Games, Fixed Points and Mathematical Economics

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Abstract

These are my Lecture Notes for a course in Game Theory which I taught in the Winter term 2003/04 at the University of Kaiserslautern. I am aware that the notes are not yet free of error and the manuscrip needs further improvement. I am happy about any comment on the notes. Please send your comments via e-mail to ce16@st-andrews.ac.uk.

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Chapter 1

Games

1.1 Introduction

Game Theory is a formal approach to study games. We can think of games as conflicts where some number of individuals (called players) take part and each one tries to maximize his utility in taking part in the conflict. Sometimes we allow the players to cooperate, in this case we speak of cooperative games as opposed to non-cooperative games, where players are not allowed to cooperate. Game theory has many applications in subjects like Economy, biology, psychology, but also in such an unpleasant subject as warfare. In this lecture we will concentrate on applications in Economics. A lot of our examples though are also motivated by classical games. However, reality is often far too complex, so we study simplified models (such as for example simplified Poker which is played with only two cards, an Ace and a two). The theory by itself can be quite abstract and a lot of methods from Functional Analysis and Topology come in. However all methods from these subjects will be explained during the course. To start with, we describe two games, which will later help us to understand the abstract definitions coming in the next section.

Example 1. Simplified Poker *: There are only two cards involved, an "Ace" and a "Two" and only two players, player 1 and player 2. At the* *beginning each one puts 1 Euro in the "pot". Both cards lie face down on the table and none of the players knows, which card is the "Ace" and which one the "Two". Then player two draws one of the cards and takes a look. If it's the "Ace" player 2 has to say "Ace". If however he has drawn the "Two", he can say two and then loses the game (in this case player 1 wins the 2 Euro in the "pot") or he can "bluff" and say "Ace". In the case , player 2 says "Ace" he has to put another Euro in the "pot". Player 1 then has two choices. Either he believes player 2 and "folds" in, in this case player 2 wins the (now) 3 Euro in the "pot", or player 1 assumes that player two has "bluffed" puts another Euro in the "pot". In this case Player 2 has to show player 1 his card. If it is indeed the "Ace" then player 2 wins the (now) 4 Euro in the pot but if it is the "Two" then player 1 wins the 4 Euro. In both cases, the game is finished.*

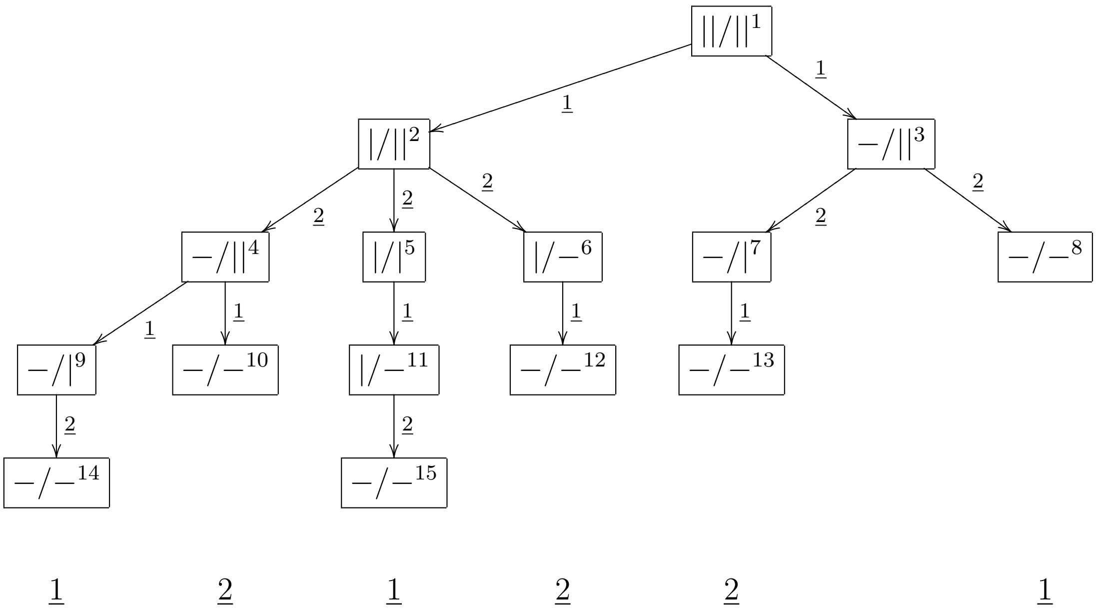
Example 2. Nim(2,2) *: There are two piles of two matches each and two players are taking part in the game, player 1 and player 2. The players take alternating turns, player 1 takes the first turn. At each turn the player selects one pile which has at least one match left and removes at least one match from this pile. The game finishes when all matches are gone. The player who takes the last match looses.*

Each of the games above is of the following structure :

1. there is a number of players taking part in the game
2. there are rules under which the player can choose their strategies (moves)
3. the outcome of the game is determined by the strategies chosen by each player and the rules

This structure will be included in the ("abstract") definition of a "Game" in Chapter 1. The two games just presented though differ under some aspect. In the second game, the players have at each time perfect knowledge about the action of their opponent. This in not the case in the first game, where player 2 can "bluff". We speak respectively of games with **perfect information** and games with **non perfect information**. Also in the first game there is a chance element, which is missing in the second game. Such informations will be considered as additional structure and considered individually. Usually a game can be considered as a tree, where the nodes are the states of the game and the edges represent the moves. For the Nim(2,2) game one has the following tree :

Example 3. Nim(2,2) Tree *:*



The number at each edge indicates which player does the move, the number on the bottom of the diagram indicates the winner.

Trees can be helpful in understanding the structure of the game. For mathematical purposes though, they are often to bulky to work with. We would like to decode the structure of the game in a more condensed form. First let us consider the individual strategies the players in Nim(2,2) can choose :

$\mathcal{S}_1$	1st turn	if at	go to	$\mathcal{S}_2$	if at	go to
s_1^1	$1 \mapsto 2$	4	9	s_2^1	2 3	4 7
s_1^2	$1 \mapsto 2$	4	10	s_2^2	2 3	5 7
s_1^3	$1 \mapsto 3$	–	–	s_2^3	2 3	6 7
				s_2^4	2 3	4 8
				s_2^5	2 3	5 8
				s_2^6	2 3	6 8

Here the strategies of player 1 are denoted with s_1^i for $i \in \{1, 2, 3\}$ and those for player 2 as s_2^j for $i \in \{1, 2, 3, 4, 5, 6\}$. If the players decide for one of their strategies, the outcome of the game is already fixed. Let us denote the outcome of the game as 1 if player 1 loses and with -1 if player 1 wins. Then the game is equivalent to the following game, which is described in matrix form :

$$L := \begin{pmatrix} -1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \end{pmatrix}$$

The value $L(i, j)$ at position (i, j) of this matrix is the outcome of the game Nim(2,2) if player 1 chooses his i-th strategy and player 2 chooses j-th strategy. We see that player 2 has a strategy which guarantees him to win, namely s_2^3.

Let us also discuss the simplified Poker in this way. The following table shows the possible strategies of the players :

s_1^1	believe $\underline{2}$ when he says "Ace"
s_1^2	don't believe $\underline{2}$ when he says "Ace"
s_2^1	say "Two" when you have "Two"
s_2^2	say "Ace" when you have "Two" ("bluff")

Since their is a chance element in the game ("Ace" or "Two" each with probability $1/2$) the losses corresponding to pairs of strategies are not deterministic. According to what card Player 2 draws, we have different losses :

$$L_{Two} := \begin{pmatrix} -1 & 1 \\ -1 & -2 \end{pmatrix} \text{ and } L_{Ace} := \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

One could now consider "Nature" as a third player and denote the losses in a three dimensional array but one usually decides not to do this and instead denote the expected losses in a matrix. Since each event "Two" and "Ace" occurs with probability $1/2$ we have for the expected losses :

$$L := \frac{1}{2}L_{Two} + \frac{1}{2}L_{Ace} = \begin{pmatrix} 0 & 1 \\ 1/2 & 0 \end{pmatrix}$$

The two examples we had so far, had the characteristic property that what one player loses, the other player wins. This is not always the case as the following example shows :

Example 4. Battle of the Sexes *A married couple are trying to decide where to go for a night out. She would like to go to the theater, he would like to go to a football match. However they are just married since a couple of weeks, so they still like to spend their time together and enjoy the entertainment only if their partner is with them. Let's say the first strategy for each is to go to the theater and the second is to go to the football match. Then the individual losses for each can be denoted in matrix form as :*

$$L := \begin{pmatrix} (-1,-4) & (0,0) \\ (0,0) & (-4,-1) \end{pmatrix}$$

Here in each entry, the first number indicates the loss for the man, the second one the loss for the woman.

The reader may find it unusual, that we always speak about losses, rather then gains (or wins). The reason is, that in convex analysis one rather likes to determine minima instead of maxima (only for formal reasons), and so since in fact we want to maximize the gains, we have to minimize the losses. The examples mentioned in this chapter will lead us directly to the formal definition of a (Two Person) Game in the next section and henceforth serve for illustrating the theory, which from now on at some points tends to be very abstract.

1.2 General Concepts of two Person Games

Definition 1.2.1. *A* **two person game** $\mathcal{G}_2$ *in normal form consists of the following data :*

1. *topological spaces* $\mathcal{S}_1$ *and* $\mathcal{S}_2$*, the so called strategies for player 1 resp. player 2,*
2. *a topological subspace* $U \subset \mathcal{S}_1 \times \mathcal{S}_1$ *of allowed strategy pairs*
3. *a biloss operator*

$$\begin{aligned} L : U &\rightarrow \mathbb{R}^2 & (1.1) \\ (s_1, s_2) &\mapsto (L_1(s_1, s_2), L_2(s_1, s_2)) & (1.2) \end{aligned}$$

$L_i(s_1, s_2)$ *is the loss of player* i *if the strategies* s_1 *and* s_2 *are played.*

For the games considered in the Introduction, the spaces $\mathcal{S}_i$ have been finite (with discrete topology) and U has to be chosen $\mathcal{S}_1 \times \mathcal{S}_2$. For the Nim(2,2) game and simplified poker we have $L_2 = -L_1$.The main problem in Game Theory is to develop solution concepts and later on find the solutions for the game. With solution concepts we mean to characterize those strategies which are optimal in some sense. We will see a lot of different approaches and also extend the definition to n-person games, but for now we stick with two person games.

Definition 1.2.2. *Given a two person game $\mathcal{G}_2$ we define its* **shadow minimum** *as*

$$\alpha = (\alpha_1, \alpha_2) \tag{1.3}$$

where $\alpha_1 = \inf_{(s_1,s_2)\in U} L_1(s_1,s_2)$ and $\alpha_2 = \inf_{(s_1,s_2)\in U} L_2(s_1,s_2)$. $\mathcal{G}_2$ is bounded from below, if both α_1 and α_2 are finite.

The shadow minimum represents the minimal losses for both players if they don't think about strategies at all. For the Nim(2,2) game the shadow-minimum is $\alpha = (-1,-1)$, for the simplified poker its $\alpha = (0,-1)$ and for the "Battle of the Sexes" it is $\alpha = (-4,-4)$. In case there exists

$$(\tilde{s}_1, \tilde{s}_2) \in U \text{ s.t. } L(\tilde{s}_1, \tilde{s}_2) = \alpha$$

then a good choice for both players, would be to choose the strategies $\tilde{s}_1$ and $\tilde{s}_2$ since they guarantee the minimal loss. However in most games, such strategies do not exist. For example in the Nim(2,2) game, there is no pair of strategies which gives biloss $(-1,-1)$.

Given a game $\mathcal{G}_2$. To keep things easy for now we assume $U = \mathcal{S}_1 \times \mathcal{S}_2$.[1] We define functions $L_1^\sharp : \mathcal{S}_1 \to \mathbb{R}$ and $L_2^\sharp : \mathcal{S}_2 \to \mathbb{R}$ as follows :

$$\begin{aligned} L_1^\sharp(s_1) &= \sup_{s_2\in\mathcal{S}_2} L_1(s_1,s_2) \\ L_2^\sharp(s_2) &= \sup_{s_1\in\mathcal{S}_1} L_2(s_1,s_2). \end{aligned}$$

[1] all coming definitions can be generalized to allow U to be a proper subset of $\mathcal{S}_1 \times \mathcal{S}_2$

$L_1^\sharp(s_1)$ is the worst loss that can happen for player 1 when he plays strategy s_1 and analogously $L_2^\sharp(s_2)$ is the worst loss that can happen for player 2 when he plays strategy s_2. If both players are highly risk aversive, the following consideration is reasonable : Player 1 should use a strategy which minimizes $L_1^\sharp$, i.e. minimizes the maximal loss and analogously player 2 should use a strategy which minimizes $L_2^\sharp$.

Exercise 1.2.1. *Compute the function $L_i^\sharp$ for the games in the Introduction.*

Definition 1.2.3. *A strategy $s_1^\sharp$ which satisfies*

$$L_1^\sharp(s_1^\sharp) = v_1^\sharp := \inf_{s_1 \in \mathcal{S}_1} L_1^\sharp(s_1) = \inf_{s_1 \in \mathcal{S}_1} \sup_{s_2 \in \mathcal{S}_2} L_1(s_1, s_2)$$

is called a **conservative strategy** *for player 1 and analogously $s_2^\sharp$ for player 2, if*

$$L_2^\sharp(s_2^\sharp) = v_2^\sharp := \inf_{s_2 \in \mathcal{S}_2} L_2^\sharp(s_2) = \inf_{s_2 \in \mathcal{S}_2} \sup_{s_1 \in \mathcal{S}_1} L_2(s_1, s_2).$$

The pair $v = (v_1^\sharp, v_2^\sharp)$ is called the **conservative value** *of the game.*

An easy computation shows that for the "Battle of the Sexes" game we have $L_1^\sharp \equiv 0 \equiv L_2^\sharp$. Hence $v_1^\sharp = 0 = v_2^\sharp$ and any strategy is a conservative strategy. However, assume the man decides he wants to see the football match, hence chooses strategy $s_1^\sharp = s_1^2$ and the woman decides she wants to go to the theater, which means she chooses strategy $s_2^\sharp = s_2^1$. Then both have chosen conservative strategies but both can do better :

$$\begin{aligned} L_1(s_1^\sharp, s_2^\sharp) = 0 \geq -1 = L_1(s_1^1, s_2^\sharp) \\ L_2(s_1^\sharp, s_2^\sharp) = 0 \geq -1 = L_2(s_1^\sharp, s_2^2). \end{aligned}$$

We say the chosen pair of strategies is not individually stable.

Definition 1.2.4. *A pair* $(s_1^\sharp, s_2^\sharp)$ *is called a* **non-cooperative equilibrium** [2] *or short* **NCE** *if*

$$L_1(s_1^\sharp, s_2^\sharp) \leq L_1(s_1, s_2^\sharp) \;\; \forall s_1 \in \mathcal{S}_1$$
$$L_2(s_1^\sharp, s_2^\sharp) \leq L_2(s_1^\sharp, s_2) \;\; \forall s_2 \in \mathcal{S}_2$$

Clearly this means that

$$L_1(s_1^\sharp, s_2^\sharp) = \min_{s_1 \in \mathcal{S}_1} L_1(s_1, s_2^\sharp)$$
$$L_2(s_1^\sharp, s_2^\sharp) = \min_{s_2 \in \mathcal{S}_2} L_2(s_1^\sharp, s_2)$$

In words : A non-cooperative equilibrium is stable in the sense that if the players use such a pair, then no one has reason to deteriorate from his strategy. For the "battle of the sexes" we have non-cooperative equilibria (s_1^1, s_2^1) and (s_1^2, s_2^2). If the strategy sets are finite and L is written as a matrix with entries the corresponding bilosses, then a non-cooperative equilibrium is a pair $(s_1^\sharp, s_2^\sharp)$ such that $L_1(s_1^\sharp, s_2^\sharp)$ is the minimum in its column (only taking into account the L_1 values) and $L_2(s_1^\sharp, s_2^\sharp)$ is the minimum in its row (only taking into account the L_2 values). Using this criterion, one can easily check that in the simplified poker, no non cooperative equilibria exist. This leads us to a crucial point in modern game theory, the extension of the strategy sets by so called mixed strategies.

Definition 1.2.5. *Let* X *be an arbitrary set and* $\mathbb{R}^X$ *the vector-space of real valued functions on* X *supplied with the topology of point wise convergence. For any* $x \in X$ *we define the corresponding Dirac measure* δ_x *as*

[2] sometimes also called individually stable

$$\begin{aligned} \delta_x : \mathbb{R}^X &\rightarrow \mathbb{R} \\ f &\mapsto f(x). \end{aligned}$$

Any (finite) linear combination $m = \sum_{i=1}^n \lambda_i \delta_{x_i}$ *which maps* $f \in \mathbb{R}^X$ *to*

$$m(f) = \sum_{i=1}^n \lambda_i f(x_i)$$

is called as **discrete measure**. *We say that* m *is positive if* $\lambda_i \geq 0$ *for all* i. *We call* m *a discrete probability measure if it is positive and* $\sum_{i=1}^n \lambda_i = 1$. *We denote the set of discrete probability measures on* X *by* $\mathcal{M}(X)$. *This space is equipped with the weak topology.*[3]

One can easily check that the set $\mathcal{M}(X)$ is convex. Furthermore we have a canonical embedding

$$\begin{aligned} \delta : X &\rightarrow \mathcal{M}(X) \\ x &\rightarrow \delta_x. \end{aligned}$$

Let us assume now that we have a two person game $\mathcal{G}_2$ given by strategy sets $\mathcal{S}_1$,$\mathcal{S}_2$ and a biloss-operator $L = (L_1, L_2)$. We define a new game $\tilde{\mathcal{G}}_2$ as follows : As strategy sets we take

$$\tilde{\mathcal{S}}_i := \mathcal{M}(\mathcal{S}_i) \ , \ i = 1, 2$$

and as biloss operator $\tilde{L} = (\tilde{L}_1, \tilde{L}_2)$ with

[3]this means $m_n \rightarrow m \Leftrightarrow m_n(f) \rightarrow m(f) \forall f \in \mathbb{R}^X$

$$\begin{array}{rcl} \tilde{L}_i : \tilde{\mathcal{S}}_1 \times \tilde{\mathcal{S}}_2 & \to & \mathbb{R} \\ (\sum_{i=1}^n \lambda_i^1 \delta_{s_1^i}, \sum_{j=1}^m \lambda_j^2 \delta_{s_2^1}) & \mapsto & \sum_{i=1}^n \sum_{j=1}^m \lambda_i^1 \lambda_j^2 L_i(s_1^i, s_2^j). \end{array}$$

Definition 1.2.6. *The sets $\mathcal{M}(\mathcal{S}_i)$ are called the* **mixed strategies** *of $\mathcal{G}$ and the game $\tilde{\mathcal{G}}_2$ is called the extension of $\mathcal{G}_2$ by mixed strategies. The strategies which are contained in the image of the canonical embedding*

$$\begin{array}{rcl} \delta : \mathcal{S}_1 \times \mathcal{S}_2 & \to & \mathcal{M}(\mathcal{S}_1) \times \mathcal{M}(\mathcal{S}_2) \\ (s_1, s_2) & \mapsto & (\delta_{s_1}, \delta_{s_2}) \end{array}$$

are called **pure strategies**.

Exercise 1.2.2. *Show that the extension of the simplified poker game has non cooperative equilibria.*

We will later see that for any zero sum game with finitely many pure strategies, the extended game has non-cooperative equilibria.

How can we interpret mixed strategies. Often it happens, that games are not just played once, but repeated many times. If player one has say 2 pure strategies and the game is repeated let's say 100 times, then he can realize the mixed strategy $0.3\delta_{s_1^1} + 0.7\delta_{s_1^2}$ by playing the strategy s_1^1 for 30 times and the strategy s_1^2 for 70 times. Another interpretation is, that every time he wants to play the mixed strategy $\lambda_1\delta_{s_1^1} + \lambda_2\delta_{s_1^2}$ he does a random experiment which has two possible outcomes, one with probability λ_1 and the other one with probability λ_2. Then he decides for one of the pure strategies s_1^1 resp. s_1^2 corresponding to the outcome of the experiment. If there are only finitely many pure strategies the mixed strategies also have a very nice geometric interpretation : Say $\mathcal{S}_1$ has $n+1$ elements. Then $\tilde{\mathcal{S}}_1$ is homeomorphic to the closed standard n simplex $\overline{\Delta}^n := \{(\lambda_0, ..., \lambda_n) \in \mathbb{R}^{n+1} | \lambda_i \geq 0, \sum_{i=0}^n \lambda_i = 1\}$ by the map

$(\lambda_0, ..., \lambda_n) \mapsto \sum_{i=1}^{n} \lambda_{i-1}\delta_{s_i}$. This relationship brings the geometry into the game.

It can also happen, that one has to many non-cooperative equilibria, in particular if one works with the extended game. In this case one would like to have some decision rule, which of the equilibria one should choose. This leads to the concept of strict solutions for a game $\mathcal{G}_2$.

Definition 1.2.7. *A pair of strategies* $(\tilde{s}_1, \tilde{s}_2)$ **sub-dominates** *another pair* (s_1, s_2) *if* $L_1(\tilde{s}_1, \tilde{s}_2) \leq L_1(s_1, s_2)$ *and* $L_2(\tilde{s}_1, \tilde{s}_2) \leq L_2(s_1, s_2)$ *with strict inequality in at least one case. A pair of strategies* (s_1, s_2) *is called* **Pareto optimal** *if it is not sub-dominated.*[4]

Definition 1.2.8. *A two person game* $\mathcal{G}_2$ *has a* **strict solution** *if :*

1. *there is a NCE within the set of Pareto optimal pairs*
2. *all Pareto optimal NCE's are interchangeable in the sense that if* (s_1, s_2) *and* $(\tilde{s_1}, \tilde{s_2})$ *are Pareto optimal NCE's, then so are* $(s_1, \tilde{s_2})$ *and* $(\tilde{s_1}, s_2)$.

The interpretation of the first condition is that the two players wouldn't choose an equilibrium strategy, knowing that both can do better (and one can do strictly better) by choosing different strategies. One can easily see, that interchangeable equilibria have the same biloss, and so the second condition implies, that all solutions in the strict sense have the same biloss. We will later discuss other solution concepts. We end this section with a definition, which is important in particular in the context of cooperative games.

Definition 1.2.9. *The* **core** *of the game* $\mathcal{G}_2$ *is the subset of all Pareto optimal strategies* $(\tilde{s}_1, \tilde{s}_2)$ *such that* $L_1(\tilde{s}_1, \tilde{s}_2) \leq v_1^\sharp$ *and* $L_2(\tilde{s}_1, \tilde{s}_2) \leq v_2^\sharp$ *where* $v = (v_1^\sharp, v_2^\sharp)$ *denotes the conservative value of the game.*

[4] in this case one sometimes speak of the collective stability property

In the end of this introductory chapter we demonstrate, how the question of existence of equilibria is related to the question of the existence of fixed points. Assume that there exist maps

$$\begin{aligned} C : \mathcal{S}_2 &\rightarrow \mathcal{S}_1 \\ D : \mathcal{S}_1 &\rightarrow \mathcal{S}_2 \end{aligned}$$

such that the following equations hold :

$$\begin{aligned} L_1(C(s_2), s_2) &= \min_{s_1 \in \mathcal{S}_1} L_1(s_1, s_2) \; \forall \; s_2 \in \mathcal{S}_2 \\ L_2(s_1, D(s_1)) &= \min_{s_2 \in \mathcal{S}_2} L_2(s_1, s_2) \; \forall \; s_1 \in \mathcal{S}_1 \end{aligned}$$

Such maps C and D are called **optimal decision rules**. Then any solution $(\tilde{s}_1, \tilde{s}_2)$ of the system

$$\begin{aligned} C(\tilde{s}_2) &= \tilde{s}_1 \\ D(\tilde{s}_1) &= \tilde{s}_2 \end{aligned}$$

is a non-cooperative equilibrium. Denoting with F the function

$$\begin{aligned} F : \mathcal{S}_1 \times \mathcal{S}_2 &\rightarrow \mathcal{S}_1 \times \mathcal{S}_2 \\ (\tilde{s}_1, \tilde{s}_2) &\mapsto (C(\tilde{s}_2), D(\tilde{s}_1)) \end{aligned}$$

then any fixed point[5] $(\tilde{s}_1, \tilde{s}_2)$ of F is a non cooperative equilibrium. Hence we are in need of theorems about the existence of fixed points. The most famous one, the Banach fixed point theorem in general does not apply, since the functions we consider are often not contractive.

[5] in general, if one has a map $f : X \rightarrow X$, then any point $x \in X$ with $f(x) = x$ is called a fixed point

The second most famous is probably the Brouwer fixed point theorem which we will discuss in the next chapter. Later we will also consider more general fixed point theorem, which apply even in the framework of generalized functions, so called correspondences.

1.3 The Duopoly Economy

In this section we try to illustrate the concepts of the previous section by applying them to one of the easiest models in Economy, the so called duopoly economy. In this economy we have to producers which compete on the market, i.e. they produce and sell the same product. We consider the producers as players in a game where the strategy-sets are given by $\mathcal{S}_i = \mathbb{R}_+$ and a strategy $s \in \mathbb{R}$ stands for the production of s units of the product. It is reasonable to assume that the price of the product on the market is determined by demand. More precisely that we assume that there is an affine relationship of the form :

$$p(s_1, s_2) = \alpha - \beta(s_1 + s_2) \tag{1.4}$$

where α, β are positive constants. This relationship says more or less, that if the total production exceeds α, then no one wants to buy the product anymore. We assume that the individual cost functions for each producer are given by :

$$\begin{aligned} c_1(s_1) &= \gamma_1 s_1 + \delta_1 \\ c_2(s_2) &= \gamma_2 s_2 + \delta_2. \end{aligned}$$

Here we interpret δ_1, δ_2 as fixed costs. The net costs for each producer are now given by

$$
\begin{aligned}
L_1(s_1, s_2) &= c_1(s_1) - p(s_1, s_2) \cdot s_1 &&= \beta s_1(s_1 + s_2 - (\frac{\alpha - \gamma_1}{\beta}) - \delta_1) \\
L_2(s_1, s_2) &= c_2(s_1) - p(s_1, s_2) \cdot s_2 &&= \beta s_2(s_1 + s_2 - (\frac{\alpha - \gamma_2}{\beta}) - \delta_2).
\end{aligned}
$$

The biloss operator is then defined by $L = (L_1, L_2)$. Assume now player i chooses a strategy $s_i \geq \frac{\alpha - \gamma_i}{\beta} - \delta_i := u_i$. Then he has positive net costs. Since no producer will produce with positive net cost we assume

$$
U = \{(s_1, s_2) \in \mathbb{R}_+ \times \mathbb{R}_+ | s_1 \leq u_1, s_2 \leq u_2\}.
$$

For simplicity we will now assume, that $\beta = 1$, $\delta_i = 0$ for $i = 1, 2$ and $\gamma_1 = \gamma_2$.[6] Then also $u_1 = u_2 =: u$ and

$$
U = [0, u] \times [0, u].
$$

Let us first consider the conservative solutions of this game. We have

$$
\begin{aligned}
L_1^\sharp(s_1) &= \sup_{0 \leq s_2 \leq u} s_1(s_1 + s_2 - u) = s_1^2 \\
L_2^\sharp(s_2) &= \sup_{0 \leq s_1 \leq u} s_2(s_1 + s_2 - u) = s_2^2.
\end{aligned}
$$

Hence $\inf_{0 \leq s_1 \leq u} L_1^\sharp(s_1) = 0 = \inf_{0 \leq s_2 \leq u} L_2^\sharp(s_2)$ and the conservative solution for this game is $(\tilde{s}_1, \tilde{s}_2) = (0, 0)$ which corresponds to the case where no one produces anything. For the conservative value of the game we obtain $v^\sharp = (0, 0)$. Obviously this is not the best choice. Let us now consider the Set of Pareto optimal strategies. For the sum of the individual net costs we have

$$
L_1(s_1, s_2) + L_2(s_1, s_2) = (s_1 + s_2)(s_1 + s_2 - u) = z(z - u)
$$

[6] it is a very good exercise to wok out the general case

where we substituted $z := s_1 + s_2$. If s_1, s_2 range within U, z ranges between 0 and $2u$ and hence the sum above ranges between $-u^2/4$ and $2u^2$. The image of L is therefore contained in the set $\{(x,y) \in \mathbb{R}^2 | -u^2/4 \leq x + y \leq 0\} \cup \mathbb{R}_+ \times \mathbb{R}_+$. For strategy pairs $(\tilde{s}_1, \tilde{s}_2)$ such that $\tilde{s}_1 + \tilde{s}_1 = u/2$ we have

$$L_1(\tilde{s}_1, \tilde{s}_2) + L_2(\tilde{s}_1, \tilde{s}_2) = (\tilde{s}_1 + \tilde{s}_2)(\tilde{s}_1 + \tilde{s}_2 - u) = -u^2/4.$$

Hence the set of Pareto optimal strategies is precisely the set

$$Pareto = \{(\tilde{s}_1, \tilde{s}_2) | \tilde{s}_1 + \tilde{s}_1 = u/2\}.$$

This set is also the core of the game. Furthermore we have $\alpha = (-\frac{u^2}{4}, -\frac{u^2}{4})$ for the shadow minimum of the game. To see what strategy pairs are non cooperative equilibria we consider the optimal decision rules C, D such that

$$\begin{aligned} L_1(C(s_2), s_2) &= \min_{s_1 \in \mathcal{S}_1} L_1(s_1, s_2) = \min_{s_1 \in \mathcal{S}_1} s_1(s1 + s_2 - u) \\ L_2(s_1, D(s_1)) &= \min_{s_2 \in \mathcal{S}_2} L_2(s_1, s_2) = \min_{s_2 \in \mathcal{S}_1} s_2(s1 + s_2 - u). \end{aligned}$$

Differentiating with respect to s_1 resp. s_2 give

$$\begin{aligned} C(s_2) &= \frac{u - s_2}{2} \\ D(s_1) &= \frac{u - s_1}{2}. \end{aligned}$$

From the last section we know that any fixed point of the map

$$(s_1, s_2) \to (C(s_2), D(s_1))$$

is a non cooperative equilibrium. Solving

$$
\begin{aligned}
\tilde{s}_1 &= C(\tilde{s}_2) &= \frac{u - s_2}{2} \\
\tilde{s}_2 &= D(\tilde{s}_1) &= \frac{u - s_1}{2}
\end{aligned}
$$

we get $(\tilde{s}_1, \tilde{s}_2) = (u/3, u/3)$. This is the only non-cooperative equilibrium. Since $\tilde{s}_1 + \tilde{s}_2 = 2/3u$ it is not Pareto optimal and hence the Duopoly game has no solution in the strict sense. However, these strategies yield the players a net loss of $-u^2/9$.

Assume now, player 1 is sure that player 2 uses the optimal decision rule D from above. Then he can choose his strategy $\tilde{s}_1$ so as to minimize

$$
s_1 \mapsto L_1(s_1, D(s_1)) \quad = \quad L_1(s_1, \frac{u - s_1}{2}) = \frac{1}{2} s_1(s_1 - u).
$$

This yields to $\tilde{s}_1 = u/2$. The second player then uses $\tilde{s}_2 = D_1(\tilde{s}_1) = \frac{u-u/2}{2} = u/4$. The net losses are then

$$
\begin{aligned}
-\frac{1}{8}u^2 < -\frac{1}{9}u^2 = \text{ NCE loss, for Player 1} \\
-\frac{1}{16}u^2 > -\frac{1}{9}u^2 = \text{ NCE loss, for Player 2.}
\end{aligned}
$$

This brings the second player in a much worse position. The pair $(\tilde{s}_1, \tilde{s}_2)$ just computed is sometimes called the **Stackelberg equilibrium**. However, if the player 2 has the same idea as player 1, then both play the strategy $u/2$ leading to a net loss of 0 for each player.

Chapter 2

Brouwer's Fixed Point Theorem and Nash's Equilibrium Theorem

The Brouwer fixed point theorem is one of the most important theorems in Topology. It can be seen as a multidimensional generalization of the mean value theorem in basic calculus, which can be stated as that any continuous map $f : [a, b] \to [a, b]$ has a fixed point, that is a point x such that $f(x) = x$.

Theorem : Let $X \subset \mathbb{R}^m$ be convex and compact and let $f : X \to X$ continuous, then f has a fixed point.

This generalization of the mean value theorem is harder to prove than it appears at first. There are very nice proofs using methods from either algebraic or differential Topology. However we will give a proof which does not depend on these methods and only uses very basic ideas from combinatorics. Our main tool will be Sperners lemma, which was proven in 1928 and uses the idea of a proper labeling of a simplicial complex. There are many applications of Brouwer's theorem. The most important one in the context of game-theory is doubtless Nash's equilibrium theorem which in its most elementary version guarantees the existence of non cooperative equilibria for all games with finitely many pure strategies. This will be proven in the last part of this chapter.

2.1 Simplices

Definition 2.1.1. *Let $x^0, ..., x^n \in \mathbb{R}^m$ be a set of linear independent vectors. The* **simplex** *spanned by $x^0, .., x^n$ is the set of all strictly positive convex combinations*[1] *of the x^i*

$$x^0...x^n := \{\sum_{i=0}^{n} \lambda_i x^i : \lambda_i > 0 \textit{ and } \sum_{i=1}^{n} \lambda_i = 1\}. \tag{2.1}$$

The x^i are called the **vertices** *of the simplex and each simplex of the form $x^{i_0}...x^{i_k}$ is called a* **face** *of $x^0...x^n$.*

If we refer to the dimension of the simplex, we also speak of an n-simplex, where n is as in the definition above.

Example 2.1.1. *Let $e^i = (0, ..., 1, ..., 0) \in \mathbb{R}^{n+1}$ where 1 occurs at the i-th position and we start counting positions with 0 then*

$$\Delta^n = e^0...e^n$$

is called the **standard n-simplex***.*

We denote with $\overline{x^0...x^n}$ the closure of the simplex $x^0...x^n$. Then

$$\overline{x^0...x^n} = co(x^0, ..., x^n)$$

where the right hand side denotes the convex closure. For $y = \sum_{i=0}^{n} \lambda_i x^i \in \overline{x^0...x^n}$ we let

[1] we consider the open simplex, note that some authors actually mean closed simplexes when they speak of simplexes

$$\chi(y) = \{i | \lambda_i > 0\}. \tag{2.2}$$

If $\chi(y) = \{i_0, ..., i_k\}$ then $y \in x^{i_0}...x^{i_k}$. This face is called the **carrier** of y. It is the only face of $x^0...x^n$ which contains y. We have

$$\overline{x^0...x^n} = \biguplus_{\{i_0,...,i_k\} \subset \{0,...n\}} x^{i_0}...x^{i_k}. \tag{2.3}$$

where k runs from 0 to n and the $\biguplus$ stands for the disjoint union. The numbers $\lambda_0, ..., \lambda_n$ are called the **barycentric coordinates** of y.

Exercise 2.1.1. *Show that any n-simplex is homeomorphic to the standard n-simplex.*

Definition 2.1.2. *Let $T = x^0...x^n$ be an n-simplex. A* **simplicial subdivision** *of $\overline{T}$ is a finite collection of simplices $\{T_i | i \in I\}$ s.t. $\bigcup_{i \in I} T_i = \overline{T}$ and for any pair $i, j \in I$ we have*

$$\overline{T}_i \cap \overline{T}_j = \begin{cases} \emptyset \\ \textit{closure of a common face} \end{cases}$$

The **mesh** *of a subdivision is the diameter of the largest simplex in the subdivision.*

Example 2.1.2.

For any simplex $T = x^0...x^n$ the **barycenter** of T denoted by $b(T)$ is the point

$$b(T) = \frac{1}{n+1} \sum_{i=0}^{n} x^i. \tag{2.4}$$

For simplices T_1, T_2 define

$$T_1 > T_2 := T_2 \text{ is a face of } T_1 \text{ and } T_2 \neq T_1.$$

Given a simplex T, let us consider the family of all simplices of the form

$$b(T_0)...b(T_k) \text{ where } T \geq T_0 > T_1 > ... > T_k$$

where k runs from 0 to n. This defines a simplicial subdivision of T. It is called the first **barycentric subdivision**. Higher barycentric subdivisions are defined recursively. Clearly, for any 0 simplex v we have $b(v) = v$.

Example 2.1.3. *(Barycentric subdivision)*

Definition 2.1.3. *Let $\overline{T} = \overline{x^0...x^n}$ be simplicially subdivided. Let V denote the collection of all the vertices of all simplices in the subdivision. A function*

$$\lambda : V \to \{0, ..., n\} \tag{2.5}$$

satisfying $\lambda(v) \in \chi(v)$ is called a **proper labeling** *of the subdivision. We call a simplex in the subdivision* **completely labeled** *if λ assumes all values $0, ..., n$ on its set of vertices and* **almost completely labeled** *if λ assumes exactly the values $0, ..., n-1$.*

Let us study the situation in the following Example :

Example 2.1.4. *(Labeling and completely labeled simplex)*

2.2 Sperners Lemma

Theorem 2.2.1. Sperner (1928) *Let $\overline{T} = \overline{x^0...x^n}$ be simplicially subdivided and properly labeled by the function λ. Then there are an odd number of completely labeled simplices in the subdivision.*

Proof. The proof goes by induction on n. If $n = 0$, then $T = \overline{T} = x^0$ and $\lambda(x^0) \in \chi(x^0) = \{0\}$. That means T is the only simplex in the subdivision and it is completely labeled. Let us now assume the theorem is true for $n-1$. Let

$$
\begin{aligned}
C &= \text{set of all completely labeled } n\text{-simplices in the subdivision} \\
A &= \text{set of almost completely labeled } n\text{-simplices in the subdivision} \\
B &= \text{set of all } (n-1) \text{ simplices in the subdivision, which lie on the} \\
&\quad\ \text{boundary and bear all the labels } 0, ..., n-1 \\
E &= \text{set of all } (n-1) \text{ simplices in the subdivision which bear all the} \\
&\quad\ \text{labels } 0, ..., n-1
\end{aligned}
$$

The sets C, A, B are pairwise disjoint. B however is a subset of E. Furthermore all simplices in B are contained in the face $x^0...x^{n-1}$. An $(n-1)$ simplex lies either on the boundary and is then the face of exactly one n-simplex in the subdivision, or it is the common face of two n-simplices.

A graph consists of edges and nodes such that each edge joins exactly two nodes. If e denotes an edge and d a node we write $d \in e$ if d is one of the two nodes joined by e and $d \notin e$ if not.

Let us now construct a graph in the following way :

$$
\begin{aligned}
\text{Edges} &:= E \\
\text{Nodes} &:= C \cup A \cup B
\end{aligned}
$$

and for $d \in D, e \in E$

$$
d \in e := \begin{cases} d \in A \cup C \text{ and } e \text{ is a face of } d \\ e = d \in B \end{cases}
$$

We have to check that this indeed defines a graph, i.e. that any edge $e \in E$ joins exactly two nodes. Here we have to consider two cases :

1. e lies on the boundary and bears all labels $0, ..., n-1$. Then $d_1 :=$

$e \in B$ and is the face of exactly one simplex $d_2 \in A \cup C$. Interpreting d_1 and d_2 as nodes we see that the definition above tells us that e joins d_1 and d_2 and no more.

2. e is a common face of two n-simplices d_1 and d_2. Then both belong to either A or C (they are at least almost completely labeled since one of their faces,e in fact, bears all the labels $0, ...n-1$) and hence by the definition above e joins d_1 and d_2 and no more.

For each node $d \in D$ the degree is defined by

$$\delta(d) = \text{ number of edges } e \text{ s.t. } d \in e.$$

Let us compute the degree for each node in our graph. We have to consider the following three cases :

1. $d \in A$: Then exactly two vertices v_1 and v_2 of d have the same label and exactly two faces e_1 and e_2 of d belong to E (those who do not have both v_1 and v_2 as vertices, and therefore bear at most $(n-2)$ different labels and hence do not belong to E). Hence $d \in e_1$ and $d \in e_2$ but no more. Therefore $\delta(d) = 2$.

2. $d \in B$: Then $e := d \in E$ is the only edge such that $d \in e$ and therefore $\delta(d) = 1$.

3. $d \in C$: Then d is completely labeled and hence has only one face which bears all the labels $0, ..., n-1$. This means only one of the faces of d belongs to E and hence using the definition we have $\delta(d) = 1$.

Summarizing we get the following :

$$\delta(d) = \begin{cases} 1 \text{ if } d \in B \cup C \\ 2 \text{ if } d \in A. \end{cases}$$

In general for a graph with nodes D and edges E one has the following relationship :

$$\sum_{d \in D} \delta(d) = 2|E|. \tag{2.6}$$

This relationship holds since when counting the edges on each node and summing up over the nodes one counts each edge exactly twice. For our graph it follows that

$$2 \cdot |A| + |B| + |C| = \sum_{d \in D} \delta(d) = 2|E|$$

and this implies that $|B| + |C|$ is even. The simplicial subdivision of $\overline{T}$ and the proper labeling function λ induce via restriction a simplicial subdivision and proper labeling function for $\overline{x^0 ... x^{n-1}}$. Then B is the set of completely labeled simplices in this simplicial subdivision with this proper labeling function. Hence by induction $|B|$ is odd and therefore $|C|$ is odd which was to prove. □

In the proof of the Brouwer fixed point theorem it will not be so important how many completely labeled simplices there are in the simplicial subdivision, but that there is at least one. This is the statement of the following corollary.

Corollary 2.2.1. *Let $\overline{T} = \overline{x^0 ... x^n}$ be simplicially subdivided and properly labeled. Then there exists at least one completely labeled simplex in the simplicial subdivision.*

Proof. Zero is not an odd number. □

2.3 Proof of Brouwer's Theorem

We will first proof the following simplified version of the Brouwer fixed point theorem.

Proposition 2.3.1. *Let $f : \overline{\Delta}^n \to \overline{\Delta}^n$ be continuous, then f has a fixed point.*

Proof. Let $\epsilon > 0$. Since $\overline{\Delta}^n$ is compact we can find a simplicial subdivision with mesh less than ϵ.[2] Let V be the set of vertices of this subdivision. Let us consider an arbitrary vertex v in the subdivision and $v \in x^{i_0}...x^{i_k}$. Let v_i and f_i denote the components of v and f. Then

$$\{i_0, ..., i_k\} \cap \{i | f_i(v) \leq v_i\} \neq \emptyset$$

since $f_i(v) > v_i$ for all $i \in \{i_0, ..., i_k\}$ would imply

$$1 = \sum_{i=0}^{n} f_i(v) \geq \sum_{j=0}^{k} f_{i_j}(v) > \sum_{j=0}^{k} v_{i_j} = \sum_{i=0}^{k} v_i = 1.$$

We define a labeling function

$$\lambda : V \to \{0, ..., n\}$$

by choosing for each vertex $v \in x^{i_0}...x^{i_k}$ one element $\lambda(v) \in \{i_0, ..., i_k\} \cap \{i | f_i(v) \leq v_i\}$. Then λ is a proper labeling function. It follows from Corollary 2.2.1 that there exists a completely labeled simplex in the simplicial subdivision. This means, there exists a simplex $x_\epsilon^0...x_\epsilon^n$ such that for any $i \in \{0, 1, .., n\}$ there exists j s.t.

$$f_i(x_\epsilon^j) \leq x_{\epsilon,i}^j \tag{2.7}$$

Where $x_{\epsilon,i}^j$ denotes the i-the component of x_ϵ^j. We now let ϵ tend to zero. Then we get a sequence of completely labeled simplexes $x_\epsilon^0...x_\epsilon^n$. Since the mesh' of the subdivisions converges to 0 and furthermore $\overline{\Delta}^n$ is compact we can extract a convergent subsequent which converges to one point in $\overline{\Delta}^n$. We denote this point with x. This point is the common limit of all the sequences (x_ϵ^j) for ϵ tending to 0. Therefore using equation (2.7) and the continuity of f we get

$$f_i(x) \leq x_i \; \forall i \in \{0, .., n\}.$$

Assume now that for one $i \in \{0, ..., n\}$ we would have $f_i(x) < x_i$.

[2] this is possible for example by using iterated barycentric subdivisions

Then

$$1 = \sum_{i=0}^{n} f_i(x) < \sum_{i=0}^{n} x_i = 1$$

Therefore we must have $f_i(x) = x_i$ for all i. This is the same as $f(x) = x$ and x is a fixed point. $\square$

In general a map $\phi : X \to Y$ is called a **homeomorphism**, if it is continuous, bijective and its inverse $\phi^{-1} : Y \to X$ is also continuous. In this case we write $X \approx Y$. We have the following corollary :

Corollary 2.3.1. *Let $X \approx \overline{\Delta}^n$ and $f : X \to X$ be continuous. Then f has a fixed point.*

Proof. Let $\phi : X \to \overline{\Delta}^n$ be a homeomorphism. Define

$$\begin{aligned} f_\phi : \overline{\Delta}^n &\to \overline{\Delta}^n \\ y &\mapsto \phi(f(\phi^{-1}(y))) \end{aligned}$$

Clearly f is continuous and hence by Proposition 2.3.1 must have a fixed point $\tilde{y}$ i.e. $f_\phi(\tilde{y}) = \tilde{y}$. Define $\tilde{x} := \phi^{-1}(\tilde{y})$. Then

$$f(\tilde{x}) = f(\phi^{-1}(\tilde{y})) = \phi^{-1}(f_\phi(\tilde{y})) = \phi^{-1}(\tilde{y}) = \tilde{x}.$$

$\square$

The following proposition will help us to prove the Brouwer fixed point theorem in its general form.

Proposition 2.3.2. *Let $X \subset \mathbb{R}^m$ be convex and compact. Then $X \approx D^n$ for some $0 \leq n \leq m$ where $D^n := \{x \in \mathbb{R}^n : ||x|| \leq 1\}$ is the n-dimensional unit-ball.*

Proof. Let us first assume that 0 is contained in the interior X° of X. Let $v \in \mathbb{R}^m$. Consider the ray starting at the origin

$$\gamma(t) := t \cdot v, \; \forall t \geq 0.$$

We claim that this ray intersects the boundary ∂X at exactly one point. Since $\gamma(0) \in X$ and X is compact, it is clear that it intersects ∂X in at least one point. Assume now that $x, y \in \partial X \cap \gamma([0, \infty))$ and $x \neq y$. Then since x and y are collinear w.l.o.g. we can assume that $||x|| > ||y||$. Since $0 \in X^\circ$ it follows that there exists $\epsilon > 0$ such that $D_\epsilon^m := \{z \in \mathbb{R}^m : ||z|| \leq \epsilon\} \subset X^\circ$. Then the convex hull $co(x, D_\epsilon^m)$ contains an open neighborhood of y. Since X is convex and closed, we have $co(x, D_\epsilon^m) \subset X$ and hence $y \in X^\circ$ which is a contradiction to $y \in \partial X$. Let us now consider the following function :

$$\begin{aligned} f : \partial X &\rightarrow S^{m-1} := \{z \in \mathbb{R}^m : ||z|| = 1\} \\ x &\mapsto \frac{x}{||x||} \end{aligned}$$

Since X contains an open ball around the origin, it follows that the map f is well defined $0 \in \partial X$ and furthermore surjective. f is clearly continuous and it follows from the discussion above that it is also injective (otherwise two elements in ∂X would lie on the same ray from the origin). Since ∂X is compact and S^{m-1} is Hausdorff it follows that f is a homeomorphism.[3] Hence the inverse map

$$f^{-1} : S^{m-1} \rightarrow \partial X$$

is also continuous. Let us define a map which is defined on the whole space X.

[3]Result from Topology : $f : A \rightarrow B$ continuous and bijective, A compact, B Hausdorff, then f is a homeomorphism.

$$\begin{array}{rcl} k : D^m & \to & X \\ x & \mapsto & \begin{cases} ||x|| \cdot f^{-1}(x/||x||) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \end{array}$$

Since X is compact, there exists $M \in \mathbb{R}$ such that $||x|| \leq M$ for all $x \in X$. Then also $||\underbrace{f^{-1}(x/||x||)}_{\in \partial X}|| \leq M$ for all $x \in X$ and hence

$$||k(x)|| \leq ||x|| \cdot M$$

It follows from this that the map k is continuous in 0. Continuity in all other points is clear so that k is a continuous map. Assume that $x, y \in X$ and $k(x) = k(y)$. Then

$$||x|| \cdot f^{-1}(x/||x||) \quad = \quad ||y|| \cdot f^{-1}(y/||y||).$$

Since $||f^{-1}(x/||x||)|| = 1 = ||f^{-1}(y/||y||)||$ we must have $||x|| = ||y||$. But then the equation above is equivalent to

$$f^{-1}(x/||x||) \quad = \quad f^{-1}(y/||y||).$$

and it follows from the injectivity of f^{-1} that $\frac{x}{||x||} = \frac{y}{||y||}$ which finally implies that $x = y$. This shows that k is injective. k is also surjective : Assume $x \in X$. Then as in the first part of the proof x can be written as $x = t \cdot \bar{x}$ where $\bar{x} \in \partial X$ and $t \in [0, 1]$ and $f(\bar{x}) = \frac{\bar{x}}{||\bar{x}||}$ which is equivalent to $\bar{x} = f^{-1}(\frac{\bar{x}}{||\bar{x}||})$. Then

$$\begin{aligned} x &= t\cdot\bar{x} = t\cdot f^{-1}\left(\frac{t\cdot\frac{\bar{x}}{||\bar{x}||}}{||t\cdot\frac{\bar{x}}{||\bar{x}||}||}\right. \\ &= ||t\cdot\frac{\bar{x}}{||\bar{x}||}||f^{-1}\left(\frac{t\cdot\frac{\bar{x}}{||\bar{x}||}}{||t\cdot\frac{\bar{x}}{||\bar{x}||}||}\right. \\ &= k\underbrace{(t\cdot\frac{\bar{x}}{||\bar{x}||})}_{\in D^m} \end{aligned}$$

Repeating the previous argument, since now D^m is compact and X is Hausdorff, we have that k is a homeomorphism. Let us now consider the general case. W.l.o.g. we can still assume that $0 \in X$ (by translation, translations are homeomorphisms) but we can no longer assume that $0 \in X^\circ$. However we can find a maximum number of linear independent vectors $v_1, .., v_n \in X$. Then $X \subset span(v_1, .., v_n)$. Easy linear algebra shows that there exists a linear map

$$\phi : \mathbb{R}^m \to \mathbb{R}^n$$

which maps $span(v_1, .., v_n)$ isomorphically to $\mathbb{R}^n$ (and its orthogonal complement to zero). This map maps X homeomorphically to $\phi(X) \subset \mathbb{R}^n$, and since linear maps preserve convexity $\phi(X)$ is still convex. Now we can apply our previous result to $\phi(X)$ and get

$$X \approx \phi(X) \approx \overline{\Delta}^n \Rightarrow X \approx \overline{\Delta}^n.$$

□

Corollary 2.3.2. *Let $X \subset \mathbb{R}^m$ be convex and compact. Then $X \approx \overline{\Delta}^n$ for some $0 \leq n \leq m$.*

Proof. Since for all n we have that $\overline{\Delta}^n$ is convex and compact, we can use the preceding proposition to follow that $\overline{\Delta}^n \approx D^n$ (take the dimensions into account). Also from the preceding proposition we know that there must exist n such that $X \approx D^n$. Then $X \approx D^n \approx \overline{\Delta}^n$. □

We are now able to proof the Brouwer fixed point theorem in its general form :

Theorem 2.3.1. *Let $X \subset \mathbb{R}^m$ be convex and compact and let $f : X \to X$ continuous, then f has a fixed point.*

Proof. By Corollary 2.3.2 $X \approx \overline{\Delta}^n$. Hence the theorem follows by application of Corollary 2.3.1. □

2.4 Nash's Equilibrium Theorem

Theorem 2.4.1. *Let $\mathcal{G}_2$ be a game with finite strategy sets $\mathcal{S}_1$ and $\mathcal{S}_2$ and $U = \mathcal{S}_1 \times \mathcal{S}_2$. Then there exists at least one non-cooperative equilibrium for the extended game $\tilde{\mathcal{G}}_2$.*

Proof. Let L be the biloss-operator of $\mathcal{G}_2$ and $\tilde{L}$ be its extension. Furthermore let $\mathcal{S}_1 = \{s_1^i : i \in \{1, .., n\}\}$, $\mathcal{S}_2 = \{s_2^j : j \in \{1, .., m\}\}$ and $l_1 = -\tilde{L}_1$, $l_2 = -\tilde{L}_2$. Clearly $\tilde{\mathcal{S}}_1 \times \tilde{\mathcal{S}}_2 \approx \overline{\Delta}^{n-1} \times \overline{\Delta}^{m-1}$ is convex and compact. Let $\tilde{s}_1 \in \tilde{\mathcal{S}}_1$ and $\tilde{s}_2 \in \tilde{\mathcal{S}}_2$ be given as

$$\begin{aligned} \tilde{s}_1 &= \sum_{i=1}^{n} \lambda_i^1 \cdot \delta_{s_1^i} \\ \tilde{s}_2 &= \sum_{j=1}^{m} \lambda_j^2 \cdot \delta_{s_2^j}. \end{aligned}$$

For $1 \leq i \leq n$ and $1 \leq j \leq m$ we define maps $c_i, d_j : \tilde{\mathcal{S}}_1 \times \tilde{\mathcal{S}}_2 \to \mathbb{R}$ as follows :

$$\begin{aligned} c_i(\tilde{s}_1, \tilde{s}_2) &= \max(0, l_1(s_1^i, \tilde{s}_2) - l_1(\tilde{s}_1, \tilde{s}_2)) \\ d_j(\tilde{s}_1, \tilde{s}_2) &= \max(0, l_2(\tilde{s}_1, s_2^j) - l_2(\tilde{s}_1, \tilde{s}_2)). \end{aligned}$$

Furthermore we define a map

$$\begin{aligned} f : \tilde{\mathcal{S}}_1 \times \tilde{\mathcal{S}}_2 &\rightarrow \tilde{\mathcal{S}}_1 \times \tilde{\mathcal{S}}_2 \\ (\tilde{s}_1, \tilde{s}_2) &\mapsto (\sum_{i=1}^{n} \underbrace{\frac{\lambda_i^1 + c_i(\tilde{s}_1, \tilde{s}_2)}{1 + \sum_{i=1}^{n} c_i(\tilde{s}_1, \tilde{s}_2)}}_{=:\tilde{\lambda}_i^1} \delta_{s_1^i}, \sum_{j=1}^{n} \underbrace{\frac{\lambda_j^2 + d_j(\tilde{s}_1, \tilde{s}_2)}{1 + \sum_{j=1}^{n} d_j(\tilde{s}_1, \tilde{s}_2)}}_{=:\tilde{\lambda}_2^j} \delta_{s_2^j}) \end{aligned}$$

Clearly $\sum_{i=1}^{n} \tilde{\lambda}_1^i = 1 = \sum_{j=1}^{m} \tilde{\lambda}_2^j$ and hence the right side in the expression defines indeed an element in $\tilde{\mathcal{S}}_1 \times \tilde{\mathcal{S}}_2$. Obviously the map f is continuous and hence by application of the Brouwer fixed point theorem there must exist a pair $(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp)$ such that

$$(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp) = f(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp).$$

Writing $\tilde{s}_1^\sharp = \sum_{i=1}^{n} \lambda_i^{1\sharp} \cdot \delta_{s_1^i}$ and $\tilde{s}_2^\sharp = \sum_{j=1}^{n} \lambda_j^{2\sharp} \cdot \delta_{s_2^j}$ we see that the equation above is equivalent to the following set of equations

$$\lambda_i^{1\sharp} = \frac{\lambda_i^{1\sharp} + c_i(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp)}{1 + \sum_{i=1}^{n} c_i(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp)}, \; \lambda_j^{2\sharp} = \frac{\lambda_j^{2\sharp} + d_j(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp)}{1 + \sum_{j=1}^{m} d_j(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp)} \; \forall\, 1 \leq i \leq n, 1 \leq j \leq m.$$

Let us assume now that for all $1 \leq i \leq n$ we have $l_1(s_1^i, \tilde{s}_2^\sharp) > l_1(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp)$. Then using that the extended biloss-operator and hence also l_1 and l_2 are bilinear we have

$$l_1(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp) = \sum_{i=1}^{n} \lambda_i^{1\sharp} l_1(s_1^i, \tilde{s}_2^\sharp) > \sum_{i=1}^{n} \lambda_i^{1\sharp} l_1(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp) = l_1(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp).$$

which is a contradiction. Therefore there must exist $i \in \{1, .., n\}$ such that $l_1(s_1^i, \tilde{s}_2^\sharp) < l_1(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp)$. For this i we have $c_i(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp) = 0$ and hence it follows from our set of equations above that for this i we have

$$\lambda_i^{1\sharp} = \frac{\lambda_i^{1\sharp}}{1 + \sum_{i=1}^{n} c_i(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp)}$$

This equation however can only hold if $\sum_{i=1}^n c_i(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp) = 0$ which by positivity of the maps c_i can only be true if

$$c_i(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp) = 0 \text{ for all } 1 \leq i \leq n.$$

By definition of the maps c_i this means nothing else than

$$l_1(s_1^i, \tilde{s}_2^\sharp) \leq l_1(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp) \; \forall \, 1 \leq i \leq n.$$

Using again the bilinearity of l_1 we get for arbitrary $\tilde{s}_1 = \sum_{i=1}^n \lambda_i^1 \cdot \delta_{s_1^i}$

$$l_1(\tilde{s}_1, \tilde{s}_2^\sharp) = \sum_{i=1}^n \lambda_i^1 l_1(s_1^i, \tilde{s}_2^\sharp) \leq \sum_{i=1}^n \lambda_i^1 l_1(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp) = l_1(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp), \; \forall \, 1 \leq i \leq n.$$

A similar argument involving $\lambda_j^{2\sharp}$ and d_j, l_2 shows that for arbitrary $\tilde{s}_2 = \sum_{j=1}^m \lambda_j^2 \cdot \delta_{s_2^j}$

$$l_2(\tilde{s}_1^\sharp, \tilde{s}_2) \leq l_2(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp), \; \forall \, 1 \leq j \leq m.$$

Using $\tilde{L}_1 = -l_1$ and $\tilde{L}_2 = -l_2$ we get

$$\begin{aligned} \tilde{L}_1(\tilde{s}_1, \tilde{s}_2^\sharp) &\geq \tilde{L}_1(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp) \\ \tilde{L}_2(\tilde{s}_1^\sharp, \tilde{s}_2) &\geq \tilde{L}_2(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp) \end{aligned}$$

which shows that $(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp)$ is a non-cooperative equilibrium for $\tilde{G}_2$. $\square$

2.5 Two Person Zero Sum Games and the Minimax Theorem

Definition 2.5.1. *A two person game $\mathcal{G}_2$ is called a* **zero sum game***, if $L_1 = -L_2$.*

Nim(2,2) and "simplified poker" are zero sum games, the "Battle of the Sexes" is not. For zero sum games one usually only denotes L_1, since then L_2 is determined by the negative values of L_1.

The Application of Theorem 2.4.1 in this contexts yields to a Theorem which is called the MiniMax-Theorem and has applications in many different parts of mathematics.

Theorem 2.5.1. *(***MiniMax Theorem***) : Let $\mathcal{G}_2$ be a zero sum game with finite strategy sets. Then for the extended game $\tilde{\mathcal{G}}_2$ we have*

$$\max_{\tilde{s}_1\in\tilde{\mathcal{S}}_1}\min_{\tilde{s}_2\in\tilde{\mathcal{S}}_2} L_1(\tilde{s}_1,\tilde{s}_2) \;=\; \min_{\tilde{s}_2\in\tilde{\mathcal{S}}_2}\max_{\tilde{s}_1\in\tilde{\mathcal{S}}_1} L_1(\tilde{s}_1,\tilde{s}_2)$$

and for any NCE $(\tilde{s}_1^\sharp,\tilde{s}_2^\sharp)$ this value coincides with $L_1(\tilde{s}_1^\sharp,\tilde{s}_2^\sharp)$. In particular all NCE's have the same biloss.[4]

Proof. Clearly we have for all $\tilde{s}_1\in\tilde{\mathcal{S}}_1, \tilde{s}_2\in\tilde{\mathcal{S}}_2$ that

$$\min_{\tilde{s}_1\in\tilde{\mathcal{S}}_1} L_1(\tilde{s}_1,\tilde{s}_2) \;\leq\; L_1(\tilde{s}_1,\tilde{s}_2).$$

Taking the maximum over all strategies $\tilde{s}_2\in\tilde{\mathcal{S}}_2$ on both sides we get for all $\tilde{s}_1\in\tilde{\mathcal{S}}_1$

$$\max_{\tilde{s}_2\in\tilde{\mathcal{S}}_2}\min_{\tilde{s}_1\in\tilde{\mathcal{S}}_1} L_1(\tilde{s}_1,\tilde{s}_2) \;\leq\; \max_{\tilde{s}_2\in\tilde{\mathcal{S}}_2} L_1(\tilde{s}_1,\tilde{s}_2).$$

Taking the minimum over all strategies $\tilde{s}_1\in\tilde{\mathcal{S}}_1$ on the right side of the last equation we get that

[4] we denote the biloss operator for the extended game with L instead of $\tilde{L}$ as we did before, just to make it readable and save energy

$$\max_{\tilde{s}_2 \in \tilde{\mathcal{S}}_2} \min_{\tilde{s}_1 \in \tilde{\mathcal{S}}_1} L_1(\tilde{s}_1, \tilde{s}_2) \quad \leq \quad \min_{\tilde{s}_1 \in \tilde{\mathcal{S}}_1} \max_{\tilde{s}_2 \in \tilde{\mathcal{S}}_2} L_1(\tilde{s}_1, \tilde{s}_2). \tag{2.8}$$

It follows from Theorem 2.4.1 that there exists at least one NCE $(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp)$. Then

$$\begin{aligned} L_1(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp) &= \min_{\tilde{s}_1 \in \tilde{\mathcal{S}}_1} L_1(\tilde{s}_1, \tilde{s}_2^\sharp) \\ L_2(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp) &= \min_{\tilde{s}_2 \in \tilde{\mathcal{S}}_2} L_1(\tilde{s}_1^\sharp, \tilde{s}_2) \end{aligned}$$

Using that $L_2 = -L_1$ and $\min(-..) = -\max(..)$ we get that the second equation above is equivalent to

$$L_1(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp) \quad = \quad \max_{\tilde{s}_2 \in \tilde{\mathcal{S}}_2} L_1(\tilde{s}_1^\sharp, \tilde{s}_2)$$

Now we have

$$\begin{aligned} \min_{\tilde{s}_1 \in \tilde{\mathcal{S}}_1} \max_{\tilde{s}_2 \in \tilde{\mathcal{S}}_2} L_1(\tilde{s}_1, \tilde{s}_2) &\leq \max_{\tilde{s}_2 \in \tilde{\mathcal{S}}_2} L_1(\tilde{s}_1^\sharp, \tilde{s}_2) = L_1(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp) \\ &= \min_{\tilde{s}_1 \in \tilde{\mathcal{S}}_1} L_1(\tilde{s}_1, \tilde{s}_2^\sharp) \leq \max_{\tilde{s}_2 \in \tilde{\mathcal{S}}_2} \min_{\tilde{s}_1 \in \tilde{\mathcal{S}}_1} L_1(\tilde{s}_1, \tilde{s}_2) \end{aligned}$$

Together with (2.7) we get

$$\max_{\tilde{s}_2 \in \tilde{\mathcal{S}}_2} \min_{\tilde{s}_1 \in \tilde{\mathcal{S}}_1} L_1(\tilde{s}_1, \tilde{s}_2) = L_1(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp) = \min_{\tilde{s}_1 \in \tilde{\mathcal{S}}_1} \max_{\tilde{s}_2 \in \tilde{\mathcal{S}}_2} L_1(\tilde{s}_1, \tilde{s}_2)$$

Since $(\tilde{s}_1^\sharp, \tilde{s}_2^\sharp)$ was an arbitrary NCE the statement of the theorem follows. $\square$

Chapter 3

More general Equilibrium Theorems

3.1 N-Person Games and Nash's generalized Equilibrium Theorem

Definition 3.1.1. *Let* $N = \{1, 2, ...n\}$. *An* N*-person (or* n*-person or* n*-player) game* $\mathcal{G}_n$ *consists of the following data :*

1. *Topological spaces* $\mathcal{S}_1, .., \mathcal{S}_n$ *so called strategies for player* 1 *to* n
2. *A subset* $\mathcal{S}(N) \subset \mathcal{S}_1 \times .. \times \mathcal{S}_n$*, the so called allowed or feasible multi strategies*
3. *A (multi)-loss operator* $L = (L_1, ..., L_n) : \mathcal{S}_1 \times ... \times \mathcal{S}_n \to \mathbb{R}^n$

All of the definitions in chapter 1 in the framework of 2 player games can be generalized to n-player games. This is in fact a very good exercise. We will restrict ourself though to the reformulation of the concept of non cooperative equilibria within n player games.

Definition 3.1.2. *A multi strategy* $s = (s_1, .., s_n)$ *is called a* **non cooperative equilibrium** *(in short NCE) for the game* $\mathcal{G}_n$ *if for all* i *we have*

$$L_i(s) = \min\{L_i(\tilde{s}) \ : \ \tilde{s} \in \{s_1\} \times .. \times \{s_{i-1}\} \times \mathcal{S}_i \times \{s_{i+1}\} \times .. \times \{s_n\} \cap \mathcal{S}(N)\}$$

The following theorem is a generalization of Theorem 2.4.1. It is also due to Nash.

Theorem 3.1.1. Nash (general Version) *: Given an N-player game as above. Suppose that $\mathcal{S}(N)$ is convex and compact and for any $i \in \{1, .., n\}$ the function $L_i(s_1, .., s_{i-1}, \cdot, s_{i-1}, .., s_n)$ considered as function in the i-th variable when $s_1, .., s_{i-1}, s_{i+1}, .., s_n$ is fixed but arbitrary is convex and continuous. Then there exists an NCE in $\mathcal{G}_n$.*

We have to build up some general concepts, before we go into the proof. It will follow later. One should mention though, that we do not assume that the game in question is the extended version of a game with only finitely many strategies and also the biloss operator does not have to be the bilinearly extended version of a biloss operator for such a game.

3.2 Correspondences

The concept of correspondences is a generalization of the concept of functions. In the context of game theory it can be used on one side to define so called generalized games and on the other side it shows up to be very useful in many proofs. For an arbitrary set M we denote with $\mathcal{P}(M)$ the power set of M, this is the set which contains as elements the subsets of M.

Definition 3.2.1. *Let X, Y be sets. A map $\gamma : X \to \mathcal{P}(Y)$ is called a* **correspondence**. *We write $\gamma : X \to\to Y$. We denote with*

$$Gr(\gamma) := \{(x, y) \in X \times Y : y \in \gamma(x)\} \subset X \times Y$$

the **graph** *of γ.*

The following example shows how maps can be considered as correspondences and that the definition above really is a generalization of the concept of maps.

Example 3.2.1. *Any map $f : X \to Y$ can be considered as the correspondence $\tilde{f} : X \to\to Y$ where $\tilde{f}(x) = \{f(x)\} \in \mathcal{P}(Y)$. In general the inverse f^{-1} of f as a map is not defined. However it makes sense to speak of the inverse correspondence $f^{-1} : Y \to\to X$ where $y \in Y$ maps to the preimage $f^{-1}(\{y\}) \in \mathcal{P}(X)$.*

Definition 3.2.2. *Let $\gamma : X \to\to Y$ be a correspondence, $E \subset Y$ and $F \subset X$. The* **image** *of F under γ is defined by*

$$\gamma(F) = \bigcup_{x \in F} \gamma(x).$$

The **upper inverse** *of E under γ is defined by*

$$\gamma^+[E] = \{x \in X : \gamma(x) \subset E\}.$$

The **lower inverse** *of E under γ is defined by*

$$\gamma^-[E] = \{x \in X : \gamma(x) \cap E \neq \emptyset\}.$$

Furthermore for $y \in Y$ we set $\gamma^{-1}(y) = \{x \in X | y \in \gamma(x)\} = \gamma^-[\{y\}]$.

It is easy to see that when γ has nonempty values one always has $\gamma^+[E] \subset \gamma^-[E]$, in general though there is no clear relation between the upper and the lower inverse unless the correspondence is given by a map. Then we have :

Example 3.2.2. *Assume the correspondence $\tilde{f} : X \to\to Y$ is given by a map as in Example 3.2.1. Then*

$$\begin{aligned}
\tilde{f}^+[E] &= \{x \in X : \{f(x)\} \subset E\} = \{x \in X : f(x) \in E\} = f^{-1}(E) \\
\tilde{f}^-[E] &= \{x \in X : \{f(x)\} \cap E \neq \emptyset\} = \{x \in X : f(x) \in E\} = f^{-1}(E).
\end{aligned}$$

This means that if one considers correspondences which are actually maps upper- and lower inverse coincide. For general correspondences this is however not the case.

Knowing that correspondences are generalized functions we would like to generalize the definition of continuity to correspondences. We assume from now on that X and Y are topological spaces.

Definition 3.2.3. *Let $\gamma : X \to\to Y$ be a correspondence. Then γ is called* **upper hemi continuous** *or short* **uhc** *at $x \in X$ if*

$$x \in \gamma^+[V] \text{ for } V \subset Y \text{ open } \Rightarrow \exists \text{ open neighborhood } U \subset X \text{ of } x \text{ s.t. } U \subset \gamma^+[V].$$

γ is called **lower hemi continuous** *or short* **lhc** *at $x \in X$ if*

$$x \in \gamma^-[V] \text{ for } V \subset Y \text{ open } \Rightarrow \exists \text{ open neighborhood } U \subset X \text{ of } x \text{ s.t. } U \subset \gamma^-[V].$$

γ is called uhc on X if γ is uhc at x for all $x \in X$ and lhc on X if γ is lhc at x for all $x \in X$.

As one can see directly from the definition, γ is uhc iff $V \subset Y$ open $\Rightarrow \gamma^+[V] \subset X$ open and lhc iff $V \subset Y$ open $\Rightarrow \gamma^-[V] \subset X$ open . Example 3.2.1 now says that a map f is continuous (as a map) if an only if it is uhc (considered as a correspondence) and this is the case if and only if it is lhc (considered as a correspondence).

Definition 3.2.4. *A correspondence $\gamma : X \to\to Y$ is called continuous if it is uhc and lhc on X.*

Definition 3.2.5. *Let $\gamma : X \to\to Y$ a correspondence. Then γ is called* **closed** *at the point $x \in X$ if*

$$x_n \to x, y_n \in \gamma(x_n)\, \forall n \text{ and } y_n \to y \Rightarrow y \in \gamma(x).$$

A correspondence is said to be **closed** *if it is closed at every point* $x \in X$*. This is precisely the case if the graph* $Gr(\gamma)$ *is closed as a subset of* $X \times Y$*.* γ *is called* **open** *if* $Gr(\gamma) \subset X \times Y$ *is an open subset.*

Definition 3.2.6. *Let* $\gamma : X \to\to Y$ *be a correspondence. We say* γ *has* **open (closed) sections** *if for each* $x \in X$ *the set* $\gamma(x)$ *is open (closed) in* Y *and for each* $y \in Y$ *the set* $\gamma^{-}[\{y\}]$ *is open (closed) in* X*.*

Proposition 3.2.1. *Let* $X \subset \mathbb{R}^m$*,*$Y \subset \mathbb{R}^k$ *and* $\gamma : X \to\to Y$ *a correspondence.*

1. *If* γ *is uhc and* $\forall x \in X$ $\gamma(x)$ *is a closed subset of* Y*, then* γ *is closed.*
2. *If* Y *is compact and* γ *is closed, then* γ *is uhc.*
3. *If* γ *is open, then* γ *is* lhc*.*
4. *If* $|\gamma(x)| = 1$ $\forall x$ *(so that* γ *can actually be considered as a map) and* γ *is* uhc *at* x*, then* γ *is continuous at* x*.*
5. *If* γ *has open lower sections (i.e* $\gamma^{-}[\{y\}]$ *open in* X $\forall y \in Y$ *), then* γ *is lhc*

Proof. 1. We have to show that $Gr(\gamma) \subset X \times Y$ is closed, i.e. its complement is open. Assume $(x, y) \notin Gr(\gamma)$, i.e. $y \notin \gamma(x)$. Since $\gamma(x)$ is closed and hence its complement is open, there exists a closed neighborhood U of y s.t. $U \cap \gamma(x) = \emptyset$. Then $V = U^c$ is an open neighborhood of $\gamma(x)$. Since γ is uhc and $x \in \gamma^+[V]$ there exists an open neighborhood W of x s.t. $W \subset \gamma^+[V]$. We have $\gamma(w) \subset V$ $\forall w \in W$. This implies that

$$Gr(\gamma) \cap W \times Y \subset W \times V = W \times U^c$$

and hence $Gr(\gamma) \cap W \times U = \emptyset$. Since y has to be in the interior of U (otherwise U wouldn't be a neighborhood of y), we have that $W \times U^\circ$ is an open neighborhood of (x, y) in $Gr(\gamma)^c$ which shows that $Gr(\gamma)^c$ is open.

2. Suppose γ were not uhc. Then there would exist $x \in X$ and an open neighborhood V of $\gamma(x)$ s.t. for all neighborhoods U of X we would have $U \not\subset \gamma^+[V]$ i.e. there exists $z \in U$ s.t. $\gamma(z) \not\subset V$. By making U smaller and smaller we can find a sequence $z_n \to x$ and $y_n \in \gamma(z_n)$ s.t. $y_n \notin V$, i.e. $y_n \in V^c$. Since Y is compact (y_n) has a convergent subsequence and w.l.o.g. (y_n) itself is convergent. We denote with $y = \lim_n y_n$ its limit. Since V^c is closed we must have $y \in V^c$. From the closedness of γ however it follows that $y \in \gamma(x) \subset V$ which is clearly a contradiction.

3. exercise !

4. exercise !

5. exercise !

$\square$

Proposition 3.2.2. *(***Sequential Characterization of Hemi Continuity***) Let $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^k$ and $\gamma : X \to\to Y$ be a correspondence.*

1. *Assume $\forall x \in X$ that $\gamma(x)$ is compact. Then γ uhc $\Leftrightarrow$ for every sequence $x_n \to x$ and $y_n \in \gamma(x_n)$ there exists a convergent subsequence $y_{n_k} \to y$ and $y \in \gamma(x)$.*

2. *γ is lhc $\Leftrightarrow$ $x_n \to x$ and $y \in \gamma(x)$ implies there exists a sequence $y_n \in \gamma(x_n)$ with $y_n \to y$.*

Proof. 1. "$\Rightarrow$" : Assume $x_n \to x$ and $y_n \in \gamma(x_n)$. Since $\gamma(x)$ is compact it has a bounded neighborhood V. Since γ is uhc at x there exists a neighborhood U of x s.t. $\gamma(U) \subset V$. Since $x_n \to x$ there exists $n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$ we have $x_n \in U$. Then since $y_n \in \gamma(x_n) \subset \gamma(U) \subset V$ for all $n \geq n_0$ and V is bounded, (y_n) has a convergent subsequence $y_{n_k} \to y$. Clearly $y \in \overline{V}$. By making $\overline{V}$ smaller and smaller (for example one can take $\overline{V}_\epsilon := \overline{\bigcup_{x\in\gamma(x)} B_\epsilon(x)}$ and let ϵ go to zero) we see that y lies in any neighborhood of $\gamma(x)$. Since $\gamma(x)$ is compact, hence also closed we must have $y \in \gamma(x)$.

"$\Leftarrow$" : Suppose γ is not uhc. Then there exists $x \in X$ and a neighborhood V of $\gamma(x)$ s.t. for any open neighborhood U of x we have $U \not\subset \gamma^+[V]$. Making U smaller and smaller we get a sequence $x_n \to x$ s.t. $\gamma(x_n) \not\subset V$. By choosing $y_n \in \gamma(x_n) \cap V^c$ we get a sequence which does not enter into V. Since V is an open neighborhood of the compact set $\gamma(x)$ such a sequence cannot converge to a limit $y \in \gamma(x)$. This however is a contradiction to the assumption on the left side in 1.)

2. exercise !

$\square$

Definition 3.2.7. *A convex set Y is a* **polytope***, if it is the convex hull of a finite set.*

Example 3.2.3. *simplices, but not only.*

Proposition 3.2.3. *Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^k$, where Y is a polytope. If for all $x \in X$ the set $\gamma(x)$ is convex and has open sections, then γ has an open graph.*

Proof. Let $(x, y) \in Gr(\gamma)$, i.e. $y \in \gamma(x)$. Since γ has open sections and Y is a polytope, there is a neighborhood U of y contained in $\gamma(x)$ s.t. U is itself the interior of a polytope. Assume more precisely that $U = (co(y_1, .., y_n))^\circ$. Since γ has open sections the sets $V_i := \gamma^-[\{y_i\}]$ are open for all i. Clearly for all $z \in V_i$ we have $y_i \in \gamma(z)$ and furthermore $x \in V_i$ for all i. The set $V := \bigcap_{i=1}^n V_i$ is nonempty and open and furthermore contains x. $W := V \times U$ is open in $X \times Y$. Let $(x', y') \in W$. Then $y_i \in \gamma(x') \forall i$. Since $\gamma(x')$ is convex we have that

$$y' \in U = co(y_1, .., y_n)^\circ \subset \gamma(x') \Rightarrow (x', y') \in Gr(\gamma).$$

Therefore $W \subset Gr(\gamma)$ and W is an open neighborhood of $(x, y) \in Gr(\gamma)$. $\square$

Definition 3.2.8. *Let $\gamma : X \to\to Y$ be a correspondence. Then $x \in X$ is called a* **fixed point** *if $x \in \gamma(x)$.*

Proposition 3.2.4. *1. Let $\gamma : X \to\to Y$ be uhc s.t. $\gamma(x)$ is compact $\forall x \in X$ and let K be a compact subset of X. Then $\gamma(K)$ is compact.*

2. *Let $X \subset \mathbb{R}^m$ and $\gamma : X \to\to \mathbb{R}^m$ uhc with $\gamma(x)$ closed $\forall\, x$. Then the set of all fixed points of γ is closed.*

3. *let $X \subset \mathbb{R}^m$ and $\gamma, \mu : X \to\to \mathbb{R}^m$ uhc and $\gamma(x), \mu(x)$ closed $\forall x$. Then*

$$\{x \in X | \gamma(x) \cap \mu(x) \neq \emptyset\} \text{ is closed in } X$$

4. *$X \subset \mathbb{R}^m$ and $\gamma : X \to\to \mathbb{R}^m$ lhc (resp. uhc). Then*

$$\{x \in X | \gamma(x) \neq \emptyset\} \text{ is open (resp. closed) in } X$$

Proof. Exercise ! □

Definition 3.2.9. *(**Closure of a Correspondence**) Let $\gamma : X \to\to Y$ be a correspondence, then*

$$\begin{aligned} \overline{\gamma} : X &\to\to Y \\ x &\mapsto \overline{\gamma(x)} \end{aligned}$$

*is called the **closure** of γ.*

Proposition 3.2.5. *Let $X \subset \mathbb{R}^m$,$Y \subset \mathbb{R}^k$.*

1. *$\gamma : X \to\to Y$ uhc at $x \Rightarrow \overline{\gamma} : X \to\to Y$ uhc at x*

2. *$\gamma : X \to\to Y$ lhc at $x \Leftrightarrow \overline{\gamma} : X \to\to Y$ lhc at x*

Proof. Exercise ! □

Definition 3.2.10. *(**Intersection of correspondences**) Let $\gamma, \mu : X \to\to Y$ be a correspondences, then define their intersection as*

$$\begin{aligned} \gamma \cap \mu : X &\to\to Y \\ x &\mapsto \gamma(x) \cap \mu(x). \end{aligned}$$

Proposition 3.2.6. *Let $X \subset \mathbb{R}^m$,$Y \subset \mathbb{R}^k$ and $\gamma, \mu : X \to\to Y$ be correspondences. Suppose $\gamma(x) \cap \mu(x) \neq \emptyset \; \forall \; x \in X$.*

1. *If γ, μ are uhc at x and $\gamma(z), \mu(z)$ are closed for all $z \in X$ then $\gamma \cap \mu$ is uhc at x.*

2. *If μ is closed at x and γ is uhc at x and $\gamma(x)$ is compact, then $\gamma \cap \mu$ is uhc at x.*

3. *If γ is lhc at x and if μ has open graph, then $\gamma \cap \mu$ is lhc at x.*

Proof. let U be an open neighborhood of $\gamma(x) \cap \mu(x)$ and define $C := \gamma(x) \cap U^c$.

1. In this case C is closed and $\mu(x) \cap C = \emptyset$. Therefore there exist open sets V_1, V_2 s.t.

$$\begin{aligned} \mu(x) &\subset V_1 \\ C &\subset V_2 \\ V_1 \cap V_2 &= \emptyset. \end{aligned}$$

Since μ is uhc at x and $x \in \mu^+[V_1]$ there exists a neighborhood W_1 of x with

$$\mu(W_1) \subset V_1 \subset V_2^c.$$

We have that

$$\gamma(x) = (\gamma(x) \cap U) \cup (\gamma(x) \cap U^c) \subset U \cup C \subset U \cup V_2.$$

Since $U \cup V_2$ is open it follows from the upper hemi continuity of γ that there exists an open neighborhood W_2 of x s.t.

$$\gamma(W_2) \subset U \cup V_2.$$

We set $W = W_1 \cap W_2$. Then W is a neighborhood of x and for all $z \in W$ we have

$$\gamma(z) \cap \mu(z) \subset (U \cup V_2) \cap V_2^c = U \cap V_2^c \subset U.$$

Hence $\gamma \cap \mu$ is uhc at x.

2. In this case C is compact and $\mu(x) \cap C = \emptyset$. For $y \notin \mu(x)$ there cannot exist a sequence $x_n \to x$, $y_n \in \mu(x_n)$ and $y_n \to y$ because of the closedness of μ. This implies that there exists a neighborhood U_y of y and W_y of x s.t. $\mu(W_y) \subset U_y^c$. Since C is compact we can find $U_{y_1}, ..., U_{y_n}$, $W_{y_1}, ..., W_{y_n}$ as above such that $C \subset V_2 := U_{y_1} \cup ... \cup U_{y_n}$. We set $W_1 := W_{y_1} \cap ... \cap W_{y_n}$. Then $\mu(W_1) \subset V_2^c$. Now we choose W_2 for x and γ as in 1.) and proceed similarly as in 1.).

3. Let $y \in (\gamma \cap \mu)(x) \cap U$. Since μ has an open graph, there is a neighborhood $W \times V$ of (x, y) which is contained in $Gr(\mu)$. Since γ is lhc at x we find that $\gamma^-[U \cap V] \cap W$ is a neighborhood of x in X and if $z \in \gamma^-[U \cap V] \cap W$ then $y \in (\gamma \cap \mu)(z) \cap U$. This however implies that $\gamma \cap \mu$ is lhc.

□

As one can do with ordinary maps one can compose correspondences.

Definition 3.2.11. *(***Composition of Correspondences***) Let $\mu : X \to\to Y$,$\gamma : Y \to\to Z$ be correspondences. Define*

$$\begin{aligned} \gamma \circ \mu : X &\to\to Z \\ x &\mapsto \bigcup_{y \in \mu(x)} \gamma(y). \end{aligned}$$

$\gamma \circ \mu$ is called the composition of γ and μ.

Proposition 3.2.7. *Let γ, μ be as above. then*

1. γ, μ *uhc* $\Rightarrow \gamma \circ \mu$ *uhc*

2. γ, μ *lhc* $\Rightarrow \gamma \circ \mu$ *lhc*

Proof. Exercise ! $\square$

Definition 3.2.12. *(***Products of Correspondences***) Let* $\gamma_i : X \to\to Y_i$ *for* $i = 1, .., k$ *be correspondences. Then the correspondence*

$$\begin{aligned} \prod_i \gamma_i : X &\to\to \prod_i Y_i \\ x &\mapsto \prod_i \gamma_i(x) \end{aligned}$$

is called the product of the γ_i*.*

Proposition 3.2.8. *Assume* γ_i *are correspondences as above.*

1. γ_i *uhc at* x *and* $\gamma_i(z)$ *compact* $\forall z \in X$ *and* $\forall\, i \Rightarrow \prod_i \gamma_i$ *uhc at* x

2. γ_i *lhc at* x *and* $\gamma_i(z)$ *compact* $\forall z \in X$ *and* $\forall\, i \Rightarrow \prod_i \gamma_i$ *lhc at* x

3. γ_i *closed at* $x\ \forall\, i \Rightarrow \prod_i \gamma_i$ *is closed at* x

4. γ_i *has open graph* $\forall\, i \Rightarrow \prod_i \gamma_i$ *has open graph*

Proof. 1.) and 2.) follow directly from Proposition 3.2.2 (sequential characterization of hemi continuity) and the fact that a sequence in a product space converges if and only if all its component sequences converge. 3.) and 4.) are clear. $\square$

Definition 3.2.13. *Let* $Y_i \subset \mathbb{R}^k$ *for* $i = 1, .., k$ *and* $\gamma_i : X \to\to Y_i$ *be correspondences. Then*

$$\begin{aligned} \sum_i \gamma_i : X &\to\to \sum_i Y_i := \{\sum_i y_i : y_i \in Y_i\} \\ x &\mapsto \sum_i \gamma_i(x). \end{aligned}$$

Proposition 3.2.9. *Let $\gamma_i : X \to\to Y_i$ be as above.*

1. *γ_i uhc and γ_i compact valued $\forall i \Rightarrow \sum_i \gamma_i$ uhc and compact valued.*
2. *γ_i lhc $\forall i \Rightarrow \sum_i \gamma_i$ lhc*
3. *γ_i has open graph $\forall i \Rightarrow \sum_i \gamma_i$ has open graph*

Proof. Follows again from Proposition 3.3.2. $\square$

Definition 3.2.14. *(* **Convex Hull of a Correspondence** *) Let $\gamma : X \to\to Y$ be a correspondence and Y be convex. then we define the* **convex hull** *of γ as*

$$\begin{aligned} co(\gamma) : X &\to\to Y \\ x &\mapsto co(\gamma(x)). \end{aligned}$$

Proposition 3.2.10. *Let $\gamma : X \to\to Y$ be a correspondence and Y be convex. Then*

1. *γ uhc at x and compact valued $\Rightarrow co(\gamma)$ is uhc at x*
2. *γ lhc at $x \Rightarrow co(\gamma)$ is lhc at x*
3. *γ has open graph $\Rightarrow co(\gamma)$ has open graph.*

Proof. Exercise ! $\square$

Proposition 3.2.11. *Let $X \subset \mathbb{R}^m$,$Y \subset \mathbb{R}^k$ and F be a polytope. If $\gamma : X \to\to Y$ has open sections, then $co(\gamma)$ has open graph.*

3.3 Abstract Economies and the Walras Equilibrium

In this chapter we build up a basic mathematical model for an Economy. As mentioned before, such models can sometimes fail to be exact replicas of the reality. However they are good to get theoretical insight into how the reality works.

We think of an economy where we have m commodities. Commodities can be products like oil, water, bread but also services like teaching, health care etc. We let

$$\mathbb{R}^m := \text{ "Commodity Space"}$$

be the space which models our commodities. A **commodity vector** is a vector in this space. Such a vector $(x_1, .., x_m)$ stands for x_1 units of the first commodity, x_2 units of the second commodity etc. . In our economy commodity vectors are exchanged (traded), manufactured and consumed in the course of economic activity. A **price** vector

$$p = \begin{pmatrix} p_1 \\ \cdot \\ \cdot \\ p_m \end{pmatrix}$$

associates prices to each commodity. More precisely p_i denotes the price of one unit of the i-th commodity. We assume $p_i \geq 0$ for all i. The price of the commodity vector $x = (x_1, .., x_m)$ the computes as

$$p(x) = \sum_{i=1}^{m} x_i \cdot p_i =< p, x > .$$

We assume that all prices are positive i.e. $p_i \geq 0$. One can interpret p as an element in the dual of the commodity space $L(\mathbb{R}^m, \mathbb{R})$. In the situation where the commodity space is finite dimensional (as in this course) this is not so much of use, it is very helpful though if one studies infinite economies where the commodity space is an infinite dimensional Hilbert-space or even more general.

As participants in our economic model we have **consumers**, **suppliers**

(also called **producers**) and sometimes **auctioneers**. Auctioneers determine the prices, one can think of a higher power like government or trade organization but sometimes they are just an artificial construct in the same way as in games with a random element, where one considers nature as an additional player. We will see later how this can be realized. The ultimate purpose of the economic organization is to provide commodity vectors for final consumption by the consumers. It is reasonable to assume that not every consumer can consume every commodity and every supplier can produce every commodity. For this reason we model **consumption sets** resp. **production sets** for each individual consumer resp. produce as

$$\begin{array}{ll} X_i \subset \mathbb{R}^m & \text{consumption set for consumer "i"} \\ Y_j \subset \mathbb{R}^m & \text{production set for consumer "j"} \end{array}$$

where we assume that we have n consumers and k suppliers in our economy and $i \in \{1, .., n\}$ as well as $j \in \{1, .., k\}$. Here X_i stands for the commodity vectors consumer "i" can consume and Y_j for the commodity vectors supplier "j" can produce.

We assume that each consumer has an initial endowment, that is a commodity vector $w_i \in \mathbb{R}^m$ he owns at initial time. Furthermore we assume that consumers have to buy their consumption at market price. Each consumer has an income at some rate which we denote with M_i. We assume that the incomes are positive i.e. $M_i \geq 0$. We assume that he cannot purchase more then his income, that is he cannot take any credit. This determines the **budget set** for player i.

$$\{x \in X_i : p(x) \leq M_i\}$$

These budget sets depend on the two parameters price p and income M_i and hence can be interpreted as correspondences

$$\begin{aligned} b_i : \mathbb{R}_+^m \times \mathbb{R}_+ &\to\to X_i \\ (p, M_i) &\mapsto \{x \in X_i | p(x) \leq M_i\} \end{aligned}$$

We do now come to a very important point in game theory and mathematical economics, so called **utility**. Utility stands for the personal gain some individual has by the outcome of some event, let it be a game, an investment at the stock-market or something similar. Often the approach is to model utility in numbers. In these approaches one usually has a utility function u which associates to the outcome a real number and outcome 1 is preferable to outcome 2 if it has a higher utility. The problem in this approach though is, that it is often very difficult to measure utility in numbers. What does it mean that outcome 2 has half the utility of outcome 1, is two times outcome 2 as good as one times outcome 1 ? However given to outcomes one can always decide which of the two one prefers. This leads one to model utility as a correspondence as we do here. We assume that each consumer has a utility correspondence

$$\begin{aligned} U_i : X_i &\to\to X_i \\ x &\mapsto \{y \in X_i : \text{ consumer "i" prefers } y \text{ to } x\} \end{aligned}$$

Here the word "prefers" is meant in the strict sense. In case where one has indeed a utility function $u_i : X_i \to \mathbb{R}$ one gets the utility correspondence as

$$U_i(x) = \{y \in X_i : u_i(y) > u_i(x)\}.$$

In our economy each consumer wants to maximize his utility, i.e. find $x \in b_i(p, M_i)$ such that

$$U_i(x) \cap b_i(p, M_i) = \emptyset.$$

Such an x is called a **demand vector** and is a solution to the consumers problem given prices p and income M_i. Since we interpret p and M_i as parameters this gives us another correspondence, the so called **demand correspondence** for consumer "i"

$$\begin{aligned} d_i : \mathbb{R}^m_+ \times \mathbb{R}_+ &\rightarrow\rightarrow X_i \\ (p, M_i) &\mapsto \{\text{ demand vectors for consumer "i" given prices } p \\ &\quad \text{and income } M_i\}. \end{aligned}$$

A supply vector[1] $y \in Y_j$ for supplier "j" specifies the quantities of each commodity supplied (positive entry) and the amount of each commodity used as an input (negative entry). The **profit** or **net income** associated with the supply vector $y = (y_1, .., y_m)$ given prices p is

$$p(y) = \sum_{i=1}^{m} p_i \cdot y_i =< p, y > .$$

The set of profit maximizing supply vectors is called the **supply set**. It depends of course on the prices p and therefore is considered as the so called **supply correspondence**

$$\begin{aligned} s_j : \mathbb{R}^m_+ &\rightarrow\rightarrow Y_j \\ p &\mapsto \{\text{ profit maximizing supply vectors for supplier "j",} \\ &\quad \text{given prices } p\}. \end{aligned}$$

In the so called **Walras Economy** which we will later study in detail, it is assumed that the consumers share some part of the profit of the suppliers as their income.[2] Let α^i_j denote consumer "i"'s share of the profit of supplier j. If supplier "j" produces y_j and prices are p, then the budget set for consumer "i" has the form

[1]commodity vectors in the context of a supplier are also called supply vectors
[2]this can be through wages, dividends etc.

$$\{x \in X_i : p(x) \leq p(w_i) + \alpha^i_j p(y_j)\}$$

The set

$$E(p) = \{\sum_{i=1}^{n} x_i - \sum_{j=1}^{k} y_j : x_i \in d_i(p, p(w_i) + \alpha^i_j p(y_j)), y_j \in s_j(p)\}$$

is called the **excess demand set**.Since it depends on p it is naturally to consider it as a correspondence the so called **excess demand correspondence**

$$\begin{aligned} E : \mathbb{R}^m_+ &\to\to \mathbb{R}^m \\ p &\to E(p). \end{aligned}$$

It would be a very good thing for the economy if the zero vector belongs to $E(p)$. In fact this means that there is a combination of demand and supply vectors which add up to zero in the way indicated above. This however means that the suppliers produce exactly the amount of commodities the consumer want to consume and furthermore the suppliers make maximum profit. A price vector p which satisfies $0 \in E(p)$ is called a **Walrasian equilibrium**. The question of course is, does such an equilibrium always exists ? We will answer this question later. As in the case of the non cooperative equilibrium for two player games in chapter 2, this has to do with fixed points. But this time fixed points of correspondences. Let us briefly illustrate why. Instead of the excess demand correspondence E one can consider the correspondence

$$\begin{aligned} \tilde{E} : \mathbb{R}^m_+ &\to\to \mathbb{R}^m \\ p &\mapsto\mapsto p + E(p). \end{aligned}$$

Then p is a Walrasian equilibrium if and only if $p \in \tilde{E}(p)$ which is the case if and only if p is a fixed point of the correspondence $\tilde{E}$. There is a slightly more general definition of a Walrasian equilibrium the so called **Walrasian free disposal equilibrium**. We will later give the precise definition and a result about when such an equilibrium exist. However before we can prove this result we need a result for correspondences which corresponds to the Brouwer fixed point theorem in the case of maps. This needs preparation and some further studies on correspondences, which will follow in the next two sections.

3.4 The Maximum Theorem for Correspondences

In the last section we learned about some correspondences which naturally occur in the framework of mathematical economics.To do analysis with those correspondences on has to know that these correspondences have some analytic properties like upper hemi continuity, lower hemi continuity or open graphs etc. This section gives the theoretical background for this.

We studied for example the budget correspondence for consumer "i" :

$$\begin{array}{rcl} b_i : \mathbb{R}_+^m \times \mathbb{R}_+ & \rightarrow\rightarrow & X_i \\ (p, M_i) & \mapsto & \{x \in X_i | p(x) \leq M_i\} \end{array}$$

The proof of the following proposition is easy and is left as an exercise :

Proposition 3.4.1. *Let $X_i \subset \mathbb{R}^m$ be closed,convex and bounded from below. Then the budget correspondence b_i as defined above is uhc and furthermore, if there exist $x \in X_i$ such that $p(x) < M_i$, then b_i is lhc at*

(p, M_i).

It is however more difficult to see that the demand correspondences

$$\begin{aligned} d_i : \mathbb{R}^m_+ \times \mathbb{R}_+ &\to\to X_i \\ (p, M_i) &\mapsto \{ \text{ demand vectors for consumer "i" given prices } p \\ &\quad \text{and income } M_i\}. \end{aligned}$$

have similar analytic properties. The reason for this is, that the demand correspondences are the result of some optimization problem (demand vectors are vectors in the budget set, which have maximum utility). The following Theorems will help us to come over this problem.

Theorem 3.4.1. *(***Maximum Theorem 1***) Let $G \subset \mathbb{R}^m$,$Y \subset \mathbb{R}^k$ and $\gamma : G \to\to Y$ be a compact valued correspondence. Let $f : Y \to \mathbb{R}$ be a continuous function. Define*

$$\begin{aligned} \mu : G &\to\to Y \\ x &\mapsto \{y \in \gamma(x) | y \textit{ maximizes } f \textit{ on } \gamma(x)\} \end{aligned}$$

and $F : G \to \mathbb{R}$ with $F(x) = f(y)$ for $y \in \mu(x)$. If γ is continuous at x, then μ is closed at x as well as uhc at x and F is continuous at x. Furthermore μ is compact valued.

Proof. Since γ is compact valued we have that $\mu(x) \neq \emptyset$ for all x. Furthermore $\mu(x)$ is closed and therefore compact (since it is a closed subset of a compact set).Let us first show that μ is closed at x. For this let $x^n \to x$ and $y^n \in \mu(x^n)$ with $y^n \to y$. We have to show that $y \in \mu(x)$. Suppose $y \notin \mu(x)$. Since γ is uhc and compact valued it follows from Proposition 3.2.1 (first part) that γ is closed at x and therefore $y \in \gamma(x)$. $y \notin \mu(x)$ however implies that there exists $z \in \gamma(x)$ s.t. $f(z) > f(y)$. Since γ is also lhc at x it follows from Proposition 3.2.2 (second part) that there exists a sequence $z^n \to z$ with $z^n \in \gamma(x^n)$. Since $y^n \in \mu(x^n)$ we have $f(z^n) \le f(y^n)$ for all n. Since f is continuous this implies that

$$f(z) = \lim_n f(z^n) \le \lim_n f(x^n) = f(x)$$

which is in contradiction to $f(z) > f(y)$. So we must have $y \in \mu(x)$ and therefore that μ is closed at x. Clearly

$$\lim_n F(x^n) = \lim_n f(y^n) = f(y) = F(x)$$

which shows that F is continuous at x. Now since $\mu = \gamma \cap \mu$ Proposition 3.2.6 (second part) implies that μ is uhc at x. □

This result corresponds to the case where the utility correspondence is in fact induced by a utility function. The following theorem corresponds to the case, where the utility correspondence is in fact an utter correspondence.

Theorem 3.4.2. *(***Maximum Theorem 2***) Let $G \subset \mathbb{R}^m$,$Y \subset \mathbb{R}^k$ and $\gamma : G \to\to Y$ a compact valued correspondence. Furthermore let $U : Y \times G \to\to Y$ have open graph. Define*

$$\begin{aligned} \mu : G &\to\to Y \\ x &\mapsto \{y \in \gamma(x) | U(y,x) \cap \gamma(x) = \emptyset\}. \end{aligned}$$

If γ is closed and lhc at x then μ is closed at x. if in addition γ is uhc at x, then μ is uhc at x. Furthermore μ is compact valued (possibly empty).

Proof. Let $x^n \to x$,$y^n \in \mu(x^n)$ with $y^n \to y$. In order to show that μ is closed at x we have to show that $y \in \mu(x)$. Assume this would not be the case, i.e. $y \notin \mu(x)$. Since $y^b \in \mu(x^n) \subset \gamma(x^n)$ and γ is closed at x we have that $y \in \gamma(x)$. However since $y \notin \mu(x)$ there must exist $z \in U(y,x) \cap \gamma(x)$. Since γ is lhc at x it follows from Proposition 3.2.2 that there exists a sequence $z^n \to z$ s.t. $z^n \in \gamma(x^n)$. Then $\lim_n (y^n, x^n, z^n) = (y, x, z) \in Gr(U)$

(since $z \in U(y,x)$). Since $Gr(U)$ is open there must exist $n_0 \in \mathbb{N}$ s.t. $(y^{n_0}, x^{n_0}, z^{n_0}) \in Gr(U)$. This however means that

$$z^{n_0} \in U(y^{n_0}, x^{n_0}) \cap \gamma(x^{n_0})$$

and therefore $y^{n_0} \notin \mu(x^{n_0})$ which is of course a contradiction. Therefore $y \in \mu(x)$ and μ is closed at x. If in addition γ is also uhc at x, then as in the previous proof Proposition 3.2.6 (second part) implies that $\mu = \gamma \cap \mu$ is uhc at x. Finally we show that μ is compact valued. Since $\mu(\tilde{x}) \subset \gamma(\tilde{x})$ for all $\tilde{x}$ and γ is compact valued it is enough to show that $\mu(\tilde{x})$ is closed in $\gamma(\tilde{x})$ (in the relative topology). Equivalently we can show that $\gamma(\tilde{x}) \setminus \mu(\tilde{x})$ is open in $\gamma(\tilde{x})$. Assume this would not be the case. Then there would exist $\tilde{y} \in \gamma(\tilde{x})$ and $z \in U(\tilde{y}, \tilde{x}) \cap \gamma(\tilde{x}) \neq \emptyset$ (i.e. $\tilde{y} \notin \mu(\tilde{x}$) as well as a sequence $y^n \in \mu(\tilde{x})$ s.t. $\lim_n y^n = \tilde{y}$ and

$$U(y^n, \tilde{x}) \cap \gamma(\tilde{x}) = \emptyset \ , \ \forall \, n.$$

Clearly $\lim_n (y^n, \tilde{x}, z) = (\tilde{y}, \tilde{x}, z) \in Gr(U)$. Since $Gr(U)$ is open there must exist $n_0 \in \mathbb{N}$ s.t. $(y^{n_0}, \tilde{x}, z) \in Gr(U)$ which implies that

$$z \in U(y^{n_0}, \tilde{x}) \cap \gamma(\tilde{x}).$$

This however contradicts the previous equation and so $\mu(\tilde{x})$ must be closed in $\gamma(\tilde{x})$. □

Proposition 3.4.2. *Let $G \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^k$ and let $U : G \times Y \to\to Y$ satisfy the following condition :*

$$z \in U(y,x) \ \Rightarrow \ \exists z' \in U(y,x) \textit{ s.t. } (y,x) \in (U^{-}[\{z'\}])^{\circ}.$$

We define $\mu : G \to\to Y$ via $\mu(x) = \{y \in Y : U(y,x) = \emptyset\}$. Then μ is closed.

Proof. Let $x^n \to x$,$y^n \in \mu(x^n)$ and $y^n \to y$. Suppose $y \notin \mu(x)$. Then there exists $z \in U(y,x)$ and by the hypothesis there exists z' s.t.

$$(y,x) \in (U^{-}[\{z'\}])^{\circ} = \{(\tilde{y},\tilde{x}) | \underbrace{U(\tilde{y},\tilde{x}) \cap \{z'\} \neq \emptyset}_{z' \in U(\tilde{y},\tilde{x})}\}^{\circ}$$

Since $\lim_n (y^n, x^n) = (y,x)$ there mist exist $n_0 \in \mathbb{N}$ s.t. $(y^{n_0}, x^{n_0}) \in (U^{-}[\{z'\}])^{\circ}$ which implies that $z' \in U(y^{n_0}, x^{n_0})$. This however is a contradiction to $y^{n_0} \in \mu(x^{n_0})$. □

Theorem 3.4.3. *Let $X_i \subset \mathbb{R}^{k_i}$ for $i = 1,..,n$ be compact and set $X = \prod_{i=1}^{n} X_i$. Let $G \subset \mathbb{R}^k$ and for each i let $S_i : X \times G \rightarrow\rightarrow X_i$ be a continuous correspondence with compact values. Furthermore let $U_i : X \times G \rightarrow\rightarrow X_i$ be correspondences with open graph. Define*

$$\begin{aligned} E : G &\rightarrow\rightarrow X \\ g &\mapsto \{x = (x_1,..,x_n) \in X : \textit{ for each } i \; x_i \in S_i(x,g) \\ &\qquad \textit{and } U_i(x,g) \cap S_i(x,g) = \emptyset\}. \end{aligned}$$

Then E has compact values, is closed and uhc.

Proof. Let us first show that E is closed at all x. This is equivalent to showing that E has closed graph. Suppose $(g,x) \notin Gr(E)$ i.e. $x \notin E(g)$. Then for some i either $x_i \notin S_i(x,g)$ or $U_i(x,g) \cap S_i(x,g) \neq \emptyset$. By Proposition 3.2.1 (first part) S_i is closed. So in the first case there exists a neighborhood V in $X \times G \times X_i$ of $(x,g,x_i) \notin Gr(S_i)$ which is disjoint from $Gr(S_i)$. Then $\tilde{V} = \{(g,x) : (x,g,x_i) \in V\}$ is an open neighborhood of (g,x) in $G \times X$ which cannot intersect $Gr(E)$ since for all $(g,x) \in \tilde{V}$ we have $x_i \notin S_i(x,g)$. In the second case there must exist i and $z_i \in U_i(x,g) \cap S_i(x,g)$. Since U_i has open graph there exist neighborhoods V of z_i and W_1 of (x,g) s.t. $W_1 \times V \subset Gr(U_i)$. Since S_i is also lhc there exists a neighborhood W_2 of (x,g) s.t. $(\tilde{x},\tilde{g}) \in W_2$ implies that $S_i(\tilde{x},\tilde{g}) \cap V \neq \emptyset$.[3] Then $W_1 \cap W_2$ is a neighborhood of (x,g) disjoint from $Gr(E)$, since for all $(\tilde{x},\tilde{g}) \in W_1 \cap W_2$ we have $U_i(\tilde{x},\tilde{g}) \cap S_i(\tilde{x},\tilde{g}) \neq \emptyset$ for the simple reason that $U_i(\tilde{x},\tilde{g})$ contains V. Since in both cases we can

[3] V is an open neighborhood of z_i, then definition lhc

find neighborhoods of (g, x) which are still contained in $Gr(E)^c$ we have that $Gr(E)^c$ is open and hence $Gr(E)$ is closed. It follows now from the compactness of X and the closedness of E as well as Proposition 3.2.1 (second part) that E is uhc. That it has compact values is clear. $\square$

Proposition 3.4.3. *Let $K \subset \mathbb{R}^m$ be compact, $G \subset \mathbb{R}^k$ and let $\gamma : K \times G \to\to K$ be closed correspondence. Define*

$$\begin{aligned} F : G &\to\to K \\ g &\mapsto \{x \in K : x \in \gamma(x, g)\}. \end{aligned}$$

Then $F : G \to\to K$ has compact values, is closed and uhc.

Proof. By Proposition 3.2.1 (second part) it is enough to show that F is closed, but this follows immediately from the closedness of γ. $\square$

Proposition 3.4.4. *Let $K \subset \mathbb{R}^m$ be compact, $G \subset \mathbb{R}^k$ and let $\gamma : K \times G \to\to \mathbb{R}^m$ have compact values and uhc. Define*

$$\begin{aligned} Z : G &\to\to K \\ g &\mapsto \{x \in K : 0 \in \gamma(x, g)\}. \end{aligned}$$

Then Z has compact values, is closed and uhc.

Proof. Exercise ! $\square$

3.5 Approximation of Correspondences

Lemma 3.5.1. *Let $\gamma : X \to\to Y$ be an uhc correspondence with nonempty convex values. Furthermore let $X \subset \mathbb{R}^m$ be compact and $Y \subset \mathbb{R}^k$ be convex. For each $\delta > 0$ define a correspondence*

$$\begin{aligned}\gamma^\delta : X &\to\to Y\\ x &\mapsto co(\bigcup_{z\in B_\delta(x)} \gamma(z)).\end{aligned}$$

Then for every $\epsilon > 0$ there exists $\delta > 0$ s.t.

$$Gr(\gamma^\delta) \subset B_\epsilon(Gr(\gamma))$$

where $B_\epsilon(Gr(\gamma))$ denotes the points in $X \times Y \subset \mathbb{R}^m \times \mathbb{R}^k$ which have (Euclidean) distance from $Gr(\gamma)$ less than ϵ.

Proof. Exercise ! □

In the proof of the next lemma we need a technique from topology called **partition of unity**. This technique is very helpful in general, however we don't give a proof here.

Proposition 3.5.1. *Let X be a topological space and $(U_i)_{i\in I}$ be an open covering of X that is each U_i is an open set and $X = \bigcup_{i\in I} U_i$ then there exists a locally finite subordinated partition of unity to this covering, that is a family of functions $f^i : X \to [0,\infty)$ s.t. for all $x \in X$ we have $f^i(x) > 0$ only for finitely many $i \in I$, $\sum_{i\in I} f_i \equiv 1$ and $supp(f^i) = \overline{\{x|f^i(x) > 0\}} \subset U_i$.*

Theorem 3.5.1. *(**von Neumann Approximation Theorem**) Let $\gamma : X \to\to Y$ be uhc with nonempty, compact and convex values. Then for any $\epsilon > 0$ there is a continuous map $f : X \to Y$ s.t.*

$$Gr(f) \subset B_\epsilon(Gr(\gamma)).$$

Proof. Let $\epsilon > 0$. By Lemma 3.5.1 there exists $\delta > 0$ s.t. $Gr(\gamma^\delta) \subset B_\epsilon(Gr(\gamma))$. Since X is compact there exist $x_1, .., x_n$ s.t. $X \subset \bigcup_{i=1}^n B_\delta(x_i)$. Choose $y^i \in \gamma(x^i)$. Let $f^1, .., f^n$ be a locally finite partition of unity subordinated to this covering. Then we have $supp(f^i) \subset B_\delta(x_i)$ and $\sum_{i=1}^n f_i \equiv 1$. We define the function f as follows :

$$\begin{array}{rcl} f : X & \to & Y \\ x & \mapsto & \sum_{i=1}^{n} f^i(x) y^i. \end{array}$$

Clearly f is continuous. Since $f^i(x) = 0$ for all $x \notin B_\delta(x_i)$ each $f(x)$ is a convex linear combination of those y^i such that $x \in B_\delta(x_i)$. Since $x \in B_\delta(x_i)$ clearly implies $x^i \in B_\delta(x)$ and therefore $y^i \in \bigcup_{z \in B_\delta(x)} \gamma(z)$ we have

$$f(x) \in co(y^i | x^i \in B_\delta(x)) \subset co(\bigcup_{z \in B_\delta(x)} \gamma(z)) = \gamma^\delta(x).$$

Hence for all $x \in X$ we have $(x, f(x)) \in Gr(\gamma^\delta) \subset B_\epsilon(Gr(\gamma))$. $\square$

Definition 3.5.1. *Let $\gamma : X \to\to Y$ be a correspondence. A selection of γ is a function $f : X \to\to Y$ s.t. $Gr(f) \subset Gr(\gamma)$, i.e. $f(x) \in \gamma(x)$ for all $x \in X$.*

In the previous proof we constructed a continuous selection for the correspondence γ^δ (in fact for $\delta > 0$ but arbitrary small). We will now show that under different assumptions we can do even better. From now on in this chapter all correspondences are assumed to be non empty valued.

Theorem 3.5.2. *(* **Browder** *) Let $X \subset \mathbb{R}^m$ and $\gamma : X \to\to \mathbb{R}^k$ be convex valued s.t. for all $y \in \mathbb{R}^k$ the sets $\gamma^{-1}(y) = \{x \in X | y \in \gamma(x)\}$ are open. Then there exists a continuous selection of γ.*

Proof. Clearly $X = \bigcup_{y \in \mathbb{R}^k \gamma^{-1}(y)}$ so that $(\gamma^{-1}(y))_{y \in \mathbb{R}^k}$ is an open covering of X. Let $f_y : X \to [0, \infty)$ denote the maps belonging to a locally finite subordinated partition of unity, so $\sum_{y \in \mathbb{R}^k} f_y \equiv 1$ and $supp(f_y) \subset \gamma^{-1}(y)$. We define the map f as follows :

$$
\begin{aligned}
f : X &\rightarrow \mathbb{R}^k \\
x &\mapsto \sum_y f_y(x)y.
\end{aligned}
$$

Then f is continuous and for each $x \in X$ $f(x)$ is a convex combination of those y s.t. $f_y(x) > 0$ which can only hold if $y \in \gamma(x)$. Since $\gamma(x)$ is convex this implies that $f(x) \in \gamma(x)$. $\square$

We state the following proposition without proof (due to time constraints, it is not more difficult to prove than those before).

Proposition 3.5.2. *Let $X \subset \mathbb{R}^m$ be compact and $\gamma : X \rightarrow\rightarrow \mathbb{R}^k$ be lhc with closed and convex values. Then there exists a continuous selection of γ.*

3.6 Fixed Point Theorems for Correspondences

One can interpret Brouwer's fixed point theorem as a special case of a fixed point theorem for correspondences where the correspondence is in fact given by a map. That this is not the only case where fixed points of correspondences are guaranteed is shown in this section. The main fixed point theorem for correspondences is the Kakutani fixed point theorem. It will follow from the following theorem.

Theorem 3.6.1. *Let $K \subset \mathbb{R}^m$ be compact, nonempty and convex and $\mu : K \rightarrow\rightarrow K$ a correspondence. Suppose there is a closed correspondence $\gamma : K \rightarrow\rightarrow Y$ with nonempty, compact and convex values where $Y \subset \mathbb{R}^k$ is also compact and convex. Furthermore assume that there exists a continuous map $f : K \times Y \rightarrow K$ s.t. for all $x \in K$ one has*

$$\mu(x) = \{f(x, y) : y \in \gamma(x)\}.$$

Then μ has a fixed point, i.e. there exists $\tilde{x} \in K$ s.t. $\tilde{x} \in \mu(\tilde{x})$.

Proof. By Theorem 3.5.1 there exists a sequence of continuous maps $g^n : K \to Y$ s.t.

$$Gr(g^n) \subset B_{\frac{1}{n}}(Gr(\gamma)).$$

We define maps h^n as follows :

$$\begin{aligned} h^n : K &\to K \\ x &\mapsto f(x, g^n(x)). \end{aligned}$$

It follows from Theorem 2.3.1 (Brouwer's fixed point theorem) that each h^n has a fixed point $x^n \in K$, i.e. a point x^n which satisfies

$$x^n = h^n(x^n) = f(x^n, g^n(x^n)).$$

Since K as well as Y are compact we can extract a convergent subsequence of (x^n) as well as $(g^n(x^n))$ and w.l.o.g. we can as well assume that these two sequences already converge and $\tilde{x} := \lim_n x^n$ as well as $\tilde{y} := \lim f^n(x^n)$. By the continuity of f we have

$$\tilde{x} = f(\tilde{x}, \tilde{y}.$$

Furthermore since γ is closed $Gr(\gamma)$ is closed and for all n we have $(x^n, g^n(x^n)) \in B_{\frac{1}{n}}(Gr(\gamma))$. Therefore $(\tilde{x}, \tilde{y}) = \lim_n (x^n g^n(x^n))$ must lie in $Gr(\gamma)$ and therefore $\tilde{y} \in \gamma(\tilde{x})$. By the assumption on the correspondence μ we have $\tilde{x} = f(\tilde{x}, \tilde{y} \in \mu(\tilde{x})$ so that $\tilde{x}$ is a fixed point of μ. $\square$

Theorem 3.6.2. *(* **Kakutani Fixed Point Theorem** *) Let $K \subset \mathbb{R}^m$ be compact and convex and $\gamma : K \to\to K$ be closed or uhc with nonempty convex and compact values. Then γ has a fixed point.*

Proof. If γ is uhc then by Proposition 3.2.1 (first part) γ is also closed, so we can just assume that γ is closed. We can then apply the previous theorem on $\mu = \gamma$ and f defined by

$$
\begin{aligned}
f : K \times K &\rightarrow K \\
(x, y) &\mapsto y.
\end{aligned}
$$

Then clearly $\mu(x) = \gamma(x) = \{y = f(x, y) | y \in \gamma(x)\}$ and therefore γ has a fixed point. $\square$

We come to another fixed point theorem, which works in the setup where the correspondence is lhc.

Theorem 3.6.3. *Let $K \subset \mathbb{R}^m$ be compact and convex and let $\gamma : K \rightarrow\rightarrow K$ be lhc with closed and convex values. Then γ has a fixed point.*

Proof. By Proposition 3.5.2 there exists a continuous selection $f : K \rightarrow K$ s.t. $f(x) \in \gamma(x)$ for all $x \in X$. Applying Theorem 2.3.1 (Brouwer's fixed point theorem) again we see that f has a fixed point, i.e. there exists $\tilde{x}$ s.t. $\tilde{x} = f(\tilde{x}) \in \gamma(\tilde{x})$. Clearly $\tilde{x})$ is also a fixed point for γ. $\square$

Theorem 3.6.4. *(**Browder**) Let $K \subset \mathbb{R}^m$ be compact and convex and let $\gamma : K \rightarrow\rightarrow K$ with nonempty convex values s.t. $\gamma^{-1}(y)$ is open for all $y \in K$. Then γ has a fixed point.*

Proof. Follows in the same way as in the previous proof by application of Theorem 3.5.2 and Brouwer's fixed point theorem. $\square$

Lemma 3.6.1. *Let $X \subset \mathbb{R}^k$ be nonempty, compact and convex and let $U : X \rightarrow\rightarrow X$ be a convex valued correspondence s.t.*

1. *$x \notin U(x)$ for all $x \in X$*
2. *$U^{-1}(x) = \{x' \in X | x \in U(x')\}$ is open in X for all $x \in X$.*

Then there exists $\overline{x} \in X$ s.t. $U(\overline{x}) = \emptyset$.

Proof. Let us define $W := \{x \in X | U(x) \neq \emptyset\}$. If $x \in W$ then there exists $y \in U(x)$ and since $x \in U^{-1}(y) \subset W$ by assumption 2.) $U^{-1}(y)$ is an open neighborhood of x in W. Therefore W is an open subset of X. We consider the restriction of U to W i.e.

$$U_{|W} : W \quad \rightarrow\rightarrow \quad X \subset \mathbb{R}^k$$

This correspondence satisfies the assumptions in the Browder Selection Theorem (Theorem 3.5.2) and therefore admits a continuous selection

$$f : W \rightarrow \mathbb{R}^k$$

which by definition of a selection has the property that $f(x) \in U(x)$ for all $x \in W$. We define a new correspondence as follows :

$$\begin{array}{ccc} \gamma : X & \rightarrow\rightarrow & X \\ x & \mapsto & \begin{cases} \{f(x)\} & \text{if } x \in W \\ X & \text{if } x \notin W \end{cases} \end{array}$$

Then γ is convex and compact valued. It follows from Proposition 3.2.1 second part that uhc if we can show that γ is closed. To show this let $x^n \rightarrow x$,$y^n \in \gamma(x^n)$ and $y^n \rightarrow y$ then we must show $y \in \gamma(x)$. If $x \notin W$ this is clear since then $\gamma(x) = X$. If however $x \in W$ then since W is open there must exist n_0 s.t. for all $n \geq n_0$ we have $x^n \in W$. For those n we have $y^n \in \gamma(x^n) = \{f(x^n)\}$ i.e. $y^n = f(x^n)$. Now it follows from the continuity of f that

$$y = \lim_n y^n = \lim_n f(x^n) = f(x)$$

and therefore $y \in \gamma(x) = \{f(x)\}$. hence γ is uhc with nonempty, convex and compact values. By application of the Kakutani Fixed Point Theorem (Theorem 3.6.2) γ has a fixed point, that is there exists $\overline{x} \in X$ s.t. $\overline{x} \in \gamma(\overline{x})$. If $\overline{x} \in W$ then by definition of γ $\overline{x} = f(\overline{x}) \in U(\overline{x})$ which is a contradiction to assumption 1.) Therefore $\overline{x} \notin W$ and therefore by definition of W $U(\overline{x}) = \emptyset$. □

Definition 3.6.1. *Let $X \subset \mathbb{R}^k$ and $f : X \to \mathbb{R}$. We say that f is* **lower semi-continuous** *if for all $a \in \mathbb{R}$ the sets $f^{-1}((a, \infty))$ are open. We call f* **quasi concave** *if for all $a \in \mathbb{R}$ the sets $f^{-1}([a, \infty))$ are convex.*

Theorem 3.6.5. *(***Ky-Fan***) : Let $X \subset \mathbb{R}^k$ be nonempty. convex and compact and $\varphi : X \times X \to \mathbb{R}$ s.t.*

1. *$\forall y \in X$ the function $\varphi(\cdot, y)$ considered as a function in the first variable is lower semi-continuous*
2. *$\forall x \in X$ the function $\varphi(x, \cdot)$ considered as a function in the second variable is quasi concave*
3. *$\sup_{x \in X} \varphi(x, x) \leq 0$.*

Then there exists $\overline{x} \in X$ s.t. $\sup_{y \in y} \varphi(\overline{x}, y) \leq 0$ i.e. $\varphi(\overline{x}, y) \leq 0$ for all $y \in X$

Proof. Define a correspondence

$$\begin{aligned} U : X &\to\to X \\ x &\mapsto U(x) := \{y \in X | \varphi(x, y) > 0\}. \end{aligned}$$

If we can find $\overline{x} \in X$ s.t. $U(\overline{x}) = \emptyset$ then we are finished. The existence of such an $\overline{x}$ follows from Lemma 3.6.1 if we can show that U satisfies the required assumptions there. First let $y_1, y_2 \in U(x)$ for an arbitrary $x \in X$. Then

$$\epsilon := \min(\varphi(x, y_1), \varphi(x, y_2)) > 0$$

and furthermore $y_1, y_2 \in \varphi(x, \cdot)^{-1}([\epsilon, \infty))$ which by assumption 2.) is convex. Therefore for all $\lambda \in [0, 1]$ we have that $\lambda \cdot y_1 + (1 - \lambda) \cdot y_2 \in \varphi(x, \cdot)^{-1}([\epsilon, \infty))$ which implies $\varphi(x, \lambda \cdot y_1 + (1 - \lambda) \cdot y_2) > 0$ and therefore

$\lambda \cdot y_1 + (1-\lambda) \cdot y_2 \in U(x)$. hence U is convex valued. Furthermore for each $y \in X$ we have that

$$\begin{aligned} U^{-1}(y) &= \{x \in X | y \in U(x)\} \\ &= \{x \in X | \varphi(x,y) > 0\} \\ &= \varphi(\cdot, y)^{-1}((0,\infty)) \end{aligned}$$

is open for all y by assumption 1.) By Assumption 3.) we also have that $\varphi(x,x) \leq 0$ for all x which by definition of U implies that $x \notin U(x)$ for all $x \in X$. Therefore U satisfies all the condition in Lemma 3.6.1. □

We are now in the position to prove the general version of the Nash Theorem :

Proof. (of Theorem 3.1.1) : We define a map $\varphi : \mathcal{S}(N) \times \mathcal{S}(N) \to \mathbb{R}$ as follows : For $s = (s_1, .., s_n), t = (t_1, .., t_n) \in \mathcal{S}(N)$ we define

$$\varphi(s,t) := \sum_{i=1}^{n} L_i(s) - L_i(s_1, .., s_{i-1}, t_i, s_{i+1}, .., s_n).$$

It follows from the convexity of the L_i in the i-th variable that $\varphi(x, \cdot)$ is quasi concave and from the continuity of the L_i that $\varphi(\cdot, y)$ is lower semi-continuous. Since $\mathcal{S}(N)$ is also convex and compact we can apply the Ky-Fan Theorem which then implies the existence of an $\overline{s} \in X$ s.t. $\varphi(\overline{s}, t) \leq 0$ for all $t \in \mathcal{S}(N)$. In particular for all $t \in \{s_1\} \times .. \times \{s_{i-1}\} \times \mathcal{S}_i \times \{s_{i+1}\} \times .. \times \{s_n\} \cap \mathcal{S}(N)$ we have

$$
\begin{aligned}
0 \geq \varphi(\overline{s}, t) &= \sum_{j=1}^{n} L_j(\overline{s}) - L_j(\overline{s}_1, .., \overline{s}_{j-1}, t_j, \overline{s}_{j+1}, .., \overline{s}_n) \\
&= \sum_{j \neq i}^{n} L_j(\overline{s}) - L_j(\overline{s}_1, .., \overline{s}_{j-1}, t_j = s_j, \overline{s}_{j+1}, .., \overline{s}_n) \\
&+ \; L_i(\overline{s}) - L_i(\overline{s}_1, .., \overline{s}_{i-1}, t_i, \overline{s}_{i+1}, .., \overline{s}_n) \\
&= L_i(\overline{s}) - L_i(\overline{s}_1, .., \overline{s}_{i-1}, t_i, \overline{s}_{i+1}, .., \overline{s}_n).
\end{aligned}
$$

where we used that by the choice of t s and t only differ in the i-th component. This however implies that

$$
L_i(\overline{s}) \leq L_i(\overline{s}_1, .., \overline{s}_{i-1}, t_i, \overline{s}_{i+1}, .., \overline{s}_n)
$$

and shows that $\overline{s}$ is am NCE for $\mathcal{G}_n$. □

3.7 Generalized Games and an Equilibrium Theorem

As before we consider a competitive environment where n players participate and act by choosing strategies which determine an outcome. We denote by $N = \{1, .., n\}$ the set of players and for each $i \in \{1, .., n\}$ with X_i the strategy set of player i. As before let $X = \prod_{i=1}^{n} X_i$ denote the multi strategy set. We do not assume that there is some kind of multi loss operator and this is where we generalize the concept of games. Instead of a multi loss operator we assume we have n correspondences $U_i : X \rightarrow\rightarrow X_i$. We think of the correspondence as follows :

$$
\begin{aligned}
U_i(x) &= \{y_i \in X_i : (x_1, .., x_{i-1}, y_i, x_{i+1}, .., x_n) \text{ determines an outcome which} \\
&\quad\; \text{player “i” prefers to the outcome determined by } (x_1, .., x_i, .., x_n) = x\}.
\end{aligned}
$$

Note that the equation above is not a definition but an interpretation.[4] The U_i can be interpreted as utilities (compare section 3.5).[5]. Furthermore we assume we have feasibility correspondences $F_i : X \to\to X_i$ for all $1 \leq i \leq n$ where the interpretation is as follows :

$$F_i(x) \;=\; \{y_i \in X_i : (x_1, .., x_{i-1}, y_i, x_{i+1}, .., x_n) \text{ is an allowed strategy } \}.$$

Then the jointly feasible multi-strategies are the fixed points of the correspondence

$$F = \prod_{i=1}^{n} X_i \to\to \prod_{i=1}^{n} X_i.$$

This correspondence generalizes the set $\mathcal{S}(N)$ in Definition 3.1.1.

Definition 3.7.1. *A* **generalized game** *(sometimes also called* **abstract economy** *) is a quadruple $(N, (X_i), (F_i), (U_i))$ where X_i, F_i, U_i are as above.*

Though the situation is much more general now, it is not harder, in fact even more natural to define what an equilibrium of a generalized game should be.

Definition 3.7.2. *A* **non cooperative equilibrium** *(short NCE) of a generalized game $(N, (X_i), (F_i), (U_i))$ is a multi-strategy $x \in X$ s.t.$x \in F(x)$ and*

$$U_i(x) \cap F_i(X) = \emptyset \;, \forall\, i.$$

The emptiness of the intersection above means that for player i, given the strategies of the other players there is no better (feasible or allowed) strategy. So in the sense of Nash, within a NCE none of the players has reason to deteriorate from his strategy. We have the

[4] The word "prefer" is always meant in the sense "strictly prefer".

[5] The case of a classic game where a multi-loss operator is given fits into this concept by setting $U_i(x) = \{y_i \in X_i : L_i(x_1, .., x_{i-1}, y_i, x_{i+1}, .., x_n) < L_i(x_1, .., x_i, .., x_n)\}$

following theorem which states the existence of such equilibria under certain assumptions on the correspondences used in the game.

Theorem 3.7.1. *(Shafer, Sonnenschein 1975) Let* $\mathcal{G} = (N, (X_i), (F_i), (U_i))$ *be a generalized game s.t. for all* i *:*

1. $X_i \subset \mathbb{R}^{k_i}$ *is nonempty, compact and convex.*
2. F_i *is continuous with nonempty, compact and convex values*
3. $Gr(U_i)$ *is open in* $X \times X_i$
4. $x_i \notin co(U_i(x))$ *for all* $x \in X$.

Then there exists a non cooperative equilibrium for $\mathcal{G}$.

Proof. Let us define for each i a map

$$\begin{aligned} \nu_i : X \times X_i &\rightarrow \mathbb{R}_+ \\ (x, y_i) &\mapsto dist((x, y_i), Gr(U_i)^c). \end{aligned}$$

Since $Gr(U_i)^c$ is closed we have

$$\nu_i(x, y_i) > 0 \Leftrightarrow y_i \in U_i(x).$$

Clearly the maps ν_i are continuous. Furthermore we define

$$\begin{aligned} H_i : X &\rightarrow\rightarrow X_i \\ x &\mapsto \{y_i \in F_i(x) : y_i \text{ maximizes } \nu_i(x, \cdot) \text{ on } F_i(x)\}. \end{aligned}$$

Since the correspondences F_i are compact valued by setting $\gamma : X \rightarrow\rightarrow X \times X_i, x \mapsto (\{x\} \times F_i(x)$ we are in the situation of the Maximum Theorem 1 (Proposition 3.4.1) where we chose $G = X$,$Y = X \times X_i$ and $f = \nu_i$. Then the Maximum Theorem says the correspondence

$$\begin{aligned}\mu(x) &= \{z \in \gamma(z) : z \text{ maximizes } \nu_i \text{ on } \gamma(z)\} \\ &= \{(x, y_i) : y_i \in F_i(x) \text{ and } y_i \text{ maximizes } \nu_i(x, \cdot) \text{ on } F_i(x)\}\end{aligned}$$

is uhc. H_i is just the composition of the correspondence μ_i with the correspondence which is given by the continuous map $pr : X \times X_i \to X, (x, y_i) \mapsto y_i$ and is therefore uhc by Proposition 3.2.7. let us define now another correspondence

$$\begin{aligned} G : X &\to\to X \\ x &\mapsto \prod_{i=1}^{N} co(H_i(x)). \end{aligned}$$

Then by Proposition 3.2.8 and Proposition 3.2.10 the correspondence G is uhc. Since furthermore X is compact and convex we are in the situation where we can apply the Kakutani Fixed Point Theorem (Theorem 3.6.2). Therefore there exists $\tilde{x} \in X$ s.t. $\tilde{x} \in G(\tilde{x})$. Since $H_i(\tilde{x}) \subset F_i(\tilde{x})$ and $F_i(\tilde{x})$ is convex, we have that

$$\tilde{x}_i \in G_i(\tilde{x}) = co(H_i(\tilde{x})) \subset F_i(\tilde{x}).$$

Since this holds for all i we have that $\tilde{x} \in F(\tilde{x})$, so that $\tilde{x}$ is a jointly feasible strategy. Let us now show that $U_i(\tilde{x}) \cap F_i(\tilde{x}) = \emptyset$ for all i. Suppose this would not be the case. Then there would exist an i and $z_i \in U_i(\tilde{x}) \cap F_i(\tilde{x})$. Then since $z_i \in U_i(\tilde{x})$ it follows from above that $\nu_i(\tilde{x}, z_i) > 0$. Since furthermore $H_i(\tilde{x})$ consists of the maximizers of $\nu_i(\tilde{x}, \cdot)$ on $F_i(\tilde{x})$ we have

$$\nu_i(\tilde{x}, y_i) \geq \nu_i(\tilde{x}, z_i) > 0 \text{ for all } y_i \in H_i(\tilde{x}).$$

This however means that $y_i \in U_i(\tilde{x})$ for all $y_i \in H_i(\tilde{x})$ and hence

$H_i(\tilde{x}) \subset U_i(\tilde{x})$. Therefore

$$\tilde{x}_i \in G_i(\tilde{x}) = co(H_i(\tilde{x}) \subset co(U_i(\tilde{x}).$$

The latter though is a contradiction to assumption 4.) in the theorem. Thus we must have $U_i(\tilde{x}) \cap F_i(\tilde{x}) = \emptyset$ for all i and this means that $\tilde{x}$ is an NCE. □

Using this theorem we can now quite easily proof Theorem 3.1.1 which is the original Nash Theorem.

Proof. (of Theorem 3.1.1).Let $\mathcal{G}_n = ((\mathcal{S}_i), \mathcal{S}(N), L)$ be an N-person game consisting of strategy-sets $\mathcal{S}_i$ for each player "i", a subset $\mathcal{S}(N)$ of allowed strategies and a multi-loss operator, which satisfy the conditions in Theorem 3.1.1. Let us make a generalized game out of this. First let us define utility correspondences $U_i : \prod_{j=1}^n \mathcal{S}_j \rightarrow\rightarrow \mathcal{S}_j$ as follows : If $s = (s_1, .., s_n) \in \mathcal{S}(N)$ then

$$\begin{aligned} U_i(s) \quad := \quad & \{\tilde{s}_i \in \mathcal{S}_i : (s_1, .., \tilde{s}_i, .., s_n) \in \mathcal{S}(N) \text{ and} \\ & L_i(s_1, .., s_{i-1}, \tilde{s}_i, s_{i+1}, .., s_n) < L_i(s_1, .., s_i, .., s_n)\} \end{aligned}$$

in the case where $s \notin \mathcal{S}(N)$ we define $U_i(s) = \emptyset$.

□

3.8 The Walras Equilibrium Theorem

In this section we reconsider the Walras economy of section 3.3 and the so called Walras Equilibrium Theorem. We refer to section 3.3 for most of the notation and also the economical interpretation.

Definition 3.8.1. *A* **Walras Economy** *is a five tuple*

$$\mathcal{WE} = ((X_i), (w_i), (U_i), (Y_j), (\alpha_j^i))$$

consisting of consumption sets $X_i \subset \mathbb{R}^m$, supply sets $Y_j \subset \mathbb{R}^m$, initial endowments w_i, utility correspondences $U_i : X_i \to\to X_i$ and shares $\alpha^i_j \geq 0$. Furthermore the prices of the commodities p_i satisfy $p_i \geq 0$ and $\sum_{i=1}^m p_i = 1$ so that the set of prices is given by the closed standard simplex $\overline{\Delta}^{m-1}$. We denote $X = \sum_{i=1}^n X_i$, $Y = \sum_{j=1}^k Y_j$ and $w = \sum_{i=1}^n w_i$.

The assumption on the price vectors to be elements in $\overline{\Delta}^{m-1}$ might look at first glance very restrictive an unrealistic. However, we didn't specify any currency or whatever. Since there is however only finitely many money in the world, we can assume that all prices lie between zero and one. By introducing a $m+1$st commodity in our economy which no one is able to consume or to produce and which can be given the price $1 - \sum_{i=1}^m p_i$ we get an economy which is equivalent to the original one such that the prices satisfy the hypotheses in the Definition above.

Definition 3.8.2. *An* **attainable state** *of the Walrasian economy $\mathcal{WE}$ is a tuple $((x_i),(y_j)) \in \prod_{i=1}^n X_i \times \prod_{j=1}^k Y_j$ such that*

$$\sum_{i=1}^n x_i - \sum_{j=1}^k y_j - w = 0.$$

We denote the set of attainable states with F.

In words : an attainable state is a state where the production of the suppliers precisely fits the demand of the consumers. let us introduce some notation which will later be of advantage. Let

$$M \;\; := \;\; \{((x_i),(y_j)) \in (\mathbb{R}^m)^{n+k} : \sum_{i=1}^n x_i - \sum_{j=1}^k y_j - w = 0\}.$$

Then the set of attainable states can be written as $F = M \cap \prod_{i=1}^n X_i \times \prod_{j=1}^k Y_j$. Let furthermore

$$\begin{aligned} pr_i : F &\rightarrow X_i \\ \tilde{pr}_j : F &\rightarrow Y_j \end{aligned}$$

denote the projections on the corresponding factors and $\tilde{X}_i := pr_i(F)$, $\tilde{Y}_j := \tilde{pr}_j(F)$.

Definition 3.8.3. *A* **Walrasian free disposal equilibrium** *is a price $\tilde{p}$ together with an attainable state $((\tilde{x}_i), (\tilde{y}_j))$ such that*

1. $< \tilde{p}, \tilde{y}_j > \geq < \tilde{p}, y_j >$ *for all* $y_j \in Y_j$ *for all* j
2. $\tilde{x}_i \in b_i(\tilde{p}, < \tilde{p}, w_i > + \sum_{j=1}^{k} \alpha_j^i < \tilde{p}, \tilde{y}_j >)$ *and*

$$U_i(\tilde{x}_i) \cap b_i(\tilde{p}, < \tilde{p}, w_i > + \sum_{j=1}^{k} \alpha_j^i < \tilde{p}, \tilde{y}_j >) = \emptyset.$$

where b_i denotes the budget correspondence for consumer i.

The first part in the definition above means that the supplier make optimal profit, the second one that the consumers get optimal utility from there consumption. The following theorem tells us about the existence of a such an equilibrium under certain assumptions.

Theorem 3.8.1. Walras Equilibrium Theorem *Assume the Walras economy $\mathcal{WE}$ satisfies the following conditions : For each $i = 1, .., n$ and $j = 1, .., k$:*

1. *X_i is closed, convex bounded from below and $w_i \in X_i$*
2. *there exists $\overline{x}_i \in X_i$ s.t. $w_i > \overline{x}_i$*[6]
3. *U_i has open graph, $x_i \notin co(U_i(x_i))$ and $x_i \in \overline{U_i(x_i)}$*
4. *Y_j is closed and convex and $0 \in Y_j$*

[6]this inequality between two vectors is meant componentwise

5. $Y \cap (-Y) = \{0\}$ *and* $Y \cap \mathbb{R}^m_+ = \{0\}$

6. $-\mathbb{R}^m_+ \subset Y$

Then there exists a Walrasian free disposal equilibrium in the economy $\mathcal{E}$.

Before we can proof this theorem we need to do some preparational work.

Definition 3.8.4. *A* **cone** *is a nonempty set* $C \subset \mathbb{R}^m$ *which is closed under multiplication by nonnegative scalars, i.e.* $\lambda \geq 0$ *and* $x \in C$ *imply* $\lambda \cdot x \in C$.

The notion of a one is well known, not so well known in general is the notion of an asymptotic cone.

Definition 3.8.5. *Let* $E\mathbb{R}^m$. *The* **asymptotic cone** *of* E *is the set* $\mathbb{A}(E)$ *of all possible limits of sequences of the form* $(\lambda_j \cdot x_j)$ *where* $x_j \in E$ *and* λ_j *is a decreasing sequence of real numbers s.t.* $\lim_j \lambda_j = 0$.

The asymptotic cone can be used to check if a subset of$\mathbb{R}^m$ is bounded as the following proposition shows :

Proposition 3.8.1. *A set* $E \subset \mathbb{R}^m$ *is bounded if and only if* $\mathbb{A}(E) = \{0\}$.

Proof. If E is bounded then there exists a constant M s.t. $\parallel x \parallel \leq M$ for all $xinE$. If $z = \lim_j \lambda_j x_j \in \mathbb{A}(E)$ then

$$0 \leq \parallel z \parallel = \lim_j |\lambda_j| \parallel x_j \parallel \leq M \cdot \lim_j |\lambda_j| = M \cdot 0 = 0$$

Therefore in this case $\mathbb{A}(E) = \{0\}$. If however E is not bounded then there exists a sequence $x_j \in E$ s.t. $\parallel x_j \parallel$ converges monotonically increasing to infinity. We set $\lambda_j = \frac{1}{\|x_j\|}$ and obtain that the sequence (λ_j) converges monotonically decreasing to zero. We have $\parallel \lambda_j \cdot x_j \parallel = 1$ for all j. Therefore the sequence $\lambda_j \cdot x_j$ is a sequence on the unit sphere $S^{m-1} \subset \mathbb{R}^m$. Since this is compact, the sequence $\lambda_j \cdot x_j$ must contain a convergent subsequence and w.l.o.g. we assume that $\lambda_j \cdot x_j$ converges itself to a point $z \in S^{m-1}$. Clearly $z \neq 0$ and $z \in \mathbb{A}(E)$. Therefore in this case $\mathbb{A}(E) \neq \{0\}$.

$\square$

Intuitively the asymptotic cone tells one about the directions in which E is unbounded. It might be difficult though to compute the asymptotic cone. For this the following rules are very helpful :

Lemma 3.8.1. *Let $E, E_i \subset \mathbb{R}^m$ and $x \in \mathbb{R}^m$ then*

1. $\mathbb{A}(E)$ *is a cone*
2. $E \subset F \Rightarrow \mathbb{A}(E) \subset \mathbb{A}(F)$
3. $\mathbb{A}(E + x) = \mathbb{A}(E)$
4. $\mathbb{A}(\prod_{i \in I} E_i) \subset \prod_{i \in I} \mathbb{A}(E_i)$
5. $\mathbb{A}(E)$ *is closed*
6. E *convex* $\Rightarrow \mathbb{A}(E)$ *convex*
7. E *closed and convex and* $x \in E \Rightarrow x + \mathbb{A}(E) \subset E$*, in particular if* $0 \in E$ *and* E *convex, then* $\mathbb{A}(E) \subset E$*.*
8. $C \subset E$ *and* C *a cone* $\Rightarrow C \subset \mathbb{A}(E)$
9. $\mathbb{A}(\bigcap_{i \in I} E_i) \subset \bigcap_{i \in I} \mathbb{A}(E_i)$

We will use these rules to prove the following result about the compactness of the set F introduced before. This compactness will later play a major role in the proof of the Walras Equilibrium Theorem.

Lemma 3.8.2. *Assume the Walras economy $\mathcal{WE}$ satisfies the following conditions : For each $i = 1, .., n$ and $j = 1, .., k$:*

1. X_i *is closed, convex bounded from below and* $w_i \in X_i$
2. Y_j *is closed and convex and* $0 \in Y_j$

3. $\mathbb{A}(Y) \cap \mathbb{R}^m_+ = \{0\}$

4. $Y \cap (-Y) = \{0\}$ *and* $Y \cap \mathbb{R}^m_+ = \{0\}$

Then the set F of attainable states is compact and nonempty. Furthermore $0 \in \tilde{Y}_j$ for $j = 1, .., k$. If more over the following two assumptions hold:

1. *for each i there exists some $\overline{x}_i \in X_i$ s.t. $w_i > \overline{x}_i$*

2. $-\mathbb{R}^m_+ \subset Y$

Then $\overline{x}_i \in \tilde{X}_i$.

Proof. Clearly $((w_i), (0_j)) \in F$ where $0_j \in Y_j$ denotes the zero vector in $0 \in Y_j$. This implies $F \neq \emptyset$. Furthermore F as the intersection of the two close sets M and $\prod_{i=1}^n X_i \times \prod_{j=1}^k Y_j$ is closed. By Proposition 3.8.1 for the compactness of F it suffices to show that $\mathbb{A}(F) = \{0\}$. By part e.) of the previous lemma we have :

$$\begin{aligned} \mathbb{A}(F) &\subset \mathbb{A}(\prod_{i=1}^n X_i \times \prod_{j=1}^k Y_j) \cap \mathbb{A}(M) \\ &\subset (\prod_{i=1}^n \mathbb{A}(X_i) \times \prod_{j=1}^k \mathbb{A}(Y_j)) \cap \mathbb{A}(M). \end{aligned}$$

Since each of the X_i is bounded from below there exist vectors $b_i \in \mathbb{R}^m$ s.t. $X_i \subset b_i + \mathbb{R}^m_+$. Applying successively parts b.), c.) and i.) of the previous lemma we get

$$\mathbb{A}(X_i) \subset \mathbb{A}(b_i + \mathbb{R}^m_+) = \mathbb{A}(\mathbb{R}^m_+) = \mathbb{R}^m_+.$$

Also by application of part d.) and assumption b.) we have

$$\mathbb{A}(Y_j) \subset \mathbb{A}(Y).$$

Let us consider the vector $\tilde{w} = (w_1, ..., w_n, 0, ..., 0) \in (\mathbb{R}^m)^{n+k}$ and

$$\tilde{M} = \{((\tilde{x}_i), (\tilde{y}_j)) | \sum_{i=1}^{n} \tilde{x}_i - \sum_{j=1}^{k} \tilde{y}_j = 0\}.$$

Then $\tilde{M}$ is a vectorspace and therefore also a cone. Furthermore we have

$$\tilde{M} + \tilde{w} = M$$

and hence by application of part c.) of the previous lemma

$$\mathbb{A}(M) = \mathbb{A}(\tilde{M} + \tilde{w}) = \mathbb{A}(\tilde{M}) = \tilde{M} = M - \tilde{w}.$$

Therefore $\mathbb{A}(F) = 0$ would follow if

$$\prod_{i=1}^{n} \mathbb{R}_+^m \times \prod_{j=1}^{k} \mathbb{A}(Y) \cap \tilde{M} = \{0\}.$$

To prove the latter we have to show that whenever $y_j \in \mathbb{A}(Y)$ for $j = 1, .., k$ and

$$\sum_{i=1}^{n} x_i - \sum_{j=1}^{k} y_j = 0 \tag{3.1}$$

for some $x_i \in \mathbb{R}_+^m$ then $x_1 = .. = x_n = y_1 = .. = y_k = 0$. Now since $\sum_{i=1}^{n} x_i \geq 0$ (componentwise, since in $\mathbb{R}_+^m$) we also must have

$$\sum_{j=1}^{k} y_j \geq 0.$$

Since $\mathbb{A}(Y)$ is convex and a cone one has $\sum_{j=1}^{k} y_j \in \mathbb{A}(Y)$. Since however by assumption 3.) we have $\mathbb{A}(Y) \cap \mathbb{R}_+^m = \{0\}$ we must have $\sum_{j=1}^{k} y_j = 0$ and therefore by equation (3.1) also $\sum_{i=1}^{n} x_i = 0$. Since $x_i \geq 0$ (componentwise) for all i we must have $x_i = 0$ for all i. Now since $y_j \in \mathbb{A}(Y)$ for all j using assumption 4.) we get for all $i = 1, .., n$

$$\sum_{j=1}^{k} y_j = 0 \Rightarrow \underbrace{y_i}_{\in \mathbb{A}(Y) \subset Y} = \underbrace{-\sum_{j \neq i} y_j}_{\in -\mathbb{A}(Y) \subset -Y} \in Y \cap (-Y) = \{0\}.$$

which finally proves that $\mathbb{A}(F) = \{0\}$ and therefore F compact. Let us now assume that in addition the assumptions a.) and b.) of the second part of the lemma hold. Choosing $\overline{x}_i \in X_i$ as in assumption a.) we get componentwise

$$\sum_{i=1}^{n} \overline{x}_i < \sum_{i=1}^{n} w_i.$$

Let us set $\overline{y} := \sum_{i=1}^{n} \overline{x}_i - \sum_{i=1}^{n} w_i$. Then $y < 0$ and by assumption b.) we must have $\overline{y} \in Y$. Therefore there must exist $\overline{y}_j \in Y_j$ s.t.

$$\overline{y} = \sum_{j=1}^{k} \overline{y}_j.$$

Clearly $\sum_{i=1}^{n} \overline{x}_i - \sum_{j=1}^{k} \overline{y}_j - \underbrace{\sum_{i=1}^{n} w_i}_{=w} = \overline{y} - \overline{y} = 0$ and therefore $((\overline{x}_i), (\overline{y}_j)) \in F$. This however implies that $\overline{x}_i \in \tilde{X}_i$.

$\square$

Proof. (of Theorem 3.8.1) Let us note first that assumption 5.) together with with 7.) of Lemma 3.8.1 imply that

$$\mathbb{A}(Y) \cap \mathbb{R}^m_+ \subset Y \cap \mathbb{R}^m_+ = \{0\}$$

and therefore all assumptions in Lemma 3.8.2 are met. The set F of attainable states is therefore compact. Since the image of compact sets under continuous maps is also compact we have that all $\tilde{X}_i, \tilde{Y}_j$ are compact. Therefore we can choose compact, convex sets $K_i, C_j \subset \mathbb{R}^m$ s.t. $\tilde{X}_i \subset K_i^\circ$ and $\tilde{Y}_j \subset C_j^\circ$. We set

$$\begin{aligned} X_i' &:= K_i \cap X_i \\ Y_j' &:= C_j \cap Y_j. \end{aligned}$$

We will set up a generalized game where these sets will serve as strategy sets for some of the participants. The participants or players will be :

1. An auctioneer : He is player "0" and his strategy set is the set of price vectors $\overline{\Delta}^{m-1}$
2. Consumers $1, .., n$ are players $1, .., n$ and their strategy sets are the X_i'
3. Suppliers $1, ..k$ are players $n+1, .., n+k$ and their strategy sets are the sets Y_j'

A typical multi-strategy therefore has the form $(p, (x_i), (y_j)) \in \overline{\Delta}^{m-1} \times \prod_{i=1}^n X_i' \times \prod_{j=1}^k Y_j'$. The utility correspondences are given as follows : For the auctioneer

$$\begin{aligned} U_0 : \overline{\Delta}^{m-1} \times \prod_{i=1}^n X_i' \times \prod_{j=1}^k Y_j' &\longrightarrow\!\!\!\!\rightarrow \overline{\Delta}^{m-1} \\ (p, (x_i), (y_j)) &\mapsto \{q \in \overline{\Delta}^{m-1} :< q, \sum_i x_i - \sum_j y_j - w > \\ &\quad > < p, \sum_i x_i - \sum_j y_j - w >\}. \end{aligned}$$

This means that the auctioneer prefers to raise the value of excess demand. The economical interpretation of this is that the prices go up, if there is more demand then supply (i.e. $\sum_i x_i - \sum_j y_j - w \geq 0$) and the prices go down if there is more supply than demand (i.e. $\sum_i x_i - \sum_j y_j - w \leq 0$). For the mathematics it is important to mention that the correspondence above has open graph, is convex valued and $p \notin$

$U_0(p, (x_i), (y_j))$. Both properties follow more or less since the inequality in the definition of U_0 is a strict one and the scalar product is bilinear. Let us define the utility correspondences for the suppliers : For supplier "l"

$$\begin{aligned} V_l : \overline{\Delta}^{m-1} \times \prod_{i=1}^{n} X_i' \times \prod_{j=1}^{k} Y_j' &\longrightarrow Y_l' \\ (p, (x_i), (y_j)) &\mapsto \{\tilde{y}_l \in Y_l' | < p, \tilde{y}_l >>< p, y_l >\}. \end{aligned}$$

Thus supplier prefer larger profits. As before it is easy to see that these correspondences have open graph and are convex valued. Furthermore $y_l \notin V_l(p, (x_i), (y_j))$. Finally the utility correspondences for the consumers[7] are : For consumer "q"

$$\begin{aligned} \tilde{U}_q : \overline{\Delta}^{m-1} \times \prod_{i=1}^{n} X_i' \times \prod_{j=1}^{k} Y_j' &\longrightarrow X_q' \\ (p, (x_i), (y_j)) &\mapsto co(U_q(x_q)) \cap X_q'. \end{aligned}$$

This correspondence is indeed well defined by the convexity of X_q. Furthermore it follows from Proposition 3.2.10. 3.) and assumption 3.) that the $\tilde{U}_q$ have open graphs and $x_q \notin \tilde{U}_q(p, (x_i), (y_j))$. They are also convex valued by the convexity of X_q'. To complete the setup of our generalized game we need feasibility correspondences for each player. For suppliers and the auctioneer this is very easy. We choose constant correspondences : In fact for supplier "l" we define

$$\begin{aligned} G_l : \overline{\Delta}^{m-1} \times \prod_{i=1}^{n} X_i' \times \prod_{j=1}^{k} Y_j' &\longrightarrow Y_l' \\ (p, (x_i), (y_j)) &\mapsto Y_l' \end{aligned}$$

[7] within our generalized game, they have to be distinguished from the utility correspondences the consumers have in the Walras Economy

and for for the auctioneer

$$\begin{aligned} F_0 : \overline{\Delta}^{m-1} \times \prod_{i=1}^{n} X_i' \times \prod_{j=1}^{k} Y_j' &\rightarrow\rightarrow & \overline{\Delta}^{m-1} \\ (p, (x_i), (y_j)) &\mapsto & \overline{\Delta}^{m-1}. \end{aligned}$$

Constant correspondences are clearly continuous and in this case they are also compact and convex valued. For the feasibility correspondences of the consumer we have to work a little bit more. Let us first define functions π_j for $j = 1, .., k$ s.t.

$$\begin{aligned} \pi_j : \overline{\Delta}^{m-1} &\rightarrow & \mathbb{R} \\ p &\mapsto & max_{y_j \in Y_j'} < p, y_j > . \end{aligned}$$

Basically these maps compute the optimal profit for the suppliers. It is not hard to see (directly) that these functions are continuous. It follows however also from Theorem 3.4.1 (Maximum Theorem 1).It follows from Lemma 3.8.2 that $0 \in \tilde{Y}_j \subset Y_j'$, so we have

$$\pi_j(p) \geq 0 \; \forall p, j.$$

We define the feasibility correspondence for consumer “q” as follows :

$$\begin{aligned} F_q : \overline{\Delta}^{m-1} \times \prod_{i=1}^{n} X_i' \times \prod_{j=1}^{k} Y_j' &\rightarrow\rightarrow & X_q' \\ (p, (x_i), (y_j)) &\mapsto & \{\tilde{x}_q \in X_l' : < p, \tilde{x}_q > \leq < p, w_q > + \sum_{j=1}^{k} \alpha_j^q \pi_j(p)\} \end{aligned}$$

This correspondence in fact only depends on the price vector p, (x_i) and (y_j) are redundant. Using assumption 2 and the second part of

Lemma 3.8.2 there exists $\overline{x}_q \in \tilde{X}_q \subset X'_q$ s.t. $\overline{x}_q < w_q$. Since also $p \geq 0$ and $\pi_q(p) \geq 0$ we have

$$< p, \overline{x}_q > < < p, w_q > + \sum_{j=1}^{k} \alpha_j^q \pi_j(p)$$

and therefore F_q is non empty valued. Furthermore it follows from Proposition 3.4.1 that F_q is lhc. Since furthermore X'_q is compact and clearly F_q has closed graph it follows from Proposition 3.2.1 b.) that F_q is uhc. Thus for each consumer the feasibility correspondences are continuous with nonempty convex values. The generalized game constructed therefore satisfies all the assumptions in the Sonnenschein-Shafer Theorem 3.7.1. Therefore there exists an NCE

$$(p^\sharp, (x_i^\sharp), (y_j^\sharp)) \in \overline{\Delta}^{m-1} \times \prod_{i=1}^{n} X'_i \times \prod_{j=1}^{k} Y'_j$$

which by definition of an NCE satisfies

1. $< q, \sum_i x_i^\sharp - \sum_j y_j^\sharp - w > \leq < p^\sharp, \sum_i x_i^\sharp - \sum_j y_j^\sharp - w >$ for all $q \in \overline{\Delta}^{m-1}$
2. $< p^\sharp, y_j^\sharp > \geq < p^\sharp, y_j >$ for all $y_j \in Y'_j$, i.e. $< p^\sharp, y_j^\sharp >= \pi_j(p^\sharp)$
3.

$$\begin{aligned} x_i^\sharp \in b_i(p^\sharp) &= \{x_i \in X'_i | < p^\sharp, x_i > \leq < p^\sharp, x_i > \leq < p^\sharp, w_i > \\ & \quad + \sum_{j=1}^{k} \alpha_j^i \underbrace{< p^\sharp, y_j^\sharp >}_{= \pi_j(p^\sharp)} \end{aligned}$$

and

$$\begin{aligned} co(U_i(x_i^\sharp)) \cap B_i(p^\sharp) &= co(U_i(x_i^\sharp)) \cap X'_i \cap B_i(p^\sharp) \\ &= \tilde{U}_i(p^\sharp, (x_i^\sharp), (y_j^\sharp)) \cap F_i((p^\sharp, (x_i^\sharp), (y_j^\sharp)) \\ &= \emptyset. \end{aligned}$$

We are now going to construct a Walras Equilibrium form the NCE $(p^\sharp, (x_i^\sharp), (y_j^\sharp))$. For notational convenience set

$$M_i :=< p^\sharp, w_i > + \sum_{j=1}^{k} \alpha_j^i < p^\sharp, y_j^\sharp >$$

for the income of consumer “i”. We show that by implying the NCE each consumer spends all of his income. Suppose not, i.e. $< p^\sharp, x_i^\sharp > < M_i$. Then since $U_i(x_i^\sharp)$ is open[8] and by assumption 3.) $x_i^\sharp \in \overline{U_i(x_i^\sharp)}$ it would follow that $U_i(x_i^\sharp) \cap b_i(p^\sharp) \neq \emptyset$ and therefore also $co(U_i(x_i^\sharp)) \cap B_i(p^\sharp)$ which is a contradiction to property 3.) of our NCE above. Therefore we have $< p^\sharp, x_i^\sharp >= M_i$ for all i or more precisely

$$< p^\sharp, x_i^\sharp >=< p^\sharp, w_i > + \sum_{j=1}^{k} \alpha_j^i < p^\sharp, y_j^\sharp > \text{ for all } i.$$

Summing up over i and using the assumption on the Walras economy that $\sum_i \alpha_j^i = 1$ for each j yields

$$< p^\sharp, \sum_i x_i^\sharp >=< p^\sharp, \sum_j y_j^\sharp + w > \quad \Rightarrow \quad < p^\sharp, \sum_i x_i^\sharp - \sum_j y_j^\sharp - w >= 0.$$

By property 1.) of our NCE we then have

$$< q, \sum_i x_i^\sharp - \sum_j y_j^\sharp - w > \leq 0 \text{ for all } q \in \overline{\Delta}^{m-1}$$

which clearly implies that $\sum_i x_i^\sharp - \sum_j y_j^\sharp - w \leq 0$. By assumption 6.) we have that $z := \sum_i x_i^\sharp - \sum_j y_j^\sharp - w \in Y$. Therefore there must exist $y_j \in Y_j$ such that $z = \sum_j y_j$. We define

$$\tilde{y}_j := y_j^\sharp + y_j \text{ for all j }.$$

[8]follows from the assumption that U_i has open graph

Then $\sum_i x_i^\sharp - \sum_j \tilde{y}_j - w = z - z = 0$, so that $\tilde{y}_j \in \tilde{Y}_j$. Further more we have

$$< p^\sharp, \tilde{y}_j >=< p^\sharp, y_j^\sharp > + < p^\sharp, y_j > \text{ for all } j.$$

Summing up these equations over j we get

$$\sum_j < p^\sharp, \tilde{y}_j >= \sum_j < p^\sharp, y_j^\sharp > + \underbrace{\sum_j < p^\sharp, y_j >}_{=< p^\sharp, z >= 0} = \sum_j < p^\sharp, y_j^\sharp > .$$

By property 2.) of our NCE and $\tilde{y}_j \in \tilde{Y}_j \subset Y_j'$we have $< p^\sharp, \tilde{y}_j >\leq< p^\sharp, y_j^\sharp >$ for all j. Therefore the equality above can only hold if $< p^\sharp, \tilde{y}_j >=< p^\sharp, y_j^\sharp >$ for all j. We have therefore shown that

$$< p^\sharp, \tilde{y}_j >\geq< p^\sharp, y_j > \text{ for all } y_j \in Y_j'.$$

We will now show that this inequality holds even for all $y_j \in Y_j$. Suppose this would not be the case, i.e. there would exist $y_j \in Y_j$ s.t. $< p^\sharp, y_j > > < p^\sharp, \tilde{y}_j >$. Since Y_j is convex we have $\lambda \cdot y_j + (1-\lambda) \cdot \tilde{y}_j \in Y_j$ for all $\lambda \in [0,1]$. Since $\tilde{y}_j \in \tilde{Y}_j \subset (Y_j')^\circ$ there exists $\lambda > 0$ s.t. $y_j' := \lambda \cdot y_j + (1-\lambda) \cdot \tilde{y}_j \in Y_j'$. Then $< p^\sharp, y_j' >>< p^\sharp, \tilde{y}_j >$ which is a contradiction to the inequality above. By construction we have that $((x_i^\sharp), (\tilde{y}_j)) \in F$. To show that $(p^\sharp, (x_i^\sharp), (\tilde{y}_j))$ is a Walrasian free disposal equilibrium it remains to show that for each i

$$U_i(x_i^\sharp) \cap \{x_i \in X_i :< p^\sharp, x_i >\leq< p^\sharp, w_i > + \sum_j \alpha_j^i < p^\sharp, \tilde{y}_j >\} = \emptyset.$$

Suppose there would be an x_i in this intersection. Then since X_i' is convex and $x_i^\sharp \in \tilde{X}_i \subset (X_i')^\circ$ there exists $\lambda > 0$ such that $\lambda \cdot x_i + (1-\lambda) \cdot x_i^\sharp \in X_i'$. Since however $x_i^\sharp \in \overline{U_i(x_i^\sharp)}$ by assumption 3.) it follows from the convexity of $b_i(p^\sharp)$ that $\lambda \cdot x_i + (1-\lambda) \cdot x_i^\sharp \in co(U_i(x_i^\sharp)) \cap b_i(p^\sharp)$. This is a contradiction to property 3.) of our NCE. Thus $(p^\sharp, (x_i^\sharp), (\tilde{y}_j))$ is indeed a Walrasian free disposal equilibrium.

$\square$

Chapter 4

Cooperative Games

4.1 Cooperative Two Person Games

In the setup of non-cooperative games the player choose independently from each other their strategies, then the game is played and delivers some output which is measured either via a loss-operator or a Utility correspondence. In the setup of cooperative games the players are allowed to communicate before choosing their strategies and playing the game. They can agree but also disagree about a joint strategy. Let us recall the "'Battle of the Sexes"' game where the strategies are given as follows :

man	woman
s_1 = "'go to theater"'	$\tilde{s}_1$ = "'go to theater"'
s_2 = "'go to soccer"'	$\tilde{s}_2$ = "'go to soccer"'

The corresponding bilosses are given by the matrix

$$L := \begin{pmatrix} (-1,-4) & (0,0) \\ (0,0) & (-4,-1) \end{pmatrix}$$

The mixed strategies of this game look as follows :

$$\begin{aligned} x \cdot s_1 + (1-x) \cdot s_2 &\leftrightarrow x \in [0,1] \\ y \cdot \tilde{s}_1 + (1-y) \cdot \tilde{s}_2 &\leftrightarrow x \in [0,1] \end{aligned}$$

and the biloss of the mixed strategy (x, y) is given by

$$\begin{aligned} L_1(x,y) &= -(5xy + 4 - 4x - 4y) \\ L_2(x,y) &= -(5xy + 1 - x - y) \end{aligned}$$

Since we have for all $x, y \in [0,1]$ that

$$\begin{aligned} L_1(1,1) &= -1 \leq -x = L_1(x,1) \\ L_2(1,1) &= -4 \leq -4y = L_2(1,y) \end{aligned}$$

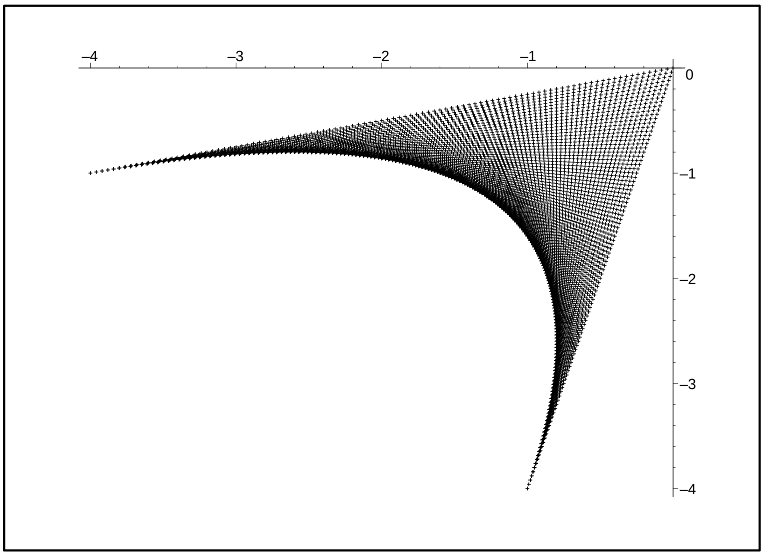
we see that the pure strategy $(1,1) \leftrightarrow (s_1, \tilde{s}_1)$ is a NCE. In the same way one can see that $(0, 0$ is an NCE. All possible outcomes of the game when using mixed strategies are given by the shaded region in the following graphic:

Assume now the man and woman decide to do the following : They throw a coin and if its head then they go both to the theater and if its number they go both to see the soccer match. The expected biloss of this strategy is:

$$\frac{1}{2} \cdot (-1, -4) + \frac{1}{2}(-4, -1) = (-\frac{5}{2}, -\frac{5}{2}).$$

We call such a strategy a jointly randomized strategy. It involves a random experiment which the to players perform together. Note that the non-cooperative mixed strategies $(\frac{1}{2}, \frac{1}{2})$ is the outcome of two random experiments which the players do independently from each other.

Figure 4.1: Biloss region for “Battle of the Sexes” :



non-cooperative setup

The (expected) biloss of this strategy is

$$
\begin{aligned}
L_1(\frac{1}{2},\frac{1}{2}) &= -\frac{5}{4} \\
L_2(\frac{1}{2},\frac{1}{2}) &= -\frac{5}{4}.
\end{aligned}
$$

One can also see that the biloss of our jointly randomized strategy is not in the biloss region of the non-cooperative game (see graphic). Hence such jointly randomized strategies give something completely new and the concept of cooperative game theory is just the extension of the original concept by these jointly randomized strategies. It is not true that the jointly randomized strategy above is in any case better than any non-cooperative strategy. In fact if both man and woman go to the soccer match, then th man is better of than with our jointly randomized strategy and vice versa the woman is better of if both go to the theater. However these to cases are unlikely to happen, if both want their will. The jointly randomized strategy is therefore in some sense a compromise. One of the main questions in cooperative game theory is to find the best compromise. Before giving a precise mathematical formulation of cooperative two person games let us mention that there are more jointly randomized strategies for the "'Battle of the Sexes"' game. In fact for $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in [0, 1]$ such that $\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 = 1$ we have the jointly randomized strategy

$$\lambda_0 \cdot (1, 0) + \lambda_1 \cdot (1, 1) + \lambda_2 \cdot (0, 1) + \lambda_3 \cdot (0, 0).$$

The expected bilosses of these strategies are

$$\lambda_0 \cdot (0, 0) + \lambda_1 \cdot (-1, -4) + \lambda_2 \cdot (0, 0) + \lambda_3 \cdot (-4, -1) = (-1 - 4\lambda_4, -4\lambda_1 - \lambda_3).$$

The possible bilosses of jointly randomized strategies are given in the following graphic :

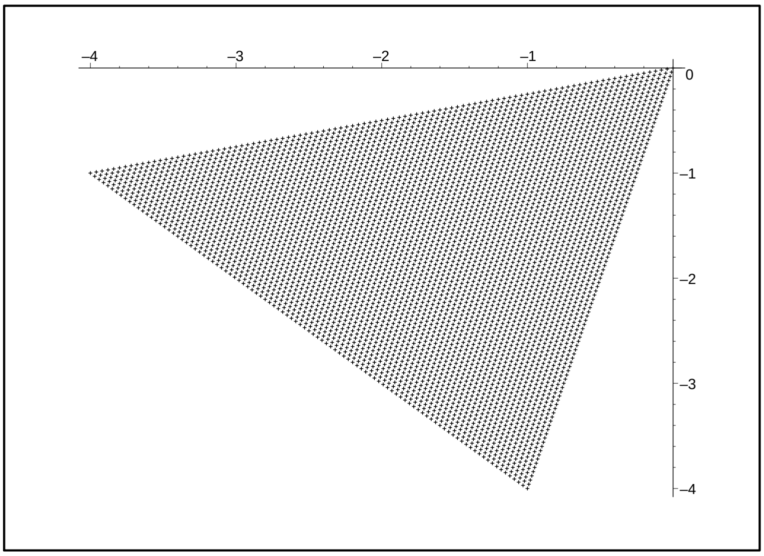
As one can see immediately, this set is th convex hull of the biloss region of th non-cooperative game. In the following we restrict ourself to games with only finite strategy sets.

Definition 4.1.1. *Let $\mathcal{G}_2$ b a (non cooperative) two person game with finite strategy sets $\mathcal{S}_1$ and $\mathcal{S}_2$ and let $L = (L_1, L_2)$ be its biloss operator. Then the corresponding cooperative game is given by the biloss operator*

$$\begin{aligned} \hat{L} : \Delta^{\mathcal{S}_1 \times \mathcal{S}_2} &\rightarrow \mathbb{R} \times \mathbb{R} \\ \sum_{i,j} \lambda_{ij}(s_i, \tilde{s}_j) &\mapsto \sum_{i,j} \lambda_{ij} L(s_i, \tilde{s}_j) \end{aligned}$$

where $\Delta^{\mathcal{S}_1 \times \mathcal{S}_2} := \{\sum_{i,j} \lambda_{ij}(s_i, \tilde{s}_j) | \sum_{i,j} \lambda_{ij} = 1, \lambda_{ij} \in [0, 1]\}$ is the (formal) simplex spanned by the pure strategy pairs $(s_i, \tilde{s}_j)$.

Figure 4.2: Biloss region for “Battle of the Sexes” :



cooperative setup

The image $im(\hat{L})$ of $\hat{L}$ is called the biloss region of the cooperative game. By definition of $\hat{L}$ it is clear that it is always convex and in fact is the convex hull of the biloss region of the corresponding non-cooperative game.

Remark 4.1.1. *If the strategy sets are not necessarily finite but probability spaces then one can can consider jointly randomized strategies as functions $\mathcal{S}_1 \times \mathcal{S}_2 \to \mathbb{R}_+$ such that $\int_{\mathcal{S}_1 \times \mathcal{S}_2} f(s, \tilde{s}) d\mathbb{P}_{\mathcal{S}_1} d\mathbb{P}_{\mathcal{S}_2} = 1$.*

Definition 4.1.2. *Given a two person game $\mathcal{G}_2$ and let $\hat{L}$ be the biloss operator of the corresponding cooperative game. A pair of losses $(u, v) \in im(\hat{L})$ is called* **jointly sub-dominated** *by a pair $(u', v') \in im(\hat{L})$ if $u' \leq u$ and $v' \leq v$ and $(u', v') \neq (u, v)$. The pair (u, v) is called* **Pareto optimal** *if it is not jointly sub-dominated.*

Let us recall the definition of th conservative value of a two person game:

$$
\begin{aligned}
u^\sharp &= \min_{s_1 \in \mathcal{S}_1} \max_{s_2 \in \mathcal{S}_2} L_1(s_1, s_2) \\
v^\sharp &= \min_{s_2 \in \mathcal{S}_2} \max_{s_1 \in \mathcal{S}_1} L_2(s_1, s_2).
\end{aligned}
$$

These values are the losses the players can guarantee for themselves, no matter what the other player does, by choosing the corresponding conservative strategy (see Definition 1.2.3).

Definition 4.1.3. *Given a two person game $\mathcal{G}_2$ and let $\hat{L}$ be the biloss operator of the corresponding cooperative game. The set*

$$B := \{(u,v) \in im(L) | u \leq u^\sharp, v \leq v^\sharp \text{ and } (u,v) \text{ Pareto optimal } \}$$

is called the **bargaining set** *(sometimes also negotiation set).*

The interpretation of the bargaining set is as follows : It contains all reasonable compromises the players can agree on. In fact no player would accept a compromise (u,v) where $u > u^\sharp$ resp. $v > v^\sharp$ because the losses $u^\sharp$ resp. $v^\sharp$ are guaranteed to him. In the same way they would not agree on a strategy pair which is jointly sub-dominated, because then by switching to the other strategy they can both do better and one of them can do strictly better. The main question however remains. What compromise in the bargaining set is the best one. Using some assumptions which can be economically motivated the so called Nash bargaining solution gives an answer to this question. This is the content of the next section.

4.2 Nash's Bargaining Solution

let us denote with $\overline{conv}(\mathbb{R}^2)$ the set of compact and convex subsets of $\mathbb{R}^n$ and with

$$\mathcal{A} := \{(u_0, v_0), P) | P \in \overline{conv}(\mathbb{R}^2) \text{ and } (u_0, v_0) \in P\}$$

Definition 4.2.1. *A* **bargaining function** *is a function*

$$\psi : \mathcal{A} \to \mathbb{R}^2$$

s.t. $\psi((u_0, v_0), P) \in P$ *for all* $(u_0, v_0) \in P$.

The economical interpretation of a bargaining function is as follows : We think of P as the biloss region of some cooperative two person game and of (u_0, v_0) as some status quo point (the outcome when the two players do not agree on a compromise). Then $\psi((u_0, v_0), P)$ gives the compromise. As status quo point one often chooses the conservative value of the game, but other choices are also possible.

Definition 4.2.2. *A bargaining function* $\psi : \mathcal{A} \to \mathbb{R}^2$ *is called a* **Nash bargaining function** *if it satisfies the following conditions : let us denote* $(u^*, v^*) := \psi((u_0, v_0, P)$. *Then*

1. $u^* \leq u_0$,$v^* \leq v_0$ *i.e. the compromise is at least as good as the status quo*
2. (u^*, v^*) *is Pareto optimal, i.e. there does not exist* $u \leq u^*, v \leq v^*$ *s.t.* $(u, v) \in P \setminus \{(u^*, v^*)\}$
3. *If* $P_1 \subset P$ *and* $(u^*, v^*) \in P_1$ *then* $(u^*, v^*) = \psi((u_0, v_0), P_1)$ *(independence of irrelevant alternatives)*
4. *Let* P' *be the image of* P *under the affine linear transformation*

$$\begin{aligned} u &\mapsto au + b \\ v &\mapsto cv + d \end{aligned}$$

then $\psi((au_0 + b, cv_0 + d), P') = (au^* + b, cv^* + d)$ *for all* $a, c > 0$ *(invariance under affine linear transformation = invariance under rescaling utility)*

5. *If P is symmetric. i.e.* $(u,v) \in P \Leftrightarrow (v,u) \in P$ *and* $u_0 = v_0$ *then* $u^* = v^*$.

We are going to prove that there is precisely one Nash bargaining function $\psi : \mathcal{A} \to \mathbb{R}^2$. For this we need the following lemma.

Lemma 4.2.1. *Let* $((u_0, v_0), P) \in \mathcal{A}$*. We define a function*

$$\begin{aligned} f_{(u_0,v_0)} : P \cap \{u \le u_0, v \le v_0\} &\to \mathbb{R}_+ \\ (u,v) &\mapsto (u_0 - u)(v_0 - v) \end{aligned}$$

If there exists a pair $(u,v) \in P$ *s.t.* $u < u_0$ *and* $v < v_0$ *then* f *takes its maximum at a unique point*

$$(u^*, v^*) := argmax(f(u,v)) \tag{4.1}$$

and $u^* < u_0, v^* < v_0$.

Proof. As $f_{(u_0,v_0)}$ is defined on a compact set and is clearly continuous, it takes its global maximum at at least one point (u^*, v^*). Let $M = f(u^*, v^*)$ then by our assumption and the definition of $f_{(u_0,v_0)}$ we have $M > 0$ and $u^* < u_0, v^* < v_0$. Assume now $f_{(u_0,v_0)}(\tilde{u}, \tilde{v}) = M$with $(\tilde{u}, \tilde{v}) \in P \cap \{u \le u_0, v \le v_0\}$. We have to show $(\tilde{u}, \tilde{v}) = (u^*, v^*)$. Assume this would not be the case. Since by definition of $f_{(u_0,v_0)}$ and M we have

$$(u_0 - u^*)(v_0 - v^*) = M = (u_0 - \tilde{u})(v_0 - \tilde{v})$$

it follows $u^* = \tilde{u} \Leftrightarrow v^* = \tilde{v}$ and we can then as well assume that $u^* \ne \tilde{u}$ and $v^* \ne \tilde{v}$. More precisely there are exactly two cases

$$(u^* < \tilde{u} \text{ and } v^* > \tilde{v}) \text{ or } (u^* > \tilde{u} \text{ and } v^* < \tilde{v}).$$

Since P is convex, it contains the point

$$(u', v') := \frac{1}{2}(u^*, v^*) + \frac{1}{2}(\tilde{u}, \tilde{v}).$$

Clearly $u' \leq u_0, v' \leq v_0$. An easy computation shows that

$$M \geq f_{(u_0,v_0)}(u',v') = \underbrace{\frac{1}{2} f_{(u_0,v_0)}(u^*,v^*)}_{=\frac{1}{2}M} + \underbrace{f_{(u_0,v_0)}(\tilde{u},\tilde{v})}_{\frac{1}{2}M} + \underbrace{\frac{(u^*-\tilde{u})(\tilde{v}-v^*)}{2}}_{>0} > M.$$

which is a contradiction. Therefore we must have $(u^*, v^*) = (\tilde{u}^*, \tilde{v}^*)$ and we are done.

□

Theorem 4.2.1. *There exists exactly one Nash bargaining function $\psi : \mathcal{A} \to \mathbb{R}^2$.*

Proof. We define a function $\psi : \mathcal{A} \to \mathbb{R}^2$ as follows . Let $((u_0, v_0), P) \in \mathcal{A}$. Then if there exists $(u, v) \in P$ such that $u < u_0, v < v_0$ then by using Lemma 4.2.1 we define

$$(u^*, v^*) := \psi((u_0, v_0), P) := argmax(f_{(u_0,v_0)}(u, v))$$

If there are no points $(u, v) \in P$ s.t. $u < u_0, v < v_0$ then the convexity of P implies that exactly two cases can occur :

1. $P \subset \{(u, v_0) : u \leq u_0\}$
2. $P \subset \{(u_0, v) : v \leq v_0\}$.

In case 1.) we define $\psi((u_0, v_0), P) := (u^*, v_0)$ where u^* is the minimal value such that $(u^*, v_0) \in P$. Similarly in case 2.) we define $\psi((u_0, v_0), P) := (u_0, v^*)$ where v^* is the minimal value such that $(u_0, v^*) \in P$. We will now show that the bargaining function ψ defined above satisfies the five conditions on Definition 4.2.2. Let us first consider the case where there exists $(u, v) \in P$ such that $u < u_0, v < v_0$. Condition 1.) is trivially satisfied. To show that 2.) is satisfied assume that $u \leq u^*, v \leq v^*$ and $(u, v) \in P \setminus \{(u^*, v^*)\}$. Then

$$f(u, v) = (u_0 - u)(v_0 - v) > (u_0 - u^*)(v_0 - v^*) = M$$

which is a contradiction. Therefore (u^*, v^*) is Pareto optimal and 2.) is satisfied. To show that condition 3.) holds let us assume that $P_1 \subset P$ and $(u^*, v^*) = \psi((u_0, v_0), P) \in P_1$. Then (u^*, v^*) maximizes the function $f_{(u_0,v_0)}$ over $P \cap \{(u, v) : u \leq u_0, v \leq v_0\}$ and therefore also over the smaller set $P_1 \cap \{(u, v) : u \leq u_0, v \leq v_0\}$. By definition of ψ we have $(u^*, v^*) = \psi((u_0, v_0), P_1)$. Now consider the affine transformation

$$u \mapsto au + b, \quad v \mapsto cv + d$$

where $a, c > 0$ and let P' be the image of P under this transformation. Since (u^*, v^*) maximizes $(u_0 - u)(v_0 - v)$ over $P \cap \{(u, v) : u \leq u_0, v \leq v_0\}$ it also maximizes

$$ac(u_0 - u)(v_0 - v) = ((au_0 + b) - (au + b))((cv_0 + d) - (cd + d)).$$

But this is equivalent to that $(au^* + b, cv^* + d)$ maximizes $(u'_0 - u)(v'_0 - v)$ over P' where $u'_0 = au_0 + b$ and $v'_0 = cu_0 + d$. Hence by definition of ψ we have $\psi((u'_0, v'_0), P') = (au^* + b, cv^* + d)$. To show that 5.) is satisfied assume that P is symmetric,$u_0 = v_0$ but $u^* \neq v^*$. Then $(v^*, u^*) \in P$ and by convexity of P also

$$(u', v') := \frac{1}{2}(u^*, v*) + \frac{1}{2}(v^*, u^*) \in P$$

and an easy computation shows that

$$f_{(u_0,v_0)}(u', v') = \frac{u^{*2} + 2u^*v^* + v^{*2}}{4} - (u^* + v^*)u_0 + u_0^2$$

where we made use of $u_0 = v_0$. Since we have $(u^* - v^*)^2 > 0$ we know $u^{*2} + v^{*2} > 2u^*v^*$ and therefore

$$f_{(u_0,v_0)}(u', v') > u^*v^* - (u^* + v^*)u_0 + u_0^2 = (u_0 - u^*)(v_0 - v^*) = f_{(u_0,v_0)}(u^*, v^*)$$

which is a contradiction and therefore we must have $u^*0 = v^*$. Therefore we have shows that the function defined in the first part of the proof is a Nash bargaining function. It remains to show that whenever $\tilde{\psi}$ is another Nash bargaining function then $\psi = \tilde{\psi}$. Let us therefore assume that $\tilde{\psi}$ is another bargaining function which satisfies the conditions 1.) to 5.) of Definition 4.2.2. Denote $(\tilde{u}, \tilde{v}) = \tilde{\psi}((u_0, v_0), P)$. Let us use the affine transformation

$$\begin{aligned} u' &:= \frac{u - u_0}{u_0 - u^*} \\ v' &:= \frac{v - v_0}{v_0 - v^*} \end{aligned}$$

and let P' denote the image of P under this transformation. We have

$$\begin{aligned} (u_0, v_0) &\mapsto (0, 0) \\ (u^*, v^*) &\mapsto (-1, -1) \\ (\tilde{u}, \tilde{v}) &\mapsto (\overline{u}, \overline{v}) \end{aligned}$$

where $(\overline{u}, \overline{v})$ is define by the relation above. Using property 4.) of ψ we see that $(-1, -1)$ maximizes $f_{(0,0)}(u, v) = u \cdot v$ over $P' \cap \{(u, v) | u \leq u_0, v \leq v_0\}$. Clearly $f_{(0,0)}(-1, -1) = 1$ Let $(u, v) \in P'$ s.t. $u \leq 0$, $v \leq 0$ and assume that $u + v < -2$. Then there exists $\epsilon > 0$ and $x \in \mathbb{R}$ s.t.

$$\begin{aligned} u &= -1 - x \\ v &= -1 + x - \epsilon. \end{aligned}$$

Then by joining $(-1, -11)$ with (u, v) with a line we see that for all $\lambda \in [0, 1]$ we have

$$(u', v') := (1 - \lambda)(-1, -1) + \lambda(-1 - x, -1 + x - \epsilon) \in P'.$$

Evaluating $f_{(0,0)}$ at this point gives

$$f_{(0,0)}(u', v') = (1 + \lambda x)(1 + \lambda(\epsilon - x)) = 1 + \lambda\epsilon + \lambda^2 x(\epsilon - x). \qquad \textbf{(4.2)}$$

Consider the last expression as a function in λ.Then evaluation at $\lambda = 0$ gives the value 1. The function is clearly differentiable with respect to λ and the derivative at point $\lambda = 0$ is $\epsilon > 0$. Therefore there exists an $\epsilon > 0$ such that the right hand side of equation 4.2 is strictly greater than one which is a contradiction. Therefore $0 \geq u + v \geq -2$. Now let

$$\tilde{P} = \{(u, v)|(u, v) \in P' \text{ or } (v, u) \in P'\}$$

be the symmetric closure of P'. Clearly $\tilde{P}$ is compact, convex and symmetric and $P' \subset \tilde{P}$. We still have though that $u + v \geq -2$ for all pairs $(u, v) \in \tilde{p}$. This means that whenever the point (u, u) lies in $\tilde{P} \cap \{(u, v)|u \leq u_0, v \leq v_0\}$ we have $0 \geq u \geq -1$. Now let

$$(\hat{u}, \hat{v}) := \tilde{\psi}((0, 0), \tilde{P})$$

then by property 5.) of $\tilde{\psi}$ we have $\hat{u} = \hat{v}$. Since $(-1, -1) \in P' \subset \tilde{P}$ by using the Pareto optimality of $(\hat{u}, \hat{v})$ we must have $(\hat{u}, \hat{v}) = (-1, -1)$. But then $(\hat{u}, \hat{v}) \in P'$ and by using property 3.) of $\tilde{\psi}$ we must have $(\overline{u}, \overline{v}) = (-1, -1)$. Computing the inverse under the affine transformation show

$$(\tilde{u}, \tilde{v}) = (u^*, v^*).$$

Therefore $\psi = \tilde{\psi}$ and we are finished. □

the consequence of Theorem 4.2.1 is that if the two players believe in the axioms of a Nash bargaining function, there is a unique method to settle the conflict once the status quo point is fixed. It follows directly from properties 1.) and 2.) that the value $(u^*, v^*) = \psi((u^\sharp, v^\sharp), Im(\hat{L})$ in the context of a cooperative game lies in the bargaining set.

Example 4.2.1. *Consider a two person game $\mathcal{G}_2$ where the biloss operator is given by the following matrix*

$$L := \begin{pmatrix} (-1,-2) & (-8,-3) \\ (-4,-4) & (-2,-1) \end{pmatrix}$$

A straightforward computation shows that the conservative values of this game are given by $u^\sharp = -3\frac{1}{3}$ and $v^\sharp = -2\frac{1}{2}$. The biloss region of the corresponding cooperative game is given in the following graphic.

The bargaining set is

$$B := \{(u,v)|v = -\frac{1}{4}u - 5, -4 \leq u \leq -8\}$$

Therefore to compute the value $\psi((u^\sharp, v^\sharp), Im(\hat{L}))$ we have to maximize the function

$$(-3\frac{1}{3} - u)(-2\frac{1}{2} + \frac{1}{4}u + 5$$

over $-4 \leq u \leq -8$. This is easy calculus and gives the values

$$u^* = -6\frac{2}{3}, \;\; v^* = -3\frac{1}{3}.$$

and therefore $\psi((u^\sharp, v^\sharp), Im(\hat{L})) = (-6\frac{2}{3}, -3\frac{1}{3})$.

The argumentation for the choice of the status quo point as the conservative value is reasonable but not completely binding as the following example shows :

Example 4.2.2. *Consider the two person game $\mathcal{G}_2$ where the biloss operator is given by the following matrix*

$$L := \begin{pmatrix} (-1,-4) & (1,4) \\ (4,1) & (-4,-1) \end{pmatrix}$$

The conservative value is $(u^\sharp, v^\sharp) = (0,0)$. this follows after a straightforward computation. The biloss region of the corresponding coopera-

tive game is given by the following graphic :

Since we have $u_0 = u^\sharp = v^\sharp = v_0$ and the biloss region is obviously symmetric we must have $u^* = v^*$. And the only point in the bargaining set which satisfies this condition is $(u^*, v^*) = (-2\frac{1}{2}, -2\frac{1}{2})$. The question is however is this compromise for player 1 as good as for player 2 ? Assume that the players are bankrupt if their losses exceed the value 3. We claim that the second player is in a stronger position than the first one and therefore deserves a bigger piece of the pie. If player 2 decides to play its first strategy, then if player 1 chooses his first strategy he wins 1, if he chooses his second strategy he is bankrupt. So he is forced to play his first strategy although he knows that he does not make the optimal profit. Player 1 has no comparable strategy to offer. If player 1 chooses strategy 1 it could happen that player 2 goes bankrupt by choosing the second strategy, but clearly player 2 would not do this and instead by choosing strategy 1 be perfectly happy with his maximum profit.The compromise $(2\frac{1}{2}, 2\frac{1}{2})$ though gives both players the same and is in this sense not fair.

Such a strategy as player twos first strategy is called a thread. We will develop a method which applies to the situation where the players thread their opponents. This method is also known as the thread bargaining solution[1] gives a solution of this problem.

Definition 4.2.3. *Let $\mathcal{G}_2$ be a non cooperative two person game with finite strategy sets $\mathcal{S}_1$ and $\mathcal{S}_2$. The corresponding* **thread game** *$\mathcal{T}\mathcal{G}_2$ is the non-cooperative two person game with mixed strategies $\Delta^{\mathcal{S}_1}$ and $\Delta^{\mathcal{S}_2}$ and biloss operator as follows :*

$$\mathcal{T}L(s, \tilde{s}) = \psi_{Nash}(L(s, \tilde{s}), Im(\hat{L}))$$

where ψ_{Nash} denotes the Nash bargaining function.

Basically the process is as follows. Start with a non-cooperative

[1] sometimes also Nash bargaining solution

game, consider then the corresponding cooperative game and look for the compromises given by the Nash bargaining function in dependence on the status quo points and you get back a non-cooperative two person game. However to apply the methods from chapter 2 and 3 we must know something about the continuity as well as convexity properties of this biloss operator.

Lemma 4.2.2. *Let $\mathcal{G}_2$ be a non cooperative two person game with finite strategy sets $\mathcal{S}_1$ and $\mathcal{S}_2$ and biloss operator L. then the function*

$$\begin{aligned} \Delta^{\mathcal{S}_1} \times \Delta^{\mathcal{S}_2} &\rightarrow \mathbb{R}^2 \\ (s, \tilde{s}) &\mapsto \psi_{Nash}(L(s, \tilde{s}), Im(\hat{L})) \end{aligned}$$

is continuous. The biloss operator in the corresponding thread game $\mathcal{TG}_2$ and satisfies the convexity assumption in Theorem 3.1.1.

Proof. The continuity property of ψ_{Nash} follows straightforward (but a bit technical) from its construction. Since the biloss operator in the thread game is basically ψ_{Nash} the continuity of the biloss operator is therefore clear. To show that it satisfies the convexity assumptions of Theorem 3.1.1 is a bit harder and we don't do it here. □

Theorem 4.2.2. *Let $\mathcal{G}_2$ be a non cooperative two person game with finite strategy sets and let $\mathcal{TG}_2$ be the corresponding thread game. Then $\mathcal{TG}_2$ has at least one non cooperative equilibrium. Furthermore the bilosses under $\mathcal{T}L$ of all non cooperative equilibria are the same.*

Proof. The existence of non cooperative equilibria follows from Theorem 3.1.1 and the previous lemma. That the bilosses under L of all the non cooperative equilibria are the same will follow from the following discussion. The thread game is a special case of a so called purely competitive game. □

Definition 4.2.4. *A two person game $\mathcal{G}_2$ is called a* **purely competitive game** *(sometimes also a pure conflict or antagonistic game) if all* *outcomes are Pareto optimal, i.e. if* (u_1, v_1) *and* (u_2, v_2) *are two possible outcomes and* $u_2 \leq u_1$ *the* $v_2 \geq v_1$.

Remark 4.2.1. *The thread game is a purely competitive game. This is clear since the values of the Nash Bargaining function are Pareto optimal.*

The uniqueness of the Nash bargaining solution does now follow from the following proposition.

Proposition 4.2.1. *Let* $\mathcal{G}_2$ *be a purely competitive two person game. Then the bilosses of all non-cooperative equilibria are the same.*

Proof. Let $(s, \tilde{s})$ and $(r, \tilde{r})$ be non cooperative equilibria. We have to prove

$$\begin{aligned} L_1(s, \tilde{s}) &= L_1(r, \tilde{r}) \\ L_2(s, \tilde{s}) &= L_2(r, \tilde{r}). \end{aligned}$$

W.l.o.g we assume $L_1(s, \tilde{s}) \geq L_1(r, \tilde{r})$. Since $(r, \tilde{r})$ is an NCE we have by definition of an NCE that

$$L_2(r, \tilde{s}) \geq L_2(r, \tilde{r}).$$

Since $\mathcal{G}_2$ is purely competitive this implies that

$$L_1(r, \tilde{s}) \leq L_1(r, \tilde{r}).$$

Since however $(s, \tilde{s})$ is also an NCE we have that

$$L_1(r, \tilde{s}) \geq L_1(s, \tilde{s})$$

and therefore by composing the two previous inequalities we get $L_1(s, \tilde{s}) \leq L_1(r, \tilde{r})$ so that in fact we have $L_1(s, \tilde{s}) = L_1(r, \tilde{r})$. The same argument shows that $L_2(s, \tilde{s}) = L_2(r, \tilde{r})$. □

Remark 4.2.2. *The proof above shows more. In fact it shows that also* $L_1(r,\tilde{s}) = L_1(s,\tilde{s})$ *and* $L_2(r,\tilde{s}) = L_2(s,\tilde{s})$ *and therefore that* $(r,\tilde{s})$ *and the similarly* $(s,\tilde{r})$ *are non-cooperative equilibria. In words, this means that all non-cooperative equilibria in a purely competitive game are interchangeable.*

Definition 4.2.5. *Let* $\mathcal{G}_2$ *be a non cooperative two person game with finite strategy sets. Then the* **Nash bargaining solution** *is the unique biloss under* $\mathcal{T}L$ *of any non cooperative equilibrium of the corresponding thread game* $\mathcal{T}\mathcal{G}_2$.

Strategies $s, \tilde{s}$ such that $(s,\tilde{s})$ is a NCE of the thread game are called **optimal threats**. Using the biloss under L of any pair of optimal threats as the status quo point for the Nash bargaining function delivers the Nash bargaining solution as a compromise. The Nash bargaining solution now now gives a compromise for any two person game with finite strategy sets which does not depend on status quo points. The difficulty however still is to find the NCE of the thread game. Let us illustrate the method first at an easy example. Assume the bargaining set is given as a line

$$B := \{(u,v)|au + v = b, c_1 \leq u \leq c_2\}.$$

Suppose player 1 threatens strategy s[2] and player 2 threatens strategy $\tilde{s}$. Then player 1's loss in the thread game is the u^* that maximizes

$$(L_1(s,\tilde{s}) - u)(L_2(s,\tilde{s}) - v) = (L_1(s,\tilde{s}) - u)(L_2(s,\tilde{s}) - b + au).$$

If this u^* happens to be in the interior of $[c_1, c_2]$ then it can be computed by setting the derivative with respect to u of the expression above equal to zero. This gives

[2]this means player 1 chooses strategy s playing the thread-game

$$0 = \frac{d}{du}((L_1(s,\tilde{s})-u)(L_2(s,\tilde{s})-b+au)) = -(L_2(s,\tilde{s})-b+au)+a(L_1(s,\tilde{s})-u).$$

This gives $u^* = \frac{1}{2a}(b + aL_1(s,\tilde{s}) - L_2(s,\tilde{s}))$. Let us denote with

$$L^{ij} = L(s_i, \tilde{s}_j)$$

where the s_i resp. $\tilde{s}_j$ denote the pure strategies of the original non cooperative game (i.e. the biloss operator L in (bi)-matrix form. Then we have for $s = \sum_i x_i \cdot s_i$ and $\tilde{s} = \sum_j x_j \cdot \tilde{s}_j$ that

$$u^* = \frac{1}{2a}(b + \sum_{i,j} x_i(aL_1^{i,j} - L_2^{i,j})y_j).$$

So as to make u^* as small as possible, player 1 has to choose $x = (x_1, ..., x_n) \in \Delta^n$ so as to minimize

$$\sum_{i,j} x_i(aL_1^{i,j} - L_2^{i,j})y_j \tag{4.3}$$

against any y.[3] Similarly substituting $au + v = b$ we get

$$v^* = \frac{1}{2}(b - \sum_{i,j} x_i(aL_1^{i,j} - L_2^{i,j})y_j).$$

and to minimize v^* player 2 chooses $y = (y_1, ..., y_m) \in \Delta^m$ so as to maximize expression 4.3. Therefore the NCE of the thread game, i.e. the Nash bargaining solution corresponds to the NCE of the Zero sum game

$$\tilde{L} = (aL_1^{ij} - L_2^{ij}) \cong (aL_1^{ij} - L_2^{ij}, -(aL_1 ij - L_2^{ij})). \tag{4.4}$$

If w^* denotes loss of player one when the NCE strategies of the fame $\tilde{L}$ are implemented, then

[3]remind that u^* is the loss player 1 suffers when the corresponding compromise is established

$$u^* = \frac{1}{2a}(b + w^*), \;\; v^* = \frac{1}{2}(b - w^*). \tag{4.5}$$

This NCE can be computed with numerical methods and some linear algebra. In general the biloss region is a polygon and the bargaining set is piecewise linear. One can then apply the method described above to each line segment in the bargaining set. We have therefore proven the following proposition which helps us to identify the Nash bargaining solution.

Proposition 4.2.2. *Let $\mathcal{G}_2$ be a two person game with finitely many pure strategies and bargaining set B. If the Nash bargaining solution lies in the interior of one of the lines bargaining set and this line is given by the equation $au + v = b$ with $c_1 \le u \le c_2$ then the Nash bargaining solution (u^*, v^*) is given by (4.5).*

If one knows on which line segment the Nash bargaining solution lies then one gets the Nash bargaining solution with this method. Moreover one also gets the optimal threads to implement the Nash bargaining solution. In the following we present a graphical method to decide on which line segment the Nash bargaining solution can lie. In general though it does also not give a clear answer to the problem.

Lemma 4.2.3. *Let $\mathcal{G}_2$ be a non-cooperative game such that the corresponding bargaining set is a line. let $(u^*, v^*) = \psi_{Nash}((u_0, v_0), Im(\hat{L})$ be the compromise given by the Nash bargaining function with status quo point $(u_0, v_0) \in Im(\hat{L})$. Assume that (u^*, v^*) is not one of the endpoints of the bargaining set. Then the slope of the line joining (u_0, v_0) and (u^*, v^*) is the negative of the slope of the bargaining set.*

Proof. Suppose (u^*, v^*) lies on the line $au + v = b$ for $c_1 < u < c_2$. Then it has to maximize the function $(u_0 - u)(v_0 - v)$ over this set, i.e. u^* must maximize $(u_0 - u)(v_0 + au - b)$ and therefore

$$\begin{aligned} 0 &= \frac{d}{du}_{u=u^*}(u_0-u)(v_0+au-b) = -(v_0+au^*-b)+a(u_0-u^*) \\ &= -v_0-2au^*+b+au_0. \end{aligned}$$

therefore

$$\begin{aligned} u^* &= \frac{b-v_0+au_0}{2a} \\ v^* &= \frac{b+v_0-au_0}{2}. \end{aligned}$$

The slope of the line from (u_0, v_0) to (u^*, v^*) is

$$\frac{v^*-v_0}{u^*-u_0} = \frac{\frac{b+v_0-au_0}{2}-v_0}{\frac{b-v_0+au_0}{2a}-u_0} = \frac{\frac{b-v_0-au_0}{2}}{\frac{b-v_0-au_0}{2a}} = a.$$

The slope of the bargaining set is clearly $-a$ since $au+v=b \Leftrightarrow v = -au+b$.

□

Example 4.2.3.

Coming back to the point where we actually want to determine the Nash bargaining solution and the optimal threats where the biloss region is a polygon, we can now try to solve all the non cooperative games (4.4) (one for each line segment in the bargaining set) and then check if which of the computed solutions satisfy the condition in Lemma 4.2.3.

Example 4.2.4. *Let us now reconsider Example 4.2.2 which looks symmetric at first glance but a closer inspection reveals that player 2 is in a stronger position then player 1. The biloss operator is*

$$L := \begin{pmatrix} (-1,-4) & (1,4) \\ (4,1) & (-4,-1) \end{pmatrix}$$

and the biloss region is drawn in Figure x.x. where the bargaining set can be identified with

$$B := \{-u + v = 5, -4 \leq u \leq -1\}$$

in equal $a = 1$ *and* $b = -5$. *Since the bargaining set is obviously one line, we can apply the method proposed before and see that the optimal threats are the equilibria of the non cooperative game with biloss operator given by*

$$\tilde{L} := \begin{pmatrix} 3 & -3 \\ 3 & -3 \end{pmatrix} \cong \begin{pmatrix} (3,-3) & (-3,3) \\ (3,-3) & (-3,3) \end{pmatrix}.$$

Since the entry 3 *at position* $(1,1)$ *of the matrix above is the biggest in its row as well the smallest in its column we have that all strategies of the form*

$$(\lambda s_1 + (1-\lambda)s_2, \tilde{s}_1)$$

are NCE's and the conservative value w^* *corresponds to the biloss of any of these. Therefore* $w^* = 3$ *and therefore using (4.5) we have*

$$(u^*, v^*) = (-1, -4)$$

Can we now say that this is the Nash bargaining solution ? It is, but it does not follow directly from the argumentation above. As mentioned for the argument above we assume that (u^*, v^*) *is in the interior of* B*, but* $(-1,-4)$ *is not. What can we say though, is that in no case the Nash bargaining solution can lie in the interior of* B *and therefore must be either* $(-1,-4)$ *or* $(-4,-1)$. *The second one is unlikely because we already saw that the second player is in a stronger position within this game. For a precise argument we assume that* $(-4,-1)$ *is the Nash bargaining solution. The thread strategies leading to this value can be identified by using Lemma 4.2.3 as* $(\lambda s_1 + (1-\lambda)s_2, \tilde{s}_2)$. *It is easy to see that those strategies are no NCE strategies for the thread game. Therefore the Nash bargaining solution is* $(-1,-4)$ *and we see that in fact it gives more to the second player than to the first.*

4.3 N-person Cooperative Games

Let us shortly reconsider the definition of an N-person game in chapter 3. An N-person game $\mathcal{G}_n$ consists of a set $N = \{1, 2, .., n\}$ of players and

1. Topological spaces $\mathcal{S}_1, .., \mathcal{S}_n$ so called strategies for player 1 to n
2. A subset $\mathcal{S}(N) \subset \mathcal{S}_1 \times .. \times \mathcal{S}_n$, the so called allowed or feasible multi strategies
3. A (multi)-loss operator $L = (L_1, ..., L_n) : \mathcal{S}_1 \times ... \times \mathcal{S}_n \to \mathbb{R}^n$

In this section we assume that $\mathcal{S}(N) = \mathcal{S}_1 \times .. \times \mathcal{S}_n$ and that $|\mathcal{S}_i| < \infty$. We will think of the strategies as pure strategies and use mixed strategies which can then be identified as simplexes in the same way as in chapter 2.

Definition 4.3.1. *A coalition is a subset $S \subset N$ which cooperates in the game. If $S = \{i_1, ..., i_k\}$ then by cooperating it can use jointly randomized strategies from the set*

$$\Delta^S := \Delta^{\mathcal{S}_{i_1} \times ... \times \mathcal{S}_{i_k}}$$

and by implying the strategy $\tilde{x} \in \Delta^S$ against the strategies of their opponents receives a **joint loss** *of*

$$\sum_{i \in S} L_i(\tilde{x}, y).$$

Writing $L_i(\tilde{x}, y)$ we mean the value of the multi-linear extended version of L_i where the components are in the right order, i.e. player i-th strategies stand at position i. We use the following notation

$$\begin{aligned} X_S &:= \mathcal{S}_{i_1} \times ... \times \mathcal{S}_{i_k} \\ Y_{N \backslash S} &:= \mathcal{S}_{j_1} \times ... \times \mathcal{S}_{j_{n-k}} \end{aligned}$$

where $N \setminus S = \{j_1, ..., j_{n-k}\}$. The worst that can happen for the coalition $S4$ is that their opponents also build a coalition $N \setminus S$. The minimal loss the coalition S can the guarantee for itself, i.e. the conservative value for coalition S is given by

$$\tilde{\nu}(S) = \min_{\tilde{x} \in \Delta^S} \max_{\tilde{y} \in \Delta^{N \setminus S}} \sum_{i \in S} L_i(\tilde{x}, \tilde{y}).$$

The following lemma says that in order to compute the conservative value one only has to use pure strategies.

Lemma 4.3.1. *In the situation above one has*

$$\tilde{\nu}(S) = \min_{x \in X_S} \max_{y \in Y_{N \setminus S}} \sum_{i \in S} L_i(x, y).$$

Proof. Due to the fact that the properly denoting of the elements in the simplices Δ^S and $\Delta^{N \setminus S}$ affords a lot of indices we do not prove the result here. Managing the complex notation the proof is in fact a straightforward result which only uses that the multi-loss operator used here is the multi-linear extension of the multi-loss operator for finitely many strategies. $\square$

Definition 4.3.2. *Let $\mathcal{G}_n$ be an N-person game. We define the characteristic function ν of $\mathcal{G}_n$ via*

$$\begin{array}{rcl} \nu : \mathcal{P}(N) & \rightarrow & [0, \infty) \\ S & \mapsto & -\tilde{\nu}(S) \end{array}$$

with the convention that $\nu(\emptyset) = 0$.

Proposition 4.3.1. *Let ν be the characteristic function of the N-person game $\mathcal{G}_n$. If $S, T \subset N$ with $S \cap T = \emptyset$ then*

$$\nu(S \cup T) \geq \nu(S) + \nu(T).$$

Proof. Since $\nu(S) = -\tilde{\nu}(S)$ we can as well prove $\tilde{\nu}(S \cup T) \leq \tilde{\nu}(S) + \tilde{\nu}(T)$. Using Lemma 4.3.1 we have

$$\begin{aligned}
\tilde{\nu}(S \cup T) &= \min_{x \in X_{S \cup T}} \max_{y \in Y_{N \setminus (S \cup T)}} \sum_{i \in (S \cup T)} L_i(x, y) \\
&\leq \min_{\alpha \in X_S} \min_{\beta \in X_T} \max_{y \in Y_{N \setminus (S \cup T)}} \sum_{i \in (S \cup T)} L_i(\alpha, \beta, y).
\end{aligned}$$

Hence for each $\alpha \in X_S$,$\beta \in X_T$ we have

$$\begin{aligned}
\tilde{\nu}(S \cup T) &\leq \max_{y \in Y_{N \setminus (S \cup T)}} \sum_{i \in (S \cup T)} L_i(\alpha, \beta, y) \\
&\leq \max_{y \in Y_{N \setminus (S \cup T)}} \sum_{i \in S} L_i(\alpha, \beta, y) + \max_{y \in Y_{N \setminus (S \cup T)}} \sum_{i \in T} L_i(\alpha, \beta, y).
\end{aligned}$$

In particular this holds for β which minimizes the first sum on the right side and α which minimizes the second sum on the right side and therefore we have

$$\begin{aligned}
\tilde{\nu}(S \cup T) &\leq \min_{\beta \in X_T} \max_{y \in Y_{N \setminus (S \cup T)}} \sum_{i \in S} L_i(\alpha, \beta, y) + \min_{\alpha \in X_S} \max_{y \in Y_{N \setminus (S \cup T)}} \sum_{i \in T} L_i(\alpha, \beta, y) \\
&\leq \min_{\beta \in X_T} \max_{(\alpha, y) \in Y_{N \setminus T}} \sum_{i \in S} L_i(\alpha, \beta, y) + \min_{\alpha \in X_S} \max_{(\beta, y) \in Y_{N \setminus S}} \sum_{i \in T} L_i(\alpha, \beta, y) \\
&\leq \min_{\beta \in X_T} \max_{y \in Y_{N \setminus T}} \sum_{i \in S} L_i(\beta, y) + \min_{\alpha \in X_S} \max_{y \in Y_{N \setminus S}} \sum_{i \in T} L_i(\alpha, y) \\
&= \tilde{\nu}(T) + \tilde{\nu}(S).
\end{aligned}$$

$\square$

Definition 4.3.3. *An N-person game $\mathcal{G}_n$ is called* **inessential** *if its characteristic function is additive, i.e. $\nu(S \cup T) = \nu(S) \cup \nu(T)$ for $S, T \subset N$ s.t. $S \cap T = \emptyset$. In this case one has $\nu(N) = \sum_{i=1}^{n} \nu(\{i\})$.*

The economical interpretation of an inessential game is that in such a game it does not pay to build coalitions, since the coalition cannot guarantee more to its members as if the individual members act for themselves without cooperating.

Definition 4.3.4. *An N-person game $\mathcal{G}_n$ is called* **essential** *if it is not inessential.*

Let us illustrate the concepts at the following example :

Example 4.3.1. Oil market game *: Assume there are three countries which we think of the players in our game.*

1. *Country 1 has oil an its industry can use the oil to achieve a profit of a per unit.*
2. *Country 2 has no oil, but has an industry which can use the oil to achieve a profit of b per unit.*
3. *Country 3 also has no oil, but an industry which can use the oil to achieve a profit of c per unit.*

We assume that $a \leq b \leq c$. The strategies of the players are as follows. The strategies of Country 2 and Country 3 are buy the oil from country 1 if it is offered to them. These are the only reasonable strategies for them, since without oil there industry does not work. Country 1 however has three strategies.

$$\begin{aligned} s_1 &= \textit{keep the oil and use it for its own industry} \\ s_2 &= \textit{sell the oil to Country 2} \\ s_3 &= \textit{sell the oil to Country 3 .} \end{aligned}$$

Denoting with $\tilde{s}, \hat{s}$ the strategies of Country 2 resp. Country 3 the multi-loss operator L of the game is given as follows :

$$\begin{aligned} L(s_1, \tilde{s}, \hat{s}) &= (a, 0, 0) \\ L(s_2, \tilde{s}, \hat{s}) &= (0, b, 0) \\ L(s_3, \tilde{s}, \hat{s}) &= (0, 0, c). \end{aligned}$$

Clearly the strategies 2 *and* 3 *for Country* 1 *do only make sense if it cooperate with the corresponding country and shares the losses (wins). From this it is easy to compute the corresponding values for* ν*, in fact we have*

$$\begin{aligned} 0 &= \nu(\emptyset) = \nu(\{2\}) = \nu(\{3\}) = \nu(\{2,3\}) \\ a &= \nu(\{1\}) \\ b &= \nu(\{1,2\}) \\ c &= \nu(\{1,3\}) == \nu(\{1,2,3\}) \end{aligned}$$

As usually when working with numerical values of utility (as is always the case when working with loss or multi-loss operators) one wished the concept to be independent of the scale of utility. For this purpose one introduces the notion of strategic equivalence.

Definition 4.3.5. *Two characteristic functions* $\nu, \hat{\nu}$ *are called* **strategically equivalent** *if there exists* $c > 0$ *and* $a_i \in \mathbb{R}^N$ *s.t.* $\forall S \in \mathbb{P}(N)$ *one has*

$$\nu'(S) = c\nu(S) + \sum_{i \in S} a_i.$$

Two N*-person games are called strategically equivalent if their characteristic functions are equivalent.*

One can check that strategical equivalence is in fact an equivalence relationship, i.e. symmetric, reflexive and transitive. We have the following proposition :

Proposition 4.3.2. *Every essential N-person game $\mathcal{G}_n$ is strategically equivalent to an N-person game $\hat{\mathcal{G}}_n$ with characteristic function $\hat{\nu}'$ satisfying*

$$\begin{aligned} \hat{\nu}(N) &= 1 \\ \hat{\nu}(\{i\}) &= 0 \end{aligned}$$

for $i = 1, 2, ..., n$.

Proof. Let ν be the characteristic function of $\mathcal{G}_n$. We define $\hat{\nu}$ as follows :

$$\hat{\nu}(S) := c\nu(S) - c\sum_{i \in S} \nu(\{i\})$$

with $c := (\nu(N) - \sum_{i=1}^{n} \nu(\{i\}))^{-1}$. One can check that $\hat{\nu}$ satisfies the condition stated in the proposition. Furthermore it follows from the definition of a characteristic function that $\hat{\nu}$ is in fact the characteristic function of the game $\hat{\mathcal{G}}_n$ which has the same strategies as $\mathcal{G}$ but with multi-loss-operator given by $\hat{L}_i = c \cdot (L_i - \nu(\{i\}))$ for $i = 1, ..., n$. □

When the players join a coalition they have to decide how they share their loss once the game is over. This leads to the definition of an imputation.

Definition 4.3.6. *An imputation in an N-person game $\mathcal{G}_n$ with characteristic function $\mathcal{G}_n$ is a vector $x = (x_1, .., x_n)^\top \in \mathbb{R}^n$ s.t.*

$$\begin{aligned} \sum_{i=1}^{n} x_i &= \nu(N) \\ x_i &\geq \nu(\{i\}). \end{aligned}$$

We denote the set of all imputations of $\mathcal{G}_n$ respectively its characteristic function with $E(\nu)$.

We interpret imputations as follows : x_i is the i-th players award (negative loss) after the game is played. The second condition says that whichever coalition player i chooses, he must do at least as good if he would act alone. This is economically reasonable, since no one would enter coalition if it would not pay for him. Furthermore from an economical point of view it is clear that $\sum_{i=1}^{n} x_i \leq \nu(N)$ because when coalition N is implemented all players work together and the sum of the individual awards must be less that the collective award. If $\sum_{i=1}^{n} x_i$ would be strictly less than $\nu(N)$ then the players in the game would be better of to work together using the strategies to implement $\nu(N)$ and then give x_i to player i and equally share the difference $\nu(N) - \sum_{i=1}^{n} x_i$. The each player does strictly better. So the second condition in the Definition of an imputation is in fact a Pareto condition.

Example 4.3.2. *For the oil market game of Example 4.3.1 we have*

$$E(\nu) = \{(x_1, x_2, x_3)^\top | x_1 \geq a, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 + x_3 = c\}$$

Can one imputation be better than another one ? Assume we have two imputations $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ then

$$\sum_{i=1}^{n} x_i = \nu(N) = \sum_{i=1}^{n} y_i.$$

Therefore if for one i we have $x_i < y_i$ then there is also a j s.t. $x_j > y_j$. So an imputation can't be better for everyone, but it is still possible that for a particular coalition x is better than y. This leads to the idea of **domination** of one imputation by another.

Definition 4.3.7. *Let x, y be two imputations and $S \subset N$ be a coalition. We say that x dominates y over S if*

$$\begin{aligned} x_i &> y_i, \textit{ for all } i \in S \\ \sum_{i \in S} x_i &\leq \nu(S). \end{aligned}$$

In this case we write $x >_S y$.

The economical interpretation of the second condition above is that the coalition S has enough payoff to ensure its members the awards x.

Definition 4.3.8. *Let x and y be two imputations of an N-person game $\mathcal{G}_n$. We say x dominates y if there exists a coalition S s.t. $x >_S y$. In this case we write $x \succ y$.*

The next definition is about the core of an N-person game. We should distinguish the core defined in the context of cooperative N-person games from the core defined in section 1 (see Definition 1.2.9).

Definition 4.3.9. *Let $\mathcal{G}_n$ be a cooperative N-person game with characteristic function ν. We define its* **core** *as the set of all imputations in $E(\nu)$ which are not dominated (for any coalition). Denoting the core with $C(\nu)$ this means that*

$$C(\nu) = \{x \in E(\nu) | \textit{ exists no } y \textit{ s.t. } y \succ x\}.$$

Theorem 4.3.1. *A vector $x \in \mathbb{R}^n$ is in the core $C(\nu)$ if and only if*

1. $\sum_{i=1}^{n} x_i = \nu(N)$
2. $\sum_{i \in S} \geq \nu(S)$ *for all* $S \subset N$.

Proof. Let us assume first that $x \in \mathbb{R}^n$ satisfies the conditions 1.) and 2.) above. Then for $S = \{i\}$ condition 2.) implies that $x_i \geq \nu(\{i\})$ so it follows from condition 1.) that x is in fact an imputation. Suppose now that x would be dominated by another imputation y. Then there exists

$S \subset N$ s.t. $y >_S X$ i.e. $y_i > x_i$ for all $i \in S$ and $\nu(S) \geq \sum_{i \in S} y_i$. Condition 2.) would then imply that

$$\nu(S) \geq \sum_{i \in S} y_i > \sum_{i \in S} x_i \geq \nu(S)$$

which of course is a contradiction. Therefore the imputation x is not dominated and hence belongs to the core $C(\nu)$. Assume now on the other side that $x \in C(\nu)$. Then x is an imputation, so condition 1.) must hold. Assume condition 2.) would not hold. Then there would exist $S \neq N$ s.t. $\sum_{i \in S} < \nu(S)$. We define

$$\epsilon := \frac{\nu(S) - \sum_{i \in S} x_i}{|S|} > 0$$

and

$$y_i := \begin{cases} x_i + \epsilon & \forall i \in S \\ \nu(\{i\}) + \frac{(\nu(N) - \nu(S) - \sum_{i \in N \setminus S} \nu(\{i\})}{|N-S|} & \forall i \in N \setminus S. \end{cases}$$

Then $\sum_{i=1}^{n} y_i = \nu(N)$ and $y_i \geq \nu(\{i\})$.[4] Moreover

$$\sum_{i \in S} y_i = \nu(S)$$

and $y_i > x_i$ for all $i \in S$. Therefore $y >_s x$ and x is dominated. This is a contradiction to $x \in C(\nu)$ and hence x must satisfy condition 2.).

□

Let us remark that the core of an N-person cooperative game is always a convex and closed set. It has one disadvantage though, in many cases it is empty.This is in particular the case for constant sum games, as we will see next.

Definition 4.3.10. *An N-person game is called a constant sum game if $\sum_{i \in N} L_i \equiv c$ where c is a constant and L_i denotes the loss operator of the i-th player.*

[4]since $\nu(N) = \nu(N \setminus S \cup S) \geq \nu(N \setminus S) + \nu(S)$ and therefore $\nu(N) - \nu(S) \geq \nu(N \setminus S) \geq \sum_{i \in N \setminus S} \nu(\{i\})$ for $i \in N \setminus S$ and $x_i \geq \nu(\{i\})$ for $i \in S$

Lemma 4.3.2. *Let ν be the characteristic function of a constant sum game, then for all $S \subset N$ we have*

$$\nu(N \setminus S)'\nu(S) = \nu(N).$$

Proof. This follows from the definition of ν by applying the MiniMax Theorem of section 2.5. to the two person zero sum game where $\mathcal{S}_1 = Y_{N\setminus S}$ and $\mathcal{S}_2 = X_S$ and the Loss-operator is given by

$$L = \sum_{i \in N\setminus S} (L_i - \frac{c}{N})$$

where $c = \sum_{i \in N} L_i$. □

Proposition 4.3.3. *If ν is the characteristic function of an essential N-person constant sum game. Then $C(\nu) = \emptyset$.*

Proof. Assume $x \in C(\nu)$ then by condition 2.) in Theorem 4.3.1 and the previous lemma we have for any $i \in N$

$$\sum j \neq ix_j \geq \nu(N \setminus \{i\}) = \nu(N) - \nu(\{i\}).$$

Since x is an imputation we also have

$$x_i + \sum j \neq ix_j = \nu(N).$$

Combining the two gives $\nu(\{i\}) \geq x_i$ for all $i \in N$. Using that ν is an essential game we get that

$$\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} \nu(\{i\}) < \nu(N)$$

which is a contradiction to x being an imputation. □

Let us study the core of the oil-market game :

Example 4.3.3. *By looking at Example 4.3.1 and Theorem 4.3.1 we see that $x = (x_1, x_2, x_3)^\top \in C(\nu)$ if and only if*

1. $x_1 + x_2 + x_3 = c$

2. $x_1 \geq a, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 \geq b, x_2 + x_3 \geq 0, x_1 + x_3 \geq c$

This however is the case if and only if $x_2 = 0$,$x_1 + x_3 = c$ *and* $x_1 \geq b$ *and therefore*

$$C(\nu) = \{(x, 0, c - x) | b \leq x \leq c\}$$

The fact that in lot of cases the core is just the empty set leads to the question what other solution concepts for cooperative N-person games are reasonable. We discuss two more, the so called stable sets and the Shapely value.

Definition 4.3.11. *A stable set* $S(\nu)$ *of an* N*-person game with characteristic function* ν *is any subset* $S(\nu) \subset E(\nu)$ *of imputations satisfying*

1. $x, y, \in S(\nu)$ *then* $x \not\succ y$ *and* $y \not\succ x$ *(internal stability)*

2. *if* $z \notin S(\nu)$ *then there is an* $x \in S(\nu)$ *s.t.* $x \succ z$ *(external stability)*

Remark 4.3.1. *We have* $C(\nu) \subset S(\nu) \subset E(\nu)$ *for any stable set* $S(\nu)$ *because undominated imputations must be within any stable set. More precisely* $C(\nu) \subset \bigcap_{S(\nu)\ \textit{stable}} S(\nu)$. *In general the inclusion is proper. It was long time a question whether all cooperative games contain stable sets. The answer to this question is no (Lucas 1968).*

Exercise 4.3.1. *Compute a stable set for the oil market game.*

Definition 4.3.12. *An* N*-person game is called* **simple** *if for all* $S \subset N$ *we have that* $\nu(S) \in \{0, 1\}$. *A* **minimum winning coalition** S *is one where* $\nu(S) = 1$ *and* $\nu(S \setminus \{i\}) = 0$ *for all* $i \in S$.

The following proposition leads to various examples of stable sets within simple games.

Proposition 4.3.4. *Let ν be the characteristic function of a simple game and S a minimum winning coalition. then*

$$V_S = \{x \in E(\nu) | x_i = 0 \forall i \in S\}$$

is a stable set.

One of the more famous solution concepts in N-person cooperative game is the so called **Shapley value**. It should be compared to the Nash bargaining solution within two person cooperative games.

Theorem 4.3.2. Shapley *: Denote with $\mathcal{V}$ the set of all characteristic functions $\nu : N \to [0, \infty)$. Then there exists exactly one function*

$$\phi = (\phi_i) : \mathcal{V} \to \mathbb{R}^n$$

which satisfies the following conditions :

1. *$\phi_i(\nu) = \phi_{\pi(i)}(\pi\nu)$ for all $\pi \in Perm(N)$[5] where $\pi\nu$ denotes the characteristic function of the game which is constructed from ν by reordering the numbers of the players corresponding to the permutation π.*
2. *$\sum_{i=1}^n \phi_i(\nu) = \nu(N)$ for all $\nu \in \mathcal{V}$.*
3. *$\mu, \nu \in \mathcal{V} \Rightarrow \phi(\mu + \nu) = \phi(\mu) + \phi(\nu)$.*

This unique function is given by

$$\phi_i(\nu) = \sum_{i \in S \subset N} \frac{(|S| - 1)!(n - |S|)!}{n!} \cdot (\nu(S) - \nu(S - \{i\})$$

$\phi(\nu)$ is called the **Shapley value** *of ν.*

We will not proof this theorem but instead motivate the formula and discuss the underlying idea. The idea of Shapley was as follows :

[5] Here $Perm(N)$ denotes the permutation-group of N, i.e. the group of bijective maps $\pi : N \to N$

the players arrive one after another at the negotiation table, but they arrive in random order. Each time a new player arrives at the negotiation table the negotiations will be extended to include the newly arrived player. If when player i arrives the players $S\setminus\{i\}$ are already sitting at the negotiation table, then the award for player i from the new negotiations should be what he brings in for the extended coalition, namely $\nu(S)-\nu(S\setminus\{i\})$. The probability that player i arrives when players $S\setminus\{i\}$ are already sitting at the negotiation table is

$$\frac{(|S|-1)!(n-|S|)!}{n!}.$$

The Shapely value can therefore be considered as the expectational award for player i. For the oil market game we get the following :

Example 4.3.4. *For the oil market game on computes* $\phi_1(\nu)=\frac{1}{2}c+\frac{1}{3}a+\frac{1}{6}b$*,* $\phi_2(\nu)=\frac{1}{6}b+\frac{1}{6}a$ *and* $\phi_3(\nu)=\frac{1}{2}c-\frac{1}{6}a-\frac{1}{3}b$*.*

Chapter 5

Differential Games

So far we considered only games which are static in the way that first the players choose their strategies, then the game is played with the chosen strategies and an output measured in loss or utility is determined by the rules of the game. in this chapter we will develop a concept of dynamic games which are played over a time interval $[0,T]$ and in which the players can choose their strategies at time t depending on the information they obtained by playing the game up to this time.

5.1 Setup and Notation

Assume we have the following ingredients : We have players "i" $i = 1,..,N$ which can choose strategies

$$\begin{aligned} \gamma^i : [0,T] &\times & Map([0,T],\mathbb{R}^n) \rightarrow \mathbb{R}^{m_i} \\ (t,x(\cdot)) &\mapsto & \gamma(t,x(\cdot)). \end{aligned}$$

We denote the i-th players strategies with Γ^i. As before an element $(\gamma^1,...,\gamma^N)$ of $\Gamma^1 \times ... \times \Gamma^N$ is called a **multi-strategy** . Furthermore we assume we have a function

$$
\begin{array}{rcl}
f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{m_1} \times ... \times \mathbb{R}^{m_N} & \to & \mathbb{R}^n \\
(t, x, u^1, ..., u^N) & \mapsto & f(t, x, u^1, ..., u^N).
\end{array}
$$

Definition 5.1.1. *A multi-strategy* $(\gamma^1, ..., \gamma^N) \in \Gamma^1 \times ... \times \Gamma^N$ *is called* **admissible** *for the initial state* $x_0 \in \mathbb{R}^n$ *if for all* $i = 1, .., N$

1. $\gamma^i(t, x(\cdot))$ *depends on* $x(\cdot)$ *only through the values* $x(s)$ *for* $0 \leq s \leq \tau^i(t)$ *where* $\tau^i \in Map([0, T], [0, T])$
2. $\gamma^i(t, x(\cdot))$ *is piecewise continuous*
3. *the following differential equation has a unique solution in* $x(\cdot)$

$$
\begin{array}{rcl}
\frac{dx(t)}{dt} & = & f(t, x(t), \gamma^1(t, x(\cdot), ..., \gamma^N(t, x(\cdot) \\
x(0) & = & x_0.
\end{array}
$$

The function τ^i *is called player* i*'s* **information structure**

Definition 5.1.1 has to be interpreted in the following way. The function f represents the rules of the games and by the differential equation above determines which strategies can actually be used by the players. The condition concerning the information structure say how much information the players can use to choose their strategies at the corresponding time. We denote the set of admissible strategies with Γ_{adm}. Clearly Γ_{adm} depends on the function f, the initial state x_0 and the information structures τ^i. Sometimes we use the notation

$$u^i(t) = \gamma^i(t, x(\cdot)$$

which has to be interpreted in the way that $(\gamma^1, ..., \gamma^N) \in \Gamma_{adm}$ and the function $x(\cdot)$ is the unique solution of the differential equation above. The payoff in our differential game is measured in values of utility (not in loss as in the previous chapter) and for a multi-strategy $(\gamma^1, ..., \gamma^N) \in \Gamma_{adm}$ player i's utility is given by an expression of the form

$$J^i(\gamma^1, ..., \gamma^N) = \int_0^T \phi^i(t, x(t), u^1(t), ..., u^N(t))dt + g^i(x(T)).$$

in this expression the time integral represents the utility taken from consumption over time, the function g^i represents the utility which is determined by the final state of the game. In the following we will make use of the following technical assumptions :

1. For $i = 1, ..N$ the functions $\phi(t, \cdot, u^1, ..., u^N)$ and $g^i(\cdot)$ are continuously differentiable in the state variable x
2. The function f is continuous in all its variables and also continuously differentiable depending the state variable x.

Remark 5.1.1. *If the function f is Lipschitz continuous admissibility of strategies is a very mild condition due to theorems from the theory of ordinary differential equations. In general though in the theory of differential games one does not assume Lipschitz continuity of f.*

The concept of Nash equilibria can now be directly applied within the concept of differential games. We just repeat, this time in the context of utility rather than loss :

Definition 5.1.2. *An admissible multi-strategy $(\gamma^1_\sharp, ..., \gamma^N_\sharp) \in \Gamma_{adm}$ is called a* **Nash Equilibrium** *if*

$$\begin{array}{ccl} J^1(\gamma^1_\sharp, ..., \gamma^N_\sharp) & \geq & J^1(\gamma^1, ..., \gamma^N_\sharp)\ \forall \gamma^1 \textit{ s.t. } (\gamma^1, ..., \gamma^N_\sharp) \in \Gamma_{adm} \\ . & . & . \qquad\qquad . \\ . & . & . \qquad\qquad . \\ J^N(\gamma^1_\sharp, ..., \gamma^N_\sharp) & \geq & J^1(\gamma^1_\sharp, ..., \gamma^N)\ \forall \gamma^N \textit{ s.t. } (\gamma^1_\sharp, ..., \gamma^N) \in \Gamma_{adm}. \end{array}$$

We denote the values on the left side with $J^i\sharp$ and call $(J^1\sharp, ..., J^N\sharp)$ the **Nash-outcome**

We discussed this Equilibrium concept in detail. the question is again whether such Equilibria exist. We do no longer have compactness of the strategy sets and therefore cannot apply the Theorems of chapter 2 and chapter 3. More techniques from Functional Analysis and Differential Equations are necessary. We will address this problem later and in the meantime introduce another Equilibrium concept which was also previously discussed in section 1.3. the so called Stackelberg Equilibrium. We end this section by introducing the most important information structures, there are many more.

Definition 5.1.3. *Player "i"'s information structure τ^i is called*

1. **open loop** *if $\tau^i(t) = 0$ for all t, that is player "i" does not use any information on the state of the game when choosing his strategies.*
2. **closed loop** *if $\tau^i(t) = t$ for all t that is player "i" uses all possible information he can gather by following the game.*
3. ϵ**-delayed closed loop** *if* $\tau^i(t) = \left\{ \begin{array}{cl} t-\epsilon & \text{if } t \in [\epsilon, T] \\ 0 & \text{if } t \in [0,\epsilon) \end{array} \right.$.

5.2 Stackelberg Equilibria for 2 Person Differential Games

In the framework of differential games another equilibrium concept has shown to be successfully, this is the concept of Stackelberg equilibria. It is normally applied within games where some players are in a better position then others. The dominant players are called the leaders and the sub-dominant players the followers. One situation where this concept is very convincing is when the players choose their strategies one after another and the player who does the first move has an advantage. This interpretation however is not so convincing in continuous time where players choose theories strategies at each time, but one could think of that one player has a larger information structure. However from a mathematical point the concept is so successful because it leads to results. In this course we will only consider Stackelberg Equilibria for 2 person differential games and first illustrate the concept in a special case.

Assume player "1" is the leader and player "2" is the follower. Suppose that player "2" chooses his strategies as a reaction of player "1"'s strategy in a way that when player "1" chooses strategy γ^1 the player "2" chooses a strategy from which he gets optimal utility given that player "1" plays γ^1. We assume for a moment that there is such a strategy and that it is unique. We denote this strategy with γ^2 and get a map

$$\begin{aligned} T : \Gamma^1 &\rightarrow \Gamma^2 \\ \gamma^1 &\mapsto T(\gamma^1) = \gamma_2. \end{aligned}$$

To maximize his own utility player "1" would the choose a strategy γ^1_* which optimizes $J^1(\gamma^1, T(\gamma^1))$ i.e.

$$J^1(\gamma^1_*, T(\gamma^1_*)) \geq J^1(\gamma^1, T(\gamma^1))$$

for all $\gamma^1 \in \Gamma^1$. $\gamma^2_* = T(\gamma^1_*)$ would be the optimal strategy for player 2 under this consideration. However we have to assume some assumptions on admissibility in the previous discussion and also in general the existence and uniqueness of the strategy $\gamma^2 = T(\gamma^1)$ is not guaranteed.

Definition 5.2.1. *The* **optimal reaction set** $R^2(\gamma^1)$ *of player "2" to player "1"'s strategy* γ^1 *is defined by*

$$R^2(\gamma^1) = \{\gamma \in \Gamma^2 | J^2(\gamma^1, \gamma) \geq J^2(\gamma^1, \gamma^2) \, \forall \gamma^2 \in \Gamma^2 \textit{ s.t. } (\gamma^1, \gamma^2) \in \Gamma_{adm}\}.$$

Definition 5.2.2. *In a 2 person differential game with player "1" as a leader and player "2" as a follower a strategy* $\gamma^1_* \in \Gamma^1$ *is called a* **Stackelberg Equilibrium Solution** *for the leader if*

$$\min_{\gamma \in R^2(\gamma_*^1)} J^1(\gamma_*^1, \gamma) \geq \min_{\gamma \in R^2(\gamma^1)} J^1(\gamma^1, \gamma)$$

for all γ^1 *in* Γ^1 *where we assume that* $\min \emptyset = \infty$ *and* $\infty \not\geq \infty$. *We denote the left hand side with* J_*^1 *and call it the Stackelberg payoff for the leader and for any* $\gamma_*^2 \in R^2(\gamma_*^1)$ *we call the pair* (γ_*^1, γ_*^2) *a Stackelberg equilibrium solution and* $(J^1(\gamma_*^1, \gamma_*^2), J^2(\gamma_*^1, \gamma_*^2))$ *the Stackelberg equilibrium outcome of the game. Even if Stackelberg solution for the leader and the optimal reaction for the follower are not unique the Stackelberg equilibrium outcome is unique.*

5.3 Some Results from Optimal Control Theory

One ccould say that optimal control problems are 1-person differential games and most of the theory of differential games uses results from there. We will therfore briefly consider the optimal control problem and state the Pontryagin principle which gives necessary conditions for a control to be optimal.

Suppose that the state of a dynamical system evolves according to a differential equation :

$$\begin{aligned} \frac{dx(t)}{dt} &= f(t, x(t =, u(t)) \\ x(0) &= 0 \end{aligned}$$

where $u(\cdot)$ denotes a control function which can be chosen within the set of admissible controls, i.e. maps $u : [0, T] \to \mathbb{R}^m$ s.t. the differential equation from above has a unique solution. The problem in optimal control theory is to choose u in a way that it maximizes a certain functional

$$J(u) = \int_0^T \phi(t, x(t), u(t))dt + g(X(T)).$$

We assume that the functions f, ϕ and g satisfy similar conditions as in section 5.1. In the economic literature a pair $(x(\cdot), u(\cdot))$ which satisfies the differential equation from above is called an **program** and an **optimal program** $(x^*(\cdot), u^*(\cdot))$ is one that maximizes J. The following principle is known as the **Pontryagin principle** and gives necessary conditions for a program to be an optimal program.

Theorem 5.3.1. *Suppose* $(x^*(\cdot), u^*(\cdot))$ *is an optimal program. Then there exists an* $\mathbb{R}^n$ *valued function*

$$p : [0, T] \to \mathbb{R}^n$$

called the costate vector (or sometimes multiplier function) s.t. if we define the function

$$H(t, x, u, p) = \phi(t, x, u) + < p, f(t, x, u) >$$

for all $(t, x, u, p) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ *then the following conditions hold for* $i = 1, .., n$ *:*

$$\begin{aligned}
\frac{dx_i^*(t)}{dt} &= = \frac{\partial H}{\partial p^i}(t, x^*(t), u^*(t), p(t)) = f_i(t, x(t =, u(t)) \\
x_i(0) &= x_{i0} \\
\frac{dp^i(t}{dt}(t) &= -\frac{\partial H}{\partial x_i}(t, x^*(t), u^*(t), p(t)) \\
p^i(T) &= \frac{\partial g}{\partial x_i})(x^*(T))
\end{aligned}$$

an if $U_{adm} \subset \mathbb{R}^m$ *denotes the set of vectors* u *s.t.* $u(\cdot) \equiv u$ *is admissible then*

$$H(t, x^*(t), u^*(t), p(t)) = \max_{u \in U_{adm}} H(t, x^*(t), u, p(t))$$

for almost all t.

The method for finding the optimal control now works basically as when using the Lagrange multiplier method in standar calculus. If one knwos from a theoretical consideration that there must exist and optimal program, then the necessary condition above help to determine it.

We will illustrate in the following example from economics how the costate vector can be interpreted. We consider an economy with a constant labour force and capital $K(t)$ at time $z \in [0,T]$ where the labour force can use some technology F in the way that it produces a product worth

$$Y(t) = F(K(t))$$

where $F(0) = 0$[1]. For the function F we assume that it has a "decreasing return to scale" property i.e.

$$\frac{dF(K)}{dK} > 0, \frac{d^2F}{dK^2} < 0$$

which from an economical point of view is very reasonable and should be interpreted in the way that one can produce more, given more capital, but that the effectivity becomes less the more capital is used in the production. We assume that the memebrs of our economy consume some of the capital and the rate of consumption at time t will be denoted with $C(t)$. Furthermore we assume that the capital depreciation is given by the constant μ. Then the evolution of capital in our economy is given by the following differential equation :

$$\begin{aligned} \frac{dK(t)}{dt} &= F(K(t)) - C(t) - \mu K(t) \\ K(0) &= K_0. \end{aligned}$$

[1]This basically mean you cannot produce anything out of nothing

where K_0 denotes the initial capital. Let U be th utility function which measures the utility taken from consumption. As always for utility functions we assume that

$$\frac{dU(C)}{dC} > 0, \frac{d^2U(C)}{dC^2} < 0.$$

The members in our economy also benefit from the final capital stock and this benefit is given by $g(K(T))$ where g is another utility function. The total utility from choosing the consumption $C(\cdot)$ over $[0,T]$ is the given by

$$W(C(\cdot)) = \int_0^T u(C(t))dt + g(K(T)).$$

Our economy would now like to hoose $C(\cdot)$ in a way that it maximizes this utility. Let us apply the Pontryagin pinciple. The Hamilton function for this problem is given by

$$H(t,K,C,p) = u(C) + p \cdot (F(K) - C - \mu K).$$

The Pontryagin principle says that the optimal consumption rate at time t satisfies $C^*(\cdot)$

$$0 = \frac{\partial H}{\partial C}(t, K^*(t), C^*(t), p(t)) = \frac{du(C)}{dC} - p(t)$$

Furthermore

$$\begin{aligned}
\frac{dP(t)}{dt} &= -\frac{\partial H}{\partial K}(t, K^*(t), C^*(t), p(t)) = -p(t) \cdot \frac{dF(K^*(t))}{dK} + \mu p(t) \\
p(T) &= \frac{\partial g(K^*(T))}{\partial K}.
\end{aligned}$$

One can show that the function p satisfies

$$p(s) = \frac{\partial}{\partial K} \int_s^T u(C^*(t))dt + g(K^*(T))_{|K=K^*(s)}$$

where C^* is considered to be a function of K^* in the way that $C^*(t) =$

$C^*(t, K^*(t))$. This representation can be obtained when solving for $C^*(\cdot)$. However the term on the right side can be interpreted as the increase in utility within the timeinterval $[s, T]$ per unit ofcapital. In the economic literature $p(s)$ is therefore called the shadowprice of capital.

5.4 Necessary Conditions for Nash Equilibria in N-person Differential Games

The following Theorem gives neccesary conditions for a multistrategy in a differential game to be a Nash equilibrium solution. The theorem heavily relies on Theorem 5.3.1.

Theorem 5.4.1. *Consider an N-person differential game as formulated in the beginning of this chapter and let* $(\gamma_\sharp^1, ..., \gamma_\sharp^N)$ *be a Nash equilibrium solution. Assume* $\gamma_\sharp^i(T, x(\cdot))$ *depends on* $x(\cdot)$ *only through* $x(t)$ *and this dependence is* C^1*-differentiable. Then there exists* N *costate vectors* $p^i(t) \in \mathbb{R}^n$ *and* N *Hamilton functions for* $i = 1, .., N$

$$H^i(t, x, u^1, ..., u^N, p^i) = \phi^i(t, x, u^1, ..., u^N) + < p^i, f(t, x, u^1, ..., u^N) >$$

s.t. the following conditions are satisfied for $k, i = 1, ..., N$

$$\begin{aligned}
\frac{dx_\sharp(t)}{dt} &= f(t, x_\sharp(t), u^1\sharp(t), ..., u_\sharp^N(t)) \\
x_\sharp(0) &= 0 \\
\frac{dP_k^i(t)}{dt} &= -\frac{\partial H}{\partial x_k}(t, x_\sharp(t), u_\sharp^1(t), ..., u_\sharp^N(t), p^i(t)) \\
&- \sum_{j \neq i} \sum_{l=1}^{m_j} \frac{\partial H}{\partial u_l^j}(t, x_\sharp(t), u_\sharp^1(t), ..., u_\sharp^N(t), p^i(t)) \cdot \frac{\partial \gamma_l^j}{\partial x_k}(t, x_\sharp(t)) \\
p_k^i(T) &= \frac{\partial g^i}{\partial x_k}(x_\sharp(T)).
\end{aligned}$$

Furthermore $u_\sharp^i(t) = \gamma(t, x_\sharp(t))$ *maximizes*

$$H^i(t, x_\sharp(t), u_\sharp^1(t), ..., u_\sharp^{i-1}(t), u, u_\sharp^{i+1}(t), ..., u_\sharp^N(t), p^i(t))$$

for all $u \in \mathbb{R}^{m_i}$ *s.t.* $u(t) \equiv u$ *is admissible.*

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