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GRAPHICAL CALCULUS

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BY

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WITH AN INTRODUCTION BY

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INTRODUCTION

ALL "up-to-date" teachers of engineering and applied sciences generally now recognize the vast superiority of graphical over purely mathematical methods of imparting instruction of almost every description. The former are much more convincing to the student, because they appeal to the eye, the training of which is one of the chief objects to be aimed at in the education of an engineer. There is no doubt that this method is capable of great extension with advantage. In this little book, for instance, we see graphical constructions of a very simple character employed to teach what, to the beginner, are somewhat abstruse mathematical principles.

The attempt to employ purely mathematical, in preference to graphical methods, seems to me quite as absurd as attempting to teach geography by giving the position of towns in terms of their latitude and longitude, and explaining the shape of a country by giving the equation to the coast-line instead of by employing the graphical method, *i.e.* exhibiting a map. The teacher who attempted the former method would indeed be considered unpractical, and would, I fear, meet with but a very limited share of success; yet, strange to say, such a method is precisely that which teachers of mathematics are trying to employ with a much more subtle subject than geography—the Calculus. Is it, then, to be wondered at that many technical students shudder at the bare sign of integration, "the long S," as they are wont to call it? Not because they cannot manipulate the symbols—far from it—but because they have not the faintest notion of the physical meaning of the processes.

I have frequently had students come under my notice who, although fairly good mathematicians as far as bookwork is concerned, yet, through not having had the advantage of a *practical* mathematical training such as we give at the Yorkshire College, were utterly at sea when they came to apply their mathematics to such a simple engineering problem as, for instance, finding the quantity of water flowing over a V notch! It is primarily to help such to acquire an intelligent *working* knowledge of the Calculus that Mr. Barker has written this little book. Even if it have only a tithe of the success that its author has had in teaching mathematics by this method, it will still be eminently successful. I can unreservedly say that this is exactly the style of book that I have been wanting to see for years, and I believe it will prove to be of very real value to those students of engineering who wish to get a stage beyond the barest elements of the subject.

Some of those who know my propensity to scoff at the mathematics so commonly drummed into technical students, may, on seeing this introduction, exclaim, "Is Saul also among the prophets?" To such I would say that if as a student it had fallen to my lot to be under such a teacher as Mr. Barker, I should always have been numbered among the prophets, though possibly the minor ones.

JOHN GOODMAN.

THE YORKSHIRE COLLEGE, LEEDS,
April, 1896.

AUTHOR'S NOTE

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GRAPHICAL CALCULUS

CHAPTER I.

INTRODUCTORY.—CURVES AND THEIR EQUATIONS.

§ 1. Co-ordinates of a Point.

The exact position of a point in a plane is completely known if its perpendicular distances from two intersecting lines in that plane are known. Thus, suppose the lines OX, OY (Fig. 1) represent to some scale two hedges of a field meeting at right angles, and we are told that an article is buried in the field at a given depth at a point A, whose perpendicular distance from the hedge OY is 20 yards, and from OX 30 yards.

It is clear that the position of the article could be immediately found by measuring 20 yards from O along OX to L, and 30 yards along LA perpendicular to OX.

Definitions.—The lines OL, LA, would be called the *co-ordinates* of the point A, with reference to the *axes* OX, OY; OL is the *abscissa*, and LA the *ordinate*; and the point A would be described in mathematical language as the point whose abscissa is 20, and whose ordinate 30, or shortly as "the point (20, 30)."

§ 2. Equation to a Line.

Suppose, however, we are told that the distance of the point A from OX is half (its distance from OY) + 20 yards. From this condition alone we could not find the exact position of the point, for there are many points in the field, in addition to the point A, of which the statement would be equally true. Thus if we take OM = 50 yards along OX, and MB at right angles = $(\frac{1}{2} \times 50 + 20)$ yards = 45 yards, we should find a point B which would also "satisfy the condition" that its distance from OX was half its distance from OY + 20 yards. Or we might have taken ON = 80 and

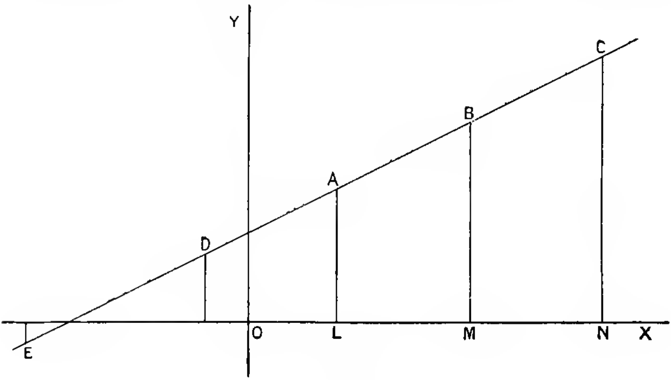

FIG. 1.

NC = 60; or, indeed, any arbitrary (or, as it is called, "independent") distance along OX, and calculated and measured the corresponding distance perpendicular to OX. We could thus find any number of points "satisfying the given condition." All these points would be found to lie on a certain straight line in the field, and no point which is not on the straight line would be found to satisfy the condition.[1] And, further, any point which is on the line will satisfy it.

We should then be sure of finding the buried article, if we were to dig a trench of the given depth along a line represented by AC.

Now let us attempt to discover the position of the article by an algebraical process.

[1] This statement should be tested by plotting the points to scale on a plan of the field.

Let x be the perpendicular distance of the buried article from OY; and y the perpendicular distance from OX.

Then we have—

$$y = \frac{x}{2} + 20$$

This is the only equation we can obtain from the data, and we are here met with the same difficulty as before, namely, that there are an infinite number of possible solutions to the equation, each solution corresponding to one particular point on the plan. Now, just as $(x = 20,\ y = 30)$ may be taken to represent the point A, so the equation $y = \frac{x}{2} + 20$ may be taken to represent the line AC. In other words, the line AC is a picture or geometrical representation of the equation $y = \frac{x}{2} + 20$. To put it in still another way, the line AC shows the relation between the value of x and that of $\left(\frac{x}{2} + 20\right)$, or y, corresponding to all values of x or y. Thus suppose we wish to find from the diagram what is the value of y or $\left(\frac{x}{2} + 20\right)$ when x is $59{\cdot}2$, say, or any other arbitrary value, we measure off to scale $59{\cdot}2$ along OX, and erect a perpendicular to OX from the point so found. The length of this perpendicular cut off by the line AC gives the required value.

The algebraical counterpart of this process is as follows: In the equation $y = \frac{x}{2} + 20$, find the value of y when x is $59{\cdot}2$. To solve this we have merely to substitute $59{\cdot}2$ for x in the equation, and solve the resulting simple equation in y; we thus find the value of y corresponding to $x = 59{\cdot}2$. This is easier and more accurate than the graphical process. Another example of the same thing is, "Find where the line $y = \frac{x}{2} + 20$ cuts the axis of y." At the required point it is obvious that

$x = 0$. Hence substituting this value of x in the equation, we obtain a simple equation in y, viz. $y = 20$, which gives the distance of the point from OX.

This is expressed by saying that—

$$y = \frac{x}{2} + 20$$

is the "equation to the line AC."

A little reflection will show the student that $y = \frac{x}{2} + 20$ is also the equation to the continuation of the straight line AC in both directions anywhere in its length, and not only of that part of it which is to the right of OY and above OX.

Convention of Signs.—In this connection it must be noticed that if distances to the right of OY are called positive, those measured to the left are to be called negative. Thus if a distance = 10 yards be measured to the *left* of OY, its distance from OY is said to be -10. The reason for this may be gathered from consideration of the following case:—

Suppose a man starts from P to walk to Q, a distance of 4 miles. After walking to S, he turns back and walks in the other direction for $2\frac{1}{2}$ miles to R. The total distance, *irrespective of direction*, which he has now walked is $1\frac{1}{2} + 2\frac{1}{2} = 4$ miles; but, considered with respect to his original destination, the *effective* distance he has walked is -1 mile, *i.e.* he has 1 mile to make up before he begins to be any nearer to Q than he was when he started. This may be conveniently expressed by prefixing the negative sign to distances walked towards the left; thus, $1\frac{1}{2} + (-2\frac{1}{2}) = -1$, which represents the total distance he has walked towards Q.

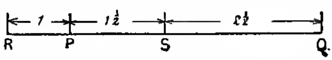

FIG. 2.

Suppose, then, we take a distance along OX $= -10$. The corresponding value of y will be $\frac{1}{2} \times (-10) + 20 = +15$. The point D $(-10, +15)$ is also on the line AC.

In the same way distances below OX are reckoned as minus quantities. Thus when $x = -50$, $y = \frac{1}{2} \times (-50) + 20 = -5$, the point E $(-50, -5)$ being also on the line AC.

Example.—Find where the given line cuts OX.

§ 3. Equation to a Curve of the Second Degree.

Suppose now that, instead of the condition $y = \frac{x}{2} + 20$, we had had the condition that the distance from OX $= \frac{1}{40}$ of (the square of the distance from OY) + 20 yards.

Dimensions of Quantities.—It may be noted, in passing, that, strictly speaking, it is as absurd to speak of one distance being = the square of another distance, as this expression is usually understood (*i.e.* that a line is equal to an area), as it would be to say, for instance, that a *square* foot is equal to 6·2 gallons. It is an absurdity of the same kind as is often committed in mechanics, when speaking of an acceleration of so many *feet per second*, instead of so many *feet-per-second per second.* Conventionally, however, it is to be understood in the following sense: We know by geometry that MP^2 (Fig. 3) = AM. MB. Now, when MB = 1″, there are as many square inches in the square on MP as there are linear inches in AM. And so, *disregarding dimensions*, we say that AM = MP^2. For instance, if MP were = 3″, AM would = 9″. This idea may also be expressed by saying that we are to regard a line as a geometrical method of representing a number, and not necessarily a number *of inches.* Thus a line 3″ long represents essentially on the simple inch scale the *number* 3, and may at pleasure stand for 3 seconds, 3 degrees, 3 feet per second, 3 square inches, or 3 units of any kind whatsoever.

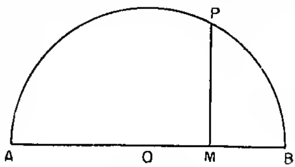

Fig. 3.

Our relation expressed algebraically is—

$$y = \frac{x^2}{40} + 20$$

Exactly as before, any value may be arbitrarily assigned to x, and the corresponding value of y or $\left(\frac{x^2}{40} + 20\right)$ calculated.

Thus, if $x = 10$—

$$y = \frac{100}{40} + 20 = 22{\cdot}5$$

$$\text{Or, } x = 20 \text{ gives } y = 30$$
$$x = 30 \text{ gives } y = 42{\cdot}5$$

and so on.

All these and similarly calculated points will be found to lie on a certain curve (Fig. 4), instead of, as before, on a straight line, and no point which is not on the curve will satisfy the equation, and any point which is on the curve will satisfy it.

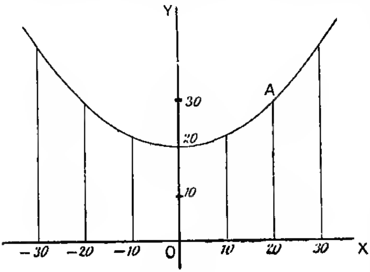

FIG. 4.

This equation, since it contains the second power of one of its "variables," is said to be of the second degree, and the curve is, as before, called "the curve $y = \dfrac{x^2}{40} + 20$," and the diagram of the curve exhibits graphically the relation between x and y, or, in other words, between x and $\dfrac{x^2}{40} + 20$, for all values of x.

The equation to any curve, then, gives a relation which must be "satisfied" by the co-ordinates of any point on the curve. We have "plotted" or "traced" these curves by arbitrarily assigning a series of values to x (*i.e.* treating x as an independent variable quantity which can assume any value we please), and calculating the value which y (the "*dependent* variable") assumes *in consequence of* x having that particular value we have assigned to it.

In this book the values of the *independent variable* are, to avoid confusion, always measured in a horizontal direction, and those of the dependent variable vertically.

§ 4. EXPERIMENTAL CURVES.

Curves may also be obtained by other means than translating an algebraical equation into geometry, which is practically the way in which we have obtained the preceding curves.

The results of series of experiments in physical or engineering science are, whenever possible, plotted on paper.[1]

This method exhibits relations between mutually dependent and therefore simultaneously varying quantities far more clearly than rows of figures or pages of symbols can possibly do. The method may be best explained by taking a simple and familiar example. Suppose a kettle of cold water is set on a fire, and the temperature of the water observed at intervals of, say, one minute by means of a thermometer, a note being taken of the time and simultaneous value of the temperature. Suppose the initial temperature of the water is 60° Fahr.

Our readings are booked as follows :—

Time (minutes).	Temperature (degrees).
After 0	60
,, 1	95
,, 2	126
,, 3	152
,, 4	174
,, 5	193
,, 6	209
,, 7	212

Mark off along OX (Fig. 5) equal distances 1, 2, 3, to represent minutes, and along OY (to scale) temperatures beginning at any convenient temperature. Then find a series of points on the paper whose abscissæ represent the observed times, and whose ordinates represent the corresponding temperature to any assumed scale. Thus the point (1 min. 95°) will be found by the intersection of a vertical through 1 min. and a horizontal through 95°. All the points are to be found in the same way, and a freehand curve drawn through them. This will be a time-temperature curve, exhibiting the result of our experiment very clearly; for not only does the height of the curve

[1] The paper most convenient for this purpose is what is called "squared paper," which is ruled in small squares of 0·1 inch side, and can be bought at any stationer's.

at any point show at once the value of the temperature at the corresponding time, but the shape of the curve conveys at a glance a general idea how the *rate of rise* of temperature varies throughout the experiment. It is clear that when the temperature is rising rapidly (as it will do at first), the curve is steeper than when it is rising less rapidly later on in the experiment. Thus the amount of rise of temperature between say 0 and 1 min. is greater than the amount of rise between 3 and 4 min., and therefore the average upward slope of the curve between 0 and 1 min. is greater than the upward slope between 3 and 4 min.

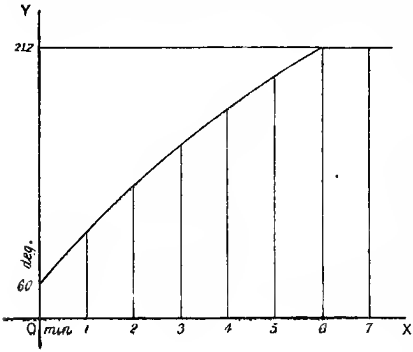

FIG. 5.

Curves of this kind are quite familiar to us. Thus we have in the daily papers curves representing time variations in the height of the barometric column or of the thermometer. The differential calculus is chiefly concerned with the slope of curves, and it is therefore important that we should get accurate ideas of how this slope is to be measured.

§ 5. Methods of measuring the Slope of a Line.

Suppose we have a sloping line AB, and we wish to determine exactly how much it slopes with respect to a horizontal AC. We can do this in several ways, any of which we can use when convenient.

(i.) We can find the number of degrees in the angle BAC by a protractor.

(ii.) We can measure the length of the arc CD by a steel tape, and also find the length of AC. By these two measurements we could easily reproduce the angle on paper.

(iii.) The method we shall always use in the differential calculus is as follows: From any point B in AB drop a perpendicular BC to AC. Measure CB and AC. Divide the length of CB by that of AC. We thus obtain the length of a line MP where AM = 1." Thus, suppose CB = 2", AC = 3". Then $\frac{CB}{AC} = \frac{2}{3} = 0{\cdot}667$. It is clear, if AC is three times AM, that therefore CB is three times MP, and therefore—

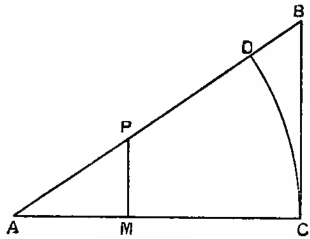

Fig. 6.

$$MP = \frac{CB}{3} = \frac{CB}{AC}$$

(Read again note on p. 5.)

This is, of course, true wherever we take B on the line AB. If AC, instead of being 3", had been 0·042" suppose, we should have found CB = 0·028, and therefore MP = $\frac{0{\cdot}028}{0{\cdot}042} = \frac{2}{3}$, as before; so that wherever B may be on AB we can always represent the ratio $\frac{CB}{AC}$ by the line MP, where AM = 1, even though B is quite close to A. The student should convince himself of the reality of this result by trial and accurate measurement. To obtain the length of MP with some accuracy, it is convenient to make AC = 10". Thus in this case CB would be 6·67. Hence MP = 0·667.

§ 6. Trigonometrical Ratios.

It is convenient to have a name for the ratio $\frac{\text{perpendicular.}}{\text{base.}}$ It is called the "tangent of the angle of slope," and the relation is expressed thus in Fig. 6—

$$\text{Tan BAC} = \frac{\text{CB}}{\text{AC}} = \text{MP}$$

There are other ratios of an angle which must be perfectly familiar to the student before he can make any considerable progress in this subject.[1] Thus $\frac{\text{CB}}{\text{AB}}$ is called the "sine of the angle BAC," written thus—

$$\text{Sin BAC} = \frac{\text{CB}}{\text{AB}}$$

also $\frac{\text{AC}}{\text{AB}}$ is called the "cosine of BAC," written thus—

$$\text{Cos BAC} = \frac{\text{AC}}{\text{AB}}$$

also "cotangent of BAC," written "cot BAC," $= \frac{\text{AC}}{\text{CB}}$

"secant of BAC," written "sec BAC," $= \frac{\text{AB}}{\text{AC}}$

"cosecant BAC," written "cosec BAC," $= \frac{\text{AB}}{\text{CB}}$

also—

$$\frac{\text{length of any arc CD}}{\text{length of its radius AC}} = \text{circular measure of the angle BAC};$$

or, in other words, the number of "radians" in that angle.

It is clear that when the arc CD, measured with a flexible steel tape along the circumference, = radius AC, the ratio $\frac{\text{arc CD}}{\text{radius AC}} = 1$. In that case the angle BAC = 1 radian = 57·03° about.

It is clear that, provided CB is always perpendicular to AC, all these ratios are quite independent of the position of C on the line AC, for as AC increases, BC and AB and the arc CD

[1] It is highly desirable that a student should have a knowledge of elementary trigonometry before commencing this subject.

also increase in the same ratio, if the angle BAC remains unaltered.

EXAMPLES.

1. Draw the following curves on both sides of the axes:—

(i.) $y = x$.
(ii.) $y = 2x$.
(iii.) $y = 2x + 3$.
(iv.) $3y = x - 6$.
(v.) $y = x^2$.
(vi.) $y^2 = x + 4$. (Notice the double sign for y: $y = \pm \sqrt{x + 4}$).
(vii.) $y = x^2 - 4$.
(viii.) $xy = 4$.

(These curves may be obtained by giving arbitrary values to *either* x or y.)

2. What are the meanings of m and c in the line $y = mx + c$?

(*Ans.* m is the tangent of the angle of slope of the line to OX. c is the distance from O, where the line cuts OY.)

3. Find where the curves

$$\text{(i.)}\ 2y = x + 3$$
$$\text{(ii.)}\ y = x^2 - 2$$
$$\text{(iii.)}\ y^2 = x - 3$$

cut the axes of X and Y.

(Find the value of x and y successively when the other variable = 0.)

4. Find the equation to a line cutting OY at a distance = 3 below O and inclined to OY at 60°. (See Example 2.)

5. Why is no part of the curve $y = x^2$ below OX?

(*Ans.* Because, whether the value of x be positive or negative, the square of it must essentially be positive, *i.e.* the ordinate must be *above* OX. Suppose any point of this curve were 2 inches below OX; we should have—

$$-2 = x^2$$
$$\text{or } x = \sqrt{-2}$$

And since the square root of a negative quantity is essentially imaginary, being neither + nor −, it is evident that we can have no real point of the curve below OX.)

6. Devise geometrical constructions for determining the value of the sine, cosine, circular measure of any angle; also for $\frac{1}{\text{sine}}$, $\frac{1}{\text{cosine}}$, etc. (See iii. p. 9.)

CHAPTER II.

GRAPHICAL DIFFERENTIATION AND INTEGRATION.

§ 7. ILLUSTRATION OF GRAPHICAL DIFFERENTIATION.

EVERY straight line, however long or short, must have a definite inclination to every other line in its plane. In this

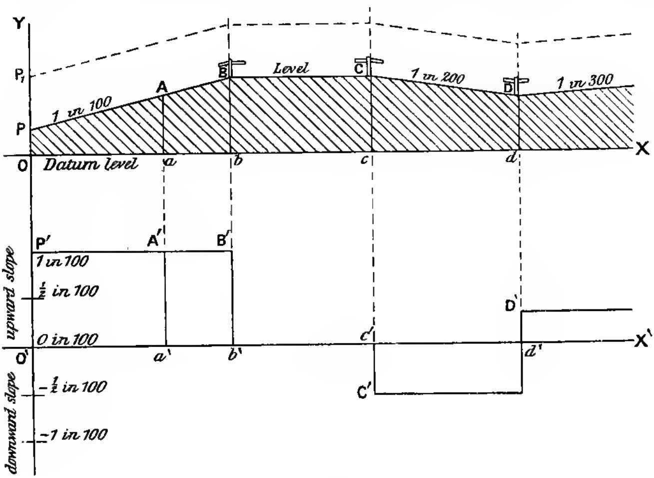

FIG. 7.

work we are chiefly concerned with that function[1] of the

[1] Any variable quantity p whose value depends on the value of another variable quantity q, is said to be a "function" of that quantity. Thus the sine of an angle is a function of that angle, because the value of the sine depends on that of the angle.

inclination of lines to the horizontal which we have called the tangent of the angle of slope (§ 6).

If we have a figure made up of straight lines, it is quite easy to determine the slope of each rectilinear part of it to a horizontal line. Suppose, for instance, we have given an elevation (drawn to scale) of a certain section of railway, such as Fig. 7, in which vertical heights are much exaggerated for the sake of clearness.

On most railways what are called the "gradients" are indicated on boards, such as that illustrated in Fig. 8, placed by the side of the line. The meaning of this is that while the line is level on the right of the board, for every 100 feet measured horizontally, towards the left the line falls 1 foot in 100; or, in other words, the tangent of the angle of slope is $\frac{1}{100}$. The direction in which the line slopes is shown by the obvious slope of the board. If the line is level, or of no slope, it might in the same way be indicated 0 in 100, but the word "level" is used instead.

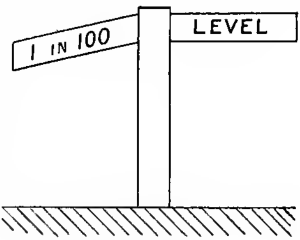

FIG. 8.

Now, we may very conveniently draw underneath the actual elevation of the railway in Fig. 7, another curve showing the *slope* at each point; or, choosing the scale of 1 inch = 1 in 100, the rate at which the line is rising just at that point for every 100 feet horizontal. It is of the highest importance that the student should thoroughly grasp the exact meaning of this lower curve, for in it is contained the very kernel of the whole subject. Suppose we take any point A on the upper curve (which we may call a curve in spite of the fact that it consists wholly of straight lines).

Draw a vertical from A to cut the lower curve in A'. Then the height of the lower curve a'A' shows the amount (1 foot) which the line would rise if it continued with the same slope as it has at A for 100 feet horizontally. The fact that the slope may change just after the point A is passed does not in the least affect the height just *at* the point A', for this is only dependent on the tangent of the angle of slope just *at* the point A. Neither is the height of the lower curve at A' any guarantee that the line will actually rise 1 foot in the next 100 feet horizontal, any more than the fact that a train may be travelling *at the rate of* 60 miles per hour is any guarantee that in the next hour it will actually travel 60 miles, or any other distance. It possibly may entirely change its velocity in the next half-second, as in the case of a collision, but this does not alter the fact that at the point in question it was travelling at 60 miles per hour. In exactly the same way, at the point B, the line is rising *at the rate of* 1 foot per 100 feet, and this is not affected by the fact that from a couple of feet to the right the line runs perfectly level. The important point to observe is that the height of the lower curve represents *a rate of rise*, and not necessarily an *actual* rise. It represents, in fact, the ratio $\dfrac{\text{corresponding small vertical rise.}}{\text{small horizontal distances.}}$ The fact that both the numerator and denominator of this fraction may be indefinitely small does not affect the value of the ratio, as explained in § 5.

Now, just to the right of point B the upper curve suddenly changes its slope from 1 in 100 to 0 in 100; and, consequently, the lower curve drops suddenly from 1 to 0. The line has no slope from near B to near C, and therefore the lower curve has no height between the same two points. Near C the rise changes into a fall, *i.e.* the rate of rise becomes negative, and so the height of the lower curve becomes negative, *i.e.* is below the axis of X (see § 2). At D the negative slope suddenly changes to a positive one, and so the lower curve suddenly jumps up above the axis.

If the student has thoroughly mastered the preceding explanation, he will have little difficulty with the rest of the subjects treated of in this book, and we have, therefore, given it a very full explanation—fuller, perhaps, than many students will find necessary. The principle here explained is exactly that running through the whole of the subject, and the student cannot be too familiar with it. The lower curve is called the "derived curve" of the upper one, and the height of the lower curve at any point gives the value of the "differential coefficient" of the upper one at the corresponding point. The student will understand these expressions better a little later on. We shall then also see that the railway companies, by means of boards, such as Fig. 8, really "differentiate" the curve of the railway for the information of the engine-drivers.

§ 8. Example of Graphical Integration.

Hereafter, in all cases, points on the derived curves are indicated by dashes, thus: P' on the first derived and P'' on the second derived correspond to P on the primary, etc.

Let us now look at the converse process. Suppose we are given this derived curve or curve of slopes, and are required to deduce from it the actual elevation of the railway. The student will find this easy if he has mastered the principle on which the derived curve was obtained. We see, in the first place, that along the line PB there must be an even uphill slope of 1 in 100; for the fact that P'B' is parallel to O'X' shows that the slope is constant. Hence PB must be perfectly straight as far as B. A question that meets us at the outset is a very suggestive one. Where are we to start to draw the curve? The bearing of this question on the subject of integration will be fully explained later on. At present it is sufficient to notice that there is nothing whatever in our derived curve to tell us where the point P is to be taken in the vertical O'O; that is to say, our curve of slopes does not give us the *height* of the point P, or any other point, above datum level. Taking, then, any arbitrary point P as starting-point, we see that the line slopes upwards 1 in 100 as far as a point corresponding to B', after which it is level as far as a point vertically above C', afterwards sloping downward $\frac{1}{2}$ in 100 as far as D, then upwards $\frac{1}{3}$ in 100. So that we can draw the actual shape of the upper curve, having given its curve of slopes.

Meaning of the Arbitrary "Constant" in Integration.—If we had taken any other point, P_1, as starting-point in the same vertical line, we should have obtained the dotted curve which is precisely similar to the one we have obtained, but shifted higher or lower, according as P_1 is higher or lower than P, and the distance between the two curves, measured perpendicularly to OX, is "constant" all along the curve. It is clear that while we cannot find the absolute height of any point on the curve so obtained, yet we can ascertain definitely the *difference of height* of two points on it, for this difference is the same wherever the curve, as a whole, may be shifted to.

§ 9. Differentiation of Continuously Varying Outline.

It is not necessary, for our process, that the real elevation of the curve should consist of straight lines. We might take any continuously curving outline, such as the hill illustrated in Fig. 9, and determine the curve of slopes for it. Now, in a rounded outline like this, the slope does not change suddenly, as it did in our assumed case of the railway, but it gradually changes from point to point. We say familiarly that the hill is steeper at one part than at another, or that as we go higher up the hill it gets steeper and steeper, or the steepness increases every step we take, and so on. Now, we have already explained how we are going to measure this steepness, viz. by the tangent of the angle of slope (§ 5). But we come again to a question which must be carefully considered in all practical applications, viz. what scale are we to use? Observe that the horizontal scale is the same for both curves, whereas a vertical height or ordinate on the upper curve has an entirely different meaning from an ordinate on the lower; the former represents an actual height, the latter a ratio. For convenience and clearness, we shall at present adopt as a unit whatever is represented by 1″ on the upper figure,[1] which is

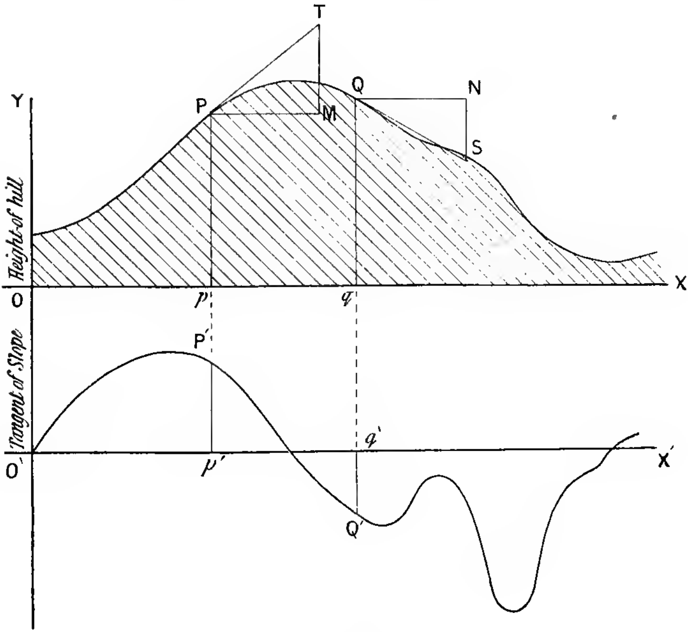

FIG. 9.

here 100′. The vertical scale of the derived curve is here full size where PM is unit length, *i.e.* the height in the given units represents the tangent of slope at the corresponding point of the upper curve.

To find the derived curve, draw a number of ordinates

[1] In the original drawing, of which Fig. 9 is a reduced copy, PM was made = 1″. This remark applies to all the figures.

to the upper curve and produce them downwards. We are about to determine the actual slope of the hill at each of the points where these ordinates cut the outline of the hill. For the sake of avoiding confusion in the figure, we shall do this for two points only, viz. P and Q; all the others are to be treated in exactly the same way. At the point P draw a line PT, touching the curve. The slope of the hill at the point P is evidently exactly the same as that of this line, for a little to the left of P the hill slopes more, and a little to the right less; so that *at* the point P the slope is the same. Through P draw PM horizontal = 1″ (representing 100′), and draw MT vertical. This line measures, say, 1·2″ = 120 feet. Then, just at the point P the hill is sloping ·120 feet in 100. Take any convenient base-line, O'X', on which to draw the derived curve, and make p'P' = MT; then p'P' represents the slope at the point P. At Q the hill is sloping downwards. Draw the tangent as before, and make QN = 1″, and draw NS vertical, and make q'Q' on the lower curve = NS, and measured in the same direction, *i.e.* downwards. Repeat this process for all the points where the ordinates cut the outline. It is convenient, before commencing, to ink in the original curve, so that any line may be removed from the figure after it is done with. It is usually more accurate to draw the tangent first, and the ordinate afterwards; the point of contact can then be more accurately found. The drawing should be made with a hard pencil, sharpened to a fine point. Unless the original curve is very accurately drawn, it is impossible, even with great care, to obtain very accurate results, owing to the difficulty of drawing in the tangents correctly. When the points are all obtained, draw an even curve through them all. This is the derived curve, or curve of slopes of the upper curve. The height of it represents, at any point, the corresponding value of the differential coefficient of the function which represents the height of the upper curve.

§ 10. Remarks on Derived Curves.

There are several important facts to be observed about two curves standing in this relation to one another.

1. At the highest point of the hill the slope is, of course, nothing, the tangent being horizontal; for suppose the slope were slightly downwards, then a point on the left of the highest point would be slightly higher than the highest point, which is absurd. (This assumes that the curve is continuous, *i.e.* that there are no angles or sharp points in it.) The height of the derived curve is therefore nothing.

2. This is also the case at the very bottom of the valley, for a similar reason.

3. Conversely, at the point corresponding to that at which the derived curve cuts the axis of X', the height of the primary curve is either a maximum or minimum.

4. In the case of a maximum, the derived curve slopes *downwards* from left to right, *i.e.* has a negative slope. In the case of a minimum, the derived curve slopes upwards from left to right, *i.e.* has a positive slope.

By a "maximum" or "minimum" we do not mean absolutely the highest or lowest point on the curve, but merely a point to the left of which the slope is in a different direction from what it is on the right.

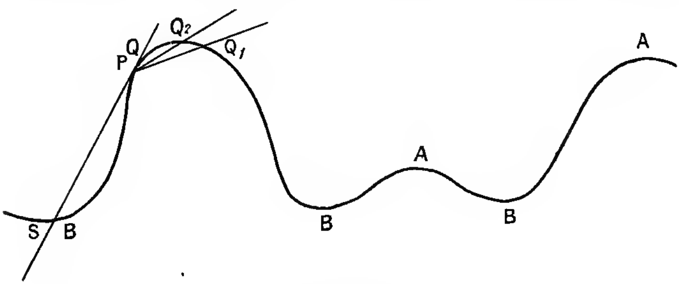

Fig. 10.

Thus in Fig. 10 the points AAA are all maxima, and BBB are all minima. For each of these points the derived curve crosses its own base-line.

5. At the point where the curve of the hill changes from convex to concave, or *vice versâ*, with respect to the axis OX its slope is greatest, and consequently the derived curve highest or lowest, as the case may be; *i.e.* at these points the derived curve has no slope (see observation 1). Such a point is called a "point of inflection" on the original curve.

§ 11. Meaning of a Tangent to a Curve.

We have spoken of "drawing a line touching a curve," but, as much of what follows depends on the relation between a curve and its tangent, it is important to get clear ideas as to the precise meaning of the expression, "a line touching a curve." Euclid's definition applied to a circle is that the line meets the curve, but, being produced, does not cut it. This definition, although applicable to a circle, would not be applicable, for instance, to such a curve as that illustrated in Fig. 10, where the tangent at P cuts the curve again at S. The following is a more modern conception of a tangent to a curve. Consider a fixed point P and a movable point Q_1 on a continuous curve of any shape. Join PQ_1, and produce it in both directions. Conceive that Q, with the line PQ always passing through it, moves gradually towards P, occupying successively such positions as PQ_1, PQ_2. "In the limit" when Q is just on the point of coinciding with P (being, in fact, "infinitely near" to P), the line PQ is a tangent to the curve at P. In this position the infinitely small portion of the curve PQ may be regarded as coinciding with the portion PQ of the touching line. This shows that a curve may be regarded as composed of an infinite number of indefinitely short straight lines joined end to end. Bearing this explanation in mind when we have to do with an extremely short bit of a curve, we shall treat it as a straight line, as the various explanations are thereby rendered much simpler and clearer.

§ 12.

We have also a remark to make with respect to the modern mathematical conception of the meaning of such expressions as "zero," "infinitely small," "infinitely great," "absolute equality," and so on. It may be stated at once that the human mind is incapable of conceiving any reality corresponding to any of these expressions. The modern conceptions of them may be briefly summed up thus: zero or 0 means "*something* smaller than *anything*." Thus, take a quantity whose weight is the thousand-millionth part of the weight of a hydrogen atom; zero weight means *some weight* smaller than this—smaller, indeed, than anything that can be named. The same idea is contained in the words "infinitely small." In the same way, "infinitely great" means "something greater than anything." "Absolute equality" between two quantities exists when the difference between them is what we have defined above as zero.

§ 13.

Let us now cease to regard the upper curve of Fig. 9 as the elevation of a hillside, and look at it simply as a curve traced on paper. The lower curve may still be called the curve of slopes of the upper one. The length of the ordinate of the lower curve, corresponding to a point P on the upper, now simply shows the *instantaneous* rate at which the upper ordinate is increasing per inch increase of the abscissa, *i.e.* the rate of increase during the instant in which the tracing point is passing through P; of course, after the point has passed through P the rate is no longer the same.

Comparing together § 5, § 9, and § 12, we see that if, in Fig. 11 (which represents a curve PB and its "first derived"), P and Q are two points on the primary curve close together, then $\frac{KQ}{PK} = p'P'$ (This is only an absolute equality when Q is infinitely near to P.) Therefore, since pq is small, KQ $= p'P' \times PK = p'P' \times p'q'$, nearly. That is to say, the number of linear inches in the short line KQ is very nearly equal to the number of square inches in the thin rectangle $P'K'q'p'$. In the same way, let $pq = qr = rs$, etc.;

$$\text{Then } LR = \text{rectangle } Q'L'r'q'$$
$$MS = R'M's'r'$$

and so on. Suppose this process continued as far as B, and the results added together; then clearly to the same approximation—

$$KQ + LR + \ldots + HB \text{ in inches} = P'K'q'p' + Q'L'r'q' + \ldots + U'H'b'u' \text{ in sq. inches } (\alpha)$$

Now, however long or short pq, qr, etc., are, the left-hand side of this equation always = CB. Imagine, then, that instead of pq, qr, etc., being finite, as in the figure, we had been able to draw the ordinates pP, qQ, etc., so close together that pq, qr, etc., are infinitely small, so small that pb contains an infinitely great number of small parts, each = pq. This will not in the least alter the total value of the left-hand side of equation (a), for this must of necessity be = CB. But when pq, qr, etc., are infinitely small, the right-hand side is equal to the area under the curve P'B', bounded on each side by the ordinates p'P' and b'B'; for it is clear that the sum of these rectangles only differs from this area by the sum of the little triangles P'K'Q', Q'L'R', etc., and these all added together are evidently considerably less than the rectangle K'D, which is infinitely small compared to the area P'B'$b'p'$, when $p'q'$ becomes infinitely small (see § 32 on "Orders of Infinitesimals").

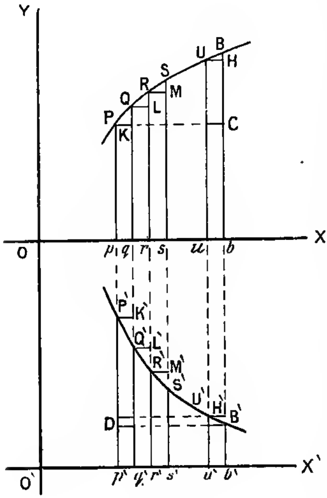

Fig. 11.

So we see that the dwindling of pq to an infinitely small quantity produces two effects simultaneously—

(1) Makes the equations such as

$$\frac{KQ}{PK} = p'P'$$

absolute equalities instead of only approximations.

(2) Makes the equation—

Sum of rectangles $P'K'q'p'$ + etc. = area of curve

an absolute equality instead of an approximation.

On the other hand, it does not interfere with the absolute equality $KQ + LR + \ldots = CB$. It therefore makes equation (α) equivalent to the assertion—

$$\text{CB in inches} = \text{area } P'B'b'p' \text{ in sq. inches} \quad . \quad (\beta)$$

(see note to § 3 and § 12).

This result must not be regarded as an approximation. It is an absolute and complete equality.

Our proof of the equality (β) would have been only an approximate one if we had imagined the strips as of finite thickness; for in that case the two equations on which (β) depends would be only approximately true. They are only absolutely true (§ 12) "in the limit" when pq, etc., are infinitely small.

Here, then, we have a most convenient way of measuring any irregular area with a curvilinear outline. Suppose, for instance, it is required to find the area of the lower figure between any two ordinates, we must regard it as the curve of slopes of some other curve, which we must proceed to draw. We shall presently explain the practical method of drawing this curve. Assuming that it has been drawn, and that the upper curve in Fig. 11 has been so obtained, then, in order to find the area of the lower curve between any two ordinates taken at random, we have only to find the difference between the two corresponding ordinates of the upper curve, and we have at once the required area in square inches.

§ 14. Practical Integration.

The method of finding the upper curve, having given the lower one, is the same in principle as that by which, in § 8, we deduced the curve of the railway from its curve of gradients.

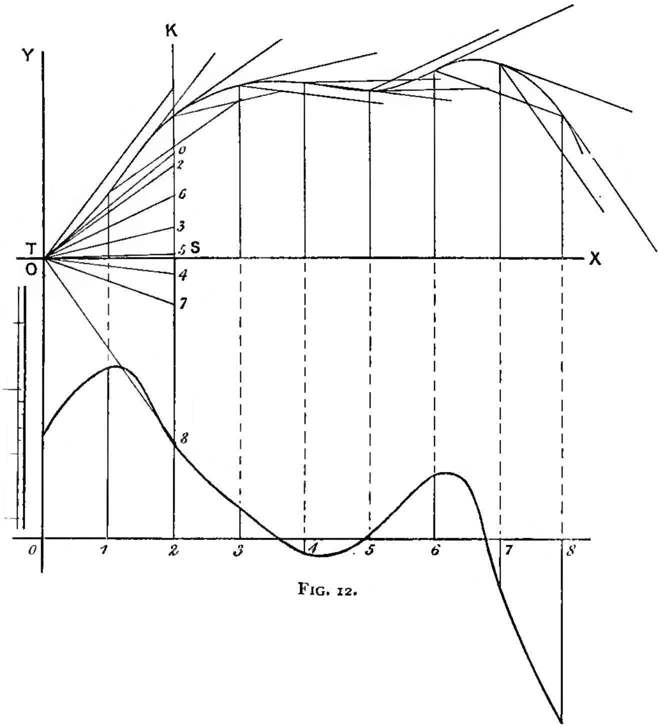

Fig. 12.

Take a curve, such as the lower one in Fig. 12, of which we are desired to find the area between any two ordinates. Then we know that the length of any ordinate of this curve in inches must = tangent of slope of required curve at the point where continuation of that ordinate cuts it. Draw a number of continued ordinates, and figure them as shown.

Set off on the axis of X, OS = 1″, and set up SK vertical. Cut off on SK, S1, S2, etc., etc., equal to the ordinates 0, 1, 2, 3, 4, respectively. Join O0, O1, etc., as shown. Now consider any one of these ordinates, say 3. If we had had the upper curve given, and had been required to deduce the height of the lower on the ordinate 3 by the process explained in § 9, it is evident that the triangle we should have drawn would have been exactly equal and similarly situated to the triangle OS3. Hence O3 must be parallel to the tangent to the upper curve at the point where ordinate 3 cuts it. Arguing in a similar way about the other points, it is clear that what we have to do is to draw a curve of which the slopes at the points where the lines 0, 1, 2, etc., cut it, are the same as the slopes of the lines O0, O1, O2, etc. On proceeding to draw the curve, we take an arbitrary point T from which to start (re-read carefully § 8 on the "Constant").

The next question is as to how we are to obtain the form of the curve between two ordinates. This we shall decide as follows. Any two neighbouring ordinates will be assumed so close together that the part of the curve between them is indistinguishable from a circular arc which has the same slopes at its ends as the actual curve would have. It will not be necessary for our purpose to find either the actual position of the centre or the length of the radius. The method of obviating this necessity is seen from Fig. 13, which is an enlarged view of two neighbouring ordinates. P is the assumed starting-point. Draw through P, PR, and PS parallel to the directions of the curve on the ordinates P and Q. These directions are obtained from O1, O2, O3, etc., in Fig. 12. Bisect RPS by PQ. It is evident, from the geometry of the

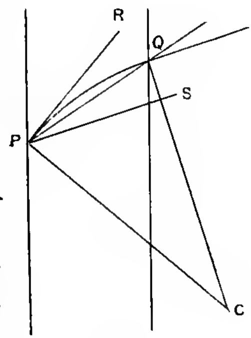

FIG. 13.

figure, that the circular arc when drawn will pass through Q. Q may next be taken as a fresh starting-point and the process repeated, and so on for the next ordinate. Thus a series of points are found through which the curve may be drawn. The difference between any two ordinates of the upper curve will, as already proved, give the area of the lower curve between the same two ordinates.

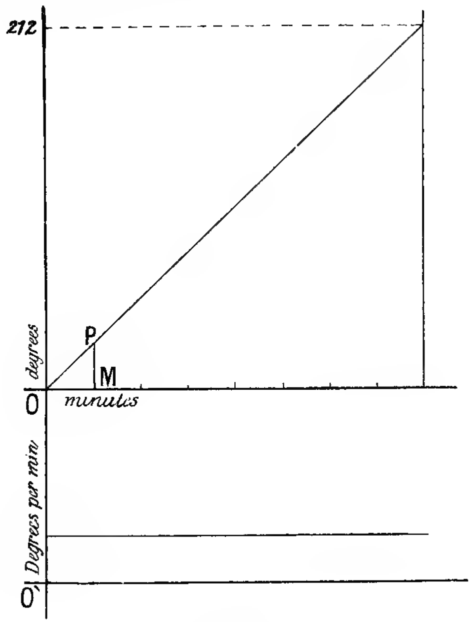

FIG. 14.

It is to be noticed that when the lower curve dips *below* the base, the corresponding portion of area is to be reckoned negative.

This is the process of "integration," and corresponds exactly to the algebraical process known by that name in mathematics.

It is necessary, for the above process, to use a hard sharp-pointed pencil, otherwise great inaccuracies may creep in. If carefully performed, the process is at least as accurate as any of the ordinary processes for finding of areas. The author of this work has devised a mechanical integrator, described in the Appendix, whereby the integral curve may be automatically drawn.

§ 15. Differential Coefficient considered as a Rate of Increase.

In cases where the curve which we are differentiating is one representing the results of a series of experiments, the derived curve is often of great importance. As an illustration of this, we will take the case of the experiment described in § 4. Suppose we had found, in that experiment, that the temperature had risen *uniformly* up to 212°—that is to say, that, during the first half-minute, if the temperature had risen $9\frac{1}{2}°$ (suppose), then during the fifth, or any other half-minute, it would also have risen $9\frac{1}{2}°$. Suppose the total time occupied = 8 min. Now, if we divide the total increase of temperature, viz. 152° (represented by O212, Fig. 14), by the time occupied, viz. 8 min., represented by ON, we shall obtain the amount of rise of temperature in 1 min. We have seen from § 5 that we shall also obtain the line MP. It is also otherwise obvious that, since OM = 1 min., MP = amount of rise of temperature in 1 min. Thus the tangent of the inclination of OP to OX represents the rate at which the temperature is rising. Our derived curve in this case would be a line parallel to O'X' at a height = MP. This would indicate that the rate of rise of temperature was constant all along the curve.

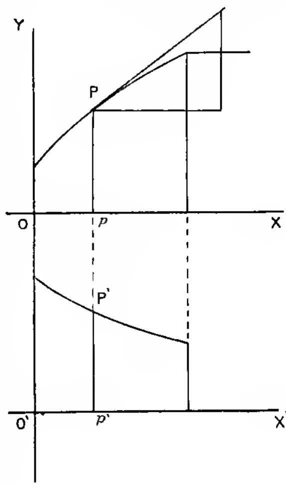

Fig. 15.

Although the actual curve was not a straight line, it may easily be seen that the tangent of inclination at a point P (Fig. 15) still represents the rate of increase at the point P, for the small "element" of curve at the point P is also part of the tangent to the curve (§ 11), and the rate of increase is therefore the same as the rate of increase along the tangent, *i.e.* = height of derived curve at the point.

§ 16. Other Applications.

Innumerable other applications of the same principle may be found. In almost every case of a curve derived from experiment, a distinct and tangible meaning may be ascribed to the height of the derived curve. One of the most important applications is the case where a curve is drawn to represent the motion of a moving body. Take the case of a man walking at a uniform rate along a road. Suppose

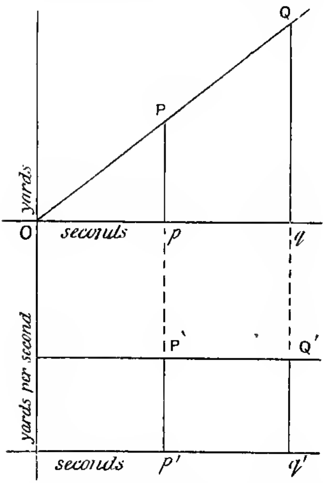

Fig. 16.

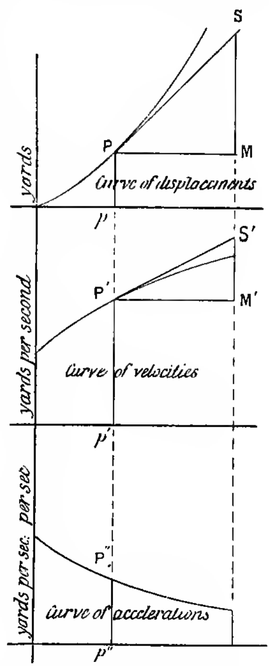

Fig. 17.

we plot vertically his distance from the starting-point, and horizontally the corresponding time. Thus after 1 sec. (represented by Op) (Fig. 16) he has walked a distance represented by pP; $\frac{pP}{Op} = \frac{qQ}{Oq} = p'P'$ = his rate of walking. The derived curve is here a horizontal line, as in the last section.

Successive Differentiation.—But suppose he walked a greater distance in the second second than in the first, and a greater still in the third, and so on, the height of the derived curve would still represent his instantaneous velocity at the corresponding point; for just at the point P, for instance, he is increasing his distance from *any* fixed starting-point at the rate of $MS = p'P'$ yards per second, so the middle curve is a curve of *velocities.* Now, if we differentiate the derived curve, we shall obtain a curve showing the time-rate at which his velocity is varying at each point, for at P' his velocity is increasing by M'S' yards-per-second every second $= p''P''$. Hence $p''P''$ represents the numerical value of his acceleration at the point P. This curve is called the second derived curve of the time-distance curve. If we differentiate again, we shall obtain a curve showing the rate at which his acceleration is changing. The dimensions of these latter units would be yards-per-second-per-second per second. This curve is called the third derived of the time-distance curve, and so on. It is easy to see that if the velocity had increased uniformly (or the time velocity curve had been an inclined straight line), the acceleration curve would have been a horizontal line, or the acceleration would have been constant or uniform, as in the case of a falling body. Thus we see that velocity = time rate of change of distance, acceleration = time rate of change of velocity, etc.

The student should think very carefully over this argument, because, in addition to its intrinsic importance, it forms, perhaps, the most perfect illustration of the application of the calculus to science that could be found. Curves may sometimes be differentiated by special constructions not involving the drawing of tangents. Several cases of this will be given later on.

EXAMPLES.

1. The space passed over by a body falling from rest is given by—

$$y = 16t^2$$

Draw this curve, selecting suitable scale, and differentiate it twice graphically. Compare the curves obtained with the curves (i.) $y' = 32t$, (ii.) $y'' = 32$. What are the meanings of y, y', y''?

2. Draw also the curve—

$$y = 10t + 16t^2$$

giving the space described by a body thrown downwards with velocity $10 \dfrac{\text{ft.}}{\text{sec.}}$;[1] differentiate it twice, and compare with curves derived from (1). Show that the first derived differs by a constant from the corresponding curve of (1), and the second derived is the same as the second derived of (1). What would have been the difference introduced if the body had been thrown upwards?

3. Draw the curve $xy = 12$. Integrate it, and find, by means of the integrated curve, the area of the given curve between the ordinates—

$x = 3$ and $x = 12$. (*Ans.*—16·63 sq. inches.)

4. Draw the curves (in the first quadrant only)—

$$y = \frac{1}{10}x^2 \text{ and } y = \sqrt{x}$$

and find their areas between the ordinates $x = 3$ and $x = 8$. *Ans.* 16·17 and 11·6.

5. Find between the same ordinates (by reducing the scale of the integrated curve) the area of—

$y^3 = x^5$ i.e. $y = x^{\frac{5}{3}}$. *Ans.* 89 sq. inches.

---

[1] This is a very convenient and suggestive notation for the unit of velocity, *i.e.* foot-per-second. It is clear that a velocity of say 6 feet per second is the same velocity as 12 feet per two seconds, or 18 feet per three seconds. This notation brings this out very clearly for $\dfrac{6 \text{ ft.}}{\text{sec.}}$ obviously $= \dfrac{12 \text{ ft.}}{2 \text{ sec.}} = \dfrac{18 \text{ ft.}}{3 \text{ sec.}}$ It will be found on examination that any quantity preceded by the word "per" is invariably a denominator. A logical extension of the same notation is used to denote unit of acceleration. An increase of velocity of s feet-per-second every second is denoted—

$$\frac{s \dfrac{\text{ft.}}{\text{sec.}}}{\text{sec.}} = s \frac{\text{ft.}}{\text{sec.}^2}$$

All physical units are treated in the same way.

CHAPTER III.

NOMENCLATURE AND GENERAL PRINCIPLES.

§ 17. To show that the Height of the Derived Curve may reasonably be denoted by $\frac{dy}{dx}$.

We have shown, in the last chapter, the geometrical meaning of the processes of differentiation and integration. We now proceed to explain the system of symbols that accompanies it. Suppose we are required to find the height of the derived curve of the curve $y = x^2$, corresponding to the point (2, 4). Now, it would obviously be very inconvenient to be compelled to draw the curve $y = x^2$ to scale, and differentiate it graphically in order to find the height of the derived curve at one point. An algebraical method of calculating it would be much more convenient, and this is what we are about to explain. The method is as follows :—

1. Calculate the height of the primary at a point Q (Fig. 18)[1] whose abscissa is slightly greater than that of P, the given point.

2. Find the difference MQ between the ordinates, and divide it by PM, the difference between the abscissæ.

3. Investigate what the result would be if PM were to gradually diminish until it became infinitely small. Now, if PM is a considerable size, the value of $\frac{MQ}{PM}$, viz. the tangent of the angle of slope of PQ, will differ considerably from the

[1] Fig. 18 is not drawn to scale.

tangent of angle of slope of the tangent to the curve at P. When PM is infinitely small, there will be no such difference. Our algebraical method of limiting values is equivalent to taking Q infinitely near to P. The algebraical result of the process must not, therefore, be regarded in the light of an approximation. It is an absolute exact truth (see § 12, p. 20).

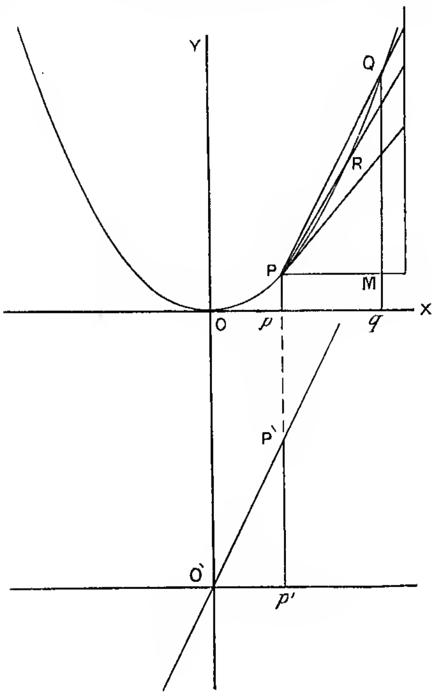

FIG. 18.

The difficulty in understanding this (if there be one) is due to an imperfect appreciation of the meaning of the expression, "when Q is infinitely near to P." Now, PM is called the "increment of x." It is written Δx, and implies the amount by which x is supposed to increase from an assumed particular value x. Similarly, MQ is called the corresponding "increment of y," and is written Δy;

$$\text{Tan QPM therefore} = \frac{\Delta y}{\Delta x}$$

Now, the point P being (2, 4), let the abscissæ of Q be $2\frac{1}{2}$, *i.e.* $\Delta x = \frac{1}{2}$. Its ordinate then $= (2\frac{1}{2})^2 = 6\frac{1}{4}$.

$$\text{Hence } \Delta y = \{(2\tfrac{1}{2})^2 - (2)^2\} = 2\tfrac{1}{4}$$

Thus when x increases by $\frac{1}{2}$ from the value 2, y increases by $2\frac{1}{4}$.

$$\text{Therefore } \frac{\Delta y}{\Delta x} = 4\tfrac{1}{2} \text{ when } \Delta x = \tfrac{1}{2}$$

Similarly, if we take $\Delta x = \frac{1}{4}$—

$$\Delta y = 1\tfrac{1}{16}$$

$$\text{and therefore } \frac{\Delta y}{\Delta x} = 4\tfrac{1}{4} \text{ when } \Delta x = \tfrac{1}{4}$$

$$\text{Similarly, if } \Delta x = 0{\cdot}000001$$

$$\text{then } \frac{\Delta y}{\Delta x} = 4{\cdot}000001$$

Again, if we take $\Delta x = -\frac{1}{4}$—

$$\text{Then } \Delta y = \{(1\tfrac{3}{4})^2 - (2)^2\} = -\tfrac{15}{16}$$

$$\frac{\Delta y}{\Delta x} = \frac{-\frac{15}{16}}{-\frac{1}{4}} = 3\tfrac{3}{4}$$

If we plot a curve (Fig. 19) showing the relation between Δx (abscissa) and $\dfrac{\Delta y}{\Delta x}$ (ordinate) for the point (2,4), we shall find it is in this case a straight line. Where this line cuts OY, the value of Δx is obviously $= 0$; *i.e.* the increment of x (PM in Fig. 18) is "indefinitely small" or "vanishes," and in this case clearly $\dfrac{\Delta y}{\Delta x} = 4$ exactly.

When this is the case, *i.e.* when Δx, and therefore Δy, diminish indefinitely, we write dx and dy instead of Δx and Δy, and the ratio (as indicated in Fig. 19) $\frac{dy}{dx}$. But $\frac{dy}{dx}$ is not like $\frac{\Delta y}{\Delta x}$ in being an actual fraction with numerator and denominator—that is to say, the dy and dx are not separate quantities which have actual numerical values; $\frac{dy}{dx}$ must be taken as a single symbol representing a definite finite quantity, although dy and dx are each infinitely small (see bottom of p. 9).

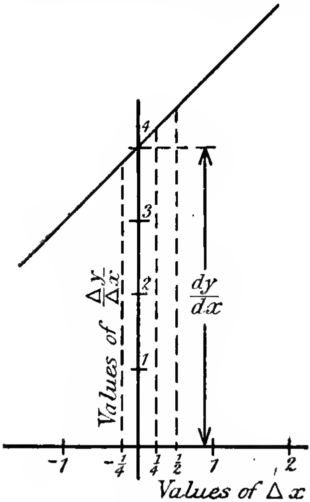

FIG. 19.

Of course, it is only for the point (2, 4) that the value of $\frac{dy}{dx} = 4$. If we had chosen the point (3, 9) instead, we should, in exactly the same way, have found $\frac{dy}{dx} = 6$. Indeed, if we had taken any point (a, a^2) on the curve, the value of $\frac{dy}{dx}$ would have been $2a$. The student should work out a few cases of this in the same manner as shown above. If he does it thoughtfully, he will probably be able to see the reason of it.

The algebraical process corresponding to that explained above is as follows—

Let $y = x^2$ . . . . (i.)

Let x increase by Δx, and in consequence y by Δy.

[In the process explained above, we took particular values $\frac{1}{2}$, $\frac{1}{4}$, $-\frac{1}{4}$, etc., for Δx, and calculated the corresponding numerical value for Δy. Here we calculate the general algebraical value for Δy in terms of Δx as follows.]

Then, since the point $(x + \Delta x, y + \Delta y)$ is by supposition on the curve $y = x^2$, we have (see p. 6)—

$$y + \Delta y = (x + \Delta x)^2 \quad . \quad . \quad \text{(ii.)}$$

Subtracting (i.) from (ii.) in order to find the difference between the ordinates, we have—

$$\Delta y = (x + \Delta x)^2 - x^2 = 2x\Delta x + (\Delta x)^2$$

Dividing by Δx the difference between the abscissæ (PM in Fig. 18)—

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x$$

which, when Δx is indefinitely diminished, becomes—

$$\frac{dy}{dx} = 2x$$

because Δx (or, as it would then be written, dx) becomes an infinitely small quantity. Therefore, as in § 12, the difference between $\frac{dy}{dx}$ and $2x$ being infinitely small, we say that $\frac{dy}{dx} = 2x$ absolutely.

(The student should compare this process with that explained above at every step. Only thus can he fully realize its meaning.)

Therefore the ordinate of the first derived curve, of the curve $y = x^2$, is always twice the abscissa, *i.e.* the derived curve is a straight line whose equation is $y' = 2x$.

Exercise.—In exactly the same way, the student can calculate the height of the derived curve for $y = x^3$. He will find it to be $y' = 3x^2$; for $y = x^4$, it is $y' = 4x^3$.

It will now be evident that the process of finding the algebraical value of $\frac{dy}{dx}$ is that of obtaining the equation to the curve which shows the relation between $\frac{\Delta y}{\Delta x}$ (ordinate) and Δx (abscissa), and finding, as at p. 4, where this curve cuts the vertical axis. This, it is clear, gives the value $\frac{\Delta y}{\Delta x}$ when $\Delta x = 0$.

We can here prove part of the general proposition that when $y = x^n$, $\frac{dy}{dx} = nx^{(n-1)}$. This is true, as a matter of fact, whatever n may be, positive or negative, integral or fractional. The reader is, however, not yet in a position to understand the complete proof, so we shall confine ourselves here to the limited case of positive integral indices, which will be necessary for purposes of illustration. The rest of the proof will be given as the reader is ready for it (§§ 24, 34).

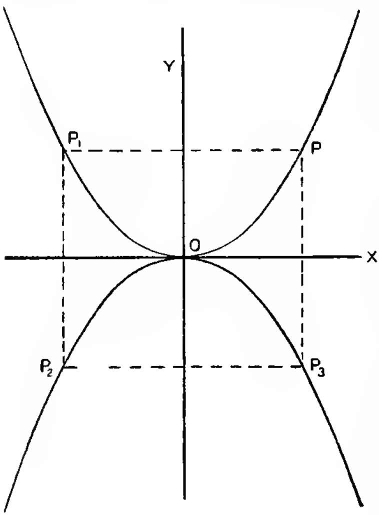

FIG. 20.

All complete curves of the form $y = x^n$ where n is positive and > 1, are in general shape similar to two of the four branches of the curves shown in Fig. 20.

Where n is an even positive integer, the curve $y = x^n$ resembles the curve P_1OP, and when n is odd, P_2OP. The reason of this is that, whether x is positive or negative, y or x^n is always positive when n is positive and even (thus $(-2)^4 = +16$), so that the ordinate y always lies above XOX. When n is odd, y or x^n is always negative when x is negative, so that in such curves as $y = x^5$ the curve on the left of OY always lies below XOX. For instance, in the curve $y = x^3$, if $x = -2$, $y = (-2)^3 = -8$. The branch OP_3 is included in such cases as $y^2 = x^3$, *i.e.* $y = \pm x^{\frac{3}{2}}$. In cases where n is < 1, but > 0, the curves resemble Fig. 20 turned through a right angle, *i.e.* looked at with OY horizontal. When n is < 0, the curves resemble hyperbolas, of which OY, OX are asymptotes.

It is a most interesting exercise to trace the variations of the curve

$y = x^n$, as n varies between $+\infty$ and $-\infty$. It affords, among other things, a beautiful illustration of the meaning of the statement that $x^0 = 1$.

Let $y = x^n$ . . . . (i.)

If x increase by Δx, and y in consequence by Δy, we have, as before—

$$y + \Delta y = (x + \Delta x)^n \quad . \quad . \quad \text{(ii.)}$$

Subtracting (i.) from (ii.), we have—

$$\Delta y = (x + \Delta x)^n - x^n$$

$$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

Expanding by the binomial theorem—

$$\frac{\Delta y}{\Delta x} = \frac{x^n + nx^{n-1}\Delta x + \dfrac{n(n-1)}{1 \cdot 2}x^{n-2}(\Delta x)^2 + \ldots - x^n}{\Delta x}$$

$$= nx^{n-1} + \Delta x \times \text{some other quantity}$$

which equation, when Δx vanishes, becomes $\frac{dy}{dx} = nx^{(n-1)}$; or, as it may be written—

$$\frac{d(x^n)}{dx} = nx^{(n-1)}$$

§ 18. Meaning of dy and dx when used alone.

In some methods of treating and writing the calculus the expressions dy and dx are used apparently alone. This seems to cause great difficulty to students, because of a sort of indefiniteness in the actual values to be assigned to the quantities denoted by dy and dx. It will be found on examination, however, that in such cases there is always an implicit reference to the ratio $\frac{dy}{dx}$. It is merely a somewhat more convenient way of referring to the ratio, and is introduced for the purpose of saving space and for convenience of printing.

Thus though dy, standing by itself and considered apart from anything else, is numerically absolutely meaningless, yet when we write, as in the curve above, $dy = 2xdx$, we really mean that $\frac{dy}{dx} = 2x$, which indicates that if x increases by a small quantity, y increases by a quantity $2x$ times as great.

So in general we may write—

$$\delta y = \frac{dy}{dx} \times \delta x \quad . \quad . \quad . \quad (a)$$

where δy, δx are corresponding small increments of y and x, which may be of definite magnitude. If, however, δx and δy are of small but finite magnitude, the equation (a) becomes an approximation, though usually an extremely close one, and not an absolute equality.

δx is here called a "differential," and $\frac{dy}{dx}$ therefore a "differential coefficient" (see illustration, § 25).

The fact that two quantities of indefinite magnitude can have a definite ratio sometimes causes students trouble. This may be got over by reflecting that the quantities $2n$ and $3n$ have always the ratio $\frac{2}{3}$ whatever the absolute value of $2n$ or $3n$.

§ 19.

It is easy to see that when $y =$ constant, *i.e.* when the primary curve is a straight line parallel to OX, since the slope is at all points of this line nothing, $\frac{dy}{dx} = 0$. Or, regarded algebraically, the statement $y =$ constant is equivalent to the statement that y does not vary, and therefore dy (which means the amount of variation of y *corresponding to* a variation dx in x) $= 0$, and therefore—

$$\frac{dy}{dx} = 0$$

§ 20. SUCCESSIVE DIFFERENTIATION.

At § 17 it was explained how it was sometimes necessary to differentiate a *derived* curve. The primary curve in that case was a time-distance curve, the first derived a time-velocity, and the second derived a time-acceleration curve. Now, the height of the primary being denoted by y, and that of the first derived by y' or $\frac{dy}{dx}$, the height of the second derived may, on the same principle, be denoted by y'' or $\dfrac{d\frac{dy}{dx}}{dx}$. The latter, however, is very inconvenient to write and print. It is therefore shortened by treating it as a simple fraction in which d stands for some definite algebraical quantity. (In reality, of course, it does not mean anything of the kind.) Thus we have—

$$\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d^2y}{(dx)^2}$$

or, omitting the bracket, $\dfrac{d^2y}{dx^2}$

The student must be careful to notice that this quasi-fraction is nothing but a shortened form of $\dfrac{d\frac{dy}{dx}}{dx}$, that the expression has absolutely nothing to do with x^2, and that at present he may regard d^2 as merely symbolical. In the same way, the height of the third derived is—

$$\frac{d\frac{d^2y}{dx^2}}{dx} = \frac{d^3y}{dx^3}$$

that of the fourth, $\dfrac{d^4y}{dx^4}$, and so on.

§ 21. Notation of Integration.

It will be seen from § 13 that the process of graphical integration consists of a construction whereby a curve is obtained of which the tangent of angle of slope is at all points = ordinate of curve we wish to integrate. The algebraical process is the exact counterpart of this, and consists in obtaining an *expression* which, when differentiated algebraically, will give as a result the expression which we wish to integrate. *There is no general method of performing this reverse operation.* Indeed, in a great number of cases it cannot be performed at all except by the aid of an infinite series. We are in all cases obliged to rely on our previous experience of differentiation. If the expression is of a type of which we have had no previous experience, we cannot do anything with it until we have twisted it into a shape which we do recognize as the result of some differentiation with which we are already acquainted.

Suppose, for instance, we wish to integrate $3x^2$. This means that $y' = 3x^2$ is to be the first derived of the curve we wish to find. The problem is stated thus for an "indefinite" integral—

$$\int 3x^2 dx$$

or, in the case of a definite integral—

$$\int_a^b 3x^2 dx$$

These expressions will be presently explained.

This symbol $\int$ may be regarded in two ways: (1) It may be taken simply as a question mark. The meaning then is—

$$\int \qquad\qquad 3x^2 \qquad d \qquad\qquad x$$

? expression will give $3x^2$ when differentiated with respect to x

(2) It may be taken to be the letter s, the first letter of the word "sum," thus—

$$\int_a^b 3x^2dx$$

The sum between the ordinates $(x=b)$ and $(x=a)$ of all such rectangles as $3x^2dx$

For it is evident that the area in square inches of a very thin vertical strip of the curve such as that shown in Fig. 21 is $3x^2dx$ (see §§ 13 and 17), and the sum of all the thin strips into which the area between b and a may be divided = whole area of curve between these two ordinates = difference between corresponding ordinates of upper curve, as already explained (§ 13).

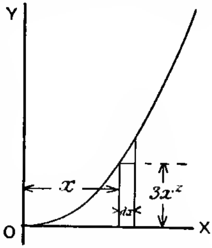

Fig. 21.

Now, let us consider what expression will give $3x^2$ when differentiating with respect to x as independent variable. (The student will understand the last expression better after reading the next chapter.)

Consider what is the rule just proved (§ 17) for differentiating x^n. We have found the differential coefficient to be $nx^{(n-1)}$. Hence the answer to the question $\int nx^{(n-1)}dx$ is x^n. It is easily seen that the given expression $3x^2$ is of the form $nx^{(n-1)}$ where $n = 3$, hence—

$$\int 3x^2dx = x^3$$

A more complete solution, as will be presently explained, is "x^3 + some constant."

Hence we see that $\int 3x^2dx = x^3$ is exactly the same equation as $\frac{d(x^3)}{dx} = 3x^2$, but put into another form. In just the same way as $\frac{20}{4} = 5$ is the same thing as $5 \times 4 = 20$. This is sometimes symbolically expressed by saying that $\int$ and d "cancel one another." Thus multiplying both sides of the equation $\frac{d(x^3)}{dx} = 3x^2$ by dx we have—

$$d(x^3) = 3x^2dx$$

Now multiply by $\int$. We obtain—

$$\int d(x^3) = \int 3x^2dx$$

or, since $\int$ and d cancel—

$$x^3 = \int 3x^2dx$$

This is not altogether a happy analogy, for $\int$ and d do not cancel on the right-hand side. The idea is that if any quantity, A (represented here by the ordinate of the upper curve), be divided into a large number of parts, and then all the parts be added together, the quantity A is reproduced.

§ 22. THE "CONSTANT" IN INTEGRATION.

The expression "$x^3 +$ constant" is known as the "indefinite integral" of $3x^2$. It is a general expression for the height of *every possible* primary which has $y' = 3x^2$ for its first derived. We have already seen (§§ 8, 13, etc.) that there are an infinite number of such curves corresponding to different starting-points on the line OY. If a value K (suppose) be assigned to the "constant," the value of $x^3 + K$ at any point also represents absolutely the area of the curve $y' = 3x^2$ between that ordinate y' and the ordinate corresponding to the point where the curve $y = x^3 + K$ cuts the axis of x. This point may be found by putting $y = 0$ in the equation and solving for x. Thus here—

$$x = -\sqrt[3]{K}$$

This may be generally explained as follows. Suppose we have any curve P'Q' (Fig. 22) of which the equation is $y' = f'(x)$ (where $f'(x)$ is a shorthand symbol for "any expression containing x"), and suppose, having integrated it, we obtain a curve PQ or TK, or some other parallel curve of which the equation is $y = f(x) + c$, where $f(x)$ is some different expression containing x. Then, in accordance with the notation already explained, we have $\int f'(x)dx = f(x) +$ constant. Then the expression $f(x) + c$ is called the indefinite integral

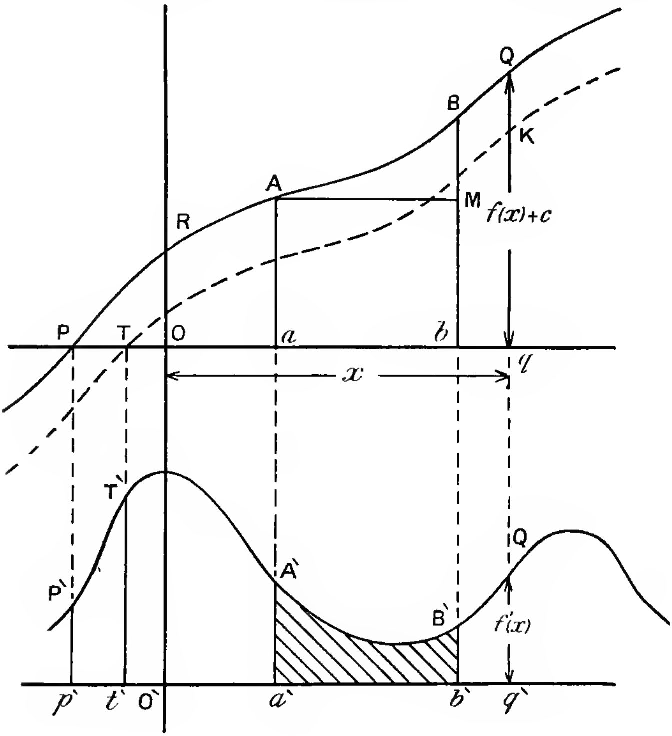

FIG. 22.

of $f'(x)$. Let $x = Oq$. Then for some particular value of c, $f(x) + c = q$Q. For some other value of c, $f(x) + c = q$K. Since the slope at K is the same as the slope at Q, the height q'Q' is the same in each case. Now, qQ = area of lower curve between the ordinate p'P' (corresponding to the point where the upper curve cuts OX) and $q'Q'$; also qK = area between $t'T'$ and $q'Q'$, and so on.

If we wish to find the (shaded) area of the lower curve between two definite ordinates, $x = 1$ and $x = 2$ (suppose), the expression for the area is $\int_1^2 f'(x)dx$.

This is called a "definite integral," and we have already seen (§ 14) that the area is found by taking the difference in length between the corresponding ordinates of the upper curve. These ordinates are found by substituting 1 and 2 in turn for x in the expression $f(x) + c$. The notation for this is $f(1) + c$ and $f(2) + c$, denoting respectively the ordinates aA and bB.

Hence clearly—

$$\begin{aligned}\int_1^2 f'(x)dx &= \{f(2) + c\} - \{f(1) + c\} \\ &= f(2) - f(1)\end{aligned}$$

Here again we see that the actual value of c is unimportant, since it disappears in the final result.

The following notation is usual as a shortened form of the expression on the right-hand side of the above equation :—

$$\int_1^2 f'(x)dx = {}_1^2\Big[f(x) + c\Big]$$

A further exposition of these facts, with actual numerical examples, will be found in Chapter IX.

The nomenclature adopted when it is necessary to differentiate any expression twice has been explained in § 20. It is also often necessary to integrate an expression twice. The notation then adopted is for an indefinite integral—

$$\iint \{f''(x)\}dxdx$$

which means—

$$\int\Big[\int \{f''(x)\}dx\Big]dx$$

Suppose the quantity inside the [ ] brackets, viz.—

$$\int \{f''(x)\}dx = \{f'(x) + c\}$$

the above expression obviously means—

$$\int \{f'(x) + c\} dx$$

It is necessary to notice that the constant c, which is required (as already explained) for the first integral, must be included under the sign of integration for the second operation. The geometrical meaning of this statement will be apparent to any one who differentiates a curve twice and then integrates the result twice. Obviously, the final result will be very much affected by the (arbitrary) height of the starting-point for the first integrated curve above O', for the height of the curve obtained by the first integration determines the slope of the curve obtained by the second integration.

For a definite integral the notation is—

$$\int_b^a \int_d^c \{f''(x)\} dxdx$$

which, as before, means—

$$\int_b^a \left[\int_d^c \{f''(x)\} dx \right] dx$$

Further explanation is not possible at this stage.

EXAMPLES.

1. Differentiate x^2, x^3, x^4, x^5, x^{16}, x^{100}, x^a, x^s, with respect to x.
2. Differentiate $(a^2)^6$, $(b^c)^{12}$, $(c^k)^n$, with respect to—
   (i.) a^2, b^c, c^k, respectively.
   (ii.) a, b, c, respectively.
   (iii.) a^p, b^q, c^r, respectively.

(Substitute in each case x for the quantity with respect to which the function is to be differentiated. Then differentiate with respect to x, and substitute again. Thus, in case iii.—

$$(b^c)^{12} = b^{12c} = (b^q)^{\frac{12c}{q}} = x^{\frac{12c}{q}}$$

and $$\frac{d\left(x^{\frac{12c}{q}}\right)}{dx} = \frac{12c}{q} x^{\left(\frac{12c}{q}-1\right)} = \frac{12c}{q} x^{\left(\frac{12c-q}{q}\right)} = \frac{12c}{q} b^{(12c-q)}\Big)$$

3. Integrate $7x^6$, $12x^{11}$, ax^{a-1}, 1.

CHAPTER IV.

GENERAL PRINCIPLES.

§ 23. Changing the Independent Variable.

We have hitherto regarded the value of the quantity denoted by y as absolutely dependent on the value we give to the independent variable denoted by x. Thus in our illustrative curve $y = x^2$ we calculated the points on it by giving arbitrary values to x, and calculating the corresponding value of y. We have also (and we shall continue to do so) always plotted the *independent* variable horizontally.

To show $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$.—Now, we might write the relation $y = x^2$ in the form $x = \pm\sqrt{y}$, and, in calculating points on the curve, give arbitrary values to y, and calculate x by finding the square roots of these values. Now, if we continued to plot y vertically and x horizontally, we should, by this process, obtain exactly the same curve as before. But by our convention we are to plot the values of the independent variable (which is now y) horizontally.

What, then, is the relation between the two curves thus obtained?

Let OPQ (Fig. 23) represent the curve $y = x^2$. Let the curve be drawn on a piece of tracing-paper held over the original curve. Holding the point O fixed, turn the tracing-paper through a right angle in the direction of the arrow; we shall thus obtain the curve OP_1Q_1, which is clearly the same curve as before, but in which the original positive values of y, such as qQ, are plotted horizontally *in the negative direction*, i.e. towards the left, as at Oq_1. If we "reflect" the curve OP_1Q_1 along OY, *i.e.* imagine the curve turned about OY as axis into the position OP_2Q_2, this defect is remedied. This latter curve is the same as would have been obtained by plotting the curve $x = \sqrt{y}$ in the ordinary way, but with y

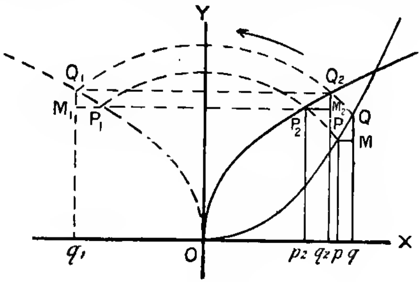

FIG. 23.

horizontally and x vertically. If we now differentiate this curve graphically in the ordinary way, what we shall obtain will be a curve showing the values of $\frac{dx}{dy}$ for all values of y Now, what we have to prove is that, taking any point P on the first curve, and *the same point* P_2 on the other curve—

$$\text{Tangent of slope of (OP)} \times \text{tangent of slope of } (OP_2) = 1$$

It will, perhaps, appear that a simpler and more direct method of exhibiting this relation would have been to find the slope of the original curve relative to the axis of y, and to treat the axis of Y exactly as we have previously treated the axis of X, plotting values of $\frac{dx}{dy}$ horizontally along a base on the left of the original axis of Y. This would, however, have involved temporarily reversing our ideas of positive and negative directions. It is easy to see that the present construction is in reality the same as this would have been if we had also rectified the signs. The additional utility of the present construction will be more apparent at a later stage.

Taking an adjacent point Q, it is clear that wherever Q is—

$$PM = M_1Q_1 = M_2Q_2 \text{ and } MQ = P_1M_1 = P_2M_2$$

Hence—

$$\frac{MQ}{PM} \times \frac{M_2Q_2}{P_2M_2} = 1$$

And as this is true wherever Q is, it is true in the limit, when it moves up to and ultimately coincides with P, *i.e.* $\frac{dy}{dx} \times \frac{dx}{dy} = 1$,[1] a result which, though it was certainly to be expected, we should not have been justified in assuming merely because $\frac{dy}{dx}$ and $\frac{dx}{dy}$ look like fractions. Thus we see that, whenever we wish to find a value for $\frac{dy}{dx}$ in any given case, if it happens to be easier to find $\frac{dx}{dy}$ from something we already know, we are at liberty to do so, and thence deduce $\frac{dy}{dx}$ by inverting the value so found.

Some students find a difficulty here which is not easy to express in words. It is as follows. This proof, as it stands, only holds when Q is at the same distance from P as Q_2 is from P_2. When this is the case, it is clear that $\frac{dy}{dx} \times \frac{dx}{dy} = 1$. But the values of dy and dx are not definite. We only know that they are indefinitely small, and, provided they are indefinitely small, they can have any order of smallness. How, then, are we to know that the dy and dx in the first factor are the same dy and dx as in the second factor? The answer is, that, provided dy and dx *are* indefinitely small, the value of the *ratio* $\frac{dy}{dx}$ is constant for any particular point, no matter what the order of smallness of dy and dx. The dy and dx of the first factor must be taken together; likewise the dy and dx of the second factor. The difficulty arises from thinking of $\frac{dy}{dx}$ as a variable fraction instead of a fixed ratio of two variable but mutually dependent quantities.

---

[1] It is easy to see, from the note on p. 47, that this result is merely another form of the trigonometrical relation $\tan\theta \times \tan\left(\frac{\pi}{2} - \theta\right) = 1$, where θ is the angle of slope of a curve at any point.

Thus we see that if we differentiate curve OPQ and OP_2Q_2 with OX as base, and take any two *corresponding* ordinates of the two derived curves (*e.g.* pP and p_2P_2 are corresponding ordinates), the rectangle formed with these two derived ordinates as sides will contain exactly one square inch.

Now, we have hitherto, for the sake of clearness, regarded the value of y as being *dependent on the* value of x. For the purposes of the calculus this is not in the least necessary. Our results are just as good if, as a matter of fact, the value of x depends on that of y, or if the value of both x and y depend on that of another variable z, which, although its variations may be the prime cause of the simultaneous variations of x and y, does not appear in the equations at all. What the calculus is really concerned with is the fact that x and y do actually vary together in such a way that every definite value of x corresponds to a definite value or values of y, and it is not concerned with what may or may not have been the ultimate cause of those simultaneous variations.

§ 24. Differentiation of $x^{\frac{1}{m}}$.

We can utilize this principle at once to further our proof of the general proposition that if $y = x^n$, $\frac{dy}{dx} = nx^{(n-1)}$.

Let n be of form $\frac{1}{m}$ where m is an integer.

Let $y = x^{\frac{1}{m}}$ where $x^{\frac{1}{m}}$ means $\sqrt[m]{x}$ (see chapter on indices and surds in any algebra).

As far as we have hitherto gone, we cannot differentiate this.

It is easy to see that the curve $y = x^{\frac{1}{m}}$ is the same as the curve $x = y^m$, for the same values of x and y satisfy both equations.

If we differentiate $x = y^m$ with respect to y, we get—

$$\frac{dx}{dy} = my^{(m-1)}$$

This operation corresponds exactly to turning our curve through a right angle, reflecting along OY, and differentiating graphically.

Hence we have—

$$\frac{dy}{dx} = \frac{1}{my^{(m-1)}}$$

But we require the value of $\frac{dy}{dx}$ in terms of x, and not of y.

Substituting $x^{\frac{1}{m}}$ for y, we get—

$$\frac{dy}{dx} = \frac{1}{m} \cdot \frac{1}{\left(x^{\frac{1}{m}}\right)^{m-1}}$$

$$= \frac{1}{m} \cdot \frac{1}{x^{\frac{m-1}{m}}}$$

$$= \frac{1}{m} \cdot x^{\left(\frac{1}{m} - 1\right)}$$

which is of the required form. Thus the proposition holds good when n is of the form $\frac{1}{m}$.

§ 25. Illustrative Example.

The following example will give an idea of the practical meaning of this principle.

Suppose we take a barometer up a mountain-side from sea-level, and note the height of the barometer at short intervals of vertical rise. (The vertical heights must, of course, be known, independently of the barometer.) Let a height-barometer reading curve be plotted from these observations. Let 100 feet = 1 unit on the diagram, and barometer-reading scale be full size. The characteristics of the curve that would be obtained by such a proceeding are shown exaggerated in Fig. 24. On differentiating this curve graphically, we shall obtain a first derived, which, since the primary always slopes downwards, lies entirely below O'X'. Thus p'P' shows the *rate of rise*, or, in other words, $P'p'$ shows the *rate of fall*, of the barometer at P per 100 feet lift.

Notice carefully the significance of the signs here. A fall is a *negative rise*. If the barometer falls $+$ 0·5 inch, it may be said to *rise* $-$ 0·5 inch.

Thus, suppose $p'P' =$ 0·5 inch. The meaning of this is that at the point P the rate of fall is $\frac{0.5 \text{ inch}}{100 \text{ feet lift}}$ (see note on p. 30).

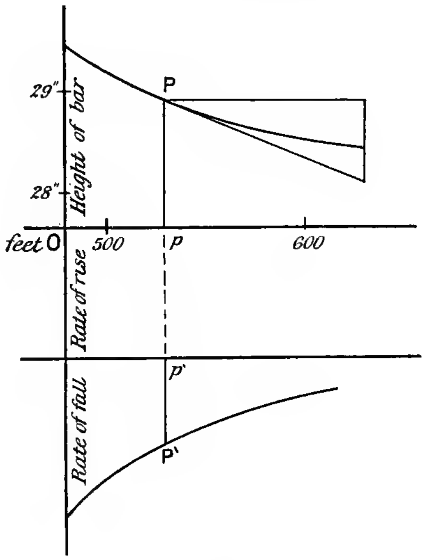

FIG. 24.

Now, we may, as at § 18, p. 38, conveniently write $\delta y = \frac{dy}{dx} \times \delta x$; *e.g.* assume that we lift the instrument through 6 inches (δx), when at an altitude given by Op. The instrument will rise (see note above) by an amount $\delta y = \frac{dy}{dx} \times \delta x$, where $\frac{dy}{dx} = p'P'$ is the current rate of rise per 100 feet lift. Hence—

$$\delta y = -\frac{0·5 \text{ inch}}{100 \text{ foot}} \times 0·5 \text{ foot}$$

The dimension "foot" cancels out

$$= -\frac{0{\cdot}5 \text{ inch}}{200} = -\tfrac{1}{400} \text{ inch}$$

that is to say, the barometer falls $\frac{1}{400}$ of an inch.

If the student be not well versed in the method of dimensions, he should carefully note the illustration given here. Of course, this only holds where δx is so small that the point on the curve whose abscissa is $(x + \delta x)$ is not any appreciable distance from the tangent to the curve at the point P. It implies that $\frac{\delta y}{\delta x} = \frac{dy}{dx}$. If $\frac{\delta y}{\delta x}$ differs sensibly from $\frac{dy}{dx}$, there will be an error introduced. This point has been already explained several times (§§ 7, 12, 13, 15, etc.) in various aspects. If the student does not understand it, he is referred to the sections quoted.

Now, suppose we turn our primary curve through a right angle and reflect it on OY. We shall obtain the upper curve in Fig. 25, which is exactly the same curve as Fig. 24 viewed under another aspect. The height of the derived curve now represents the vertical distance through which we must lift the barometer in order that it may fall 1 inch, assuming that the rate of vertical lift per inch fall of the barometer remains constant; or, in other words, $p'P'$ represents the instantaneous rate of lifting per inch rise of barometer. Now, what we have just proved amounts in this case to this—

Rate of vertical lift (in hundreds of feet) per inch fall of barometer

$$= \frac{1}{\text{rate of fall of barometer in inches per 100 feet lift}}$$

In the particular case considered above, rate of fall of barometer per 100 feet lifted through was 0·5 inch per 100 feet at a certain point. Hence, at that point rate of lifting per 1 inch fall of barometer $= \frac{1}{0{\cdot}5} = 2$; *i.e.* the instrument must be lifted 2 units or 200 feet if the barometer is to fall 1 inch (assuming constant rate of falling), which is, of course, otherwise obvious.

This illustration may probably present some difficulty to the student, partly owing to the essential difficulty (to a beginner) of viewing the same ratio under two aspects, and partly from the confusion introduced by the practically

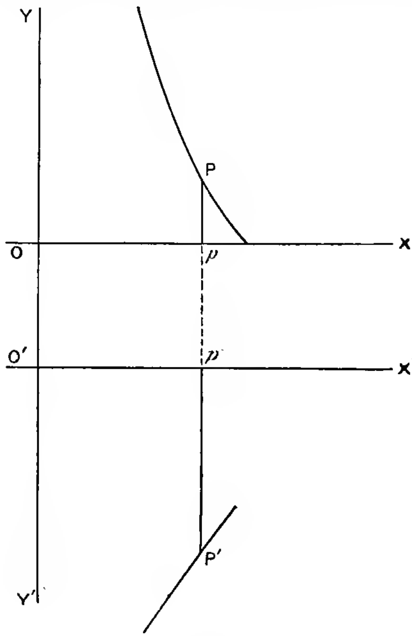

FIG. 25.

necessary difference of scale in the vertical and horizontal directions. We have already had several instances in which the scale was intentionally arranged so as to be as simple as possible. This is intended as an exercise in variation of scale. The best way to understand confusing examples of this kind is to keep the mind fixed on the diagram rather than on the form of the words.

Exercise.—Draw a curve of any shape and differentiate it. Turn it through a right angle, as explained in § 23. Reflect on OY, and differentiate again. Mark corresponding points on the two curves, and show by the method explained in § 2, Fig. 3, that the mean proportional between the heights of the derived curves is always 1 inch. (This is, of course, merely an application of the principle that $\tan\theta \times \cot\theta = 1$.)

§ 26. Differentiation of Sum and Difference of Functions.

Suppose we have given two elementary curves; for example, (1) and (2) in Fig. 26, which represent $y = x^2$ and $y = \sqrt{x}$. Draw another curve whose ordinates are = the sums of the corresponding ordinates of the given curves. This may easily be done graphically. In Fig. 26, all pairs of corresponding ordinates (such as Pp, qQ) of (1) and (2) are together equal to the ordinate (such as rR) of (3). Differentiate (1) and (2), and place the derived curves on the immediate right of the corresponding primaries; thus (4) is the first derived of (1), (5) of (2), and (6) of (3). We shall now show that there is the same relation between the ordinates of (4), (5), and (6) as there is between those of (1), (2), and (3), *e.g.* that $p'P' + q'Q' = r'R'$.

Proof.—By construction—

$$pP + qR = rR$$

$$\text{that is, } sL + tM = uN;$$

also by construction—

$$sS + tT = uU$$

Hence by subtraction—

$$LS + MT = NU$$

Dividing through by PL = QM = RN, we have—

$$\frac{LS}{PL} + \frac{MT}{QM} = \frac{NU}{RN}$$

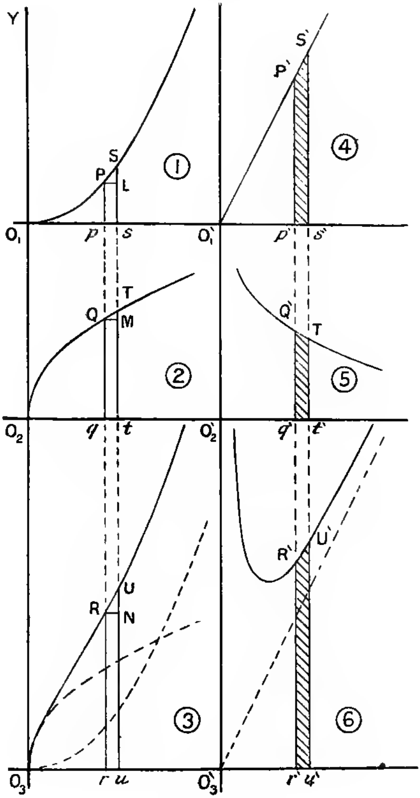

FIG. 26.

that is, when S is close to P, and therefore T and U close to Q and R respectively—

$$p'P' + q'Q' = r'R'$$

(see note on p. 48).

Roughly speaking, the meaning of this is, that if we have two slopes (which we may imagine as wedges cut out of a pack of cards) of the same length piled on top of one another in the way shown in Fig. 27, the resulting slope (tangent of angle) is the same as that of the other two added together. This can be easily seen.

Now, the equation to curve (3) in Fig. 26 is evidently—

Ord. of (3).		Ord. of (1).		Ord. of (2).
y	$=$	x^2	$+$	$x^{\frac{1}{2}}$

Our result tells us that its derived equation is—

Slope of (3).		Slope of (1).		Slope of (2).
y^1	$=$	$2x$	$+$	$\frac{1}{2}x^{-\frac{1}{2}}$

as already shown in §§ 17 and 24.

Exactly similar reasoning applies to the differential co-efficient of differences of functions. In this case the ordinates of curve (3) are to be made = the differences of the ordinates of (1) and (2). The figure can be easily made from Fig. 26, by exchanging the places occupied by curves (1) and (3), and also those of (4) and (6). It is then easily seen that the former proof applies also to this. Indeed, the proof given above for the sum will hold throughout independently if we change the + sign into −. If the wedges (1) and (3) in Fig. 27 also change places, the result may be seen to be the same as the + result viewed under another aspect.

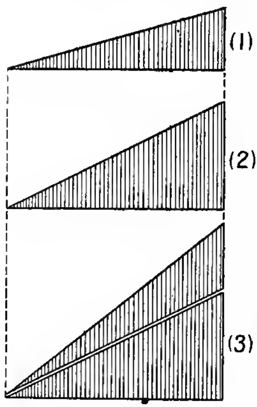

Fig. 27.

Generally speaking, our result may be written thus:

If $y = u + v - w + t - r$, etc., where u, v, etc., are any functions of x, we have—

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx} + \frac{dt}{dx} - \frac{dr}{dx}$$

There is no difficulty in extending the proof in this manner. It is left as an exercise for the student.

§ 27. ILLUSTRATIVE EXAMPLE.

A simple practical example of this principle, which, though not scientifically quite accurate, as will presently be explained, is instructive and easy to understand, is as follows.

A man holds two appointments, in one of which his salary is £180, with an annual increase of £20 per annum. In the other his salary is £85, with an annual increase of £15 per annum. His income tax at this time is £7 per annum, and is increasing at the rate of £1 per year per year. It is required to find what is the total net rate of increase of income.

Call the salaries y_1, y_2, the income tax y_3, and total income y. Let x represent the number of years reckoned from this time. We have $y = y_1 + y_2 - y_3$. It is clear that, if we take Δx to be any integral number of years—

$\frac{\Delta y_1}{\Delta x}$ = rate of increase on first appointment

$\frac{\Delta y_2}{\Delta x}$ = " " second "

$\frac{\Delta y_3}{\Delta x}$ = " " income tax

In this simple case it is easy to see that $\frac{\Delta y}{\Delta x} = \frac{\Delta y_1}{\Delta x} + \frac{\Delta y_2}{\Delta x} - \frac{\Delta y_3}{\Delta x}$, which is another way of saying that total net increase of income = rate of increase of salary on first appointment + rate of increase of salary on second appointment − rate of increase of income tax.

This notation is not strictly applicable to the case in point, because the rate of pay does not increase every instant, but step by step, each step being one year broad, and we have, therefore, to assume such particular values for Δx as will make the expressions $\frac{\Delta y}{\Delta x}$, etc., give a correct idea of the rate of increase. If the salary or rate of pay increased every instant, this stepped figure (28) would merge into a straight line as shown, and the notation $\frac{dy}{dx}$ could then have been applied to it.

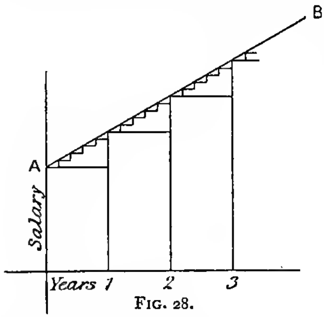

FIG. 28.

Exercise.—The line AB, Fig. 28, is itself the derived curve of another curve. What do the ordinates of this other curve represent? *Ans.* The aggregate earnings of the man.

Another illustration of a more scientific character will be found in the following. A man in a corridor train commences to walk along the corridor in the same direction as the train is moving. Suppose the ordinate of curve (1), Fig. 26, represents the distance travelled by the train in a time (after the moment of starting) represented by the abscissa. Let curve (2) represent in the same way the distance walked by the man along the corridor (or, as it is expressed in kinematics, "relative to the train"). Then curve (3) represents the total distance moved by the man through space (relatively to the ground) in time represented by the corresponding abscissa. Our principle states that ordinate of derived curve of (1) + ordinate of derived curve of (2) = ordinate of derived curve of (3), which, as we see from § 17, is equivalent to stating that at any instant velocity of train + velocity of man along corridor = total velocity of man relative to rails. Differentiating again, we have acceleration of train + acceleration of man along corridor = acceleration of man relative to rails.

Let the student follow out the case where the man walks in the opposite direction along the corridor.

§ 28. To Differentiate $nf(x)$.

An important analytical principle can be deduced from a special case of this result.

Suppose each of the two curves (1) and (2) in Fig. 26 had been exactly alike; then curve (3) would be twice as high as either (1) or (2), and (6) would be twice as high as (4) or (5). Thus—

$$\text{if } y = 2u$$

$$\text{then } \frac{dy}{dx} = 2\frac{du}{dx}$$

$$\text{or if } y = 3v, \frac{dy}{dx} = 3\frac{dv}{dx}$$

This law evidently holds for any integer whatsoever. It also clearly holds for any fraction. For in this case curve (1) is half as high as curve (3), so that the derived of (1) is half as high as the derived of (3). Proceeding in this way, we can prove the principle for any positive or negative integer or fraction.

Hence, when n is *any* quantity—

$$\text{if } y = nu, \frac{dy}{dx} = n\frac{du}{dx}$$

Exercise.—Prove the result when n is a negative quantity.

§ 29. Integrals of Sums and Differences of Functions.

It is obvious that the principle proved in § 26 applies equally well to integrations, for in Fig. 26 the curves (1), (2), and (3) are respectively the integrals of (4), (5), and (6). Now, (6) is the sum of (4) and (5), and the ordinate of (3) represents its area. The ordinate of (2) represents the area of (5) (reckoned as explained at p. 42), and that of (1) the area of (4). Hence since (1) + (2) = (3), it is clear that—

Area of (4) + area of (5) = area of (6)

This proof assumes as self-evident the fact that a curve cannot have more than one first derived curve.

This may easily be proved independently, for if we take a corresponding "element of area" of each curve (as an infinitely thin strip is called), such as that shaded in the figure, it is clear that since—

$$p'P' + q'Q' = r'R'$$

area of strip of (4) + area of strip of (5) = area of strip of (6)

The same relation holds for all the common strips into which each curve may be divided. It therefore holds for the sum of all the strips, *i.e.* for the whole areas of the curves. The same is obviously true *mutatis mutandis* for the difference of two curves.

As a special case of this, we see that—

$$\int nf(x)dx = n\int f(x)dx$$

where n is any quantity whatever, and $f(x)$ has the meaning already explained on p. 42.

For since $nf(x) = f(x) + f(x) + \ldots$ to n terms, we have as above $\int nf(x)dx = \int f(x)dx + \int f(x)dx \ldots$ to n terms $= n\int f(x)dx$.

We can now easily find the integral of any multiple of any power of x. Suppose we require $\int 3x^5dx$. The power of the integral must clearly be x^6. Let us differentiate x^6, and compare the result with the proposed expression. We obtain $6x^5$. This is clearly twice too great, so the desired integral is $\frac{1}{2}x^6$.

Similarly, required $\int \frac{p}{\sqrt{q}}x^r dx$ (where p, q, r represent either numerical quantities or expressions whose values do not depend on that of x), we find that the d.c. of x^{r+1} is $(r+1)x^r$. This must be multiplied by $\frac{p}{\sqrt{q}(r+1)}$ to obtain the desired result; hence the required integral is $\frac{p}{\sqrt{q}(r+1)}x^{(r+1)}$.

Many other expressions can be reduced by simple algebraical or trigonometrical operations to forms which can be differentiated or integrated by means of these rules.

For instance, to find $\int (x+a)^2 dx$, we have—

$$(x-a)^2 = x^2 - 2ax + a^2, \text{ and therefore}$$
$$\int (x-a)^2dx = \int x^2dx - 2a\int xdx + \int a^2dx$$
$$= \frac{x^3}{3} - ax^2 + a^2x$$

EXAMPLES.

1. Differentiate (by expanding) $(x+3)^2$, $(x+a)^4$, $(1+16x^2+64x^4)^{\frac{1}{2}}$, $(x+a)(x-b)$. *Ans.* $2(x+3)$, $4(x+a)^3$, $16x$, $2x+a-b$.

2. Integrate—

(i.) $\left(\frac{a\sqrt{b}}{c\sqrt{d}}x^p \times \frac{e}{\sqrt{f}}x^q\right)$. *Ans.* $\frac{ae}{(p+q+1)c}\sqrt{\frac{b}{df}}x^{(p+q+1)}$.

(ii.) $(x+3)^3$. *Ans.* $\dfrac{x^4}{4}+3x^3+\dfrac{27}{2}x^2+27x$.

(iii.) $(x-3)^3$. *Ans.* $\dfrac{x^4}{4}-3x^3+\dfrac{27}{2}x^2-27x$.

(iv.) $(x+a)(x-a)$. *Ans.* $\dfrac{x^3}{3}-a^2x$.

(v.) $\sqrt{x^8(1+2a^4+a^8)}$. *Ans.* $\dfrac{1+a^4}{5}x^5$.

CHAPTER V.[1]

GENERAL PRINCIPLES (*continued*).

§ 30. Products of Functions.

It must be carefully noticed that the principles explained in the last chapter cannot be extended by analogy to multiplication and division of functions.

For instance, if—

$$y = uv$$

where uv stand for any expressions involving x, it by no means follows that—

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dv}{dx}$$

or that if—

$$y = \frac{u}{v}$$

that therefore—

$$\frac{dy}{dx} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

[1] In this and the succeeding chapters almost all the diagrams are reduced copies of drawings made to scale. In many cases the scale with which the curves are to be measured is given. The student should always measure the curves, and make as large accurate drawings to scale for himself as possible. Three times full size is a convenient scale for a half-imperial sheet.

The student must be very much on his guard against assuming results from analogies of this kind. He should in all cases return to the curves, and think each principle out on its own merits.

The graphical proof of the formula for $\frac{d(uv)}{dx}$ will be best understood if we first give a brief algebraical one.

Suppose we have three variable and mutually related quantities, which we shall denote (1), (2), (3) respectively, of which the values of (1) and (2) both depend directly on the value of an independent variable x, so that curves can be obtained which show the relations between (1) and x and between (2) and x. When x has any value we like to give it, (1) and (2) each assume definite corresponding values.

Now (3) is to vary in such a way that, whatever the value of x, its value is always equal to the product of the corresponding values of (1) and (2). Thus it is clear that the value of (3) must also depend entirely on the value of x, and the problem is—if the value of x changes slightly at a time when the values of (1), (2), and (3) are respectively u, v, and y—to find what is the relative magnitude of the consequent change in the value of (3).

Obviously, from the data—

$$y = uv \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

Now, if x changes from the value it now has, it is clear that the values of (1), (2), and (3) will also change from the values u, v, and y respectively.

Suppose a change in the value of x of the magnitude Δx causes the three quantities to become $(u + \Delta u)$, $(v + \Delta v)$, $(y + \Delta y)$, respectively. Then, since (3) always = (1) × (2), we must have—

$$y + \Delta y = (u + \Delta u)(v + \Delta v)$$

On multiplying out, this becomes—

$$y + \Delta y = uv + u\Delta v + v\Delta u + \Delta u\Delta v$$

but since $y = uv$, we have, on subtraction—

$$\Delta y = u\Delta v + v\Delta u + \Delta u\Delta v \quad . \quad . \quad \text{(ii.)}$$

and therefore—

$$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}\Delta u \quad . \quad . \quad \text{(iii.)}$$

If this change in x had been infinitely small, all the quantities, Δy, Δu, Δv, would have been infinitely small too, and the equation (iii.) would have been—

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} + \frac{dv}{dx}du \quad . \quad . \quad \text{(iv.)}$$

Now, $\frac{dy}{dx}$, $\frac{dv}{dx}$, and $\frac{du}{dx}$ are all of finite magnitude, although dy, du, dv, dx are infinitely small (as already explained in §§ 5, 17, 18, etc.). Hence the last term in (iv.), being the product of a finite quantity with an infinitely small one, does not affect the equation at all, as it is infinitely small compared to the other terms (see § 12). Therefore we have—

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

That is to say, when the variable quantities (1), (2), (3) have the values u, v, y, the rate of change of (3) per unit increase of $x = u \times$ rate of change of (2) $+ v \times$ rate of change of (1).

This result may be very clearly demonstrated graphically.

The ordinates of curves (1), (2), (3) (Fig. 29) represent the values of the three variables for all values of x. The length of ordinate of (3) always = product of corresponding ordinates of (1) and (2). Curves (1) and (2) being given, and the scale with which they are to be measured, (3) can always be found. Thus if pP = 0·4, and qQ = 2·0, then rR = 0·8, and so on.

Let pP, qQ, rR be the definite values u, v, y respectively, and let $PL = QM = RN$ be the value Δx.

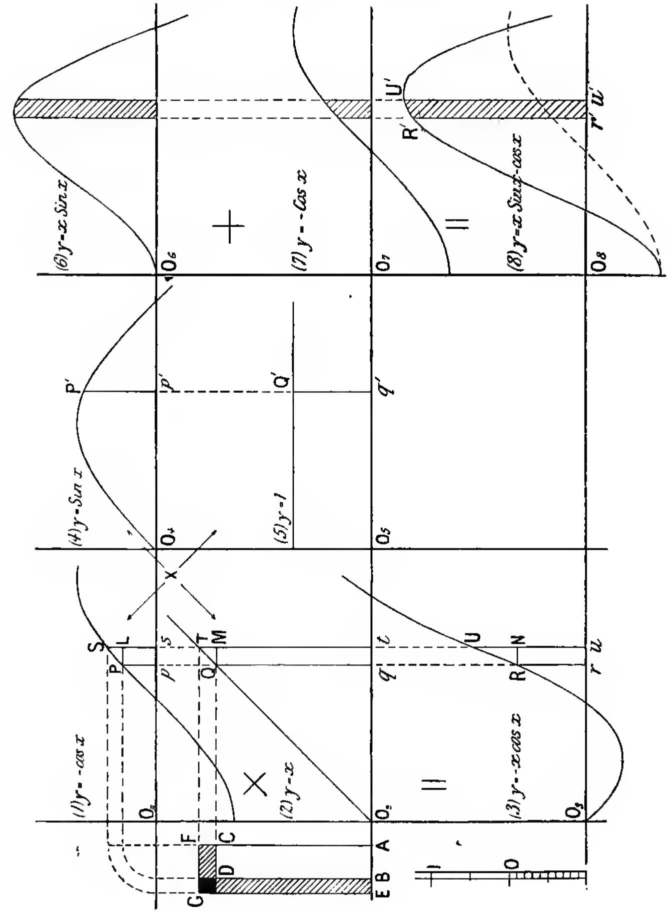

FIG. 29.

Then $LS = \Delta u$, $MT = \Delta v$, $NU = \Delta y$

Draw a rectangle ACDB, of which $AB = pP$, $AC = qQ$.

It is clear that the number of square inches (see note on p. 17) in this rectangle = number of inches in rR. Produce AB, AC, so that AE = sS, AF = tT. Then—

$$\text{Rectangle AFGE} = u\text{U}$$
$$\text{hence gnomon EDF} = \text{NU} = \Delta y$$
$$\text{But gnomon EDF} = \text{rect. ED} + \text{rect. DF} + \text{rect. DG}$$
$$= \text{BD . BE} + \text{CD . CF} + \text{CF . BE}$$
$$i.e.\ \text{NU} = \Delta y = y_2 \Delta y_1 + y_1 \Delta y_2 + \Delta y_1 \Delta y_2$$

All these increments have been produced by an increment PL = Δx in x.

This is true however near Q is to P, or however far off it is. It is therefore true when PL is infinitely small. But when PL is infinitely small, the small rectangle DG is infinitely small compared to the rectangles ED, DF; for, comparing the area of rectangle DG with that of, say, FD, when PL and therefore also CF and BE have become infinitely small, we see that, although each of these rectangles has the same breadth, CF, yet the length of DF, viz. CD, being of finite magnitude, contains an infinite number of lines = EB, which is the length of the rectangle DG, and therefore rectangle DG is infinitely smaller than DF.

Thus $\Delta y_1 \Delta y_2$ vanishes in the equation for Δy, being infinitely small compared to the other terms, and our equation may be written $dy = udv + vdu$ (see pp. 37 and 21), remembering that all these increments have been produced by an increment dx in x. This may be signified to the eye by writing equation in the form—

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

To illustrate this, differentiate curves (1) and (2), placing each derived curve on the immediate right of its primary at (4) and (5). Then multiply curves (1) and (5) together, just as (1) and (2) were multiplied together to produce (3), and place the result at (7). Do the same with (2) and (4), placing the resulting curve at (6). Now, on adding together (6) and (7), we shall, as the proof shows, obtain (8), which is the first derived of (3), as may be found by trial.

The actual equations to the curves are shown in the diagram.

§ 31. Illustration of D.C. of Product.

Before proceeding to a more general case of products, we shall consider an example of the application of this principle. Given that at the present time the total number of poor-houses in the country is 7251, and the average number of paupers in each 112, the rate of increase in the number of poor-houses in the country is 8 per year, and the annual decrease of the average number of paupers in each is 1·7. Find how the total number of paupers in the country is varying.

It is clear that—

Total number of paupers = total number of poor-houses × average number of persons in each.

Let y = total number of paupers.

y_1 = total number of poor-houses.

y_2 = average number of paupers in each.

$$\text{Then } y = y_1 \times y_2$$

Let x be the number of years reckoned from the present time.

Now—

$$\frac{dy_1}{dx} = \text{rate of increase of poor-houses}$$

$$= +8$$

$$\frac{dy_2}{dx} = \text{rate of increase of average persons in each}$$

$$= -1{\cdot}7$$

$\frac{dy}{dx}$ = required rate of increase of total number of paupers.

Hence we have, from what we have just proved—

$$\frac{dy}{dx} = -7251 \times 1 \cdot 7 + 112 \times 8$$
$$= -11,430 \cdot 7.$$

Hence, from these data, the total number of paupers is decreasing at the rate of 11,430·7 annually.

Exercise.—Why is this number not exactly the same as that which would have been obtained by finding the value of $(7251 \times 112 - 7259 \times 110 \cdot 3)$?

Ans. Because the rate of increase $\frac{dy}{dx}$ is not constant. The value here found for it is only the *momentary* rate of increase (see remark on p. 14).

§ 32. D.C. of a Continued Product.

We can easily proceed from this result to a more general expression for the continued product of a number of functions. Suppose, for instance, $y = uvw$, where u, v, and w are, as before, shorthand symbols for "any expressions involving x."

Here our primary curves are $y_1 = u$, $y_2 = v$, $y_3 = w$, etc.

The above equation may be written as a product of two quantities, thus—

$$y = (uv) \times w$$

In this form we can differentiate it by the previous section, thus—

$$\frac{dy}{dx} = w\frac{d(uv)}{dx} + uv\frac{dw}{dx}$$

We can differentiate (uv) as before, so as to get the first term of the right-hand side in a simpler form—

$$\frac{dy}{dx} = w\left(v\frac{du}{dx} + u\frac{dv}{dx}\right) + uv\frac{dw}{dx}$$

Removing the bracket, we obtain—

$$\frac{dy}{dx} = uv\frac{dw}{dx} + vw\frac{du}{dx} + wu\frac{dv}{dx}$$

In the same way we can proceed to the differentiation of the continued product of four functions, the result being as follows—

$$\text{If } y = rstu$$

$$\frac{dy}{dx} = rst\frac{du}{dx} + stu\frac{dr}{dx} + tur\frac{ds}{dx} + urs\frac{dt}{dx}$$

These results are often more conveniently and symmetrically written—

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{s}\frac{ds}{dx}, \text{ etc.}$$

the result being obtained by dividing each side of the upper equation by the corresponding side of the equation $y = urst$.

Exercise.—Draw three curves at random, and a fourth showing the value of the product of the three. Differentiate them graphically, and exhibit the truth of the above result as accurately as possible. Prove the result independently of the proof in § 30.

The chief difficulty in understanding this result is due to the multitude of different symbols which are often (as the student is prone to think) needlessly introduced into the proof. He is apt, for instance, to stumble over the symbols u, v, etc., and to ask himself, in the case of such an equation as $y = u$, "What is the use of introducing u at all, if we are already dealing with a quantity y, which denotes exactly the same thing?" The answer to this is that, whereas y stands for the ordinate to the curve, u is used for brevity, instead of (fx), and means "some expression involving x," and may stand for any such expression; and the equation $y = u$ means "there are, for any value of x, one or more definite values for y;" i.e. "x and y are dependent on one another." If the student finds other difficulties of this kind, the best plan is to express his difficulty in words and write it out. It will, in most cases, be found that the very exercise of explaining accurately to himself what his difficulty is (besides being of high educational value in itself), will enable him to explain the difficulty away. To obtain a clear understanding of any point, there is nothing like seeking for a geometrical explanation by assuming curves about which to reason instead of symbols. It is much easier to reason about the curves themselves than about the symbols denoting them.

§ 33. Orders of Infinitesimals.

The case of the d.c. of the continued product of three functions may be proved independently of § 30, and by the same method as was adopted in that section. This proof illustrates very well the principle of what are called "orders of infinitesimals." While leaving the working out of the complete proof as an easy exercise for the student, we shall give as much of it as will enable us to show the meaning of this expression.

Suppose the ordinates of curve (O_4), in Fig. 30, represent the products of the corresponding ordinates of (O_1), (O_2), (O_3). I imagine a rectangular block made, the lengths of whose edges are equal to a particular set of corresponding ordinates of three given curves.

Let $O_1 p = O_2 q =$, etc., be the current value of the independent variable.

Let $ab = p\text{P}$, $bc = q\text{Q}$, $mn = r\text{R}$.

Then the number of cubic inches in the white block (of which only one face, *abcd*, can be seen in this view) = the number of inches in $s\text{S}$.

Now, if x increases by $\Delta x = pt =$, etc., then a consequent simultaneous increase, Δy_1, Δy_2, Δy_3, and Δy, will take place in y_1, y_2, y_3, and y. These increments are respectively KT, LU, MV, NW. The block, therefore, increases to *amzjhe*. The number of cubic inches in this increment = NW. This increment consists of all the shaded parts of the block, together with a slab at the back, which in this view is entirely concealed. This cubical increment is made of several parts.

(i.) Three large flat plates (shaded light in Fig. 30), *degfc*,

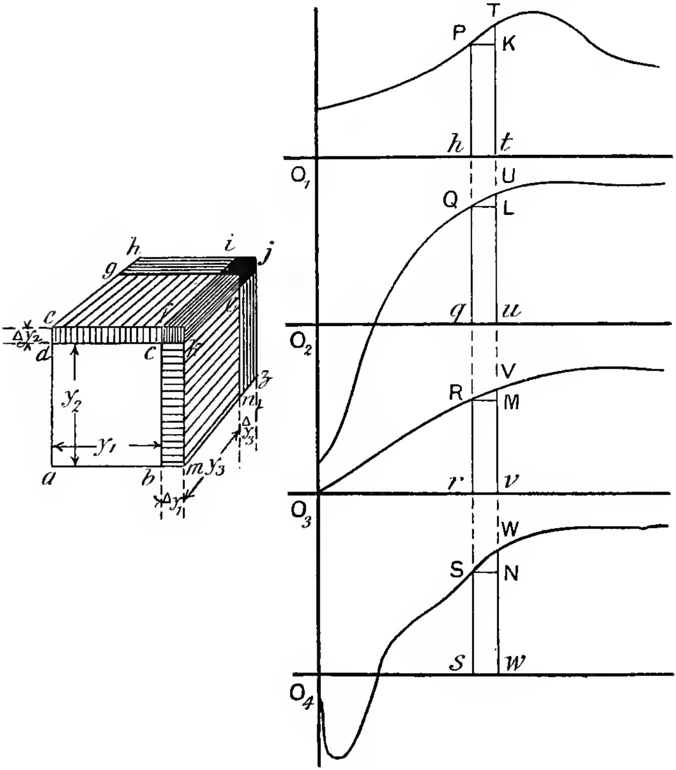

FIG. 30.

cbmnlk, and the hidden plate at the back. These are represented by $y_3 y_1 \Delta y_2$, $y_2 y_3 \Delta y_1$, $y_1 y_2 \Delta y_3$, respectively.

(ii.) Three rectangular four-sided prisms (shaded darker in the figure), *ghi*, *lnz*, *fckl*; the magnitudes of these are respectively, $y_1 \Delta y_2 \Delta y_3$, $y_2 \Delta y_3 \Delta y_1$, $y_3 \Delta y_1 \Delta y_2$.

(iii.) The small rectangular piece *ijl* (black), whose magnitude is $\Delta y_1 \Delta y_2 \Delta y_3$.

Now, suppose Δx, and consequently also Δy_1, Δy_2, Δy_3, and Δy, to dwindle indefinitely till they are infinitely small $= dx$, dy_1 dy_2, dy_3, and dy. Then—

(1) The flat plates, such as *gedcf*, become indefinitely thin in one direction (*de*); and although the other edges, such as *eg*, *ef*, are the same size as the original block, yet these flat plates are clearly, as regards their cubical contents, infinitely small *compared to the block abcd.*

(2) In the same way, the prismatic pieces (such as *fckl*) though in two directions (*cf*, *kl*) they are just the same size as the plates (*gedcf*), nevertheless become cubically infinitely small compared to the plates.

(3) Again, the small piece (black), though in every two directions it is the same size as one of the darkly shaded prismatic pieces, is nevertheless infinitely small compared to any of them.

Hence we see that though we may have any number of different quantities of the same kind, and all infinitely small, yet they may have "orders" of smallness among themselves, *i.e.* one quantity (3) may be infinitely small compared to another quantity (2), which is itself infinitely small compared to another quantity (1), which in turn may be infinitely small compared to a quantity y, and so on. This would be expressed by saying that the plates are an "infinitesimal of the first order," the prismatic pieces "an infinitesimal of the second order," and the small cubical piece "of the third order," and so on.

We have already had several instances of infinitesimals of different orders. Thus in § 13, Fig. 11, what we showed with respect to each of the infinitely small "elementary" vertical rectangles of which the lower curve was composed was in reality, that each rectangle differed from the corresponding strip of the curve by a small triangle, which was infinitely small compared to the infinitely thin rectangle; in other words, that the difference was an "infinitesimal of the second order," and the sum of an infinite number of these infinitesimals of the second order was, *comparable in size* with the rectangle K′D, an infinitesimal of the first order.

§ 34. D.C. OF x^n.

We are now, for the first time, in a position to complete the proof of the formula for d.c. of x^n.

We have not hitherto proved that it is true either for such expressions as $x^{\frac{p}{q}}$ or $x^{(-m)}$.

$$\text{If } y = x^{\frac{p}{q}}$$

$$y = x^{\frac{1}{q}} \times x^{\frac{1}{q}} \times x^{\frac{1}{q}} \ldots \text{ to } p \text{ factors}$$

We can differentiate it in this form from the rule for continued products, for we have seen (§ 24) that when $y = x^{\frac{1}{q}}$ the rule holds good.

Hence we have, from § 32—

$$\frac{dy}{dx} = x^{\frac{1}{q}} \times x^{\frac{1}{q}} \times \ldots \text{ to } p - 1 \text{ factors} \times \frac{1}{q}x^{\left(\frac{1}{q}-1\right)}$$

+ the same quantity $(p - 1)$ times repeated since all the factors are alike

$$= p \times \left\{ \left(x^{\frac{1}{q}}\right)^{p-1} \times \frac{1}{q}x^{\left(\frac{1}{q}-1\right)} \right\}$$

$$= \frac{p}{q}\left(x^{\frac{p-1}{q}} \times x^{\frac{1-q}{q}} \right)$$

$$= \frac{p}{q}x^{\left(\frac{p}{q}-1\right)}$$

which is of the required form.

Now, let us suppose that—

$$y = x^{-m}$$

where m itself, apart from its sign, is any integral or fractional positive quantity.

Another way of writing the equation is—

$$y = \frac{1}{x^m}$$

Now, we shall find $\frac{dy}{dx}$ *indirectly* thus. Find *an expression for* the differential coefficient of $x^m \times \frac{1}{x^m}$, which we know *from other sources* = 0 (for $x^m \times \frac{1}{x^m} = 1$, and from § 19 we know that when $y = 1$ $\frac{dy}{dx} = 0$).

Having found this expression, if we equate it to zero we shall have a simple equation involving $\frac{d\left(\frac{1}{x^m}\right)}{dx}$, which by solution will give us what we require in terms of the d.c. of x^m, which we know:

$$\frac{d\left(x^m \times \frac{1}{x^m}\right)}{dx} = x^m \frac{d\left(\frac{1}{x^m}\right)}{dx} + \frac{1}{x^m} mx^{(m-1)}$$

Hence—

$$x^m \frac{d\left(\frac{1}{x^m}\right)}{dx} + \frac{1}{x^m} mx^{(m-1)} = 0$$

therefore—

$$\frac{d\left(\frac{1}{x^m}\right)}{dx} = \frac{-\frac{1}{x^m} mx^{(m-1)}}{x^m}$$

$$= -m\frac{x^{(m-1)}}{x^{2m}}$$

$$= -mx^{(-m-1)}$$

As this proof is usually given, it involves a difficulty to the beginner which he often finds difficult to express in words. It is usual to write—

$$\text{If } y = \frac{1}{x^m}$$

$$\text{then } yx^m = 1$$

Differentiating both sides of this equation, we have, etc. Though the student cannot find anything to actively object to in the words in italics, and though he may understand the process of differentiating a product, yet, because he does not understand the meaning of the reasoning, the proof fails to convince him. If he compares the reasoning given above with what is usually given, *i.e.* if he substitutes for y in terms of x in the product to be differentiated, he will find it easier to understand.

Taking the meaning of the italicized words, literally they may be assumed to mean that if two curves, $z = yx^m$ and $z = 1$, are the same, then their derived curves are also the same with respect to any variable whatever. The fact that one of the factors of the product yx^m, viz. y, does not *contain* x, need not trouble us, for we know that although y does not appear to contain x at all, yet it does so in reality, for the value of y may be expressed, if we please, in terms of x. Indeed, if it did not depend on x the expression $\frac{dy}{dx}$ would be utterly meaningless.

§ 35. D.C. of a Quotient.

We can easily find the d.c. of a quotient of two functions of x by an application of this principle; for suppose $y = \frac{u}{v}$, where u and v are any functions of x—

Then $yv = u$

Differentiating both sides of this equation by the product-rule on p. 65—

$$y\frac{dv}{dx} + v\frac{dy}{dx} = \frac{du}{dx}$$

This is a simple equation to find $\frac{dy}{dx}$, giving—

$$\frac{dy}{dx} = \frac{\frac{du}{dx} - y\frac{dv}{dx}}{v}$$

or, substituting $\frac{u}{v}$ for y—

$$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

If this is not clear to the student, let him substitute, as an example, say, $(x^{\frac{5}{7}})$ for u, and $(x^{\frac{2}{3}})$ for v, and $\left(\frac{x^{\frac{5}{7}}}{x^{\frac{2}{3}}}\right)$ or $(x^{\frac{1}{21}})$ for y, throughout the proof here given.

The expression for $\frac{dy}{dx}$ should be committed to memory.

The same thing could be proved directly from the curves Fig. 29. The bracketed numbers refer to the ordinates of the corresponding curves, and $d(1)$ means the "first derived of (1)."

Given $(2) = \frac{(3)}{(1)}$, required the equation of (5). We have—

$$(5) = \frac{(7)}{(1)} = \frac{(8) - (6)}{(1)}$$

$$= \frac{d(3) - \frac{(3)}{(1)}d(1)}{(1)}$$

$$= \frac{(1)d(3) - (3)d(1)}{(1)^2}$$

Or we might prove the same thing thus:

$$y = \frac{u}{v} = u\left(\frac{1}{v}\right)$$

$$= uv^{-1}$$

$$\frac{dy}{dx} = u\frac{d(v^{-1})}{dx} + v^{-1}\frac{du}{dx}$$

$$= u \times \left\{(-1)v^{-2}\frac{dv}{dx}\right\} + \frac{1}{v}\frac{du}{dx}$$

(The student will not understand this last step till Chapter VIII. is reached.)

$$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

As an example, we might have $y = \frac{\sin x}{\log x}$ to find $\frac{dy}{dx}$. We can write at once—

$$\frac{dy}{dx} = \frac{\log x\frac{d(\sin x)}{dx} - \sin x\frac{d(\log x)}{dx}}{(\log x)^2}$$

which we cannot as yet further simplify.

Every practical example of the product-rule furnishes also an example of this rule.

It is to be noticed that, since in the plate ordinate of (6) + ordinate of (7) = ordinate of (8), therefore—

area of (6) + area of (7) = area of (8)
i.e. area of (6) = area of (8) − area of (7)
i.e. area of (6) = ordinate of (3) − area of (7)

all areas being taken between corresponding ordinates, or reckoned as explained at p. 42. The bearing of this on the integration of expressions will be explained later on.

EXAMPLES.

1. From the illustrative example given in § 31, find, by inversion of this, the rate at which the poor-houses in the country are increasing, given total number of paupers (= 112 × 7251) and the average in each poor-house (112) and the rates of increase of these (−11,430 and −1·7 respectively).

2. Find from the rule for the d.c. of products the result for $\frac{d(x^2)}{dx}$, $\frac{d(x^3)}{dx}$, etc. (Thus $y = x \times x \times x$, therefore $\frac{dy}{dx} =$, etc.) Prove the rule for positive integers in this way by induction.

3. The length, width, and height of a cubical block of crystal are given respectively by the equations—

$$L_1 = l_1 (1 + a_1 \tau)$$
$$L_2 = l_2 (1 + a_2 \tau)$$
$$L_3 = l_3 (1 + a_3 \tau)$$

where l_1, l_2, l_3 are the length, width, and height at a temperature at 0° C, a_1, a_2, a_3 are constants, and τ the temperature centigrade.

Find (1) the rate of cubical expansion of the whole crystal per degree rise of temperature. (2) The rate of expansion of a block which is 1 cubic inch at 0° C. (3) The rate of expansion of a block which is 1 cubic inch at a temperature τ.

Is the rate of (1) cubical expansion, (2) linear expansion, constant at all temperatures?

CHAPTER VI.

DIFFERENTIAL COEFFICIENTS OF TRIGONOMETRICAL FUNCTIONS.

§ 36. D.C. OF SIN x.

WE have proved that for all values of n $\frac{d(x^n)}{dx} = nx^{(n-1)}$, and we now proceed to deduce expressions and derived curves for other functions of x.

Let $y = \sin x$.

When $\sin x$, $\sin \theta$, and similar trigonometrical expressions are used in abstract mathematics, the quantities x, θ, etc., invariably refer to an angle of x or θ *radians*, and not degrees. When degrees are meant, the symbol $^\circ$ is never omitted. Thus $\sin 2^\circ$ means the sine of 2 degrees; but $\sin 2$ would mean the sine of 2 radians, or $2 \times 57{\cdot}295^\circ$. The reason for this will appear as we proceed.

The following practical process will give a tangible conception of the meaning of the curve $y = \sin x$.

Draw a circle (Fig. 31) with 1 unit radius,[1] and divide and number the circumference, starting at A counter-clockwise, into, say, 32 equal parts. Draw a horizontal line OX through the centre of the circle. Drop perpendiculars from each of the points of division to BOA. Take OX = circumference of circle $= 2 \times 3{\cdot}142 = 6{\cdot}284$, and divide it into 32 equal parts. Set up perpendiculars and number them

[1] The unit may conveniently be 3 inches long.

corresponding to the numbers on the circle, through each of the points of division, equal in length to the corresponding perpendicular to BOA (thus 11 = 11, 22 = 22, etc.; these numbers are not shown on Fig. 31), and in the same direction as these are drawn. Draw, with great care, a smooth curve through all the points thus found. This is the curve $y = \sin x$, for the number of units in the abscissa (*e.g.* Op) = number of radians in the corresponding angle (AOP), and the ordinate pP represents to scale the numerical value of the sine of that angle.

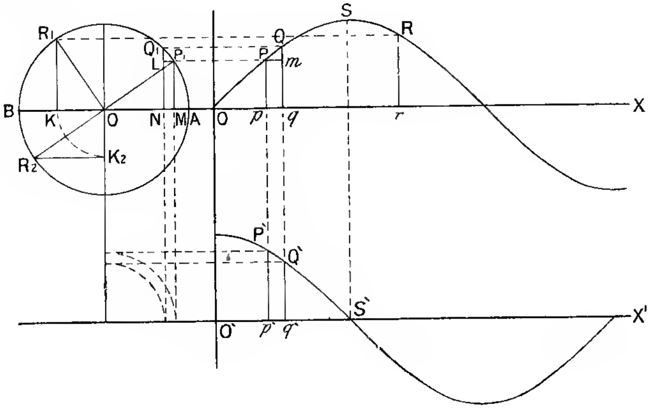

FIG. 31.

It should be noticed that when we speak of the angle AOR_2, we refer to the whole amount of angle (in this case greater than two right angles) included by the arc AP_1R_2, and not the smaller angle included by the other part of the circumference.

Now differentiate this curve graphically. If the work is accurately done, a curve will be obtained precisely similar to the original curve, but moved to the left by a distance = 1·57, which is half the length of one of the loops.

(Considerable accuracy may be obtained if a large number of points be taken, and the scale of the drawing increased.)

The reason for this peculiarity will now be shown.

Take two points P_1Q_1 on the circle (Fig. 31), and find the corresponding points PQ on the curve. Draw through ordinates Pp', Qq'. We have then—

$$Op = \text{arc } AP_1; \quad pP = MP_1, \text{ etc.}$$
$$Oq = \text{arc } AQ_1; \quad qQ = NQ_1$$
$$\text{therefore } mQ = LQ_1; \quad Pm = P_1Q_1$$
$$\text{therefore } \frac{mQ}{Pm} = \frac{LQ_1}{P_1Q_1}$$

But when Q moves up to P in the limit, the figure P_1Q_1L becomes a small right-angled triangle, similar and similarly situated to the triangle R_1KO, where OR_1 is perpendicular to OP_1. Also $\frac{mQ}{Pm}$ becomes $\frac{dy}{dx}$, or the tangent of the angle of slope of the curve at the point P.

$$\text{Hence } \frac{dy}{dx} = p'P' = \frac{KR_1}{OR_1} = KR_1$$

$$\text{since } OR_1 = 1 \text{ unit}$$
$$\text{Hence we have } p'P' = rR$$

Now, pr is evidently $= P_1R_1 = 1{\cdot}57 = \frac{\pi}{2}$. Hence the height of the derived curve at any point P_1 is the same as the height of the original curve at a point $1{\cdot}57$ to the right of P; in other words, the derived curve is exactly similar to the original curve moved $1{\cdot}57$ to the left. Its equation must therefore be—

$$y' = \sin\left(x + \frac{\pi}{2}\right)$$

Now, if we turn the triangle OR_1K round the point O as centre, through a right angle into the position OR_2K_2, each side becomes parallel to a side of the triangle P_1OM, and since $OR_2 = P_1O$, we have—

$$R_2K_2 = OM = \cos x \text{ (see § 6)}$$

Hence we have—

$$\frac{d(\sin x)}{dx} = \cos x$$

For instance, the tangent of the angle of slope of the curve $y = \sin x$ where $x = 1{\cdot}3$ units, suppose, $= \cos 1{\cdot}3$ radian $= \cos 74{\cdot}5^\circ = 0{\cdot}267$.

If we differentiate again, we shall have the original curve moved $3{\cdot}14$ to the right; we have, therefore—

$$\frac{d^2y}{dx^2} = \sin (x + \pi) = -\sin x$$

$$\text{or } \frac{d(\cos x)}{dx} = -\sin x$$

which result may also be proved independently in the same way. This should on no account be omitted by the student.

Again—

$$\frac{d^2(\cos x)}{dx^2} = \frac{d^3(\sin x)}{dx^3} = -\cos x$$

$$\frac{d^4(\sin x)}{dx^4} = \sin x$$

and so on.

The meaning of the negative sign in the result $\frac{d(\cos x)}{dx}$ $= -\sin x$ is to be carefully noted. It affords a most instructive example of the meaning of derived functions.

Consider the angle AOP_1 as being "generated" by the line OP_1 turning round O in the "counter-clockwise" direction. As x increases in value, $\cos x$ or OM *decreases*, *i.e.* the increment of $\cos x$ (corresponding to a positive increment of x) is negative. Thus—

$$\frac{d(\cos x)}{dx} = \frac{\text{increment of } \cos x}{\text{increment of } x} = \frac{\text{negative quantity}}{\text{positive quantity}}$$
$$= \text{negative quantity}$$

The negative sign indicates that when x increases $\cos x$ diminishes in the first quadrant. Now, in the second quadrant the *arithmetical* magnitude of $\cos x$ increases; but as its sign is negative, since M is then on the left of O (see § 2), we could no more say that the absolute value of $\cos x$ is increasing under these circumstances, than we could say that the value of a man's estate is increasing because his debts are increasing. So that, in the second quadrant, $(-\sin x)$ is still negative, *i.e.* $\cos x$ diminishes, while x increases. In the third quadrant, $\sin x$ being negative (since P is below BOA) $-\sin x$, or $\frac{dy}{dx}$, is positive, as it should be, because in this quadrant $\cos x$ increases algebraically along with x, just as the value of a man's estate increases when his debts decrease. In the fourth quadrant $\cos x$ increases as x increases, because $(-\sin x)$ is positive.

§ 37. Motion of Mechanism of Direct-acting Engine.

An example of the use of these results is found in the investigation of the motion of the mechanism of an ordinary steam-engine. Neglect, for the sake of simplicity, the effect of the obliquity of the connecting-rod, or assume that the crank-pin works in a slot perpendicular to the stroke of the piston.

Let OP_1 (Fig. 32) represent the crank of an engine of 2-feet stroke working at 60 revolutions per minute. Required the piston velocity when the crank is inclined at 30°, suppose.

Let a curve be plotted to scale, showing the distance of the cross-head from its central position, corresponding to the total distance travelled by the crank-pin, starting at far dead centre.

The horizontal scale is to be the same as the vertical.

Thus at any point P, corresponding to P_1 on the circle, the crank-pin has moved through a distance $AP_1 = Op$, and its displacement from the central position is clearly $OM_1 = pP$.

When the crank-pin has reached Q_1 the piston displacement is $ON = qQ$ on the other side of centre, and distance moved by crank-pin is $AP_1Q_1 = Oq$, and so on. Plotting all such values on the curve, we clearly get a curve of cosines to a certain scale. Now, if we differentiate the curve graphically, the meaning of our derived curve will depend on the length we take lines such as PL. (The crank-pin is assumed to have a constant velocity.)

FIG. 32.

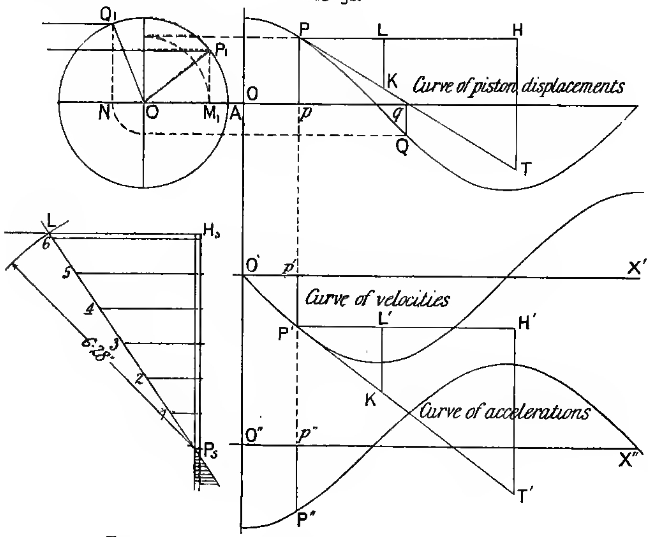

FIG. 33. FIG. 34.

(1) If we take them equal to 1 inch, we shall get a curve the length in inches of whose ordinates show the values of the *ratio*—

$$\frac{\text{small displacement of piston}}{\text{displacement of crank-pin in same time}} = \frac{\text{piston velocity}}{\text{crank-pin velocity}}$$

(2) If we take PL to scale = distance travelled by crank-pin in 1 second, the ordinates in inches will show the absolute velocity of the piston in feet per second.

(3) If PL is taken = crank radius, LK will give us the velocity of the piston on the same scale as OP_1, represents the crank-pin velocity. In any case, whatever the length of PL, we shall always get a negative curve of sines to *some* scale or other. The height of the derived curve also represents the velocity of the piston to some scale, which it is necessary to determine from common-sense principles. Thus, suppose the linear scale is *a quarter full size*, and we take PH any arbitrary distance, say 10 inches. Then HT represents, on the given linear scale, the distance that the piston would have moved during the time taken by the crank-pin to describe 10 inches × 4 = 40 inches if the piston speed had remained the same as it is at the point P. Thus HT represents the velocity of the piston at the point P to the same scale as PH represents the velocity of the crank-pin. From this we can easily find the scale of velocities. The crank-pin moves at the rate of 2 × 1 × 3·14 feet per second = 6·28 feet per second, and the scale is such that PH represents this velocity. The method, therefore, of constructing the scale is as follows—

Make P_sH_s (Fig. 33) = PH, and describe an arc of circle with radius PL = 6·28 units, cutting H_sL in L.

Mark off the points of the scale as shown, and project to PH.

We thus get a scale of piston velocities applicable to the derived curve, which would be obtained by making all lines such as PH of the given arbitrary length.[1]

Now, it is clear, from what has been said, that distances along the line OX may be taken to represent to some scale either (1) displacement of crank-pin, or (2) time occupied in making that displacement; for since the velocity of the crank-pin is constant = in this case 6·28 feet in 1 second, the same distance which represents 6·28 feet on the linear scale along

[1] The ordinates of the curves of velocity and acceleration have been reduced in the figure owing to want of space. The student should draw a larger figure for himself.

OX will also represent 1 second. Thus, if we differentiate the derived curve, bearing this in mind,[1] we shall be able easily to find the scale of accelerations applicable to the curve thus produced (§ 16).

Thus, suppose P'H' represents 0·25 second. It is clear that H'T' = p''P'' represents a rate of change of velocity of H'T' (measured by the *velocity* scale already constructed) in 0·25 second, *i.e.* four times that change of velocity in 1 second. Thus, suppose H'T', when measured by the velocity scale, to represent 4·52 feet-per-second. Then the acceleration scale is such that p''P'' represents a rate of change of velocity of 4·52 feet-per-second per $\frac{1}{4}$ of a second, *i.e.* 18·08 feet-per-second per second, and we can proceed as before to construct a complete scale with which to measure accelerations on the second derived curve.

The student who desires to understand the subject thoroughly should on no account omit to perform the complete process himself, and think out himself *ab initio* all the principles involved in the construction of his own scales. It need not be pointed out to him that the whole process is utterly useless unless he can construct exact scales for himself by which to measure his curves. He should be able to alter his scale at pleasure, in case he has not room enough to adhere to one.

The derivation of the second derived curve is of the highest importance in calculations respecting the inertia of moving parts in high-speed engines. It will be found, in the process of graphical differentiation, unless very great care be taken in the exact determination of a large number of points on the curves, that great and cumulative errors may be made in the drawing of the tangents to the curves. For this reason, other and more accurate methods are preferable where it is possible to find them. In particular, very simple and accurate methods are known for determining the curves here found by the process just explained. Those processes also take account of the varying obliquity of the connecting-rod. We might

[1] It is best to mark off O'X' in seconds or half or quarter seconds.

have done the same by a modification of the construction for determining the curve of displacements; but if we had done so, the process would not have corresponded with the algebraical investigation to be given shortly. A proof is here given which shows the real though obscure connection of the following process with that of graphical differentiation. The student should not fail to perform the operation by both processes and compare them.

Divide the circle representing (to scale) the path of the crank-pin into a number (32) of equal parts. Draw the line of centres, and put in the centre lines of the various positions of the connecting-rod with one end on the line of centres and the other end on the circle at the points of division. Produce

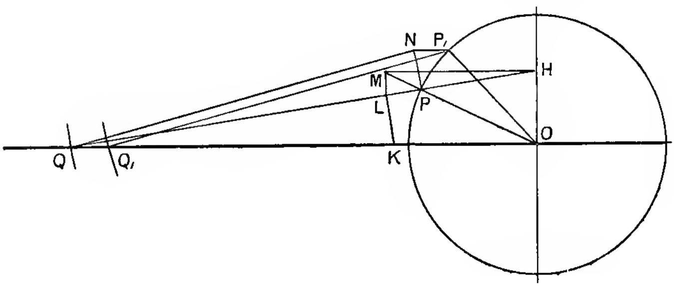

FIG. 35.

the connecting-rod, if necessary, till it cuts the vertical through O in H; then OH represents the velocity of the piston to the same scale as OP represents the velocity of the crank-pin. For consider two infinitely near positions of the crank-pin P and P_1. Draw in the two positions of the connecting-rod QP and Q_1P_1. Draw a horizontal P_1N, and with centre Q and radius QP describe a small arc of a circle PN. Then NQQ_1P_1 is a parallelogram, for $Q_1P_1 = QN$, and NP_1 is parallel to QQ_1. Therefore $NP_1 = QQ_1$. Thus, while the crank-pin describes PP_1, the piston describes NP_1. Now, in the limit when PP_1 is indefinitely small, the small figure

NPP_1 becomes a triangle of which PP_1 is perpendicular to OP, PN to PH, and NP_1 to OH. Hence if we turn the small triangle NPP_1 through a right angle round the point P in the right-handed direction, each of its sides will be parallel to one of the sides of the triangle POH. Hence we have by similar triangles—

$$\frac{P_1N}{P_1P} = \frac{OH}{OP}$$

$$i.e. \ \frac{\text{velocity of piston at point P}}{\text{velocity of crank-pin}} = \frac{OH}{OP}$$

i.e. on the same scale as that on which OP represents the velocity of the crank-pin, OH represents that of the piston. Hence, plotting the values of OH on a base representing the path of the crank-pin unrolled, we get a curve of velocities of the piston which is in practice more accurate than that obtained by direct differentiation.

In the same way it may be proved that if HM be drawn horizontal to meet OP produced, and ML vertical to cut QP in L, and LK perpendicular to the connecting-rod, then OK represents the acceleration of the piston to the same scale as OP represents the radial acceleration of the crank-pin, viz. $\omega^2 r$, or $\frac{v^2}{r}$, where ω represents the angular velocity of the crank in radians per second, and v the linear velocity of the pin. A curve plotted on a similar base to the preceding and vertically underneath it, shows the value of the piston acceleration.

The student should not fail to draw this curve, and to demonstrate to himself that it is the same curve as would have been derived by graphical differentiation from the curve of velocities, as obtained by the method of Fig. 34.

It is impossible for any one to properly appreciate the extremely instructive points involved in these constructions without thoughtfully drawing the curves to scale.

Now, the algebraical investigation of the same thing is the exact counterpart of the process first described, neglecting the effect of obliquity of the rod.

Let y represent the displacement of the piston from its central position;

x the angular displacement of the crank, starting from A (Fig. 32) in radians;

And r the radius of the crank-pin circle.

$$\text{Then } y = r \cos x$$

$$\frac{dy}{dx} = -r \sin x$$

$$\frac{dy}{rdx} = -\sin x$$

Now, rdx = distance moved by crank-pin, while the crank describes the angle dx radians (for, as in § 6, since angle $= \frac{\text{arc}}{\text{radius}}$, therefore arc = angle × radius); dy = distance moved by piston in same time.

Hence $\frac{dy}{rdx}$ = ratio of velocities of piston and crank-pin.

Take any particular value for the angle, say $30^\circ = \frac{\pi}{6}$ = AOP.

$$\text{Velocity ratio at P} = -\sin \frac{\pi}{6} = -\tfrac{1}{2}$$

i.e. the piston is moving *backwards* half as fast as the crank-pin is moving.

§ 38. D.C of $y = \text{Sin}^{-1} x$.

From the result already obtained—

$$\frac{d(\sin x)}{dx} = \cos x$$

combined with the principle deduced in § 23, we can at once find an expression giving the height of the first derived curve of the curve $y = \sin^{-1} x$.

As already shown, this means "y is equal to the angle (in radians) whose sine is x."

Let the curve OQ_1P_1 in Fig. 36 be $Y = \sin X$. Rotate this curve through a right angle into position dotted, and reflect it, and we obtain the curve $x = \sin y$, or, as it may be written, $y = \sin^{-1} x$.

x is thus geometrically substituted for Y, and y for X. x and X are both plotted horizontally.

This is the curve $OPQP_2$. Consider the point P.

$$\frac{dy}{dx} = \text{limit of } \frac{NQ}{PN} \text{ when Q is infinitely near to P,}$$

$$= \text{limit of } \frac{P_1N_1}{N_1Q_1}$$

$$= \frac{dX}{dY} = \frac{1}{\cos X} = \frac{1}{\cos y}$$

for every value of X = corresponding value of y. This result is perfectly satisfactory, and is all we require if y is to be the independent variable; but if x is, as usual, the independent variable, we can immediately find the value of this in terms of x by substituting; thus—

$$\frac{1}{\cos y} = \pm \frac{1}{\sqrt{1 - \sin^2 y}} = \pm \frac{1}{\sqrt{1 - x^2}}$$

For the derived curve, therefore, we get—

$$y' = \pm \frac{1}{\sqrt{1 - x^2}}$$

which is usually written without the double sign, because writing a double sign may be considered as part of the process of finding a square root.

The shape of this curve is shown in the figure. It consists of two infinite branches as shown. At P the slope is $p'P'$; but at P_2, of which the abscissa is the same as that of P, the slope

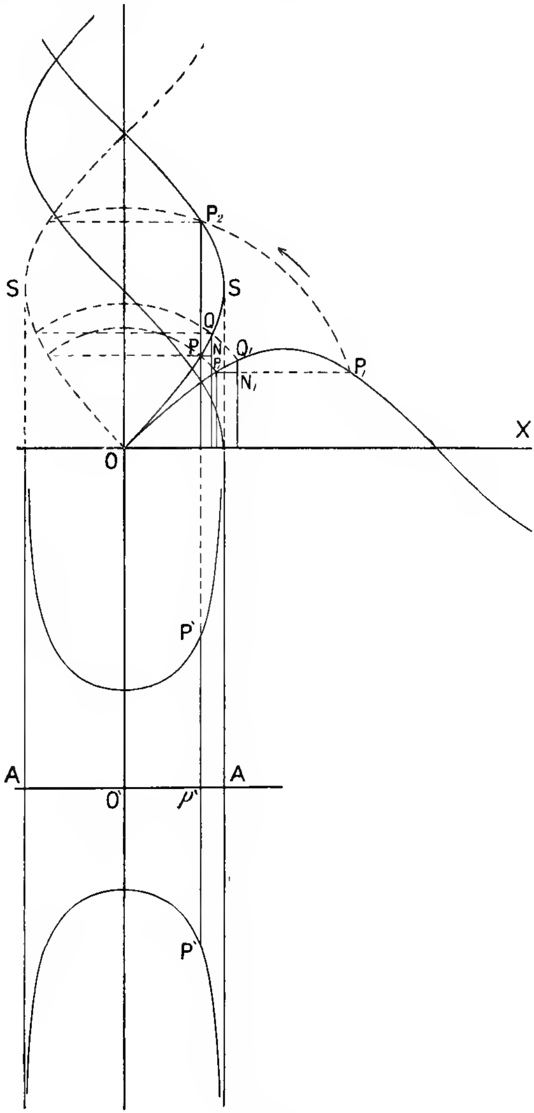

FIG. 36.

is $p'P_2'$. The meaning of the double sign is thus rendered evident. Each of these two branches approach the lines SA, but it is clear they never actually touch it in finite space; for at the point S, $\frac{dy}{dx}$ is infinitely great, and though by taking the point P near to S we can make the distance of the point P' from the line SA as small as we please, yet the ordinate of P' becomes enormously great, and the actual co-ordinates of the point S' would be $(1, \infty.)$ When a line and a curve have this relation to one another, *i.e.* the curve continually approaches as near as we please to the line, but never actually meets it in finite space, the line is said to be an "asymptote"[1] to the curve. These asymptotes are of great importance in the general tracing of curves. In general, both co-ordinates of the point of contact are infinite.

D.C. of Cos^{-1} x.—It is clear that by moving the vertical curve downwards through a distance $= \frac{\pi}{2}$, so that the point S is on the line OX, we shall obtain the curve $y = \cos^{-1} x$, since we obtain the curve $Y = \cos X$ by moving the horizontal curve to the left through the same distance. Now, it is clear that this does not in the least alter the derived curve, so that—

$$\frac{d(\cos^{-1} x)}{dx} = \frac{d(\sin^{-1} x)}{dx} = \pm \frac{1}{\sqrt{1 - x^2}}$$

as may be proved independently, thus—

$$y = \cos^{-1} x$$
$$x = \cos y$$
$$\frac{dx}{dy} = -\sin y = \pm \sqrt{1 - x^2}$$
$$\frac{dy}{dx} = \pm \frac{1}{\sqrt{1 - x^2}}$$

the same expression as before.

[1] From three Greek words, signifying "not falling together."

It will, of course, be seen that neither of the curves $y = \sin^{-1} x$ nor $y = \cos^{-1} x$ can have an abscissa > 1 or < -1; for there is no possible angle which has a sine or cosine > 1 or < -1. The same thing may be seen in the equation to the derived curve; for if x becomes greater than 1, say 2, we have—

$$y = \frac{1}{\sqrt{-3}}$$

an imaginary expression, for it is impossible that a negative quantity should have a real square root, since the square of *any* real quantity, positive or negative, has a positive sign.

§ 39.

In a similar way the d.c.'s of $\tan x$, $\cot x$, $\sec x$, and cosec x can be obtained. The principles involved have in previous

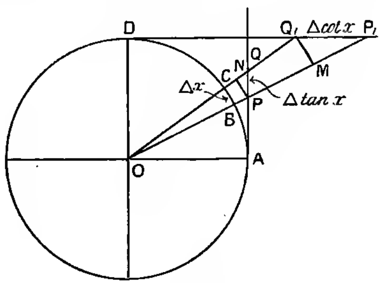

FIG. 37.

sections been fully explained, and as these can also be easily obtained from the d.c.'s of $\sin x$ and $\cos x$, we shall merely give brief geometrical proofs. The student should in all cases draw the actual curves.

Make OA = 1 inch (Fig. 37). Draw tangents at A and D. Consider the point B on the circle.

Let AB, or the angle AOB in radians, $= x$.

$$\text{Then AP} = \tan x, \ \text{PQ} = \Delta(\tan x), \ \text{BC} = \Delta x$$

$$\frac{d(\tan x)}{dx} = \text{limit of } \frac{PQ}{BC}$$

$$= \quad ,, \quad \frac{PQ}{PN} \cdot \frac{PN}{BC}$$

$$= \quad ,, \quad \frac{PQ}{PN} \cdot \frac{OP}{OB}$$

$$= \quad ,, \quad \frac{PQ}{PN} \cdot \frac{OP}{OA}$$

$$= \sec^2 x \text{ in limit}$$

since angle QPN = angle AOP

Also $DP_1 = \cot x$, $P_1Q_1 = \Delta \cot x$.

$$\frac{d(\cot x)}{dx} = \text{limit of } \frac{P_1Q_1}{BC}$$

$$= \quad ,, \quad -\frac{Q_1P_1}{BC}$$

$$= \text{limit of } -\frac{Q_1P_1}{MQ_1} \cdot \frac{MQ_1}{BC}$$

$$= \quad ,, \quad -\frac{Q_1P_1}{MQ_1} \cdot \frac{OQ_1}{OD}$$

$$= -\operatorname{cosec}^2 x$$

Again, $\sec x = OP$, $\Delta(\sec x) = NQ$.

$$\frac{d(\sec x)}{dx} = \text{limit of } \frac{NQ}{BC}$$

$$= \quad ,, \quad \frac{NQ}{PN} \cdot \frac{PN}{BC}$$

$$= \quad ,, \quad \frac{NQ}{PN} \cdot \frac{OP}{OA}$$

$$= \tan x \,.\, \sec x$$

Again, $\operatorname{cosec} x = OP_1$, $\Delta(\operatorname{cosec} x) = -MP_1$.

$$\frac{d(\operatorname{cosec} x)}{dx} = \text{limit of } -\frac{MP_1}{BC}$$

$$= \quad ,, \quad -\frac{\mathrm{MP}}{\mathrm{MQ_1}} \cdot \frac{\mathrm{MQ_1}}{\mathrm{BC}}$$

$$= \quad ,, \quad -\frac{\mathrm{MP_1}}{\mathrm{MQ_1}} \cdot \frac{\mathrm{OQ_1}}{\mathrm{OD}}$$

$$= -\cot x \,.\, \operatorname{cosec} x$$

Exercises.—(These exercises are of the highest importance.) Prove each of these results from the d. c.'s of $\sin x$ and $\cos x$ on the principles explained in Chapter V. in the following manner :—

$$\frac{d(\tan x)}{dx} = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Prove also the following results by the same method as that explained for $y = \sin^{-1} x$, drawing the curve in each case.

$$\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1 + x^2}$$

$$\frac{d(\cot^{-1} x)}{dx} = -\frac{1}{1 + x^2}$$

$$\frac{d(\sec^{-1} x)}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d(\operatorname{cosec}^{-1} x)}{dx} = -\frac{1}{x\sqrt{x^2 - 1}}$$

Thus, if $y = \tan^{-1} x$, then $x = \tan y$.

$$\frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

Prove these results graphically by tracing the curves and inverting them.

CHAPTER VII.

DIFFERENTIAL COEFFICIENTS OF LOGARITHMIC FUNCTIONS.

§ 40. D.C. OF LOG x.

WE will now consider the curve $y = \log x$. A remark similar to the one we made in defining the meaning of such expressions as $\sin x$ applies here, viz. that in abstract mathematics $\log x$ with no suffix signifies, not the ordinary logarithm as found in log tables, but the "natural" logarithm to base "e" where e is the value of the infinite series—

$$1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots = 2{\cdot}7167 \ldots, \text{etc.}$$

which is the value which the expression $\left(1 + \frac{1}{n}\right)^n$ assumes when n is infinitely great. The student cannot hope to understand this fully unless he be acquainted with the algebraical theory of logarithms, which is found in any fairly advanced book on algebra, such as Hall and Knights' "Higher Algebra." He may, nevertheless, obtain approximate values of the natural logarithm of a number by multiplying its ordinary logarithm (to base 10) by the log of 10 to base e, viz. 2·303 about.

Calculate in this way the natural logarithm of 0·25, 0·5, 0·75, 1·25, 1·50, 2·0, 3·0, 4·0, 5·0, 7·5, 10. Plot points whose abscissæ are the numbers here given in inches or other units, and ordinates the calculated logarithms. Carefully draw a smooth curve through these points. This curve crosses the line OX at a point whose abscissa is 1; for with any base whatever $\log 1 = 0$.

On the left of point (1,0) care must be taken: thus from the tables we can find $\log_{10} 0{\cdot}5 = \bar{1}{\cdot}69897$, which for our

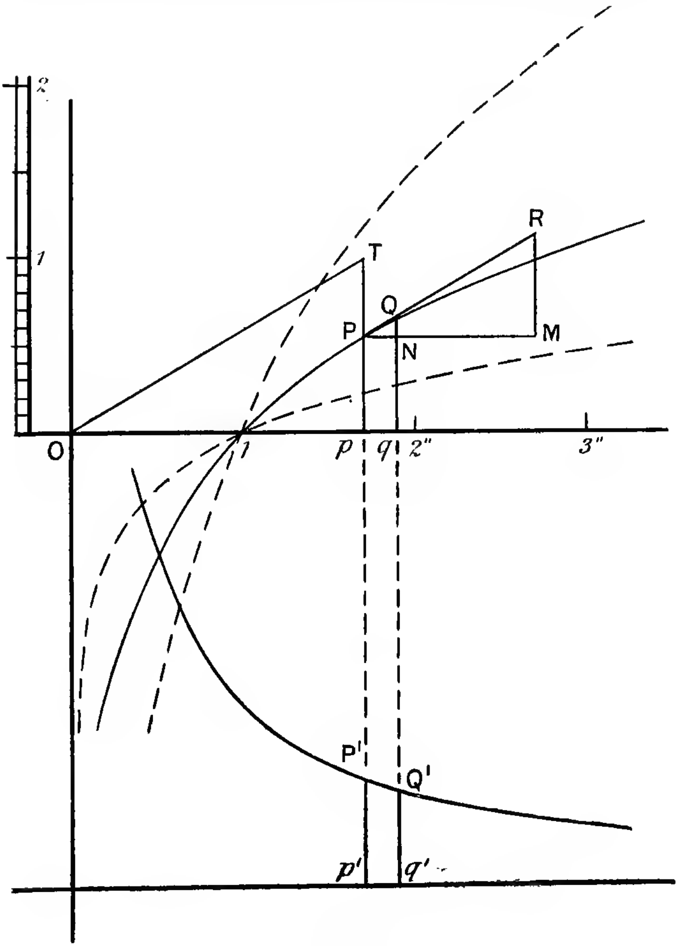

FIG. 38.

purpose is practically equivalent to 0·700, since we cannot plot correct to $\frac{1}{1000}$ inch.

Hence we have—

$$\log_{10} 0{\cdot}5 = -1 + 0{\cdot}70 = -0{\cdot}3$$
$$\text{hence hyp. } \log 0{\cdot}5 = -0{\cdot}3 \times 2{\cdot}303 = -0{\cdot}69$$

and similarly for the other points.

Draw a curve through the points. This curve is shown in a full line in Fig. 38.

Now differentiate this curve graphically. The general shape of the curve obtained will be as shown in the lower part of Fig. 38. Take a number of points such as P′ on the curve, measure with a decimal scale p'P′ and O′p', multiply their lengths together, and the result will be found to be always 1 if the work is accurately done. Its equation must therefore be $xy' = 1$, or $y' = \frac{1}{x}$.

Another way of exhibiting this fact very clearly is to take a number of points P on the primary, through which erect pT perpendicular = 1 inch. Join OT. Then OT will be found parallel to the tangent PR at T.

Exercise.—The whole curve may therefore be drawn by the method explained in § 14. Draw it in this way, and compare it with the curve just plotted.

These exceedingly important facts may be proved algebraically as follows. Consider another ordinate qQ near pP, distance h from it.

Let Op = x; pP = $\log x$.

$$\text{Then } Oq = x + h$$
$$qQ = \log(x + h)$$

Therefore, with the usual notation—

$$\frac{\Delta y}{\Delta x} = \frac{\log(x+h) - \log x}{h}$$

which, from the nature of logarithms

$$= \frac{1}{h} \log \left(\frac{x+h}{x} \right)$$

$$= \frac{1}{x} \cdot \frac{x}{h} \log \left(1 + \frac{h}{x} \right)$$

$$= \frac{1}{x} \log \left(1 + \frac{h}{x} \right)^{\frac{x}{h}}$$

Write n instead of $\frac{x}{h}$. This becomes—

$$\frac{1}{x} \log \left(1 + \frac{1}{n} \right)^n$$

When h or Δx diminishes indefinitely, it is clear that n increases indefinitely. When, therefore, this takes place, $\log \left(1 + \frac{1}{n} \right)^n$ becomes, from the definition above, $\log e = 1$.

At the same time, when Δx or h dwindles indefinitely, $\frac{\Delta y}{\Delta x}$ becomes $\frac{dy}{dx}$, or height of derived curve; hence $\frac{dy}{dx} = y' = \frac{1}{x}$, as before.

§ 41. Illustrations.

Some interesting and instructive results may be derived from these equations. On the same base as before, plot the logarithms as found in ordinary tables. A curve will be obtained similar to the other in general character, but flatter. As we have seen, it is $\frac{1}{2 \cdot 3}$ as high at all points. It is the lower dotted curve in the figure. Its equation is $y = \mu \log_e x$. It crosses the line OX at the same point as the other. Its slope at this point may be approximately found from the tables, for we have—

$$\log_{10} 1 = 0 \cdot 000{,}0000$$
$$\log_{10} 1 \cdot 0001 = 0 \cdot 000{,}0434$$

$$\text{Hence at this point } \Delta y = 0{\cdot}000{,}0434$$
$$\Delta x = 0{\cdot}0001$$

$\frac{\Delta y}{\Delta x}$, which, when Δx is so small as $0{\cdot}0001$, will be very nearly $= \frac{dy}{dx} = 0{\cdot}434$.

This is also evident from the equation—

$$\text{if } y = \mu \log_e x$$
$$\frac{dy}{dx} = \frac{\mu}{x} \text{ (see § 28)}$$
$$\text{which (when } x = 1) = \mu.$$

It is interesting to notice that, in an ordinary book of logarithms, the height of the derived curve of the curve of ordinary logarithms is given by the side of the tables, so as to enable any one using the tables to "interpolate." This height is called "difference" in the tables. The principle made use of in the calculation of intermediate logarithms is $\delta y = \frac{dy}{dx} \cdot \delta x$. The value of $\frac{dy}{dx}$ is given as a "difference."

We can obtain another curve of the same character by plotting the lengths taken from a slide rule on the same base. This is the upper dotted curve in Fig. 38. The graduations of a slide rule are ruled proportional to the logarithms of the numbers engraved on the rule, so that addition and subtraction on the rule, which are easily performed mechanically by sliding one scale over the other, are equivalent to multiplication and division respectively. On the ordinary small "Gravet" rule, log 10 is represented by $12{\cdot}5$ c.m. $= 4{\cdot}921$ inches.

Hence this latter curve is $\frac{4{\cdot}921}{2{\cdot}302} = 2{\cdot}135$ times as high as the e curve, and its slope at the point (1,0) is $2{\cdot}135$ (§ 28).

The result we have obtained may also be written $\int \frac{dx}{x} = \log x + c$. In this form it is extremely useful to the engineer in enabling him to find the work done by a gas (such as air) in expanding isothermally or at constant temperatures. This will be fully considered in the next chapters.

§ 42. D.C. of e^x.

By inverting the curve of logarithms, as explained in § 23, we can prove a result of great importance.

The curve P_1 (Fig. 39) is the curve $Y = \log X$. Rotate it about point O into the position dotted, and reflect on OY, and we get a curve whose equation might be written conformably with those of $\sin x$ and $\cos x$, etc., $y = \log^{-1} x$, or y is the number whose logarithm is x.

It is usual, however, to write the equation $y = e^x$, for e^x is obviously, from the definition, the number whose logarithm to base e is x.

If the student cannot understand this, he is referred to any book on algebra which contains a chapter on logarithms.

If this curve is differentiated graphically, the result will be a curve which is exactly in every respect like the primary curve. In other words, the peculiarity of this curve is that if the tangent at a point P be produced so as to meet OX in S, then, wherever P is on the curve, Sp will be exactly 1 inch, for the triangle SpP is evidently exactly equal and similarly situated to the triangle we should have drawn for the point P in differentiating the curve in the ordinary way. This is expressed by saying that the "subtangent" is constant, Sp being the subtangent.

This result may be proved as follows :—

$$\text{If } y = e^x$$

$$\text{then } x = \log y$$

$$\therefore \frac{dx}{dy} = \frac{1}{y}$$

$$\therefore y' = \frac{dy}{dx} = y = e^x$$

An interesting property of this curve is that, if a series of abscissæ are taken in arithmetical progression, the corresponding ordinates are in geometrical progression. The

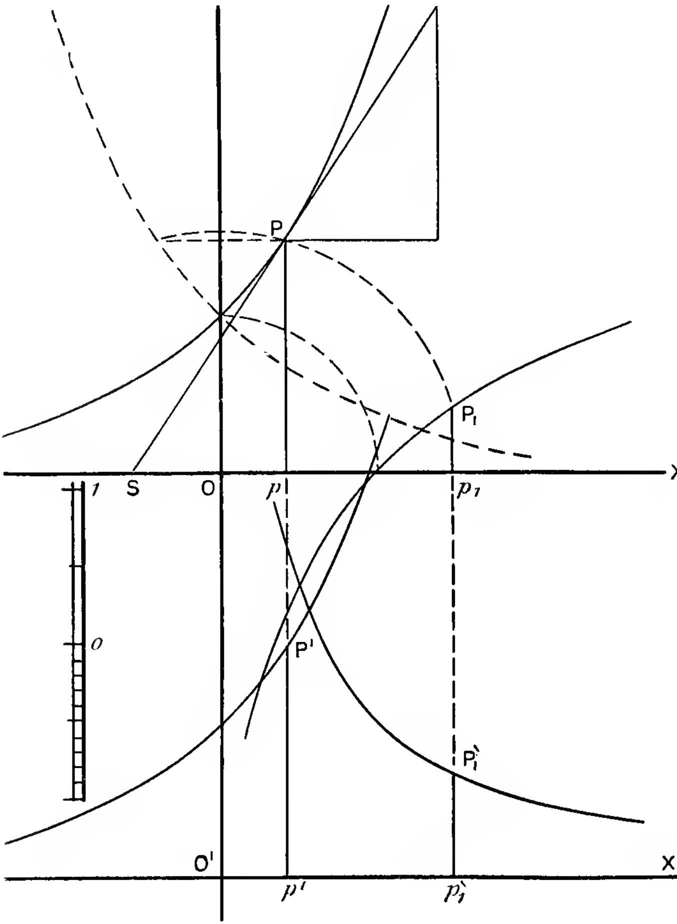

FIG. 39.

student should prove this algebraically from the equation to the curve.

The whole curve should be obtained by this method, by taking ordinates 0·25 inch apart. The ratio of the ordinates will be $e^{0·25}$, the value of which must be calculated, and the successive ordinates found geometrically by a construction similar to that of Fig. 3. All logarithms can be graphically obtained from this curve by measuring the abscissæ corresponding to an ordinate whose length = number whose log is required.

The whole of the results in this chapter and the last must be thoroughly learnt off by heart. The student who wishes to proceed with the subject will save himself much time and annoyance by making himself perfectly familiar with them at the outset. It is not too much to say that one-half of the difficulty usually met by elementary students of the integral calculus is due to an imperfect knowledge of these few simple results. The student can best learn them by deducing them for himself once every day, and constantly picturing to himself the curves representing the functions and their differential coefficients. He thus obtains a practical and real familiarity with the functions, such as he could not get by studying the symbols only. Unless he is gifted with an exceptional memory, he will find even the few here collected difficult to remember otherwise than by understanding what they mean. The results should be as familiar forwards as they are backwards; *e.g.* he should know that $\int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x$ just as well as that $\frac{d(\sec^{-1} x)}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$

Direct.		Inverse.	
Function.	Diff. coefficient.	Function.	Diff. coefficient.
x^n	nx^{n-1}		
$\sin x$	$\cos x$	$\sin^{-1} x$	$\pm \dfrac{1}{\sqrt{1-x^2}}$
$\cos x$	$-\sin x$	$\cos^{-1} x$	$\pm \dfrac{1}{\sqrt{1-x^2}}$
$\tan x$	$\sec^2 x$	$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\cot x$	$-\operatorname{cosec}^2 x$	$\cot^{-1} x$	$\dfrac{-1}{1+x^2}$
$\sec x$	$\sec x \tan x$	$\sec^{-1} x$	$\dfrac{1}{x\sqrt{x^2-1}}$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	$\operatorname{cosec}^{-1} x$	$\dfrac{-1}{x\sqrt{x^2-1}}$
$\log x$	$\dfrac{1}{x}$	e^x	e^x

EXAMPLES.

1. What is the equation to the inverse curve of the lower dotted curve in Fig. 38? Is the subtangent constant in this curve? Is the first derived curve like the primary curve? Prove your answer graphically and analytically.

2. Assuming the result for the d.c. of e^x, prove algebraically by inversion the result for $y = \log x$.

3. Differentiate $y = a^x$.

CHAPTER VIII.

DIFFERENTIATION OF A FUNCTION OF A FUNCTION OF A VARIABLE WITH RESPECT TO THAT VARIABLE.

§ 43.

WE have considered, in the preceding chapters, the process of differentiation of simple functions of a variable x (such as $\sin x$, $\log x$, etc.) with respect to that variable—that is, the relative magnitude of the change produced in the value of the function by a small change in the value of the variable. Now, this small change in the value of the variable may have been itself produced by a change in some other variable (z, suppose), on the value of which x depends, and it is often necessary to know the ratio between a change in the value of the given function of x and a small change in the value of z (which latter produces a certain change in x, and in consequence a change in $f(x)$, the function to be differentiated). In other words, we have to differentiate some function of x (say $\log x$) with respect, not to x, but to z, *i.e.* to find the value of $\frac{d(\log x)}{dz}$. Of course, this would not be possible unless there were some relation subsisting between x and z, such that x takes up a definite value corresponding to any given value of z (see the note at the end of § 23, on p. 49).

As the meaning of this process is usually very confusing to the beginner, and as it is important that he gets clear ideas on it, we shall illustrate it by an everyday example.

Suppose a tradesman starts in business for himself at the beginning of the year 1870. At the beginning of that year he earns profit at the rate of £200 per annum, or about 11*s.* per day, or 1*s.* 4½*d.* an hour.

Suppose this rate of profit *gradually* and regularly increases by £20 per annum every year, so that, for instance, in the middle of 1870 he is earning 11*s.* 6*d.* per day, or £210 per year; and at the beginning of 1871 he is earning £220 per year, or about 12*s.* a day. It is clear that his *average* rate of profit throughout 1870 has been £210 per annum, which sum also represents his total earnings for the year. At the beginning of 1872 he is earning £240 a year, and so on (i.).

Let the current rate of profit at any time be denoted by £z per annum, and suppose his current rate of living expenditure at the same time is given by £$z^{\frac{4}{5}}$ per annum, denoted by y (ii.).

It is required to find the rate per annum at which his rate of living expenditure is increasing.

This example can most easily be understood by following the curves on Fig. 41. The "dimensions" of this rate of increase will be "pounds-per-annum every year," in the same way as the dimensions of an acceleration are "feet-per-second every second." It would be incorrect to measure this rate of increase in "pounds-per-annum," because "pounds-per-annum" are the dimensions of an income or annual expenditure, and not a rate of annual *increase* of income or of annual expenditure (cf. Fig. 17).

Let y be his rate of living per year, and z his rate of earning profit (both in pounds per annum), at a time represented by x years counted from the beginning of 1870.

It is evident that the relation between y and z is—

$$y = z^{\frac{4}{5}} \quad . \quad . \quad . \quad . \quad (a)$$

This being the algebraical expression of supposition (ii.) above.

Also we have—

$$z = £200 + £20x \quad . \quad . \quad . \quad (b)$$

which expresses supposition (i.).

We have then to find the value of $\frac{dy}{dx}$. Now, from equation (*a*) we can (§ 34) easily determine $\frac{dy}{dz}$, or his rate of increase of expenditure per £1 increase of income;[1] but this is not what we want. Also from (*b*) we can find $\frac{dz}{dx}$, or his rate of increase of income per year (§ 16); but neither is this what we require.

Now, from these two equations, (*a*) and (*b*), we can obtain another involving only y and x, for we can substitute £200 + £$20x$ instead of z in the equation—

$$y = z^{\frac{4}{5}}$$

This process is called "eliminating z between (*a*) and (*b*)." We thus obtain—

$$y = (£200 + £20x)^{\frac{4}{5}} \quad . \quad . \quad . \quad (c)$$

Here we are fixed, for we have hitherto proved no rule which will enable us to differentiate this expression with respect to x. We have, in fact, come to a point where we must differentiate a function (viz. the power $\frac{4}{5}$) of a function (viz. $200 + 20x$) of a variable (x) with respect to that variable.

If the student has followed the previous reasoning carefully, he will probably suspect that we shall find what we require by multiplying together the two d.c.'s already found; that is—

$$\text{that } \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

but he must be very careful to notice that he has no right whatever to take this result as proved merely because $\frac{dy}{dz}$ and $\frac{dz}{dx}$ look like fractions. He should know already that dx, dy, and dz are not quantities to which

---

[1] If we take £100 as the unit, $\frac{dy}{dz}$ represents the amount by which his rate of expenditure increases per £100 increase of income. In this case, however, z would represent the profit in *hundreds of pounds* per annum, and we should take account of this algebraically by modifying equations (*a*) and (*b*) according to the units we are working in.

definite values can be assigned, and therefore to cancel out one dz with another without inquiring into the meaning of the process is an operation which is quite as illegitimate as it would be to cancel out the d in $\frac{dy}{dx}$, and to put $\frac{dy}{dx} = \frac{y}{x}$. In certain cases the latter might be true, but in the great majority of cases it would not be. It would signify that the tangent at a point P of a curve passes through the origin O, which is obviously generally untrue.

The student's aim should be to grasp the *meaning* underlying all these symbols. He should never perform algebraical operations of this kind in

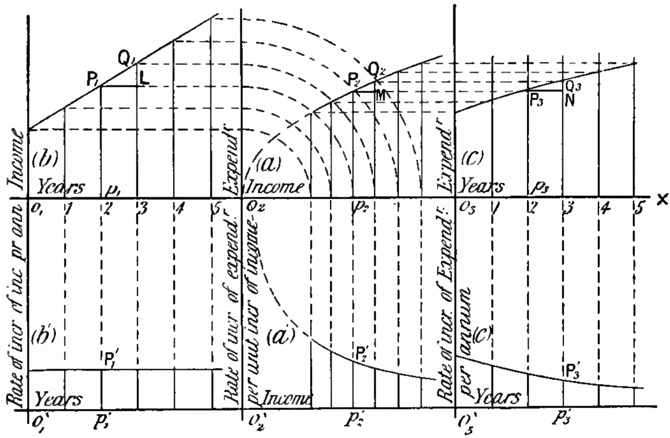

FIG. 40.

a haphazard fashion without making himself acquainted with the principle involved. In this case it is perfectly true that $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$, but it requires proof before it can be accepted, and it is only to be taken as another analogy between the laws relating to differential coefficients and those relating to fractions.

Draw the two curves representing relations (*a*) and (*b*) as shown in Fig. 40. (The curves in the figure are not drawn to scale.) Notice that curve (*a*) does not involve the idea of time, but simply shows the expenditure corresponding to any income. Also that, since, during the period under consideration, the rate of profit is always greater than £200 per annum, we have nothing to do with the dotted part of the curve.

Curve (b) shows, in the length of its ordinate, the income corresponding to a time given by the abscissa. To obtain a curve showing time—annual expenditure—we must combine the abscissæ of (b) with the *corresponding* ordinates of (a). Thus, consider a time two years after January, 1870, *i.e.* January, 1872. The income is given by p_1P_1. Transfer this to O_2p_2 as shown. Then p_2P_2 gives the living expenditure *at this date*. Take a base, O_3X, divided exactly like O_1X, and transfer the ordinates such as p_2P_2 to p_3P_3, where $O_3p_3 = O_1p_1$. This curve, when drawn, is the result of graphically eliminating z between (a) and (b).

Consider *corresponding* ordinates, Q_1, Q_2, Q_3, adjacent to P_1, P_2, P_3, where Q_2, Q_3 are obtained from Q_1, exactly as P_2, P_3 were obtained from P_1.

Then clearly—

$$\begin{aligned} LQ_1 &= P_2M \\ MQ_2 &= NQ_3 \\ P_3N &= P_1L \end{aligned}$$

Hence we have—

$$\frac{NQ_3}{P_3N} = \frac{MQ_2}{P_1L} = \frac{MQ_2}{P_2M} \cdot \frac{LQ_1}{P_1L}$$

From the way in which the curves were constructed, this is true wherever Q may be. Now, when Q_1 approaches P_1, so that P_1L dwindles indefinitely, it is clear that all the other quantities in the above equation do the same; and when this is the case, the equation becomes—

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

for the three ratios which are contained in the equation become respectively $\frac{dy}{dx}$, $\frac{dy}{dz}$, and $\frac{dz}{dx}$, whatever actual values the quantities denoted by NQ_3, P_3N, P_2M may have, provided always these are infinitely small (since the part of the curve along which Q may move consistently with this condition is an infinitely short *straight line*, as already explained in § 13; see also note on p. 48).

Differentiating all three curves, then we see that any ordinate of $(b') \times$ *corresponding* ordinate of (a') = corresponding ordinate of (c'). The criterion of correspondence is, of course, not the same as that in the case of the curves multiplied together in the ordinary sense. Thus $p_1'P_1'$, $p_2'P_2'$, $p_3'P_3'$, are corresponding ordinates, although $O_1'p_1'$ is not $= O_2'p_2'$.

It is clear that the curve (c') in this case is represented by—

$$y' = \tfrac{4}{5}z^{\frac{4}{5}-1} \times 20$$

$$= \tfrac{4}{5}(200 + 20x)^{-\frac{1}{5}} \times 20 = \frac{16}{\sqrt[5]{200 + 20x}}$$

§ 44.

(i.) On the same principle, we can differentiate such expressions as $(\sin x)^2$.

The curves in this case are—

$$y = z^2 \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$z = \sin x \quad . \quad . \quad . \quad . \quad (b)$$

By eliminating z, we obtain—

$$y = (\sin x)^2 \quad . \quad . \quad . \quad . \quad (c)$$

Here—

$$\frac{dy}{dz} = 2z \quad . \quad . \quad . \quad . \quad (a')$$

$$\frac{dz}{dx} = \cos x \quad . \quad . \quad . \quad (b')$$

Hence—

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} - 2z \cos x$$
$$= 2 \sin x \cos x \quad . \quad . \quad . \quad (c')$$

The letters denoting the equation correspond to the same letters in the illustration.

(ii.) A frequent application of the same principle occurs in the differentiation of such expressions as $\sin \frac{x}{a}$. Here we might be tempted to think that the d.c. was $\cos \frac{x}{a}$. But it must be carefully noticed that this would be the d.c. with respect to $\frac{x}{a}$, and not to x (see Examples II. at end of Chapter III.).

$$\text{Here } z = \frac{x}{a} \quad . \quad . \quad . \quad . \quad . \quad (b)$$
$$y = \sin z \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$
$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{a} \cos \frac{x}{a}$$

The mistake in this case arises from the fact that $\frac{x}{a}$ is, for convenience, not usually enclosed in a bracket, although it might be if desired.

(iii.) Take another case: $y = \log (\sin x)$.

$$\text{Let } z = \sin x \quad . \quad . \quad . \quad . \quad (b)$$
$$\text{then } y = \log z \quad . \quad . \quad . \quad . \quad (a)$$
$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$
$$= \frac{1}{z} \cdot \cos x = \frac{1}{\sin x} \cos x$$
$$= \cot x$$

(iv.) Take a more complicated case: $y = (e^x \cos x)^n$.

$$\text{Let } z = e^x \cos x \quad . \quad . \quad . \quad (a)$$

$$y = z^n \quad . \quad . \quad . \quad . \quad (b)$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$\frac{dy}{dz} = nz^{n-1}$$

$\frac{dz}{dx}$ must be found by the rule for products of functions given in § 30, thus—

$$\frac{dz}{dx} = e^x \frac{d(\cos x)}{dx} + \cos x \frac{d(e^x)}{dx}$$

$$= -e^x \sin x + e^x \cos x$$

$$= e^x(\cos x - \sin x)$$

Hence—

$$\frac{dy}{dx} = n(\epsilon^x \cos x)^{n-1} \times \epsilon^x(\cos x - \sin x)$$

After a certain amount of practice, the student will find that he is able to dispense with the z substitution, and to write down the result without any intermediate step.

(v.) A difficulty arises to beginners when they have to differentiate such an expression as, say, q^3 with respect to x. They are tempted to write down as the result $3q^2$, forgetting that this is the d.c. with respect to q, and not to x. They are often unable to trace the meaning of differentiating q^3, which does not appear to contain x, with respect to x. If so, they should read again the note at the end of § 23, and remember that there could not be such a thing as a d.c. of q^3 with respect to x unless a relation such as is there described subsisted between q and x.

$$\frac{d(q^3)}{dx} \text{ is therefore } 3q^2 \cdot \frac{dq}{dx}$$

$$i.e. \frac{dy}{dx} = \frac{dy}{dq} \cdot \frac{dq}{dx}$$

The RULE, therefore, is as follows :—

To differentiate any function of a quantity enclosed in a bracket with respect to a variable x (*e.g.* cos (log x))—

(i.) Differentiate the expression, treating the *whole* quantity in the bracket as an independent variable. This would give us $-\sin(\log x)$.

(ii.) Multiply this by the differential coefficient with respect to the variable of the quantity enclosed in the bracket. Here the d.c. of (log x) is $\frac{1}{x}$. Hence—

$$\frac{d\{\cos(\log x)\}}{dx} = \frac{-\sin(\log x)}{x}$$

If there are two or more brackets enclosed one within the other, it is easy to see by induction that we must first treat the whole of the *outside* bracket as an independent variable, and proceed inwards, treating each bracket in turn as the independent variable, multiplying all the successive results together. Careful attention to the following example will enable the student to understand this.

Let $y = \left[\log\{\log(\sin e^x)\}\right]^n$.

(1) Differentiate as though the quantity contained in the [ ] brackets were an independent variable. This gives us—

$$n\left[\log\{\log(\sin e^x)\}\right]^{n-1}$$

(2) From our rule, it is clear that this must be multiplied by the d.c. of the quantity contained *in* the [ ] brackets. Hence we have, as it were, to start the same process over again, absolutely neglecting everything outside the [ ] brackets. This, according to our rule, will involve treating the quantity in the { } brackets as an independent variable. Thus far we have—

$$n[\log\{\log(\sin e^x)\}]^{n-1} \times \frac{1}{\{\log(\sin e^x)\}}$$

Now, in order to obtain the d.c. mentioned in (2) above, we must multiply $\frac{1}{\{\log(\sin e^x)\}}$ by the d.c. of the quantity *in* the $\{\ \}$ brackets, which in turn involves treating the expression in the ( ) brackets as independent variable. This gives—

$$n\left[\log\{\log(\sin e^x)\}\right]^{n-1} \times \frac{1}{\{\log(\sin e^x)\}} \times \frac{1}{(\sin e^x)}$$

which we must then multiply by the d.c. of the quantity *in* the ( ) brackets. This involves treating e^x as an independent variable. (The student is apt to stumble at the last step, because e^x is not enclosed in visible brackets.) Finally, the whole expression must be multiplied by the d.c. of e^x with respect to x. The whole expression is then

$$n\left[\log\{\log(\sin e^x)\}\right]^{n-1} \times \frac{1}{\{\log(\sin e^x)\}} \times \frac{1}{(\sin e^x)} \times \cos e^x \times e^x$$

The student should not be satisfied till he can write out any complicated result like this at sight, without any substitutions. He must learn to fix his attention on each bracket in turn, treating it quite apart from anything else, and regarding the next bracket proceeding inwards as the independent variable. If he finds it impossible at first to avoid getting the thread of his thoughts entangled among the brackets, he should get a separate piece of paper and cross each bracket out as it is done with. He will thus find an apparently extremely complicated expression quite simple to differentiate.

§ 45. Applications.

The application of this rule is the source of much of the difficulty which the student meets with in applying elementary calculus to science. Differential coefficients of quantities are sometimes treated of with respect to variables, with which the quantities have no apparent connection. New variables are often arbitrarily introduced, and d.c.'s assumed with respect to them; so the student is quite bewildered by the multiplicity of symbols. He is again reminded that the very existence of a d.c. of any quantity with respect to a variable involves the existence of a definite relation such that, other variables being constant, the assumption of a particular value by one fixes the value of the other.

For instance, suppose that each of the following variables, (a), (b), (c), (d), (e), etc., are exclusively dependent on (A), the temperature during the winter:—

(a) The number of unemployed workmen.
(b) The demand for overcoats.
(c) The amount of railway traffic.
(d) The sale of skates.
(e) The death rate, etc.

We are assuming that we have curves given, representing the value of each of these variables, corresponding to values of (A). Derived curves could be obtained representing their rates of increase or decrease per degree-rise of the thermometer. From these curves we could find a relation such as $\frac{d(b)}{d(e)}$, for $\frac{d(b)}{d(e)} = \frac{d(b)}{d(\mathrm{A})} \cdot \frac{d(\mathrm{A})}{d(e)}$, although the demand for overcoats might have no apparent connection with the death rate. Or, again, we might introduce the arbitrary variable *time*, although in our original curves the idea of time did not enter; but, in order to make such a relation as $\frac{dc}{dt}$ have any determinable value, we must have given a curve showing the relation between any one of these variables and the time. Suppose the primary and first derived of the time-temperature curve had been given. Then we have, say—

$$\frac{d(d)}{dt} = \frac{d(d)}{d\mathrm{A}} \cdot \frac{d\mathrm{A}}{dt}$$

and so on.

Direct-acting Engine.—We have already had a disguised example of the application of this principle in the case of the engine in § 37, which we now proceed to explain more fully. What we actually wish to find in the problem is the *velocity* of the piston, and this, as we have seen in § 16, is the first derived function of the time-displacement relation, or $\frac{dp}{dt}$.

Now, the geometrical relation between crank angle θ and piston position p furnishes us with the means of finding the value of $\frac{dp}{d\theta}$ for any value of θ. This quantity (neglecting obliquity) we have seen to be $-r \sin \theta$. Hence we have only to multiply by the corresponding value of the relation $\frac{d\theta}{dt}$ (*i.e.* the height of the first derived of the time-angle curve) in order to find $\frac{dp}{dt}$, for $\frac{dp}{dt} = \frac{dp}{d\theta} \cdot \frac{d\theta}{dt}$.

Now, we know from the data of the problem that the time-angle curve is a sloping straight line, since the motion of the crank is a *uniform* rotation, *i.e.* the amount of angle described is proportional to the time; hence the first derived is a horizontal line, or the "angular velocity is constant." The height of this first derived is given in the problem, for we are told the crank turns at 60 revolutions a minute, or 2π radians per second. Hence for the time-piston displacement first derived curve we have—

$$\frac{dp}{dt} = \frac{dp}{d\theta} \cdot \frac{d\theta}{dt} = -r \sin \theta \times 2\pi$$

Turning back to § 37, we find the assumption that—

$$\frac{\text{small displacement of piston}}{\text{corresponding small displacement of crank-pin}} = \frac{\text{velocity of piston}}{\text{velocity of crank-pin}}$$

It is easily seen that this is the same thing as—

$$\frac{dp}{rd\theta} = \frac{\frac{dp}{dt}}{r\frac{d\theta}{dt}}$$

or that—

$$\frac{dp}{dt} = \frac{dp}{d\theta} \cdot \frac{d\theta}{dt}$$

In that section we avoided the *general* assumption by showing, from other considerations, that in that particular case the result held good.

It is now quite easy to correct this investigation for obliquity of the connecting rod. It is clear that, corresponding to the position C of the crank-pin, the actual displacement of the

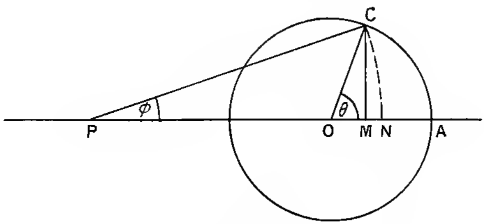

FIG. 41.

piston p is not OM, but ON, where CN is a circle with P as centre.

Let angle MOC = θ.
MPC = ϕ.
PC = l.
OC = r.

$$\text{Then } p = \text{ON} = \text{OM} + \text{MN}$$

$$\text{or } p = r\cos\theta + (l - l\cos\phi) \quad . \quad . \quad \text{(i.)}$$

$$\text{Now CM} = r\sin\theta = l\sin\phi$$

$$\text{Therefore } \sin\phi = \frac{r}{l}\sin\theta \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

Differentiating (i.) with respect to t (see note at end of § 34), we have (see p. 112 (v.))—

$$\frac{dp}{dt} = -r \sin \theta \cdot \frac{d\theta}{dt} + l \sin \phi \cdot \frac{d\phi}{dt}$$

because the d.c. of $l = 0$ (see § 19). But from (ii.) this

$$= r \sin \theta \left(\frac{d\phi}{dt} - \frac{d\theta}{dt} \right) \quad . \quad . \quad . \quad \text{(iii.)}$$

We also have from (ii.)—

$$r \cos \theta \frac{d\theta}{dt} = l \cos \phi \frac{d\phi}{dt}$$

$$\text{or } \frac{d\phi}{dt} = \frac{r \cos \theta}{l \cos \phi} \cdot \frac{d\theta}{dt}$$

$$= \frac{r \cos \theta}{\sqrt{l^2 - r^2 \sin^2 \theta}} \cdot \frac{d\theta}{dt}$$

since $l \cos \phi = l\sqrt{1 - \sin^2 \phi} = l\sqrt{1 - \frac{r^2}{l^2} \sin^2 \theta} = \sqrt{l^2 - r^2 \sin^2 \theta}$

Substituting this value of $\frac{d\phi}{dt}$ in equation (iii.) above, we obtain—

$$\frac{dp}{dt} = r \sin \theta \frac{d\theta}{dt} \left(\frac{r \cos \theta}{\sqrt{l^2 - r^2 \sin^2 \theta}} - 1 \right)$$

which is the exact value of the velocity of the piston. If the connecting rod $= n \times$ length of crank, this becomes—

$$\frac{dp}{dt} = -r \sin \theta \frac{d\theta}{dt} \left(1 - \frac{\cos \theta}{\sqrt{n^2 - \sin^2 \theta}} \right)$$

This expression is rather complicated. It is simplified as follows: Sin² θ can never be $>$ 1, whereas n^2 is always comparatively large, usually about 25. Hence $\sqrt{n^2 - \sin^2 \theta}$, being very nearly $= \sqrt{n^2}$, is put $= n$. The maximum error in doing this is very small, for $\sqrt{24} = 4 \cdot 9$, and $\sqrt{25} = 5$; but when $\sin^2 \theta = 1$, $\cos \theta = 0$, so that even this estimated error of 2 per cent. in the value of the fraction appears to have a much greater effect than it actually has. Making this approximation we have, since $2 \sin \theta \cos \theta = \cos 2\theta$ (see any book on trigonometry).

$$\frac{dp}{dt} = -\text{V} \sin \theta + \frac{\text{V} \sin 2}{2n}$$

$$\text{where V} = r\frac{d\theta}{dt} = \text{velocity of crank-pin}$$

$$= \text{a constant}$$

Differentiating again with respect to the time, we obtain the acceleration—

$$\frac{d^2p}{dt^2} = -\text{V} \cos \theta \frac{d\theta}{dt} + \frac{2\text{V}}{2n} \cos 2\theta \frac{d\theta}{dt} \text{ (see p. 111 (ii.))}$$

It is well to test the accuracy of equations of this kind by a process known as "taking dimensions." It is clear that in any equation whatever all the terms must be of the same kind. It would, for instance, be absurd to have an equation such as the following :—

$$2\frac{\text{ft.}}{\text{sec.}} + 3\frac{\text{ft.}}{\text{sec.}^2} = 5 \text{ seconds}$$

for a velocity can by no conceivable process be added numerically to an acceleration, much less can the sum of the two be equated to a time. Similarly, if our equation is correct, it is certain that all its terms, however obtained, must be of the same kind. This we can test by finding what are the dimensions of each of the terms (see note on p. 30, also p. 51). If these are not alike, it is very certain our equation must be wrong. Now, the left-hand side of the equation is obviously an acceleration, being the time-rate of variation of a velocity. This is also suggested by the form in which it is written, since d^2p would naturally suggest a length. If this had been written (dp^2) we should have expected it to represent a small area. dt^2 represents, naturally, the square of a small element of time.

Hence $\frac{d^2p}{dt^2}$ has for its dimensions $\frac{\text{ft.}}{\text{sec.}^2}$, *i.e.* the dimensions of an acceleration. Now consider the right-hand side. V is a velocity having for its dimensions $\frac{\text{ft.}}{\text{sec.}}$. Cos θ has for its dimensions $\frac{\text{length}}{\text{length}} = \frac{1}{1}$; or, in other words, has zero dimensions, or the dimensions of a simple number or "numeric." Now, $\frac{d\theta}{dt}$ has for its dimensions $\frac{\text{angle}}{\text{time}}$, but an angle has no dimensions, for the same reason that a cosine has none. Hence the total dimensions of the first term are $\frac{\text{ft.}}{\text{sec.}} \times \frac{1}{1} \times \frac{1}{\text{sec.}} = \frac{\text{ft.}}{\text{sec.}^2}$ = an acceleration.

Let the student work out for himself the dimensions of the second term on the right-hand side. In an ordinary algebraic equation, such as $x^3 + 3x^2 + \ldots = 0$, each of the letters must be assumed to be of zero dimensions, *i.e.* to represent numerics (see note on p. 5).

Both the expressions for the velocity and the acceleration are given in terms of θ and $\frac{d\theta}{dt}$, and can therefore be found numerically by substitution.

EXAMPLES.

1. By the method of Fig. 40 obtain the curves—

(i.) $y = \sin (x^2)$.

(ii.) $y = \sin (\log x)$.

(iii.) $y = \log (\sin x)$.

(iv.) $y = \log (\log x)$.

(v.) $y = \log (\cos e^x)$ [1].

(vi.) $y = \log \{\log (a + bx^n)\}$.

Differentiate them by (i.) multiplying together *corresponding* ordinates (as explained on p. 110) of the respective derived curves; (ii.) by the method of Fig. 9; (iii.) algebraically, and plot the curve by calculation. Compare the results.

2. Differentiate—

(i.) $\log (\sqrt{x-a} + \sqrt{x-b})$. *Ans.* $\frac{1}{2\sqrt{(x-a)(x-b)}}$

(ii.) $l^{ax} \sin^m rx$. *Ans.* $l^{ax} \sin^{m-1} rx\,(a \sin rx + mr \cos rx)$.

---

[1] For this example and the next, the process must be a compound one. Thus: Find $y = \cos e^x$ by the method of Fig. 40, and $y = \log \cos e^x$ by a repetition of the process. Find by induction which derived curves must be taken for the factors of the result.

(iii.) x^x. Take logs thus—

$$\text{Let } y = x^x$$
$$\text{therefore } \log y = x \log x$$
$$\text{Differentiating } \frac{1}{y}\frac{dy}{dx} = \log x + \frac{x}{x}$$
$$\frac{dy}{dx} = y(\log x + 1)$$
$$= x^x(\log x + 1)$$

(iv.) $\dfrac{1 - \tan x}{\sec x}$. (*Ans.* $-(\cos x + \sin x)$.)

(v.) l^{x^x}. *Ans.* $l^{x^x} \times x^x \log l \times (1 + \log x)$. Take logs twice in succession.

(vi.) $[\log\{\log(\log x)\}]$. *Ans.* $\dfrac{1}{x \log x \cdot \log(\log x)}$

(vii.) $\sqrt{2ax - x^2}$. *Ans.* $\left(\dfrac{a - x}{\sqrt{2ax - x^2}}\right)$

(viii.) $\tan^{-1}\dfrac{2x}{1 - x^2}$. *Ans.* $\dfrac{2}{1 + x^2}$

3. Find the exact velocity and acceleration of a piston of an engine, given crank 8 inches, connecting rod 30 inches, revolutions 95 per minute, at angles 30°, 45°, 60°, 90°, 120°, and 150°.

CHAPTER IX.

INTEGRATION.

§ 46. Examples of Integration.

We have already explained, in Chapters II. and III., the real nature and nomenclature of the process called integration, and have obtained the integral of one function of x, viz. x^n, which is—

$$\int x^n dx = \frac{1}{n+1} x^{(n+1)}$$

It has also been pointed out that, to effect any proposed integration, it is essential that we have a previous knowledge of the process of differentiation; and it is only by working backwards from this knowledge that we can obtain *an expression for* an integrated curve, though we can graphically find the curve itself independently of its equation.

There are many simple integrals which we can write down at once if we know the corresponding proposition of the differential calculus; but it should be clearly understood that there is no general method by which we can deduce the integral of a function from first principles in the same way as we have deduced the d.c. of various functions. The process of integration is essentially a tentative process depending on a previous knowledge of the differential calculus, just as the process of division in arithmetic is a tentative process depending on a previous knowledge of multiplication. It is impossible, therefore, to attain proficiency, or even facility, in integration without a previous familiarity with the differential calculus. The expressions the integrals of which we can write down at once are the *results* of the differentiations explained in Chapters VI. and VII. Thus we have—

$$\int \cos x \, dx = \sin x + c$$

which means precisely the same thing as—

$$\frac{d(\sin x + c)}{dx} = \cos x$$

in much the same way as $\frac{12}{4} = 3$ means precisely the same thing as $3 \times 4 = 12$.

Again—

$$\int (-\sin x) dx = \cos x + c$$

which means the same thing as—

$$\frac{d(\cos x + c)}{dx} = -\sin x$$

This may also be written—

$$\int \sin x \, dx = -\cos x - c$$

$$\text{for } \frac{d(-\cos - c)}{dx} = \sin x$$

Also $\iint \sin x \, dx dx$, which means, as already pointed out—

$$\int (\int \sin x \, dx) dx = \int (-\cos x + c) dx$$
$$= -\sin x + cx + e \text{ (see § 36 and p. 45).}$$

Also—

$$\iiint \sin x \, dx dx dx = \int \Big[\int \Big\{ \int (\sin x \, dx) dx \Big\} \Big] dx$$
$$= \int \Big[\int \Big\{ (-\cos x + c) dx \Big\} \Big] dx$$
$$= \int \big(-\sin x + cx + e \big) dx$$
$$= \cos x + \frac{c}{2} x^2 + ex + f$$

Also we have $\int \frac{1}{\sqrt{1-x^2}} dx$, which is usually shortened into—

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \text{ or } = \cos^{-1} x$$

corresponding to—

$$\frac{d(\sin^{-1} x)}{dx} = \pm \frac{1}{\sqrt{1-x^2}} = \frac{d(\cos^{-1} x)}{dx}$$

This result is sometimes confusing to the student. How is it that the same expression can have two different integrals? To answer this we must refer to §§ 8, 16, 22, in which it was shown that in graphically integrating a curve we have to assume some arbitrary point to commence from, which point may be at any height above or below the base-line OX. At whatever point we start from in the same vertical line, we shall obtain the same shape of curve. If we draw two such curves starting at different points, any ordinate of one is greater than the corresponding ordinate of the other by a definite and constant amount.

Now, bearing this in mind, let us look at Fig. 35, which shows the curve $y = \sin^{-1} x$ and its derived curve $y' = \pm \frac{1}{\sqrt{1-x^2}}$. If we integrate the latter graphically, starting at O, we shall, of course, obtain the curve $y = \sin^{-1} x$; but if we start at a point P at a distance $\frac{\pi}{2} = 1{\cdot}57$ units lower down, we shall obtain the precisely similar curve $y = \cos^{-1} x$ as shown. In fact, the angle whose sine is x is greater by exactly $\frac{\pi}{2}$, or 90° than the angle whose cosine is x, for all values of x. It will be evident from this example that a complete solution of the integral $\int \frac{dx}{\sqrt{1-x^2}}$ is not $\sin^{-1} x$ or

$\cos^{-1} x$, but it must be some function of x which will include both these functions and an infinite number of other similar ones, for we may start to draw our curve from any one of an infinite number of points on the vertical OY. Now, any curve which would answer to the description $y = \int \pm \frac{dx}{\sqrt{1-x^2}}$, *i.e.* which might be obtained from the curve $y' = \pm \frac{1}{\sqrt{1-x^2}}$ by graphical integration would be included in the equation $y = \sin^{-1} x +$ some constant.

For different values of the constant we should get different curves, but *all of exactly the same shape.* If the constant were $\left(-\frac{\pi}{2}\right)$ we should obtain $y = \sin^{-1} x - \frac{\pi}{2}$, which we have shown above to be the same thing as $y = \cos^{-1} x$, whereas if the constant were 0 we should have $y = \sin^{-1} x$.

Now, in every case of an "indefinite integral," *i.e.* without any limits specified (see § 22), this unknown constant must be represented by a letter, though it is often omitted for convenience, unless more than one successive integration is required, when it must never be omitted (see Examples 1 and 2, p. 30; also p. 45). It will also be evident that we can in general find the exact value of this constant, if we know one point through which the integrated curve passes. But in the above case, if we know that $x = 0$ when $y = 0$, we know that the constant must be either 0 or $n\pi$ where n is an integer. If we know, in addition, that $\frac{dy}{dx}$ at O $= +1$, we know that the constant is either 0 or $2n\pi$, *i.e.* $\pi \times$ an even number, either of which would give us exactly the same result. If $\frac{dy}{dx}$ at O is -1, then the constant must be $(2n + 1)\pi$, *i.e.* $\pi \times$ any odd number. This is perfectly definite, for all odd numbers would give the same result.

§ 47. Example of Quadrature of Area.

It has been already shown, both graphically (§ 10) and algebraically (§ 22), how the constant disappears if the integral is taken between definite limits.

As an illustration, let us find the area of the curve $y = \sin x$ (Fig. 42) between the limits $x = 1$ and $x = 2$. It has been already shown (§ 21) that this area would be represented by—

$$\int_1^2 \sin x dx = {}_1^2\Big[-\cos x + \text{constant}\Big]$$

This is the solution usually employed, the constant being represented by "C." For the sake of definiteness, let us assume a particular value for this constant, say $1\frac{1}{2}$ units. Now, if this constant had been 0, the integrated curve $y = -\cos x$ would have cut OY at a point where $y = -\cos 0 = -1$, as shown in the dotted line in Fig. 42; but since we have arbitrarily added $1\frac{1}{2}$ to this value, the curve lies as shown, where $OA = 0{\cdot}5$. Draw in the limiting ordinates $p'P$ and $q'Q$ at distances of 1 and 2 units from OY. The upper curve is $y = -\cos x + 1\frac{1}{2}$, and we know (§ 13) that $qQ - pP$ in inches = number of square inches in area $P'p'q'Q'$. Now—

$$\begin{aligned} qQ &= -\cos 2 \text{ radians} + 1\tfrac{1}{2} \\ &= -\cos 114^\circ\ 39' + 1\tfrac{1}{2} \\ &= 0{\cdot}401 + 1{\cdot}5 = 1{\cdot}90 \\ pP &= -\cos 1 \text{ radian} + 1{\cdot}5 \\ &= -\cos 57^\circ\ 19' + 1{\cdot}5 \\ &= -0{\cdot}540 + 1{\cdot}5 = 0{\cdot}96 \end{aligned}$$

Hence—

$$\begin{aligned} qQ - pP &= (0{\cdot}401 + 1{\cdot}5) - (-0{\cdot}540 + 1{\cdot}5) \\ &= 0{\cdot}401 + 0{\cdot}540 + 1{\cdot}5 - 1{\cdot}5 \\ &= 0{\cdot}941 \text{ sq. units} \end{aligned}$$

The constant, it will be seen, disappears entirely in the result, so that its absolute magnitude is a matter of no importance.

We shall often find that the adoption of two different methods of integration will give us a different result for the

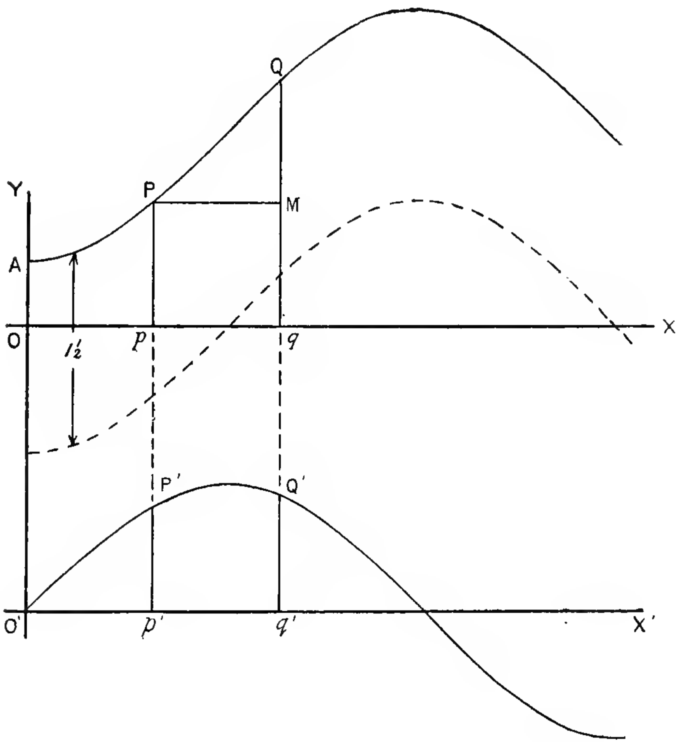

FIG. 42.

same function. If, however, *the work has been correct*, it will invariably be found, on examination, that the two curves represented by the two results are of exactly the same shape and are exactly alike in every particular, except that one is higher than the other by a definite and constant quantity all along its length, and that the one algebraical expression for value of y can be expressed as the sum of the other and a constant.

§ 48. Work done by Expanding Gas.

Another function for which we found the d.c. was $\log x$. The result was $\frac{1}{x}$ (see § 40).

Hence—

$$\int \frac{dx}{x} = \log x$$

or, as we have seen the general integral to be, $\log x + c$.

This result is of great importance. It is constantly occurring in engineering problems. It furnishes, for instance, a solution of the question as to the amount of work done by compressed air or steam in expanding from one pressure or volume to another.

Take the case of air. Boyle's law tells us that if air expands or contracts at a constant temperature, the pressure varies inversely as the volume, or, in other words, pv = constant. This constant can easily be calculated from the mass of air and the temperature. For 1 lb. of air at 32° Fahr. the value of the constant is 26,214, when the pressure is measured in pounds weight per square foot, and the volume in cubic feet. For half this quantity of air the constant is, of course, 13,107; for at the same pressure the volume is half what it was before, and therefore the product pv has half its previous value. In the same way, since the volume varies directly as the absolute temperature (*i.e.* temperature Fahr. + 461° nearly), pressure being constant, this product must vary according to the same law, as may easily be seen by imagining the pressure kept constant, while the temperature, and therefore the volume, varies. The constant may in all cases be calculated by finding the value of the expression, $53{\cdot}2 \times m \times \tau$, where m = mass of gas in pounds, τ = absolute temperature.

Now, if all these values of p and v for a given mass of gas at a given constant temperature be plotted on a curve (pressures-vertical), the resulting curve will be a rectangular hyperbola, whose equation is $p = \frac{\text{constant}}{v}$. It is the "indicator card" of the expansion, and it is shown in all works on the steam-engine, in a similar way to that adopted in § 14, that the area under the curve between any two ordinates represents the amount of work done during the expansion between the corresponding volumes.

The chief difficulty in understanding the working of these problems is that of units, which will continually harass the student till he masters it once for all. He must here *imagine* the curve drawn to full inch scale, *i.e.* 1 inch vertical = 1 lb. per sq. foot, 1 inch horizontal = 1 cub. foot. Under these circumstances, 1 sq. inch on the diagram represents $\frac{1 \text{ lb.}}{\text{ft.}^2} \times \text{ft.}^3 = 1$ ft.-lb. If the scale had been 1 inch vertical = 1000 lb., and 1 inch horizontal = 10 ft.^{3}, an area of 1 sq. inch on the diagram would have represented $1000 \frac{\text{lb.}}{\text{ft.}^2} \times 10 \text{ ft.}^3$ = 10,000 ft.-lbs.

This may be taken as an example of the general method of finding the scale in which an area, such as an indicator diagram, measures a quantity, whose value we require. The rule is, consider what quantity would be represented by a square figure one inch long and one inch high.

Taking, then, the full-size diagram, we have its equation—

$$p = \frac{26214}{v}$$

for 1 lb. of air at 32° Fahr.

Its integrated curve, as we have seen, is a curve of logarithms, each of whose ordinates is × 26214 (see § 40), whose equation is therefore—

$$y = 26{,}214 \log v + c$$

The area, then, of the lower curve between any two ordinates (say, where the volumes are 5 cubic feet and 9 cubic feet) is the difference between the two corresponding ordinates of the upper curve—

$$\begin{aligned}
&= 26{,}214 \log 9 + c - 26{,}214 \log 5 - c \\
&= 26{,}214 (\log 9 - \log 5) \\
&= 26{,}214 \log \tfrac{9}{5} = 26{,}214 \times 0{\cdot}588 \\
&= 15{,}413{\cdot}8
\end{aligned}$$

This is the area of the curve, and since each square inch represents 1 ft.-lb., the total work done is 15,413·8 ft.-lbs. If any constant other than 26,214 had been given with the same ratio of expansion, this constant, instead of 26,214, would have been multiplied by $\log \frac{9}{5}$.

Thus, suppose in an air-compressor, diameter of cylinder = 10 inches, stroke = 2 feet; required the work done per stroke in compressing air *isothermally* up to 6 atmospheres.

Here volume of air compressed per stroke $= 10^2 \times 0{\cdot}7854 \times 24$
$= 1884{\cdot}96$ cub. in.

The corresponding pressure is that of the atmosphere, viz. 14·7 lbs. per square inch;

$$\text{The constant therefore} = 1885 \times 14{\cdot}7 = 27709{\cdot}5$$

Notice very carefully the effect of altering the units from ft.^{3} and $\frac{\text{lb.}}{\text{ft.}^2}$ to in.^{3} and $\frac{\text{lb.}}{\text{in.}^2}$. If we plot this expansion curve in these units, one square inch of the diagram will represent $1 \frac{\text{lb.}}{\text{in.}^2} \times 1 \text{ in.}^3 = 1$ in.-lb. Therefore we must divide the area by 12 to get foot-pounds. The result is—

$$27709{\cdot}5 \log \tfrac{6}{1} \text{ in.-lbs.} = \frac{27709{\cdot}5 \times 1{\cdot}7918}{12} \text{ ft.-lbs.}^{[1]}$$

[1] Part of this work, viz. $\frac{1884{\cdot}96 \times 14{\cdot}7}{12}$ ft.-lbs., has been done by the atmosphere which presses on the suction side of the piston.

The student should in every case pay great attention to the units in which he is working, otherwise he will find himself hopelessly confused. For instance, if in the above case he had taken pressures in $\frac{\text{lbs.}}{\text{in.}^2}$ and volumes in ft.^{3}, then one square inch of diagram would have represented $\frac{1 \text{ lb.}}{\text{in.}^2} \times \text{ft.}^3 = 1728$ in.-lbs.

It may be noted that in an actual air-compressor the work would have been greatly in excess of this, because a large amount of heat is developed in the air by the process of compression, which increases the pressure, and therefore also the work done.

If the compression is effected without any heat being lost, as will very nearly be the case if it is done very rapidly, it may be shown that—

$$pv^{1{\cdot}408} = \text{constant}$$

The constant here, also, will have to be calculated either from known simultaneous values of the pressure and volume, with the help of a table of logarithms, or from the temperature and volume and mass. In this latter case the constant $= m \times 53{\cdot}2 \times \tau \times v^{0{\cdot}408} = c$, suppose. Here, as before, the work done is—

$$\int_{v_1}^{v_2} pdv\,^{1}$$

$$\text{but } p = \frac{c}{v^{1{\cdot}408}}$$

$$\text{hence integral } = \int_{v_1}^{v_2} \frac{cdv}{v^{1{\cdot}408}} = \int_{v_1}^{v_2} cv^n dv = c\int_{v_1}^{v_2} v^n dv$$

$$\text{where } n = -1{\cdot}408$$

[1] Here it will be noticed that it is impossible to integrate this as it stands, because the expression to be integrated, viz. p, does not contain v at all, and dv tells us that the expression has to be integrated with respect to v. Hence our object is to change p into an expression containing no other *variable* except v. This we must do by "eliminating" p between the two equations, or, in other words, substituting for p an equal value in terms of v, *i.e.* $\frac{c}{v^{1{\cdot}408}}$.

Hence the work done is—

$$c\left[\frac{1}{n+1}v^{n+1}\right]_{v_1}^{v_2}$$

or, substituting $p_1v_1^{1\cdot408}$ for c and $-1\cdot408$ for n—

$$p_1v_1^{1\cdot408}\left[\frac{1}{-0\cdot408}v^{-0\cdot408}\right]_{v_1}^{v_2}$$

Hence, substituting v_2 and v_1 in turn for v, and subtracting, we obtain—

$$-\frac{1}{0\cdot408}(p_1v_1^{1\cdot408}\times v_2^{-0\cdot408} - p_1v_1^{1\cdot408}\times v_1^{-0\cdot408})$$

but since we know that $p_1v_1^{1\cdot408} = p_2v_2^{1\cdot408}$, we can write this—

$$+\frac{1}{0\cdot408}(-p_2v_2^{1\cdot408}\times v_2^{-0\cdot408} + p_1v_1) = \frac{1}{0\cdot408}(p_1v_1 - p_2v_2)$$

which represents the work done.

§ 49. Integrals to be Learnt.

The following integrals must be learnt by heart. The corresponding differentials are also given in § 42.

$$\int x^n dx = \frac{1}{n+1}x^{(n+1)}$$

$$\int \frac{dx}{x} = \log_e x$$

$$\int e^x dx = e^x$$

$$\int a^x dx = \frac{a^x}{\log_e a}$$

$$\int \sin x dx = -\cos x$$

$$\int \cos x dx = \sin x$$

$$\int \frac{dx}{\cos^2 x} = \tan x$$

$$\int \frac{dx}{\sin^2 x} = \cot x$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a}$$

These are the most important elementary integrals. The integration of any expression which can be integrated is effected by transforming it by processes which will be shortly explained, into forms of which the integral is known.

Many simple cases can be so transformed at once. For instance, required—

$$\int \frac{dx}{\sqrt{x}}$$

This can be written—

$$\int x^{-\frac{1}{2}} dx$$

This is evidently an example of the x^n integral given above, where $n = -\frac{1}{2}$. Since we know that $\int x^n dx = \frac{1}{n+1} x^{n+1}$, we have evidently—

$$\int x^{-\frac{1}{2}} dx = \frac{1}{-\frac{1}{2} + 1} x^{-\frac{1}{2}+1} = 2x^{\frac{1}{2}}$$

$$= 2\sqrt{x}$$

To test the accuracy of this, let us differentiate the latter expression—

$$\frac{d(2\sqrt{x})}{dx} = \frac{d(2x^{\frac{1}{2}})}{dx} = 2 \times \tfrac{1}{2} \times x^{(\frac{1}{2}-1)}$$

$$= x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$$

or again, required—

$$\int \frac{dx}{(px)^{\frac{q}{r}}} = \int \frac{1}{p^{\frac{q}{r}}} x^{-\frac{q}{r}} dx$$

$$\text{Here } -\frac{q}{r} = n$$

hence required integral $= \frac{1}{p^{\frac{q}{r}}} \times \frac{1}{1 - \frac{q}{r}} \times x^{1-\frac{q}{r}}$.

Again, required $\int \cos mxdx$—

Here, if we try sin mx as the resulting integral, we shall find, on differentiating it—

$$\frac{d(\sin mx)}{dx} = m \cos mx$$

It is evident, therefore, that sin mx is m times too great; therefore, instead of taking sin mx, we evidently ought to have had $\frac{1}{m}$ sin mx, which on differentiating gives cos mx. This is, therefore, the correct result.

It is to be noticed that functions of $(x + a)$, where a is any constant, can usually be treated exactly like the same function of x. For instance, $\frac{d \sin (x + a)}{dx} = \cos (x + a)$, and $\frac{d \log (x+a)}{dx} = \frac{1}{x + a}$, for the d.c. of $(x + a)$ with respect to $x = 1$ (see § 43). This principle does not, of course, extend to such expressions as $\log (x^2 + a^2)$, whose d.c. is evidently $\frac{1}{x^2 + a^2} \times 2x$.

Similarly, we are continually dependent on our previous experience of the differential calculus to enable us to effect any proposed integration, and it is thus evident that we must have the d.c.'s of the elementary functions at our finger-ends before we can hope to attain facility in integration.

Suppose, for instance, we have to find $\int \frac{dx}{a^2 + x^2}$. Even in simple case of this kind we are at once hopelessly lost, unless we happen to know that $\frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right) = \frac{a}{a^2 + x^2}$.

Now, if we tried $\tan^{-1}\frac{x}{a}$, to see if it would produce $\frac{1}{a^2 + x^2}$, on differentiation we should find it was a times too large; hence the correct result is $\frac{1}{a}\tan^{-1}\frac{x}{a}$.

Such a procedure is, no doubt, extremely unsatisfying to the student; at the same time it is the only one that is open to him, and he must be content to make the best of it by continued practice. He will find even the elementary integrals and a few easy applications to be of great service to him in practical work.

Examples.

Integrate a^{3x}, $a^{2x} \times c^x$, $\sin x \cos x$, $\left(\frac{1}{1 + \sin x} + \frac{1}{1 - \sin x}\right)$, $\left(\frac{1}{\sqrt{x + a} - b} \cdot \frac{1}{\sqrt{a - x + b}}\right)$, $\left(\frac{1}{x - \sin x}\right)$, $\left\{\frac{(x - p)^2 + 2px}{(x^2 + p^2)^2}\right\}$, $(\sqrt[3]{x} - \log p)$, $\left\{\frac{x^4}{(x - a)} \cdot \frac{5\sqrt{x - a}}{x^5\sqrt{x + a}}\right\}$.

CHAPTER X.

METHODS OF INTEGRATION.

§ 50. Integration by Expansion.

Many expressions can be integrated by expanding them into separate terms by some algebraical or trigonometrical process. This method should always be tried before any other.

Take, for instance, $\int (a^2 + x^2)^3 dx$. On expansion, this becomes—

$$\int (a^6 + 3a^4x^2 + 3a^2x^4 + x^6)dx$$

This expression, as we have shown (§ 29)—

$$= \int a^6 dx + \int 3a^4x^2 dx + \int 3a^2x^4 dx + \int x^6 dx$$
$$= a^6x + a^4x^3 + \tfrac{3}{5}a^2x^5 + \tfrac{1}{7}x^7$$

Again—

$$\int \frac{dx}{x^2 - a^2} = \int \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx$$
$$= \frac{1}{2a}\int \frac{dx}{x-a} - \frac{1}{2a}\int \frac{dx}{x+a}$$
$$= \frac{1}{2a}\log(x-a) - \frac{1}{2a}\log(x+a)$$
$$= \frac{1}{2a}\log\frac{x-a}{x+a}$$

This is a very important result.

Again—

$$\int \sin^2 x dx = \tfrac{1}{2}\int 2\sin^2 x$$

$$= \tfrac{1}{2}\int(1-\cos 2x)dx = \tfrac{1}{2}\int dx - \tfrac{1}{2}\int \cos 2x dx$$
$$= \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x$$

Examples.—Integrate $(x+a)^2$, $(px+q)^3$, $(x+a)(x-a)$, $\dfrac{x^3-a^3}{x-a}$, $(\cos x + \sin x)^2$, $\dfrac{1}{x^2+5x+6}$, $\dfrac{1}{x^2-6}$. (Split this denominator into $(x+\sqrt{6})(x-\sqrt{6})$.)

It is not necessary or desirable to give the answers. It is always possible to find whether the result is correct by differentiating the result obtained. The d.c. of the answer should, of course, be the same as the function to be integrated.

§ 51.

If it is found impossible to reduce the proposed expression into a series of simple integrable forms, the next thing to be done is to try whether it is possible to write it in the form of two factors, of which one is the d.c. of the other, or of some power or root of the other. If so, the expression can be immediately integrated.

Thus to find $\int(x^4 + a^2x^2)^{\frac{1}{2}}dx$. Here we see that we can write this in the form—

$$\int x\sqrt{x^2+a^2}dx = \tfrac{1}{2}\int 2x \times (x^2+a^2)^{\frac{1}{2}}dx$$

Now, $2x$ is the differential coefficient of x^2+a^2.

Now, obviously, if we differentiate $(x^2+a^2)^{\frac{1}{2}+1}$, we shall obtain (§ 43)—

$$\tfrac{3}{2}(x^2+a^2)^{\frac{1}{2}} \times 2x = 3x\sqrt{x^2+a^2}$$

This is three times too large. Hence the required function is $\frac{1}{3}(x^2+a^2)^{\frac{3}{2}}$.

Again, required $\int(px^2+2qx+r)^{\frac{5}{2}}(px+q)dx$. This becomes—

$$\tfrac{1}{2}\int(px^2+2qx+r)^{\frac{5}{2}}(2px+2q)dx$$

By a similar process, we obtain—

$$\tfrac{1}{2} \times \tfrac{2}{7}(px^2 + 2qx + r)^{\frac{7}{2}}$$

A similar case is—

$$\int \frac{ax + b}{ax^2 + 2bx + c} dx = \tfrac{1}{2} \int \frac{2ax + 2b}{ax^2 + 2bx + c} dx$$

Here the numerator is the d.c. of the denominator. When this is the case, we at once write down the integral $\log (ax^2 + 2bx + c)$. The student will see the reason for this if he differentiates this expression.

Examples.—Integrate $\dfrac{\log x}{x}$ $\left(\text{this expression} = (\log x) \times \dfrac{1}{x}\right)$, $\dfrac{1}{x \log x}$, $\dfrac{x}{px^2 + q^2}$, $\dfrac{x}{\sqrt{px^2 + q^2}}$, $\tan x \left(= \dfrac{\sin x}{\cos x}\right)$, $\sec x \tan x$ $\left(= \dfrac{\sin x}{\cos^2 x}\right)$, $\dfrac{\sin 2x}{\cos^2 x}$.

§ 52. Integration by Substitution.

The next method to be tried is more difficult to understand. It consists in changing the variable from x to some other, usually z, by substituting z for some function of x contained in the expression to be integrated. By this means, as will presently be shown, we can often, by judicious substitution, reduce a complicated expression in x to a simple one in z. This is really treating the expression exactly as in the last case, but we shall be able to deal by this method with more difficult examples.

Take a simple case for the purposes of illustration. Find the area of the curve $y = \left(1 + \dfrac{x}{2}\right)^{\frac{3}{2}}$ between the ordinates $x = 0{\cdot}5$ and $x = 2{\cdot}0$. As we have seen, this area is represented by $\displaystyle\int_{0\cdot5}^{2\cdot0} \left(1 + \frac{x}{2}\right)^{\frac{3}{2}} dx$. Now, suppose we substitute z instead

of the expression in the brackets, we obtain $\int_{x = 0\cdot5}^{x = 2\cdot0} z^{\frac{3}{2}}dx$.
At first sight, the meaning of this is far from clear. The student will have seen that before we attempt to integrate any expression we must first of all get it into some form in which functions of one variable only are present, otherwise we cannot be sure of what we are doing. Since we cannot integrate this expression with respect to x, we are going to make z or $\left(1 + \frac{x}{2}\right)$ the independent variable, to see whether it is any easier to integrate in that way. To do this we must *completely* change every x in the expression into the corresponding value in terms of z by means of the known relation between z and x. In doing this we must not omit to change the dx into some multiple of dz. In order to explain the method geometrically, draw the curve $y = \left(1 + \frac{x}{2}\right)^{\frac{3}{2}}$. The easiest way of doing this is as follows:—

Draw a curve (Fig. 43) showing the relation between x and z, or $1 + \frac{x}{2}$, for all values of x between the given limits. For convenience of reference, call the ordinate of this curve z. This curve, which is shown at (a) in the diagram, has for its equation $z = 1 + \frac{x}{2}$. Now the ordinates of our given curve $y = \left(1 + \frac{x}{2}\right)^{\frac{3}{2}}$ are the $\frac{3}{2}$ power of the ordinates of curve (a). Transfer the ordinates of curve (a) to another horizontal base O_bX_b as shown, and on this base draw the curve $y = z^{\frac{3}{2}}$ by calculating the ordinates with a table of logs or a slide rule, and erect ordinates to it from all the points p_2, q_2, etc. This curve consists of two branches (as shown) with a "cusp" at the origin (see Fig. 20), *i.e.* the tangent at the origin touches two branches of the curve at the origin. Next draw another base O_cX as shown, and divide it exactly as O_aX_a is divided, and erect ordinates from each of the points of division. Transfer each of the ordinates of curve (b) to the *corresponding* ordinate of this new curve as shown. Draw a smooth curve through each of the points so found (much of the actual construction is left out in the figure for the sake of clearness). This curve is easily seen to be the given curve $y = \left(1 + \frac{x}{2}\right)^{\frac{3}{2}}$. Draw in the limiting ordinates $x = 0{\cdot}5$ and $x = 2{\cdot}0$, and the *corresponding* ordinates of curve (b). It is clear that, although we cannot at once find the area of curve (c), yet we can at once find

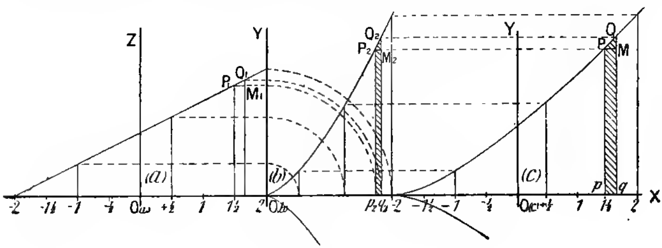

FIG. 43.

that of curve (b) between the corresponding ordinates, for it is $\int z^{\frac{3}{2}} dz$, taken between proper limits, these limits being the values of $1 + \frac{x}{2}$ when x is $0{\cdot}5$ and $2{\cdot}0$ respectively, which values are $1{\cdot}25$ and $2{\cdot}0$.

Our object, then, is to find a relation between the area of curve (c) and that of (b). Consider any shaded element of area of (c), and the corresponding elements of (b) and (a). These elements of (b) and (c) are the same height, and their areas are directly as their breadths, and clearly $\frac{P_2M_2}{PM} = \frac{M_1Q_1}{P_1M_1}$ $= \frac{dz}{dx}$.

Now $\frac{dz}{dx} = \frac{1}{2}$ always, that is to say, $dz = \frac{1}{2}dx$. Hence the area of curve (b) between the ordinates 1·25 and 2·0 = $\frac{1}{2}$ area of given curve (c) between the ordinates 0·5 and 2·0 for each element of (b) = $\frac{1}{2}$ corresponding element of (c). The reasoning would have been exactly the same if $\frac{dz}{dx}$ had not been constant, as in the next example, for instance. The above shows the geometrical meaning of the following reasoning, which is that given in most text-books :—

To find $\int\left(1+\frac{x}{2}\right) dx.$

Let $1+\frac{x}{2} = z.$

$$\text{Then } \tfrac{1}{2}dx = dz$$
$$dx = 2dz$$

$$\text{Hence } \int\left(1+\frac{x}{2}\right)^{\frac{3}{2}} dx = \int 2z^{\frac{3}{2}}dz$$
$$= 2 \times \tfrac{2}{5}z^{\frac{5}{2}} = \tfrac{4}{5}z^{\frac{5}{2}}$$
$$= \tfrac{4}{5}\left(1+\frac{x}{2}\right)^{\frac{5}{2}}$$

Take another example (the student should not fail to draw the curves in this example and the next) : $\int\frac{xdx}{\sqrt{a^2+x^2}}.$

Put $\sqrt{a^2+x^2} = z$;

$$\text{Then } a^2+x^2 = z^2$$

$$\text{And therefore } 2xdx = 2zdz, \text{ or } dx = \frac{z}{x}dz$$

$$\text{Hence } \int\frac{xdx}{\sqrt{a^2+x^2}} = \int\frac{x\cdot\frac{z}{x}\cdot dx}{z} = \int dz = z$$
$$= \sqrt{a^2+x^2}$$

Again, if we desire to find $\int\frac{dx}{\sqrt{x^2+a^2}}$

Assume $\sqrt{x^2 + a^2} = z - x$.

$$\text{Then } x^2 + a^2 = z^2 - 2zx + x^2$$

$$\therefore z^2 - a^2 = 2zx$$

$$\therefore 2z\frac{dz}{dx} = 2z + 2x\frac{dz}{dx}$$

$$\text{or } \frac{dz}{dx} = \frac{z}{z - x}$$

$$\text{or } \frac{dz}{z} = \frac{dx}{z - x}$$

$$\text{Hence } \int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{dx}{z - x} = \int \frac{dz}{z}$$

$$= \log z = \log (x + \sqrt{x^2 + a^2})$$

The student can only hope to learn what substitution will be required in any given case by continued practice. He is referred to a larger book for further examples.

Examples. — Integrate $\frac{1}{\sqrt{x + a}}$, $\frac{1}{(x + 3)^{\frac{3}{5}}}$, $\frac{a}{(x - b)^p}$, $\frac{x - a}{\sqrt{(x - c)^2 + b^2}}$, $\frac{ax + b}{\sqrt{ax^2 + 2bx + c}}$, $\frac{1}{\sqrt{x^2 + 4}}$, $\frac{1}{\sqrt{(x + a)^2 + 16}}$.

§ 53. Integration by Parts.

Another method of great importance is known as "integration by parts," which presents considerable difficulty to the beginner, because of the large number—eight—of different functions which have to be simultaneously borne in mind. The process is, in reality, the opposite of that explained in § 30 for differentiating a product.

Before commencing the following explanation, the student should carefully read § 30. That article showed how to find the d.c. of an expression represented by curve (3), which was the product of two other expressions represented by (1) and (2). It was there seen that if—

(4) be the first derived of (1)
(5) be the first derived of (2)
(6) be the product of (2) and (4)
(7) be the product of (1) and (5)
(8) be the sum of (6) and (7)

Then (8) is the first derived of (3)

But suppose we had been given curves (6), (2), and (4) in their correct places, and had been required to complete Fig. 44, we should evidently have proceeded as follows—

(*a*) integrated (4), thereby producing (1)
(*b*) multiplied together (1) and (2), producing (3)
(*c*) differentiated (2), producing (5)
(*d*) multiplied together (1) and (5), producing (7)
(*e*) addedtogether (6) and (7), producing (8)

Now, it was shown on p. 75, that since ordinate of (6) + ordinate of (7) = ordinate of (8), therefore area of (6) + area of (7) = area of (8) (§ 29, p. 59), all areas being taken between corresponding ordinates.

But area of (8) is represented by the difference of corresponding ordinates of (3), since (8) is the first derived of (3) (§ 13).

Hence we have—

(*f*) area of (6) = ordinate of (3) − area of (7)

Now, suppose that, instead of the curves (6), (2), and (4), we had given their equations, and had been required to find the area of (6) (or, in other words, its integral), we might proceed exactly as at (*a*), (*b*), (*c*), (*d*) above, by writing down the *equations to the curves* instead of drawing the curves themselves, and by the same processes as there described finding the equations (3) and (7). Then by the use of relation (*f*) we can make the integral of (6) depend on the integral of (7). The use of the process consists in this, that sometimes (7) is an easier expression to integrate than (6). If, in any given case, it is not so, then the process is of no assistance, and some other must be tried.

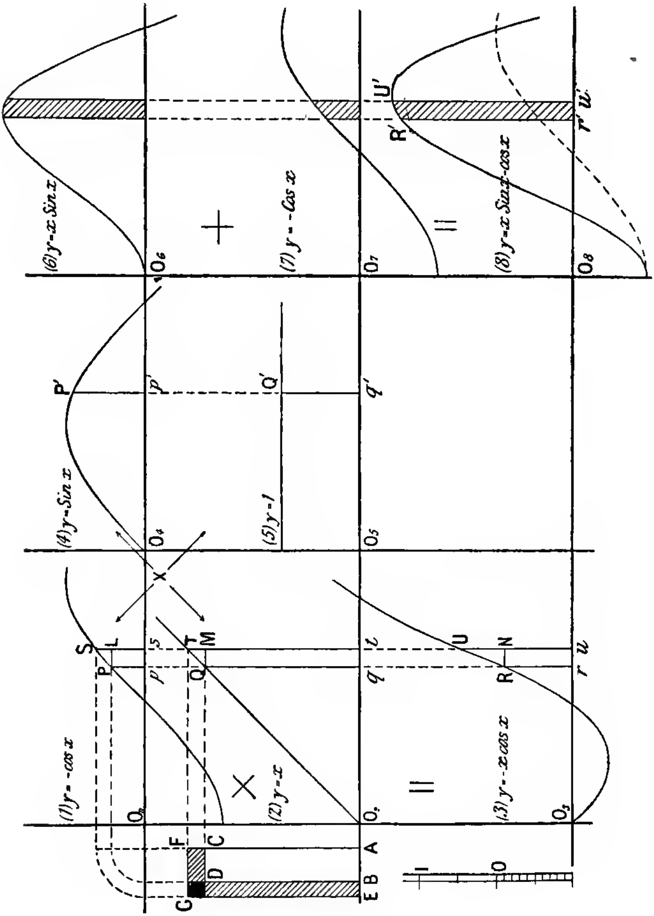

FIG. 44.

An example will make the meaning of this clear.

Required the integral of $x^n \log x$, an expression which we cannot integrate immediately.

Here equation (6) is $y = x^n \log x$
equation (4) is $y = x^n$
equation (2) is $y = \log x$

The product of any ordinate of (4) with the corresponding one of (2) is then equal to that of (6). Now, obviously, as at above, (1), being the integral of (4), must be $y = \frac{1}{n+1} x^{(n+1)}$.

(*b*) . . . (3), being (1) × (2), is $y = \frac{\log x}{n+1} \times x^{(n+1)}$

(*c*) . . . (5) being the first derived of (2) is $y = \frac{1}{x}$

(*d*) . . . (7), being (1) × (5), is $y = \frac{x^{(n+1)}}{n+1} \times \frac{1}{x}$

$$= \frac{x^n}{(n+1)}$$

$$(f) \text{ We have } \overset{\text{Area of (6).}}{\int x^n \log x\, dx} = \overset{\text{Ordinate of (3).}}{\frac{\log x}{n+1} x^{(n+1)}} - \overset{\text{Area of (7).}}{\int \frac{x^n dx}{(n+1)}}$$

$$= \frac{x^{(n+1)} \log x}{(n+1)} - \frac{x^{(n+1)}}{(n+1)^2}$$

Exactly the same method may be applied to find $\int x \sin x$, which is curve (6) in Fig. 44. This is left as an example for the student.

In working examples, the beginner will save himself a great deal of confusion if he writes the symbols down in the same relative positions as the corresponding curves in Fig. 44, till he has become thoroughly familiar with the process.

Thus $-\cos x$	$\sin x$	$x \sin x$
x	1	$-\cos x$
$-x \cos x$		$(x \sin x - \cos x)$

The symbols corresponding to the curves in the diagram, the order of writing down is as follows: (6), (4), (2), (1), (5), (3), (7). (8) may be left out, as it is not needed.

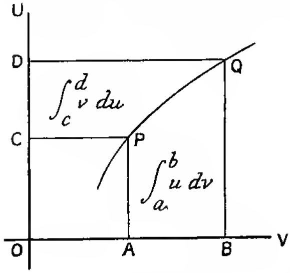

FIG. 45.

Another geometrical illustration which the student should completely analyze for himself is as follows. In Fig. 45, PABQ represents the area $\int_a^b udv$, and the area CPQD represents the area $\int_c^d vdu$ when c and d are respectively the values which v assumes when u has the values a and b. Here it is clear that area PABQ = area CPABQDC − area CPQD, or $\int udv = uv - \int vdu$, all taken between corresponding limits. u and v are both supposed to be dependent on the independent variable x, which does not appear in the diagram (see § 45 and note to § 23 on p. 49). Hence the above equation is really a shortened form of

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Of course, it is only by trial that the student can discover whether or not the process is of any use to him; that is to say, whether or not equation (7) is any simpler to integrate than curve (6); if not, of course the process is useless. It is to be noticed that integration by parts can almost always be tried in two ways, viz. by putting each factor in turn in position (4). Thus, in this instance, if we had written the process thus—

$\frac{1}{2}x^2$	x	$x \sin x$
$\sin x$	$\cos x$	$\frac{1}{2}x^2 \cos x$
$\frac{1}{2}x^2 \sin x$		$(x \sin x + \frac{1}{2}x^2 \cos x)$

it is evident that $\frac{1}{2}x^2 \cos x$ is no easier to integrate than $x \sin x$.

Take another case, $\int \log xdx$. Here there is, apparently, only one factor, but we can use 1 for the other factor, and proceed thus—

$$\begin{array}{l|c|l} x & 1 & \log x \\ \log x & \frac{1}{x} & 1 \\ x \log x & & \end{array}$$

$$\text{Hence} \int \log x dx = x \log x - \int 1 dx$$

$$= x \log x - x$$

It is sometimes possible to integrate an expression by integrating by parts twice in succession. Care must be taken which expression is placed in position (4) in the second operation, otherwise the only result will be to reproduce the original expression. Several of the following examples must be treated in this way.

Examples.

Integrate $x \cos px$, xe^x, x^3e^x, $x \sin x \cos x$, $x^n \log x$, $x^n(\log x)^2$, $e^{ax} \sin bx$. (Integrate by parts twice, and reduce the required integral by the principles of simultaneous equations from the equations so obtained.)

Sin $\sqrt{x}$ (put $\sqrt{x} = z$, as in § 52; then integrate by parts), $(\log x)^2$, $x^3 \log x^2$.

CHAPTER XI.

MISCELLANEOUS APPLICATIONS OF DIFFERENTIATION.

§ 54. Maxima and Minima.

In § 13 (i.), (ii.), (iii.), which should be re-read, it was shown that the height of a derived curve corresponding to a maximum or minimum on the primary (which terms were there explained) was 0, or at that point the primary curve was of no slope. Conversely, if at any point the first derived cuts OX, then the corresponding point of the primary is a maximum or minimum. It was also shown in (iv.) that we can distinguish between a maximum and a minimum by the *direction of slope* of the first derived curve. In other words, if the second derived curve is at that point above the axis of X, then the point on the primary is a minimum. If at that point the height of the second derived curve is negative, then the point on the primary is a maximum.

These principles are of great use in practice from the ease with which we can find algebraical expressions for the height of a derived curve. If we have an expression involving x which, as x increases, first increases and then diminishes, we can find that value of x at which the function has attained its maximum value, by finding that value of x which makes the first derived function $= 0$. In other words, differentiate the function and equate the d.c. to zero, thus giving an equation to find x (see Example in § 2). Differentiating the d.c. thus found with respect to the same variable, we evidently obtain an expression for the height of the second derived curve at any point. Substitute in this expression the value of x found by equating the first derived function to zero. If the result is negative, the height of the primary is at the corresponding point a maximum; and if positive, a minimum.

The successive derived curves of sin x form a very intelligible illustration of this. Suppose we wish to find at what points the value of sin x is a maximum or minimum. The height of the first derived curve is clearly (Fig. 46) given by cos x for all values of x. Now, equating this to zero, we

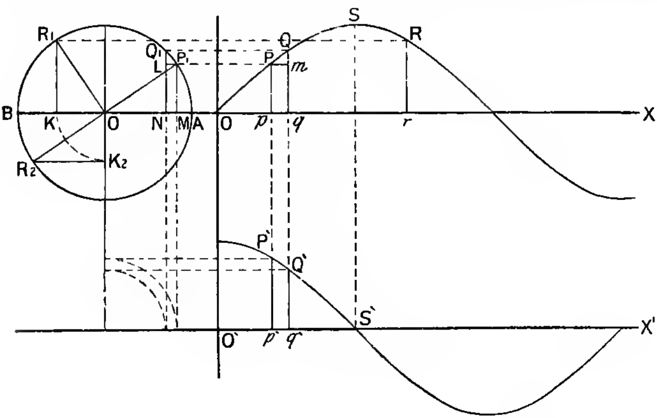

FIG. 46.

find a value such as O'S' of x for which the first derived curve cuts the axis of x.

These values are found from the equation cos $x = 0$.

Hence $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, etc., each of which corresponds to either a maximum or a minimum.

To find which is which, differentiate cos x, giving $-$ sin x.

Now substitute the above values of x in the expression $-$ sin x, and see whether the result is a positive or negative value.

$-\sin\frac{\pi}{2} = -1$, showing that at a point distant $\frac{\pi}{2}$ from OY the primary curve attains a maximum.

$-\sin\frac{3\pi}{2} = +1$,, ,, minimum.

$-\sin\frac{5\pi}{2} = -1$,, ,, maximum,

etc., etc.

These can easily be verified in the diagram.

The way in which this principle can be applied in practical case will be seen by considering an example. Suppose a man running along a footpath AO (Fig. 47) wishes to reach a house at H, in the middle of a ploughed field, in the shortest possible time. Suppose that he can run on the footpath at the rate of 15 feet per second, but only at 10 feet per second in the ploughed field. What will be his quickest route across the field? If he ran across AH, he would have the shortest possible distance to go, but his speed would be only 10 feet per second instead of 15 along the footpath, whereas if he went to O along the footpath he would have a greater distance to go, but his average speed would be greater than in the other case. It is clear that the shortest time would be occupied by taking some intermediate route, ApH, where he would partly utilize his superior speed along the footpath, and partly the advantage of cutting off the corner pOH. It is our object to determine the position of the point p.

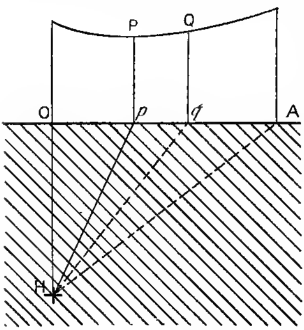

FIG. 47.

Suppose the distances are OA = 500 feet, OH = 150 feet.

Let Op (the distance to be found) = x.

Then distance to be run in the field = $\sqrt{x^2 + (150)^2}$

$$\text{Time occupied} = \frac{\sqrt{x^2 + (150)^2}}{10} \text{ seconds}$$

$$\text{Distance to be run along road} = 500 - x$$

$$\text{Time occupied} = \frac{500 - x}{15} \text{ seconds}$$

$$\text{Total time occupied} = y = \frac{\sqrt{x^2 + (150)^2}}{10} + \frac{500 - x}{15}$$

We have now to find the minimum value of y.

Plot various values of y along OA as base, and draw a curve through the points so found. Thus qQ represents the time occupied if the man leaves the path at q. Now, this curve is obviously parallel to OA at the point where the derived curve (a line not shown in the figure) cuts O'X'. This value of x will be found by equating the d.c. of y to zero (see example in § 2).

Simplifying the equation, we have—

$$y = \frac{3\sqrt{x^2 + (150)^2} + 1000 - 2x}{30}$$

Now, since we only wish to find for what value of x this is a minimum, it is clear that we may simply consider the numerator, for if the numerator is a minimum, the whole fraction will be a minimum. (If the denominator contained x, or anything dependent on x, we could not legitimately disregard it.) Similarly, we may disregard the 1000, since it is always the same, whatever be the value of x, and our problem becomes to find for what value of x, $3\sqrt{x^2 + (150)^2} - 2x$ is a minimum.

Exercise.—Let the student determine for himself the graphical interpretation of the process of disregarding these constants.

Differentiating this, we obtain (§ 43)—

$$\frac{3}{2\sqrt{x^2 + (150)^2}} \times 2x - 2$$

This represents a multiple of the tangent of slope of the upper curve, and must therefore = o at a point corresponding to P. Therefore, to find x, we have the equation—

$$\frac{3x}{\sqrt{x^2 + (150)^2}} - 2 = 0$$

or, simplifying—

$$5x^2 = 4 \times (150)^2$$
$$\text{whence } x^2 = \tfrac{4}{5}\,150^2$$
$$\text{therefore } x = 134 \text{ yards about}$$

the negative value being obviously inadmissible.

Another familiar example shortly worked out may serve to further illustrate the process.

According to the post-office regulations for the size of parcels which may be sent by parcels post, the length of parcel + girth must not be greater than 6 feet. Required the greatest volume which can be sent.

Let x feet be the girth; then $6 - x =$ length.

Now, with any given perimeter, the figure containing the greatest area is well known to be a circle.

The area of a circle of girth $x = \dfrac{x^2}{4\pi}$

$\therefore$ volume of a parcel of girth x and length $(6-x) = \dfrac{x^2}{4\pi}(6-x)$

Therefore the question becomes for what value of x has $6x^2 - x^3$ a maximum value.

Differentiating and equating to zero, we have—

$$12x - 3x^2 = 0$$

Neglecting the solution $x = 0$, which obviously gives a minimum, we have—

$$x = 4$$

$$\text{The volume is therefore } \frac{8}{\pi} = 2{\cdot}55 \text{ cubic feet}$$

Examining these three examples, the student will see that the rule is—Express the quantity of which we have to find the maximum or minimum value in terms of one variable; differentiate the expression with respect to that variable; equate to zero, and solve the resulting equation. The solution gives the value of the variable for which the expression has either a maximum or a minimum value. If it is not apparent on inspection whether the value so found gives a maximum or minimum, differentiate the first differential coefficient, and substitute in the expression thus found the value in question. If this gives a negative result, the value gives a maximum; and if a positive result, a minimum (see § 10, 4).

Example 1.—A man weighs 160 lbs.; he attaches a rope to the top of a post 20 feet high, with the object of pulling it over: what is the greatest bending moment he can produce at the base of the post, assuming that the pull he can exert on the rope varies as the sine of the angle which the rope makes with the ground? *Ans.* 1600 lbs. feet.

Example 2.—Find the minimum weight of a cylindrical boiler made of $\frac{1}{2}''$ plate necessary to hold 200 cubic feet of water. Neglect overlap of plates. One cubic inch of iron weighs 0·28 lb. *Ans.* 3850 lbs. (about).

Example 3.—One leg of a pair of compasses is held vertical with its point stuck in a board, and the compasses are rotated about this leg as axis: find what angle the other leg must make with the vertical, in order that the bending moment tending to open the compasses may be a maximum, given that the legs are uniform, each $5''$ long and each weighing $\frac{1}{2}$ oz. *Ans.* $\cos\theta = \dfrac{0{\cdot}5g \pm \sqrt{\dfrac{g^2}{4} + 12{\cdot}5\omega^4}}{5\omega^2}$ where $g = 32{\cdot}2$, and ω = angular velocity.

§ 55. Indeterminate Forms.

It sometimes happens that we have to find the value of some function of x which cannot be obtained by simple substitution, because on giving x the required value the function assumes the form $\frac{0}{0}$, 0^0, $0 \times \infty$, or some such form. In these cases the application of the principles of the differential calculus enables us to effect a very simple solution. For instance, we might have to find the value of—

$$\frac{x^3 - 12x + 9}{x^3 - 4x^2 - 5x + 24} \text{ when } x = 3$$

It is easy to see that this fraction assumes the form $\frac{0}{0}$ when $x = 3$, and we can therefore not determine its value by simple substitution. Assume that the two curves APB, CPD represent the values of the numerator and denominator respectively of any fraction $\left(\frac{f_1(x)}{f_2(x)}\right)$, which assumes the form $\frac{0}{0}$ when $x = \text{OP}$. Now suppose, for the sake of definiteness, that we have found $\frac{dy}{dx}$ for the curve APB to have the value 1·5 at the point P, and for the curve CPD the value 2·5 at the same point. Draw the tangents ST and RQ at the point P. Take two verticals TQM and SRN near to P, but on opposite sides of it. It is clear that $\frac{\text{MT}}{\text{MQ}}$ is constant wherever we take T on the line ST. Now, when very near the point P, the points T and Q (as has been frequently explained) are respectively on the curves representing the value of the numerator and denominator of the given fraction. Now, when T *has* travelled through P to an extremely small distance the other side of P, it is clear that the value of the *ratio* has not altered either in sign or magnitude, because the numerator and denominator have *both* changed sign, and therefore their ratio has the same sign as before. Thus we have $\frac{MT}{MQ} = \frac{NS}{NR} =$ constant whatever be the sign and magnitude of the values PM, PN. It is therefore true when they are "infinitely small." Now, it is impossible to imagine what happens *at* the point P in just the same way as it is impossible to imagine an infinitely great or an infinitely small

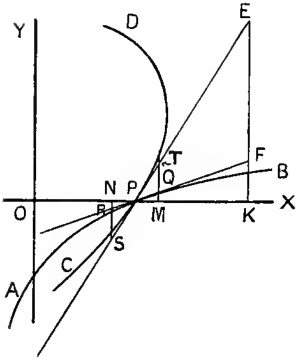

FIG. 48.

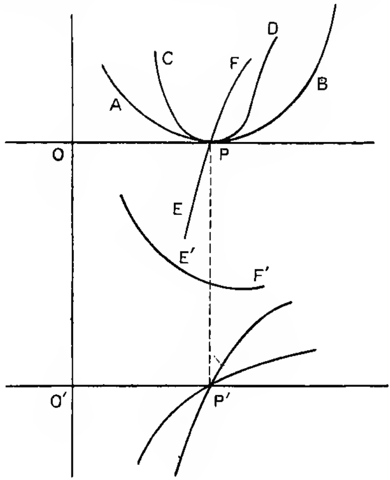

FIG. 49.

quantity. Assuming the curve representing the value of the ratio at all points along OX is continuous, *i.e.* does not make a sudden vertical jump *at* the point P—and there is no reason for supposing it would—we say that *at the point P* it has the same value as it has at a point infinitely near to P. We therefore express the fact that, however small PM is, the value of the fraction $\frac{MQ}{MT} = \frac{KF}{KE} = \frac{\frac{df_1(x)}{dx}}{\frac{df_2(x)}{dx}}$, by saying that the value of the fraction at the point P = ratio of the values of the d.c.'s at that point.

If the curves AB, CD are of the form shown at APB, CPD in Fig. 49, it will happen that $\frac{df_1(x)}{dx}$ and $\frac{df_2(x)}{dx}$ are both equal to zero, in which case, by the same reasoning as before, the ratio will be that of the values of the *second* derived functions at the point. Again, one curve may be of the form APB, and the other of the form EPF, in which case the ratio will be $= 0$ or ∞. This will be shown by one derived function vanishing, while the other is finite.

Again, if the curves, in addition to having no slope at P, have a point of inflection at that point, the second derived functions will vanish at that point, and the ratio required will be that of the heights of the third derived curves at the point. In general the required ratio will be that of the first *pair of* derived functions which do not *both* vanish at the point.

Examples.—Find the value of—

(i.) $\dfrac{\tan x - \sec x + 1}{\tan x - \sec x + 1}$ when $x = 0$. (*Ans.* 1.)

(ii.) $\dfrac{\log x}{x-1}$ when $x = 1$. (*Ans.* 1.)

(iii.) $\dfrac{x - \sin x}{x^3}$ when $x = 0$. (*Ans.* $\frac{1}{6}$.)

(iv.) $\dfrac{e^{ax} - e^{ap}}{(x-p)^n}$ when $x = p$. (*Ans.* ∞.)

§ 56. Equation to Tangent to a Curve.

It is clear that, given any equation to a curve, and any point on it, we can at once write down the equation to the tangent and normal at that point; for we know that the equation to any line passing through a point (ab) whose tangent of slope is m, is $(y-b) = m(x-a)$; for this equation, being of the first degree, clearly represents a straight line. It is also satisfied by the point $(a, b,)$, and since it is of the form $y = mx + c$ when simplified, it represents a line inclined at $\tan^{-1} m$. Hence, substituting $\frac{dy}{dx}$ for m, we have—

$$(y-b) = \frac{dy}{dx}(x-a) \text{ for the tangent}$$

$$\text{and } (y-b) = \frac{a-x}{\frac{dy}{dx}} \text{ for the normal}$$

Example 1.—Find the equation to a tangent and normal of the curve $y = \frac{x^2}{10}$, at a point on it whose abscissa is 5.

Here it is clear $y = 2{\cdot}5$; $\frac{dy}{dx} = 1$. Hence equation to tangent is $(y-2{\cdot}5) = (x-5)$, and to normal $y - 2{\cdot}5 = 5 - x$.

2.—Prove the subnormal in the curve $y^2 = 2mx$ is equal to m.

Take any point $(x_1 y_1)$ on the curve. Since the point is on the curve we have—

$$y_1^2 = 2mx_1$$
$$\text{hence } y_1 = \sqrt{2mx_1}$$

hence the point $(x_1, \sqrt{2mx_1})$ is on the curve.

Find where the normal at the point $(x_1, \sqrt{2mx_1},)$ cuts OX by putting $y = o$ in its equation, and show that this point is at a constant distance from the point whose abscissa is x_1.

§ 57. Radius of Curvature.

Consider any curve APQ (Fig. 49). It is clear that y, $\frac{dy}{dx}$, etc., are not the only quantities in connection with the curve which assume definite values for any assumed value of x. Other such quantities are S, the length of the curve reckoned from A; ϕ, the angle of slope in radians, etc. We are, therefore, quite within our rights in speaking of such differential coefficients as $\frac{dx}{ds}$, $\frac{d\phi}{ds}$ (see § 45). It would be easy to plot, for instance, a curve showing in its ordinate the length of the curve reckoned from A corresponding to each value of x. This should be done by measurement. The first derived of this curve would represent the value of $\frac{ds}{dx}$. But we can see that another geometrical function

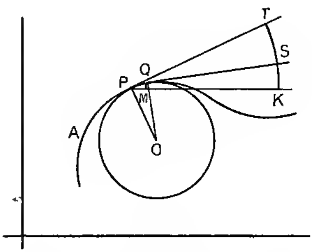

FIG. 50.

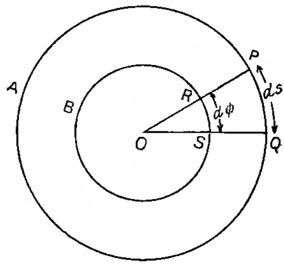

FIG. 51.

would represent this value independently of such a curve. It is clearly the limit of $\frac{PQ}{PM}$, or the secant of the slope at P, which function will vary when the slope of the curve varies.

Consider what is the geometrical meaning of $\frac{ds}{d\phi}$. It is the limit of the ratio of the length PQ (Fig. 50) to the difference (measured in radians) between the angle of slope of the curve at P and at Q. Consider first what this ratio means for the circle APQ (Fig. 51).

We have by definition—

$$OPd\phi = ds, \text{ where } OR = 1 \text{ inch}$$

$$i.e. \; OP = \frac{ds}{d\phi}$$

This holds good just as well for the circle in Fig. 50, for it is clear that the angle $d\phi$ between PT and QS = the angle POQ. OP is here clearly the radius of curvature of a circle, which most nearly coincides at P with the curve. Hence, to find the value of the radius of curvature at any point on the curve, we have to direct our attention to finding the value of $\frac{ds}{d\phi}$ at that point; that is, the height of the first derived curve of the curve representing the values of ϕ (abscissa) and s (ordinate).

This curve can easily be drawn by measurement. For the co-ordinates of any point corresponding to P, Fig. 50, we must take AP (the arc) for ordinate, and TK (arc) for abscissa where PT = 1 inch. Differentiating this curve graphically, we obtain a curve showing in its ordinate the length of the radius of curvature.

The algebraical process is directed towards obtaining the value of $\frac{ds}{d\phi}$ from the successive derived coefficients of the primary curve with respect to x.

We have—

$$r = \frac{ds}{d\phi} = \frac{1}{\frac{d\phi}{ds}} \quad . \quad . \quad . \quad (\S\ 23)$$

$$= \frac{1}{\frac{d\phi}{dx} \cdot \frac{dx}{ds}} \quad . \quad . \quad . \quad (\S\ 43)$$

Now, $\tan \phi = \frac{dy}{dx}$.

Differentiating this with respect to x (§§ 43, 44), we obtain—

$$\frac{d \tan \phi}{dx} = \frac{d\left(\frac{dy}{dx}\right)}{dx}$$

$$\text{or } \sec^2 \phi \frac{d\phi}{dx} = \frac{d^2y}{dx^2}$$

$$\text{Hence } \frac{d\phi}{dx} = \frac{d^2y}{dx^2} \cos^2 \phi$$

$$\text{Also } \frac{dx}{ds} = \cos \phi$$

$$\text{Hence } r = \frac{1}{\dfrac{d\phi}{dx} \cdot \dfrac{dx}{ds}} = \frac{1}{\dfrac{d^2y}{dx^2} \cos^3 \phi} = \frac{\sec^3 \phi}{\dfrac{d^2y}{dx^2}}$$

Now, we know that $\sec^2 \phi = 1 + \tan^2 \phi$.

$$\text{Hence } \sec \phi = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\text{Hence } r = \frac{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$$

which gives the value of r in terms of the corresponding height of the first derived and second derived curve.

This process seems confusing at first. The student should bear in mind that the object is to express $\frac{ds}{d\phi}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. In order to do this, we must first separate ds and $d\phi$ by means of § 43. By that section we know that—

$$\frac{d\phi}{ds} = \frac{d\phi}{dx} \cdot \frac{dx}{ds}$$

Now, we can find $\frac{d\phi}{dx}$ by expressing the relation between ϕ and x, and differentiating it with respect to x. This relation is—

$$\tan \phi = \frac{dy}{dx}$$

$$\text{or } \phi = \tan^{-1} \frac{dy}{dx}$$

On differentiating either of these, we obtain $\frac{d\phi}{dx}$ in terms of $\frac{dy}{dx}$, and

$\frac{d^2y}{dx^2}$. We also know that $\frac{dx}{ds} = \cos\phi$. These equations are combined, as shown in the above article.

There are several other expressions for this most important function, but the use of them involves ideas beyond the scope of the present work.

Example 1.—Find the smallest radius of curvature of the curve $y = e^x$.

Example 2.—Prove that, when a ball is projected obliquely upwards, the centrifugal force due to the curvature of the path at the highest point just balances the weight of the ball.

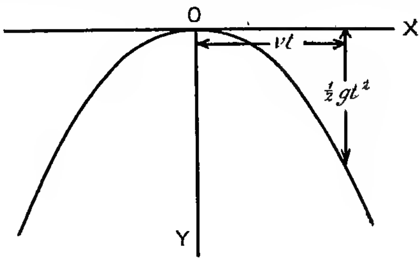

FIG. 52.

At its highest point the ball is moving horizontally (§ 10) with velocity $v\frac{\text{ft.}}{\text{sec.}}$, suppose. Now, after t seconds the co-ordinates of the position of the ball are $x = vt$, and $y = \frac{1}{2}gt^2$.

The equation of the path of the ball, therefore, is—

$$-y = \tfrac{1}{2}g\frac{x^2}{v^2} = \frac{g}{2v^2}x^2$$

This is found by eliminating t between $x = vt$ (i.) and $y = \frac{1}{2}gt^2$ (ii.), *i.e.* substituting $\frac{x}{v}$ for t in (ii.).

$$\text{Hence } \frac{dy}{dx} = -\frac{g}{v^2}x$$

$$\frac{d^2y}{dx^2} = -\frac{g}{v^2}$$

Hence radius of curvature at the point (0, 0,)

$$= \frac{(1 + 0^2)^{\frac{3}{2}}}{-\frac{g}{v^2}} = -\frac{v^2}{g}$$

i.e. a distance $\frac{v^2}{g}$ vertically downwards.

Let m = mass of ball.

Centrifugal force on ball due to curvilinear path

$$= \frac{mv^2}{r} = \frac{mv^2}{\frac{v^2}{g}}$$

$$= mg = \text{weight of ball}$$

§ 58. Illustration of Taylor's Theorem.

It may not be out of place in the present work, without going into the proof of what is called "Taylor's theorem" (which will be found in any book on the differential calculus), to give an illustration of the meaning of that very comprehensive proposition, which will, perhaps, enable the student to grasp its meaning better.

The proposition is as follows: If any curve is represented by the equation $y = f(x)$, and if we know the height of the curve and all its derived curves corresponding to one value x of the variable, then assuming that the function is "continuous," *i.e.* that neither the function nor its derived functions become infinite for any of the values of x under consideration, the height of the curve at a point whose abscissa is $(x + h)$ is—

$$f(x) + hf'(x) + \frac{h^2}{1 \cdot 2}f''(x) + \frac{h^3}{1.2.3}f'''x + \ldots \quad \text{(i.)}$$

where $f'(x)$, $f''(x)$, etc., are the heights of the first, second, etc., derived curves at the point whose abscissa is x.

The function we shall take for illustration will be x^3. The curve is $y = x^3$, and the height which we shall calculate corresponds to an abscissa $(x + h)$.

Assume that the curve (Fig. 53) represents the distance travelled by a particle, as in Fig. 17.

Let $Op = x$, $pq = h$.

Then if OPQ represent the curve, $y = f(x)$, it is clear that qQ represents $f(x + h)$.

Of course, we could in this case arrive at the height qQ by cubing $(x + h)$; but we can also arrive at it by another process, which has the advantage that it is applicable to all other functions of x. Differentiate the primary function. Then it is clear that the area $P'p'q'Q' = MQ$.

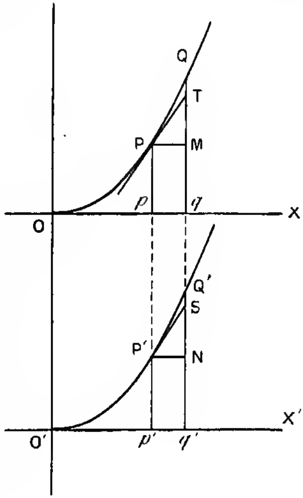

FIG. 53.

$$
\begin{aligned}
qQ = qM + MQ &= f(x) + \text{area } P'p'q'Q' \\
&= f(x) + P'p'q'N + SP'N + Q'P'S \\
&= f(x) + hf'(x) + \tfrac{1}{2}h \times h \tan SP'N + \text{area } Q'P'S \\
&= f(x) + hf'x + \tfrac{1}{1\cdot 2} h^2 f''(x) + \text{area } Q'P'S'
\end{aligned}
$$

Now, this small area $Q'P'S'$ is called "the remainder after three terms of the series." An expression is found for it in all books on the calculus. Its actual value in this case is h^3. Working the above formula (i.) out, we find—

$$
\begin{aligned}
f(x) &= x^3 \\
f'(x) &= 3x^2 \\
f''(x) &= 3.2x \\
f'''(x) &= 3.2.1 \\
f''''(x) &= 0
\end{aligned}
$$

$$f(x+h) = x^3 + 3x^2h + \frac{3.2x}{1.2}h^2 + h^3\frac{3.2.1}{1.2.3} + 0$$
$$= x^3 + 3x^2p + 3xp^2 + h^3$$

which agrees with the ordinary formula.

Again, suppose we are given $\sin 30^\circ = 0{\cdot}5$, and are required to find, say, $\sin 35^\circ$. We have—

$$x = 30^\circ = \frac{\pi}{6} \text{ radians}$$
$$h = 5^\circ = \frac{5\pi}{180}$$
$$= \frac{\pi}{36} \text{ radians}$$

$$\text{Here } f(x) = \sin x = \tfrac{1}{2}$$
$$f'(x) = \cos x = \frac{\sqrt{3}}{2}$$
$$f''(x) = -\sin x = -\tfrac{1}{2}$$
$$f'''(x) = -\cos x = -\frac{\sqrt{3}}{2}$$
$$f''''(x) = \sin x, \text{ etc.}$$
$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{1.2}f''(x) + \frac{h^3}{1.2.3}f'''(x)\ldots$$
$$= 0{\cdot}5 + \frac{\pi}{36} \times \frac{\sqrt{3}}{2} - \frac{\pi^2}{36^2} \times \tfrac{1}{2} \times 0{\cdot}5 - \frac{\pi^3}{36^3}$$
$$\times \frac{1}{1.2.3} \times \frac{\sqrt{3}}{2} + \frac{\pi^4}{36^4} \times \frac{1}{1.2.3.4} \times \tfrac{1}{2}, \text{ etc.}$$
$$= 0{\cdot}5 + 0{\cdot}075 - 0{\cdot}0022$$
$$\textit{i.e. } \sin 35^\circ = 0{\cdot}573\ldots$$

The process being carried to any desired degree of accuracy.
Again, given $\log_{10} 2 = 0{\cdot}301$, required $\log_{10} 3$.

$$\text{Here } (x) = \mu \log x$$
$$= 0{\cdot}434 \log x$$
$$h = 1$$

$$f'(x) = \frac{\mu}{x}$$

$$f''(x) = -\frac{\mu}{x^2}$$

$$f'''(x) = +2\mu x^{-3} = \frac{2\mu}{x^3}$$

$$f''''(x) = -\frac{6\mu}{x^4}, \text{ etc.}$$

Now, $x = 2$, $h = 1$; and $\log_{10} 2 = 0{\cdot}301$.

$$\text{Hence } \log_{10}(x+h) = 0{\cdot}301 + \frac{0{\cdot}434}{2} - \frac{0{\cdot}434}{1.2.4} + \frac{0{\cdot}434 \times 2}{1.2.3.8}, \text{ etc.}$$

$$= 0{\cdot}536 - 0{\cdot}061 = 0{\cdot}475 = \log 3$$

The process being, as before, carried to any desired degree of accuracy.

Examples.

(1) Given $\log 1 = 0$, find the distance between marks 1 and 2 on a slide rule where the distance between 1 and 10 is 12·5 cm. (The equation to the curve is $y = p \log x$, find the value of p.)

(2) Find by calculation *hyp* log 3·25, *hyp* log 4·21, *hyp* log 7, given $\log 1 = 0$.

(3) Find by calculation cos 4°, cos 66°, sin 72°, etc.

(4) Find the value of $5^{2{\cdot}15}$, $3^{4{\cdot}31}$, etc., by calculation from the values of 5^2, 3^4, etc. (The equations are here $y = \epsilon^{(x \log 5)}$, etc.)

CHAPTER XII.

MISCELLANEOUS APPLICATIONS OF INTEGRATION.

§ 59. THE CUBATURE OF SOLIDS.

WE have already shown the application of integration to the finding of areas.

It was shown in Chapter I., § 3, that a line may represent an area. If a line in one direction represents an area, and if a line at right angles to it represents a linear distance, then it is clear that the area of the rectangle formed on these two lines as sides will represent a volume. Thus, if the number of inches in AB (Fig. 54) represents the sectional area of a prism in square inches, and AD = length of prism, it is clear that the number of *square* inches in ABCD represents the number of *cubic* inches in the substance of the prism.

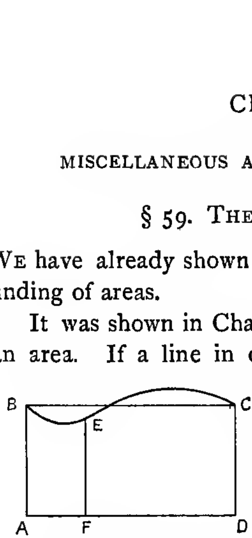

FIG. 54.

If the sectional area of the prism is not constant all along the length, but varies from point to point, then if a curve BEC be drawn so that the length of the ordinate FE at any point F represents the value of the sectional area at that point, then it is easy to show by splitting the area up into vertical elements, exactly as explained in § 13, that the area of the figure BECDA still represents the volume of the irregular solid.

Exercise.—Draw any irregular curve about 10 inches long, and a straight line of similar length. Imagine a solid generated by the curve revolving about the line as axis. Find graphically the whole volume generated by the method of sectional areas.

Now, the algebraical method of obtaining the volume of a solid is the counterpart of this process. It consists in obtaining the equation to the line of sectional areas BEC (Fig. 54), and integrating it with respect to x between the ordinates AB

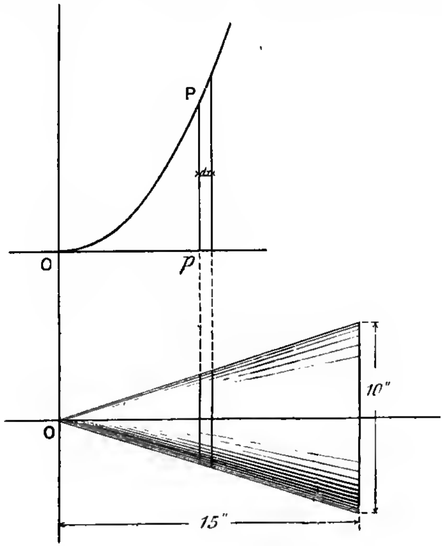

FIG. 55.

and DC. Let it be required to obtain the volume of a cone of the dimensions shown in Fig. 55.

To obtain the equation of the line of sectional areas, consider what will be the height of the curve at a distance x from O. Now, the radius of the circular section of the cone at that point will clearly be $x \tan \theta = \frac{5}{15} \times x = \frac{x}{3}$, and the area of this circle $\frac{\pi x^2}{9}$ = height of curve pP at that point. In other words, the equation to the curve of sectional areas is $y = \frac{\pi}{9}x^2$. Hence we require the area of this curve between the limits 0 and 15. This is given by—

$$\int_0^{15} \frac{\pi x^2}{9} dx = \frac{\pi}{9} \int_0^{15} x^2 dx$$
$$= \frac{\pi}{9} \left[\frac{x^3}{3}\right]_0^{15}$$
$$= \frac{\pi}{9}\left(\frac{15^3}{3} - \frac{0^3}{3}\right)$$
$$= \frac{15^3\pi}{27} = \frac{5 \times 5 \times 5\pi}{1} = 125\pi \text{ cub. in.}$$

It is easy to show that the formula $\frac{\pi}{9} \times \frac{x^3}{3}$ gives exactly the same result as that derived from the common rule, "$\frac{1}{3}$ of volume of cylinder on same base;" for, taking r = radius of base, h = height, we have, as a result—

$$\int_0^h \pi\left(\frac{rx}{h}\right)^2 dx = \frac{\pi r^2}{h^2} \times \tfrac{1}{3}h^3 = \tfrac{1}{3}\pi r^2 h$$

The formula for the volume of a sphere of radius a is obtained in the same way.

Taking the origin at the centre of the elevation of the sphere, the equation to the curve of sectional areas is clearly $y = \pi(a^2 - x^2)$. The integral is therefore—

$$\int_{-a}^{a} \pi(a^2 - x^2)dx = \pi \left[a^2x - \frac{x^3}{3}\right]_{-a}^{a}$$
$$= \tfrac{4}{3}\pi a^3$$

The same result may also be obtained by imagining the sphere split up into concentric spherical shells, remembering that area of surface of sphere is $4\pi r^2$.

Example 1.—Obtain the volume of a cone of height h, the base being an ellipse whose semi-axes are a, b (area of an ellipse $= \pi ab$). *Ans.* $\frac{1}{3}\pi abh$.

2.—Obtain the volume of a solid paraboloid generated by the revolution of the parabola, $y^2 = 4ax$, round its axis, between the planes $x = 5$ and $x = 9$. *Ans.* $112\pi a$.

§ 60.

By a slight extension of this principle, we can obtain such results as the following.

Find the total mass of a sphere of radius 10 inches, whose density varies as the square of the distance from the centre, the density (mass per unit volume) at the surface being 0·25 lb. per inch³. Consider an elementary spherical shell of infinitely small thickness dx, and of radius x. The surface of this shell is $4\pi x^2$. The volume of it is $4\pi x^2 dx$. The quantity of matter in it is clearly $4\pi x^2 \rho dx$, where ρ is the density or quantity of matter per unit volume at a distance x from the centre. Now we have $10^2 : x^2 :: \frac{0{\cdot}25 \text{ lb.}}{\text{in.}^3} : \rho$, since the density varies as the square of the distance from the centre.

$$\text{Therefore } \rho = \frac{0{\cdot}25 \times x^2}{10^2} \cdot \frac{\text{lbs.}}{\text{in.}^3}$$

Hence, substituting this value of ρ in the above expression, we see that required total mass is the result of adding together all such small masses as $\left(4\pi x^4 \times \frac{0{\cdot}25}{10^2}\right) dx$ between the given limits; that is—

$$\int_0^{10} \left(4\pi x^4 \times \frac{0{\cdot}25}{10^2}\right) dx = \left(\frac{\pi}{100} \times \tfrac{1}{5}x^5\right)_0^{10}$$

$$= \frac{\pi}{500} \times 10^5 = 200\pi \text{ lbs.}$$

§ 61. Graphical Solution of Differential Equations.

Problems sometimes arise in which we are given a relation subsisting between two or more of the primary or successive derived functions of a quantity, and we are required to find either the primary or some other function connected with it. These problems are very confusing to the beginner, and we shall show in what way many of them can be attacked graphically by the careful application of the principles already explained. For example, a train weighs 50 tons exclusive of the engine. The resistance to motion due to mechanical friction alone is constant at all velocities, and is of the magnitude of say 8 lbs. per ton. The resistance due to other causes (such as that due to the atmosphere) varies directly as the 1·7th power of the velocity in $\frac{\text{ft.}}{\text{sec.}}$, being, let us say, $= \cdot 0025 \times v^{1\cdot 7}$. Suppose we are also given a curve showing the magnitude of the pull in the drawbar as the speed varies, and are required to find—

(i.) The maximum velocity attainable on the level.

(ii.) The time occupied in attaining it.

(iii.) Distance travelled in that time.

A method similar to the following may be used in attacking problems of this kind. Suppose AHB (Fig. 56) is the given velocity-pull curve, of which both scales must be given (the figure is not drawn to scale) where the drawbar pull is plotted vertically in tons suppose, and velocity horizontally in $\frac{\text{ft.}}{\text{sec.}}$.

We have now to draw on the same base the curve of resistances. This resistance consists of two parts—

(*a*) Frictional resistance, which is constant whatever the velocity.

(*b*) Other resistances, which vary as the 1·7th power of the velocity.

Find the total value of (*a*) for the whole train.

This is—

$$\frac{8 \text{ lbs.}}{\text{ton}} \times 50 \text{ tons} = 400 \text{ lbs.}$$

Set this value off at O_1D to the same scale as the drawbar

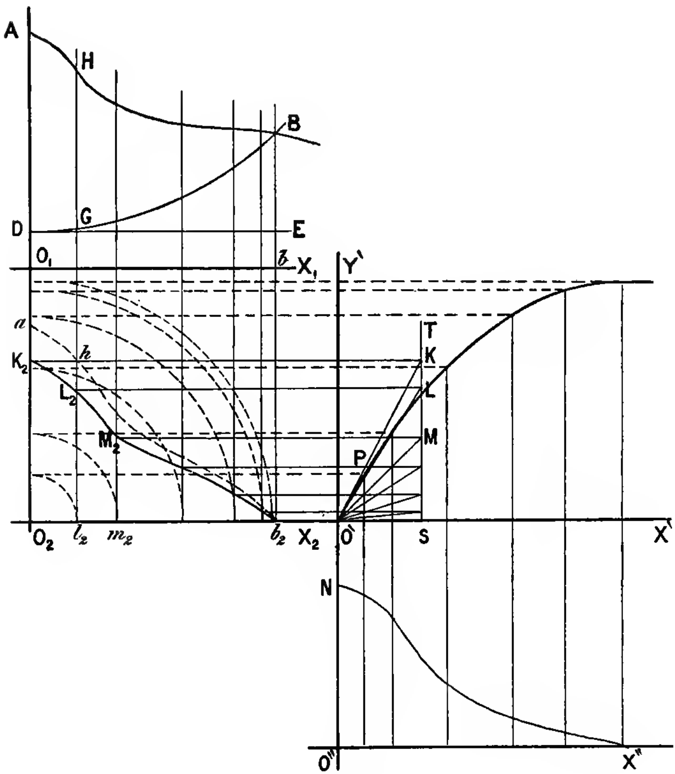

FIG. 56.

pull is plotted in, and draw a horizontal line DE. This is the curve of frictional resistances.

On DE as base plot a curve DB, whose ordinates show the corresponding values of $\cdot 0025 v^{1\cdot 7}$ to the given scale. Then it is clear that the height of the curve DB above O_1X_1 at any point shows the total resistance to the motion of the train at a constant velocity represented by the abscissa.

Now, of the total force in the drawbar pulling the train only part is required to overcome the actual constant-velocity resistance. The whole force over and above this part is employed in increasing the velocity of the train, *i.e.* in producing acceleration. This latter surplus force is clearly given by the length of the ordinates between the two curves AB and DB. At the point B this surplus vanishes; the maximum velocity is therefore given by O_1b, for here the total force in the drawbar is absorbed in overcoming constant-velocity resistances, and there is none left to increase that velocity. Transfer these lengths of ordinate between the two curves to corresponding positions on the base O_2X_2, thus $GH = l_2h$, etc. This gives a curve ahb_2, which shows the net force producing acceleration at all velocities.

Now, from this curve we can easily deduce another showing the actual value of the acceleration produced. We have from Dynamics—

$$\text{Force in tons} = \frac{\text{mass in tons} \times \text{acceleration in } \dfrac{\text{ft.}}{\text{sec.}^2}}{g}$$

where g is the acceleration due to gravity, $= 32 \dfrac{\text{ft.}}{\text{sec.}^2}$ about. Hence—

$$\text{Acceleration in } \frac{\text{ft.}}{\text{sec.}^2} = \text{force in tons} \times \frac{32}{50}$$

It is obvious that we need not consider the mass of the engine in this equation, because any force necessary to accelerate the engine does not appear in the drawbar at all, being absorbed in increasing the velocity of the engine. It is only the surplus force not absorbed in the engine itself that appears as a pull in the drawbar.

Hence if we reduce all ordinates of curve *ahb* in the ratio $\frac{32}{60}$ we shall obtain a curve $K_2L_2M_2b_2$, which gives the actual acceleration in $\frac{\text{ft.}}{\text{sec.}^2}$ corresponding to any velocity.

Now, we know that if the velocity and the acceleration were each plotted separately on the same time base, the former would be the integral of the latter (§ 16), and the problem resolves itself into the finding of the time base, *i.e.* from known simultaneous values of the velocity and acceleration to deduce a time velocity and a time acceleration curve. This we may do in the following manner: Divide the curve $K_2L_2M_2$ into small parts K_2L_2, L_2M_2, etc., such that each part is nearly straight, and draw ordinates at all these points. Take a base O'X' as shown collinear with O_2X_2. Let the time from the instant of starting be reckoned from O'. It is obvious that the point O' is on the curve of velocities. The height of the acceleration curve at this point is clearly O_2K_2 = O''N, which lines also represent the tangent of slope of the velocity curve. Take O'S to represent 1 second, and set up ST vertical. Transfer the ordinate O_2K_2 to this line as shown; then, as in § 14, O'K must be tangential to the curve of velocities at the point O'. Next consider the point L_2. Here the acceleration is l_2L_2 and the velocity O_2l_2. Hence wherever the ordinate of the time-velocity curve corresponding to the point L_2 may be, the slope of that curve at the point where that ordinate cuts it, must be the slope of the line O'L (where L is projected from L_2). Hence, as shown in § 14, the point on the time-velocity curve corresponding to L_2 must lie on the line bisecting KO'L, and since the height of the point above O'X' is given by O_2l_2, we can find the point P by projecting as shown. This gives another point on the velocity curve, and we can proceed exactly as before to find a third point. Thus the whole curve may be drawn and the acceleration curve plotted as we go along. The distance from O'' of the point where the acceleration curve cuts O''X'', gives the required time. The space passed over in this time can be found by integrating the time-velocity curve in the usual manner.

Similar problems can often be solved algebraically if we can find the equations to the curves involved, as in the following example :—

A smooth tube 15 feet long turns round a vertical axis at 2 revolutions per second, so as to be always horizontal. A smooth marble is placed in the tube at the vertical axis of rotation : find its velocity when it is swirled out at the other end.

It is clear that if a piece of string of length x were tied to the marble, the radial acceleration would be $\omega^2 x$, and the tension in the string $m\omega^2 x$, so that when the marble is free in the tube, at a distance x from the axis it has an acceleration $\omega^2 x$ along the tube. Now, if we plot values of $\omega^2 x$ along a line representing the tube we obtain a curve of accelerations, but *not a curve of time-acceleration*, so that the area of this curve does not represent the velocity, as we see to be the case from §§ 14–17 together. If, however, we could by any means transform this curve, as in § 43, to a time acceleration curve by transferring the ordinates to a time base, then we could integrate it graphically.

We are, in fact, required to integrate $\omega^2 x$ with respect to t, the time, in order to find the velocity. Hence the problem is to find some function which, when differentiated with respect to t, will produce $\omega^2 x$.

Now, we have that—

$$\omega^2 x = \frac{d^2x}{dt^2} = \frac{d\left(\frac{dx}{dt}\right)}{dt} \quad . \quad . \quad . \quad \text{(i.)}$$

$$\text{that is, } \int \omega^2 x dt = \frac{dx}{dt}$$

Consider what is the relation of the element of area $\omega^2 x dt$ to the element of area $\omega^2 x dx$, which would be the corresponding element on an x base, *i.e.* on the length of the tube.

It is clear from § 51 that—

$$\omega^2 x dx = \omega^2 x \frac{dx}{dt} dt$$

and therefore $\int \omega^2 x dx = \int \omega^2 x \frac{dx}{dt} dt$ . . . (ii.)

Now, if we multiply each side of (i.) above by $\frac{dx}{dt}$, we shall find that we are able to integrate both sides with respect to t, and thereby obtain an equation for $\frac{dx}{dt}$.

$$\text{Thus } \int \omega^2 x \frac{dx}{dt} \cdot dt = \int \frac{d^2x}{dt^2} \cdot \frac{dx}{dt} \cdot dt$$

which we see, from (ii.), becomes $\int \omega^2 x dx = \int \frac{d^2x}{dt^2} \cdot \frac{dx}{dt} \cdot dt.$

It is easy to see that the expression $\frac{d^2x}{dt^2} \cdot \frac{dx}{dt}$ is the differential coefficient with respect to t of $\frac{1}{2}\left(\frac{dx}{dt}\right)^2$, for, as in § 50, we have split the expression $\frac{d^2x}{dt^2} \cdot \frac{dx}{dt}$ into two factors, one of which $\left(\frac{d^2x}{dt^2}\right)$ is the d.c. of the other $\left(\frac{dx}{dt}\right)$.

Therefore we have—

$$\tfrac{1}{2}\omega^2 x^2 = \tfrac{1}{2}\left(\frac{dx}{dt}\right)^2$$

There is no constant required, since, as explained in § 22, the line of velocities cuts OX when $x = 0$.

$$\text{Hence } \frac{dx}{dt} = \omega x$$

The velocity is therefore—

$$4 \times \pi \times 15 = 60 \, \pi \text{ feet per second}$$

Dividing by x, we obtain—

$$\frac{1}{x} \cdot \frac{dx}{dt} = \omega$$

Integrating again with respect to t—

$$\log x = \omega t$$
$$\text{or } x = e^{\omega t} + \text{const.}$$

which constant in this case is -1, as may be seen by considering the instant of starting.

If this latter equation be differentiated twice with respect to t, the original equations (i.) will be reproduced.

A bird's-eye view of the whole of this problem may be obtained by considering that $\frac{d^2x}{dt^2} = \omega^2 x$, *i.e.* the height of the second derived of the time-distance curve is a positive constant multiple of the height of the time-distance curve itself. The only curve that satisfies this condition is $x = e^{ct}$, where c is a constant.

Examples.—(1) Given $\frac{d^2y}{dx^2} = \frac{1}{4}\left(\frac{dy}{dx}\right)^2$, find y in terms of x, (i.) graphically and (ii.) analytically. (For the latter, divide both sides of the equation by $\frac{dy}{dx}$ and integrate. Obtain resulting equation for $\frac{dy}{dx}$ in the e form. Invert it, and integrate with respect to y.) *Ans.* $x = -4e^{\left(-\frac{y}{4}\right)}$.

(2) Solve the tube problem graphically.

§ 62. Rectification of Curves.

It is sometimes very useful to be able to find the length of curves.

We saw in the last chapter, § 57, that we might have a curve plotted on the x base showing in its ordinate the length of the curve measured from a fixed point on it. The tangent of slope of this curve will clearly be the secant of the angle of slope of the original curve.

The student will understand this without difficulty if he works the following exercise. Draw a curve (a) of any shape, and take a point P on it; make another curve (b) on an x base, by measurement from A, whose ordinates represent the corresponding length of curve (a) measured from P. Differentiate it graphically, and show that the derived curve so obtained is the same as would have been obtained by plotting values of the secant of the angle of slope of (a) on an x base. Thus it is clear that the integrated curve of a curve showing the values of the secant of the angle of slope of the original curve is a curve showing the *length* of the original curve.

Now, if ϕ be the angle of slope of the original curve, we have, from trigonometry—

$$\sec^2 \phi = 1 + \tan^2 \phi.$$

$$\text{that is, } \sec \phi = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Sec ϕ we have called $\frac{ds}{dx}$. Hence—

$$\text{length of curve} = S = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

the limits being taken as required.

Thus, find the length of wire rope required to hang between two pillars 120 yards apart, assuming the curve of the rope is given by the equation—

$$y = \frac{m}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right)$$

This is the actual curve in which a rope hangs, and is called the "catenary curve." The axis of x is a horizontal line at a depth m below the lowest point of the curve; m also represents the length of the same kind of rope which weighs as much as the tension at the lowest point of the rope.

N

Assume $m = 100$.

The sag of the rope is then $= y - 100$, where $y =$ greatest ordinate.

$$\text{Sag of rope} = 50\left(e^{\frac{60}{100}} + e^{-\frac{60}{100}}\right) - 100$$
$$= 50\left(1{\cdot}82 + \frac{1}{1{\cdot}82}\right) - 100$$
$$= 50 \times 0{\cdot}37 = 18{\cdot}5 \text{ yards}$$

$$\text{Now } \frac{dy}{dx} = \tfrac{1}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right)$$
$$\frac{ds}{dx} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = \tfrac{1}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right)$$

Therefore the whole length, being twice the half-length, is—

$$2 \times \tfrac{1}{2}\int_0^{60}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right)dx = \left\{m\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right)\right\}_0^{60}$$

since s is measured from the vertex,

$$= 100\left(1{\cdot}82 - \frac{1}{1{\cdot}82}\right)$$
$$= 127 \text{ yards nearly}$$

Example.—Plot a catenary from the given equation.[1] Draw a curve on an x base representing the secant of its angle of slope, and obtain the above result by integrating this curve.

[1] This may best be done by first obtaining the curve $y = e^{\frac{x}{m}}$ (i.) by the method described on pp. 102, 103, *i.e.* find geometrically the lengths of a series of equidistant ordinates in geometrical progression, *i.e.* with a constant common ratio which must be calculated from the equation. The curve $y = e^{-\frac{x}{m}}$ (ii.) can then be found by combining this curve with the curve $y = \frac{1}{x}$ by the method of Figs. 40, 43. These two curves (i.) and (ii.) should then be added together (Fig. 26), and the result divided by $\frac{m}{z}$.

§ 63. Centres of Gravity.

The finding of centres of gravity and moments of inertia is an application of the integral calculus of very great service to engineers. The principle of these methods is exactly the same as that adopted in elementary mechanics, but the proofs are very much simplified by the application of the calculus. The proposition relied on in all these methods is that the resultant of a system of forces acting on a body has the same tendency to twist it about any arbitrary point or line as the sum of the twisting tendencies of each of the forces taken separately. This proposition, as applied to the special purpose before us, is embodied in the following rule. Find the mass of each of the separate parts of the object, and multiply each by the algebraical distance (+ or −) of the centre of gravity of that part from any convenient straight line. Add all these products together, and divide the sum by the sum of the masses. The result is the distance of the centre of gravity from the line.[1]

The following graphical process embodies this rule, and applies the principles already explained.

Let it be required to find the centre of gravity of a piece of plate cut into the shape of a curve of sines between the ordinates $x = 0$ and $x = 2$ (Fig. 58). It will be seen that, for the graphical method, any arbitrary curve of any shape whatever might be used; but as we shall also give the corresponding analytical process, it will be convenient to consider a curve of which the equation is known.

Draw a number of ordinates to the curve at convenient and well-defined distances from O, such as 0·2 inch, 0·4 inch, 0·6 inch, etc. Measure the length of each ordinate, such as rR, with a decimal scale, and multiply it by the scale length of the corresponding abscissa Or.

[1] A very lucid explanation of this and similar propositions will be found in Professor Goodman's treatise on "Applied Mechanics."

There is no difficulty in devising a method whereby this may be done graphically, if desired. Thus to multiply together the lines AB, AC (Fig. 57), complete the rectangle, and mark off DE = 1 inch, and complete as shown; then AF represents the product required. The actual ordinate, however, may be obtained much more rapidly and accurately with a slide rule, an instrument with which every engineer or scientist should be familiar.

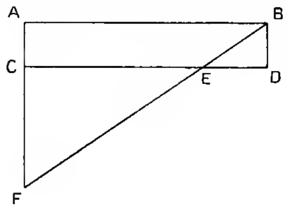

Fig. 57.

Plot the value thus found along the same ordinate as at rR_1 and carefully draw a curve, OR_1P_1, through all the points so obtained. Now, the area of this curve will be the moment of the whole area about O in inch units to the same scale as the area of ORQ represents the actual area. That is to say, suppose OX, OY are both held horizontally, then the tendency in inch units which the force exerted by gravity on the plate

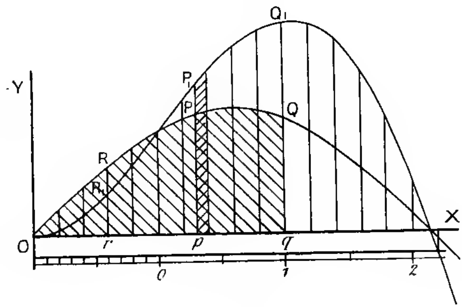

Fig. 58.

ORQ has to twist the plate about the line OY (supposed held fixed) is represented by the area of $OR_1P_1Q_1Y$ in square inches, to the same scale as the area of ORQ represents the weight of the plate.

Thus if the area of ORQ is = (a) sq. inches, and OR_1Q_1 = (b) sq. inches, and weight of the plate = 2 ounces; then, since (a) sq. inches represents 2 ounces, 1 sq. inch $= \frac{2}{a}$ ounces, and the moment of the weight about OY is therefore $b \times \frac{2}{a}$ ounce-inch units. For consider an element of area pP_1 of breadth dx. It is clear that the moment of the element pP about OY is $\mu \times pP \times dx \times Op$, where μ represents the mass of a square inch of the plate. Now, $pP \times Op = pP_1$, and $pP_1 \times dx =$ area of element pP_1; hence the area of the element $pP_1 \times \mu =$ moment of element pP about OY.

Now, the quantity μ, as we have seen, $= \frac{\text{weight of plate}}{\text{area of plate}}$ $= \frac{2}{a} \cdot \frac{\text{oz.}}{\text{in.}^2}$ in above example. Hence, to the same scale as the area of pP represents the weight of the corresponding strip of plate, pP_1 represents the moment of that strip about OY. The same may be said of all corresponding strips, and is therefore true of them all taken together.

Now, let $\overline{X}$ be the distance of centre of gravity from OY.

Then we have area of $OR_1P_1Q_1 =$ area of OPQ $\times \overline{X}$.

$$\text{Hence } \overline{X} = \frac{\text{area of } OR_1P_1Q_1}{\text{area OPQ}}$$

Integrate both the curves graphically, as already explained, or find their areas by the planimeter or otherwise; set off a line representing the area of the curve OP_1Q_1 vertically, as at CB (Fig. 6); set off a line representing the area OPQ to the same scale horizontally, as at AC. Join AB, make AM = 1 inch, and draw MP vertical; then MP represents the distance of the centre of gravity from OY.

The process must be repeated with OX vertical to get the distance of the centre of gravity from OX.

We thus get two intersecting lines, each of which contains the centre of gravity, which point is therefore found at the point of intersection of the lines.

The student will now have no difficulty in understanding the following process, which is inserted without explanation; as it is the same step by step as the graphical process just described.

$$\bar{X} = \frac{\int_0^2 x \sin x dx}{\int_0^2 \sin x dx}$$

$$= \frac{{}_0^2[\sin x - x \cos x]}{{}_0^2[-\cos x]} \quad \text{(see § 53)}$$

$$= \frac{\sin 2 - 2 \cos 2 - 0}{-\cos 2 + \cos 0}$$

$$= \frac{\sin 114{\cdot}59^\circ - 2 \cos 114{\cdot}59^\circ}{1 - \cos 114{\cdot}59^\circ}$$

$$= \frac{\sin 65{\cdot}41^\circ + 2 \cos 65{\cdot}41^\circ}{1 + \cos 65{\cdot}41}$$

$$= \frac{\sin 65^\circ 24' + 2 \cos 65^\circ 24'}{1 + \cos 65^\circ 24'} \text{ nearly}$$

$$= \frac{0{\cdot}90921 + 0{\cdot}83252}{1{\cdot}41626}$$

$$= 1{\cdot}23'' \text{ nearly}$$

Examples.—Find the centre of gravity of a triangle, a cone, a frustum of a cone, an arc of a circle, a slice of a sphere, a rod whose density varies as the *n*th power of its distance from one end, any section of a parabolic plate, a theoretical indicator card (no compression).

§ 64. Moments of Inertia.

Moments of inertia may be found in a similar way to that employed for centres of gravity. The moment of inertia of an area about an axis in its plane is analogous to what we have already described to be the moment of an area about an axis; but whereas each ordinate in Fig. 57 = ordinate of area × distance of ordinate from axis of Y, each ordinate in the corresponding Fig. 59 for the moment of inertia = ordinate $S_1S_2 \times (Os)^2$.[1]

Let PQRS be any area of which it is desired to find the moment of inertia about the axis OY. Set up ordinates

[1] There is a good deal of confusion in the minds of students as to the exact connection between the moment of inertia of an area about a line in its plane, and what is called the moment of inertia of a solid body imagined spinning about an axis. In the discussion on centres of gravity, the same point arose in connection with the relation between the geometrical first moment of an area about a line and its mechanical analogue. Without going fully into a question which has more to do with rigid dynamics than calculus, we may point out that a clear conception of the meaning of the moment of inertia of a body may be obtained by considering it as the *angular mass* of the body and a couple as angular force. The meaning of this will be clear from the following analogy. If a force acts on a mass perfectly free to move—

Force in poundals = mass in pounds × acceleration in feet per second per second

Similarly, if a couple acts on a mass perfectly free to turn round—

Couple in ft.-poundals = moment of inertia × angular acceleration in radians per second per second

or angular force = angular mass × angular acceleration

Again—

Momentum = mass × velocity

angular momentum = moment of inertia × angular velocity

$$\text{kinetic energy} = \begin{cases} \tfrac{1}{2}\ \text{mass} \times (\text{velocity})^2 \\ \tfrac{1}{2}\ \text{I} \times (\text{angular velocity})^2 \end{cases}$$

A conception of the magnitude of unit moment of inertia in foot and pound units may be derived from the consideration that if a body has unit moment of inertia round an axis, and is rotating at unit angular velocity, it will do $\frac{1}{2}$ ft.-poundal of work before being brought to rest. The connection between the geometrical moment of inertia and the mechanical one consists simply in the introduction of the factor ρ, or the mass per unit area, into the expression for the element. The mechanical significance of this is easily seen from the above remarks. The geometrical moment of inertia is, as it were, a skeleton which we may endow with life either by multiplying by pounds mass per square inch; it then comes in for calculating kinetic energies, etc.; or if used for such purposes as the determination of stresses in beams, we multiply it by pounds weight per square inch (tension or compression), and in other ways which need not be here mentioned.

parallel to the given axis, as in § 63, and multiply the length S_1S_2 of each by the square of its distance from the given axis $(Os)^2$, and set up the length so obtained on each of the ordinates; thus $sS = S_1S_2 \times OS^2$. The area of the curve so obtained, found in any manner, is the moment of inertia of the area about that line in inch units. The proof of this is almost identical with that given for the similar method used in connection with finding centres of gravity in § 63, and may easily be completed by the student himself.

Just as before, if $y = f(x)$ be the equation to the boundary line of the curve, all we have to do is to find the value of $\int_b^a yx^2dx$, which gives us at once the value of the moment of

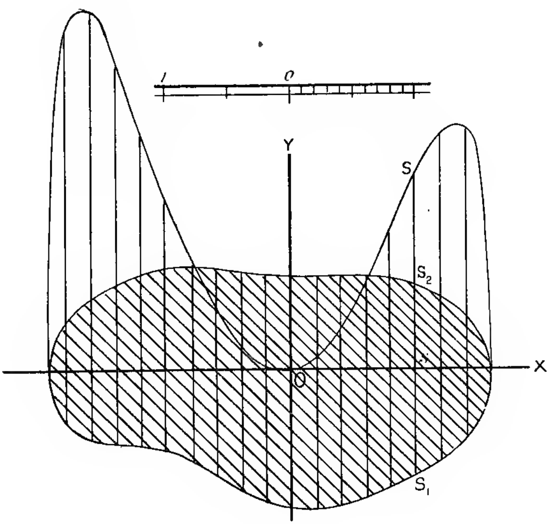

Fig. 59.

inertia. The value of this area or definite integral, divided by the greatest distance from the axis of the boundary of the figure, gives us what is known as the modulus of the section.

It may be found graphically, in the manner explained in § 63 as the length of a line. Also the value of this area or integral divided by the area of the figure gives the value of the square of the radius of gyration.

Thus, required the dynamical moment of inertia of a flat circular plate of mass m lbs., whose radius $= r$, about an axis passing through its centre perpendicular to its plane. Consider a circular element of the plate, radius x, breadth dx;

$$\text{Its area is } 2\pi x dx$$

$$\text{its mass is } 2\pi x dx \times \mu = 2\pi x dx \times \frac{m}{\pi r^2}$$

where μ is mass of unit area of plate—

$$\text{its moment of inertia about O obviously} = \frac{2mx^3dx}{r^2}$$

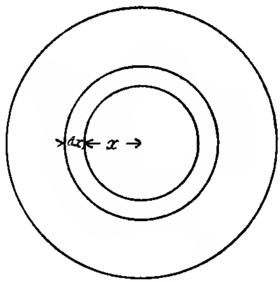

FIG. 60.

Hence, whole moment of inertia—

$$= \frac{2m}{r^2}\int_0^r x^3dx$$

$$= \frac{2m}{r^2}\left(\tfrac{1}{4}x^4\right)_0^r$$

$$= \tfrac{1}{2}mr^2$$

The geometrical moment of inertia is $\frac{1}{2}ar^2$ or $\frac{\pi}{2}r^4$, which, multiplying by the mass per unit area, becomes $\frac{1}{2}mr^2$ as above.

Consider the case of a cylindrical shaft under torsional stress. Suppose f is the shearing stress per square inch at a distance = 1 inch from the axis. Clearly the stress at any distance x from the centre is fx. Since the stress varies as the strain, and the strain varies as the distance from the axis.

Area of an elementary annular ring of radius $x \times$ breadth dx

$$= 2\pi x \times dx$$

total stress on this layer $2\pi x dx \times fx = 2\pi f x^2 dx$

moment of this stress about axis $= 2\pi f x^2 dx \times x = 2\pi f x^3 dx$

Plot values of this along the horizontal radius, and find the area of the curve so obtained, and compare result with the result of the integral of $2\pi f x^3 dx$ between limits r and o.

EXAMPLES.

1. Find the moment of inertia of a fly-wheel, outside diameter 6 feet, sectional area of rim 4×5, inside diameter of rim 5 feet 4 inches, six arms each of sectional area an ellipse of axes $3\frac{1}{2}$ and 2. Boss, a cylinder 8 inches diameter $\times$ 8 inches long, with a 3-inch hole for the shaft. Mass of cubic inch of iron, 0·26 lb. $= \rho$. (*Method.*—Plot a curve on the horizontal radius of the wheel as base, showing the value of $\rho a x^2$, where a is the area of metal cut through by an imaginary cylinder of radius x, concentric and coaxal with the wheel.) Find the scale on which the area of this curve represents the moment of inertia. Find the radius of gyration by a graphical process (find the weight of the wheel graphically as the area of a curve, showing the values of ρa). Then—

$$I = MR^2$$

$$R^2 = \frac{I}{M}$$

$$R = \sqrt{\frac{I}{M}}$$

Find this graphically by the process of Fig. 3.

2. Find the moment of inertia, by graphical method, of a rectangular section, a box section, a triangular section, and a circular section about axes in their planes and passing through their centres of gravity, and compare your results with that given by the formula nah^2—

where $n = \frac{1}{12}$ for rectangular section.
$\frac{1}{18}$ for triangular section.
$\frac{1}{16}$ for circular section.
a = area of section.
h = height in plane of bending.

APPENDIX.

BARKER'S PLANIMETER.

THIS instrument was devised by the author for the purpose of mechanically drawing the integral curves, on the principle explained in § 14. It is here described for the first time. It consists of a horizontal slide AB, carrying a slider DF, to which is rigidly attached the vertical slide CE, which is fitted accurately perpendicular to AB. The vertical slide CE carries a long slider GH, to which is fitted the tracer P, and to which is pivoted at L the rod KL, which slides through the piece M. M is itself pivoted on a clamp as shown, and the clamp can be secured to any part of the piece D, which is graduated. By means of a double parallelogram of jointed rods, KL is kept parallel to the piece N, one point of which is pivoted on a slider Q, on which a vernier is engraved; at the same time Q is allowed to assume any position on the vertical slide. This piece N carries a wheel with points on its periphery, and the bearings of the wheel are so attached to the piece N that its plane is always parallel to the axis of KL. It is clear that when the pointer P traces out any curve, the wheel will roll out its integral, for the tangent of angle of slope of the upper curve is clearly proportional to the ordinate of the curve traced out by P. The adjustments required are, in the case of a curve on a base, that the instrument must be so placed that AB is parallel to OX, the given base; also KL must be parallel to AB when the tracer P is on the base OX. However, it may be used for finding areas independently of this adjustment, for if the pointer P be placed on the curve whose area is required—such as an indicator diagram—and the reading of Q taken, and the tracer be then carried round the curve to the starting-point, and Q read again, the difference of the readings gives the area required, in units which depend on the position of M on D. This may be adjusted to read in any units within the working limits of the instruments.

The instrument may also be used for differentiating a curve, by tracing out a curve with the wheel ; but it is rather difficult to hold the wheel sufficiently steady.

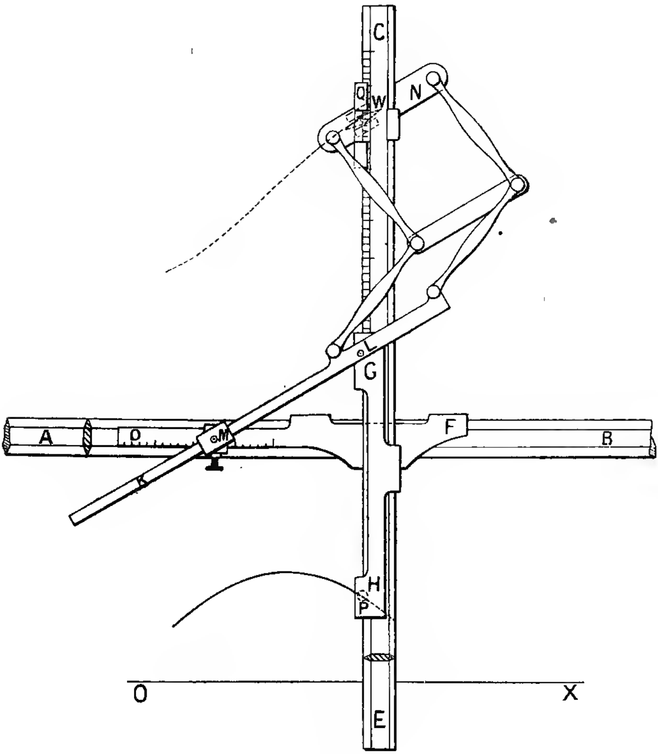

Fig. 61.

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