ANALYTICAL GEOMETRY

FOR

BEGINNERS

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ANALYTICAL GEOMETRY

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BY

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PREFACE.

The following pages embrace, in the main, the substance of lectures which for some years past I have been giving to students of applied science. Fragments of this work have also been given to students to whom a general knowledge of the principles of Analytical Geometry was part of a liberal education.

It is important that the beginner should not think the terms "Analytical Geometry" and "Conic Sections" are synonymous. Analytical Geometry is the application of Analysis, or algebra, to Geometry, the principal quantities involved in the equations having reference to and receiving their meaning from certain lines known as axes of co-ordinates, or their equivalents. The principles of Analytical Geometry are developed in the first two chapters of this book. It is usual to illustrate these principles by applying them to the straight line, and to obtaining the properties of the simplest yet most important curves with which we are acquainted,—the Conic Sections. Hence the remainder of the book is occupied in applying the principles and methods of Analytical Geometry to the straight line, circle, parabola, etc.

Throughout the effort has been to limit the size of the book, while omitting nothing that seemed essential. Many important properties of the Conics are given as exercises, the solutions being made simple by the results of previous exercises, as well as by hints and suggestions. These hints and suggestions will be found of very frequent occurrence in the exercises; they seem necessary to students beginning a subject, with whom the usual question is, "How shall I start the problem?" In addition it seems wise to make the exercises easy by offering suggestions, rather than to make them easy through their being mere repetitions of the same problem.

Several of the articles in the chapters on the parabola and ellipse will be found to be almost verbatim copies of the corresponding articles in the chapter on the circle, the object being to impress on the student the essential uniformity of the methods employed.

I take the liberty of suggesting that institutions where the conics are studied should be provided with accurately-constructed metal discs for drawing the curves. A large part of the beauty and attractiveness of the subject is lost when figures are rudely and carelessly represented. The majority of students can best realize and be made to feel an interest in the analytical demonstration of a proposition, when it has been preceded or followed by an instrumental proof of the probability of its truth.

A. B.

UNIVERSITY OF TORONTO,
December, 1904.

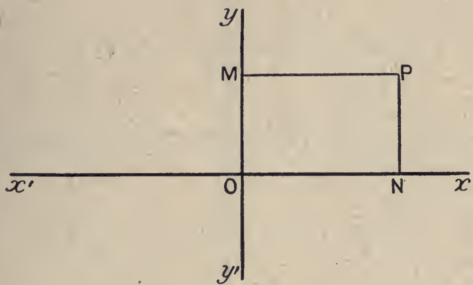
CONTENTS.

ANALYTICAL GEOMETRY.

CHAPTER I.

POSITION OF A POINT IN A PLANE. CO-ORDINATES.

1. On a sheet of paper draw two lines xOx', yOy', intersecting at O. On Ox measure ON of length 23 millimetres; and through N draw NP, parallel to Oy, and of length 16 millimetres. We arrive evidently,



in this way, at a definite point P, i.e., *definite so far as its position with respect to the lines* xOx', yOy' *is concerned.*

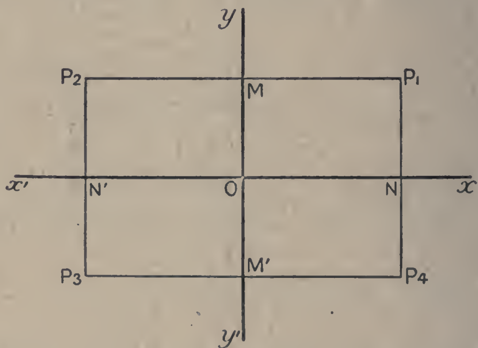
Again, we reach the same point P, if we take on Ox, $ON = 23$ millimetres, on Oy, $OM = 16$ millimetres, and through N and M draw NP, MP, parallel to Oy, Ox respectively, intersecting in P.

We have thus a means of representing the position of a point in a plane, with reference to two intersecting lines in that plane.

Draw two lines xOx', yOy', intersecting at any angle, and locate with reference to them, as above, the point P in the following cases:

$ON = \frac{3}{4}$ in., $NP = 1\frac{1}{2}$ in.; $ON = 48$ mm., $NP = 35$ mm.; $ON = 0$, $NP = 1\frac{3}{4}$ in.; $ON = 27$ mm., $NP = 0$; $ON = 0$, $NP = 0$; $ON = 1\frac{1}{4}$ in., $NP = 1\frac{7}{16}$ in.; etc.

2. Each of the lines xOx', yOy', however, has two sides; and if we are told only the distances of a point from each of these lines, in direction parallel to the other, then the point may occupy any one of four different positions, P_1, P_2, P_3, P_4, as illustrated in the following figure:



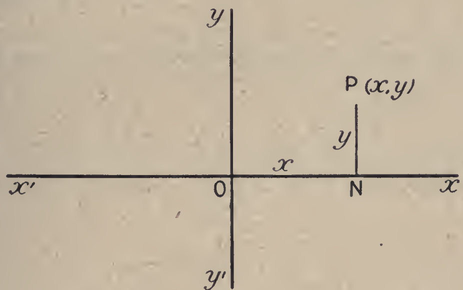
To get rid of this ambiguity the signs + and − are introduced to indicate contrariety of direction. Thus lines measured to the right, in the direction Ox, are considered *positive*, and lines measured in the opposite direction Ox' are considered *negative*; lines measured upwards, in the direction Oy, are considered *positive*, and lines measured in the opposite direction Oy' are considered *negative*.

Hence for P_1, ON is $+ve$, and $NP_1 + ve$; for P_2, ON' is $-ve$, and $N'P_2 + ve$; for P_3, ON' is $-ve$, and $N'P_3 - ve$; for P_4, ON is $+ve$, and $NP_4 - ve$.

Draw two lines xOx', yOy', intersecting at any angle, and locate with reference to them the point P in the following cases, PN being the line from P to xOx', parallel to yOy':

$ON = 47$ mm., $NP = -23$ mm.; $ON = -2$ in., $NP = 1\frac{1}{4}$ in.; $ON = 0$, $NP = -1\frac{3}{4}$ in.; $ON = -52$ mm., $NP = -63$ mm.; $ON = -\frac{3}{4}$ in., $NP = 0$; $ON = -\frac{1}{2}$ in., $NP = -1\frac{1}{2}$ in., etc.

3. The line xOx' is called the **axis of** x, and yOy' the **axis of** y; together these lines are called the **axes of co-ordinates.**



When xOx', yOy' are at right angles to each other, they are spoken of as **rectangular axes**; when not at right angles, as **oblique axes.** Throughout the following pages the axes will be supposed rectangular unless the contrary is indicated.

The point O is called the **origin.**

The length ON is called the **abscissa** of the point P, and is generally denoted by x; the length NP is called the **ordinate** of P, and is generally denoted by y. Together x and y are called the **co-ordinates** of P.

The point P is indicated by the form (x, y), the abscissa being written first. Thus $(-3, 2)$ means the point reached by measuring 3 units along Ox', and 2 units upwards, in direction parallel to Oy.

The preceding method of representing the position of a point in a plane, with reference to two axes, is known as the method of **Cartesian co-ordinates.**

Exercises.

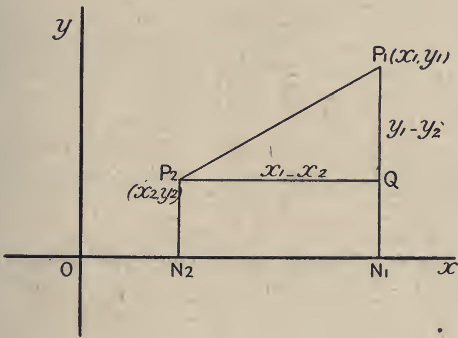
1. Draw two axes of co-ordinates at right angles to each other, and locate the following points, the unit of length being a centimetre:

$$(2, -3); (5, 1); (-4, -5); (0, -4); (3, 0); (-3, 1).$$

2. In the preceding question the origin and axis of x remaining unchanged, but the angle between the axes, *i.e.*, the positive directions of the axes, being 60°, find the new co-ordinates of the above points already placed.

3. Keeping to the origin and axes of Exercise 2, place the points $(2, -3)$; $(5, 1)$; $(-4, -5)$; $(0, -4)$; $(3, 0)$; $(-3, 1)$.

4. To express the distance between two points in terms of the co-ordinates of the points.



Let P_1 (x_1, y_1) and P_2 (x_2, y_2) be the two points. Draw P_2Q parallel to Ox.

Then $ON_1 = x_1$, $N_1P_1 = y_1$, $ON_2 = x_2$, $N_2P_2 = y_2$.

$$\therefore\ P_2Q = N_2N_1 = x_1 - x_2;$$

$$QP_1 = P_1N_1 - QN_1 = P_1N_1 - P_2N_2 = y_1 - y_2.$$

Hence $P_1P_2^2 = P_2Q^2 + QP_1^2,$

$$= (x_1 - x_2)^2 + (y_1 - y_2)^2;$$

or $P_1P_2 = \sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}}.$

Exercises.

1. Find the distance between the points (1, 2) and (– 3, – 1).

2. Show that the points (2, – 2), (– 2, 2), ($2\sqrt{3}$, $2\sqrt{3}$) form the angular points of an equilateral triangle, as do also the points (2, – 2), (– 2, 2), (– $2\sqrt{3}$, – $2\sqrt{3}$).

3. Two points (4, 0) and (0, 4) being given, find two other points which with these are the angular points of two equilateral triangles.

4. If the point (x, y) be equidistant from the points (4, – 5), (– 3, 2), then are x and y connected by the relation $x - y = 2$.

5. Show that the points (3, 1), (0, – 3), (– 4, 0), (– 1, 4) form the angular points of a square. [Prove that sides are equal, and also diagonals.]

6. Show that the points (5, 3), (6, 0), (0, – 2), (– 1, 1) are the angular points of a rectangle. [Prove that opposite sides are equal, and also diagonals.]

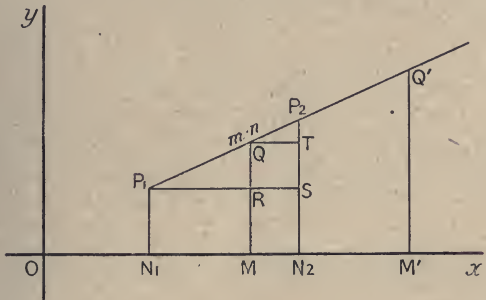
7. Show that the points (5, 1), (2, – 2), (0, – 1), (3, 2) form the angular points of a parallelogram. [Prove that opposite sides are equal, and diagonals unequal.]

8. Express by an equation the condition that the point (x, y) is at a distance 3 from the point (– 1, 2).

9. A line whose length is 13 has one end at the point (8, 3), and the other end at a point whose abscissa is – 4. What is the ordinate of this end?

10. Find the distance between two points $P'(x', y')$ and $P''(x'', y'')$, the axes being oblique and inclined to one another at an angle ω.

5. To find the co-ordinates of a point which divides the straight line joining two given points in a given ratio.



Let Q (x, y) be a point dividing the straight line joining the points P_1 (x_1, y_1) and P_2 (x_2, y_2) in the ratio $m : n$. Complete the figure as in the diagram.

$$\text{Then } m : n = P_1Q : QP_2 = P_1R : RS = N_1M : MN_2,$$
$$= x - x_1 : x_2 - x;$$
$$\therefore m\ (x_2 - x) = n\ (x - x_1);$$
$$\text{or } x = \frac{mx_2 + nx_1}{m+n}.$$

$$\text{Also, } m : n = P_1Q : QP_2 = ST : TP_2 = RQ : TP_2,$$
$$= y - y_1 : y_2 - y;$$
$$\therefore m\ (y_2 - y) = n\ (y - y_1);$$
$$\text{or } y = \frac{my_2 + ny_1}{m+n}.$$

COR. If Q bisect the line P_1P_2, then $x = \frac{1}{2}\ (x_1 + x_2)$, $y = \frac{1}{2}\ (y_1 + y_2)$.

If P_1P_2 be divided externally in Q' (x, y), so that $P_1Q' : P_2Q' = m : n$, then

$$m : n = P_1Q' : P_2Q' = N_1M' : N_2M' = x - x_1 : x - x_2;$$

$$\therefore m(x - x_2) = n(x - x_1);$$

$$\text{or } = \frac{mx_2 - nx_1}{m - n}.$$

$$\text{Similarly, } y = \frac{my_2 - ny_1}{m - n}.$$

The co-ordinates of Q' may also be obtained as follows:

$$\text{Since } P_1Q' : P_2Q' = m : n;$$

$$\therefore P_1Q' : Q'P_2 = m : -n;$$

i.e., P_1P_2 is divided at Q' in the ratio $-\frac{m}{n}\cdot$ Hence, substituting $-\frac{m}{n}$ for $\frac{m}{n}$ in the expressions for the co-ordinates of Q, the co-ordinates of Q' are obtained.

Exercises.

1. Find the co-ordinates of the middle points of the sides of the triangle whose angular points are $(2, -3)$, $(3, 1)$, $(-4, 2)$.

2. A straight line joins the points $(3, 4)$ and $(5, -2)$; find the co-ordinates of the points which divide the line into three equal parts.

3. Two points, $P_1 (x_1, y_1)$ and $P_2 (x_2, y_2)$, are joined; find the co-ordinates of the $n - 1$ points which divide $P_1 P_2$ into n equal parts.

4. From a figure, without using the formulas, find the point which divides the line joining $(2, 1)$, $(-4, -5)$ in the ratio $3 : 4$. Verify your result from formulas.

5. From a figure, and also from the formulas, find the points which divide the line joining $(3, 1)$ and $(7, 4)$, internally and externally, in the ratio $3 : 2$.

6. Show by analytical geometry that the figure formed by joining the middle points of the sides of any quadrilateral is a parallelogram.

[Suppose the angular points of the quadrilateral are (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4). Then the middle points are $\{ \frac{1}{2}(x_1+x_2), \frac{1}{2}(y_1+y_2) \}$, etc.]

7. Prove analytically that the straight lines which join the middle points of the opposite sides of any quadrilateral bisect each other. [The co-ordinates of the middle point of the line joining the middle points of one pair of opposite sides are $\frac{1}{2}\{ \frac{1}{2}(x_1+x_2)+\frac{1}{2}(x_3+x_4) \}$, $\frac{1}{2}\{ \frac{1}{2}(y_1+y_2)+\frac{1}{2}(y_3+y_4) \}$; etc.]

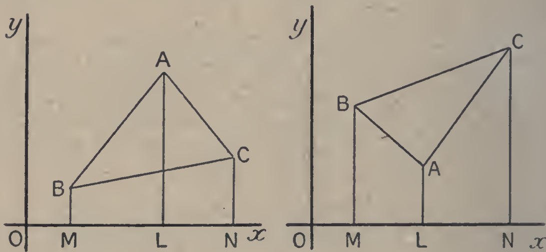
8. Prove analytically that the straight line joining the middle points of two sides of a triangle, is half the third side.

9. Prove analytically that the middle point of the hypotenuse of a right-angled triangle is equidistant from the three angles. [Take the sides of the triangle in the co-ordinate axes, so that the right angle is at the origin.]

10. If the angular points of a triangle ABC be (x_1, y_1), (x_2, y_2), (x_3, y_3), show that the co-ordinates of the point dividing in the ratio $2:1$ the line joining A to the middle point of BC are $\frac{1}{3}(x_1+x_2+x_3)$, $\frac{1}{3}(y_1+y_2+y_3)$.

Hence show analytically that the medians of any triangle pass through the same point.

6. To express the area of a triangle in terms cf the co-ordinates of its angular points.



Let the angular points of the triangle, A, B, C, be (x_1, y_1), (x_2, y_2), (x_3, y_3) respectively. Complete the figure by drawing the ordinates of the angular points.

Then $\pm \triangle ABC = BMLA + ALNC - BMNC.$

But $BMLA = \frac{1}{2}(BM + AL)\, ML = \frac{1}{2}(y_2 + y_1)(x_1 - x_2).$

Similarly $ALNC = \frac{1}{2}(y_1 + y_3)(x_3 - x_1)$;

and $BMNC = \frac{1}{2}(y_2 + y_3)(x_3 - x_2).$

$$\text{Hence } \pm \triangle ABC = \tfrac{1}{2}\left\{(y_2 + y_1)(x_1 - x_2) + (y_1 + y_3)(x_3 - x_1) - (y_2 + y_3)(x_3 - x_2)\right\},$$

$$= \tfrac{1}{2}\left\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right\},$$

that sign being selected which makes the expression for the area a positive quantity.

Exercises.

1. Find the area of the triangle whose angular points are (4, 3), (−2, 1), (−5, −6).

2. Find the area of the triangle whose angular points are (−4, −5), (−3, 6), (1, −1).

3. Verify the formula of § 6 when A is in the second quadrant, B in the third, and C in the fourth. [Here x_1, x_2, y_2, y_3 are intrinsically negative quantities, and therefore $-x_1$, $-x_2$, $-y_2$, $-y_3$ are positive quantities. Thus $y_1 - y_2$ is the height of A above B measured parallel to the axis of y; and $x_3 - x_2$ is the distance between B and C measured parallel to the axis of x.]

4. Without using the formula, but employing the method of § 6, find the area of the triangle whose angular points are (1, 3), (4, 6), (−3, −2). Verify by use of formula.

5. By finding the area of the triangle whose vertices are (2, 3), (4, 7), (−2, −5), show that these points must lie on one straight line.

6. Find the area of the triangle whose angular points are (1, 2), (−3, 4), (x, y).

Hence express the relation that must hold between x and y that the point (x, y) may lie anywhere on the straight line joining the points (1, 2), (−3, 4).

7. Employ the formula of § 6 to verify the truth that triangles on the same base and between the same parallels are equal in area. [Take two of the vertices on the axis of x, say (a, o) and (a', o), and the third at the point (x, b) where x is a variable and b a constant.]

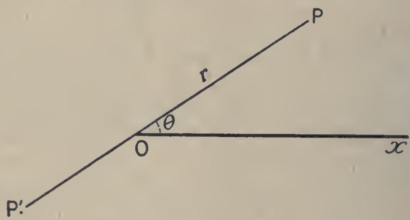
8. Show that the points $(a, b+c)$, $(b, c+a)$, $(c, a+b)$ are in one straight line.

9. If the point (x, y) be equidistant from the points (−2, 1), (3, 2), then $5x+y=4$; also if the point (x, y) be equidistant from the points (1, −2), (3, 2), then $x+2y=2$.

Prove this, and hence show that the point $(\frac{2}{3}, \frac{2}{3})$ is equidistant from the three points (−2, 1), (3, 2), (1, −2).

Polar Co-ordinates.

7. There are several ways of representing the position of a point in a plane, in addition to the method of Cartesian co-ordinates. One of these is the following, known as the method of **Polar Co-ordinates.**



Let Ox be a fixed line in the plane, called the **initial line**; and O a fixed point in this line, called the **origin.** Then the position of a point P in the plane is evidently known, if the angle POx and the length OP be known.

OP is called the **radius vector,** and is usually denoted by r; POx is called the **vectorial angle,** and is usually denoted by θ. Together r and θ are called the **polar co-ordinates** of P; and the point P is indicated by the form (r, θ).

The angle θ is considered positive when measured from Ox in a direction contrary to that in which the hands of a watch revolve, and negative if measured in the opposite direction.

The radius vector r is considered positive if measured from O along the bounding line of the vectorial angle, and negative if measured in the opposite direction.

Thus the co-ordinates of P are r, θ; or $r, -(2\pi - \theta)$; or $-r, \pi + \theta$; or $-r, -(\pi - \theta)$. The co-ordinates of P' are $r, \pi + \theta$; or $r, -(\pi - \theta)$; or $-r, \theta$; or $-r, -(2\pi - \theta)$.

Exercises.

1. Locate the following points:

$$(3,\ 30^\circ);\ (-3,\ 30^\circ);\ \left(2,\ \frac{\pi}{3}\right);\ \left(2,\ -\frac{\pi}{3}\right);\ (-4,\ 135^\circ);\ (4,\ -45^\circ);$$

$$\left(-6,\ \frac{\pi}{4}\right);\ \left(6,\ -\frac{3\pi}{4}\right).$$

2. If r, θ be the polar co-ordinates of a point, and x, y the Cartesian co-ordinates, then $x = r\cos\theta$, $y = r\sin\theta$.

3. If $(r,\ \theta)$, $(r',\ \theta')$ be two points, the square of the distance between them is

$$r^2 + r'^2 - 2rr'\cos(\theta - \theta').$$

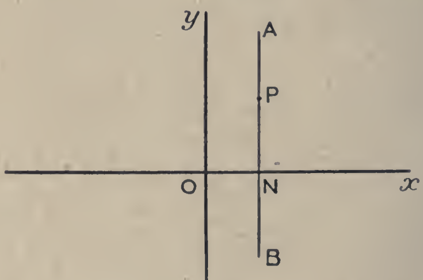
CHAPTER II.

EQUATIONS AND LOCI.

I. Equations of Loci or Graphs.

8. Any line constructed under fixed instructions, or *in accordance with some law*, consists of a series of points whose positions are determined by such instructions or law. To a line so regarded, the term **locus** or **graph** is usually applied. The expression of this law in algebraic language, under principles suggested by the preceding chapter, creates an equation which we speak of as the equation of the line, locus, or graph. The thought here expressed in general language, will be made clearer by a series of illustrations, which we proceed to give:

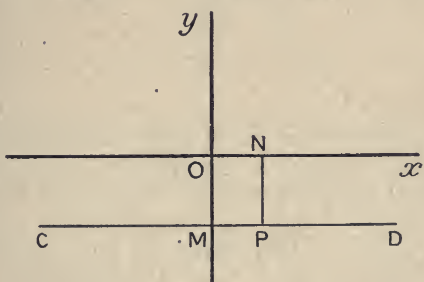
Ex. 1. Through a point N on the axis of x, at distance $+2$ from the origin, draw AB parallel to the



axis of y; i.e., construct a locus every point of which is at distance $+2$ from the axis of y. Then while the ordinates of points on AB vary, every point has the same abscissa, namely $+2$. If therefore we consider the line AB in connection with the equation $x=2$, or $x-2=0$, we see that the co-ordinates of every point on AB satisfy this equation. Hence $x-2=0$ is said to be the equation of the line AB, since it is an algebraic representation of AB in the sense that the co-ordinates of every point on AB satisfy this equation, and the value of x (and values of y) which satisfies this equation corresponds to points on AB.

If we write the equation in the form $x+0y-2=0$, the satisfying of it by $x=2$ and varying values of y perhaps becomes clearer.

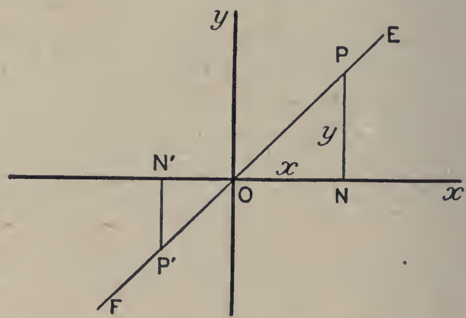
Ex. 2. Through a point M on the negative part of the axis of y, at distance 3 from the origin, draw CD parallel to the axis of x. Then while the abscissas of



points on CD vary, every point has the same ordinate, namely -3. If therefore we consider the line CD in connection with the equation $y=-3$, or $y+3=0$, we see that the co-ordinates of every point on CD satisfy this equation. Hence $y+3=0$ is said to be the equation of the line CD, since it is an algebraic representation of

CD in the sense that the co-ordinates of every point on *CD* satisfy this equation, and the value of y (and values of x) which satisfies this equation corresponds to points on *CD*.

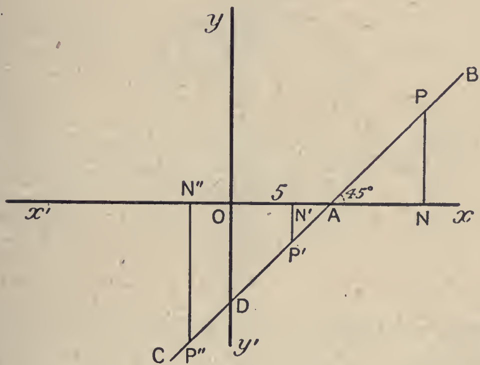
Ex. 3. If through the origin we draw a line *EF* making an angle of 45° with the axis of x, at every point of this line y is equal to x, being of the same magnitude and sign. Therefore the co-ordinates of every point on *EF* satisfy the equation $x=y$, or



$x-y=0$. Hence $x-y=0$ is said to be the equation of *EF*, since $x-y=0$ is an algebraic (*i.e.*, analytical) representation of *EF*, and *EF* is a geometrical representation of $x-y=0$, in the sense that the co-ordinates of every point on *EF* satisfy $x-y=0$, and each pair of real values of x and y which satisfy the equation are the co-ordinates of a point on *EF*.

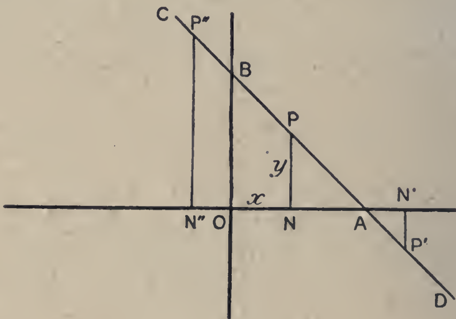
It thus appears that at all events in the cases which we have considered, **lines may be represented by means of equations between two variables,** the meaning being that each pair of real roots of an equation represent the co-ordinates of a point on its line, and the co-ordinates of any point on a line satisfy its equation. We proceed still further to illustrate and generalize the statement in black face.

Ex. 4. Take a point A on the positive direction of the axis of x, at distance 5 from the origin, and through A draw BC making an angle of 45° with Ox, and cutting Oy' in D. Then for the point P, which is any



point on the section AB, $5 = ON - AN = ON - NP = x - y$. For the point P', which is any point on the section AD, $5 = ON' + N'A = ON' + P'N' = x + (-y)$, since at P', y is negative, and therefore $-y$ positive. For P'', which is any point on the section DC, $5 = N''A - N''O = P''N'' - N''O = -y - (-x)$, since at P'', x and y are both negative, and therefore $-x$, $-y$ both positive; hence $5 = x - y$. Therefore throughout the line BC the equation $x - y = 5$ represents the relation between the x and y of any point. And on the other hand all pairs of real values of x and y which satisfy the equation represent points on BC; for give to x in the equation any value, say that represented by ON', then the equation gives for y the value $y = ON' - 5$, which is $N'P'$. We say then that $x - y = 5$ is the equation of BC.

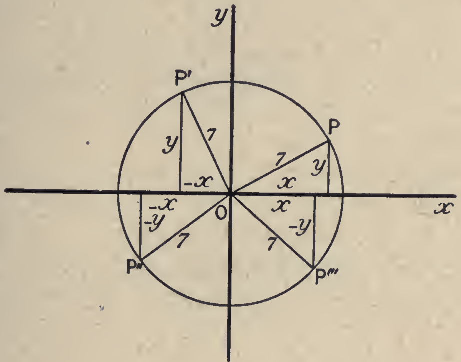
Ex. 5. Take points A and B, on Ox and Oy respectively, at distance 6 from the origin, and through them draw the straight line AB. Then for the point



P, which is any point on the section AB, $6 = ON + NA = ON + NP = x + y$. For the point P', which is any point on the section AD, $6 = ON' - AN' = ON' - PN' = x - (-y) = x + y$, since at P', y is negative, and therefore $-y$ positive. For the point P'', which is any point on the section BC, $6 = N''A - N''O = N''P'' - N''O = y - (-x) = y + x$, since at P'', x is negative, and therefore $-x$ positive. Therefore throughout the line CD, the equation $x + y = 6$ represents the relation between co-ordinates of any point. And on the other hand all pairs of real values of x and y which satisfy the equation, when viewed as co-ordinates, conduct us to points on CD; for give to x in the equation any value, say that represented by ON'', then the equation gives for y

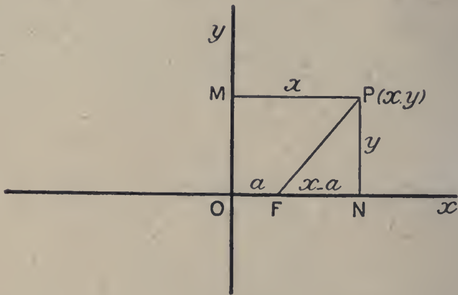
the value $y = 6 - ON''$, which is $N''P''$. We say then that $x + y = 6$ is the equation of CD.

Ex. 6. With the origin as centre and radius 7 describe a circle. Then while the abscissa and ordinate of a point in this circle vary continually as the point



travels along the curve, they are nevertheless always connected by the relation $x^2 + y^2 = 49$. Hence $x^2 + y^2 = 49$ is said to be the equation of this circle, for it is the analytical representation of the circle in the sense that the co-ordinates of every point on the circle are bound by the relation $x^2 + y^2 = 49$, and every pair of real values of x and y which satisfy the equation are represented by the co-ordinates of some point on this circle.

Ex. 7. Frequently, without constructing the locus geometrically, we may express algebraically the law of the locus, and so obtain its equation. Thus, suppose we are required to find the equation of the locus of a point which moves so that its distance from the point (a, o) is equal to its distance from the axis of y. Let $P(x, y)$ be



any point on the locus, and F the point (a, o). Then we express the law of the locus when we write

$$FP = MP,$$

$$\text{or } FP^2 = MP^2,$$

$$\text{or } (x - a)^2 + y^2 = x^2.$$

Hence $y^2 - 2ax + a^2 = o$ is the equation of the locus, whose geometrical form is a matter for further investigation.

It may here be stated that, as we advance in the subject, it will appear that **all the properties of a curve are latent in its equation,** and will reveal themselves as suitable examinations or analyses of the equation are made.

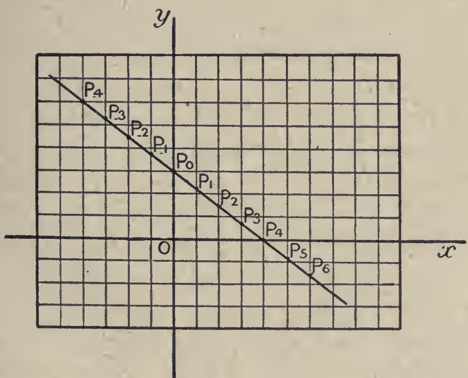
II. Loci or Graphs of Equations.

9. In the preceding illustrations we have placed a locus subject to certain conditions, and have shown that there is an equation between two variables corresponding to it.

The converse operation is, given an equation between two variables, to show that there is a locus, or graph, corresponding to it, *i.e.*, such that all pairs of real roots of the equation correspond to the co-ordinates of points on the locus, there being no points on the locus whose co-ordinates are not pairs of real roots of the equation.

In dealing with this other side of the proposition we shall begin by solving the equation, thus finding a succession of pairs of real values of x and y. We shall then construct the corresponding points, in this way arriving at individual members of an infinite series of points which form a locus or graph. After a few illustrations we shall feel ourselves justified in saying that to every equation involving two variables there corresponds a locus or graph.

Ex. 1. Let us consider the graph of the equation $3x+4y=12$, *i.e.*, the succession of points whose co-ordin-



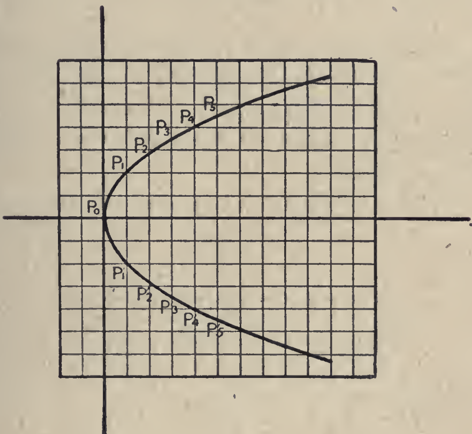
ates are pairs of roots of this equation. Both variables, x and y, appear in this equation. By giving to x a succession of values, 0, 1, 2, 3, . . . , and solving for y, we shall obtain the pairs of values we seek, as follows: $x=0$, $y=3$; $x=1$, $y=2\frac{1}{4}$; $x=2$, $y=1\frac{1}{2}$; $x=3$, $y=\frac{3}{4}$; $x=4$, $y=0$; $x=5$, $y=-\frac{3}{4}$; $x=6$, $y=-1\frac{1}{2}$; etc.; $x=-1$, $y=3\frac{3}{4}$; $x=-2$, $y=4\frac{1}{2}$; $x=-3$, $y=5\frac{1}{4}$; $x=-4$, $y=6$; etc.

Plotting the corresponding points we obtain P_0, P_1, P_2, ... , P_{-1}, P_{-2}, ... The use of squared paper will enable one to locate the points rapidly and accurately.

Evidently between any two of the points, say P_2, P_3, there exists an indefinite number of points of the locus, whose co-ordinates are obtained by giving to x values between 2 and 3, separated each from the preceding and succeeding by indefinitely small intervals, and finding the corresponding values of y. The graph or locus corresponding to the equation $3x+4y=12$, thus consists of a succession of points, infinite in number, each being indefinitely close to the preceding and succeeding; *i.e.*, the graph is continuous. Plainly also there is no limit to the extension of the graph in either direction; *i.e.*, it is infinite in length.

In this case the graph is a straight line. It will subsequently be shown that all equations of the first degree in x and y represent straight lines. The fact that, in the graph of $3x+4y=12$, equal increments of 1 in x give equal decrements of $\frac{3}{4}$ in y shows that the points determined must lie on a straight line.

Ex. 2. Again let us consider the graph of the equation $y^2=4x$. Here $y=\pm 2\sqrt{x}$; and for each value of x there are two values of y which are equal in magnitude but with opposite signs. Since y^2 is necessarily positive, x cannot be negative, *i.e.*, no part of the graph can lie to the left of the origin. By giving to x a succession of values, 1, 2, 3, 4, ..., and calculating the corresponding values of y from a table of square roots, we obtain the pairs of roots we seek, as follows: $x=0$, $y=0$; $x=1$, $y=\pm 2$; $x=2$, $y=\pm 2{\cdot}83$; $x=3$, $y=\pm 3{\cdot}46$; $x=4$, $y=\pm 4$; $x=5$, $y=\pm 4{\cdot}47$; $x=6$, $x=\pm 4{\cdot}90$; $x=7$, $y=\pm 5{\cdot}29$; $x=8$, $y=\pm 5{\cdot}66$; $x=9$, $y=\pm 6$; $x=10$, $y=\pm 6{\cdot}32$; etc.



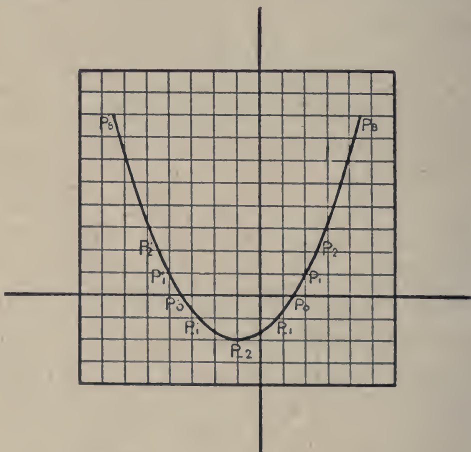
Plotting the corresponding points we obtain P_0, P_1, P_2, ..., and P'_1, P'_2, P'_3, ...

Here again between any two of these points, say P_3, P_4, there exists an indefinite number of points of the locus whose co-ordinates are obtained by giving to x values between 3 and 4, separated each from the preceding and succeeding by indefinitely small intervals, and calculating the corresponding values of y. The locus or graph corresponding to the equation $y^2=4x$ thus consists of a succession of points, infinite in number, each being indefinitely close to the preceding and succeeding; *i.e.*, the locus is continuous.

Evidently the locus extends without limit to the right, receding as it does so from the axis of x.

In this case, as will subsequently appear, the locus is the curve called the parabola,—one of the class known as conic sections. The values of y are given to the second decimal, that it may be seen the graph is not a straight line; for while the values of x proceed by equal increments of 1, the successive increments of y are 2, ·83, ·63, ·54, ·47, ·43, ·39, ·37, ·34, ·32. In the case of a straight line, equal increments in x give equal increments in y, as is evident from the principle of similar triangles.

Ex. 3. To construct the graph of the equation



$x^2+2x-3y-5=0$. Here $x=-1\pm\sqrt{3(y+2)}$. Evidently y cannot be less than -2; *i.e.*, the graph cannot be more than 2 units below the axis of x. With a table of square roots we readily obtain the following pairs of values: $y=-2$, $x=-1$; $y=-1$, $x=\cdot73$ or $-2\cdot73$; $y=0$, $x=1\cdot45$ or $-3\cdot45$; $y=1$, $x=2$ or -4; $y=2$, $x=2\cdot46$ or $-4\cdot46$; $y=3$, $x=2\cdot87$ or $-4\cdot87$; $y=4$, $x=3\cdot24$ or $-5\cdot24$; $y=5$, $x=3\cdot58$ or $-5\cdot58$; $y=6$, $x=3\cdot90$ or $-5\cdot90$; $y=7$, $x=4\cdot20$ or $-6\cdot20$; $y=8$, $x=4\cdot48$ or $-6\cdot48$; etc.

Plotting the corresponding points we obtain P_{-2}, P_{-1}, P_0, ...; P'_{-1}, P'_0, ...

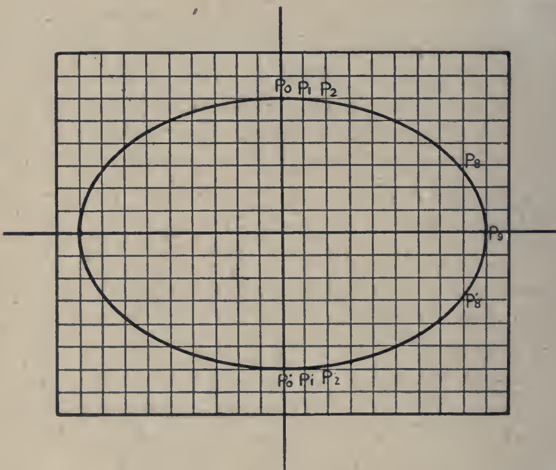
Here also between any two of these points there exists an indefinite number of other points whose coordinates are roots of the equation, and the graph is continuous. The graph manifestly extends upwards without limit in two branches, receding from the axis of y as it ascends.

This curve also is a parabola. The values of x are given to the second decimal, that it may be seen the graph is not a straight line; for while the values of y proceed by equal increments of 1, the successive increments in the value of x are $1\cdot73$, $\cdot72$, $\cdot55$, $\cdot46$, $\cdot43$, $\cdot37$, $\cdot34$, $\cdot32$, $\cdot30$, $\cdot28$. In the case of a straight line equal increments in y give equal increments or decrements in x, as is evident from the principle of similar triangles.

Ex. 4. To construct the graph of the equation $\frac{x^2}{9^2}+\frac{y^2}{6^2}=1$. Here $y=\pm\frac{2}{3}\sqrt{9^2-x^2}$. Evidently x can never be >9, nor <-9; *i.e.*, the graph cannot extend more than 9 units of length to right and left of the origin. The following will be found to be the pairs of values derived from this equation: $x=0$, $y=\pm6$; $x=1$, $y=\pm5\cdot96$; $x=2$, $y=\pm5\cdot85$; $x=3$, $y=\pm5\cdot66$; $x=4$,

$y = \pm 5{\cdot}37$; $x=5$, $y = \pm 4{\cdot}99$; $x=6$, $y = \pm 4{\cdot}47$; $x=7$, $y = \pm 3{\cdot}77$; $x=8$, $y = \pm 2{\cdot}75$; $x=9$, $y=0$.

Plotting the corresponding points we obtain $P_0, P_1, P_2, \ldots P'_0, P'_1, P'_2 \ldots$ If we give to x the values $-1, -2, \ldots -9$, we obtain exactly the same values for y as above, since it is the form x^2 which occurs under the radical sign. Hence the construction of the graph to the left of the axis of y is a repetition



of that to the right. For the reason stated in discussing the other graphs, this also is continuous.

In this case, as will subsequently appear, the graph is the curve called the ellipse,—one of the class known as conic sections. Here again, while the values of x proceed by equal increments of 1, the successive decrements of y are ·04, ·09, ·19, ·29, ·38, ·52, ·70, 1·02, 2·75, and the graph, for the reason stated in Exs. 2 and 3, cannot be a straight line.

10. The illustrations which we have given point to the general conclusion that **when the law of a locus, or graph, is given, there corresponds to it a certain equation; and, conversely, when an equation is given, there corresponds to it a certain locus, or graph.**

11. If we have two equations, say $x+y=6$ and $\frac{x^2}{9^2}+\frac{y^2}{6^2}=1$, each represents a locus, and the co-ordinates of the points of intersection of these loci must satisfy both equations. But we find the values of x and y which satisfy both equations by solving them as simultaneous. Hence **to find the points of intersection of two loci, solve their equations as simultaneous.** In the preceding loci the points of intersection will be found to be (0, 6) and $(8\frac{4}{13}, -2\frac{4}{13})$.

12. Suppose that from the equations

$$\left.\begin{aligned} x+y-6&=0 \\ \frac{x^2}{9^2}+\frac{y^2}{6^2}-1&=0 \end{aligned}\right\} \ldots (1)$$

we form the equation

$$(x+y-6)+k\left(\frac{x^2}{9^2}+\frac{y^2}{6^2}-1\right)=0, \ldots (2)$$

where k is any quantity involving the variables, or independent of them.

Then each of the equations in (1) represents a locus, and at the points of intersection of these loci both $x+y-6$ and $\frac{x^2}{9^2}+\frac{y^2}{6^2}-1$ vanish. Hence at these points of intersection (2) is satisfied.

But (2) also represents some locus.

Therefore (2) must be the equation of a locus passing through the intersections of the loci represented by the equations in (1).

13. **The locus of an equation is not changed by any transposition of the terms of the equation, or by the multiplication of both members of the equation by any finite constant.** For manifestly the equation after such modification is still satisfied by the co-ordinates of points on the locus of the original equation, and by the co-ordinates of such points only.

Exercises.

1. A point moves so as to be at a constant distance -5 from the axis of y. Find the equation of its locus.

2. A point moves so as to be always at equal distances numerically from the axes. Find the equation of its locus or graph.

3. Find the equation of the path traced by a point which is always at equal distances from the points

(1). $(0, 0)$ and $(5, 0)$; (2). $(3, 0)$ and $(-3, 0)$;
(3). $(-2, 3)$ and $(5, -4)$; (4). $(a+b, a-b)$ and $(a-b, a+b)$.

4. In the preceding exercise place accurately in each case the fixed points, and construct the graph of the moving point.

5. Find the equation of the graph of a point which moves so that its ordinate is always greater than the corresponding abscissa by a given distance a. Construct the graph.

6. A point moves so that its abscissa always exceeds $\frac{3}{4}$ of its ordinate by 2. Find the equation of its graph, and construct a series of points on it.

7. A point moves so that the excess of the square of its distance from $(-a, 0)$ above the square of its distance from $(a, 0)$ is constant, and equal to c^2. Find the equation of its path.

Solve the problem also by constructing the locus by synthetic Geometry, and then forming its equation.

8. A point moves so that the square of its ordinate is always twice its abscissa. Find the equation of its locus.

9. A point moves so that the square of its abscissa is always twice its ordinate. Find the equation of its locus.

Find the co-ordinates of the points in which this locus intersects that in Exercise 8.

10. A point moves so that its distance from the axis of y is one-third its distance from the origin. Find the equation of its locus.

11. A point moves so that its distance from the axis of y is equal to its distance from the point (3, 2). Find the equation of its locus.

12. A is a point on the axis of x at distance $2a$ from the origin. P (x, y) is any point, the foot of whose ordinate is M. Find the equation of the locus of P when it moves so that PM is a mean proportional between OM and MA.

13. A point moves so that the ratio of its distances from two fixed points, (0, 0) and $(a, 0)$, is constant and equal to 2 : 1. Find the equation of the point's graph.

14. A point moves so that one-half of its ordinate exceeds one-third of its abscissa by 1. Find the equation of its locus and construct a series of points on it.

15. Are the points (4, 3), (− 4, 9), (6, 2) on the locus of $3x+4y=24$?

16. Are the points (2, 3), (2, − 3), (− 2, − 3), $(\sqrt{7}, \frac{3}{2})$, (3, 2) on the locus of $\frac{x^2}{4}+\frac{y^2}{9}=2$?

17. The abscissa of a point on the locus of $4x^2+9y^2=10$ is $\frac{1}{2}$; what is the point's ordinate?

18. Find a series of points on the graph of the equation $2x-5y-10=0$, and trace the line as far as these points suggest its position.

19. Trace a part of the graph of the equation $4x+3y+12=0$. Find the points at which it cuts the axes of x and y.

20. Find the points where the graph of the $\frac{x^2}{4}+\frac{y^2}{9}=2$ cuts the axes of x and y.

21. Trace a part of the graph of the equation $y^2-2x+2y-3=0$, beginning at the point where $x=-2$.

Is it cut by the line whose equation is $x+3=0$?

22. Trace a part of the graph of the equation $x^2-4x-2y+6=0$. Does the axis of x cut it?

23. Find the points where the curve of the preceding exercise is cut by the line $x-y=0$.

For what different values of m will the line $y-mx=0$ meet this curve in two points that are coincident, *i.e.*, that have the same ordinate?

24. Trace a part of the graph of the equation $y=x^2-2x+1$.

25. Find the co-ordinates of the points in which the loci whose equations are $\frac{x^2}{16}+\frac{y^2}{9}=1$ and $\frac{x^2}{9}+\frac{y^2}{16}=1$ intersect.

26. Show that the locus whose equation is $x^2+y^2=\frac{288}{25}$ passes through the intersections of the two preceding loci, both from the mode of formation of this equation, and also because it is satisfied by the co-ordinates of the points in which the preceding loci intersect.

CHAPTER III.

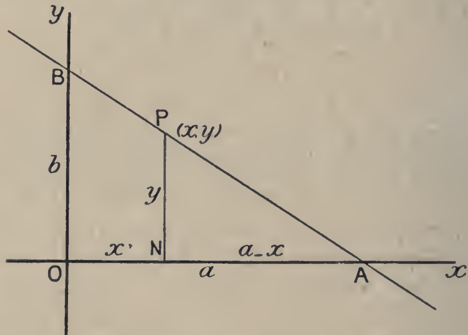
THE STRAIGHT LINE.

In Chapter II., § 8, Exs. 1, 2, 3, 4, 5, illustrations have been given showing that fixed relations exist between the co-ordinates of points on straight lines whose positions are defined by certain numerical conditions. Such relations between the co-ordinates have been called the equations of the lines. It is proposed in the present chapter to deal with the equations of straight lines in a general manner. The position of a straight line may be defined in two ways, either (1) by two points through which it passes being given, or (2) by one point through which it passes and its direction being given. This classification will be followed in arriving at the various forms of the equation.

A straight line is individualized by the data which fix its position,—lengths of intercepts on axes, length of intercept on axis of x and direction, etc.; we shall find that the equation which is the analytical representative of the line, is individualized by these data forming the coefficients of x and y, and the constant term. The student who wishes to follow the subject of Analytical Geometry intelligently must never lose sight of the absolute correlation, or correspondence, which exists between the line, straight or curved, and the equation which is its algebraic equivalent.

I. Line defined by two points through which it passes.

14. **To find the equation of a straight line in terms of the intercepts which it makes on the axes.**



Let $OA, = a$, and $OB, = b$, be the intercepts of the line on the axes of x and y respectively. Let $P (x, y)$ be any point on the line, whose ordinate is PN.

Then, by similar triangles PNA, BOA,

$$\frac{y}{b} = \frac{a - x}{a},$$

$$= 1 - \frac{x}{a};$$

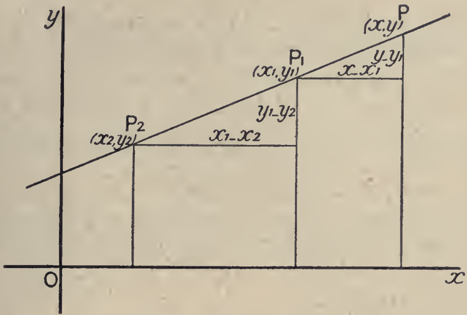
$$\text{or } \frac{x}{a} + \frac{y}{b} = 1,$$

which is the equation required.

This is the equation of a line through two given points (a, o), (o, b), the points being in particular positions, *i.e.*, on the axes.

In the following article the given points through which the line passes are any fixed points.

15. To find the equation of a line which passes through two given points.



Let P_1 (x_1, y_1) and P_2 (x_2, y_2) be the two fixed points through which the line passes; and let P (x, y) be any point on the line. Complete the figure as in the diagram. Then the lines in the figure have evidently the values indicated; and by similar triangles

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2},$$

which is the equation required.

The equation of § 14 is a particular case of this, and may therefore be deduced from it. In § 14 the line passes through the points (a, o), (o, b). Substituting these co-ordinates for (x_1, y_1), (x_2, y_2) in the equation of the present article, we have

$$\frac{x - a}{a - o} = \frac{y - o}{o - b},$$

$$\text{or } \frac{x}{a} - 1 = -\frac{y}{b},$$

$$\text{or } \frac{x}{a} + \frac{y}{b} = 1.$$

Exercises.

1. Obtain the equation of § 14 when the point P is taken in the second quadrant; when it is taken in the fourth quadrant.

2. Obtain the equation of § 14 when the intercepts on the axes are $-a$, $-b$, the point P in the demonstration being taken in the second quadrant. Verify your result by substituting $-a$, $-b$ for a, b in the formula of § 14.

3. The intercepts a straight line makes on the axes of x and y are -3 and 2, respectively. Find its equation from a figure, taking the point P in the third quadrant. Verify by obtaining the equation from the formula of § 14.

4. From the fact that

$$\Delta PBO + \Delta PAO = \Delta BOA,$$

obtain the equation of § 14.

5. Find expressions for the intercepts on the axes derived from the equation of § 15.

6. From the expression for the area of a triangle in terms of the co-ordinates of its angular points (Ch. I., § 6), obtain the equation of a straight line through two given points. [Take as angular points (x, y), (x_1, y_1), (x_2, y_2).]

Reduce your equation to the form in § 15.

7. From a figure obtain the equation of a straight line whose intercepts on the axes of x and y are -4 and -5, respectively. Verify by obtaining the equation also from the formula of § 14.

8. Find from a figure, and also from the formula of § 15, the equation of a straight line through the points $(-3, -1)$, $(4, -5)$. Find the intercepts on the axes.

9. A straight line which passes through the origin has its equation satisfied by the values $x = -3$, $y = 5$. Find its equation both from a figure and from a formula.

10. The intercept of a straight line on the axis of y is -4, and it passes through the point $(-2, 5)$. Find its equation.

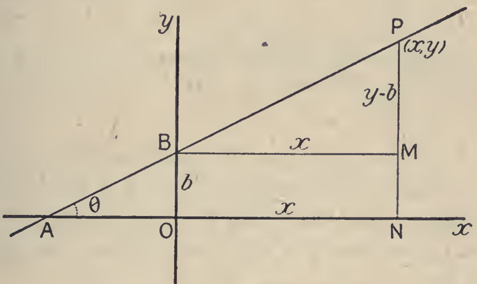
11. A straight line passes through the point $(-4, -5)$, and makes equal intercepts on the axes. Find its equation. [Assume $\frac{x}{a} + \frac{y}{a} = 1$ as

its equation. Since it passes through $(-4, -5)$, $\frac{-4}{a}+\frac{-5}{a}=\pm 1$; whence a is found.]

12. **When the equation of a straight line is given, what in general is the easiest way of constructing the line,** *i.e.*, placing it properly with respect to the co-ordinate axes?

II. Line defined by one point through which it passes, and by its direction.

16. **To find the equation of a straight line in terms of the angle it makes with the axis of x and its intercept on the axis of y.**



Let PBA be the straight line, θ the angle BAO, which it makes with the axis of x, and b the intercept it makes on the axis of y. Let $P(x, y)$ be any point on the line. Complete the figure as in the diagram.

Then $MP = NP - NM = NP - OB = y - b$; and $BM = ON = x$.

$$\text{Hence } \frac{y-b}{x} = \frac{MP}{BM} = \tan PBM = \tan\theta;$$

$$\text{or } y = x\tan\theta + b,$$

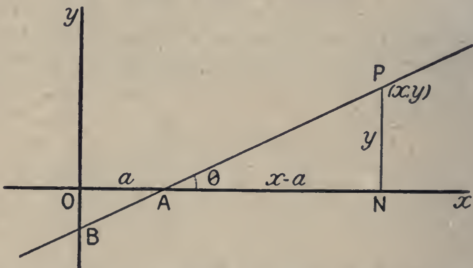
which is the equation required.

It is convenient to replace $\tan\theta$ by m, so that the equation becomes

$$y = mx + b,$$

where m is the tangent of the angle which the line makes with the axis of x.

17. **To find the equation of a straight line in terms of the angle it makes with the axis of x and its intercept on the axis of x.**



Let PAB be the straight line, θ the angle PAN which it makes with the axis of x, and a its intercept OA on the axis of x. Let $P(x, y)$ be any point on the line. Complete the figure as in the diagram.

Then $NP = y$; and $AN = ON - OA = x - a$

Therefore $\dfrac{y}{x-a} = \dfrac{NP}{AN} = \tan PAN = \tan\theta$;

or $y = (x - a)\tan\theta$,

which is the equation required.

The equation may conveniently be written

$$y = m(x - a),$$

where m is the tangent of the angle which the line makes with the axis of x.

In this and the preceding articles the straight line and the point P have been taken so that the quantities

a, b, m, x, y are all positive. The results obtained in this way are absolutely general, while we are spared the trouble of considering quantities intrinsically negative.

18. If two equations represent the same straight line, they must be, in effect, the same equation; and therefore the coefficients of x and y, and the constant term, in one are either equal or proportional to the coefficients of x and y, and the constant term, in the other. If the coefficient of one of the variables, say of y, in one be equal to the coefficient of y in the other, then the coefficients of x must be equal, and the constant terms must be equal.

Thus if we write the standard forms of the equation of the straight line, obtained in §§ 14, 15, 16, 17, as follows,—

$$y = -\frac{b}{a}x + b,$$

$$y = \frac{y_1 - y_2}{x_1 - x_2}x - \frac{y_1 - y_2}{x_1 - x_2}x_1 + y_1,$$

$$y = mx + b,$$

$$y = mx - ma,$$

since they may be supposed to represent the same straight line, and since the coefficients of y are equal, we see that

$$-\frac{b}{a} = \frac{y_1 - y_2}{x_1 - x_2} = m.$$

But m is the tangent of the angle which the line makes with the axis of x. Hence $-\frac{b}{a}$ and $\frac{y_1 - y_2}{x_1 - x_2}$ represent also the tangent of the line's inclination to the axis of x.

Again we must also have

$$b = -\frac{y_1 - y_2}{x_1 - x_2}x_1 + y_1 = -ma.$$

But b is the intercept on the axis of y. Hence $-\frac{y_1 - y_2}{x_1 - x_2}x_1 + y_1 \left(= \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2}\right)$ and $-ma$ represent also the line's intercept on the axis of y.

We shall often find ourselves able, in future, to obtain important results by thus comparing equations of the same line, which differ in form.

Exercises.

1. Obtain the equation of § 16 when the point P is taken in the second quadrant; when it is taken in the third quadrant.

2. Obtain the equation of § 17 when the point P is taken in the third quadrant; when it is taken in the fourth quadrant.

3. The inclination of a straight line to the axis of x is 45°, and its intercept on the axis of y is -5. Find its equation from a figure, taking the point P in either the first or third quadrant. Verify by obtaining the equation from the formula of § 16.

4. The inclination of a straight line to the axis of x is 60°, and its intercept on the axis of x is -7. Find its equation from a figure, taking the point P in the second quadrant. Verify by obtaining the equation from the formula of § 17. What is the line's intercept on the axis of y?

5. A straight line passes through the origin, and makes an angle of 30° with the axis of x. Find its equation.

6. A straight line makes an angle of 120° with the axis of x, and its intercept on the axis of y is -3. Construct the line, and find its equation from the figure, and also from the formula of § 16.

7. A straight line makes an angle of 150° with the axis of x, and its intercept on the axis of x is -3. Construct the line, and find its equation from the figure, and also from the formula of § 17. Verify your construction by finding from your equation the intercepts on the axes.

8. A straight line passes through the point $(4, \sqrt{3})$, and makes an angle of 60° with the axis of x. Find its equation. [Assume $y = x \tan 60^\circ + b$ as its equation. Since it passes through $(4, \sqrt{3})$, $\sqrt{3} = 4\sqrt{3} + b$; whence b is found.]

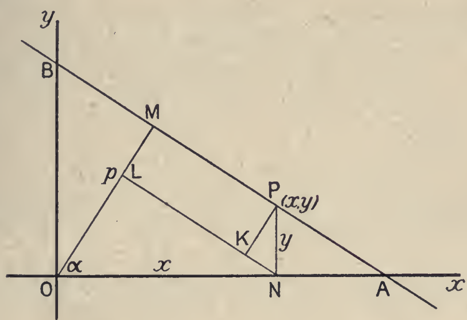
9. What angle does the line through the points $(5, 3)$, $(-2, -4)$ make with the axis of x? What is its intercept on the axis of y?

10. What is the characteristic of the system of lines obtained by varying b in the equation $y = 2x + b$?

11. Obtain the equation of § 15 by supposing (x_1, y_1), (x_2, y_2) to be points on the line $y = m(x - a)$, and thence finding m and a.

12. Two lines pass through the point $(1, 5)$, and form, with the axis of x, an equilateral triangle. Find the equations of the lines.

19. To find the equation of a straight line in terms of the length of the perpendicular upon it from the origin and the angle the perpendicular makes with the axis of x.



Let AB be the given straight line, p the perpendicular OM upon it from the origin, and α the angle MOA which the perpendicular makes with the axis of x. Let $P(x, y)$ be any point on the line. Complete the figure as in the diagram.

Then $\angle KNP = 90° - \angle KNO = \alpha$.

Hence $p = OM = OL + KP$,

$$= ON \cos\alpha + NP \sin\alpha,$$

$$= x\cos\alpha + y\sin\alpha;$$

and $x\cos\alpha + y\sin\alpha = p$ is the equation required.

If we have obtained the equation of a straight line from given data, *e.g.*, that it passes through two given points, and wish to find the perpendicular upon it from the origin, and the angle the perpendicular makes with the axis of x, we follow the method suggested in § 18:

Ex. A straight line passes through the points (1, 5), and (7, 2). Find p and α for this line.

The equation of the line through the given points will be found to be (§ 15.)

$$x + 2y = 11.$$

Suppose that $x\cos\alpha + y\sin\alpha = p$ represents the same line.

$$\text{Then } \frac{\cos\alpha}{1} = \frac{\sin\alpha}{2} = \frac{p}{11}.$$

$$\text{But } \frac{\cos\alpha}{1} = \frac{\sin\alpha}{2} = \frac{\sqrt{\cos^2\alpha + \sin^2\alpha}}{\sqrt{1^2+2^2}} = \frac{1}{\sqrt{5}};$$

$$\therefore \cos\alpha = \frac{1}{\sqrt{5}}, \ \sin\alpha = \frac{2}{\sqrt{5}}, \text{ and } p = \frac{11}{\sqrt{5}}.$$

In selecting the sign for the square root we are guided by the consideration that p is always positive, and select the sign which makes it so.

Thus the line through the points (1, 5), $(-7, 2)$ is $3x - 8y = -37$. We select the negative sign for the square root, for it gives $p = \frac{-37}{-\sqrt{73}} = \frac{37}{\sqrt{73}}$. Also we have $\cos\alpha = -\frac{3}{\sqrt{73}}$, $\sin\alpha = \frac{8}{\sqrt{73}}$. These values for $\cos\alpha$, $\sin\alpha$ show that α lies between 90° and 180°, which is confirmed if we accurately construct the line from the original data.

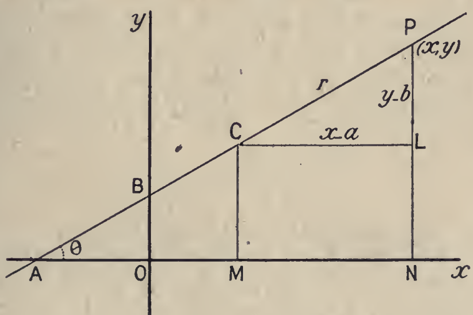
The line whose intercept on the axis of x is 4, and which makes an angle of 30° with this axis, has for equation $x - y\sqrt{3} = 4$. We select the positive sign which gives

$$p=\frac{4}{2}=2\text{; also }\cos\,\alpha=\frac{1}{2},\ \sin\,\alpha=-\frac{\sqrt{3}}{2}.$$

These expressions for $\cos\,\alpha$, $\sin\,\alpha$ show that α is a negative angle, which is confirmed if we construct the line from the original data.

In §§ 16, 17 the straight line was defined by its direction and by its passing through a point specially placed,—on one of the axes. In § 19 the line was again defined by a special point—foot of perpendicular from origin on line—and by its direction which was fixed by the direction of the perpendicular. In the article which follows, the line, as before, will be defined by its direction, and by a point through which it passes, but this point will be any point on the line. The article, therefore, may be regarded as giving the general proposition of which the three preceding were special cases.

20. **To find the equation of a straight line in terms of the angle it makes with the axis of x and the coordinates of a point through which it passes.**



Let AB be the given straight line, $P\,(x,\,y)$ any point on it, $C\,(a,\,b)$ the given point through which it passes, and BAO (θ) the angle it makes with the axis of x. Let $CP=r$, a variable quantity, since P moves along the line. Complete the figure as in the diagram.

$$\text{Then } \cos\theta=\cos PCL=\frac{CL}{CP}=\frac{x-a}{r}\text{; } \therefore \frac{x-a}{\cos\theta}=r.$$

$$\text{Also } \sin\theta=\sin PCL=\frac{LP}{CP}=\frac{y-b}{r}\text{; } \therefore \frac{y-b}{\sin\theta}=r.$$

$$\text{Hence } \frac{x-a}{\cos\theta}=r=\frac{y-b}{\sin\theta}\text{; and}$$

$$\frac{x-a}{\cos\theta}=\frac{y-b}{\sin\theta}$$

is the equation required.

The equation may be written

$$y-b=\tan\theta\,(x-a),$$
$$\text{or } y-b=m\,(x-a).$$

Possibly, however, the most useful form of the equation is

$$\frac{x-a}{\cos\theta}=\frac{y-b}{\sin\theta}=r,\ldots(1)$$

where r is a variable, and is the distance between the fixed point (a, b) and the moving point (x, y). It is to be noted that in (1) there are two equations and three variables, x, y, r. These equations may be written

$$\frac{x-a}{l}=\frac{y-b}{m}=r,$$
$$\text{or } x=a+lr,\ y=b+mr,$$

where $l=\cos\theta$, $m=\sin\theta$, so that $l^2+m^2=1$. Here l and m are called the *direction-cosines* of the line; for l is the cosine of the angle which the line makes with the axis of x, and m is the sine of the same angle, and therefore the cosine of the angle which the line makes with the axis of y. It is scarcely necessary to remind the student that m in this equation represents $\sin\theta$, while in the equation $y-b=m(x-a)$ it represents $\tan\theta$.

Exercises.

1. From the equation $y-b=m(x-a)$ of § 20 deduce the equations of §§ 16, 17.

2. From the data of equation $\frac{x-a}{\cos\theta}=\frac{y-b}{\sin\theta}$ of § 20 deduce the data of equation $x\cos\alpha+y\sin\alpha=p$ of § 19, (1) from the figure, (2) by comparing the equations. [(1). In figure of § 20 from O and M draw OX, MY perpendicular to the line; and from M draw MZ perpendicular to XO produced. Then $p=OX=MY-ZO=b\cos\theta-a\sin\theta$. Also $\alpha=90+\theta$.]

3. The perpendicular from the origin on a straight line is 6, and this perpendicular makes an angle of 30° with the axis of x. Find the equation of the line.

4. The perpendicular from the origin on a straight line is 4, and this perpendicular makes an angle of 120° with the axis of x. Find the equation of the line.

5. A straight line passes through the points $(-3, 2)$, $(-1, -1)$. Find its equation, and write it in the form $x\cos\alpha+y\sin\alpha=p$. [Here α will be found to be an angle between 180° and 270°, and both its cosine and sine are negative.]

6. A straight line passes through the point $(3, -2)$, and makes an angle of 45° with the axis of x. Find its equation, and write it in the form $x\cos\alpha+y\sin\alpha=p$. [$p$ makes an angle of $-45°$ with Ox.]

7. Form the equation of the line through the point $(7, 1)$, and making an angle of 60° with the axis of x.

8. Form the equation of the line whose intercepts on the axis of x and y are 3 and 5 respectively; and write it in the form $\frac{x-a}{\cos\theta}=\frac{y-b}{\sin\theta}=r$. [The equation is $\frac{x}{3}+\frac{y}{5}=1$, which may be written $\frac{x}{-3}=\frac{y-5}{5}$, or $\dfrac{x-0}{-\frac{3}{\sqrt{34}}}=\dfrac{y-5}{\frac{5}{\sqrt{34}}}=r$. It is only when the denominators become the values of $\cos\theta$ and $\sin\theta$ that we are at liberty to put

the fractions equal to r. Here in selecting the sign of the square root we take that which will make $\sin\theta$ positive, for θ being less than 180°, its sine is necessarily positive.]

9. A straight line passes through the point $(5, -2)$, and the perpendicular on it from the origin makes an angle of 30° with Ox. Find the equation of the line. [Let $x\cos 30° + y\sin 30° = p$ be the equation. Since the line passes through $(5, -2)$, we have $5\cdot\frac{\sqrt{3}}{2} - 2\cdot\frac{1}{2} = p$, which determines p.]

10. **Find the distance from the point (7, 1) to the line $x - y = 5$, measured in a direction making an angle of 60° with the axis of x.** [The equation of the line through (7, 1) is $\dfrac{x-7}{\frac{1}{2}} = \dfrac{y-1}{\frac{\sqrt{3}}{2}} = r$; whence $x = 7 + \frac{1}{2}r$, $y = 1 + \frac{\sqrt{3}}{2}r$, where r is the distance from (7, 1) to (x, y). But if we substitute these values of x and y in the equation $x - y = 5$, x and y must have reference to the point of intersection of the lines, and r becomes the distance required. Substituting, $7 + \frac{1}{2}r - 1 - \frac{\sqrt{3}}{2}r = 5$; and $r = 1 + \sqrt{3}$.]

III. General Equation of First Degree.

The straight line, defined in the preceding articles by various data, has been found in each case to be represented analytically by an equation of the *first degree* in x and y. In the following article it will be shown that an equation of the first degree in x and y must always represent a straight line.

21. **To show that every equation of the first degree represents a straight line.**

The equation of the first degree, in its most general form, may be expressed by

$$Ax + By + C = 0.$$

This represents some locus. Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be any three points on the locus it represents. Hence,

$$Ax_1 + By_1 + C = 0,$$
$$Ax_2 + By_2 + C = 0,$$
$$Ax_3 + By_3 + C = 0.$$

Therefore

$$A\,(x_2 - x_3) + B\,(y_2 - y_3) = 0,$$
$$A\,(x_3 - x_1) + B\,(y_3 - y_1) = 0,$$
$$A\,(x_1 - x_2) + B\,(y_1 - y_2) = 0.$$

Multiplying these equations by x_1, x_2, x_3, respectively, and adding, the term involving A disappears, B divides out, and we have

$$x_1\,(y_2 - y_3) + x_2\,(y_3 - y_1) + x_3\,(y_1 - y_2) = 0.$$

But the left-hand side of this equation represents twice the area of the triangle whose angular points are (x_1, y_1), (x_2, y_2), (x_3, y_3). Hence the area of this triangle is zero, and the three points must lie in a straight line.

But these are *any three* points on the locus which $Ax + By + C = 0$ represents.

Hence $Ax + By + C = 0$ must represent a straight line.

Though three constants A, B, C, appear in the equation $Ax + By + C = 0$, there are in reality only two; for we may write the equation in the form $\frac{A}{C}x + \frac{B}{C}y + 1 = 0$, without any loss of generality (§ 13). Thus $Kx + Ly + 1 = 0$ is just as general a representation of the equation of the first degree as $Ax + By + C = 0$, and may be used as such. The student may see this more clearly if stated in a concrete form: Two points, say (2, 3), (4, 7), are sufficient to fix a line. Hence their co-ordinates must be sufficient to determine the coefficients in the equation which represents the line. But these give only two equations, $2A+3B+C=0$, $4A+7B+C=0$; and these are sufficient to determine only two unknowns, $\frac{A}{C}$, $\frac{B}{C}$.

Exercises.

1. From the equations of the preceding article obtain the form

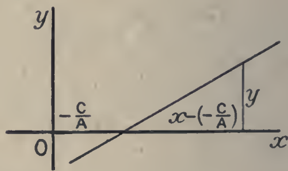
$$\frac{x_3-x_1}{x_1-x_2}=\frac{y_3-y_1}{y_1-y_2};$$

and hence show that (x_3, y_3) must lie in the straight line joining (x_1, y_1), (x_2, y_2); *i.e.*, that the three points lie in a straight line which, therefore, $Ax+By+C=0$ must represent.

2. Convert the equation $Ax+By+C=0$ into the form

$$\frac{y}{x-(-\frac{C}{A})}=-\frac{A}{B}, \text{ a constant};$$

and hence show that $Ax+By+C=0$ must represent a straight line.



3. From the fact that the equations

$$Ax+By+C=0,\ A'x+B'y+C'=0,$$

when solved as simultaneous, give only one pair of values for x and y, show that each must represent a straight line.

4. Use the general equation of the first degree, $Ax+By+C=0$ or $Kx+Ly=1$, to find the equation of the straight line through the points $(5, 1)$, $(-3, -1)$.

22. To find the general equation of all straight lines through a fixed point.

Let (a, b) be the fixed point. Any straight line whatever is represented by

$$Ax+By+C=0.$$

If this pass through (a, b),

$$Aa+Bb+C=0.$$

Combining this condition to which A, B, C are subject, with the equation of the line, we get

$$A\,(x-a)+B\,(y-b)=0,$$

$$\text{or}\quad K\,(x-a)+(y-b)=0,$$

which is the equation of *any* straight line through the point (a, b), $A : B$, or K, being still an undetermined quantity.

It will be seen that the following article is a general discussion involving methods and principles that have already been appealed to in special cases.

23. To reduce the general equation of the first degree, $Ax+By+C=0$, to the standard forms.

(1). Since multiplication of an equation by a constant and transposition of terms do not change the locus which the equation represents (§ 13), therefore whatever locus is represented by $Ax+By+C=0$, the same locus is represented by

$$\frac{x}{-\dfrac{C}{A}}+\frac{y}{-\dfrac{C}{B}}=1.$$

Comparing this with

$$\frac{x}{a}+\frac{y}{b}=1,$$

we have $a=-\dfrac{C}{A}$, $b=-\dfrac{C}{B}$, giving the intercepts on the axes in terms of A, B, C.

We may, of course, also find the intercepts on the axes by putting in succession $y=0$, $x=0$ in the equation $Ax+By+C=0$.

(2). The equation $Ax+By+C=0$, may be written

$$Ax+C=-By,$$

$$\text{or } \frac{Ax+C}{-C}=\frac{By}{C},$$

$$\text{or } \frac{x+\frac{C}{A}}{-\frac{C}{A}}=\frac{y}{\frac{C}{B}},$$

$$\text{or } \frac{x-\left(-\frac{C}{A}\right)}{-\frac{C}{A}-0}=\frac{y-0}{0-\left(-\frac{C}{B}\right)}.$$

Comparing this with

$$\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2},$$

we see that $Ax+By+C=0$ passes through the points

$$\left(-\frac{C}{A},\ 0\right),\ \left(0,\ -\frac{C}{B}\right).$$

The equation $Ax+By+C=0$ may, however, be reduced to the form of an equation of a straight line through two points, in an endless variety of ways. Thus if $x=2$, $y=\dfrac{-2A-C}{B}$; and if $x=-1$, $y=\dfrac{A-C}{B}$; we shall then find ourselves able to express $Ax+By+C=0$ in the form of the equation of a straight line through the points $\left(2,\ -\dfrac{2A+C}{B}\right)$, $\left(-1,\ \dfrac{A-C}{B}\right)$.

(3). The equation $Ax+By+C=0$ may be written

$$y=-\frac{A}{B}x+\left(-\frac{C}{B}\right).$$

Comparing this with

$$y = mx + b,$$

we have $m = -\frac{A}{B}$, $b = -\frac{C}{B}$, giving the tangent of the angle which $Ax + By + C = 0$ makes with the axis of x, and the intercept on the axis of y.

(4). The equation $Ax + By + C = 0$ may be written

$$y = -\frac{A}{B}\left\{x - \left(-\frac{C}{A}\right)\right\}.$$

Comparing this with

$$y = m\,(x - a),$$

we have $m = -\frac{A}{B}$, $a = -\frac{C}{A}$, giving $\tan\theta$ again, and the intercept on the axis of x.

(5). Comparing the equation

$$Ax + By + C = 0$$

with the equation $x\cos\alpha + y\sin\alpha - p = 0$, we have

$$\frac{\cos\alpha}{A} = \frac{\sin\alpha}{B} = \frac{-p}{C}.$$

But $$\frac{\cos\alpha}{A} = \frac{\sin\alpha}{B} = \frac{\sqrt{\cos^2\alpha + \sin^2\alpha}}{\pm\sqrt{A^2 + B^2}} = \frac{1}{\pm\sqrt{A^2 + B^2}}.$$

Therefore

$$\cos\alpha = \frac{A}{\pm\sqrt{A^2 + B^2}},\quad \sin\alpha = \frac{B}{\pm\sqrt{A^2 + B^2}},\quad p = \frac{-C}{\pm\sqrt{A^2 + B^2}}.$$

Now p is necessarily a positive quantity; hence that sign is selected for $\sqrt{A^2 + B^2}$ which will make p a positive quantity.

Hence

$$\frac{A}{\pm\sqrt{A^2 + B^2}}\,x + \frac{B}{\pm\sqrt{A^2 + B^2}}\,y = \frac{C}{\pm\sqrt{A^2 + B^2}},$$

represents the equation $Ax + By + C = 0$ reduced to the form $x\cos\alpha + y\sin\alpha = p$.

(6). The equation $Ax+By+C=0$ may be written in the form

$$\frac{x-\left(-\frac{C}{A}\right)}{-B}=\frac{y-0}{A}.$$

Comparing this with

$$\frac{x-a}{\cos\theta}=\frac{y-b}{\sin\theta},$$

we have

$$\frac{\cos\theta}{-B}=\frac{\sin\theta}{A}=\frac{\sqrt{\cos^2\theta+\sin^2\theta}}{\pm\sqrt{A^2+B^2}}=\frac{1}{\pm\sqrt{A^2+B^2}};$$

whence $\cos\theta=\dfrac{-B}{\pm\sqrt{A^2+B^2}}$, $\sin\theta=\dfrac{A}{\pm\sqrt{A^2+B^2}}$,

where that sign is selected which will make $\sin\theta$ a positive quantity, for θ being less than 180°, its sine is necessarily positive.

We have thus found the direction-cosines of the line.

The general equation then, reduced to the required form, is

$$\frac{x-\left(-\frac{C}{A}\right)}{\frac{-B}{\pm\sqrt{A^2+B^2}}}=\frac{y-0}{\frac{A}{\pm\sqrt{A^2+B^2}}}.$$

When the equation is expressed in this form, the denominators being the direction-cosines of the line, we may put its terms equal to r, where r is the distance from $\left(-\frac{C}{A}, 0\right)$ to (x, y).

Retaining the direction-cosines in the denominators, we may replace $-\frac{C}{A}, 0$ by the co-ordinates of any point through which the line passes.

Exercises.

1. Find the equation of the straight line through the intersection of the straight lines $x-3y=0$, $2x+y-1=0$, and passing through the point $(-2, 3)$. [$(x-3y)+k(2x+y-1)=0$ represents a locus through the intersection of these straight lines (§ 12); since it is of the first degree in x and y, it represents a straight line. Putting $x=-2$, $y=3$, we find k.]

2. Find the equation of a straight line through the intersection of the lines $x+y+b=0$, $y=b$, and through the origin.

3. Show that the lines $2x-3y+4=0$, $3x-y-1=0$, $4x-3y+2=0$, all pass through one point. [All straight lines through the intersection of the first two are represented by $(2x-3y+4)+k(3x-y-1)=0$, or by $(2+3k)x-(3+k)y+4-k=0$; and of this system one is $4y-3y+2=0$, provided a value of k can be found that will satisfy $\frac{2+3k}{4}=\frac{3+k}{3}=\frac{4-k}{2}$. The value $k=\frac{6}{5}$ does this. Hence the three lines pass through a point. The concurrence of the lines can also be proved by finding values of x and y which satisfy the first two equations, and showing that these values satisfy the third.]

4. A straight line cuts off variable intercepts, a, b, on the axes, which, however, are such that $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}$, a constant. Show that all such lines pass through a fixed point. [The lines are represented by $\frac{x}{a}+\frac{y}{b}=1$; but $\frac{c}{a}+\frac{c}{b}=1$; introducing this relation, the equation of the lines becomes $\frac{1}{a}(x-c)+\frac{1}{b}(y-c)=0$, which always passes through the fixed point (c, c).]

5. A straight line slides with its ends on the axes of x and y, and the difference of the intercepts, a, b, on the axes is always proportional to the area enclosed; *i.e.*, $b-a=Cab$, where C is a constant. Show that the line always passes through a fixed point. [Line is $\frac{x}{a}+\frac{y}{b}=1$; also $\frac{1}{a}\cdot\frac{1}{C}-\frac{1}{b}\cdot\frac{1}{C}=1$; etc.]

6. **Find the direction-cosines of the line** $-5x+2y+4=0$.

7. The points (2, 3), (4, 8) both lie on the line $5x-2y-4=0$ of the previous exercise. Write the line in the form $\frac{x-a}{l}=\frac{y-b}{m}=r$, using these points.

8. **Find the distance from the point (2, 3) along the line** $5x-2y-4=0$ (of the previous exercise), **to the line** $x+2y-10=0$. [See Ex. 10, p. 50.]

9. Reduce the equation $3x+4y+12=0$ to the standard form $\frac{x-a}{l}=\frac{y-b}{m}=r$, the point (a, b) being the point where the line cuts the axis of x.

10. In the standard form $\frac{x-a}{\cos\theta}=\frac{y-b}{\sin\theta}=r$, for what direction from the point (a, b) are the values of r positive, and for what direction negative? [Discuss the equation $y-b=r\sin\theta$, where $\sin\theta$ is always positive.]

11. Find the direction-cosines of the line $2x-y=2$.

12. **Employ the standard form** $\frac{x-a}{l}=\frac{y-b}{m}=r$ **to find the middle point of that segment of the line** $2x-y=2$ **which is intercepted by the lines** $x+2y=4$, $3x+4y=12$. [Let the point be (a, b), so that $2a-b=2$. Then the line $2x-y=2$ may be written $\frac{x-a}{\frac{1}{\sqrt{5}}}=\frac{y-b}{\frac{2}{\sqrt{5}}}=r$, and $x=a+\frac{r}{\sqrt{5}}$, $y=b+\frac{2r}{\sqrt{5}}$. Substituting these values in $x+2y=4$, we get, as distance from middle point to $x+2y=4$, $r=\frac{1}{\sqrt{5}}(4-a-2b)$. Similarly, distance from middle point to line $3x+4y=12$ is $r=\frac{\sqrt{5}}{11}(12-3a-4b)$. These values of r are equal with opposite signs, being measured in opposite directions; hence $\frac{1}{\sqrt{5}}(4-a-2b)=-\frac{\sqrt{5}}{11}(12-3a-4b)$, or $13a+21b=52$; also $2a-b=2$; whence $a=\frac{94}{55}$, $b=\frac{78}{55}$.]

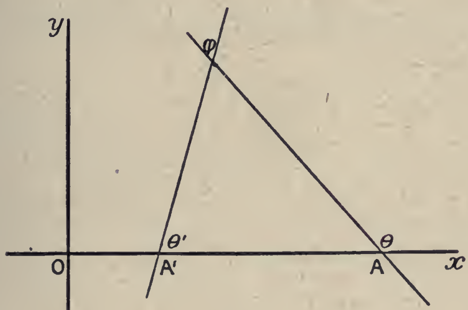
13. **Find the equation of the locus of the bisections of all lines terminated by the lines** $x+2y=4$, $3x+4y=12$, **and having the same direction as the line** $2x-y=2$. [If (a, b)

be the bisection of *any* of such lines, by the preceding exercise we get the relation $13a+21b=52$. Hence $13x+21y=52$ is the equation of the locus.]

14. Lines are drawn parallel to the line $2x-y=2$. Find the loci of points which divide the parts of such lines intercepted by the lines $x+2y=4$, $3x+4y=12$, both internally and externally in the ratio $2:1$, *i.e.*, distance to $x+2y=4$ double that to $3x+4y=12$.

What do these loci, with the lines $x+2y=4$, $3x+4y=12$, form?

24. To find the angle between two straight lines whose equations are given.



(1). Let the equations of the lines be $y=mx+b$, $y=m'x+b'$, where $m=\tan\theta$ and $m'=\tan\theta'$, θ and θ' being the angles which the lines make with the axis of x. Let ϕ be the angle between the lines.

$$\text{Then } \tan\phi=\tan(\theta-\theta'),$$
$$=\frac{\tan\theta-\tan\theta'}{1+\tan\theta\tan\theta'},$$
$$=\frac{m-m'}{1+mm'},$$

which determines the angle between the lines.

(2). If the equations of the lines be $Ax+By+C=0$, $A'x+B'y+C'=0$, these may be written

$$y=-\frac{A}{B}x-\frac{C}{B},\ y=-\frac{A'}{B'}x-\frac{C'}{B'};\ \text{and}\ m=-\frac{A}{B},\ m'=-\frac{A'}{B'}.$$

$$\text{Hence}\ \tan\phi=\frac{-\frac{A}{B}+\frac{A'}{B'}}{1+\frac{A}{B}\cdot\frac{A'}{B'}}$$

$$=\frac{A'B-AB'}{AA'+BB'}.$$

25. **When the equations of two straight lines are given, to find the conditions for parallelism and perpendicularity.**

If the lines be parallel, $\phi=0°$, and $\tan\phi=0$; if perpendicular $\phi=90°$, and $\tan\phi=\infty$. Hence, referring to the forms for $\tan\phi$ in § 24, we see that,—

(1). If the equations of the lines be $y=mx+b$, $y=m'x+b'$,

condition for parallelism is $m=m'$;

" " perpendicularity is $1+mm'=0$, or $m'=-\frac{1}{m}$.

(2). If the equations of the lines be $Ax+By+C=0$, $A'x+B'y+C'=0$,

condition for parallelism is $A'B-AB'=0$, or $\frac{A}{A'}=\frac{B}{B'}$;

" " perpendicularity is $AA'+BB'=0$.

Exercises.

1. If ϕ be the angle between the lines $Ax+By+C=0$, $A'x+B'y+C'=0$, show that

$$\cos\phi=\frac{1}{\sec\phi}=\frac{1}{\sqrt{1+\tan^2\phi}}=\ldots=\frac{AA'+BB'}{\sqrt{A^2+B^2}\ \sqrt{A'^2+B'^2}}.$$

2. **If the lines** $Ax+By+C=0$, $A'x+B'y+C'=0$ **be converted into the form** $x\cos\alpha+y\sin\alpha=p$, **so becoming**

$$\frac{A}{\sqrt{A^2+B^2}}x+\frac{B}{\sqrt{A^2+B^2}}y+\frac{C}{\sqrt{A^2+B^2}}=0, \text{ etc., } \textbf{show that}$$

$$\cos\phi=\cos(\alpha-\alpha')=\ldots=\frac{AA'+BB'}{\sqrt{A^2+B^2}\ \sqrt{A'^2+B'^2}}.$$

3. **Find the equation of the straight line through the point** $(2, -3)$, **and parallel to the line** $2x-5y-1=0$. [All lines through $(2, -3)$ are represented by $A(x-2)+B(y+3)=0$, (§ 22); and for that which is parallel to $2x-5y-1=0$ we must have $\frac{2}{A}=\frac{-5}{B}$. Hence $\frac{2}{A}\cdot A(x-2)+\frac{-5}{B}\cdot B(y+3)=0$, or $2x-5y-19=0$ is the equation required.]

4. **Find the equation of the straight line through the point** $(2, -3)$ **and perpendicular to the line** $2x-5y-1=0$. [Here again all lines through $(2, -3)$ are represented by $A(x-2)+B(y+3)=0$, and condition of perpendicularity is $2A-5B=0$.]

5. The angular points of a triangle are $(3, -4)$, $(4, 5)$, $(-2, -6)$. Find the equations of the lines through the angular points and perpendicular to the opposite sides. [Line through $(3, -4)$, $(4, 5)$ is $9x-y-31=0$, and, transposing coefficients and changing sign, evidently $x+9y+k=0$ is perpendicular to this. Then find k from the fact that this line passes through $(-2, -6)$.]

6. The equations of the sides of a triangle are $x+2y-5=0$, $2x+y-7=0$, $x-y+1=0$. Find the equations of the straight lines through the angular points and perpendicular to the opposite sides. [All straight lines through the intersection of $x+2y-5=0$, $2x+y-7=0$ are represented by $(x+2y-5)+k\,(2x+y-7)=0$; that which is perpendicular to $x-y+1=0$ requires $1(1+2k)-1(2+k)=0$; whence k.]

7. Find the co-ordinates of the foot of the perpendicular from the origin on the line $7x-5y-6=0$.

8. Find the angle between the lines $3x+y-2=0$, $2x-y-3=0$.

9. Find the equation of the straight line perpendicular to $\frac{x}{a}+\frac{y}{b}=1$, and passing through the point (a, b).

10. Find the equation of the straight line which makes an intercept a on the axis of x, and is perpendicular to $Ax+By+C=0$.

11. Find the equations of the lines through (2, 3), and making an angle of 30° with $x-2y+6=0$.

12. Find the equations of the straight lines through the origin, and making an angle of 45° with the line $x+y=5$. [Let line be $y=mx$. Then $\pm 1=\dfrac{m+1}{1-m}$; etc.]

13. Write down the equations of the straight lines perpendicular to $x \cos \alpha + y \sin \alpha = p$, the perpendiculars from the origin on them being both p'.

14. Form the equations to two lines through the points (3, -2) and (4, 3), respectively, and at right angles to each other.

Why do undetermined constants appear in the equations?

15. Find the angle between the lines $3x+y+12=0$, and $x+2y-1=0$. Infer the angle between the lines $6x+2y-1=0$ and $3x+6y+5=0$; also between the lines $3x-9y-5=0$ and $4x-2y+7=0$.

16. What relation exists between the lines $5x-3y+10=0$, $6y-10x+9=0$? What is the distance between them?

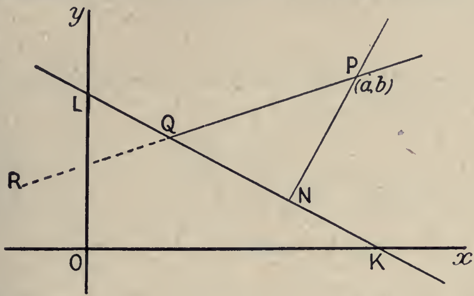
17. The equations of the sides of a triangle are $x-y+1=0$, $7x-4y+1x=0$, $8x-5y-1=0$. Find the equations of the lines through the angular points parallel to the opposite sides.

18. Find the condition that the line $y=mx+b$ may be perpendicular to the line $x \cos \alpha + y \sin \alpha = p$.

19. Find the equation of the line through (4, 5) and parallel to $2x-3y-5=0$. Find the distance between these parallel lines. What therefore is the perpendicular distance from (4, 5) to $2x-3y-5=0$? [p in $2x-3y-5=0$ is $\dfrac{\sqrt{5}}{13}$; and for parallel line is $\dfrac{7}{\sqrt{13}}$; and these are on opposite sides of the origin.]

20. The equations of two sides of a triangle are $3x+4y-12=0$ and $2x-y+4=0$. Find the equation of the third side that the origin may be the orthocentre of the triangle.

26. To find the distance from a given point to a given straight line, estimated in a given direction.



Let $P(a, b)$ be the given point, and $KL\,(Ax+By+C=0)$ the given straight line. Suppose PQR the direction in which the distance to the given line is to be estimated, and let this direction be determined by its direction-cosines l, m.

Then the equation of PQ is

$$\frac{x-a}{l}=\frac{y-b}{m}=r;\ldots(1)$$

whence $x=a+lr,\ y=b+mr,\ldots(2)$

where r is the distance from (a, b) to (x, y).

If we substitute the expressions of (2) for x and y in $Ax+By+C=0$, we are assuming that the x and y of (1) are the same as the x and y of $Ax+By+C=0$. Hence (x, y) must be the point Q where the lines intersect, being the only point for which x and y have the same values for both PQ and KL.

Substituting

$$A(a+lr)+B(b+mr)+C=0;$$

$$\text{and } PQ=r=-\frac{Aa+Bb+C}{Al+Bm},$$

which is the distance required.

Numerical exercises embodying the principle of this proposition have already been given. Exs. 12, 13, 14, pp. 58 and 59.

27. To find the perpendicular distance from a given point to a given straight line.

Let (a, b) be the given point, and $Ax+By+C=0$ the given straight line.

Then the direction given by the line

$$\frac{x-a}{l}=\frac{y-b}{m}=r$$

in § 26, becomes now the direction PN, which is perpendicular to $Ax+By+C=0$. The condition for perpendicularity requires

$$\frac{1}{l}\cdot A-\frac{1}{m}\cdot B=0;$$

$$\text{whence } \frac{l}{A}=\frac{m}{B}=\pm\frac{\sqrt{l^2+m^2}}{\sqrt{A^2+B^2}}=\pm\frac{1}{\sqrt{A^2+B^2}};$$

$$\text{and } l=\pm\frac{A}{\sqrt{A^2+B^2}},\quad m=\pm\frac{B}{\sqrt{A^2+B^2}}.$$

Substituting these values for l and m in the expression for r, § 26, we have

$$PN=r=\pm\frac{Aa+Bb+C}{\sqrt{A^2+B^2}},$$

which is the perpendicular distance required.

If only the *magnitude* of the perpendicular distance is required, the algebraic sign before the expression for PN may be neglected. The question of the sign of PN will be considered in § 28.

If the given point be the origin, a and b are both zero, and the perpendicular distance becomes

$$\frac{C}{\sqrt{A^2+B^2}},$$

an expression we have already met with.

Exercises.

1. Find the distance from the origin to the line $2x+5y-10=0$, in a direction making an angle of 45° with the axis of x.

2. Find the distance from the point (7, 1) to the axis of y, in a direction making an angle of 60° with the axis of x.

3. Find l, m, the direction-cosines of the line $\frac{x-a}{l}=\frac{y-b}{m}=r$, such that the part of this line intercepted by the axes of x and y shall be bisected at the point (a, b). [$x=a+lr$, $y=b+mr$; and equations of axes are $y=0$, $x=0$. Hence substituting and putting values of r equal with opposite signs, $\frac{a}{-l}=\frac{b}{m}$; etc.]

4. The equations of the sides of a triangle are $2x+9y+17=0$, $7x-y-38=0$, $x-2y+2=0$. Employ the method of § 26 to find the length of the side whose equation is $7x-y-38=0$. [Intersection of last two is (6, 4); and $7x-y-38=0$ may be written in form

$$\frac{x-6}{\frac{1}{5\sqrt{2}}}=\frac{y-4}{\frac{7}{5\sqrt{2}}}=r;$$

whence distance from (6, 4) to intersection with $2x+9y+17=0$.].

5. In the triangle of the preceding exercise, find the lengths of the perpendiculars from the angles on the opposite sides.

6. Find the orthocentre of the triangle in Exercise 4.

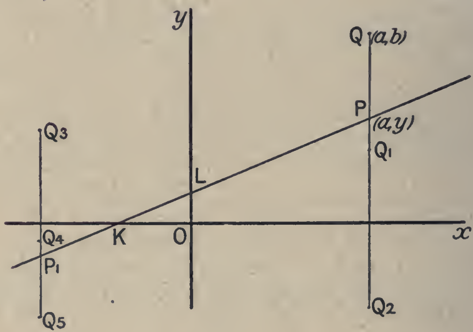
7. Find the distance along the line through the points (3, –1), (–4, 5) to the line $A(x+4)+B(y-5)=0$.

8. Find the equations of the lines perpendicular to the line $x-2y+8=0$, and at distance 3 from the origin.

9. Show that the locus of a point which moves so that the sum or difference of its distances from two given straight lines is constant, is itself a straight line.

10. Express by an equation the relation that must hold between a and b that the point (a, b) may be equally distant from the lines $3x-4y+5=0$ and $x+2y-7=0$. Give two results.

28. It is, of course, only when (a, b) is a point on the line $Ax+By+C=0$ that the expression $Aa+Bb+C$ vanishes. A little consideration will show that $Aa+Bb+C$ is positive for all points (a, b) on one side of the line $Ax+By+C=0$, and negative for all points on the other:



Let Q be the point (a, b), and KL the line $Ax+By+C=0$.

Since when $P\ (a, y)$ is a point on the line, $Aa+By+C$ vanishes, therefore

$$Aa+Bb+C=Aa+Bb+C-(Aa+By+C)$$
$$=B\,(b-y).$$

An examination of the sign of $(b-y)$ for different positions of (a, b) as $Q, Q_1, \ldots,$ when (a, y) is P or P_1, will show that $b-y$ is positive on one side of the line, and negative on the other side. Hence $Aa+Bb+C$ is positive on one side of the line and negative on the other, the factor B being of constant sign.

Or we may say at once that when Q is on one side of the line, $b-y$ represents a distance measured in direction PQ, and on the other side, in direction QP; and that therefore $b-y$ has opposite signs on opposite sides of the line.

Hence $Aa+Bb+C$ has opposite signs on opposite sides of the line $Ax+By+C=0$, (a, b) being a point not on this line.

We may thus speak of the *positive* and *negative sides* of a line.

If we can ascertain the sign of $Aa+Bb+C$ for any point (a, b) outside the line, we know at once the positive and negative sides of $Ax+By+C=0$. We naturally select the origin ($a=0$, $b=0$) for this examination. If then C be positive, the origin is on the positive side of $Ax+By+C=0$; and if C be negative, the origin is on the negative side.

If the line be of the form $Ax+By=0$, we may substitute the co-ordinates of any point on either of the axes, and so determine the positive and negative sides. Thus for the line $3x-4y=0$, the positive part of the axis of x is on the positive side, and the positive part of the axis of y is on the negative side.

If we change the form of the equation of the line from $Ax+By+C=0$ to $-Ax-By-C=0$, the former positive side now becomes the negative, and the negative side the positive.

If in the expression for the perpendicular from the point (a, b) on the line $Ax+By+C=0$,

$$\frac{Aa+Bb+C}{\sqrt{A^2+B^2}},$$

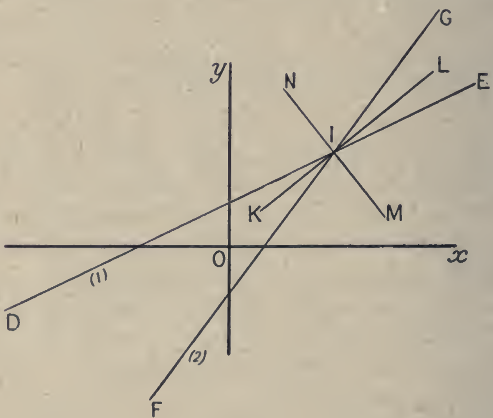
we agree to consider $\sqrt{A^2+B^2}$ always positive, the sign of the perpendicular is the same as the sign of $Aa+Bb+C$, and therefore is positive on the positive side of the line, and negative on the negative side.

The preceding considerations are of importance in forming the equations of the loci of points equidistant from two given straight lines; *i.e.*, the equations of lines bisecting the angles between two given straight lines. We can best illustrate this by a numerical example:

Suppose the equations of two given lines are

$$x-2y+4=0 \ldots (1)$$
$$4x-3y-6=0 \ldots (2)$$

Evidently the origin is on the positive side of (1), and on the



negative side of (2); *i.e.*, the region within the angle DIF is positive for (1) and negative for (2); within the angle GIE, negative for (1) and positive for (2); within the angle DIG, negative for (1) and negative for (2); within the angle FIE, positive for (1) and positive for (2). We have thus the regions within which the perpendiculars on (1) and (2) are of the same or of opposite signs. Hence if we write

$$\frac{x-2y+4}{\sqrt{5}} = -\frac{4x-3y-6}{5} \quad .. \ (3),$$

we condition that (x, y) should be a point within either of the angles *DIF*, *GIE*, such that the perpendiculars from it on (1) and (2) are numerically equal. Thus (3) is the equation of the line *KIL* which bisects the angles *DIF*, *GIE*.

If we write

$$\frac{x-2y+4}{\sqrt{5}} = \frac{4x-3y-6}{5} \quad .. \ (4),$$

we condition that (x, y) shall be a point within either of the angles *DIG*, *FIE*, such that the perpendiculars from it on (1) and (2) are numerically equal. Thus (4) is the equation of the line *NIM* which bisects the angles *DIG*, *FIE*.

Exercises.

1. On which side, or sides, of the line $4x-3y-5=0$ do the points (1, 2), (4, 2) lie? On which side, or sides, of the line $4x-9y=0$? On which side, or sides, of the line $9y-4x=0$?

2. Find the equations of the lines bisecting the angles between the lines $x-3y=0$, $4x-3y=0$; and having constructed these lines, from the mode of derivation of each of the bisectors, place them properly with respect to $x-3y=0$ and $4x-3y=0$.

From the equations of the bisectors how must they be situated with respect to one another?

3. Find the equations of the lines bisecting the angles between the lines $x-y+6=0$, $x-5y+30=0$; and distinguish between the bisectors. Verify your result from a figure in which the bisectors are placed by finding approximately their intercepts on the axes.

4. The angular points of a triangle are (1, 2), (4, 3), (3, 6). Find the equations of the lines bisecting the interior angles of this triangle. Prove that these bisecting lines all pass through a point.

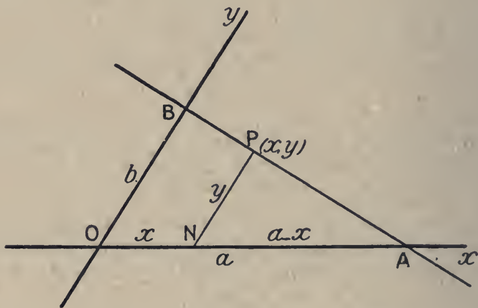
5. In the triangle of the preceding exercise find the bisectors of the exterior angles at (1, 2), (4, 3).

Prove that they intersect on the bisector of the interior angle at (3, 6).

IV. Oblique Axes.

29. In certain cases it is convenient to employ oblique axes, and though the detailed consideration of such is beyond the purposes of the present work, the following propositions may be of service.

(1). To find the equation of a straight line in terms of the intercepts which it makes on the axes, supposed oblique.



The diagram being analogous to that of § 14, where the axes are rectangular, by similar triangles *PNA*, *BOA*,

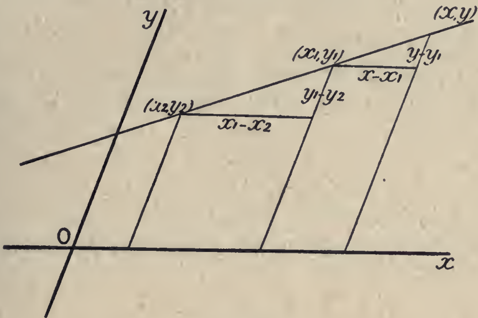
$$\frac{y}{b} = \frac{a-x}{a},$$

$$= 1 - \frac{x}{a};$$

$$\text{or} \quad \frac{x}{a} + \frac{y}{b} = 1,$$

which is the equation required, the form being the same as when the axes are rectangular.

(2). **To find the equation of a straight line which passes through two given points, axes oblique.**



The diagram and notation being analogous to that of § 15, where the axes are rectangular, by similar triangles

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2},$$

which is the equation required, the form being the same as when the axes are rectangular.

(3). **The general equation of the first degree,** $Ax + By + C = 0$, **always represents a straight line, axes oblique.**

It represents some locus. Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be *any* three points on the locus it represents.

$$\text{Then} \quad Ax_1 + By_1 + C = 0,$$
$$Ax_2 + By_2 + C = 0,$$
$$Ax_3 + By_3 + C = 0.$$

$$\text{Hence} \quad A(x_3 - x_1) + B(y_3 - y_1) = 0,$$
$$A(x_1 - x_2) + B(y_1 - y_2) = 0;$$

$$\text{and} \therefore \quad \frac{x_3 - x_1}{x_1 - x_2} = \frac{y_3 - y_1}{y_1 - y_2};$$

i.e., by § 29, (2), the point (x_3, y_3) must lie on the straight line through (x_1, y_1), (x_2, y_2). Hence, since any three points on the locus which $Ax+By+C=0$ represents are in a straight line, the locus must be a straight line.

(4). The equation $Ax+By+C=0$ represents any straight line. If it pass through the point (a, b) we have $Aa+Bb+C=0$. Therefore, combining this condition between A, B, C with the equation of the line, we have

$$A(x-a)+B(y-b)=0,$$

as the equation of any straight line through (a, b), the form being the same as when the axes are rectangular.

Exercises.

1. A straight line cuts off intercepts on oblique axes, the sum of the reciprocals of which is a constant quantity. Show that all such straight lines pass through a fixed point.

2. A straight line slides along the axes (oblique) of x and y, and the difference of the intercepts is always proportional to the area it encloses. Show that the line always passes through a fixed point. [Here $b-a=kab \sin \omega$, where k is a constant, and ω is the angle between the axes. Hence $\frac{1}{a} \cdot \frac{1}{k \sin \omega} - \frac{1}{b} \cdot \frac{1}{k \sin \omega} = 1$, etc.]

3. The base $(2a)$ and the straight line (b) from the vertex to the middle point of the base of a triangle being axes, find the equations of the straight lines which join the middle points of the other sides to the opposite angles. Find also their point of intersection.

4. Show that the straight lines $x-y=0$, $x+y=0$ are perpendicular at whatever angle the axes be inclined to one another.

5. OA, OB are two fixed straight lines inclined at any angle, and A, B are fixed points on them. Q and R are variable points on OA and OB, such that AQ is to BR in the constant ratio k. Show that the locus of the middle point of QR is a straight line. [Take OA, OB as axes of x and y. Let $P(x, y)$ be the middle point of QR; and let $AQ=\alpha$. Then $2x=a+\alpha$, $2y=b+k\alpha$, where $a=OA$, $b=OB$; eliminate α.]

6. Ox, Oy are two fixed straight lines intersecting at any angle. From a point P perpendiculars PN, PM are drawn to Ox, Oy, and through N and M lines are drawn parallel to Oy, Ox, meeting in Q. Show that if the locus of P be a straight line, the locus of Q is also a straight line. [Take Ox, Oy as axes. Let co-ordinates of P be x, y, and of Q, α, β. Let ω be the angle between Ox, Oy. Then since P moves in a straight line, $Ax+By+C=0$. Also $\alpha=x+y\cos\omega$, $\beta=y+x\cos\omega$; whence find x and y, and substitute in $Ax+By+C=0$.]

7. If the angular points of a triangle lie on three fixed straight lines which meet in a point, and two of the sides pass through fixed points, then the third side also passes through a fixed point. [Let A, B, C be the triangle, and Ox, Oy, $y=mx$ the three fixed straight lines, A lying on Ox, B on Oy, and C on $y=mx$. Let BC pass through a fixed point $P(f, g)$, and CA through a fixed point $Q(h, k)$. Let $OA=\alpha$, $OB=\beta$. Then forming the equations of BP, AQ, and introducing the condition that they intersect on $y=mx$, we shall obtain a relation between α and β which may be put in form

$$\frac{1}{\alpha}\cdot\text{const.}+\frac{1}{\beta}\cdot\text{const.}=1.]$$

CHAPTER IV.

CHANGE OF AXES.

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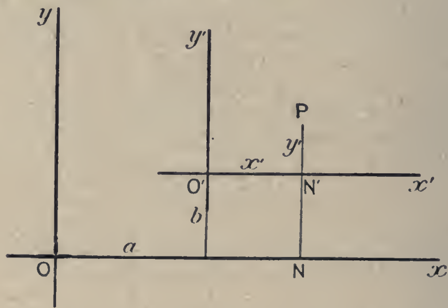
30. To change the origin of co-ordinates without changing the direction of the axes.



Let Ox, Oy be the original axes; $O'x'$, $O'y'$ the new axes having the same directions as the former. Let the point P have the co-ordinates x, y with respect to the original axes, and the co-ordinates x', y' with respect to the new axes.

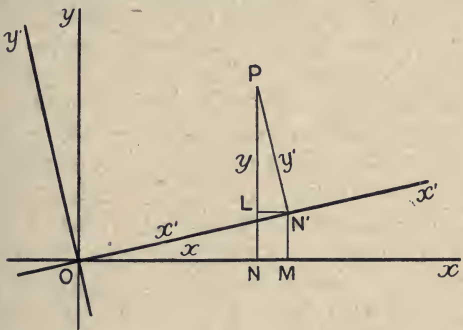
Let the co-ordinates of O' with respect to the original axes be a, b.

$$\text{Then } x = ON = a + O'N' = a + x';$$
$$y = NP = b + N'P = b + y'.$$

Hence the old co-ordinates of any point are expressed in terms of the new ones. Therefore, if in the equation of any locus we substitute $a+x'$ for x, and $b+y'$ for y, the new equation will be the equation of the same locus but referred to $O'x'$, $O'y'$ as axes.

In applying the forms we may save time by writing $x+a$ for x, and $y+b$ for y.

31. To change the directions of the axes without changing the origin, both systems being rectangular.



Let Ox, Oy be the original axes; Ox', Oy' the new axes; and θ the angle xOx' through which the axes have been turned. Let the point P have the co-ordinates x, y with respect to the original axes, and the co-ordinates x', y' with respect to the new axes.

Draw PN, PN' perpendicular to Ox, Ox' respectively, and $N'M$, $N'L$ perpendicular to Ox, PN respectively.

Then $\angle N'PL = x'Ox = \theta$; and

$$x = ON = OM - LN' = x' \cos\theta - y' \sin\theta;$$
$$y = NP = MN' + LP = x' \sin\theta + y' \cos\theta.$$

Hence the old co-ordinates of any point are expressed in terms of the new ones. Therefore, if in the equation of any locus we substitute $x' \cos\theta - y' \sin\theta$ for x, and $x' \sin\theta + y' \cos\theta$ for y, the new equation will be the equation of the same locus, but referred to Ox' Oy' as axes.

In applying the forms we may save time by writing $x\cos\theta - y\sin\theta$ for x, and $x\sin\theta + y\cos\theta$ for y.

32. These formulas for change of axes are chiefly used for the purpose of getting rid of certain terms of the equation of a locus, so simplifying the equation and making the discussion of the nature and properties of the locus less laborious. It will usually be found that the simplifying of the equation in this way is represented geometrically by the placing of the origin, or axes, or both, more symmetrically with respect to the locus. Indeed we might naturally expect that making the equation a more symmetric function of x and y, i.e., simplifying it, would be accompanied or represented geometrically by placing the corresponding locus more symmetrically with respect to the lines (axes) which give to x and y their meaning and values.

These general statements may well be illustrated by a numerical example.

Ex. To simplify the equation

$$13x^2 - 10xy + 13y^2 - 58x - 22y + 37 = 0. \ldots (1)$$

(a). Let us first examine whether we can transfer the origin to a point such that the terms of the first degree in x and y shall disappear from the equation,—

Transferring the origin to the point (h, k), the equation becomes

$$13(x+h)^2 - 10(x+h)(y+k) + 13(y+k)^2 - 58(x+h) - 22(y+k) + 37 = 0,$$

or

$$13x^2 - 10xy + 13y^2 + (26h - 10k - 58)x + (-10h + 26k - 22)y$$
$$+ 13h - 10hk + 13k^2 - 58h - 22k + 37 = 0. \ldots (2)$$

The terms involving first powers of x and y disappear if

$$26h - 10k - 58 = 0,$$
$$10h + 26k - 22 = 0.$$

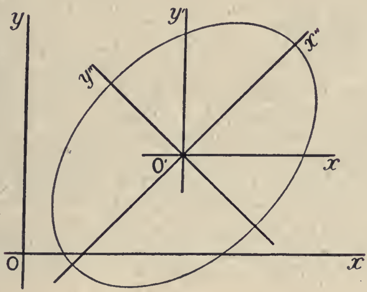
These equations are satisfied by $h = 3$, $k = 2$.

Substituting these values for h and k in (2), the equation becomes

$$13x^2 - 10xy + 13y^2 - 72 = 0, \ldots (3)$$

which represents what equation (1) becomes when the origin is transferred to the point (3, 2).

A knowledge of the locus we are dealing with would show that transferring the origin to the point (3, 2) is placing it at the point O' with respect to which the locus has *central symmetry*, the axes becoming $O'x'$ and $O'y'$.



(b). Let us next enquire whether we can turn the axes through an angle such that the term involving the product xy shall disappear from the equation.

Turning the axes through the angle θ, the equation becomes

$$13\,(x \cos\theta - y \sin\theta)^2 - 10\,(x \cos\theta - y \sin\theta)\,(x \sin\theta + y \cos\theta) + 13(x \sin\theta + y \cos\theta)^2 - 72 = 0,$$

or

$$(13 \cos^2\theta - 10 \sin\theta \cos\theta + 13 \sin^2\theta)\,x^2 + (-10 \cos^2\theta + 10 \sin^2\theta)xy + (13 \sin^2\theta + 10 \sin\theta \cos\theta + 13 \cos^2\theta)y^2 - 72 = 0. \ldots (4)$$

The term involving xy disappears if

$$-10 \cos^2\theta + 10 \sin^2\theta = 0,$$

or if $\tan\theta = 1$, *i.e.*, if $\theta = 45°$.

Substituting then $\frac{1}{\sqrt{2}}$ for $\sin\theta$ and $\frac{1}{\sqrt{2}}$ for $\cos\theta$ in (4), the equation becomes

$$4x^2+9y^2=36,$$

$$\text{or}\quad \frac{x^2}{9}+\frac{y^2}{4}=1.$$

which represents what equation (3) becomes when the axes at O' are turned through the angle 45°.

A knowledge of the locus we are dealing with would show that turning the axes at O' through 45° means making Ox'' and $O'y''$ the axes, with respect to both of which the locus has *axial symmetry.*

Exercises.

1. What substitutions for x and y are made in a given equation when the origin is transferred to the point (3, 4), the directions of the axes remaining the same?

2. What substitutions for x and y are made in a given equation when the origin is transferred to the point $(-4, -3)$, the directions of the axes remaining the same?

3. What substitutions for x and y are made in a given equation when the axes (rectangular) are turned in a positive direction through the angle 60°, the origin remaining the same?

4. What substitutions for x and y are made in a given equation when the axes (rectangular) are turned through the angle $-\frac{\pi}{6}$, the origin remaining the same?

5. What substitutions for x and y are made in a given equation when the axes are turned through 180°, the positive directions thus becoming the negative, and the negative, the positive?

6. Transform the origin to the point where the axis of y cuts the locus whose equation is $3x-5y-15=0$, and find what this equation becomes when referred to the new axes.

7. Find what the equations $x-y=0$ and $x+y=0$ become when the axes are turned through 45°.

8. Find what the equation $y^2-4x+4y+8=0$ becomes when the origin is transferred to the point $(1, -2)$; and trace a part of the graph of the equation when referred to the new axes.

9. Find what the equation $x^2+y^2-4x+6y=0$ becomes when the origin is transferred to the point $(2, -3)$.

10. Does the point $(0, -\frac{C}{B})$ lie on the locus of the equation $Ax+By+C=0$?

When the origin is transferred to this point what does this equation become?

11. What does the equation $x^2+y^2=r^2$ become when the origin is transferred to the point $(-a, -b)$, the directions of the axes being unchanged?

12. What does the equation $x^2+y^2-2x+4y+1=0$ become when the origin is transferred to the point $(1, -2)$? Trace the graph of the equation.

13. What does the equation $x^2+y^2=r^2$ become when the axes are turned through any angle θ, the origin remaining the same?

14. **Find the point to which the origin must be transferred that the equation $3x^2+4y^2-12x+8y+15=0$ may involve no terms of the first degree in x and y.**

[Let (h, k) be the point required. Then transferring origin to this point the equation becomes $3(x+h)^2+4(y+k)^2-12(x+h)+8(y+k)+15=0$, or $3x^2+4y^2+(6h-12)x+(8k+8)y+3h^2+4k^2-12h+8k+15=0$. Hence terms of first degree disappear if $6h-12=0$, and $8k+8=0$; *i.e.*, if $h=2$, $k=-1$; and the point to which origin must be transferred is $(2, -1)$.]

What does the equation become when the origin is transferred to this point?

15. Does the point $(0, -2)$ lie on the locus of the equation $x^2+xy-3x+y+2=0$?

Transfer the origin to a point which will make the constant term disappear from this equation. What does the equation become?

When no constant term appears in the equation of a locus, through what point must the locus pass?

16. **Through what angle must the axes be turned that the term involving xy in the equation $6x^2-4\sqrt{3}xy+10y^2=4$ may disappear?**

[Let θ be the angle required. Then substituting $x \cos \theta - y \sin \theta$ for x, and $x \sin \theta + y \cos \theta$ for y, the equation becomes $6(x \cos \theta - y \sin \theta)^2 - 4\sqrt{3}(x \cos \theta - y \sin \theta)(x \sin \theta + y \cos \theta) + 10(x \sin \theta + y \cos \theta)^2 = 4$, or $(6 \cos^2 \theta - 4\sqrt{3} \cos \theta \sin \theta + 10 \sin^2 \theta)x^2 + (-12 \cos \theta \sin \theta - 4\sqrt{3} \cos^2 \theta + 4\sqrt{3} \sin^2 \theta + 20 \sin \theta \cos \theta)xy + (6 \sin^2 \theta + 4\sqrt{3} \sin \theta \cos \theta + 10 \cos^2 \theta)y^2 = 4$. Hence that the term involving xy may disappear, we must have $8 \sin \theta \cos \theta - 4\sqrt{3}(\cos^2 \theta - \sin^2 \theta) = 0$, or $\sin 2\theta - \sqrt{3} \cos 2\theta = 0$, or $\tan 2\theta = \sqrt{3}$. Therefore $2\theta = 60°$, and $\theta = 30°$.]

What does this equation become when the axes are turned through this angle?

17. If the equation of a locus be $5x - 4y + 1 = 0$, through what angle must the axes be turned that the term involving x may disappear? What does the equation become?

18. Through what angle must the axes be turned that the term xy may disappear from the equation $xy = k^2$?

What does the equation become?

CHAPTER V.

THE CIRCLE.

In Synthetic Geometry the circle is defined, and from its definition the properties of the curve are deduced. In Analytical Geometry we define the circle, and from the definition form the equation of the curve, the equation being nothing more than the translation of the definition into analytic language. The equation being thus a special form of the definition of the circle, a consideration of the equation will reveal the properties of the curve.

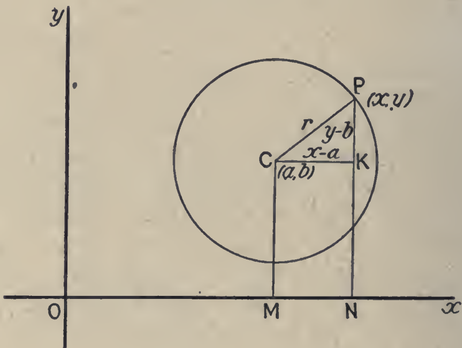
DEFINITION,—A circle is the locus of a point which moves in a plane so as to be always at a constant distance from a fixed point.

The constant distance is called the *radius*, and the fixed point the *centre* of the circle.

I. Equation of the Circle.

33. To find the equation of a circle whose centre and radius are given.

Let $C(a, b)$ be the centre, and r the radius of the circle; and suppose $P(x, y)$ any point on the circum-



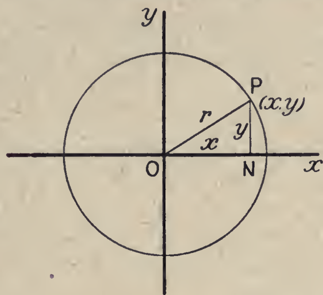
ference. Complete the figure as in the diagram. Then evidently $CK = x - a$, $KP = y - b$; and

$$(x - a)^2 + (y - b)^2 = r^2,$$

which is therefore the equation of the circle (§ 8).

It will be seen that the preceding equation expresses in algebraic language the characteristic property, or law, of the circle, namely, the constancy of the distance (r) of the moving point (x, y) from the fixed point (a, b).

COR. 1. If the centre of the circle be at the origin, $a=0$, $b=0$; and the preceding equation reduces to

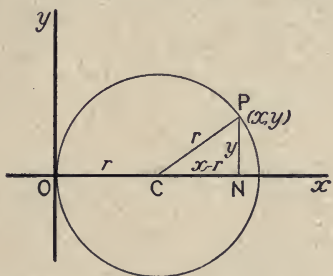


the form

$$x^2+y^2=r^2,$$

as may also be seen at once from the preceding figure. This form of the equation of the circle is the one generally used, being the simplest.

COR. 2. If the centre of the circle be at the point $(r, 0)$, then $a=r$, $b=0$; and the equation becomes



$$(x-r)^2+(y-0)^2=r^2,$$

$$\text{or } x^2+y^2-2rx=0,$$

a form which may also be obtained from the preceding figure.

34. To interpret geometrically the equation

$$x^2 + y^2 + 2Ax + 2By + C = 0,$$

where A, B and C are constants, axes being rectangular.

This equation may be written in the form

$$x^2 + 2Ax + A^2 + y^2 + 2By + B^2 = A^2 + B^2 - C,$$

or $$\{x - (-A)\}^2 + \{y - (-B)\}^2 = A^2 + B^2 - C.$$

The left side of this equation expresses (§ 4) the square of the distance of the moving point (x, y) from the fixed point $(-A, -B)$; and the equation declares that the square of this distance is equal to $A^2 + B^2 - C$, which is constant.

Hence the equation

$$x^2 + y^2 + 2Ax + 2By + C = 0,$$

when the axes are rectangular, is the equation of a circle whose centre is $(-A, -B)$, and whose radius is $\sqrt{A^2 + B^2 - C}$.

Exercises.

1. Find the equations of the following circles from the formula of § 33. Also construct the circles, and find their equations from the figures without using the formula:

(1). Centre $(4, -3)$; radius 5.
(2). Centre $(3, 2)$; radius 4.
(3). Centre $(-4, 0)$; radius 3.
(4). Centre $(-5, -5)$; radius 5.
(5). Centre $(-3, 2)$; radius $\sqrt{13}$.

2. Find the co-ordinates of the centre and the radius of each of the following circles:

(1). $x^2 + y^2 - 6x - 2y + 6 = 0$.
(2). $x^2 + y^2 + 6x + 2y + 6 = 0$.
(3). $x^2 + y^2 + 8x = 0$.
(4). $(x + y)^2 + (x - y)^2 = 4$.
(5). $x^2 + y^2 = ax + by$.
(6). $x^2 + y^2 + 2fx + 2gy + f^2 + g^2 = c^2$.
(7). $a(x^2 + y^2) = bx + cy$.

3. Find the equation of the circle which has for diameter the line joining the points (1, 2), (5, 5).

4. Find the points at which the circle $x^2+y^2-7x-11y+10=0$ intersects the axis of x. [At such points $y=0$. Putting $y=0$ in the equation of the circle, $x^2-7x+10=0$; whence the values of x at the points of intersection.]

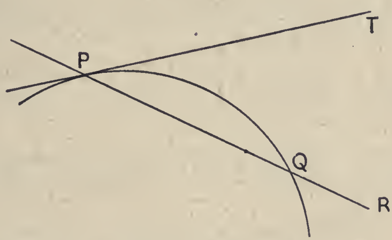
5. In the equation of a circle, $x^2+y^2+2Ax+2By+C=0$, what must be the value of C if the circle passes through the origin? [If the circle passes through the origin, the equation must be satisfied by $x=0$, $y=0$.]

6. In the equation of the preceding question what are the intercepts on the axis, the circle passing through the origin?

7. Find the equation of the circle which passes through the origin, and cuts off lengths a, b, from the axes.

II. Tangents and Normals.

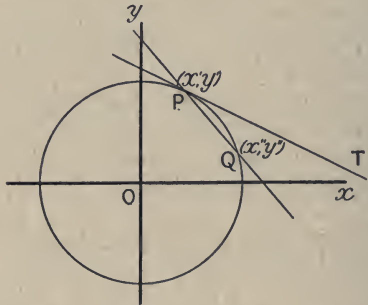
35. DEFINITIONS. A straight line which meets a curve will in general intersect it in two or more points. Such a line is called a **secant** to the curve, as PQR.



P and Q being successive points of intersection of the secant with the curve, if Q move along the curve so as to approach indefinitely close to P, the limiting position of PQR, say PT, is called the **tangent** at P; and P is called the **point of contact** of the tangent PT.

The tangent is thus a straight line passing through two points on the curve which are indefinitely close to one another.

36. To find the equation of the tangent to the circle $x^2+y^2=r^2$ in terms of the co-ordinates of the point of contact (x', y').



Let PQ be a secant through the points $P(x', y')$ and $Q(x'', y'')$ on the circle $x^2+y^2=r^2$.

The equation of the line through (x', y'), $(x''. y'')$ is

$$\frac{x-x'}{x'-x''}=\frac{y-y'}{y'-y''},$$

$$\text{or } y-y'=\frac{y'-y''}{x'-x''}(x-x')\ldots\ldots(1).$$

Also since (x', y'), (x'', y'') lie on the circle $x^2+y^2=r^2$, therefore

$$x'^2+y'^2=r^2,$$
$$x''^2+y''^2=r^2;$$

and therefore

$$(x'^2-x''^2)+(y'^2-y''^2)=0\ldots\ldots(2),$$

$$\text{or } \frac{y'-y''}{x'-x''}=-\frac{x'+x''}{y'+y''}\ldots\ldots(3).$$

Hence (1) becomes

$$y - y' = -\frac{x' + x''}{y' + y''}(x - x') \ldots\ldots (4).$$

Let now the point (x'', y'') move up indefinitely close to (x', y'); then PQ becomes PT, the tangent at P; also $x'' = x'$, $y'' = y'$, and (4) becomes

$$y - y' = -\frac{x'}{y'}(x - x') \ldots\ldots (5).$$

Hence $xx' + yy' = x'^2 + y'^2 = r^2 \ldots\ldots (6)$; and

$$xx' + yy' = r^2$$

is the equation of the tangent to the circle at the point (x', y').

It is important to note the significance of the different stages in arriving at the equation $xx' + yy' = r^2$:

(1) is the equation of the line through *any* two points (x', y'), (x'', y''), the coefficient $\frac{y' - y''}{x' - x''}$, being the tangent of the angle which the line makes with the axis of x (§18).

(2) is the condition that the points (x', y'), (x'', y'') are two points on *any* circle whose centre is at the origin, for r does not appear in this condition.

(3), which is another form of condition (2), gives the tangent of the angle which the secant through (x', y'), (x'', y'') makes with the axis of x.

(4) is therefore the equation of a line through two points (x', y'), (x'', y'') or *any* circle whose centre is at the origin, provided (3) is true;

and (5) is therefore the equation of a line through (x', y') and a contiguous point, both points being on *any* circle whose centre is at the origin; and $-\frac{x'}{y'}$, in this is the *ultimate* value of the tangent of the angle at which PQ, now PT, is inclined to the axis x.

But when in (6) we put $x'^2 + y'^2$ equal to r^2, the circle ceases to be *any* circle, and becomes that whose radius is r.

× 37. **To find the equation of the tangent in terms of its inclination to the axis of x.**

Let θ be the angle which the tangent makes with the axis of x; and let $\tan\theta = m$.

Then the tangent may be represented by $y = mx + b$, where b is yet to be found.

If we treat the equations

$$y = mx + b,$$
$$x^2 + y^2 = r^2,$$

as simultaneous, the resulting values of x and y must be the co-ordinates of the points in which the straight line intersects the circle (§ 11).

Hence the values of x in

$$x^2 + (mx + b)^2 - r^2 = 0,$$
$$\text{or } (1 + m^2)x^2 + 2mbx + b^2 - r^2 = 0 \quad \ldots . \quad (1)$$

must be the values of x at the points where the straight line intersects the circle. If these values of x are equal, the points of intersection coincide, and the straight line is a tangent.

The condition for equal values of x is

$$(1 + m^2)(b^2 - r^2) = m^2 b^2,$$
$$\text{or } b = \pm r\sqrt{1 + m^2}.$$

Hence

$$y = mx \pm r\sqrt{1 + m^2}$$

is the equation of the tangent to the circle, having an inclination θ to the axis of x ($m = \tan\theta$). The double sign refers to the parallel tangents at the extremities of any diameter; these though differing in position have the same inclination to the axis of x.

Equation (1), being a quadratic, shows that a straight line cuts a circle in two points.

The following is an alternative demonstration of the preceding proposition:

We have shown that the equation $xx'+yy'=r^2$ is the tangent at the point (x', y'). If now the equations

$$xx'+yy'-r^2=0,$$
$$mx-y+b=0,$$

represent the same straight line, then

$$\frac{m}{x'}=\frac{-1}{y'}=\frac{b}{-r^2}.$$

$$\text{But } \frac{m}{x'}=\frac{-1}{y'}=\frac{\mp\sqrt{1+m^2}}{\sqrt{x'^2+y'^2}}=\mp\frac{\sqrt{1+m^2}}{r};$$

$$\text{therefore } \frac{b}{-r^2}=\frac{\mp\sqrt{1+m^2}}{r},$$

$$\text{or } b=\pm r\sqrt{1+m^2};$$

and $y=mx\pm r\sqrt{1+m^2}$ is a tangent to the circle $x^2+y^2=r^2$.

38. The straight line drawn through any point on a curve, perpendicular to the tangent at that point, is called the **normal.**

To find the equation of the normal to the circle $x^2+y^2=r^2$ at the point (x', y').

The equation of *any* straight line through the point (x', y') is

$$A(x-x')+B(y-y')=0 \ldots . (1).$$

If this be the normal at (x', y') it is perpendicular to the tangent

$$xx'+yy'=r^2;$$

and the condition for perpendicularity (§ 25) is

$$Ax'+By'=0 \ldots\ldots (2).$$

Introducing in (1) the relation between A and B given by (2), and so making (I) the normal, we have for the equation of the normal at (x', y')

$$x-x'-\frac{x'}{y'}(y-y')=0,$$

$$\text{or } \frac{x}{x'}=\frac{y}{y'}.$$

The form of this equation shows that the line it represents passes through the origin, which for the circle $x^2+y^2=r^2$ is the centre. Hence *the normal at any point of a circle passes through the centre.*

Exercises.

1. Write down the equations of the tangents to the circle $x^2+y^2=25$ at the points

$(-5, 0)$; $(3, -4)$; $(-1, 2\sqrt{6})$; also at the points whose abscissa is -2.

2. Write down the equations of the tangents to the circle $x^2+y^2=r^2$ which have the following inclinations:

30° to axis of x; 60° to axis of x; 45° to axis of x.

3. Find the points of contact of the tangents in the preceding exercise. [Identifying $x-y\sqrt{3}\pm 2r=0$ with $xx'+yy'-r^2=0$, we have $\frac{x'}{1}=\frac{y'}{-\sqrt{3}}=\frac{-r^2}{\pm 2r}$; &c.]

4. Find the equations of the tangents to the circle $x^2+y^2=r^2$ which are

(1) parallel to $\frac{x}{a}+\frac{y}{b}=1$;

(2) parallel to $Ax+By+C$;

(3) perpendicular to $Ax+By+C$.

[The first equation may be written $y=-\frac{b}{a}x+b$, so that $m=-\frac{b}{a}$.]

5. Find the equations of the tangents to the circle $x^2+y^2=r^2$ which pass through the point $(a, 0)$. [If $xx'+yy'=r^2$ be such a tangent, $ax'=r^2$, and $x'=\frac{r^2}{a}$. Also $x'^2+y'^2=r^2$; &c.]

6. Find the values of k that the line $y=mx+k$ may touch the circle $x^2+y^2=4$. [Either of methods of § 37.]

7. Find the condition that the line $Ax+By+C=0$ may be a tangent to the circle $x^2+y^2=r^2$. [Either of methods of § 37.]

8. Find the equations of the tangents to the circle $x^2+y^2=r^2$ which pass through the point $\{(1+\sqrt{3})r, (1+\sqrt{3})r\}$. [Use the equation $y=mx+r\sqrt{1+m^2}$.]

9. Find the equation of the circle whose centre is at the origin, and which touches the line $x+y\sqrt{3}-6=0$. [Assume $x^2+y^2=r^2$ as circle, and obtain condition that line touches it.]

10. Find the equations of the circles which touch the positive directions of the axes of co-ordinates, and also the line $x+2y=4$. [The equation of the circle touching the axes is $x^2+y^2-2rx-2ry+r^2=0$; for in this putting $y=0$, the values of x are equal, etc.]

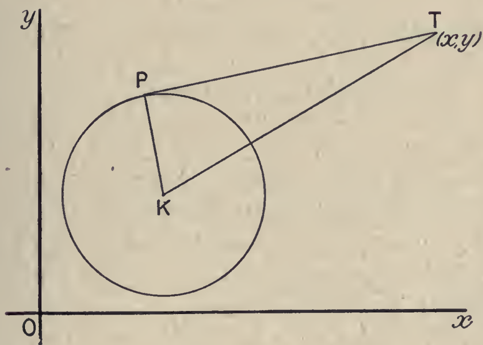
III. Radical Axes.

✗ **39. To interpret geometrically the expression**

$$x^2+y^2+2Ax+2By+C$$

when (x, y) is not a point on the circle

$$x^2+y^2+2Ax+2By+C=0.$$



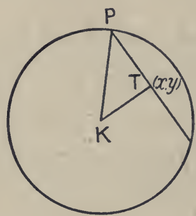
Let the circle of the diagram be the circle $x^2+y^2+2Ax+2By+C=0$. Then K, its centre, is the point $(-A, -B)$; and KP, its radius, is $\sqrt{A^2+B^2-C}$ (§ 34). Let T be the point (x, y), and let PT be a tangent passing through T. Then

$$x^2+y^2+2Ax+2By+C=\{x-(-A)\}^2+\{y-(-B)\}^2 - \{A^2+B^2-C\},$$

$$=KT^2-KP^2,$$

$$=PT^2.$$

Hence when (x, y) is a point external to the circle $x^2+y^2+2Ax+2By+C=0$, the expression $x^2+y^2+2Ax+2By+C$ is the square of the tangent from (x, y) to the circle.



If the point $T(x, y)$ be within the circle, in like manner

$$\begin{aligned} x^2+y^2+2Ax+2By+C &= KT^2-KP^2, \\ &= -(KP^2-KT^2), \\ &= -PT^2, \end{aligned}$$

PT being the chord which is bisected at T, and the angle PTK being therefore a right angle (§ 46, i).

Hence in this case $x^2+y^2+2Ax+2By+C$ is the square of half the chord bisected at (x, y), with negative sign prefixed.

In both cases PT^2 is equal to the product of the segments of any chord through T (§ 45, Cor.).

Hence in both cases we may say that $x^2+y^2+2Ax+2By+C$ represents the product of the segments of any chord through (x, y), the negative sign occurring when the point is within the circle, since then the segments are measured in opposite directions from T.

40. The equations of two circles being

$$x^2+y^2+2Ax+2By+C=0, \quad \ldots \quad (1)$$
$$x^2+y^2+2A'x+2B'y+C'=0, \quad \ldots \quad (2)$$

if we place the left sides of these equations equal to one another, so obtaining the equation

$$x^2+y^2+2Ax+2By+C=x^2+y^2+2A'x+2B'y+C', \dots . (3)$$

then (x, y) in (3) must be a point such that the squares of the tangents (or products of segments of chords), and therefore the tangents, from (x, y) to (1) and (2) are equal to one another.

But (3) reduces to

$$2(A-A')x+2(B-B')y+C-C'=0, \dots . (4)$$

which, being of the first degree, represents a straight line.

Hence *the locus of points from which the tangents to two given circles are equal is a straight line.* Such a locus is called the **radical axis** of the two circles. Equation (4) is the radical axis of the circles (1) and (2).

A convenient notation for (1) and (2) is $S=0$, $S'=0$; so that $S-S'=0$ is the radical axis. If $S=0$, $S'=0$ intersect, then (§ 12) the points of intersection lie on $S-S'=0$. Hence *when two circles intersect, their radical axis passes through the points of intersection.*

41. The equation of the straight line joining the centres $(-A, -B)$, $(-A', -B')$ of the circles (1) and (2) in § 40 is

$$\frac{x+A}{-A+A'}=\frac{y+B}{-B+B'},$$

$$\text{or } \frac{x}{A-A'}-\frac{y}{B-B'}+\frac{A}{A-A'}-\frac{B}{B-B'}=0;$$

and this (§ 25) is evidently perpendicular to the radical axis

$$2(A-A')x+2(B-B')y+C-C'=0.$$

Hence *the radical axis of two circles is at right angles to the line joining their centres.*

42. The radical axes of three given circles are concurrent.

Let the three circles be represented by

$$S=0,\ S'=0,\ S''=0.$$

Taking these two and two together, their radical axes are

$$S-S'=0,\ S'-S''=0,\ S''-S=0.$$

$$\text{But}\quad (S-S')+(S'-S'')=0$$

is (§ 12) a straight line through the intersection of the first two radical axes; and this reduces to $S''-S=0$, which is the third radical axis.

The point in which the radical axes of three circles intersect is called the **radical centre** of the circles.

Exercises.

1. Find the centre and radius of the circle $Kx^2+Ky^2+2Ax+2By+C=0$.

2. In the case of the circle of the preceding exercise, what is the expression for the square of the tangent from the point (x, y), this point being without the circle?

3. Find the expression for the square of the tangent from the point (x, y) to the circle

$$x^2+y^2+2Ax+2By+C+\lambda(x^2+y^2+2A'x+2B'y+C')=0.$$

4. Show that, as λ varies, all the circles of the form given in the preceding exercise have the same radical axis. What is its equation? [Take any two values of λ, say λ_1 and λ_2; etc.]

5. Granted that this series of circles has but one radical axis, how do you explain the fact that this radical axis is the same as that of the circles

$$x^2+y^2+2Ax+2By+C=0,$$
$$x^2+y^2+2A'x+B'y+C'=0.$$

6. Prove that the square of the tangent that can be drawn from any point on one circle to another is proportional to the perpendicular from this point to the radical axis of the circles. [Square of tangent from (a, b) on $x^2+y^2+2A'x+2B'y+C'=0$ to $x^2+y^2+2Ax+By+C=0$

is $x^2+y^2+2Aa+2Bb+C$. Also $x^2+y^2+2A'a+B'b+C'=0$. Hence square of tangent $=x^2+y^2+2Aa+2Bb+C-(x^2+y^2+2A'a+2B'b+C')$ $=2(A-A')a+2(B-B')b+C-C'$; etc.]

7. Show that any pair of the system of circles represented by $x^2+y^2+2\lambda x+c=0$, where λ is variable, has the same radical axis.

8. As a point moves round one of the circles of the preceding system, tangents are drawn to two other circles of the same system. Show that the ratio of these tangents is constant. [Let k be value of λ for first circle, and g, h for two other circles. Then squares of tangents are $2(g-k)x$, $2(h-k)x$, Ex. 6.]

9. There is a series of circles all of which pass through two fixed points, and also a fixed circle. Show that the radical centre is a fixed point. [The series is represented by $x^2+y^2+2Ax+2\lambda y+C=0$, where λ is variable, and A, C constants; for such pass through two fixed points on axis of x. Axis of x is their radical axis. Let $x^2+y^2+2Px+2Qy+R=0$ be fixed circle. Radical axis of it and any one of series is $2(P-A)x+2(Q-\lambda)y+R-C=0$, etc.]

10. Three circles have fixed centres, and their radii are $r_1+\lambda$, $r_2+\lambda$, $r_3+\lambda$, where λ is a variable. Show that their radical centre lies on a fixed straight line. [Let circles be $x^2+y^2=(r_1+\lambda)^2$, $(x-a)^2+y^2=(r_2+\lambda)^2$, $(x-b)^2+(y-c)^2=(r_3+\lambda)^2$. The rad. axis of first and second is $2ax-a^2=(r_1-r_2)(r_1+r_2+2\lambda)$; and of first and third $2bx+2cy-b^2-c^2=(r_1-r_3)(r_1+r_3+2\lambda)$. Eliminate λ.]

IV. Poles and Polars.

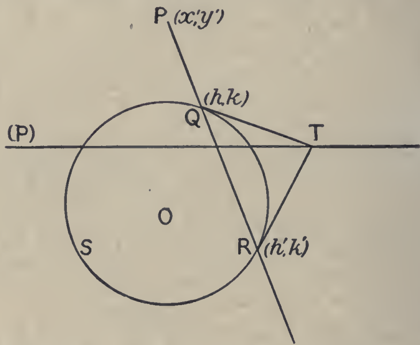
DEFINITION. The **polar** of any point P with respect to a circle is the locus of the intersection of tangents drawn at the ends of any chord which passes through P.

The point P is called the **pole** of the locus. It may be either within or without the circle; if it be on the circumference, the locus is evidently the tangent at P.

43. To find the polar of any given point (x', y') with respect to the circle $x^2+y^2=r^2$.

Let QRS be the circle $x^2+y^2=r^2$, and P the given point (x', y'). Let a chord through P cut the circle in

$Q(h, k)$ and $R(h', k')$; and let QT, RT be the tangents at (h, k), (h', k'). Then as the chord through P assumes different positions, and in consequence T changes its position, the locus of T is the polar of P.



The tangents at (h, k) and (h', k') are

$$xh + yk = r^2,$$
$$xh' + yk' = r^2.$$

Hence the co-ordinates of T satisfy these equations, and therefore the co-ordinates of T satisfy

$$x(h - h') + y(k - k') = 0.$$

But since (x', y') lies on the straight line through (h, k), (h', k'), therefore (§ 15)

$$\frac{x' - h}{h - h'} = \frac{y' - k}{k - k'}.$$

Hence the co-ordinates of T always satisfy

$$x(h - h')\frac{x' - h}{h - h'} + y(k - k')\frac{y' - k}{k - k'} = 0;$$

therefore they always satisfy

$$x(x' - h) + y(y' - k) = 0,$$
$$\text{or } xx' + yy' = xh + yk = r^2;$$

that is, $xx' + yy' = r^2$ is the equation of the locus of T, and is therefore the equation of the polar of (x', y')

The polar of P may conveniently be denoted by enclosing P in brackets, —(P).

COR. 1. The equation of the line joining (x', y') to the origin is

$$\frac{x}{x'} = \frac{y}{y'},$$

and (§ 25) this is evidently perpendicular to

$$xx' + yy' = r^2.$$

Hence *in the circle the polar is perpendicular to the line joining the centre to the pole.*

COR. 2. If O be the centre of the circle, and ON the perpendicular from O on (P), $xx' + yy' = r^2$, then (§ 27)

$$ON = \frac{r^2}{\sqrt{x'^2 + y'^2}};$$

$$\text{also } OP = \sqrt{n'^2 + y'^2};$$

$$\therefore OP \,.\, ON = r^2.$$

COR. 3. Let the pole $P(x' y')$ be without the circle, and let the point of contact of the tangent through P be (α, β). Then, the equation of the tangent being $x\alpha + y\beta = r^2$, since it passes through (x', y'),

$$x'\alpha + y'\beta = r^2 \; \ldots\ldots \; (1).$$

Also the equation of the polar of (x', y') is

$$xx' + yy' = r^2 \; \ldots\ldots \; (2).$$

It is evident therefore from (1) that (α, β) satisfies (2), *i.e.*, that the point of contact of the tangent from the pole lies on the polar.

This conclusion, however, is evident from the figure. For when PQR becomes a tangent, the points Q, R and T coincide in the point of contact of the tangent from P.

Hence, when the pole is without the circle, we may construct the polar by drawing from the pole two tangents to the circle, and drawing a straight line through the points of contact. If the pole be within the circle, the relation (Cor. 2) $OP \cdot ON = r^2$ suggests the construction: Draw a chord through P at right angles to OP, and at its extremity draw a tangent meeting OP in N. The line through N parallel to the chord is the polar.

44. **If $Q(x'', y'')$ lies on the polar of $P(x', y')$, then P lies on the polar of Q.**

For the polar of $P(x', y')$ is

$$xx' + yy' = r^2.$$

If $Q(x'', y'')$ lies on this, then

$$x''x' + y''y' = r^2.$$

But this is the condition that $P(x', y')$ may lie on $xx'' + yy'' = r^2$, which is the polar of $Q(x'', y'')$.

Cor. 1. If therefore a point Q moves along the polar of P, the polar of Q always passes through P; *i.e.*, **if a point moves along a fixed straight line, the polar of the point turns about a fixed point,** such fixed point being the pole of the fixed straight line.

Cor. 2. A special case of the preceding corollary is,—**The straight line which joins two points P and Q is the polar of the intersection of the polars of P and Q.**

45. **A chord of a circle is divided harmonically by any point on it and the polar of that point.**

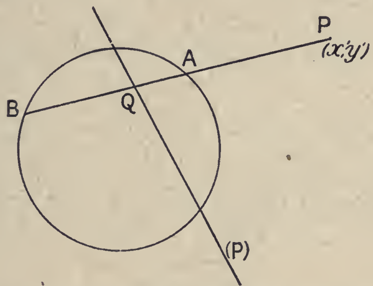
Let (x', y') be the pole P; then $xx' + yy' = r^2$ is the polar (P). Also a chord PAB through (x', y') is represented by

$$\frac{x-x'}{l}=\frac{y-y'}{m}=k,$$

$$\text{or } x=x'+lk,\ y=y'+mk, \ \ldots\ldots \ (1)$$

where k represents the distance from (x', y') to (x, y).

In combining (1) with the equation of the circle, (x, y) must be the point which is common to chord and circle, *i.e.*, must be A or B; and therefore k must be PA or PB.



Similarly in combining (1) with the equation of the polar, k must be PQ.

Combining (1) with the equation of the circle $x^2+y^2=r^2$,

$$(x'+lk)^2+(y'+mk)^2=r^2,$$

$$\text{or } k^2+2(lx'+my')k+x'^2+y'^2-r^2=0,$$

$$\text{since } l^2+m^2=1.$$

Hence, since PA, PB are the roots of this quadratic in k,

$$PA+PB=-2(lx'+my');\ PA\,.\,PB=x'^2+y'^2-r^2;$$

and therefore $\dfrac{1}{PA}+\dfrac{1}{PB}=-\dfrac{2(lx'+my')}{x'^2+y'^2-r^2} \ \ldots\ldots \ (2).$

Again, combining (1) with the equation of the polar, $xx'+yy'=r^2$,

$$(x'+lk)x'+(y'+mk)y'=r^2,$$

$$\text{or } (lx'+my')k+x'^2+y'^2-r^2=0.$$

Hence, since PQ is the root of this equation in k,

$$PQ = -\frac{x'^2 + y'^2 - r^2}{lx' + my'} \quad \ldots\ldots (3).$$

Therefore from (2) and (3)

$$\frac{1}{PA} + \frac{1}{PB} = \frac{2}{PQ},$$

and AB is divided harmonically in P and Q.

COR. Since incidentally $PA \,.\, PB$ has been shown equal to $x'^2 + y'^2 - r^2$, an expression independent of the direction-cosines l, m, which give the direction of the chord, therefore the product $PA \,.\, PB$ is constant for all directions through P. Hence **if a chord of a circle pass through a fixed point, the rectangle contained by the segments of the chord is constant.**

Exercises.

1. Find the polars of the points (3, 6), (2, 5), (-6, -8), with respect to the circle $x^2 + y^2 = 25$.

2. Find the points of contact of tangents from the point (7, 1) to the circle $x^2 + y^2 = 25$. [Find points of intersection of polar with circle.]

3. Find the pole of the line $3x - 2y + 5 = 0$ with respect to the circle $x^2 + y^2 = 17$. [Let $xx' + yy' = 17$, which is the polar of (x', y'), be the line $3x - 2y + 5 = 0$. Then $\frac{x'}{3} = \frac{y'}{-2} = \frac{-17}{5}$; etc.]

4. Find the poles of the lines

$$Ax + By + C = 0, \quad \frac{x}{a} + \frac{y}{b} = 1,$$

with respect to the circle $x^2 + y^2 = r^2$.

5. Find the locus of the pole of the line $y = mx + b$ with respect to the circle $x^2 + y^2 = r^2$, m being variable and b constant. [This line always passes through the fixed point $(0, b)$; therefore its pole always moves along, etc.]

6. Find the locus of the pole of the line $y=mx+b$ with respect to the circle $x^2+y^2=r^2$, m being constant and b variable. [Let pole be (x', y'). Then $xx'+yy'-r^2=0$ and $mx-y+b=0$ represent the same line; therefore $\frac{x'}{m}=$ etc.]

7. If the poles lie on the line $Ax+By+C=0$, obtain the general equation of the polars, the circle being $x^2+y^2=r^2$. [The polars all pass through the point $\left(-\frac{A}{C}r^2, -\frac{B}{C}r^2\right)$.]

8. When does the polar become a chord of contact; and when a tangent?

9. Show that if the point (x', y') lies on the circle $x^2+y^2=k^2$, its polar with respect to the circle $x^2+y^2=r^2$ touches the circle $x^2+y^2=\frac{r^4}{k^2}$.

10. Show that if the polar of (x', y') with respect to the circle $x^2+y^2=r^2$ touches the circle $x^2+y^2=k^2$, then (x', y') lies on the circle $x^2+y^2=\frac{r^4}{k^2}$.

11. Prove that the distances of two points (x', y'), (x'', y'') from the centre of a circle are proportional to the distance of each from the polar of the other with respect to the circle. [Distance of (x', y') from the polar of (x'', y'') is $\frac{x'x''+y'y''-r^2}{\sqrt{x''^2+y''^2}}$; etc.]

12. Find the conditions that must be fulfilled that the line $Ax+By+C=0$ may be the polar of (a, b) with respect to the circle $x^2+y^2=r^2$.

46. The following are analytical solutions of propositions familiar in synthetic geometry:

(i). The line from the centre of a circle to the middle point of a chord is perpendicular to the chord.

Let (x', y'), (x'', y'') be the extremities of the chord. Then $\frac{1}{2}(x'+x'')$, $\frac{1}{2}(y'+y'')$ are the co-ordinates of its middle point. Hence the equation of the line through the centre of the circle $x^2+y^2=r^2$ and the middle point of the chord is

$$\frac{x}{\frac{1}{2}(x'+x'')}=\frac{y}{\frac{1}{2}(y'+y'')}. \quad \ldots (1)$$

Also, § 36, (4), the equation of the chord through (x', y'), (x'', y'') is

$$(x-x')(x''+x')+(y-y')(y'+y'')=0. \quad \ldots (2)$$

And evidently (1) and (2) fulfil the condition of perpendicularity (§ 25).

(ii). **The perpendicular from the centre of a circle on a chord bisects the chord.**

Let (x', y'), (x'', y'') be the extremities of the chord. Its equation is, § 36, (4),

$$(x-x')\,(x'+x'')+(y-y')\,(y'+y'')=0.$$

Also the equation of a line through the centre of the circle and perpendicular to this is (§ 25)

$$x(y'+y'')-y(x'+x'')=0\,;$$

and this latter line evidently passes through $\{\tfrac{1}{2}(x'+x''),\ \tfrac{1}{2}(y'+y'')\}$, which is the middle point of the chord.

(iii). **To find the locus of the bisections of parallel chords of a circle.**

Let $x^2+y^2=r^2$ be the circle; also let l, m be the direction-cosines of the chords, and (a, b) the middle point of any one of them.

The equation of the chord is

$$\frac{x-a}{l}=\frac{y-b}{m}=k,$$

$$\text{or}\quad x=a+lk,\ y=b+mk.$$

If we combine this with the equation of the circle, k will represent the distance from (a, b) to the points of intersection of the chord with the circle; and since (a, b) is the middle point of the chord, the values of k must be equal with opposite signs. Combining the equations,

$$(a+lk)^2+(b+mk)^2=r^2,$$

$$\text{or}\quad k^2+2(al+bm)k+a^2+b^2-r^2=0.$$

Since the values of k are equal with opposite signs

$$al+bm=0.$$

Now (a, b) is the middle point of *any* chord of those which are parallel. Hence the relation holding between the co-ordinates of every middle point of the set is

$$lx+my=0,$$

which therefore is the locus required.

Its form shows that it passes through the origin which is the centre of the circle; and also that it is perpendicular to the chords it bisects; *i.e.*, the locus of the bisections of a system of parallel chords of a circle is a diameter perpendicular to the chords. Evidently this may be inferred from (ii), and the preceding is given for the sake of the method.

(iv). To find the locus of the middle points of the chords of a circle which pass through a fixed point.

Let the circle be $x^2+y^2=r^2$, and the fixed point $P(a, b)$. Let a chord through P cut the circle in A and B, and let $N(\alpha, \beta)$ be the middle point of AB.

Then the chord may be represented by the equation

$$\frac{x-a}{l}=\frac{y-b}{m}=k,$$

$$\text{or} \quad x=a+lk, \ y=b+mk.$$

If we combine this with the equation of the circle, k will represent PA or PB. Hence PA, PB are the values of k in the equation

$$k^2+2(al+bm)k+a^2+b^2-r^2=0.$$

$$\therefore \ PA+PB=-2(al+bm);$$

$$\text{and} \quad PN=\tfrac{1}{2}(PA+PB)=-(al+bm).$$

$$\text{Hence} \quad PN^2+alPN+bmPN=0.$$

$$\text{But} \quad PN^2=(\alpha-a)^2+(\beta-b)^2;$$

$$\text{also} \quad lPN=\alpha-a, \ mPN=\beta-b.$$

$$\therefore \ (\alpha-a)^2+(\beta-b)^2+a(\alpha-a)+b(\beta-b)=0,$$

$$\text{or} \quad \alpha^2+\beta^2-a\alpha-b\beta=0,$$

which is the relation always holding between α, β, the co-ordinates of the middle point of AB. Hence

$$x^2+y^2-ax-by=0$$

is the equation of the locus required.

Putting this equation in the form

$$(x-\tfrac{1}{2}a)^2+(y-\tfrac{1}{2}b)^2=\tfrac{1}{4}(a^2+b^2),$$

we see that the locus is a circle whose centre is the middle point of the line joining P to the centre of $x^2+y^2=r^2$, and which passes through $P(a, b)$ and the centre of $x^2+y^2=r^2$; *i.e.*, the locus is the circle described on OP as diameter.

(v). The angle in a semicircle is a right angle.

Let the circle be $x^2+y^2=r^2$, and $A(r, 0)$, $A'(-r, 0)$, the extremities of the diameter which cuts off the semicircle. Also let $P(\alpha,\beta)$ be any point on the circle.

The equation of PA, through (α, β), $(r, 0)$ is

$$\frac{x-\alpha}{\alpha-r}=\frac{y-\beta}{\beta-0},$$

$$\text{or} \quad \beta x-(\alpha-r)y-\beta r=0. \ \ldots\ldots \ (1)$$

Similarly, the equation of PB, through (α, β) $(-r, 0)$ is

$$\beta x-(\alpha+r)y+\beta r=0. \ \ldots\ldots \ (2)$$

The condition for the perpendicularity of (1) and (2) is

$$\beta^2+\alpha^2-r^2=0,$$

which holds, since (α, β) is on the circle $x^2+y^2=r^2$.

(vi). **Angles in the same segment of a circle are equal to one another.**

Let the axis of x be taken parallel to the chord of the segment. Then, § 46, (iii), the extremities of the chord, C, C', may be represented by $(a, b,)$, $(-a, b)$. Let $P(\alpha, \beta)$ be any point on the arc.

The equation of PC, through (α, β), (a, b) is

$$\frac{x-\alpha}{\alpha-a}=\frac{y-\beta}{\beta-b}.$$

The equation of PC', through (α, β), $(-a, b)$ is

$$\frac{x-\alpha}{\alpha+a}=\frac{y-\beta}{\beta-b}.$$

Therefore (§ 24)

$$\tan CPC'=\frac{\dfrac{\beta-b}{\alpha-a}-\dfrac{\beta-b}{\alpha+a}}{1+\dfrac{\beta-b}{\alpha-a}\cdot\dfrac{\beta-b}{\alpha+a}},$$

$$=\frac{2a(\beta-b)}{2b(b-\beta)}, \text{ since } \alpha^2+\beta^2=r^2=a^2+b^2.$$

$\therefore \tan CPC'=-\dfrac{a}{b}$, and is the same whatever be the position of $P(\alpha, \beta)$.

Exercises.

1. Find the locus of a point which moves so that the sum of the squares of its distances from two fixed points, say $(a, 0)$, $(-a, 0)$, is constant and equal to c^2.

2. A point moves so that the square of its distance from the base of an isosceles triangle is equal to the rectangle under its distances from the sides. Show that its locus is a circle. [Let base $=2a$, perp. ht. $=b$. Take centre of base for origin, and base for axis of x. Then equations of sides are $\dfrac{x}{a}+\dfrac{y}{b}-1=0$, $-\dfrac{x}{a}+\dfrac{y}{b}-1=0$, etc.]

3. A straight line moves so that the product of the perpendiculars on it from two fixed points is constant. Prove that the locus of the feet of these perpendiculars is a circle, being the same circle for both

feet. [Let constant product $=c^2$; line be $y=mx+k$; points be $(a, 0)$, $(-a, 0)$. Then $c^2=\frac{k^2-m^2a^2}{1+m^2}$, or $c^2+a^2=\frac{k^2+a^2}{1+m^2}$. Also line through $(a, 0)$ perpendicular to $y=mx+k$ is $x+my-a=0$; and at intersection $x=\frac{a-mk}{1+m^2}$, $y=\frac{ma+k}{1+m^2}$; etc.]

4. Find the locus of a point within the circle $x^2+y^2=r^2$, which moves so that the rectangle under the segments of the chords through it is equal to r times the perpendicular on the line $x=r$.

5. Find the locus of a point without the circle $x^2+y^2=r^2$, which moves so that the rectangle under the segments of the chords through it (or the square of the tangent from it) is equal to r times the perpendicular on the line $x=r$.

6. A point moves so that the sum of the squares of its distances from the sides of an equilateral triangle is constant $(=c^2)$. Show that the locus of the point is a circle. [Let sides be $2a$; origin at centre of base; base be axis of x. Then sides are

$$\frac{x}{a}+\frac{y}{a\sqrt{3}}=1,\ -\frac{x}{a}+\frac{y}{a\sqrt{3}}=1;\ \text{etc.}]$$

7. From a fixed point O a straight line OP is drawn to a fixed straight line. In OP a point Q is taken such that $OP.OQ=c^2$, a constant. Find the locus of Q. [Let fixed point be origin, and fixed line $x-a=0$.]

8. From a fixed point $A(a, 0)$ within the circle $x^2+y^2=r^2$ a straight line AP is drawn to the circumference, and produced to Q, so that $AQ=nAP$. Find the locus of Q.

9. Find the locus of a point which moves so that the length of the tangent from it to the fixed circle $x^2+y^2=r^2$ is in a constant ratio to the distance of the moving point from a fixed point (a, b).

10. Two lines through the points $(a, 0)$, $(-a, 0)$ intersect at an angle θ. Find the equation of the locus of their point of intersection.

CHAPTER VI.

THE PARABOLA.

47. Definitions. A **Conic Section,** or **Conic,** is the locus of a point which moves so that its distance from a fixed point is in a constant ratio to its distance from a fixed straight line.

The fixed point is called the **focus**; the fixed straight line is called the **directrix**; the constant ratio is called the **eccentricity,** and is usually denoted by e.

When the eccentricity, e, is equal to unity, the conic is called a **parabola**; when less than unity, an **ellipse**; when greater than unity, an **hyperbola.**

A conic section is so called because if a right circular cone be cut by any plane, the curve of section will, in all cases, be a conic as defined above. It was as sections of a cone that these curves were first known and their properties investigated.

It will be observed as we proceed that, in determining the equations of the conics from their definitions, we select as axes lines specially placed with respect to the curves. This is done that the equations may be obtained in their simplest forms. See § 32.

The parabola ($e = 1$) is the simplest of these curves, and it will be first considered.

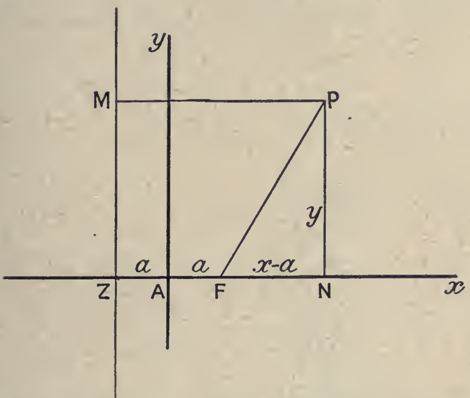
As in the case of the circle, we shall form the equation of the parabola from its definition, the equation being thus the expression of the law of the curve in algebraic language. An examination of the equation will then reveal the properties of the parabola.

I. Equation and Trace of the Parabola.

× **48. To find the equation of the Parabola.**

In the parabola the eccentricity, e, is unity, and the distance of a point on the curve from the focus is equal to its distance from the directrix.

Let $P(x,y)$ be any point on the curve; F the focus; ZM the directrix; PM perpendicular to ZM. Then $PF=PM$.



Let ZAx, through the focus and at right angles to the directrix, be the axis of x. Let A be the bisection of ZF, so that $AF=AZ$, and A is a point on the curve. Take A as the origin, and Ay as the axis of y. Then $AN=x$, $NP=y$. Let $ZA=AF=a$.

$$\text{Then } FP=MP=ZN;$$

$$\therefore FP^2=ZN^2,$$

$$\text{and } FN^2+NP^2=(ZA+AN)^2;$$

$$\therefore (x-a)^2+y^2=(a+x)^2,$$

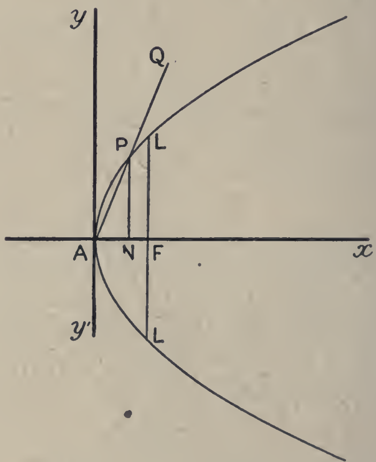
$$\text{or } y^2=4ax,$$

which therefore is the equation of the parabola.

In the above the positions of the directrix and focus are supposed to be given, *i.e.*, the distance ($ZF=2a$) of the focus from the directrix. This distance is the quantity which individualizes the curve, or distinguishes it from other parabolas. Thus a is a known quantity, just as r, the radius of the circle, is a known quantity in the equation $x^2+y^2=r^2$. Such quantities as r and a are called *parameters*.

49. To trace the form of the Parabola from its equation.

(1). $y^2=4ax$. When $x=0$, $y=0$, *i.e.*, the curve passes through the origin.



(2). $y^2=4ax$. Since y^2 is always positive, therefore x is always positive, a being supposed positive. Hence the curve does not exist to the left of the origin A.

(3). $y = \pm 2\sqrt{ax}$. Therefore for a given value of x, the values of y are equal with opposite signs. Hence the curve is symmetrical with respect to Ax. This line Ax, with respect to which the parabola is symmetrical, is called its **axis.**

(4). Also as x increases, y increases; and when x becomes indefinitely great, y is indefinitely great. Thus as the generating point recedes from yAy', both above and below Ax, it recedes from Ax; and the curve consists of infinite branches above and below its axis.

(5). If we suppose the straight line $y = mx + b$ to cut the parabola $y^2 = 4ax$, we shall have for the x's of the points of intersection the equation $(mx+b)^2 - 4ax = 0$, or $m^2x^2 + 2(mb - 2a)x + b^2 = 0$,—a quadratic, giving two values of x. Hence a straight line can cut a parabola in only two points.

(6). If P be any point on the curve, and it be supposed to move along the curve indefinitely close to A, the line APQ is ultimately the tangent at A, and the angle PAN is then the angle at which the curve cuts the axis of x.

$$\tan PAN = \frac{NP}{AN} = \frac{y}{x} = \frac{4a}{y}, \text{ since } y^2 = 4ax.$$

Therefore ultimately $\tan PAN = \frac{4a}{0} = \infty$; and the angle PAN in the limit is $90°$. Hence the curve cuts the axis of x at right angles.

Collating these facts, we see that the parabola has the form given in the diagram. In § 9, Ex. 2, we plotted the graph of the parabola $y^2 = 4x$, *i.e.*, for which $a = 1$.

Later on, when the equation of the tangent is reached, we shall be in a position to show that, as we recede from the origin, the direction of the curve becomes less and less inclined to the axis of x, and "at infinity" is parallel to the axis of x.

The point A is called the **vertex** of the curve.

50. To find the distance of any point on the parabola from the focus.

In the diagram of § 48,

$$\text{distance of } P \text{ from focus} = PF = MP = ZN,$$
$$= a + x.$$

DEF. The double ordinate through the focus, LFL' of the diagram of § 49, is called the *latus rectum*.

51. To find the length of the latus rectum of the parabola.

The co-ordinates of L are a, FL. Substituting these in $y^2 = 4ax$,

$$FL^2 = 4a \cdot a = 4a^2,$$
$$\text{and } FL = 2a;$$
$$\therefore \text{latus rectum } L'FL = 4a.$$

Exercises.

1. Find the equation of the parabola, taking the directrix as the axis of y, and the axis of the curve as the axis of x.

2. Find the equation of the parabola, taking the axis of the curve as the axis of y, and the tangent at the vertex as the axis of x.

3. Trace the curve whose equation is $x^2 = 4by$, the axes being placed as usual.

4. Find the equation of the parabola, taking the vertex as origin, and the tangent at the vertex as the axis of y, the curve existing only to the left of the origin.

5. If the distance of a point from the focus of the parabola $y^2 = 4ax$ be $4a$, find its co-ordinates.

6. Find where the line $12x - 7y + 12 = 0$ cuts the parabola $y^2 = 12x$.

7. Find where the line $y=x-a$, which passes through the focus, *i.e.*, is a *focal chord*, cuts the parabola $y^2=4ax$.

8. Place the following curves correctly with respect to the axes,—

$$y^2=-4ax\,;\ x^2=4ay\,;\ x^2=-4ay.$$

9. Find the co-ordinates of the point, other than the origin, where the parabolas $y^2=4ax$, $x^2=4by$ intersect. If AN, NP be these co-ordinates, show that

$$\frac{4a}{NP}=\frac{NP}{AN}=\frac{AN}{4b},$$

i.e., that NP, AN are two geometric means between $4a$ and $4b$.

10. A chord through the vertex of the parabola $y^2=4ax$ makes an angle of 30° with the axis of x. Find the length of the chord.

11. If the straight line $y=m(x-a)$, which as m varies represents any line through the focus, cuts the parabola $y^2=4ax$ in points whose ordinates are y_1, y_2, show that $y_1y_2=-4a^2$, *i.e.*, that the product of the ordinates of the ends of a focal chord is constant.

12. If the directrix be $Ax+By+C=0$, and the focus (a, b), show that the equation of the parabola is $(x-a)^2+(y-b)^2=\dfrac{(Ax+By+C)^2}{A^2+B^2}$.

NOTE. This equation may be reduced so that the terms of two dimensions take the form $(Bx-Ay)^2$, *i.e.*, a perfect square. Thus, the above being the general equation of the parabola, the characteristic of all forms of its equation is that the terms of two dimensions form a perfect square.

13. Chords are drawn from the vertex of a parabola at right angles to each other. Show that the line joining the other ends of the chords passes through a fixed point on the axis of the parabola. [The perpendicular chords may be represented by $y=mx$, $y=-\dfrac{1}{m}x$. Then the other ends are $\left(\dfrac{4a}{m^2}, \dfrac{4a}{m}\right)$ and $(4am^2, -4am)$. Forming the line through these, and putting $y=0$; etc.]

14. Find the length of the side of an equilateral triangle of which one angle is at the focus, and the others lie on the parabola $y^2=4ax$. [Line through focus making 30° with axis is $\dfrac{x-a}{\frac{\sqrt{3}}{2}}=\dfrac{y-0}{\frac{1}{2}}=r$; or $x=a+r\dfrac{\sqrt{3}}{2}$, $y=r\dfrac{1}{2}$. Combine with $y^2=4ax$ for values of r.]

15. A circle passes through the fixed point $(a, 0)$, and touches the fixed straight line $x+a=0$. Show that the locus of its centre is the parabola $y^2=4ax$. [Evident at once from definition of parabola.]

16. A circle passes through the origin and the extremities of the latus rectum of the parabola $y^2=4ax$. Find the equation of the circle. [Equation of circle must be of form $x^2+y^2-2Ax=0$.]

17. Through any fixed point $C(b, 0)$ on the axis of a parabola $y^2=4ax$ a chord PCP' is drawn. Show that the product of the ordinates of P, P' is constant, and also the product of the abscissas. Compare Ex. 10. [Combining equations $y=m(x-b)$, $y^2=4ax$, we get $y^2-\frac{4a}{m}y-4ab=0$; also $x^2-2\left(b+\frac{2a}{m^2}\right)x+b^2=0$. Apply theory of quadratics.]

18. **If x_1, y_1 be the co-ordinates of one end of a focal chord, find the co-ordinates of the other end.** [Use results of previous exercise, or obtain independently.]

19. Find the locus of a point which moves so that its shortest distance from a given circle is equal to its distance from a given straight line.

20. Given the focus F and two points P, P', on a parabola, obtain a geometrical construction for its directrix.

21. A circle is described on a focal chord PFP' of a parabola as diameter. Show that it touches the directrix.

II. Tangents and Normals.

52. **To find the equation of the tangent to the parabola $y^2=4ax$ in terms of the co-ordinates of the point of contact (x', y').**

Let PQ be a secant through the points $P(x', y')$ and $Q(x'', y'')$ on the parabola $y^2=4ax$.

The equation of the line through (x', y'), (x'', y'') is

$$\frac{x-x'}{x'-x''}=\frac{y-y'}{y'-y''},$$

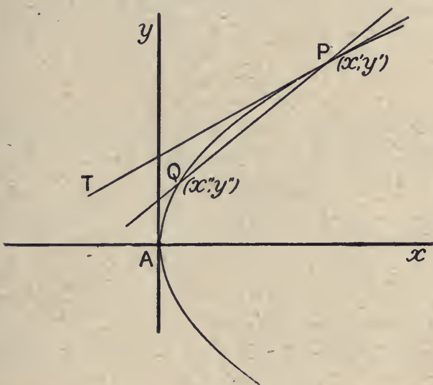
or
$$y-y'=\frac{y'-y''}{x'-x''}(x-x'). \quad \ldots\ldots \quad (1)$$

Also since (x', y'), (x'', y'') lie on the parabola $y^2 = 4ax$, therefore

$$y'^2 = 4ax',$$
$$y''^2 = 4ax'';$$
$$\text{and } \therefore\ y'^2 - y''^2 = 4a(x' - x''),$$
$$\text{or } \quad \frac{y' - y''}{x' - x''} = \frac{4a}{y' + y''}.$$

Hence (1) becomes

$$y - y' = \frac{4a}{y' + y''}(x - x') \quad \ldots\ldots \quad (2)$$



Let now the point (x'', y'') move up indefinitely close to (x', y'); then PQ becomes PT, the tangent at P; also y'' becomes y', and (2) becomes

$$y - y' = \frac{4a}{2y'}(x - x').$$

$$\begin{aligned} \text{Hence } yy' &= 2ax - 2ax' + y'^2, \\ &= 2ax - 2ax' + 4ax', \\ &= 2a(x + x'); \end{aligned}$$

and $yy' = 2a(x + x')$ is the equation of the tangent to the parabola $y^2 = 4ax$, at the point (x', y').

It is important to observe the complete analogy of the methods in determining the equations of the tangents, normals, etc., to the circle and to the parabola, the demonstration in the one case being, in effect, a transcription of that in the other. Still further illustrations of this correspondence will be met with when the ellipse is under consideration. In synthetic geometry the drawing of a tangent to the circle throws no light on the drawing of a tangent to the parabola: for each curve a fresh method must be devised. In analytical geometry the methods are marked by generality.

53. **To find the equation of the tangent to the parabola $y^2 = 4ax$ in terms of its inclination to the axis of x.**

Let θ be the angle which the tangent makes with the axis of x; and let $\tan\theta = m$.

Then the tangent may be represented by $y = mx + b$, where b is yet to be found.

If we treat the equations

$$y = mx + b,$$
$$y^2 = 4ax,$$

as simultaneous, the resulting values of x and y must be the co-ordinates of the points in which the straight line intersects the parabola (§ 11).

Hence the values of x in

$$(mx + b)^2 - 4ax = 0,$$
$$\text{or} \quad m^2x^2 + 2(mb - 2a)x + b^2 = 0, \quad \ldots\ldots \quad (1)$$

must be the values of x at the points where the straight line intersects the parabola. If these values of x are equal, the points of intersection coincide, and the straight line is a tangent.

The condition for equal values of x is

$$m^2b^2 = (mb - 2a)^2,$$

$$\text{or} \quad b = \frac{a}{m}.$$

$$\text{Hence} \quad y = mx + \frac{a}{m}$$

is the equation of the tangent to the parabola $y^2 = 4ax$, having an inclination θ to the axis of x ($m = \tan\theta$).

The following is an alternative demonstration of the preceding proposition:

We have shown that the equation $yy' = 2a(x + x')$ is the tangent at the point (x', y'). If now the equations

$$2ax - yy' + 2ax' = 0,$$
$$mx - y + b - 0,$$

represent the same straight line, then

$$\frac{m}{2a} = \frac{1}{y'} = \frac{b}{2ax'}.$$

$$\text{Hence} \quad b = \frac{4ax'}{2y'} = \frac{y'^2}{2y'} = \frac{y'}{2}.$$

$$\text{Also} \quad y' = \frac{2a}{m};$$

$$\therefore \ b = \frac{a}{m};$$

and $y = mx + \dfrac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$.

54. **To find the equation of the normal to the parabola $y^2 = 4ax$ at the point (x', y').**

The equation of *any* straight line through the point (x', y') is

$$A(x - x') + B(y - y') = 0. \quad \ldots \quad (1)$$

If this be the normal at (x', y'), it is perpendicular to the tangent

$$2ax - yy' + 2ax' = 0;$$

and the condition for perpendicularity (§ 25) is

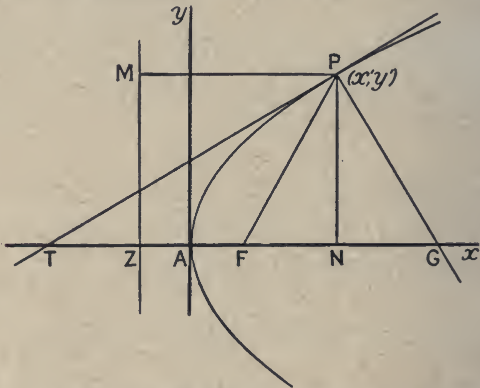
$$2aA - y'B = 0. \quad \ldots \quad (2)$$

Introducing in (1) the relation between A and B

given by (2), and so making (1) the normal, we have for the equation of the normal

$$x - x' + \frac{2a}{y'}(y - y') = 0.$$

55. **At any point on the parabola the tangent bisects the angle between the focal distance and the perpendicular on the directrix.**



Let AP be the parabola $y^2 = 4ax$; $P(x', y')$ the point of contact; PT, $yy' = 2a(x + x')$, the tangent; PF the focal distance; PM the perpendicular on the directrix. Then the angles FPT, MPT are equal.

The co-ordinates of T are AT, 0. Introducing these in the equation of the tangent,

$$0 = AT + x'; \text{ or } AT = -x'.$$

$$\therefore\ TA = AN.$$

$$\text{And } AF = ZA;$$

$$\therefore\ TF = ZN = MP = PF.$$

$$\therefore\ \triangle FTP \text{ is isosceles;}$$

$$\text{and } \angle FPT = \angle FTP = \angle MPT;$$

so that PT bisects the angle FPM.

COR. To draw the tangent at any point P on a parabola, measure FT on the axis equal to FP, and join PT. Then PT is the tangent at P.

DEFNS. The line TN is called the **subtangent.**

If PG be the normal at P, the line NG is called the **subnormal.**

56. **To show that in the parabola the subnormal is constant.**

The co-ordinates of G are AG, 0. Introducing these in the equation of the normal,

$$AG - x' + \frac{2a}{y'}(0 - y') = 0,$$

$$\text{or} \quad AG = x' + 2a.$$

$$\text{But} \quad AN = x';$$

$$\therefore \quad NG = 2a, \text{ and is constant.}$$

57. **To find the locus of the foot of the perpendicular from the focus on the tangent.**

The tangent being

$$y = mx + \frac{a}{m},$$

the straight line FY, through the focus $(a, 0)$ and perpendicular to this is (§§ 22, 25)

$$y = -\frac{1}{m}(x - a).$$

If we combine these equations, we obtain an equation which is true at Y, the intersection of the lines. Therefore at Y

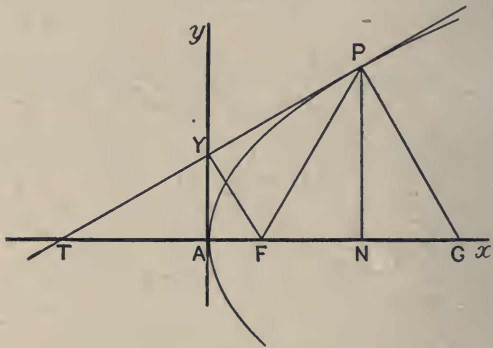
$$mx + \frac{a}{m} = -\frac{x}{m} + \frac{a}{m},$$

$$\text{or} \quad (m^2 + 1)x = 0.$$

But m^2+1 cannot vanish; therefore

$$x=0;$$

i.e., Ay, the tangent at the vertex, is the locus of Y, the foot of the perpendicular from the focus on the tangent.



Hence FTP being an isosceles triangle, and FY perpendicular to the base, FY bisects the angle TFP. Hence the triangles AFY, YFP are similar,

$$\text{and} \quad \frac{AF}{FY}=\frac{FY}{FP},$$

$$\text{or} \quad FY^2=AF \, . \, FP.$$

58. The results of §§ 56–7 may be obtained, possibly more simply, as follows:

The triangle TPG being right-angled, and $\angle FTP$ being equal to $\angle FPT$, therefore $\angle FPG=\angle FGP$.

$$\therefore \; FG=FP=a+x'. \quad (\S\, 50)$$

$$\text{Also} \; FN=x'-a.$$

$$\therefore \; NG=a+x'-(x'-a)=2a.$$

Again (§ 55) $TA = AN$. Also Ay is parallel to NP. Therefore Y is the bisection of TP, the base of the isosceles triangle FTP. Hence FY is perpendicular to PT; and the locus of Y is the tangent at the vertex.

Exercises.

1. **From the equation of the tangent, $yy' = 2a(x + x')$, show that "at infinity" the parabola becomes parallel to its axis.** [Tangent is $y = \frac{2a}{y'}x + \frac{y'}{2}$; $\therefore$ tan $\theta = \frac{2a}{y'} = 0$, when y is indefinitely great.]

2. Find the co-ordinates of the point of contact of the tangent $y = mx + \frac{a}{m}$. Identifying the lines $y = mx + \frac{a}{m}$, $yy' = 2a(x + x')$, we have $\frac{2a}{m} =$ etc.]

3. Prove that $y = mx + \frac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$ by combining these equations as for finding points of intersection.

4. Find the equations of the tangents to the parabola $y^2 = 4ax$, drawn at the extremities of its latus rectum.

5. Find the equations of the normals to the parabola $y^2 = 4ax$, drawn at the ends of its latus rectum.

6. Find the equation of that tangent to the parabola $y^2 = 4x$, which makes an angle of 60° with the axis of x.

7. Find the point of contact of that tangent to the parabola $y^2 = 4x$, which makes an angle of 60° with the axis of x.

8. Show that the tangent to the parabola $y^2 = 4ax$ at the point (x', y') is perpendicular to the tangent at the point $\left(\frac{a^2}{x'}, -\frac{4a^2}{y'}\right)$. [Ex. 18, p. 112.]

9. Find the equations of the tangents to the parabola $y^2 = 4ax$ which pass through the point $(-2a, a)$. [Let tangent be $y = mx + \frac{a}{m}$. Then $a = -2am + \frac{a}{m}$; whence values of m.]

10. Find the equations of the tangents which touch both the circle $x^2+y^2=a^2$ and the parabola $y^2=4a\sqrt{2}x$. [The line $y=mx+\frac{a\sqrt{2}}{m}$ is a tangent to the parabola. Find condition that this may be tangent to circle.]

11. For what point on the parabola $y^2=4ax$ is the normal PG equal to the subtangent?

12. If the ordinate PN at any point P on the parabola meet LT, the tangent at the end of the latus rectum, in T, then TN is equal to PF. [Tangent at L is $y=x+a$.]

13. If FQ be the perpendicular from the focus on the normal at P, then $FQ^2=AN.PF$. [Perpendicular from $(a, 0)$ on $x-x'+\frac{2a}{y'}(y-y')=0$ is $\dfrac{a+x'}{\sqrt{1+\frac{4a^2}{y'^2}}}=\dfrac{a+x'}{\sqrt{1+\frac{a}{x'}}}=$ etc.]

14. The locus of the vertices of all parabolas which have a common focus and a common tangent is a circle. [If F be focus, PT the tangent, and P the point of contact, then $FT(=FP)$ is axis. If FY be perp. to PT and YA to FT, A is vertex.]

15. From *any* point on the directrix, say $(-a, k)$, two tangents are drawn to the parabola $y^2=4ax$. Show that these tangents are at right angles to one another. [$y=mx+\frac{a}{m}$ is any tangent. If it pass through $(-a, k)$, $m^2+\frac{k}{a}m-1=0$; etc.]

16. Tangents to the parabola $y^2=4ax$ pass through the point $\left(-a, -\frac{2a}{\sqrt{3}}\right)$. Find their points of contact. See Ex. 2. [The values of m for these tangents will be found to be $\sqrt{3}$ and $-\frac{1}{\sqrt{3}}$.]

17. Show that the points of contact in the preceding exercise lie on a line through the focus. [In § 59, Cor. 2, this is shown to be true for points of contact of pairs of tangents from any point on the directrix.]

18. Show also that the line through the points of contact in Exercise 16 is perpendicular to the line from the intersection of the tangents to the focus. [In § 60 this is shown to be true in case of tangents from any point on directrix.]

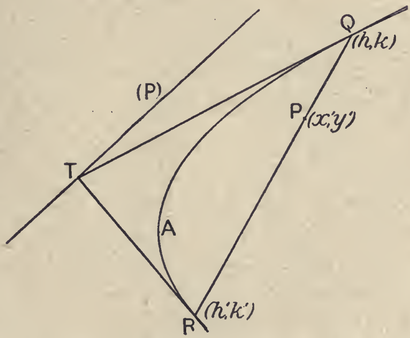
19. Through the vertex A of a parabola a perpendicular is drawn to any tangent, meeting it in Q and the curve in R. Show that $AQ.AR=4a^2$. [Use for tangent $y=mx+\frac{a}{m}$; then perp. line through A is $my+x=0$.]

20. Two tangents are drawn from any point $R(h, k)$ to the parabola $y^2=4ax$. If $p_1.p_2$ be the perpendiculars from the focus on these tangents, show that $p_1.p_2=a.RF$. [Use for tangent $y=mx+\frac{a}{m}$; $\therefore k=mh+\frac{a}{m}$, or $m^2-\frac{k}{h}m+\frac{a}{h}=0$. Hence $m_1.m_2=\frac{a}{h}$, $m_1+m_2=\frac{k}{h}$; $p_1=\frac{a}{m_1}\sqrt{1+m_1^2}$; etc.]

21. Find the equation of the common tangent to the parabolas $y^2=4ax$, $x^2=4by$. [Tangents to each may be represented by $y=mx+\frac{a}{m}$, $x=m'y+\frac{b}{m'}$. Identifying these $\frac{m}{1}=\frac{1}{m'}=-\frac{am'}{bm}$; etc.]

III. Poles and Polars.

59. To find the polar of any given point (x', y') with respect to the parabola $y^2=4ax$.



Let QAR be the parabola, and P the given point (x', y'). Let a chord through P cut the parabola in $Q(h,k)$ and $R(h', k')$; and let QT, RT be the tangents at (h, k), (h', k'). Then as the chord through P assumes different positions, and in consequence T changes its position, the locus of T is the polar of P.

The tangents at (h, k) and (h', k') are

$$yk = 2a(x+h),$$
$$yk' = 2a(x+h').$$

Hence the co-ordinates of T satisfy these equations; and therefore the co-ordinates of T satisfy

$$y(k-k') = 2a(h-h').$$

But since (x', y') lies on the straight line through (h, k), (h', k'), therefore (§ 15)

$$\frac{x'-h}{h-h'} = \frac{y'-k}{k-k'}.$$

Hence the co-ordinates of T always satisfy

$$y(k-k')\frac{y'-k}{k-k'} = 2a(h-h')\frac{x'-h}{h-h'};$$

therefore they always satisfy

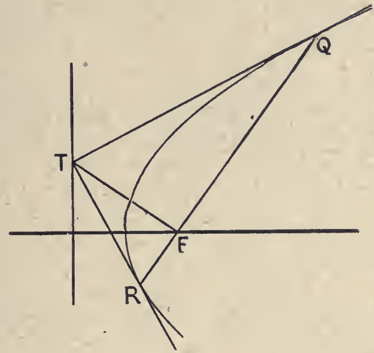
$$y(y'-k) = 2a(x'-h),$$
$$\text{or } \; yy' - 2a(x+h) = 2a(x'-h),$$
$$\text{or } \; yy' = 2a(x+x');$$

that is, $yy' = 2a(x+x')$ is the equation of the locus of T, and is therefore the equation of the polar of (x', y').

COR. 1. If the pole P is without the parabola, when the chord PQR becomes a tangent, the points Q, R and T coincide, and the point of contact is a point on the polar. Hence when the pole is without the parabola, the line joining the points of contact of tangents from it is the polar.

COR. 2. If the pole be the focus $(a, 0)$, the polar is $0 = x + a$, which is the directrix. Hence **the directrix is the locus of the intersection of tangents at the extremities of focal chords.**

60. In the parabola (1) the tangents at the extremities of any focal chord are at right angles to each other; and (2) the focal chord is at right angles to the line joining its pole to the focus.



(1). Let $(-a, \beta)$ be the point T on the directrix, where the tangents QT, RT at the ends of the focal chord QFR intersect. Then QT, RT are at right angles:

Let $y = mx + \frac{a}{m}$ be either of these tangents. Since it passes through $(-a, \beta)$, therefore

$$\beta = -ma + \frac{a}{m},$$

$$\text{or} \quad m^2 + \frac{\beta}{a}m - 1 = 0,$$

which therefore is the relation connecting the m's of QT, RT. If m_1, m_2 be the roots of this equation

$$m_1 . m_2 = -1,$$

i.e., QT, RT are at right angles.

(2). Also QR, TF are at right angles:

The equation of QR, which is the polar of $(-a, \beta)$ is

$$y\beta = 2a(x - a).$$

Also the equation of TF, through $(a, 0)$ $(-a, \beta)$ is

$$\frac{y}{\beta} + \frac{x-a}{2a} = 0;$$

and these lines satisfy the condition for perpendicularity (§ 25).

61. **In the parabola if $Q(x'', y'')$ lies on the polar of $P(x', y')$ then P lies on the polar of Q.**

For the polar of $P(x', y')$ is

$$yy' = 2a(x + x').$$

If $Q(x'', y'')$ lies on this, then

$$y''y' = 2a(x'' + x').$$

But this is the condition that $P(x', y')$ may lie on $yy'' = 2a(x + x'')$, which is the polar of $Q(x'', y'')$.

COR. 1. If therefore a point Q moves along the polar of P, the polar of Q always passes through P; *i.e.*, **if a point moves along a fixed straight line, the polar of the point turns about a fixed point,** such fixed point being the pole of the fixed straight line.

COR. 2. A special case of the preceding corollary is,— **The straight line which joins two points P and Q is the polar of the intersection of the polars of P and Q.**

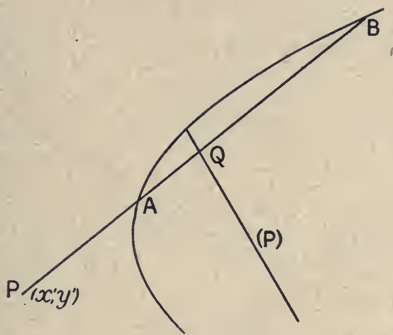
62. **A chord of a parabola is divided harmonically by any point on it and the polar of that point.**

Let (x', y') be the pole P; then $yy' = 2a(x + x')$ is the polar (P). Also a chord PAB through (x', y') is represented by

$$\frac{x-x'}{l}=\frac{y-y'}{m}=r,$$

$$\text{or}\quad x=x'+lr,\ y=y'+mr,\ \ldots . \ (1)$$

where r represents the distance from (x', y') to (x, y).



In combining (1) with the equation of the parabola, (x, y) must be the point which is common to chord and parabola, *i.e.*, must be A or B; and therefore r must be PA or PB.

Similarly in combining (1) with the equation of the polar, r must be PQ.

Combining (1) with the equation of the parabola $y^2=4ax$,

$$(y'+mr)^2=4a(x'+lr),$$

$$\text{or}\quad m^2r^2+2(my'-2al)r+y'^2-4ax'=0.$$

Hence since PA, PB are the roots of this quadratic in r,

$$PA+PB=-\frac{2(my'-2al)}{m^2};\quad PA\,.\,PB=\frac{y'^2-4ax'}{m^2};$$

and therefore $\dfrac{1}{PA}+\dfrac{1}{PB}=-\dfrac{2(my'-2al)}{y'^2-4ax'}$. ... (2).

Again, combining (1) with the equation of the polar $yy'=2a(x+x')$,

$$(y'+mr)y'=2a(x'+lr+x'),$$
$$\text{or}\quad (my'-2al)r+y'^2-4ax'=0.$$

Hence, since PQ is the root of this equation in r,

$$PQ=-\frac{y'^2-4ax'}{my'-2al}.\quad\ldots\ (3).$$

Therefore from (2) and (3)

$$\frac{1}{PA}+\frac{1}{PB}=\frac{2}{PQ},$$

and AB is divided harmonically in P and Q.

Exercises.

1. Show that the polars of all points on a line parallel to the axis of the parabola, which points may therefore be represented by (x', β) where β is a constant, are parallel to one another. [The polar of (x', β) is $y\beta=2a(x+x')$.]

2. In the preceding exercise show that the portions of the polars intercepted by the parabola (which form a set of parallel chords) are all bisected by the line $y=\beta$. [The points of intersection of the polar with the parabola are given by combining the equations $y\beta=2a(x+x')$, $y^2=4ax$; whence $y^2-2\beta y+4ax'=0$. If y_1, y_2 be the roots of this, $y_1+y_2=2\beta$; etc.]

3. **Find the direction-cosines of the chord of the parabola $y^2=4ax$, which is bisected at the point (x', y'); and thence obtain the equation of this chord.** [Let chord be $\dfrac{x-x'}{l}=\dfrac{y-y'}{m}=r$, or $x=x'+lr$, $y=y'+mr$. Combining this with $y^2=4ax$, $m^2r^2+2(my'-2al)r+y'^2-4ax'=0$. Since (x', y') is middle point, values of r are equal with opposite signs. Therefore $my'-2al=0$; etc.]

4. **Show that the polar of any point within the parabola is parallel to the chord which is bisected at that point.** [Polar of (x', y') is $yy'=2a(x+x')$; equation of chord bisected at (x', y') is, Ex. 3, $\frac{y-y'}{2a}=\frac{x-x'}{y'}$; etc.]

5. Find the poles of the lines

$$Ax+By+C=0, \ \frac{x}{h}+\frac{y}{k}=1$$

with respect to the parahola $y^2=4ax$. [Identify these lines with $yy'=2a(x+x')$.]

6. Two tangents make angles $\tan^{-1}m$, $\tan^{-1}m'$ with the axis of the parabola $y^2=4ax$. Find the polar of their intersection, *i.e.*, the chord of contact.

7. **Chords are drawn to a parabola through the intersection of the directrix with the axis. Show that the tangents at the points where a chord cuts the curve intersect on the latus rectum.**

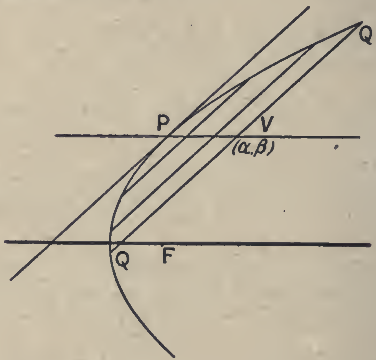
8. The pole of any tangent to the parabola $y^2=4ax$ with respect to the circle $x^2+y^2=r^2$, lies on the parabola $y^2=-\frac{r^2}{a}x$. [Tangent to $y^2=4ax$ is $y=mx+\frac{a}{m}$, and the pole of this with respect to $x^2+y^2=r^2$ is $x=-\frac{r^2m^2}{a}$, $y=\frac{r^2m}{a}$. Eliminate m.]

9. The pole of any tangent to the circle $x^2+y^2=r^2$ with respect to the parabola $y^2=4ax$, lies on the locus whose equation is $\frac{x^2}{r^2}-\frac{y^2}{4a^2}=1$. [Tangent to circle is $y=mx\pm r\sqrt{1+m^2}$. Identifying this with $yy'=2a(x+x')$, $y'=\frac{2a}{m}$, $x'=\pm\frac{r}{m}\sqrt{1+m^2}$. Eliminate m.]

10. Two tangents drawn to a parabola make complementary angles with the axis of the parabola. Show that their chord of contact must pass through the foot of the directrix. [The two tangents are represented by $y=mx+\frac{a}{m}$, $y=\frac{x}{m}+am$. Their intersection is $\left\{a, \frac{a}{m}(1+m^2)\right\}$; etc.]

IV. Parallel Chords and Diameters.

63. To find the locus of the bisections of parallel chords in the parabola $y^2=4ax$.



Let the direction-cosines of the parallel chords be l, m; and let (α, β) be the middle point of *any* one of them: its equation is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=r;$$

$$\text{whence } x=\alpha+lr,\ y=\beta+mr.$$

Combining these with $y^2=4ax$, we have

$$(\beta+mr)^2=4a(\alpha+lr),$$

$$\text{or } m^2r^2+2(\beta m-2al)r+\beta^2-4a\alpha=0,$$

where r is now the distance from (α, β) to Q or Q'.

Since (α, β) is the middle point of QQ', the values of r are equal with opposite signs. This requires

$$\beta m-2al=0,$$

$$\text{or } \beta=2a\frac{l}{m}.$$

But l and m are the same for all these chords, since they are parallel. Hence the ordinates of the bisections of all these chords are subject to the above relation.

The locus of the bisections of the set of parallel chords whose direction-cosines are l, m is therefore

$$y = 2a\frac{l}{m},$$

and is a straight line parallel to the axis of the parabola.

DEF. The straight line bisecting a set of parallel chords is called a **diameter.**

Evidently all diameters, being parallel to the axis of the parabola, are parallel to one another.

Let the chord QVQ' move parallel to itself towards P where the diameter cuts the curve. Then VQ, VQ' remains always equal to one another, and therefore vanish together; and the chord prolonged becomes the tangent at P. Hence **the tangent at the extremity of a diameter is parallel to the chords which the diameter bisects.**

The equation of the diameter may be written

$$y = 2a \cot\theta,$$

since $l = \cos\theta$, $m = \sin\theta$, where θ is the angle which the chords make with the axis of the parabola.

Exercises.

1. In the figure of § 63 show that $FP = \frac{a}{m^2}$. [If Y be foot of perp. from F on tangent, $FY = FP \sin FPY = FP \sin\theta = FP.m$. Also $FY^2 = a \,.\, FP$; etc.]

2. In the figure of § 63 find the co-ordinates of P in terms of l and m.

3. In the same figure show that

$$PV = \alpha - a\frac{l^2}{m^2}.$$

4. In the same figure show that

$$QV^2 = 4FP \, . \, PV.$$

[From equation of § 63,

$$QV^2 = \frac{4a\alpha - \beta^2}{m^2} = \frac{4a\alpha - 4a^2\dfrac{l^2}{m^2}}{m^2} = 4\frac{a}{m^2}(\alpha - a\frac{l^2}{m^2}) = \text{etc.} \quad \text{Exs. 1 and 3.]}$$

NOTE. If we refer the curve to oblique axes, namely, PV as axis of x, and the tangent at P as the axis of y, the result $QV^2 = 4FP \, . \, PV$ shows that we may write the equation of the parabola in the form $y^2 = 4a'x$, where a' is FP, the distance of the present origin P from the focus.

5. If tangents be drawn from any point on PV (§63), show that the points of contact are at the ends of one of the chords which PV bisects. [Any point on PV may be represented by $(x', 2a\frac{l}{m})$, and its polar is $\frac{y}{m} = \frac{x + x'}{l}$, etc.]

6. Find the locus of the middle points of the ordinates of the parabola $y^2 = ax$. [$2y$ of locus $= y$ of parabola.]

7. Find the locus of the middle points of all the radius vectors drawn from the focus of the parabola $y^2 = 4ax$. [If (α, β) be the middle point of any vector drawn to (x, y) on parabola, $\alpha = \frac{1}{2}(a + x)$, $\beta = \frac{1}{2}(0 + y)$; etc.]

8. Two tangents to the parabola make angles θ, θ' with the axis. Find the equation of the diameter which bisects their chord of contact. [Tangents are $y = mx + \frac{a}{m}$, $y = m'x + \frac{a}{m'}$; whence the y of their point of intersection.]

9. QQ' is any one of a system of parallel chords of the parabola $y^2 = 4ax$, and O is a point on QQ' such that the rectangle $OQ.OQ'$ is equal to a constant $\pm c^2$, according as O is without or within the parabola. Show that the locus of O is given by $y^2 = 4ax \pm m^2c^2$, where m is the sine of the angle which the chords make with the axis.

10. Parallel chords are drawn in a parabola. Show that the locus of the intersection of normals at the ends of the chords is a straight line. [Normals are $x - x' + 2a\frac{y}{y'} - 2a = 0$, $x - x'' + 2a\frac{y}{y''} - 2a = 0$; whence $\frac{2ay}{y'y''} = -\frac{x' - x''}{y' - y''} = -\frac{l}{m}$, and $x - 2a = \frac{x'y' - x''y''}{y' - y''} = \frac{1}{4a}(y'^2 + y'y'' + y''^2)$. Also $y'' = 4a\frac{l}{m} - y'$.]

11. Parallel chords are drawn in a parabola $y^2 = 4ax$. Show that the locus of the intersection of tangents at the ends of the chords and the locus of the intersection of normals at the ends of the chords, give by their intersection, as the direction of the chords varies, the locus $y^2 = a(x - 3a)$.

64. To show that the equation

$$y = a + bx + cx^2$$

represents a parabola; and to determine its vertex, axis, focus and directrix.

$$y = a + bx + cx^2; \ldots .(1)$$

$$\therefore x^2 + \frac{b}{c}x = \frac{y}{c} - \frac{a}{c};$$

$$\therefore x^2 + \frac{b}{c}x + \frac{b^2}{4c^2} = \frac{y}{c} + \frac{b^2 - 4ac}{4c^2};$$

$$\therefore \left(x + \frac{b}{2c}\right)^2 = 4 \cdot \frac{1}{4c}\left(y + \frac{b^2 - 4ac}{4c}\right) \ldots .(2).$$

Now transfer the origin to the point $\left(-\frac{b}{2c}, -\frac{b^2 - 4ac}{4c}\right)$ by writing $x - \frac{b}{2c}$ for x, and $y - \frac{b^2 - 4ac}{4c}$ for y (Chap. iv. § 30). Then (2) becomes

$$x^2 = 4 \cdot \frac{1}{4c}y;$$

which is the equation of a parabola whose vertex is at the origin (o, o), and whose axis is the axis of y, i.e., $x = 0$. Its focus is $\left(0, \frac{1}{4c}\right)$, and directrix $y + \frac{1}{4c} = 0$.

Hence, reverting to the original axes, we see that

$$y = a + bx + cx^2$$

represents a parabola

whose vertex is at the point $\left(-\frac{b}{2c}, -\frac{b^2-4ac}{4c}\right)$;

" axis " the line $x+\frac{b}{2c}=0$;

" focus " at the point $\left(-\frac{b}{2c}, -\frac{b^2-4ac}{4c}+\frac{1}{4c}\right)$;

" directrix is the line $y+\frac{b^2-4ac}{4c}+\frac{1}{4c}=0$.

Exercises.

1. Determine the vertex, axis, focus and directrix of the parabola $y=3+2x+x^2$.

2. Determine the vertex, axis, focus and directrix of the parabola $y=4-2x+3x^2$.

3. Determine the vertex, axis, focus and directrix of the parabola $y=-5+3x-2x^2$.

4. Determine the vertex, axis, focus and directrix of the parabola $y^2-6y+3=x$.

5. The focus of a parabola is $(4, -3)$ and its directrix is $y+2=0$. Find its equation, and also its vertex and axis.

In each of the preceding cases construct the parabola, placing it correctly with respect to the original axes.

CHAPTER VII.

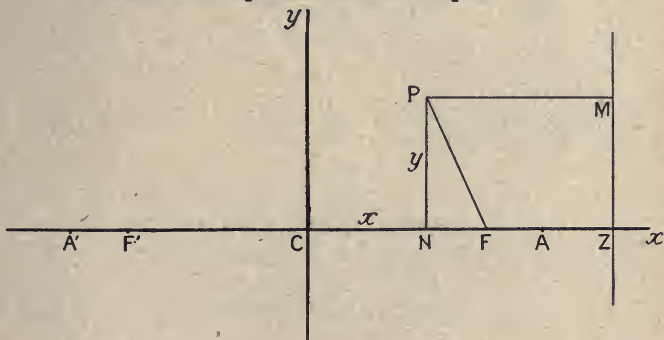
THE ELLIPSE.

Definition. An **Ellipse** is the locus of a point which moves so that its distance from a fixed point, called the focus, is in a constant ratio ($e<1$) to its distance from a fixed straight line, called the directrix.

As in the case of the circle and of the parabola, we shall form the equation of the ellipse from its definition, the equation being thus the accurate and complete expression, in algebraic language, of the definition of the curve. The properties of the curve must therefore be latent in its equation, and a suitable examination of the equation will reveal them.

I. Equation and Trace of the Ellipse.

65. To find the equation of the Ellipse.



Let F be the focus, and MZ the directrix; and let FZ be perpendicular to MZ.

Divide FZ internally at A and externally at A' so that

$$\frac{FA}{AZ}=e, \text{ and } \frac{A'F}{A'Z}=e;$$

$$\text{or } FA=e.AZ, \text{ and } A'F=e.A'Z;$$

then A and A' are points on the locus.

Bisect $A'A$ at C; and let $A'A=2a$, so that $A'C=CA=a$.

$$\begin{aligned} \text{Then} \quad 2CF &= A'F-FA, \\ &= e(A'Z-AZ), \\ &= e.2a; \\ \therefore\ CF &= ae. \\ \text{Also} \quad CZ &= \frac{1}{2}(A'Z+AZ), \\ &= \frac{1}{2e}(A'F+FA), \\ &= \frac{1}{2e}.2a, \\ &= \frac{a}{e}. \end{aligned}$$

Let now C be taken as origin, and CZ as axis of x. Also let Cy, perpendicular to CZ, be the axis of y. Let $P(x, y)$ be any point on the locus, and PM the perpendicular to MZ.

Then, by definition of ellipse,

$$\frac{PF}{PM}=e;$$

$$\therefore PF^2=e^2.PM^2=e^2.NZ^2;$$

$$\therefore\ y^2+(ae-x)^2=e^2\left(\frac{a}{e}-x\right)^2;$$

$$\therefore\ x^2(1-e^2)+y^2=a^2(1-e^2);$$

$$\therefore \frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1,$$

which is the equation of the ellipse.

The equation is usually written

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where $b^2=a^2(1-e^2)$, or $e^2=\dfrac{a^2-b^2}{a^2}$.

The results

distance from C to focus, $CF,=ae$,

" " " " directrix, $CZ,=\dfrac{a}{e}$,

square of eccentricity, $e^2,=\dfrac{a^2-b^2}{a^2}$,

are important, and should be remembered.

It would perhaps have been more natural to have taken the distance from the focus to the directrix, FZ, as the parameter, along with e, in terms of which to express the equation. However, $FZ=\dfrac{a}{e}-ae=a\dfrac{1-e^2}{e}$, or $a=\dfrac{e}{1-e^2}\cdot FZ$; so that a is fixed when FZ is, e being known. For subsequent work a is a more convenient parameter than FZ.

In future, unless the contrary is stated, the equation of the ellipse will be supposed to be of the form $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

66. To trace the form of the Ellipse from its equation.

(1). If $y=0$, $x=\pm a$; if $x=0$, $y=\pm b$. Hence if on the axes we take $CA=a$, $CA'=-a$, $CB=b$, $CB'=-b$, the curve passes through the points A, A', B, B'.

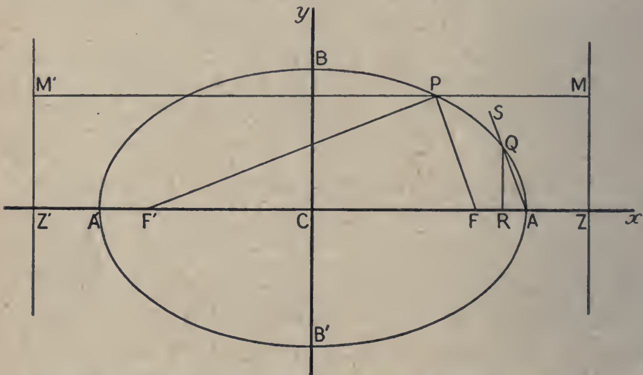
AA' is called the **axis major,** and BB' the **axis minor.** CB is less than CA, since $b^2=a^2(1-e^2)$. A and A' are called the **vertices** of the ellipse.

(2). $y=\pm\dfrac{b}{a}\sqrt{a^2-x^2}$, $x=\pm\dfrac{a}{b}\sqrt{a^2-y^2}$. Hence x cannot exceed, numerically, $\pm a$, nor can y exceed, numerically,

$\pm b$; and the curve falls entirely within the rectangle whose sides pass through A, A', B, B', and are parallel to the axes.

(3). $y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$. Hence as x increases from 0 to a, y continually decreases.

(4). For a given value of x, the values of y are equal with opposite signs. Hence the curve is symmetrical with respect to the axis of x. Similarly it is symmetrical with respect to the axis of y.



(5). If we suppose the straight line $x = mx + k$ to cut the ellipse, we shall have for the x's of the points of intersection the equation $\frac{x^2}{a^2} + \frac{(mx+k)^2}{b^2} = 1$, or $\left(\frac{1}{a^2} + \frac{m^2}{b^2}\right)x^2 + 2\frac{mk}{b^2}x + \frac{k^2}{b^2} - 1 = 0$,—a quadratic, giving two values of x. Hence a straight line can cut an ellipse in only two points.

(6). If Q be any point on the curve, and it be supposed to move indefinitely close to A, the line AQS is ultimately the tangent at A, and the angle QAR is the angle at which the curve cuts the axis of x. Now

$$\tan QAR = \frac{RQ}{RA} = \frac{y}{a-x} = \frac{b^2}{a^2} \cdot \frac{a+x}{y}.$$

Therefore ultimately $\tan QAR = \frac{b^2}{a^2} \cdot \frac{a+x}{0} = \infty$; and the angle QAR in the limit is 90°. Hence the curve cuts the axis of x at A at right angles; similarly it cuts the axis of y at B at right angles; and by the symmetry of the curve therefore at A' and B'.

Collating these facts, we see that the ellipse has the form given in the diagram. In § 9, Ex. 4, we plotted the graph of the ellipse $\frac{x^2}{9^2} + \frac{y^2}{6^2} = 1$, *i.e.*, for which $a = 9$, $b = 6$.

The symmetry of the curve shows that, since there is a focus F and a direction ZM to the right of the origin, **there is a focus F' and a directrix $Z'M'$, at the same distances to the left of the origin.** Hence we have not only the constant relation $PF = e.PM$ for all positions of P, but also the constant relation $PF' = e.M'P$.

67. DEFINITIONS. The point C is called the **centre** of the ellipse.

Any chord through the centre is called a **diameter.**

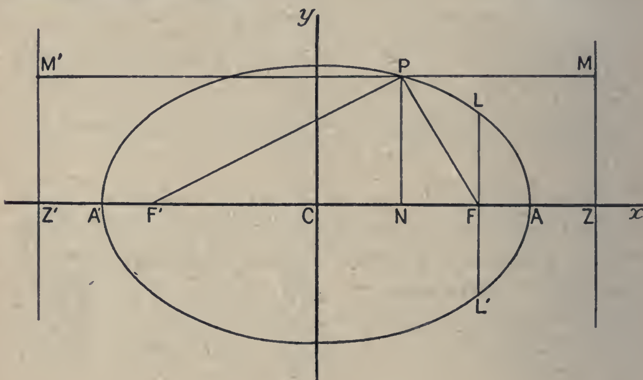
Every chord through the centre of the ellipse is there bisected.

For if (α, β) be any point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which therefore satisfies this equation; then $(-\alpha, -\beta)$ also satisfies this equation, and therefore is a point on the ellipse. But (α, β), $(-\alpha, -\beta)$ lie on the line $\frac{x}{\alpha} - \frac{y}{\beta} = 0$ which passes through the origin, and are at equal distances from the origin since $\alpha^2 + \beta^2 = (-\alpha)^2 + (-\beta)^2$. Hence every diameter is bisected at the centre of the ellipse.

68. **To find the distances of any point (x, y) on the ellipse from the foci.**



Let P be the point (x, y).

Then
$$PF = e\,.\,PM = e\,.\,NZ,$$
$$= e(CZ - CN),$$
$$= e\left(\frac{a}{e} - x\right) = a - ex.$$

Also
$$PF' = e\,.\,M'P = e\,.\,Z'N,$$
$$= e(Z'C + CN),$$
$$= e\left(\frac{a}{e} + x\right) = a + ex.$$

Hence
$$PF + PF' = (a - ex) + (a + ex),$$
$$= 2a.$$

Therefore **the sum of the focal distances is constant for all points on the ellipse, and is equal to** $2a$.

The preceding property, $PF + PF' = 2a$, enables us to describe an ellipse mechanically. For if an inextensible string, its ends fastened together, be thrown over two pins at F and F', a pencil P, moved so as to keep the string always tense, will describe an ellipse of which F and F' are foci; for $PF + PF'$ will be constant.

DEFINITION. A double ordinate through a focus, as LFL', is called a **latus rectum.**

69. To find the length of the latus rectum of the ellipse.

The co-ordinates of L are ae and FL. Hence, substituting these in the equation of the ellipse, we have

$$\frac{a^2e^2}{a^2} + \frac{FL^2}{b^2} = 1;$$

$$\therefore\ FL^2 = b^2(1 - e^2),$$

$$= b^2\left(1 - \frac{a^2 - b^2}{a^2}\right),$$

$$= \frac{b^4}{a^2};$$

$$\text{and } FL = \frac{b^2}{a};$$

$$\therefore\ L'FL = 2\frac{b^2}{a}.$$

Exercises.

1. Find the axes, major and minor, the eccentricity, the distances from centre to focus and directrix, and the latus rectum of the ellipse $3x^2 + 4y^2 = 12$.

2. Determine the same quantities for the ellipse $x^2+12y^2=4$.

3. If l be the entire length of the string referred to in § 68, and d the distance between the pins, show that $d=2\sqrt{a^2-b^2}$, and $l=2a+2\sqrt{a^2-b^2}$.

4. The latus rectum of an ellipse being $2l$, and the eccentricity being e, express the axes in terms of these quantities.

5. If in an ellipse the angle $FBF'=90°$, find the eccentricity, and the relation between the axes. [Here $b=ae$; $\therefore b^2=a^2e^2=a^2-b^2$; etc.]

6. If in an ellipse r be a semi-diameter whose inclination to the axis major is α, and e be the eccentricity, find the axes in terms of these quantities. [$x=r\cos\alpha$, $y=r\sin\alpha$; substitute in equation of ellipse.]

7. Find the equation of the ellipse whose foci are at the points (3, 0), (−3, 0), and whose eccentricity is $\frac{1}{3}$. [$ae=3$; $\frac{a^2-b^2}{a^2}=\frac{1}{9}$.]

8. Find the equation of the ellipse whose latus rectum is $\frac{9}{2}$, and eccentricity $\frac{\sqrt{7}}{4}$, the axes of the curve being the axes of coordinates.

9. Without reference to the results of § 65, show that, if $FZ=k$, and FZ be divided in A and A' in the ratio $e:1$,

$$FA=\frac{e}{1+e}k;\ AZ=\frac{1}{1+e}k;\quad A'F=\frac{e}{1-e}k;\ A'Z=\frac{e}{1-e}k.$$

10. Hence show that

$$CZ=\frac{k}{1-e^2},\quad CF=\frac{e^2k}{1-e^2},$$

where C is the middle point of AA'.

11. Hence express the equation of the ellipse in the form

$$\frac{x^2}{e^2k^2}(1-e^2)^2+\frac{y^2}{e^2k^2}(1-e^2)=1.$$

12. From the figure of § 68 evidently $PF^2=y^2+(ae-x)^2$. Use this to show that $PF=a-ex$. Also use $PF'^2=y^2+(ae+x)^2$ to show that $PF'=a+ex$.

13. Find the locus of a point which moves so that the sum of its distances from two fixed points $(ae, 0)$, $(-ae, 0)$, is constant and equal to $2a$. [$\sqrt{y^2+(ae-x)^2}+\sqrt{y^2+(ae+x)^2}=2a$; etc.]

14. Find the equation of the ellipse whose focus is (4, 3), eccentricity $\frac{1}{\sqrt{2}}$, and directrix $4x+3y-50=0$. $\left[(x-4)^2+(y-3)^2=\frac{1}{2}\left(\frac{4x+3y-50}{5}\right)^2\text{; etc.}\right]$

15. A straight line AB has its extremities on two lines OA, OB at right angles to each other, and slides between them. Show that the locus of a point C on AB is an ellipse whose semi-axes are equal to CB and CA. [If C be (x, y), and AB make an angle α with OA, then $\sin\alpha=\frac{y}{CA}$, $\cos\alpha=\frac{x}{CB}$; etc.]

16. Find the locus of the vertex of a triangle, having given its base $2c$, and the product, k^2, of the tangents of the angles at the base. [Take base as axis of x, and centre of base as origin.]

17. In the ellipse show that CB is a mean proportional between $A'F$ and FA.

18. Ellipses are described on the same axis major, *i.e.*, a is constant for all. Show that the locus of the extremities of their latera recta is $x^2=a(a-y)$, a parabola. [If (x, y) be the extremity of a latus rectum, $x=ae, y=\frac{b^2}{a}$; $\therefore x^2=a^2e^2=a^2-b^2$; etc.]

19. If two circles touch each other internally, the locus of the centres of circles touching both is an ellipse whose foci are the centres of the given circles and whose axis major is the sum of their radii.

20. If P be any point (x, y) on an ellipse, show that $\tan\frac{1}{2}PF'F=\frac{(a-x)(1-e)}{y}$. [$\tan PF'F=\frac{y}{ae+x}$.]

21. If P be any point on an ellipse, show that

$$\tan\frac{1}{2}PF'F\tan\frac{1}{2}PFF'=\frac{1-e}{1+e}.$$

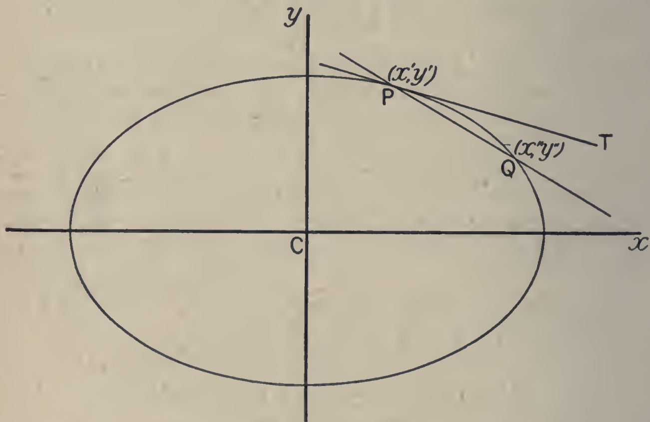
22. If P be any point on an ellipse, show that the locus of the centre of the circle inscribed in the triangle PFF' is an ellipse. [If P be (x', y'), the bisectors at F', F are $y=\frac{(a-x')(1-e)}{y'}(x+ae)$, and $y=-\frac{(a+x')(1-e)}{y'}(x-ae)$; whence at intersection $x=ex'$, $y=\frac{e}{1+e}y'$; etc.]

23. If P be any point on an ellipse, show that the circles described on $F'P$ as diameter touch the circle described on the axis major. [If M be the middle point of $F'P$, show that $F'M+MC=a$. If P be (x, y), then M is $\{\frac{1}{2}(x-ae), \frac{1}{2}y\}$.]

24. In the ellipse if $a=b$ the equation becomes $x^2+y^2=a^2$. which is the equation of the circle. The circle being thus a special form of the ellipse, find its eccentricity, and the distances of its foci and directrices from the centre.

II. Tangents and Normals.

70. To find the equation of the tangent to the ellipse in terms of the co-ordinates of the point of contact (x', y').



Let PQ be a secant through the points $P(x', y')$ and $Q(x'', y'')$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equation of the line through $(x', y'), (x'', y'')$ is

$$\frac{x-x'}{x'-x''}=\frac{y-y'}{y'-y''},$$

$$\text{or}\quad y - y' = \frac{y' - y''}{x' - x''}(x - x'). \quad \ldots . \ (1).$$

Also since (x', y'), (x'', y'') lie on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, therefore

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1,$$

$$\frac{x''^2}{a^2} + \frac{y''^2}{b^2} = 1;$$

$$\text{and} \ \therefore \ \frac{1}{a^2}(x'^2 - x''^2) + \frac{1}{b^2}(x'^2 - y''^2) = 0,$$

$$\text{or}\quad \frac{y' - y''}{x' - x''} = -\frac{b^2}{a^2} \cdot \frac{x' + x''}{y' + y''}.$$

Hence (1) becomes

$$y - y' = -\frac{b^2}{a^2} \cdot \frac{x' + x''}{y' + y''}(x - x'). \quad \ldots . \ (2).$$

Let now the point (x'', y'') move up indefinitely close to (x', y'); then PQ becomes PT, the tangent at P; also x'' becomes x', and y'', y'; and (2) becomes

$$y - y' = -\frac{b^2}{a^2} \cdot \frac{2x'}{2y'}(x - x').$$

$$\text{Hence}\quad \frac{yy'}{b^2} - \frac{y'^2}{b^2} = -\frac{xx'}{a^2} + \frac{x'^2}{a^2},$$

$$\text{or}\quad \frac{xx'}{a^2} + \frac{yy'}{b^2} = \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1;$$

and $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$ is the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, at the point (x', y').

71. To find the equation of the tangent to the ellipse in terms of its inclination to the axis of x.

Let θ be the angle which the tangent makes with the axis of x; and let $\tan\theta = m$.

Then the tangent may be represented by $y=mx+k$, where k is yet to be found.

If we treat the equations

$$y=mx+k,$$
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

as simultaneous, the resulting values of x and y must be the co-ordinates of the points in which the straight line intersects the ellipse (§ 11).

Hence the values of x in

$$\frac{x^2}{a^2}+\frac{(mx+k)^2}{b^2}=1,$$

$$\text{or}\quad \left(\frac{1}{a^2}+\frac{m^2}{b^2}\right)x^2+2\frac{mk}{b^2}x+\frac{k^2}{b^2}-1=0, \quad \ldots\ldots (1)$$

must be the values of x at the points where the straight line intersects the ellipse. If these values of x are equal, the points of intersection coincide, and the straight line is a tangent.

The condition for equal values of x is

$$\left(\frac{1}{a^2}+\frac{m^2}{b^2}\right)\left(\frac{k^2}{b^2}-1\right)=\frac{m^2k^2}{b^4},$$
$$\text{or}\quad k=\pm\sqrt{m^2a^2+b^2}.$$
$$\text{Hence}\quad y=mx\pm\sqrt{m^2a^2+b^2}$$

is the equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, having an inclination θ to the axis of x ($m=\tan\theta$). The double sign refers to parallel tangents on opposite sides of the ellipse.

The following is an alternative demonstration of the preceding proposition:

We have shown that the equation $\frac{xx'}{a^2}+\frac{yy'}{b^2}=1$ is the tangent at the point (x', y'). If now the equations

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}-1=0,$$
$$mx-y+k=0$$

represent the same straight line, then

$$\frac{m}{\frac{x'}{a^2}}=\frac{-1}{\frac{y'}{b^2}}=\frac{k}{-1}.$$

Hence $$k=\frac{-ma}{\frac{x'}{a}}=\frac{b}{\frac{y'}{b}}=\frac{\pm\sqrt{m^2a^2+b^2}}{\sqrt{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}}},$$
$$=\pm\sqrt{m^2a^2+b^2};$$

and $y=mx\pm\sqrt{m^2a^2+b^2}$ is a tangent to the ellipse.

72. The equation $y=mx+\sqrt{m^2a^2+b^2}$ may be written

$$-x\sin\theta+y\cos\theta=\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}.$$

If α be the angle the perpendicular from the origin on the tangent makes with the axis of x, then $\theta=\alpha-90°$, and the preceding equation of the tangent becomes

$$x\cos\alpha+y\sin\alpha=\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha}.$$

× 73. **To find the equation of the normal to the ellipse at the point (x', y').**

The equation of *any* straight line through the point (x', y') is

$$A(x-x')+B(y-y')=0. \quad \ldots (1)$$

If this be the normal at (x', y') it is perpendicular to the tangent

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1;$$

and the condition of perpendicularity (§ 25) is

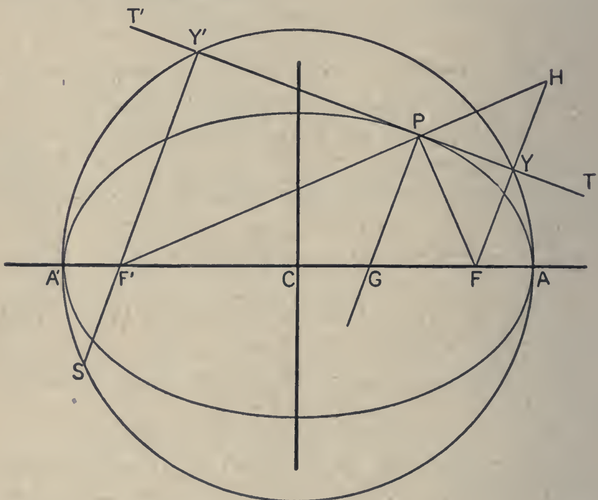
$$\frac{x'}{a^2}\cdot A+\frac{y'}{b^2}\cdot B=0. \quad \ldots . (2)$$

Introducing in (1) the relation between A and B given by (2), and so making (1) the normal, we have for the equation of the normal

$$\frac{x-x'}{\frac{x'}{a^2}}=\frac{y-y'}{\frac{y'}{b^2}},$$

or $$\frac{a^2}{x'}x-\frac{b^2}{y'}y=a^2-b^2.$$

74. In the ellipse the normal bisects the angle between the focal distances.



Let the normal at P cut the axis major in G. Then the co-ordinates of G are CG and 0. Substituting these in the equation of the normal (§ 73),

$$\frac{a^2}{x'}\cdot CG-\frac{b^2}{y'}\cdot 0=a^2-b^2;$$

$$\therefore\ CG = \frac{a^2 - b^2}{a^2} x' = e^2 x'.$$

Hence $\quad F'G = ae + e^2x'$, and $GF = ae - e^2x'$.

$$\therefore\ \frac{F'G}{GF} = \frac{ae + e^2x'}{ae - e^2x'} = \frac{a + ex'}{a - ex'} = \frac{F'P}{PF};$$

and therefore PG, the normal, bisects the angle $F'PF$ between the focal distances.

Also since the angles GPT, GPT' are right angles, and the angles GPF, GPF' are equal, therefore the angles FPT, $F'PT'$ are equal; *i.e.*, **the tangent makes equal angles with the focal distances.**

75. In the ellipse the product of the perpendiculars from the foci on the tangent is constant and equal to b^2.

Let FY, $F'Y'$ be the perpendiculars from the foci on the tangent, and let the equation of the tangent be expressed in the form

$$y = mx + \sqrt{m^2a^2 + b^2}.$$

Then since FY is the perpendicular from $(ae, 0)$ on this line, therefore

$$FY = \frac{mae + \sqrt{m^2a^2 + b^2}}{\sqrt{m^2 + 1}}.$$

Similarly, $F'Y'$ being the perpendicular from $(-ae, 0)$,

$$F'Y' = \frac{-mae + \sqrt{m^2a^2 + b^2}}{\sqrt{m^2 + 1}}.$$

Hence

$$\begin{aligned} FY.F'Y' &= \frac{-m^2a^2e^2 + m^2a^2 + b^2}{m^2 + 1}, \\ &= \frac{-m^2(a^2 - b^2) + m^2a^2 + b^2}{m^2 + 1}, \\ &= b^2. \end{aligned}$$

76. In the ellipse the locus of the foot of the perpendicular from either focus on the tangent is a circle on the axis major as diameter.

Let the equation of the tangent be expressed in the form

$$y = mx + \sqrt{m^2a^2 + b^2};$$

then the equation of FY, through $(ae, 0)$ and perpendicular to this, is

$$y = -\frac{1}{m}(x - ae).$$

These equations may be written

$$y - mx = \sqrt{m^2a^2 + b^2},$$

$$\text{and} \quad my + x = ae = \sqrt{a^2 - b^3}.$$

If we square these equations we are including in the one case the tangent parallel to the above, and in the other the perpendicular $F'Y'$ through the other focus. If we then add we shall have a result which holds at Y and Y', and also at the corresponding points on the parallel tangent; and if m disappears, we shall have a result which holds at Y and Y' for all positions of the tangent, *i.e.*, the locus of Y and Y'.

Squaring and adding,

$$(1 + m^2)(x^2 + y^2) = (1 + m^2)a^2,$$

$$\text{or} \quad x^2 + y^2 = a^2;$$

i.e., the locus of Y, Y' is a circle on the axis major as diameter.

This circle is called the **auxiliary circle.**

The following alternative proof of this proposition may be noted:

Let $F'P$, FY produced meet in H. Then the triangles PYF, PYH are equal in all respects.

$$\therefore F'H = F'P + PF = 2a.$$

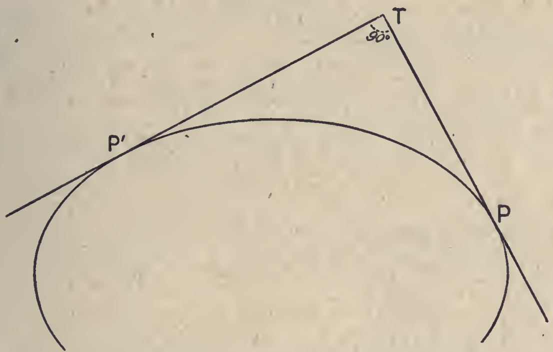
Also C being the middle point of FF', and Y of FH, therefore CY is half of $F'H$, and therefore is equal to a. Hence the locus of Y is a circle with centre C and radius a.

The auxiliary circle furnishes a simple proof that $FY.F'Y'=b^2$:

For let $Y'F'$ produced meet the circle in S. Then

$$\begin{aligned} FY.F'Y' &= SF'.F'Y' \\ &= A'F'.F'A \\ &= (a-ae)(a+ae) \\ &= a^2-a^2e^2 \\ &= a^2-(a^2-b^2) \\ &= b^2. \end{aligned}$$

77. To find the locus of the intersection of tangents at right angles to each other.



The tangents PT, $P'T$, at right angles to one another, are represented by

$$y=mx+\sqrt{m^2a^2+b^2},$$

$$y=-\frac{1}{m}x+\sqrt{\frac{a^2}{m^2}+b^2};$$

which may be written

$$y-mx=\sqrt{m^2a^2+b^2}.$$

$$my+x=\sqrt{a^2+m^2b^2}.$$

If we square these equations we are including tangents parallel to these; if we then add we shall have a result which holds at the intersections of these perpendicular tangents; and if m disappears, the result holds at the intersections of all pairs of perpendicular tangents, *i.e.*, we have the equation of the locus of T.

Squaring and adding,

$$(1+m^2)(x^2+y^2)=(1+m^2)(a^2+b^2),$$

$$\text{or}\quad x^2+y^2=a^2+b^2;$$

which therefore is the equation of the locus of T. The locus is evidently a circle.

Exercises.

1. Find the tangents to the ellipse $3x^2+4y^2=12$ at the points whose abscissa is 1.

2. Show that the lines $y=x\pm\frac{7}{2}$ are tangents to the ellipse $4x^2+3y^2=21$. What are the points of contact?

3. Find the condition that the line $\frac{x}{m}+\frac{y}{n}=1$ may touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

4. Find the equations of the tangents drawn from the point $\left(2,\frac{1}{\sqrt{3}}\right)$ to the ellipse $\frac{x^2}{3}+\frac{y^2}{2}=1$. [$y=mx+\sqrt{3m^2+2}$ is the equation of any tangent. If it passes through $\left(2,\frac{1}{\sqrt{3}}\right)$, $\frac{1}{\sqrt{3}}=2m+\sqrt{3m^2+2}$; whence values of m.]

5. The normal at (x', y') on the ellipse divides the axis major into segments whose product is equal to $a^2-e^4x'^2$.

6. Find the locus of the middle point of that part of a tangent to an ellipse, which is intercepted between the tangents at A and A'.

7. Find the point of contact at which the tangent $y=mx+\sqrt{m^2a^2+b^2}$ touches the ellipse. [Identifying the lines $mx-y+\sqrt{m^2a^2+b^2}=0$, $\frac{xx'}{a^2}+\frac{yy'}{b^2}-1=0$, we have $\dfrac{\frac{x'}{a^2}}{m}=\dfrac{\frac{y'}{b^2}}{-1}=\dfrac{-1}{\sqrt{m^2a^2+b^2}}$.]

8. Show that the parallel tangents $y=mx\pm\sqrt{m^2a^2+b^2}$ touch the ellipse at opposite ends of a diameter. [Find points of contact.]

9. Find the equation of that tangent to the ellipse, which cuts off equal intercepts from the positive directions of the co-ordinate axes. [The tangent is of form $x+y=k$.]

10. If θ, θ' be the angles which the two tangents drawn to the ellipse from the external point (h, k) make with the axis major, then

$$\tan\theta\tan\theta'=\frac{k^2-b^2}{h^2-a^2}.$$

Substitute h and k for x and y in $y=mx+\sqrt{m^2a^2+b^2}$, and form the quadratic in m.]

11. From the result of the preceding exercise deduce an alternative proof of the proposition in §77. [Since the tangents are at right angles $\theta'=90+\theta$, and $\tan\theta\tan\theta'=-1$.]

12. The equation of the tangent to an ellipse may be written in the form $x\cos\alpha+y\sin\alpha=\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha}$. Employ this to find the locus of the intersection of perpendicular tangents.

13. Two tangents are such that the product of the tangents of the angles they make with the axis major is $-\frac{b^2}{a^2}$. Show that they intersect on the ellipse $\frac{x^2}{2a^2}+\frac{y^2}{2b^2}=1$. [See Ex. 10.]

14. The tangent at a point P on an ellipse meets the tangent at A in K. Show that CK is parallel to $A'P$. [$AK=\frac{b^2}{a}\cdot\frac{a-x'}{y'}$; $\tan KCA$ $=\frac{b^2}{a^2}\cdot\frac{a-x'}{y'}=\frac{y'}{a+x'}$; etc.]

15. Find the equations of the tangents to the ellipse which are parallel to the line $\frac{x}{a}+\frac{y}{b}=1$. [$y=-\frac{b}{a}x+b$; $\therefore m=-\frac{b}{a}$; etc.]

16. In the preceding exercise find the co-ordinates of the point of contact which lies in the positive quadrant. [Compare equation of tangent with equation $\frac{xx'}{a^2}+\frac{yy'}{b^2}=1$.]

17. If r be the radius vector CP of an ellipse, and p be the perpendicular from the centre C on the tangent at $P(x', y')$, then

$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}-\frac{r^2}{a^2b^2}.$$

[From equation $y=mx+\sqrt{m^2a^2+b^2}$, $p^2=\frac{m^2a^2+b^2}{m^2+1}$; also $x'=\frac{-ma^2}{\sqrt{m^2a^2+b^2}}$, $y'=\frac{b^2}{\sqrt{m^2a^2+b^2}}$; $\therefore r^2=\frac{m^2a^4+b^4}{m^2a^2+b^2}$; etc. More readily, from §§ 85, 87, $CD^2+r^2=a^2+b^2$, $CD.p=ab$; etc.]

18. The tangent at P on an ellipse cuts the axis major in T. The normal at P cuts the axis major in G. Find the position of P that PT and PG may be equal. [Since $PT=PG$, $\therefore \left(\frac{a^2}{x'}-x'\right)^2+y'^2=(x'-e^2x')^2+y'^2$; etc.]

19. Find the condition that the line $lx+my+n=0$ may be a normal to the ellipse. [Identifying the equations $\frac{a^2}{x'}x-\frac{b^2}{y'}y-(a^2-b^2)=0$ and $lx+my+n=0$, we have $\dfrac{\frac{a}{l}}{\frac{x'}{a}}=\dfrac{-\frac{b}{m}}{\frac{y'}{b}}=\dfrac{-(a^2-b^2)}{n}$; etc.]

20. If (x', y') be the point of intersection of the curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $x^2+y^2=k^2$; and θ be the angle at which the curves cut one another; show that

$$\tan\theta=x'y'\frac{a^2-b^2}{a^2b^2}.$$

[Form equations of tangents at (x', y').]

21. Find the common tangents to the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$.

III. Poles and Polars.

78. To find the polar of any given point (x', y') with respect to the ellipse.

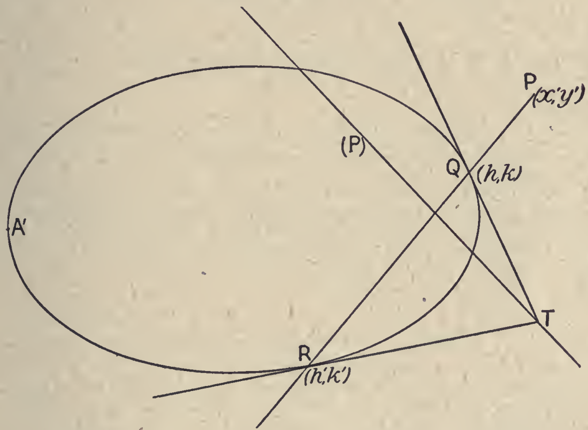
Let QRA' be the ellipse, and P the given point (x', y'). Let a chord through P cut the ellipse in $Q(h, k)$ and $R(h', k')$; and let QT, RT be the tangents at (h, k),

(h', k'). Then as the chord through P assumes different positions, and in consequence T changes its position, the locus of T is the polar of P.

The tangents at (h, k) and (h', k') are

$$\frac{xh}{a^2}+\frac{yk}{b^2}=1,$$

$$\frac{xh'}{a^2}+\frac{yk'}{b^2}=1.$$



Hence the co-ordinates of T satisfy these equations; and therefore the co-ordinates of T satisfy

$$\frac{x}{a^2}(h-h')+\frac{y}{b^2}(k-k')=0.$$

But since (x', y') lies on the straight line through (h, k), (h', k'), therefore (§ 15)

$$\frac{x'-h}{h-h'}=\frac{y'-k}{k-k'}.$$

Hence the co-ordinates of T always satisfy

$$\frac{x}{a^2}(h-h')\frac{x'-h}{h-h'}+\frac{y}{b^2}(k-k')\frac{y'-k}{k-k'}=0;$$

therefore they always satisfy

$$\frac{x}{a^2}(x'-h)+\frac{y}{b^2}(y'-k)=0,$$

$$\text{or}\quad \frac{xx'}{a^2}+\frac{yy'}{b^2}=\frac{xh}{a^2}+\frac{yk}{b^2}=1;$$

that is $\frac{xx'}{a^2}+\frac{yy'}{b^2}=1$ is the equation of the locus of T, and therefore is the equation of the polar of (x', y').

COR. 1. If the pole P is without the ellipse, when the chord PQR becomes a tangent, the points Q, R and T coincide, and the point of contact is a point on the polar. Hence when the pole is without the ellipse, the line joining the points of contact of tangents from it is the polar.

COR. 2. If the pole be the focus $(ae, 0)$, the polar is $\frac{xae}{a^2}+\frac{y.0}{b^2}=1$, or $x=\frac{a}{e}$, which is the directrix. Hence **the directrix is the locus of the intersection of tangents at the extremities of focal chords.**

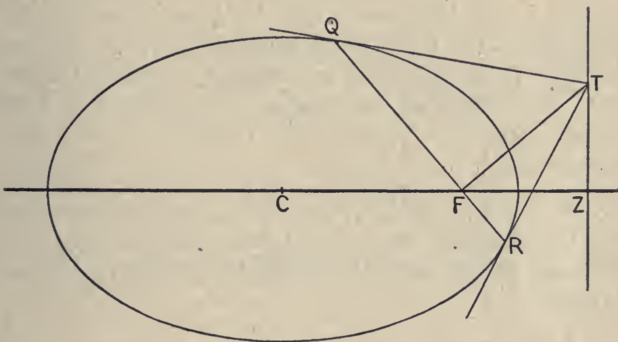
79. In the ellipse any focal chord is at right angles to the line joining its pole to the focus.

Let QR be any focal chord; and let T on the directrix (§ 78, Cor. 2) be the pole of QR. Then QR, TF are at right angles.

For let $TZ=\beta$, so that the co-ordinates of T are $\frac{a}{e}$, β. Then the equation of QR, which is the polar of $\left(\frac{a}{e}, \beta\right)$, is

$$\frac{x \cdot \frac{a}{e}}{a^2}+\frac{y\beta}{b^2}=1,$$

$$\text{or} \quad \frac{x}{ae}+\frac{y\beta}{b^2}=1. \quad \ldots.(1)$$



Also the equation of TF, through $(ae, 0)$, $\left(\frac{a}{e}, \beta\right)$ is

$$\frac{x-ae}{ae-\frac{a}{e}}=\frac{y-0}{0-\beta},$$

$$\text{or} \quad aex-\frac{b^2}{\beta}y=a^2-b^2; \quad \ldots.(2)$$

and equations (1) and (2) evidently represent two lines at right angles to each other (§ 25).

80. In the ellipse if $Q(x'', y'')$ lies on the polar of $P(x', y')$, then P lies on the polar of Q.

For the polar of $P(x', y')$ is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1.$$

If $Q(x'', y'')$ lies on this, then

$$\frac{x''x'}{a^2}+\frac{y''y'}{b^2}=1.$$

But this is the condition that $P(x', y')$ may lie on $\frac{xx''}{a^2}+\frac{yy''}{b^2}=1$, which is the polar of $Q(x'', y'')$.

COR. 1. If therefore a point Q moves along the polar of P, the polar of Q must always pass through P; *i.e.*, **if a point moves along a fixed straight line, the polar of the point turns about a fixed point,** such fixed point being the pole of the fixed straight line.

COR. 2. A special case of the preceding corollary is,—**The straight line which joins two points P and Q is the polar of the intersection of the polars of P and Q.**

81. **A chord of an ellipse is divided harmonically by any point on it and the polar of that point.**

Let (x', y') be the pole P; then $\frac{xx'}{a^2}+\frac{yy'}{b^2}=1$ is the polar (P). Also a chord PAB through (x', y') is represented by

$$\frac{x-x'}{l}=\frac{y-y'}{m}=r,$$

$$\text{or } x=x'+lr,\ y=y'+mr, \ldots . (1)$$

where r represents the distance from (x', y') to (x, y).

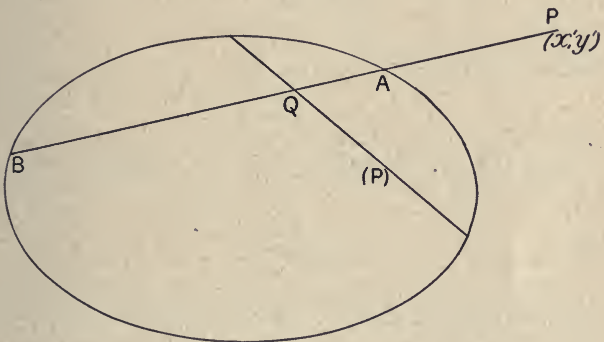
In combining (1) with the equation of the ellipse, (x, y) must be the point which is common to chord and ellipse, *i.e.*, must be A or B; and therefore r must be PA or PB.

Similarly in combining (1) with the equation of the polar, r must be PQ.

Combining (1) with the equation of the ellipse, we have

$$\frac{(x'+lr)^2}{a^2}+\frac{(y'+mr)^2}{b^2}=1,$$

$$\text{or}\quad \left(\frac{l^2}{a^2}+\frac{m^2}{b^2}\right)r^2+2\left(\frac{lx'}{a^2}+\frac{my'}{b^2}\right)r+\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1=0.$$



Hence, since PA, PB are the roots of this quadratic in r,

$$PA+PB=-2\frac{\frac{lx'}{a^2}+\frac{my'}{b^2}}{\frac{l^2}{a^2}+\frac{m^2}{b^2}};\quad PA\,.\,PB=\frac{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1}{\frac{l^2}{a^2}+\frac{m^2}{b^2}};$$

$$\text{and}\ \therefore\ \frac{1}{PA}+\frac{1}{PB}=-2\,\frac{\frac{lx'}{a^2}+\frac{my'}{b^2}}{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1}.\quad\ldots(2)$$

Again, combining (1) with the equation of the polar $\frac{xx'}{a^2}+\frac{yy'}{b^2}=1$, we have

$$\frac{(x'+lr)x'}{a^2}+\frac{(y'+mr)y'}{b^2}=1,$$

$$\text{or}\quad \left(\frac{lx'}{a^2}+\frac{my'}{b^2}\right)r+\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1=0.$$

Hence, since PQ is the root of this equation in r,

$$PQ=-\frac{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1}{\frac{lx'}{a^2}+\frac{my'}{b^2}}. \quad \ldots . (3)$$

Therefore from (2) and (3)

$$\frac{1}{PA}+\frac{1}{PB}=\frac{2}{PQ},$$

and AB is divided harmonically in P and Q.

Exercises.

1. Chords are drawn to an ellipse through the intersection of the directrix with the axis major. Show that the tangents at the points where a chord cuts the curve intersect on the latus rectum produced.

2. Find the pole of the line $Ax+By+C=0$ with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. [Identify the equations $Ax+By+C=0$ and $\frac{xx'}{a^2}+\frac{yy'}{b^2}-1=0$.]

3. Find the locus of a pole with respect to the ellipse when the perpendicular p from the origin on the polar is constant.

4. Find the pole of the normal to the ellipse at the point (x', y').

5. If the pole of the normal at $P(x', y')$ to an ellipse lies on the normal at $Q(x'',y'')$, then the pole of the normal at Q lies on the normal at P.

6. Find the condition that the polar of (x', y') with respect to the ellipse may be a tangent to the parabola $y^2 = -2\frac{b^2}{a}x$. [Identify the equations $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$, or $y = -\frac{b^2}{a^2} \cdot \frac{x}{y'}x + \frac{b^2}{y'}$, and $y = mx - \frac{b^2}{2am}$.]

7. Show that the polar of any point on the auxiliary circle with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a tangent to the ellipse $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2}$. [Identify the equations $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$, or $y = -\frac{b^2}{a^2} \cdot \frac{x'}{y'}x + \frac{b^2}{y'}$, and $y = mx + \sqrt{m^2a^2 + \frac{b^4}{a^2}}$. Then $m = -\frac{b^2}{a^2} \cdot \frac{x'}{y'}$, $m^2a^2 + \frac{b^4}{a^2} = \frac{b^4}{y'^2}$; eliminate m.]

8. If from a point on the directrix, say $\left(\frac{a}{e}, \beta\right)$, a pair of tangents be drawn to the ellipse and also to the auxiliary circle, the chords of contact intersect on the axis major.

9. A, B, C, D **are four points taken in order on an ellipse.** AD, BC **intersect at** P; AC, BD **at** Q; **and** AB, CD **at** R. **Show that the triangle** PQR **is such that each vertex is the pole of the opposite side.** [Let BD meet PR in T. Then, from property of complête quadrilateral, D, Q, B, T form a harmonic range, and RD, RQ, RA, RP a harmonic pencil. Hence if RQ cut AD in X and BC in Y, then D, X, A, P form a harmonic range, and also C, Y, B, P. But polar of P cuts AD and BC harmonically; etc.]

The triangle PQR, each of whose sides is the polar of the opposite vertex, is said to be **self-conjugate,** or **self-polar** with respect to the ellipse.

10. **Employ the preceding to draw tangents to an ellipse from a given point, using a ruler only.**

11. Find the direction-cosines of the chord of the ellipse, which is bisected at the point (x', y'); and thence obtain the equation of this chord. [Follow method suggested in Ex. 3, p. 126.]

12. **Show that the polar of any point within the ellipse is parallel to the chord which is bisected at that point.**

13. Find the pole with respect to the ellipse of the line $Ax + By = 0$, which passes through the centre.

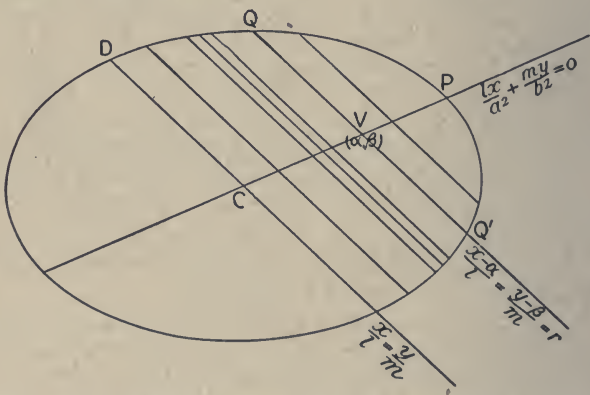
14. Show that the locus of the poles of the parallel lines represented by $y = mx + k$, where m is constant and k varies, is a straight line through the centre of the ellipse. [Identify the lines $mx - y + k = 0$ and $\frac{xx'}{a^2} + \frac{yy'}{b^2} - 1 = 0$.]

NOTE. The preceding result is reconciled with that of § 79, Cor. 1, by thinking of the polars (parallel) as turning about a point at infinity.

15. The equation of the chord joining the points (x', y'), (x'', y'') on the ellipse is $\frac{(x - x')(x' + x'')}{a^2} + \frac{(y - y')(y' + y'')}{b^2} = 0$, (§ 70, Eq. 2). Find the pole of this line; and show that it lies on the line joining the centre of the ellipse to the middle point of the chord.

IV. Parallel Chords and Conjugate Diameters.

82. To find the locus of the bisections of parallel chords in the ellipse.



Let the direction-cosines of the parallel chords be l, m, so that $\frac{x}{l} = \frac{y}{m}$ is that which passes through the centre; and let (α, β) be the middle point of *any* one of them: its equation is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=r;$$

$$\text{whence} \quad x=\alpha+lr,\ y=\beta+mr.$$

Combining these with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have

$$\frac{(\alpha+lr)^2}{a^2}+\frac{(\beta+mr)^2}{b^2}=1,$$

$$\text{or} \quad \left(\frac{l^2}{a^2}+\frac{m^2}{b^2}\right)r^2+2\left(\frac{l\alpha}{a^2}+\frac{m\beta}{b^2}\right)r+\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1=0,$$

where r is now the distance from (α, β) to Q or Q'.

Since (α, β) is the middle point of QQ', the values of r are equal with opposite signs. This requires

$$\frac{l\alpha}{a^2}+\frac{m\beta}{b^2}=0.$$

But l, m are the same for all these chords, since they are parallel. Hence the co-ordinates of the bisections of all these chords are subject to the above relation.

The locus of the bisections of the set of parallel chords whose direction-cosines are l, m is therefore

$$\frac{lx}{a^2}+\frac{my}{b^2}=0,$$

and is a straight line through the centre of the ellipse, *i.e.*, is a diameter (§ 67).

Conversely, any straight line through the centre, *i.e.*, any diameter, bisects a set of parallel chords. For $Ax+By=0$ represents any diameter. It may be written in the form

$$\frac{\dfrac{Aa^2}{\sqrt{A^2a^4+B^2b^4}}x}{a^2}+\frac{\dfrac{Bb^2}{\sqrt{A^2a^4+B^2b^4}}y}{b^2}=0,$$

which is of the form $\frac{lx}{a^2}+\frac{my}{b^2}=0$, and therefore bisects the chords whose direction-cosines are

$$\frac{Aa^2}{\sqrt{A^2a^4+B^2b^4}},\quad \frac{Bb^2}{\sqrt{A^2a^4+B^2b^4}}.$$

COR. Let the chord QVQ' move parallel to itself towards P, where the diameter cuts the curve. Then QV, VQ' remain always equal to one another, and therefore vanish together; and the chord prolonged becomes the tangent at P. Hence **the tangent at the extremity of a diameter is parallel to the chords which the diameter bisects.**

83. **Conjugate Diameters.** Since all chords parallel to $\frac{x}{l}=\frac{y}{m}$ are bisected by $\frac{lx}{a^2}+\frac{my}{b^2}=0$, *i.e.*, by $\frac{x}{\frac{a^2}{l}}=\frac{y}{-\frac{b^2}{m}}$;

therefore all chords parallel to $\frac{x}{\frac{a^2}{l}}=\frac{y}{-\frac{b^2}{m}}$ are bisected by

$$\frac{\frac{a^2}{l}x}{a^2}+\frac{-\frac{b^2}{m}y}{b^2}=0, \text{ i.e., by } \frac{x}{l}=\frac{y}{m}.$$

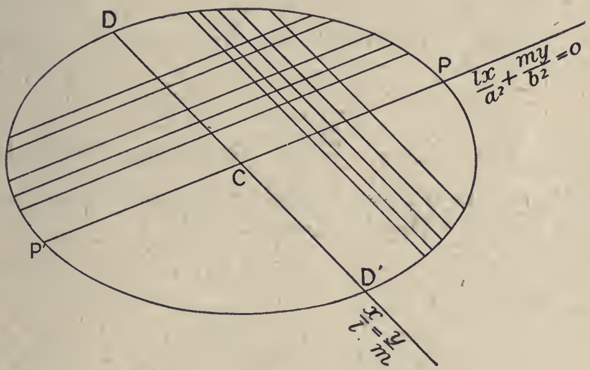
Hence the diameters PCP', DCD' are such that each bisects all chords parallel to the other. Such diameters are called **conjugate diameters.** They exist in pairs; and since DCD' is *any* line through the centre, it is evident that there is an infinite number of pairs of conjugate diameters. The axes of the ellipse, ACA', BCB', are a special case of conjugate diameters.

Since (§ 32) the tangent at the extremity of a diameter is parallel to the chords which the diameter bisects, therefore **the tangents at the extremities of each of a pair of conjugate diameters are parallel to the other diameter of the pair.**

Since the conjugate diameters $\frac{x}{l}=\frac{y}{m}$ and $\frac{lx}{a^2}+\frac{my}{b^2}=0$ may be written $y=\frac{m}{l}x$, $y=-\frac{b^2}{a^2}\cdot\frac{l}{m}x$, therefore, if θ, θ' be the angles these lines make with the axis of x,

$$\tan\theta=\frac{m}{l}, \text{ and } \tan\theta'=-\frac{b^2}{a^2}\cdot\frac{l}{m}.$$

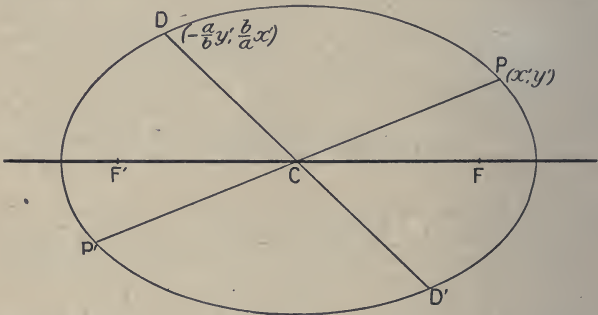
$$\text{Hence } \tan\theta \,.\, \tan\theta'=-\frac{b^2}{a^2},$$



a relation which in the ellipse always connects the tangents of the angles which any pair of conjugate diameters make with the axis major. The negative sign in the preceding expression shows that in the ellipse conjugate diameters fall on opposite sides of the axis minor.

84. The co-ordinates of the extremity of any diameter being given, to find those of the extremity of the diameter conjugate to it.

The equation of the tangent at P is $\frac{xx'}{a^2}+\frac{yy'}{b^2}=1$; and therefore the equation of CD, which is parallel to the tangent at P (§83) and passes through the origin,



is $\frac{xx'}{a^2}+\frac{yy'}{b^2}=0$, or $y=-\frac{b^2}{a^2}\cdot\frac{x'}{y'}x$. Combining this with the equation of the ellipse, we shall obtain the co-ordinates of D and D'. The combination gives

$$\frac{x^2}{a^2}+\frac{1}{b^2}\cdot\frac{b^4}{a^4}\cdot\frac{x'^2}{y'^2}x^2=1,$$

$$\text{or}\quad \frac{x^2}{a^2}\cdot\frac{b^2}{y'^2}\left(\frac{y'^2}{b^2}+\frac{x'^2}{a^2}\right)=1,$$

$$\text{or}\quad \frac{x^2}{a^2}\cdot\frac{b^2}{y'^2}=1;$$

$$\therefore x=\pm\frac{a}{b}y'.$$

Substituting these values in $y=-\frac{b^2}{a^2}\cdot\frac{x'}{y'}x$, we get

$$y=\mp\frac{b}{a}x'.$$

Hence the co-ordinates of D are $-\frac{a}{b}y', \frac{b}{x}x'$.

The co-ordinates $\frac{a}{b}y', -\frac{b}{a}x'$ evidently have reference to the point D'.

85. The sum of the squares of any pair of conjugate semi-diameters is constant, and equal to a^2+b^2.

$$\text{For } CP^2+CD^2 = x'^2+y'^2+\frac{a^2}{b^2}y'^2+\frac{b^2}{a^2}x'^2,$$
$$= a^2\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)+b^2\left(\frac{y'^2}{b^2}+\frac{x'^2}{a^2}\right),$$
$$= a^2+b^2.$$

86. The product of the focal distances is equal to the square of the conjugate semi-diameter.

$$\text{For } PF.PF' = (a-ex')(a+ex'),$$
$$= a^2-e^2x'^2,$$
$$= a^2-\frac{a^2-b^2}{a^2}x'^2,$$
$$= a^2-x'^2+\frac{b^2}{a^2}x'^2,$$
$$= \frac{a^2}{b^2}y'^2+\frac{b^2}{a^2}x'^2,$$
$$= CD^2.$$

87. If a circumscribing parallelogram be formed by drawing tangents at the extremities of conjugate diameters, its area is constant and equal to $4ab$.

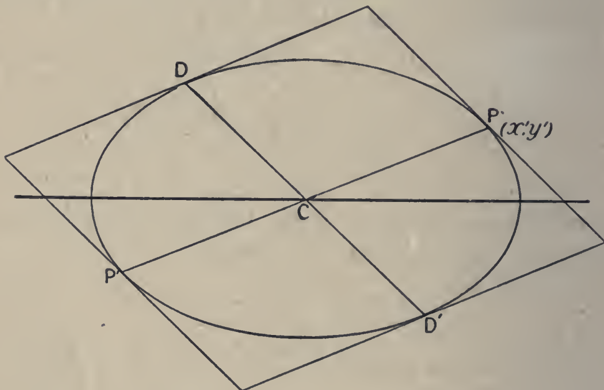
Area of circumscribing $\|^{m} = 4CD \times$ perp. from C on tangent at P,

$$= 4CD.\frac{1}{\sqrt{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}}},$$

$$= 4CD \cdot \frac{ab}{\sqrt{\frac{b^2}{a^2}x'^2 + \frac{a^2}{b^2}y'^2}},$$

$$= 4CD \cdot \frac{ab}{CD},$$

$$= 4ab.$$



88. **To find the equation of the ellipse when referred to conjugate diameters as axes of co-ordinates.**

Reverting to § 82, since V is the middle point of QQ', $\frac{l\alpha}{a^2} + \frac{m\beta}{b^2} = 0$, and r is QV. Therefore

$$\left(\frac{l^2}{a^2} + \frac{m^2}{b^2}\right) QV^2 + \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1. \quad \ldots .(1)$$

Also combining $\frac{x}{l} = \frac{y}{m} = r$, or $x = lr$, $y = mr$ with the equation of the ellipse, r now being CD, we have

$$\frac{l^2}{a^2} + \frac{m^2}{b^2} = \frac{1}{CD^2}. \quad \ldots .(2)$$

Again the equation of CP, through (α, β), is $\frac{x}{\alpha}=\frac{y}{\beta}$, or, making the denominators direction-cosines,

$$\frac{x}{\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}}=\frac{y}{\frac{\beta}{\sqrt{\alpha^2+\beta^2}}}=r;$$

$$\text{whence}\quad x=\frac{\alpha r}{\sqrt{\alpha^2+\beta^2}},\quad y=\frac{\beta r}{\sqrt{\alpha^2+\beta^2}}.$$

Combining this with the equation of the ellipse, r now being CP,

$$\frac{\alpha^2 . CP^2}{a^2(\alpha^2+\beta^2)}+\frac{\beta^2 . CP^2}{b^2(\alpha^2+\beta^2)}=1,$$

$$\text{or}\quad \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=\frac{CV^2}{CP^2}, \quad \ldots . (3)$$

$$\text{since}\quad \alpha^2+\beta^2=CV^2.$$

Substituting the results (2) and (3) in (1),

$$\frac{QV^2}{CD^2}+\frac{CV^2}{CP^2}=1. \quad \ldots . (4)$$

Now suppose the ellipse referred to CP and CD as oblique axes, CP being the axis of x and CD the axis of y. Then for the co-ordinates of any point Q we have $x=CV$, $y=VQ$. Let $CP=a'$, $CD=b'$. Then (4) becomes $\frac{y^2}{b'^2}+\frac{x^2}{a'^2}=1$, and

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1$$

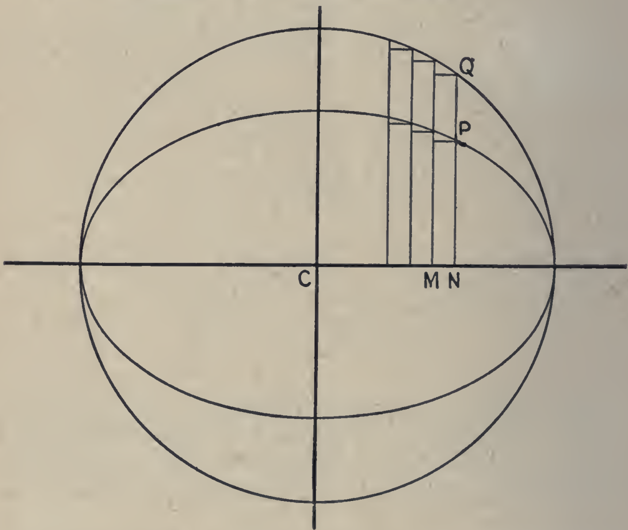
is the equation of the ellipse referred to conjugate diameters as axes of co-ordinates.

Another proof of this proposition will be found in Ch. IX., §115.

89. **The area of the ellipse is πab.**

In the ellipse $y=\frac{b}{a}\sqrt{a^2-x^2}$; in the auxiliary circle $y=\sqrt{a^2-x^2}$. Hence CN being the x for both P and Q, $PN=\frac{b}{a}QN$; and therefore

rectangle PM : rectangle $QM = b : a$.



This proportion holds for all such corresponding rectangles. Hence the sum of all such rectangles as PM is to the sum of all such rectangles as QM in the ratio $b : a$.

But if the number of these rectangles be increased indefinitely, their width being indefinitely diminished, their sums become the areas of the ellipse and circle respectively.

Hence

$$\text{Area of ellipse} : \text{area of circle} = b : a;$$

$$\therefore \text{area of ellipse} = \frac{b}{a}\cdot\pi a^2,$$

$$= \pi ab.$$

Exercises.

1. Find the equations of the diameters which are conjugate to the following:

$$x+y=0;\ \frac{x}{a}+\frac{y}{b}=0;\ ax-by=0.$$

[The equation $\tan\theta\tan\theta' = -\frac{b^2}{a^2}$ may conveniently be used (§ 83).]

2. The length of a semi-diameter is k, and it lies in the second quadrant; find the equation of the conjugate diameter. [If (x', y') be the extremity of k, $x'^2+y'^2=k^2$ and $\therefore x' = -a\sqrt{\frac{k^2-b^2}{a^2-b^2}}$, $y' = b\sqrt{\frac{a^2-k^2}{a^2-b^2}}$; hence equation of this diameter is $\frac{x}{-a\sqrt{k^2-b^2}} = \frac{y}{b\sqrt{a^2-k^2}}$; and thence equation of its conjugate.]

3. If the extremity P of a diameter be $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$, show that D, the extremity of the diameter conjugate to it, is $\left(-\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$.

4. Show that the conjugate diameters in the preceding exercise are equal in length, each semi-diameter being $\sqrt{\frac{1}{2}(a^2+b^2)}$.

5. Show that the equation of the ellipse when referred to equal conjugate diameters as axes of co-ordinates is $x^2+y^2=\frac{1}{2}(a^2+b^2)$. [§88.]

This has the form of the ordinary equation of the circle; but it must be remembered that the axes here are not rectangular. The equation of the circle referred to oblique axes, origin being at centre, is $x^2+y^2+2xy\cos\omega=r^2$.

6. A point moves so that the sum of the squares of its distances from two intersecting straight lines Ox, Oy, inclined at an angle ω, is constant and equal to c^2. Prove that its locus is an ellipse.

7. Show that tangents at the extremities of any chord parallel to $\frac{x}{l}=\frac{y}{m}$ intersect on the line $\frac{lx}{a^2}+\frac{my}{b^2}=0$, (§82); *i.e.*, **tangents at the ends of any chord parallel to a given diameter intersect on the conjugate diameter.** [Any chord parallel to $\frac{x}{l}=\frac{y}{m}$ is represented by $\frac{x-\alpha}{l}=\frac{y-\beta}{m}$; identifying this with $\frac{xx'}{a^2}+\frac{yy'}{b^2}=1$, we find pole of chord, which will be found to satisfy $\frac{lx}{a^2}+\frac{my}{b^2}=0$.]

8. If tangents be drawn from any point on the line $\frac{lx}{a^2}+\frac{my}{b^2}=0$, (§82), their chord of contact is parallel to $\frac{x}{l}=\frac{y}{m}$; *i.e.*, **the chord of contact of tangents from any point on a diameter is parallel to the conjugate diameter.** [Let (x', y') be any point on $\frac{lx}{a^2}+\frac{my}{b^2}=0$, so that $\frac{a^2}{lx'}=-\frac{b^2}{my'}$. Chord of contact is $\frac{xx'}{a^2}+\frac{yy'}{b^2}=1$; modify this by preceding relation.]

9. If CD, the diameter conjugate to CP, cuts the focal distances PF, PF' in L and M, then $PL=PM$. [The normal at P is perpendicular to CD.]

10. Find the locus of the middle points of chords joining the extremities of conjugate diameters. [If (x', y') be P, the end of one diameter, and (x, y) the middle point of PD, then $x=\frac{1}{2}(x'-\frac{a}{b}y')$, $y=\frac{1}{2}(y'+\frac{b}{a}x')$; $\therefore\ \frac{x'}{a}-\frac{y'}{b}=\frac{2x}{a}$, $\frac{x'}{a}+\frac{y'}{b}=\frac{2y}{b}$; $\therefore\ \frac{x'}{a}=\frac{x}{a}+\frac{y}{b}$, $\frac{y'}{b}=\frac{y}{b}-\frac{x}{a}$; etc.]

11. If PF' meet CD in E, then $PE=a$. [For (§76) CY is parallel to PF', and CD to PY.]

12. If (α, β) and (α', β') be the intersections of the tangents and normals respectively at the extremities of a pair of conjugate diameters, then $a^2\alpha'\beta=b^2\alpha\beta'$. [Normals at (x', y'), $(-\frac{a}{b}y', \frac{b}{a}x')$ are $\frac{a^2}{x'}x-\frac{b^2}{y'}y=a^2-b^2$, and $-\frac{ab}{y'}x-\frac{ab}{x'}y=a^2-b^2$. Hence, subtracting, at intersection $ax(ay'+bx')-by(bx'-ay')=0$, and $\therefore\ a\alpha'(ay'+bx')-b\beta'(bx'-ay')=0$. Treat tangents similarly.]

13. A pair of conjugate diameters, CP and DC, are produced to meet the directrix. Show that the orthocentre of the triangle formed by these conjugate diameters and the directrix, is the focus. [§78, Cor. 2; §79; and Ex. 7 of these Exercises.]

14. Through the foci F, F', lines FQ, $F'Q$ are drawn parallel respectively to the conjugate diameters CD, CP. Show that the locus of Q is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=e^2$. [$FQ$, through $(ae, 0)$ and parallel to $\frac{x}{l}=\frac{y}{m}$ is $y=\frac{m}{l}(x-ae)$; $F'Q$, through $(-ae, 0)$ and parallel to $\frac{lx}{a^2}+\frac{my}{b^2}=0$, is $y=-\frac{b^2}{a^2}\cdot\frac{l}{m}(x+ae)$; etc.]

15. PCP', DCD' are conjugate diameters of an ellipse, and P, P', D, D' are joined to a point Q on the circle $x^2+y^2=r^2$. Show that $PQ^2+P'Q^2+DQ^2+D'Q^2=2(a^2+b^2)+4r^2$, *i.e.*, is constant.

16. If the tangent at the vertex A cut any two conjugate diameters in T and T', then $AT \,.\, AT'=b^2$.

17. If the conjugate diameters CP, CD make angles θ, θ' with the axis major CA, show that $\cos PCD=e^2\cos\theta\cos\theta'$. [If (x',y') be P, then $\cos\theta=\frac{x'}{CP}$, $\sin\theta=\frac{y'}{CP}$, $\cos\theta'=-\frac{ay'}{b.CD}$, $\sin\theta'=\frac{bx'}{a.CD}$; $\cos PCD$ $=\cos(\theta'-\theta)=$etc.]

18. The angle between the equal conjugate diameters being 120°, show that the eccentricity of the ellipse is $\sqrt{\frac{2}{3}}$. [Use results of Exs. 3, 4 and 17, remembering that $\frac{b^2}{a^2}=1-e^2$.]

19. Show that the angle between any pair of conjugate diameters is obtuse, except when they become the axes CA, CB. [See Ex. 17; $\cos PCD$ is negative.]

20. Show that the angle PCD, between the conjugate diameters, is a maximum when these diameters are equal. [§87, $ab=$ area of DP $=CP.CD\sin PCD$. Hence PCD, being obtuse, will be greatest when $\sin PCD$ is least, *i.e.*, when $CP.CD$ is greatest. Also $2CP.CD=CP^2$ $+CD^2-(CP-CD)^2=a^2+b^2-(CP-CD)^2$; etc.]

21. If θ, θ' be the angles which the conjugate diameters CP, CD make with the axis major CA, then $CP^2\sin 2\theta+CD^2\sin 2\theta'=0$. [Use results for $\cos\theta$, etc., in Ex. 17.]

22. Find the locus of the middle points of chords of an ellipse which pass through the point $A(a, 0)$. [Any chord AP may be represented by $y=m(x-a)$. Substituting in equation of ellipse $\frac{y}{m}+a$ for x, we get ordinate of $P=-\frac{2ab^2m}{m^2a^2+b^2}$; and $\therefore$ if (x, y) be middle point of AP, $y=-\frac{ab^2m}{m^2a^2+b^2}$, $x=\frac{y}{m}+a=\frac{a^3m^2}{m^2a^2+b^2}$. Hence $\frac{x}{y}=-\frac{a^3m^2}{ab^2m}$, or $m=-\frac{b^2x}{a^2y}$; etc.]

23. CP, CD are conjugate diameters, and diameters CK, CL are drawn parallel to the focal distances DF, DF'. Show that CP bisects the angle between CK and CL. [See note on Ex. 9.]

24. CP, CD are conjugate diameters, and a diameter CK is drawn parallel to the focal distance DF. Show that PN, the perpendicular from P on CK, is equal to b. [Let KC meet the tangent at D in T, and let CM be perpendicular to tangent DT. Then $CT=a$, (§ 76). Also triangles PCN, CTM are similar. Hence $\frac{PN}{CP}=\frac{CM}{a}$, or $a.PN=CP.CM$; etc.]

25. If r, r' be any two semi-diameters at right angles to one another, show that

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{r^2}+\frac{1}{r'^2}.$$

CHAPTER VIII.

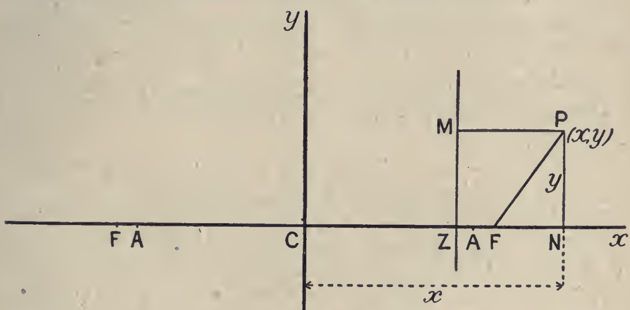
THE HYPERBOLA.

DEFINITION. An **Hyperbola** is the locus of a point which moves so that its distance from a fixed point, called the focus, is in a constant ratio ($e>1$) to its distance from a fixed straight line, called the directrix.

We shall form the equation of the hyperbola from its definition, the equation being thus the translation of the definition into analytic language. The properties of the hyperbola, all of which spring from its definition, will then be contained in the equation of the curve, and will appear on a suitable examination or analysis being made of this equation.

I. Equation and Trace of the Hyperbola.

90. To find the equation of the Hyperbola.



Let F be the focus, and MZ the directrix; and let FZ be perpendicular to MZ.

Divide FZ internally at A, and externally at A', so that

$$\frac{AF}{ZA}=e, \text{ and } \frac{A'F}{A'Z}=e,$$

$$\text{or} \quad AF=e.ZA, \text{ and } A'F=e.A'Z;$$

then A and A' are points on the locus.

Bisect $A'A$ at C; and let $A'A=2a$, so that $A'C=CA=a$.

$$\begin{aligned} \text{Then} \qquad 2CF &= A'F+AF, \\ &= e(A'Z+ZA), \\ &= e.2a; \\ \therefore\ CF &= ae. \end{aligned}$$

$$\begin{aligned} \text{Also} \qquad CZ &= \frac{1}{2}(A'Z-ZA), \\ &= \frac{1}{2e}(A'F-AF), \\ &= \frac{1}{2e}.2a, \\ &= \frac{a}{e}. \end{aligned}$$

Let now C be taken as origin, and CZ as axis of x. Also let Cy, perpendicular to CZ, be the axis of y. Let $P(x, y)$ be any point on the locus, and PM, PN the perpendiculars to MZ, Cx, respectively.

Then by definition of the hyperbola,

$$\frac{PF}{MP}=e;$$

$$\therefore\ PF^2=e^2.MP^2=e^2.ZN^2;$$

$$\therefore\ y^2+(x-ae)^2=e^2\left(x-\frac{a}{e}\right)^2;$$

$$\therefore\ x^2(e^2-1)-y^2=a^2(e^2-1);$$

$$\therefore\ \frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1,$$

which is the equation of the hyperbola.

The equation is usually written

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

$$\text{where} \quad b^2 = a^2(e^2 - 1), \text{ or } e^2 = \frac{a^2 + b^2}{a^2}.$$

The results,

distance from C to focus, CF, $= ae$,

" " " " directrix, CZ, $= \dfrac{a}{e}$,

Square of eccentricity, e^2, $= \dfrac{a^2 + b^2}{a^2}$,

are very important, and should be remembered.

The student may ask why, in the preceding demonstration, the directrix ZM was taken to the left of F, having, in the case of the ellipse, been taken to the right of F. In the hyperbola FA is greater than AZ; and therefore, when Z is taken to the left of F, the external point of division A' will occur to the left also. Hence C, the origin, falls to the left, and the point P, with the associated lines, comes to be in the first or positive quadrant, *i.e.*, we have the usual convenience of dealing with positive quantities x, y, etc.

Unless the contrary is stated, the equation of the hyperbola will be supposed to be of the form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

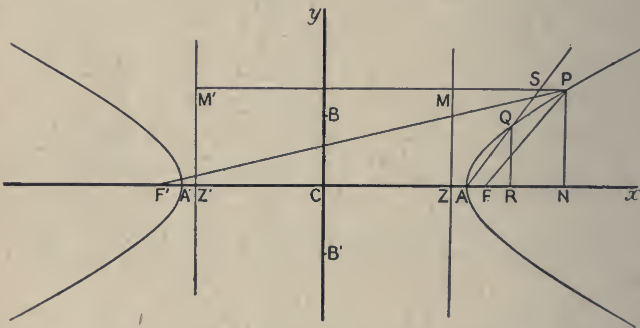
91. To trace the form of the hyperbola from its equation.

(1). If $y = 0$, $x = \pm a$. Hence if on the axis of x we take $CA = a$, $CA' = -a$, the curve passes through A and A'. If, however, we put $x = 0$, $y = \pm b\sqrt{-1}$, which, being imaginary, shows that the axis of y does not cut the curve.

AA' is called the **transverse axis** of the hyperbola. If points B, B' be taken on the axis of y, such that $CB=b$, $CB'=-b$, then BB' is called the **conjugate axis.** A and A' are called the **vertices** of the hyperbola.

(2). $y=\pm\frac{b}{a}\sqrt{x^2-a^2}$. Hence x cannot be numerically less than $\pm a$; and the curve falls entirely beyond two lines drawn through A and A' at right angles to the axis of x.

(3). $y=\pm\frac{b}{a}\sqrt{x^2-a^2}$. Hence as x increases numerically beyond $\pm a$, y increases; and when x becomes indefinitely great, y also becomes indefinitely great. Thus the curve has infinite branches on both sides of the origin, and above and below the axis of x.



(4). For a given value of x, the values of y are equal with opposite signs. Hence the curve is symmetrical with respect to the axis of x. Similarly from $x=\pm\frac{a}{b}\sqrt{y^2+b^2}$ it appears that the curve is symmetrical with respect to the axis of y.

(5). If we suppose the straight line $y=mx+k$ to cut the hyperbola, we shall have for the x's of the points of intersection the equation $\frac{x^2}{a^2}-\frac{(mx+k)^2}{b^2}=1$, or $\left(\frac{1}{a^2}-\frac{m^2}{b^2}\right)x^2$ $-2\frac{mk}{b^2}x-\frac{k^2}{b^2}-1=0$, a quadratic, giving two values of x. Hence a straight line can cut an hyperbola in only two points.

(6). If Q be any point on the curve, and it be supposed to move along the curve indefinitely close to A, the line AQS is ultimately the tangent at A, and the angle QAR is then the angle at which the curve cuts the axis of x. Now

$$\tan QAR=\frac{RQ}{AR}=\frac{y}{x-a}=\frac{b^2}{a^2}\cdot\frac{x+a}{y}.$$

Therefore ultimately $\tan QAR=\frac{b^2}{a^2}\cdot\frac{a+a}{0}=\infty$; and the angle QAR in the limit is 90°. Hence the curve cuts the axis of x at A at right angles; and by symmetry therefore at A' also.

Collating these facts, we see that the hyperbola has the form given in the diagram.

The symmetry of the curve shows that, since there is a focus F and a directrix ZM to the right of the origin, **there is a focus F' and a directrix $Z'M'$ at the same distances to the left of the origin.** Hence we have not only the constant relation $PF=e.MP$ for all positions of P, but also the constant relation $PF'=e.M'P$.

92. The point C is called the **centre** of the hyperbola.

Any chord through the centre is called a **diameter.**

Every chord through the centre of the hyperbola is there bisected. This proposition may be proved for the hyperbola in the same way as for the ellipse (§ 67).

93. **To find the distances of any point** (x, y) **on the hyperbola from the foci.**

Let P be the point (x, y).

Then $PF = e.MP = e.ZN,$

$$= e(CN - CZ),$$

$$= e\left(x - \frac{a}{e}\right) = ex - a.$$

Also $PF' = e.M'P = e.Z'N,$

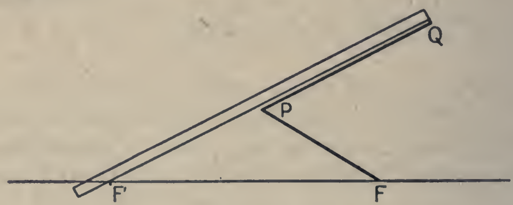
$$= e(CN + Z'C),$$

$$= e\left(x + \frac{a}{e}\right) = ex + a.$$

Hence $PF' - PF = (ex + a) - (ex - a),$

$$= 2a.$$

Therefore **the difference of the focal distances is constant for all points on the hyperbola, and is equal to** $2a$.



The preceding property, $PF - PF = 2a$, suggests a method of describing the hyperbola mechanically. For if a straight-edge $F'Q$ be capable of revolution about F'; and the ends of an inextensible string, of length less than $F'Q$, be fastened at F and Q, and the string be kept taut by a pencil at P, the pencil will trace out an hyperbola as the straight-edge revolves about F'. For l being the length of the string, let $2a+l$ be the length of FQ. Hence PQ being common to straight-edge and string, $PF'-PF=2a$, and the locus of P obeys the law of the hyperbola.

94. Following the method of §69, we may show that in the hyperbola, as in the ellipse, the double ordinate through a focus, called the latus rectum, is $2\frac{b^2}{a}$.

Exercises.

1. Find the axes, transverse and conjugate, the eccentricity, the distances from centre to focus and directrix, and the latus rectum of the hyperbola $3x^2-4y^2=12$.

2. In an hyperbola the distance from the centre to a focus is $\sqrt{34}$, and to a directrix $\frac{25}{\sqrt{34}}$; find the equation of the curve.

3. The equation of an ellipse being $2x^2+7y^2=14$, find the equation of an hyperbola which is confocal with it and whose conjugate axis is equal to the minor axis of the ellipse.

4. In the hyperbola $2x^2-3y^2=6$ find the distances from the point $(3, 2)$ to the foci.

5. If the crack of a rifle and the thud of the ball on the target be heard at the same instant, show that the locus of the hearer is that branch of an hyperbola for which the rifle is the farther and the target the nearer focus.

6. Find the locus of a point which moves so that its distance from the origin is a mean proportional between its distances from the points $(c, 0)$, $(-c, 0)$.

7. Find the equation of the hyperbola whose eccentricity is 2, directrix $3x+4y-12=0$, and focus (3, 2). [See Ex. 14, p. 141.]

8. A semi-diameter of an hyperbola, whose length is 2, makes an angle of 30° with the transverse axis. The eccentricity is $\sqrt{3}$. Find the equation of the hyperbola. [Hyperbola passes through point $(\sqrt{3}, 1)$.]

9. PNP' is a double ordinate of an hyperbola, and $PA, P'A'$ intersect at Q. Show that as PNP' varies in position, the locus of Q is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. [If (α, β), $(\alpha, -\beta)$ be P and P', then equations of PA, $P'A'$ are $\beta(x-a)-\alpha y+ay=0$, $\beta(x+a)+\alpha y+ay=0$; whence α and β; etc.]

10. Show that the locus of the centre of a circle which touches externally each of two given circles is an hyperbola.

11. In the preceding exercise r, r' being the radii of the given circles, and $2k$ the distance of their centres apart, find the equation of the hyperbola referred to.

12. In an hyperbola a line from the centre to an extremity of a latus rectum makes an angle of 45° with the transverse axis. Find the eccentricity.

Most of the propositions established in the previous chapter for the ellipse hold good for the hyperbola also; and the proofs in the case of the hyperbola are repetitions of the demonstrations of Chapter VII., with $-b^2$ substituted for $+b^2$. It is sufficient therefore in what follows to give merely the enunciations.

II. Tangents and Normals.

95. (1). **The equation of the tangent to the hyperbola in terms of the co-ordinates of the point of contact (x', y') is**

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=1 \ldots (\S\ 70).$$

(2). **The equation of the tangent to the hyperbola in terms of its inclination to the axis of x is**

$$y=mx \pm \sqrt{m^2a^2-b^2} \ldots (\S\ 71).$$

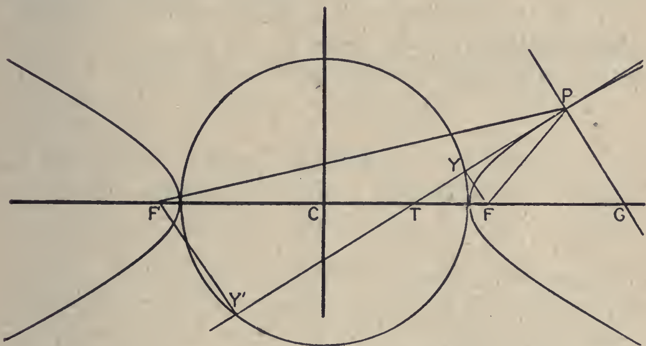
(3). **If α be the angle which the perpendicular from the origin on the tangent makes with the axis of x, the equation of the tangent becomes**

$$x \cos \alpha + y \sin \alpha = \sqrt{a^2 \cos^2\alpha - b^2 \sin^2\alpha} \ldots (\S\ 72).$$

(4). **The equation of the normal to the hyperbola at the point (x', y') is**

$$\frac{a^2}{x'}x + \frac{b^2}{y'}y = a^2 + b^2 \ldots . (\S\ 73).$$

(5). **In the hyperbola the tangent bisects the angle between the focal distances.** (§ 74).



Here putting $y=0$ in the equation of the tangent, $CT = \frac{a^2}{x'}$. Hence

$$F'T = ae + \frac{a^2}{x'} = \frac{a}{x'}(ex' + a);$$

$$TF = ae - \frac{a^2}{x'} = \frac{a}{x'}(ex' - a);$$

$$\therefore \frac{F'T}{TF} = \frac{ex' + a}{ex' - a} = \frac{F'P}{PF}, \ (\S\ 93);$$

and therefore PT bisects the angle $F'PF$.

Evidently **the normal makes equal angles with the focal distances.**

(6). **In the hyperbola the product of the perpendiculars from the foci on the tangent is constant and equal to** b^2. (§75).

Here, in following the method of §75, we shall get the result $FY.F'Y' = -b^2$, the negative sign being explained by the incidence of the perpendiculars on opposite sides of the tangent. See §28.

(7). **In the hyperbola the locus of the foot of the perpendicular from either focus on the tangent is a circle on the axis major as diameter,** *i.e.*, **the circle**

$$x^2 + y^2 = a^2 \quad . . \quad (\S 76).$$

(8). **In the hyperbola the locus of the intersection of tangents at right angles to each other is the circle**

$$x^2 + y^2 = a^2 - b^2. \quad . . (\S 77).$$

This circle reduces to a point when $a = b$; i.e., when $a = b$ only one pair of tangents are at right angles to each other, namely the asymptotes in the case of the rectangular hyperbola, (§ 100). The circle becomes imaginary when $a < b$, i.e., no tangents are then at right angles to each other.

Exercises.

1. Find the tangents to the hyperbola $3x^2 - 4y^2 = 12$ at the points whose ordinates are $+3$.

2. Find the value of m that the line $y = mx$ may be a tangent to the hyperbola.

3. Find the values of m that the line $y - k = m(x - h)$, which passes

through the point (h, k), may be a tangent to the hyperbola. [Identify $mx - y + k - mh = 0$ with $\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1$, giving $\frac{x'}{ma^2} = \frac{y'}{b^2} = \frac{-1}{k - mh}$;

whence $\frac{x'}{a} = \frac{-am}{k - mh}$, $\frac{y'}{b} = \frac{-b}{k - mh}$; etc.]

4. Is it possible for all values of m to draw a tangent to the hyperbola parallel to the line $y = mx$? [The tangents parallel to this are $y = mx \pm \sqrt{m^2a^2 - b^2}$.]

5. Is it possible for all positions of the point (h, k), (Ex. 3), to draw tangents from it to the hyperbola?

6. If P be any point on the hyperbola, and circles be described on PF, PF' as diameters, show that they will touch the circle described on AA' as diameter. [See Ex. 23, p. 142.]

7. The line $y = mx + \frac{c}{m}$ touches the parabola $y^2 = 4cx$, (§53). Show that it will also touch the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$, if $c^2 = m^2(m^2a^2 - b^2)$.

8. Show that the ellipse, and hyperbola

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a^2 - k} - \frac{y^2}{k - b^2} = 1,$$

are confocal.

9. Show that the confocal conics in the preceding exercise cut one another at right angles. [If (x', y') be their point of intersection $\frac{x'^2}{a^2(a^2 - k)} - \frac{y'^2}{b^2(k - b^2)} = 0$. Form equations of tangents at intersection.]

10. If α, β be the intercepts on the axes of any tangent to an hyperbola, show that $\frac{a^2}{\alpha^2} - \frac{b^2}{\beta^2} = 1$.

11. Two tangents are drawn to the hyperbola from the point (α, β), such that the product of the tangents of the angles they make with the transverse axis is λ. Show that the locus of (α, β) is $y^2 + b^2 = \lambda(x^2 - a^2)$. [$\beta = m\alpha \pm \sqrt{m^2a^2 - b^2}$.]

12. Find the condition that the line $lx + my + n = 0$ may be a normal to the hyperbola, [Identify this equation with $\frac{a^2}{x'}x + \frac{b^2}{y'}y - (a^2 + b^2) = 0$,

obtaining $\frac{a^2}{lx'} =$, etc. Thence x', y'; etc.]

13. Show that the line through the centre perpendicular to the normal at any point does not meet the hyperbola. [Line in question is $\frac{xx'}{a^2}-\frac{yy'}{b^2}=0$.]

14. In the equilateral hyperbola $(a=b)$ if PG, the normal at P, meet the transverse axis in G, then PCG is an isosceles triangle.

15. If the tangent and normal at P cut the transverse axis in T and G respectively, show that F', T, F, G form a harmonic range.

III. Poles and Polars.

96. (1). **The polar of any given point (x', y') with respect to the hyperbola is**

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=1 \ldots (\S\, 78).$$

The directrix is the polar of the focus (§ 78, Cor. 2).

(2). **In the hyperbola any focal chord is at right angles to the line joining its pole to the focus.** (§ 79).

(3). **In the hyperbola if Q lies on the polar of P, then P lies on the polar of Q.** (§ 80).

If a point moves along a fixed straight line, its polar turns about the pole of this line. (§ 80, Cor. 2).

(4). **A chord of an hyperbola is divided harmonically by any point on it and the polar of that point.** (§ 81.)

Exercises.

1. Find the pole of the line $lx+my+n=0$ with respect to the hyperbola. [Identify this line with $\frac{xx'}{a^2}-\frac{yy'}{b^2}=1$, obtaining $\frac{x'}{la^2}=\frac{-y'}{mb^2}$ $=\frac{-1}{n}$.]

2. Chords to the hyperbola are drawn through the intersection of the directrix with the axis. Show that tangents at their ends intersect on the latus rectum.

3. Find with respect to the hyperbola the pole of its normal $\frac{a^2}{x'}x+\frac{b^2}{y'}y=a^2+b^2$.

4. In the hyperbola find the locus of the pole of the normal. [Put x and y equal to results in previous exercise, and eliminate x', y' by means of equation of hyperbola.]

5. Find the locus of the poles with respect to the hyperbola, of tangents to the circle $x^2+y^2=a^2+b^2$. [Tangents to the circle are represented by $y=mx+\sqrt{(a^2+b^2)(m^2+1)}$. Identifying this with $\frac{xx'}{a^2}-\frac{yy'}{b^2}=1$, we get $\frac{x'}{ma^2}=\frac{y'}{b^2}=\frac{-1}{\sqrt{(a^2+b^2)(m^2+1)}}$. Eliminate m.]

6. A tangent to the hyperbola $x^2-y^2=a^2$ is drawn, whose inclination to the axis of x is $\tan^{-1}m$. Find its pole with respect to the parabola $y^2=4ax$. [Tangent is represented by $y=mx+a\sqrt{m^2-1}$. Identifying this with $yy'=2a(x+x')$ we get $\frac{2a}{m}=\frac{y'}{1}=\frac{2x'}{\sqrt{m^2-1}}$.]

7. In the preceding exercise find the locus of the pole as m varies.

8. If (x', y') be a point on the hyperbola $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, show that its polar with respect to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ touches the former hyperbola. [Here $-\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$. Also polar is $\frac{xx'}{a^2}-\frac{yy'}{b^2}=1$. Find condition that this touches $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.]

9. Find the direction-cosines of the chord of the hyperbola which is bisected at the point (x', y'); and thence obtain the equation of this chord. [Follow method suggested in Ex. 3, p. 126.]

10. In the hyperbola show that the polar of any point is parallel to the chord which is bisected at that point.

11. In the hyperbola find the pole of the chord which is bisected at the point (x', y') (Ex. 9); and show that it lies on the line joining (x', y') to the centre of the hyperbola.

Hence tangents at the ends of any chord of an hyperbola intersect on the diameter which bisects that chord.

12. If tangents be drawn from (x', y') to the hyperbola, show that they touch the same or opposite branches of

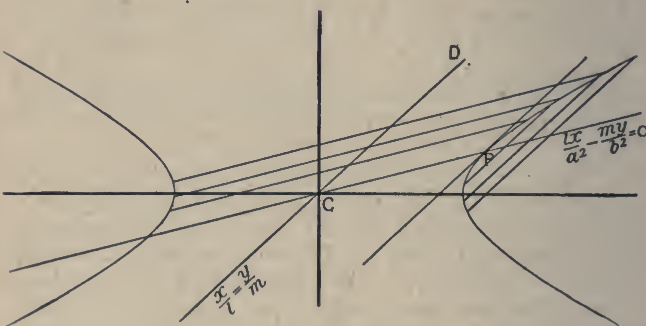
the hyperbola according as $b^2x'^2 - a^2y'^2$ is positive or negative. [Combining $\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1$, the polar of $(x'\ y')$, with equation of hyperbola, we get a quadratic for x's of points of contact. Product of roots will be found to be $\frac{a^4(b^2+y'^2)}{b^2x'^2 - a^2y'^2}$.]

Later it may be seen that $b^2x'^2 - a^2y'^2$ is positive or negative, according as (x', y') is within or without the asymptotes. **Hence the tangents from a given point touch the same or opposite branches of the hyperbola, according as the given point is between or outside the asymptotes.**

IV. Parallel Chords and Conjugate Diameters.

97. (1). **In an hyperbola the locus of the bisections of all chords parallel to the diameter** $\frac{x}{l} = \frac{y}{m}$ **is the diameter** $\frac{lx}{a^2} - \frac{my}{b^2} = 0$. (§ 82).

The tangent at the extremity of a diameter is parallel to the chords which that diameter bisects. (§ 82, Cor.)



(2). **If** CP **bisects the chords parallel to** CD, **then** CD **bisects all chords parallel to** CP. (§ 83).

CP and CD are called **conjugate diameters**.

If θ, θ' be the angles which CP, CD make with the axis of x, then

$$\tan\,\theta \cdot \tan\,\theta' = +\frac{b^2}{a^2}, \quad \ldots . (\S\,83)$$

a relation which in the hyperbola always connects the tangents of the angles which any pair of conjugate diameters make with the transverse axis. The positive sign before $\frac{b^2}{a^2}$ shows that in the hyperbola conjugate diameters fall on the same side of the conjugate axis.

98. The equation of the tangent at P (x', y') is $\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1$; and therefore the equation of CD, which is parallel to the tangent at P [§ 97, (1)] and passes through the origin, is $\frac{xx'}{a^2} - \frac{yy'}{b^2} = 0$, or $y = \frac{b^2}{a^2} \cdot \frac{x'}{y'}x$. For the points where CD cuts the hyperbola we must combine this with the equation of the curve. The combination gives

$$\frac{x^2}{a^2} - \frac{1}{b^2} \cdot \frac{b^4}{a^4} \cdot \frac{x'^2}{y'^2}x^2 = 1,$$

$$\text{or} \quad \frac{x^2}{a^2} \cdot \frac{b^2}{y'^2}\left(\frac{y'^2}{b^2} - \frac{x'^2}{a^2}\right) = 1,$$

$$\text{or} \quad \frac{x^2}{a^2} \cdot \frac{b^2}{y'^2}(-1) = 1,$$

$$\text{or} \quad x = \pm\frac{a}{b}y'\sqrt{-1}.$$

Hence CD does not meet the hyperbola; *i.e.*, **in the hyperbola one of a pair of conjugate diameters does not meet the curve.**

Exercises.

1. Find for the hyperbola the equations of the diameters which are conjugate to the following;

$$x+y=0\,;\ \frac{x}{a}-\frac{y}{b}\,;\ ax+by=0\,;\ bx-2ay=0.$$

2. If a be greater than b, find which of each pair of conjugate diameters in the preceding exercise, cuts the hyperbola.

3. If $\frac{m}{l}$ be positive, show that $\frac{m}{l}$, $\frac{b}{a}$, $\frac{b^2}{a^2}\cdot\frac{l}{m}$ are in order of magnitude; *i.e.*, **that the pair of conjugate diameters** $\frac{x}{l}=\frac{y}{m}$, $\frac{lx}{a^2}-\frac{my}{b^2}=0$, **lie on opposite sides of the line** $\frac{x}{a}-\frac{y}{b}=0$. [If $\frac{m}{l}<\frac{b}{a}$, then $1<\frac{b}{a}\cdot\frac{l}{m}$, and $\frac{b}{a}<\frac{b^2}{a^2}\cdot\frac{l}{m}$.]

If $\frac{m}{l}$ **be negative, show that the pair of conjugate diameters** $\frac{x}{l}=\frac{y}{m}$, $\frac{lx}{a^2}-\frac{my}{b^2}=0$ **lie on opposite sides of** $\frac{x}{a}+\frac{y}{b}=0$.

4. The length of a semi-diameter to the hyperbola is k and it lies in the first quadrant; find the equation of the conjugate diameter. [See Ex. 2, p. 169.]

5. In the hyperbola show that tangents at the extremities of any chord parallel to $\frac{x}{l}=\frac{y}{m}$ intersect on the line $\frac{lx}{a^2}-\frac{my}{b^2}=0$; *i.e.*, **in the hyperbola, as in the ellipse, tangents at the ends of any chord parallel to a given diameter intersect on the conjugate diameter.** [See Ex. 7, p. 170.]

6. In the hyperbola, if tangents be drawn from any point on the line $\frac{lx}{a^2}-\frac{my}{b^2}=0$, their chord of contact is parallel to $\frac{x}{l}=\frac{y}{m}$; *i.e.*, **in the hyperbola, as in the ellipse, the chord of contact of tangents from any point on a diameter is parallel to the conjugate diameter.** [See Ex. 8, p. 170].

7. In the hyperbola, if CD, the diameter conjugate to CP, cuts the focal distances PF, PF' in L and M, then $PL=PM$. [See Ex. 9, p. 170.]

8. In the hyperbola, if PF' meet CD in E, then $PE=a$. [See Ex. 11, p. 170.]

9. Through the foci F, F' of the hyperbola lines FQ, $F'Q$ are drawn parallel to the conjugate diameters CD, CP. Show that the locus of Q is the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=e^2$. [See Ex. 14, p. 171.]

10. Show that the radius-vectors r, r' from the centre to the hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=e^2$, and making the same angle θ with the transverse axis, are in the constant ratio $1:e$. Hence show that the latter hyperbola lies entirely on the concave side of the former. $\left[r^2\left(\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}\right)=1\text{; etc.}\right]$

V. Asymptotes and Conjugate Hyperbola.

99. DEF. An **asymptote** to any curve is the limiting position of the tangent as the point of contact moves off to an infinite distance, the tangent itself remaining at a finite distance from the origin.

In the parabola the tangent $yy'=2a(x+x')$ may be written

$$y=\frac{2a}{y'}x+2a\cdot\frac{x'}{y'}.$$

Here, when the point of contact (x', y') is "at infinity," the tangent of the angle which the tangent makes with the axis of x, i.e., $\frac{2a}{y'}$, is $\frac{2a}{\infty}=0$; and the tangent "at infinity" is parallel to the axis of x. Also, the intercept of the tangent on the axis of y

$$=2a\cdot\frac{x'}{y'}=2a\cdot\frac{y'}{4a}, \text{ (since } y'^2=4ax'\text{, and } \therefore \frac{x'}{y'}=\frac{y'}{4a}\text{)}, =\frac{1}{2}y'=\infty.$$

Hence in the parabola, when the point of contact moves off to an infinite distance, the tangent is at an infinite distance from the origin. Thus though the parabola has tangents at infinity, it is not usual to speak of such as asymptotes.

In the next article we shall see that in the hyperbola the tangents "at infinity" are not at an infinite distance from the origin, and the curve has asymptotes.

Ex. In the preceding explain why, when the ordinate of the point of contact is y' and the intercept on the axis of y is $\frac{1}{2}y'$, it is possible for the tangent "at infinity" to be parallel to the axis of x.

X 100. To find the asymptotes of the hyperbola.

The tangent to the hyperbola, $\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1$, may be written

$$\frac{x}{a^2} \cdot \frac{x'}{y'} - \frac{y}{b^2} = \frac{1}{y'}. \quad \ldots (1)$$

When the point of contact (x', y') moves off to an infinite distance, $\frac{1}{y'} = \frac{1}{\infty} = 0.$

Also since

$$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1;$$

$$\therefore \frac{x'^2}{y'^2} = \frac{a^2}{b^2} + \frac{a^2}{y'^2},$$

$$= \frac{a^2}{b^2}, \text{ when } y' = \infty;$$

$$\therefore \frac{x'}{y'} = \pm \frac{a}{b}, \text{ ultimately.}$$

Hence when the point of contact (x', y') becomes infinitely distant, the tangent (1) becomes

$$\frac{x}{a^2} \cdot \pm \frac{a}{b} - \frac{y}{b^2} = 0,$$

$$\text{or } \pm \frac{x}{a} - \frac{y}{b} = 0,$$

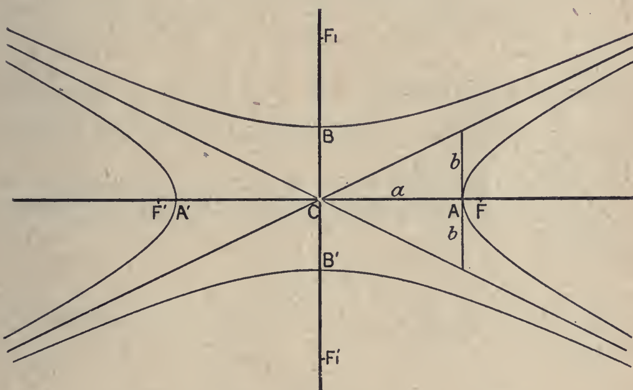
which represent two lines through the origin. Therefore the asymptotes of the hyperbola are

$$\frac{x}{a} \pm \frac{y}{b} = 0.$$

They may be included in the single form

$$\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = 0, \text{ or } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

Since the equations of the asymptotes are $y = \pm \frac{b}{a} x$, the asymptotes are two lines equally inclined to the axis of x at angles whose tangents are $\pm \frac{b}{a}$.



The preceding diagram illustrates the method of drawing the asymptotes, and their positions. The upper and lower curves are the branches of the conjugate hyperbola, to be referred to in the next article.

If $a = b$, the asymptotes make angles of 45° with the transverse axis, and are at right angles to each other.

The curve is then called the **rectangular** or **equilateral hyperbola.** Its equation is $x^2 - y^2 = a^2$.

101. CONJUGATE HYPERBOLA. The equation

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

evidently represents an hyperbola which cuts the axis of y at points B and B', distant $\pm b$ from C. It does not cut the axis of x. Thus BCB' is its transverse axis, and ACA' its conjugate axis,—lines which are respectively the conjugate and transverse axes of the original or primary hyperbola. Two such hyperbolas are said to be **conjugate** with respect to each other.

We may conveniently speak of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. as the **primary hyperbola,** and of $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as the **conjugate hyperbola.** Both curves are represented in the diagram of § 100.

If e' be the eccentricity of the conjugate hyperbola, evidently

$$e'^2=\frac{a^2+b^2}{b^2}.$$

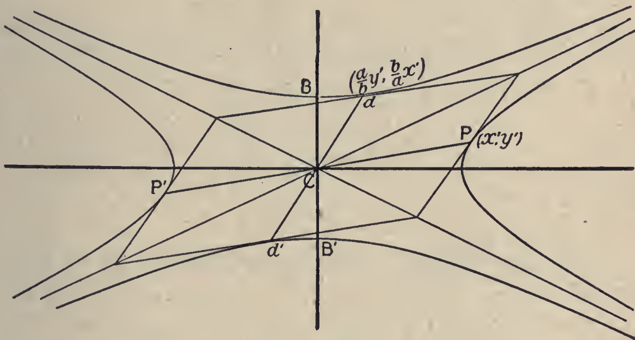
If F_1, F_1' be its foci, $CF_1' = CF_1 = e'b = \sqrt{a^2+b^2} = CF = CF'$.

The distance of its directrices from C is $\frac{b}{e'}=\frac{b^2}{\sqrt{a^2+b^2}}$;

and their equations are $y = \pm \frac{b^2}{\sqrt{a^2+b^2}}$.

Its asymptotes evidently are $\frac{x}{a}\pm\frac{y}{b}=0$, and **are the same as those of the primary hyperbola.**

102. The extremity P of any diameter being (x', y'), then d, the point in which the conjugate diameter cuts the conjugate hyperbola, is $\left(\frac{a}{b}y',\ \frac{b}{a}x'\right)$. (§84).



The equation of the tangent at P is $\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1$; and therefore the equation of Cd, which is parallel to the tangent at P [§97, (1)], and passes through the origin, is $\frac{xx'}{a^2} - \frac{yy'}{b^2} = 0$, or $y = \frac{b^2}{a^2} \cdot \frac{x'}{y'} x$. For the point d, when Cd cuts the conjugate hyperbola, $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we must combine these equations:

$$-\frac{x^2}{a^2} + \frac{1}{b^2} \cdot \frac{b^4}{a^4} \cdot \frac{x'^2}{y'^2} x^2 = 1,$$

$$\text{or}\quad \frac{x^2}{a^2} \cdot \frac{b^2}{y'^2}\left(-\frac{y'^2}{b^2} + \frac{x'^2}{a^2}\right) = 1,$$

$$\text{or}\quad \frac{x^2}{a^2} \cdot \frac{b^2}{y'^2} = 1;$$

$$\therefore\ x = \pm \frac{a}{b} y',$$

$$\text{and } y = \pm\frac{b}{a}x'.$$

Since, §97, (2), CP, Cd lie in the same quadrant, the positive signs have reference to the point d, and the negative to d'.

103. **In the hyperbola** $CP^2 - Cd^2 = a^2 - b^2$. (§85).

$$\text{For } CP^2 - Cd^2 = x'^2 + y'^2 - \frac{a^2}{b^2}y'^2 - \frac{b^2}{a^2}x'^2,$$

$$= a^2\left(\frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right) - b^2\left(\frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right),$$

$$= a^2 - b^2.$$

104. **The product of the focal distances** PF, PF', **is equal to** Cd^2. (§86).

$$\text{For } PF.PF' = (ex' - a)(ex' + a),$$

$$= e^2x'^2 - a^2,$$

$$= \frac{a^2 + b^2}{a^2}\cdot x'^2 - a^2,$$

$$= x'^2 - a^2 + \frac{b^2}{a^2}x'^2,$$

$$= \frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2,$$

$$= Cd^2.$$

105. **The tangents at** P **and** d **intersect on the asymptote** $\frac{x}{a} - \frac{y}{b} = 0$.

For the tangent at P (x', y') is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1\,; \quad \ldots.(1)$$

and the tangent at d $\left(\frac{a}{b}y', \frac{b}{a}x'\right)$ is

$$-\frac{x\cdot\frac{a}{b}y'}{a^2}+\frac{y\cdot\frac{b}{a}x'}{b^2}=1,$$

$$\text{or}\quad -\frac{xy'}{ab}+\frac{yx'}{ab}=1. \quad \ldots.(2).$$

Subtracting (1) and (2) we have (§ 12) the equation of a line through their intersection.

$$\text{Hence}\quad \frac{x}{a}\left(\frac{x'}{a}+\frac{y'}{b}\right)-\frac{y}{b}\left(\frac{y'}{b}+\frac{x'}{a}\right)=0,$$

$$\text{or}\quad \left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x'}{a}+\frac{y'}{b}\right)=0,$$

$$\text{or}\quad \frac{x}{a}-\frac{y}{b}=0,$$

which is the asymptote, is a line through the intersection of these tangents.

In like manner we may show that the tangents at d and P' intersect on $\frac{x}{a}+\frac{y}{b}=0$; etc.

Thus the asymptotes are the diagonals of the parallelogram formed by drawing tangents to the hyperbola and to its conjugate at the extremities of conjugate diameters.

106. **If a parallelogram be formed by drawing tangents at P, P', d, d', its area is constant, and equal to** $4ab$. (§ 87).

For Area of $\|^m = Cd \times$ perp. from C on tangent at P,

$$=4Cd.\,\frac{1}{\sqrt{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}}},$$

$$=4Cd.\,\frac{ab}{\sqrt{\frac{b^2}{a^2}x'^2+\frac{a^2}{b^2}y'^2}},$$

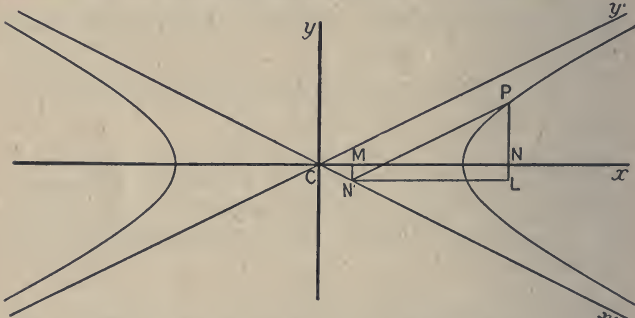
$$= 4Cd \cdot \frac{ab}{Cd},$$
$$= 4ab.$$

107. Following the method of § 88, we may show that the equation of the hyperbola referred to a pair of conjugate diameters as axes of co-ordinates is

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1,$$

where $a' = CP$, and $b' = Cd$. See also § 115.

108. **To find the equation of the hyperbola when referred to its asymptotes as axes of co-ordinates.**



Let Cx', Cy' be the asymptotes of the hyperbola, on which P (x, y) is any point. Let PN' be parallel to Cy', so that $CN' = x'$, $N'P = y'$ are the co-ordinates of P when the asymptotes Cx', Cy' are the axes of co-ordinates. PNL, MN' are parallel to Cy, and $N'L$ to Cx.

Let α be the angle $xCy' = xCx'$, so that (§ 100) $\tan \alpha = \frac{b}{a}$.

Hence $\frac{\sin \alpha}{\cos \alpha} = \frac{b}{a}$; or $\frac{\sin^2 \alpha}{b^2} = \frac{\cos^2 \alpha}{a^2} = \frac{1}{a^2 + b^2}$.

Then $x = CN = CM + MN = CN' \cos\alpha + N'P \cos\alpha =$
$(x' + y') \cos\alpha.$

$y = NP = LP - N'M = N'P \sin\alpha - CN' \sin\alpha =$
$(y' - x') \sin\alpha.$

Substituting these values of x and y in $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have

$$\frac{(x'+y')^2\cos^2\alpha}{a^2} - \frac{(y'-x')^2\sin^2\alpha}{b^2} = 1,$$

$$\text{or} \quad \frac{(x'+y')^2}{a^2+b^2} - \frac{(y'-x')^2}{a^2+b^2} = 1,$$

$$\text{or} \quad 4x'y' = a^2 + b^2.$$

Hence, dropping accents,

$$xy = \tfrac{1}{4}(a^2 + b^2)$$

is the equation of the hyperbola when referred to the asymptotes as axes of co-ordinates.

109. **To find the equation of the tangent to the hyperpola $xy = \frac{1}{4}(a^2 + b^2)$ at the point (x', y').**

The equation of the secant through the points $P(x', y')$ and $Q(x'', y'')$ is

$$\frac{x - x'}{x' - x''} = \frac{y - y'}{y' - y''}. \quad ..(1)$$

But since (x', y'), (x'', y'') lie on the curve $xy = \frac{1}{4}(a^2 + b^2)$,

$$x'y' = \tfrac{1}{4}(a^2 + b^2) = x''y'';$$

$$\therefore \frac{x'}{x''} = \frac{y''}{y'};$$

$$\therefore \frac{x' - x''}{x''} = -\frac{y' - y''}{y'}. \quad ..(2)$$

From (1) and (2)

$$\frac{x - x'}{x''} = -\frac{y - y'}{y'}.$$

When Q moves up to P, and PQ becomes the tangent at P, $x''=x'$. Hence the equation of the tangent at (x', y') is

$$\frac{x-x'}{x'}+\frac{y-y'}{y'}=0,$$

$$\text{or} \quad \frac{x}{x'}+\frac{y}{y'}=2.$$

COR. 1. The intercepts of the tangent on the asymptotes are $2x'$, $2y'$. Hence the part of the tangent intercepted by the asymptotes is bisected at the point of contact (x', y').

COR. 2. The area of the triangle formed by the asymptotes and the tangent is evidently $\frac{1}{2}\times 2x'\times 2y' \sin 2\alpha = 2x'y' \sin 2\alpha = (a^2+b^2)\sin\alpha\cos\alpha =$

$$(a^2+b^2)\cdot\frac{b}{\sqrt{a^2+b^2}}\cdot\frac{a}{\sqrt{a^2+b^2}}=ab;$$

and is constant for all positions of the tangent.

Exercises.

1. In the equilateral hyperbola $x^2-y^2=a^2$, show that CP of the primary hyperbola is equal to Cd of the conjugate hyperbola (§102); also that CP, Cd are equally inclined to the common asymptote $x-y=0$. [§103. Also, §97, (2), $\tan\theta.\tan\theta'=\frac{a^2}{a^2}=1$; $\therefore\ \theta=\frac{\pi}{2}-\theta'$. Or co-ordinates of P are x', y', and those of d, y', x'.]

2. In the hyperbola show that the locus of the middle point of the line Pd is the asymptote $\frac{x}{a}-\frac{y}{b}=0$. [§102.]

3. Show that the line Pd is parallel to the asymptote $\frac{x}{a}+\frac{y}{b}=0$.

4. From Exercises 2 and 3 show that the asymptotes and any pair of conjugate diameters form a harmonic pencil. [Geometry for Schools. Prop. 24 of Addl. Props.]

5. If r be a semi-diameter of the primary, and r' a semi-diameter of the conjugate hyperbola, r and r' being at right angles, show that $\frac{1}{a^2}-\frac{1}{b^2}=\frac{1}{r^2}-\frac{1}{r'^2}$. [For primary $x=r\cos\theta$, $y=r\sin\theta$; for conjugate $x=-r'\sin\theta$, $y=r'\cos\theta$; etc.]

6. The equation of an hyperbola which has its asymptotes as co-ordinate axes, and which passes through the point (h, k), is $xy=hk$ [§ 108.]

7. If the line $\frac{x}{a}+\frac{y}{b}=1$ touch the hyperbola $xy=k^2$, then $4k^2=ab$.

8. **If any line meet the hyperbola in Q, Q' and the asymptotes in R, R', show that** $QR=Q'R'$. [If tangent parallel to QQ' touch hyperbola in P, then chords parallel to QQ' are all bisected by CP, §97, (1). Also since (§ 109, Cor. 1) tangent at P is there bisected, the parts of all lines parallel to the tangent, and intercepted by asymptotes, are bisected by CP.]

9. Given the two conjugate diameters CP, Cd in magnitude and direction, find by construction the asymptotes and axes. [§97, (1), and § 105. Or Exs. 2 and 3.]

10. In the preceding exercise does the fixing of CP, Cd, in magnitude and direction, completely determine the hyperbola? [§ 107.]

11. A variable circle passes through two fixed points A and A', where $AA'=2a$. Show that the locus of a point on the circle where the tangent is perpendicular to AA' is $x^2-y^2=a^2$, AA' being the axis of x, and the middle point of AA' the origin.

12. If from the point P on the hyperbola, PN be drawn perpendicular to the transverse axis, and the tangent at P cut the transverse axis at T, show that CA is a mean proportional between CN and CT. [Use equation $\frac{xx'}{a^2}-\frac{yy'}{b^2}=1$.]

13. Given two conjugate diameters CP, Cd of the hyperbola, find by geometric construction the transverse and conjugate axes of the curve, *i.e.*, a and b. (Use Exs. 9 and 12.]

14. Two tangents are drawn to the hyperbola and produced to meet the asymptotes. Show that the lines joining the points of intersection with the asymptotes are parallel to one another. [Take asymptotes for axes; (x', y'), (x'', y'') for points of contact; and therefore $\frac{x}{x'}+\frac{y}{y'}=2$, etc., for tangents. $x'y'=x''y''$.]

15. If the hyperbola be referred to its asymptotes as axes of co-ordinates, the lines $\frac{x}{x'} - \frac{y}{y'} = 0$, $\frac{x}{x'} + \frac{y}{y'} = 0$ represent conjugate diameters. [The tangent at P (x', y') is parallel to the diameter conjugate to CP.]

16. If on any chord of an hyperbola as diagonal a parallelogram be constructed whose sides are parallel to the asymptotes, show that the other diagonal passes through the centre. [Refer curve to asymptotes as axes, and let (x', y'), (x'', y'') be co-ordinates of ends of chord; then (x', y''), (x'', y') co-ordinates of ends of other diagonal.]

17. In the preceding exercise show that the diagonal through the centre, and a diameter parallel to the given chord are conjugate diameters with respect to each other. [Equation of diagonal through centre is $\frac{x}{x' - x''} + \frac{y}{y' - y''} = 0$. Form equation of diameter parallel to chord. Use Ex. 15.]

18. If two hyperbolas have the same asymptotes, and r, r' be the lengths of semi-diameters to them making the same angle θ with the transverse axis, then $r:r'$ is a constant ratio for all values of θ. [If a, b, a', b', be the semi-axes, then $\frac{b}{a} = \frac{b'}{a'}$. Also $r^2 \left(\frac{b^2}{a^2}\cos^2\theta - \sin^2\theta\right) = b^2$; etc.]

19. Two hyperbolas have common asymptotes, and any tangent is drawn to the inner one. Show that as a chord of the outer it is bisected at the point of contact. [Use Ex. 8.]

20. If 2α be the angle between the asymptotes of the hyperbola $xy = k^2$, and a, b be its transverse and conjugate semi-axes, show that $a = 2k\cos\alpha$, $b = 2k\sin\alpha$.

CHAPTER IX.

THE GENERAL EQUATION OF THE SECOND DEGREE.

110. In finding the equations of the parabola, ellipse, and hyperbola, the positions of the origin and axes with respect to the curves were specially selected, that the equations of the curves might be obtained in forms the simplest and therefore most convenient for discussion. From Ex. 12, page 111, however, it appears that when the directrix is *any* line, $Ax + By + C = 0$, and the focus *any* point, (a, b), the equation of the parabola consists of terms involving x^2, xy, y^2, x, y, and a term with no variable in it. Indeed in the case of any conic, if we suppose the directrix, say $Ax + By + C = 0$, and the focus, say (α, β), to have any positions with respect to the axes, and the eccentricity to be any ratio $e:1$, the definition of a conic is expressed by the equation

$$(x-\alpha)^2 + (y-\beta)^2 = e^2 \cdot \frac{(Ax + By + C)^2}{A^2 + B^2},$$

which consists of terms involving x^2, xy, y^2, etc.

This equation may be written in the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

which is the general equation of the second degree; and in the present chapter it is proposed to show that **conversely the general equation of the second degree must represent a conic,** under which name is included the circle, as well as two intersecting, coincident, or parallel straight lines. See accompanying note.

This converse proposition will be established by showing that, by a proper selection of axes and origin, the general equation of the second degree may be reduced to forms which we shall recognize as the equations of the ellipse (including the circle), hyperbola, parabola, or of two intersecting, coincident, or parallel straight lines.

NOTE. (1). In the hyperbola $\frac{b^2}{a^2}x^2 - y^2 = b^2$, if a and b become indefinitely small, but yet remain in a finite ratio, so that $\frac{b}{a} = k$, a finite quantity, the equation becomes $k^2x^2 - y^2 = 0$, or $kx - y = 0$, $kx + y = 0$, which represent two straight lines intersecting at the origin. Here $e = \sqrt{\frac{a^2+b^2}{a^2}} = \sqrt{1+k^2}$, and is finite. Also $ae = 0 . e = 0$; $\frac{a}{e} = \frac{0}{e} = 0$. Thus two intersecting lines are a conic whose foci are at the intersection of the lines, and whose directrices coincide and bisect the angle between the lines.

(2). Writing the equation of the hyperbola in the form $x^2 - \frac{a^2}{b^2}y^2 = a^2$, we see that if b remain finite and a become indefinitely small, it takes the form $x^2 = 0$, which represents two straight lines coincident with the axis of y. Here $e = \sqrt{1 + \frac{b^2}{a^2}} = \infty$; also $ae = \sqrt{a^2+b^2} = b$; $\frac{a}{e} = \frac{0}{\infty} = 0$. Thus two coincident lines are a conic whose foci are distant $\pm b$ on each side of them, and whose directrices coincide with them.

(3). If b become infinite and a remain finite, the equation of (2) becomes $x^2 - a^2 = 0$, or $x - a = 0$, $x + a = 0$, representing two parallel straight lines. Here $e = \sqrt{1 + \frac{b^2}{a^2}} = \infty$; $ae = \infty$; $\frac{a}{e} = 0$. Thus two parallel straight lines are a conic whose foci are at infinity, and directrices two coincident lines midway between the parallel lines.

(4). The parabola $y^2 = 4ax + b^2 = 4a\left(x + \frac{b^2}{4a}\right)$, as a becomes indefinitely small while b remains finite, represents two parallel straight lines,

$y = \pm b$. If b also become indefinitely small, $\frac{b^2}{4a}$ remaining infinite, the equation represents two coincident straight lines. The term $\frac{b^2}{4a}$ is the distance of the vertex to the left of the origin.

111. By turning the axes of co-ordinates through a certain angle, retaining the same origin, the term involving the product xy may always be made to disappear from the general equation of the second degree.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

For (§ 31) we turn the axes through the angle θ by substituting $x\cos\theta - y\sin\theta$ for x, and $x\sin\theta + y\cos\theta$ for y.

Making these substitutions, the preceding equation becomes

$$\begin{aligned} a(x\cos\theta - y\sin\theta)^2 &+ 2h(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) \\ &+ b(x\sin\theta + y\cos\theta)^2 + 2g(x\cos\theta - y\sin\theta) \\ &\quad + 2f(x\sin\theta + y\cos\theta) + c = 0. \quad \ldots.(1). \end{aligned}$$

In this equation the coefficient of xy is

$$2(b-a)\sin\theta\cos\theta + 2h(\cos^2\theta - \sin^2\theta),$$
$$\text{or } (b-a)\sin 2\theta + 2h\cos 2\theta;$$

and putting this equal to zero, we see that it will vanish if

$$\tan 2\theta = \frac{2h}{a-b}.$$

But the tangent of an angle may have any value from $+\infty$ to $-\infty$. Hence a value of θ can always be found which will satisfy the above equation.

Introducing this value of θ in (1), the term involving xy disappears, and (1) takes the form

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0,$$

with which therefore we may now deal, with the certainty that it includes all the geometric forms that the (in appearance) more general equation of the enunciation can possibly represent.

112. **To show that the equation**

$$Ax^2+By^2+2Gx+2Fy+C=0\ldots.(1)$$

must represent a conic.

I. If in this equation A or B be zero, suppose A, it may be written

$$B\left(y+\frac{F}{B}\right)^2=-2Gx+\frac{F^2}{B}-C,\ \ldots.(2)$$

$$=-2G\left\{x-\left(\frac{F^2}{2BG}-\frac{C}{2G}\right)\right\}\ldots..(3)$$

(α) Transferring the origin to the point $\left(\frac{F^2}{2BG}-\frac{C}{2G}, -\frac{F}{B}\right)$, equation (3) becomes

$$By^2=-2Gx,$$

$$\text{or}\quad y^2=-\frac{2G}{B}x,$$

which represents a **parabola.**

(β). If in equation (2), $G=0$, the equation reduces to

$$y=-\frac{F}{B}\pm\sqrt{\frac{F^2}{B^2}-\frac{C}{B}},$$

which represents **two parallel straight lines.**

(γ). If in equation (2), $G=0$ and also $\frac{F^2}{B}-C=0$, the equation reduces to

$$\left(y+\frac{F}{B}\right)^2=0,$$

$$\text{or}\quad y=-\frac{F}{B},\ y=-\frac{F}{B},$$

which represent **two coincident straight lines.**

II. If in equation (1) neither A nor B be zero, it may be written

$$A\left(x+\frac{G}{A}\right)^2+B\left(y+\frac{F}{B}\right)^2=\frac{G^2}{A}+\frac{F^2}{B}-C. \quad \ldots .(4)$$

Transferring the origin to the point $\left(-\frac{G}{A}, -\frac{F}{B}\right)$, equation (4) becomes

$$Ax^2+By^2=\frac{G^2}{A}+\frac{F^2}{B}-C. \quad \ldots (5)$$

(α). If in (5) $\frac{G^2}{A}+\frac{F^2}{B}-C=0$, the equation becomes

$$Ax^2+By^2=0,$$

which represents **two straight lines intersecting** at the origin. The lines are *real* if A and B have different signs, and *imaginary* if A and B have the same sign.

(β). If the right side of (5) be not zero, the equation may be written

$$\frac{x^2}{\frac{1}{A}\left(\frac{G^2}{A}+\frac{F^2}{B}-C\right)}+\frac{y^2}{\frac{1}{B}\left(\frac{G^2}{A}+\frac{F^2}{B}-C\right)}=1.$$

Here, if the denominators of x^2 and y^2 be both positive, the equation represents an **ellipse**; if the denominators be of opposite signs, the equation represents an **hyperbola.**

If the denominators be both negative, evidently no real values of x and y can satisfy the equation, and the locus may be said to be an **imaginary ellipse.**

NOTE. The equation of a conic, when the axes are oblique, must be of the second degree. For if it were of the third degree, on combining it with the equation of a straight line, we should get three points of intersection.

113. **When the equation**

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a conic referred to its centre, then $g = 0$ **and** $f = 0$.

If (α, β) be any point on the conic, since the centre is the origin, $(-\alpha, -\beta)$ must also be a point on the curve. Hence

$$a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0,$$
$$\text{“} \quad \text{“} \quad \text{“} \quad - 2g\alpha - 2f\beta + c = 0.$$

Subtracting

$$g\alpha + f\beta = 0.$$

Such an equation would make $\beta : \alpha$ a constant ratio $-g : f$. But as (α, β) moves along the curve, $\beta : \alpha$ cannot be a constant ratio. The necessary inference is that the above equation is true, because separately $g = 0$ and $f = 0$.

Hence the equation of a conic when referred to its centre as origin is of the form

$$ax^2 + 2hxy + by^2 + c = 0.$$

114. **To find the centre of the conic**

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Let (x', y') be the centre. Transferring the origin to the centre (x', y'), the equation becomes (§ 30)

$$a(x + x')^2 + 2h(x + x')(y + y') + b(y + y')^2 + 2g(x + x') + 2f(y + y') + c = 0,$$

or

$$ax^2 + 2hxy + by^2 + 2(ax' + hy' + g)x + 2(hx' + by' + f)y + ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c = 0.$$

But now, since the conic is referred to its centre as origin, the coefficients of x and y must vanish (§ 113).

Hence

$$ax' + hy' + g = 0,$$
$$hx' + by' + f = 0;$$

whence

$$x' = \frac{hf - bg}{ab - h^2}, \quad y' = \frac{gh - af}{ab - h^2}, \quad \dots.(1)$$

which give the co-ordinates of the conic's centre.

Evidently the equation of the conic when referred to its centre as origin is

$$ax^2 + 2hxy + by^2 + c' = 0,$$

where $c' = ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c$. In this expression for c' we must substitute for x', y' the values given by (1).

115. To find the form of the equation of the ellipse, or of the hyperbola, when the curve is referred to its conjugate diameters as axes of co-ordinates.

From the note to §112 we see that the equation is included in the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

But since the intersection of the conjugate diameters is the centre, the centre is the origin. Hence (§113) $g = 0$, $f = 0$; and the equation reduces to

$$ax^2 + 2hxy + by^2 + c = 0.$$

Again, since one conjugate diameter bisects chords parallel to the other, if (α, β) be a point on the curve, $(\alpha, -\beta)$ is also a point on the curve. Therefore

$$a\alpha^2 + 2h\alpha\beta + b\beta^2 + c = 0,$$
$$a\alpha^2 - 2h\alpha\beta + b\beta^2 + c = 0.$$

Subtracting, $h\alpha\beta = 0$; and therefore $h = 0$. Hence the equation takes the form

$$ax^2 + by^2 + c = 0,$$

which may be written $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = \pm 1$, if a and b have the same sign; or $\dfrac{x^2}{a'^2} - \dfrac{y^2}{b'^2} = \pm 1$, if a and b have different signs. Compare §§88, 107.

116. The general equation of the second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \ldots (1)$$

represents a parabola, an ellipse, or an hyperbola according as $ab - h^2$ is equal to, greater than, or less than zero.

For when the focus is any point (α, β), the directrix any line $Ax + By + C = 0$, and the eccentricity any ratio e, the equation of a conic, derived at once from its definition, is

$$(x - \alpha)^2 + (y - \beta)^2 = e^2 \cdot \frac{(Ax + By + C)^2}{A^2 + B^2},$$

or

$$(A^2 + B^2 - e^2A^2)x^2 - 2e^2ABxy + (A^2 + B^2 - e^2B^2)y^2 + \ldots . = 0.$$

From the mode of its derivation this is the equation of *any* conic. But all conics are also represented by (1). Therefore, since the two equations may be regarded as representing the same curve,

$$\frac{a}{A^2 + B^2 - e^2A^2} = \frac{h}{-e^2AB} = \frac{b}{A^2 + B^2 - e^2B^2} = \lambda, \text{ say.}$$

Hence

$$\begin{aligned} ab - h^2 &= \lambda^2(A^2 + B^2 - e^2A^2)(A^2 + B^2 - e^2B^2) - \lambda^2e^4A^2B^2, \\ &= \lambda^2(A^2 + B^2)^2(1 - e^2). \end{aligned}$$

Now the curve is a parabola, ellipse, or hyperbola according as $e = 1$, < 1, or > 1.

Hence the curve represented by (1) is

a parabola if $ab - h^2 = 0$;
an ellipse if $ab - h^2$ is positive;
an hyperbola if $ab - h^2$ is negative.

Exercises.

1. Find the nature of the conic $7x^2 - 2xy + 7y^2 - 24 = 0$. [Here, §111, $\tan 2\theta = \frac{-2}{7-7} = \infty$; and $\theta = 45°$. Hence for x substitute $\frac{x-y}{\sqrt{2}}$, and for y, $\frac{x+y}{\sqrt{2}}$. Or §116 will show at once its class.]

2. In the conic of the preceding exercise, find the semi-axes, the co-ordinates of the foci, and the equations of the directrices.

3. What curve does the equation $3x^2 - 2x + 5y + 7 = 0$ represent? Find the co-ordinates of its focus, and the equations of its axis and directrix. Draw the curve, placing it correctly with respect to the original axes. [§116 shows at once that it is a parabola. The equation may be written $(x - \frac{1}{3})^2 = -\frac{5}{3}(y + \frac{4}{3})$. [Transfer origin to $(\frac{1}{3}, -\frac{4}{3})$, and it becomes $x^2 = -\frac{5}{3}y$; etc.]

4. Find what curve is represented by the equation $(x - y)^2 = 2(x + y)$, and place it with respect to the axes. [§111, $\tan 2\theta = \frac{-2}{1-1} = \infty$, and $\theta = 45°$. Turn axes through 45°.]

5. Interpret the equation $2x^2 - 5xy + 3y^2 + 6x - 7y + 4 = 0$. [The centre, §114, is given by $4x - 5y + 6 = 0$, $5x - 6y + 7 = 0$; and centre is (1, 2). Transferring to this point the equation becomes $2x^2 - 5xy + 3y^2 = 0$; etc.]

6. Interpret the equation $2xy - x - y = 0$. Place the curve. [§116 shows it to be an hyperbola. The left side cannot be factored, and ∴ it cannot be two st. lines. Centre, §114, is given by $2x - 1 = 0$, $2y - 1 = 0$. Transfer to centre; etc.]

7. Find nature of curve $x^2 - 4xy + 4y^2 - 2x + y - 6 = 0$. Find its vertex and axis; and place it properly with respect to the original axes of co-ordinates. [Turning axes (§111) through θ, where $\sin\theta = \frac{1}{\sqrt{5}}$, $\cos\theta = \frac{2}{\sqrt{5}}$, equation reduces to $\left(y + \frac{2}{5\sqrt{5}}\right)^2 = \frac{3}{5\sqrt{5}}\left(x + \frac{154}{15\sqrt{5}}\right)$. Transferring origin to $\left(-\frac{154}{15\sqrt{5}}, -\frac{2}{5\sqrt{5}}\right)$, equation becomes $y^2 = \frac{3}{5\sqrt{5}}x$.]

8. Find nature of curve $x^2-2xy+y^2+4x+4y-4=0$. Find its vertex and axis; and place it properly with respect to the original axes of co-ordinates. [Turning axes through 45° (§ 111), equation becomes $y^2=-2\sqrt{2}\left(x-\frac{1}{\sqrt{2}}\right)$. Transferring origin to $\left(\frac{1}{\sqrt{2}}, 0\right)$, equation becomes $y^2=-2\sqrt{2}x$.]

9. Find the centre of $x^2-6xy+9y^2+4x-12y=0$, and interpret your result.

10. Find the centre of $3x^2-2xy+y^2-10x-2y+19=0$. By transferring the origin to this point show that this equation represents two imaginary straight lines.

11. Find the centre of the locus $x^2+5xy+y^2-12x-9y+10=0$. Transfer the origin to the centre, and turn the axes through such an angle that the term involving xy will disappear. Draw the curve, placing it correctly with respect to the original axes.

12. When the equation of the second degree can be reduced to the form $(hx-ky)(h'x-k'y)=c$, the factors being real, show that it represents an hyperbola whose asymptotes are $hx-ky=0$, $h'x-k'y=0$. [The form $ab-h^2$ of §116 here is $-\frac{1}{4}(hk'-h'k)^2$, a negative quantity; the curve is $\therefore$ an hyperbola. Also these lines pass through the origin which is the centre of the curve, and cut the curve in points given by $x^2=\frac{c}{0}$, $y^2=\frac{c}{0}$, *i.e.*, at infinity. They are $\therefore$ the asymptotes.]

13. Find the centre of $y^2-xy-5x+5y=1$; transfer the origin to it; interpret the equation, and place the curve correctly with respect to the original axes.

14. Find the centre of $y^2-xy-6x^2+27x-4y=0$. Transfer the origin to it; interpret the equation, and place the curve.

15. Interpret the equation $3x^2+4y^2+12x-8y+8=0$.

16. The equation $y=x\tan\theta-\frac{g\sec^2\theta}{2v^2}x^2$ represents the path of a projectile in a vacuum, the origin being the point of projection, axis of x horizontal, axis of y vertical and upwards, θ the angle the direction of projection makes with axis of x, and v the initial velocity. Show that the curve is a parabola. Find co-ordinates of vertex and focus;

also the equation of the directrix. [For discussion write equation in form $y=ax-bx^2$. Whence $\left(x-\frac{a}{2b}\right)=-\frac{1}{b}\left(y-\frac{a^2}{4b}\right)$; etc.]

17. Interpret the equation $9x^2-24xy+16y^2+6ax-8ay=0$.

18. Interpret the equation $4x^2-4xy+y^2+\sqrt{5}(3x+y)=0$. Construct the co-ordinate axes to which in succession it is referred, and finally place the curve, so showing its position with respect to the original axes. [A parabola by §116. By §111, $\tan 2\theta=\frac{-4}{3}$; $\therefore$ $\tan\theta=2$, and $\sin\theta=\frac{2}{\sqrt{5}}$, $\cos\theta=\frac{1}{\sqrt{5}}$. Turning axes through θ, equation becomes $y^2-y+x=0$, or $(y-\frac{1}{2})^2=-(x-\frac{1}{4})$; etc.]

19. Interpret the equation $3x^2+2xy+3y^2-14x+6y+11=0$, and place it with respect to the axes.

20. Interpret the equation $3x^2+8xy-3y^2+10x-20y-50=0$. Construct the co-ordinates axes to which in succession it is referred, and finally place the curve, so showing its position with respect to the original axes. [First transfer to centre.]

ANSWERS.

Chapter I. Point in a Plane. Co-ordinates.

Page 10.

2. $(2+\sqrt{3},\ -2\sqrt{3})$; $\left(5-\frac{1}{\sqrt{3}},\ \frac{2}{\sqrt{3}}\right)$; $\left(-4+\frac{5}{\sqrt{3}},\ -\frac{10}{\sqrt{3}}\right)$;
$\left(\frac{4}{\sqrt{3}},\ -\frac{8}{\sqrt{3}}\right)$; $(3,\ 0)$; $\left(-3-\frac{1}{\sqrt{3}},\ \frac{2}{\sqrt{3}}\right)$.

Page 12.

1. 5. **2.** Side, $4\sqrt{2}$. **3.** $(2+2\sqrt{3},\ 2+2\sqrt{3})$, $(2-2\sqrt{3},\ 2-2\sqrt{3})$.
5. Sides, 5; diagonals, $5\sqrt{2}$.
6. Sides, $\sqrt{10}$, $2\sqrt{10}$; diagonals, $5\sqrt{2}$.
7. Sides, $3\sqrt{2}$, $\sqrt{5}$; diagonals, $\sqrt{17}$, $\sqrt{29}$.
8. $x^2+y^2+2x-4y-4=0$. **9.** 8 or -2.
10. $(P'P'')^2=(x'-x'')^2+(y'-y'')^2+2(x'-x'')(y'-y'')\cos\omega$.

Pages 14-15.

1. $\left(\frac{5}{2},\ -1\right)$; $\left(-\frac{1}{2},\ \frac{3}{2}\right)$; $\left(-1,\ -\frac{1}{2}\right)$. **2.** $\left(\frac{11}{3},\ 2\right)$; $\left(\frac{13}{3},\ 0\right)$.
3. $\left\{x_1+\frac{1}{n}(x_2-x_1),\ y_1+\frac{1}{n}(y_2-y_1)\right\}$; $\left\{x_1+\frac{2}{n}(x_2-x_1),\ y_1+\frac{2}{n}(y_2-y_1)\right\}$; etc.
4. $\left(-\frac{4}{7},\ -\frac{11}{7}\right)$. **5.** $\left(\frac{27}{5},\ \frac{14}{5}\right)$; $(15,\ 10)$.

Page 17.

1. 18. **2.** $25\frac{1}{2}$. **4.** $1\frac{1}{2}$. **6.** $5-x-2y$; $x+2y=5$.

Chapter II. Equations and Loci.

Pages 34-36.

1. $x+5=0$. **2.** $x-y=0$, or $x+y=0$.
3. (1). $2x-5=0$; (2). $x=0$; (3). $x-y=2$; (4). $x-y=0$. **5.** $y=x+a$.
6. $4x-3y=8$. **7.** $4ax=c^2$. **8.** $y^2=2x$. **9.** $x^2=2y$; $(0,\ 0)$, $(2,\ 2)$.
10. $8x^2-y^2=0$. **11.** $y^2-6x-4y+13=0$. **12.** $x^2+y^2-2ax=0$.

13. $3x^2+3y^2-8ax+4a^2=0$. **14.** $2x-3y+6=0$.
15. $(4, 3)$ and $(-4, 9)$ are; $(6, 2)$ is not. **16.** All are except the last.
17. ± 1. **19.** $(-3, 0)$; $(0, -4)$. **20.** $(\pm 2\sqrt{2}, 0)$; $(0, \pm 3\sqrt{2})$. **21.** No.
22. No. **23.** $(3-\sqrt{3}, 3-\sqrt{3})$; $(3+\sqrt{3}, 3+\sqrt{3})$. $m=-2\pm\sqrt{6}$.
25. $\left(\frac{12}{5}, \frac{12}{5}\right)$; $\left(\frac{12}{5}, -\frac{12}{5}\right)$; $\left(-\frac{12}{5}, \frac{12}{5}\right)$; $\left(-\frac{12}{5}, -\frac{12}{5}\right)$.

Chapter III. The Straight Line.

Pages 40-41.

2. $\frac{x}{a}+\frac{y}{b}=-1$. **3.** $2x-3y+6=0$. **5.** $x_1-\frac{x_1-x_2}{y_1-y_2}y_1$, $y_1-\frac{y_1-y_2}{x_1-x_3}x_1$.
7. $5x+4y+20=0$. **8.** $4x+7y+19=0$. $-\frac{19}{4}, -\frac{19}{7}$. **9.** $5x+3y=0$.
10. $9x+2y+8=0$. **11.** $x+y+9=0$. **12.** Find its intercepts on axes.

Pages 44-45.

3. $y=x-5$. **4.** $y=\sqrt{3}(x+7)$. $7\sqrt{3}$. **5.** $x-y\sqrt{3}=0$.
6. $x\sqrt{3}+y+3=0$. **7.** $x+y\sqrt{3}+3=0$. **8.** $x\sqrt{3}-y-3\sqrt{3}=0$.
9. Equation is $y=x-2$; angle, $45°$; intercept, -2.
10. They are parallel.
11. $m=\frac{y_1-y_2}{x_1-x_2}$; $a=\frac{x_2y_1-x_1y_2}{y_1-y_2}$. Then substitute in $y=m(x-a)$.
12. $y=x\sqrt{3}+5-\sqrt{3}$; $y=-x\sqrt{3}+5+\sqrt{3}$.

Pages 49-50.

3. $x\sqrt{3}+y=12$. **4.** $x-y\sqrt{3}+8=0$.
5. $3x+2y+5=0$; $-\frac{3}{\sqrt{13}}x-\frac{2}{\sqrt{13}}y=\frac{5}{\sqrt{13}}$.
6. $x-y=5$; $\frac{x}{\sqrt{2}}-\frac{y}{\sqrt{2}}=\frac{5}{\sqrt{2}}$.
7. $\frac{x-7}{\cos 60°}=\frac{y-1}{\sin 60°}$, or $x\sqrt{3}-y=7\sqrt{3}-1$. **9.** $x\sqrt{3}+y=5\sqrt{3}-2$.

Page 52.

4. $K=1$, $L=-4$; and line is $x-4y=1$.

Pages 57-59.

1. $20x+17y-11=0$. **2.** $x+2y=0$. **5.** Fixed point is $\left(\frac{1}{C}, -\frac{1}{C}\right)$.

6. $\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}.$

7. $\frac{x-2}{\frac{2}{\sqrt{29}}} = \frac{y-3}{\frac{5}{\sqrt{29}}} = r$; $\frac{x-4}{\frac{2}{\sqrt{29}}} = \frac{y-8}{\frac{5}{\sqrt{29}}} = r.$ 8. $\frac{\sqrt{29}}{6}.$

9. $\frac{x-(-4)}{-\frac{4}{5}} = \frac{y-0}{\frac{3}{5}} = r.$

10. If (a, b) be below axis of x, positive for direction toward axis of x, and beyond axis of x; if (a, b) be above axis of x, positive for direction from axis of x; i.e., positive for direction in which y increases.

11. $\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}.$ 14. $41x+62y=164$; $19x+18y=76$. A harmonic pencil.

Pages 60-62.

4. $5x+2y-4=0$. 5. $x+9y+56=0$; $6x+11y+26=0$; $5x+2y-30=0$.

6. $x+y-4=0$; $2x-y-1=0$; $x-2y+3=0$. 7. $\frac{21}{37}, -\frac{15}{37}.$ 8. $45°$.

9. $ax-by=a^2-b^2$. 10. $Bx-Ay-aB=0$.

11. $(8 \pm 5\sqrt{3})(x-2)=11(y-3)$. 12. $x=0$; $y=0$.

13. Writing $90°+\alpha$ and $-(90-\alpha)$ for α, we get $-x\sin\alpha+y\cos\alpha=p'$, and $x\sin\alpha-y\cos\alpha=p'$.

14. $A(x-3)+B(y+2)=0$; $B(x-4)-A(y-3)=0$. Because the data are not sufficient to fix the lines geometrically.

15. $45°$. 16. $\frac{29}{2\sqrt{34}}.$ 17. $8x-5y+2=0$; $7x-4y-2=0$; $x-y-2=0$.

18. $m=\tan\alpha$; i.e., $\theta=\alpha$. 19. $2x-3y+7=0$; $\frac{12}{\sqrt{13}}$; $\frac{12}{\sqrt{13}}.$

20. Perpendiculars through origin on given sides are $4x-3y=0$, $x+2y=0$. These with given sides give angular points $(-6, -8)$, $(12, -6)$; and third side is $x-9y-66=0$.

Pages 65-66.

1. $\frac{10\sqrt{2}}{7}.$ 2. 14. 3. $l=\frac{-a}{\sqrt{a^2+b^2}}$, $m=\frac{b}{\sqrt{a^2+b^4}}.$ 4. $5\sqrt{2}$.

5. $\frac{65}{\sqrt{85}}$; $\frac{13}{\sqrt{2}}$; $\frac{18}{\sqrt{5}}.$ 6. It is the intersection of the lines $9x-2y-46=0$, $x+7y+11=0$, $2x+y-7=0$; and is $\left(\frac{60}{13}, -\frac{29}{13}\right)$.

7. $\sqrt{85}$. 8. $2x+y\pm 3\sqrt{5}=0$.

10. $\dfrac{3a-4b+5}{5}=\pm\dfrac{a+2b-7}{\sqrt{5}}$. This is of first degree in a and b, and therefore the locus of (a, b) must be one or other of the straight lines $(3\mp\sqrt{5})x+(-4\mp 2\sqrt{5})y+(5\pm 7\sqrt{5})=0$.

Page 69.

1. Opposite; negative; positive.

2. Bisector of angles in first and third quadrants is $(4\sqrt{2}+\sqrt{5})x-(3\sqrt{2}+3\sqrt{5})y=0$; the other is $(4\sqrt{2}-\sqrt{5})x-(3\sqrt{2}-3\sqrt{5})y=0$. Perpendicular.

3. The origin lies on the positive side of each line, and $(1-\sqrt{13})x-(5-\sqrt{13})y+(30-6\sqrt{13})=0$ is the bisector of angle in which origin lies. The other bisector is $(1+\sqrt{13})x-(5+\sqrt{13})y+(30+6\sqrt{13})=0$.

4. $(1+2\sqrt{2})x-(3+\sqrt{2})y+5=0$ through $(1,2)$; $x+2y-10=0$ through $(4, 3)$; $(3+2\sqrt{2})x+(1-\sqrt{2})y-15=0$ through $(3, 6)$. Their intersection is $(2\sqrt{2}, 5-\sqrt{2})$.

5. $(1-2\sqrt{2})x-(3-\sqrt{2})y+5=0$ through $(1, 2)$; $2x-y-5=0$ through $(4, 3)$. Point of intersection $(4-\sqrt{2}, 3-2\sqrt{2})$.

Pages 72-73.

1. If $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{1}{c}$, lines pass through (c, c).

2. Line always passes through $\left(\dfrac{1}{k\sin\omega}, -\dfrac{1}{k\sin\omega}\right)$.

3. $\dfrac{x+a}{3a}=\dfrac{y}{b}$; $\dfrac{x-a}{3a}=-\dfrac{y}{b}$; $\left(0, \dfrac{1}{3}b\right)$. 5. $kx-y=\frac{1}{2}(ak-b)$.

6. Locus of Q is $(A-B\cos\omega)x+(B-A\cos\omega)y+C\sin^2\omega=0$.

7. Fixed point through which AB passes is $\left\{\dfrac{f(k-mh)}{k-mf}, \dfrac{k(g-mf)}{k-mf}\right\}$.

Chapter IV. Change of Axes.

Pages 78-80.

1. $x+3$ for x; $y+4$ for y. 2. $x-4$ for x; $y-3$ for y.

3. For x, $x\cos 60^\circ-y\sin 60^\circ=\frac{1}{2}(x-y\sqrt{3})$; for y, $x\sin 60^\circ+y\cos 60=\frac{1}{2}(x\sqrt{3}+y)$.

4. For x, $x\cos(-30°)-y\sin(-30°)=\frac{1}{2}(x\sqrt{3}+y)$; for y, $x\sin(-30°)+y\cos(-30°)=\frac{1}{2}(-x+y\sqrt{3})$.

5. For x, $x\cos 180° - y\sin 180° = -x$; for y, $x\sin 180° + y\cos 180° = -y$

6. $3x-5y=0$. 7. $y=0$, $x=0$. 8. $y^2=4x$. 9. $x^2+y^2=13$.

10. Yes. $Ax+By=0$. 11. $(x-a)^2+(y-b)^2=r^2$. 12. $x^2+y^2=4$.

13. It remains $x^2+y^2=r^2$. 14. $3x^2+4y^2-1=0$.

15. Yes. Transfer to $(0, -2)$. $x^2+xy-5x+y=0$. Locus must pass through origin.

16. $x^2+3y^2=1$. 17. The angle whose tangent is $\frac{5}{4}$. $y\sqrt{41}-1=0$.

18. 45°. $x^2-y^2=2k^2$.

Chapter V. The Circle.

Pages 84-85.

1. (1). $x^2+y^2-8x+6y=0$. (2). $x^2+y^2-6x-4y-3=0$.
(3). $x^2+y^2+8x+7=0$. (4). $x^2+y^2+10x+10y+25=0$.
(5). $x^2+y^2+6x-4y=0$.

2. (1). $(3, 1)$ and 2. (2). $(-3, -1)$ and 2. (3). $(-4, 0)$ and 4.
(4). $(0, 0)$ and $\sqrt{2}$. (5). $\left(\frac{a}{2}, \frac{b}{2}\right)$ and $\frac{1}{2}\sqrt{a^2+b^2}$. (6). $(-f-g)$ and c.
(7). $\left(-\frac{b}{2a}, -\frac{c}{2a}\right)$ and $\frac{\sqrt{b^2+c^2}}{2a}$.

3. $x^2+y^2-6x-7y+15=0$. 4. $(2, 0)$, $(5, 0)$. 5. $C=0$.

6. $-2A$, $-2B$. 7. $x^2+y^2-ax-by=0$.

Pages 90-91.

1. $x+5=0$; $3x-4y-25=0$; $x-2\sqrt{6}y+25=0$; $2x\pm\sqrt{21}y+25=0$.

2. $x-y\sqrt{3}\pm 2r=0$; $x\sqrt{3}-y\pm 2r=0$; $x-y\pm r\sqrt{2}=0$.

3. $\left(-\frac{r}{2}, \frac{r\sqrt{3}}{2}\right)$, $\left(\frac{r}{2}, -\frac{r\sqrt{3}}{2}\right)$; $\left(\frac{r\sqrt{3}}{2}, -\frac{r}{2}\right)$, $\left(-\frac{r\sqrt{3}}{2}, \frac{r}{\sqrt{2}}\right)$;
$\left(-\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right)$, $\left(\frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}}\right)$.

4. $bx+ay\pm r\sqrt{a^2+b^2}=0$; $Ax+By\pm r\sqrt{A^2+B^2}=0$;
$Bx-Ay\pm r\sqrt{A^2+B^2}=0$.

5. $rx\pm\sqrt{a^2-r^2}y=ar$. 6. $k=\pm 2\sqrt{2}$. 7. $C^2=r^2(A^2+B^2)$.

8. $x - y\sqrt{3} + 2r = 0$; $x\sqrt{3} - y - 2r = 0$. **9.** $x^2 + y^2 = 9$. **10.** $r = 3 \pm \sqrt{5}$.

Pages 94–95.

1. $\left(-\frac{A}{K}, -\frac{B}{K}\right)$; $\sqrt{\frac{A^2}{K^2} + \frac{B^2}{K^2} - \frac{C}{K}}$. **2.** $x^2 + y^2 + 2\frac{A}{K}x + 2\frac{B}{K}y + \frac{C}{K}$.

3. $x^2 + y^2 + 2\frac{A + \lambda A'}{1 + \lambda}x + 2\frac{B + \lambda B'}{1 + \lambda}y + \frac{C + \lambda C'}{1 + \lambda}$.

4. $2(A - A')x + 2(B - B')y + C - C' = 0$.

5. These circles are circles of the series, namely when $\lambda = 0$ and $\lambda = \infty$.

7. Radical axis is $x = 0$. **8.** Ratio $= \sqrt{\frac{g - k}{h - k}}$.

9. Radical centre is $\left\{-\frac{R - C}{2(P - A)}, 0\right\}$.

10. Locus of radical centre is $\frac{2ax - a^2}{r_1 - r_2} - \frac{2bx + 2cy - b^2 - c^2}{r_1 - r_3} = r_2 - r_3$.

Pages 100–101.

1. $3x + 6y - 25 = 0$; $2x + 5y - 25 = 0$; $6x + 8y + 25 = 0$. **2.** $(3, 4)$; $(4, -3)$.

3. $\left(-\frac{51}{5}, \frac{34}{5}\right)$. **4.** $\left(-\frac{A}{C}r^2, -\frac{B}{C}r^2\right)$; $\left(\frac{r^2}{a}, \frac{r^2}{b}\right)$. **5.** $by = r^2$.

6. $x + my = 0$. **7.** $(x + \frac{A}{C}r^2) + \lambda(y + \frac{B}{C}r^2) = 0$, where λ is variable.

8. When pole is without circle; when pole is on circle.

12. We must have $\frac{a}{A} = \frac{b}{B}$, and then $r^2 = -C\frac{a}{A} = -C\frac{b}{B}$.

Pages 104–105.

1. The circle $x^2 + y^2 = \frac{1}{2}(c^2 - 2a^2)$.

2. Locus is circle $x^2 + y^2 + 2\frac{a^2}{b}y - a^2 = 0$.

3. The circle is $x^2 + y^2 = c^2 + a^2$. **4.** The circle $x^2 + y^2 - rx = 0$.

5. The circle $x^2 + y^2 + rx - 2r^2 = 0$.

6. Locus is circle $x^2 + \left(y - \frac{a}{\sqrt{3}}\right)^2 = \frac{2}{3}(c^2 - a^2)$.

7. The circle $x^2 + y^2 - \frac{c^2}{a}x = 0$. **8.** The circle $\left\{x + (n - 1)a\right\}^2 + y^3 = n^2r^2$.

9. The circle $x^2 + y^2 - r^2 = n^2\left\{(x - a)^2 + (y - b)^2\right\}$.

10. The circle $x^2 + y^2 - 2a \cot\theta . y = a^2$.

Chapter VI. The Parabola.

Pages 110-112.

1. $y^2=4a(x-a)$. 2. $x^2=4ay$. 4. $y^2=-4ax$. 5. $(3a, \pm 2a\sqrt{3})$.

6. $\left(\frac{3}{4}, 3\right)$, $\left(\frac{4}{3}, 4\right)$.

7. $\{a(3+2\sqrt{2}), a(2+2\sqrt{2})\}$; $\{a(3-2\sqrt{2}), a(2-2\sqrt{2})\}$.

9. $\left(4a^{\frac{1}{3}}b^{\frac{2}{3}}, 4a^{\frac{2}{3}}b^{\frac{1}{3}}\right)$. 10. $8a\sqrt{3}$. 13. The point $(4a, 0)$.

14. $4a(\sqrt{3}\pm 2)$. Note that one value is $+ve$, and other $-ve$.

16. $x^2+y^2-5ax=0$.

17. Product of ordinates $=-4ab$; product of abscissas $=b^2$.

18. $\frac{a^2}{x_1}$; $-\frac{4a^2}{y_1}$.

19. A parabola whose focus is centre of circle, and directrix parallel to given line, at distance from it equal to radius of circle.

20. Describe two circles with centres P, P', both passing through F, and draw a double tangent.

21. PP' is equal to sum of distances of P, P' from directrix, and therefore middle point of PP' is $\frac{1}{2}PP'$ from directrix.

Pages 119-121.

2. $x'=\frac{a}{m^2}$, $y'=\frac{2a}{m}$. 3. To find x's of points of intersection we get $m^2x^2-2ax+\frac{a^2}{m^2}=0$, a complete square. 4. $x\mp y+a=0$.

5. $x\pm y-3a=0$. 6. $3x-y\sqrt{3}+1=0$. 7. $\left(\frac{1}{3}, \frac{2}{\sqrt{3}}\right)$.

9. $x+y+a=0$; $x-2y+4a=0$. 10. $y=\pm(x+a\sqrt{2})$.

11. $x=\frac{a}{2}(1+\sqrt{5})$, $y=\pm a\sqrt{2+2\sqrt{5}}$. 14. Circle on FY as diameter.

16. $\left(\frac{a}{3}, \frac{2a}{\sqrt{3}}\right)$; $(3a, -2a\sqrt{3})$.

18. Lines in question are $x\sqrt{3}+y-a\sqrt{3}=0$, $x-y\sqrt{3}-a=0$.

21. $a^{\frac{1}{3}}x+b^{\frac{1}{3}}y+a^{\frac{2}{3}}b^{\frac{2}{3}}=0$.

Pages 126-127.

3. $l=\frac{y'}{\sqrt{y'^2+4a^2}}$, $m=\frac{2a}{\sqrt{y'^2+4a^2}}$. Equation of chord is $\frac{x-x'}{y'}=\frac{y-y'}{2a}$.

5. $\left(\frac{C}{A}, -\frac{2aB}{A}\right)$; $\left(-h, -\frac{2ah}{k}\right)$.

6. Intersection is $\left\{\frac{a}{mm'}, \frac{a}{mm'}(m+m')\right\}$; chord of contact is $(m+m')y=2(mm'x+a)$.

7. Polar of foot of directrix $(-a, 0)$ is $x-a=0$.

10. Chord of contact is $(1+m^2)y=2m(x+a)$, which always passes through $(-a, 0)$.

Pages 129-131.

2. $\left(a\frac{l^2}{m^2}, 2a\frac{l}{m}\right)$. 6. $y^2=ax$. 7. $y^2=2a\left(x-\frac{a}{2}\right)$. 8. $y=a\left(\frac{1}{m}+\frac{1}{m'}\right)$.

10. The straight line $x-2a=4a\frac{l^2}{m^2}+\frac{m}{2l}y$.

Page 132.

1. $(-1, 2)$; $x+1=0$; $(-1, \frac{9}{4})$; $y-\frac{7}{4}=0$.
2. $(\frac{1}{3}, \frac{11}{3})$; $x-\frac{1}{3}=0$; $(\frac{1}{3}, \frac{15}{4})$; $y-\frac{43}{12}=0$.
3. $(\frac{3}{4}, -\frac{31}{8})$; $x-\frac{3}{4}=0$; $(\frac{3}{4}, -4)$; $y+\frac{15}{4}=0$.
4. $(-6, 3)$; $y-3=0$; $(-\frac{23}{4}, 3)$; $x+\frac{25}{4}=0$.
5. $2y=-21+8x-x^2$; $(4, -\frac{5}{2})$; $x-4=0$.

Chapter VII. The Ellipse.

Pages 139-142.

1. $a=2$; $b=\sqrt{3}$; $e=\frac{1}{2}$; $ae=1$; $\frac{a}{e}=4$; $2\frac{b^2}{a}=3$.

2. $a=2$; $b=\frac{1}{\sqrt{3}}$; $e=\sqrt{\frac{11}{12}}$; $ae=\sqrt{\frac{11}{3}}$; $\frac{a}{e}=4\sqrt{\frac{3}{11}}$; $2\frac{b^2}{a}=\frac{1}{3}$.

4. $a=\frac{l}{1-e^2}$; $b=\frac{l}{\sqrt{1-e^2}}$. 5. $e=\frac{1}{\sqrt{2}}$; $a=b\sqrt{2}$.

6. $a^2=\frac{1-e^2\cos^2\alpha}{1-e^2}r^2$; $b^2=(1-e^2\cos^2\alpha)r^2$. 7. $\frac{x^2}{81}+\frac{y^2}{72}=1$.

8. $\frac{x^2}{16}+\frac{y^2}{9}=1$. 13. $\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$. 14. $34x^2-24xy+41y^2=1250$.

15. Ellipse is $\frac{x^2}{CB^2}+\frac{y^2}{CA^2}=1$. 16. The ellipse $\frac{x^2}{c^2}+\frac{y^2}{k^2c^2}=1$.

22. Ellipse is $\frac{x^2}{a^2e^2}+\frac{(1+e)^2y^2}{b^2e^2}=1$. **24.** $e=0$; $ae=0$; $\frac{a}{e}=\infty$.

Pages 150-152.

1. $x\pm 2y=4$. **2.** $\left(-\frac{3}{2}, 2\right)$; $\left(\frac{3}{2}, -2\right)$. **3.** $\frac{a^2}{m^2}+\frac{b^2}{n^2}=1$.

4. $x+y\sqrt{3}-3=0$; $5x-y\sqrt{3}+9=0$. **6.** The axis minor produced.

7. $\left(-\frac{ma^2}{\sqrt{m^2a^2+b^2}}, \frac{b^2}{\sqrt{m^2a^2+b^2}}\right)$. **9.** $x+y=\sqrt{a^2+b^2}$. **15.** $\frac{x}{a}+\frac{y}{b}=\pm\sqrt{2}$.

16. $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$. **18.** $\left(\frac{\pm a^2}{\sqrt{a^2+b^2}}, \frac{\pm b^2}{\sqrt{a^2+b^2}}\right)$. **19.** $\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$.

21. $x\pm y=\pm\sqrt{a^2-b^2}$.

Pages 158-160.

1. Polar of foot of directrix $\left(\frac{a}{e}, 0\right)$ is $x-ae=0$. **2.** $\left(-\frac{Aa^2}{C}, -\frac{Bb^2}{C}\right)$.

3. $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{p^2}$. **4.** $\left\{\frac{a^4}{x'(a^2-b^2)}, -\frac{b^4}{y'(a^2-b^2)}\right\}$.

5. The condition in each case is $\frac{a^6}{x'x''}+\frac{b^6}{y'y''}=(a^2-b^2)^2$. **6.** $y'^2=2\frac{b^2}{a}x'$.

7. The result is $x'^2+y'^2=a^2$, which is the condition that (x', y') lies on the auxiliary circle. **8.** Both pass through focus.

11. $l=\frac{\frac{y'}{b^2}}{\pm\sqrt{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}}}$, $m=\frac{-\frac{x'}{a^2}}{\pm\sqrt{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}}}$. $\frac{xx'}{a^2}+\frac{yy'}{b^2}=\frac{x'^2}{a^2}+\frac{y'^2}{b^2}$.

13. The pole is a point at infinity in a direction making an angle $\tan^{-1}\frac{Bb^2}{Aa^2}$ with Ox. **14.** $\frac{x}{a^2}+\frac{my}{b^2}=0$.

15. $\left(\frac{x'+x''}{\frac{x'x''}{a^2}+\frac{y'y''}{b^2}+1}, \frac{y'+y''}{\frac{x'x''}{a^2}+\frac{y'y''}{b^2}+1}\right)$; and line joining centre to middle point of chord is $\frac{x}{\frac{1}{2}(x'+x'')}=\frac{y}{\frac{1}{2}(y'+y'')}$.

Pages 169-172.

1. $\frac{x}{a^2}-\frac{y}{b^2}=0$; $\frac{x}{a}-\frac{y}{b}=0$; $\frac{x}{a^3}+\frac{y}{b^3}=0$. **2.** $\frac{x}{a}\sqrt{k^2-b^2}-\frac{y}{b}\sqrt{a^2-k^2}=0$.

6. The equation of the locus referred to Ox, Oy as axes is $x^2+y^2=\frac{c^2}{\sin^2\omega}$, Ox, Oy being (Ex. 5) the directions of equal conjugate diameters.

8. Chord of contact is $\frac{x}{l}-\frac{y}{m}=\frac{a^2}{lx'}$. **10.** The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$.

22. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{x}{a}$.

Chapter VIII. The Hyperbola.

Pages 179-180.

1. $a=2$; $b=\sqrt{3}$; $e=\frac{\sqrt{7}}{2}$; $ae=\sqrt{7}$; $\frac{a}{e}=\frac{4}{\sqrt{7}}$; $2\frac{b^2}{a}=3$.

2. $\frac{x^2}{25}-\frac{y^2}{9}=1$. **3.** $2x^2-3y^2=6$. **4.** $(\sqrt{5}-1)\sqrt{3}$, $(\sqrt{5}+1)\sqrt{3}$.

5. The $2a$ of the hyperbola is the distance sound travels while the ball is in flight.

6. $x^2-y^2=\frac{c^2}{2}$. **7.** $11x^2+96xy+39y^2-138x-284y+251=0$.

8. $2x^2-y^2=5$. **10.** The centres of the given circles are the foci of the hyperbola.

11. In it $a^2=\frac{1}{4}(r-r')^2$, $b^2=k^2-\frac{1}{4}(r-r')^2$. **12.** $\frac{1}{2}(\sqrt{5}+1)$.

Pages 182-184.

1. $\pm x-y=1$. **2.** $m=\pm\frac{b}{a}$. **3.** The values of m are given by the quadratic $m^2a^2-b^2=(k-mh)^2$.

4. Impossible when $m^2<\frac{b^2}{a^2}$. **5.** Impossible when $\frac{h^2}{a^2}-\frac{k^2}{b^2}-1$ is positive, i.e., when (h, k) is on concave side of hyperbola.

12. $\frac{a^2}{l^2}-\frac{b^2}{m^2}=\frac{(a^2+b^2)^2}{n^2}$.

Pages 184-186.

1. $\left(-\frac{l}{n}a^2, \frac{m}{n}b^2\right)$. **2.** Polar of $\left(\frac{a}{e}, 0\right)$ is $x=ae$.

3. $\left\{\frac{a^4}{x'(a^2+b^2)}, \frac{-b^4}{y'(a^2+b^2)}\right\}$. 4. $\frac{a^6}{x^2}-\frac{b^6}{y^2}=(a^2+b^2)^2$.

5. $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}$. 6. $\left(\frac{a}{m}\sqrt{m^2-1}, \frac{2a}{m}\right)$. 7. $4x^2+y^2=4a^2$.

9. $l=\frac{a^2y'}{\pm\sqrt{b^4x'^2+a^4y'^2}}$, $m=\frac{b^2x'}{\pm\sqrt{b^4x'^2+a^4y'^2}}$; $\frac{xx'}{a^2}-\frac{yy'}{b^2}=\frac{x'^2}{a^2}-\frac{y'^2}{b^2}$.

11. $\left\{\frac{x'}{\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}, \frac{y'}{\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}\right\}$; and this lies on line $\frac{x}{x'}=\frac{y}{y'}$.

Pages 188-189.

1. $\frac{x}{a^2}+\frac{y}{b^2}=0$; $\frac{x}{a}-\frac{y}{b}=0$; $\frac{x}{a^3}+\frac{y}{b^3}=0$; $2bx-ay=0$.

2. $\frac{x}{a^2}+\frac{y}{b^2}=0$; $\frac{x}{a}-\frac{y}{b}=0$ is conjugate to itself, and meets curve at infinity; $\frac{x}{a^3}+\frac{y}{b^3}=0$; $bx-2ay=0$. 4. $\frac{x}{a}\sqrt{k^2+b^2}-\frac{y}{b}\sqrt{k^2-a^2}=0$.

6. Chord of contact is $\frac{x}{l}-\frac{y}{m}=\frac{a^2}{lx'}$, or $=\frac{b^2}{my'}$, where (x', y') is a point on $\frac{lx}{a^2}-\frac{my}{b^2}=0$.

Pages 198-200.

10. Yes. Its equation is $\frac{x^2}{CP^2}-\frac{y^2}{Cd^2}=1$.

Chapter IX. General Equation of Second Degree.

Pages 209-211.

1. When axes are turned through 45°, $3x^2+4y^2=12$, an ellipse.

2. 2 and $\sqrt{3}$; $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$; $x+y\pm4\sqrt{2}=0$.

3. A parabola. Focus $(\frac{1}{3}, -\frac{7}{4})$; axis, $x=\frac{1}{3}$; directrix, $y+\frac{11}{12}=0$. It falls below point $(\frac{1}{3}, -\frac{4}{3})$.

4. Reduced equation is $y^2=\sqrt{2}x$, a parabola. Hence with original axes, tangent at vertex is $x+y=0$, and axis is $x-y=0$.

5. The straight lines $2x-3y+4=0$, $x-y+1=0$.

6. Transferred to centre equation becomes $xy=\frac{1}{4}$, hyperbola referred to asymptotes (present axes) as axes. Curve lies in first and third quadrants of axes through $(\frac{1}{2}, \frac{1}{2})$.

7. Parabola. The successive changes of axes, with final equation, show positions of vertex and axis. **8.** As in Ex. 7.

9. Equations giving centre (§114) are both $x-3y+2=0$, i.e., there is a line of centres. The equation represents two parallel lines $x-3y=0$, $x-3y+4=0$; and a point on $x-3y+2=0$ bisects any line intercepted by the parallels.

10. (3, 4). Transformed equation is $3x^2-2xy+y^2=0$, or $x-\frac{1+\sqrt{-2}}{3}y=0$, $x-\frac{1-\sqrt{-2}}{3}y=0$. **11.** Centre (1, 2). Origin being transferred, and axes turned through 45°, equation becomes $7x^2-3y^2=10$, an hyperbola.

13. Centre $(-5, -5)$. Origin being transferred, equation becomes $y^2-xy=1$, an hyperbola, asymptotes (Ex. 12) being $y=0$, $x-y=0$. We see in what angle between asymptotes curve lies by putting $x=0$, which gives $y=\pm 1$.

14. Centre (2, 3). Transferring, equation becomes $y^2-xy-6x^2+21=0$, an hyperbola with asymptotes $3x-y=0$, $2x+y=0$. Curve lies in angle in which axis of x is.

15. Transferred to centre $(-2, 1)$, the ellipse $3x^2+4y^2=8$.

16. Transferring to $\left(\frac{a}{2b}, \frac{a^2}{4b}\right)$, the equation is $x^2=-\frac{1}{b}y$, a parabola. Hence vertex is $\left(\frac{a}{2b}, \frac{a^2}{4b}\right)$; focus $\left(\frac{a}{2b}, \frac{a^2-1}{4b}\right)$; directrix $y=\frac{a^2+1}{4b}$.

17. The parallel lines $3x-4y=0$, $3x-4y+2a=0$.

19. An ellipse; centre $(3, -2)$; axis minor at angle 45° to Ox; axis major $=2\sqrt{2}$, minor $=2$.

20. An equilateral hyperbola; centre $(1, -2)$; transverse axis inclined at angle $\tan^{-1}\frac{1}{2}$ to Ox; semi-axis $=5$.