A. S. Troelstra
Lectures on Linear Logic.
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Linear logic, the invention of Jean-Yves Girard, is a "resource-conscious" logic in which the formulas represent types of resource, and resources cannot be used ad libitum. It can be thought of as a logic of actions, where classical logic is a logic of truth. It seems to have some promise as a way of representing and understanding parallel computations.

In classical logic, the lack of consciousness of resources manifests itself in the fact that an assumption in a proof, once introduced, can be re-used indefinitely many times. In proof-theoretical terms, this corresponds to Gentzen's structural rule of contraction. In linear logic, assumptions can be re-ordered freely, but not re-used indefinitely; one may think of assumptions as resources that once used are destroyed or unavailable.

If we start from a sequent system for classical logic in the style of Gentzen with introduction and elimination rules for each connective and omit the rule of contraction, the result is an interesting and important subsystem of linear logic. Conjunction and disjunction bifurcate into four different connectives. Conjunction, for example, becomes two connectives, a parallel conjunction and an additive conjunction. A similar phenomenon occurs in relevant logics, where 'and' comes in both intensional and extensional flavours. Girard distinguishes between 'multiplicative' and 'additive' operators; this corresponds roughly to the distinction drawn by Anderson and Belnap between 'intensional' and 'extensional' operators. The resulting calculus is elegant and possesses many appealing symmetries. For a reader familiar with relevant logics, it can be described as the result of omitting contraction and distribution from the logic **R** of relevant implication.

The omission of the contraction rule, though, results in a logic lacking in sufficient expressive power. In compensation, Girard introduced special connectives restoring the possibility of re-use of resources. These operators, somewhat resembling modal operators in their behaviour, are one-place sentence operators $!A$ and $?A$. The formula '$!A$', read as 'store A', is a formula allowing the indefinite re-use of the assumption A; the operator $?A$ is the dual of $!A$. The addition of these two connectives, called 'exponentials' by Girard, results in a richly expressive logic in which both classical and intuitionistic logic can be exactly embedded.

Troelstra's lectures, written in the lucid style familiar from his earlier books and papers, is the best available introduction to the rapidly expanding field of resource-conscious logics. In the first part of the book, Troelstra introduces the basic sequent calculus for linear logic, gives some elementary syntactic result, including cut elimination, then proves the embedding results for classical and intuitionistic logic mentioned above. Accounts of natural deduction systems and Hilbert-style axiomatic systems follow.

The second part of the book is devoted to semantical interpretations of the system and of its computational aspects. Troelstra begins with algebraic semantics; linear logic is complete with respect to certain algebraic models defined on complete lattices. There follows a chapter describing the close connection with certain categories. Next Troelstra explains what are perhaps the most interesting of the models for linear logic, the type-theoretic models he calls 'Girard domains.' Here propositions are interpreted as 'webs', sets with a reflexive symmetric relation representing compatibility between atomic bits of information. The connectives are certain natural operations on these webs; the approach is closely related to the Scott domains used in denotational semantics of programming languages.

The great interest aroused by linear logic is in part due to its apparent promise as a tool for analysing computations. Troelstra devotes four chapters to computational aspects of the formalism. The 'propositions as types' idea, due to Curry and Howard, provides the link to computation. Following this idea, we can think of an implication $A \rightarrow B$ (for example) as a set of constructions, each of which maps a construction in A into a construction in B. A proof of a logically valid formula will then be a closed term in a certain language of functions. Troelstra discusses the evaluation of terms in this language, also a kind of abstract machine in which the details of this evaluation can be implemented.

The book concludes with some proof-theoretical material. Proof-nets, a way of analysing proofs in linear logic in terms of paths through the derivations, are described. A short and elegant proof of the undecidability of propositional linear logic is given; a clever coding is used to mimic the actions of an abstract computer in the derivations of linear logic. The last chapter, written by Dirk Roorda, is a proof of strong normalization for linear logic.

Linear logic since its first publication by Girard in 1987 has aroused a considerable degree of enthusiasm among many logicians and category theorists. The response among computer scientists has been mixed. Many European computer scientists have embraced it with fervour, but the research community in North America has not shown comparable enthusiasm, with the exception of people strongly influenced by category-theoretical methods. The literature on the subject is already diverse and expanding rapidly. Troelstra's monograph is clearly written and an ideal introduction to the field for logicians, philosophers or computer scientists anxious to know what all the fuss is about.

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