If $b_1 = b_2 = b_3$ …… $= b_m = 0$, the set of equation is said to be homogeneous. Thus AX = 0

For such a system the rank of the matrix A and augmented matrix [A : B] are equal. Hence a system of homogeneous linear equations is always consistent.

If $x = 0$ (zero) *i.e.* $x_1 = x_2 \ldots x_n = 0$, then the solution is a trival solution. Thus condition for a trival solution of the system of linear equation is $\rho(A) = n$.

If $\rho(A) < n$, then the solution will be non-trival.

> **Note**—A homogeneous system of linear equation is n unknowns whose determinant of coefficient does not equal to zero, has only the trival solution.

Eigen values and eigen vectors—If V is a vector space over the field F and T is a linear operator on V. An eigen value of T is a scalar c in F such that there is a non-zero vector $\alpha \in V$ with $T\alpha = C\alpha$.

If c is an eigen value of T, then

(a) Any α such that $T\alpha = c\alpha$ is called eigen vector of T associated with the eigen value c;

(b) The collection of all c such that $T\alpha = c\alpha$ is called the eigen space associated with c.

Eigen value of matrix A over F—If A is an $n \times n$ matrix over the field F, an eigen value of A over F is a scalar c is F such that the matrix $(A - cI)$ is singular (not invertible)

Eigen Polynomial—

$$f(c) = |A - cI|.$$

Diagonalisation—If T is a linear operator on the finite dimensional space V. The T is diagonalisation if there is a basis for V each vector of which is an eigen vector of T.

Some Important Theorems

1. If T is a linear operator on a finite dimensional space V and c is any scalar. Then following are equivalent—

   (a) c is an eigen value of T

   (b) The operator $(T - cI)$ is singular (not invertible)

   (c) $\det(T - cI) = 0$

2. Similar matrices have the same eigen polynomial.

3. If $T\alpha = c\alpha$ and F is any polynomial, then $F(T)\alpha = F(c)\alpha$.

4. Suppose T is a linear operator on the finite dimensional space V, c_1 …… c_k are K-distinct eigen values of T and ω_1 is the space of eigen vector associated with the eigen value c_i,

   If $\omega = \omega_1 + \omega_2 + \ldots\ldots + \omega_k$, then

   $\dim \omega = \dim \omega_1 + \dim \omega_2 + \ldots + \dim \omega_k$.

   In fact, if B_i is an ordered basis for ω_i then $B = (B_1 \ldots\ldots B_k)$ is an ordered basis for ω.

5. If T is a linear operator on a finite dimensional space V, and c_1 …… c_k are k characteristic values of T and w_i is a null space of $(T - c_iI)$. Then the following are equivalent.

   (i) T is diagonalisation

   (ii) The eigen polynomial for T is

   $$F = (x - c_i)^{di} \ldots\ldots (x - c_k)^{dk}$$

   with $\dim \omega_1 = d_i$

   $$i = 1 \ldots\ldots, k$$

   (iii) $\dim V = \dim \omega_1 + \dim \omega_2 + \ldots + \dim \omega_K$.

Minimal Polynomials—If T is a linear operator on a finite dimensional vector space V over the field F. The minimal polynomial for T is the (unique) monic generator of the ideal of polynomials over F which annihilate T.

Theorem—If T is a linear operator on an n dimensional vector space V. Then eigen and minimal polynomials for T have the same roots, except for multiplicities.

Cayley-Hamilton theorem—If T is a linear operator on a finite dimensional vector space V and f is the eigen polynomial for T, then

$$f(T) = 0$$

i.e., the minimal polynomial divides the eigen polynomial for T.

Hermitian Matrix—A square matrix A is said to be a hermitian matrix if the transpose of the conjugate matrix is equal to the matrix itself

i.e. $A^* = A$

$\Rightarrow \overline{a_{ij}} = a_{ji}$

where $A = [a_{ij}]_{n \times n}$

> **Note**—In hermitian matrix, the elements on the principal diagonal must be all real numbers.
>
> *i.e.* $\overline{a_{ii}} = a_{ii}$

Important Facts

1. If A is a hermitian matrix, then KA is also hermitian for any real number K.
2. If A and B are hermitian matrices of same order, then $\lambda_1 A + \lambda_2 B$ also hermitian for any real number as λ_1, λ_2 etc.
3. If A be any square matrix, then AA^* and A^*A are also hermitian.
4. If A and B are hermitian, then AB is also hermitian, iff AB = BA.
5. If A is a hermitian matrix, then $\overline{A}$ is also hermitian.
6. If A and B are hermitian matrices of same order, then AB + BA is also hermitian.
7. If A is a square matrix then $A + A^*$ is a hermitian matrix.
8. Any square matrix can be uniquely expressed as $A + iB$, where, A and B are hermitian matrices.
9. If A is any hermitian matrix, then all positive integral powers of A are hermitian.

Skew-Hermitian Matrix—A square matrix A is said to be skew-hermitian, its

$$A^* = -A \Rightarrow \overline{a_{ij}} = -a_{ji}$$

> **Note**—The elements on the principal diagonal must be purely imaginary number or zero.

Important Facts

1. If A is a skew-hermitian matrix, then KA is also skew-hermitian for any real number K.
2. If A and B are skew-hermitian matrices of same orders, then $\lambda_1 A + \lambda_2 B$ is also skew-hermitian for any real number as λ_1, λ_2 etc.
3. If A and B are hermitian matrices of same order, then AB – BA is skew-hermitian.
4. If A is any square matrix, then $A - A^*$ is a skew-hermitian matrix.
5. Every square matrix can be uniquely represented as the sum of a hermitian and a skew-hermitian matrices.
6. If A is a skew-hermitian matrix, then iA is a hermitian.
7. If A is a skew-hermitian matrix, then $\overline{A}$ is also skew-hermitian.

Unitary Matrix—A square matrix A is said to be unitary matrix, iff

$$AA^* = I = A^*A.$$

Important Facts

1. If A is a unitary matrix, then A' is also unitary.
2. For any two unitary matrices A and B. AB and BA are also unitary matrices.
3. If A is a unitary matrix, then A^{-1} is also unitary.

Finite dimensional inner product spaces

Inner product—Let F be the field of real or complex numbers and V vector space over F. An inner product on V is a function which assigns to each ordered pair of vectors $\alpha, \beta \in V$ a scalar $(\alpha/\beta) \in F$, defined as

For $\alpha, \beta, \gamma \in V$, and all scalar

(a) $(\alpha + \beta/\gamma) = (\alpha/\gamma) + (\beta/\gamma)$

(b) $(c\,\alpha/\beta) = c\,(\alpha/\beta)$

(c) $(\beta/\alpha) = \overline{(\alpha/\beta)}$ (a complex conjugate)

(d) $(\alpha/\alpha) = 0$ if $\alpha \neq 0$

Inner product spaces—An inner product sapce is a real or complex space with a specified inner product on that space.

Norm—If $\|\alpha\|^2 = (\alpha/\alpha)$, then $\|\alpha\|$ is called norm of (α/α).

Euclidean space—A finite dimensional real inner product space.

Unitary space—A complex inner product space.

Orthogonal set—Let α and β be vectors in an inner product space V. Then α is orthogonal to β if $(\alpha/\beta) = 0$

Orthogonal set—Let V be a vector space $S \subset V$. The set S is orthogonal set if $\forall\ \alpha, \beta \in S$

$$\Rightarrow \quad (\alpha/\beta) = 0, \alpha \neq \beta.$$

Orthonormal set—The orthogonal set S with property

$$\|\alpha\| = 1, \forall\ \alpha \in S.$$

Orthogonal Complement—Let V be a inner product space and $S \subset V$. The orthogonal complement of S is the set $S^{\perp}$ of all vectors in V which are orthogonal to every vector in S.

Standered inner product—On F^n, $\alpha = (x_1 \ldots\ldots x_n)$, $\beta = (y_1 \ldots\ldots y_n)$ and $(\alpha/\beta) = \sum_i x_i \overline{y_i}$

when F = R,

$$(\alpha/\beta) = \sum_i x_i\, y_i = \alpha\beta$$

(the dot product)

Some Important Theorems

1. If V is an inner product space, then for any α, $\beta \in$ V and c any scalar.
   (i) $\| c\alpha \| = |c| \, \|\alpha\|$
   (ii) $\|\alpha\| > 0$ for $\alpha \neq 0$
   (iii) $|(\alpha/\beta)| \leq \|\alpha\| \, \|\beta\|$ (Cauchy-Schwarz-inequality)
   (iv) $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$.
2. An orthogonal set of non-zero vectors is linearly independent.
3. If a vector β is a linear combination of an orthogonal sequence of non-zero vectors α_1 …… α_n, then β is the particular combination
$$\beta = \sum_{i=1}^{n} \frac{\left(\frac{\beta}{\alpha}\right)}{\|\alpha\|^2} \alpha_i.$$
4. Let V be an inner product space and let β_1 … β_n be any independent vectors in V. then one may construct orthogonal vectors α_1 …… α_n $\in$ V such that each K = 1, 2, ……, n the set $\{\alpha_1 \ldots\ldots \alpha_k\}$ is a basis for the subspace spanned by β_1 …… β_k.
5. Every finite dimensional inner product space has an orthonormal basis.
6. Let V be an inner product space, W is finite dimensional subspace and E the orthogonal projection of V on W. Then the mapping $\beta \to \beta - E\beta$ is the orthogonal projection of V on W^1.
7. Let W be a finite dimensional subspace of an inner product space V and E be the orthogonal projection of V on W, then E is an idempotent linear transformation of V onto W, $W^{\perp}$ is the null space of E and.
$$V = W \oplus W^{\perp}.$$
8. Let $\{\alpha_1 \ldots\ldots \alpha_n\}$ be an orthogonal set of non-zero vectors in an inner product space V. If $\beta \in$ V, then
$$\sum_{K} = \frac{|(\beta/\alpha_K)|^2}{\| \alpha \|^2} \leq \| \beta \|^2.$$
and the equality holds iff
$$\beta = \sum_{K} \frac{(\beta/\alpha_K)}{\| \alpha_K \|^2} \alpha_K$$

- Orthogonal complement of inner product space V is zero subspace {0} and $\{0\}^{\perp}$ = V
- If $S \subset V$, then $S^{\perp}$ is always a subspace of V.

Gram-Schmidt Orthonormalization Process

Let $S = \{x_1, x_2, \ldots\ldots\}$ be a linearly independent sequence in an inner product space. Then there exists an orthonormal sequence
$$T = \{y_1, y_2, \ldots\ldots\}$$
such that span (S) = span (T)

Since S is linearly independent $x_K \neq 0$ for each K. Define
$$y_1 = \frac{x_1}{\| x_1 \|} \text{ so that } \| y_1 \| = 1$$
Define $\quad V = x_2 - (x_2, y_1)y_1$

Then $V \perp y_1$ and $V \neq 0$, since $\{x_1, x_2\}$ is linearly independent.

Hence $y_2 = \frac{V}{\|V\|}$ is orthogonal to y_1 and $\|y_2\|$ = 1. We now inductively define.
$$V = x_n - \sum_{K=1}^{n-1} (x_n, y_K)\, y_K$$
and
$$y_n = \frac{V}{\|V\|}$$
It is clear from the construction that span (S) = span (T).

Self adjoint operators

Adjoint—Let T be a linear operator on an inner product space V. Then we say T has an adjoint on V if there exists a linear operator T^* on V such that
$$\left(T\frac{\alpha}{\beta}\right) = \left(\frac{\alpha}{T^*}\beta\right) \text{ for all } \alpha, \beta \in V.$$

Some Important Theorem

1. Let V be a finite dimensional inner product space and f a linear functional on V. Then there is a unique vector $\beta \in$ V such that
$$f(\alpha) = \left(\frac{\alpha}{\beta}\right) \text{ for } \alpha \in V.$$
2. Let V be a finite dimensional inner product space and let $B = \{\alpha_1 \ldots\ldots \alpha_n\}$ be an orthonormal basis for V. Let T be a inner operator on V and let A be the matrix of T in theordared basis B. Then $A_{Kj} = \left(\frac{T\alpha_j}{\alpha_K}\right)$
3. Let V be a finite dimensional inner product space, and let T be a linear operator on V. In any orthonormal basis for V, the matrix of T^* is the conjugate transpose of the matrix of T.

4. Let V be a finite dimensional inner product space. If T and U are linear operators on V and c is a scalar.
   (i) $(T+U)^* = T^* + U^*$
   (ii) $(cT)^* = c(T)^*$
   (iii) $(TU)^* = U^*T^*$
   (iv) $(T^*)^* = T.$

Some Solved Examples

Example 1. Find the dimension of V where V $= \{a_0 + a_1x + a_2x^2 + a_3x_i^3, x \in R\}$.

Solution : S $= \{1, x, x^2, x^3\}$ is a basis and V has therefore, dimension.

Example 2. Standard basis of c^n is an orthonormal set with respect to standard inner product.

Solution :

$$S = \{\varepsilon_1, \varepsilon_2, \ldots\ldots \varepsilon_n\}$$

is a standard basis of c^n, where

$$\varepsilon_i = \{0, \ldots\ldots 1, 0, 0\ldots\ldots 0)$$

For standard inner product $(a)\ \forall \left(\dfrac{\varepsilon_i}{\varepsilon_j}\right) = 0$

$$i \neq i,\ i = 1, \ldots\ldots n$$

and $\left(\dfrac{\varepsilon_i}{\varepsilon_j}\right) = 1\ \forall\, i = 1 \ldots\ldots n$

$\therefore$ Set S $= \{\varepsilon_1 \ldots\ldots, \varepsilon_n\}$ is an orthonormal set.

Example 3. Find the dimension of the vector space $c(R)$ of the complex number over real numbers ?

Solution : Set $\{i, j\} \subset c$ forms a basis for c

$\because a.1\,(+)\,b.\,i = 0 \Rightarrow a = 0, b = 0, a, b \in R$

$\therefore$ 1, i are linearly independent

$\forall\ a + b_i \in c,\ a, b \in R \Rightarrow \{1, i\}$ span $c(R)$.

Example 4. Find the condition $\{\alpha + i\beta, a + ib\}$, $\alpha, \beta, a, b \in R$ is a basis for vector space c over R.

Solution : $(\alpha + i\beta)$ and $(a + ib)$ to be linearly inde-pendent

$$\Rightarrow \quad \theta_1(a+ib) + \theta_2(\alpha + i\beta) = 0$$

$$\theta_1 a + \theta_2 \alpha = 0$$

$$\theta_1 b + \theta_2 \beta = 0$$

$$\Rightarrow \quad \theta_1\,(a\beta - b\alpha) = 0$$

$$\theta_2(a\beta - b\alpha) = 0$$

$\Rightarrow \theta_1 = 0 = \theta_2 \Leftrightarrow a\beta - b\alpha = 0$

$\therefore$ If $a\beta = b\alpha$, then $\{\alpha + i\beta, a + ib\}$ is a basis.

Example 5. Let V be a finite dimensional vector space, W_1, …… W_K be subspaces of V such that

$$V = W_1 + \ldots\ldots + W_K$$

and $\dim V = \dim W_1 + \ldots\ldots + \dim W_K$

Prove that

$$V = W_1 \oplus \ldots\ldots \oplus W_K.$$

Solution :

$$V = W_1 + \ldots\ldots\ W_K$$

$\Rightarrow \quad \dim V \leq \dim W_1 + \ldots\ldots + \dim W_K$

This inequality converts into equality if W_1 …… W_K are linearly independent

$\Rightarrow W = W_1 \oplus W_2 \oplus \ldots\ldots \oplus W_K$ is the direct sum of W_1 …… W_K.

OBJECTIVE TYPE QUESTIONS

1. Let V be a vector space, T is a linear transform on V into V such that $T\alpha = 0$, $\forall\ \alpha \in V$—
   (A) T is identity transform
   (B) T is zero transform
   (C) T is invertible
   (D) T is orthogonal

2. Let dimension of a vector space V be dim V $= n$. If any set S $\subset$ V and shave m elements, $m > n$, then—
   (A) S is linearly independent
   (B) S is linearly dependent
   (C) S is zero subspace
   (D) None of these

3. If A is a matrix of order n, then A is invertible iff—
   (A) $A \neq 0$ (B) $A^{-1} = 0$
   (C) $|A| \neq 0$ (D) $|A| = 0$

4. Let S be an orthonormal set, $\alpha \in S$, then—
   (A) $\left(\dfrac{\alpha}{\alpha}\right) = 1$ (B) $\left(\dfrac{\alpha}{\alpha}\right) = 0$
   (C) $\left(\dfrac{\alpha}{\alpha}\right) > 0$ (D) $\left(\dfrac{\alpha}{\alpha}\right) < 1$

5. Let S be an orthonormal set, then for $\alpha \in S$—
   (A) $\|\alpha\| = 0$ (B) $\|\alpha\| > 0$
   (C) $\|\alpha\| = 1$ (D) $\|\alpha\| < 1$

6. If V is a vector space, f is a linear functional on V, then the vector space V^* is dual space of V if V^* is—
   (A) A collection of all linear functional f on V.
   (B) A collection of all vector spaces on which f is defined
   (C) Collection of all linear operators
   (D) None of these

7. Let A and B, C are two matrices of order n, then—
   (A) $|AB| = |A|\,|B|$ (B) $|AB| \neq |A|\,|B|$
   (C) $|AB| > |A|\,|B|$ (D) $|AB| < |A|\,|B|$

8. If A is square matrix of order n, then—
   (A) $|\text{adj } A| \neq A$ (B) $|\text{adj } A| = |A|$
   (C) $|A| \neq |A^T|$ (D) None of these

9. The orthogonal complement of inner product space V is—
   (A) Zero subspace {0}
   (B) V itself
   (C) Any subspace
   (D) None of these

10. If {0} is a zero subspace of inner product space V, then $\{0\}^{\perp}$ is equal to—
    (A) {0} (B) V
    (C) ϕ (D) None of these

11. The zero subspace of inner product space consist—
    (A) Zero element only
    (B) Non-zero elements
    (C) Identity
    (D) None of these

12. If V is an inner product space, then for $\alpha\ \beta \in V$—
    (A) $\left|\left(\frac{\alpha}{\beta}\right)\right| = \|\alpha\|\,\|\beta\|$
    (B) $\left|\left(\frac{\alpha}{\beta}\right)\right| \geq \|\alpha\|\,\|\beta\|$
    (C) $\left|\left(\frac{\alpha}{\beta}\right)\right| \leq \|\alpha\|\,\|\beta\|$
    (D) None of these

13. If V is an inner product space, then for $\alpha, \beta \in V$—
    (A) $\|(\alpha + \beta)\| \geq \|\alpha\|\,\|\beta\|$
    (B) $\|(\alpha + \beta)\| = \|\alpha\|\,\|\beta\|$
    (C) $\|(\alpha + \beta)\| \leq \|\alpha\|\,\|\beta\|$
    (D) None of these

14. If V is an inner product space, then for $\alpha \in V$—
    (A) $\|\alpha\| \leq 0$ for $\alpha \neq 0$
    (B) $\|\alpha\| \geq 0$ for $\alpha \neq 0$
    (C) $\|\alpha\| = 0$ for $\alpha \neq 0$
    (D) $\|\alpha\| > 0$ for $\alpha \neq 0$

15. The Cauchy-Schwarz inequality states—
    (A) $\left|\left(\frac{\alpha}{\beta}\right)\right| \geq \|\alpha\|\,\|\beta\|$
    (B) $\left|\left(\frac{\alpha}{\beta}\right)\right| \leq \|\alpha\|\,\|\beta\|$
    (C) $\|\alpha + \beta\| \geq \|\alpha\| + \|\beta\|$
    (D) None of these

16. If V is an inner product space $\alpha, \beta \in V$ and c any scalar, then—
    (A) $\|c\alpha\| = \|c\|\,\|\alpha\|$ (B) $\|c\alpha\| \geq \|c\|\,\|\alpha\|$
    (C) $\|c\alpha\| \leq \|c\|\,\|\alpha\|$ (D) None of these

17. For vectors $\alpha, \beta \in V$ and scalar c, the inner product $\left(\frac{\alpha}{\alpha}\right)$ is—
    (A) Greater than zero, for $\alpha \neq 0$
    (B) Equals to zero, for $\alpha \neq 0$
    (C) Less than zero, for $\alpha \neq 0$
    (D) None of these

18. For vector $\alpha, \beta, \gamma \in V(F)$, the inner product of $(\alpha + \beta/\gamma))$ is equal to—
    (A) $\left(\frac{\alpha}{\gamma}\right)\beta$ (B) $\alpha\gamma + \beta\gamma$
    (C) $(\alpha/\gamma) + (\beta/\gamma)$ (D) None of these

19. For the vectors $\alpha, \beta, \gamma \in V(F)$ and non zero scalar, the inner product of $(\alpha/c\beta + \gamma)$ is equal to—
    (A) $\bar{c}\left(\frac{\alpha}{\beta}\right) + \left(\frac{\alpha}{\gamma}\right)$ (B) $\bar{c}\left(\frac{\beta}{\alpha}\right) + \left(\frac{\gamma}{\alpha}\right)$
    (C) $\bar{c}\,(\alpha) + \beta$ (D) $\bar{c}\ \alpha\gamma + \alpha\beta$

20. The norm of α with respect to inner product (α/α) is—
(A) $\|\alpha\| = \left(\frac{\alpha}{\alpha}\right)$ (B) $\|\alpha\|^2 = \left(\frac{\alpha}{\alpha}\right)$
(C) $\|\alpha\| = \left(\frac{\alpha}{\alpha}\right)^2$ (D) None of these

21. The equality in Cauchy-Schwarz inequality $(\alpha/\beta) \le \|\alpha\| \, \|\beta\|$ occurs, when—
(A) α and β are linear independent
(B) α and β are linear dependent
(C) α and β are non zero
(D) None of these

22. An orthogonal set of non-zero vectors—
(A) Linearly independent
(B) Linearly dependent
(C) Constant
(D) None of these

23. Let V be an inner product space and $\alpha, \beta \in$ V, then α and β are the orthogonal to each other if—
(A) $\left(\frac{\alpha}{\beta}\right) > 0$ (B) $\left(\frac{\alpha}{\beta}\right) < 0$
(C) $\left(\frac{\alpha}{\beta}\right) = 0$ (D) None of these

24. If V is a inner product space then for every $\alpha \in$ V—
(A) Zero vector is orthogonal to α
(B) Zero vector is not orthogonal to α
(C) Zero vector does not exist
(D) None of these

25. Let A and B are two matrix A and B are similar matrix, then—
(A) A, B have same characteristic polynomial
(B) A,B may have different characteristic polynomial
(C) A, B have same value
(D) A, B have different value

26. Let V be a vector space and T a linear operator on V. If W is a subspace of V, W is invariant under T if—
(A) $T(W) \subset W$
(B) $W \subset T(W)$
(C) $T(W) = W$
(D) None of these

27. The vectors $\alpha_1 \ldots\ldots \alpha_n$ are linearly dependent if for scalars $c_1 \ldots\ldots c_n$, $c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots + c_n\alpha_n = 0$ implies—
(A) $c_1 \ldots\ldots c_n$ are not zero
(B) $c_1 = c_2 = \ldots\ldots = c_n = 0$
(C) $c_1 = c_2 = \ldots\ldots = c_n$
(D) None of these

28. If B and B' are two basis of vector space V, then—
(A) B and B' have some numbers of elements
(B) B and B' have distinct number of elements
(C) B = B'
(D) None of these

29. If V is a finite dimensional vector space, then if W is a subspace of V, then—
(A) W is finite dimensional
(B) W is infinite dimensional
(C) The dimensional of W is greater than V
(D) None of these

30. If A is a matrix, then—
(A) Row rank (A) = column rank (B)
(B) Row rank (A) $\neq$ column rank (B)
(C) Row rank (A) > column rank (B)
(D) None of these

31. If w_1 and w_2 are subspaces of V, then following is false—
(A) $w_1 \cup w_2$ is a subspace of V
(B) $w_1 \cap w_2$ is a subspace of V
(C) $w_1 + w_2$ is a subspace of V
(D) $w_1 \cup w_2$ is not a subspace ofV.

32. If dim W = m, dim V = n, and $W \subset V$ dim $\left(\frac{V}{W}\right)$ is—
(A) $m + n$ (B) $n - m$
(C) $m - n$ (D) None of these

33. A linear transformation T from V onto W, is non-singular if—
(A) $T\alpha = 0 \Rightarrow \alpha = 0$
(B) $T\alpha = 0 \Rightarrow \alpha \neq 0$
(C) $T\alpha = c \Rightarrow \alpha = 0$
(D) None of these

34. If T is a linear transformation from vector space V into vector space W. Let dim V = m, dim W = n, then rank of T is—
(A) m (B) n
(C) $m - n$ (D) $m + n$

35. Let 0 be a zero vector in vector space V, then $\{0\}$ is—
(A) Zero subspace of V
(B) Null space of V
(C) Identity space of V
(D) None of these

36. The dimension of the vector space is—
(A) Number of elements in vector space
(B) Number of elements in basis of the vector space
(C) Subspace of vector space
(D) None of these

37. If w_1 and w_2 are finite dimensional subspaces of vector space V, then—
(A) $\dim (w_1 + w_2) = \dim w_1 + \dim w_2$
(B) $\dim (w_1 + w_2) = \dim w_1 + \dim w_2 + \dim (w_1 \cup w_2)$
(C) $\dim (w_1 + w_2) = \dim w_1 + \dim w_2 - \dim (w_1 \cap w_2)$
(D) $\dim (w_1 + w_2) = \dim w_1 + \dim w_2 + \dim (w_1 \cap w_2)$

38. Let V and w be finite dimensional vector spaces dim V = n and dim $w = n$, then dim L(V, w) is—
(A) mn (B) $m + n$
(C) m/n (D) $m - n$

39. If V is a vector space with dim V = n, then dimension of hyper space of V is—
(A) n (B) $n - 1$
(C) $n + 1$ (D) 0

40. If V is a finite dimensional vector space and let w is a subspace of V then—
(A) $\dim w + \dim w^0 = \dim V$
(B) $\dim w - \dim w^0 = \dim V$
(C) $(\dim w)(\dim w^0) = \dim V$
(D) $\dfrac{\dim w}{\dim w^0} = \dim V$

41. If w_1 and w_2 are subspaces of a finite dimensional vector space. Then $w_1 = w_2$ iff—
(A) $w_1{}^0 = w_1{}^0$
(B) $w_1{}^0 - w_1{}^0 = \{0\}$
(C) $w_1{}^0 \neq w_1{}^0$
(D) None of these

42. If w_1 and w_2 are subspaces of finite dimensional space, then $w_1{}^0 = w_2{}^0$ iff—
(A) $w_1 = w_2$ (B) $w_1 \neq w_2$
(C) $w_1 \subset w_2$ (D) $w_1 \supset w_2$

43. If w is the proper subspace of a vector space v, then—
(A) $\dim v < \dim w$ (B) $\dim w < \dim v$
(C) $\dim w = \dim v$ (D) None of these

44. If v is a vector space, then dimension of v is equal to—
(A) Number of element of vector space v
(B) Number of element in a basis for v
(C) Number of non zero elements of v
(D) None of these

45. If w_1 and w_2 are two subspaces of vector space, then—
(A) $\dim (w_1 + w_2) \leq \dim w_1 + \dim w_2$
(B) $\dim (w_1 + w_2) \geq \dim w_1 + \dim w_2$
(C) $\dim (w_1 + w_2) = \dim w_1 + \dim w_2$
(D) None of these

46. If w_1 and w_2 are disjoint subspaces of vector space, then—
(A) $w_1 \cap w_2 = 0$ (B) $w_1 \cap w_2 = \{0\}$
(C) $w_1 \cup w_2 = \{0\}$ (D) None of these

47. If v is a vector space, a hyper space in v is—
(A) Maximal subset of v
(B) Maximal subspace of v
(C) Minimal subspace of v
(D) Minimal subset of v

48. If v is n dimensional vector space and w is m dimensional vector space over the same field then space L(v, w) has the dimension—
(A) $m + n$ (B) $m - n$
(C) mn (D) m/n

49. If T is linear operator on v, then—
(A) $T^3 = T.T.T.$ (B) $T^3 = T + T + T$
(C) $T^3 = T^2 + T$ (D) None of these

50. For identity transformation I, on finite dimensional vector space, the rank of I is—
(A) dim V (B) 0
(C) 1 (D) None of these

51. Let V be finite dimensional vector space, T is a zero transformation on V, then null space of T is—
(A) V (B) {0}
(C) ϕ (D) None of these

52. Let V be a finite dimensional space. T is zero transformation on V. Then range of T is—
(A) {0} (B) V
(C) ϕ (D) None of these

53. If T is a zero transformation on finite vector space V. The rank of T is—
(A) 0 (B) 1
(C) dim V (D) None of these

54. If T is a zero transformation on finite vector space V. Then nullity of T is—
(A) dim V (B) Zero
(C) 1 (D) None of these

55. Let T be a linear transformation on finite vector space V. Then—
(A) Rank T < dim V
(B) Rank T = dim V
(C) Rank T > dim V
(D) Rank T = nullity (V)

56. Let V be a vector space and (1) an inner product on V, then $(0/\beta)$ is equal to, for all $\beta \in$ V—
(A) Zero (B) Greater than zero
(C) Less than zero (D) None of these

57. Let V be a vector space, (1) an inner product on V, then If $(\alpha/\beta) = 0$ for all $\beta \in$ V—
(A) $\alpha \neq 0$ (B) $\alpha = 0$
(C) $\alpha > 0$ (D) $\alpha < 0$

58. Let V be a vector space over F.
(a) The sum of two inner product is an inner product.
(b) The difference of two inner product is an inner product.
(A) (a) and (b) both false
(B) (a) and (b) both true
(C) (a) is true, (b) is false
(D) (a) is false, (b) is true

59. Let (1) an standard inner product on R^2. If $(\alpha/\gamma) = -1$ and $(\beta/\gamma) = 3$, given $\alpha = (1, 2)$, $\beta = (-1, 1)$ the value of γ is—
(A) (0, 1) (B) (0, 3)
(C) $\left(0, \frac{2}{3}\right)$ (D) (0, 0)

60. If A = $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then characteristic polynomial for A is—
(A) $x^2 + 1$ (B) $x + 1$
(C) $x - 1$ (D) $x^2 - 1$

61. Let A = $\begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$, then the characteristic polynomial for A is—
(A) $x^3 + 5x^2 + 8x + 4$ (B) $x^2 + 5x$
(C) $x^3 - 5x^2 + 8x - 4$ (D) None of these

62. Let T be a linear operator on a finite dimensional space V and c_1 …… c_K be the distinct characteristic values of T. Let w_i be null spaces of $(T - c_iI)$. If T is diagonal *lizable* then—
(A) (dim w_1) (dim w_2) (dim w_3) …… (dim w_K) = dim V
(B) dim w_1 + dim w_2 + … + dim w_K = dim V
(C) dim w_1 – dim w_2 + … – dim w_K = dim V
(D) None of these

63. A linear operator E on vector space V is projection if—
(A) $E(\alpha) = \alpha$ (B) $E^2(\alpha) = E(\alpha)$
(C) $E^3(\alpha) = E(\alpha)$ (D) None of these

64. The zero subspace have the dimension—
(A) One (B) Two
(C) Three (D) Zero

65. The vector (x, y) and $(-y, x)$ with respect to standard inner product are—
(A) Orthonormal (B) Continuous
(C) Orthogonal (D) None of these

66. Let V be a finite dimensional vector space. If V^* is the dual of V, then—
(A) dim V = dim V^*
(B) dim V > dim V^*
(C) dim V < dim V^*
(D) None of these

67. If two vectors α and β are linearly dependent then for some scalar c—
(A) $\alpha = c\beta$ (B) $\alpha = c + \beta$
(C) $\alpha = c - \beta$ (D) None of these

68. If $\{\alpha_1 \ldots\ldots \alpha_n\}$ is a basis of V and if $\{\beta_1 \ldots \beta_n\}$ are linearly dependent in V then—
(A) $m = n$ (B) $m \geq n$
(C) $m \leq n$ (D) None of these

69. Let A be a $m \times n$ matrices with row rank $= r =$ column rank the dimension of the space of solutions of the system of linear equations $AX = 0$ is—
(A) r (B) $n - r$
(C) $m - r$ (D) $\min(m, n) - r$

70. A matrix M has eigen values 1 and 4 with corresponding eigen vectors $(1, -1)^T$ and $(2, 1)^T$, respectively. then M is—
(A) $\begin{bmatrix} -4 & -8 \\ 5 & 9 \end{bmatrix}$ (B) $\begin{bmatrix} 9 & -8 \\ 5 & -4 \end{bmatrix}$
(C) $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ (D) $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

71. Let PID, ED, UFD denote the set of all principal ideal domains Euclidean domains, unique factorization domain respectively then—
(A) $UFD \subset ED \subset PID$
(B) $PID \subset ED \subset UFD$
(C) $ED \subset PID \subset UFD$
(D) $PID \subset UFD \subset ED$

72. The Hermite interpolating polynomial for the function $f(x) = x^6$ based on -1, 0 and 1 is—
(A) $x^4 - 2x^2$ (B) $2x^4 - x^2$
(C) $x^4 + 2x^2$ (D) $2x^4 + x$

73. The system of equations
$$\begin{aligned} 3x + 2y &= 4{\cdot}5 \\ 2x + 3y - z &= 5{\cdot}0 \\ -y + 2z &= -0{\cdot}5 \end{aligned}$$
is to be solved by successive over relaxation method. The ortimal relaxation factor w_{opt}, rounded upto two decimal places is given by—
(A) 1·23 (B) 0·78
(C) 1·56 (D) 0·63

74. In a matric space (x, d)—
(A) Every infinite set E has a limit point in E
(B) Every closed subset of a compact set is compact
(C) Every closed and bounded set is compact
(D) Every subset of a compact set is closed

75. Let (x, d) be a complete matric space and $f : x \to x$ satisfies $d\, f(x), f(y) \leq \alpha\, (x, y)$ for some α, $0 \leq \alpha < 1$ for all $x, y, \in x$, then—
(A) f is bonnded function an x
(B) f need not be continuous on x
(C) $\{f(x_n)\}_{n=1}^{\infty}$ may not be a couchy sequence even though $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in x
(D) $f(P) = P$ for some $P \in x$.

76. Let $A \in c^{m \times n}$ and A', A^* denote respectively the transpose and conjugate transpose of A, then—
(A) Rank (AA^*A) = rank (A)
(B) Rank (A) = rank (A^2)
(C) Rank (A) = rank (A'A)
(D) Rank (A^2) – rank (A) = rank (A^3) – rank (A^2)

77. Consider 2×2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
if $a + d = 1 = ad - bc$, then A^3 equals—
(A) 0 (B) $-I$
(C) 31 (D) None of these

78. The sequence $\{x_n\}$ of $m \times m$ matrices difined by the iterations
$x_{n+1} = 2x_n - x_n, Ax_n, n = 0, 1, 2$
when $x_0 = I$ the identity matrix converges to A^{-1}, if and only if each eigen value λ of A satisfies—
(A) $|\lambda^\circ| < 1$ (B) $|\lambda - 1| < 1$
(C) $|\lambda + 1| < 1$ (D) None of the above

79. Let P be a matrix of order $m \times n$ and Q be a matrix of order $n \times p$, $n \neq p$. if rank (P) = n and rank (Q) = P, then rank (PQ) is—
(A) n (B) p
(C) np (D) $n + p$

80. Let P and Q be square matrices such that PQ = I, the identity matrix. Then zero is an eigen value of—
(A) P but not of Q (B) Q but not of P
(C) both P and Q (D) Neither P nor Q

81. Let W be the space spanned by $f = \sin x$ and $g = \cos x$, then for any real of θ, $f_1 = \sin(x + \theta)$ and $g_1 = \cos(x + \theta)$—
(A) are vectors in W
(B) are linearly independent
(C) do not form a basis for W
(D) Form a basis for W

82. Consider the basis $S = \{v_1, v_2, v_3\}$ for R^3 where $v_1 = (1, 1, 1)$, $v_2 = (1, 1, 0)$, $v_3 = (1, 0, 0)$ and let $T : R^3 \to R^2$ be a linear transformation such that
$Tv_1 = (1, 0)$, $Tv_2 = (2, -1)$, $Tv_3 = (4, 3)$. Then $T(2, -3, 5)$ is—
(A) $(-1, 5)$ (B) $(3, 4)$
(C) $(0, 0)$ (D) $(9, 23)$

83. For $0 < \theta < \pi$ the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
(A) Has no real eigen value
(B) Is orthogonal
(C) Is symmetric
(D) Is skew symmetric

84. The eigen values of a 3×3 real matrix P are $1, -2, 3$. Then—
(A) $P^{-1} = \frac{1}{6}(5I + 2P - P^2)$
(B) $P^{-1} \frac{1}{6}(5I - 2P + P^2)$
(C) $P^{-1} = \frac{1}{6}(5I - 2P - P^2)$
(D) $P^{-1} = \frac{1}{6}(5I + 2P + P^2)$

85. Let $T : c^n \to c^n$ be a linear operator having n distinct eigen values then—
(A) T is inversible
(B) T is inversible as well is digonalizable
(C) T is not diagonalizable
(D) T is diagonalizable

86. Let U be a 3×3 complex Hermitian matrix which is unitary then the distinct eigen values of U are—
(A) $\pm i$ (B) $1 \pm i$
(C) ± 1 (D) $\frac{1}{2}(1 \pm i)$

87. The polynomial $f(x) = x^5 + 5$ is—
(A) Irreducible over C
(B) Irreducible over R
(C) Irreduciable over Q
(D) Not irreducible over Q

where Q denotes the field of rational number.

88. Let T be the matrix (occurring in a typical transportation problem) given by

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Then—
(A) Rank T = 4 and T is unimodular
(B) Rank T = 4 and T is not unimodular
(C) Rank T = 3 and T is unimodular
(D) Rank T = 3 and T is not unimodular

89. Let A be an $n \times n$ complex matrix whose characteristic polynomial is given by
$f(t) = t^n + c_{n-1}t^{n-1} + \ldots\ldots + c_1 t + c_0$, then
(A) $\det(A) = c_{n-1}$
(B) $\det(A) = c_0$
(C) $\det(A) = (-1)^n c_{n-1}$
(D) $\det(A) = (-1)^n c_0$

90. Let A be any $n \times n$ non singular complex matrix and let $B = (\bar{A})^t A$, where $(\bar{A})^t$ is the conjugate transpose of A. If λ is an eigen value of B, then—
(A) λ is real and $\lambda < 0$
(B) λ is real and $\lambda \leq 0$
(C) λ is real and $\lambda \geq 0$
(D) λ is real and $\lambda > 0$

91. Let $T : c^n \to c^n$ be a linear operator of rank $n - z$ then—
(A) 0 is not an eigen value of T
(B) 0 must be an eigen value of T
(C) 1 can never be an eigen value of T
(D) 1 must be an eigen value of T

92. The dimension of the vector space of all 3×3 real symmetric matrices is—
(A) 3 (B) 9
(C) 6 (D) 4

93. Let A be a non zero upper triangular matrix all of whose eigen values are 0, then I + A is—
(A) Invertible (B) Singular
(C) Idempotent (D) Nilpotent

94. The eigen values of a skew symmetric matrix are—
(A) Negative
(B) Singular
(C) Of absolute value 1
(D) Purely imaginary or zero

95. Which of the following Banach spaces is not separable ?
(A) L^1 [0, 1] (B) L^∞ [0, 1]
(C) L^2 [0, 1] (D) C [0, 1]

96. For a subset A of a metric space, which of the following implies the other three ?
(A) A is closed
(B) A is bounded
(C) Closure of B is compact for every $B \subseteq A$
(D) A is compact

97. Let T be an arbitrary linear transformation form R^n which is not one-one, then—
(A) Rank T > 0 (B) Rank T = n
(C) Rank T < n (D) Rank T = $n - 1$

98. Let T be a linear transformation form $R^3 \to R^2$ defined by $T(x, y, z) = (x + y, y - z)$. Then the matrix of T with respect to the ordered bases {(1, 1, 1), (1, – 1, 0), (0, 1, 0)} and {(1, 1), (1, 0)} is—
(A) $\begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$
(C) $\begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 1 & 0 \end{bmatrix}$

99. Let the characteristics equations of a matrix M be $\lambda^2 - \lambda - 1 = 0$, then—
(A) M^{-1} does not exist
(B) M^{-1} exist but cannot be determined from the data
(C) $M^{-1} = M + 1$
(D) $M^{-1} = M - 1$

100. Consider the matrix $M = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ and let S_M be the set of 3 × 3 matrices N such that MN = 0. Then the dimension of the real vector S_M is equal to—
(A) 0 (B) 1
(C) 2 (D) 3

101. Choose the correct matching from A, B, C and D for the transformation T_1, T_2 and T_3 (mappings from R^2 to R^3) as defind in group 1 with the statements given in group 2.

Group 1

P. $T_1(x, y) = (x, x, 0)$
Q. $T_2(x, y) = (x, x + y, y)$
R. $T_3(x, y) = (x, x + 1, y)$

Group 2

1. Linear transformation of rank 2
2. Not a linear transformation
3. Linear transformation of rank 1

(A) P – 3, Q – 1, R – 2
(B) P – 1, Q – 2, R – 3
(C) P – 3, Q – 2, R – 1
(D) P – 1, Q – 3, R – 2

102. Let $M = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & -4 & 0 & 0 \end{bmatrix}$. Then—
(A) $MM^T = I$ where M^T is the transpose of M and I is the identify matrix
(B) Column vectors of M from an orthogonal system of vectors
(C) Column vectors of M from an orthonormal system of vectors
(D) $(Mx, My) = (x, y)$ for all x, y in R^4 where (,) is the standard inner product on R^4.

103. Let $M = \begin{bmatrix} 1 & 1+i & 2i & 9 \\ 1-i & 3 & 4 & 7-i \\ -2i & 4 & 5 & i \\ 9 & 7+i & -i & 7 \end{bmatrix}$. Then—
(A) M has a purely imaginary eigen values
(B) M is not diagonalizable
(C) M has eigen values which are neither real nor purely imaginary
(D) M has only real eigen values

104. Consider the matrix $M = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$ where a, b and c are non-zero real numbers. Then the matrix has—
(A) Three non-zero real eigen values
(B) Complex eigen values
(C) Two non-zero eigen value
(D) Only one non-zero eigen value

105. The minimal polynomial of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ is—

(A) $(x-1)^2(x-2)$
(B) $(x-1)(x-2)^2$
(C) $(x-1)(x-2)$
(D) $(x-1)^2(x-2)^2$

106. The set of all $x \in R$ for which the vectors $(1, x, 0)$, $(0, x^2, 1)$ and $(0, 1, x)$ are linearly independent in R^3 is—

(A) $\{x \in R : x = 0\}$
(B) $\{x \in R : x \neq 0\}$
(C) $\{x \in R : x \neq 1\}$
(D) $\{x \in R : x \neq -1\}$

107. Consider the vector space R^3 and the maps f, $g : R^3 \to R^3$ defined by $f(x, y, z) = (x, |y|, z)$ and $g(x, y, z) = (x+1, y-1, z)$. Then—

(A) Both f and g are linear
(B) Neither f nor g is linear
(C) g is linear but not f
(D) f is linear but not g.

108. Let $M = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}$. Then—

(A) M is diagonalizable but not M^2
(B) M^2 is diagonalizable but not M
(C) Both M and M^2 are diagonalizable
(D) Neither M nor M^2 is diagonalizable

109. Let M be a skew symmetric, orthogonal real matrix, the only possible eigen values are—

(A) $-1, 1$ (B) $-i, i$
(C) 0 (D) $1, i$

110. Let S and T be two linear operators on R^3 defined by $S(x, y, z) = (x, x+y, x-y-z)$
$T(x, y, z) = (x+2z, y-z, x+y+z)$. Then—

(A) S is invertible but not T
(B) T is invertible but not S
(C) Both S and T are invertible
(D) Neither S nor T is invertible

111. Let V, W and X be three finite dimensional vector spaces such that dim V = dim X. Suppose $S : V \to W$ and $T : W \to X$ are two linear maps such that to $S : V \to X$ is injective. Then—

(A) S and T are surjective
(B) S is surjective and T is injective
(C) S and T are injective
(D) S is injective and T is surjective

112. If a square matrix of order 10 has exactly 4 distinct eigen values, then the degree of its minimal polynomial is—

(A) At least 4 (B) At most 4
(C) At least 6 (D) At most 6

113. Consider the matrix $M = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$.

Then—

(A) M has no real eigen values
(B) All real eigen values of M are positive
(C) All real eigen values of M are negative
(D) M has both positive and negative real eigen values

114. Consider the real inner product space p [0, 1] of all polynomials with the inner product $\langle f, g \rangle = \int_0^1 f(x)\, g(x)\, dx$. Let M = span {1} the orthogonal projection of x^2 on to M is—

(A) 1 (B) $\frac{1}{2}$
(C) $\frac{1}{3}$ (D) $\frac{1}{4}$

115. The matrix of T^{-1} with respect to the basis $\{1, x, x^2\}$ is—

(A) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

116. The dimension of the eigen space of T^{-1} corresponding to the eigen value 1 is—

(A) 4 (B) 3
(C) 2 (D) 1

Answers with Hints

1. (B)	2. (B)	3. (C)	4. (A)	5. (C)
6. (A)	7. (A)	8. (B)	9. (A)	10. (B)
11. (A)	12. (C)	13. (C)	14. (D)	15. (B)
16. (A)	17. (A)	18. (C)	19. (A)	20. (B)
21. (B)	22. (A)	23. (C)	24. (A)	25. (A)
26. (A)	27. (B)	28. (A)	29. (A)	30. (A)
31. (A)	32. (B)	33. (A)	34. (A)	35. (A)
36. (B)	37. (C)	38. (A)	39. (B)	40. (A)
41. (A)	42. (A)	43. (B)	44. (B)	45. (A)
46. (B)	47. (B)	48. (C)	49. (A)	

50. (A) Rank $\quad T = \text{dim (range of I)}$
$$= \text{dim V.}$$

51. (A) Null space of
$$T = \{\alpha : T\alpha = 0, \alpha \in V\}$$
$$= V$$

52. (A) Range of $T = \{\beta : T\beta, \beta \in V]$
$$= \{0\}$$

53. (A) Rank of $\quad T = \text{dim \{range of T\}}$
$$= \dim\{0\}$$
$$= 0$$

54. (A) Nullity $\quad T = \text{dim \{null space of T\}}$
$$= \dim(V)$$

55. (A) Rank T + nullity
$$T = \dim V$$
$$\Rightarrow \quad \text{Rank } T < \dim V$$

56. (A)
$$\left(0+\frac{\beta}{\beta}\right) = \left(\frac{0}{\beta}\right)+\left(\frac{\beta}{\beta}\right)$$
$$\Rightarrow \quad \left(\frac{\beta}{\beta}\right) = \left(\frac{0}{\beta}\right)+\left(\frac{\beta}{\beta}\right)$$
$$\Rightarrow \quad \left(\frac{0}{\beta}\right) = 0$$

57. (B)
$$\left(\alpha+\frac{\beta}{\beta}\right) = \left(\frac{\alpha}{\beta}\right)+\left(\frac{\beta}{\beta}\right)$$
$$\because \quad \left(\frac{\alpha}{\beta}\right) = 0$$
$$\Rightarrow \quad \left(\alpha+\frac{\beta}{\beta}\right) = \left(\frac{\beta}{\beta}\right)$$
$$\Rightarrow \quad \alpha+\beta = \beta$$
$$\Rightarrow \quad \alpha = 0$$

58. (B) (a)
$$\left(\frac{\alpha}{\beta}\right)+\left(\frac{\gamma}{\beta}\right) = \left(\alpha+\frac{\gamma}{\beta}\right)$$

(b)
$$\left(\frac{\alpha}{\beta}\right)-\left(\frac{\gamma}{\beta}\right) = \left(\frac{\alpha}{\beta}\right)+\left(-1\frac{\gamma}{\beta}\right)$$
$$= \left(\alpha-\frac{\gamma}{\beta}\right)$$

59. (C)
$$\left(\frac{\alpha}{\gamma}\right)+\left(\frac{\beta}{\gamma}\right) = \left(\alpha+\frac{\beta}{\gamma}\right)$$
$$\Rightarrow \quad -1+3 = \left(\alpha+\frac{\beta}{\gamma}\right)$$
$$\Rightarrow \quad \left(\alpha+\frac{\beta}{\gamma}\right) = 2$$
$$\because \quad \alpha+\beta = (1, 2)+(-1, 1)$$
$$= (0, 3)$$
$$\therefore \quad \left(\alpha+\frac{\beta}{\gamma}\right) = 0\cdot y_1 + 3y_2$$
$$= 3y_2 = 2$$
$$\Rightarrow \quad y_2 = \frac{2}{3} \text{ and } y_1 = 0$$
$$\therefore \quad y = (y_1, y_2)$$
$$= \left(0, \frac{2}{3}\right)$$

60. (A)
$$f(x) = \det(xI + A)$$
$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} x - \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$
$$= \begin{vmatrix} x & 0 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$
$$= \begin{vmatrix} x & 1 \\ -1 & x \end{vmatrix} = x^2+1$$

61. (C)
$$f(x) = |xI - A|$$
$$= \begin{vmatrix} x-3 & -1 & -1 \\ 2 & x-2 & 1 \\ -2 & -2 & x \end{vmatrix}$$
$$= x-3\begin{vmatrix} x-2 & 1 \\ -2 & x \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -2 & x \end{vmatrix} - \begin{vmatrix} 2 & x-2 \\ -2 & -2 \end{vmatrix}$$
$$= x^3 - 5x^2 + 8x - 4$$

62. (B) 63. (C)

64. (D) $\because$ $\{0\}$ is a linear dependent set
$\therefore$ It cannot be a basis
$\therefore$ $\phi \subset \{0\}$, and ϕ spans $\{0\}$
$\therefore$ $\dim\{0\} = \dim \phi = 0$

65. (C) (x, y) and $(-y, x)$ are orthogonal to each other to standard inner product.

$$\because \quad \left(\frac{(x, y)}{(-y, x)}\right) = 0 = -xy + yx = 0$$

66. (A) 67. (A) 68. (C)

69. (B) Let the equation be

$$\sum_{i=1}^{n} a_{ij} x_j = 0, i = 1, 2, \ldots\ldots, m.$$

Since the rank of A which is $m \times n$ matrix is r, the number of solutions (linearly independent) is $n - r$.

This is because you are looking at the annidilator of the subspace W generated by $(a_{i1}, a_{i2}, \ldots\ldots, a_{in})$.

$i = 1, \ldots\ldots, m$ vectors in R^n, which is an r dimensional vector space.

$$\dim A(W) = \dim R^n - \dim W = n - r$$

70. (D) The matrix is $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Given $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\therefore \quad a_{11} - a_{12} = 1 \quad \ldots$(i)

and $\quad a_{12} - a_{22} = -1 \quad \ldots$(ii)

and $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$

$\therefore \quad 2a_{11} + a_{12} = 8 \quad \ldots$(iii)

and $\quad 2a_{21} + a_{22} = 4 \quad \ldots$(iv)

solving equations (i), (ii), (iii) and (iv)

We get $\quad a_{11} = 3,$

$a_{12} = 2,$

$a_{21} = 1,$

$a_{22} = 2$

$\therefore$ Matrix is $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

71. (C) Euclidean domain is a PID but the converse is not true.

$\therefore \quad ED \subseteq PID$

PID and ED are unique factorisation domains

$\therefore$ ED $\subseteq$ PID $\subseteq$ UFD, UFD need not be PID or ED.

72. (B) Since $f(x) = x^6$, we take the hermite interpolating polynomial to be even function.

$$P(x) = a_0 + a_1 x^2 + a_2 x^4$$

$$P(0) = f(0),$$

$$P'(0) = f'(0)$$

$P(1) = f(1)$ and $P'(1) = f'(1)$ yield the equations

$a_0 = 0, a_1 + a_2 = 1, 2a_1 + 4a_2 = 6$

Solving, we get

$$a_1 = -1, a_2 = 2$$

$$\therefore \quad P(x) = 2x^4 - x^2.$$

73. (A) $\quad A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4{\cdot}5 \\ 5 \\ -0{\cdot}5 \end{bmatrix}$

where, $\quad A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$$A = D + L + U$$

where, $\quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix},$

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$H = -D^{-1}(L + U)$$

$$= \begin{bmatrix} 0 & -\frac{2}{3} & 0 \\ -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

Characteristic equation is,

$$\lambda^3 - \lambda\frac{11}{18} = 0$$

$$\therefore \lambda = 0, \pm\sqrt{\frac{11}{18}}$$

$$\therefore \quad P(H) = \sqrt{\frac{11}{18}}$$

and $\quad \frac{2}{\mu^2}\sqrt{1 - \mu^2} = 1{\cdot}23$ (approx.)

74. (B) 75. (D) 76. (C) 77. (D) 78. (A)

79. (B) By a theorem which states "If A and B be $m \times n$ and $n \times p$ matrices respectively, then rank (AB) $\leq$ min (rank A, rank B)",

We have $\quad$ rank (P) $= n$

rank (Q) $=$ P

Since P in a $m \times n$ matrix with rank n, $n \leq m$ and Q is $n \times p$ matrix with rank P. P $\leq n$

$\therefore \quad$ rank (PQ) $= \min(n, p) =$ P

80. (D) It $\lambda = 0$ is an eigen value of P, then eigen value of a is $\frac{1}{\lambda}$ *i.e.*, ∞. It is not possible.

Hence neither P nor Q.

81. (D)	82. (D)	83. (B)	84. (A)	85. (B)
86. (C)	87. (A)	88. (C)	89. (B)	90. (D)
91. (B)	92. (B)	93. (D)	94. (D)	95. (C)
96. (D)	97. (D)			

98. (B) $f_1 = (1, 1, 1)$, $f_2 = (1, -1, 0)$, $f_3 = (0, 1, 0)$, $g_1 = (1, 1)$, $g_2\,(1, 0)$

$$\begin{aligned}(a, b) &= xg_1 + yg_2\\ &= x(1, 1) + y(1, 0)\\ &= (x + y, x + 0.y)\end{aligned}$$

$$\left.\begin{aligned}\therefore\ x + y &= a\\ \therefore\ x + 0.y &= b\end{aligned}\right[\text{ and } x = b \text{ and } y = a - b$$

$$\begin{aligned}\therefore\quad (a, b) &= bg_1\,(a - b)\,g_2\\ T(x, y, z) &= (x + y, y - z)\\ F(f_1) &= F(1, 1, 1) = (2, 0)\\ &= 0\cdot g_1 + 2g_2\\ F(f_2) &= F(1, -1, 0) = (0, -1)\\ &= -g_1 + g_2\\ F(f_3) &= F(0, 1, 0) = (1, 1)\\ &= g_1 + 0\cdot g_2\\ \therefore\quad [F]_f^g &= \begin{bmatrix}0 & -1 & 1\\ 2 & 1 & 0\end{bmatrix}\end{aligned}$$

99. (D) Given

$$\begin{aligned}\lambda^2 - \lambda - 1 &= 0\\ \Rightarrow\quad 1 &= \lambda^2 - \lambda\\ \Rightarrow\quad \lambda^{-1} &= \lambda - 1\\ \Rightarrow\quad M^{-1} &= M - 1\end{aligned}$$

100. (D)

101. (A) P : $\quad T = \begin{bmatrix}1 & 0\\ 1 & 0\\ 0 & 0\end{bmatrix}$

$\Rightarrow \quad$ Rank $= 1$

Q : $\quad T_2(x, y) = (x, x + y, y)$

Linear transformation of rank 2.

102. (B) Let A_1 and A_2 be two complex n-vectors, then A_1 is said to be orthogonal to A_2, if $(A_1, A_2) = 0$

i.e. $\quad A_1^TA_2 = 0$

Here, $\quad A_1 = \begin{bmatrix}0\\ 2\\ 0\\ 0\end{bmatrix}$,

$$A_2 = \begin{bmatrix}0\\ 0\\ 0\\ -4\end{bmatrix}$$

$$A_1^T = [0, 2, 0, 0]$$

$$\therefore\quad (A_1A_2) = [0\ \ 2\ \ 0\ \ 0]\begin{bmatrix}0\\ 0\\ 0\\ -4\end{bmatrix} = 0$$

Similarly

$$(A_2A_3) = (A_3A_4) = 0$$

Thus column vectors from the orthogonal set of system.

103. (D) Since M is Hermitian matrix, so it will have only real eigen values.

104. (A)

105. (C) The monic polynomial of lowest degree that arihilates a matrix A is called the minimal polynomial of A.

$$\text{Here } [A - \lambda I] \begin{vmatrix}1-\lambda & 0 & 0 & 0\\ 1 & 1-\lambda & 0 & 0\\ 0 & 0 & 2-\lambda & 0\\ 0 & 0 & 0 & 2-\lambda\end{vmatrix}$$

$$= (1 - \lambda)^2\,(2 - \lambda)^2$$

$\therefore$ Roots of the equation $|A - \lambda I| = 0$ are 1, 1, 2, 2.

Each characteristic root of A is also a root of minimal polynomial. If $m(x)$ is the minimal polynomial of A, then both $(x - 1)$ and $(x - 2)$ are factors of $m(x)$.

106. (A) $a_1(1, x, 0) + a_2(0, x^2, 1) + a_3(0, 1, x) = 0$

This is for linearly independence

on comparing, we get

$$\begin{aligned}a_1 &= 0 &&\text{...(i)}\\ a_1x + a_2x^2 + a_3 &= 0 &&\text{...(ii)}\\ a_2x^2 + a_3 &= 0 &&\text{...(iii)}\\ \Rightarrow\quad a_2 + a_3x &= 0\end{aligned}$$

from equations (i), (ii), (iii)

$a_1 = a_2 = a_3 = 0$ for linear independent

This is true only for $x = 0$.

107. (C) Let $\quad f = a_1x + a_2\,|y| + a_3z$

Here f are two functions one for y and other for $-y$.

$$f = a_1x + a_2y + a_3z$$

and $\quad f' = a_1x - a_2y + a_3z$

Now $\quad g = a_1(x+1) + a_2(y-1) + a_3 z$
$= a_1 x + a_2 y + a_3 z + (a_1 - az)$
$= f + (a_1 - a_2)$

i.e., g is linear but not f.

108. (D) $\quad M = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}$

Then $\quad M' = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 3 & 5 & 9 \end{bmatrix}$

$\Rightarrow MM' \neq I$

Now $\quad M^2 = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}$

$= \begin{bmatrix} 1 & 15 & 45 \\ 0 & 16 & 45 \\ 0 & 0 & 81 \end{bmatrix}$

$M^2 M'^2 \neq I$

Hence neither M nor M^2 is diagonalizable.

109. (A) Let $\quad M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

But M is skew symmetric matrix

i.e. $\quad M' = -M$

$\Rightarrow \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ -b_1 & -b_2 & -b_3 \\ -c_1 & -c_2 & -c_3 \end{bmatrix}$

On comparing, we get

$a_1 = -a_1 \Rightarrow a_1 = 0$

$a_2 = -b_1$ and so on

Then $\quad M = \begin{bmatrix} 0 & -b_1 & -c_1 \\ -a_2 & 0 & -c_2 \\ -a_3 & -b_3 & 0 \end{bmatrix}$

But M is also orthogonal real matrix

i.e. $\quad MM' = I$

$\Rightarrow \begin{bmatrix} b_1^2 + c_1^2 & c_1c_2 & b_1b_3 \\ c_1c_2 & a_2^2 + c_2^2 & a_2a_3 \\ b_1b_3 & a_2a_3 & a_3^2 + b_3^2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

on comparing, we get

$M = \begin{bmatrix} b_1^2 + c_1^2 & 0 & 0 \\ 0 & a_2^2 + c_2^2 & a_2a_3 \\ b_1b_3 & a_1a_3 & a_3^2 + b_3^2 \end{bmatrix}$

For eigen values

$|M - \lambda I| = 0$

Hence, $\quad \lambda = -1, 1.$

110. (A) Let $\quad \alpha = (a_1, b_1, c_1),$
$\beta = (a_2, b_2, c_2)$

then $\quad S(\alpha) = S(\beta)$

$\Rightarrow (a_1, a_1 + b_1, a_1 - b_1 - c_1)$
$= (a_2, a_2 + b_2, a_2 - b_2 - c_2)$

$\Rightarrow a_1 = a_2,$
$a_1 + b_1 = a_2 + b_2,$
$a_1 - b_2 - c_1 = a_2 - b_2 - c_2$

$\Rightarrow a_1 = a_2, b_1 = b_2, c_1 = c_2$

Hence, S is one-one.

Therefore, S must be onto also and thus S is invertible.

Similarly,

$T(\alpha) = T(\beta)$

$\Rightarrow (a_1 + 2c_1, b_1 - c_1, a_1 + b_1 + c_1)$
$= (a_2 + 2c_2, b_2 - c_2, a_2 + b_2 + c_2)$

$\Rightarrow a_1 \neq a_2, b_1 \neq b_2, c_1 \neq c_2$

$\therefore$ T is not one-one.

Hence, T is not onto and thus T is not invertible.

111. (C) $\quad \dim V = \dim X$

where V, W and X be three finite dimensional vector spaces.

$S : V \to W,$
$T : W \to X$

such that $ToS \to V \to X$ is one-one

Hence, S and T both are injective.

112. (A) A square matrix of order 10 has exactly 4 distinct eigen values, then the degree of its minimal polynomials must be at least 4.

113. (A) $\quad |M - \lambda I| = \begin{vmatrix} -\lambda & 1 & 2 & 0 \\ 1 & -\lambda & 1 & 0 \\ 2 & 1 & -\lambda & 2 \\ 0 & 0 & 2 & -\lambda \end{vmatrix}$

On expanding

$= -\lambda \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{vmatrix}$

$-1 \begin{vmatrix} 1 & 1 & 0 \\ 2 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{vmatrix} + 2 \begin{vmatrix} 1 & -\lambda & 0 \\ 2 & 1 & 2 \\ 0 & 0 & -\lambda \end{vmatrix}$

$= -\lambda^4 - 9\lambda^2 - 2\lambda + 4$

$\Rightarrow \quad |M - \lambda I| = 0$

$\Rightarrow \quad \lambda^4 + 9\lambda^2 + 2\lambda - 4 = 0,$

$\Rightarrow$ M has no real eigen values.

114. (C)

115. (B) $T(x_0, x_1, x_2) = (x_0, x_0 + x_1, x_0 + x_1 + x_2)$

Let basis are $(1,0,0)$ $(0,x,0)$ and $(0,0,x^2)$

Then $T(1, 0, 0) = (1, 1, 1)$

$T(0, x, 0) = (0, x, x)$

$T(0, 0, x^2) = (0, 0, x^2)$

$$\Rightarrow \quad T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & x & x \\ 0 & 0 & x^2 \end{bmatrix},$$

$$|T| = 1(x^3) = x^3.$$

At $x = 1$,
$$T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and $|T| = 1.$

Cofactors of T

$T_{11} = 1, T_{12} = 0, T_{13} = 0$

$T_{21} = -1, T_{22} = 1, T_{23} = 0$

$T_{31} = 0, T_{32} = -1, T_{33} = 0$

$\therefore$ adj T = Transpose of Co-factors matrix.

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence $T^{-1} = \frac{1}{|T|}\text{adj } T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

116. (B) $|T^{-1} - \lambda I| = \begin{bmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$

$$= (1 - \lambda)^3$$

Hence, dimension of eigen space of T^{-1} is 3.