How to Teach Elementary Mathematics

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Introduction

This book has been written as a resource for elementary school teachers. The concepts and calculations of basic arithmetic and geometry are laid out in a gradual, step-by-step fashion. Emphasis is placed on how to explain the usual procedures of arithmetic with toys, games, pictures, exercises, and illustrative examples. The general goal is to give students very simple and basic exercises to conduct and discuss. In the process of the class discussion (prompted by the teacher, as necessary), the hope is that the class will notice the patterns that form the basic rules of mathematics. When children make discoveries like these, they develop interest, excitement, and a very solid understanding of the underlying principles. The key is for the teacher to use exercises and examples that most clearly illustrate and explain the usual procedures and algorithms of elementary mathematics.

The ideas in this book are not intended to replace a teacher's curriculum. Every teacher has certain favorite examples, exercises, and games that give character to his or her teaching style and should not be abandoned. Also, there are a nearly endless variety of different types of mathematical problems. No book would be capable of covering all the possible problems and situations that an elementary school teacher might face. Instead, this book focuses upon the core basics of elementary mathematics: whole numbers, addition, subtraction, shapes, multiplication, division, fractions, decimals, area, and volume. When students have a solid understanding of these basics, their teachers will not have much difficulty teaching them more advanced topics and problems that might appear in textbooks, exams, or real-world situations.

There is no reference to grade levels or ages in this book. The only requirement for each section is that the students be comfortable with the preceding material. In an ideal situation, those children who find the material easy should be allowed to move on to more complicated mathematics. Those children who struggle should be taught from the last place where they fully understood the material. Of course, few educators find themselves in ideal situations. It is hoped that this book will at least provide some ideas and insights for those front-line warriors who daily struggle with the challenges of elementary education.

I would like to extend my gratitude to all of those without whom this work would not be possible. My mother taught me enthusiasm and daring. My father taught me patience and reasoning. Richard Lavers and Ernst Fandreyer showed me the fun of teaching math. Loring Tu taught, by example, how to write a book. Mauricio Gutierrez and Zbigniew Nitecki taught me the necessity of hard work. Most of all, this book would not be possible without Mahesh Sharma, who taught me the area model of multiplication, the components of the concepts of whole numbers and fractions, and many other ideas which were essential for this book.

Chapter 0: Mathematics

Mathematics is the study of relationships. Arithmetic studies the relationships among the numbers, the way numbers can be combined and compared. Geometry studies the relationships between lines, shapes, lengths, areas, and volumes. Algebra is nearly the same as arithmetic, but with some of the numbers unknown and replaced by variables. Trigonometry studies the relationships between the angles and lengths of triangles. Calculus studies the relationships between the slopes of curved lines and the areas beneath them. In more advanced mathematics, the objects which are compared and related can be extremely abstract, but the overall process is much the same.

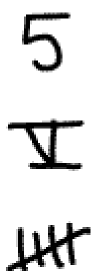
First we describe the objects and properties which we are about to study. Next, we talk about how they can be compared. We try to find natural ways to combine the objects. At every step, we see if it is possible to undo what we just did, if we can reverse each process of combining things. Finally, we make entirely new objects and start all over. This, in a nutshell, is mathematics.

If mathematics is the study of relationships, then it would be most interesting to jump straight to the most important and useful relationships of all: human relationships. Unfortunately, human beings change a lot and are quite unpredictable. It is hard to say anything certain or definite about all people, much less make predictions about what an individual will do. Thus, we will leave the study of human individuals to psychology and the study of groups of people to sociology. Instead, we will begin mathematics, the study of basic relationships, with objects that are basic, universal, and do not change. We begin with numbers.

Chapter 1: The Natural Numbers

Our first objects of study will be the *natural numbers*, also known as the *counting numbers*, which begin with 1, 2, 3, 4, and so on. When you teach children about numbers, there is something important to remember: numbers are at the same time very weird and very natural and innate.

Why are numbers weird? Well, what are they made of? Are numbers made of wood or metal? Do numbers have weight or height? Can we pick up a number? Can we sit on a number? The answer to most of these questions is "no." Numbers, we see, are not physical objects, and so they are quite different from cats and buttons and most of the other usual things in our lives. If we want to explain what an awl is, all we need to do is find one and then show how it can be used for making holes. If we want to explain what the number 5 is, however, we cannot bring one out because it is not a physical object.

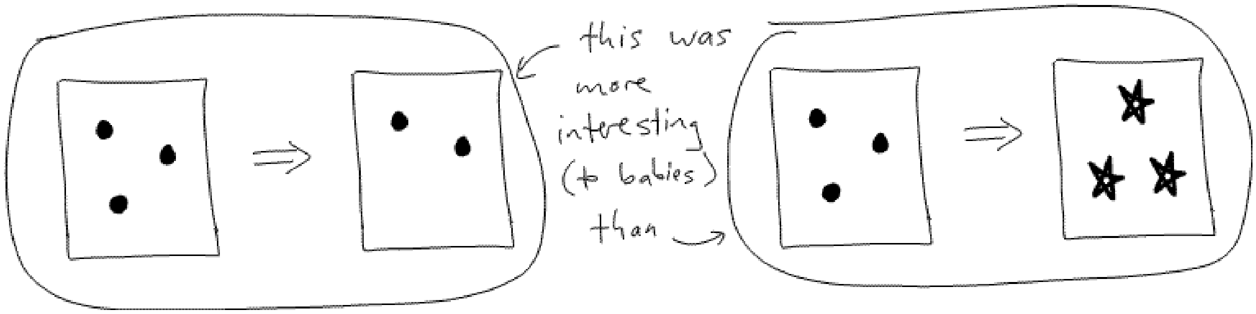


This leads to a place where mathematics is often taught incorrectly. Many people confuse the number five with the symbol that represents it. Each of the symbols illustrated on the left represents the number five (with a Hindu-Arabic numeral, a Roman numeral, and tally marks respectively), but none of them is actually what the number five represents. A number is a concept, and should not be confused with the manner with which it is represented.

We shall get into the meaning of a number shortly, but first let us look at an analogous situation. Love, like the number 5, is a concept and not a physical object. You cannot weigh love in pounds, cannot measure its height, or any other such thing as that. Suppose a person told us that he knew all about love and that it was the object illustrated on the right. We would immediately realize that this man does not really know what love is. Sure, he might be able to read a sentence written like "I ♥ New York," but he does not fully understand the concept.

The concept of love can take a very long time to properly understand. Rare are the teenagers who understand the extent of their parents' love. Couples who have been married a long time often say that there was much they did not know about love after only five years of marriage. This is natural because love is a complex and abstract concept.

Fortunately for teachers, recent studies have hinted that numbers are innately known and understood, even by infants. Newborn babies were shown cards with various numbers of objects on them. At regular intervals, the cards would be replaced. The researchers kept track of how long the babies paid attention to the cards before they looked away. It turned out that the infants paid the most attention when the number of objects on the cards was changed. A baby shown a card with three dots, for example, would pay more attention to a subsequent card with two dots than a card with three rectangles or a card with three dots in a different arrangement. Similarly, a baby shown a card with two dots would react more significantly to the subsequent sound of three drumbeats than to the sound of two drumbeats.[1]



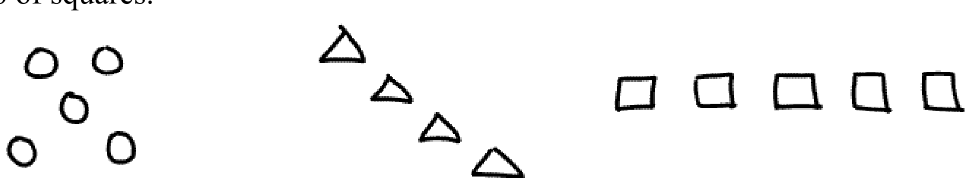
The most natural conclusion from these experiments is that people are born with an innate ability to recognize different numbers. If babies did not have the ability to discern the difference between small numbers, they would not be able to react to these differences. A small percent of people, however, seem to be born without these basic and innate mathematical abilities. Just as people with dyslexia read more slowly than most people, people with dyscalculia have impaired number sense and take much longer than most people to count the number of dots on a page.

It is for these reasons that we can say that numbers are weird (they are abstract concepts made out of ideas and not tangible objects) and yet natural and innate (people are generally born with the ability to recognize the differences in small numbers). Now, let us examine what numbers actually are. For the sake of simplicity, we will use the word "number" to mean "natural number" up until the point where we begin to examine fractions.

A natural number is a concept that has three parts:

(1) A number is a property of a group of objects.
(2) Two groups have the same number if their objects can be paired up.
(3) Numbers can be put in order.

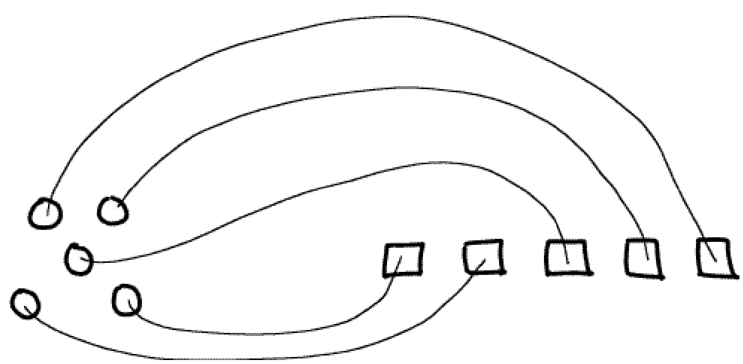
The first aspect of numbers is that every group of objects has a number. Thus, given any group of objects, it is reasonable to ask "what is the number of these objects?" Note that the number is a property of all the objects put together and not something individually owned by any one of the objects. The group of circles below has a number, as does the group of triangles and the group of squares.



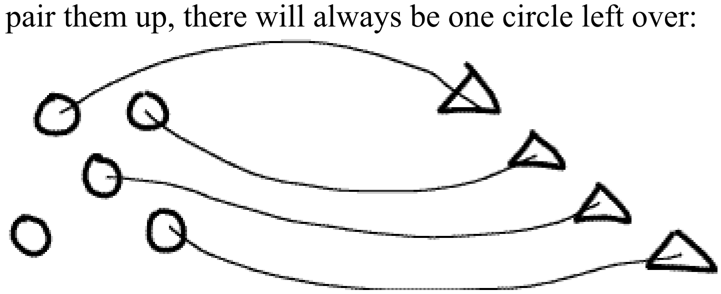
[1] "Easy as 1, 2, 3." The Economist 30 Dec. 2008

Analogously, the property of being lined up is another group property. The triangles are lined up and the squares are lined up, but the circles are not. This property of being lined up is a collective one; if even one square were moved up or down, then the group of squares would no longer have the property of being lined up. Similarly, there is no circle in the group that has the number 5; the number 5 is a property shared by the whole group.

The second aspect of number is very intriguing. It states that even though every group has a number, some of these numbers are the same. Two groups have the same number if we can take the objects in the two groups and pair them up completely, with nothing left over in either group. One way to pair objects up is to connect them with lines. For example, the circles and squares above have the same number, because we can pair them up as follows:

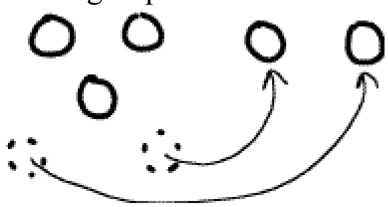


The circles and triangles, however, have different numbers because however we try to pair them up, there will always be one circle left over:

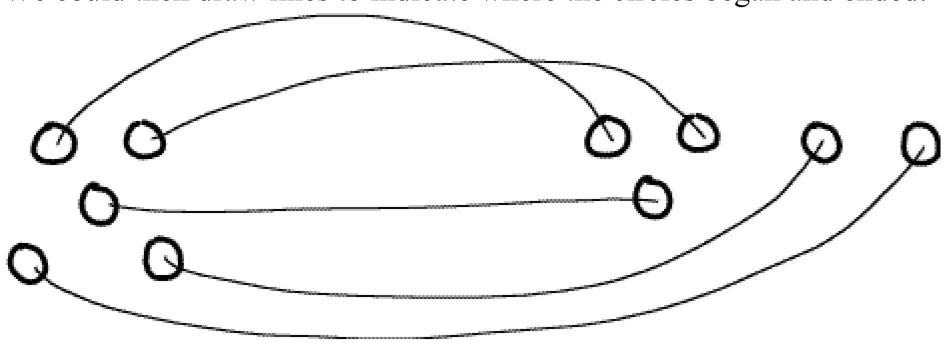


The number five, thus, is not the shape of the symbol "5," but a property shared by all the groups of objects which can be paired up exactly with the group of circles pictured above.

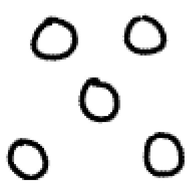
If we were to take a picture of a group of objects, rearrange them, and then take another picture, there is a natural way to pair up the objects in the two pictures: draw a line from where each object started to where it ended up. For example, suppose the bottom two circles of the above group were moved to the right as illustrated:

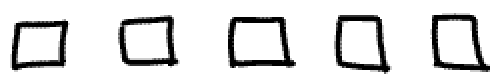


We could then draw lines to indicate where the circles began and ended:

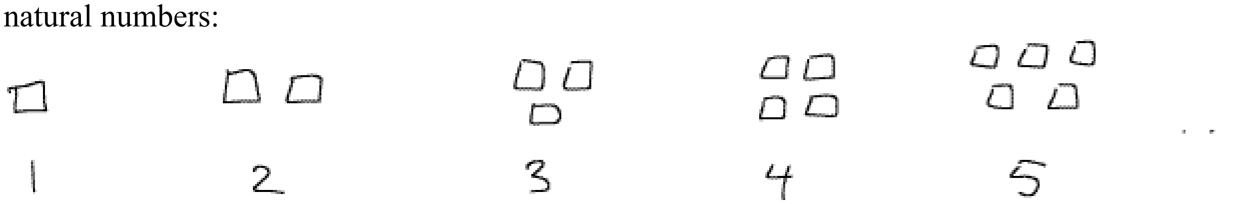


This is a natural way to pair up the objects in the two groups. We can conclude that moving objects around does not change their number. This might sound obvious, but it is an important aspect of the concept of number. There are some children, for example, who feel that the squares below have the number 5, but the circles do not. Such children must be taught to separate the idea of arrangement from that of number.





The third and final aspect of numbers is that they come in an order. The smallest natural number is the number 1. You cannot have a group of fewer objects (it may be debatable if 1 object really constitutes a group, but 0 things are certainly not a group). Whenever you add one more object to a group, you end up with the next larger number. In this sense, we develop the natural numbers:



It is curious that this last concept of a number is often the very first which is taught. Before teaching a child what a number is (a property of a group) or what it means for two groups to have the same number (they can be paired up), most parents teach their children how to count. It is very valuable for a child to learn how to count, because it teaches the names of the numbers and reinforces the concepts of order and sequence. However it is also important for a child to learn the other two aspects of the concept of number.

In order to teach the idea that a number is a property of a group, have the child count objects. This is a learning exercise that can occur anywhere – in a car, in a room, while reading a story, or anywhere. How many birds are in this picture? How many people are in this room? How many cars can you see? By counting things like this, the child will be able to connect the names of numbers with groups of objects.

When a child is good at counting the number of objects in a group, try reversing the process. Have the child separate out a given number of objects from a pile. For example, have the child give you 6 pennies from a large pile of coins. Similarly, you can have the child ask you for a certain number of objects (give me four buttons) and then correct you if you are wrong ("no, that is five buttons. Here, let me show you four..."). You could also show a child a number of different groups of items (some pencils, some paper clips, some buttons, etc.) and then ask the child to tell you which was a group of eight. As an extra challenge, you could set up the piles so that there were several correct answers, working to show the child that it is possible for several groups of objects to have the number eight.

There are a number of tricks to teaching a child about the pairing concept of numbers. One is to have the child count a number of objects, then move the objects around and have the child count them again. Similarly, you can have a child count a row of objects, first from left-to-right, and then from right-to-left. Ideally, after a few tries the child will announce that there is no need to keep counting the items, that the number will remain the same. Such a child has just made a fundamental step forward in understanding numbers.

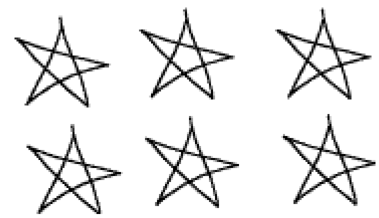
Another way to develop the pairing concept of number is to give have a child compare two groups of items, each one too numerous for him or her to count. For example, give a child a large box with empty bottles and bottle caps, then ask whether the number of bottles and caps are the same. Ideally, the child will think to screw the caps on the bottles. If there are screw caps left over, then there are more caps. If there are bottles left over, then there are more bottles. Otherwise, the numbers are the same. A simlar exercise would be to show a child a large picture with many people on it, most of them wearing hats, and ask the child if there are more hats or more people. Even if there are too many people to count, a child ought to be able to compare the numbers of hats and people. These are ways to reinforce the pairing concept of number. While it initially might help the child if the items go together (bottles and bottlecaps, hats and heads), the child should eventually be able to compare two large groups of unrelated items (for example: pennies and popsicle sticks) by pairing them up.

Questions:

(1) Name 5 objects which are physical and 5 (non-mathematical) objects which are made of ideas. Discuss how these ideas are generally introduced to children.

(2) Name 5 properties, other than number, that a group of objects might have.

(3) Suppose that a child, when asked the number of stars to the right, says "1... 2... 3... 4... 5... 6." When asked for the number of stars a second time, the child says the same thing: "1... 2... 3... 4... 5... 6." Which part(s) of the concept of number is this child missing? Explain.



(4) A child is shown an arrangement of buttons and, after counting them, declares that there are 7 buttons. The buttons are then rearranged. When asked how many buttons are there, the child counts them again. Which part(s) of the concept of number is this child missing? Explain.

(5) Suppose that a child, when shown 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, is able to point to them in order and say their names: one, two, three, etc. This same child, when shown 1, 2, 3, 4, 7, 5, 6, 9, 10, 8, makes several mistakes, saying "five" when pointing at the "7" and so on. What does this say about the child's understanding of number?

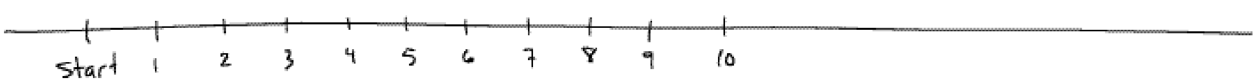
(6) Suppose a child is able to write out all the numbers from 1 to 10, name them, and recognize them out of order (say "five" when shown a "5," etc.). What further challenges ought you pose the child in order to identify if she really understands the numbers from one to ten?

Chapter 2: The Number Line

As a teaching tool and a reinforcement of the order property of numbers, few things can beat the number line. Many elementary and middle school teachers use either a small number line at the top of a piece of paper or else a larger number line across the front of the room (often above the blackboard). Both miss out one of the crucial teaching components: the ability for students to physically act out mathematics. Many of the people who find mathematics easy are able to imagine moving things around in their heads (rotating shapes, moving numbers from one side of an equation to another, etc.). It is good to encourage students to think in these ways by giving them the opportunity to move things about on a large scale.

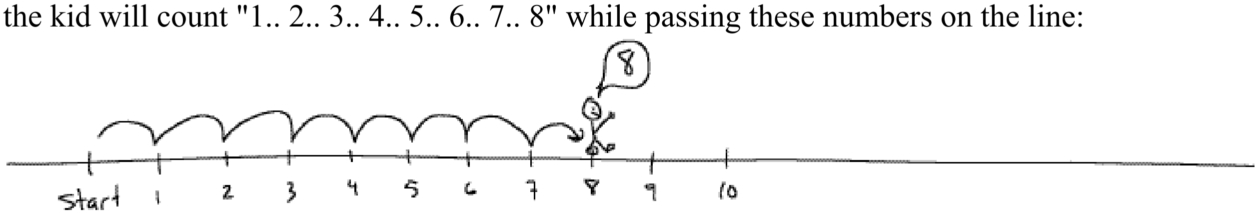
To begin, draw a long line on the ground. If it is summertime, go outside to a paved area in the shade. Parking lots and sidewalks work especially well. If it is cold or rainy out, you can make a line with masking tape on the ground. If you do not want to mark up the floor (or if you work in a carpeted area), a length of clothesline works very well – just draw the numbers on index cards and attach them at regular intervals with clothespins. The most important detail is that the line be clearly marked and placed so that your students can comfortably stand and walk on the line.

Mark off a "start" near one end of the line, but not at the exact beginning of the line. Then, at equal distances, mark off the numbers 1, 2, 3, and so on. The distance between the numbers should be the length of a big step for your students – small enough for everyone to step from one mark to the next, but too big for anyone to skip over a mark without jumping. In an ideal world, the line will be long enough to go up to 20, but most classrooms are less than 20 paces long. All in all, it is much better to make the marks further apart and have the line weave like a snake up and down aisles than to make the a perfectly straight line with the marks too close together. Even if the line is long enough to accommodate 20 marks, only label the first 10. You should thus have something that looks like:



At first, you should draw out the line and all the numbers, but as soon as they are able, the students should be allowed to draw the line, mark off the regular intervals, and draw the numbers (especially outside where the line will need to be redrawn regularly).

The first number line activity is called "walk the line." Let the kids line up behind start and take turns walking forward a given number of marks. For example, if you say "walk 8," then the kid will count "1.. 2.. 3.. 4.. 5.. 6.. 7.. 8" while passing these numbers on the line:

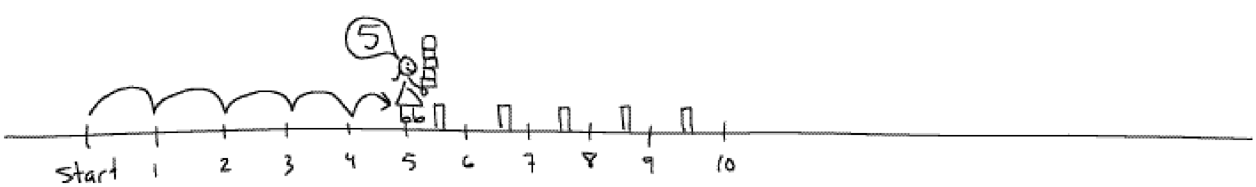


This activity not only reinforces the order property of numbers, but also the symbols and their meanings. A child who is told to walk 8, for example, will take eight steps and end up standing beside the symbol "8" on the ground. The child will be connecting the verbal and symbolic forms of the number 8 with a kinesthetic one: the act of taking 8 steps. It is for this reason that the marks should be spaced appropriately – a child who crosses several marks in one stride or walks directly to the number 8 without counting steps will be missing out on one of the key aspects of the activity.

Unless your children are very young, they will probably tire of this game in a reasonably short period of time. On the one hand, this is excellent – boredom is an indication that your students have understood the lesson and are ready for something else. On the other hand, the activity gets them up and out of their seats, but is a controlled exercise, and thus good practice for following instructions, taking turns, and so on. If you want to spice up the game a little, you can give it a pirate theme and have the kids "walk the plank 7 spaces," perhaps while wearing a pirate hat. Feel free to let the kids come up with their own ideas – hop like a rabbit 5 spaces, lumber like a dinosaur 9 spaces, or so on.

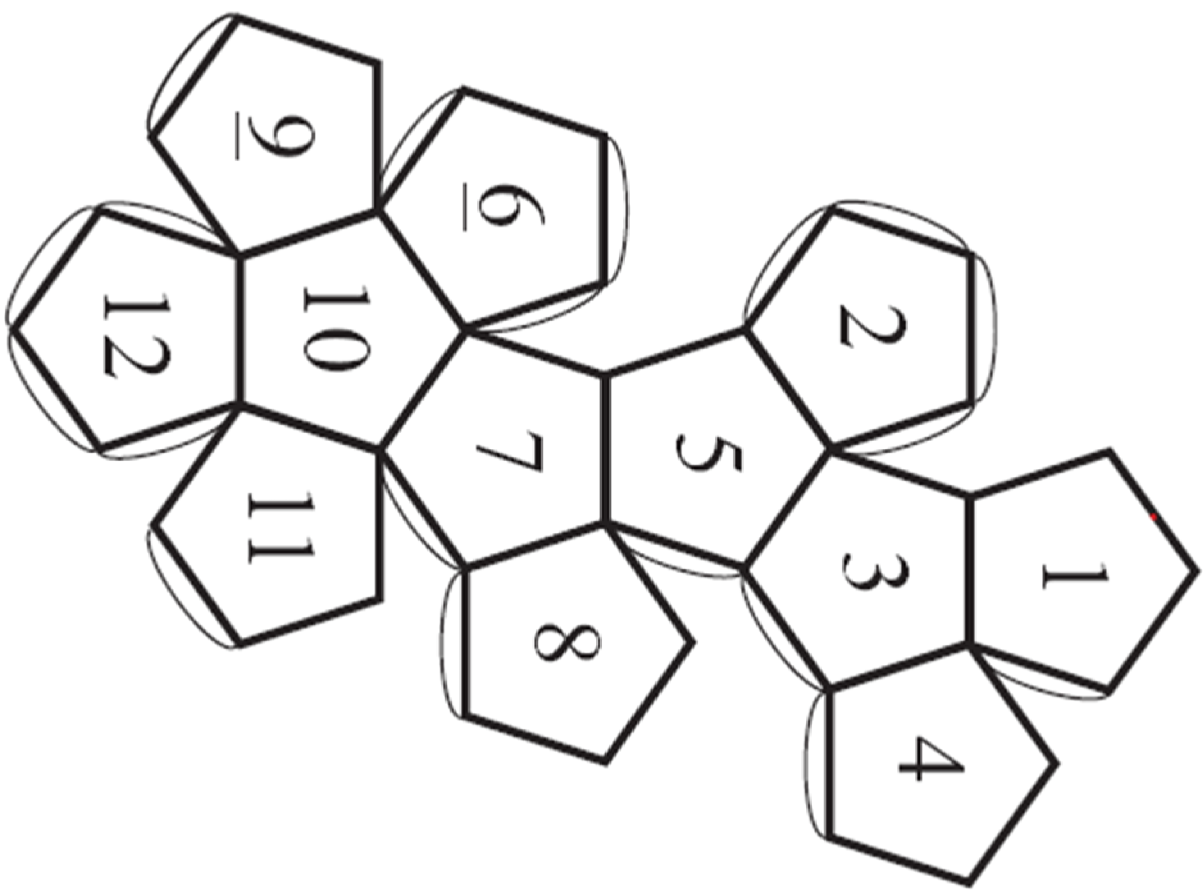
It is very important that the students begin at a place called start. Later on, this will be labeled "0" and called the "origin," a synonym for "start." If a child starts at the number 1 (a very common mistake), then one step will take him or her to the number 2. The child will count "one!," but end up at 2, which will be confusing. The numbers should correspond with the steps that each child takes, thus walking forward five steps should leave a student standing at the number 5. This helps to introduce and reinforce the symbols for the numbers, in case the kids know how to count aloud, but do not know the symbols used to represent the numbers. At first, the numbers will correspond with the number of steps: the number 7 involves doing seven things – taking steps. In the future, the numbers will correspond to the distances between the marks – the number 1 is one length away from start, the number 3 is three lengths away, and so on. Representing numbers with lengths will help the students to learn fractions in the future.

As an additional reinforcement, you can place objects between the marks for the children to pick up as they walk along. Bottles and cardboard bricks are good objects to use, so long as they are light enough for a child to carry several and tall enough to be picked up easily while walking. In this version of the activity, a child who is told to "walk five" will not only end up saying five numbers, taking five steps, and saying "five" while ending up at the notch marked 5, but will also find herself holding five objects:



This ought to help all the students in the class come to understand the natural numbers, no matter what their particular learning style. Note that the objects should be placed between the marks and not at the marks. In the future, the number 5 will be represented by the length between the origin and the 5 mark. We prepare for this by having the objects placed on the lengths and not at the marks. Between start and the 5 mark, there are five blocks:

In order to pick out the numbers for the students to act out, you can have a stack of index cards with various numbers written on them, shuffle them up, and then call out the numbers for the children in order. This would enable you to add more numbers to the stack as the children got better at counting out the smaller numbers, and perhaps take out the numbers that the children found too easy. You could also make a big dodecahedron out of poster board (you could enlarge the planar net illustrated below) and label the 12 sides with the numbers from 1 to 12. The kid at start will then get to roll the die to determine how many spaces to walk. Of course, this would require adding the numbers 11 and 12 to the number line.



It might happen that some of your students will find this game too easy and get bored before the others have mastered it. This can lead to them being disruptive and troublesome. The trick to deal with this is to praise them for mastering the game and "deputize" them as your personal assistants. Draw a new number line for each deputy (or let them make their own lines). Next, divide the class up among the various lines and give the deputies index cards or dice to coordinate their own copies of the game. Perhaps you will no longer run a number line yourself, but just walk around and make sure that everyone is playing properly, taking turns, and counting out the spaces correctly.

Questions:
(1) Why is it important that the marks on the number line be equally spaced?

(2) Suppose a child walks straight to the answer instead of counting steps. What knowledge does this indicate? What aspects of the concept of number might the child still be missing?

(3) Is there a learning style that the number line game does not incorporate? If so, can you think of a way to add this to the game?

(4) Suppose a teacher illustrates the number line for a class by moving cars attached to clothespins on a long strip of wood marked with numbers at the front of the room. Name the learning styles that this version of the game does not facilitate.

Chapter 3: Cuisenaire Rods

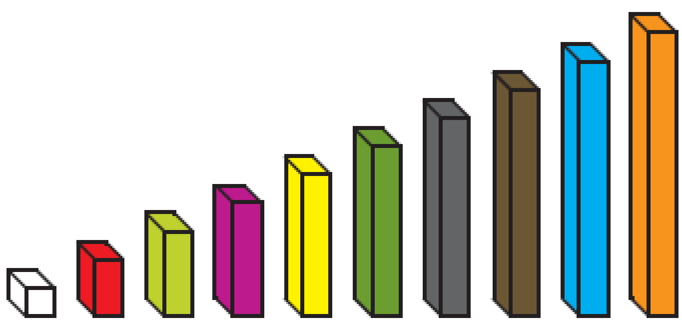
Teaching numbers with a number line has one main problem: it does not emphasize that a number is a property of a grouping. Through the activity, children are encouraged to look at the number 5 as "1.. 2.. 3.. 4.. 5" instead of as a single concept. This will lead to two difficulties for the child in the future.

The first problem is that the child will be inclined to rely upon counting to solve addition and even multiplication problems. Even though the child will be able to obtain correct answers, these will be found slowly. In advanced mathematics like algebra, trigonometry, and calculus, it will be essential to add, subtract, and multiply quickly and without resorting to counting. Success in mathematics is not only determined by accuracy and precision, but also speed and ease of recall.

The second problem is that the child will not understand that a number is a single thing, a concept. A child who thinks that the number 5 is five things will be very confused by a number like $\frac{4}{7}$ which, in one sense, is four things and, in another sense, is less than one thing.

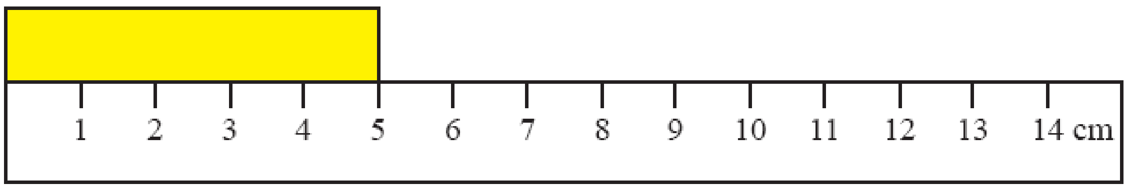
In general, a child who relies upon counting to solve basic arithmetic problems will have a great deal of difficulty with fractions and will find algebra bewildering.

The solution is to reinforce the grouping concept of numbers. An excellent tool to this end are *Cuisenaire rods*, developed by Georges Cuisenaire in 1952. A set of Cuisenaire rods consists of a large number of rectangular blocks which vary in length from 1cm to 10cm:



These rods represent the numbers from 1 to 10 in three different ways: with their lengths, their side areas, and with their volumes. For now, however, we will view them as lengths.

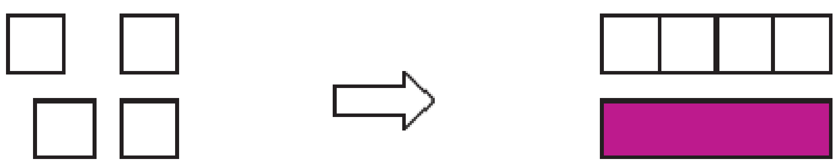
For children who have played on a number line, the next step is to have them measure Cuisenaire rods with a metric ruler. The yellow rod, for example, represents the number 5 because it goes from the starting point of the ruler to the number 5, just like the 5 walk on the number line:



If emphasis was put on the importance of always beginning at the "start" mark on the number line, then your students might avoid the common mistake of trying to measure objects by setting the beginning of the object at 1 instead of at the beginning of the ruler. Children who are taught numbers largely through counting believe that the numbers start at 1. It is important, however, to realize that number 1 already counts something – one inch, one centimeter, one penny, etc.

The most valuable aspect of this exercise is that a Cuisenaire rod is a single object, just like the number that it represents. A five is like a yellow block, a nine like a blue one. Also, do notice that there are no notches on the sides of the rods. There is no way to count "1.. 2.. 3.. 4.. 5" along the side of a yellow rod. Working with Cuisenaire rods, thus, can help students advance beyond counting.

As another transistion between counting and number concept, you can have children count out a number of the white cubes and then find the single rod with the same overall length. For example, four cubes put together have the same length as the purple 4-rod:



This is another way of reinforcing the manner in which the purple rod represents the number 4.

There are a large number of math activities which can be conducted with Cuisenaire rods, and this book will discuss several of them. The most valuable, in my opinion, is that these rods can move students away from a counting understanding of number and toward a more abstract understanding of numbers as individual objects.

In order to work with Cuisenaire rods, it is important for your students to develop a familiarity with them. You should have them measure and compare rods until they have associated the numbers and colors. If you ask "what number is the orange rod?" your students should be able to answer "10" immediately, without having to measure it. Similarly, if you say "show me the 6 rod," your students should fetch out a dark green. As well, your students should be able to connect the rods with the symbols for numbers, 1 through 10. To some degree, the colors are arbitrary and thus ultimately unrelated to mathematics. However, the benefit, both in giving students tangible objects with which to work and leading them toward a more abstract understanding of number, is great.

Questions:

(1) Name the colors of the Cuisenaire rods in order: 1, 2, 3, 4, etc. without looking at a chart.

(2) Name the numbers of the rainbow: red, orange, yellow, blue, light green, dark green, blue, purple, again without looking at a chart.

(3) Discuss the benefits and disadvantages to having students associate numbers with concrete objects like Cuisenaire rods.

Chapter 4: Equality

We have already explained the concept of equality; two groups have equal number if their objects can be paired up exactly. With Cuisenaire rods, two rods are equal if they line up exactly when placed side-by-side. This idea of pairing up and having the same length can be used to introduce the equals sign. The equals sign originally was designed to represent two lines of the same length, but the manner in which the these lines are lined up also implies pairing.

Too often, the equals sign is introduced briefly or not at all. This leads students to view it as a mere piece of punctuation, to be placed before an answer just as a period ends a sentence. For one thing, this leads students to misuse the equals sign, putting it in situations where it does not belong. For another, this cheats students out of an appreciation of the most important concept in algebra. The algebra involved in solving and rearranging equations can be almost entirely explained with a proper understanding of the concept of equality.

There are three main aspects to the concept of equality:

(1) Reflexivity: everything is equal to itself.

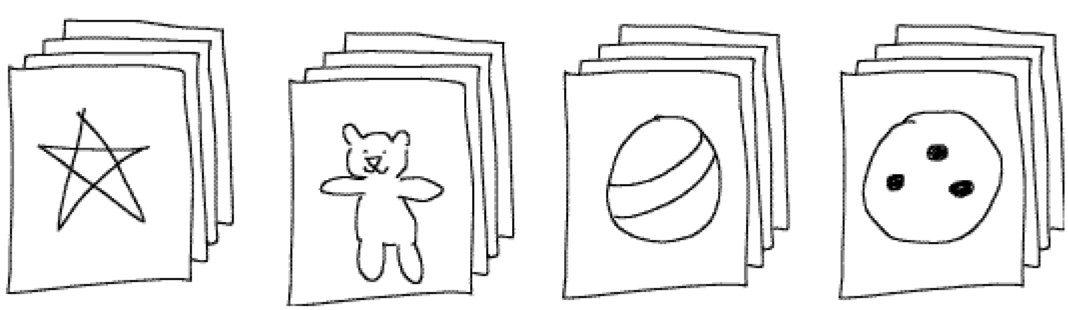
(2) Symmetry: if two things are equal, then they are equal in both ways: $a = b$ and $b = a$.

(3) Transitivity: if two things are both equal to a third, then they are equal to each other. Thus if $a = b$ and $c = b$, then $a = c$.

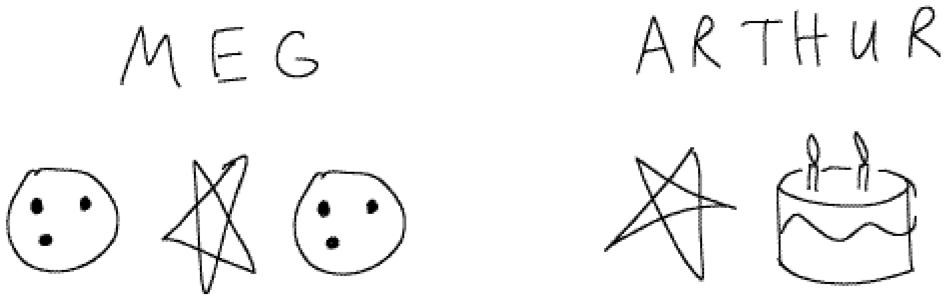
Not only is equality a key concept in mathematics, but it also can be used to introduce several very important other concepts in education. Discuss what the Declaration of Independence means by "We hold these Truths to be self-evident, that all Men are created equal..." Discuss the concepts of fairness, of equal treatment, equal turns, and equal portions.

There is a basic exercise in equality that can be played with very young children. While this game need not take very long to master, it develops one of the main skills necessary to find common denominators for fractions.

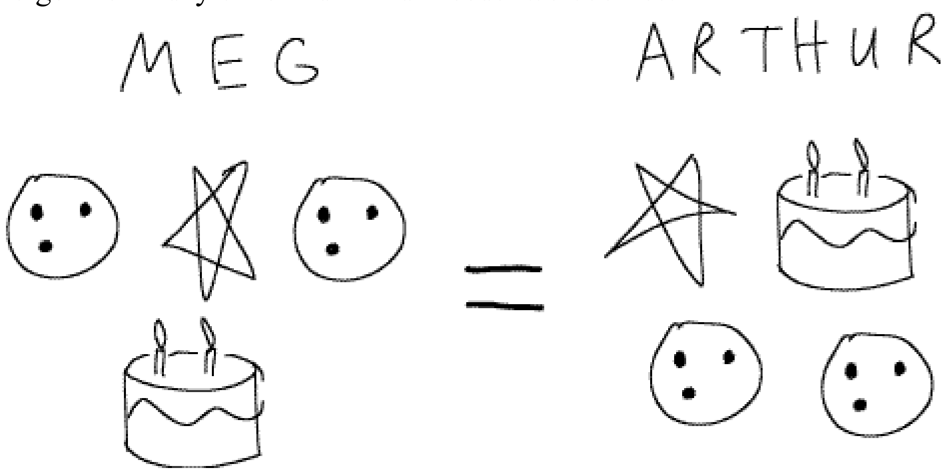
To play this game, you need several piles of items which are generally uniform – a stack of pennies, a stack of identical buttons, a stack of pencils, etc. This can also be played with symbols drawn on cards: several cards with identical stars, several with identical teddy bears, several with identical balls, cookies, etc. The game might be more fun with actual physical items, but the cards could be used to practice the names of shapes or colors at the same time.



If you are working with several children, put them in groups of two or three. Each child is then given a few of the items (or cards). The activity then proceeds as follows: the children have to decide which items each one of them must receive in order for everything to be fair (everyone ending up with the same items). For example, suppose you are working with two children, Meg and Arthur, and they begin with the following items:



The goal is for them to decide that, in order for them to end up with the same objects, Meg needs to get a birthday cake and Arthur needs two cookies:



If you are working with only one child, then give the items to dolls or stuffed animals and let the child add what is necessary to make everything fair.

When you begin, start with only two people (imaginary or otherwise) and a few items. Praise the child for any work that makes the two piles equal. Later, begin to insist that the child add only as few items as possible. For example, in the above set-up you could give both Meg and Arthur another star each and end up with equal piles, but this would be unnecessary. Eventually, you can present the child with three or more piles, perhaps each with a large number of items. The idea, as with every mathematical activity, is to present the child with a challenge that he or she will succeed at.

This is a very basic game, but it reinforces the concept of equality and prepares children for success with fractions in the future. Because the order of the objects do not matter, this game also prepares students for the concept of commutivity.

Questions:

(1) Suppose Amy has ♦♦♥♣♣ and Mark has ♥♥♥♦♪. Make this fair, using as few new items as possible.

(2) Suppose Kim has ☺☼☺♥, Jenna has ☼☼♪, and Miguel has ☺♥♥♥♪. Make this fair, using as few new items as possible.

(3) Suppose Terrell has 3 blue bears and one red bear. Rachel has 1 blue bear, 4 red bears, and a green bear. What should they be given to make everything fair?

(4) How many different kinds of equality do you think elementary children should be taught? Name them, and briefly explain what sorts of lessons and activities might be used to introduce them.

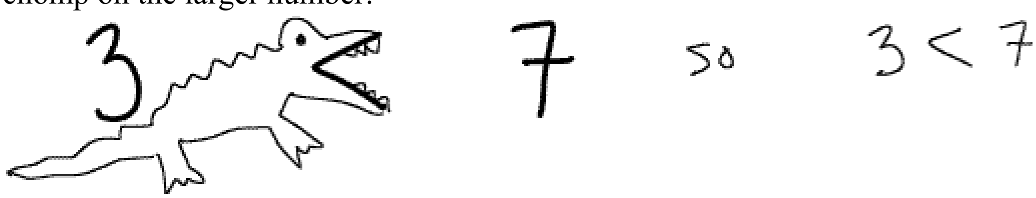
Chapter 5: Inequalities

When children understand the concepts of numbers and equality, they are ready for inequalities: the concepts of more and less.

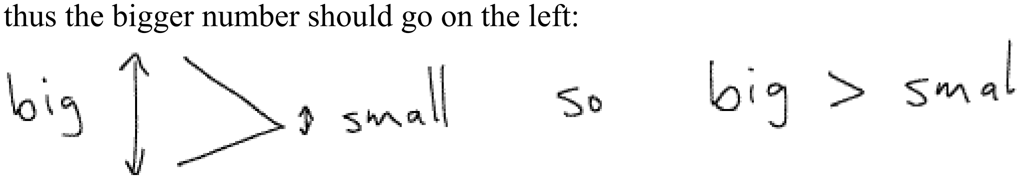
There are many ways to understand and explain inequalities. If two groups of objects cannot be paired up, then the one that has items left over has the larger number. On the number line, a number that you walk is bigger than all the numbers that you pass along the way. With Cuisenaire rods, a larger number corresponds with a longer rod.

As with all mathematical concepts, however, it is ideal if you can get the students to figure them out on their own. You might begin by giving examples instead of definitions. For example, say "9 is more than 4" and "7 is more than 2" and see if they can figure out which is more: "5 or 3." It is possible that different students will come up with different ways to look at the concept. Perhaps one will say that you have to count to 3 before you count to 5. Perhaps one will say that 5 is further away from start than 3 on the number line. Perhaps one will say that a person with 5 cookies has more than a person with 3 cookies. A brainstorming session like this can be incredibly useful in showing the students that there are lots of different ways to look at things. Also, a student who has ownership over an idea, having come up with it independently, is much more likely to remember it and help share the knowledge with others than one who is given a definition to memorize.

You can teach the students that > is the symbol for "is more than," that < is the symbol for "is less than," and = is the symbol for "is equal to." You can teach them the trick that the > and < symbols can be drawn to look like the mouth of a greedy crocodile who always prefers to chomp on the larger number:



You could also teach the students that the ">" symbol is taller on the left than it is on the right, thus the bigger number should go on the left:



You can also explain that the "=" sign is drawn to look like two sticks of the same length laid next to each other, thus represents two things that are the same size. On the other hand, it is much more fun to write a large number of examples on the board and let the students come up with their own explanations. Don't let them settle on incorrect explanations, of course, but let them play with their imaginations.

A lot of people feel that math has only one correct answer and lots of wrong answers and thus they are terrified to venture a guess. In advanced mathematics (as well as science, engineering, and more), however, there is a great need for people to come up with new ideas and approaches, so it is important to teach children that it is useful to try different things. Even if it turns out not to work, or not to work in all situations, a child should be praised for coming up with an idea to try out. Similarly, a good question or interesting challenge problem is worth a great deal, so children who pose them should be encouraged and taken seriously.

As soon as possible, children should be encouraged to write mathematical statements. For example, $3 < 5$ and $9 > 1$ are mathematical statements, as is $4 = 4$. Ideally, children will realize that mathematics is a language with which a great many ideas may be expressed. You could show a pile of pennies and a pile of pencils and ask the students to express the relationship of their numbers with an inequality, perhaps $5 < 8$. You could give a child the numbers 6 and 3 written individually on small squares of paper, along with the symbol $<$, and see how many different true statements could be arranged with them (only $3 < 6$, unless you flip some of the squares upside down to make $6 > 3$, $3 < 9$, and $9 > 3$).

To encourage creativity, you can let your students choose some of the symbols on their own. For example, have them "make a math statement with the symbols 4 and $<$." There are many possible answers, for example "$4 < 8$," "$4 < 5$," "$2 < 4$" and so on. The dream is to show your students that math is a language in which there is the freedom to say a great number of things, and not just an exercise in getting the correct answer desired by the teacher.

Remember that, just as with a foreign language, it is important to be able to translate in both directions: from math into English and from English into math. A child should be able to read the statement "$4 > 2$" aloud, thus converting math symbols into English words. Also, you should be able to tell a child "five is less than nine" and have the child write it out with symbols: $5 < 9$, thus translating from English into math.

Another useful exercise is to have students construct untrue statements. While it might be discomforting to write "$2 > 7$" on the board, it is very important for students to not believe everything they see, but to look at things critically and discern truth from falsehood. Few things practice this skill as quickly or in a more entertaining fashion than to assign each child in a class to write three true number statements and two false ones. They can then take turns saying their statements or writing them on the board, and having the class evaluate which ones are true and which are false. In some classrooms, the only person who ever marks things wrong is the teacher. It is far better for maturity and mathematical thinking, however, for a child to learn how to do this independently. The only disadvantage to playing this game is that you will forever afterward have to ask your students to "make a *true* statement with the symbols 3 and $>$" because otherwise you'll have to accept just about anything.

It often happens in teaching that some students learn quickly while others take longer to grasp the concepts. This leaves the teacher with a dilemma – do you move on to more material and leave some kids in the dark, or do you go over the material again while a number of the students are bored? Because so much of mathematical knowledge is sequential and critical for future concepts, it is generally important to wait until nearly everyone has a proper understanding of the material. A useful way to keep the stronger students from being bored (and potentially disruptive) is to assign them special roles as "arithmetic experts," "math medics," or "deputy math marshals." People of all ages like to be useful and enjoy special recognition. For elementary students, you could have them make special symbols of their status, perhaps cone hats for "math wizards," construction-paper headbands for "math pros," or sashes for "math champions." A child who has demonstrated the mastery of a particular concept of mathematics can get an emblem drawn onto his or her hat, or perhaps a sort of scout merit badge for a sash:

the class "less-than" expert,
ready to help

A child who is struggling with an assignment can cry out "medic!" or "math help!" and call over one of the other students to help. This has to be regulated to prevent chaos, of course, but the act of teaching and explaining provides excellent practice for both a marshal's math and communication skills. The students with difficulties benefit from more one-on-one instruction than a single teacher can provide. The ultimate goal, of course, is for all the students to become experts before the class moves on to a new topic. If someone has been out sick, you can have the students of the class take turns explaining the missed material. This is an excellent form of review. Furthermore, if the students understand some aspect incorrectly, it will become quickly apparent.

An excellent game to reinforce and practice comparing small numbers is the card game War. Pair students up and give one of each pair a deck of cards to deal out entirely between them. The students then flip over the top cards of their stacks simultaneously. The student whose card is higher wins both cards. If the cards are equal, then each student puts three cards face down on the equal cards, then flip over a new card. The winner here gets all ten of the cards. In the event of subsequent ties, the process is repeated. Any child who plays this game for even half an hour should have no problem comparing small numbers. If some in the class are having trouble, pair them up with better students, so that they can teach each other. The only problem with this game is that it often can run for hours without a winner. Thus, it is good to set a kitchen timer and have the pairs of students play the game for only a set time, like 5 or 10 minutes. When the bell goes off, have each kid count the number of cards he or she has – the one with the most cards wins. You could even have a class champion, the one with more cards than anyone else. A five-minute game of war can serve as an excellent warm-up for math time or a special reward for a well-behaved class. If your students are relying too much on the number of the symbols on each card (counting out the six spades on the 6♠, for example), then give them a deck of index cards with only the numbers written on them. The idea is to develop an automatic recall of math facts and not to exercise counting.

Questions:

(1) If you have the symbols "6," "3," "5," and "<" written on four squares of paper, how many different true mathematical statements can you make? (Flipping squares over is okay!)

(2) Name all the different ways that the statement $6 > 4$ could be demonstrated.

(3) Name all the different ways that $8 < 5$ could be shown to be a false statement.

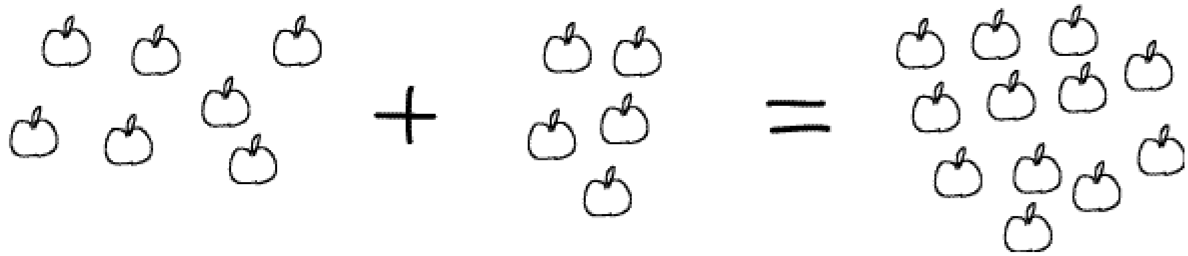
Chapter 6: Basic Addition

When your kids are all good at counting, comparing, and recognizing small numbers, give one of the smarter students in the class a number one or two beyond the highest mark on the line. For example, if your number line goes up to 12, ask this student to walk to 14. The dream, of course, is that the student will think to draw two more marks, label the first one 13 and the second one 14. If this isn't obvious to the student, see if someone else in the class can figure out what to do. Drop some hints if necessary (give the student the chalk or tape, etc.), but try very hard to have the innovation come from the class. If someone does figure it out, call that student a trailblazer! Make a big deal about this! As with everything else, this is independent thinking and innovation, going beyond the usual limits. If Martha was the student who figured it out, maybe her name should be a verb, and that to "Martha" something is to extend it to make it more useful (the children might wish to Martha recess, for example).

There are some math teachers who only know one way to solve certain problems, often a multi-step approach. Unfortunately, this tends to make them suspicious of students who find short-cuts. Sometimes they even mark students wrong for "skipping steps" or "not following the rules," even when the student has solved the problem correctly. Students find this very unfair, and they tend to resent the entire subject of mathematics after receiving this sort of treatment. In your classes, students should be encouraged to find new ways to solve problems. Ideally, a student with a new idea should present it to the class for them to examine. If you can find some situations where the technique does not work, then lead the student to contemplate such situations. But again, innovation should be encouraged, even if it is found to be ultimately incorrect. Of course, the end result of mathematics is to be able to do it quickly and easily, so excessively complicated techniques, while praised as creative, should be gently set aside in favor of methods that work faster and are easier for the whole class to understand. It is not unheard-of for a student to actually teach the teacher a better method for figuring things out!

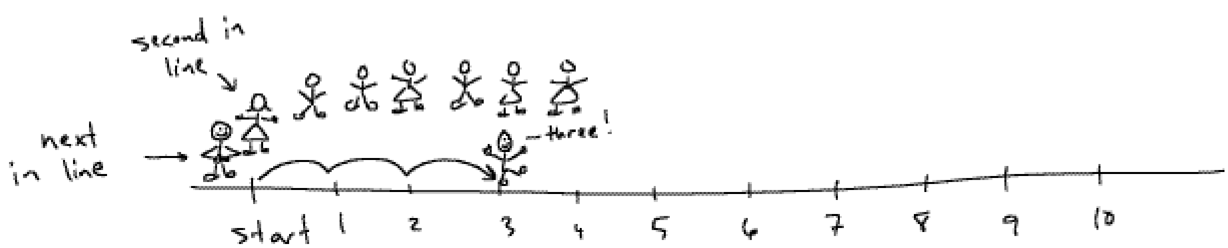
When the students in your class are able to count, read, and compare numbers up to 20, it is time to introduce them to addition. Many teachers use apples, or other concrete objects, to introduce addition. This has worked to teach children for ages, but it is not the best technique. Students who learn how to add with apples tend to have difficulty in the future learning how to add fractions and negative numbers. Fractional apples (they don't slice easily into equal-sized pieces) and negative apples (what are these?) do not make much sense. Also, when a number is represented by a pile of objects, the idea of number order is lost. Traditionally, to add 7 apples to 5 apples, a student must first count out a pile of 7 apples, then count out a pile of 5 apples, then put the two piles together, and then count the whole pile over again. Adding two numbers thus requires counting three times.

Even a teacher, for example, would most likely need to count three times in order to verify that the following number sentence was true:



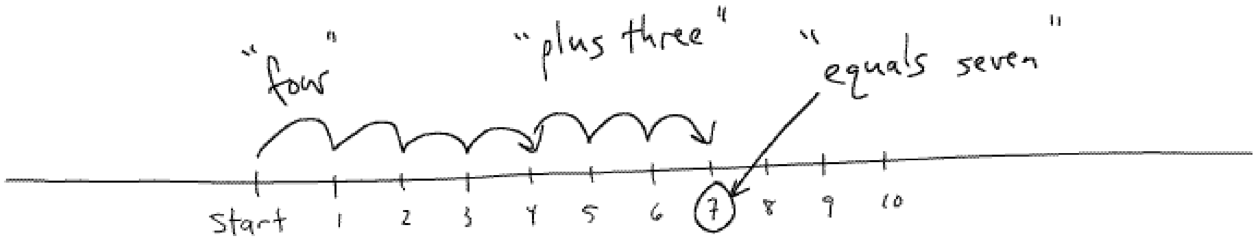
A better method for introducing numbers is to use the number line. To add 7 + 5, a student begins at start and counts 7 steps forward. As before, the student can verify that this was done correctly by looking down and seeing the number 7 where he or she is standing. The student then says "plus!" and proceeds to count 5 more steps. If this is done correctly (no skipping steps or counting too quickly), the student will end up standing at the number 12, the answer.

To begin with, line the students up at start and have them take turns acting out addition problems. If you bend the line around to run parallel to the number line, you can make it easy for the kids in the back of the line to watch the kids take their turns. After a child finishes acting out an addition problem, he or she can then take a place at the end of the line (to cheers and praise, ideally). With young children, it might help to draw this waiting line on the ground as well.



To introduce and reinforce the "+" symbol, write out each addition problem. For example, after a student acts out 2 + 6, write out "2 + 6 = 8" on the board, on a piece of paper, or on the ground (if you're outdoors). It will pay out future dividends if you also write "8 = 2 + 6" sometimes. Again, you can either roll two dice or draw cards from a deck of numbers to choose the problem. It helps to begin with easy problems, so start with a deck of index cards that only have the numbers 1 through 5 on them, and then slowly add larger numbers to the mix. If you are using dice to randomly select numbers, then the usual six-sided cube dice are best to begin with.

It is a good idea to regiment a procedure for acting out each addition problem. For example, suppose the next child in line is shown a card with "4 + 3" written on it. The child then steps up to the start mark, says "four" and then counts out the four steps "1, 2, 3, 4" to the four mark. Next, she says "plus three" and counts "1, 2, 3" more steps. When finished, she says "equals seven," reading the number off of the ground:



This is why we made a big deal about children counting their steps from the very beginning, to get them ready for this procedure. If a kid is able to recognize the number 8, then he might be tempted to say "eight!" and then hurry over to the mark on the floor, using any number of steps. This is the recognition of a symbol, but does not involve doing 8 of anything. The kid will think that he is advanced and solving the problem quickly, but he might be missing the point of the exercise. Such a child might also try to solve 4 + 5 by first hurrying over to the number 4 and then trying to figure out what 4 + 5 might be, in order to then skip over to the right answer. The idea is not to get kids to jump straight to the answer (known by some other method), but to give them a tool for figuring out the answer. If you do 4 of something (steps of a certain size) and then 5 more, you will have ended up doing 9 of them.

Do not hold the class back and bore them by forcing them to act out all their additions after they have gotten good at them. If a child asks "what is 5 + 2?" then encourage the kid to go up to the line on the classroom floor and figure it out on his own. If a child is given "6 + 1" at the starting line and can immediately say "6 + 1 = 7," then give that child a different problem. If the child automatically can add any two numbers from 1 to 9, then make that child an addition deputy to help run another number line. In this manner, the whole class will gather around a single line on the first day or two that you introduce it, thus all of them can learn the process directly from you. On later days, however, after the stronger students have caught on, you will be able to run several number lines all at the same time, and the children will not have to wait very long for their turns to come around again. Ideally, you will eventually have so many addition marshals helping out (each with either dice or index cards) that you can walk around and make sure that everyone is playing correctly, rather than running a line of your own. Unless, of course, you want to keep in charge of the students who are having the most difficulty (with either mathematics or behavior).

Probably the most useful form of educational technology, extensively used by future doctors and engineers in graduate school, are flash cards. Nothing helps to establish and reinforce the cerebral connections between two ideas than to put them on the two sides of an index card, then look at one side and try to remember what is written on the other. The basic addition facts (1 + 1 = 2, 1 + 2 = 3, up to 10 + 10 = 20) are so very useful and important that you want your students to be able to recall them immediately, without needing to think or count. Thus, it is a great idea to make up a set of flash cards with these facts. Begin with only the easiest ones, perhaps using only the numbers from 1 to 4. On the one side, write an addition problem, for example 3 + 2. As a class, use the number line to figure out that 3 + 2 = 5 should go on the other side. Many classrooms incorporate flash cards, of course, but most hand them to the students as "math facts to memorize." This leads children to think of mathematics as a long series of rules that must be remembered, rules that originate from the teacher. It would be much better for the students of the class to decide upon the answers on their own, thus to view the arithmetic facts as "math discoveries" and not "rules to remember." A child only needs to know that "adding a number means walking further along the line." This single concept will enable a child to discover all of the standard addition facts.

After the class has collectively made up a set of addition flash cards, let each student make up a personal set to take home for practice. You can make up a set of cards for each kid with all 100 addition facts from 1 + 1 up to 10 + 10 written on one side (and the other sides left blank). Each child can take a card out of the deck, act it out on a number line, and then write the answer (subject to your approval that the answer is correct) on the back. Encourage your children to start with the easiest ones and only make a few at a time – the process of memorizing should be done at the individual's pace. The end result is that each child will have a deck of flash cards which can be practiced either at home or at school, until the basic facts can be instantly recalled. Also, by making the deck of cards personally, it becomes easy to add in new math facts and to take out the ones that are too easy. A little geometry can be added in, for example, by drawing basic shapes on one side and their names on the other. With commercially-made decks, the costs are high and the flexibility low.

Once the students have had several days of experience with acting out additions on a large number line, they can draw their own small number lines on a piece of paper. Because of their previous experience, the kids will most likely be imagining themselves walking along the line as they hop from one mark to the next with the tip of their pencils. This can enable a whole class to make up their set of flash cards quickly, without having to wait for a turn on a large number line. This is also an excellent exercise in representing a real-life situation with an abstract illustration, thus teaching the class to be able to act things out on figures and in their imaginations.

When a child has mastered all of the basic addition facts and can go through a deck of flash cards without making a mistake, then award a very special badge for his or her math cap, sash, etc. This is a level of math mastery that every child should attain.

If the class has been enjoying playing War, it can be extended into an addition game. First, take out all the face cards from the deck and declare that Aces represent the number 1 (contrary to the 14 they generally represent when playing War). Next, have each pair of players deal out the deck between them as previously. This time, however, each player draws two cards and adds their numbers together. The player with the higher sum is the winner and takes all four cards. Again, it helps to pair weaker math students up with stronger ones, so that teaching can occur while the game goes on. If you notice that the children are adding by counting the symbols on the cards, switch them for index cards with the numbers 1 through 10 on them (and no accompanying symbols).

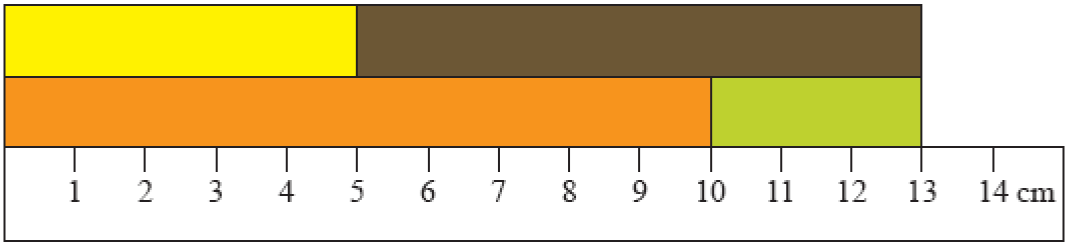
It is useful and important to try to use as many different words for addition as possible. The problem "4 + 7" does not always have to be phrased: "what is 4 plus 7?" You can say "what is the sum of 4 and 7?", "what do you get when you combine 4 and 7?", "what is 7 more than 4?", "what do you get when you add 4 and 7?", "how much is 4 together with 7?", or any number of other ways. Different words lead to different ways to conceptualize the process. Asking what is 7 more than 4 makes a person think about starting with 4 things and then putting 7 more to it. Asking what is 7 and 4 together makes a person think about taking two groups of things and putting them all together. While it might seem the same to a grown-up who has mastered addition, these are slightly different ideas. Furthermore, in word problems and real-life applications, all sorts of different words are clues that addition is needed. It is best to start practicing early.

It is also a great exercise to get the class to be on the lookout for patterns among the basic addition facts. Encourage them to discuss their strategies and observations, both as an exercise in communication and critical thinking, and as a means of helping the others learn. The fact that order in which numbers are added doesn't matter, for example that $4 + 6 = 6 + 4$, is called the *commutative property of addition*. It is best not to tell this to the students, but see if they can notice it on their own. As usual, be very excited when someone notices this, and name it the "Omar rule" or the "switcheroo principle" or whatever the class decides to call it. It is probably not important to have young children worry about remembering the word "commute," but it wouldn't hurt to introduce the word, either.

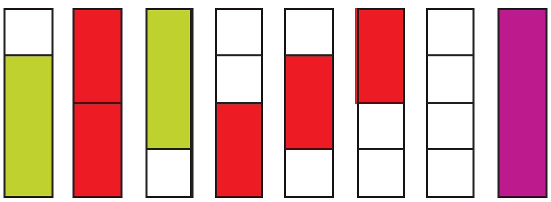
Similarly, a child might come up with the rule that "adding one to a number gives you the next number." If the class seems to have trouble remembering a particular fact, for example the sum of 4 and 5, then invite them to brainstorm a trick for remembering it. This sort of work empowers children to think creatively and reinforces the idea that mathematics is a place where there are lots of different ways to view and approach solutions.

Cuisenaire rods are a useful tool for reinforcing basic addition facts, especially if your students are having trouble progressing from figuring out sums by counting to automatically recalling the basic addition facts. To add two numbers with Cuisenaire rods, put the two rods end-to-end. Their overall length is the sum. If the sum is 10 or less, then there will be a single rod which has the same length – the answer. For example, $4 + 3 = 7$ because the purple 4 rod and the light green 3 rod together have the same length as the black 7 rod:

If the sum is over 10, then you can either use a combination of Cuisenaire rods (ideally including the orange 10 rod) or else a centimeter ruler. For example, 5 + 8 = 13, as illustrated:

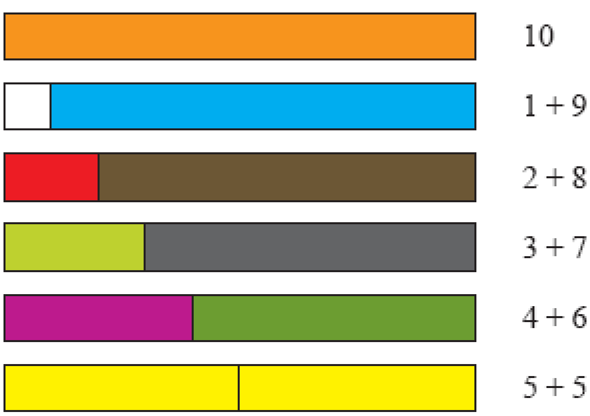


A useful exercise is a game called "how can we make a number?" Start with a small number, for example 4, and then ask "what numbers add to make 4?" Using only two numbers, the possibilities are 1 + 3, 2 + 2, and 3 + 1. However, you should allow students to use any number of numbers, so additional possibilities are 1 + 1 + 2, 1 + 2 + 1, 2 + 1 + 1, 1 + 1 + 1 + 1, and the number 4 all by itself. It helps to let students play with Cuisenaire rods while brainstorming:

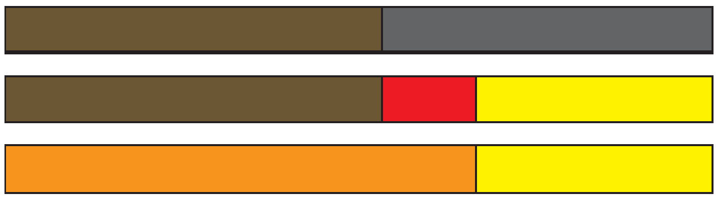


It is not so terribly important for students to find all the possibilities, although this can be set as a challenge for the more talented students. Instead, this is a way to reinforce addition while preparing students for the concept of subtraction.

The most important number to be able to break down is 10 because it forms the basis of our number system. Every student should be able to break 10 into a pair of addends in every possible way:

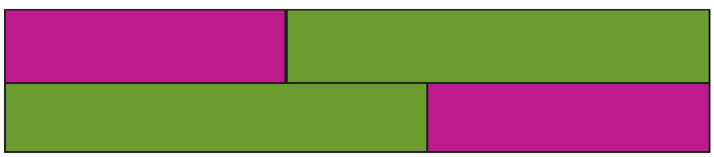


The ability to break down numbers can be used to help with the quick recall of basic addition facts. For example, to add 8 + 7, one can think that it takes 2 more than 8 to make 10. If the student knows that 7 = 2 + 5, then it is easy to compute 8 + 7 = 8 + 2 + 5 = 10 + 5 = 15:

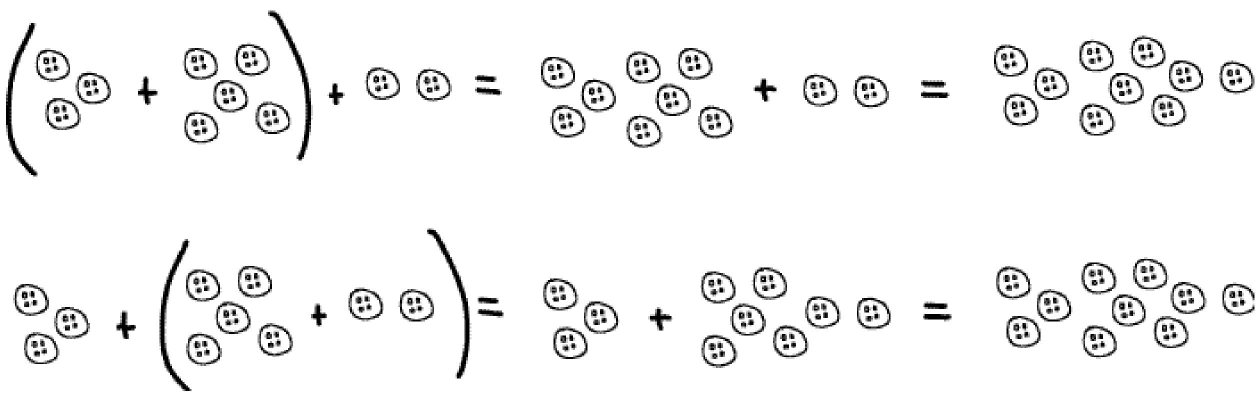


In fact, merely thinking that an 8 "wants" a 2, a 9 "wants" a 1, a 7 "wants" a 3, and a 6 "wants" a 4 can make adding into the teens quite easy.

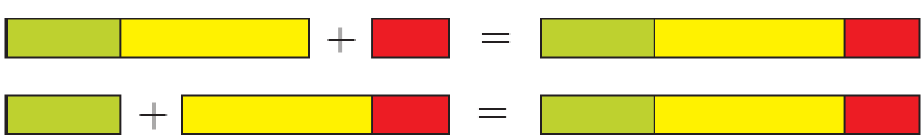
Cuisenaire rods are also an ideal tool for illustrating the commutative property of addition. The fact that 4 + 6 = 6 + 4 can be seen just by reversing the order of the two blocks:



Another property of addition is the *associative property*, that when three or more numbers are added together, it does not matter which are added first. For example if we add 3 + 5 + 2, we can first add the 3 and the 5 and then add the 2 or else first add the 5 and 2 and then add the 3. This is most easily illustrated with parentheses: (3 + 5) + 2 = 3 + (5 + 2). We can show this with buttons:



With Cuisenaire rods, this is even easier. Three rods put together will have the same length, no matter which two are put together first:



Children can practice and reinforce their addition facts by testing them out with small objects. For example, it might be enlightening for one student to count out 5 pennies and another to count out 7 pennies and the whole class to predict that the combined pile will have exactly 12 pennies. Rather than begin with pennies (or apples) as the explanation of addition, it is nice to use such examples to validate and reaffirm the mathematical facts.

This leads to a particularly wonderful property of mathematics: certainty. Children love to be experts, whether memorizing facts about the dinosaurs, the top speeds of land animals, or anything else they may have read. Unfortunately, these facts are almost always found in books and not via experimentation, and thus are subject to debate and correction. There are few subjects, whether history, science, or reading where a child can know everything about the subject and be certain that it is correct. In mathematics, however, there can be absolute certainty. A child who knows the basic addition facts can not only tell you with certainty that "$5 + 6 = 11$," but the child can prove it in a variety of ways (on a number line, with buttons, etc.). It is a sad truth that children these days sometimes grow up in very uncertain settings. Some do not know where they will sleep each night, whether their parents will both be home, or what sorts of conditions they will face when they get off the bus. It can help such children immensely to have something, anything, in life upon which they can rely for certainty and consistancy. Mathematics, especially if the children are led to discover the facts and not merely told to believe them, can therefore be a cornerstone in a child's development of self. No matter what might go wrong, $5 + 6$ will always be 11. With a firm foundation in the verifiable facts of mathematics, a child can take further steps in education with confidence. Many people in history have worked their way out of unfortunate situations via a solid education, and math is one of the best places to begin learning.

Questions:

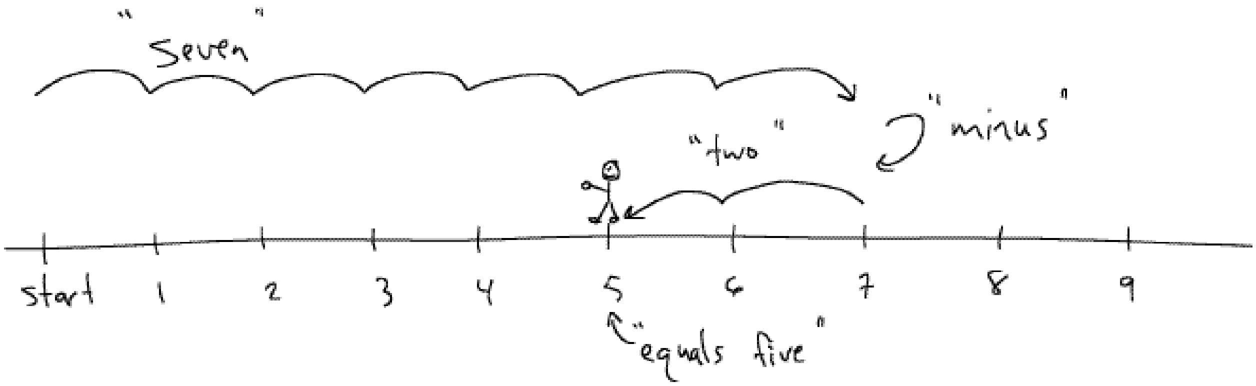
(1) Name and illustrate all the possible ways that 3 + 5 could be calculated and demonstrated.

(2) Name some games which could be used with the flash cards the children create.

(3) Which sorts of math deputy do you think would appeal to the students most (math wizard, math doctor, math scout, etc.)? Discuss the potential advantages and disadvantages to this system. Can you think of ways to amend the system to avoid the disadvantages you imagine?

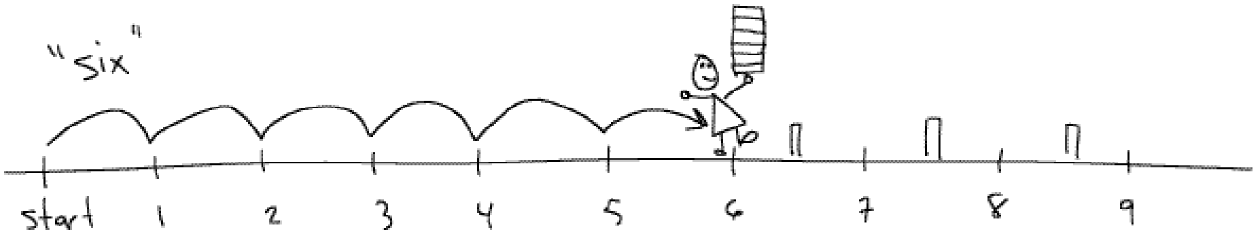
Chapter 7: Basic Subtraction

When children have learned how to add by walking on a number line, it is very easy to introduce the concept of subtraction. To subtract, you walk from start to the first number, then turn around when you say "minus," and walk the second number. For example, to act out 7 – 2, a child steps up to the start line, says "seven," walks forward 7 steps to the mark labeled 7, says "minus" while turning around, then says "two" and walks 2 steps back toward start. The child will then say "equals 5" upon finishing at that mark:

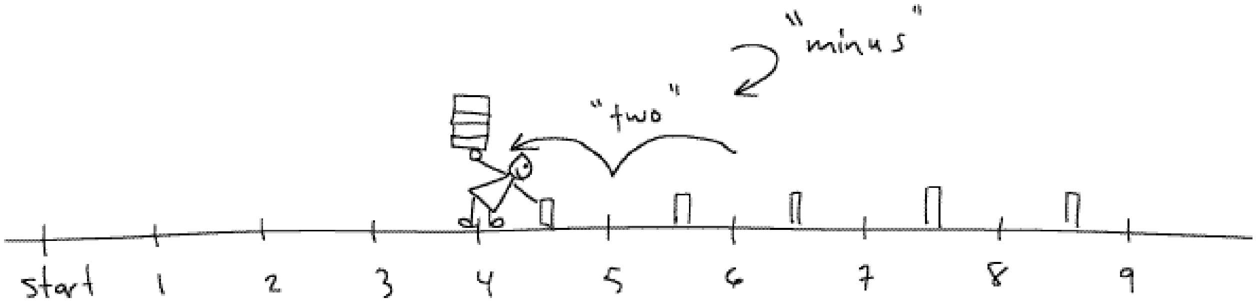


All of the tricks and procedures for practicing and memorizing addition facts also apply here. At first, have the whole class form a line up to the starting mark where they can see the whole line and then take turns acting out subtraction problems. The children who master this first then get to be in charge of parallel lines, reading out problems off of index cards. (In this case, write the whole problems out on the cards, for example 5 – 2 and 9 – 3.) When everyone understands the process, have the class collectively make a deck of flash cards for practice. After that, have each student make up a personal deck of cards for practice at home.

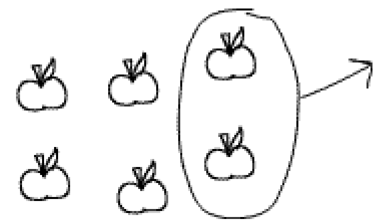
If the children seem to want or need concrete objects to associate with the numbers, you can put the blocks/bottles back between the numbers. When students add, they pick up all the blocks along the way. To subtract, the children pick up all the blocks on the way to the first number, and then put some of them back as they return towards start. For example, a student who was acting out 6 – 2 would begin by saying "six" and picking up the 6 blocks along the way as she walked 6 steps to the number 6:



She would then say "minus" as she turned around. Now, as she says "two" and takes two steps back, she is passing empty lengths without blocks. As she passes each, she should put a block back:

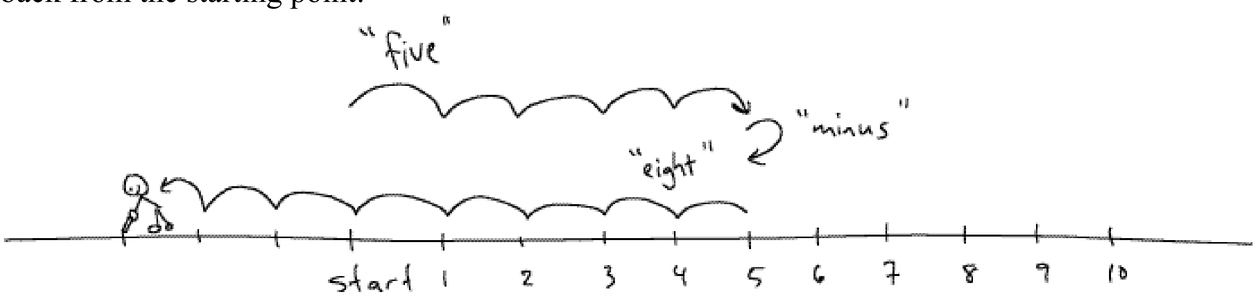


In this fashion, the child has not only ended up with the correct answer (6 – 2 = 4), but she has acted it out in two different ways. She has taken 6 steps forward and then 2 steps back, ending up exactly 4 steps away from start. She has also collected 6 objects and then put 2 back, ended up with 4 objects.



This models the usual way that subtraction is introduced, as "take-aways." Traditionally, 6 – 2 is taught with a problem like "if you have 6 apples and then eat 2, how many apples will you have left?" There are many advantages of the number line model over this traditional apples model, however.

The traditional model does not explain negative numbers very well at all. Many teachers teach their students that, for example, "you can subtract 5 from 8, but you cannot subtract 8 from 5." With apples, this is very clear – if you have 5 apples, it is impossible to eat 8 of them. Unfortunately, this sort of rational explanation can lead to students having great difficulties with the concept of integers and negative numbers when they are introduced in middle school. Some teachers try to explain that negative apples are "apples owed," but this still does not explain how a person with 5 apples can eat 8 of them. With the number line, however, it is very possible to walk forward 5 steps, turn around, and then walk 8 steps back. It will be necessary to draw in more marks, but this concept was already introduced when a student had to add 5 + 6 on a line that only was marked up to 10. The answer to 5 – 8 is thus the name of the mark which is 3 steps back from the starting point:



This number is called negative 3, meaning that it is 3 steps from start taken in the opposite of the usual direction. It is certainly not necessary to worry elementary school children with the concepts of negative numbers, but they shouldn't be avoided either. If a curious student wants to subtract 7 from 3 or wants to know about the negative numbers on the outdoor thermometer, then give a quick explanation and make it clear that these are advanced ideas that will be taught in the later grades. Rather than prepare students to have difficulty with future mathematics, give them something to look forward to: a world on the other side of start called the negative numbers.

Similarly, the number line explains the addition and subtraction of fractions in ways that adding and subtracting apples cannot. This will be discussed in greater detail later in the book.

While negative numbers are not necessary for elementary mathematics, the number zero is quite important. It was a huge breakthrough in the evolution of human thought to consider "nothing" to be "a thing" that could be discussed in relation to the other numbers. The number zero is not technically a counting number, so there a new term was introduced to refer to "all the natural counting numbers along with zero." These are the *whole numbers*: 0, 1, 2, 3, etcetera. Rather than teach your students about the concept of zero, however, it would be best to have them discover it on their own. While acting out the subtraction game on the number line, give a problem like 4 – 4 to one of the students with either strong math skills or a creative mind. If she plays the game correctly, she should end back up at the starting mark. This is, of course, the correct answer. However, see if she recognizes that this is the same as nothing. If you are working with blocks between the numbers, she should find herself at start with no blocks left.

In a pinch, it is all right to explain a concept like zero to your class. It might be asking too much for a class to reinvent one of humanity's greatest discoveries in a matter of minutes. On the other hand, children as young as 4 years old have figured out the idea of zero from the number line. If this should happen, it certainly calls for celebration! You might name the number after the discoverer (Kara's number), or at least refer to her every time it comes up in conversation. For a while, label the first mark on the line with both "start" and "0." This will help prepare the class for when fractions come into play and all of the marks have several names.

To help the class get comfortable with the concept of zero, revisit the "walk the number" game, but shuffle a lot of zeros into the deck. When a student need to walk to the number 5, he or she must take 5 steps forward. What must a student do to end up at the number zero? Nothing! When a student is given the number zero, thus, he or she merely says "zero" and is finished. Make sure to give the child a few moments, to be sure that he or she understands that doing nothing is the answer, and not interrupt the student from walking forward and illustrating a lack of understanding.

A child who understands what it means to "walk zero" will have no trouble adding and subtracting with zero. To act out 4 + 0, for example, the child should say "four," walk forward 4 steps from start, then say "plus zero," and not walk any more. The answer is thus 4 + 0 = 4. Similarly, to subtract zero, a student merely turns around after the first number and then does nothing. Hopefully, the class will soon reach the general understanding that "adding and subtracting zero does not change a number." However, as with everything else, it pays greatly to not tell this to the class, but instead have them act out various problems with zero until they come to this realization on their own. Not only will this lead to a solid understanding of the concept, but it gives the class an opportunity to conduct an intellectual discussion with abstract mathematical ideas. For future debating and critical thinking skills, few things offer better introductory subjects than sharing and comparing ways to envision mathematics.

Another crucial concept in mathematics is that of *inverse operations*, processes that undo one another. As with everything else, it would be incredible for your students to notice this on their own, that if you add a number and then subtract it, you will end up with what you started with. For example, if you start with 7 and add 5, you get 12. If you take this 12 and then subtract 5, you are back with the your original 7. On a number line, adding numbers takes you further away from start and subtracting numbers involves turning around and walking back toward start. With groups of objects, if you subtract and then add the same number, you end up replacing the same items you just removed. If your students do not notice this after a lot of work with addition and subtraction, then you can explain it to them, but do try to be patient and give them time to figure out the pattern.

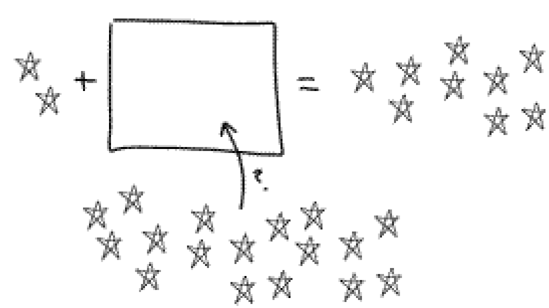
Addition and subtraction are, to some degree, the nicest of all mathematical operations because they are completely reversable. In the future, it will be possible to reverse almost every multiplication with a division. The one exception will be multiplication by zero, which cannot be reversed. This will lead to the famous prohibition: do not divide by zero! The operation of squaring a number will be reversed by taking square roots, but this will have even more restrictions. In general, the concept of an inverse function is one of the most important in all of mathematics. Thus, if a student in your class can figure it out on their own, some serious celebration should ensue!

One trick to lead your students to this realization is to ask the class questions like "2 plus what is 9?" As they have worked with breaking 9's up, this should sound like a recall of addition knowledge (which it is), while it actually is a subtraction question. At the time, though, you can write the question with a box to be filled in with the answer:

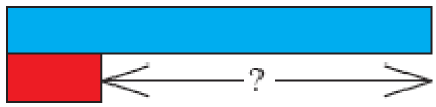
$$2 + \square = 9$$

In fact, this is really an algebra problem, using a box to represent a variable, but it is not necessary to tell the students this!

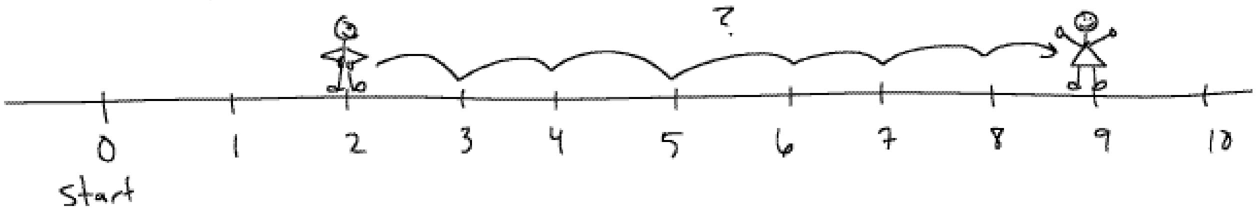
You can act this out with small objects by putting down 2 and then counting the number that must be added until you have 9:



With Cuisenaire rods, this is figured out by putting the 2 rod beside a 9 rod and then finding the rod which makes up the missing space:



On the number line, this is acted out by having one student stand on the number 2 and a second student on the number 9. The question is thus "how many steps must the first student take in order to get to the second student?"



This is a very useful exercise because it prepares students for the future concept of a *vector*, a length with a direction. The child at 2 must take 7 steps *forward* to get to the child at 9. The child at 9, on the other hand, must take 7 steps *backward* to get to the child at 2. You need not explain these fine details or introduce the concept of a vector, but working with the idea in elementary school will plant the seeds for future ease of understanding.

As with the concept of addition, there are a great number of different vocabulary words used to refer to subtraction. These refer to subtly different mental pictures, so it is useful to use as many of them as possible.

The words "subtract," "take-way," and "minus" generally lead someone to imagine a number of things from which some are being removed. These words, thus, lead one to the apples model of subtraction. If students are comfortable with the concept of a step as a thing, however, these can also be envisioned with the number line in that you go forward a number of steps and then take some of them back.

The phrases "how much more," "how much fewer," and the concept of "difference," on the other hand, lead one to imagine the comparison of two numbers. With Cuisenaire rods, we imagine two rods, side-by-side, and try to figure out what rod will make up the difference. With small objects, we have two piles of objects and need to figure out how many need to be added (or subtracted) to make the piles equal. On the number line, we imagine two students standing at different numbers and wonder about the distance between them.

Questions:

(1) Explain in detail all the ways that 7 – 3 could be computed and illustrated.

(2) Name some non-mathematical processes which can be reversed or undone. How could these be used to help explain the relationship between addition and subtraction?

(3) Write several different subtraction word problems, each using a different phrase to indicate subtraction.

Chapter 8: Units

The *unit* of a number is what the number 1 represents. For example, if a person says that he "ate five," it is reasonable to ask "five what?" Were these 5 jelly beans, 5 hamburgers, or what? In order to understand what the number 5 means, it is necessary to know what 1 of them is, to know what units are being used.

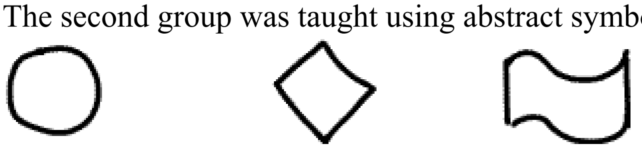
There are a great many different units in common use. The units which measure length include feet, inches, miles, meters, centimeters, and kilometers. The units of time include hours, minutes, seconds, days, weeks, months, and years. There are units for weight, area, volume, cost, and much more.

The importance of units in teaching has been the subject of much recent debate. Traditionally, students were first taught to memorize their math facts (basic addition, subtraction, multiplication, and division tables) and then were later given word problems which used these facts in context. More recently, teachers have been encouraged to do everything with concrete real-life examples and avoid "naked numbers" as much as possible.

The evidence in favor of concrete learning is largely anecdotal. Students appear more engaged when playing with manipulatives than when listening to the recitation of math facts. Colorful classrooms with large tables and lots of teaching tools seem to produce much better students. Common sense says that students who view mathematics as something that is real, tangible, and useful will appreciate it more, invest more effort in learning it, and understand it better in the long run.

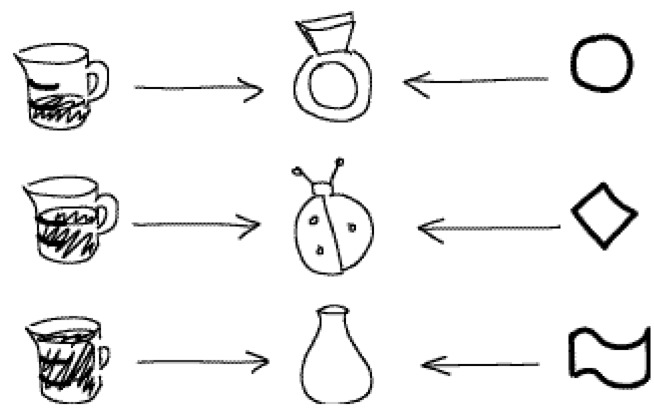
A team of researchers at Ohio State University (Kaminski, Sloutsky, and Heckler) decided to test this hypothesis with a controlled scientific experiement.[2] A number of sixth-grade students were divided into two groups. Both were then taught an unusual, yet basic form of mathematics (modulo 3 arithmetic – addition with only three numbers). The first group was taught in a concrete fashion that used symbols that made sense: measuring cups that were one, two, or three-thirds full:

The second group was taught using abstract symbols that had no relation to the concepts:



[2] Kaminsky, Sloutsky, and Heckler. "Do Children Need Concrete Instantations to Learn an Abstract Concept?" *Proceedings of the XXVIII Annual Conference of the Cognitive Science Society*, 2006 < http://cogdev.cog.ohio-state.edu/fpo644-Kaminski.pdf>

The students were then given an exam which tested their knowledge, using the symbols with which they had been taught. The students who were taught with the concrete symbols did a little bit better on this exam than the students who were taught with the abstract symbols. Next, however, both sets of symbols were replaced with completely different symbols. The students were told that the new system used the same knowledge that they had just learned. The students who were taught with concrete symbols were baffled and answered questions no better than randomly. The students who were taught with abstract symbols, however, did even better than they did on their first exam.



It was suggested that most elementary school teachers use many different concrete models to illustrate concepts, not just one. The experiment was then repeated: half of the sixth-graders were taught with three different sets of concrete models in succession while the other half were taught with the abstract symbols for the same amount of time. Again, the students who were taught abstractly were able to transfer their knowledge to the novel situation while the students taught with multiple concrete models were not.

Finally, it was suggested that teaching students first with concrete models and then with abstract symbols would do more than teaching with abstract symbols alone. Subsequent research showed that this was not the case. The students who had seen concrete models did as well, but no better, than those who were only taught abstractly.

If these results run counter to your common sense, there is an easy experiment you can run on your own. Ask an adult for the combined length of 7 ropes, each 25 feet long. Most adults will be able to tell you that the answer is 7 × 25, but only some will be able to give you an immediate answer. Next, ask the same person for the value of 7 quarters. It is very likely that the person will be able to answer $1.75 automatically. This is because people work with the concrete units of money on a regular basis, and thus are familiar with calculations in dollars and cents. This knowlege, unfortunately, does not transfer automatically to situations with the same computation but different units. A student who is taught how to mentally calculate 7 × 25, however, will be able to answer either of these questions with ease. Abstract knowledge can be put into context easily, but specific knowledge is difficult to generalize.

Perhaps the most encouraging aspect of this research is that it means less work overall for teachers. Rather than have to come up with a vast array of different contexts, manipulables, and word problems to each each new concept, teachers can teach the concepts abstractly. Not only will this take less time, but the benefit is greater.

It is still useful to use physical objects and activities to teach mathematics, but the best ones are the most general. Small piles of buttons or marbles, for example, are necessary to introduce children to the concept of number, but replacing them with Cuisenaire rods helps abstract these concepts. A number line is little more than a straight line (an abstract geometric object) labeled with the symbols for numbers (abstract concepts) which illustrates the order property of the numbers. It is not difficult for a child to use the number line to discover the facts of addition and subtraction, and yet it does not burden them with excessive context. The most abstract of all the teaching techniques are the flash cards which relate two concepts without any explanation. By having the children write their own flash cards, however, this avoids the meaninglessness of traditional rote memorization.

Mathematics is an *a priori* field, one whose knowledge comes from basic principles, and not an *empirical* one whose knowledge comes from observations. In mathematics, we can figure out from our concepts of number and the base-ten system that 37 + 25 = 62. There is no need to actually combine a pile of 37 buttons with one of 25 buttons in order to verify that this is true. In fact, if a person were to conduct such an experiment and conclude that there were only 61 buttons in the combined pile, we would not doubt mathematics, but rather question whether the person had counted correctly. It is for this reason that we should teach children how to add first, and then how to add things second.

A first exercise in units is to ask the class to add various things together. A child who knows, from the number line and flash cards, that 4 + 3 = 7 should find little difficulty in adding 4 apples + 3 apples, 4 dollars + 3 dollars, 4 feet + 3 feet, or any similar calculation. You can verify one of these, showing how 4 pencils plus 3 pencils equal 7 pencils, but for the most part you want the students to know the answers without verification. Have them add elephants, airplanes, unicorns, wishes, dinosaurs – anything that the children find fun to work with. In a short period of time, they should learn that their knowledge can be applied to absolutely any sorts of units.

A useful next exercise is to have the students try to add different sorts of objects together. If a child has 4 apples and is given 3 dollars, what will this be all together? Hopefully, your students will realize that there will not be seven of anything, but merely 4 apples and 3 dollars, things which cannot be added. Problems like these reinforce the most crucial detail of units: you can only add and subtract numbers that have the same units. To reinforce this, go around the room and ask the students a variety of addition problems, some with the same units and some with different. It is curious that this knowledge, so easily figured out by common sense, often escapes students studying algebra. Just as 4 dogs plus 5 dogs make 9 dogs, $4x + 5x = 9x$, not $9x^2$.

As a next step, add several compound groups together. For example, if a child has 4 apples and 3 dollars, then receives 5 more apples and 2 more dollars, what will she have in the end? Here, we can add together 4 apples plus 5 apples (because they have the same units) and end with 9 apples. Similarly, 3 dollars plus 2 dollars are 5 dollars. The child will end up with 9 apples and 5 dollars. This is the same reasoning which, in algebra, will explain why $4x + 3y + 5x + 2y = 9x + 5y$. If you want to prepare your students in advance for algebra, you can abbreviate your units. For example, write $5A + 2D$ to represent 5 apples and 2 dollars. As with all of these advanced preparations, it is useful to introduce them early, to take away the fear and the mystique, but it is important to make it clear that this is advanced knowledge for which the students will not be responsible anytime soon.

As a final step, have students work with a large number of units all at the same time. One possible scenario is to run a hypothetical shop, with 8 or so different kinds of inventory. You could keep track of pencils, pens, erasers, paper clips, staplers, rulers, notebooks, and crayons, for instance, or a much more wild assortment of goods dreamed up by your students. Begin with several of each item, and then receive a shipment containing more of some of the goods. Have some imaginary customers purchase some of the goods. At first, write out everything with words. For example, you might begin writing:

5 pencils + 4 pens + 12 erasers + 14 paper clips + 3 staplers + 4 rulers + 6 notebooks + 10 crayons

and then add a shipment of 5 crayons and 2 staplers:

5 pencils + 4 pens + 12 erasers + 14 paper clips + 3 staplers + 4 rulers + 6 notebooks + 10 crayons + 5 crayons + 2 staplers = 5 pencils + 4 pens + 12 erasers + 14 paper clips + 5 staplers + 4 rulers + 6 notebooks + 5 crayons

Ideally, this will seem a bit confusing for the class, so abbreviate and arrange things like:

5 pencils	+ 4 pens	+ 12 erasers	+ 14 clips	+ 3 staplers	+ 4 rulers	+ 6 books	+ 10 crayons
				+ 2 staplers			+ 5 crayons
5 pencils	+ 4 pens	+ 12 erasers	+ 14 clips	+ 5 staplers	+ 4 rulers	+ 6 books	+ 15 crayons

As a final improvement, make columns which correspond with each of the inventory items, and work only with numbers. For example, the above calculation can be written as:

pencils	pens	erasers	paper clips	staplers	rulers	note-books	crayons
5	4	12	14	3 + 2	4	6	10 + 5
5	4	12	14	5	4	6	15

For extended computations (several incoming shipments and several purchases), it will hopefully be clear to all the class that this column method is by far the most convenient and efficient. You can tell them, in fact, that real businesses use this method to keep track of their inventory with special computer programs.

Questions:

(1) Name ten different units used for measuring volume.

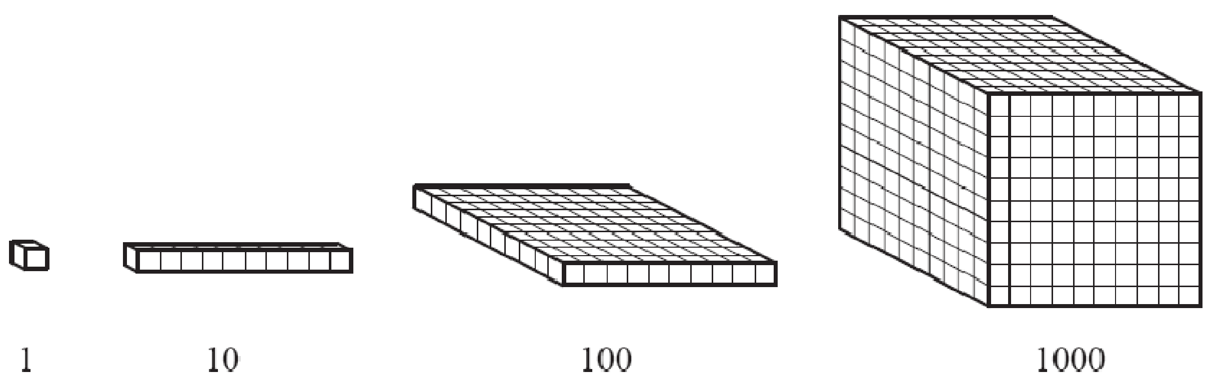
(2) What is another situation, other than a store, where it is necessary to keep track of the quantities of several different items?

Chapter 9: The Base Ten Number System

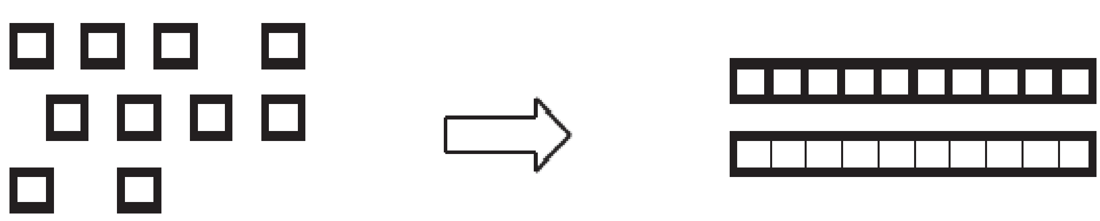
The key to the algorithms for adding, subtracting, multiplying, and dividing large numbers is the base-ten number system. Quite likely, this was the biggest breakthrough in all the history of mathematics. It is therefore very important to introduce the base-ten system to elementary students and show how it explains all the usual algorithms.

The first trick to the base ten number system is to convince your students that groups of items can serve as units. Ask them to add 2 apples + 7 apples, then 2 bears + 7 bears, and then 2 dozen + 7 dozen. Ideally, they will realize that 2 dozen + 7 dozen = 9 dozen, just as 2 of anything plus 7 more equals 9. Next, ask them to add 2 tens plus 7 tens, then 2 hundreds and 7 hundreds, then 2 thousands and 7 thousands.

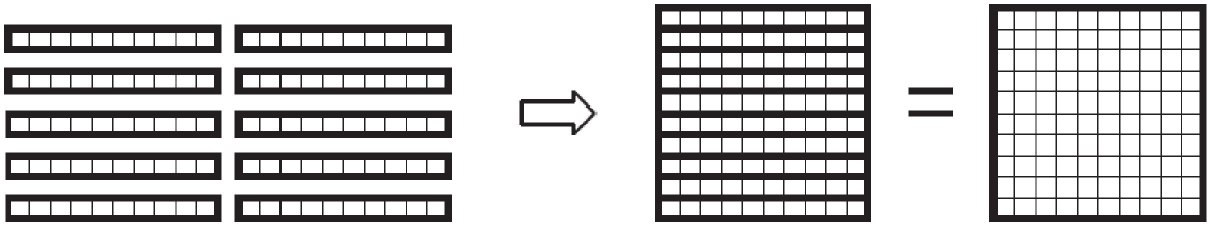
For a tangible presentation of the relative values of 1, 10, 100, and 1000, few things work better than base-ten blocks. These are something of an extension of Cuisenaire rods. The 1 blocks are cubes, 1 centimeter in each dimension, just as with the smallest of the Cuisenaire rods. The next size of base-ten block are the rods, which are 1cm by 1cm by 10cm, just like the orange Cuisenaire 10-rods. These also represent the number 10. Bigger than this are the "flats" which are 10cm by 10cm by 1cm. Finally, there are the "big cubes" which are 10cm in every dimension:



When you ask the class to compute 4 hundreds plus 7 hundreds, you can illustrate this with a handful of flats. Four of the big squares plus 7 more makes a total of 11. The biggest advantage of the base-ten blocks is that they illustrate the relative and cumulative values. For example, 10 one-cubes can be lined up and shown to be the same length (and volume) as a 10 rod:



Similarly, ten 10-rods can line up to take the same size and shape of a 100-flat:



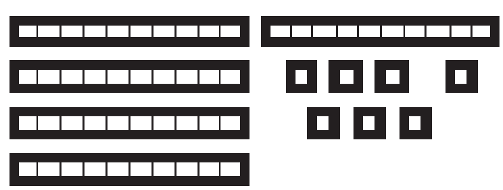
Finally, ten of the 100-flats can be stacked upon one another to form a cube with the same dimensions as the 1000-big cube.

All of these activities should be played out by the class. Have them put 10 little cubes end-to-end, then match the result up with a 10-rod. Similarly, have them compare 10 rods to a flat and 10 flats to a big cube. Another invaluable exercise is to practice exchanging these equivalent objects: either trading in 10 identical blocks for one of the next size up or else breaking down a block for 10 of the next size down.

When the students are comfortable making these sorts of exhanges, make the situation more abstract by playing the same game with fake money. At first use only \$1, \$10, \$100, and \$1000 bills – coins and bills of other denominations can be confusing and distract from the base-ten system. Assign a child to serve as the class banker (a little green visor would add a nice touch), ready to break a \$100 bill into ten \$10 bills, sell a \$10 bill for ten \$1, or any other such conversion.

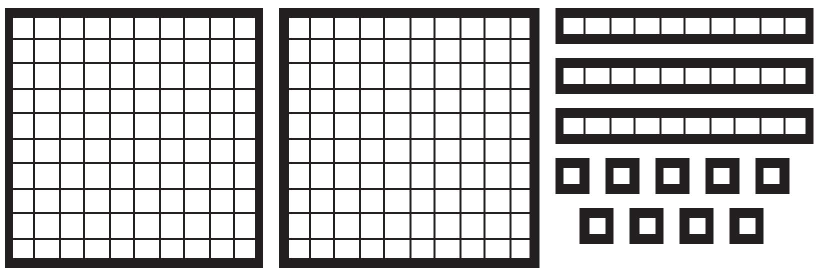
As an extra incentive, you can make a habit of giving out these bills as awards for winning games, asking good questions, being on good behavior, or other such things you would like to reward. Try to avoid the temptation to exchange these fake bills for real goods (pencils, etc.) unless you can get someone else (the principal, say) to pick up the costs. Instead, try to come up with other things which could be purchased. Perhaps a child could purchase a round of applause for \$20, for example, or the right to go to the head of a line. Perhaps a child could earn the right to wear a special hat. In a classroom like this, children can practice adding, subtracting, and converting between denominations of bills without the downfalls (hoarding, expense, jealousy, etc.) of a real economy. In any case, refuse to allow a student to keep more than 9 of any denomination – this will force students to keep going to the banker for conversions.

As the class comes to understand how ten small cubes can be converted into a long rod, teach them how to count up to 100. Show them how to use the base-ten blocks to count out large numbers. It would be very tedious, for example, to count out 57 small cubes. With base-ten blocks, however, you can use 5 rods to represent 50 of the cubes, and then count out only 7 small cubes to finish the number. This is a very valuable skill for your children to learn.



Similarly, students should be able to recognize a number that is represented with base-ten blocks. See if the students can notice the similarities between the names of the numbers and the base-ten blocks which represent them. Ideally, a child will realize that the word "forty" sounds a lot like "four," but means that you will need four rods and not four small cubes. It is as if the "ty" at the end stands for "tens." For example, "sixty" means "six tens" and "ninety" means "nine tens." As with everything else, teach it if you must, but do try to let the class work out numerous examples and discuss the results before you give up and tell them.

When the children are comfortable with the numbers up to 100, try numbers up to 1000. Hopefully they will quickly catch on that the number 239 is formed just as it sounds, with 2 hundreds, three tens, and nine ones:



As you explain how to read and form numbers with base-ten blocks, you can also introduce the base-ten notation. Just as the class shop kept track of pencils, pens, and staplers with numbers in different columns, the base-ten number system keeps track of numbers by making columns for the different powers of ten:

thousands	hundreds	tens	ones

For example, to represent the number "two-hundred thirty-nine," we need two hundreds, three tens, and nine ones, as illustrated above. The short-hand method to represent this is with the table:

thousands	hundreds	tens	ones
	2	3	9

Another way to represent this number is by writing it out as a sum: 200 + 30 + 9. A final, and useful way, is to write this out as a sum of words: 2 hundreds + 3 tens + 9 ones.

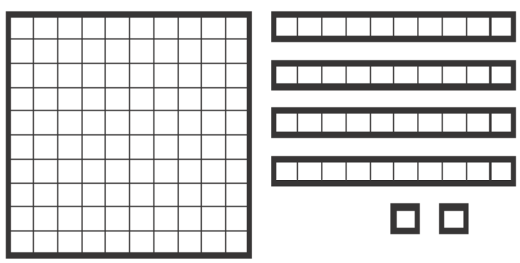
A student should be able to tell you what each digit in a number represents. In the number 4,824, for example, the 8 is really the number 800 because it is in the hundreds column. Until your students get the hang of it, have them write out the above chart with "thousands," "hundreds," etc. This is an excellent place to introduce graph paper. Children love graph paper!

Use something like the following as an anecdote for the importance of place value: suppose there is a toy you want to buy. The sales clerk will not tell you the price, only that there is a "9" in it. See how unhelpful this is! Does the toy cost $9 or $19? Does the toy cost $900? Is it only $1.99? Place value makes all the difference!

The one key to emphasize here is that we are only allowed to have 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 in any column. As soon as we have 10 of something, we must change it in for one of the next larger size. An excellent way to demonstrate this is by incrementing a number (adding one at a time) in base-ten form, much in the same way that a car's odometer counts the miles. Many teachers count the days of the school year, adding one every morning and representing the number with both base-ten blocks and in base-ten form. This is an excellent exercise, especially each time the last digit rolls over and you cash in 10 one-cubes for a single ten rod.

A student who understands the base-ten number system should be able to take a number in any of the following forms and convert it into any of the others:

(a) with base-ten numbers, for example: 142
(b) in words, for example: "one-hundred and forty-two
(c) with base-ten blocks, for example:



(d) with fake money, for example: a $100 bill, four $10 bills, and two $1 bills
(e) written out as a sum, for example: 100 + 40 + 2
(f) written out as a sum with words, for example:
1 hundred + 4 tens + 2 ones

Practice with these skills will pay off great dividends when it comes to explaining the addition, subtraction, multiplication, and division of large numbers. This understanding will also be essential for explaining the algorithm for long division. Also, do not forget to include numbers with zeros, for example 405, 2000, and 850.

Questions:
(1) Represent the number 425 in all 5 of the other forms listed above.
(2) What are some other rewards which would cost a poorly-paid teacher nothing and yet serve as motivation for students to acquire fake class money?
(3) Write a number with the digits 1 and 9 in it, where the 1 represents more than the 9.
(4) In the number 40,296, what does the digit 9 represent?

Chapter 10: The Addition of Large Numbers

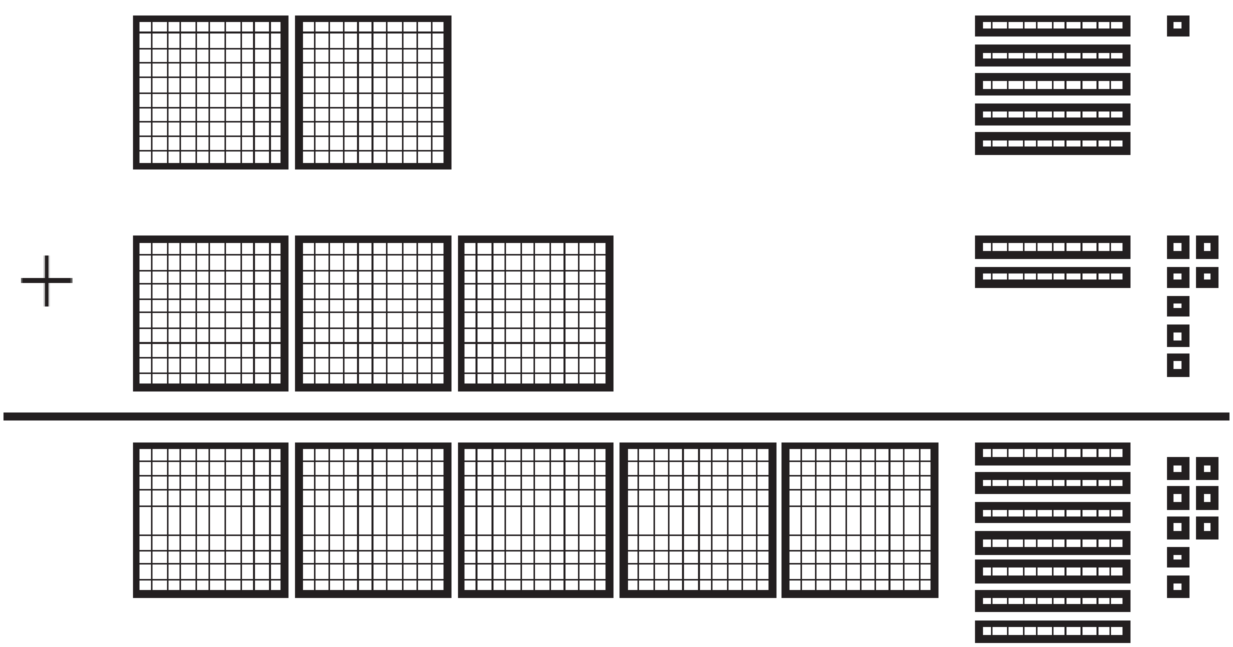
The usual algorithm for adding large numbers involves writing one number above the other, aligning the one's digits, then adding each column separately and "carrying the one" where necessary. For example, the completed calculation of adding 452 + 275 will look like the figure below.

$$
\begin{array}{cccc}
 & 1 & & \\
 & 4 & 5 & 2 \\
+ & 2 & 7 & 5 \\
\hline
 & 7 & 2 & 7
\end{array}
$$

Traditionally, this algorithm was taught as a multi-step procedure, without any detailed reasoning or explanation. Students were expected to be able to duplicate the process and to be content that the end result was correct. This is not satisfactory, however, because it teaches students, incorrectly, that mathematics is an arbitrary set of rules and procedures which might not make sense but must be followed and obeyed. It is much better to show students how this algorithm was developed. Not only will this help their understanding, but it will also make the other algorithms of arithmetic and algebra make more sense.

The trick to understanding this algorithm lies entirely with the base-ten number system, and so it is best to begin with base-ten blocks. It is also best to explain the "adding columns" and "carrying the one" concepts separately, so begin with problems that do not involve any carrying.

The process is simple: first represent each number with base-ten blocks and then group the blocks by size. For example, to add 251 + 327:



The 1 cube plus the 7 cubes become 8 cubes, the 5 rods plus the 2 rods become 7 rods, and the 2 flats plus the 3 flats become 5 flats. In practice, these will probably all end up in a jumble when the children put them together. However, when you demonstrate (transparent base-ten blocks can be used with an overhead projector), try to arrange them as above, to help lead the class toward the usual algorithm.

When this same arrangement is written out with words, it will look like:

$$\begin{array}{lccccc} & \text{2 hundreds} & + & \text{5 tens} & + & \text{1 one} \\ + & \text{3 hundreds} & + & \text{2 tens} & + & \text{7 ones} \\ \hline & \text{5 hundreds} & + & \text{7 tens} & + & \text{8 ones} \end{array}$$

This can be understood simply by treating "hundreds," "tens," and "ones" as separate units.

To save time in writing out these units over and over, we can use a base-ten table. These columns are implied by numbers written in base-ten, which explains the "adding columns" aspect of the addition algorithm:

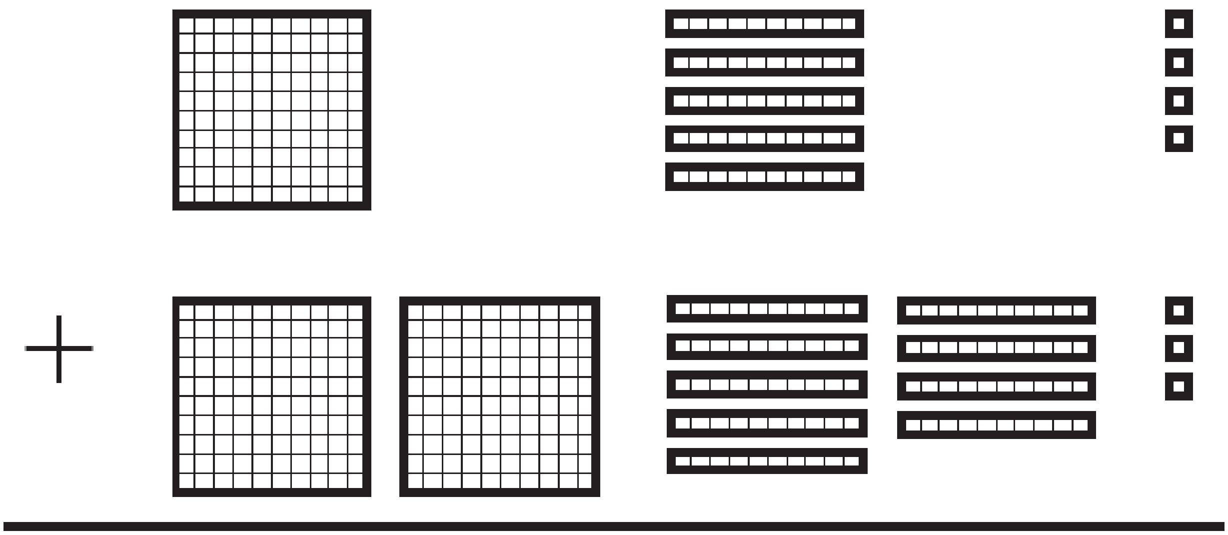
thousands	hundreds	tens	ones
	2	5	1
	+ 3	+ 2	+ 7
	5	7	8

$$\begin{array}{rccc} & 2 & 5 & 1 \\ + & 3 & 2 & 7 \\ \hline & 5 & 7 & 8 \end{array}$$

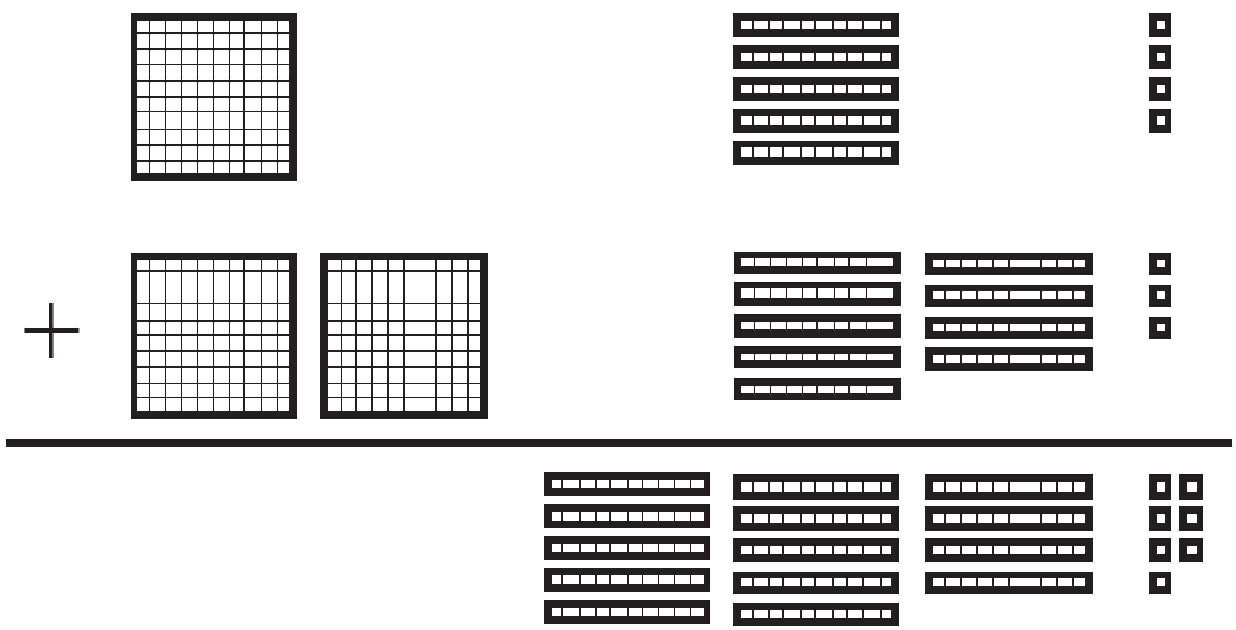
All of this, of course, is dependent upon the class having a solid understanding of the base-ten number system.

When the students have come to understand the "adding columns" approach to adding large numbers, introduce a problem that involves carrying. It is best to start slowly, with a problem that has only one incident of carrying. As with everything else, it pays to work out each problem before class, just to ensure that it is not more difficult than you originally intended.

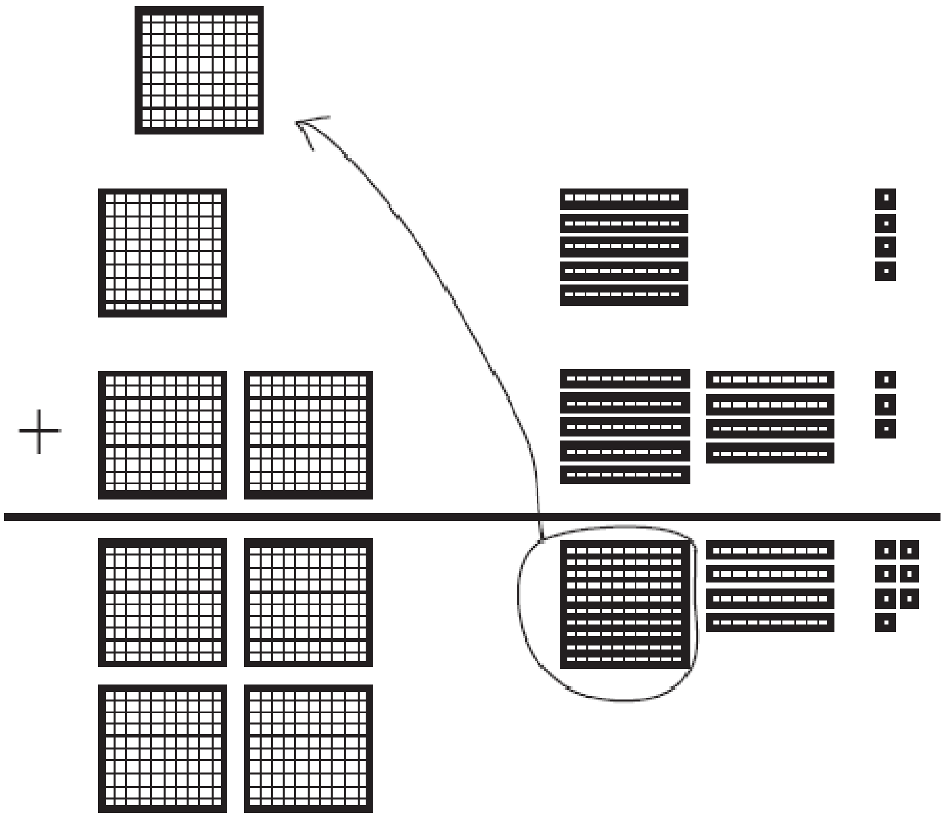
For example, to add 154 + 293 we begin as before, with base-ten blocks:



It would make some sense to begin with the largest units first, as these mean the most, and combine the three hundreds. However, we are trying to teach the usual algorithm, so we begin with the smallest units first. This makes things work most smoothly when carrying occurs several times in the problem. Just as before, we add the 4 cubes to the 3 cubes to get 7 cubes. When we add the 5 rods with the 9 rods, however, we end up with 14 rods:



Here, we use the skill practiced before and convert 10 ten-rods into 1 hundred flat. It helps to reinforce this by lining the 10 rods up and comparing it to a flat before replacing. To further prepare the students for the usual algorithm, put the new flat up on top to where we will later "carry the one." This is added to the other 3 flats, for a total of 4:



We thus conclude that 154 + 293 = 447.

When we represent the numbers in base-ten columns, we move closer to the short-cut algorithm:

thousands	hundreds	tens	ones
	1	5	4
	+ 2	+ 9	+ 3
		14	7

We cannot have ten or more in a column, so we break up the 14 tens into 10 tens + 4 tens. Next we argue, just as with the base-ten blocks, that 10 tens is 1 hundred. This one is again placed up above the other numbers in the hundreds column:

thousands	hundreds	tens	ones
	$^{1}1$	5	4
	+ 2	+ 9	+ 3
		~~14~~	7

10 + 4

With this done, we can finish adding the hundreds:

thousands	hundreds	tens	ones
	$^{1}1$	5	4
	+ 2	+ 9	+ 3
	4	~~14~~	7

10 + 4

We would have gotten the same result with less writing if we had just "carried the one" from the 14 over into the next column. This explains the usual short-cut algorithm for adding large numbers:

thousands	hundreds	tens	ones
	$^{1}1$	5	4
	+ 2	+ 9	+ 3
	4	14	7

$$\begin{array}{cccc} & 1 & & \\ & 1 & 5 & 4 \\ + & 2 & 9 & 3 \\ \hline & 4 & 4 & 7 \end{array}$$

This is rather slick. The 1 that we are carrying is not a number but a digit. The 1 in 14 represents a ten. Because the 14 is in the tens column, this thus represents 10 tens. The little 1 we put in the hundreds column represents 1 hundred. If anything, we are carrying a hundred, not a one. However, this is a testament to the beauty of the base-ten system. Each column represents a group that is exactly ten times bigger than the one to the right of it. This means that ten of any column will always equal one of the next column over. Thus, any time the numbers in a column add to 10 or more, we can take the "one" off of the beginning of the number and move it over to the next column.

Because this short-cut is so clever, it is important to explain it properly and thoroughly to your students. It is a trick that takes advantage of a quirk of the base-ten number system and not something that is immediately obvious to most people. You can use the expression "carry the one," but try to also point out what the 1 represents in each calculation. This reinforces the base-ten system, both the meaning of each column and the manner in which ten of one column can be traded in for one of the next.

After students become comfortable with carrying, you can introduce problems that involve two or more carries, for example 295 + 348 or 2,952 + 6,499. Make sure to include some problems where a whole new column is opened up, as with 65 + 73 or 950 + 425. Also, it is important to include some numbers with zeros among their digits, like 4,002 + 208. Finally, don't forget to include adding numbers of fundamentally different sizes, for example 54 + 271 or 8,293 + 407.

Questions:

(1) Explain in detail the addition 176 + 218 with:

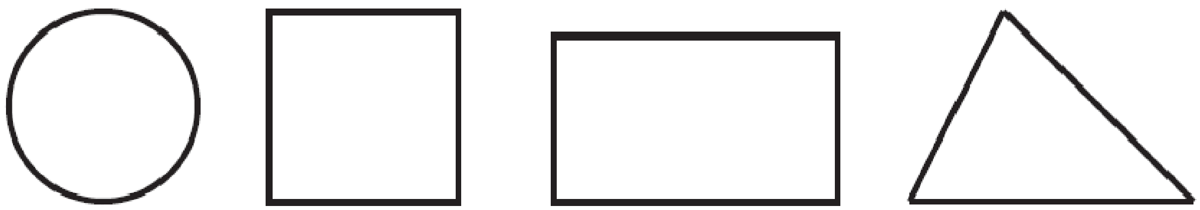
(a) base-ten blocks (b) words (c) base-ten columns (d) the usual short-cut

(2) When you "carry the one" in the above problem, what are the two (equivalent) things that the the 1 represents?

Chapter 11: Basic Geometry

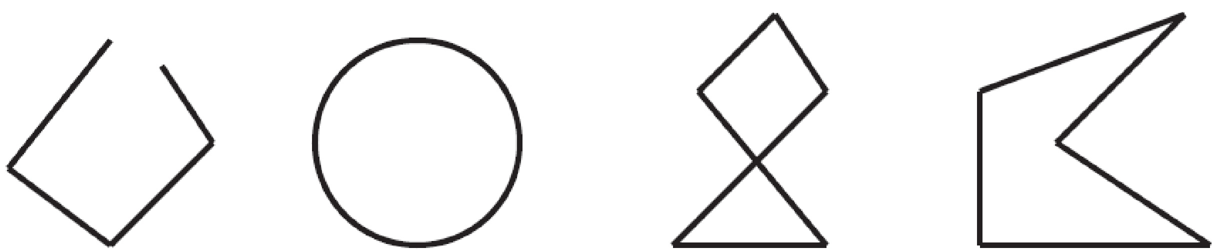
Children should be introduced to the basic shapes as early as possible, in order for them to practice discernment, classification, and vocabulary. Do not overwhelm them, however, because it will be a long time before this knowledge is critical for their education. Introduce a few of the concepts at first, and add more when the kids are comfortable and ready for more.

The basic shapes are circles, squares, rectangles, and triangles:



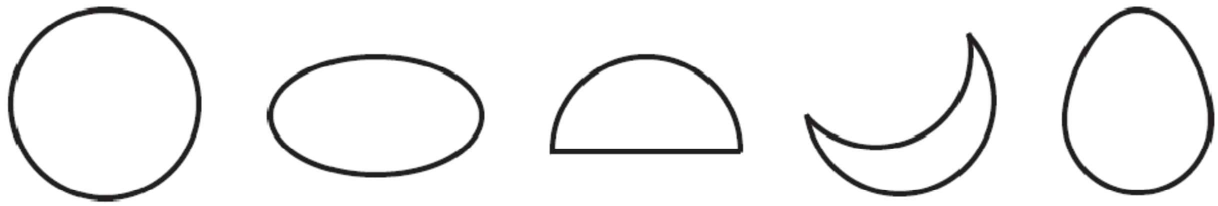
Even these can be a bit tricky for very young children, for the difference between a square and a rectangle is rather subtle. Ask the children to point out all the examples of these they can find, in the room, outside, and elsewhere.

When a child has an easy time with these shapes, you can begin to introduce the concept of a polygon. A *polygon* is a closed loop made of straight lines that does not intersect itself. These aspects are generally easiest to understand by looking at counter-examples:



Again, this technical definition is something that a high school student ought to know. In elementary school, it is enough for a child to know the names of basic polygons.

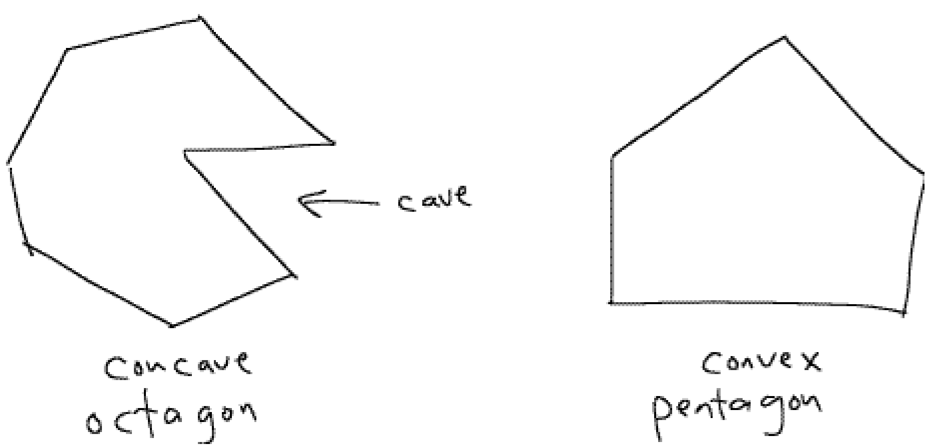
Most of geometry concerns itself with lines and polygons. There are a few shapes with curved sides that children ought to know:



circle | ellipse, or oval | semi-circle, or half-circle | crescent, or lune | egg

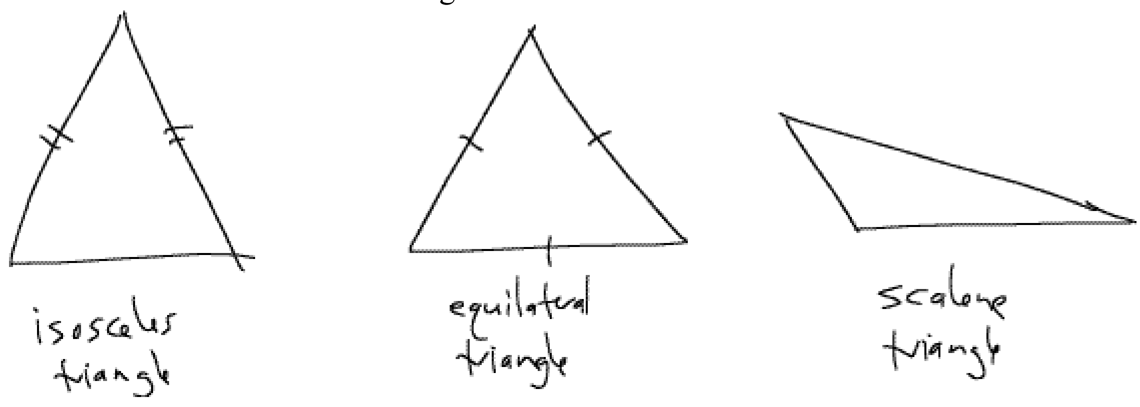
Polygons are most generally classified by the number of straight parts (called either *edges* or *sides*) which make up the loop. A 3-sided polygon is called a *triangle*. A 4-sided polygon is called a *quadrilateral*. A 5-sided polygon is called a *pentagon*. A 6-sided polygon is called a *hexagon*. A polygon with 7 sides is a *heptagon*, with 8 sides an *octagon*, with 9 sides a *nonagon*, with 10 sides a *decagon*, and with 12 sides a *dodecagon*. For young children, it is sufficient to count the sides and not necessary to learn the Greek and Latin prefixes for numbers. In fact, most polygons have names which come straight from the number. For example, a polygon with 25 sides is a 25-gon, and one with 43 sides is a 43-gon.

A polygon that looks like it has a dent in it (a bit like a cave) is called *concave*. Otherwise, it is called *convex*:

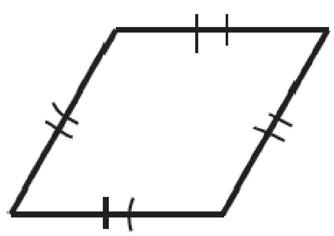


Nearly all of the shapes used in mathematics are convex, but it is useful to know these words.

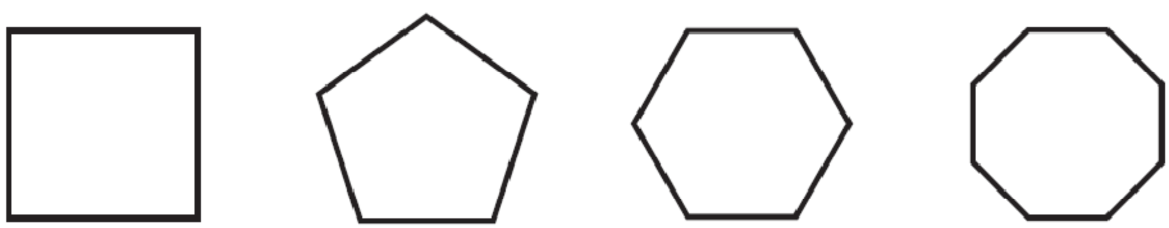
Polygons are also classified depending upon how many of their sides have the same length. A triangle with two sides the same length is called *isosceles*, with all three sides the same length *equilateral*, and with three different lengths *scalene*. We use little tick-marks to indicate when two sides have the same length:



A quadrilateral with all sides the same length is called a *rhombus*:

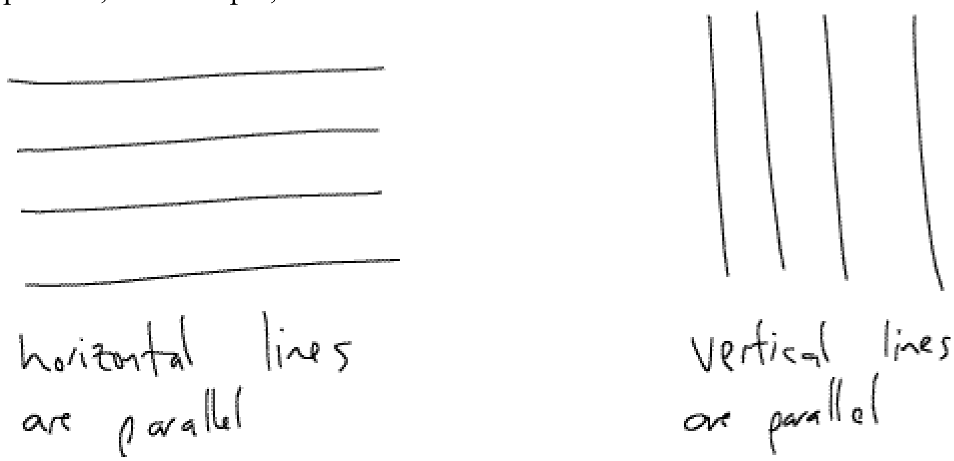

rhombus

Nearly all of the polygons with 5 or more sides that appear in mathematics are *regular polygons*, where all the sides have the same length and all the angles are the same:

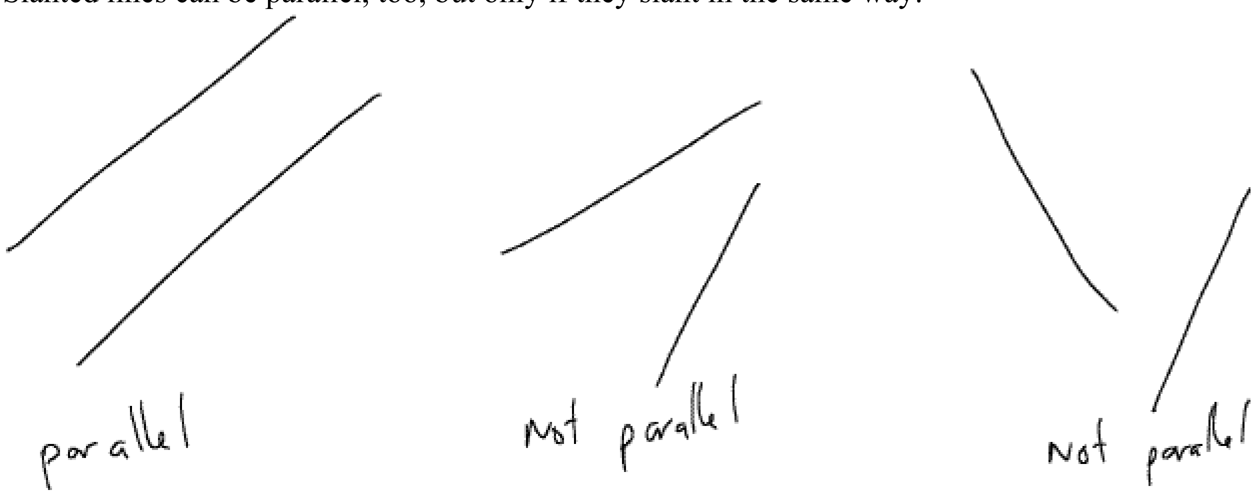

regular quadrilateral regular pentagon regular hexagon regular octagon

A square, for example, is a regular quadrilateral because it has four sides all the same length and the angles at the four corners are all the same.

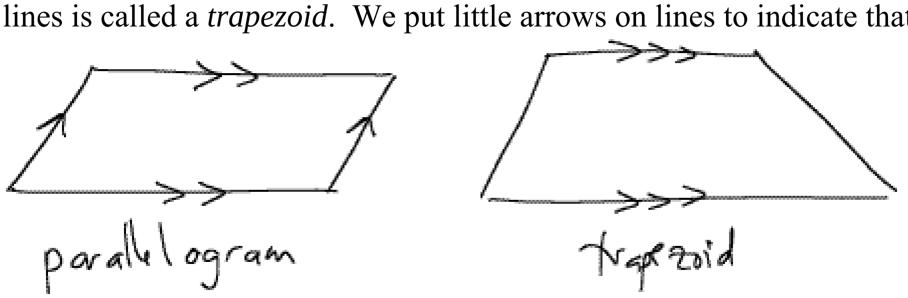
Another means of classifying a polygon depends on the concept of parallel lines. Mathematically, this is a very tricky concept, but it can be introduced rather simply to children. Basically, two lines are parallel if they run in the same direction. All horizontal lines are parallel, for example, as are all vertical lines:



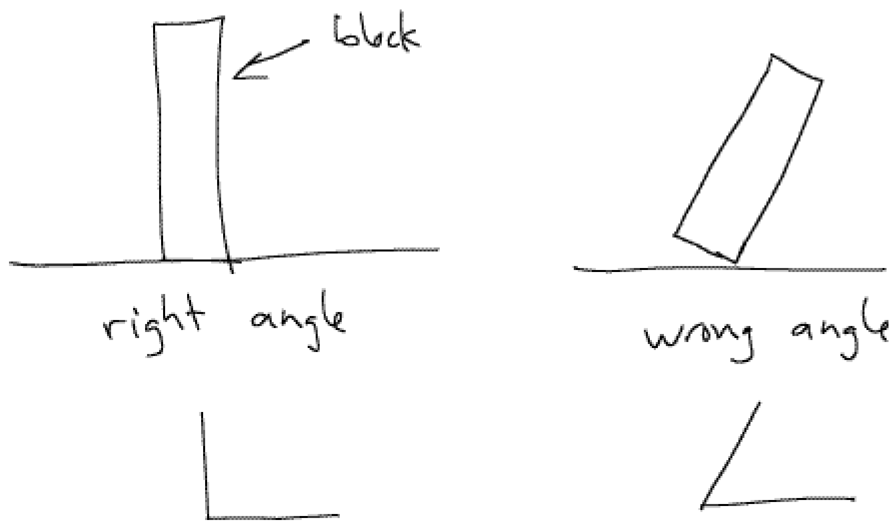
Slanted lines can be parallel, too, but only if they slant in the same way:



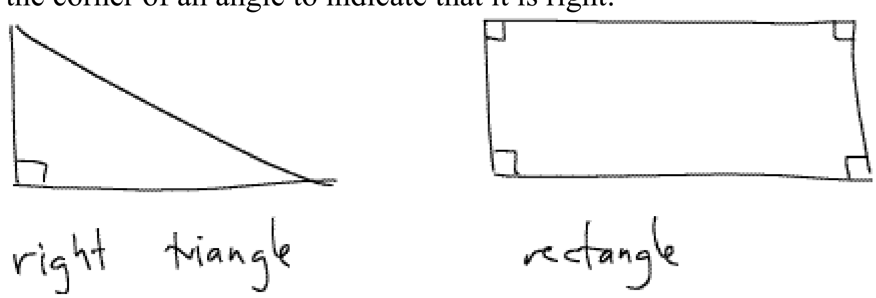
Actually, only quadrilaterals are classified by parallel lines. A quadrilateral with two pairs of parallel lines is called a *parallelogram*. A quadrilateral with only one set of parallel lines is called a *trapezoid*. We put little arrows on lines to indicate that they are parallel:



The last way to classify polygons depends on right angles. The easiest way to explain a *right* angle is that it is the right way to place a block on the ground if you don't want it to fall over. There are technically no "wrong" angles in math, but the idea is useful in introducing right angles:



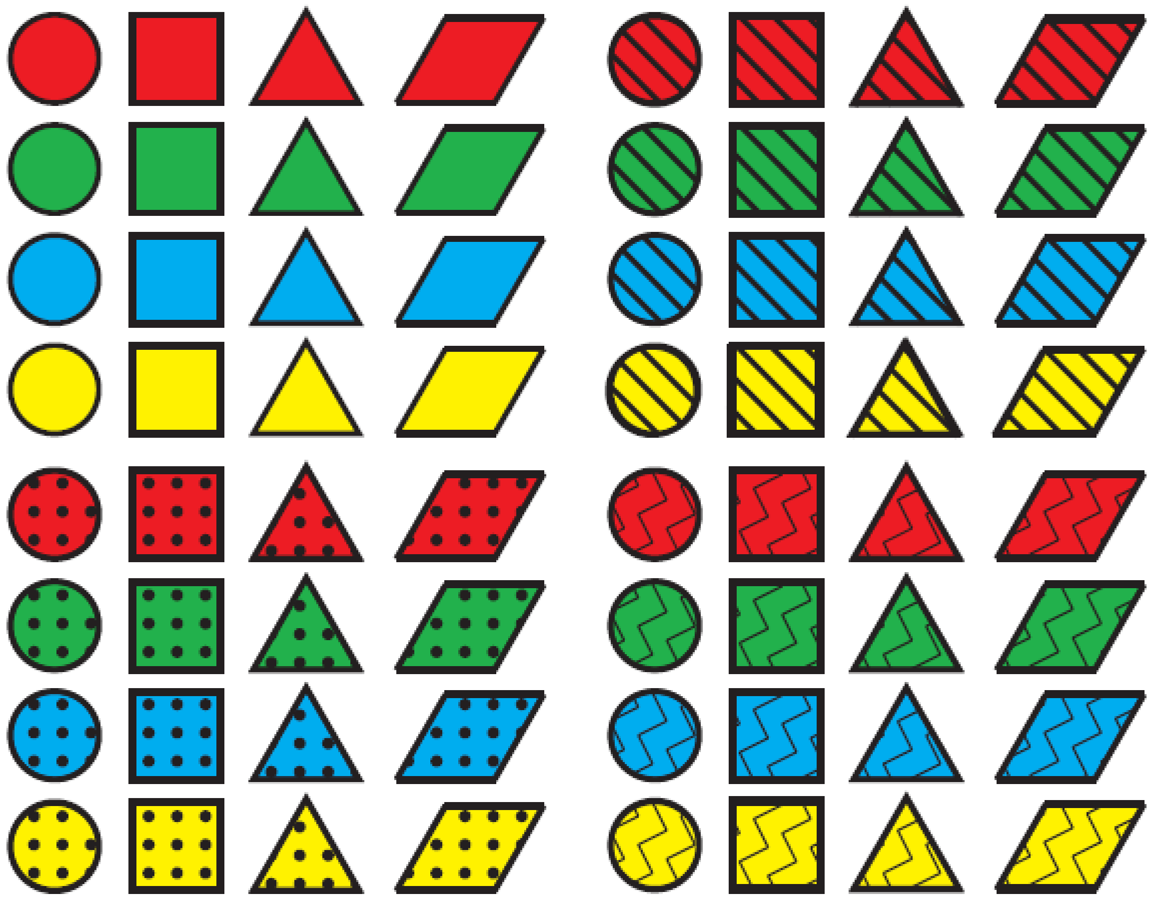
A triangle with a right angle is called a *right triangle*. A quadrilateral with all angles right is called a *rectangle*. Because the angles of a square are all right angles, we put a little square in the corner of an angle to indicate that it is right:



For the most part, the shapes confronting elementary school children are all drawn to scale, so they should recognize when an angle is right or when two sides are the same length, even if the squares and tick-marks are absent. It is only in later grades when figures are drawn contrary to their information, just to test analytical skills.

Once again, the vocabulary listed in this chapter should not be given to a child all at once, but only gradually as the child shows a readiness and capacity for learning more. Start with circles, squares, rectangles, and triangles. Move on to regular polygons – saying "eight-sided shape" instead of "octagon" at first. Later introduce the idea of a right angle as something found in squares and rectangles, then define right triangles. When the child is comfortable with all of this, you can introduce the concept of parallel lines, parallelograms, and trapezoids. Concavity and convexity are not too difficult to differentiate. The different kinds of triangles, rhombi, and other properties based on length can be introduced as children work through the edge-measuring exercises discussed later in this chapter.

As an early exercise in discernment, make a set of index cards with something like the following drawn on them, one symbol per card:



Shuffle these up and deal out a dozen or so to each kid and have them separate them into groups. There are a variety of ways in which this can be done, so accept a lot of different answers. Some might separate them by color, some by shape. Some might put the ones with a similar pattern together. Some might put the four-sided shapes all together. Have the students discuss the way they did this, to share all the ways it could be done. If everyone misses an obvious and important category, you can mention it, but try to let them figure them all out on their own.

As a later exercise, deal out 2 cards to each kid. The new challenge is to name all the ways in which the two objects are similar (shape, color, pattern, number of sides, etc.) and all the ways in which they are different. For a more advanced challenge, deal out 3 cards to each kid and see if they can find something that all the cards have in common.

As another exercise, have a child draw out a card from one of these decks. The child must name one of the aspects of the figure and then go through the deck and pull out all the cards that share that attribute. To make this more challenging, have the child name two aspects (green triangle, for example, or red and striped) and then find only those cards that share both properties.

For children who have played extensively with this deck, you can give them a much bigger challenge: take one of the cards out of the deck. See if the child can deduce the card you took, by sorting out the cards and deciding which is missing. If a kid wants a bigger challenge, take two or three cards out and see if he or she can figure out which they are. A disciplined child could even play this independently, by dealing the card face-down and only flipping it back over to verify if the guess is correct.

If the children enjoy the challenge of these games, you (or they) can add more shapes, colors, and patterns to the deck. You could also introduce size with smaller versions of the figures.

Games like these are excellent for introducing deductive reasoning, the process of elimination, the ability to recognize attributes, and the ability to classify, among other things. All of these are essential skills for success in mathematics, logic, science, and more.

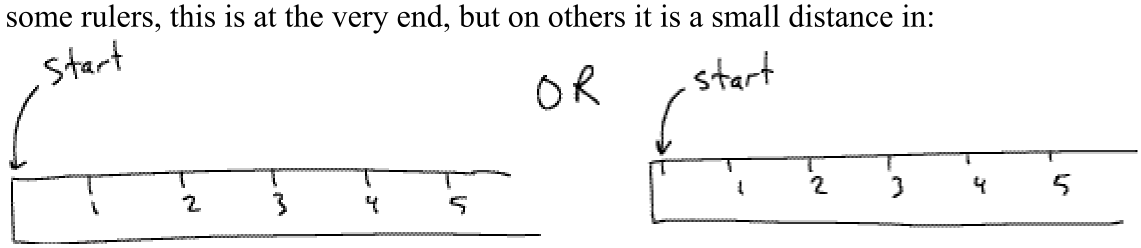
This game also teaches children that things can have a large number of different attributes. A square, for example, is a rectangle, a quadrilateral, a parallelogram, a rhombus, and a regular polygon. To force a child to recite this as a math fact would be a pointless and cruel exercise in rote memorization. With experience playing the pattern game (and gradual exposure to the definitions), however, a child might recognize these attributes as easily as noticing that an index card had a small, blue, striped circle drawn on it.

As soon as children have learned how to count on the number line and add multi-digit numbers, they are ready for the concepts of measurement and perimeter.

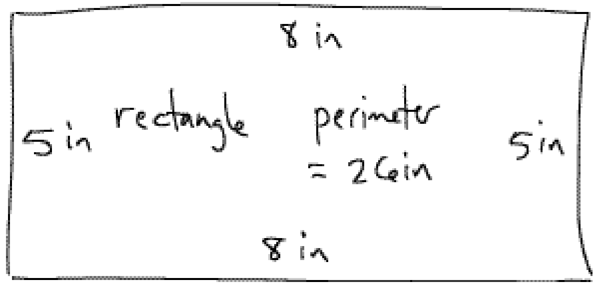
To do this, make a number of different shapes (each fairly large) out of a durable material like oak-tag. For a first assignment, have the children label each one with as many attributes as they know (polygon, triangle, etc.).

Next, give them each a ruler and teach them how to measure the length of each side. Initially, have them measure only to the nearest inch. Only later on should you worry about half-inches and other fractions of inches, unless, of course, your kids ask about them. It is always a good idea to answer questions honestly, but make it clear that the knowledge is advanced and not to be expected of them at the time. To avoid confusion, though, try to provide your students with shapes whose lengths are whole-inch lengths.

One important trick is to teach your students how to find the "start" mark on a ruler. For some rulers, this is at the very end, but on others it is a small distance in:

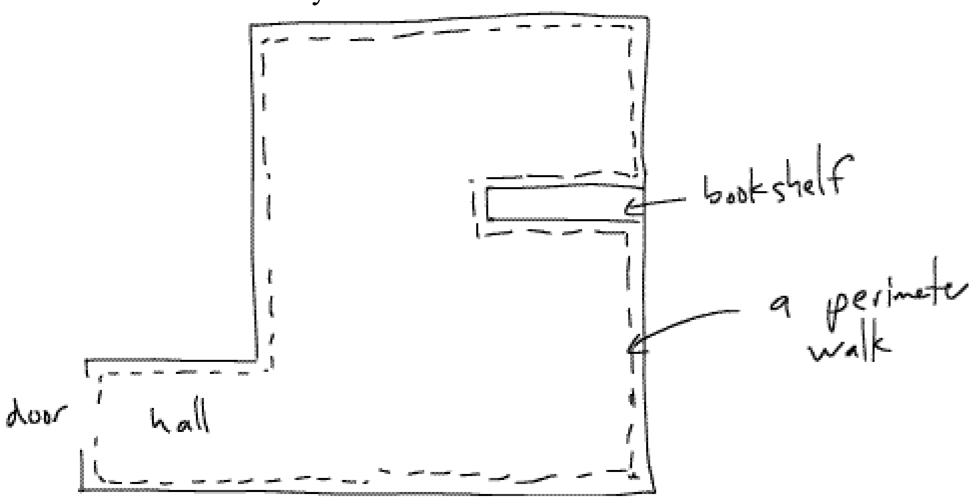


Have the children write the length along each edge of the figure. They can also calculate the *perimeter* of the shape – the sum of all the sides. For example:



Many children confuse the concepts of perimeter and area. To help avoid this, use the word "perimeter" as much as possible.

One fun exercise is called "walk/scout/check the perimeter." A fidgety child could be asked to walk the perimeter of a rug, of the room, or even of the whole building, if you trust the child out of your sight for that long. This, of course, means to walk entirely around the outside edge of the area, following the corners as closely as possible. For example, the following might be an overhead-view of your classroom:



To "mark off a perimeter," have children lay out objects all the way along the perimeter walk, either yarn, clothesline, sticks, or something of the sort. You could also have them mark off the perimeter with cones or blocks, but it is better to use a long one-dimensional object like string or rope because the perimeter is a one-dimensional object. If this gets out of control or tangles up the room with yarn, you can have kids mark off the perimeter of a smaller things: a throw rug, a beach blanket, or a shape drawn with masking tape on the ground. Ideally, try to get your hands on the yellow tape used by construction workers to mark off the perimeter of a worksite, and have the students use this.

As a group project, you should have the class measure the perimeter of the room. Have them draw a rough map of the shape of the classroom (don't worry if it is not to scale – that comes much later) and then copy down the lengths of each side as they measure it. Have them measure several times, to catch any mistakes (and practice). It is generally best to measure only to the nearest foot. This way, when a child walks the perimeter of the classroom, they can tell you the distance walked! As an extra assignment, have the children make rough maps of their bedrooms and measure out the dimensions. If they do not have their own rulers, you can have them make rulers out of oak-tag – a very useful exercise in itself – by copying the measurements off a classroom ruler.

As a warm-up for multiplication, you can have children walk the perimeter of the room several times (or a rug, or a shape taped to the floor, if that is more manageable) and then calculate the total distance walked.

Another excellent thing to measure is the kids themselves. Measure their heights with several units – to the nearest foot, to the nearest inch, and to the nearest centimeter. This helps to illustrate how a single measurement can have several different numbers, depending upon the unit used. Small units lead to big numbers, and big units lead to small numbers. Of course, see if you can get the class to discover this pattern on their own, perhaps with a little prompting, after measuring something with several units.

You can also measure all sorts of different parts of the kids – the length of a foot, the distance from thumb to pinky-finger on an outstretched hand – by tracing these first onto paper. Practice in measurement is not only a good geometry exercise, but it also reinforces the number line. By making the objects very personal, you can increase the children's interest significantly.

For a particularly fun exercise, you can have them lie down on large sheets of paper (or on a sidewalk) and trace their own perimeter. You could invite the students to try to measure their personal perimeter, but be kind – this is nearly impossible to do accurately.

In general, geometry offers all sorts of possibilities for entertaining, hands-on activities for students.

Questions:

(1) Write out all the definitions for all vocabulary words in this chapter.

(2) Drawn a concave decagon.

(3) Draw a hexagon that has exactly one pair of parallel lines.

(4) Draw two figures that have 2 attributes in common and 2 dissimilar attributes.

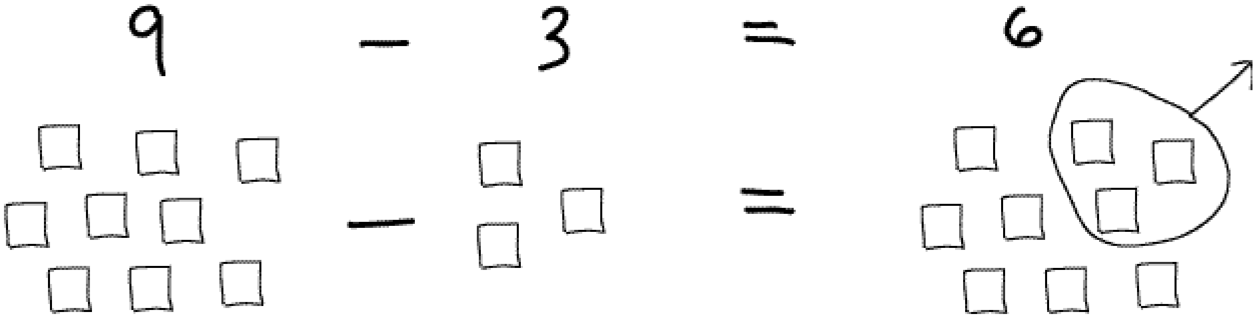
(5) Draw (to scale!) an isosceles (non-equilateral) triangle with a perimeter of 12 inches.

(6) Name all of the different geometry vocabulary terms that apply to squares.

Chapter 12: The Subtraction of Large Numbers

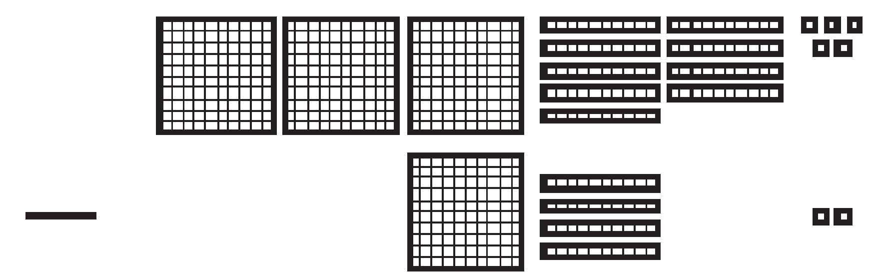
The process of subtracting numbers with several digits is very similar to the process of adding multi-digit numbers. Again, we line up the numbers so that their digits correspond – the ones place with the ones place, the tens place with the tens place etc. – and subtract by column. Instead of carrying over when we have 10 or more in one place, we sometimes have to do the exact opposite and borrow from the next larger place when we do not have enough to subtract. As before, this is all best explained by starting with base-ten blocks, moving to base-ten columns, and then ending with the usual short-cut algorithm for long subtraction.

Just as a warm-up for subtracting with base-ten blocks, illustrate a problem that the student already know automatically, like 9 – 3:



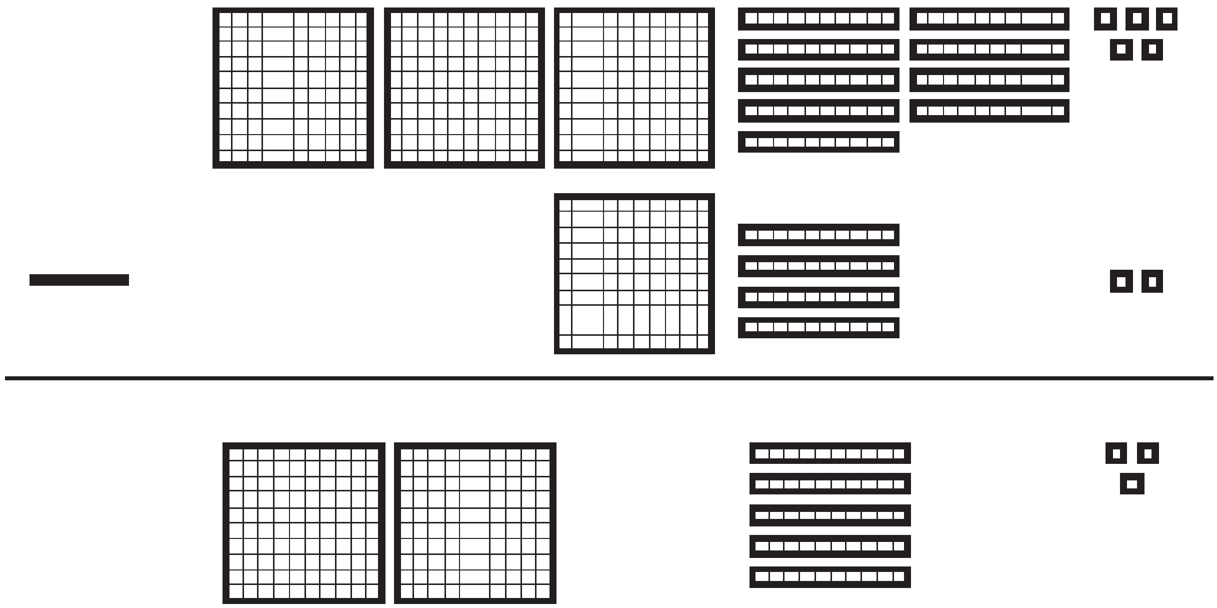
Basically, the first pile of blocks represents how much we begin with. The second pile of blocks represents the number that we have to take away from that pile.

In order to teach the "do everything by columns" aspect of the algorithm, it is a good idea to begin with some problems that will not involve any borrowing, for example 395 – 142. First, we illustrate the two numbers with base-ten blocks. Again, it is a good idea to line things up by size, just as we will with the usual subtraction algorithm:

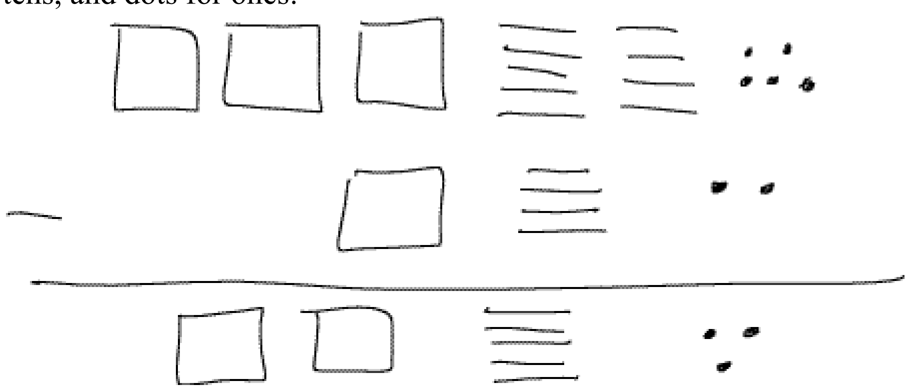


It helps if you illustrate this as the students go along. First give them the numbers to form with base-ten blocks. When they accomplish this, put the above on the blackboard or an overhead projector.

We thus begin with 3 hundreds, 9 tens, and 5 ones. From this, we want to remove 1 hundred, 4 tens, and 2 ones. This can be easily acted out, resulting in the correct answer, 2 hundreds, 5 tens, and 3 ones:



The students, after they pull away the 1 hundred, 4 tens, and 2 ones, will only have the answer remaining. This is why it is so useful to have illustrated the problem on the board or overhead for them. If you do not have transparent base-ten blocks and an overhead projector, you can agree upon a short-hand notation for illustrating base-ten blocks: squares for hundreds, lines for tens, and dots for ones:



There are two different ways to represent in words what the students have just worked out with base-ten blocks. We could say that we are taking away 1 hundred, 4 tens, and 2 ones, thus all three of these are subtracted:

$$\begin{array}{rlclcl} & \text{3 hundreds} & + & \text{9 tens} & + & \text{5 ones} \\ - & \text{1 hundred} & - & \text{4 tens} & - & \text{2 ones} \\ \hline & \text{2 hundreds} & + & \text{5 tens} & + & \text{3 ones} \end{array}$$

We could also say that we are subtracting the entire 142 at one time, and thus use parentheses to indicate this:

$$\begin{array}{rlclcll} & \text{3 hundreds} & + & \text{9 tens} & + & \text{5 ones} & \\ -\ (& \text{1 hundred} & + & \text{4 tens} & + & \text{2 ones} &) \\ \hline & \text{2 hundreds} & + & \text{5 tens} & + & \text{3 ones} & \end{array}$$

It is not correct, however, to use the latter notation without the parentheses, although some teachers do this:

$$\begin{array}{rlclcl} & \text{3 hundreds} & + & \text{9 tens} & + & \text{5 ones} \\ - & \text{1 hundred} & + & \text{4 tens} & + & \text{2 ones} \\ \hline & \text{2 hundreds} & + & \text{5 tens} & + & \text{3 ones} \end{array} \qquad \text{WRONG!}$$

Sure, you can suppose that the big minus sign to the left makes it clear that this is a subtraction problem, and that we subtract each of the parts. However, this leads to a very common misake that people make in algebra, confusing expressions like $4x - 5 + x$ and $4x - (5 + x)$. We need parentheses to express precisely what is being subtracted. Putting the parentheses in now should help your students avoid this pitfall in the future. Otherwise, use a minus sign for each part of the number which is being subtracted.

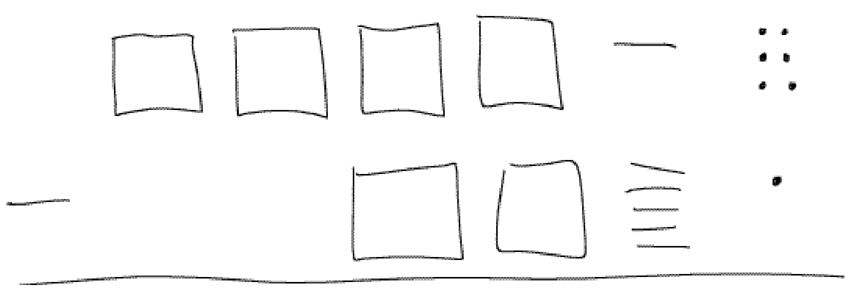
Next up, we rewrite the same information in a more compact form with base-ten columns, and then again in the usual base-ten fashion:

thousands	hundreds	tens	ones
	3	9	5
	- 1	- 4	- 2
	2	5	3

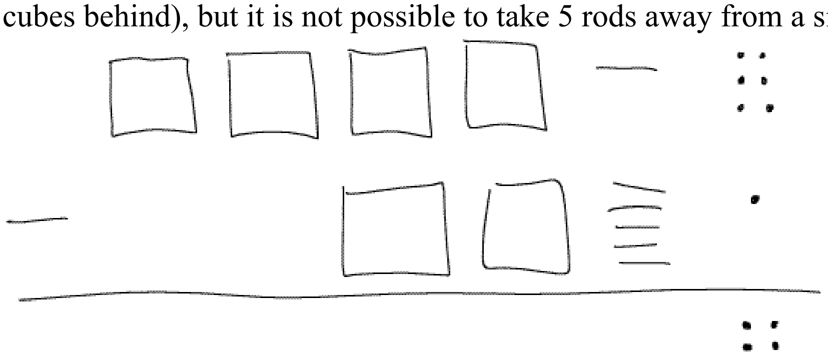
$$\begin{array}{r} 3\;9\;5 \\ -\;\;1\;4\;2 \\ \hline 2\;5\;3 \end{array}$$

By this time, or with another example or two, your students ought to understand that we can subtract one column at a time because each column represents the same sort of object (tens, hundreds, etc.). Hopefully, the fact that addition works the exact same way will make this easier to grasp.

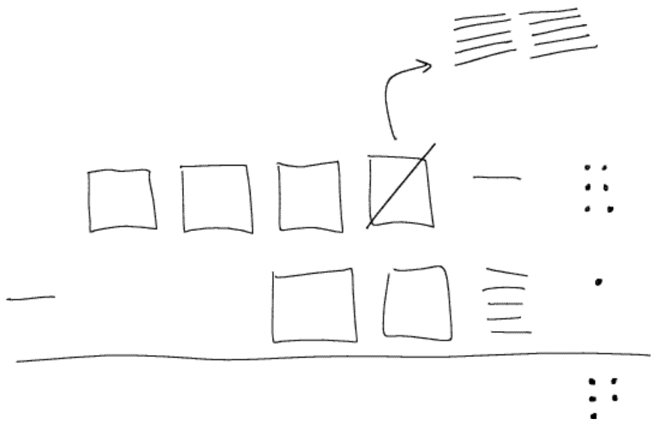
Next, we have to work the class through some problems to introduce the idea of borrowing. Start with a problem where only one borrowing will be necessary, like 416 – 251. Again, begin by having the students represent the numbers with base-ten blocks. When they are done, draw out their answer on the board as we have done before:



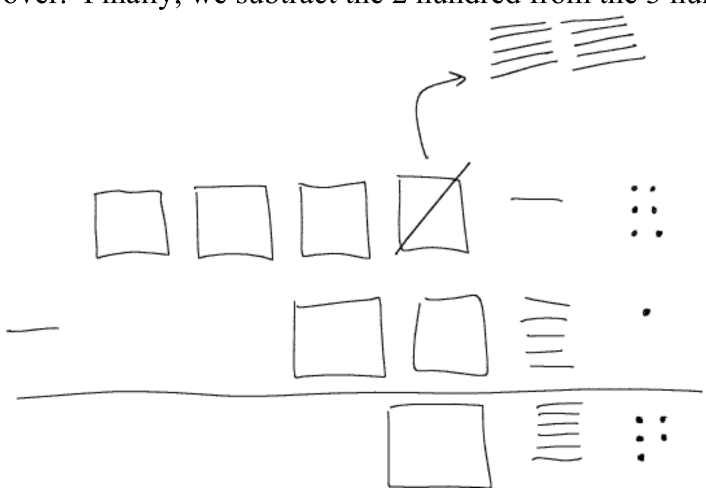
Next, the students should take the blocks representing the top number and take away as many blocks as the bottom number. The 1 cube can be taken away from the 6 cubes easily (leaving 5 cubes behind), but it is not possible to take 5 rods away from a single rod:



This is why we practiced converting between the different sizes of base-ten blocks (and between the different denominations of fake money). What we need to do is borrow one of the hundred-flats and break it into 10 ten-rods:



This makes it so that we have a total of 11 rods up top, enough to subtract 5 and still have 6 left over. Finally, we subtract the 2 hundred from the 3 hundreds and finish the problem:



Thus 416 – 251 = 165. Notice that we put a slash through the square representing the flat which was broken down, hinting of the way it is done in the usual algorithm.

Next, as before, we do the same problem over again, using words instead of blocks. Again, we get stuck where we need to subtract the 5 tens from the 1 ten:

	4 hundreds	+	1 tens	+	6 ones
-	2 hundred	-	5 tens	-	1 ones
					5 ones

We solve the problem in the exact same way as before, by taking one of the hundreds and turning it into 10 tens:

3 hundreds + ~~1 hundred~~

10 tens

~~4~~ hundreds	+	1 tens	+	6 ones
- 2 hundred	-	5 tens	-	1 ones
				5 ones

Now, instead of 4 hundreds, 1 ten, and 6 ones, we have 3 hundreds, 11 tens, and 6 ones. This makes it easy to complete the subtraction:

3 hundreds	+	11 tens	+	6 ones
- 2 hundred	-	5 tens	-	1 ones
1 hundreds	+	6 tens		5 ones

If we illustrate this with base-ten columns, there are two ways to show the borrowing. We could cross out one of the hundreds and turn it into ten in the ten's column:

thousands	hundreds	tens	ones
	3 + 1	10	
	~~4~~	1	6
	- 2	- 5	- 1
			5

thousands	hundreds	tens	ones
	3 + 1	10 + 1 =	
	~~4~~	11	6
	- 2	- 5	- 1
	1	6	5

The sneaky way to illustrate what happens is to subtract 1 hundred from the hundred's column and put this 1 directly to the left of the 1 in the ten's place, making it eleven. Just as with the "borrowing the one" trick in long addition, this is only possible because of the way our number system is positional and based on powers of ten. However, it also explains the short-cut trick, in which we borrow one from the next column over by reducing the number by 1 and then putting this 1 to the left of the column which is having trouble:

thousands	hundreds	tens	ones
	3 + 1 $\cancel{4}$ - 2	$^{1}1$ - 5	6 - 1
	1	6	5

$$\Rightarrow \quad \begin{array}{rccc} & \overset{3}{\cancel{4}} & {}^{1}1 & 6 \\ - & 2 & 5 & 1 \\ \hline & 1 & 6 & 5 \end{array}$$

With a few examples, your students ought to be able to understand the process of borrowing when subtracting multi-digit numbers. It is not a good idea to force the children to do everything out with base-ten blocks after they understand, but it is a good idea to use them to motivate and explain the process. Your end goal, of course, is for students to subtract large numbers quickly and accurately, which generally involves as little writing as possible, using the usual short-cut algorithm.

There is one more short-cut which you might want to teach. Often, students have the most trouble when trying to borrow from a column with zeros, for example when subtracting 302 – 85. Let us skip straight to the base-ten columns representation:

thousands	hundreds	tens	ones
	3	0 - 8	2 - 5

We have a problem right away – we have only 2 ones and must remove 5. We want to borrow from the tens column, but nothing is there. The long way to solve this problem is to take the 3 hundreds, break it into 2 hundreds + 1 hundred, then break the 1 hundred into 10 tens:

thousands	hundreds	tens	ones
	2 + 1 ~~3~~	10 0 - 8	2 - 5

We can now borrow one from the ten's place (leaving behind 9 tens), and turn it into 10 ones, turning the 2 ones into 12 ones:

thousands	hundreds	tens	ones
	2 + 1 ~~3~~	9 + 1 ~~10~~ 0 - 8	12 - 5
	2	1	7

$\Rightarrow$

$$\begin{array}{rccc} & \overset{2}{\cancel{3}} & \overset{9}{\cancel{{}^{1}0}} & {}^{1}2 \\ - & & 8 & 5 \\ \hline & 2 & 1 & 7 \end{array}$$

We are able to subtract the 5 ones from the 12 ones, leaving 7 ones. Next, we are able to subtract the 8 tens from the 9 tens, leaving 1 ten. Finally, there are 2 hundreds from which nothing is subtracted, and so they remain. The answer is 302 – 85 = 217.

The short-cut for this is as follows. Rather than break the 1 hundred into 10 tens, and then 1 of the tens into 10 ones, we could do all of this at once. Break the 1 hundred into 9 tens and 10 ones. Think of this as the way a cashier at a bank might break a hundred dollar bill. Hopefully, as the class banker converts fake money for your students, this sort of thing will come up. If a student has a fake \$100 and wants to spend \$7, the banker might turn the \$100 bill into 9 \$10 bills and 10 \$1 bills. In real life, of course, a banker will surely use some \$20 bills and \$5 bills, but we avoid these denominations to make this simpler and focus the student's minds on the base-ten number system. In any case, if one of our hundreds is broken in this fashion, then the calculation will look like this:

thousands	hundreds	tens	ones
	2 + 1	9 + 10	
	$\cancel{3}$	0	$^{1}2$
		- 8	- 5
	2	1	7

$$\Rightarrow \quad \begin{array}{rrrr} & {}^{2}\cancel{3} & {}^{9}\cancel{0} & {}^{1}2 \\ - & & 8 & 5 \\ \hline & 2 & 1 & 7 \end{array}$$

This is, of course, just a short-hand trick and not essential for children to know. However, it does make calculations quicker, for example when subtracting 2003 – 17:

$$\begin{array}{rrrrr} & 2 & 0 & 0 & 3 \\ - & & & 1 & 7 \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{rrrrr} & {}^{1}\cancel{2} & {}^{9}\cancel{0} & {}^{9}\cancel{0} & {}^{1}3 \\ - & & & 1 & 7 \\ \hline & 1 & 9 & 8 & 6 \end{array}$$

Here, we have to borrow a thousand, and break it into 9 hundreds, 9 tens, and 10 ones.

As with long addition, it is a good idea to slowly increase the difficulty of the problems you pose your students. Begin with problems that do not involve borrowing, then move to problems with only one borrowing. Next, offer problems where there are two borrowings, but separated, for example 5,193 – 2,634. Then offer problems with multiple borrowing all together. Only at the end should you offer problems when you need to borrow from a column with zero in it.

An excellent warm-down from a long subtraction problem is to check your answer with long addition. If 2003 – 17 = 1986, then 1986 + 17 ought to = 2003. Have them check! This not only teaches students that they can check their work, also reinforces the idea that addition and subtraction are inverse operations.

Questions:

(1) Show how to compute 493 – 27 with

- (a) base-ten blocks (or the short-hand notation for them)
- (b) words (4 hundreds + 9 tens, etc.)
- (c) base-ten columns
- (d) the usual short-cut algorithm

(2) In the last problem, what exactly was borrowed? What did it become?

(3) Explain, in words, how a person who had $423 might give $185 to a friend. Assume that all the money is in $100, $10, and $1 bills, and that there is a nearby banker, ready to offer change in those same denominations. Make sure you explain the amount of money the generous person will end up with.

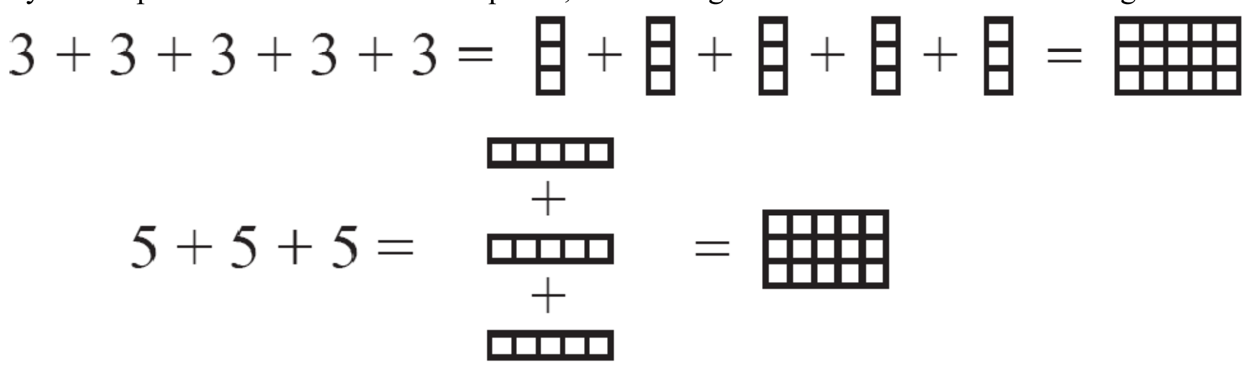
Chapter 13: Models of Multiplication

There are several ways to introduce, explain, and model multiplication. Children should be shown all of them, but the area model will prove to be the most capable of explaining more advanced forms of multiplication (large numbers, fractions, algebraic expressions, etc.).

The most common way to introduce multiplication is to represent it as a short-cut for repeated addition. For example, the calculation 3 + 3 + 3 + 3 + 3 can be viewed as "three added five times" and written 3 × 5. This explains why the word "times" is used to mean multiplication. Children can easily relate to this concept, for they understand what it means to "kick a ball 5 times," "go to the zoo three times," and similar situations where "times" indicates a multiple.

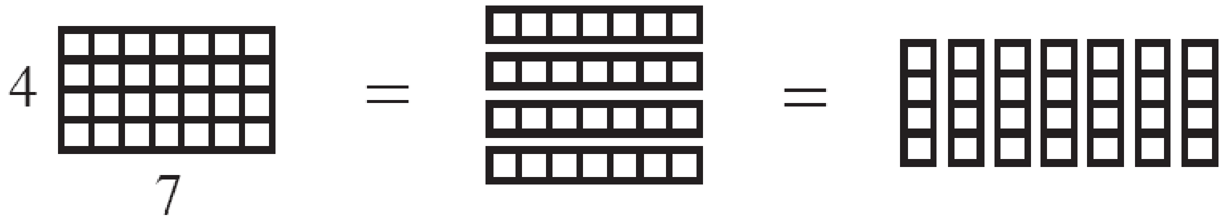
A similar, but slightly-different, model for multiplication involves the concept of grouping. A multiplication like 3 × 5 can be viewed as "three groups of 5 all added together." Using this model, 3 × 5 = 5 + 5 + 5.

Both of these models result in the same answer. However, it is not immediately obvious that 3 + 3 + 3 + 3 + 3 = 5 + 5 + 5. Even if your children have an easy time adding both sides to get 15, this does not illustrate why this would be true if the numbers were changed. You could teach your students to simply accept this as a fact, but it helps to show them an illustration. The key is to represent the numbers with squares, then arrange them to form the same rectangle:



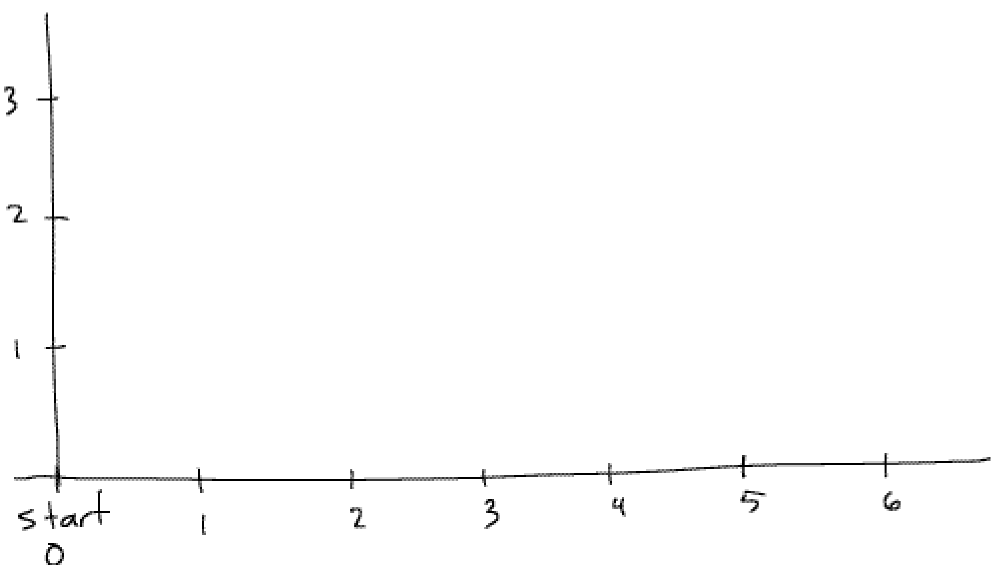
Hopefully, it is clear that this can be done with absolutely any numbers. This means that the order in which numbers are multiplied does not matter, that multiplication is *commutative.* For example, 3 × 5 = 5 × 3 and 57 × 29 = 29 × 57.

The idea of using rectangles to represent multiplications is the heart of the *area model for multiplication* (also called the *array model*). This states that the product of two numbers is the area of a rectangle with the two lengths as sides. For example, 4 × 7 is the area of a rectangle with sides of length 4 and 7:



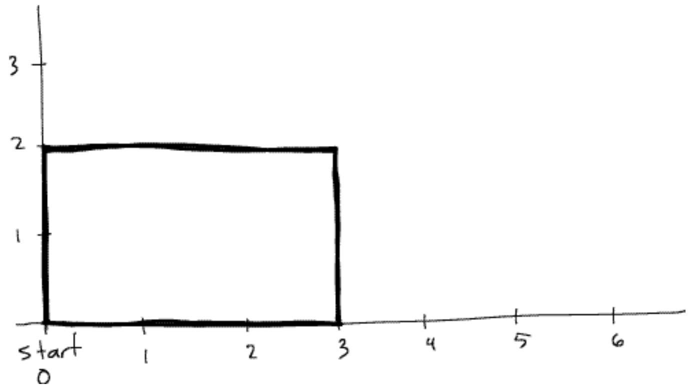
As discussed before, this rectangle can be broken up to illustrate either 4 groups of 7 (7 + 7 + 7 + 7) or 7 groups of 4 (4 + 4 + 4 + 4 + 4 + 4 + 4). However, this model of multiplication can also be used in a large number of other situations, as will be developed over the course of this book.

The best way to introduce the area model of multiplication is by extending the number line into a new direction. From the start-0 mark, make another number line with the same start and the same distances between marks:

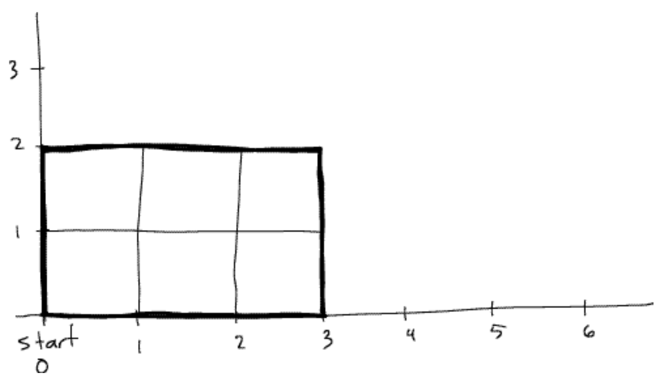


While the number line can be constructed in a large number of ways (chalk, tape, clothesline, etc.), this is best done with chalk.

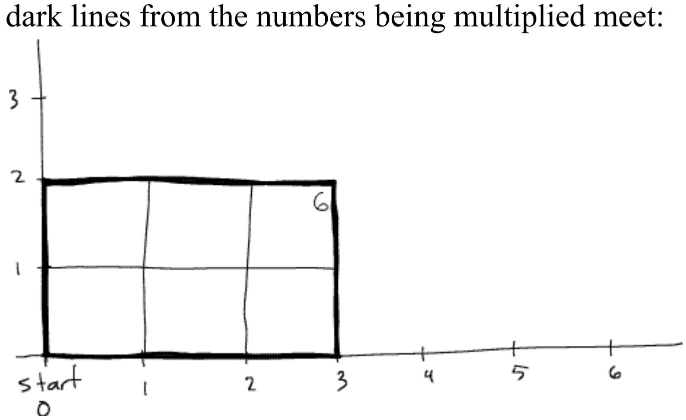
To illustrate a multiplication, first draw a rectangle with one corner at the starting point. (It helps to begin introducing the word "origin" at this time.) On each of the two number lines (you can call them "axes," the "x-axis" and the "y-axis," if this doesn't bother or confuse the class) the rectangle should go to one of the numbers being multiplied. Make the rectangle dark, preferably with a color of chalk different from that used to draw the number lines. For example, to illustrate 2 × 3, draw the following big rectangle:



Next, with a different color of chalk (white would probably work best) divide the big rectangle up into a number of squares by drawing lines parallel to the axes through the smaller numbers:

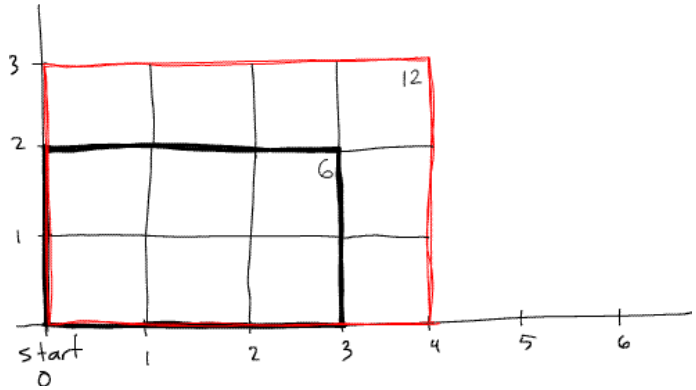


The students should be encouraged to walk around the figure on the ground and help count out the number of squares. This number should be written up in the corner, where the two dark lines from the numbers being multiplied meet:

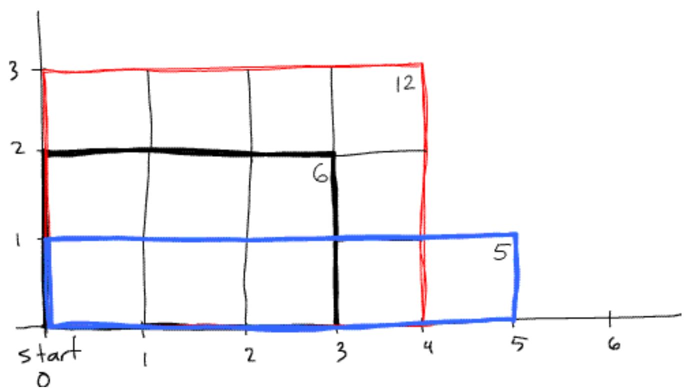


This process, you should tell them, is called multiplication. In the example, we have shown how 2 × 3 = 6. At this point, it is an excellent time to show how this rectangle illustrates the other ways of looking at multiplication. This is the number 2 added 3 times. This is the number 3 added 2 times. This rectangle can be broken into 2 groups of 3, or into 3 groups of 2.

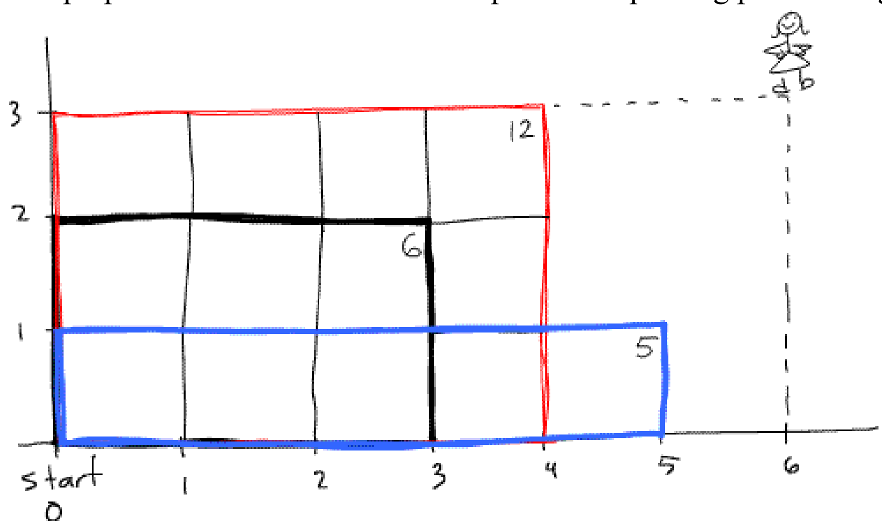
Next, offer a slightly larger multiplication, for example 3 × 4, and see if anyone from the class is able to pick up the process. Pick a new color for the rectangle, to help differentiate it from the first:



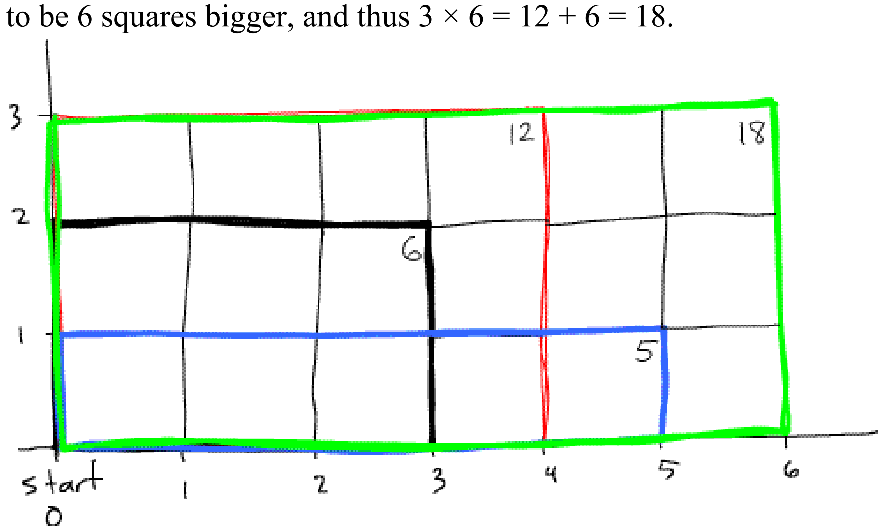
You can also show how the rectangles are allowed to cross over each other, for example with 1 × 5:



As soon as your students pick up on process, give them chalk and let them proceed. At first, have the children volunteer to try and go one at a time. Let the child pick a multiplication (for example 3 × 6), and then have another child stand at the corner of the rectangle which will be furthest from start. This will help to ensure that the rectangle is drawn properly. This also will prepare students for the identical process of plotting points and graphing:



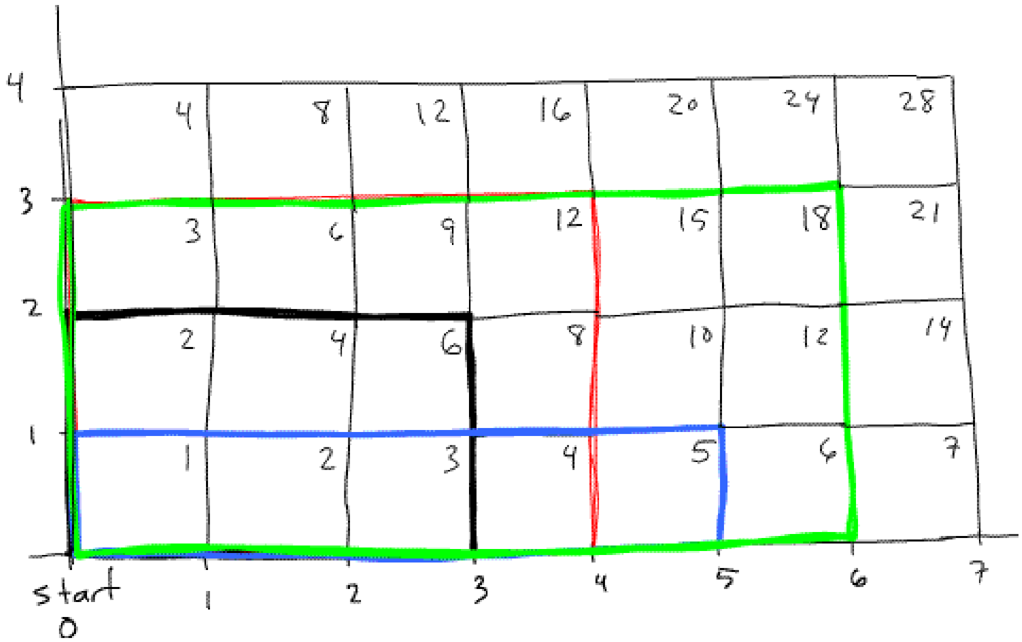
Have the other students in the class decide if she is standing in the right spot, or if she should move over a bit in one direction or another. This will make it more of a team sport, and also help to reinforce the concept. When it is agreed that she is in the right spot, the big rectangle can be drawn, the inside lines can be extended, and the number of squares can be counted. Challenge the students to count the squares in as many ways as possible (by 3's, by 6's, in various blocks). Only when a large number of kids have counted the squares and all come up with the same answer should the answer be filled in. Notice (and prompt) the students to use the previously-calculated areas as a short-cut. For example, if 3 × 4 = 12 has been computed with a red rectangle (as above), then the bigger green 3 × 6 rectangle (drawn below) can be easily seen to be 6 squares bigger, and thus 3 × 6 = 12 + 6 = 18.



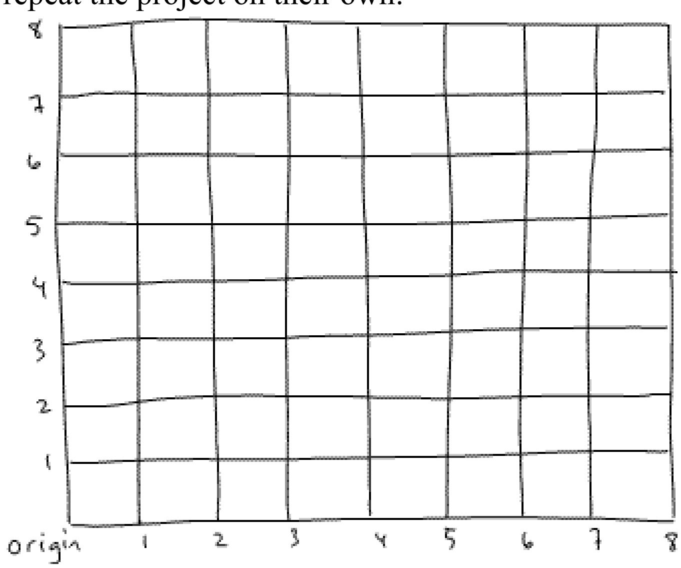
Have the students say the new multiplication fact several times, to help reinforce it: 3 × 6 = 18.

If your students have already mastered working with the number line and adding double-digit numbers, they ought to be able to pick up this process without too much difficulty. At first, have the students take turns drawing out the rectangles, counting out the squares, and writing down the numbers. Have the students calculate the multiplications of smaller numbers before they more on to the larger ones.

The end result of this will be a large multiplication table drawn on the ground:



When you have finished with this activity (limit the numbers to a size that can be filled entirely in the available time), bring the class back inside. Hand out sheets like the following (size it so as to not go beyond what has been done outside on the playground) and have the class repeat the project on their own:



This is another example of making a concrete exercise more abstract. This also gives the opportunity to complete the game for those students who didn't feel that they got enough turns. For the students who are having trouble with the concept, have them outline each rectangle with crayon, then count out the squares and write in the number. For the students who have caught on, make it a contest to see who can complete the table first without making a single mistake. Again, it is better to start with easier grids, perhaps only up to 5 × 5, than to make the process seem daunting and tedious. If many students are having difficulty, you can work on a copy of their grid at the board or overhead and do the first few problems collectively.

It is important that the students also write out the problems using multiplication notation. For example, have them write 1 × 4 = 4 and 3 × 5 = 15 as they figure these out. This will introduce and reinforce the meaning of the whole exercise.

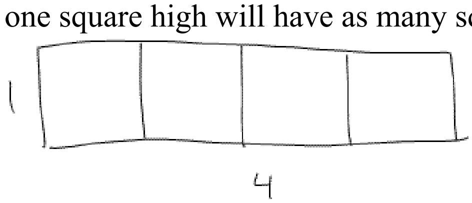
Similarly, you should show them a completed times table (on an overhead, etc.) and have them read multiplications off of it. The ability to read information off of a table is an essential skill in our world which, like anything else, requires practice.

As you work with multiplication, try to use as many of the different terms as possible. The first number is called the *multiplier* and the second the *multiplicand*. For example, in 5 × 7 = 35, the multiplier is 5 and the multiplicand is 7. The result of multiplying, in this example 35, is called the *product*.

In any case, this single concept of the area model is all a student needs in order to figure out all the multiplications from 1 × 1 to 9 × 9 and beyond. Once again, this is an excellent opportunity to teach students that mathematics is all about exploration and discovery, and not about believing and remembering. Children should be able to illustrate and explain basic multiplication facts that they have found on their own. There is a great pride in certainty and discovery. Fortunately, there are enough basic math facts for every child in your class to discover and explain several for the rest.

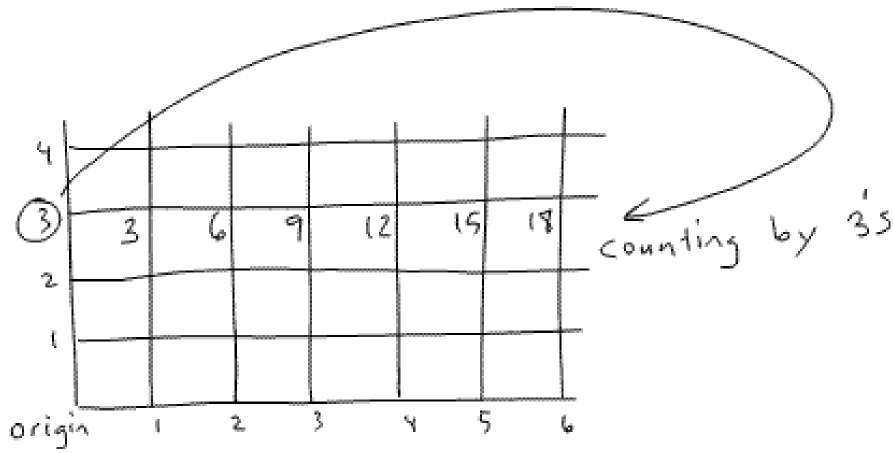
The dream in all of this is that the students will begin to notice patterns as the multiplication table takes shape.

They may realize that the numbers along the edges are the same as the numbers on the number line. If so, challenge the students to phrase this generally. Ideally, they will say that 1 times any number is that number. Further challenge them to explain why. A rectangle that is one square high will have as many squares as its width. For example, 1 × 4 = 4, as shown:

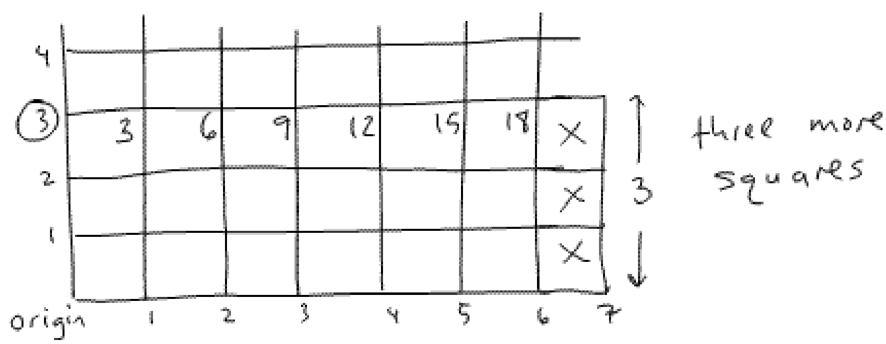


There are other ways to look at this. To add the number 4 "one time" will leave you with just the number 4, for example. This is an excellent topic for a class discussion. Ideally, of course, the students will notice this on their own. However, as with everything, you should prompt them toward this discovery if no one seems to notice, or thinks to mention it. Because of this property, the number 1 has a special name. It is called the *multiplicative identity*. A number's identity is its value. When you multiply 1 by a number, the value does not change, and thus it results in the number's identity.

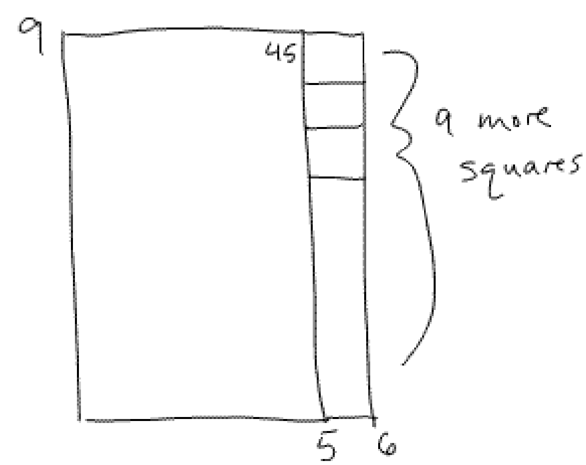
Students might notice patterns as they go up or across the figure. You add the same number each time, for example 2, 4, 6, 8, etc. or 5, 10, 15, 20, etc. The number that you add, in fact, is the number at the end of the row:



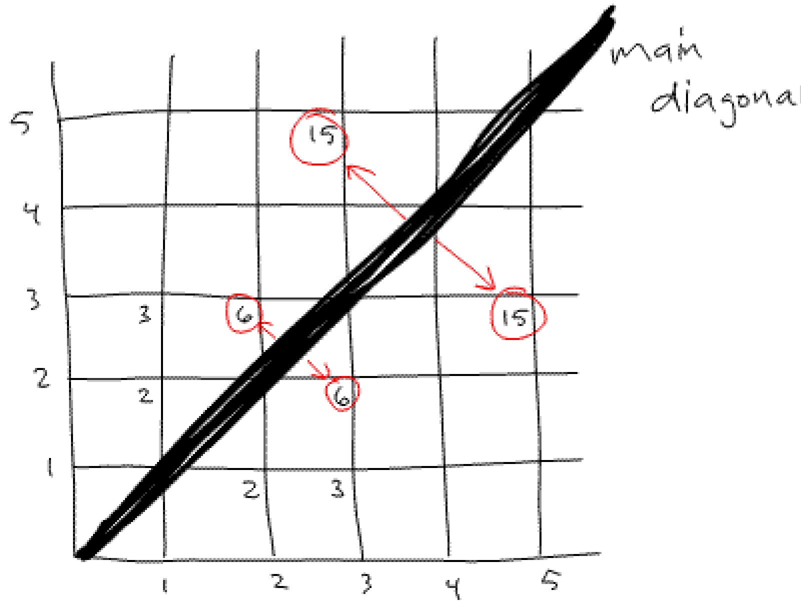
This makes sense, because each successive rectangle will have that many more squares:



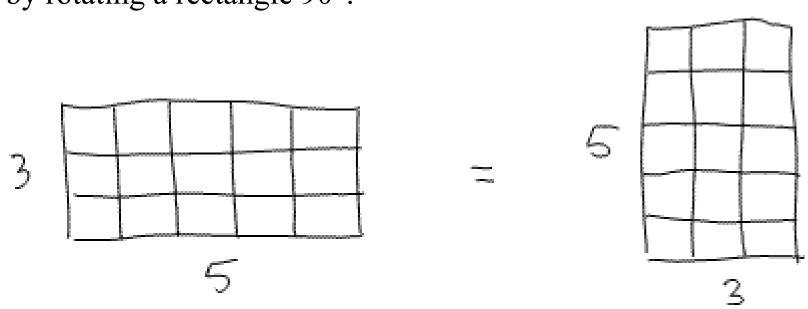
This property can be used to help remember multiplication facts. For example, suppose a student remembers that $9 \times 5 = 45$, but cannot remember 9×6. We are going from 5 nines to 6 nines, and thus will need to add 1 more nine. The answer will be $45 + 9 = 54$.



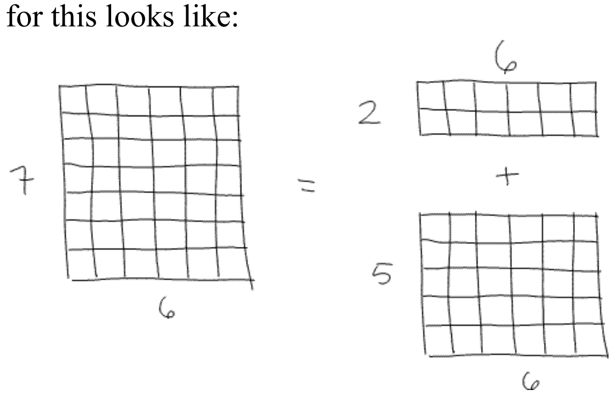
Another useful observation is that the completed multiplication table will be *symmetric*, that everything repeats on either side of the main diagonal:



When these corresponding math facts are written out, they illustrate the commutative property of multiplication: $3 \times 5 = 5 \times 3$, etc. Of course, this can be illustrated even more clearly by rotating a rectangle 90°:



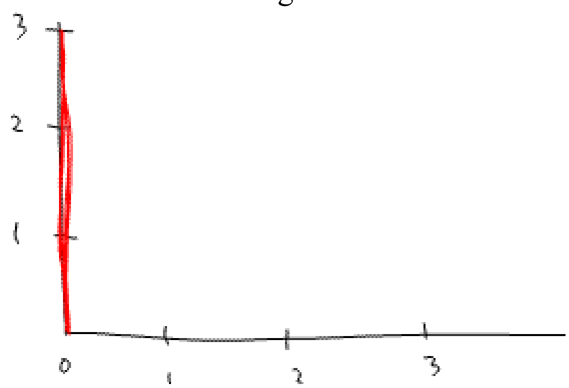
Another property to point out (unless your students figure it out on their own) is called the *distributive property*. This is best illustrated with an example. We can break up a multiplication like 7×6 into $(2 + 5) \times 6$ and then multiply separately: $2 \times 6 + 5 \times 6$. The picture for this looks like:



For someone who knows easily that $2 \times 6 = 12$ and $5 \times 6 = 30$, this makes calculating 7×6 easy: $7 \times 6 = 12 + 30 = 42$.

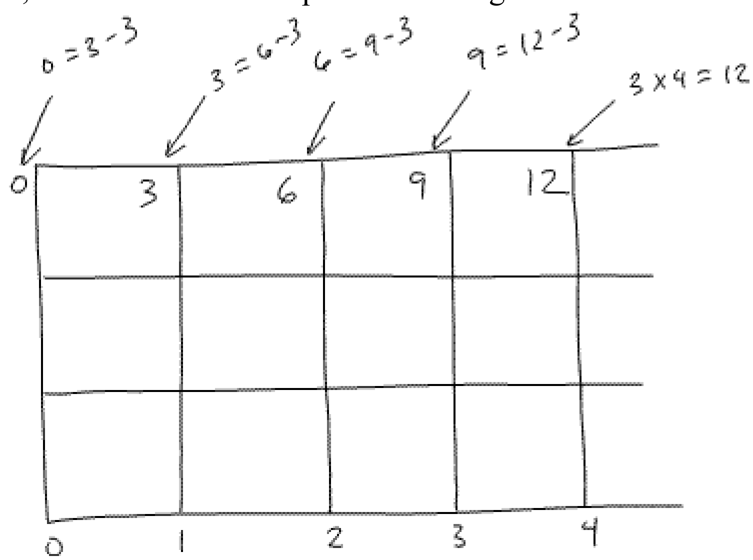
The distributive property of multiplication is something cool to share with the students and something fantastic for them to notice on their own (even with prompting), but not a concept they should worry about, much like the term "commutative property." These will come up later in algebra, so it is handy for your students to see them early, but these are not words or concepts to be drilled on homework or exams.

A slightly complicated concept that the students should come to learn and undertand, however, is multiplication by zero. If one of the two numbers multiplied is zero, the corresponding rectangle will actually be just a straight line. For example 3×0 is a straight line that runs from the origin to the number 3:



It is a bit of a stretch to call this a rectangle, but it does run as far as the number 3 on the vertical axis and as far as the number zero on the horizontal axis. Certainly there are no squares inside this rectangle, and thus $3 \times 0 = 0$.

Another way to figure out multiplication by zero is to follow the patterns. In the third row up, for example, each step to the right involves adding 3 to the number. Each step to the left, thus, ought to subtract 3. If we start at $3 \times 4 = 12$ and work our way to the left, we see 12, 9, 6, and 3. One more step to the left ought to thus subtract 3 more, resulting in zero:



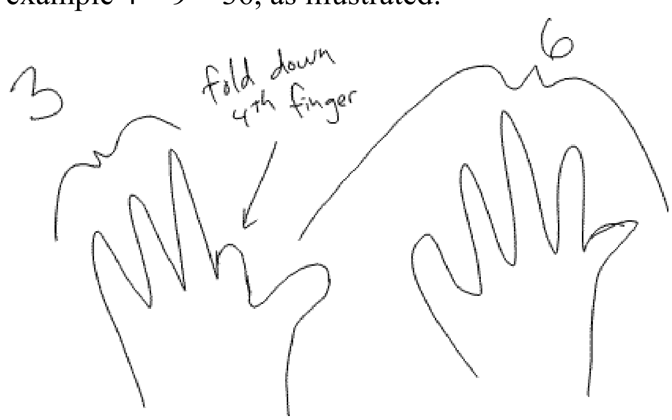
This technique isn't quite as pretty as the straight-line-rectangle illustration, but it can be extended even further in the future to explain the multiplication of negative numbers.

As with the basic addition and subtraction facts, the multiplication facts from 0×0 up to 9×9 (the product of any two digits) are very important for students to be able to recall rapidly. Have them make up their own flash cards, based on the times tables they construct, and have them practice with them, either straight or in conjuction with flash-card games. Another option is to have them play multiplication war – each player draws two cards and the one with the largest product wins the round.

As with the addition facts, it is a good idea to have the class brainstorm mnemonics to help remember the facts. Multiplying by 1 is easy – it does not change the number. Multiplying by 0 is easy – the answer is always zero. Multiplying by 2 is the same as adding the number to itself.

When you multiply an even number by 5, the result is ten times half the number. For example, $8 \times 5 = 4 \times 10 = 40$. This is because $8 \times 5 = 4 \times 2 \times 5 = 4 \times 10$.

There are a number of tricks for multiplying by 9. All of them ultimately involve looking at the 9 as $10 - 1$. For example $7 \times 9 = 7 \times (10 - 1) = 70 - 7 = 63$. When you multiply the numbers from 1 to 9 by 9, the two digits of the answer will always sum to 9. For example, $7 \times 9 = 63$ and $6 + 3 = 9$. Similarly, $3 \times 9 = 27$ and $2 + 7 = 9$. Because of this, you can multiply by 9 by holding out your hands and lowering the finger corresponding by other multiplicand. For example $4 \times 9 = 36$, as illustrated:



Ultimately, of course, students should be able to instantly recall any math fact without having to consult their fingers.

Have students share their techniques for remembering the math facts. Have the class collectively vote on the top 5 most difficult multiplications to recall (8×7? 6×9?). Sometimes, the mere act of isolating problems in this way can make them special, and thus easier to remember. Extra praise should be awarded to students who "defeat the monsters" by remembering the hardest ones to recall.

Questions:

(1) Draw the rectangle which illustrates the product 4×5. Explain how this rectangle can be broken up to illustrate both the "grouping" and "repeated addition" models of multiplication.

(2) Make a 10×10 grid and use bold rectangles in different colors to illustrate 2×7, 3×9, 4×2, and 9×5.

(3) Draw rectangles to show how the products 4×5 and 4×6 are related.

(4) Use the area model of multiplication to illustrate the distribution $8 \times 6 = (5 + 3) \times 6 = 5 \times 6 + 3 \times 6$

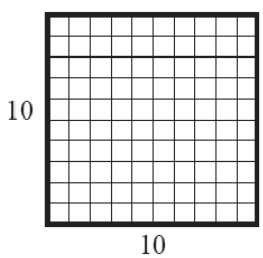
Chapter 14: Multiplying by Powers of Ten

Students who are able to multiply any two single digits together will likely find it easy to multiply by the various powers of ten (10, 100, 1000, 10000, etc.). This is because of the way our number system is set up.

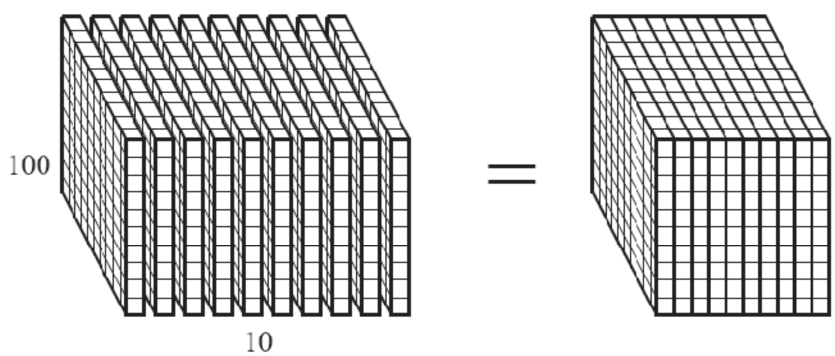
Our number system is based on the number 10. Each number column represents the number 10 multiplied by itself a certain number of times. This can be illustrated:

ten-thousands	thousands	hundreds	tens	ones
10000	1000	100	10	1
10 x 10 x 10 x 10	10x 10 x 10	10 x 10	10	1

The 100-flat block from a set of base-ten blocks, for example, looks exactly like the rectangle which represents the multiplication 10 × 10:



The 1000-big cube block is not shaped as a rectangle. However, it can be formed from 10 hundred-flats stacked up together, and thus represents 100 + 100 + 100 + 100 + 100 + 100 + 100 + 100 + 100 + 100 = 100 × 10 = 10 × 10 × 10:



Following this pattern, students ought to accept that 10,000 = 10 × 10 × 10 × 10, that 100,000 = 10 × 10 × 10 × 10 × 10, and that 1,000,000 = 10 × 10 × 10 × 10 × 10 × 10. Ideally, some student will notice that the number of zeros in the number is the same as the number of tens multiplied together, but otherwise you can prompt them or point this out.

A digit in one of the base-ten columns represents the product of that digit and the column quantity. For example, a 6 in the hundreds column represents 6×100, which in turn represents $6 \times 10 \times 10$:

$600 =$

hundreds	tens	ones
6	0	0

$= 6$ hundreds
$= 6 \times 100$
$= 6 \times 10 \times 10$

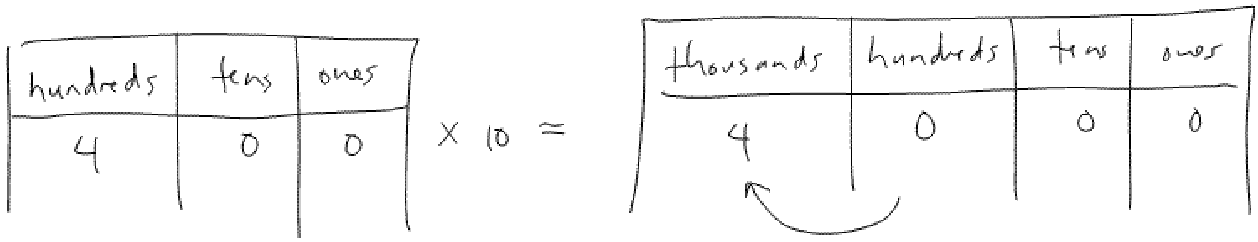
Similarly, the number $50 = 5 \times 10$, the number $4000 = 4 \times 1000 = 4 \times 10 \times 10 \times 10$, and so on. It is useful to run through a number of exercises like this with the class. When you present them with a number like 700, they should be able to write it as $7 \times 10 \times 10$. You should also be able to show them something like $2 \times 10 \times 10 \times 10$ and have them recognize this as 2000.

When the class is comfortable with this, walk them through the process of multiplying by 10. For example, to multiply 600×10, we first expand out the 600, so that the multiplication becomes $600 \times 10 = 6 \times 10 \times 10 \times 10$. This, the students should recognize, is 6000. Similarly, $4{,}000 \times 10 = 4 \times 10 \times 10 \times 10 \times 10 = 40{,}000$.

There are two notes to introduce here. First, if your students are not ready for such large numbers, then keep things simple. Even though there is nothing conceptually more difficult about numbers in the ten-thousands than numbers in the hundreds, it is best not to overwhelm your students. Of course, if anyone asks, feel free to quickly illustrate how a million can be reduced to a product of tens, but make it clear that this is advanced material for which they are not responsible.

Second, when we break up the 600 in 600×10, we should technically use parentheses to separate the 600 from the 10. It should look like $600 \times 10 = (6 \times 10 \times 10) \times 10$. However, there is a property of multiplication called the *associative property of multiplication* which states that we can rearrange these parentheses. Thus $(6 \times 10 \times 10) \times 10 = 6 \times (10 \times 10 \times 10) = 6 \times 1000 = 6{,}000$. In advanced mathematics, this is a very important property. A math teacher ought to be aware of it, but elementary school students do not need to learn this term. They should, however, be taught that a long sequence of multiplications can be performed in any order. For example, when multiplying $6 \times 2 \times 5$, it might be easier to multiply the 2×5 first, to get $6 \times 2 \times 5 = 6 \times 10$ rather than to multiply the 6×2 first, to get 12×5. No matter how it is done, however, the answer will be the same.

Hopefully, as your students work out problems like 400 × 10 = 4 × 10 × 10 × 10 = 4000, 50 × 10 = 5 × 10 × 10 = 500, and 3 × 10 = 30, they will notice a clear pattern. Multiplying one of these numbers by 10 only moves it's non-zero digit over to the next column. For example 400 × 10 = 4,000 looks on the base-ten columns like:



Another way to look at this is that multiplying by ten "adds a zero" to the end of the number. For example, when you multiply 70 by 10, you end up with 700, which looks just like 70, but with an extra zero on the end. Ideally, your students will notice this pattern on their own. Try to encourage this by having them discuss thoughts and ideas. It helps if you list out all the problems you have worked out thus far on the board, for example:

50 × 10 = 500
400 × 10 = 4000
7 × 10 = 70
2000 × 10 = 2000
600 × 10 = 6000

Perhaps a student will notice part of the pattern, for example that multiplying a number in the hundreds by 10 results in a number in the thousands. Encourage these ideas, write them out on the board, have the students share and discuss them, and then encourage them to find more. As with everything, though, if no one seems to notice the big picture, you can begin to prompt them in the right direction, and ultimately point it out if necessary.

The next step is to multiply more complicated numbers by 10. The trick is to break up the numbers via the base-ten system and then use the distributive property. For example, to multiply 705 × 10, we first break up 705 = 700 + 5. Because of the distributive property, we can multiply each of these by 10 separately. Thus, the complete problem is worked out as 705 × 10 = (700 + 5) × 10 = (700 × 10) + (5 × 10) = 7000 + 50 = 7050.

As another example, 824 × 10 = (800 + 20 + 4) × 10 = (800 × 10) + (20 × 10) + (4 × 10) = 8000 + 200 + 40 = 8240.

By this point, your students should notice that the "stick a zero at the end" rule still seems to apply. What is really happening is that each digit in the number is being moved over to the next column on the left. For example, in 824 × 10 = 8240, the 8 moves from the 100's place to the 1000's place, the 2 moves from the 10's place to the 100's place, and the 4 moves from the 1's place to the 10's place. While the "stick a zero at the end" rule is much easier to remember, it is important to reinforce the base-ten number system as much as possible. It is far better for your students' education for them to be taught the long way first, and then the short-cut second. Also, when the students work with decimals later on, the "add zeros" technique will not longer apply, but the "shift all the digits over a place" will still work.

Because $100 = 10 \times 10$ (and multiplication is associative), multiplying by 100 is the same as multiplying by 10 twice. For example, $25 \times 100 = 25 \times 10 \times 10 = 250 \times 10 = 2500$. Similarly, $826 \times 100 = 826 \times 10 \times 10 = 8260 \times 10 = 82{,}600$. After running through several examples like these, see if your students notice the pattern, that multiplying by 100 results in "adding two zeros to the end of the number" or, equivalently, "moving all digits two columns to the left."

This is a good time to help explain a common form of casual speech. The price of a used car, for example, might be described as "twelve hundred dollars." This means 12 hundreds, which is the same as $12 \times 100 = \$1200$. The proper way to say this number is "one thousand, two-hundred dollars." However, these expressions are frequently encountered and mathematically useful, so it is a good idea to discuss them with your students. With base-ten number columns, this can be illustrated as:

thousands	hundreds	tens	ones
	12	0	0

=

thousands	hundreds	tens	ones
1	2	0	0

Notice that a situation like the above would occur if we added two numbers and got a 12 in the hundreds column: we would "carry the one" over to the thousands column.

This concept can be used to explain how to compute multiplications like 600×7. Because we know that $6 \times 7 = 42$, we conclude that 6 hundreds $\times$ 7 = 42 hundreds = 4200. Similarly, $3 \times 8000 = 3 \times 8$ thousands = 24 thousands = 24,000.

If you students question this, there are a few ways to justify it. For one, if you have four people with 3 apples each, then all together there are 4×3 apples = 12 apples. This is the same process as 4×3 hundreds = 12 hundreds. For another approach, you could say that 4×3 hundreds = 3 hundreds + 3 hundreds + 3 hundreds + 3 hundreds = 12 hundreds. In any event, the ability to make calculations like these will be essential in order to explain the process of long multiplication, as will be discussed in the next chapter.

Your students will also need to know how to multiply numbers like 40×60. To do this, we break apart the numbers and then rearrange them so that all of the tens are at the end. We are able to do this because of the commutative and associative properties of multiplication. In this example, we write $40 \times 60 = 4 \times 10 \times 6 \times 10 = 4 \times 6 \times 10 \times 10 = 2400$. Similarly, $800 \times 60 = 8 \times 10 \times 10 \times 6 \times 10 = 8 \times 6 \times 10 \times 10 \times 10 = 48{,}000$. Hopefully your students will notice that these sorts of multiplications only involve multiplying the two main digits together and then adding as many zeros as the two numbers have in total.

If you students have grown comfortable with multiplying numbers by 10 and by 100, you can try out multiplying by 1000 or even 10,000. Again, you do not want to overwhelm them with huge numbers before they are ready, but it is very beneficial to see how all of these follow the same pattern. For example, $43 \times 1000 = 43 \times 10 \times 10 \times 10 = 430 \times 10 \times 10 = 4300 \times 10 = 43{,}000$. It might be easier to look at this as the number 43 in the thousands column. There are many, many ways that this can be figured out – encourage your students to discuss the ways that they like best. Hopefully, they will decide that "multiplying by 1000 results in adding three zeros to the end of the number" or "shifting all digits three columns to the left."

One place where students often have trouble is when the two leading digits multiply to a number that ends in 0. For example, many students will understand that $500 \times 700 = 350{,}000$. We multiply $5 \times 7 = 35$ and then add as many zeros to the end (4) as there were in the beginning (two at the end of 500 and two at the end of 700). However, many of the same students will try to say that $500 \times 800 = 40000$, to make sure that both sides of the equation have 4 zeros. Unfortunately, we must start with $5 \times 8 = 40$ and *then* add the four more zeros, resulting in $500 \times 800 = 400{,}000$. If $500 \times 700 = 350{,}000$ but $500 \times 800 = 40{,}000$, then things would not make sense.

If you work out problems up to multiplying by 10,000, then hopefully your students will catch on to the general pattern and short-cuts. If not, have your students work out each problem the long way. It is far more important for your students to actually understand what is going on in a problem than for them to remember the tricks that serve as short-cuts.

Questions:

(1) Write out the numbers 400 and 50,000 each as a single digit multiplied by a number of tens.

(2) Use the associative and commutative properties of multiplication to write out (with parentheses) all of the possible ways that $4 \times 7 \times 5$ could be calculated.

(3) Write out the multiplication 4000×10 by first breaking down the 4000 as a single digit multiplied by a number of tens.

(4) Show how the distributive property is necessary to explain 278×10.

(5) Explain 43×1000 by breaking 1000 up into a product of tens and then multiplying by each separately.

(6) Explain what is meant by "fifteen hundred" both with base-ten number columns and with multiplication.

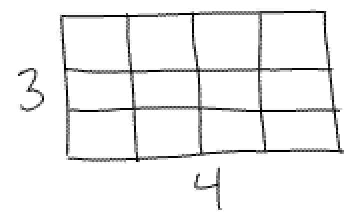
(7) Work out 3×900 both with units and with repeated addition.

(8) Show how 700×400 can be illustrated by breaking the numbers up into digits multiplied by a number of tens, and then using the commutative and associative properties of multiplication.

Chapter 15: Basic Areas

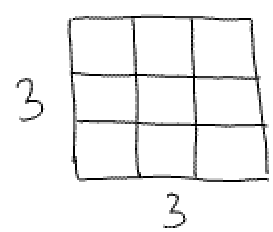
One of the wonderful benefits of using the area model to introduce multiplication is that it introduces the geometric concept of area at the same time.

When first working with multiplication, we have students count the number of squares in a variety of rectangles, for example the 12 squares in a 3 by 4 rectangle. As early as possible, we should teach the students the vocabulary word *area*. The area of a 3 by 4 rectangle is 12, for example, because it consists of 12 squares.



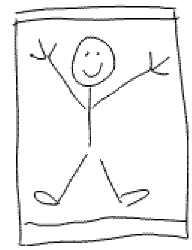
When students have developed some comfort with multiplication, ask them questions in a variety of ways, including area. For example, you could ask "what is 4 times 8?" or "what is the product of 5 and 3?" or "how many is 7 groups of 6?" or "what would you get if you added five 8's all together?" or "what is the area of a 4 by 6 rectangle?" The answers to all of these come from multiplication, but it develops a child's vocabulary and understanding of the concepts to think about these.

As a special case of rectangle, have your students consider the area of a square. The area of a 3 by 3 square, for example, is 9. A square with a length of 5, similarly, will have an area of 25. Have your students draw squares of different sizes and calculate their areas. These are called *square numbers* and are very important in mathematics: 1, 4, 9, 16, 25, 36, etc. You should tell your students that to *square* a number is to multiply that number by itself, to find the area of a square with that length for each side. It is not critical for the class to memorize this term, but exposure to the idea early will help them later on. There is no need to mention at this time that mathematicians represent "five squared" by 5^2. It is enough for them to be able to find the area of a square when given the length of one side.

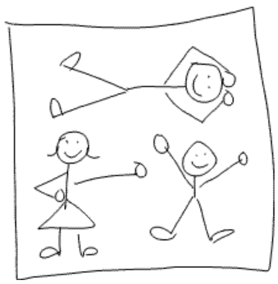


As was discussed earlier, children tend to have difficulties differentiating between "perimeter" and "area." Perhaps it is because both concepts tend to have a rectangle as their mental image, and the word "perimeter" is not one used frequently in everyday speech, and thus does not come with solid associations. In any case, it is thus the teacher's job to make these associations by using the word frequently. We have already described the "walk the perimeter" game in some detail. Now that area has been introduced, this game should be mixed up with one called "cover the area."

The "cover the area" game is best played on a rug, but a towel, tarp, piece of cloth, or similar object will work as well. For a child to "cover the area" of an object, he or she must lay down and attempt to cover it entirely without spilling over the sides. For example, a child covering the area of a towel might look like the picture to the right. Such a towel will be described as having the area of one child.



For a larger object, like a rug, the child assigned to "cover the area" will need to enlist the help of fellow students. The following rug has an area of three children.



If you do not have enough children to cover a given area, you can have a kid roll over a number of times, keeping count of how many times he has laid out flat, in order to get an estimate of the object's area.

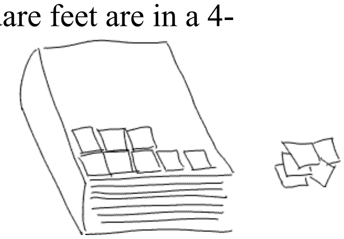
After your students are good at playing both "walk the perimeter" and "cover the area," abbreviate these commands to just "show me the perimeter" (walk around the object counting steps) and "show me the area" (count all the non-overlapping places you can lay on it). This ought to teach them the difference between area and perimeter. This also ought to be a great way to work off excess energy after a long bout of sitting and listening.

Hopefully, your students will notice that the "child" unit for measuring area is not very precise. A rug, for example, might have an area of 6 Jims but only 5 Alices. If they do not notice this, have a small child measure an area and then a large child measure the same area, then point out the difference. To remedy this, explain that mathematicians use squares to measure areas precisely. Now, repeat the "cover the area" game, but have the children lay out 1-foot squares instead of lying down on the ground. This would be ideally done with durable materials: squares of carpet, linoleum, or heavy cardboard.

Have the children cover rectangles in different ways. First of all, have them cover a towel, rug, or piece of cloth with the squares. Second, have them construct a rectangle with a certain number of squares, for example 12 or 20. Third, have them construct a rectangle with specific dimensions, for example a 3-foot by 5-foot rectangle. Make sure you have enough squares for these tasks!

While doing this, take care to use the term "square foot" as much as possible. Ask the students "how many square feet is the area of this rug?" and "how many square feet are in a 4-foot by 5-foot rectangle?"

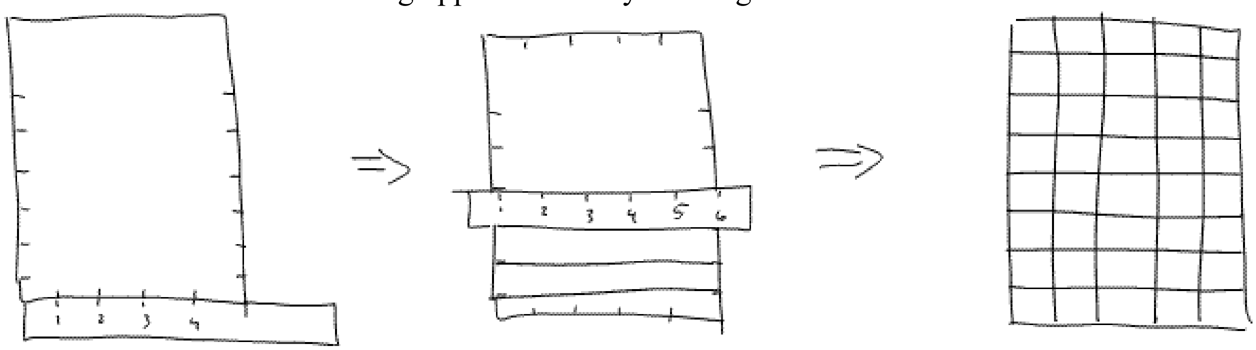
You can also have students work at their desks with square inches, cut out of oak tag or construction paper. Have them count the area of the



cover of a book or a rectangle drawn on a piece of paper by lining up squares of paper on them.

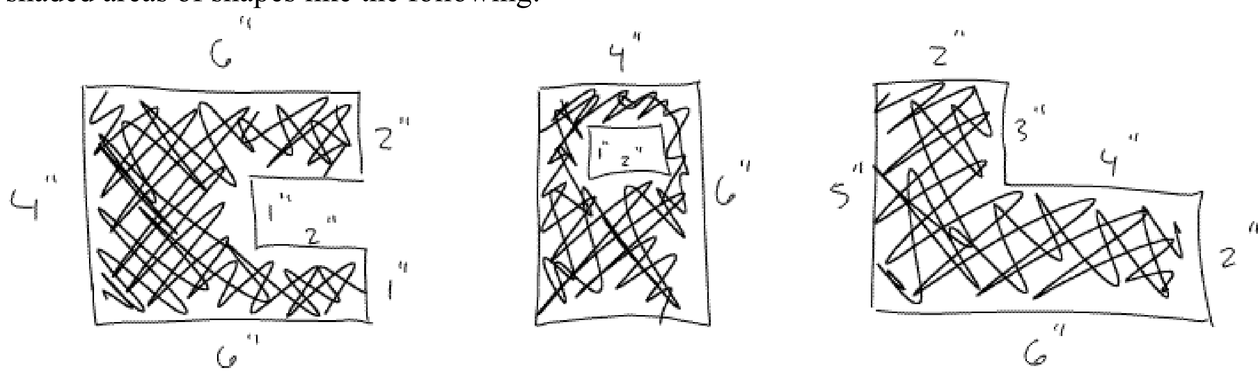
When you students have counted the squares, have them measure the two lengths of the rectangle with a ruler and compare the number of squares with the product of the lengths. While they should technically already understand that these will always come out equal, this concrete demonstration ought to reinforce the concept and increase the certainty and confidence they have in mathematics. It helps to make sure in advance that the rectangles and squares have lengths which are whole-number multiples of inches. Thus, it is a good idea to have them work with shapes carefully measured and drawn, then photocopied and distributed.

You can also have yourstudents make their own 1-inch squares by cutting out ones photocopied on paper (saving you a lot of time and trouble). As an added challenge, you can have them draw their own grid on paper by making the sides of piece of paper at 1-inch intervals with a ruler and then connecting opposite lines by drawing with a ruler.

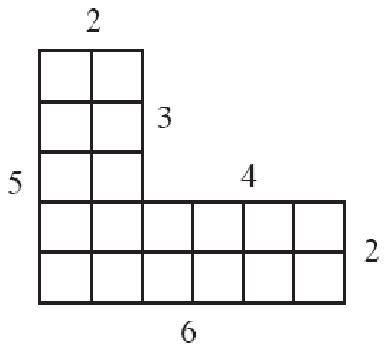


This can be difficult for small children, but it is excellent for practicing measuring, following directions, and precision work. Make sure to point out all the lines which are parallel.

When your students are comfortable with finding the areas of squares and rectangles, challenge them to find the areas of odd shapes. For example, you can have them try to find the shaded areas of shapes like the following:

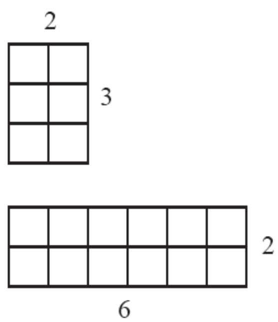


At first, have them find the areas by covering them with their 1-inch squares. For example, the following shape has an area of 18 square inches:

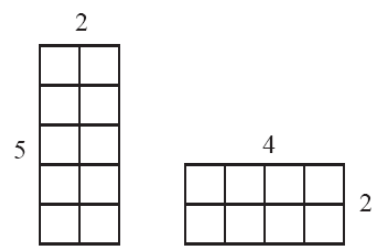


Encourage your class to discuss how this area might be calculated without resorting to little squares. If no one comes up with the idea of cutting the shape apart into rectangles, give them scissors and have them cut out the shape, as a hint.

Ideally, your students will see that a single cut can break this shape into two rectangles:

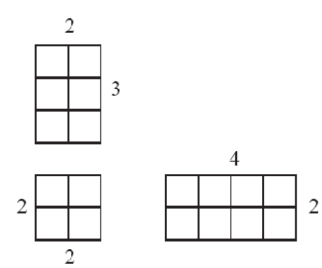


OR

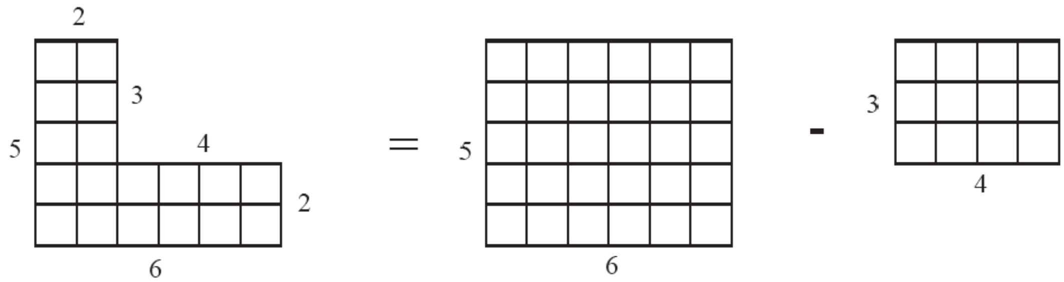


The figure on the left consists of rectangles with area 2 × 3 = 6 and 2 × 6 = 12, for a total of 18. The figure on the right consists of rectangles with area 2 × 5 = 10 and 2 × 4 = 8, also for a total of 18.

Of course, there is no reason why it cannot be broken up into even more rectangles. In the figure to the right, the shape has been broken up into three rectangles, with areas 2 × 3 = 6, 2 × 2 = 4, and 2 × 4 = 8, for the same grand total of 18.



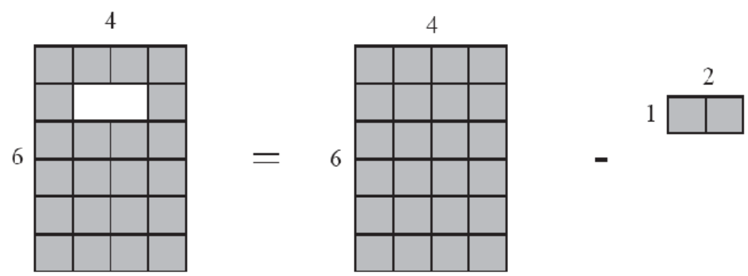
Perhaps a clever student will also notice that the whole figure can be formed, not by adding rectangles, but by subtracting them:

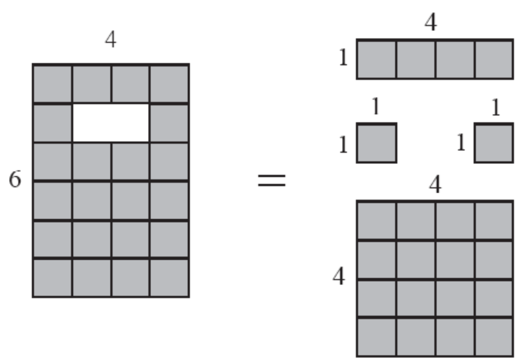


Here, the total area is calculated to be 5 × 6 – 3 × 4 = 30 – 12 = 18.

Encourage your students to share their techniques and ideas. This is a wonderful place for your students to discover that there are many ways in mathematics to find the answer to a problem. No method is any better or worse than another, although the faster techniques should be recognized as such.

The subtraction method, for example, is much faster in the following situation than the cut-into-rectangles one. With subtraction, we calculate the area as 4 × 6 – 1 × 2 = 24 – 2 = 22.

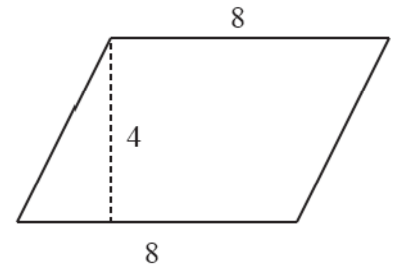




With the the cut-into-rectangles method, we have to add together four different rectangles: 1 × 4 + 1 × 1 + 1 × 1 + 4 × 4 = 4 + 1 + 1 + 16 = 22.

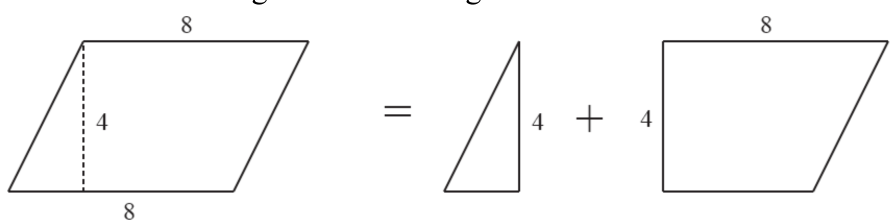
However, both of these methods (as well as many others) all work to calculate that the area of the shape is 22 square inches. Encourage any sort of creative thinking by singling out every new idea for special praise.

Notice that we have begun to use the *order of operations* in these examples. This is to say that when an equation involves addition, subtraction, and multiplication, we do all of the multiplication first, and then do the addition and subtraction, from left to right. Some students will wonder why this must be, and will wonder why we do not do everything from left to right, as we do when reading. As an example, present the children with the following problem: "Suppose pencils cost 5 cents each and pens cost 7 cents each. How do you write the cost of 3 pencils and 4 pens?" The most reasonable answer is to multiply the costs by the quantities and then add them together: 5 × 3 + 7 × 4. There is no way to write out the correct answer to this problem with addition and multiplication *unless* we do the multiplications first. If we do everything from left-to-right, there is no way to get the right answer.

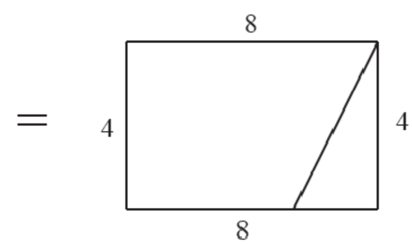


When your students are comfortable with cutting apart shapes to calculate their areas, introduce them to the area of a parallelogram with the following exercise. First, give them a non-rectangular paralellogram (on photocopied paper) and have them try to measure its area with square inches. For example, give them a parallelogram with a base that is 8 inches wide and with a vertical height of 4.

Your students will find it very difficult (if not impossible) to tile this figure with squares. Next, offer them scissors and challenge them to cut it up into rectangles. This will again be quite difficult. If no one is able to figure it out, show them how only one cut is necessary – along the dotted line indicating the vertical height:



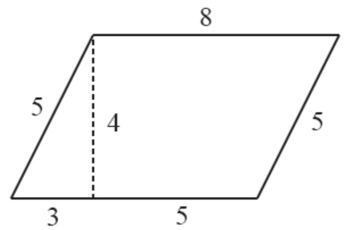
Once this cut is made, the two pieces can be rearranged into a rectangle. The rectangle in this example has an area of $4 \times 8 = 32$, and thus the original parallelogram has an area of 32 as well.

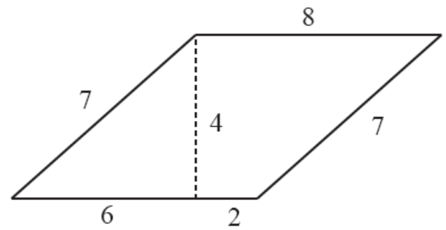


Try this a few times more with different-shaped parallelograms (always with whole-number widths and heights, of course) until everyone in the class knows how to cut them apart and rearrange them into rectangles. This is an excellent way for the students to practice "breaking down a problem." Parallelograms will also be the key to calculating the area of triangles, which will be discussed later.

When you discuss the results of several parallelogram computations (you might break the class into groups and have each group find the area of a different parallelogram), hopefully someone will notice an easy way of finding the area. All you need to do is multiply the length of the base with the vertical height.

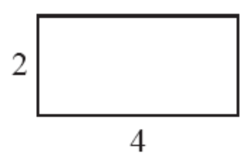
Do be careful that the students realize that the vertical height is necessary and not the slant height (which is part of the perimeter). To emphasize this, make sure that two of your examples have the same height and base, but different slant heights. For example, the following two parallelograms can both be cut along the dotted lines to form rectangles with base 8 and height 4. Thus, these both have an area of 32 square inches.

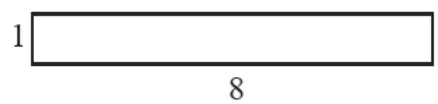




If we multiply the base times the slant height, we get 40 for the first parallelogram and 56 for the second – numbers which have no meaning in these figures. (Note: the slant height of the second parallelogram is actually slightly more than 7.)

In order to calculate the perimeters of these figures, however, we need to add up the lengths of the four sides, which comes out to be 26 inches for the first parallelogram and 30 for the second. The distance around the two figures is different, even though their areas are the same. Discuss this curiosity with the class, and see if they can make sense with it. As a comparison, challenge them to find two rectangles which have the same area but different perimeters. There are many possible answers. For example, the following rectangles both have an area of 8, but the first has a perimeter of 12 and the second has a perimeter of 18.

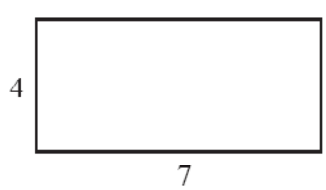


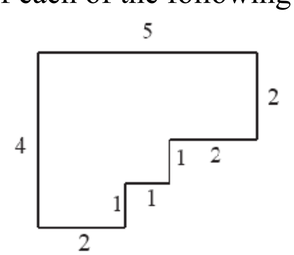


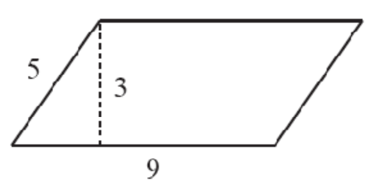
Similarly, you can challenge you students to find two rectangles with the same perimeters but different areas. These exercises will help your class to differentiate between perimeter and area.

Questions:

(1) Find the perimeter and area of each of the following figures:



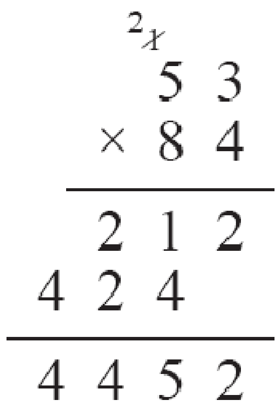


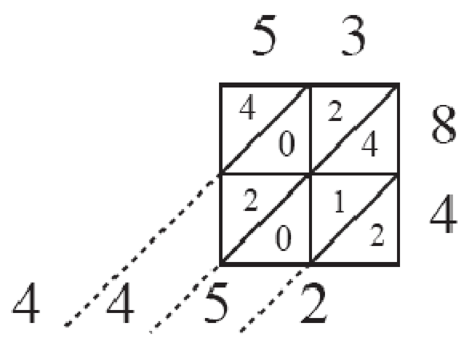


(2) Illustrate how the last two figures above can be cut apart into rectangles.

Chapter 16: The Multiplication of Large Numbers

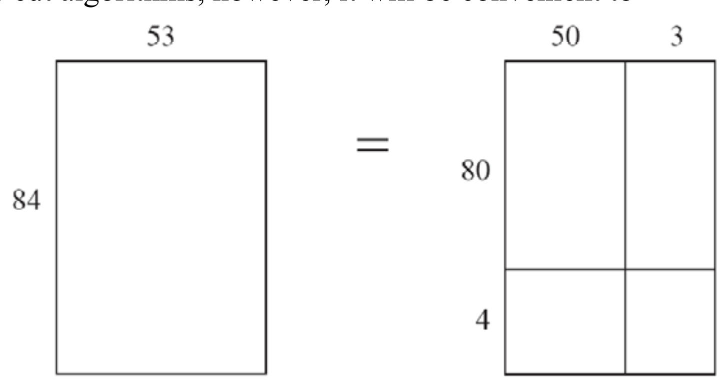
While the repeated addition model for multiplication easily explains that 7 × 3 = 7 + 7 + 7 = 21, it is a very poor model for multiplying large numbers together. Few teachers would ever expect students to compute 53 × 84 by adding 53 to itself 84 times. The students, of course, would most likely make some careless mistakes in such a long and tedious computation. Instead, most teachers teach what we will call the short-cut techniques for long multiplication. For example, the two most common short-cut techniques for calculating that 53 × 84 = 4,452 are illustrated below:



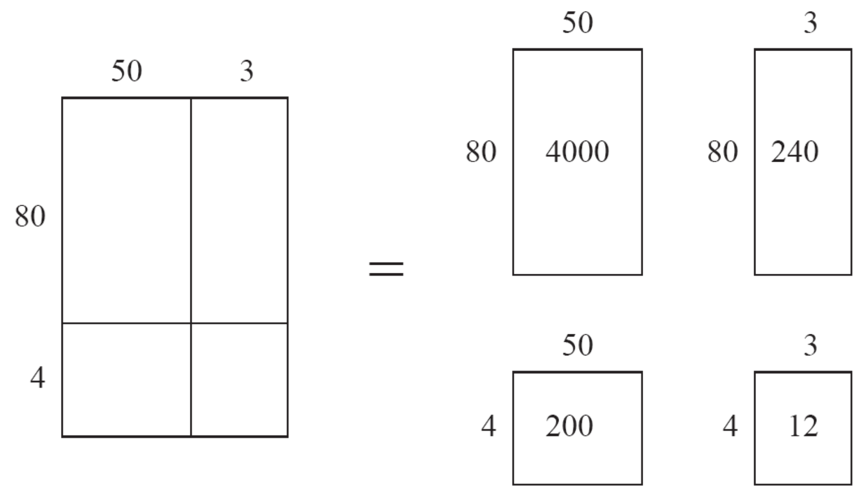


It is possible to teach these computational techniques by breaking the whole process down to a series of steps which only require the knowledge of how to multiply one-digit numbers together. However, when a student is taught to follow a computational procedure, the sense and meaning behind the steps is often neglected. Such teaching reinforces the misconception that mathematics is a collection of facts and procedures which must be blindly believed, remembered, and obeyed.

The area model for multiplication is capable of quickly providing an illustration and explanation for both of the short-cut techniques. For example, the area model for multiplication explains that 53 × 84 is the number of squares in a rectangle with height 53 and width 84. For the purposes of comparing this to the short-cut algorithms, however, it will be convenient to rotate the rectangle so that it has a height of 84 and a width of 53. If we break up these two dimensions into their base-ten components, 53 = 50 + 3 and 84 = 80 + 4, we can see that this rectangle can be naturally divided into four pieces. For the purposes of illustration, we will not worry about drawing the lengths of rectangles to scale.



Each of the four pieces is a rectangle, thus their areas are obtained by multiplying. In the last chapter, we reviewed how to compute 50 × 80 = 5 × 10 × 8 × 10 = 5 × 8 × 10 × 10 = 40 × 10 × 10 = 4000. Similarly, the areas of the other three rectangles are 80 × 3 = 240, 4 × 50 = 200, and 4 × 3 = 12.



The full area of the rectangle is thus 53 × 84 = 4000 + 240 + 200 + 12 = 4,452. It will be convenient to write this sum in the following way:

$$
\begin{array}{rrl}
 & 53 & \\
\times & 84 & \\
\hline
 & 12 & = 4 \times 3 \\
 & 200 & = 4 \times 50 \\
 & 240 & = 80 \times 3 \\
+ & 4000 & = 80 \times 50 \\
\hline
 & 4452 &
\end{array}
$$

This form is convenient, because it leads to a clear explanation of the usual method of long multiplication. The only difference, in fact, is that some of the addition is done along the way. If we add some of the columns together, we end up with almost the exact same figure as at the beginning of this chapter:

$$\begin{array}{r} 53 \\ \times\ 84 \\ \hline 12 \\ 200 \\ 240 \\ +\ 4000 \\ \hline 4452 \end{array} \qquad \begin{array}{r} 53 \\ \times\ 84 \\ \hline 212 \\ 4240 \\ + \quad \\ \hline 4452 \end{array}$$

In the short-cut process, we first say "multiply the 4 by the 3 and get 12, put the 2 below the 4 and then carry the 1":

$$\begin{array}{r} {}^{1}53 \\ \times\ 84 \\ \hline 2 \end{array}$$

This calculation is indeed 4 × 3, but rather than write down the 12 all at once, we are going to try to add it to the next number before writing it down. Next, we say "multiply the 4 by the 5 and add the 1 (which we held from the earlier 12) to get 21, which we write beside the 2":

$$\begin{array}{r} {}^{1}53 \\ \times\ 84 \\ \hline 212 \end{array}$$

This is a bit misleading, because we are not actually multiplying 4 and 5. The 5 in this case represents 50, so we are really multiplying 4 × 50 = 200. The "one" that we carried was not really a 1, but the 10 from the 12. When we multiply 4 and 5 tens, we get 20 tens, and when we add 1 more ten, we get 21 tens. This is why we shift the 21 over when we write it, so that it is actually in the tens place where it belongs. This is all fairly complicated. Would it be such a waste of paper to write out the 12 and the 200 separately?

Next, we say "multiply the 8 by the 3 to get 24, put the 4 underneath the 8, and then carry the 2." It helps to cross out the 1 that we carried earlier so that it doesn't cause any confusion:

$$
\begin{array}{rrrr}
 & 2 & & \\
 & \cancel{1} & & \\
 & & 5 & 3 \\
\times & & 8 & 4 \\
\hline
 & 2 & 1 & 2 \\
 & & 4 & \\
\end{array}
$$

Once again, this is rather misleading. The "8" in 84 actually stands for 80, so we are really multiplying 80 × 3 = 240. We skip writing the 0 (to save pencils?) and just write the 4. By putting the 4 "underneath the 8," we put it in the tens place where it belongs, as it represents 40. The 2 that we carry is really a 200.

Next, we say "multiply the 8 by the 5 to get 40, add the 2, and then write down the 42 to the left of the 4." When we add the resulting numbers all up, we obtain the answer 4,452:

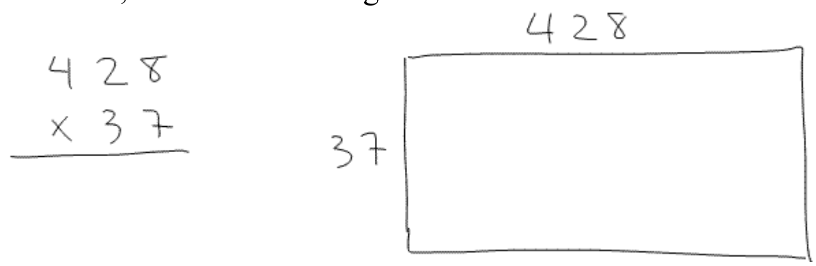
$$
\begin{array}{rrrrr}
 & & 2 & & \\
 & & \cancel{1} & & \\
 & & & 5 & 3 \\
 & \times & & 8 & 4 \\
\hline
 & & 2 & 1 & 2 \\
+ & 4 & 2 & 4 & \\
\hline
 & 4 & 4 & 5 & 2 \\
\end{array}
$$

As before, we are misleading the student into thinking that we are multiplying 8 × 5. We are really multiplying 80 × 50 = 4000, adding the 200 that was carried earlier, and then writing down 42 in the hundreds column (or rather, the 2 in the hundreds column and the 4 in the thousands column just to the left of it).

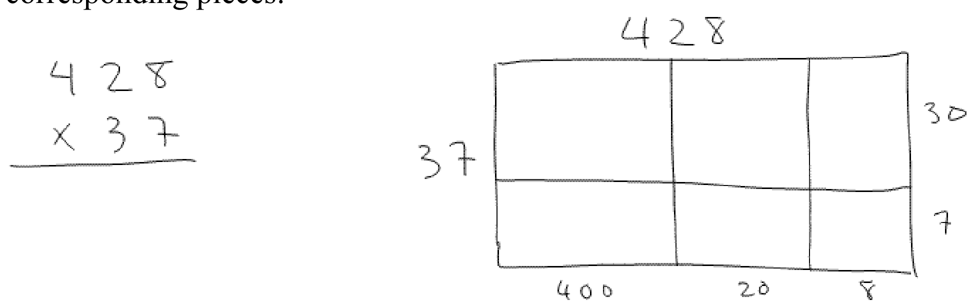
All of this can make a lot more sense when illustrated with the big 53 by 84 rectangle which has been broken into the four natural sub-rectangles.

This short-cut process is very clever and only works because of the manner in which the base-ten system underlies our number system. Because it is so abbreviated and so clever, it is difficult for children to understand what it means and why it ends up with the correct answer. This is why it is best to begin with teaching students the "multiply it all out" form of long multiplication.

In the "multiply it all out" form of multiplication, you begin by representing the multiplication in both the standard form and as a rectangle. For example, to begin explaining 428 × 37, draw the following on the board:

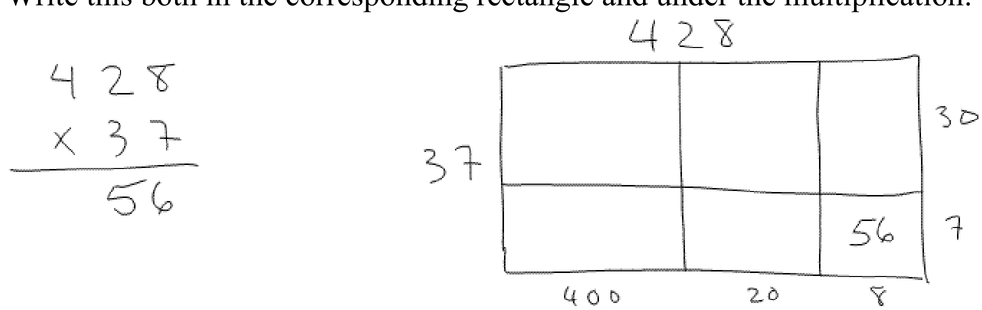


Next, have the class state the meaning of each digit in the two numbers. For example, the "4" represents 400, the "2" represents 20, etc. As they do this, break up the rectangle into the corresponding pieces:



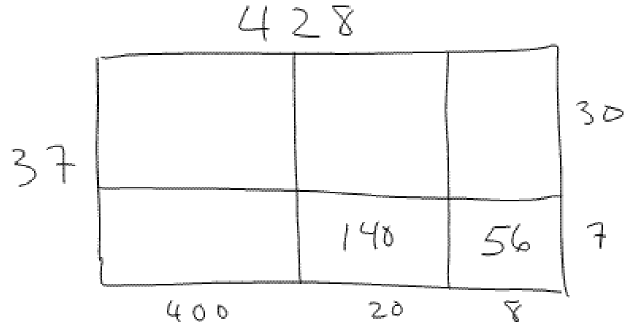
The class ought to recognize that the answer to the multiplication will be the area of the rectangle, which can be found by adding together the area of each of the 6 rectangular pieces.

Begin, as you would with the short-cut method, by multiplying 7 × 8 and obtaining 56. Write this both in the corresponding rectangle and under the multiplication:



Next, multiply 7 × 20 = 140 and write this answer in both places:

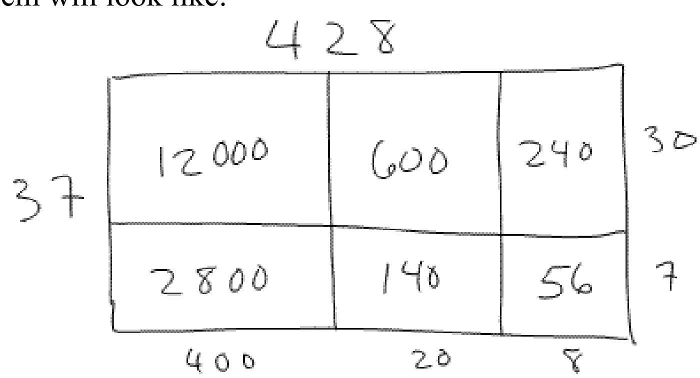
```
  428
 x 37
 ----
   56
  140
```



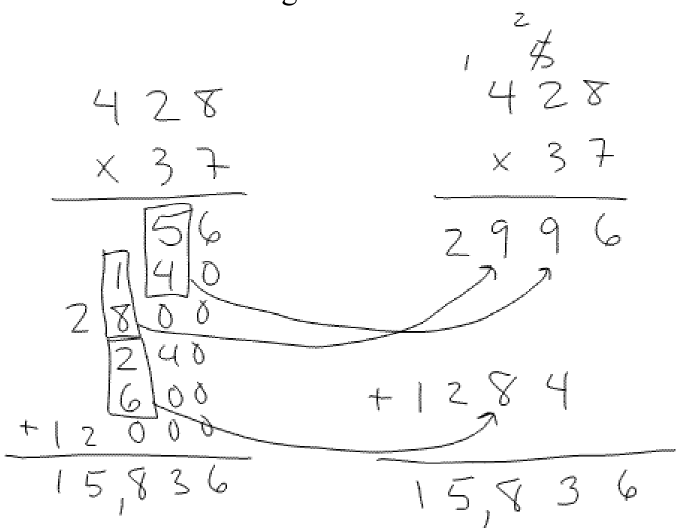
After you have gone through this whole process with the class a few times, you might begin to explain how the short-cut method will abbreviate this a bit. Rather than write the whole 56, for example, you will put down only the 6 and then save the 5 aside to add to the tens place of the next answer. When you get 140, you add the 5 tens to get 190, put down the 9, and then save the 1, representing the 100 that will be added later to the 7 × 400 = 2800. If they find this confusing, just imagine how much more difficult it is without any explanation at all!

When all is said and done, the problem will look like:

```
    428
   x 37
  -----
     56
    140
   2800
    240
    600
+ 12000
 ------
 15,836
```

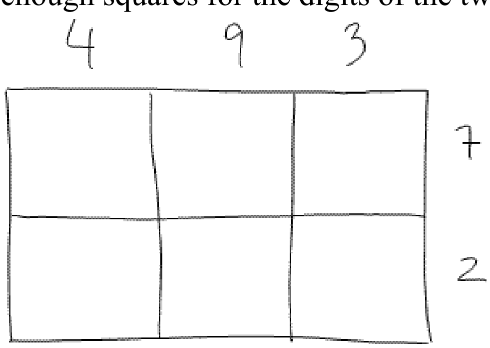


This ought not to be a difficult process for the students to understand and follow. When they get comfortable with it, encourage them to do it without the rectangle. Later still, show them how the usual algorithm saves even more time:

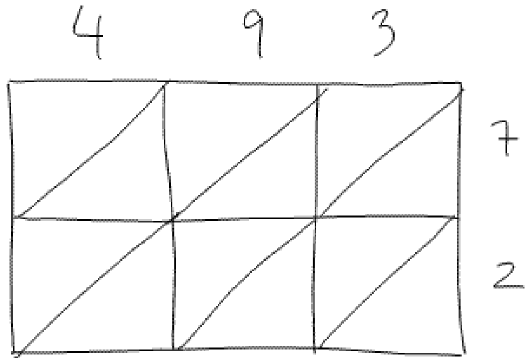


If clarity and comprehension is what you want most for your students, then perhaps it is best to stay with the "multiply it all out" form of multiplication, which makes sense and obtains the correct answer in a reasonable amount of time. If speed, however, is desired, then you should teach your student the fastest pencil-and-paper multiplication technique, called either "lattice multiplication" or the "multiplication boxes" method. This is best explained with an example.

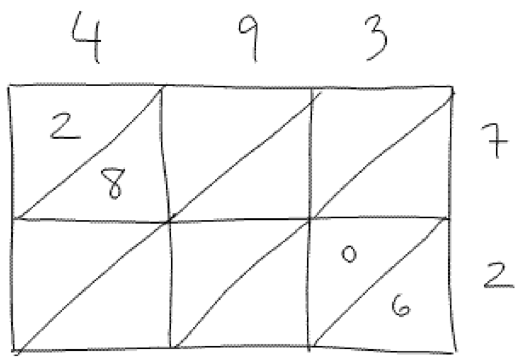
To multiply 493 × 72 with multiplication boxes, begin with a rectangle divided into enough squares for the digits of the two numbers:



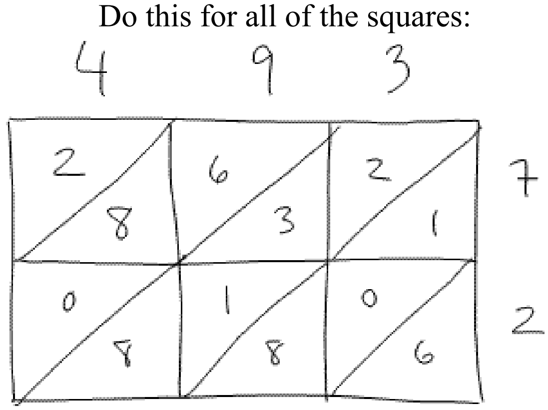
Draw a slash vertically through each square, from the top right to bottom left:



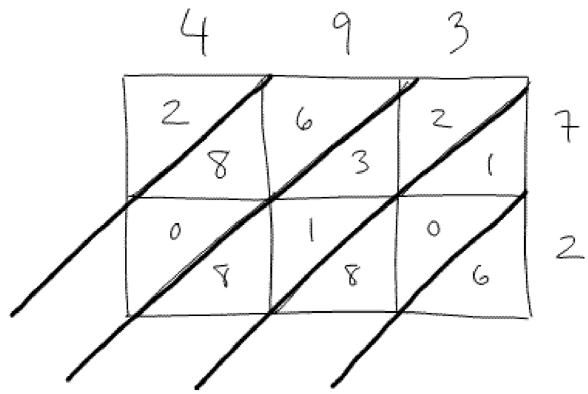
In each square, put the product of the numbers labeling its column and row. Use the space above the slash for the ten's digit, if there is one. If there is no ten's digit, you can either put a zero or leave it blank. For example, the upper-left square is in the column labeled 4 and the row labeled 7, so we write the product $4 \times 7 = 28$ by putting the 2 above the slash and the 8 below. For the lower-right square, the product is $3 \times 2 = 6$, so we put a zero in the ten's place above the slash:



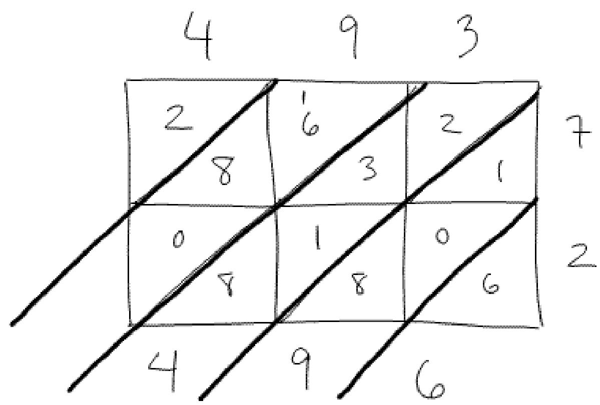
Do this for all of the squares:



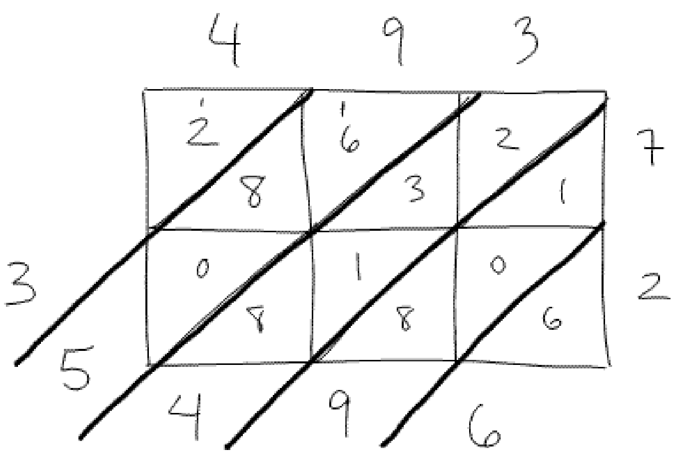
Next, trace the slashes boldly, extending down and to the left:



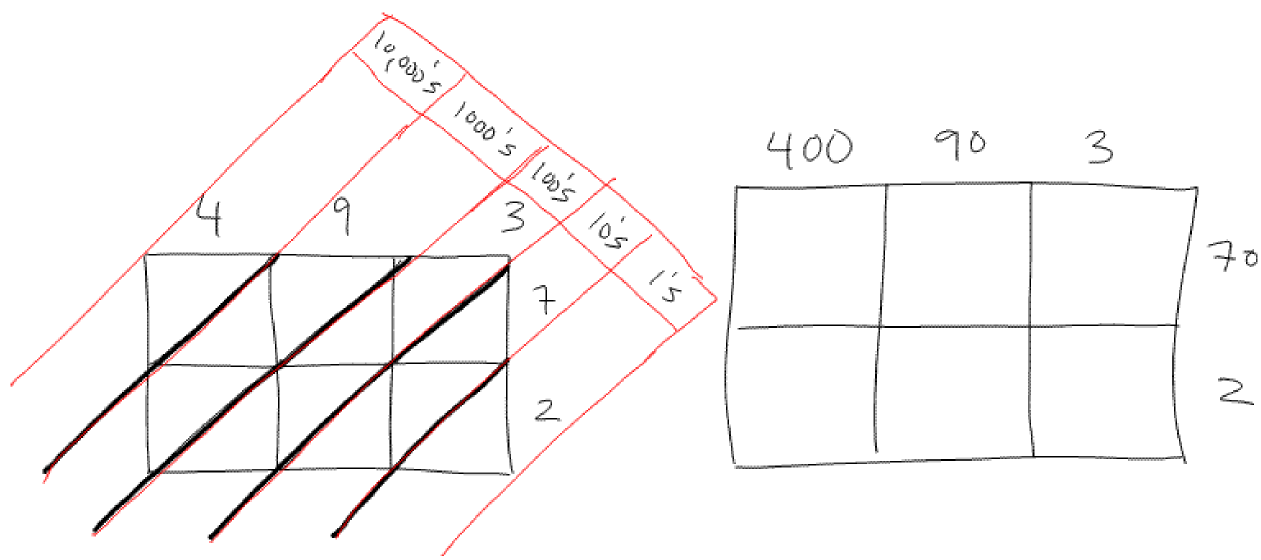
Finally, add the numbers up in each slanted column, carrying over into the next column if necessary. For example, the right-most digit will be 6. The next will be 1 + 0 + 8 = 9. The next column adds to 2 + 3 + 1 + 8 = 14, so we put down a 4 and carry a 1:



When we finish this process, these digits will form the answer. In this example, 493 × 72 = 35,496.

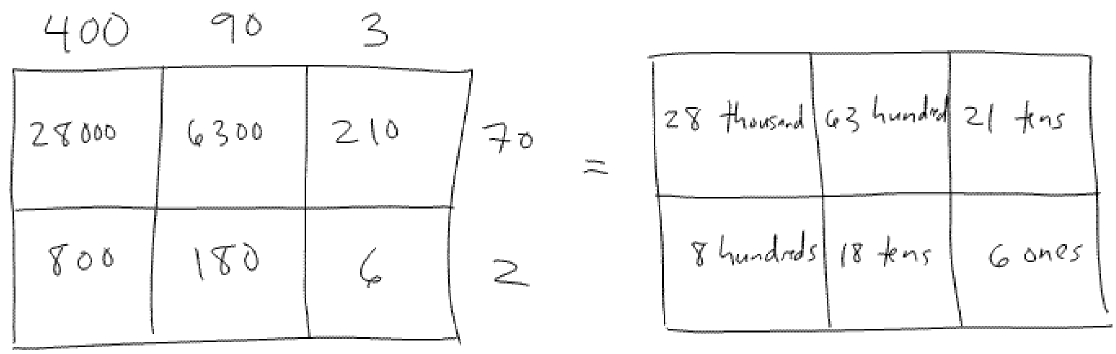


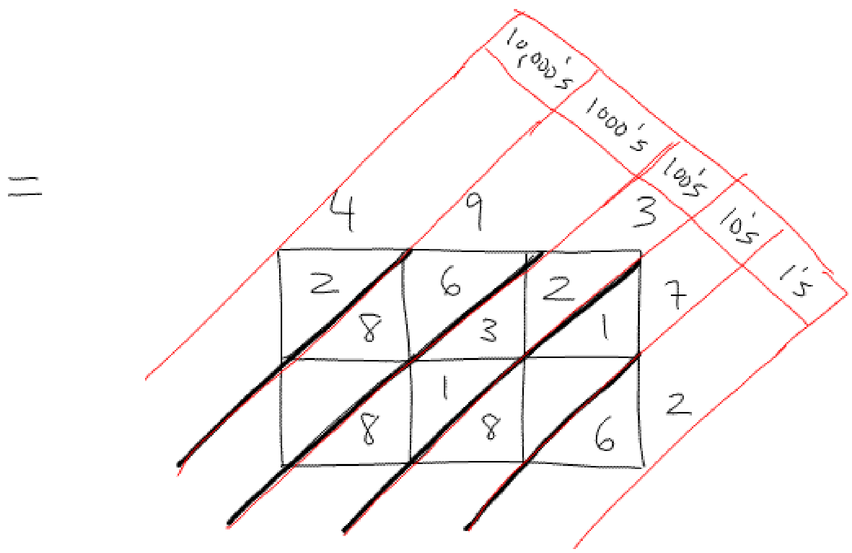
Not only is this method much faster than the usual short-cut method for long multiplication, but it can also be explained very clearly using the area model of multiplication. All we need to do is draw the corresponding rectangle and label the base-ten columns in the multiplication boxes:



The upper-left multiplication box corresponds to the upper-right box of the rectangle, which has area 400 × 70 = 28,000. This number can certainly be represented with a 2 above the slashed line (in the 10,000's column) and an 8 below (in the 1,000's column). Similarly, the upper-right multiplication box corresponds to the upper-right in the rectangle, with area 3 × 70 = 210. This number can be represented by a 2 in the 100's column (above the slash) and a 1 in the 10's column (below the slash).

The parallels between the two become even more clear when we write out the numbers as 28 thousand, 63 hundreds, 21 tens, etc.:





This method of multiplication is very clever and not very intuitive. Very few people would be able to come up with this idea on their own. Children can be taught to follow its process without too much difficulty, but it is important for them to understand what they are doing. Thus, it is important that they be first shown the area model of multiplication and the "multiply everything together" methods. Not only can these be used to explain the multiplication boxes, but they will also be used later on to explain the multiplication of fractions and algebraic expressions.

Questions:

(1) Illustrate the following multiplications using the area model, the multiply-everything out model, the traditional short-cut, and the multiplication box method:

(a) 57×39

(b) 256×71

(c) 715×806

Chapter 17: Mental Math

In all areas of mathematics, the most important step is to understand the underlying principles. Often, this involves having the right mental picture for the mathematical process. So far, we have seen how the number line is a good way to envision addition and subtraction. We used base-ten blocks to envision our positional number system. We have used rectangles to envision multiplication.

In order to take the next step in mathematics, however, a student needs to be able to perform operations quickly. In order to learn multiplication, for example, a child needs to be able to add quickly. For long multiplication, a child needs to know how to instantly recall the product of any two single-digit numbers. Drawing figures and counting blocks, however useful they are to understanding, are very slow processes. It is for this reason that when children are comfortable with a concept, we move on to teach them the most efficient methods of computation. We have so far examined in detail the algorithms for adding, subtracting, and multiplying large numbers.

An even bigger step toward fast computation is called *mental math* – computations done without even pencil and paper. Some people have mastered a whole host of complicated and involved mental mathematics processes. We don't want to overwhelm our students, so we'll stick with two of the easier sorts of mental math: addition and multiplication of certain two-digit numbers.

When adding two-digit numbers together, first look to see if there will be carrying from the one's digits. For example, when adding 27 + 45, the ones place will have 7 + 5 = 12 and thus require carrying. When adding 23 + 16, on the other hand, there will not be any carrying from the one's place because 3 + 6 = 9 has only one digit. It should not require writing anything down to recognize if the one's place will carry or not.

If there is no carrying from the one's place, simply add the ten's place and one's place separately. For example to add 23 + 16, we merely add the 20 + 10, say "thirty" and then add 3 + 6 and say "nine." We can thus calculate 23 + 16 = 39 mentally, without ever writing anything down. If there is carrying in the ten's place, this doesn't add much complication. For example, 71 + 54 = 125. We add the 70 + 50, say "one-hundred twenty" and then add the 1 + 4 and say "five."

If we recognize right away that there will be carrying from the one's place, then we do the exact same thing, but add one more ten to the ten's place. For example, to add 27 + 45, we recognize that there will be carrying and thus add 20 + 40 + 10 and say "seventy" then add 7 + 5 and just say the last digit: "two." Thus 27 + 45 = 72. Similarly, 84 + 67 will involve carrying, so we add 80 + 60 + 10 and say "one hundred fifty" and then add 4 + 7 and say "one," the last digit of this sum.

There are several benefits to teaching these sorts of strategies to students. The first, and most obvious, is that it speeds up the process of calculation. A student who can add mentally, will be able to work through problems quickly.

Another benefit is that mental calculation will give the student a better chance to see the big picture. Rather than being bogged down by a process – getting out pencil and paper, writing out the problem, following the steps of computation – the student will be able to focus on the entire problem at once. This is especially useful when working on multi-step and word problems.

Another benefit is that this process exercises short-term memory. When working on logic puzzles, critical thinking, and other advanced topics (including non-mathematical ones), people often need to keep several things in mind at once. In the instance of mental addition, there really is only one thing that must be kept in mind aside from the computation – whether or not the one's place involves carrying. In later problems, a student will need to keep in mind even more details. This is a skill to be practiced!

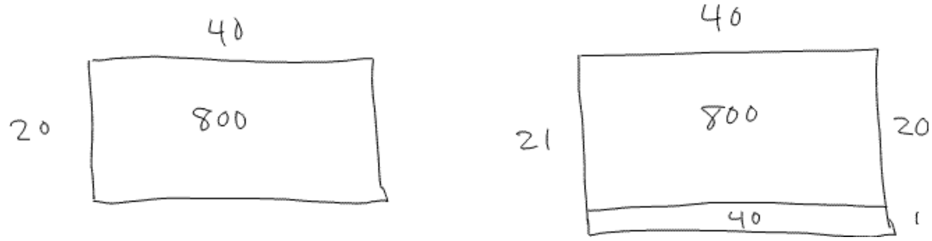
The modern American school system often demands very little memorization on the part of its students. In former times, students were expected to recall numerous poems, quotes, and famous speeches. This skill, like so many others, is developed with practice. When children are not expected to memorize things, their ability to memorize remains undeveloped. While I am a firm opponent of having students memorize things which are meaningless to them (computational procedures, quotes in foreign languages, etc.), I do believe that it is important to exercise the ability to precisely recall and retain information. Young children should be encouraged to memorize their favorite poems (including ones they write themselves). Exercises in mental mathematics are another good place to develop these skills.

A final benefit of mental mathematics is that these exercises teach students to innovate. A student should not have to use pencil and paper to add 400 + 800, although many will, provided that they were not encouraged to try working things out mentally first. There are many patterns to mathematics which can be detected by thinking about them, patterns which are not as easily noticed by a student who is set on a single process involving pencil and paper. The ability to think about things critically is perhaps the most crucial in all of education, up to and beyond graduation from college. Exposure to mental mathematics is a useful step in this direction.

A slightly more advanced form of mental mathematics involves multiplication. The first step we have already discussed, multiplying numbers like 40 × 600 and 300 × 500. In both cases, we need only multiply the two non-zero digits and then attach as many zeros to the end as the two numbers have in total. Because 4 × 6 = 24, we know that 40 × 600 = 24000 (one added zero for the 40 and two more for the 600). Similarly, 300 × 500 = 150000. While it might be challenging for a student to recognize that 150000 is pronounced "one-hundred, fifty thousand" off the top of her head, she should be able to see 300 × 500 and immediately write down 150000 and then put the comma in the right place: 150,000. Not only would it be a waste of time and paper to compute this problem with long multiplication, but all of the zeros involved can lead to mistakes.

$$\begin{array}{r} 300 \\ \times\ 500 \\ \hline 000 \\ 000 \\ +\ 1500 \\ \hline 150{,}000 \end{array}$$

When children are comfortable with multiplying numbers like these in their heads, they are ready for the next sort of mental multiplication: problems *close* to these. We will consider two different multiplications to be close if only one of the numbers is different, and different by only one. For example, the multiplication 21 × 40 will be considered close to 20 × 40. The second one is easy to compute mentally: 20 × 40 = 800. Thus, the answer to 21 × 40 ought to be close to 800. Because 21 > 20, it should make sense that our answer will be a little bit more than 800. There are a few different ways to figure this out. One way is to imagine the corresponding rectangles:



When we increase the multiplier from 20 to 21, the rectangle gets 1 unit taller. This adds a little sliver with dimensions 1 × 40 onto the rectangle, adding 40 more square units to the area.

Another way to look at this problem is to think about the grouping method of adding. Because 20 × 40 = 800, we know that 20 groups of 40 things makes a total of 800 things. To calculate 21 × 40, we will need 21 groups of 40. This is one more group of 40.

In either case, we look at 21 × 40, we think "20 × 40 = 800" is close, and then figure that 21 × 40 will be 40 more, thus 21 × 40 = 800 + 40 = 840.

As another example, try 30 × 51. Here, the nearest easy multiplication is 30 × 50 = 1500. Rather than 50 groups of 30, however, we want 51 groups of 30, thus we will need 30 more. Thus 30 × 51 = 1,530.

In short, if you increase one of the numbers of a multiplication by one, the answer increases by the amount of the number that *did not change*. For example, if 200 × 60 = 12000, then 201 × 60 = 12,060. We add 60 more because we increased the other number by 1. On the other hand, 200 × 61 = 12,200, because the 200 was the number that didn't change. Similarly, because we know that 70 × 40 = 2800, we can conclude that 71 × 40 = 2840 and 70 × 41 = 2870.

When one of the numbers is one less than in an easy multiplication, we follow the same process, but subtract instead of adding. For example, 19 × 6 is very close to 20 × 6 = 120. Instead of 20 groups of 6, however, we only have 19 groups of 6, which is 6 less. We conclude that 19 × 6 = 120 – 6 = 114. Similarly, 30 × 99 is very close to 30 × 100 = 3,000. However, we have 99 groups of 30, which is one group less than 100 groups of 30. We thus subtract one group of 30, resulting in 30 × 99 = 3,000 – 30 = 2,970. These are more difficult because mental subtraction requires a bit more thinking than mental addition. The idea is not to get students to solve all problems mentally, but to show them some speed tricks and short-cuts while exercising their minds.

Again, we can remember that "the number that doesn't change is the one you add or subtract." For example, because 30 × 50 = 1500, we know that 30 × 49 will be 30 (the number that doesn't change) less, so 30 × 49 = 1500 – 30 = 1470. Similarly, 29 × 50 = 1500 – 50 = 1450.

When one of the numbers changes by 2 or more, we add or subtract that many times the number that doesn't change. For example 32×40 is two 40's more than $30 \times 40 = 1200$, thus $32 \times 40 = 1200 + 2 \times 40 = 1280$. Similarly, 60×95 is five 60's less than $60 \times 100 = 6000$, thus $60 \times 95 = 6000 - 5 \times 60 = 6000 - 300 = 5700$.

Some numbers are not too difficult to multiply by 2 or by 3 because they do not involve any carrying. For example $14 \times 2 = 28$ and $21 \times 3 = 63$. This makes certain other multiplications easy to compute mentally. For example, $13 \times 30 = 390$ and $43 \times 200 = 8600$.

We can take these to the next step, but only with students whose mental addition and multiplication are quite sharp. For example, to compute 23×21, we can compare it to $23 \times 20 = 460$ and then add one more 23 (the number that doesn't change), to get $23 \times 21 = 460 + 23 = 483$.

A problem like 12×42 could be computed in a number of ways. We could think about $10 \times 42 = 420$ and then add two more 42's, resulting in $12 \times 42 = 420 + 84 = 504$. We could also have looked at $12 \times 40 = 480$ and then added two more 12's, for $12 \times 42 = 480 + 24 = 504$.

As a final detail, two-digit numbers that end in "5" are generally easy to double and triple. For example $15 \times 2 = 30$ is not so hard to remember. This means that $15 \times 20 = 300$ and $15 \times 21 = 300 + 15 = 315$. Similarly, we can compute 26×30 by thinking about $25 \times 30 = 750$ and then adding one more 30, to obtain $26 \times 30 = 750 + 30 = 780$.

To mentally compute the problem mentioned much earlier in the book, 7×25, we could look at this as close to $10 \times 25 = 250$ and then subtract three 25's, resulting in $7 \times 25 = 250 - 75 = 175$. We could also have seen this as close to $7 \times 20 = 140$ and added five more 7's, resulting in $7 \times 25 = 140 + 35 = 175$. These are not easy, but not impossible for a student with some practice.

All of these are short-hand tricks that only work for certain types of problems. A problem like 36×47 is not very close to anything easy to compute, and thus is best left to pencil and paper. The goal of mental math is not to take away these tools altogether, but to make students look for an easy and immediate answer before working a problem out on paper. A little bit of mental math practice can sharpen a student's math skills, speed up their computations, get them thinking abstractly, and exercise their memory.

As a final demonstration, here are all of the examples from this chapter, rewritten. It often helps a student to glean the overall pattern by seeing things lined up like this:

$20 \times 40 = 800$
$21 \times 40 = 800 + 40 = 840$

$30 \times 50 = 1500$
$30 \times 51 = 1500 + 30 = 1530$

$200 \times 60 = 12000$
$200 \times 61 = 12000 + 200 = 12{,}200$

$70 \times 40 = 2800$
$71 \times 40 = 2800 + 40 = 2840$
$70 \times 41 = 2800 + 70 = 2870$

$20 \times 6 = 120$
$19 \times 6 = 120 - 6 = 114$

$30 \times 100 = 3000$
$30 \times 99 = 3000 - 30 = 2970$

$30 \times 50 = 1500$
$30 \times 49 = 1500 - 30 = 1470$
$29 \times 50 = 1500 - 50 = 1450$

$30 \times 40 = 1200$
$32 \times 40 = 1200 + 2 \times 40 = 1280$

$60 \times 100 = 6000$
$60 \times 95 = 6000 - 5 \times 60 = 6000 - 300 = 5700$

$23 \times 20 = 460$
$23 \times 21 = 460 + 23 = 483$

$10 \times 42 = 420$
$12 \times 42 = 420 + 84 = 504$

$12 \times 40 = 480$
$12 \times 42 = 480 + 24 = 504$

$15 \times 20 = 300$
$15 \times 21 = 300 + 15 = 315$

$25 \times 30 = 750$
$26 \times 30 = 750 + 30 = 780$

$10 \times 25 = 250$
$7 \times 25 = 250 - 75 = 175$

$7 \times 20 = 140$
$7 \times 25 = 140 + 35 = 175$

Questions:

(1) Add the following numbers mentally, without writing anything down but the final answer:

(a) 54 + 9
(b) 15 + 32
(c) 27 + 18
(d) 31 + 95
(e) 67 + 14
(f) 84 + 67
(g) 39 + 62

(2) Multiply the following numbers mentally, writing down only the final answers:

(a) 400 × 70
(b) 120 × 30
(c) 20 × 31
(d) 41 × 70
(e) 32 × 70
(f) 49 × 20
(g) 60 × 19
(h) 99 × 8
(i) 201 × 40
(j) 23 × 300
(k) 20 × 140
(l) 25 × 21
(m) 15 × 5
(n) 14 × 30

Chapter 18: Division

Only when children are comfortable with long multiplication and have all of their basic multiplication facts memorized should they be introduced to division. These skills are as important to division as a solid understanding of addition is for learning multiplication.

To warm students up for division, ask them "times what" sorts of questions that only require the recall of basic multiplication facts. For example, ask "5 times what is 15?" or "what do you have to multiply by 6 to get 48?" You can also phrase things with the other multiplication vocabulary, for example "how many groups of 7 will make 28?" and "20 is what number added up 5 times?" You can also give written assignments with blanks and boxes left for the students to fill in, for example:

$$9 \times \boxed{} = 36 \qquad \underline{} \times 5 = 45$$

In order to answer any of these problems, your students will have to follow the mental process of division, even though the concepts and vocabulary are from multiplication. Questions like these also refresh your students' recall of the basic multiplication facts.

The most useful way to introduce the concept of division is as an analogy: the relationship between division and multiplication is the same as the relationship between subtraction and addition. Subtraction "undoes" addition. For example, if someone adds 5 to a number, then you will need to subtract 5 in order to undo this and return to the original number. Similarly, in order to undo a subtraction of 3, you will need to add 3 back. The processes of addition and subtraction are called *inverses* because of this property.

By the same logic, because division is the inverse of multiplication, it undoes multiplication. For example, if we multiply 9 by 4, we get 36. In order to get back to the 9, we will need to *divide* the 36 by 4. Thus, 36 ÷ 4 = 9. Because multiplication is commutative, it also follows that 36 ÷ 9 = 4. It will help the students to see a number of examples, paralleled with the situation from addition and subtraction. Show them several examples like:

4 + 3 = 7	so	7 – 3 = 4	and	7 – 4 = 3
6 × 9 = 54	so	54 ÷ 9 = 6	and	54 ÷ 6 = 9
2 × 7 = 14	so	14 ÷ 7 = 2	and	14 ÷ 2 = 7

Following this pattern, have your students name various basic multiplication facts (anything from 1 × 1 up to 9 × 9) and then write the equivalent division facts. Have them practice pronouncing the division facts out loud, for example "14 divided by 2 equals 7." This will introduce the vocabulary to them while making it clear that they have already learned most of the necessary information.

In terms of vocabulary, there are only three main terms that need to be learned. In a problem like 14 ÷ 2 = 7, the 14 is the *dividend* (the number being divided), the 2 is the *divisor* (the number doing the dividing) and the 7 is the *quotient* (the end result). These words are useful when discussing mathematics and how to teach it, but not as crucial for students to learn right away. Use these words periodically in class, but do not quiz your students on them. The concepts are far more important than the words used to convey them.

It would not be a bad idea to use these division facts to make up flash cards, with things like 16 ÷ 2 on one side and 8 on the other. When you review these with the students, have them solve a problem like 16 ÷ 2 by inserting a box (but only if they can't remember the answer right away):

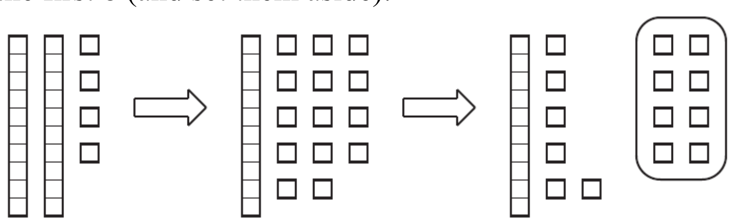
$$16 \div 2 = \square \quad \Longrightarrow \quad 2 \times \square = 16$$

This reinforces the fact that division is the inverse of multiplication, gives the students a process for calculating division problems, and helps them begin remembering the basic division facts.

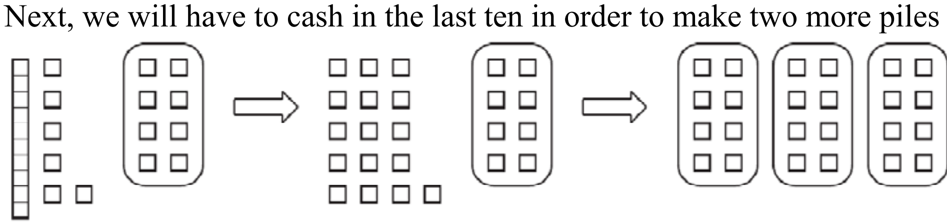
Merely being able to remember that 16 ÷ 2 = 8, however, does not mean that a student has a proper grasp on the concept of division. Each of the different models of multiplication should be recalled and used to add new light upon the concept of division.

One way to view multiplication is repeated addition. For example 3 × 5 = 3 + 3 + 3 + 3 + 3. The opposite of repeated addition is repeated subtraction. Thus, one way to look at a division problem is to ask "how many times can we subtract the divisor from the dividend?" For example, the answer to 24 ÷ 8 will be the number of times 8 can be subtracted from 24.

The repeated-subtraction model of division can be acted out with base-ten blocks. This will reinforce the base-ten number system and borrowing, while at the same time adding to the concept of division. For example, to divide 24 ÷ 8, we start with 24 blocks, then subtract 8 as many times as possible. We will need to cash in one of the tens into 10 ones in order to subtract the first 8 (and set them aside):

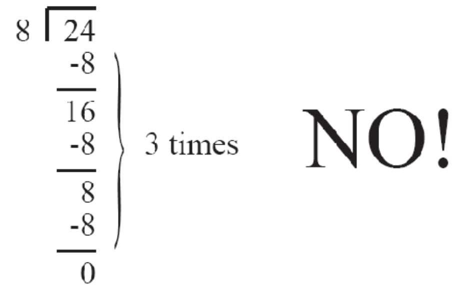


Next, we will have to cash in the last ten in order to make two more piles of 8:



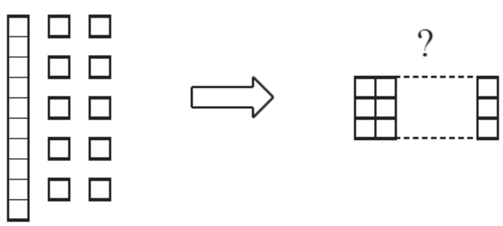
Because we were able to subtract 8 three times from 24, this means that 24 ÷ 8 = 3. Hopefully, your students will be impressed that this demonstration came up with the same answer that they could have obtained by viewing division as the inverse of multiplication.

Even though your students, at this point, ought to be able to subtract comfortably, it is a very bad idea to have them perform repeated subtraction in the following manner:

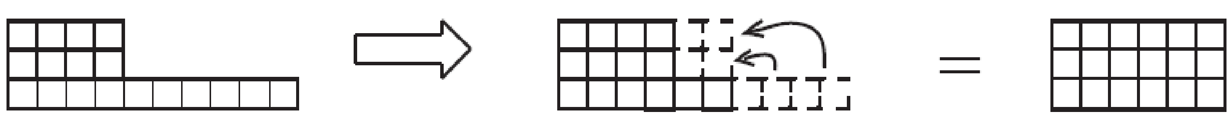


This does look like a very convenient way to illustrate repeated subtraction. However, students who are shown this technique early on will have to unlearn it in order to learn the algorithm for long division, which starts out remarkably similarly. It is very important that a student's mathematical education progress smoothly, without the need to midlead them, even for the sake of convenience. For example, many teachers explain that "even numbers are those which can be divided by 2 and odd numbers are those which cannot be divided by 2." This sets up a child to believe that 5 cannot be divided by 2, which will make 2.5 come as an unpleasant shock later on. Similarly, teachers often tell students that "you can subtract 5 from 8, but you cannot subtract 8 from 5," leading to difficulties later on with negative numbers. Teaching students to approach long division problems incorrectly, as illustrated above, will also lead to difficulties later on. Remember that you are not just teaching the content of a single grade, but that you are teaching a human being who will hopefully grow up to be an excellent adult – teach for the long haul, even if this requires abandoning convenient short-cuts.

The area model can also be used to illustrate division. With the area model, a multiplication like $4 \times 6 = 24$ means that the numbers 4 and 6 represent the height and width of a rectangle while the number 24 represents the area of the rectangle. An equivalent division problem is $24 \div 4 = 6$. Again, the 24 represents the area of a rectangle, the 4 represents the height, and the 6 represents the width. Put in this fashion, a question like $18 \div 3 = ?$ can be phrased as "if a rectangle has an area of 18 squares and a height of 3, what will its width be?" With base-ten blocks, this challenge looks like:

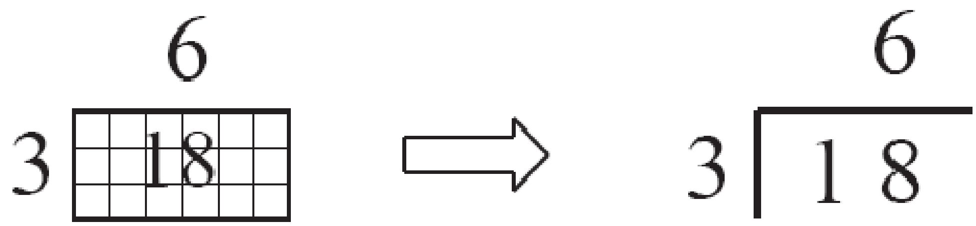


In other words, if you are given 18 blocks and arrange them to make a rectangle (cashing in some of the bigger ones as necessary), how wide will the rectangle be? Clearly the ten will need to be broken down, and the 18 individual blocks can be made into a rectangle with height 3 and width 6:



This can also be illustrated with Cuisenaire rods, trading in rods for equivalent smaller ones until they can all be put together in the shape of a rectangle with the right height.

By far, the cutest aspect of the area model for division is how it can be used to explain one of the ways in which division is represented. The division symbol $\overline{)}$ can be made to look like the top and left of a rectangle. The number inside represents the area of the rectangle. The number to the left represents the height of the rectangle. The answer to the problem, on the top, gives the width of the rectangle.



It can be seen that the process of making a rectangle of height 3 from 18 squares is the same as repeated subtraction. We subtract 3 as many times as possible from the 18 squares, and each time we are able, we make another column of the rectangle.

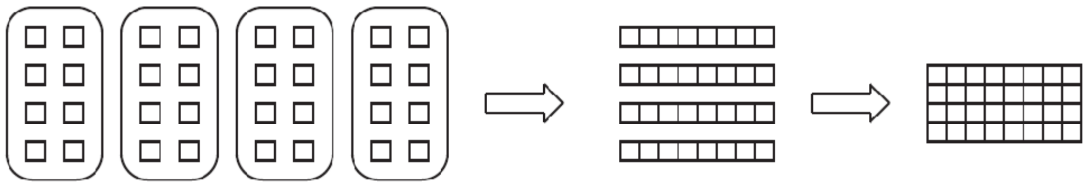
The final way to view division conceptually is as the inverse to the grouping model of multiplication. The multiplication 4 × 5 = 20 can be thought of as "four groups of 5 add up to 20." The number 4 is thus the number of groups, the 5 is the number of things in each group, and the 20 is the total number of things. The inverse of this problem, 20 ÷ 4 = 5, says that if you have 20 things and split them up into 4 groups, there will be 5 items in each group. In the short run (before fractions and decimals), this will be the most useful model for division. This even makes sense with everyday language. Children will know that to divide a number of objects is to share them equally. If you have 20 blocks and want to share them equally among 4 children, the answer to 20 ÷ 4 will be the number of blocks that each child gets.

The best way to use the sharing and dividing model of division is for a child to count out the correct number of items (cards work well, but so do pennies, etc.) and then share them out into the requisite number of piles. To divide 28 ÷ 4, for example, you can count out 28 cards and then "deal" them out to 4 different people. When this is done, each one should have 7 cards. This is thus the answer: 28 ÷ 4 = 7.

Each of the different models is useful for different aspects of division. The sharing model is very intuitive and will be used in explaining the algorithm for long division. The repeated subtraction (and sharing) models both work well to explain *remainders*: something left over that is not enough for another subtraction (or another round of one for each pile). The area model, on the other hand, will be the only one that can explain the division of fractions.

Here is an activity for students which will link all of the models of division together with one exercise. Begin with a slightly challenging, but not too complicated, division problem. For example, begin with 32 ÷ 4. First, have each student count out 32 squares. The small base-ten blocks can be used, as can the Cuisenaire 1-blocks, or the square inches used from area activities. Square blocks work best, so if there aren't enough, have the students gather in small groups.

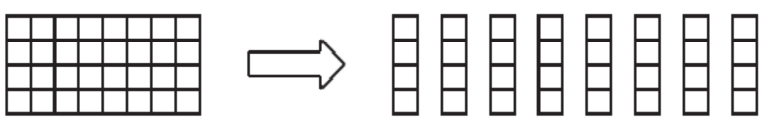
Next, have the students "divide the blocks into four groups." From this, the students ought to be able to see that there are 8 blocks in each group, thus 32 ÷ 4 = 8. This activity has been played before. Now, however, take it a step further – have the students use the same blocks to make a rectangle with a height of 4. Hopefully, everyone will recognize that this can be done easily and immediately by making a row out of each group:



This shows that the "divide into groups" model is essentially the same as the area model for division.

Next, have the students name the two multiplications illustrated by this rectangle. In this example, the answers should be 4 × 8 = 32 and 8 × 4 = 32. This helps to reinforce the inverse nature relationship between multiplication and division.

Next, with the rectangles of squares still in front of the students, have them subtract the divisor as many times as possible. In this example, to compute 32 ÷ 4 the students should subtract 4 as many times as possible from the 32 blocks. Ideally, students will realize that each column of the rectangle contains exactly 4 blocks, and thus the subtraction can occur as many times as the width of the rectangle:



In this manner, your students can see how all of the different models of division all ultimately illustrate and describe the same process.

Questions:

(1) Write the two division problems which are inverse to the equation 6 × 7 = 42. Pick four more basic multiplication facts and show the inverses to these as well.

(2) Show how the problem 27 ÷ 9 can be approached by equating it to a box (□), writing the inverse multiplication problem, and then solving it with a basic multiplication fact.

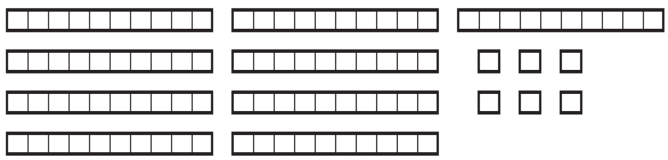
(3) Illustrate how 36 ÷ 9 can be solved with (a) repeated subtraction, (b) the area model, and (c) the sharing model.

(4) Name as many things as you can recall which you were taught in the early grades, but were later told were not entirely true.

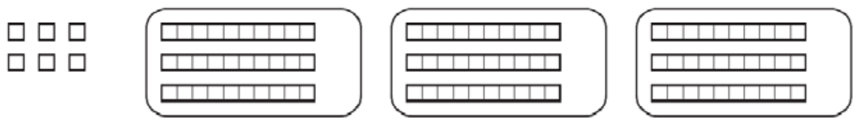
Chapter 19: Dividing Large Numbers

When students know the various models of division and are able to compute division problems that are inverse to the basic multiplication facts, they are ready for basic long division.

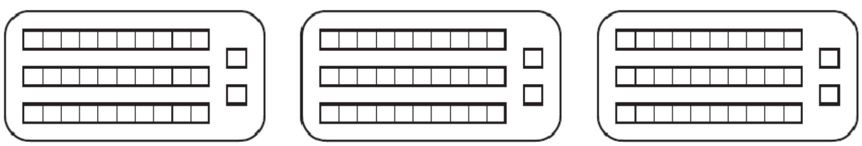
To begin, start with a medium-sized problem that will come out nicely, for example 96 ÷ 3. We will use the sharing-out method, taking 96 things and dividing them evenly into 3 groups. The easiest way to get 96 objects is by using base-ten blocks. Have the whole class watch as you count out 96 objects (working with transparent base-ten blocks on an overhead projector would work well):



The easiest way to divide these up into three equal piles would be to share out as many of the 10-rods as possible first. When these are shared out, we put three in each pile, using them all up:



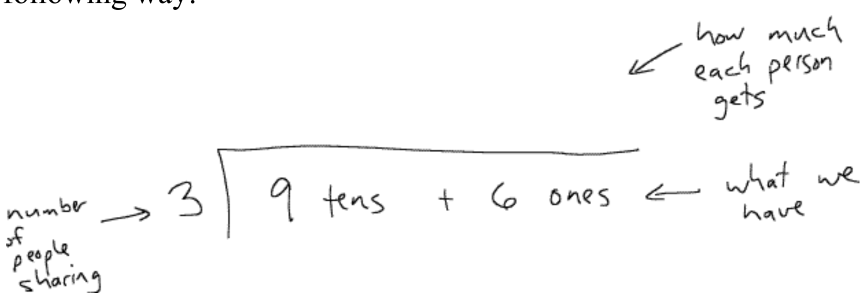
Next, we share out the remaining 6 blocks by putting 2 in each pile:



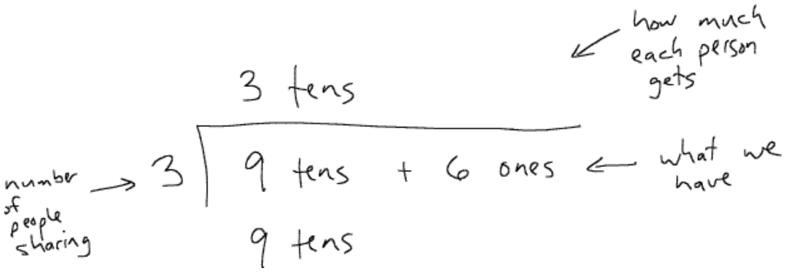
We thus conclude that 96 ÷ 3 = 32 because each pile has 32 blocks in it.

Next, repeat this exercise with fake money. Put all the students in the class into groups of 3. Give each a stack with $96 (nine $10 bills and six $1 bills), and then have them share the money out evenly among themselves. Hopefully, everyone will realize that the easiest way to do this is to share out the $10 bills evenly and then the $1 bills evenly. Each child should end up with $32 in fake money. This ought to be fairly easy, now that everyone has just seen it done with base-ten blocks.

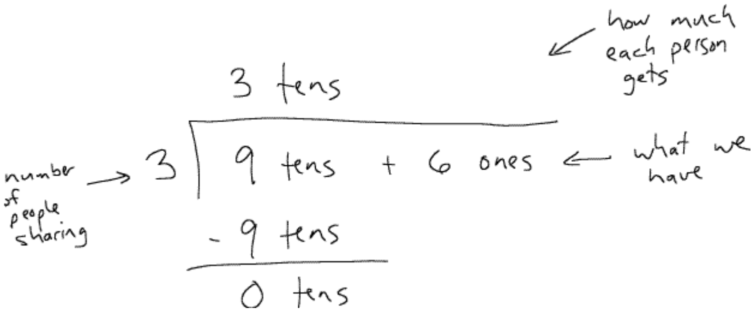
As soon as this exercise has been completed, write out what has just happened in the following way:



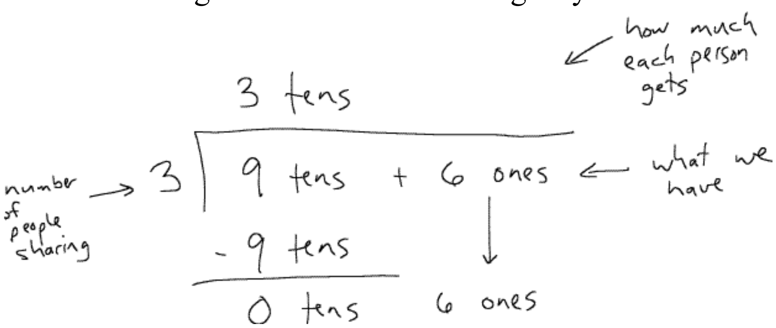
The first thing we do is share out the tens among the three people as much as is possible. Point out that because 9 ÷ 3 = 3, this means that we can give 3 ten-dollar bills to each person. This will use up 3 × 30 = $90 of the dollars. This is written out in the following fashion:



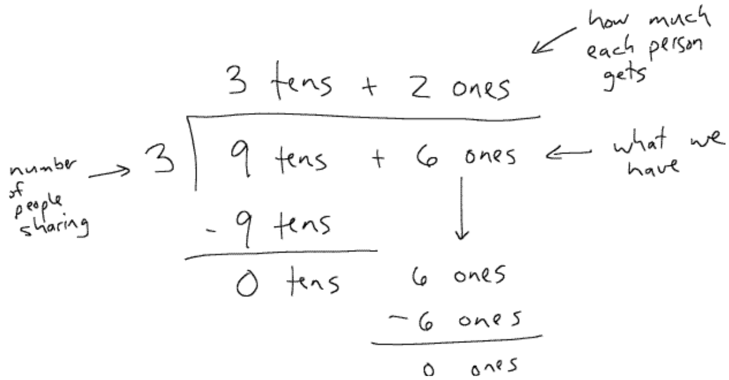
When we subtract these 9 tens, they are all used up:



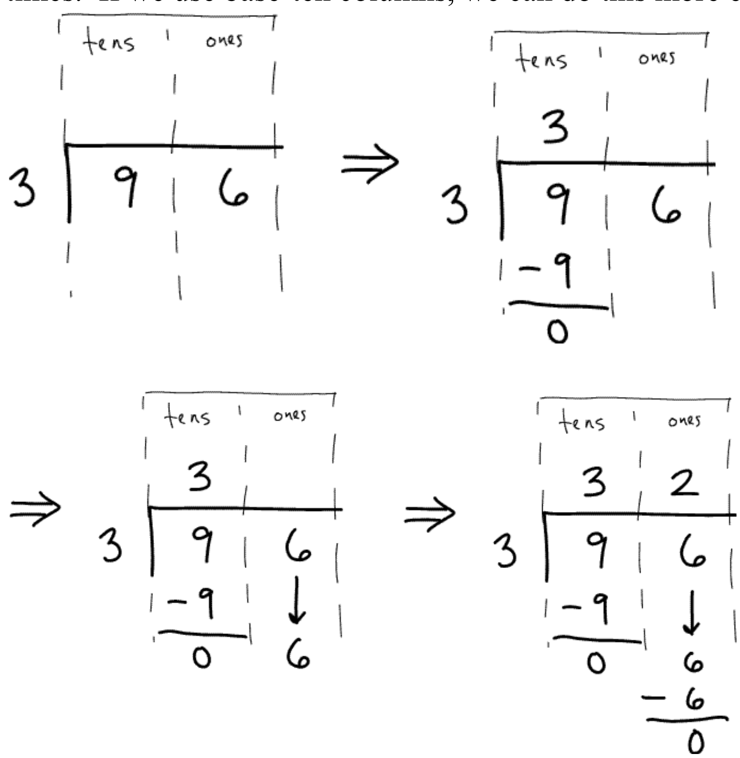
At this point, we now turn our attention to the 6 one-dollar bills that we have left to share. The number is brought down in the following way:



Because 6 ÷ 3 = 2, we know that these 6 one-dollar bills can be divided among the 3 people by giving $2 to each. This uses up all 6 of the one-dollar bills, so there is nothing left:



Next, point out how tedious it was to have to write "tens" four times and "ones" five times. If we use base-ten columns, we can do this more efficiently:



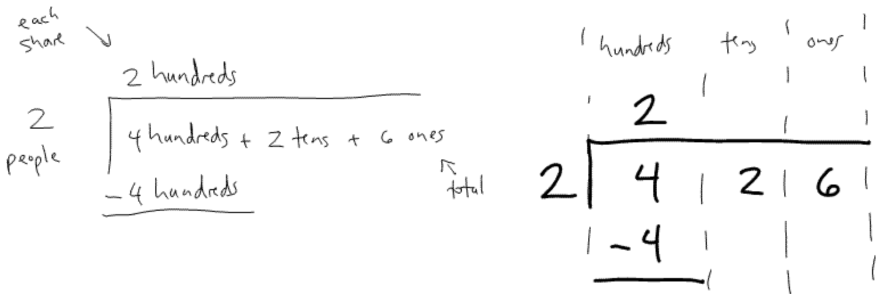
This is, of course, the process of long division, explained from basic principles.

Have students discuss the different methods of solving this single problem, in order to reinforce their similarities.

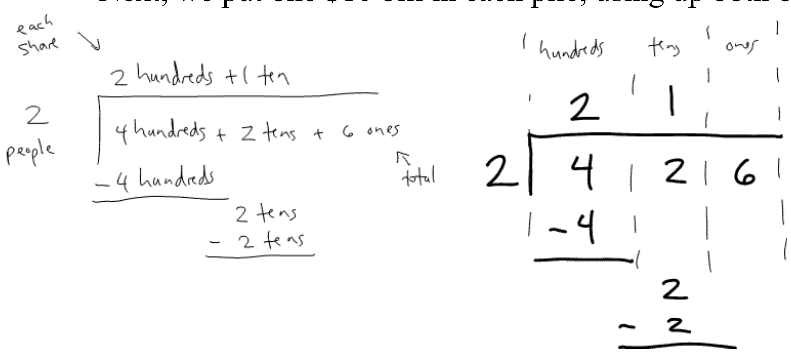
As a next example, present the class with a more difficult problem, perhaps one that extends into the hundred's place. Have the students work it out with money while you work it out with base-ten blocks on the overhead. As each step progresses, write out what has happened, both with words and with base-ten columns. For example, have the class work on 426 ÷ 2.

The students (in groups) should begin with $426, made up with 4 hundred-dollar bills, 2 ten-dollar bills, and 6 one-dollar bills. On the overhead, have the corresponding 4 hundred-flats, 2 ten-rods, and 6 one-cubes.

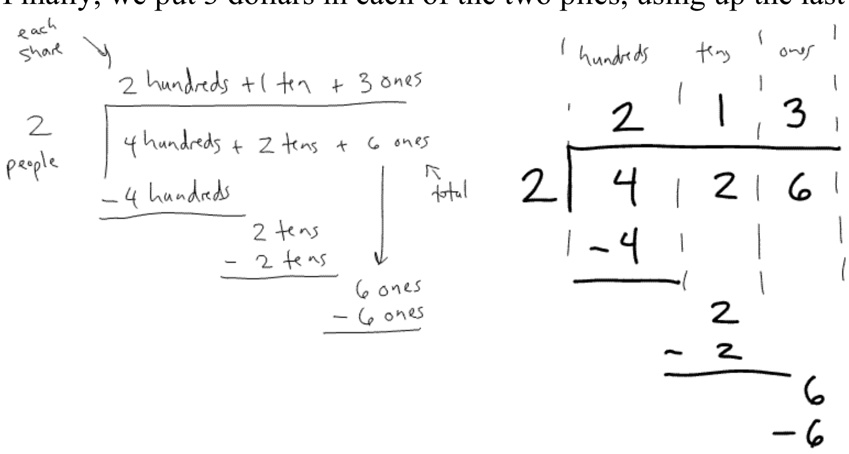
First of all, encourage the students to share out the larger bills first. In this case, there is no problem with sharing out the tens first, then the ones, and then the hundreds, but we want to lead them toward the general process of long division. Thus, we first take the 4 hundreds and divide them into two equal groups, with 2 hundreds in each. This uses up all 4 hundreds, and so we write on the board:



Next, we put one $10 bill in each pile, using up both of the 2 tens:

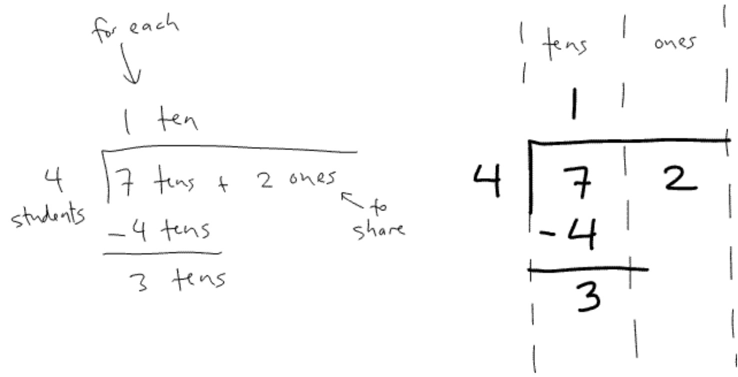


Finally, we put 3 dollars in each of the two piles, using up the last of the money.

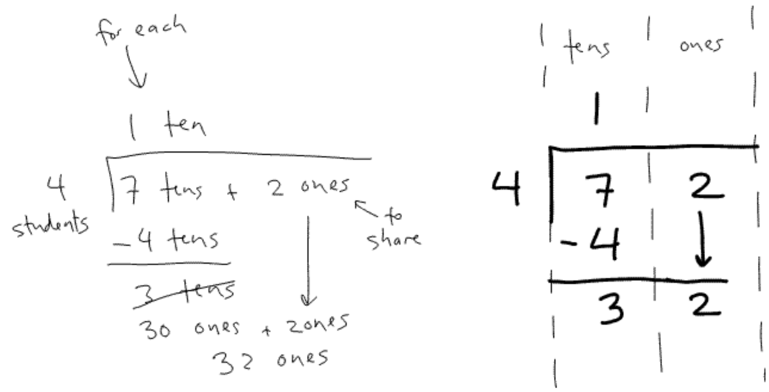


We conclude that 426 ÷ 2 = 213.

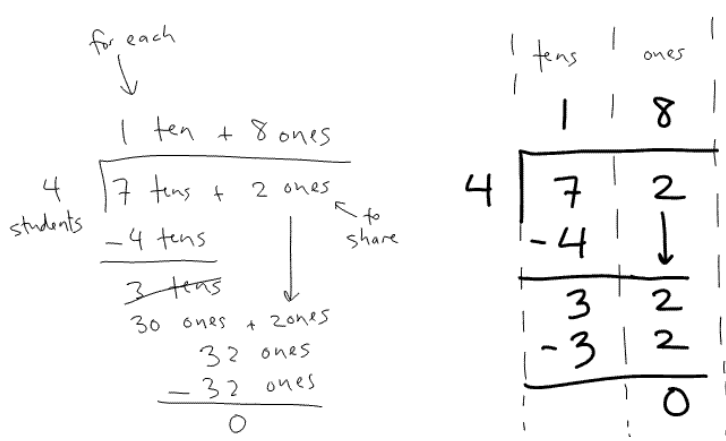
Next, introduce a problem where some breaking-down will be necessary. For example, try a problem like 72 ÷ 4. In this case, when the 7 ten-dollar bills are divided up among the 4 students in each group, we will be able to give each student only one $10 bill. There will be 3 left over, which is not enough for each student to get one more. Using the base-ten column notation, we write that each student has been given 1 ten-dollar bill, which uses up a total of 4 × $10 = $40, and leaves $30 left:



Hopefully, as the students contemplate how to share the 3 ten-dollar bills among the four recipients, they realize that it is necessary to make change and break the tens into smaller bills. We thus turn the 3 ten-dollar bills into 30 one-dollar bills. When we "bring down" the 2 one-dollar bills and add them to the stack of 30, we end up with 32 one-dollar bills:

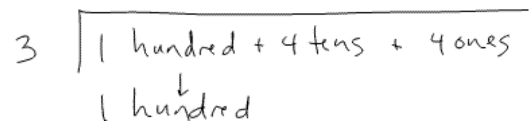


This is yet another amazing feature that comes naturally from our base-ten positional numbering system – rather than convert the 3 tens into 30 ones and then add them to the 2 ones, we can just slide the 2 down to the right of the 3 and automatically have the 32 ones, ready to divide. If each student gets 8 one-dollar bills, this will use up all of the money:

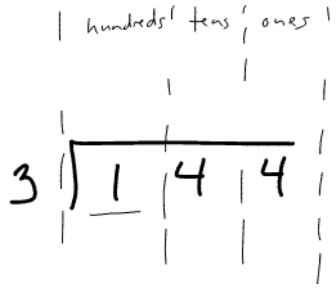


We conclude that 72 ÷ 4 = 18. As we know that multiplication and division are opposite operations, we can even double-check our answer by multiplying 18 × 4 = 72. We can do this out the long way, of course, or we could use mental math. We know 18 is 2 less than 20, so 18 × 4 = 20 × 4 – 2 × 4 = 80 – 8 = 72.

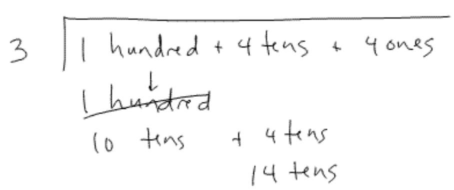
A similar sort of problem to work out would be something like 144 ÷ 3. Here, students in groups of 3 must share 1 hundred dollar bill, 4 ten dollar bills, and 4 ones among themselves. We begin by trying to share the largest denomination of money, the $100 bill:

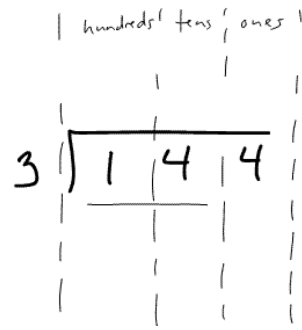


Clearly, we cannot give all three people a $100 bill when there is only one. This is the same thing as saying "3 doesn't go into 1" when we look at the short-hand notation:

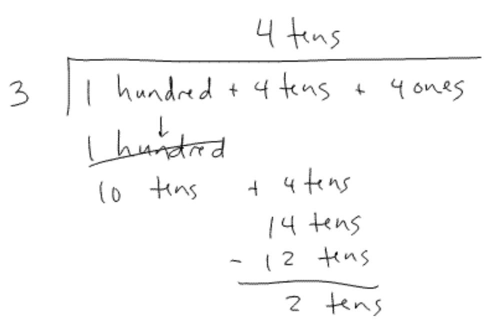


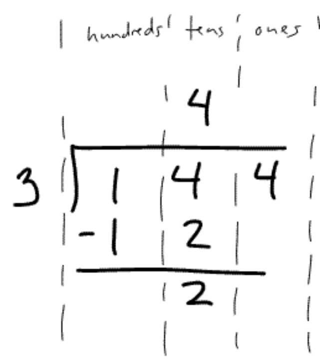
With paper money (or the written-out method) we need to break the hundred-dollar bill into 10 tens and add them to the 4 tens we already had for a total of 14 ten-dollar bills. When we work with the short-cut base-ten columns method, we need only to look at the first two digits to find this very 14:



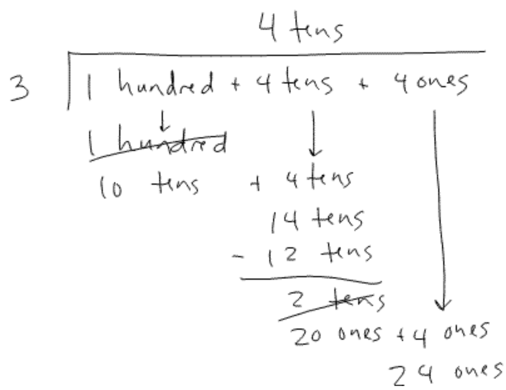


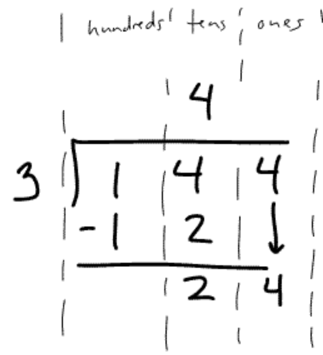
With 14 tens, we can give each of the 3 students 4 ten-dollar bills, using up a total of 12 tens ($120), with 2 tens left over:



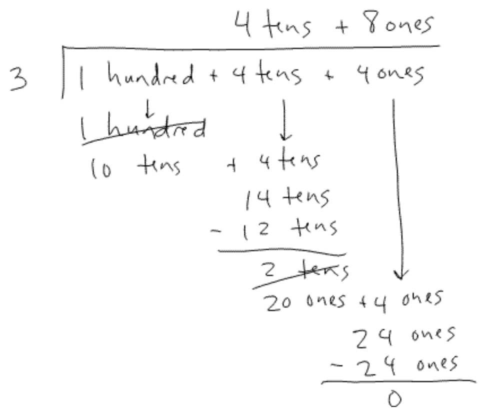


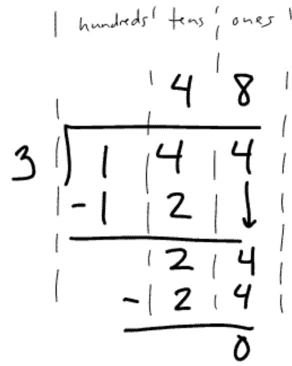
When we write things out the long way (or work with fake money), we need to break the 2 ten-dollar bills into 20 one-dollar bills and add them to the 4 ones we began with for a total of 24 one-dollar bills. With the short-cut method, all we need to do is "bring down the 4" and we'll have 24 ones all ready:





These 24 ones can be evenly divided by giving 8 to each student. This will use up the last of the money, so everything comes out evenly:

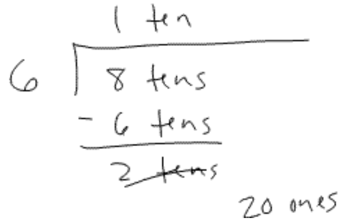


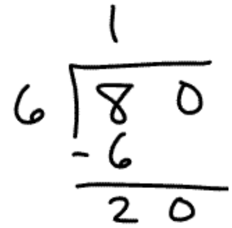


Once again, we can verify that 144 ÷ 3 = 48 by checking the equivalent multiplication equation, that 3 × 48 = 144. As 3 × 50 = 150 and 48 = 50 – 2, we use mental math to see that 3 × 48 = 3 × (50 – 2) = 3 × 50 – 3 × 2 = 150 – 6 = 144. This proves that we arrived at the right answer.

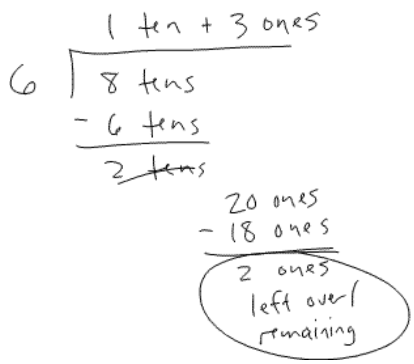
In this manner, first working out the problem with base-ten blocks or fake money, then writing out what has happened with words (hundreds, tens, and ones), and then writing everything out with base-ten columns, we slowly introduce long division to students in a way that clearly explains the algorithm. We only say "how many times does 3 go into 14," as in the last problem, as a short-cut way of looking at the problem. The 14, in this case, represents 14 tens and not 14 ones. This does not matter – as long as everything is kept in tidy columns, the process will result in the correct answer.

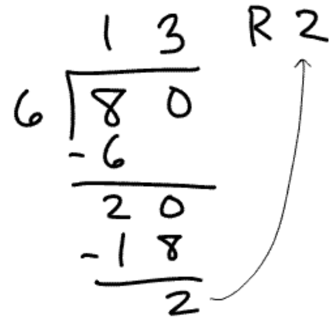
When students are comfortable with this process, try one that ends with a remainder, for example, 80 ÷ 6. Each of the 6 students can get 1 ten-dollar bill, leaving 2 tens that are converted into 20 ones:





From the 20 ones, we can give each of the 6 students 3 ones, and there will be 2 ones left over. We write this as a remainder:



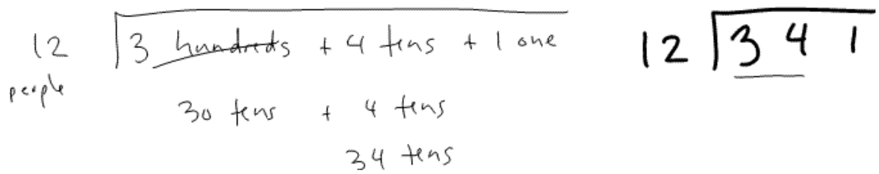


We say "when you divide 80 by 6, you get 13 with 2 left over" or "... with a remainder of 2." When we get into decimals later, we will break these 2 dollar bills into 20 dimes and then break the leftover dimes into pennies, but for now it is enough to say that we have 2 one-dollar bills which remain.

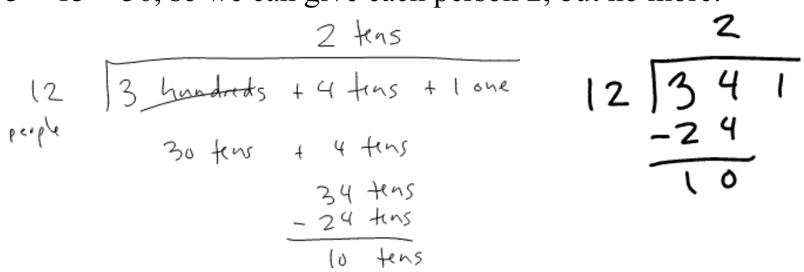
As students grow more comfortable with this process, try some examples where you do not actually hand out fake money or base-ten blocks, but merely talk about how it would be done and write the problems out with both words, with base-ten columns, and with the usual method of long-division (without labeling the base-ten columns). Hopefully, the students will be able to imagine the money and be able to see how the algorithm follows the sharing process.

When students are comfortable with this, you can slowly introduce them to long division by two and three-digit numbers. This is where the skills of mental math and approximation come in very handy.

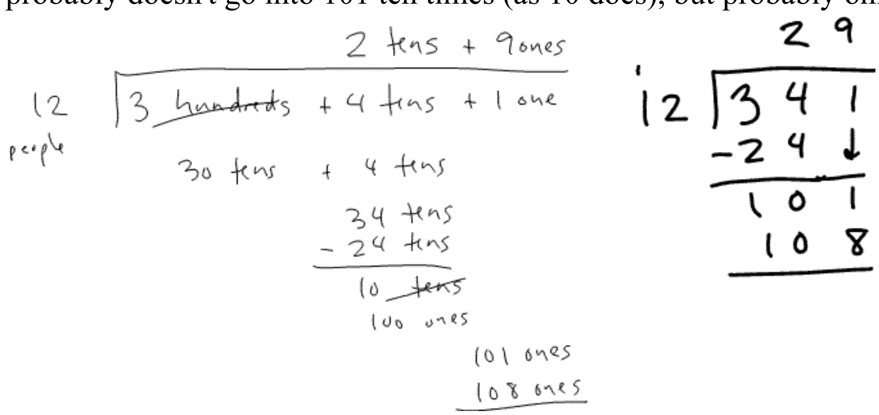
For example, to compute 341 ÷ 12, you students will hopefully see that the answer will be the amount of money that each of 12 friends will get when sharing $341 evenly. There are only 3 hundred-dollar bills, not enough to go around even once, so we convert the 3 hundreds into 30 tens and add them to the 4 tens:



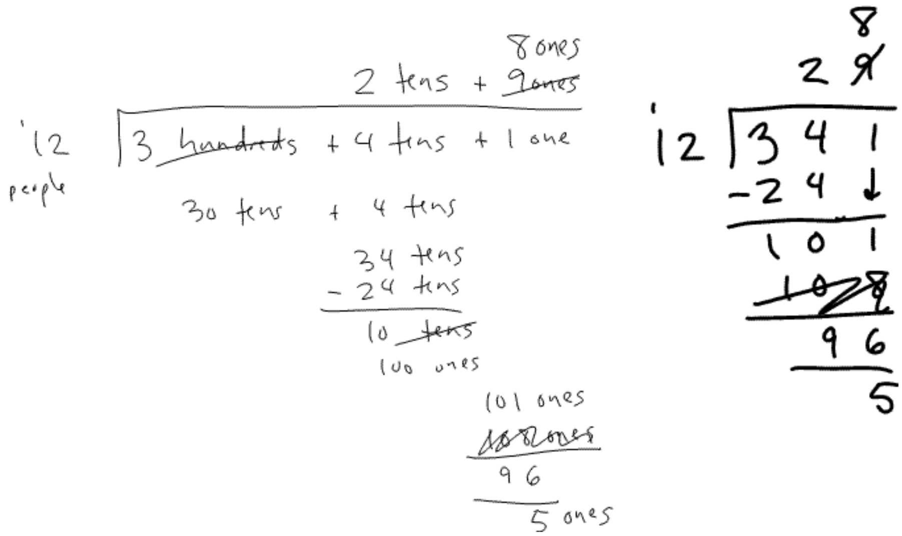
We need to figure out how many times we can give each of 12 people a $10 bill from a stack of 34. Hopefully, you students will be able to recognize that 1 × 12 = 12, 2 × 12 = 24, and 3 × 13 = 36, so we can give each person 2, but no more:



This is by far the hardest part of the entire process of long division: having to figure out how many times one number goes into another without going over. Some teachers encourage their students to write out all the multiples of the divisor from 1 to 9 on the side of the paper (in this example: 1 × 12 = 12, 2 × 12 = 24, 3 × 12 = 36, 4 × 12 = 48, 5 × 12 = 60, 6 × 12 = 72, 7 × 12 = 84, 8 × 12 = 96, 9 × 12 = 108). This does work, and saves the student from having to do much thinking, estimating, or mental math. However, this is also a very slow process. It is much better for students to practice estimating. For example 12 is a little bit more than 10, so 12 probably doesn't go into 101 ten times (as 10 does), but probably only 9 times instead:



Ooops! We can't give out 9 ones to each of the 12 people. There isn't enough to go around. Thus we have to change our guess to 8 instead:



Thus, we have obtained our answer: 341 ÷ 12 = 28 with a remainder of 5.

Note that it is a good idea to make mistakes like the above with your students. They are certainly going to make mistakes when they try these problems, so it is important for you to show them how to recognize and fix a mistake. These are critical skills, and they are best taught if you make mistakes yourself!

Also, you might want to point out that when you realized that 9 × 12 = 108 was more than the 101, it is not necessary to erase everything and then try to figure out what 8 × 12 might be. Using mental math, it is clear that 8 × 12 will be 12 less than 108, thus 8 × 12 = 108 – 12 = 96. In other words, if you guess too high, you can subtract 1 from your guess and subtract the divisor (here it was 12) from the number you are taking away. If you guess too low, you can add 1 to your guess and then add the divisor to the number you are taking away. There will still be plenty of trial and error when doing a problem like 2892 ÷ 37, but this trick can help them find a way through.

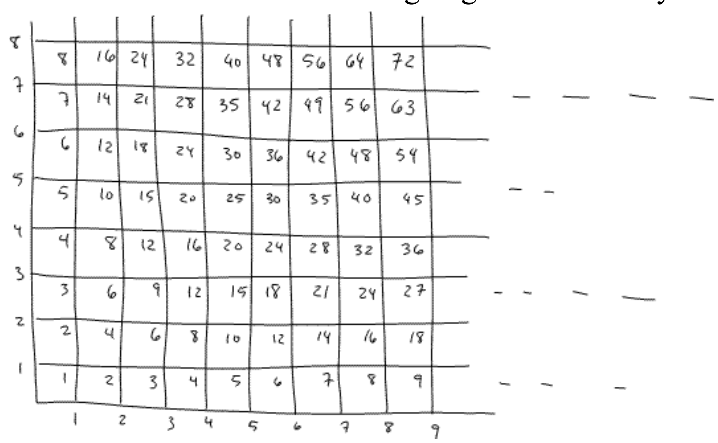
Questions:

(1) Explain how to divide 256 ÷ 6 using (a) fake money, (b) the same process written out with words (hundreds, tens, ones), and (c) the same process, utilizing the base-ten column notation. Go step-by-step, just like the examples from this chapter.

Chapter 20: Factoring

When children have first gotten comfortable with the concept of long division, they are ready to learn one of the most important skills in mathematics: factoring. This will be essential for success with fractions, and will help to make algebra easy to understand.

To begin, students will need to know about prime numbers. The best way to introduce this concept is with a multiplication table. You can draw out the grid with sidewalk chalk and have your students fill it out, or else have them take turns telling what to fill in on a projected grid (if the weather is bad). It will help to keep not restrict the grid to 10 × 10, so that your students can envision the table going on indefinitely:



While the whole class has gathered around the big table (or all looking up at a projected one), discuss the different numbers which will appear on the table. It should be clear that if the table continues on forever, all of the natural numbers will appear on the leftmost column and the bottom row. Not all of the numbers, however, will appear in the squares outside this row and column. The numbers which appear outside are called the *composite numbers*, the numbers which can be formed (composed) by multiplying two numbers together (numbers which are 2 or more):

It is a good exercise to have the students write out the first composite numbers in order: 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, etc.

The *prime numbers* are the numbers bigger than one which are not composite. The number 1 is the multiplicative identity, and is so special that it is neither prime nor composite. Some students find this confusing, but often because they are taught poorly. Because composite numbers are formed by multiplying two numbers together (numbers 2 or greater), some teachers say that a prime number can only be formed by multiplying 1 and itself (using whole numbers). This is true for all the prime numbers except for 1, because $1 = 1 \times 1$. It is generally easier to break all the natural numbers into three categories: the multiplicative identity, the composite numbers, and the prime numbers. This helps to explain why 1 is not a prime number.

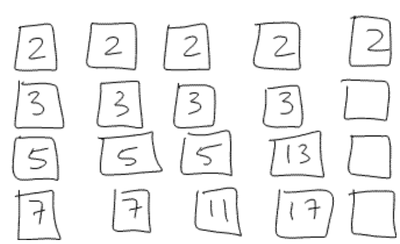
A clever way to define *prime number* is "a number with exactly two factors, one and itself." The number 1 has only one factor, and thus is not prime. This distinction is rather subtle, however, and will need plenty of explanation and discussion if you are to use it in your class. Remember that making a definition more clever does not usually make it easier to understand.

When this is cleared up, you can have your students list out the prime numbers. If you are working on a big table drawn in chalk, have the students walk around and figure out the numbers (other than 1) which will never appear inside the first row and column of the chart. These numbers begin 2, 3, 5, 7, 11, 13, 17, 19, 23, etc. These can also be figured out by looking at the table of composite numbers and writing out the missing ones (except 1).

Prime numbers are important because they are the building blocks of all the natural numbers (other than 1). Every number 2 or greater is either a prime number or else can be written as a product of prime numbers. The process of breaking down a number until it is represented as a product of prime numbers is called *factoring*. The ability to do this is so important that mathematicians call it the *Fundamental Theorem of Arithmetic* – the most important property in all of arithmetic.

Pretty much all of the numbers that your students will ever need to factor will require only the first ten prime numbers: 2, 3, 5, 7, 11, 13, 17, 19, 23, and 27. Thus, it will be useful for your students to learn to recognize these numbers. Show them a few numbers between 2 and 30, then have them pick out the ones that are prime from the ones that are composite. Most elementary factoring, in fact, will use only the first four primes: 2, 3, 5, and 7.

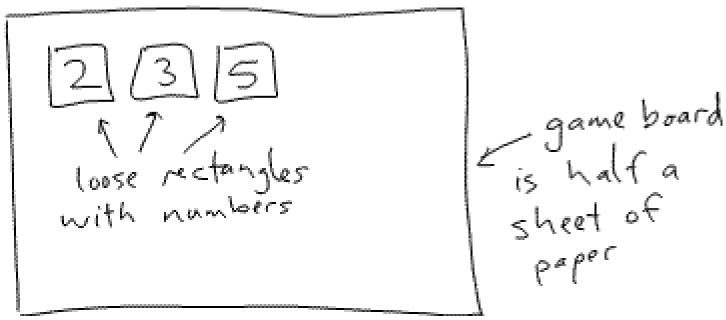
An excellent exercise to run with your students at this point is the factoring game. In order to play this, each student will need a set of pieces of paper with the first few prime numbers written on them. The pieces of paper ought to be small enough to fit ten or so across a student's desk, and yet bigger than a breath can easily disrupt. Ideally, these would be made of decent card stock and cut into nice rectangles. However, the exercise will work just as well with the rectangles formed by folding and ripping a piece of paper in half four or five times. On five of these pieces of paper, the students should write the number 2, on four they



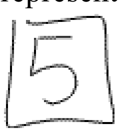
should write the number 3, on three they should write the number 5, and on two they should write 7. They should also make one 11, one 13, and one 17. These quantities are not very important. It is only important that the pieces of paper only have prime numbers on them, and that there be several extra of the smallest prime numbers. There should also be several blank rectangles for students to add more prime numbers as necessary.

These rectangles will be the pieces of the game. The game board will consist of half a piece of paper.

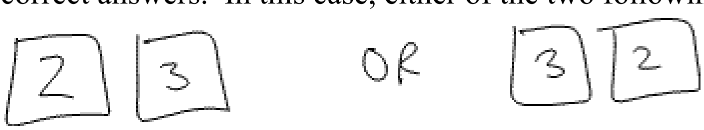
The game is quite simple – the students must put some of numbers on the paper in order to make various numbers. The numbers on the paper will all be multiplied together. For example, the following combination will represent the number 30 because 2 × 3 × 5 = 30:



In order to represent a prime number, a single card is used. For example, the number 5 is represented by putting just a single rectangle on the board:



In this way, have your students form, one at a time, the numbers from 2 up to 30. You should announce the number, for example 6, and then walk around the room praising all the correct answers. In this case, either of the two following answers would be correct:



Try to encourage your students to put the cards side-by-side like this. In advanced mathematics, any time two symbols are put side-by-side like this without a symbol in-between (like 5*x* or *xy*), it will mean multiplication.

When you reach prime numbers that haven't been written out on a card yet (19 and 23 will probably be the first two), see if your students can recognize this. The only solution is to write these numbers on new cards and then display those numbers alone.

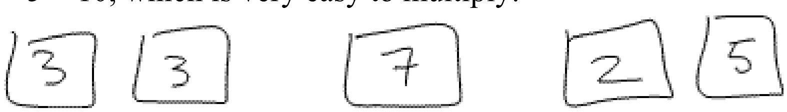
This is an excellent exercise for many reasons. The basic multiplication and division facts are practiced. The commutative property of multiplication is reinforced every time there are multiple answers formed by rearranging the factors. The prime numbers stand out as very special – requiring only a single card. Everyone gets to work on the problems at once. You can see right away the students who understand and those who need extra help. Best of all, this introduces and reinforces the important skill of factoring.

When playing this game, some students will want to represent numbers with addition instead of multiplication. Explain to them that there are many ways in which numbers can be formed by adding. The number 9, for example, could be formed with 3 + 3 + 3 or with 7 + 2 or with 5 + 2 + 2 or with 3 + 2 + 2 + 2. When using multiplication, however, there is only one way to form 9 with multiplication: $9 = 3 \times 3$. It is for these unique answers that we look.

When your students get the hang of factoring small numbers, challenge them with larger numbers, like 36, 45, 60, 75, 100, 150, 180, and more. Teach them to try to mentally break down the number first into easy parts, for example $180 = 18 \times 10$, and then to break down the remaining parts.

At the same time, it is a good idea to have your students practice working in the other direction. Show them a series of factors and have them put them back together into a single number, for example:

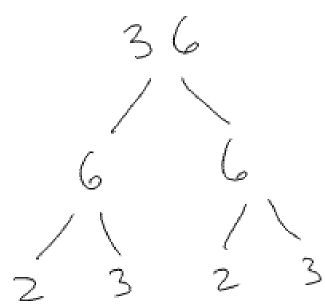
Have your students to form this number (or another example) with the cards in front of them. Encourage them to rearrange and group the squares in order to make the multiplication easier. If we work this from left to right, for example, we will first compute $2 \times 3 = 6$, then $6 \times 3 = 18$, but then 18×7 is not very easy. Instead, it works a lot better to pull the 2 and the 5 aside, because $2 \times 5 = 10$, which is very easy to multiply:

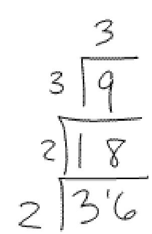


Now, we see that this number is $9 \times 7 \times 10 = 63 \times 10 = 630$. Impress upon your students that this is the *only* way that 630 can be made by multiplying prime numbers – there will always be a 2, a 5, a 7, and two 3's, although they can be put in any order.

This exercise is a great way to practice mental mathematics. Furthermore, having the students play around with rearranging factors provides very useful practice with the commutative property of multiplication.

It is common to teach students how to factor with either factor trees or factor cakes. As an example, the number 36 is factored in both formats to the right. In the first, we break 36 into 6×6, and then break each of the 6's into 2×3. In the second, we divide 36 first by 2, then the resulting 18 by 2, then the resulting 9 by 3. In either case, we see $36 = 2 \times 2 \times 3 \times 3$. We read the factors off the factor tree by looking at the "leaves" – the numbers at the very ends of the "branches." On a factor cake, the factors are the "frosting" on the outer edge and top of the "cake." (Think of this as a multi-layer wedding cake.)





These are both perfectly valid and useful ways to factor numbers. However, they are time-consuming and do not teach children the very valuable skill of being able to factor quickly in their heads. When we work later with fractions, we will want students to be able to reduce a fraction like $\frac{15}{36}$ by automatically recognizing that "there is a 3 in both 15 and 36." A student who learns to go straight from numbers to factors by playing the factor game will likely have an easier time with this than a student who learns factoring only through factor trees.

As you begin to challenge your students with ever-larger numbers to factor, this is a great time to introduce tricks for recognizing divisibility.

A number is divisible by 2 when its last digit is even. That is, any number that ends in 0, 2, 4, 6, or 8 can be evenly divided by 2. This is because any number can be broken into tens and ones, and tens can always be divided by 2. For example, the number 754 can be viewed as 75 tens and 4 ones. Any number of tens can be divided by 2, and so it only matters that the last digit, representing ones, can also be divided by 2.

A number is divisible by 5 when its last digit is divisible by 5 (that is, the number ends in 0 or 5). This follows by the same reasoning as above – any number of tens can be divided by 5, so it is only the one's digit that determines whether a number is divisible by 5 or not.

These two numbers, 2 and 5, are two of the nicest numbers with which to multiply and divide. The only nicer number for multiplying and dividing is 10. This is because our number system is based on the number 10, and $10 = 2 \times 5$.

A number is divisible by 3 when the sum of all the digits is divisible by 3. For example, the digits of the number 7,925 add up to $7 + 9 + 2 + 5 = 23$. Because 23 cannot be evenly divided by 3, the number 7,925 cannot be evenly divided by 3. On the other hand, the number 84 can be divided evenly by 3 because its digit sum is $8 + 4 = 12$, which is evenly divisible by 3. The reason why this is true requires some advanced mathematics which we will omit here.

For the most part, these three tricks are enough to factor most numbers up into the hundreds. First, divide the number by 2 as many times as is possible (repeat until the last digit is no longer even). Next, divide by 5 as many times as possible. Next, divide by 3 as many times as possible. The result ought to be a reasonably small number which is either prime or the product of a few small prime numbers like 7 or 11.

Of course, if the prime numbers are large, factoring can be extremely difficult. This, incidentally, is the key to most modern secret codes – two large prime numbers are multiplied together and only those who know how to factor the result can read the secret message.

In any case, factoring is a very important skill for students to know. Start your class off with small numbers, then slowly challenge them to factor larger and larger numbers. These skills will come in very handy when working with fractions and algebra.

Questions:

(1) Identify each of the following as prime, composite, or the multiplicative identity:

(a) 7, (b) 18, (c) 25, (d) 27, (e) 1, (f) 37, (g) 52, (h) 17, (i) 87, (j) 61

(2) Factor each of the following numbers into a product of primes:

(a) 20, (b) 35, (c) 42, (d) 84, (e) 130, (f) 175, (g) 240, (h) 365, (i) 876

(3) Name the three different 6-digit numbers which begin with 51047 and are evenly divisible by 3 (your three answers will only differ in their last digit).

(4) Explain how you would rearrange factors in order to calculate $2 \times 2 \times 3 \times 5 \times 5 \times 11$ in your head.

(5) Show how to factor 120 using (a) a factor tree and (b) a factor cake.

Chapter 21: Multiplying and Dividing Factored Numbers

At this point, we have now come up with two different ways to represent numbers: with the base-ten numbering system and with products of factors. The first can be viewed as the "adding way to represent a number." For example, 270 means 200 + 70, a sum. When we factor 270, we get 2 × 5 × 3 × 3 × 3, a product. Thus, the factored form of a number can be viewed as the "multiplying way to represent a number."

When we add and subtract numbers, it works best to have the number represented in the addition format. For example, 572 + 261 is easily computed by adding the digits in each column and carrying where necessary:

$$\begin{array}{rccc} & {}^{1} & & \\ & 5 & 7 & 2 \\ + & 2 & 6 & 1 \\ \hline & 8 & 3 & 3 \end{array}$$

If we are given two numbers in factored format, for example 2 × 3 × 7 and 3 × 5 × 5 × 2, it would be very difficult to add them. The only reasonable way would be to multiply them all out into addition format (2 × 3 × 7 = 42 and 3 × 5 × 5 × 2 = 150) and then add them (to get 192).

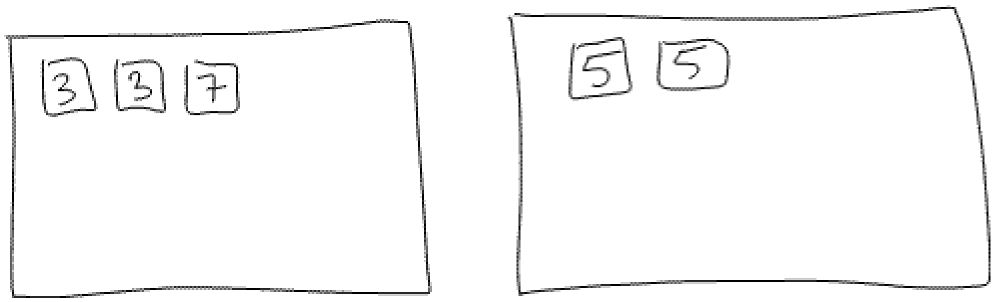
On the other hand, it can be very difficult to multiply two numbers which are represented in the addition format. If we want to multiply 362 = 300 + 60 + 2 by 74 = 70 + 4, we will need to perform 6 different multiplications (the 4 with the 2, the 4 with the 60, the 4 with the 300, the 70 with the 2, the 70 with the 60, and the 70 with the 300):

$$\begin{array}{rrrrr} & & 3 & 6 & 2 \\ \times & & & 7 & 4 \\ \hline & & & & 8 \\ & & 2 & 4 & 0 \\ & 1 & 2 & 0 & 0 \\ & & 1 & 4 & 0 \\ & 4 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ \hline 2 & 6, & 7 & 8 & 8 \end{array}
\qquad\qquad
\begin{array}{rrrrr} & {}^{4}\cancel{{}^{2}} & & {}^{1} & \\ & & 3 & 6 & 2 \\ & & \times & 7 & 4 \\ \hline & 1 & 4 & 4 & 8 \\ 2 & 5 & 3 & 4 & \\ \hline 2 & 6, & 7 & 8 & 8 \end{array}$$

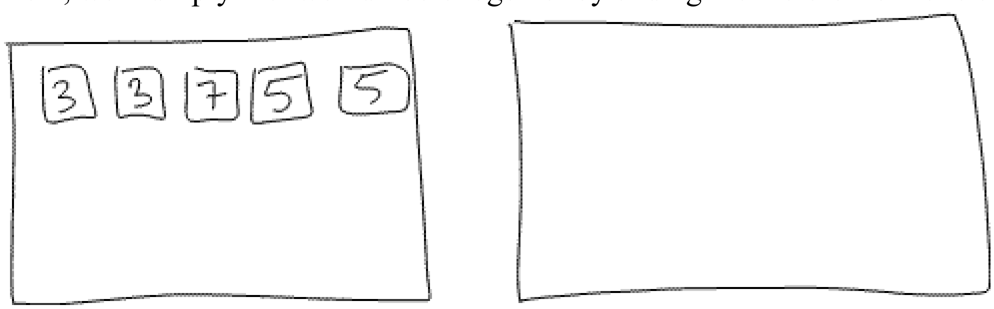
Both the "multiply everything" and traditional short-cut methods involve making all six of these multiplications. The short-cut, however, incorporates some of the addition as we go along.

Multiplying factored numbers, on the other hand, is incredibly easy. In order to multiply 2 × 3 × 7 and 3 × 5 × 5 × 2, we only need to multiply all of the factors together: 2 × 3 × 7 × 3 × 5 × 5 × 2. Of course, many people will then ask "what is this number?" because we are so familiar with the additive base-ten numbering system. However, if we want the factored form of the product, it is very easy: (2 × 3 × 7) × (3 × 5 × 5 × 2) = 2 × 3 × 7 × 3 × 5 × 5 × 2.

Because this is so easy, it can be a fun and short exercise for your students. Have them play the factor game with two different game boards (half-sheets of paper). Have them form a number on each board, and then multiply the two numbers together. For example, to find the factored form of 63 × 25, we first factor each of the numbers:



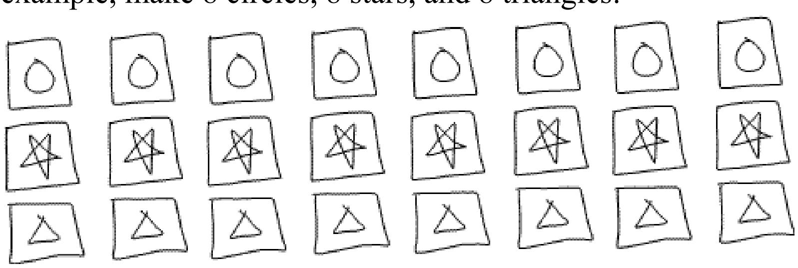
Next, we multiply the two numbers together by sliding the factors from one sheet onto the other:



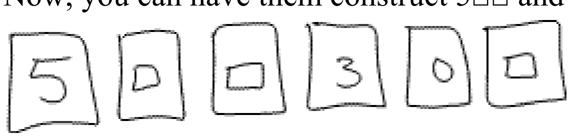
So whatever 63 × 25 may be, its factored form is 3 × 3 × 7 × 5 × 5.

Many students at this point will be uncomfortable leaving off with answers like this. They will have been taught that 3 × 3 × 7 × 5 × 5 is a *problem* and not an *answer*, and feel the need to multiply everything back together.

To overcome this very natural inclination, it will help to make the process more abstract. To do this, have each student make some cards with abstract symbols on them. Have them, for example, make 8 circles, 8 stars, and 8 triangles:



Now, you can have them construct 5□□ and 3○□, then multiply them together to get:



Yes, we can multiply the 5 × 3 to get 15, but the circle and squares will have to be left as factors.

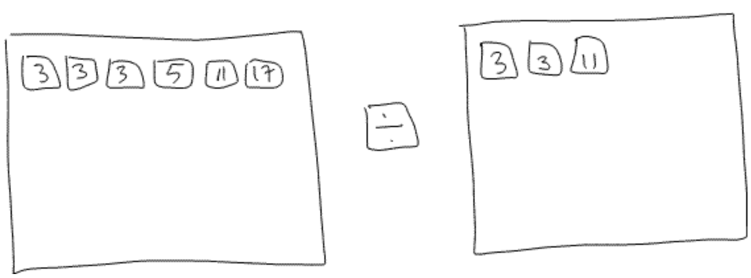
This would be an excellent time to introduce your students to the concept of exponents. If, for example, you wanted the factored form of 16 × 64, you would end up with 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2. There are so many 2's here that it is a bit hard to keep track of them all. Thus, we can put a little symbol up in the corner to indicate how many 2's we have. Thus $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^{10}$. There is no need to introduce the various rules for multiplying and dividing numbers with exponents – it is enough at this point for the students to know what these little numbers mean.

When your students are comfortable multiplying numbers in factored form, have them figure out what dividing must mean. To multiply a factored number by another factored number, we slide the new factors over to form one big product of factors. Division is the inverse, opposite, process. Thus, to divide by a factored number, we must take those factors away.

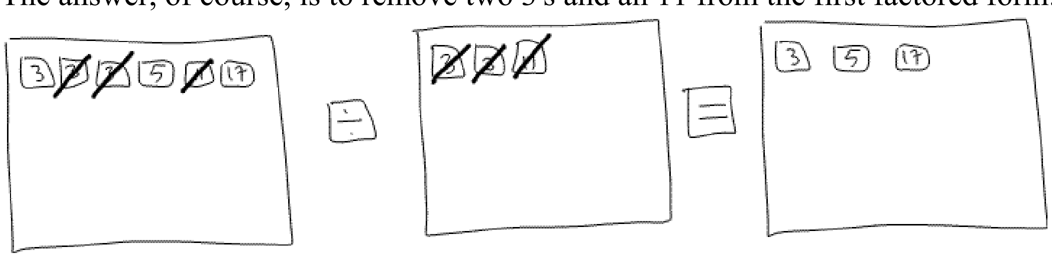
It might help at this point to refer back to easier problems, to remember the general pattern. Start with a basic multiplication problem like 6 × 7 = 42 and then have the students name the two division equations equivalent to this: 42 ÷ 6 = 7 and 42 ÷ 7 = 6. Next, introduce a multiplication problem in factored form, for example, (2 × 2 × 5) × (2 × □) = 2 × 2 × 5 × 2 × □. The two division problems equivalent to this are thus (2 × 2 × 5 × 2 × □) ÷ (2 × 2 × 5) = 2 × □ and (2 × 2 × 5 × 2 × □) ÷ (2 × □) = 2 × 2 × 5.

Once your students have arrived at (2 × 2 × 5 × 2 × □) ÷ (2 × □) = 2 × 2 × 5, have them dicsuss how they might figure this out. Hopefully, they will realize that, while multiplying by 2 × □ involves tacking on the additional factors 2 and □, dividing by 2 × □ will involve taking away these factors. If they do not get this right away, run through another example, and then gently prompt them toward this discovery. It could help to work through these problems with the prime number cards – have the students slide the 2 and □ over to multiply, and then slide them away to "un-multiply."

Next, have your students make up a division symbol and then work out various problems. For example, have them work out (3 × 3 × 3 × 5 × 11 × 17) ÷ (3 × 3 × 11) in the following manner:



The answer, of course, is to remove two 3's and an 11 from the first factored form:

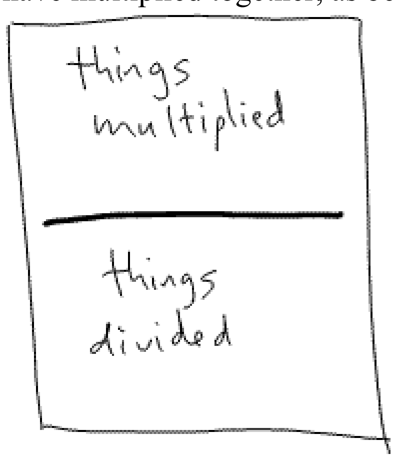


After your students do a few examples like this, show them how this gives the same answers as long division. For example, 50 ÷ 2, when put into factored form, becomes (2 × 5 × 5) ÷ 2 = 5 × 5, which is 25. Your students will hopefully be able to compute 50 ÷ 2 = 25 immediately in their heads, but it is nice to see how the factored form agrees with earlier results. Similarly, 72 ÷ 6 becomes (2 × 2 × 2 × 3 × 3) ÷ (2 × 3) = 2 × 2 × 3.

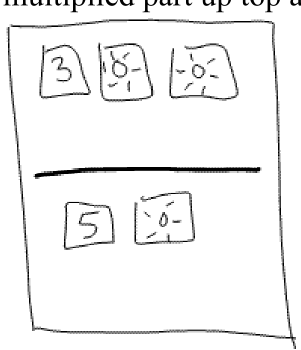
At this point, all of multiplication and division with factored numbers ought to be very easy for your students. To multiply, you combine all the factors together. To divide, you remove the divided factors. Have a discussion with your class to see if they can recognize the place where this process will run into trouble.

The problem, of course, comes in when we do not have all the necessary factors in the first number. We can divide (2 × ☼ × ☼ × 5) ÷ (5 × ☼) = 2 × ☼ just by taking away the 5 and the ☼. We cannot do the same thing with (3 × ☼ × ☼) ÷ (5 × ☼) because there is no 5 in the first number to remove. We can remove the ☼, at least, so (3 × ☼ × ☼) ÷ (5 × ☼) = (3 × ☼) ÷ 5.

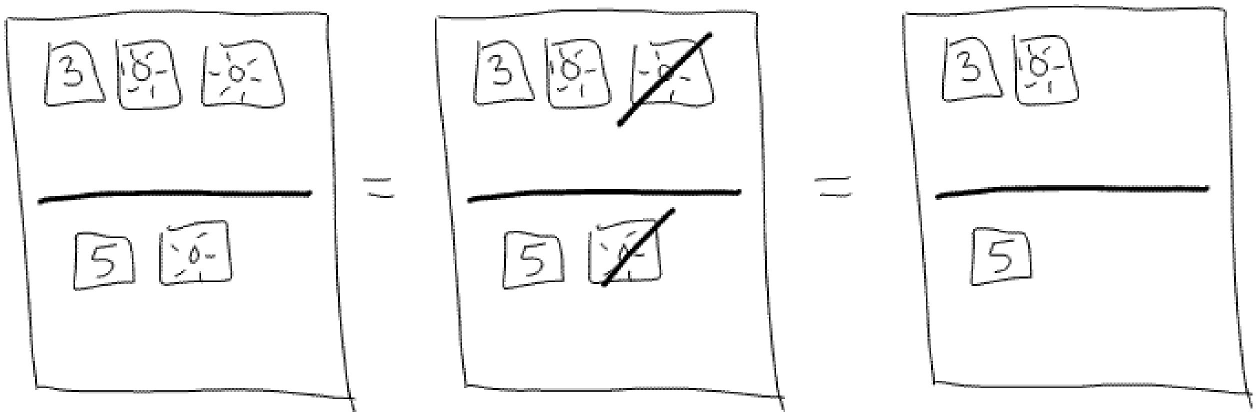
There is no way to solve the problem (3 × ☼) ÷ 5. All that we can do is remember that we need to remove a factor of 5, that our 3 × ☼ still needs to be divided by 5. In order to do this, we expand the game board. Rather than work with a half-sheet of paper, we work with a full sheet of paper that has a line drawn across the bottom. The top will contain all of the things we have multiplied together, as before, and the bottom will contain all the things we are dividing:



For example, in order to represent the problem (3 × ☼ × ☼) ÷ (5 × ☼), we put the multiplied part up top and the divided part down below:



We are able to divide by the sun, but not the 5, thus we we can only do the following:



This "game" (not too much fun and not at all competative, but it does involve moving pieces around on a board) is not very hard for students to learn. However, this single concept will be pretty much all that will be needed for your students to master all of the technical computations involved in working with fractions. Rather than try to explain what fractions are first and then teaching all the computations second, you will find that the computations can be taught very easily as an abstract game. This will be discussed at length in the next chapter.

Questions:

(1) Illustrate how the multiplication of (2 × 2 × 5) × (2 × 3 × 7) will look when performed with factor cards.

(2) Illustrate how the division of (3 × 3 × 5 × 5 × 5 × 11) ÷ (3 × 5 × 3) will look when performed with factor cards.

(3) Illustrate how the division of (2 × □ × □ × ☼) ÷ (3 × □ × □ × □) will look when performed with factor cards.

(4) Show how to divide 20☼ ÷ 4□ using factor cards.

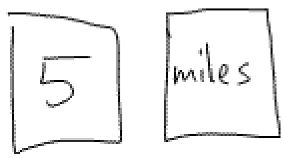
Chapter 22: Multiplying and Dividing with Units

When students have had some experience with factoring numbers, it is very useful to revisit the concept of units. This is because units are factors.

So far, we have talked about how units tell what a number actually means. There is a huge difference, for example, between 5 feet and 5 miles. These are both the number 5, but one of them if far longer than the other.

The only lesson we have taught about units is that numbers can only be added and subtracted if they have the same units. We can add 5 feet + 7 feet = 12 feet, for example, but we cannot add 5 feet + 5 miles and get 10 of anything. If the units are different, we merely leave them as separate things added together, like 5 feet + 5 miles. Later on we will have students convert miles into feet so that these can be added, but for the time being we will consider this as intractable as 5 feet + 5 oranges.

Now, we can explain to students that units are actually just factors. When you have something like 5 miles, this actually means 5 × miles, which would be represented with symbol cards as:

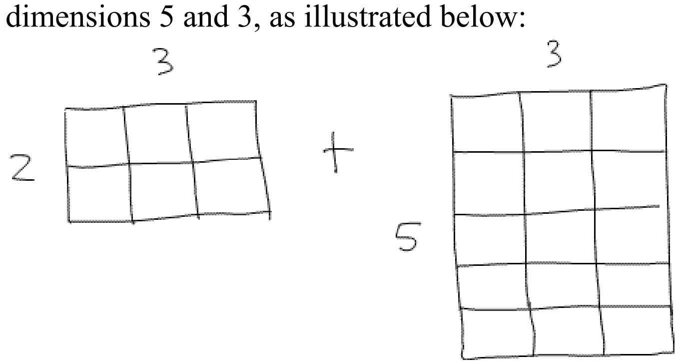


To walk 5 miles is to walk the distance of 1 mile 5 times in a row. It is for this reason that this represents a multiplication. We say "five miles" instead of "five times miles" merely because everyone already knows what we are talking about.

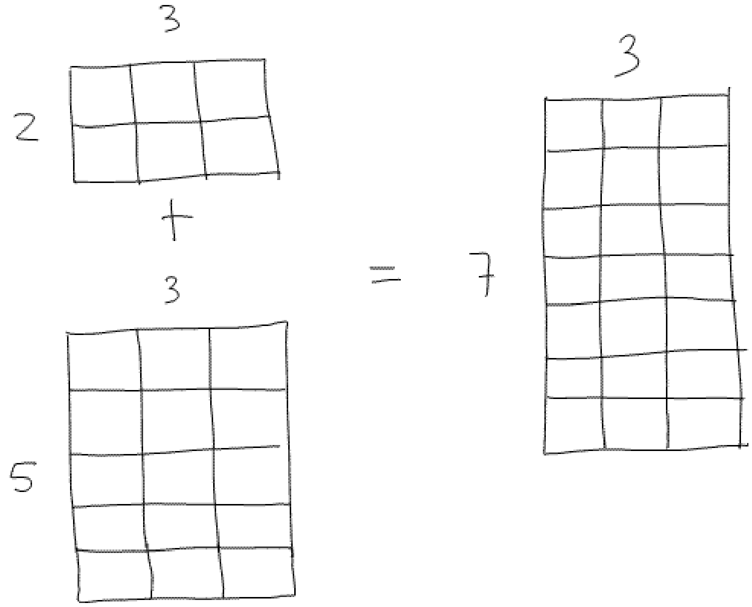
If 5 miles really meant 5 + miles, then 5 feet would mean 5 + feet and we would be able to add 5 miles + 5 feet = 5 + miles + 5 + feet = 10 + miles + feet, which would be ridiculous.

If 5 miles really meant something other than multiplication between the 5 and the miles, then it is curious that all of the mathematics with units works exactly as if these two things were multiplied together.

For example, take a situation like 2 × 3 + 5 × 3. Using the area model of multiplication, we can represent these two multiplications with rectangles. The 2 × 3 represents the area of a rectangle with dimensions 2 and 3, and the 5 × 3 represents the area of a rectangle with dimensions 5 and 3, as illustrated below:



We can slide these rectangles together to form one big rectangle with the combined area. This big rectangle will have a height of 7 and a width of 3:

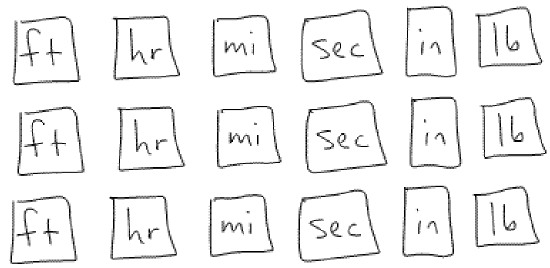


We conclude that 2 × 3 + 5 × 3 = 7 × 3. This is just an illustration of the inverse of the distributive law, that 2 × 3 + 5 × 3 = (2 + 5) × 3. We call this "factoring out a 3." Things will always work this way when the second factor of the two products is the same – because the second factor is 3, we can stack the two rectangles on top of one another and make a bigger rectangle. This will also work when the second factor is not a number, like 3, but a unit, like feet. If we take 2 feet + 5 feet, we can look at this as 2 × feet + 5 × feet, use the distributive property to "factor out the feet" to get (2 + 5) × feet, and end up with 7 × feet = 7 feet.

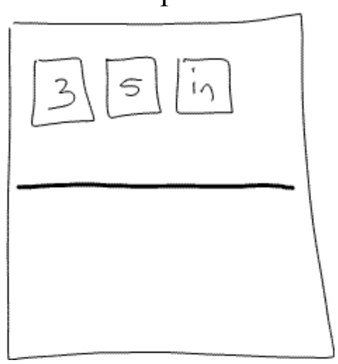
It is not necessary to explain this long and complicated process to young students. It is enough, as was done earlier, to explain that 2 feet + 5 feet = 7 feet can be added because the "units are the same." However, this sort of reasoning will be necessary later on in order to make algebra understandable. Right now, it is enough to realize that units can be treated as though they were factors multiplied by the preceeding numbers.

The main difference between 2 × 4 and 2 × feet is that we can simplify the first expression into 2 × 4 = 8, but we cannot do anything with 2 × feet except write it without the multiplication symbol: 2 × feet = 2 feet. It is for this reason that we have prepared students by using factor cards with both prime numbers and abstract symbols:

At this point, we should introduce factor cards with units on them. Have each student make several cards with the abbreviations for common units: feet, miles, seconds, pounds, inches, etc.

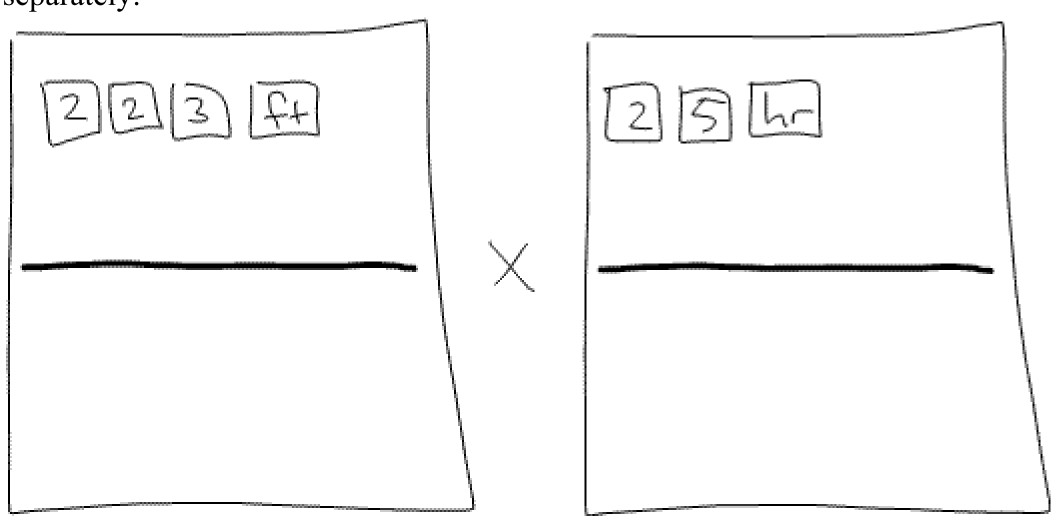


Now, have students factor numbers that have units. For example, to factor 15 inches, a student will put the following symbols on the factor board:

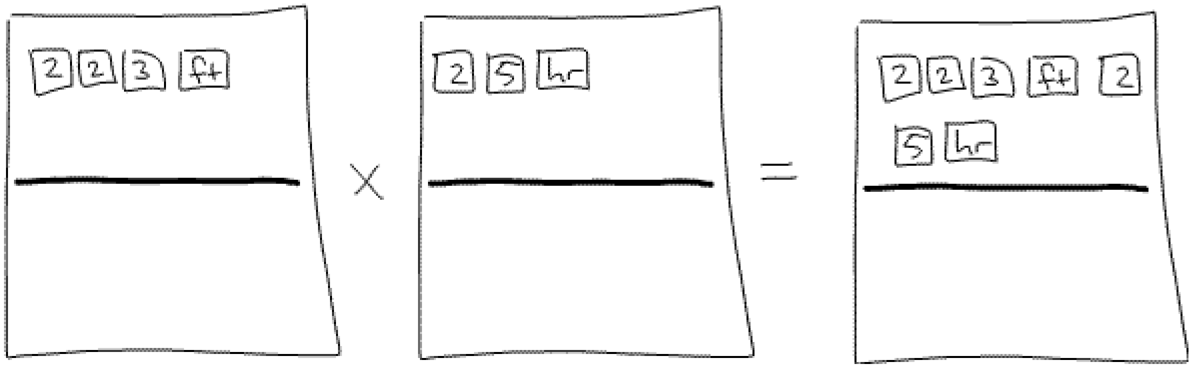


We are able to break up the 15 into 3 × 5. There is no way to break up the inches, so we use a factor card with the abbreviation for inches on it.

After a few examples like the last one, have your students try to factor and then multiply into factored form. For example, factor and multiply 12 feet × 10 hours. First, we factor these separately:

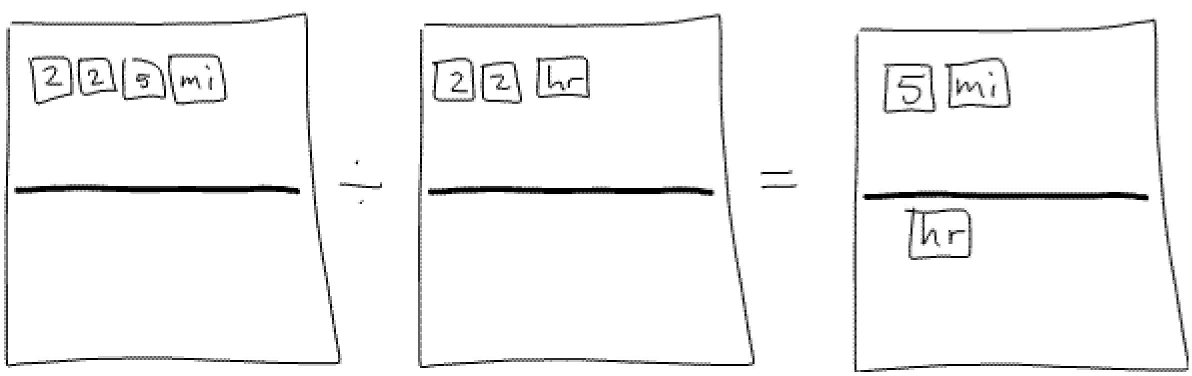


Next just as with the multiplication of any two numbers in factored form, we just bring all the factors together:



In other words, 12 feet × 10 hours = 2 × 2 × 3 × ft × 2 × 5 × hr, in factored form. We could , if we wanted multiply all the numbers back together, but not the units. The end result will be 120 × feet × hours, which can also be written 120 feet-hours. We say that the units of this number are "feet hours" (or perhaps "foot-hours"). These are certainly funny units which do not come up in everyday life very often. However, if we ever multiply a number of feet by a number of hours, feet-hours is what we will end up with.

Similarly, we can divide something like 20 miles ÷ 4 hours and simplify the result as follows:



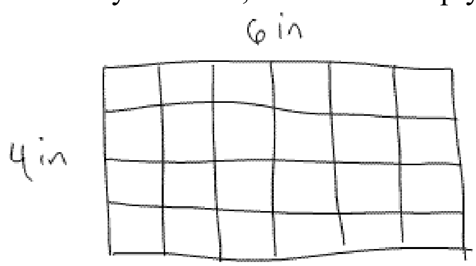
The 20 miles factors into 2 × 2 × 5 × miles and the 4 hours factors into 2 × 2 × hours. In order to divide, we have to take the second set of factors away from the first set. We can easily take away both of the 2's. However, there is no "hours" in the first factored number for us to remove. Thus, we put the "hours" down below in the still-needs-to-be-divided section. The end result is 5 × miles with hours yet-to-be-divided. The number of our result is 5. The units of our result is "miles divided by hours" which can also be pronounced "miles per hour."

That last problem could be related to a real-life word problem, for example: "How fast did you travel if you went 20 miles in 4 hours?" However, word problems are generally quite difficult for students. Instead, we will put the real-life aspects off until later and have the students approach these problems abstractly. This problem is absolutely no different from the problem 20☼ ÷ 4□ given at the end of the last section, except that ☼ is pronounced "miles" and □ is pronounced "hours." Because these computations are so much easier to imagine abstractly, we will have the students master them abstractly before introducing them in word problems and real-life situations.

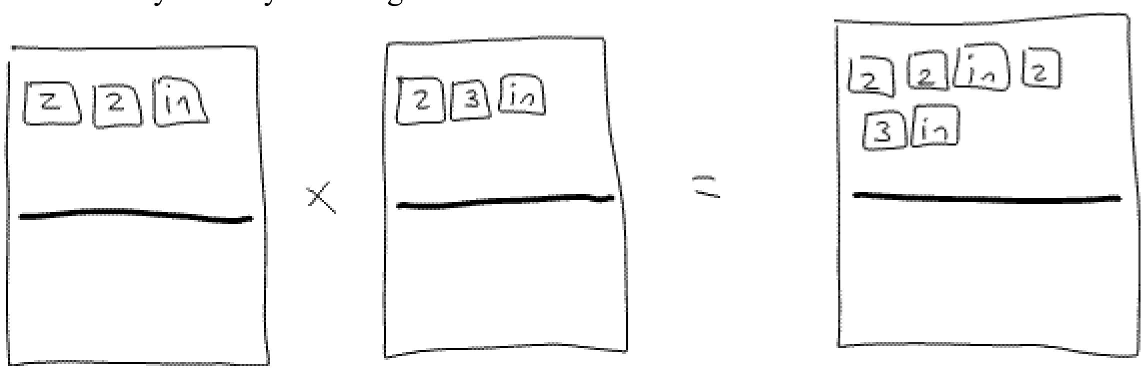
In many ways, this is like the story of "The Karate Kid." For a long time, the boy who wanted to learn karate was told instead to wash cars, paint houses, polish floors, and do other menial, repetative tasks in a prescribed manner. Eventually, he grew upset at this treatment and insisted on being taught karate. The master instructor then showed the boy how these repetitive gestures were actually karate moves, and that the hours of work had developed the strength to use them.

Here, we will teach children to perform very simple tasks with small slips of paper. When they have mastered these skills, we will then show them how easily and immediately they can be put to use in working with fractions and solving word problems. Later on, the same skills will make algebra easy. The key is to teach the children how think abstractly, to treat prime factors and units as "things" no different from stars and rectangles.

At this point, we can explain the units of areas. To find the area of a rectangle which is 4 inches by 6 inches, we must multiply these two lengths together: 4 inches × 6 inches.



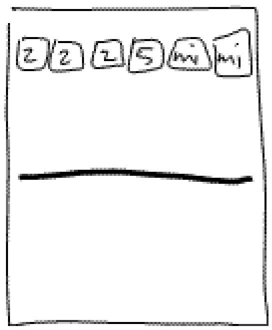
This is easily done by factoring:



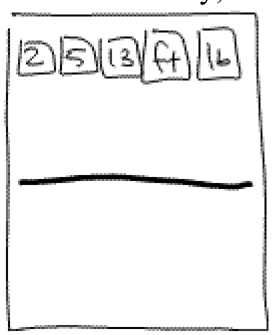
The answer is 2 × 2 × inches × 2 × 3 × inches. If we multiply as much of this back together as possible (all the numbers), we get 24 × inches × inches = 24 inches × inches. One way to avoid having to write "inches" twice is to use the exponents briefly explained earlier. We can either write 24 inches^2 or, to be even more abbreviated, 24 in^2. Another way to describe this is to remind the students that multiplying something by itself is called *squaring* it. Thus inches × inches = *square inches*. A final way to look at this is to go back to the rectangle and see that it is made of 24 squares, each one with all sides 1 inch long. Thus, 4 inches × 6 inches = 24 square inches = 24 in^2.

Ideally, your students will realize that this whole process can be greatly simplified by just multiplying the numbers together and then combining the units. For example, 15 feet × 4 seconds = 60 feet-seconds. There really is no need to factor out the 15 and the 4, so long as you know to treat the feet and seconds as factors.

Have your students work the opposite of this process by factoring numbers with more complicated units. Have them, for example, factor 40 square miles into:



Similarly, challenge your students to factor 130 foot-pounds into:



Hopefully, your students will catch on to treating units as factors and find these sorts of problems no more difficult than any other sort of factoring.

Questions:

(1) Illustrate how each of the following would look when factored out with factor cards:

(a) 30 feet

(b) 27 hours

(c) 144 mile-seconds

(d) 50 square inches

(2) Show how to multiply out 14 feet × 6 pounds using factor cards.

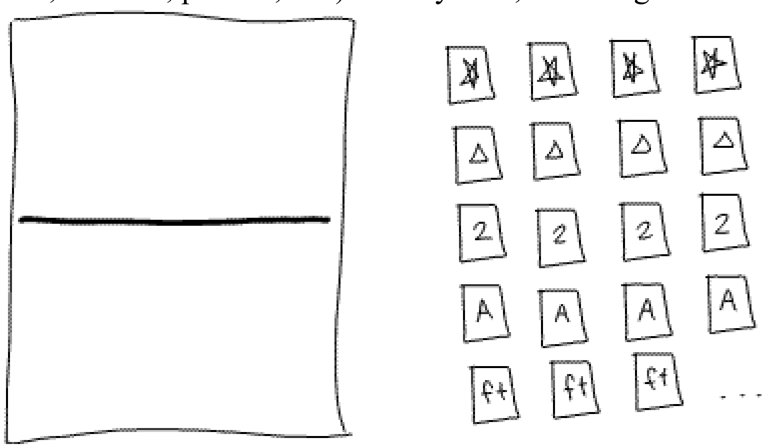
(3) Show how to divide 60 pounds ÷ 40 foot-pounds using factor cards.

(4) Use factoring to show why 8 seconds + 5 seconds = 13 seconds.

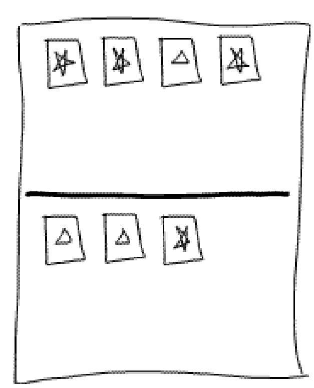
Chapter 23: The Symbols Game

The symbols game is an exercise which abstractly teaches children the rules of fractions. With the skills taught by this game and the ability to factor numbers, children will be able to add, subtract, multiply, divide, reduce, and compare fractions. However, this game is so simple and easy to learn that it can be taught to children well before they learn how to add or multiply. The only disadvantage is that the game is too simple to be much fun – children will likely master the game and bore of it in a very short period of time.

The game board consists of a piece of paper with a dark line drawn across the middle. The game pieces consist of small rectangles with symbols drawn on them. At first, the rectangles will have basic shapes and symbols on them – stars, triangles, smiley faces, etc. Later, we will use prime numbers (2, 3, 5, 7, etc.), variables (*x*, *y*, *A*, *B*, etc.), and units (miles, feet, seconds, pounds, etc.). In any case, there ought to be 8 to 10 copies of each symbol:

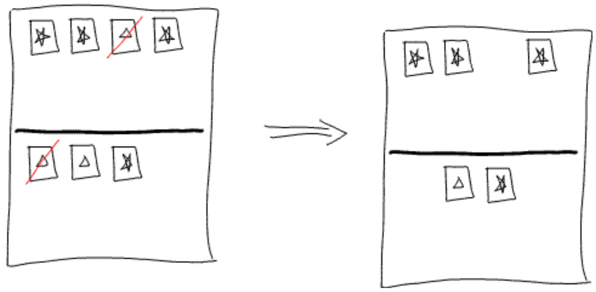


The choice of symbols is not very important. However, by starting with stars and triangles, we will teach the students to view them as abstract symbols, as *things*, which will help them to think abstractly. If you begin with numbers and letters, the children will likely try to follow rules they have already been taught to associate with these objects. The whole point of the symbols game is that it has very few rules and can be mastered quickly and easily.



To begin the game, some of the symbols will be put into the put into the top half of the game board and other symbols will be put into the bottom half. You will illustrate this starting position up on the board for all the students to copy. For example, a beginning board might look like the figure to the right.

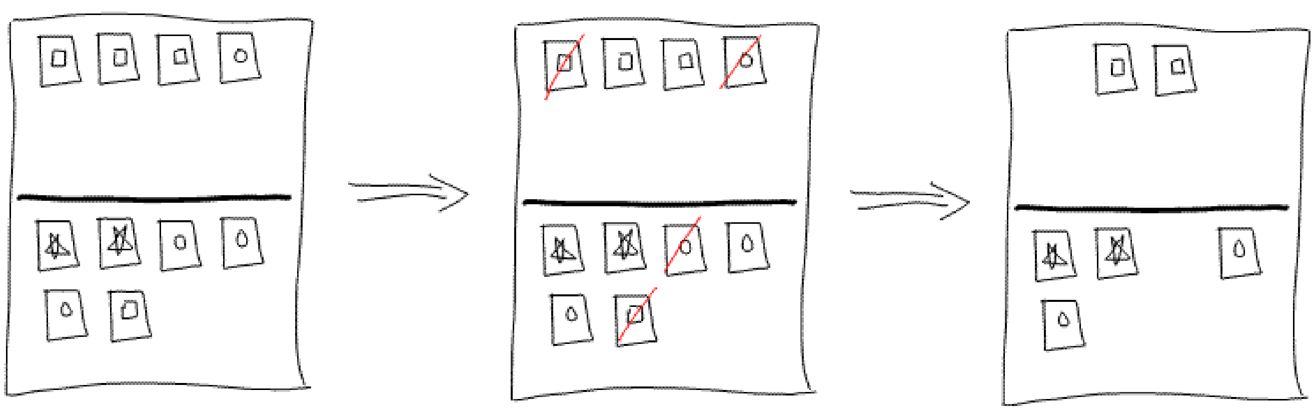
The main rule for the game is this: if the same symbol appears in both the top and bottom of the board, then both of them may be removed. In our current example, for instance, there is a triangle both in the top and bottom of the board. We may remove this pair of symbols:



Notice that we cannot remove the second triangle from the bottom because there is no corresponding triangle up top to remove with it. We could, if we wish, take away a star from both the top and the bottom.

The order in which the symbols appear in either the top or the bottom is not important: you can rearrange and reorder the symbols in the top and the symbols in the bottom. It is not allowed, however, to move a symbol from the top to the bottom or from the bottom to the top.

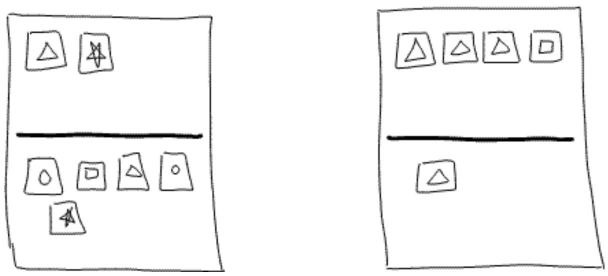
The first game to play with these symbols is called "take as many pieces away as possible." The teacher tells all the students how to set up the starting board (either draw it on the blackboard, or else just tell the students verbally to "put three squares in the top..." etc. Next, the students try to remove as many pieces as possible, always in identical pairs with one symbol coming from the top and the other from the bottom. For example, a game might be played as illustrated:



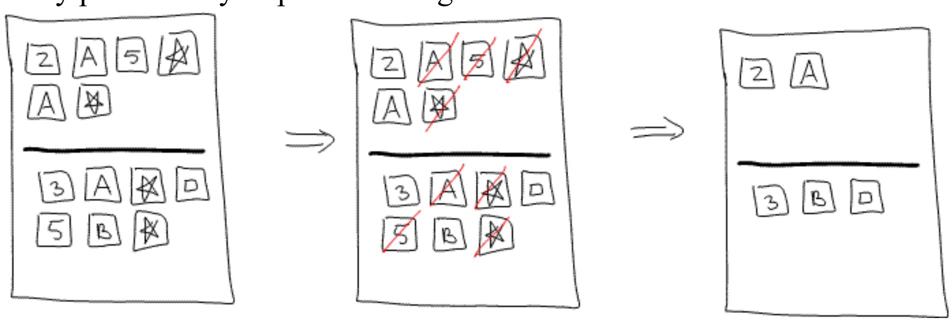
The starting board began with three squares and a circle up top. In the bottom, there were two stars, three circles, and a square. To win the game, a student needs to remove a pair of squares, one from the top and one from the bottom, and a pair of circles in the same fashion. The winning board thus has two squares up top and down below: two stars and two circles.

Hopefully, this game will become boringly easy in a short period of time. When your students have caught on, give them a board with a large number of pieces in both the top and the bottom. It should be clear that this does not make the game much more difficult. In general, it will take longer to set up the board than to play!

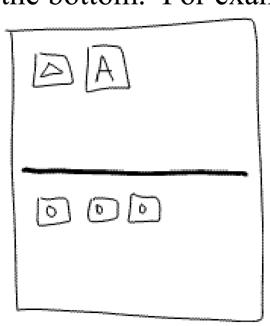
Before you move on, make sure to introduce two special scenarios: one in which the entire top will be taken away, and one in which the entire bottom will go away. Such starting boards could look like:



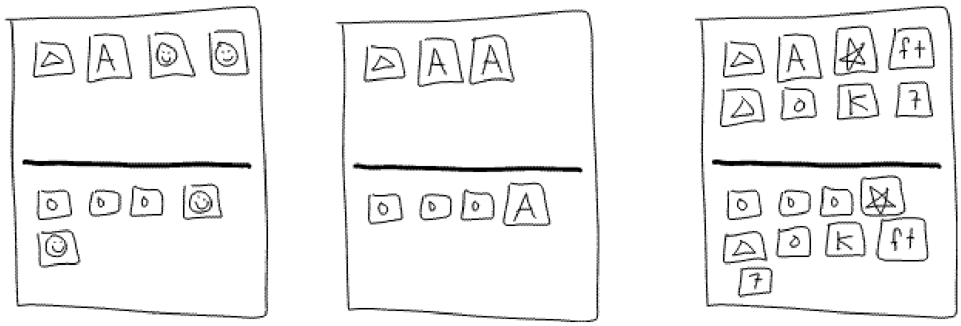
At this point, you can begin to introduce some numbers and letters. Ideally, your students will treat these as abstractly as they did the symbols. For example, a round of playing "take as many pieces away as possible" might look like:



When this has become too easy, introduce the inverse, opposite operation to the one move so far in the game. The opposite of "take away an identical pair, one from the top and one from the bottom," is "add in an identical pair, one to the top and one to the bottom." The only thing potentially curious about this new rule is that the children have the freedom to add any symbols they desire, so long as they are both the same and one goes to the top and one goes to the bottom. For example, suppose you give the students the following starting board:



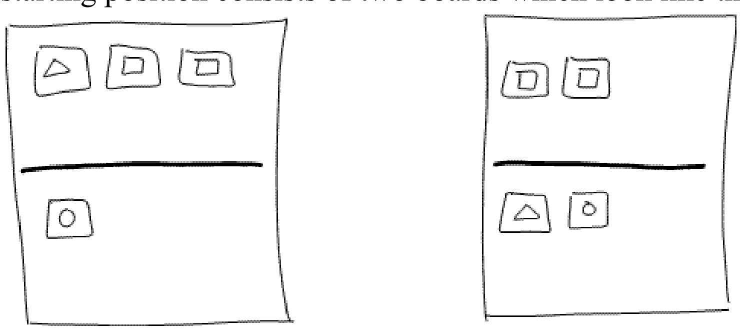
Next, tell them to do whatever they like with the board, so long as they follow the two rules. There will be many possible boards that the students might end up with. Some examples are:



After students have added and subtracted whatever they wish, you can have the students swap boards (or, if this causes too many pieces of paper to slide and fall off the boards, have the students temporarily swap seats). Have the students then play "take away as many pieces as possible" with the various boards. Everyone should end up with the same board. This is an excellent time to talk about how inverse processes undo each other.

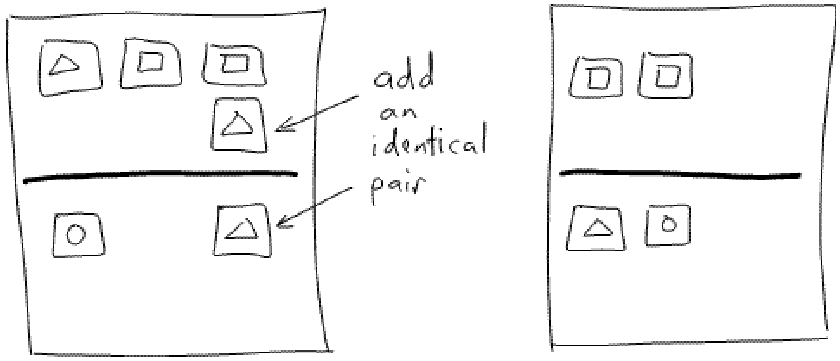
When students have grown comfortable with the rules for adding and removing identical pairs of symbols, they are ready for the next step. This next game requires each student to have two separate game boards. The rules will apply to each board separately. For example, a student may add or subtract a pairs from the left board without having to worry about the right board. Similarly, anything done to the right board will not affect the left board.

The name of this next game is "make the bottoms the same." For example, suppose the starting position consists of two boards which look like the following:



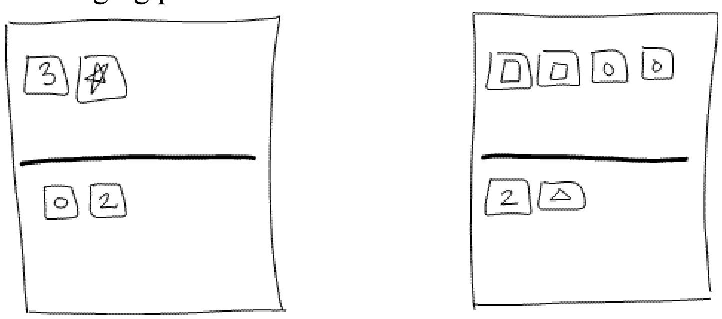
There are no cancellations possible (we will call it a *cancellation* when we take away a pair of symbols) in either of these two boards. There is a triangle in the top of the left board and one in the bottom of the right board, but we cannot remove these. We can only perform operations (adding and subtracting pairs of symbols can be called *operations*) to one board at a time. Because no board has the same symbol in its top and bottom, we cannot remove anything.

The challenge before the students, now, is to "make the bottoms the same." The first board has a circle in the bottom. The second board has both a triangle and a circle in the bottom. These bottoms are not the same. Because we cannot remove any pieces, the only thing we can do is add more pieces, using the "add the same symbol to the top and bottom" rule. As the whole class ponders this problem, hopefully someone will flash onto the answer: we will need to add a triangle to the top and bottom of the first board. When this is done, the bottoms will both contain a circle and a triangle, and thus will be the same. Remember, the order isn't important:

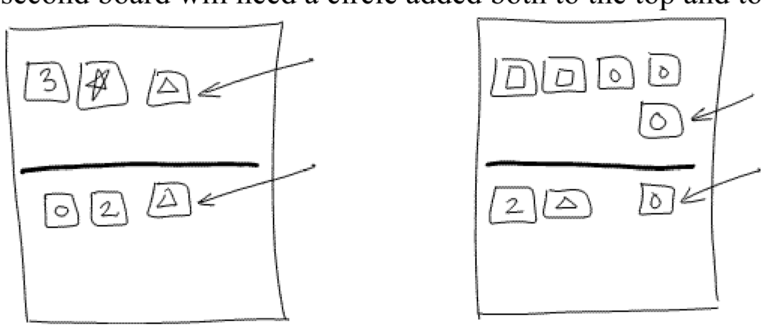


Also remember that we are not allowed to just add a single triangle to the bottom – we are only allowed to add symbols in pairs, one to the top and one to the bottom.

Once the whole class has understood this answer, give them all a slightly more challenging problem for each student to work individually, something like:



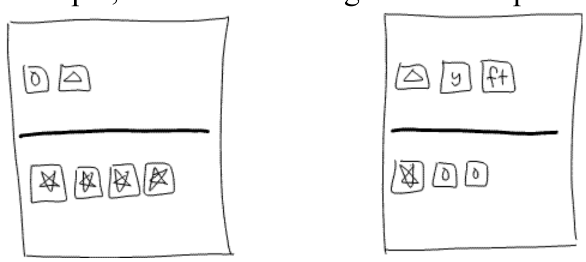
Recall that the thinking required for this game is identical to the equality game described back in chapter 4. Back then, we would give one child a circle and a 2, and another child a 2 and a triangle. We would then ask "what more do we have to give each child in order for everything to be fair?" It was not difficult then for the student to realize that the first child needed a triangle and the second child needed a circle. The only thing different now is that, in order to give the first bottom a triangle, we need to put a matching on in the top of that board. Similarly, the second board will need a circle added both to the top and to the bottom:



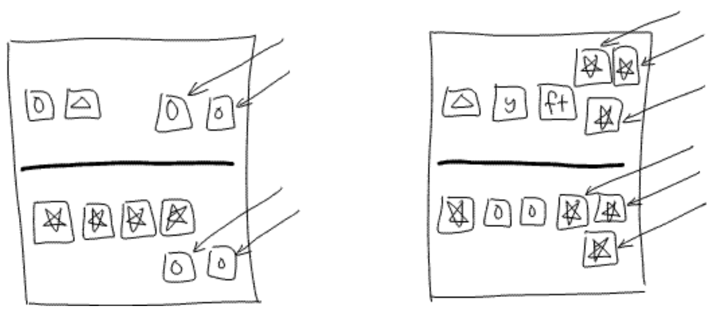
We have now completed the challenge because the bottoms are the same (a circle, a 2, and a triangle in each).

Some students will feel as though it is necessary to add four of a symbol at a time, one to the top and bottom of both boards. Explain to them that this is not necessary, that each board can be dealt with individually. Other students will want to add symbols only to the bottoms, as with the old equality game. Explain to these that all the rules still apply, that pieces must be added and removed in pairs.

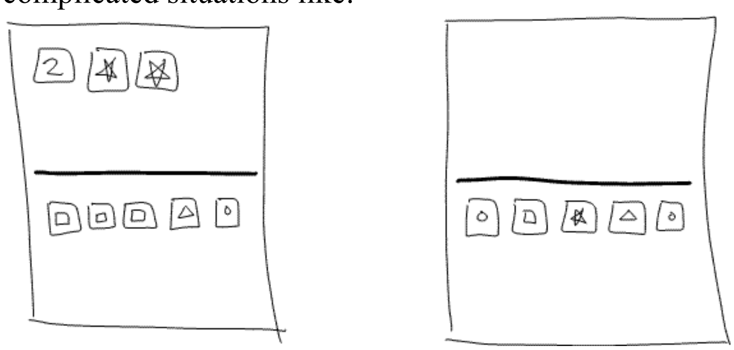
Another way to make the game trickier is to require multiple symbols to be added. For example, take the following initial set-up:



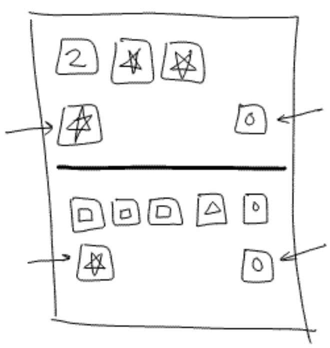
The bottoms are certainly different. The one on the left has 4 stars and the one on the right has only 1. Thus, we will need to add three stars to the right board. Similarly, we will need to add two circles to the left board:

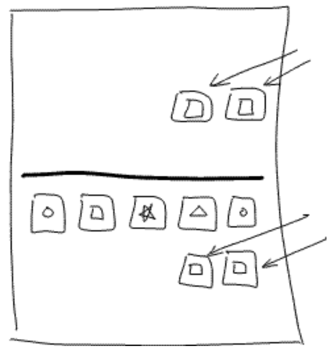


Ideally, your students will quickly catch on, especially if they have previous experience with the equality game. Even very young children ought to be able to pick up the game without too much difficulty. Soon, they ought to be able to "make the bottoms the same," even for very complicated situations like:



The fact that the second board has nothing in the top might bother some students, but it ought not (similarly, you could introduce a board with nothing in the bottom). The first board will need a star and another circle. The second board will need two squares:

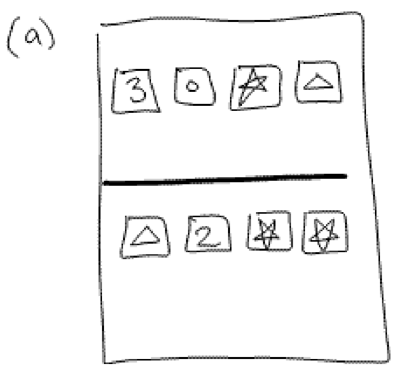


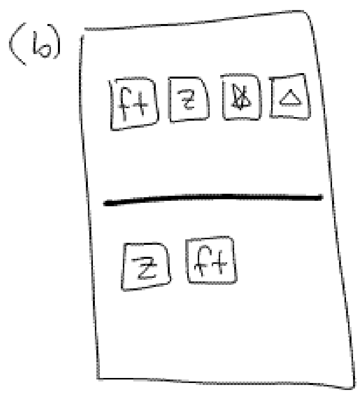


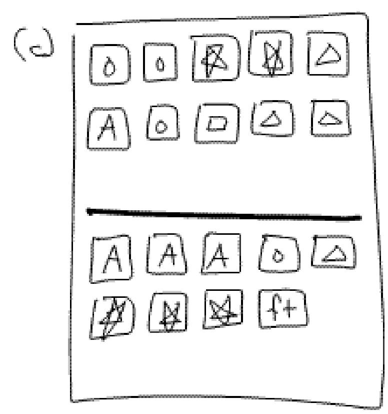
Hopefully, your students will soon tire of this game because it is too easy and does not offer any challenge. This is excellent, for it means that your students have mastered the skills necessary to work with fractions. The "take as many pieces away as possible" game is exactly what is necessary to reduce a fraction whose numerator and denominator have been factored. The "make the bottoms the same" game is exactly what is necessary to find a common denominator for adding, subtracting, and comparing fractions whose numerators and denominators have been factored. This will be discussed in greater detail in the chapters to come.

Questions:

(1) Show the end result of playing "take as many pieces away as possible" with the following initial boards:

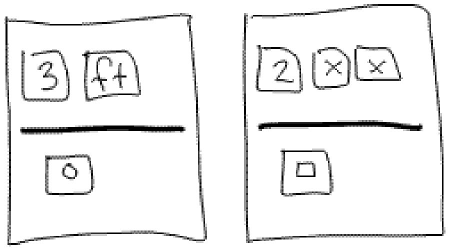




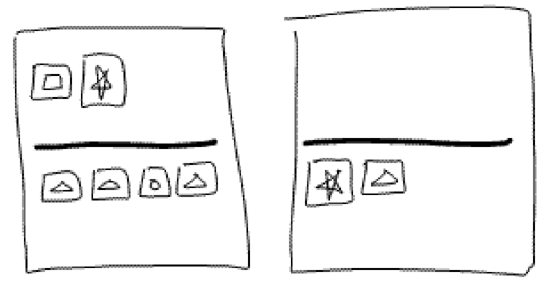


(2) Play the "make the bottoms the same" game with each of the following sets of initial boards. Use arrows to indicate the pieces which must be added.

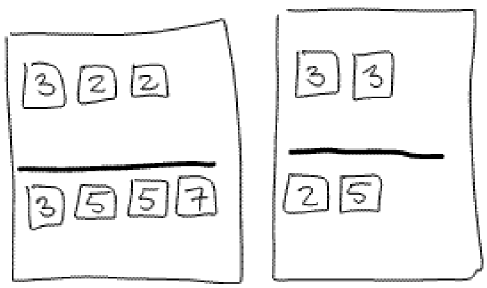
(a)



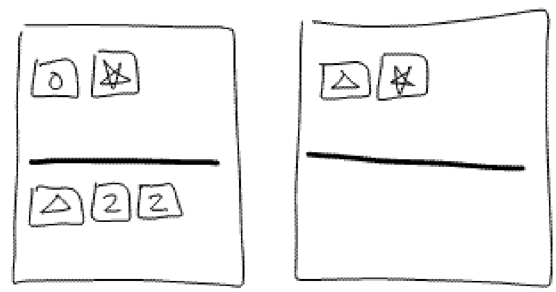
(b)



(c)



(d)



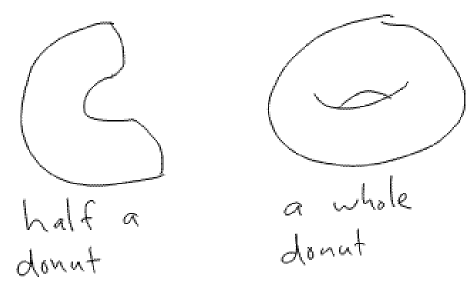
Chapter 24: Fractions

As soon as we introduce the concept of division, it is only a matter of time before we must explain fractions. We can say, for example, that 20 ÷ 3 is "6 with a remainder of 2," for when 20 marbles are shared among 3 children we can give 6 marbles to each child and then put aside the 2 left over. However, it does not seem nearly as reasonable to say that 3 ÷ 5 is "0 with 3 left over." If you have 3 cookies which must be shared among 5 children, it is just not fair to give nothing to each of the children and then put all three cookies aside.

The origins of the word *fraction* are similar to the word *fracture* – both come from a Latin word meaning "to break" (into pieces). This, of course, is the only solution for sharing 3 cookies among 5 children – the cookies must be broken into pieces!

Children often have a lot of difficulty with fractions, and so we should introduce the concept gradually. The computational skills necessary to add, subtract, multiply, divide, reduce, and compare fractions have already been mastered by children who know how to factor and how to play the symbols game. However, as with everything, we want the children to understand what is happening from the concepts, and not just follow set procedures.

The first fraction to introduce is the concept of "half." Have your students give examples of halves. They might suggest half a donut, half a sandwich, half an hour, a half-day, to walk half-way to a destination, or similar things. When you have written out a number of these examples, go back and discuss each one individually. Ask them to explain what a whole would be. For example, with a half-donut, have them talk about what a whole donut would be. For halves like this (which are physical objects), you could have the class draw a half and a whole. Another good question is to ask "which is bigger: a half or a whole?" Hopefully, this will be an easy question. Finally, ask the class "how many halves make a whole?" Hopefully, they will all realize that two halves make a whole.



There are three parts to the concept of a fraction:

(1) that the fraction is relative to a whole,
(2) that the whole has been broken into parts, and
(3) that all the parts are the same size.

It is to emphacize the importance of the whole what we ask children who mention "half a donut" to then illustrate what "a whole donut" will look like in comparison."

As a first exercise, divide your class up into small groups and give each group a bag with 60 identical objects in it. Poker chips will work nicely, but so will anything else that won't roll off the tables (like marbles), get stolen (like coins), or get stuck together (like paperclips). Have each group divide the pile into halves. This activity might appear identical to earlier exercises where the students acted out division with piles of objects. The main difference here will be the language used. We will not refer to "one" as "one chip" but as "one bag of chips." Thus, when the students have divided the bag out into two piles of equal size, we will call each of these "half a bag of chips" and not "thirty chips."

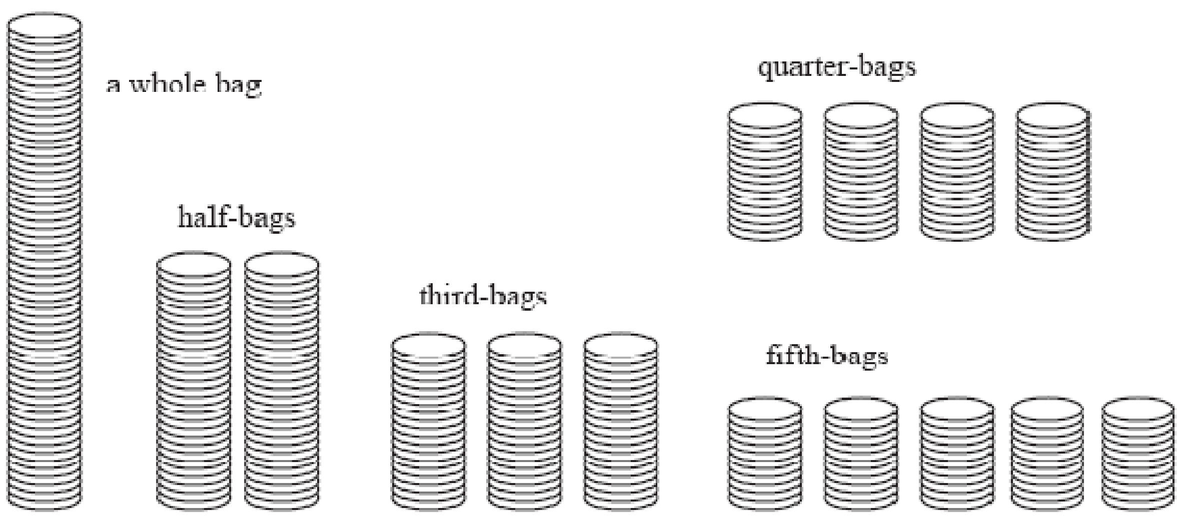
If your students find this uncomfortable, offer them the following example. A foot consists of 12 inches. Half a foot is 6 inches. Thus, the same length could be described as either "a half" (of a foot) or "six" (inches). Similarly, half an hour could also be described as thirty minutes.

Have one of the groups of students present their "half bags" up at the front of the room, where they should be displayed prominently beside a full bag (all sixty chips from a bag in one pile). These students should split up and join other groups.

Each group should now recombine their chips back into the bag. Next, they should divide their bag up into thirds. Explain to the class that a third of a bag is the amount that each person will get when it is divided evenly into three piles. Again, this exercise is not different from what your class has done before, except that the language is different. When everyone has made three piles of 20 chips, have one of these be presented up at the front of the room with the other piles. These students should then split up to join other groups.

Next, have the students repeat the process by dividing the chips from each bag into four equal piles. These piles will be called "fourths of a bag." Again, one of the examples will be transferred up to the front of the room.

Have the students do the same thing to make "fifths of a bag" and "sixths of a bag." When all is done, the result might look something like this:



The piles need not be as neat as these. The whole point to emphasize is that a "third" is what you get when you divide a whole evenly into three parts, a "fourth" is something you get when you divide a whole evenly into four parts, a "fifth" is something you get when you divide something evenly into five parts, etc.

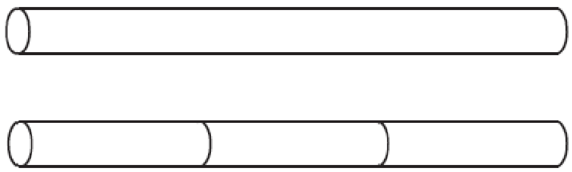
An important thing to point out now is how halfs, thirds, fourths, and fifths all compare with one another. The smallest is the fifth, then the fourth, then the third, and then the half. The whole is largest of all. Children often get confused when comparing fractions like $\frac{1}{3}$ and $\frac{1}{5}$ and think that a fifth must be larger than a third because 5 > 3. It is not necessary to introduce fraction notation until we can explain what it means. However, it is enough at the moment for students to understand that sharing something among more people means that each will get less. Show the students a big bag of small objects (bigger than the ones they are working with) and ask which would be bigger – a tenth of the bag or a fourth of the bag. Have them discuss this as a class. Ideally, at least one student will reason that a tenth must be the result of dividing the contents into 10 equal-sized piles. Hopefully, they will all realize that sharing the contents among 10 people will result in smaller piles than sharing them among 4. One useful anecdote to use here is the birthday cake analogy – more kids at a party means less cake for each kid (but more presents for the birthday child).

When your students are comfortable with dividing a pile of things into halves, thirds, fourths, and such, they are ready for the next exercise. This will require some advanced preparation. You will need a large number of objects cut to various lengths. You can let the children work in small groups, but each group will need a set. The best objects to use would be wooden dowels, but this can be expensive. Teriyaki sticks are much cheaper and easier to cut (a pair of wire cutters will do the job), but can contain splinters. Pipe cleaners (also known as "fuzzy sticks" and "chenille stems") can work, but won't survive much abuse at the hands of students. Drinking straws are both cheap and easy to cut with scissors, so these might be your best bet. A nice set made out of wooden dowels, however, ought to last through many years of teaching.

To make a set, you want to cut your straws/dowels/rods to a variety of lengths. The longest will be considered a whole, then there will be rods with half that length, then thirds, fourths, fifths, sixths, and eights. If the longest one is cut to be 24cm, then the halves will be 12cm, the thirds 8cm, the fourths 6cm, the fifths 48mm, the sixths 4cm, and the eighths 3cm. You could also start with a whole length of a foot, making the halves six inches, the thirds four inches, the sixths two inches, and the eighths $1\frac{1}{2}$ inches. The fifths would be a bit trickier to measure, as they would be $2\frac{2}{5}$ inches each, which is hard to measure on a standard ruler. There should be enough of each length to make a whole (three thirds, eight eighths, etc.), but there should be a few more of each in addition, just to make things more challenging.

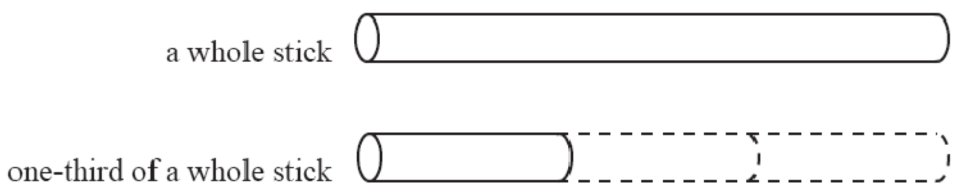
In any case, for the exercise, put the children into small groups and give each one a set of sticks. First, have them sort the sticks by length (put all the sticks of the same length into one pile). If you want to begin with an extra-easy version, you can start them out with sets with only a few different sizes, say wholes, halves, thirds, and fourths. Next, have the children sort the piles by size, putting the pile with the longest sticks on the left, then the pile with the next longest sticks beside it, and so on.

Next, have the children take some of the sticks from a pile and put them end-to-end and then compare the result to the longest stick, called the whole stick. Have them figure out how many sticks it takes to be as long a whole stick. For example, with the third-sticks, this should look like:



Now have the class decide if the sticks they are using are half sticks, third sticks, fourth sticks, fifth sticks, or what. If they struggle with this, remind them about the exercise with the bag of poker chips. When the bag was divided into four equal piles, each pile was a fourth of a bag. Hopefully, they will realize that the same reasoning works here, that three equal-sized sticks which together make up a full stick are each a third.

The point of this exercise is for the students to get away from dividing a large number of things and begin thinking about what it would be to divide a single thing into pieces. A third is a single thing, but only makes sense when there is a whole with which to compare it:



You might need to walk the class through one of these examples, explaining the reason why the above stick is called "one-third of a whole stick." Hopefully, however, the students will be able to catch on to the pattern and identify the others. Once again, have the students compare the different sizes (the eighth stick is smaller than the sixth stick, etc.).

This is a useful exercise to repeat. Give the students a set with only some of the different sizes, and then have them race to identify the sizes that they have. Ideally, they will first find all the sticks of a given length, and then put them end-to-end to compare with the longest stick (always have one of these in a set). This will help to reinforce in their minds that a fraction is formed when a whole is divided up into pieces of equal length.

As a next exercise, have students work with measuring cups. Give each group of students a 1-cup, a half-cup, a third of a cup, and a fourth cup. Make sure that each of these is well-labeled with the corresponding fraction. Probably the best thing for your students to measure would be water. They could dip the cups into a big tub or bowl with water and then pour from one cup to another (over the bowl). This has many potential downsides, however, ranging from a lot of accidental spilling to a lot of intentional throwing and splashing. Sand works well, but will also run the risk of being thrown. You do not want to anger your custodial staff. If there is a sand box outside, you could run the exercise there. Similarly, water could be played with more safely outside.

Still, for a safe, inexpensive, and reasonably-easy-to-clean thing to measure, it is hard to beat uncooked rice or oatmeal.

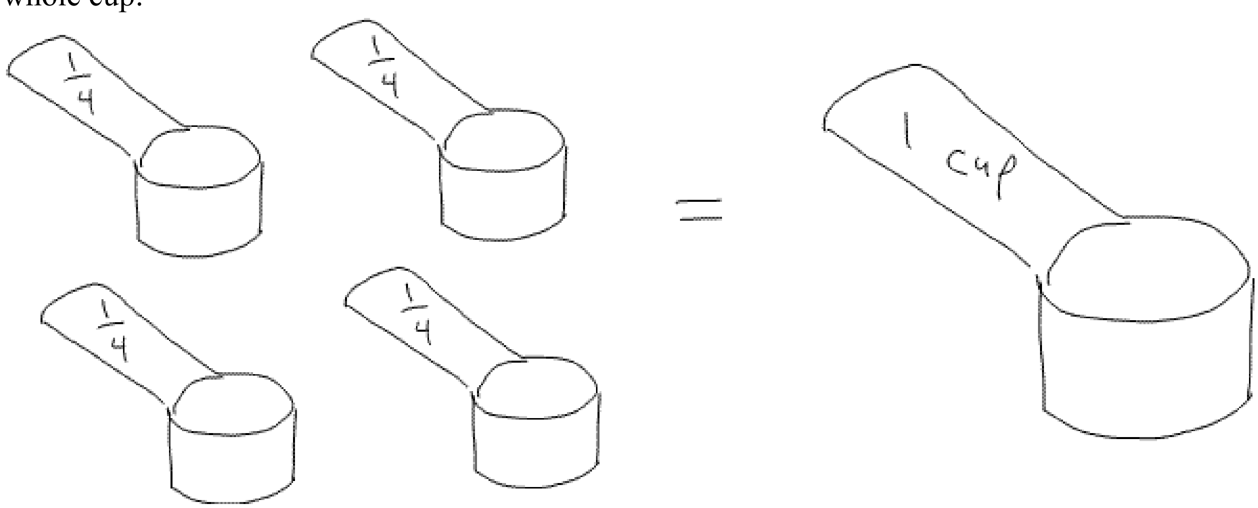
Basically, this exercise is much like the rest – the students see how many level scoops it takes for each measuring cup to fill up the 1-cup. This time, however, you will not only be using the math language (one-half, one-third, and one-fourth), but you will also be able to reference the symbols drawn on the cups. After all the similar games you have played so far, your students will hopefully be able to identify the various cups as one-half, one-third, and one-fourth. The one-third cup, for example, can be filled (level) and poured into the largest one-cut exactly three times before the one-cup is level-full itself. This would be an excellent time to ask, one more time, which of the fractional cups is bigger. Hopefully everyone will say that the half cup is bigger than either the third or the fourth cups.

When the class has figured out this last exercise, ask them if they can explain why the half-cup has a 1/2 written on it while the third-cup has 1/3 and the fourth cup 1/4. Hopefully, they will guess (correctly) that the 4 in 1/4 means that it takes 4 of them to make up a whole cup, that the 3 in 1/3 means that it takes 3 of them to make up a whole cup, and so on.

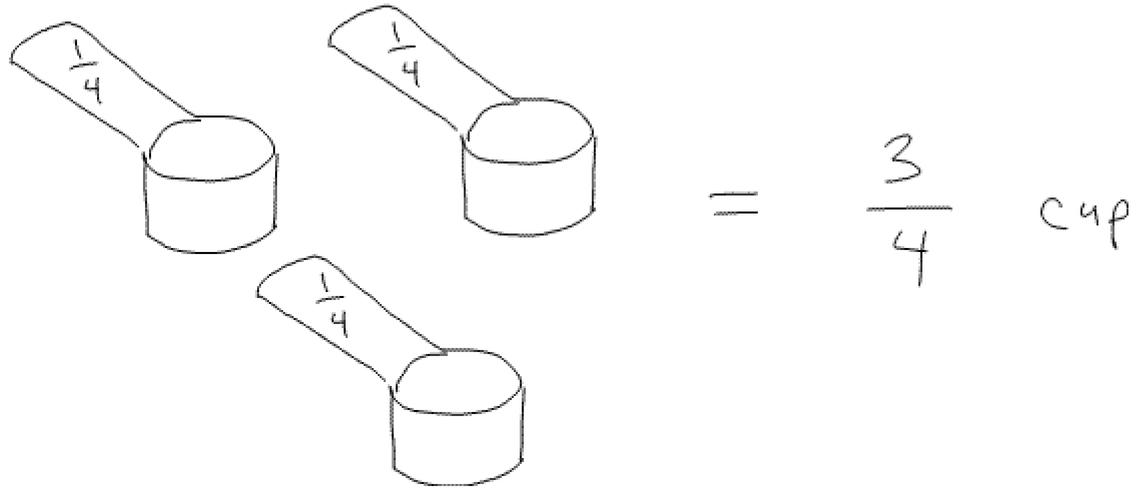
This is an excellent time to explain the notation for fractions.

A *fraction* is a new kind of number, one used to describe a number that has been broken up into pieces. Each fraction is described by two numbers, a *numerator* up top and a *denominator* down below a horizontal line. The numerator tells us how many pieces we have. The denominator tells us the size of each piece. Specifically, the denominator tells us how many pieces it will take to make up a whole.

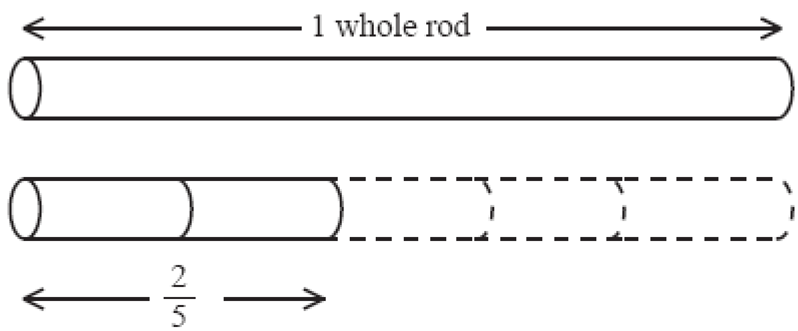
The 1/4 cup, for example, has a 1 in the numerator and a 4 in the denominator. This means that we have 1 piece (of a cup), and that it will take 4 of these (small cups) to make up a whole cup:



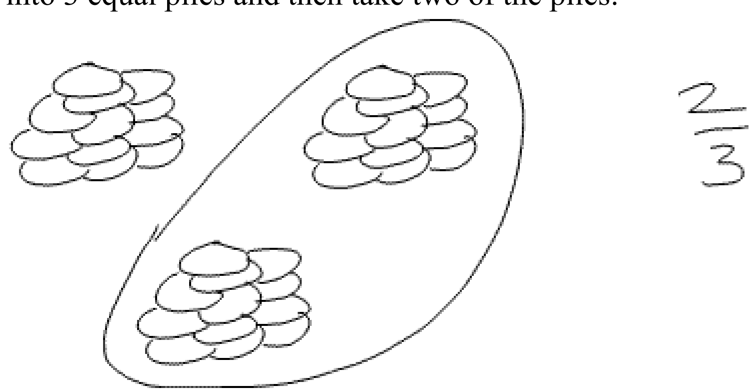
If a recipe calls for 3/4 of a cup of sugar, for example, then we will need three of these little cups. The 4 gives the size and the 3 gives the number:



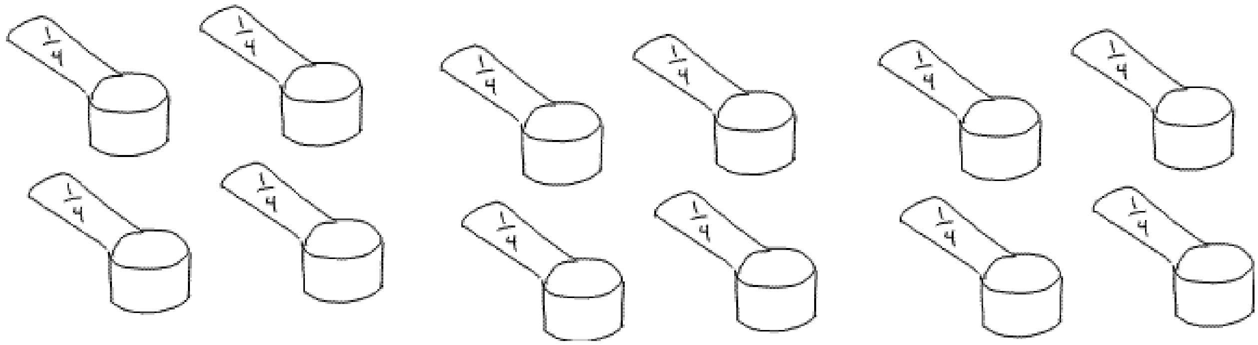
The number 2/5, as another example, consists of 2 pieces, because the numerator is 2. These pieces are fifths, meaning that it will take 5 of them together to make up a whole. If we represent this with sticks, it will look like:



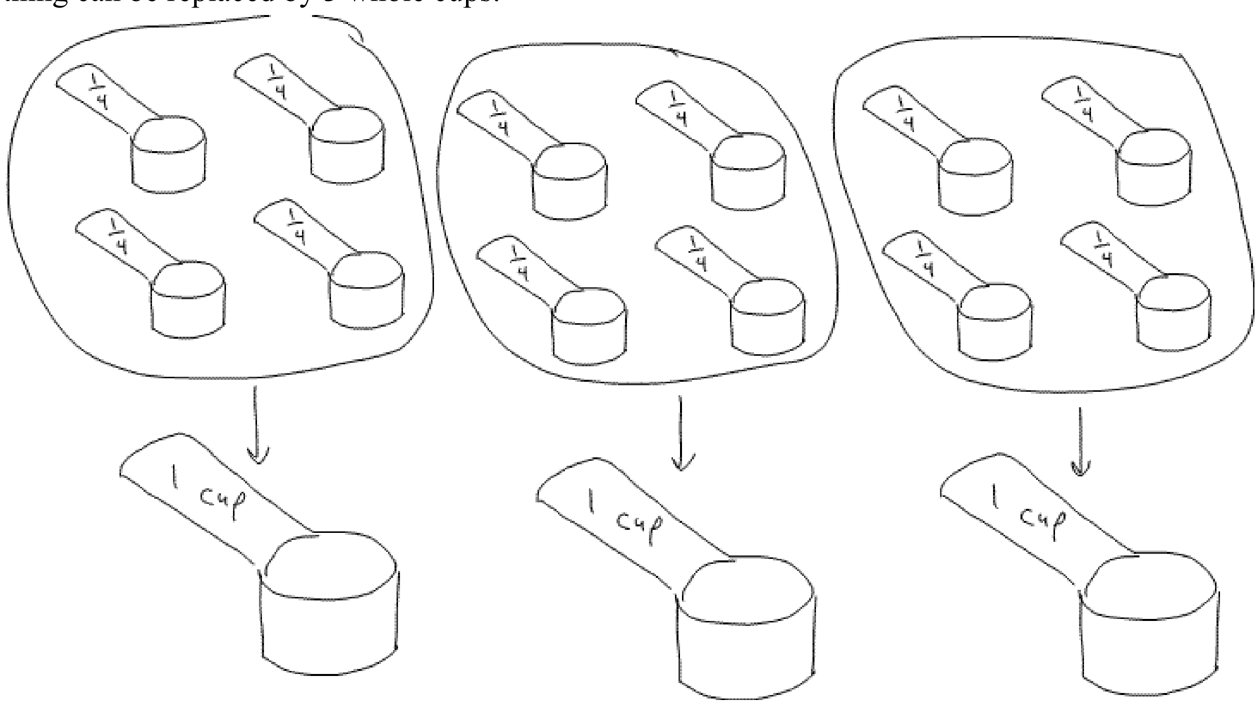
Similarly, if we want to count out 2/3 of a bag of poker chips, we first divide the bag up into 3 equal piles and then take two of the piles:



The little horizontal bar in a fraction is the same as the division symbol. We can demonstrate this with something easy to divide, for example 12 ÷ 4. We know that 12 ÷ 4 = 3. Now, let us look at the fraction 12/4. This means that we have 12 pieces and that it takes 4 of those pieces to make a whole. If we look at this with measuring cups, 12 fourths will look like:

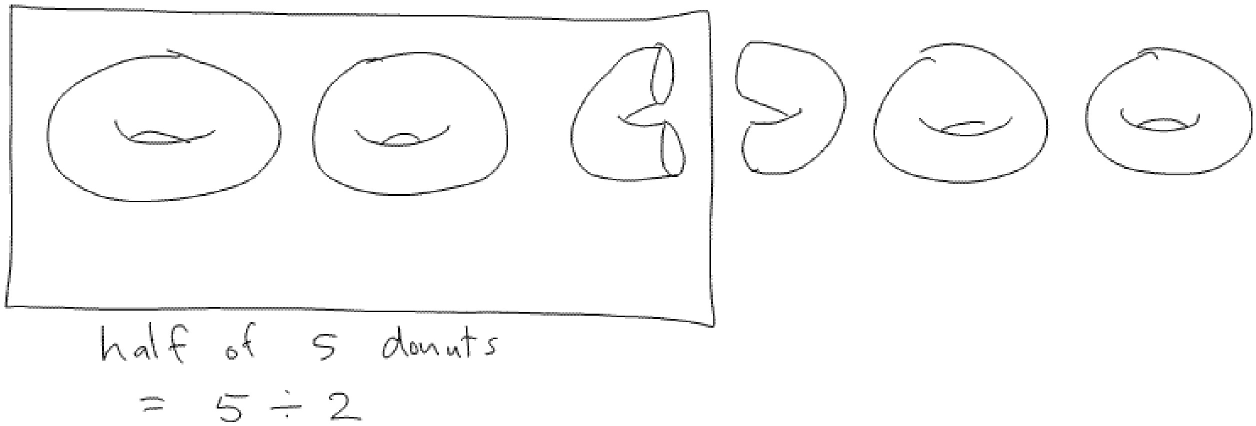


We know that four of these together is the same as 1 whole cup, so this means that the whole thing can be replaced by 3 whole cups:

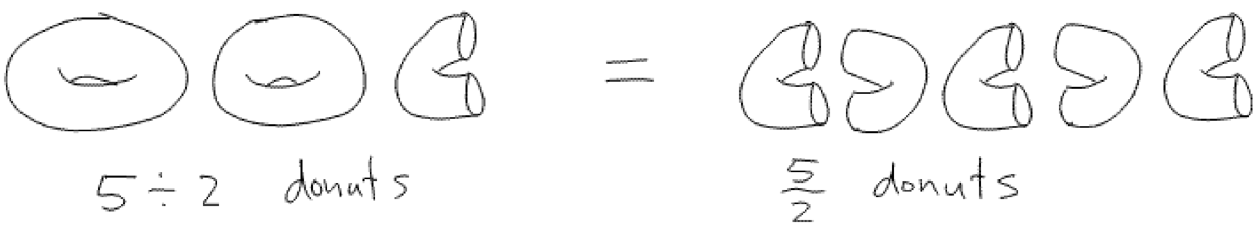


Thus $\frac{12}{4}$ of a cup is the same thing as 3 cups. This indicates that $\frac{12}{4}$ is the same thing as 12 ÷ 4.

Another way to see that "fractions are the same as division" is to look at a simple situation like "divide 5 donuts between 2 kids." The natural way to do this is to give each kid 2 donuts, and then split the last one in half:

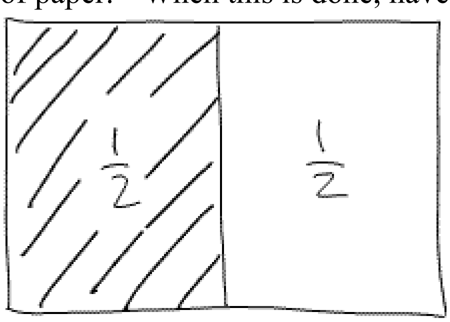


If we look at the fraction 5/2 donuts, we see that this means we have 5 pieces (the numerator). The denominator tells us that it will take 2 of these pieces to make a whole donut. In other words, we have 5 half-donuts. This is exactly as much donut as the 5 ÷ 2 donuts illustrated above:

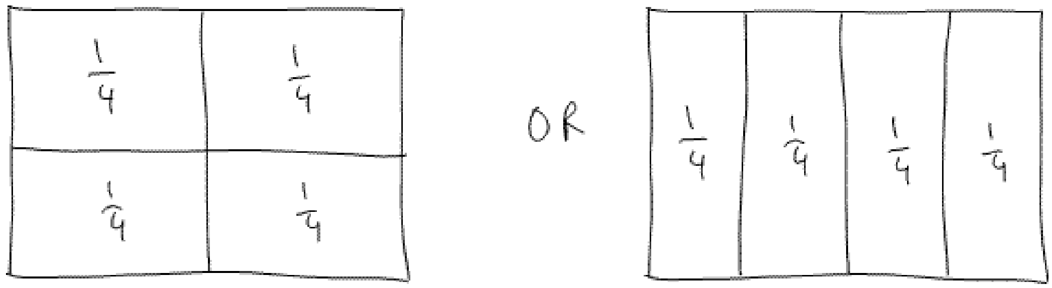


We have so far looked at fractions in many different ways. We have looked at fractions of a group of objects (fractions of a bag of poker chips). We have looked at fractions of lengths (a third of a rod). We have looked at fractions of volumes (half a cup).

It is also very popular to look at fractions of areas. The easiest example would be to give each child a piece of paper and have them "fold it in half." Then have the children unfold the papers and then trace along the fold-line. Have each child next "color (or shade) half of the piece of paper." When this is done, have the children use the fraction notation to label both parts:

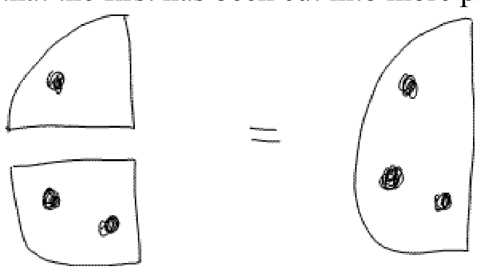


Next, with a new piece of paper, have the children fold the paper in half, then half again. Next, they can unfold the paper, trace the fold-lines, and then label each of the areas:



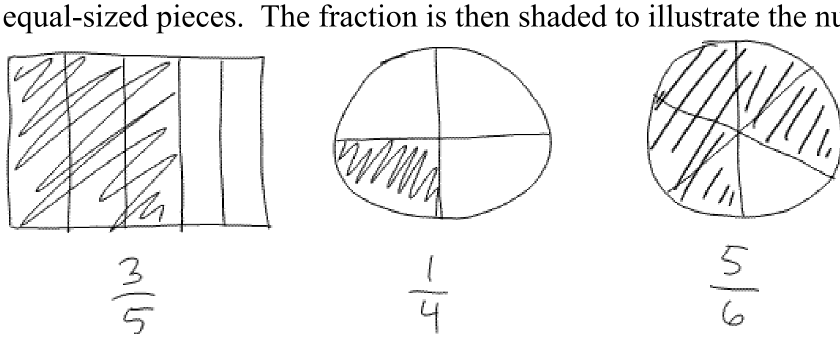
There are different ways this can come out, depending on how the paper is folded. In fact, it would be excellent if both ways came out. You can reinforce the concept of fraction by emphasizing that 1/4 is piece of a whole. Have the students "cover the whole paper with their hands" to emphasize the concept of the whole. Have them discuss the relative sizes of the 4 pieces. If the pieces are not the same size, then they will not be fourths.

At this point, you can have your students cover up various fractions with their hands. Have them "cover up 3/4" for example. When you have them "cover up 2/4," see if anyone will notice that this is the same as 1/2. If so, explain to them that there are many different ways to describe the same fraction. For example, 2/4 of a cookie is the same as 1/2 of a cookie, except that the first has been cut into more pieces:



Tell the class not to worry too much about this – we will cover it later.

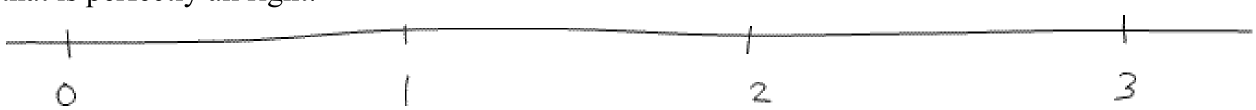
Teaching students to understand fractions of areas is very useful. This is because so many books, examples, and exams use areas to depict fractions. The whole is represented by a single shape, usually either a rectangle or a circle, which has been divided up into a number of equal-sized pieces. The fraction is then shaded to illustrate the number of pieces in question:



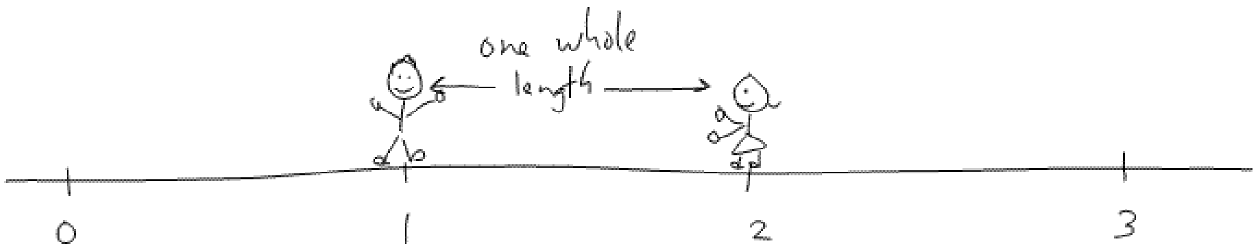
Because your students will see so many of these, it is useful to go through many examples. Each time, make sure that the students identify (1) what the whole is, (2) how many pieces make up a whole, and (3) how many pieces are shaded. The answer to (3) is numerator, the answer to (2) is the denominator, and (2) can only be answered when (1) is made clear.

The opposite process is also useful – having students draw areas to illustrate various fractions. Do not be too fussy about the pieces being of equal size, so long as none are too huge or too small. Merely have each student draw a shape to represent the whole, break is up into as many equal parts as the denominator, and then shade in as many as the numerator. Not only will this help them to prepare for all the fraction illustrations they will see in their lives, but it also reinforces the concept and notation for fractions.

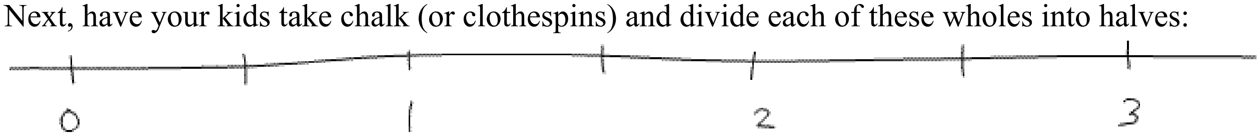
For a final exercise in fractions, we go back to the number line. Draw a new number line, but this time make the distance between marks much bigger. If the line only goes up to 3 or 4, that is perfectly all right:



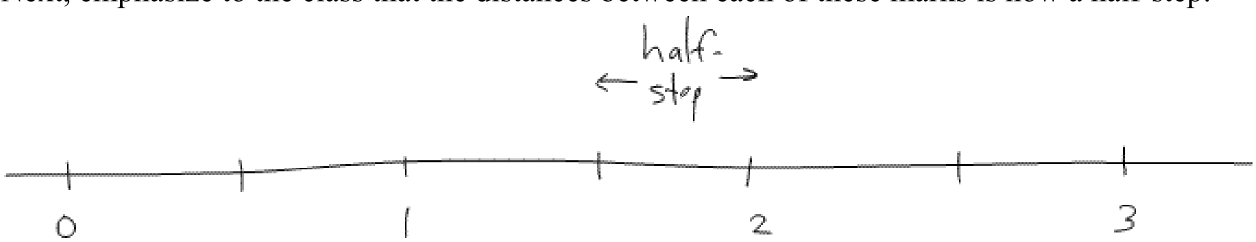
First of all, ask the class to identify what a "whole" would be on the number line. If you are working with a line drawn in chalk or a clothesline pinned with numbers, have the kids walk around and figure this out. If they are having trouble, ask them to say what "one" is. The number 1 drawn next to the second mark is a 1, true, but this is meant to indicate one unit distance from start. The best answer is that a "whole" is a "whole step," the distance between any two marks:



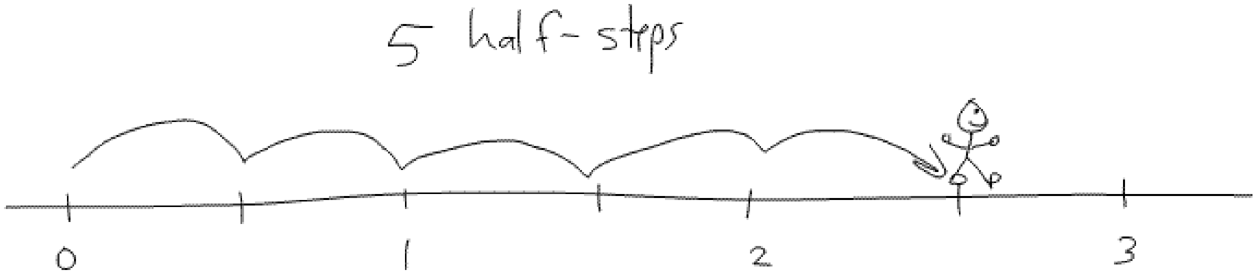
Next, have your kids take chalk (or clothespins) and divide each of these wholes into halves:



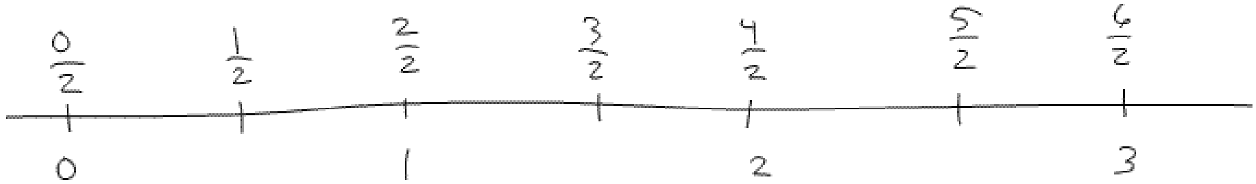
Next, emphasize to the class that the distances between each of these marks is now a half-step:



As a beginning exercise, have the students play the "walk the number" game, but with half steps. For example, you could ask a student to walk 5 half-steps:

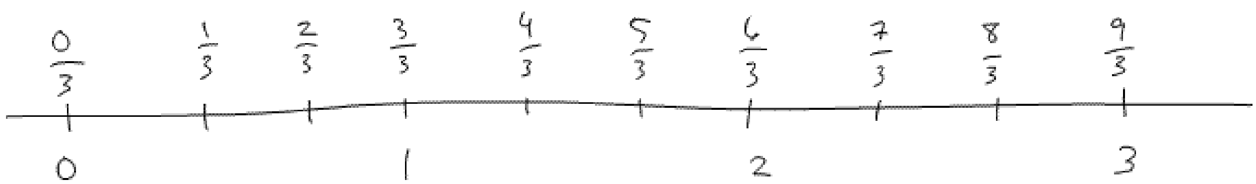


After a few rounds of this game, the children should be ready to label the new marks. Remember, that just as the starting mark is labeled 0 (no steps from start), this can also be called 0/2 (no half-steps from start):

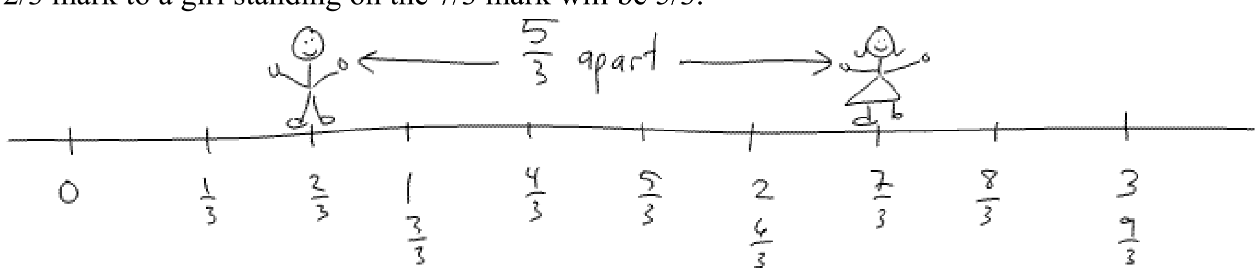


Your students might notice here that some marks have the same name. For example, the 2 mark is also labeled 4/2. This is because taking 2 steps forward will take you to the same spot as 4 half-steps. Once again, every number have many different fractional names, which we will discuss in a later chapter.

On another day (and on another number line), you can repeat the exercise, but work with thirds instead of halves. The end result will be a line that looks like:



You will want to emphasize to your students that these numbers label the distance from start (marked 0). If you want to know the distance between two different notches, you will have to count the number of spaces between them. For example, the distance from a boy standing on the 2/3 mark to a girl standing on the 7/3 mark will be 5/3.



Your students might conjecture (correctly) that the 5 is found by subtracting the 2 from the 7, but we will hold off on subtracting fractions until later. For the meantime, the number line is an excellent place to illustrate fractions which are bigger than 1 (fractions like 5/3 and 7/2).

Questions:

(1) Represent the fraction 4/5 with:

(a) rods

(b) the area of a rectangle

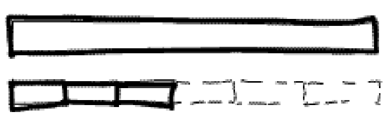
(c) the contents of a bag of small items

(2) Draw a number line with all the marks from 0 up to 4 labeled, including all fourths.

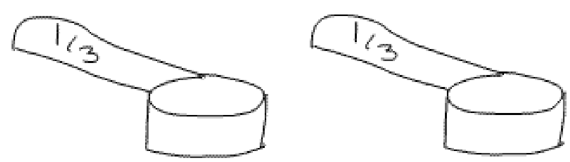
(3) Illustrate the similarity between 10 ÷ 2 and 10/2.

(4) Identify each of the fractions illustrated:

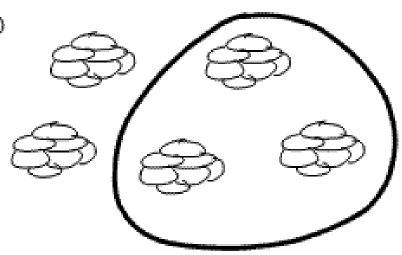
(a)



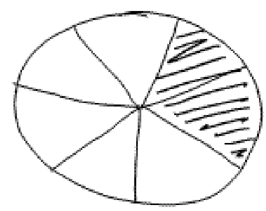
(b)



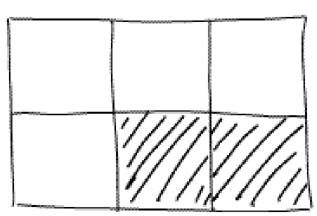
(c)



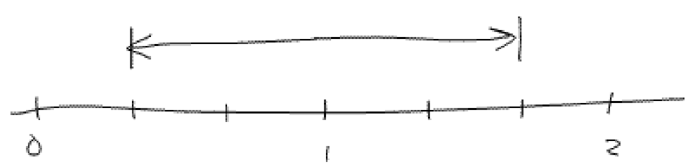
(d)



(e)



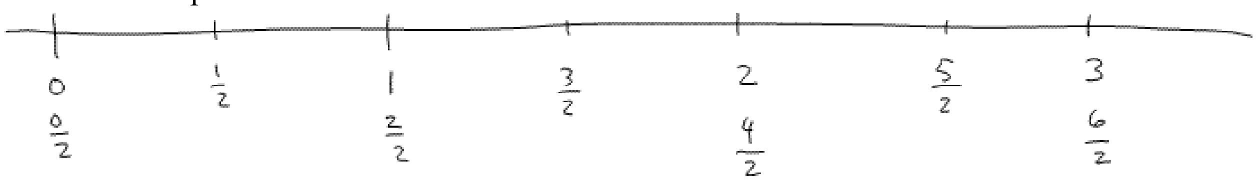
(f)



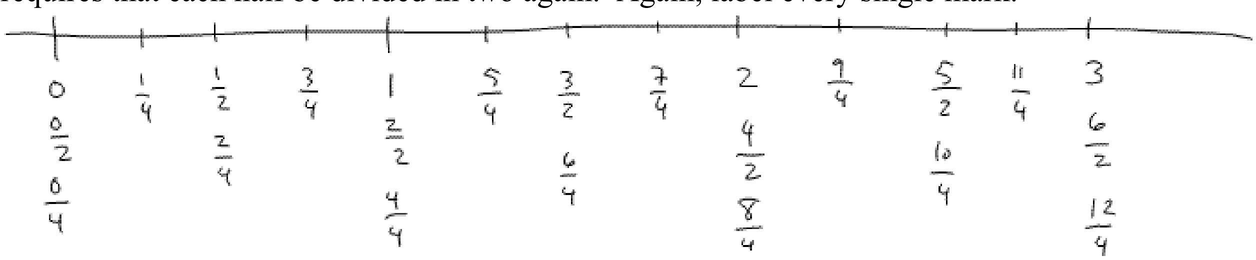
Chapter 25: Reducing Fractions

When students begin to encounter fractions, one of the first things to explain and explore is the fact that each number can be represented by many different fractions. When two fractions represent the same number, they are called *equivalent fractions*. An excellent way to introduce this (if someone in the class has not already noticed this and pointed it out) is the following exercise.

First, draw a number line up to 3 or 4 (use very large steps) and label all the marks. Next, divide each step into halves and label them all:

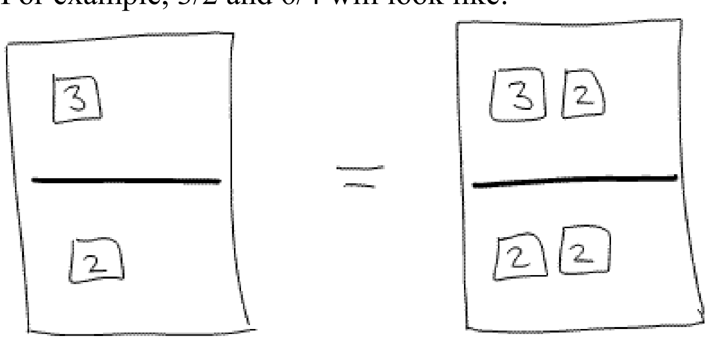


Next, divide each of the steps into fourths. Hopefully, your students will realize that this only requires that each half be divided in two again. Again, label every single mark.



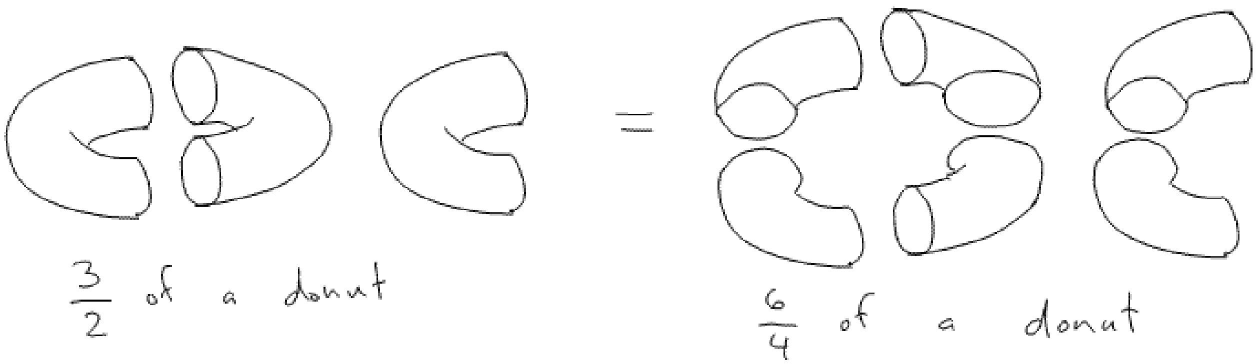
This will now provide us with a number of equivalent fractions. For example, 3/2 and 6/4 are both equivalent.

To begin analyzing a pair of equivalent fractions, have the students factor both the numerator and denominator of each, then represent each fraction on a separate symbol board. For example, 3/2 and 6/4 will look like:



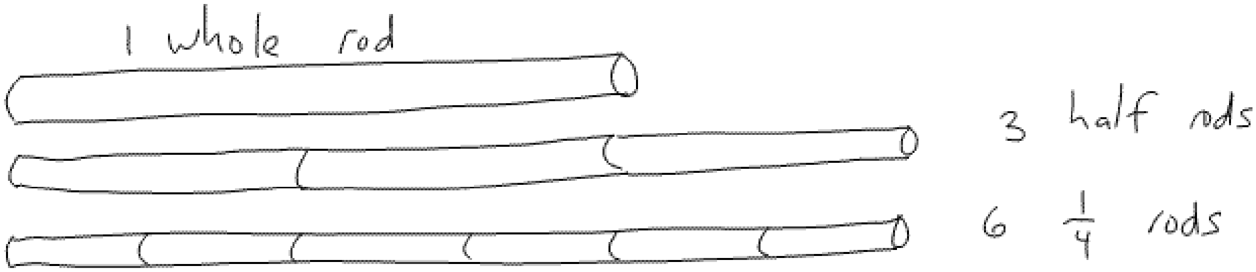
Hopefully your students will immediately recognize that these are also equivalent via the symbols game. The fraction 3/2 can be turned into 6/4 just by inserting a factor of 2 into both the numerator and denominator.

It is easy to explain how this happens with fractions. When you multiply both the numerator and denominator by 2, you are simultaneously doubling the number of pieces and making each piece half as big. If we did this with donuts, for example, then 3/2 of a donut (three half pieces) could be turned into 6/4 of a donut (six fourth pieces) just by cutting each one in half:

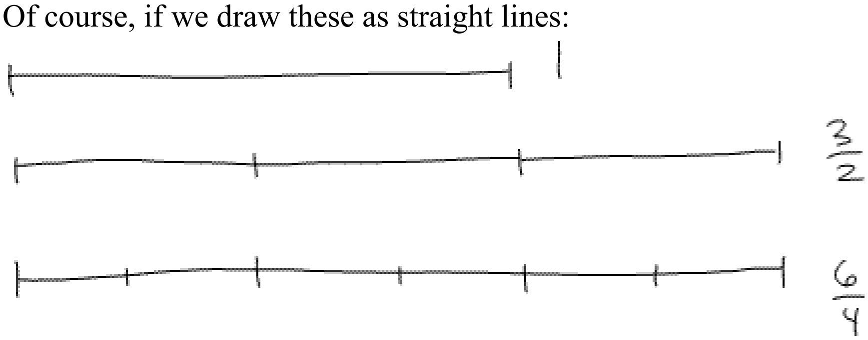


Similarly, we can turn 6/4 into 3/2 by taking away a factor of 2 from both the numerator and denominator. With donuts, this would be like gluing (with frosting?) donut-fourths together in pairs to turn them into donut-halves.

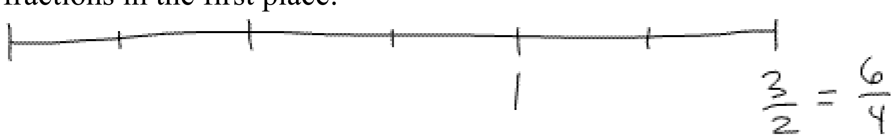
With fraction rods, this is even easier to illustrate. If we put three of the half-rods end-to-end, they have the exact same length as six of the fourth-rods put together:



Of course, if we draw these as straight lines:

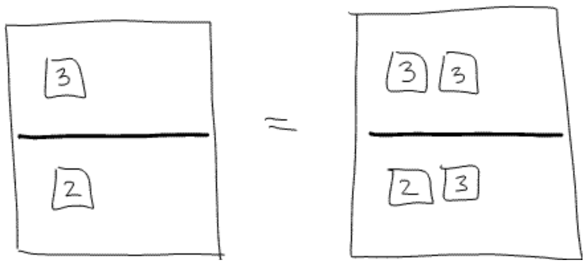


And then overlap them, we get the exact same number line which brought us these equivalent fractions in the first place:

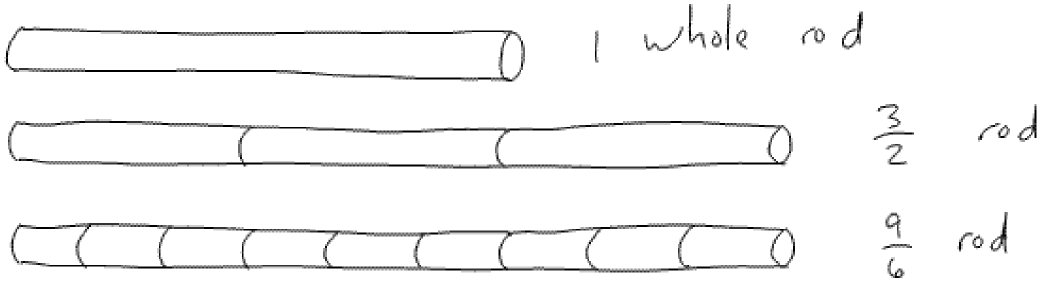


The general rule to remember is "if you break everything in half, you will get twice as many pieces, but it will take twice as many to make a whole." Because the numerator is the number of pieces and the denominator is the number of pieces required to make a whole, this means that we can multiply the numerator and denominator by 2 whenever we want.

With this in mind, see if your students can explain what the following symbol game equivalence would mean in terms of fractions:

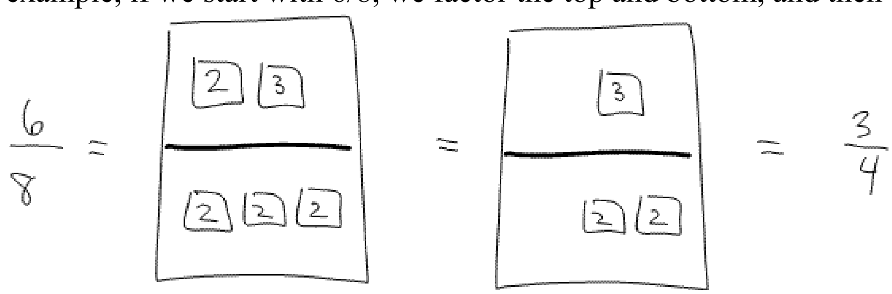


In terms of fractions, the symbol game says that 3/2 = 9/6. This is because we inserted a factor of 3 to both the top and bottom of the fraction. The first fraction consists of 3 pieces, each one half of a whole. The second fraction consists of 9 pieces, each one a sixth of a whole. Again, we can lay these out with the factor rods:

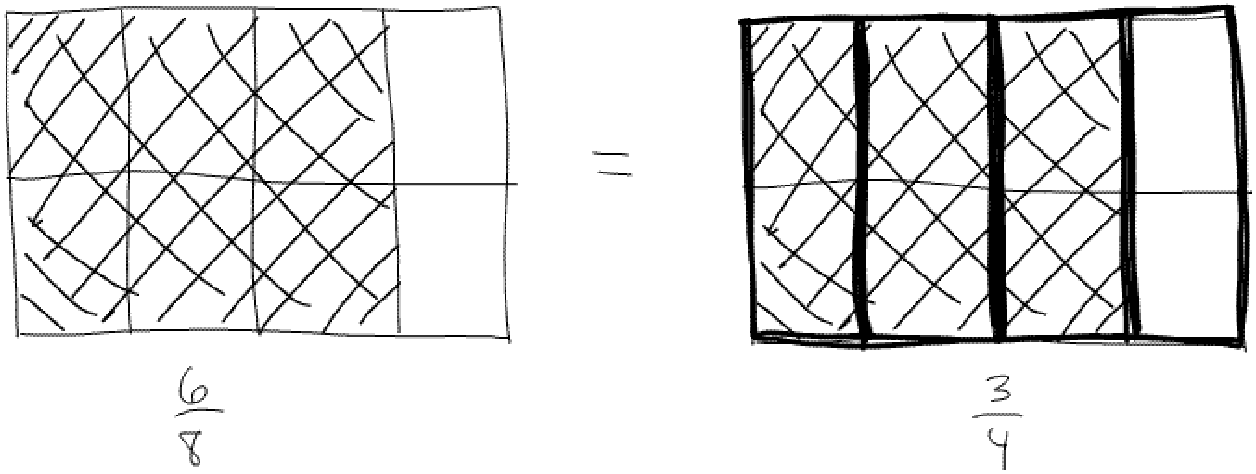


We can see that it takes 3 sixth-rods to be the same length as a half-rod. Thus, to turn 3/2 into 9/6, all we have to do is break each half rod into three equal pieces. This will multiply the number of pieces by 3 (from 3 to 9) and also multiply the number it takes to make a whole by 3 (from 2 to 6). Similar to before: "when you break everything in three, we end up with three times as many pieces, but it will also take three times as many to make a whole." This is exactly what multiplying the top and bottom of a fraction by 3 accomplishes.

The symbol game called "take away as many pieces as possible," when played on a fraction, *reduces* the fraction. This means that the fraction is made as simple as possible. For example, if we start with 6/8, we factor the top and bottom, and then take away a two from each:



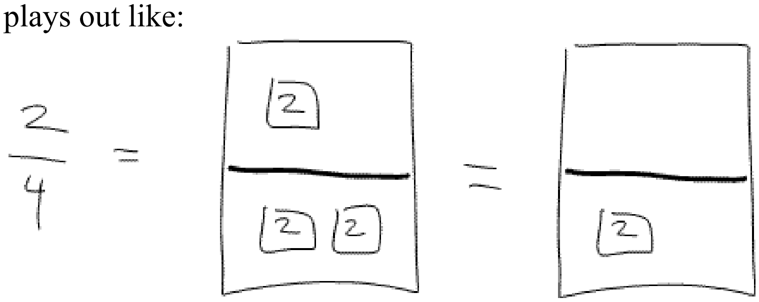
We could demonstrate this with folded paper, if we wish. Fold the paper in half three times, then trace the fold-lines and shade in 6 of the 8 equal-sized rectangles. With a marker, we can show how these 6 eights are the same as 3 fourths:



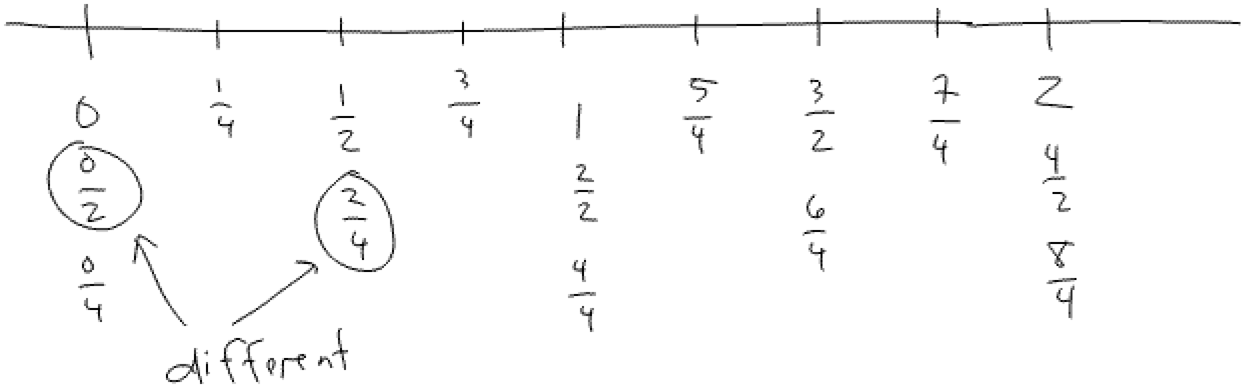
With some practice, students will be able to reduce fractions without having to go through all the trouble of drawing pictures or using the symbols game. When given a fraction like 15/25, students who are good at factoring will be able to see, in their heads, that the 15 = 5 × 3 and the 25 = 5 × 5, so the only factor that can be removed will be a 5. It is often said that "there is a 5 in both 15 and 25." To remove a factor of 5 from each is to divide the 15 by 5 and the 25 by 5. These are basic division facts that a student should be able to do automatically. Thus, each of the two numbers can be crossed out and replaced by that number divided by 5. The work will generally look as illustrated to the right. Thus 15/25 = 3/5. If we had 3 pieces, each of which were one-fifth of a whole, and then broke each into 5 equal parts, then we would have 15 pieces and it would take 25 of them to make a whole. In reducing 15/25 to 3/5 we do the opposite process – we group the small 25ths pieces together by 5's to make 3 larger pieces, each a fifth of the whole.

$$\frac{\cancel{15}^{\,3}}{\cancel{25}_{\,5}}$$

Students can run into difficulty where either the numerator or denominator run out of symbols. For example, suppose a student wants to reduce 2/4 by factoring. The symbol game plays out like:



What does it mean when the numerator is empty? Some students want to say that "nothing is zero" and thus write that 2/4 = 0/2. However, a look at the number line shows that taking 2 fourth-steps is very different than taking no half-steps:



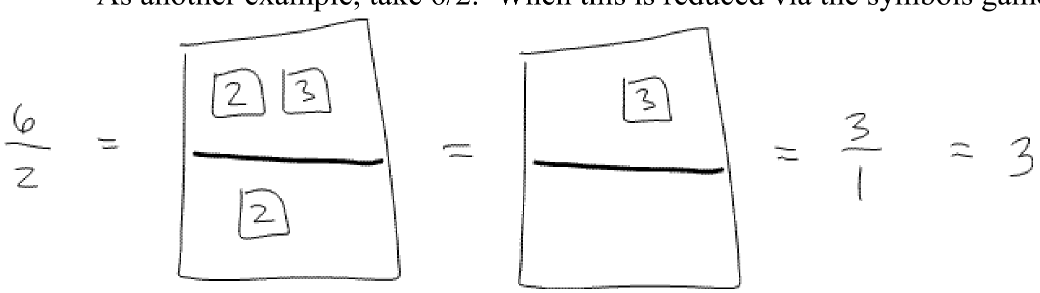
Other students let the 2 on the bottom float up to the top, saying that 2/4 = 2. This is also clearly wrong, for the number 2 is way over to the right. No, the number line makes it quite clear that the number of halves equivalent to 2/4 is 1/2. As strange as it may sound, when either the top or the bottom of the symbols board is empty, this means that there is a 1 there, and not a 0.

Actually, this is not as strange as it may initially seem. This is because "doing nothing" is different when multiplying than when adding. If you have just added something to a number and that number did not change, you could say that you "added nothing." The only number that could do this is zero. For example, 5 + 0 does nothing to the number 5. Because zero has this special property, it is called the *additive identity*.

On the other hand, if you have just multiplied something with a number, and that number did not change, you could say that you "multiplied nothing." The only number that does this is the number 1. For example, 7 × 1 does nothing to the number 7. This is why the number 1 is called the multiplicative identity.

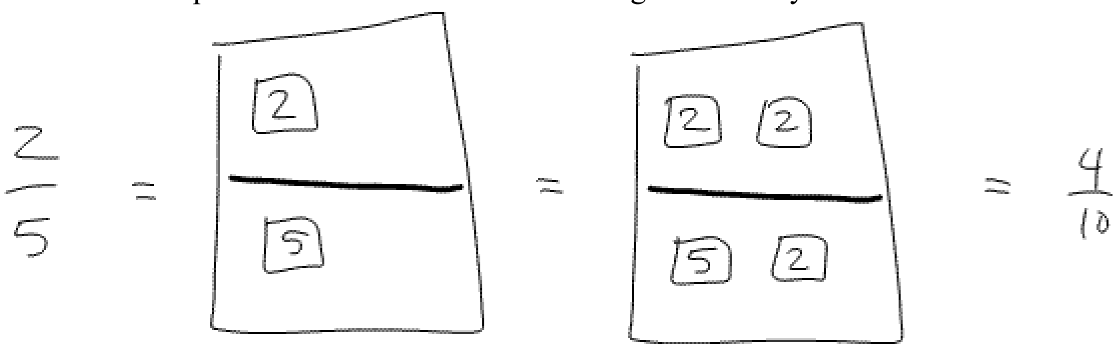
When we are working with the symbols game, we are working with multiplying (up top) and dividing (down below). Thus, when there is "nothing" in one of these places, it represents the number 1.

As another example, take 6/2. When this is reduced via the symbols game, we get:

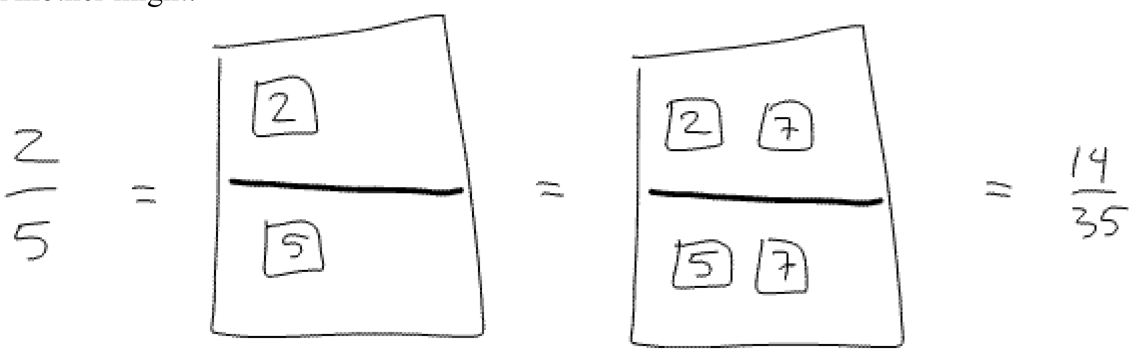


We can see this on the number line, where taking 6 half-steps forward is the same as taking 3 whole steps forward. We can also see this by viewing 6/2 = 6 ÷ 2 = 2. We could also see this by saying that 3/1 consists of 3 pieces where it only takes 1 piece to make a whole. This means that we have 3 whole pieces, which is simply written 3.

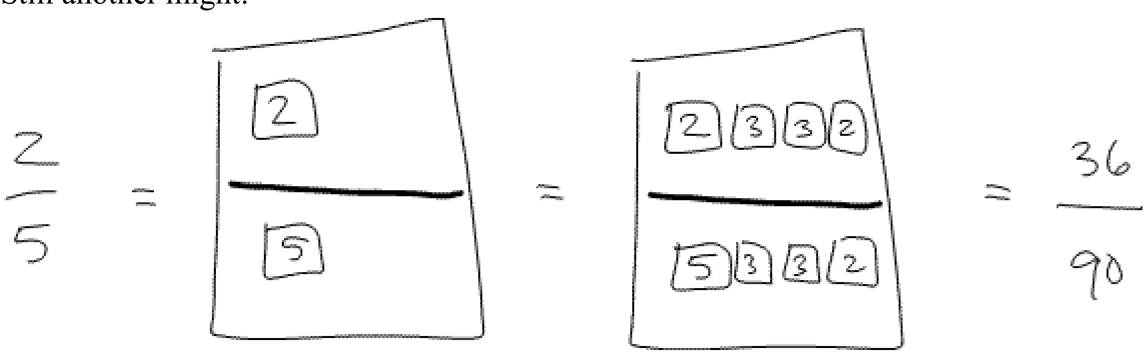
It is important for students to reverse the process of reducing. This is called "finding equivalent fractions." For example, you could give your students a fraction like 2/5 and have them find an equivalent fraction. One student might do this by:



Another might:

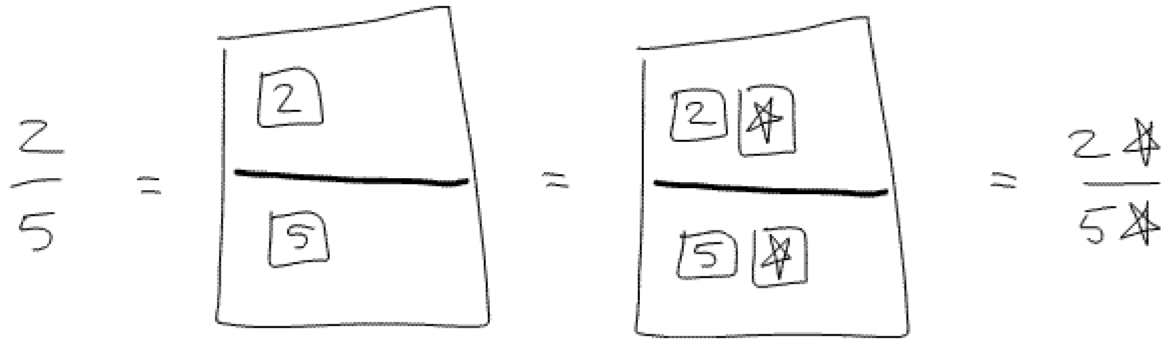


Still another might:



All of these are correct answers. The fraction 2/5 is the same as 4/10, as 14/35, as 36/90, and as many other fractions as well. Working through exercises like this might help students who are having trouble grasping the concept of reducing fractions. These are all equivalent (taking 4 tenth steps is the same as taking 2 fifth steps). However, the 2/5 fraction is the simplest one of all (it has the fewest factors), and thus is considered the reduced version of all these fractions.

Note that a student could just as easily do the following:



This is also equivalent, no matter what the star might mean. It is not important to stress examples like this now, but a little exposure to them now might make algebra make more sense later down the line.

Questions:

(1) Draw a number line out from 0 to 3 and divide the steps into halves, then fourths, and then eighths. Label all of these marks in all the ways possible. Use this to find several fractions which are equivalent to 10/4.

(2) Illustrate with rods why 2/3 = 4/6. Make sure to include a picture of the whole rod for reference.

(3) Illustrate with the area of a rectangle why 4/8 = 1/2.

(4) Show, step-by-step, how the symbol game can be used to reduce 20/36.

(5) Show how the symbol game can find 3 different fractions which are all equivalent to 9/4.

(6) Explain how 3/9 can be reduced with (a) rods and (b) the symbols game.

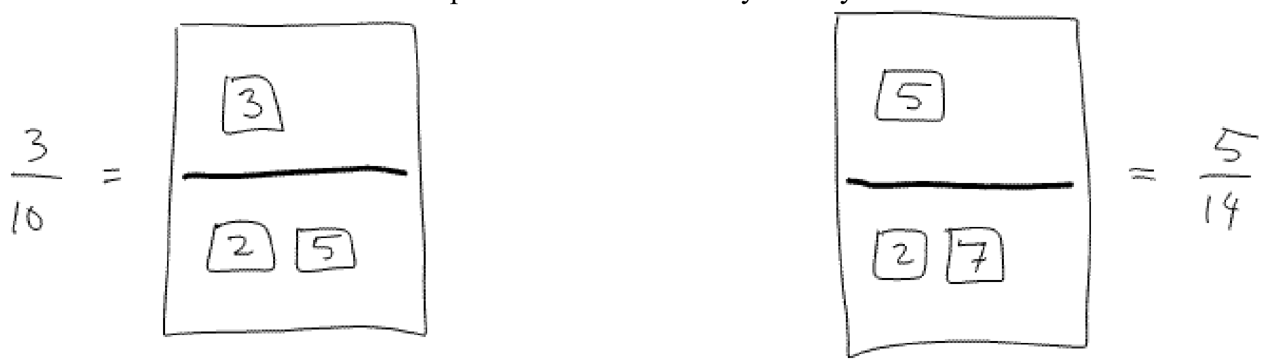
(7) Find 3 different fractions which are all equivalent to the number 5.

Chapter 26: Comparing Fractions

When students are comfortable with reducing fractions and finding equivalent fractions, they are ready for comparing fractions. The key to everything lies in the famous expression: "you can't compare apples to oranges." In order to compare fractions (or add and subtract them for that matter), we will need to make them represent the same sorts of things.

As an initial question, ask the class which is more: 3 medium-sized pieces of cake or 5 small pieces of cake. This may or may not strike up a lively debate, but hopefully the discussion will end with a request for more information. Unless your students can actually see how large a medium-sized piece is compared to a small piece, there is no way to be sure which one is a better choice.

Here is where we bring fractions into the game. A fraction not only tells you the number of pieces (the numerator), but also the amount of a whole that each piece denotes (the denominator). Suppose, for example, we want to compare 3/10 and 5/14. This is not an easy question. The first fraction has fewer pieces (only 3 compared with 5). However, those pieces are bigger (it only takes 10 to make a whole, rather than 14). To find out which one is bigger, we first factor the two fractions and represent them on side-by-side symbols boards:

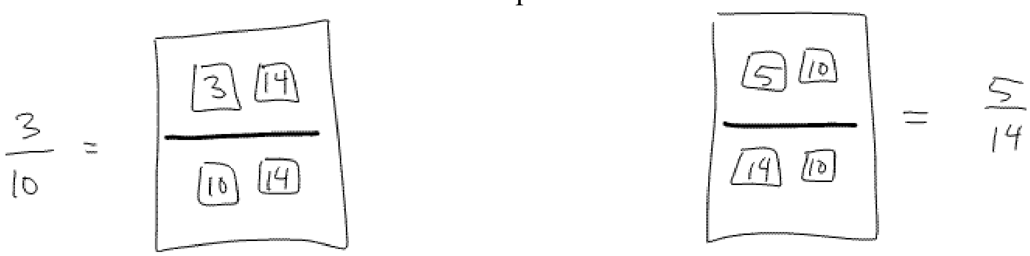


Next, we play the "make the bottoms the same" game. As they both already have a 2 in the denominator, we only need to put a 7 in the top and bottom of the first fraction and a 5 in the top and bottom of the right:

When we multiply everything back together, we can now see that the first fraction is equivalent to 21/70 and the second fraction is the same as 25/70. Now these two fractions have the same denominator, called a *common denominator*. It is now quite easy to compare the two fractions. All of the pieces are now seventieths of a cake (it takes 70 of them to make a whole cake). It is clear that 25 of these pieces will be bigger than 21 pieces of the same size. Thus, 25/70 > 21/70. When we reduce the two fractions, this inequality will read 5/14 > 3/10.

If we had been lazy, we could have not bothered to factor the numbers into prime numbers. We could have made new symbol cards for 10 and 14, then laid out the two boards like:

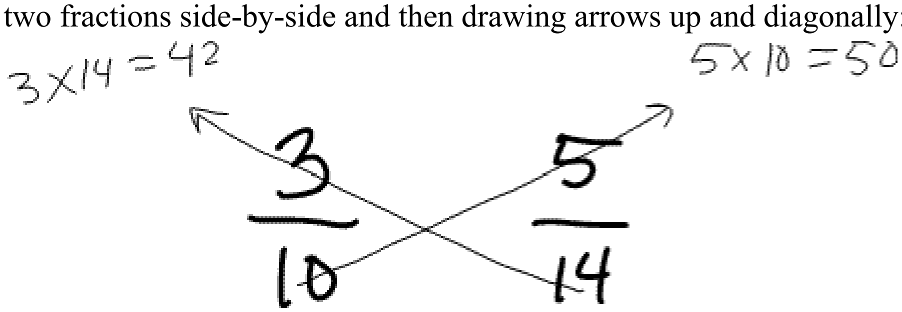
Next, we could play the "make the bottoms the same" game just by putting a 14 in the top and bottom of the first and a 10 in the top and bottom of the second:



The first fraction multiplies out to 42/140 and the second one to 50/140. Thus, the second fraction is bigger: 5/14 > 3/10.

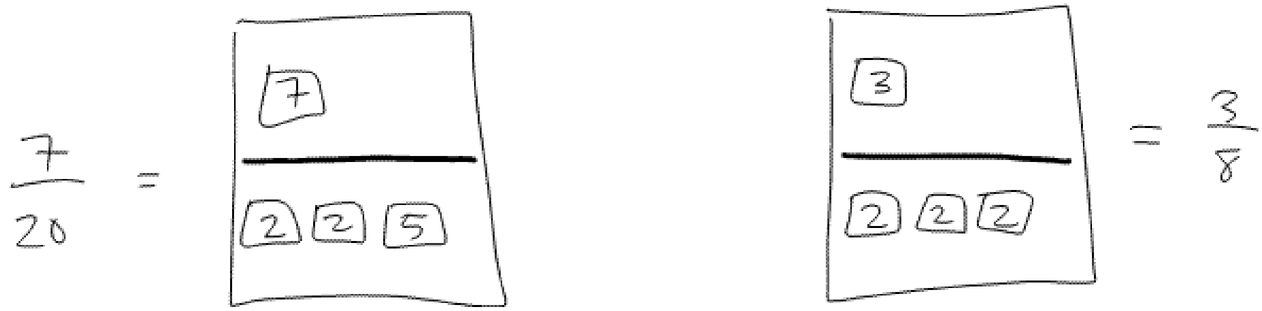
To be even lazier, there is no need to multiply the two denominators together. It does not matter that the two fractions have been converted into 140ths. All that matters is that the fractions have a common denominator. This will be guaranteed – the common denominator will be the product of the original two denominators (in this example 10 and 14). All we really need to do is compare what we get when we multiply the numerator of each fraction by the denominator of the other. In this example, this means we need to compare 3 × 14 to 5 × 10.

The quickest short-cut of all to compare two fractions is simply to do that: multiply the numerator of each fraction by the denominator of the other. We can illustrate this by putting the two fractions side-by-side and then drawing arrows up and diagonally:

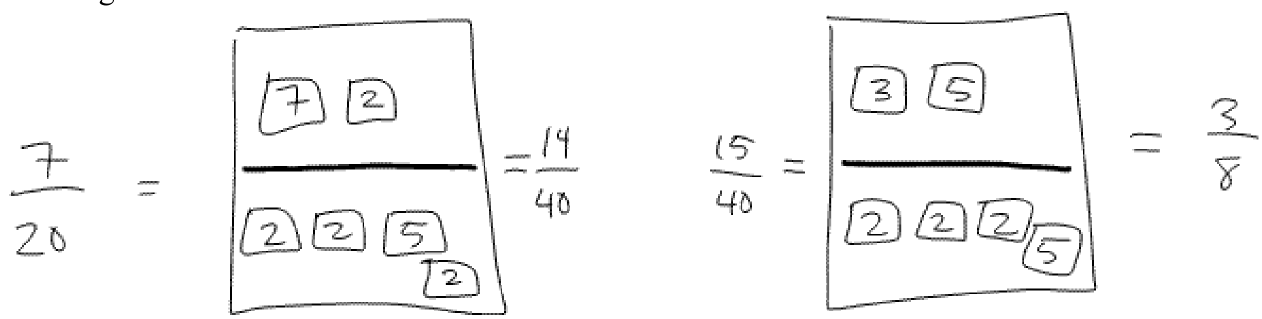


Because 3 × 14 = 42 and 5 × 10 = 50, we can tell right away that 3/10 < 5/14.

As another example, let us compare 7/20 and 3/8 using the two different methods. With the symbols boards, we set them up like:

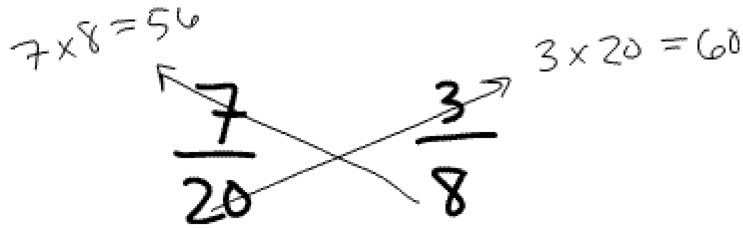


We play the "make the bottoms the same" game by putting another 2 in the top and bottom of the first and a 5 in the top and bottom of the second. We then multiply everything back together:



We conclude that 7/20 < 3/8 because 14/40 < 15/40/

With the short-cut method, we multiply the numerator of each fraction with the denominator of the other:



Because 7 × 8 = 56 is less than 3 × 20 = 60, we conclude that 7/20 < 3/8.

Even though the short-cut method is generally much faster, it is still a good idea to introduce the concept through the symbols game. This explains the short-cut method, thus avoids the danger of students following a procedure without really understanding it.

If you run through a number of examples like these, your students ought to catch on rather quickly. Not only is it useful to compare fractions, but this reinforces both finding equivalent fractions and the concept of a fraction's denominator. Also, the skill of finding a common denominator will be essential for the upcoming work with adding and subtracting fractions.

As well, you should try to develop some fraction common sense with your students, in order that they might be able to compute quickly. Rather than go through either the symbols method or the short-cut for comparing fractions, a child who properly understands the concepts of numerator and denominator ought to be able to compare some fractions immediately.

For example, it should be easy to compare 4/11 and 6/11 right away. These two fractions describe pieces of the same size (elevenths of a whole) and thus 4 of them is less than 6, so 4/11 < 6/11.

Similarly, if you ask a child whether he or she would like 5 big pieces of cake or 3 medium-sized pieces, the answer should be easy – more pieces of a bigger size will clearly be together bigger than fewer smaller pieces. Thus, a fraction like 5/6 will definitely be bigger than 3/11. Similarly, to compare 6/21 and 7/10 is also easy – the second fraction has more pieces (7 versus 6) of a bigger size (it only takes 10 to make a whole, not 21). Thus 6/21 < 7/10. Students can verify these estimates with either of the fraction-comparison techniques, but it would be good for them to develop this sort of automatic common sense.

Also, it is good for a child to be able to recognize right away whether a number is bigger than1. If you have more than enough pieces to make a whole, then you have more than a whole. In terms of a fraction, this means that if the numerator is bigger than the denominator, then we have more than 1. For example, the fractions 6/5, 7/3, and 21/15 all are larger than 1. If we took 7 third-steps from start on the number line, for example, we would reach the number 1 after 3 steps, and then continue beyond it for 4 more steps. This means that 7/3 > 1.

Similarly, if the numerator of a fraction is less than the denominator, then we do not have enough pieces to make a whole, and so our fraction is less than one. The fraction 4/5, for example, has only 4 pieces. Because it would take 5 pieces to make a whole, we have less than a whole. Thus 4/5 < 1.

When students are able to recognize these properties, it becomes very easy to compare fractions like 6/5 and 8/15. We see that 6/5 is more than 1 because 6 > 5 and 8/15 is less than 1 because 8 < 15. Thus it is quite clear that 6/5 > 8/15.

As a last level of sophistication, it is useful for people to be able to immediately recognize whether a fraction is more or less than 1/2. A fraction is more than 1/2 if the numerator is more than half of the denominator, and less otherwise. For example, 3/10 < 1/2 because 3 is less than half of 10. On the other hand, 4/7 > 1/2 because 4 is a little more than half of 7. Because it is relatively easy to mentally divide a number by 2, comparing a fraction to a half can be done quickly, without having to write anything down.

This can make comparing fractions like 2/9 and 11/16 quite easy. We see right away that 2/9 < 1/2 (because 2 is less than half of 9) and 11/16 > 1/2 (because 11 is more than half of 16). Thus 2/9 < 11/16.

Working through these short-cuts and strategies for comparing fractions will help enhance the mental agility of your students and reinforce the essential concepts of fractions.

On a separate note, many teachers (and textbooks) teach a different method for obtaining a common denominator for two fractions. First, they have students look at the *multiples* (found by multiplying by 1, 2, 3, etc.) of each of the two denominators. Next, they have students identify the smallest multiple that can be obtained from either of the denominators. This is called either the *least common multiple* or the *least common denominator* (often abbreviated LCD). Next, the students must multiply the top and bottom by the multiple necessary to obtain this LCD. The result will be two fractions with a common denominator.

For example, in order to compare 3/10 and 5/14, we first begin by listing out the multiples of 10, the first denominator. These are 1 × 10 = 10, 2 × 10 = 20, 3 × 10 = 30, 4 × 10 = 40, 5 × 10 = 50, 6 × 10 = 60, 7 × 10 = 70, 8 × 10 = 80, 9 × 10 = 90, 10 × 10 = 100, and so on. Next, we list out all the multiples of 14, the second denominator: 1 × 14 = 14, 2 × 14 = 28, 3 × 14 = 42, 4 × 14 = 56, 5 × 14 = 70, 6 × 14 = 84, 7 × 14 = 98, etc. We then look through the lists of multiples to recognize that 70 is a multiple of both 10 and 14. Furthermore, this is the smallest multiple of both (the next one will be 140). We then go back and see that 70 = 7 × 10, so we will multiply the top and bottom of our first fraction by 7. Similarly, because 70 = 14 × 5, we multiply the top and bottom of the second fraction by 5. The work required to thus compare these two fractions will thus look like:

$\frac{3}{10}$ vs. $\frac{5}{14}$

multiples of 10 are 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110...

multiples of 14 are 14, 28, $\frac{14 \times 3}{42}$ $\frac{14 \times 4}{56}$ $\frac{14 \times 5}{70}$ $\frac{14 \times 6}{84}$

LCM

$$\frac{3}{10} = \frac{3 \times 7}{10 \times 7} = \frac{21}{70} \qquad \text{because } \frac{25}{70} > \frac{21}{70}$$

$$\frac{5}{14} = \frac{5 \times 5}{14 \times 5} = \frac{25}{70} \qquad \text{we know } \frac{5}{14} > \frac{3}{10}$$

There are many disadvantages to this method. First of all, it is very long and tedious. In order to compare 5/21 and 7/15, for example, this method will require the students to perform at least 12 multiplications. To compare 8/25 and 7/23 requires at least 48 multiplications. In the process of all these multiplications, it is very possible that a student will make a small mistake. This makes the process take even longer. In the resulting mess, it is often very difficult for the student to find the original mistake. Furthermore, teaching a technique like this makes even simple problems take a very long time. This takes away from a student's ability to make quick, mental calculations and develop an intuition and familiarity with fractions. Finally, this process does not work with algebraic fractions like 3/x and 5/2x, so it does not help prepare students for algebra. In the end, it is no surprise that students who are taught this method frequently grow up to be uncomfortable with fractions.

By comparison, a little bit of time invested in teaching students how to factor and play the symbols game will make finding the least common denominator for 3/10 and 5/14 much easier, as illustrated earlier in the chapter. This method should not only make fractions more easy to work with, but will also benefit the students later on when they enounter algebra.

Questions:

(1) Show how to compare each of the following sets of fractions, both by the symbols game and by the short-cut method:

(a) 4/9 and 5/11
(b) 7/10 and 17/24
(c) 8/27 and 4/15
(d) 4/5 and 3/4

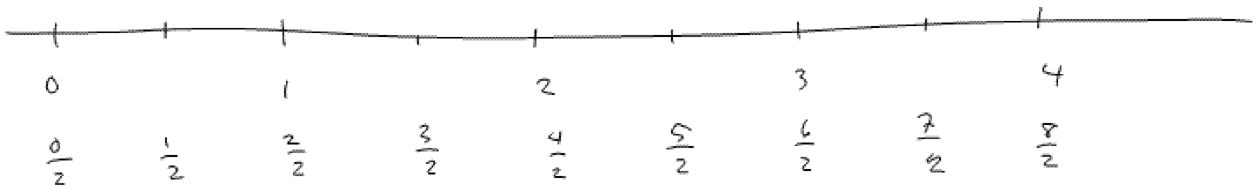
(2) Identify the larger fraction from each pair mentally. Then explain each is easy to do.

(a) 9/20 and 7/20
(b) 3/14 and 5/12
(c) 9/11 and 8/15
(d) 2/3 and 7/6
(e) 7/9 and 6/11
(f) 1/5 and 5/7
(g) 2/3 and 3/7

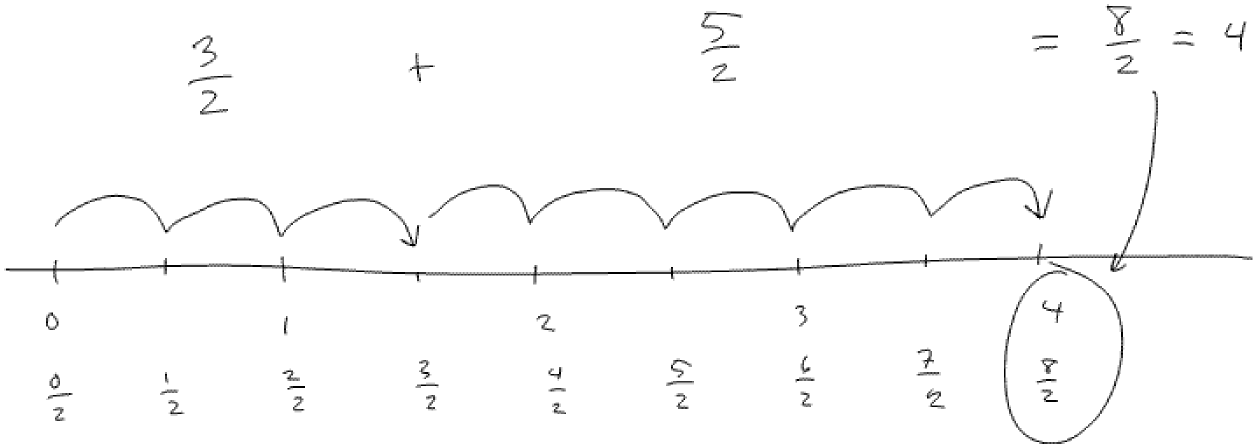
Chapter 27: Adding and Subtracting Fractions

When students are able to compare fractions by finding a common denominator, they have everything they need to add and subtract fractions.

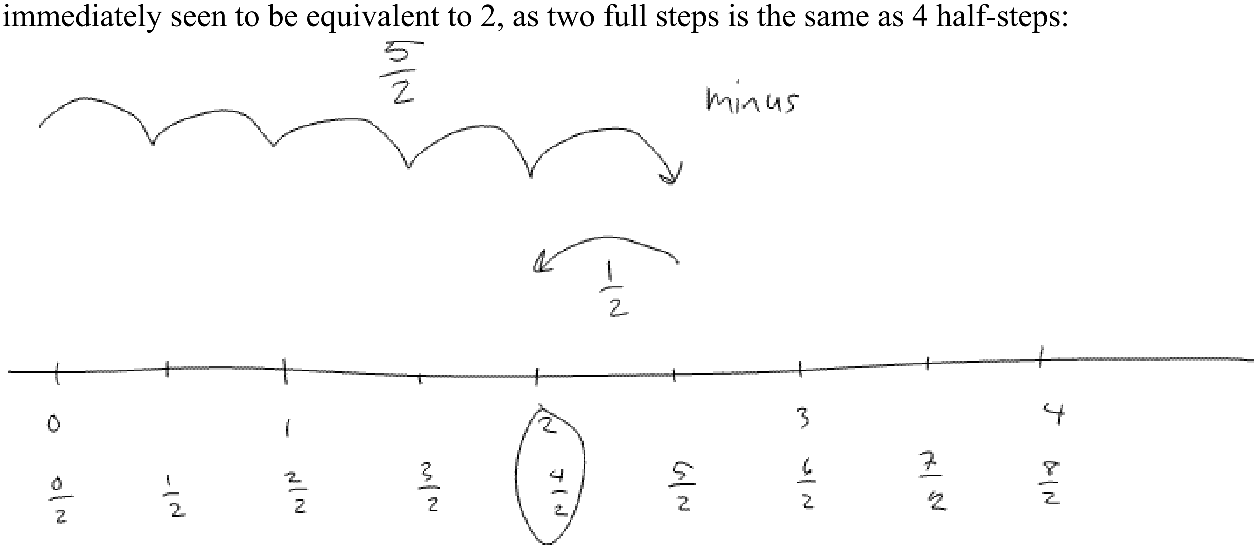
Begin by having students work out problems on the number line. For example, draw out a number line in whole and half-steps:



Next, have the students add simple fractions with a common denominator. The process is exactly the same as that for adding whole numbers, except that fractional steps are used. For example, to add 3/2 + 5/2 will look like:



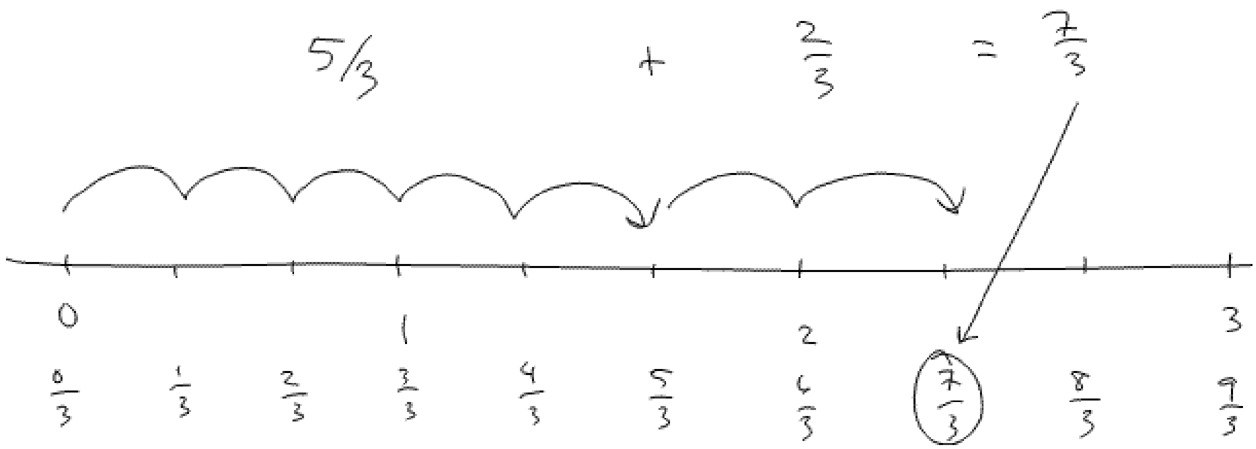
Similarly, to subtract 5/2 – 1/2, a student will first take 5 half-steps to the 5/2 mark, then turn around and walk 1 half-step back to the 4/2 mark. Thus 5/2 – 1/2 = 4/2. This can be immediately seen to be equivalent to 2, as two full steps is the same as 4 half-steps:



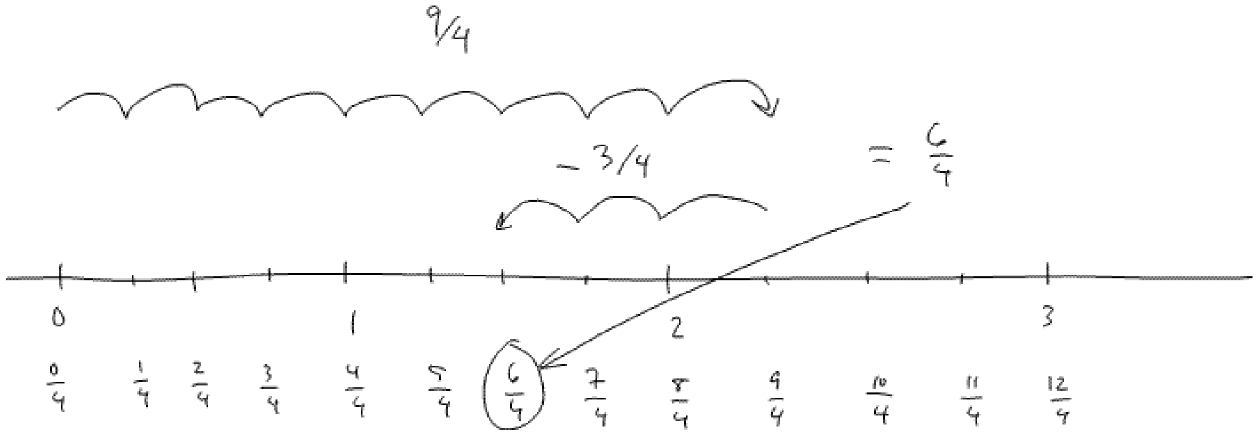
After the class has worked out a number of problems like these, they will hopefully catch on to the pattern. Just as 7 steps plus 5 steps equals 12 steps, for example, they should notice that 7 half-steps plus 5 half-steps will equal 12 half-steps. Thus 7/2 + 5/2 = 12/2. As the students line up to work out problems, you might even notice that some students are able to give the answer even without having walked out a single on on the number line. As they catch on, ask them to solve more challenging ones in their heads. Ask them what 11/2 + 12/2 should be, or what 20/2 – 3/2 equals. Ideally, they will realize that you merely need to add or subtract the numerators, and leave the size of the steps alone.

The students will likely catch on to the pattern well before they are able to phrase it as "to add two fractions with a common denominator, you add the numerators and keep the denominators the same." It is not necessary that they be able to come up with these words, for it is the understanding of the concept and the process that are most important. However, if the children are excited by this discovery, it might be a worthwhile exercise in vocabulary to see (with some prodding and prompting) that they can recognize that it is the numerators which are added and the denominators which are left alone.

In order for students to fully catch on to the pattern, you may need to run a few more exercises with different number lines. For example, try a number line divided into thirds. To act out 5/3 + 2/3 will thus look like:



If work with both half-steps and third-steps does not make things click with your students, try out quarter-steps. For example, 9/4 – 3/4 = 6/4, as can be acted out as:



As your students work out these various problems, write down each computed equation in a place for later reference. Thus, when everything is done (and you are back inside), you can show the list to your students and have them discuss it. For example, a partial list might look like:

3/2 + 5/2 = 8/2
2/2 + 1/2 = 3/2
7/2 + 2/ 2 = 9/2
5/2 – 1/2 = 4/2
8/2 – 3/2 = 5/2
5/3 + 2/3 = 7/3
1/3 + 1/3 = 2/3
3/4 + 5/4 = 8/4
9/4 – 3/4 = 6/4

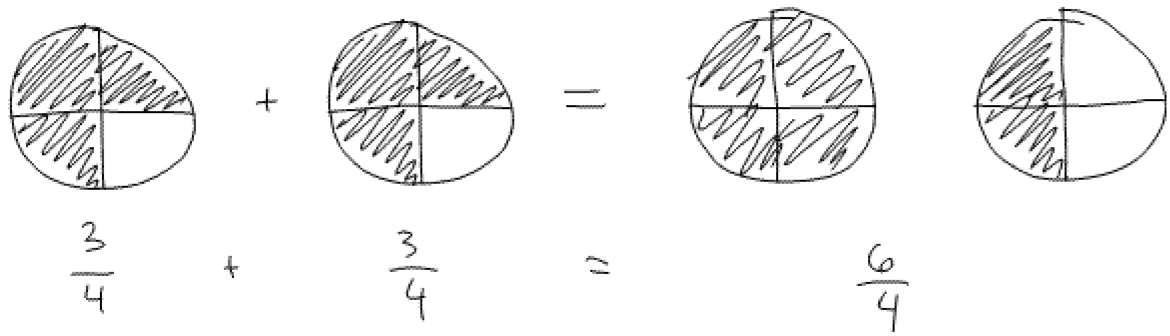
At this point, it is best to keep the fractions un-reduced (write 8/2 instead of 4, for example). This is a wonderful exercise in analyzing a list of data, looking for patterns, suggesting hypotheses, and drawing conclusions. In some sense, an exercise like this can be compared to the scientific method. Should there be any debate, you can conduct a new experiment by going back to the number lines and see if something turns out to be true.

Ideally, your students will not only notice the pattern, but will be convinced that it is correct. Hopefully, some students might protest that it is unnecessary to go back to the number line to verify 7/4 + 1/4 = 8/4, for example, because one quarter more that 7 quarters will clearly be eight quarters. When your students are first able to be certain of something, using common sense without having to go back and check, then they have attained a new level of abstract understanding. This is a monumental step in the development of their intellect.

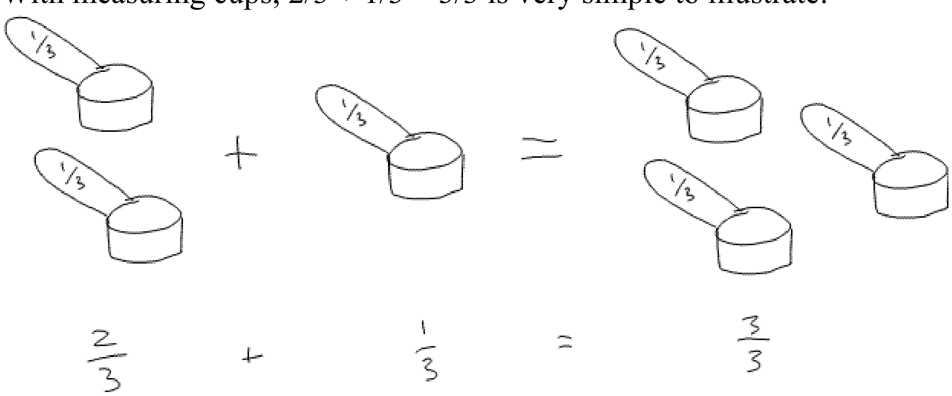
Challenge the class by giving them fractions with sizes that you have not yet worked with. For example, ask them to calculate 7/10 + 2/10 or 8/25 – 3/25. When problems like these become easy, challenge them further to explain what is happening. The best answer for the first is probably that "when you have 7 pieces and add 2 more of the same size, you get 9 pieces of the same size." However, it is quite possible that your students will have clever, creative, and equally correct ways to view what is going on. These should be shared among the class! Each can look at things however he or she may like, so long as it leads to the correct answers.

Another way to view this process is to imagine that the denominator of a fraction is actually a unit. For example, instead of 3/4 + 5/4, we could write this as "3 fourths + 5 fourths." Because the units are the same, we are able to add: 3 fourths + 5 fourths = 8 fourths.

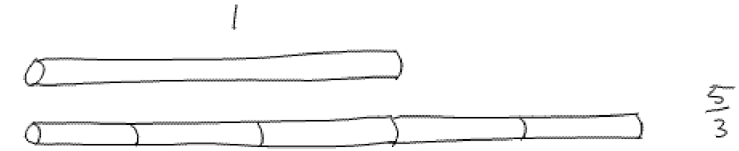
It is easy to illustrate the addition and subtraction of fractions with common denominators. For example, 3/4 + 3/4 = 6/4 can be illustrated using the areas of circles which have been subdivided into quarters:

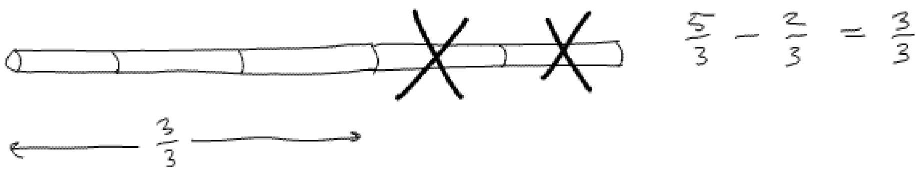


With measuring cups, 2/3 + 1/3 = 3/3 is very simple to illustrate:



To subtract is just as easy – just represent the first fraction with cups, rods, or areas, and then take away as many pieces as the numerator of the second fraction. For example, to subtract 5/3 – 2/3 with fraction rods looks like:

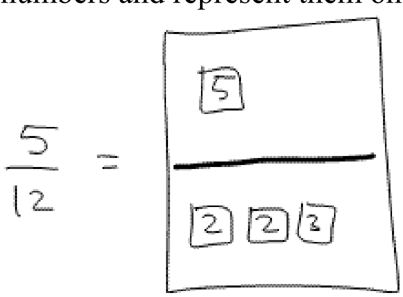


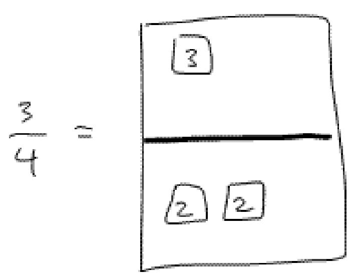


It is important for children to properly understand adding fractions with common denominators before moving on to adding fractions with different denominators. They must have a good understanding of what is happening and why, not just a procedure to follow blindly (if the bottoms are the same, add the tops and leave the bottom alone). This will enable them to understand the concepts better. Even children who get the answers correctly should be asked to explain the way they view the problem.

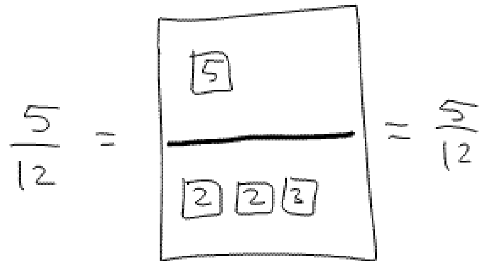
Next, introduce problems where the denominators are different, for example 5/12 + 3/4. Have the class discuss what to do with this problem. There will likely be some in the class who want to add the numerators. However, if the answer to this problem is to be 8, ask them to explain "8 what?" Should it be 8/12, for example, or 8/4? Hopefully, they can immediately see that these are very different answers. The fraction 8/12 is less than 1 while 8/4 is greater than 1, for example.

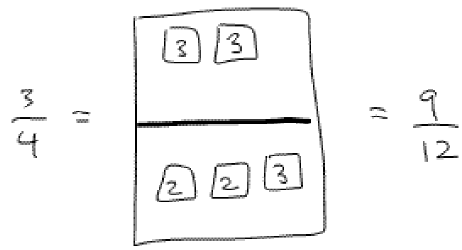
No, just as we convert fractions to a common denominator to compare them, we do the same to add or subtract them. Thus we factor the numerators and denominators into prime numbers and represent them on symbols boards:





Next, we play the "make the bottoms the same" version of the symbols game, and multiply the new fractions back together:

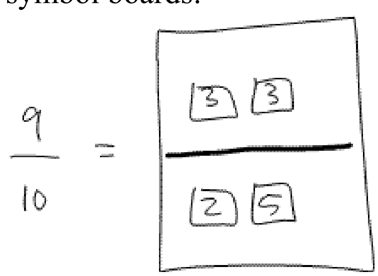


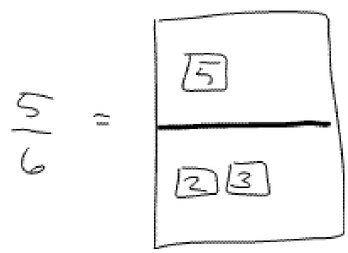


Now that we have equivalent fractions with the same denominators, we can add them: 5/12 + 3/4 = 5/12 + 9/12 = 14/12.

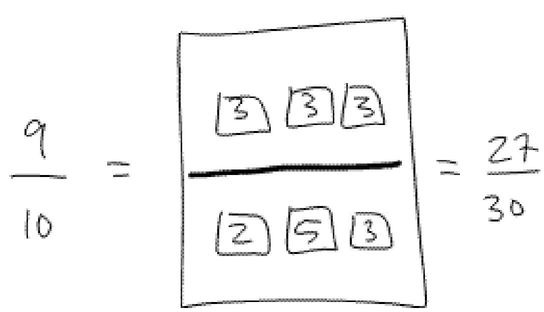
Make sure that the students are comfortable with the process of finding the common denominator and adding the fractions before worrying about reduced answers. We have now reached the point in mathematics where problems can take many steps. It is crucial that each step be easy and well-understood before expecting a student to do all of them in a row successfully. However, when you work out a problem like this with the whole class, it might be a good idea to see if anyone notices that "there is a 2" in both 14 and 12, thus the fraction 14/12 can be reduced.

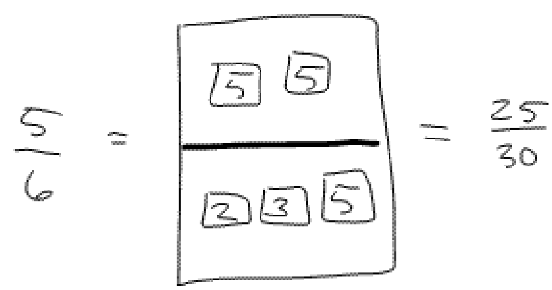
As another example, let us work out 9/10 – 5/6. We first factor the numbers out onto symbol boards:





Next, we find a common denominator by "making the bottoms the same":





After multiplying everything back together, we have obtained equivalent fractions with the same denominator. These can now be easily subtracted. Thus 9/10 – 5/6 = 27/30 – 25/30 = 2/30. This can then be reduced to 1/15.

When the class is comfortable with factoring and using the symbol boards like this, they can begin to work without them. Rather than break out the prime number cards, they can play the game on paper. For example, to work out 3/8 + 7/10, the students can first write out the problem:

$$\frac{3}{8} + \frac{7}{10}$$

Next, they can write out the problem in factored form. Encourage them to leave a good bit of space after each fraction, so that there will be room to insert more factors:

$$\frac{3}{8} + \frac{7}{10} = \frac{3}{2 \times 2 \times 2} + \frac{7}{2 \times 5}$$

From here, the students can recognize that, just as in the symbols game, the first fraction will need a 5 in denominator while the second fraction will need two 2's. As with the symbols game, this can only be done by putting the same factors up top as well:

$$\frac{3}{8} + \frac{7}{10} = \frac{3 \times 5}{2 \times 2 \times 2 \times 5} + \frac{7 \times 2 \times 2}{2 \times 5 \times 2 \times 2}$$

We can now put the fractions back together, resulting in a fraction problem with common denominators:

$$\frac{3}{8} + \frac{7}{10} = \frac{3 \times 5}{2 \times 2 \times 2 \times 5} + \frac{7 \times 2 \times 2}{2 \times 5 \times 2 \times 2}$$

$$= \frac{15}{40} + \frac{28}{40} = \frac{43}{40}$$

This is a bit of work, but it is far less than that required by using the "least common multiples" approach. Some teachers skip the least common multiples altogether and have the students just multiply each fraction by the denominator of the other. In the above case, that means multiplying the first fraction by 10 over 10 and the second by 8 over 8:

$$\frac{3}{8} + \frac{7}{10} = \frac{3 \times 10}{8 \times 10} + \frac{7 \times 8}{10 \times 8}$$

$$= \frac{30}{80} + \frac{56}{80} = \frac{86}{80}$$

This does obtain the correct answer, but usually ends up with a fraction that requires significant reduction. Also, this tends to require large multiplications, which can lead to mistakes. For example, when 4/15 +6/25 is worked out with factoring, the work looks like:

$$\frac{4}{15} + \frac{6}{25} = \frac{2 \times 2 \times 5}{3 \times 5 \times 5} + \frac{2 \times 3 \times 3}{5 \times 5 \times 3}$$

$$= \frac{20}{75} + \frac{18}{75} = \frac{38}{75}$$

In this case, the answer is already in reduced form. To work it out with multiplying each fraction by the denominator of the other, the work required is:

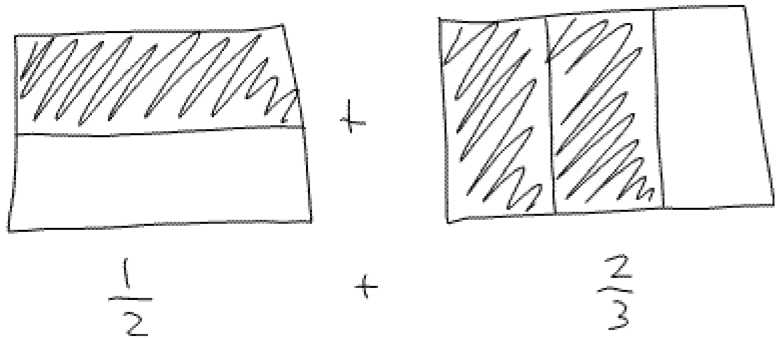
$$\frac{4}{15} + \frac{6}{25} = \frac{4 \times 25}{15 \times 25} + \frac{6 \times 15}{25 \times 15}$$

$$\begin{array}{r} {}^{2} \\ 25 \\ \times 15 \\ \hline 125 \\ 25 \\ \hline 375 \end{array}$$

$$= \frac{100}{375} + \frac{90}{375} = \frac{190}{375}$$

Here, the multiplications are a bit more difficult (25 × 15 is not easily done mentally) and the end result will need to be reduced. When this problem is worked out by using the least common multiple approach, the student will have to list out the multiples of 15 (15, 30, 45, 60, 75, 90...) and 25 (25, 50, 75...) in order to find the common denominator, which certainly takes more time than factoring 15 = 3 × 5 and 25 = 5 × 5.

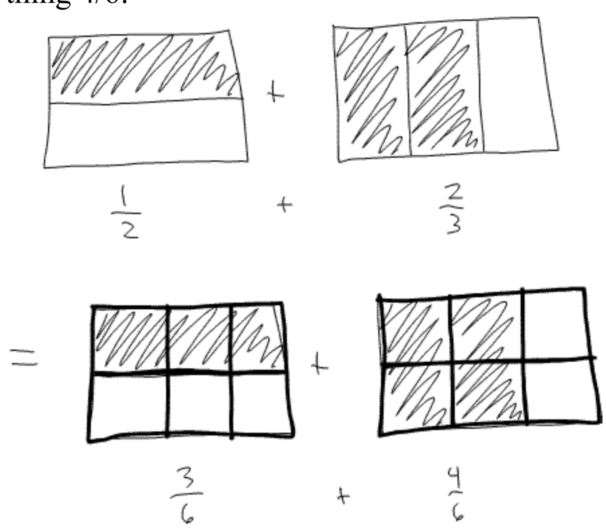
If students want a visual illustration of the entire process of combining fractions with different denominators, you can work out something like 1/2 + 2/3 with areas. First, we illustrate 1/2 and 1/3 as fractions of a standard-sized whole rectangle:



With the symbols game, we will find a common denominator by putting a 3 in the top and bottom of the 1/2 and a 2 in the top and bottom of the 2/3:

$$\frac{1}{2} + \frac{2}{3} = \frac{1 \times 3}{2 \times 3} + \frac{2 \times 2}{3 \times 2} = \frac{3}{6} + \frac{4}{6}$$

When we put a 3 in the top and bottom of the 1/2 fraction, what we are really doing is splitting each piece into three pieces. This gives us three times more pieces (from 1 to 3 in the numerator) but makes each piece three times smaller (from 2 to make a whole to 6 in the denominator). Thus, the 1/2 becomes 3 sixth-sized pieces. Similarly, when we put the 2 in the top and bottom of the 2/3, we are splitting each of these third-pieces in half, making the whole thing 4/6:



Now we are adding pieces of the same size (fractions with a common denominator). Adding 4 of these pieces to 3 of the same size, we end up with 7. Thus, we have illustrated that 1/2 + 2/3 = 3/6 + 4/6 = 7/6.

A very common mistake that students make with fractions is confusing addition with multiplication. This can be surprising at first, for people rarely confuse the two operations when combining whole numbers. They rarely, for example, confuse 2×5 with $2 + 5$, except occasionally and when working very quickly. It is very likely that $2/3 + 5/4$ is confused with $2/3 \times 5/4$ because neither operation makes much sense to the children. When the procedures but not the reasons behind them are given to students to copy and memorize, the result is only occasional success with no real thinking or understanding underneath. It is for reasons like this that fraction operations ought to be taught slowly and illustrated in a variety of ways, so that the students can imagine what is really going on.

In truth, nearly everything comes down to the definition of *numerator* and *denominator*. If children understand that the number underneath the line gives the size of each piece, written as the number of these pieces required to make a whole, then so much of fractions will make sense. You can only add, subtact, and compare fractions with the same denominator, because these things can only be done with pieces of the same size. Take your time to introduce fractions, and make sure to emphasize the role of the denominator in each instance. When the concept truly clicks for a student, suddenly everything ought to make a lot of sense. And when the concept is taught through factoring, even algebra will be easy to understand.

Questions:

(1) Illustrate $5/4 + 7/4$ on a number line.

(2) Illustrate $5/4 - 2/4$ with measuring cups.

(3) Illustrate $2/5 + 4/5$ with fraction rods.

(4) Illustrate $7/3 - 2/3$ with shaded-in areas.

(5) Show how $4/15 + 5/6$ can be computed with the symbols game.

(6) Show how $5/14 - 3/10$ can be computed with the symbols game.

(7) Show how $3/4 + 1/2$ can be illustrated with areas and then solved by subdividing those areas.

(8) Show how $17/30 + 5/12$ can be computed by playing the symbols game on paper.

Chapter 28: Proper and Improper Fractions

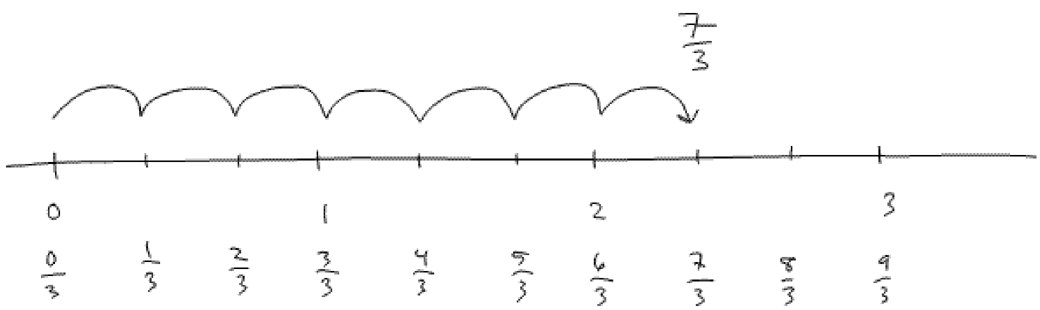
A fraction is called *improper* if the numerator is greater than the denominator. For example, 7/3 is improper because the numerator 7 is greater than the denominator 3.

A fraction is *proper* if it is not improper. For example, 3/5 and 2/7 are proper fractions.

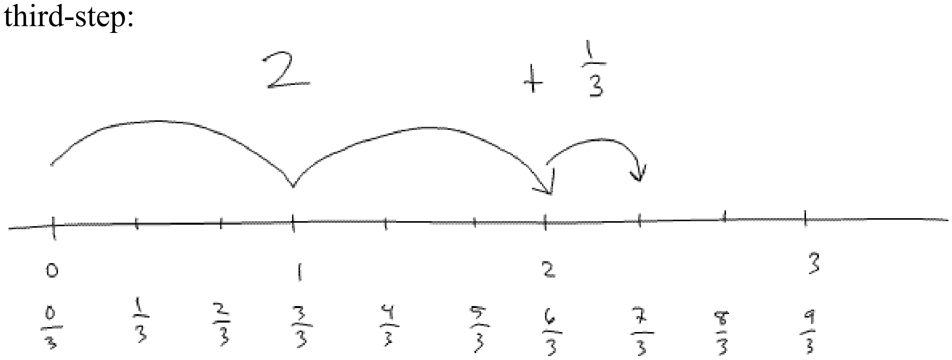
A *mixed* number consists of a whole number added to a proper fraction. For example 4 + 1/5 is a mixed number. It is common to skip the "+" in mixed numbers. Thus 4 + 1/5 is commonly written $4\frac{1}{5}$. This notation is very unfortunate because in nearly every other situation in mathematics, when two things are put side-by-side like this, it represents multiplication and not addition. However, this is common usage and thus must be accepted and explained to students.

Students should be taught how to convert improper fractions into mixed numbers and vice-versa. This will help them to emphasize that fractions are just another way to represent division. This will also give students a feel for the size of a fraction, relative to the nearest whole numbers.

The best place to begin is on a number line. For example, look at the improper fraction 7/3 on a number line that has been divided into thirds:



The usual way to look at this number is as 7 third-steps from start. However, we could also reach this same mark by first taking as many whole steps as possible, and then as many third-steps as necessary. In this case, we can reach this mark with 2 whole steps and then 1 third-step:

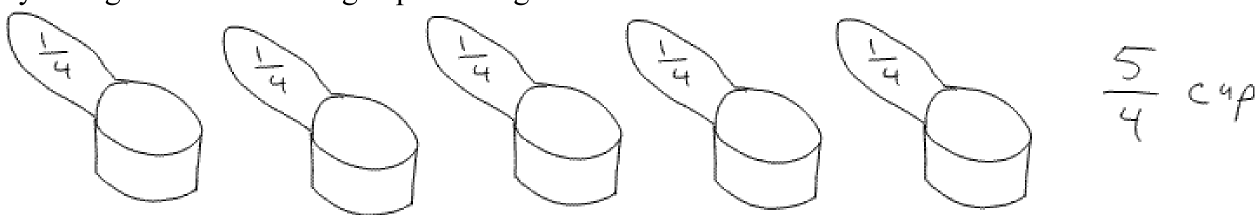


This explains why the mixed number equivalent to 7/3 is 2 + 1/3, also written $2\frac{1}{3}$.

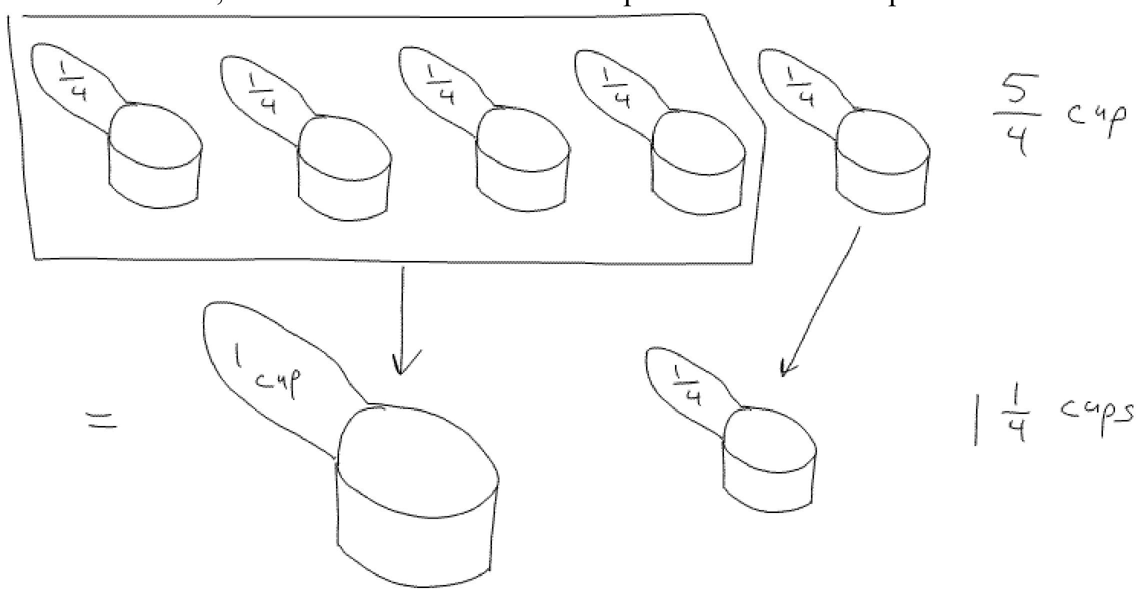
When we multiply, divide, and compare fractions, it is easiest if our fractions are written in improper form. This is why the word "improper" is a bit unfair. It would be better to call such fractions "working fractions," much like the work clothes that we might put on to do yard work. When we want to display a fraction in its prettiest form, however, we use mixed and proper fractions, like the proper clothes we wear to weddings and formal occasions.

The fraction 7/3 is explained with its two numbers – there are 7 pieces and it takes 3 of these pieces to make a whole. The equivalent mixed fraction $2\frac{1}{3}$ is explained with three numbers. This mixed fraction emphasizes that the number is a little bit bigger than 2.

We could use measuring cups to illustrate the conversion of an improper fraction to proper form. Suppose a recipe calls for 5/4 cups of sugar. We certainly could measure this out by filling the 1/4 measuring cup with sugar 5 times:



However, we know that 4 of these 1/4 cups makes a whole cup:



Instead of filling a 1/4 cup 5 times, we could fill a 1-cup and a 1/4-cup instead. This shows why $5/4 = 1\frac{1}{4}$.

The best short-cut for converting a fraction from improper to proper form is division. For example, suppose we want to represent the improper fraction 73/5 as a mixed number. We know that this number has 73 parts, and it takes 5 of those parts to make a whole. To figure out how many wholes we can make from 73, we divide 73 ÷ 5. With long-division, this looks like:

$$\begin{array}{r} 14 \\ 5\overline{)73} \\ -5 \\ \hline 23 \\ -20 \\ \hline 3 \end{array}$$

We began by dividing the 7 tens by 5. We were able to put aside 1 group of 5 tens, with 2 tens left over. We then converted the 2 tens into 20 ones, added them to the 3 ones, and got 23. We then divided these 23 ones by 5 and got 4 groups of 5 with a remainder of 3.

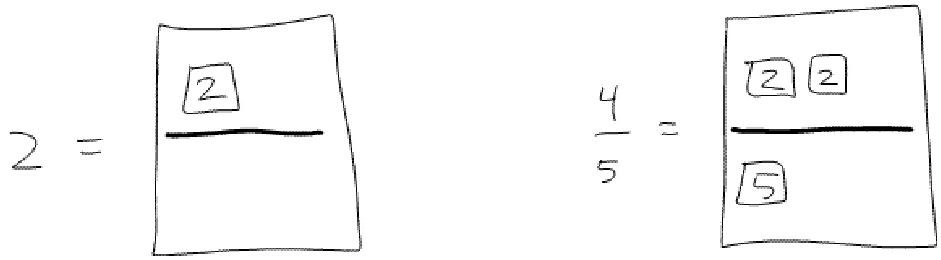
At this point in the process, we should divide the 3 ones by 5, if possible. Before we had fractions, we did not know how to divide 3 by 5, and so we would say that 73 ÷ 5 = 14, "with a remainder of 3". Now, however, we know that 3 ÷ 5 = 3/5. Thus, we can say that 73 ÷ 5 = $14\frac{3}{5}$. This illustrates how division can convert an improper fraction like 73/5 into its equivalent mixed number $14\frac{3}{5}$. The improper fraction 73/5 describes a number formed by 73 pieces which are each 1/5 of a whole, while the mixed-number $14\frac{3}{5}$ makes it clear that this number is a bit more than 14, but not quite 15.

With a little practice, students ought to be able to mentally convert improper fractions into mixed numbers. For example, 22/7 should not require setting up a long-division problem. If we try to divide 22 by 7, we will get 3 with a remainder of 1 because 7 × 3 = 21 is 1 shy of 22. Thus, we can immediately see that 22/7 = $3\frac{1}{7}$. Similarly, 11/4 = $2\frac{3}{4}$ because 4 goes into 11 twice, with a remainder of 3.

It is important to emphasize that the denominator will never change. We are putting together as many wholes as possible, but the remaining pieces will stay the same size. If we start with sixths, for example, then the remainder will be in sixths as well.

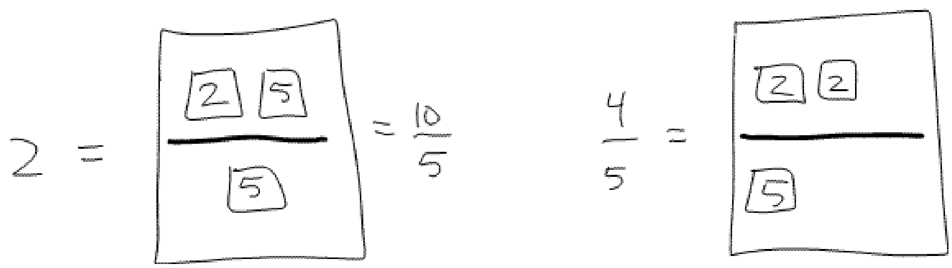
It is also useful to be able to convert back from mixed numbers to improper fractions. This sounds a bit strange when we use the word "improper," because children are generally discouraged from behaving improperly. However, if we look at improper fractions as "working fractions," it makes more sense. When we want to multiply or divide fractions, for example, it will be much easier when the fractions are written in improper form. In advanced mathematics, in fact, fractions are almost always kept in improper form.

To convert a mixed number into improper form, it is only necessary to remember the unwritten "+" sign. For example, $2\frac{4}{5}$ really means 2 + 4/5. When we represent these two numbers on a symbol board, it will look like:



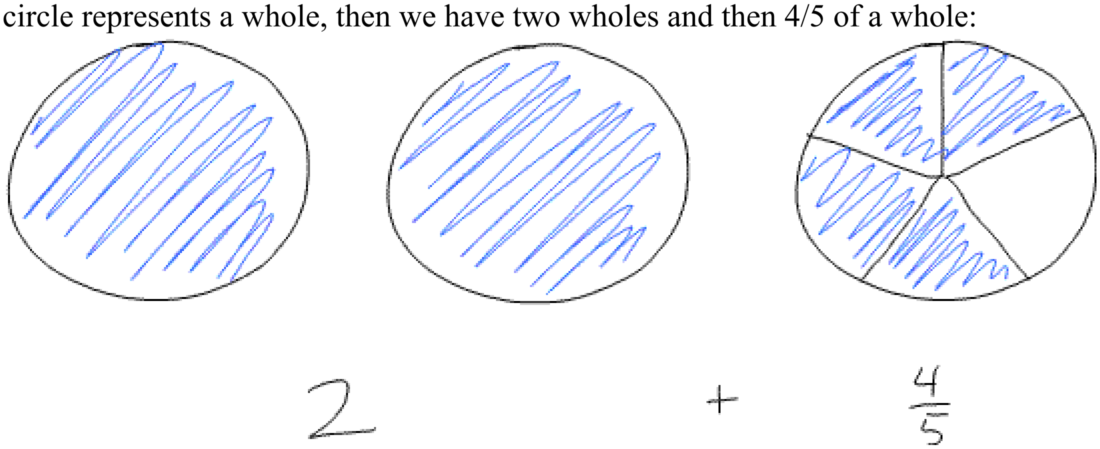
Some students get confused when trying to view a whole number like 2 as a fraction. However, this really need not be difficult. The two is not being divided by anything, and thus there is nothing in the denominator. Of course, by "nothing" we mean the number 1, the number which "does nothing" when multiplying and dividing.

When we perform the "make the bottoms the same" game, we need only put a 5 in the top and bottom of the 2 fraction:

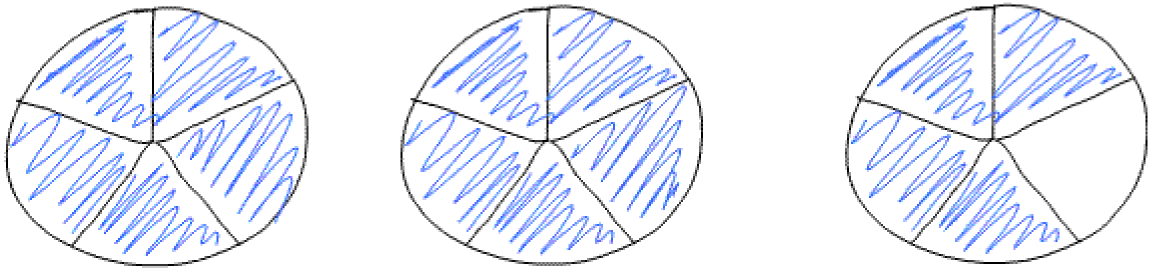


We can now add the two fractions: 10/5 + 4/5 = 14/5. This is the improper fraction equivalent to $2\frac{4}{5}$.

We can illustrate what has just happened by representing the number $2\frac{4}{5}$ with areas. If a circle represents a whole, then we have two wholes and then 4/5 of a whole:



When we put the 5 in the top and bottom of the 2 fraction, we split everything into 5 equal pieces. This divided each of the whole circles into 5 fifths:



$$\frac{10}{5} \qquad + \qquad \frac{4}{5}$$

The end result is that $2\frac{4}{5}$ has been converted into 14/5.

Whenever a mixed-number is converted into an improper fraction, the process will be remarkably similar to this. For example, let us convert $6\frac{3}{4}$ into an improper fraction by using the pencil-and-paper version of the symbols game:

$$6\frac{3}{4} = 6 + \frac{3}{4} = \frac{6 \times 4}{1 \times 4} + \frac{3}{4} = \frac{24}{4} + \frac{3}{4} = \frac{27}{4}$$

Here, we convert the 6 wholes into 24/4 by chopping each whole into 4 equal-sized pieces. These 24/4 can then be added to the 3/4 to make a total of 27/4.

Each time we convert a mixed number, we will have to break the wholes up into pieces of the same size as those in the fraction. This will result in as many pieces as the whole times the denominator. As a short-cut, we can do this directly: multiply the whole number by the denominator and then add the numerator:

$$6\frac{3}{4} = \frac{6 \times 4 + 3}{4} = \frac{24 + 3}{4} = \frac{27}{4}$$

$6\frac{3}{4}$ and then add

multiply

For example, to convert $9\frac{2}{5}$ into an improper fraction, we first multiply $9 \times 5 = 45$, next we add $45 + 2 = 47$, and finally we put this new numerator over our old denominator: 47/5. What we are really doing, of course, is converting the 9 into 45/5 and then adding the 2/5. As always, make sure your students really understand what is happening before teaching them this short-cut.

With a little practice and a lot of emphasis on the underlying reasons and meaning, students ought to be able to convert between proper and improper fractions without much difficulty.

Questions:

(1) Give examples of three proper fractions, three improper fractions, and three mixed numbers.

(2) Illustrate how a number line can convert 5/2 into a mixed number.

(3) Show how measuring cups can illustrate the conversion of 8/3 into a mixed number.

(4) Use long division to convert 225/7 into a mixed number.

(5) Use the symbols game to convert $7\frac{5}{6}$ into an improper fraction.

(6) Use the pencil-and-paper symbols game to convert $3\frac{1}{5}$ into an improper fraction.

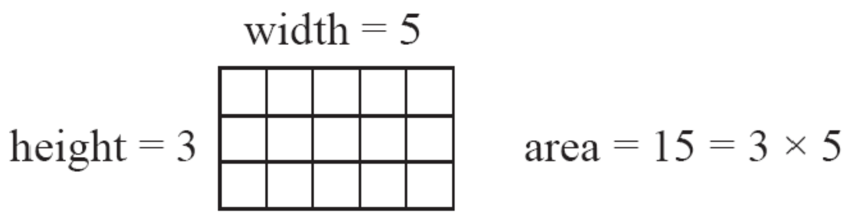
(7) Use areas to show how to convert $1\frac{3}{4}$ into an improper fraction.

(8) Compare the short-cut method and the pencil-and-paper symbols method for converting $5\frac{4}{9}$ into an improper fraction.

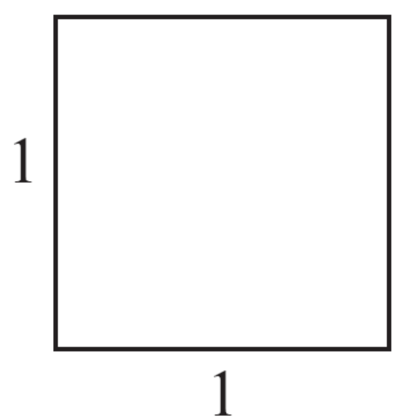
Chapter 29: Multiplying Fractions

In order to explain the multiplication of fractions, it is important to choose the right model for multiplication. The "repeated addition" model is useless, for there is no way to compute $\frac{2}{5} \times \frac{3}{4}$ by adding 2/5 to itself 3/4 of a time. Similarly, the grouping model makes no sense, as there is no easy way to compute 2/5 of a group with 3/4 in each group.

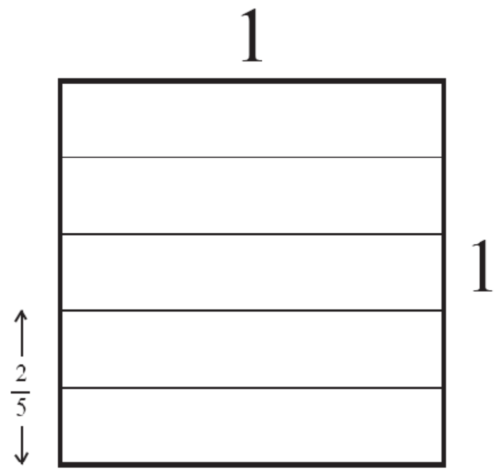
The area model is the best for explaining the multiplication of fractions. Remember that the area model explains 3 × 5 by looking at the area of a rectangle which has a height of 3 and a width of 5. There are 15 squares in this rectangle, and thus 3 × 5 = 15:

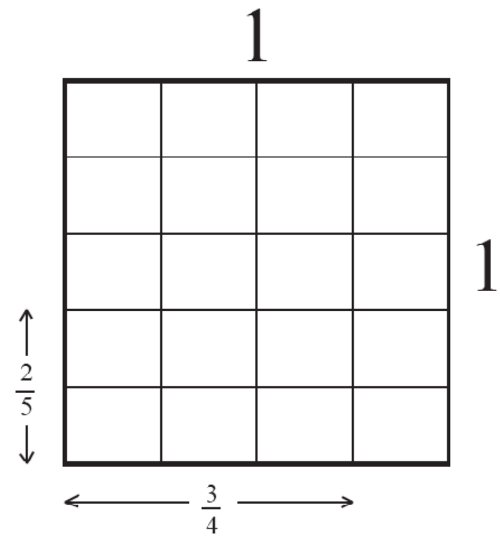


To explain the multiplication of $\frac{2}{5} \times \frac{3}{4}$, we will thus need to calculate the area of a rectangle with a height of 2/5 and a width of 3/4. We begin with a single square, with dimensions 1 × 1. This is because, as always with fractions, it is essential to identify what a "whole" represents.

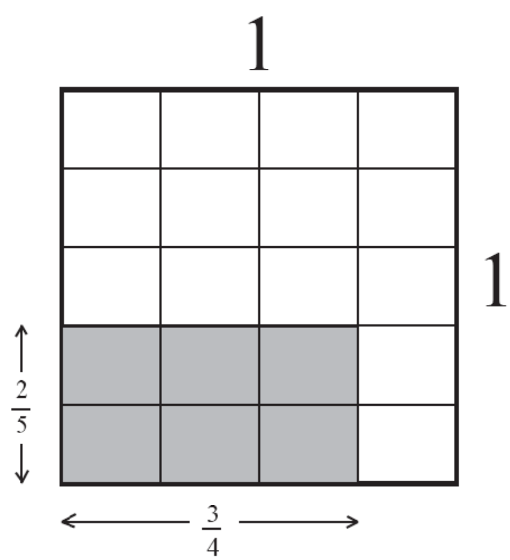


In order to get a height of 2/5, we divide the height into equal fifths. Similarly, in order to get a width of 3/4, we divide the width into equal quarters:



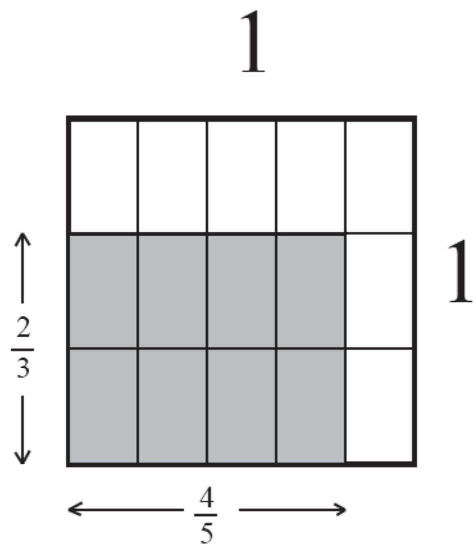


The product $\frac{2}{5} \times \frac{3}{4}$ is thus the area of the rectangle with a height of 2/5 and width of 3/4:



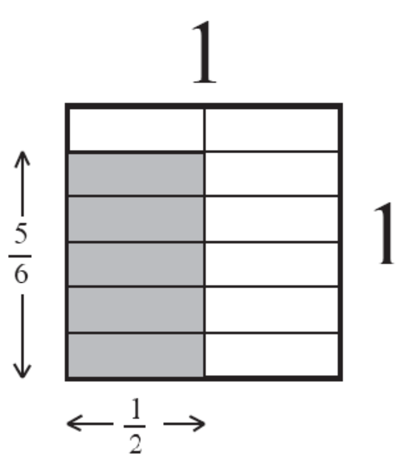
The answer is a fraction, because the entire big square is one whole 1 × 1 and this is clearly only part of it. This whole has been broken up into 20 pieces. The 6 shaded pieces make up the rectangle in question. Thus, the area of our rectangle is 6/20. We have thus computed that $\frac{2}{5} \times \frac{3}{4} = \frac{6}{20}$.

It is a good idea to go through a first example slowly, as above. The fact that we start with a single square is very important for the calculation of the final fraction. Afterward, we can go a bit more quickly. For example, to compute $\frac{2}{3} \times \frac{4}{5}$, we take a single 1×1 square, divide its height into thirds, and divide its width into fifths. The answer to our multiplication is the area of the interior rectangle with a height of 2/3 and width of 4/5:



Here, we can see that a rectangle with a height of 2/3 and a width of 4/5 is comprised of 8 parts, and that it takes 15 of these parts to make up the whole original square. Thus, $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$.

Similarly, we illustrate $\frac{5}{6} \times \frac{1}{2}$ with the following rectangle:



This shows that $\frac{5}{6} \times \frac{1}{2} = \frac{5}{12}$.

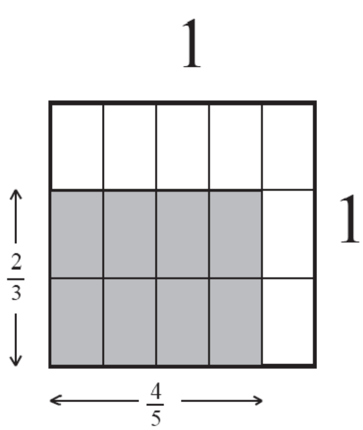
With a few examples like these worked out, show them all to your students and see if they can detect a pattern:

$$\frac{2}{5} \times \frac{3}{4} = \frac{6}{20}$$
$$\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$$
$$\frac{5}{6} \times \frac{1}{2} = \frac{5}{12}$$

Hopefully, they will notice that we obtain the answers by multiplying the numerators together to form the new numerator, and the denominators together to form the new denominator:

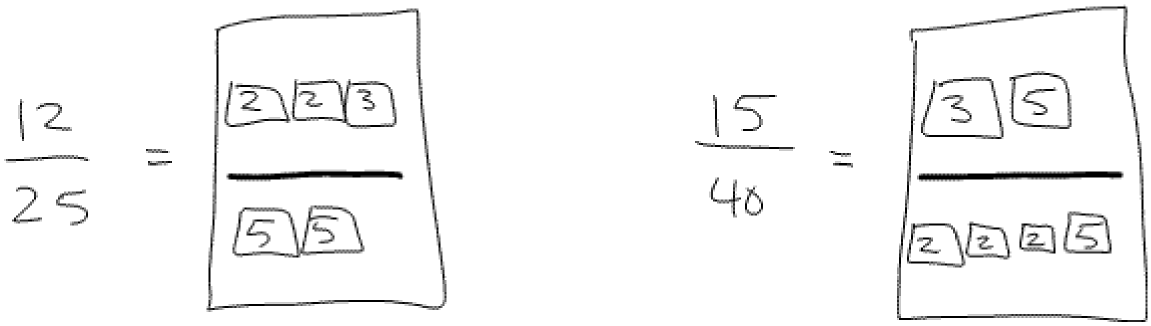
$$\frac{2}{5} \times \frac{3}{4} = \frac{2 \times 3}{5 \times 4} = \frac{6}{20}$$
$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}$$
$$\frac{5}{6} \times \frac{1}{2} = \frac{5 \times 1}{6 \times 2} = \frac{5}{12}$$

This is the short-cut for multiplying fractions: "multiply the tops together and the bottoms together." If we look closely at an illustrated example, we can see why this is the case. For example, look at $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$:

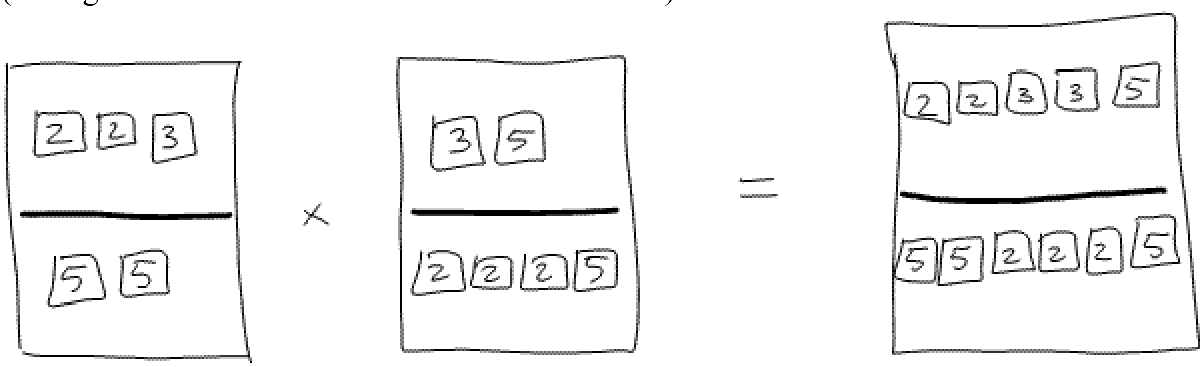


The shaded-in rectangle is 2 pieces high (the numerator of 2/3) and 4 pieces wide (the numerator of 4/5). This means that we are guaranteed to have 2 × 4 = 8 total pieces. The unit square has been divided by 3 along one side (the denominator of 2/3) and divided by 5 (the denominator of 4/5) along the bottom. This chops the whole square into 3 × 5 total pieces. Thus, our answer will consist of 2 × 4 pieces of a size that takes 3 × 5 to make a complete whole. This spells out why $\frac{2}{3} \times \frac{4}{5}$ equals $\frac{2 \times 4}{3 \times 5}$. Any other example will work the same way.

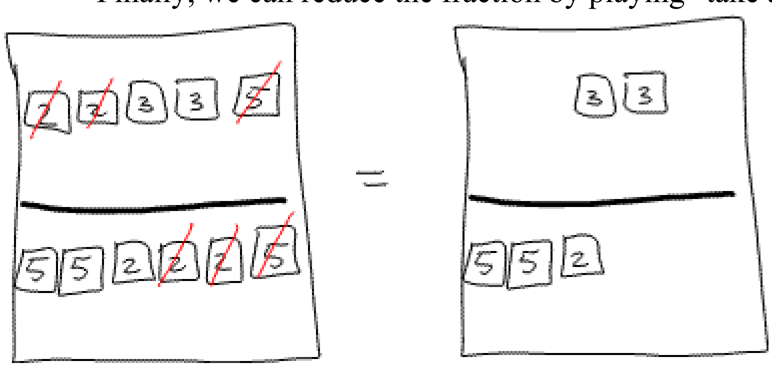
When students get the hang of this, we can introduce a new version of the symbols game. In this version, we represent two different fractions, and then multiply them together. All we do is slide all the top factors together and bottom factors together. For example, to multiply $\frac{12}{25}\times\frac{15}{40}$, we first represent each fraction on its own symbols board (by factoring):



Next, we multiply these two fractions together by combining the tops and bottoms (sliding all the factors from one sheet onto the other):



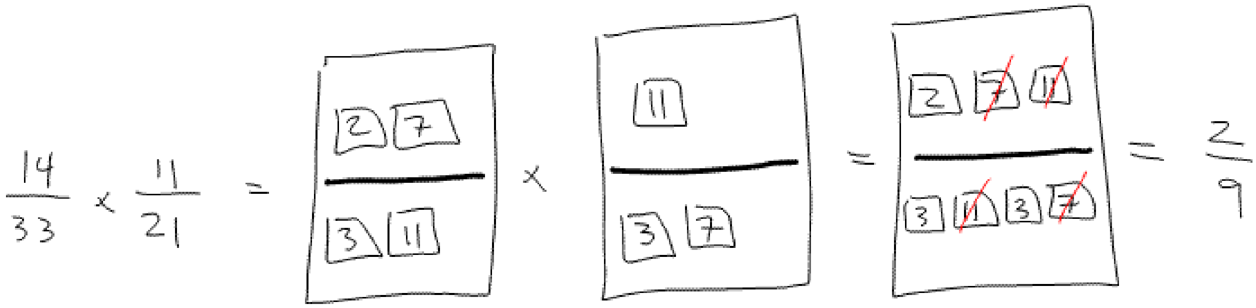
Finally, we can reduce the fraction by playing "take away as much as possible":



The answer, thus, in reduced form is: $\frac{12}{25}\times\frac{15}{40}=\frac{9}{50}$.

With a few examples like this, hopefully your students will notice that any of the numerator factors can be cancelled with any of the denominator factors. Thus, we can remove pairs of factors even before combining everything together. For example, when we multiply $\frac{14}{33}\times\frac{11}{21}$, we will end up with a numerator of $2 \times 7 \times 11$ and a denominator of $3 \times 11 \times 3 \times 7$.

When we reduce, we will be able to take away a pair of 7's (one from the top and one from the bottom) as well as a pair of 11's. With the symbols game, this looks like:



This is far preferable to multiplying out the numerators and denominators first, and then trying to reduce the result: $\frac{14}{33}\times\frac{11}{21}=\frac{14\times 11}{33\times 21}=\frac{154}{693}$.

The shortest method of all is to immediately recognize common factors and cancel them right away. For example, a child who knows how to factor 14 and 21 ought to notice that there is a 7 in both of them. Thus, we can divide the 14 by 7 (leaving 2) and the 21 by 7 (leaving 3). When this is done on paper, it will look like:

$$\frac{\overset{2}{\cancel{14}}}{33}\times\frac{11}{\underset{3}{\cancel{21}}}$$

If we further recognize that 11 goes evenly into both 11 and 33, we can divide these in just the same fashion, resulting in:

$$\frac{\overset{2}{\cancel{14}}}{\underset{3}{\cancel{33}}}\times\frac{\overset{1}{\cancel{11}}}{\underset{3}{\cancel{21}}}$$

Now, all that is necessary is to multiply the remaining factors in the numerator (2 and 1) and the factors in the denominator (3 and 3) to obtain the answer: 2/9.

Questions:

(1) Illustrate $\frac{2}{3}\times\frac{3}{4}$ with the area model for multiplication.

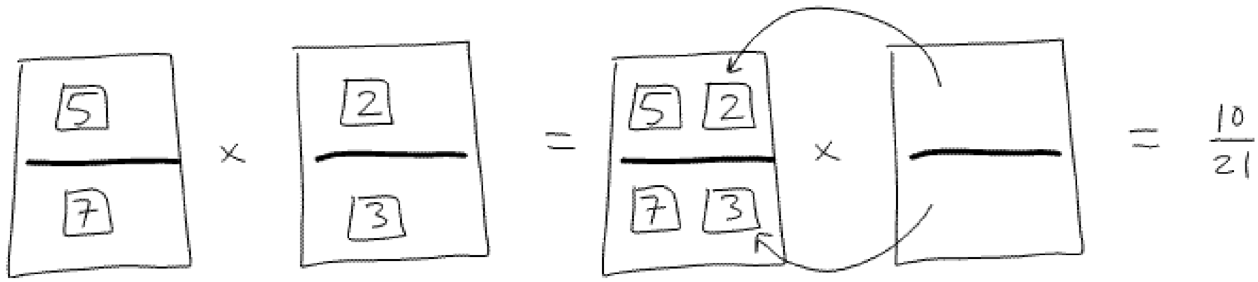
(2) Illustrate $\frac{3}{5}\times\frac{5}{6}$ with the area model for multiplication. Explain in words how this explains the short-cut method for multiplying fractions.

(3) Show how $\frac{24}{35}\times\frac{21}{16}$ can be computed using the symbols game.

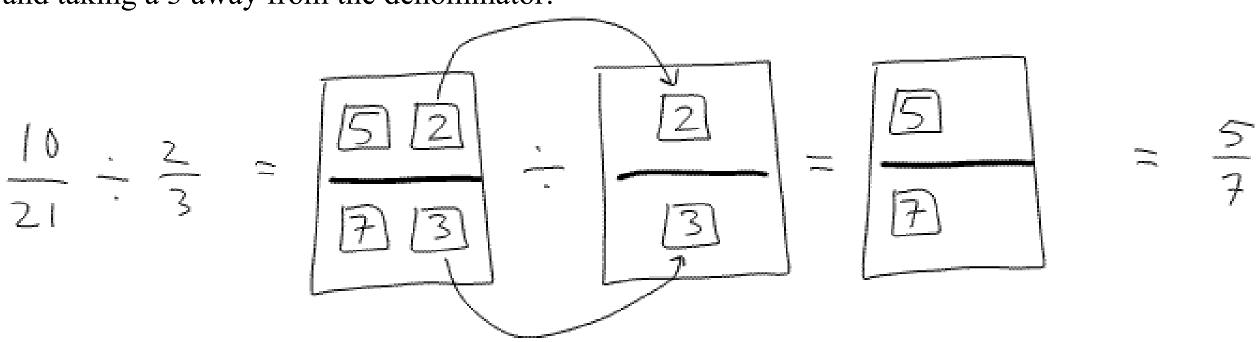
(4) Show how cancellation can occur before multiplication when computing $\frac{25}{36}\times\frac{9}{20}$ via the shortest possible method.

Chapter 30: Dividing Fractions

To explain the division of fractions, we go back to the original definition of division: the opposite of multiplication. When we multiply two fractions together, we slide the factors from the second fraction over into the first, combining numerators and denominators separately. For example, we compute $\frac{5}{7} \times \frac{2}{3}$ with a symbol board as follows:

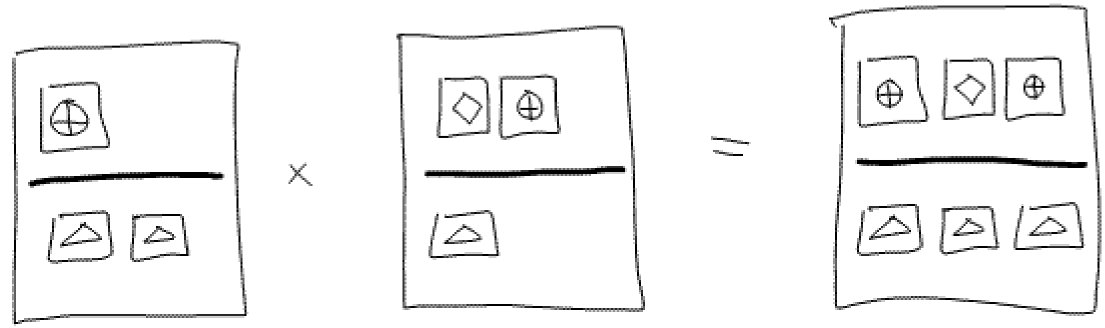


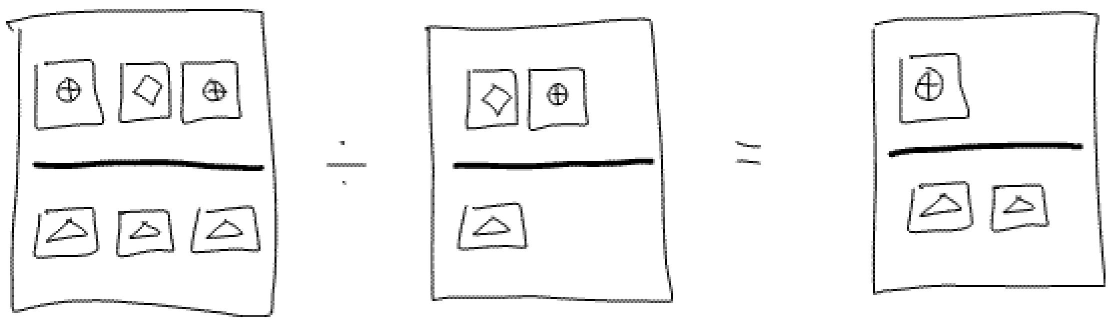
Because $\frac{5}{7} \times \frac{2}{3} = \frac{10}{21}$ and division is the opposite of multiplication, we can deduce that $\frac{10}{21} \div \frac{2}{3} = \frac{5}{7}$. Just as multiplying by 2/3 involves sliding a 2 into the numerator and a 3 into the denominator, dividing by 2/3 involves the exact opposite – taking a 2 away from the numerator and taking a 3 away from the denominator:



In some sense, this is very much the same process as multiplying fractions. When we multiply two fractions, we multiply the numerators and multiply the denominators. For example, $\frac{9}{10} \times \frac{3}{7} = \frac{9 \times 3}{10 \times 7} = \frac{27}{70}$. When we divide two fractions, we divide the numerators and divide the denominators. For example, $\frac{27}{70} \div \frac{3}{7} = \frac{27 \div 3}{70 \div 7} = \frac{9}{10}$. We can see that our answer is correct because we have two inverse equations: $\frac{9}{10} \times \frac{3}{7} = \frac{27}{70}$ and $\frac{27}{70} \div \frac{3}{7} = \frac{9}{10}$.

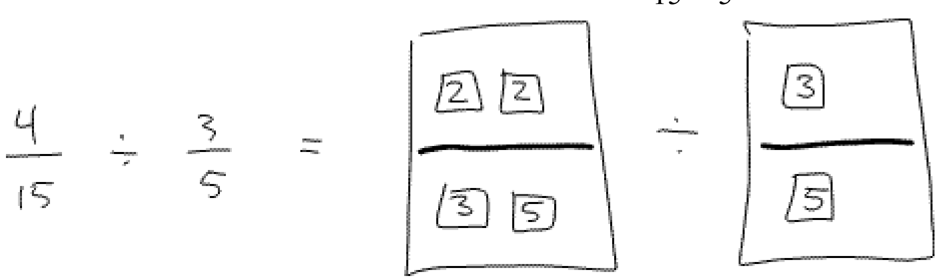
It may help your students understand the process if you present a pair of inverse examples worked out with abstract symbols. For example, consider the following:





When we multiply by $\frac{\lozenge\oplus}{\Delta}$, we put a $\lozenge$ and $\oplus$ in the numerator and a Δ in the denominator. When we divide by $\frac{\lozenge\oplus}{\Delta}$, we take a $\lozenge$ and $\oplus$ away from the numerator and take a Δ away from the denominator.

The only problem with this is that the first fraction might not have the factors which must be taken away. For example, if we want to divide $\frac{4}{15} \div \frac{3}{5}$, then we are stuck with the following:



We will be able to take the 5 away from the denominator, however, there is no 3 in the numerator of the first fraction for us to take away. This is the problem.

The trick is that, instead of taking away a factor from the numerator, we can just put a copy of it in the denominator. For example, suppose we want to divide $\oplus\Delta\div\oplus$. To do this, we take the $\oplus$ away from the numerator: $\oplus\Delta\div\oplus=\Delta$. However, we could put a $\oplus$ in the denominator instead for the same effect:

$$\oplus\triangle \div \oplus = \frac{\oplus\triangle}{\oplus} = \frac{\cancel{\oplus}\triangle}{\cancel{\oplus}} = \triangle$$

The answer comes out the same when we reduce.

In order to divide $\frac{4}{15}\div\frac{3}{5}$, we can use the same process. We take the 5 away from the denominator of 4/15. Because there is no 3 factor in the numerator of 4/15 for us to remove, we put a 3 in its denominator instead:

$$\frac{4}{15} \div \frac{3}{5} = \frac{\boxed{2}\,\boxed{2}}{\boxed{3}\,\boxed{5}} \div \frac{\boxed{3}}{\boxed{5}} = \frac{\boxed{2}\,\boxed{2}}{\boxed{3}\,\boxed{3}} = \frac{4}{9}$$

remove

bring down

We can verify that $\frac{4}{15}\div\frac{3}{5}=\frac{4}{9}$ by checking the inverse multiplication problem: $\frac{4}{9}\times\frac{3}{5}$.

$$\frac{4}{9} \times \frac{3}{5} = \frac{2\times 2}{3\times 3} \times \frac{3}{5} = \frac{2\times 2\times \cancel{3}}{3\times \cancel{3}\times 5} = \frac{4}{15}$$

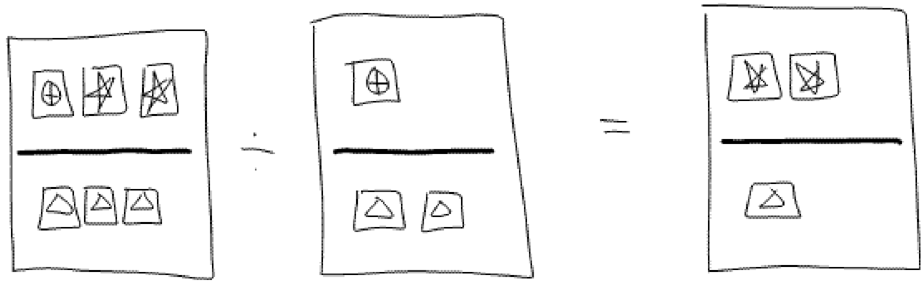
This confirms that $\frac{4}{15}\div\frac{3}{5}=\frac{4}{9}$.

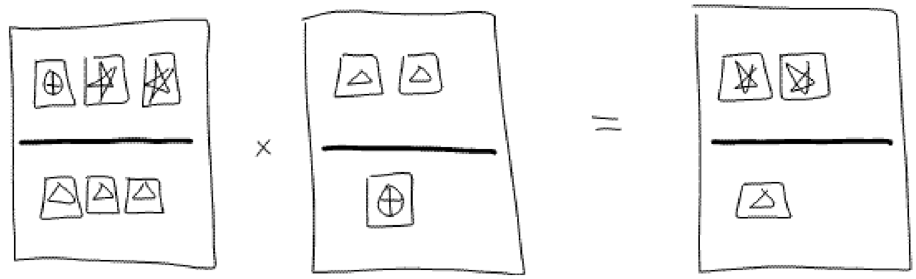
There is an even easier way to divide fractions: multiply by the *reciprocal* of the divisor. The reciprocal is just the "flip" of the fraction, with the numerator and denominator exchanged. For example, the reciprocal of 3/7 is 7/3.

As an example, look at $\frac{4}{15}\div\frac{3}{5}$ again. This is the same as $\frac{4}{15}\times\frac{5}{3}$, which can be multiplied and reduced as:

$$\frac{4}{15} \times \frac{5}{3} = \frac{\boxed{2}\,\boxed{2}}{\boxed{3}\,\cancel{\boxed{5}}} \times \frac{\cancel{\boxed{5}}}{\boxed{3}} = \frac{4}{9}$$

One way to see that this works is by reasoning. If we want to remove a factor from the numerator, we can put a copy of it in the denominator and then reduce. If we want to remove a factor from the denominator, we can put a copy of it in the numerator and reduce. For example, both of the following operations result in removing a $\oplus$ from the numerator and two Δ's from the denominator.

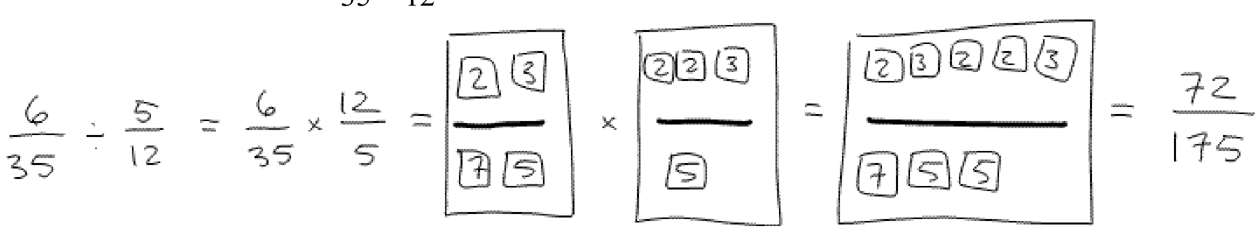




In the first, we use the fact that dividing by $\frac{\oplus}{\Delta\Delta}$ means taking those factors away from the numerator and denominator. In the second, we put everything together and reduce the resulting fraction, cancelling out a pair of $\oplus$ (one from the top and one from the bottom) and two pairs of Δ.

This example should make it clear that dividing by $\frac{\oplus}{\Delta\Delta}$ does the same thing as multiplying by $\frac{\Delta\Delta}{\oplus}$, the reciprocal. This will always be the case.

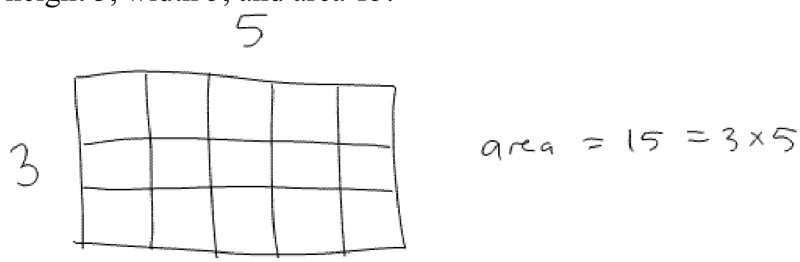
Best of all, this trick works even when the factors are not available for being taken away. For example, to calculate $\frac{6}{35} \div \frac{5}{12}$, all we have to do is multiply by the reciprocal 12/5 of 5/12:



We can verify that this is true by multiplying $\frac{72}{175} \times \frac{5}{12}$ to get $\frac{6}{35}$.

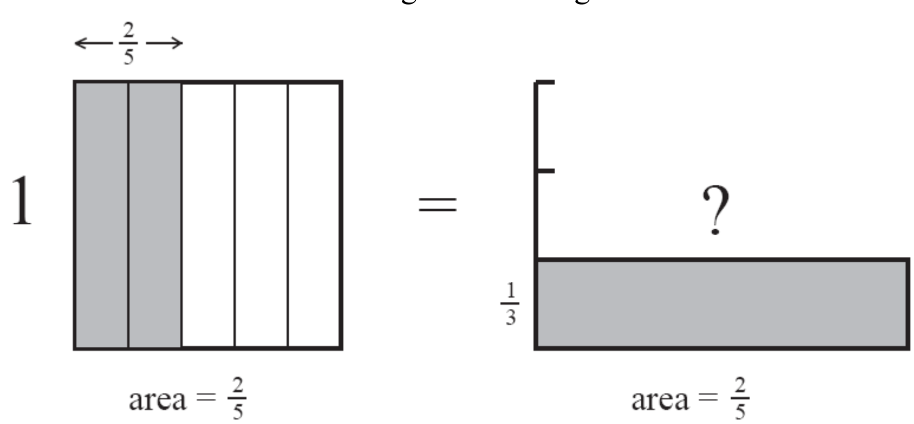
We now have two ways to explain to students why dividing by a fraction can be computed by multiplying by the reciprocal. First of all, we can run several examples using this method and then verify our answers by checking the inverse multiplication problems. For example, we can prove that $\frac{2}{3} \div \frac{3}{7} = \frac{2}{3} \times \frac{7}{3} = \frac{14}{9}$ by checking that $\frac{14}{9} \times \frac{3}{7} = \frac{2}{3}$. Second of all, we can reason our way through the symbols game, showing that multiplying by 7/3 will serve to take away a 7 from the denominator and a 3 from the numerator, which is what dividing by 3/7 serves to do.

For students who want yet another proof that dividing by a fraction will result in multiplying by its reciprocal, we can sketch out a visual proof. To do this, we go back to the area model of multiplication. The multiplication 3 × 5 = 15 is illustrated by a rectangle with height 3, width 5, and area 15:



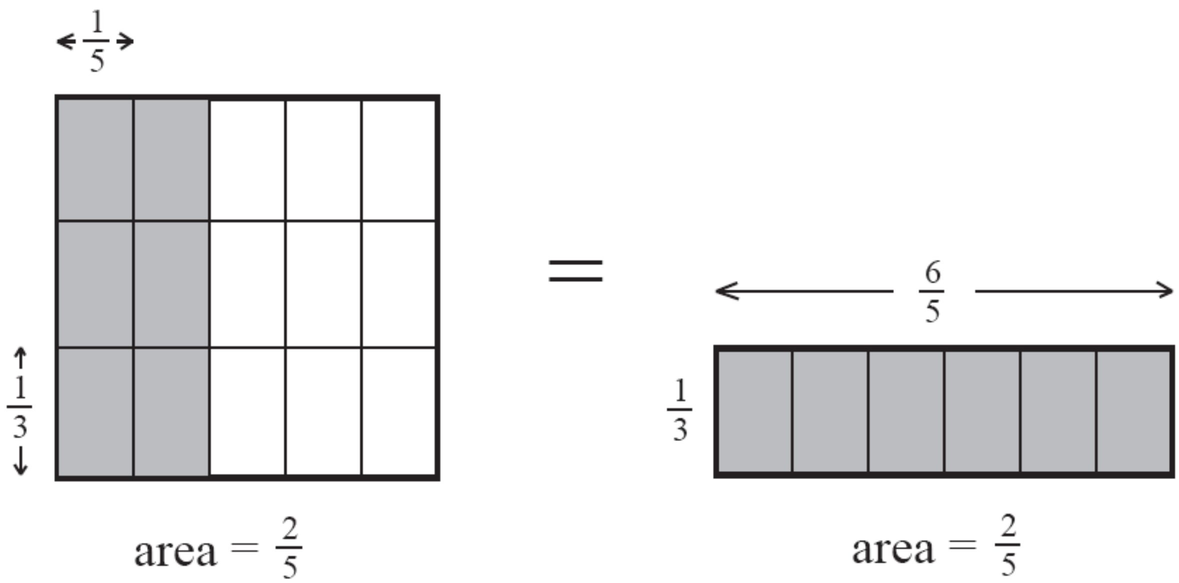
The corresponding division problem 15 ÷ 3 asks "what is the width of a rectangle that has an area of 15 and a height of 3?"

To compute a division problem like $\frac{2}{5} \div \frac{1}{3}$, we ask a similar question: "what is the width of a rectangle that has an area of 2/5 and a height of 1/3? To begin, we must start with an area of 2/5. The easiest way to do this is to take a square, divide it into 5 equal parts, and then shade in of them. This results in a rectangle with a height of 1 and a width of 2/5:



In order to divide this by 1/3, we need to somehow find a rectangle that has the same area, 2/5, but a height of 1/3 instead. The answer to $\frac{2}{5} \div \frac{1}{3}$ will be the width of that rectangle.

The most reasonable way to rearrange our 1 by 2/5 rectangle into something with a height of 1/3 is to split it into thirds horizontally. This chops our 2 fifth pieces into six pieces, each one with a height of 1/3 and a width of 1/5. We can put these 6 pieces side-by-side to form a rectangle with a height of 1/3 and a width of 6/5. Because we have only cut apart an area and rearranged it, this new rectangle will have the same area of 2/5 as the original one.



From this, we can see that a rectangle with an area of 2/5 and a height of 1/3 will have a width of 6/5. Thus, $\frac{2}{5} \div \frac{1}{3} = \frac{6}{5}$. This produces the same result as multiplying by the reciprocal, and thus offers a third way to verify that this works: $\frac{2}{5} \times \frac{3}{1} = \frac{6}{5}$. This also helps to make clear that dividing by 1/3 is the same thing as multiplying it by 3, because it involves slicing a rectangle into three parts (vertically) and then laying these three parts end-to-end.

There are thus many ways to verify that dividing by a fraction can be computed by multiplying by the fraction's reciprocal. All of them, however, ultimately rely upon the fact that division serves to do the opposite of multiplication. You could do this directly with your students, by first calculating a handful of fraction multiplications and then listing out all of the corresponding divisions.

For example, you might start with:

$$\frac{3}{5}\times\frac{2}{3}=\frac{2}{5} \qquad \frac{4}{9}\times\frac{8}{5}=\frac{32}{45} \qquad \frac{7}{10}\times\frac{2}{5}=\frac{7}{25} \qquad \frac{1}{3}\times\frac{1}{5}=\frac{1}{15} \qquad \frac{2}{1}\times\frac{1}{7}=\frac{2}{7}$$

Next, have the students name all of the division problems which correspond to the ones listed. Continuing our example, we get 10 different division problems:

$$\frac{2}{5}\div\frac{2}{3}=\frac{3}{5} \qquad \frac{2}{5}\div\frac{3}{5}=\frac{2}{3} \qquad \frac{32}{45}\div\frac{8}{5}=\frac{4}{9} \qquad \frac{32}{45}\div\frac{4}{9}=\frac{8}{5} \qquad \frac{7}{25}\div\frac{2}{5}=\frac{7}{10}$$

$$\frac{7}{25}\div\frac{7}{10}=\frac{2}{5} \qquad \frac{1}{15}\div\frac{1}{5}=\frac{1}{3} \qquad \frac{1}{15}\div\frac{1}{3}=\frac{1}{5} \qquad \frac{2}{7}\div\frac{1}{7}=\frac{2}{1} \qquad \frac{2}{7}\div\frac{2}{1}=\frac{1}{7}$$

At this point, open up a class-wide discussion. See if the students can detect any patterns which might be used to find a short-cut pattern for computing these divisions directly.

For example, a sharp-eyed student might notice that $\frac{1}{15}\div\frac{1}{3}=\frac{1}{5}$ could be calculated by dividing across the bottom like $\frac{1}{15}\div\frac{1}{3}=\frac{1}{15\div 3}=\frac{1}{5}$. This is legitimate.

Another student might extend this further by looking at $\frac{32}{45}\div\frac{8}{5}=\frac{4}{9}$ and seeing the pattern continue: we can divide the numerators and denominators separately. Thus $\frac{32}{45}\div\frac{8}{5}=\frac{32\div 8}{45\div 5}=\frac{4}{9}$. Again, this is perfectly correct.

If your student have noticed these patterns, ask them for ideas as to what might be happening in $\frac{2}{5}\div\frac{2}{3}=\frac{3}{5}$. It seems as though the 2's are disappearing and the 3 is moving up from the denominator to the numerator. It might be asking too much for a student to suggest multiplying by the reciprocal, but you might want to prod them along by mentioning that there is a special math term called a reciprocal, and explaining what it is. If all else fails, have them "flip and multiply" to get $\frac{2}{5}\times\frac{3}{2}=\frac{3}{5}$. Hopefully, your students will not only notice that this results in the right answer, but that it solves all the other ones as well.

As always, it is best for the students in your class to figure out the rules of mathematics by themselves. This is not always possible, and fraction division is certainly one of the trickiest operations in elementary mathematics to figure out. However, this is still no excuse for merely telling your students "this is how it is" and making them accept the "flip and multiply" without proof. You can certainly expect your students to come up with many fraction multiplication problems on their own. You can also expect them to know that division is the opposite of multiplication and thus name the division problems which correspond to their multiplication problems. This is an excellent first step forward, to have your students begin with a large number of division problems of which they are already certain. You can then proceed by pattern recognition, the symbols game, or the area model, depending upon the interest and ability of your students. In the end, you might need to suggest "flip and multiply" as a computational short-cut.

However, just throw this out as a possibility and have your students test it out to see if it works. They should be the judges of what does and does not work.

Questions:

(1) Name all the division problems which correspond to the following:

$$\frac{3}{4}\times\frac{2}{7}=\frac{3}{14} \qquad \frac{5}{12}\times\frac{7}{5}=\frac{7}{12} \qquad \frac{4}{5}\times\frac{7}{11}=\frac{28}{55} \qquad \frac{1}{3}\times\frac{2}{5}=\frac{2}{15}$$

(2) Show how to compute $\frac{\oplus\oplus}{\Diamond\Delta}\div\frac{\oplus}{\Delta}$ using the symbols game. Verify your answer with a multiplication.

(3) Show how to compute $\frac{\Delta\Delta\Delta}{\otimes\Diamond}\div\frac{\otimes}{WW\Diamond}$ using "flip and multiply." Verify your answer with a multiplication.

(4) Show how to compute $\frac{1}{2}\div\frac{1}{3}$ by using the area model for division.

(5) Compute each of the following with "flip and multiply" and verify the answer with multiplication:

(a) $\frac{25}{36}\div\frac{15}{4}$

(b) $\frac{2}{7}\div\frac{4}{5}$

(c) $\frac{2}{3}\div 6$

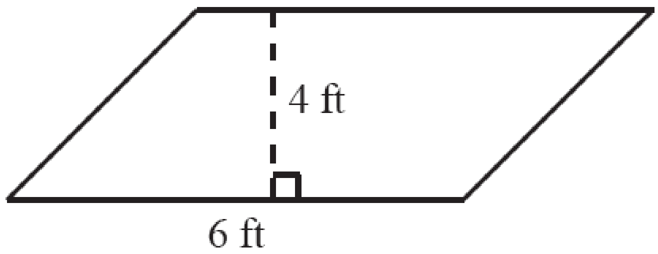
(d) $10\div\frac{2}{3}$

(e) $35 \div 5$

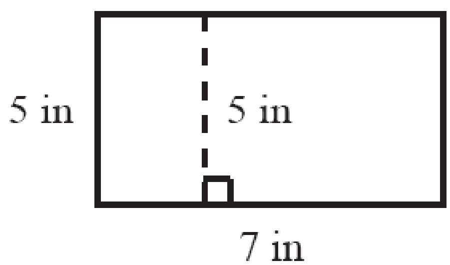
Chapter 31: More Areas

When students are able to divide, they are ready to learn how to calculate the areas of two new shapes: triangles and trapezoids.

The key to everything lies in knowing that the area of a parallelogram is found by multiplying the length of the base by the height, as discussed in detail in chapter 15. For example, the area of the following parallelogram will be 6 ft × 4 ft = 24 ft^2.



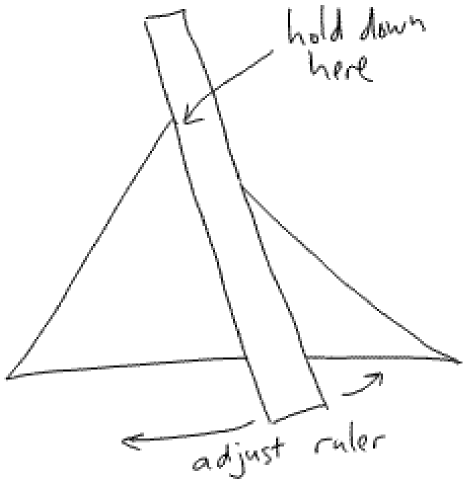
A rectangle, of course, is just a special kind of parallelogram, where the height is the same as one of the sides. For example, the area of the following rectangle is 5 in × 7 in = 35 in^2.

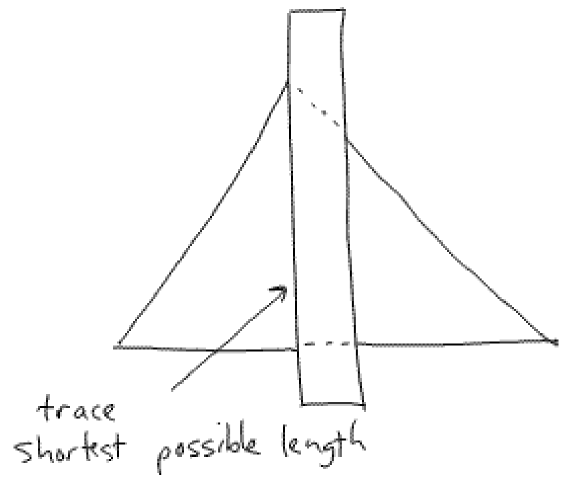


Review these two examples before you go on to cover triangles and trapezoids with your class.

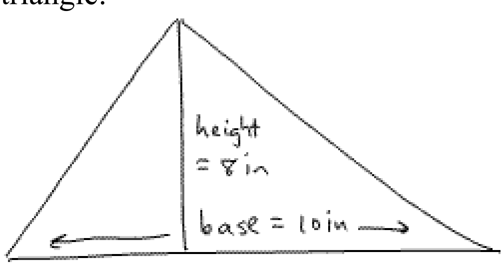
The formula for the area of a triangle is not generally something that a child will figure out on his or her own. Thus, it will be a good idea to prompt them with the following exercise. Each student will need two pieces of paper, a ruler, a pencil, and a pair of scissors. The scissors can be shared, but it would be best for each student to have a ruler.

First, have every student draw a nice, big triangle on a piece of paper. Have them use a ruler to make sure the lines are straight. Next, have them cut out their triangles with the scissors. Next, have them measure the length of the longest side and turn this down to make the base. Finally, have them try to draw the shortest line from the top vertex of the triangle down to the base. To do this, they should hold one edge of the ruler to this point, and swivel the ruler about until it seems to be the shortest possible length:

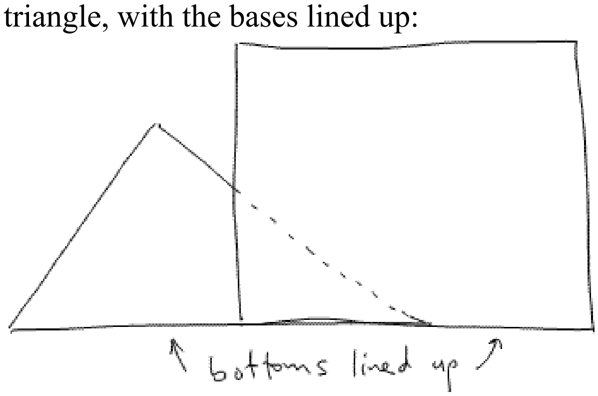




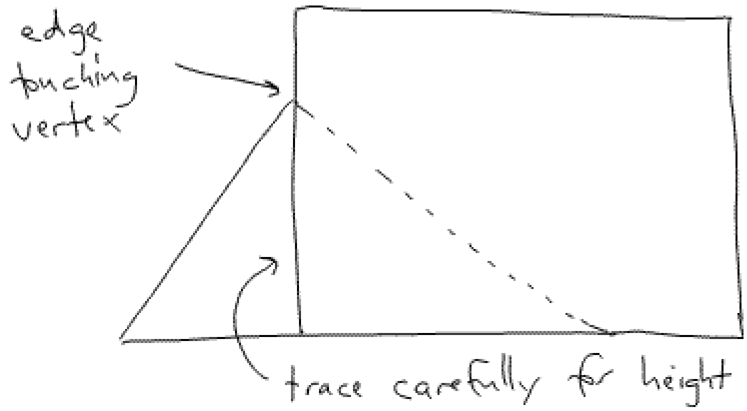
When they are done, have your students write the measurement of the height and base on the triangle:



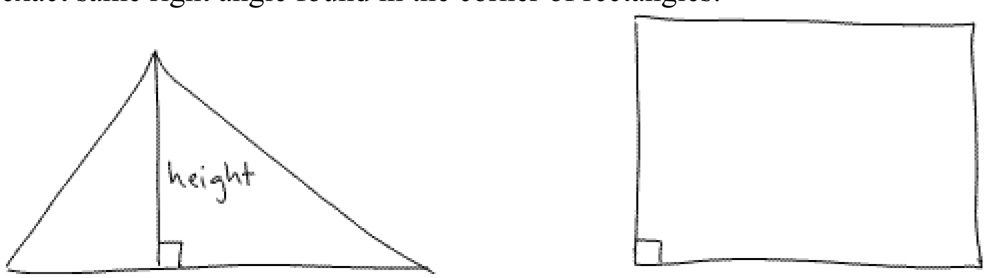
Let your students struggle with finding the height and making their best approximation of the shortest line from the top vertex to the base. Then, show them the following technique for finding it more easily. To do this, take the second piece of paper and have it partially overlap the triangle, with the bases lined up:



Next, slide the second piece of paper over, always with the bottoms lined up, until one edge just touches the top vertex of the triangle:

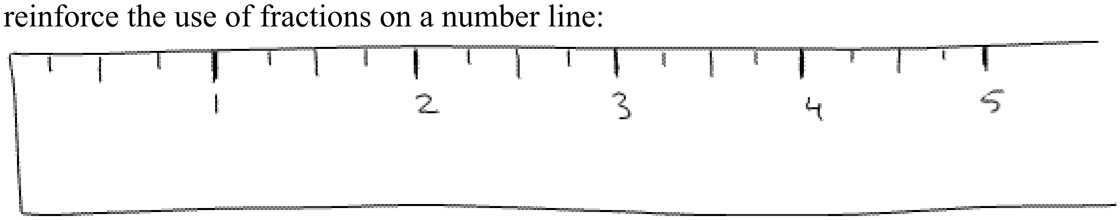


This will serve several purposes. First of all, some students will be frustrated by not being able to draw a perfect height. This exercise ought to be a relief to them. Second, this helps to illustrate that the shortest distance from a point to a line will make a right angle. Finally, this helps to explain why we use a little right-angle square at the corner of a height line – it is the exact same right angle found in the corner of rectangles:



If your students have trouble with this exercise, you can provide them with ready-made triangles photocopied onto sheets of paper for them to cut out. This will not help the students very much with developing their motor skills but it can save you some time. When you do this, you can ensure that the base and height of the triangles come out to have nice measurements.

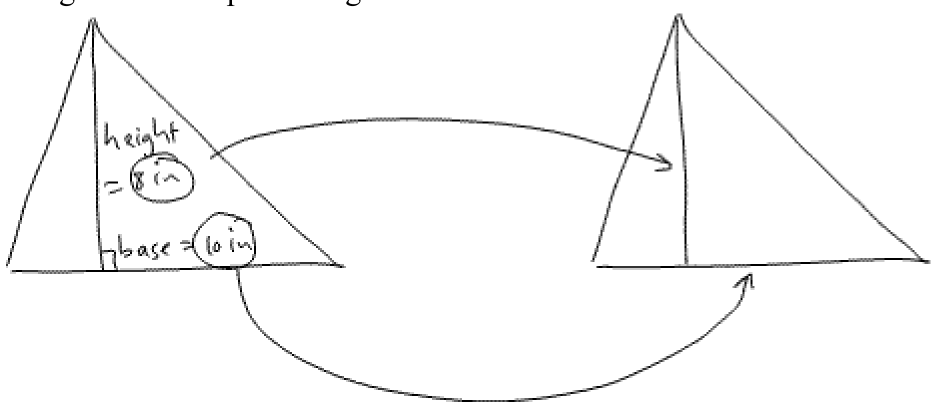
Incidentally, have the students measure only as accurately as they can do so comfortably. If they are not good at fractions, you can have them round their measurements to the nearest whole inch. Otherwise, have them measure to the nearest half, quarter, or eighth inch, depending on their level of ability. Measuring to a fraction of an inch, by the way, is an excellent way to reinforce the use of fractions on a number line:



When each student has cut out a triangle, found its height, and measured its dimensions, have them trace the triangle onto the second piece of paper. It can be a bit tricky to trace something as flexible as paper, so you can either do this exercise with firm paper like oak tag or else urge your students to trace gently and carefully. When they are done, they should then cut out the second triangle.

These two triangles are called *congruent* because they have the same size and shape. They can be placed to exactly overlap one another. We do not call them equal triangles because they aren't really the same. You can hold one triangle in your left hand, for instance, and the other in your right. Every measurement about the two triangles will be the same, but they are still technically different. This is why mathematicians came up with a special word: congruent.

After you explain this concept to the class, have them label the base and height of the second triangle. Hopefully, at least some students will realize that this can be done without a ruler. As long as the two triangles are aligned the same way, the dimensions from the first triangle can be copied straight onto the second:

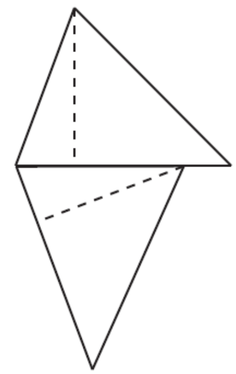


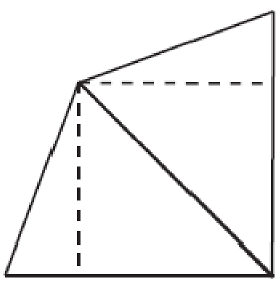
The sides which match up in this fashion are called *corresponding sides*. The bases correspond, for example. Similarly, the two sides on the left in the above figure correspond, and the side on the right of each triangle correspond.

Best of all, these two triangles have the same area.

At this point, challenge each student to put the two triangles together in a way that makes a shape whose area you know how to measure. Give them some time to play around. Incidentally, the acts of sliding around a shape, flipping it over, and rotating it are called *transformations*. To slide the triangle left, right, up, or down is called a *translation*. To flip the triangle over is called a *reflection*. To turn the triangle is called a *rotation*. Having students play in this way will help them prepare for advanced geometry and help them develop useful visualization skills.

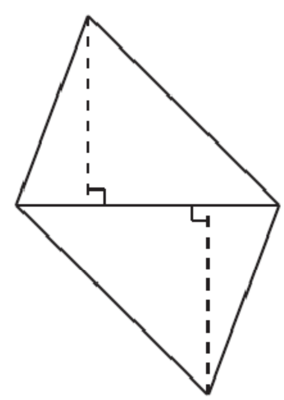
If sides of a different length (sides which do not correspond) are put together, the shape can be quite bizarre. This is not a shape for which we can easily calculate the area:



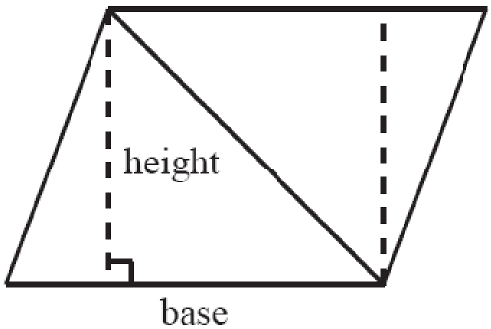


If one of the triangles is flipped over before two sides of the same length are put together, the result can also be quite odd. These shapes will also generally not be ones whose area we can calculate.

If the two bases are put together, without flipping the triangles, then the result will be a parallelogram, but not one whose base and height can be immediately identified, thus we will still have trouble finding the area.

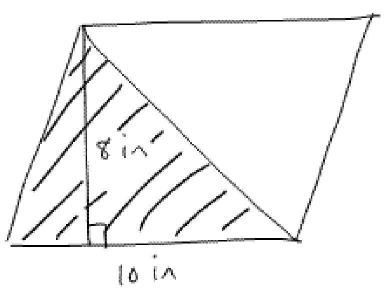


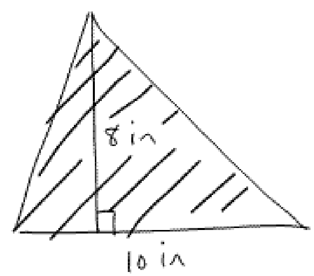
However, if corresponding sides (other than the bases) are put together without flipping over the triangles, the result will be a parallelogram with the same height and base as the original triangle:



The area of the whole parallelogram can be immediately calculated as base × height. Because this parallelogram is formed by two triangles with the same area, each of these triangles must have exactly half the area of the parallelogram.

In this way, you can lead your class to discover the formula for the area of a triangle: half of the product of the base and height. For example, if a triangle has a base of 10 inches and a height of 8 inches, then two of them can be put together to form a parallelogram with base 10 in and height 8 in. The parallelogram has area 10 in × 8 in = 80 in^2. The triangle thus has half this area, which can be either calculated as $\frac{1}{2} \times 80\,in^2 = 40\,in^2$ or 80 in^2 ÷ 2 = 40 in^2.



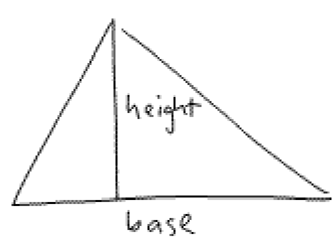


area = 8 in × 10 in = 80 in^2

area = $\frac{1}{2}$ × 8 in × 10 in = 40 in^2

We could write this as:

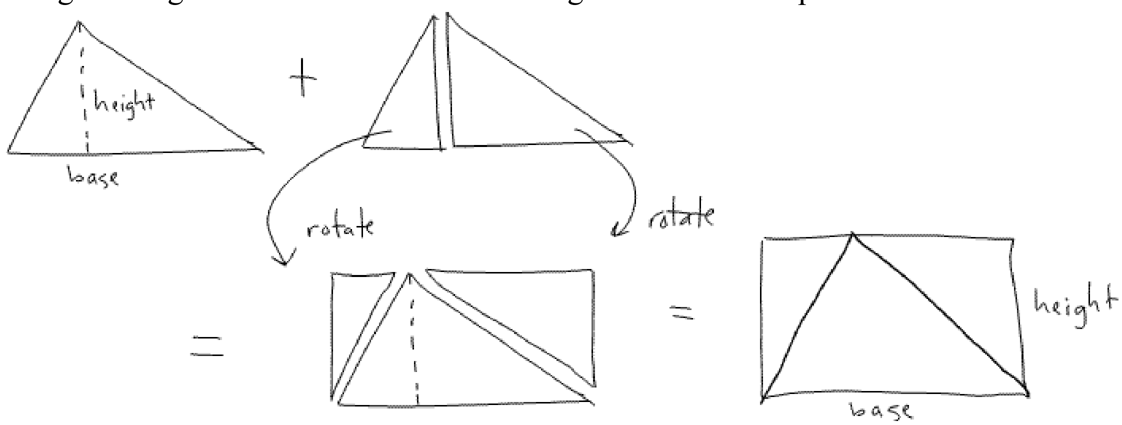
Area of a triangle = $\frac{1}{2}$ x base x height



We could also introduce a little bit of algebra by writing $A = \frac{1}{2} \times b \times h$ where *A* represents the area of the triangle, *b* represents the length of the base, and *h* represents the height. In general, it helps students to first see variables that make sense like this (*b* is the first letter of the word "base") than to see variables that do not, like *x* and *y*.

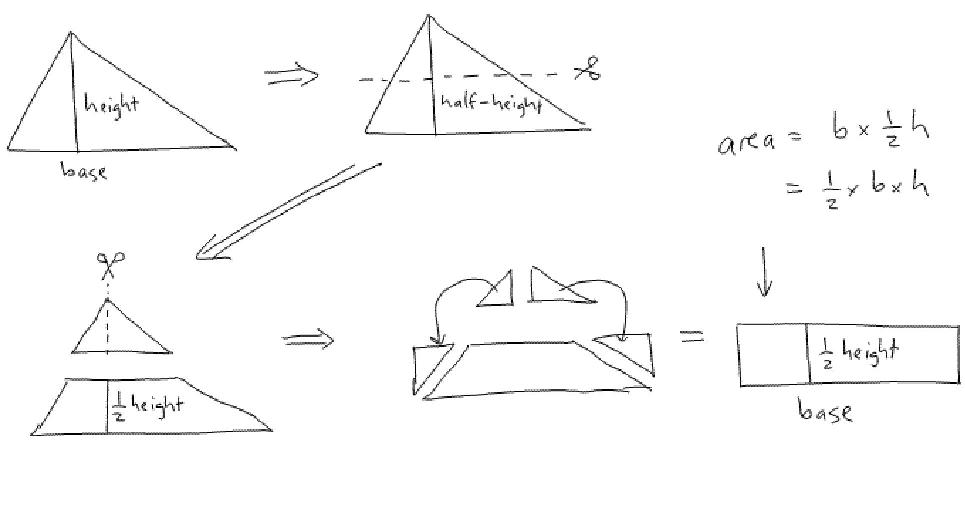
Remind students that because multiplication is commutative and associative, we can multiply the 1/2 by either the base or the height first – generally one that is even. For example, a triangle with base 5 in and height 4 in will have an area of $\frac{1}{2} \times 5in \times 4in$. If we multiply the 1/2 and the 5 in first, we will get $2.5in \times 4in = 10in^2$, though this might require some long multiplication. If we multiply the 1/2 and the 4in first, we will get $5in \times 2in = 10in^2$. This results in the same answer, but with less work.

It is possible that your students will not be as comfortable with the area of a parallelogram as they are with the area of a rectangle. If so, they can cut one of the triangles along the height line and then make a rectangle with the three pieces:



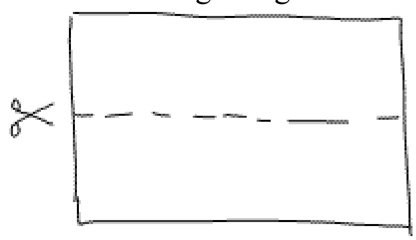
This time, we have taken the areas of two congruent triangles and made a rectangle. As this rectangle has an area of base × height, this further illustrates that the area of one of the triangles would be half of the product of base and height.

There are many other ways to verify this, so your students might enjoy cutting apart triangles and trying to put them back together into new shapes. Another way that works, for example, involves cutting the height of a single triangle in half. When the tip is cut along the height line, the three pieces can be put together in the shape of a rectangle with the same base but only half the height of the original triangle. Thus, the area of the triangle is $A = b \times \frac{1}{2}h$

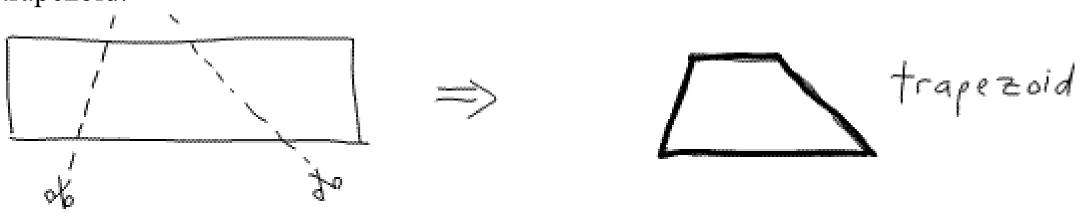


After the exercise with the triangles, your students ought to be up for the challange of figuring out the area of a trapezoid. This begins the same way – have students cut out a trapezoid and then trace it on a second piece of paper in order to cut out a congruent trapezoid.

One way to do this is to have them start by folding a piece of paper in half length-wise, and then cutting along this line:

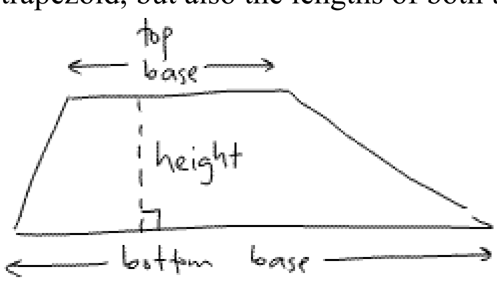


If they take one of these pieces and cut off two ends diagonally, this will result in a trapezoid:



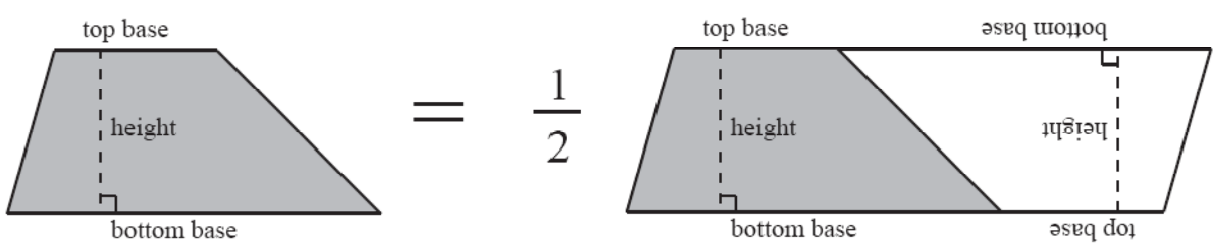
Furthermore, this trapezoid can be easily traced onto the other half of the piece of paper, so that only a single piece of paper is needed for this exercise.

This time, the students will not only need to draw and measure the height of the trapezoid, but also the lengths of both the bottom base and the top base:

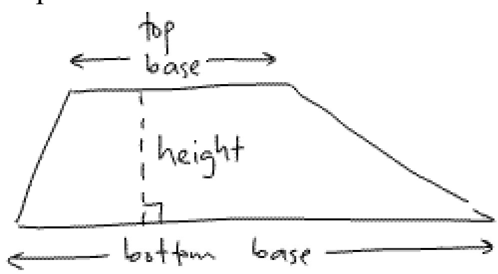


When all of this is done, the same challenge is given to the students: find a way to put the two trapezoids together in order to form a shape whose area you know how to calculate.

If the students understood what happened with the triangles, they ought to have less difficulty putting the two trapezoids together to form a parallelogram:



Once again, because the two trapezoids have the same area, they will each have half the area of the parallelogram they form. This parallelogram has the same height as the trapezoid. The base of the parallelogram will be the sum of the top and bottom bases of the trapezoid. Thus, the area of the parallelogram will be (bottom base + top base) × height and the area of the trapezoid will be:



$$\text{area} = \frac{1}{2} \times (\text{top base} + \text{bottom base}) \times \text{height}$$

For some reason, mathematicians like to refer to the top base as b_1 and the bottom base as b_2 (or vice-versa), and the height as *h*. Thus, the area of a trapezoid = $\frac{1}{2} \times (b_1 + b_2) \times h$.

There are other ways in which this can be done. There are ways, for example, in which a single trapezoid can be cut apart and rearranged into a rectangle. There is also a way to cut a single trapezoid into a triangle and a parallelogram – two objects whose areas we know how to calculate. Challenge your students to find these ways, and more, to calculate the area of a trapezoid.

The area of a trapezoid is not a terribly important formula for your students to study and memorize. These areas rarely come up outside of math books and math exams. However, the area of a triangle is something that appears in a large number of situations. Furthermore, the activities of this chapter ought to help introduce and develop a number of useful concepts and skills: precise measurement, the height of a shape, congruent objects, transformations, and abstract formulas. Best of all, students who come to understand the area of a triangle through hands-on work should generally have a more solid understanding of what the formula means and how it can be applied.

Questions:

(1) Cut out a scalene triangle, measure its base and height, and trace it onto another piece of paper. Show how the two triangles can be placed together to form three different parallelograms.

(2) Cut out a trapezoid, measure its height and bases, and then trace it onto another piece of paper. Show how the two trapezoids can be put together to establish the formula for the area of a trapezoid.

(3) Show how a single trapezoid can be cut into a triangle and a parallelogram. Measure the heights and bases of these objects, then calculate their areas. Verify that the sum of the areas gives the same answer as that obtained by the formula for the area of a trapezoid.

Chapter 32: Decimals

Students will need to understand fractions fairly well before they will be ready to understand decimals. On the one hand, this is unfortunate because decimal numbers are much easier to work with than fractions. On the other, this can come as a relief to students who might think that math only gets harder after fractions. All in all, decimals are not only a powerful new sort of numbers for children to come to understand, but work with them helps to reinforce understanding of the base-ten number system.

The base-ten number system has a column for 1's, 10's, 100's, 1000's, and so on. Now that we have fractions, we can introduce columns for $\frac{1}{10}$'s, $\frac{1}{100}$'s, $\frac{1}{1000}$'s, and so on. You get all the first numbers by multiplying 1 by some number of 10's, for example $1000 = 1 \times 10 \times 10 \times 10$. The 1's column is 1 multiplied by no tens. The new columns will be what we get when we divide 1 by some number of tens. If we divide $1 \div 10 \div 10$, for example, we get $\frac{1}{100}$.

This is an excellent opportunity to check your students understandings of the relative sizes of fractions. Have them put the following numbers in order, from largest to smallest: 1, $\frac{1}{100}$, 100, $\frac{1}{10}$, 1000, 10, $\frac{1}{1000}$, and 10,000. At the same time, you can introduce some new vocabulary: $\frac{1}{10}$'s are called *tenths*, $\frac{1}{100}$'s are called *hundredths*, and $\frac{1}{1000}$'s are called *thousandths*. With this, they should be able to draw out the new, expanded, base-ten columns:

thousands 1000's	hundreds 100's	tens 10's	ones 1's	tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s

Have the class practice converting between numbers written in number-columns and numbers in expanded notation. For example, the number $400 + 20 + 7 + \frac{5}{10} + \frac{3}{100}$ looks like:

thousands 1000's	hundreds 100's	tens 10's	ones 1's	tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
	4	2	7	5	3	

Similarly, the following number-column:

thousands 1000's	hundreds 100's	tens 10's	ones 1's	tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
3		9	6	8		

represents the expanded number $3000 + 90 + 6 + \frac{8}{100}$.

Similarly, the number $20 + 5 + \frac{3}{10} + \frac{5}{1000}$ will be represented, in base-ten column notation, as:

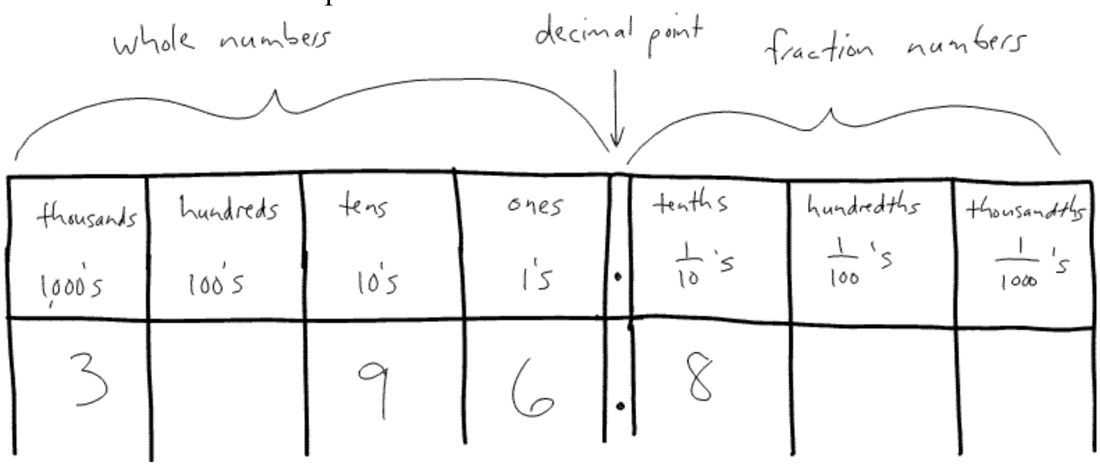
thousands 1000's	hundreds 100's	tens 10's	ones 1's	tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
		2	5	3		5

Run through a number of examples like this with your class. This should not be too difficult.

It is likely, however, that your students will be tempted to condense numbers like 3000 + 90 + 6 into 3,096. This is perfectly acceptable, thus $3000 + 90 + 6 + \frac{8}{10} = 3{,}096 + \frac{8}{10}$.

At this point, you can introduce your students to the usual decimal notation. First, show them the problem that will arise if we just use the sequence of digits as before. The number $3000 + 90 + 6 + \frac{8}{10}$ would be written 30968, which will look exactly the same as 30,000 + 900 + 60 + 8. These are very different numbers, with one less than 4,000 and the other greater than 30,000. Similarly, if we write $20 + 5 + \frac{3}{10} + \frac{5}{1000}$ as 25305, this will look exactly like 20,000 + 5,000 + 300 + 5. Hopefully, examples like these will show your class that there is a need for something to differentiate these numbers.

The key to everything is the *decimal point*, a single dot that goes after the digit in the 1's place. In column notation, we can put the point in to separate the old whole-number places from the new fraction-number places:



Thus, the number $3000 + 90 + 6 + \frac{8}{10}$ will be written as 3,096.8, and the number $20 + 5 + \frac{3}{10} + \frac{5}{1000}$ will be written as 25.305. You might need to emphasize the importance of using zeroes in the columns between digits, so that $20 + 5 + \frac{3}{10} + \frac{5}{1000}$ is not written as 25.35, just as 3000 + 90 + 6 is not written as 396.

When you first introduce this, use the column notation as an in-between step. For example, to convert 290.14 into expanded notation, first have the class write it in column notation:

thousands 1000's	hundreds 100's	tens 10's	ones 1's	.	tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
	2	9	0	.	1	4	

and then have them write this out as $200 + 90 + \frac{1}{10} + \frac{4}{100}$.

Similarly, to convert a number from $500 + \frac{3}{1000}$ into decimal form, have them first write it into column notation:

thousands 1000's	hundreds 100's	tens 10's	ones 1's	.	tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
	5			.			3

and make sure to put in zeroes between the digits:

thousands 1000's	hundreds 100's	tens 10's	ones 1's	.	tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
	5	0	0	.	0	0	3

before the number is written into decimal form: 500.003.

As the class gets more comfortable with this, you can slowly take them away from the column notation and encourage them to convert quickly between a decimal number (for example, 20,135.076) and its expanded notation ($20{,}000 + 100 + 30 + 5 + \frac{7}{100} + \frac{6}{1000}$). Students who are still a bit uncertain, however, should be allowed to continue using the column notation. As with anything else, play this up as speed issue – that being able to think and work quickly is an advantage, but only so long as you understand what is going on.

Also remember to teach your students that a number with no whole part should be written with a zero in the one's place, but often is not. For example, $\frac{1}{10}+\frac{2}{100}$ should be written 0.12, but is often written as .12 when there is no danger of a person not noticing the decimal point.

Not only should children know the meaning of decimal numbers, in converting to and from expanded form, but it is also important for them to learn how to pronounce them.

The most common way to pronounce decimals is the easy and lazy way, in which the decimal point is pronounced "point" and the following digits are named individually. For example, 27.6 can be pronounced "twenty-seven point six." Similarly, the number 390.105 is often pronounced "three-hundred ninety point one oh five," using the short-cut "oh" instead of "zero." Students should know how to read and understand these numbers because most of the decimals they encounter in everyday life will be pronounced in this fashion. To practice, you should have students read decimal numbers out loud. Also, you should be able to tell the class a decimal number and have them all write it down correctly.

In addition to this, students should be taught the correct way to pronounce decimal numbers. Not only will this be proper for formal occasions, but it will also increase their understanding of the decimal number system. To prepare a decimal number for its formal name, we expand it out and then combine the whole parts and fractional parts separately.

For example, the number 318.27 expands out into $300 + 10 + 8 + \frac{2}{10} + \frac{7}{100}$. We combine the whole parts as before: $300 + 10 + 8 = 318$.

In order to combine the fractional parts, $\frac{2}{10} + \frac{7}{100}$, we need to find a common denominator. The trick here is to use a symbols board (or the short-hand notation), but factor things into tens and not into primes. Thus $\frac{2}{10}+\frac{7}{100}=\frac{2}{10}+\frac{7}{10\times 10}$. All we need to do is multiply the 2/10 by 10/10 to get $\frac{2\times 10}{10\times 10}+\frac{7}{10\times 10}=\frac{20}{100}+\frac{7}{100}=\frac{27}{100}$.

Thus, the number $300 + 10 + 8 + \frac{2}{10} + \frac{7}{100} = 318 + \frac{27}{100}$ and is pronounced "three-hundred eighteen and twenty-seven hundredths."

Similarly, $16.134 = 10 + 6 + \frac{1}{10}+\frac{3}{100}+\frac{4}{1000}$

$= 16 + \frac{1\times 10\times 10}{10\times 10\times 10}+\frac{3\times 10}{10\times 10\times 10}+\frac{4}{10\times 10\times 10} = 16 + \frac{100}{1000}+\frac{30}{1000}+\frac{4}{1000} = 16 + \frac{134}{1000}$ and is pronounced "sixteen and one-hundred thirty-four thousandths."

Hopefully, after working out a few examples like this, your students notice the standard short-cut. All you have to do is read the digits after the decimal point as if they were whole number and then name the smallest fraction-column used by the number. For example, 25.89 is "twenty-five and eighty-nine hundredths." The digits after the decimal place are 89, and the last place used is the hundredths place. Similarly, 212.046 is "two-hundred twelve and forty-six thousandths," and 3041.1025 is "three-thousand forty-one and one-thousand, twenty-five ten-thousandths."

By running through exercises like these, your students should develop the ability to convert numbers easily between spoken, written, and expanded-out form.

Questions:

(1) Write the following number in (a) expanded form, (b) decimal form, and (c) formal words:

thousands 1000's	hundreds 100's	tens 10's	ones 1's	.	tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
	3		6	.	2		5

(2) Write the number $8{,}000 + 200 + 70 + \frac{5}{100} + \frac{4}{1000}$ in (a) base-ten column notation, (b) decimal form, and (c) formal words.

(3) Write the number 87.63 in (a) expanded form, (b) base-ten column notation, and (c) formal words.

(4) Write the number "two-thousand, eight-hundred four and seventeen thousands" in (a) base-ten columns, (b) expanded form, and (c) decimal form.

(5) In the number 32.1975, what does the digit 7 represent?

(6) Show the computations that convert $1 \div 10 \div 10 \div 10$ into 1/1000.

(7) Show the computations which convert $\frac{3}{10} + \frac{7}{100} + \frac{9}{1000}$ into a single fraction.

Chapter 33: Adding and Subtracting Decimals

When students are able to expand out decimal numbers and write them in base-ten column notation, they are ready to learn how to add and subtract decimals. The process is nearly identical to the adding and subtraction of large whole numbers.

To add decimals, we begin by converting the numbers into base-ten column form, and then add the numbers one column at a time. It is best, as before, to begin with some examples that do not involve carrying.

For example, suppose we want to add 37.254 + 5,201.62. First, we make an extended base-ten column and put both of the numbers in it, one on top of the other:

thousands 1000's	hundreds 100's	tens 10's	ones 1's	.	tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
		3	7	.	2	5	4
5	2	0	1	.	6	2	

Next, we add the columns individually. Empty spaces count as zeroes:

thousands 1000's	hundreds 100's	tens 10's	ones 1's	.	tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
		3	7	.	2	5	4
5	2	+ 0	+ 1	.	+ 6	+ 2	
5	2	3	8	.	8	7	4

This results in the answer: 37.254 + 5,201.62 = 5,238.874.

This also makes clear the biggest key to adding decimals: we have to line up the decimal points. Because the decimal points help us to put the digits in the right columns, when they line up, we are assurred that vertically-aligned digits will represent the same sort of number (hundreds, tens, ones, tenths, etc.). Thus, the above calculation can be performed much more quickly without the base-ten number columns, just by lining up the digits of the numbers. As before, empty spaces count as zeroes:

$$\begin{array}{r} 37.254 \\ +\,5201.62 \\ \hline 5238.874 \end{array}$$

It is best to start off with an example or two like this one, where the columns might be tricky (the two numbers do not have the same number of decimal digits) but the addition does not involve any addition. This will help to make clear that the key is to line up the decimal points and not the right-hand ends of the numbers (as was the case with whole numbers).

If students are getting confused and trying to line up the right-hand ends, then run an example with whole numbers. For example, 327 + 21, on the expanded base-ten columns, will look like:

thousands 1000's	hundreds 100's	tens 10's	ones 1's	.	tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
	3	2	7	.			
		+ 2	+ 1	.			
	3	4	8	.			

Emphasize that we are not only lining up the right-hand ends of the numbers, but we are also lining up the decimal points (which just happen to be at the right-hand end of whole numbers):

$$\begin{array}{r} 327. \\ +\,21. \\ \hline 348. \end{array}$$

Next, your students should be ready for an addition problem that involves carrying. Begin with one that involves carrying in one of the fractional-valued columns. For example, have your students work through 12.27 + 3.294. We begin by putting everything together in base-ten number columns:

thousands 1000's	hundreds 100's	tens 10's	ones 1's	.	tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
		1	2	.	2	7	
			+3	.	+2	+9	4
						16	4

The 4 in the thousandths column is added to nothing, so it stays a 4. The 7 hundredths + 9 hundredths, however, become 16 hundredths. Just as with adding whole numbers, we cannot leave 16 in a column, so we split it into 10 + 6. If we write this with fractions, this will look like: $\frac{16}{100} = \frac{10}{100} + \frac{6}{100}$. When we reduce the $\frac{10}{100}$, it becomes $\frac{1}{10}$. Thus, we can bring the 10 hundreds out of the hundredths column and turn it into 1 tenth in the tenths column:

thousands 1000's	hundreds 100's	tens 10's	ones 1's	.	tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
		1	2	.	^{1}2	7	
			+3	.	+2	+9	4
						~~16~~	4
						10 + 6	

At this point, the rest of the problem can be added together as before:

thousands 1000's	hundreds 100's	tens 10's	ones 1's	.	tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
		1	2	.	$^{1}2$	7	
			+3	.	+2	+9	4
		1	5	.	5	16	4
						10 + 6	

It will help to then run the same problem without using the base-ten columns:

$$
\begin{array}{r}
{}^{1} \\
12.27 \\
+\ 3.294 \\
\hline
15.564
\end{array}
$$

Hopefully, your students will see that this looks exactly like the addition and carrying that we worked out with large whole numbers. This is because of the way the columns are set up. We were able to "carry the 1" with whole numbers because 10 ones make 1 ten, 10 tens make 1 hundred, 10 hundreds make 1 thousand, and so on. Similarly, 10 thousandths make 1 hundredth, 10 hundredths make 1 tenth, and 10 tenths make 1 one. In other words, any time the digits in one column add to 10 or more, we can take the tens digit of that sum and bring it over to the next column on the left. If our number system were designed differently, then tricks like this might not work. However, this sort of thing is exactly why our number system is laid out the way it is.

The base-ten columns notation is only intended to help students to understand how our number system works. If students catch on quickly, do not slow them up by forcing them to work with the base-ten columns. Those who are having some trouble (especially with lining up the numbers properly) can encouraged to use the columns, but the others should be allowed to fly ahead to the fastest method they are capable of using and understanding.

Everyone, ultimately, should be able to add numbers like 98.72 + 154.618 with only the following work on their paper:

$$
\begin{array}{cccccccc}
 & & 1 & & & & & \\
 & & & 1 & & & & \\
1 & & 9 & 8 & . & 7 & 2 & \\
 & 1 & 5 & 4 & . & 6 & 1 & 8 \\
\hline
 & 2 & 5 & 3 & . & 3 & 3 & 8
\end{array}
$$

If the students have been taught the base-ten number system and why it enables us to "carry the one" while adding large numbers, then adding decimals should not pose much challenge.

Subtraction is exactly the same: we line up the decimal points and subtract with borrowing, just as was done with large whole numbers. Begin with an example that does not involve borrowing, for example 37.98 – 14.2. Put the two numbers up in base-ten columns and subtract one column at a time:

thousands 1000's	hundreds 100's	tens 10's	ones 1's		tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
		3	7	.	9	8	
		- 1	- 4	.	- 2		

When a column is empty, we put a zero in its place:

thousands 1000's	hundreds 100's	tens 10's	ones 1's		tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
		3	7	.	9	8	
		-1	-4	.	-2	-0	
		2	3	.	7	8	

Thus, 37.98 – 14.2 = 23.78. As with addition, show your students how the same thing is done without the number columns:

$$\begin{array}{r} 37.98 \\ -\ 14.2 \\ \hline 23.78 \end{array}$$

Notice that it is not really necessary to put a zero after the 2, though it is perfectly acceptable to do so.

Next, introduce a problem that will require one instance of borrowing, preferably in a fraction-denominated column. For example, have the class subtract 38.4 – 5.16. When laid out, we see that there are no hundredths from which we can subtract 6 hundredths:

thousands 1000's	hundreds 100's	tens 10's	ones 1's		tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
		3	8	.	4		
			-5	.	-1	-6	

First of all, we can make the 0 hundredths in the first number more clear by putting a zero in that column. Next, we can "borrow" 1 tenth from the column to the left. When we convert 1/10 into hundredths, it becomes $\frac{1}{10} = \frac{10}{100}$. Thus, this becomes 10 hundredths, from which we are able to subtract 6:

thousands 1000's	hundreds 100's	tens 10's	ones 1's		tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
		3	8	.	3 + 1 ~~4~~	10	
			−5	.	−1	−6	
		3	3	.	2	4	

Thus 38.4 – 5.16 = 33.24. We can verify this, if we wish, by checking the inverse operation, by checking that 33.24 + 5.16 = 38.4:

$$\begin{array}{r} {}^{1} \\ 33,24 \\ +\ \ 5.16 \\ \hline 38.40 \end{array}$$

Some students might wonder if there is a difference between 38.40 and 38.4. Have them expand out the two numbers, just to check. The first comes out to $3 \times 10 + 8 \times 1 + 4 \times \frac{1}{10} + 0 \times \frac{1}{100} = 30 + 8 + \frac{4}{10}$, which is exactly what the second number represents. Thus, these two numbers are the same.

This is a very valuable exercise to run by your students. When we were dealing with whole numbers, "adding a zero to the end" was a short-cut technique for multiplying the number by 10. When a number has a decimal point, however, "adding a zero to the end" does not change the number at all! In order to "add a zero" to a number without a decimal point, we had to shift all the digits to new base-ten columns. With a decimal, however, "adding a zero" does not change the positions of digits in the columns.

At this point, your students ought to be ready for more complicated decimal subtractions, for example 284.163 – 32.705. We begin with the base-ten column notation:

thousands 1,000's	hundreds 100's	tens 10's	ones 1's	.	tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
	2	8	4	.	1	6	3
		- 3	- 2	.	- 7	- 0	- 5

Because we cannot subtract 5 thousandths from 3 thousandths, we have to borrow from the next column over. We break the 6 hundredths into 5 hundredths + 1 hundredth and turn the 1 hundredth into 10 thousandths. We can then add the 10 thousands to the 3 thousandths to make 13 thousands. Of course, if we want to be clever, we could take 1 hundredth from the 6 and then put it directly to the left of the 3, to make the 13 thousandths. However it is done, we are then able to subtract the 5 thousandths from the 13 thousandths to end up with 8 thousandths:

thousands 1,000's	hundreds 100's	tens 10's	ones 1's	.	tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
						5 + 1	10
	2	8	4	.	1	~~6~~	3
		- 3	- 2	.	- 7	- 0	- 5
						5	8

We can also subtract the 0 hundredths from the 5 remaining hundredths.

The problem now is in the tenths column. We cannot subtract 7 tenths from 1 tenth. Thus, we break the 4 ones into 3 ones + 1 one. The 1 one is the same as 10 tenths, for a total of 11 in the tenths column. After this, we will be able to subtract in each column:

thousands 1000's	hundreds 100's	tens 10's	ones 1's	.	tenths $\frac{1}{10}$'s	hundredths $\frac{1}{100}$'s	thousandths $\frac{1}{1000}$'s
			3 + 1		10	5 + 1	10
	2	8	~~4~~	.	1	~~6~~	3
		−3	−2	.	−7	−0	−5
	2	5	1	.	4	5	8

When this makes sense to your class, you can show them how it looks with the short-cut notation:

$$\begin{array}{r} 2\;8\;\overset{3}{\cancel{4}}.{}^{1}1\;\overset{5}{\cancel{6}}\;{}^{1}3 \\ -\;3\;2.7\;0\;5 \\ \hline 2\;5\;1.4\;5\;8 \end{array}$$

Basically, as long as your students know that it is the decimal points which need to line up (and not the rightmost digits) and that missing digits count as zeros, they should find adding and subtracting with decimals to be pretty much the same as adding and subtracting whole numbers.

Questions:

(1) Show that 10 thousands is the same as 1 hundredth.

(2) Add 305.27 + 9.295 with base-ten columns. Take care to explain the carrying in detail.

(3) Subtract 288.72 – 35.197 with base-ten columns. Explain the borrowing in detail.

Chapter 34: Multiplying and Dividing by Decimals by Powers of Ten

The famous short-cut for multiplying and dividing decimals by powers of ten can be easily explained using base-ten column notation.

First of all, a *power* of ten is what we get when we multiply the number 1 by some number of tens. This could be 1 itself, or 1 × 10 = 10, or 1 × 10 × 10 = 100, or 1000, or so on. The reason these are called powers is because they can be easily represented with exponential powers. For example, one-thousand is called "ten to the third power" because $1000 = 10 \times 10 \times 10 = 10^3$. Similarly, when we divide 1 by a number of tens, we get a negative power of 10. For example, $\frac{1}{100} = 1 \div 10 \div 10 = 10^{-2}$. The pattern of exponents can make a lot of sense when illustrated on base-ten number columns:

thousands	hundreds	tens	ones	.	tenths	hundredths	thousandths	ten-thousandths
1000	100	10	1		$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	$\frac{1}{10000}$
10^3	10^2	10^1	10^0		10^{-1}	10^{-2}	10^{-3}	10^{-4}

This is only to explain the use of the phrase "powers of ten" and its connection to the base-ten numbering system. Children will not need to concern themselves with exponents, especially negative exponents, until they learn algebra in middle or high school. For the rest of this chapter, we will use "a power of ten" to only mean a positive power of ten, a number of tens multiplied together.

We saw earlier that when a whole number is multiplied by 10, each of its digits moves to the left. This is because ones become tens, tens become hundreds, and so on when multiplied by 10. For example, a number like 504:

thousands	hundreds	tens	ones	.	tenths	hundredths	thousandths	ten-thousandths
	5	0	4					

represents 500 + 4. When multiplied by 10, this becomes (500 + 4) × 10 = 500 × 10 + 4 × 10 = 5000 + 40. It is as if each of the digits moved to the left one space.

thousands	hundreds	tens	ones	.	tenths	hundredths	thousandths	ten-thousandths
5	0	4						

The short-cut we discussed earlier for this was called "add a zero to the end of the number." For example, $67 \times 10 = 670$, as if a single zero were just tacked onto the end of the 67.

The extended short-cut was called "add a zero at the end for every zero in the power of ten." When we multiply by 1000, for example, we are really multiplying by $1000 = 10 \times 10 \times 10$. Each of these tens "adds a zero," so multiplying by 1000 involves "adding three zeros to the end," just as the number 1000 itself has 3 zeros. For example, $67 \times 10{,}000 = 670000$. In general, it is a good idea to add all the zeros first, and then worry putting in a comma after every third digit: $67 \times 10{,}000 = 670{,}000$.

With decimal numbers, we have to go back to the "shift all the digits over" concept rather than the "add zeros to the end." For example, let us multiply 54.29×10. We begin by representing 54.29 on a base-ten number chart:

thousands	hundreds	tens	ones	•	tenths	hundredths	thousandths	ten-thousandths
		5	4	.	2	9		

This means that $54.29 = 50 + 4 + \frac{2}{10} + \frac{9}{100}$. When we multiply by 10, each of these parts must be multiplied by 10 (because of the distributive property). Thus $54.29 \times 10 = (50 + 4 + \frac{2}{10} + \frac{9}{100}) = 50 \times 10 + 4 \times 10 + \frac{2}{10} \times 10 + \frac{9}{100} \times 10 = 500 + 40 + 2 + \frac{9}{10}$:

thousands	hundreds	tens	ones	•	tenths	hundredths	thousandths	ten-thousandths
	5	4	2	.	9			

Just as before, this number consists of the exact same digits as before, each shifted one column to the left.

Without the number columns, however, things look a bit different. When we multiply $54.29 \times 10 = 542.9$, it does not look as though anything has moved to the left. Instead, it looks as though the decimal point has moved to the right:

$$54.29 \times 10 = 54.29 = 542.9$$

This, of course, is the exact same thing.

Similarly, when we multiply by a larger power of ten, we have as many shifts as there are zeros in the power of ten. For example, look at 3.125 × 100. The number 3.125 looks like:

thousands	hundreds	tens	ones	.	tenths	hundredths	thousandths	ten-thousandths
			3	.	1	2	5	

This represents $3.125 = 3 + \frac{1}{10} + \frac{2}{100} + \frac{5}{1000}$.

It follows by distribution that $3.125 \times 100 = (3 + \frac{1}{10} + \frac{2}{100} + \frac{5}{1000}) \times 100$

$= 3 \times 100 + \frac{1}{10} \times 100 + \frac{2}{100} \times 100 + \frac{5}{1000} \times 100$

$= 300 + 10 + 2 + \frac{5}{10}$. This looks like:

thousands	hundreds	tens	ones	.	tenths	hundredths	thousandths	ten-thousandths
	3	1	2	.	5			

As before, we can look at this as each digit moving over two columns to the left. We can also look at 3.125 × 100 = 312.5 and imagine that the decimal point has moved two spaces to the right:

3.125 × 100 = 3.125 = 312.5

In either case, the "two columns" or "two spaces" comes from the fact that 100 is formed by two 10's being multiplied together (100 = 10 × 10), thus multiplying by 100 is the same as multiplying by 10 twice.

Similarly, multiplying by 1000 is the same as multiplying by 10 three times because 1000 = 10 × 10 × 10. Thus, for example, 128.91065 × 1000 = 128,910.65. We'll show this with the "moving decimal point" notation because it takes less space. Of course, there is no need for a child to write out the number three times. The "moving decimal" can be drawn right on the starting number. It would be nice, however, for the final answer to be written out separately.

128.91065 × 1000

= 128.91065

= 128910.65

As usual, the most kind and educational thing you can do is to not point out the "moving decimal point" trick at first. Instead, do a number of problems out with base-ten column notation and distribution. Hopefully, one of your students will notice the pattern and announce it to the rest of the class. In this manner, the student will get all the honor and glory associated with "making math easy for everyone." As always, this trick will be named after the student, both to praise that person and to help the others recall (ask, for example: "remember Samantha's trick for these problems?"). In an ideal class, the students keep their eyes peeled for tricks, patterns, and short-cuts. This is so much more fun and educational than telling the students what to do and penalizing them for deviations.

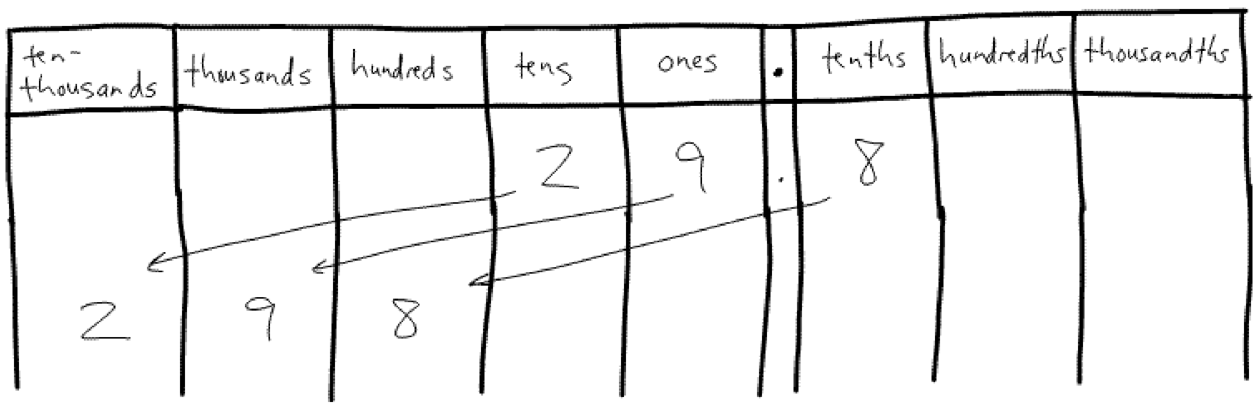
If no one in the class notices the short-cut, however, you should slowly prompt them toward it, and ultimately just tell them. One useful trick is to line up a whole series of examples, to focus the class on pattern recognition. For example, show them:

$3.17 \times 10 = 31.7$ $\quad$ $23.905 \times 10 = 239.05$ $\quad$ $282.096 \times 10 = 2820.96$

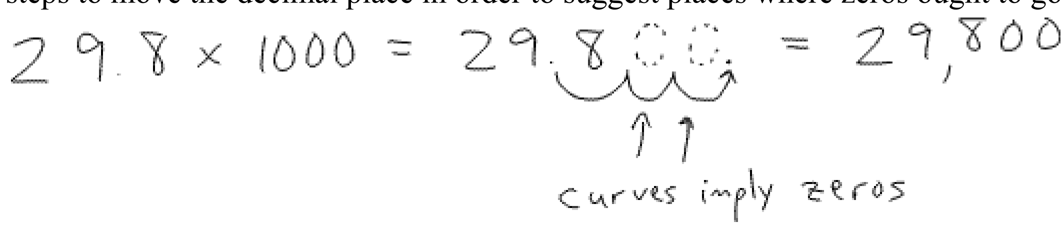
$3.17 \times 100 = 317$ $\quad$ $23.905 \times 100 = 2{,}390.5$ $\quad$ $282.096 \times 100 = 28{,}209.6$

etc.

Be sure to include examples where you multiply by more tens than you have decimal places. For example, look at 29.8×1000. The 29.8 has only one decimal place, but it is being multiplied by $1000 = 10 \times 10 \times 10$. With base-ten columns, this involves moving each digit to the left three times:



This number is 20,000 + 9,000 + 800, which can only be written with two more zeros at the end: 29,800. With the "moving the decimal place" notation, we use low, swinging, curvy steps to move the decimal place in order to suggest places where zeros ought to go:



Because division is the inverse operation to multiplication, the process of dividing by a power of ten is the exact opposite of multiplying. Rather than moving each digit to the left in base-ten columns, the digits move to the right. Rather than the decimal point moving to the right, it moves to the left. The number of spaces moved is still the number of zeros in the power of ten.

For example, to divide 38.97 ÷ 10, we start with 38.97:

ten-thousands	thousands	hundreds	tens	ones	.	tenths	hundredths	thousandths
			3	8	.	9	7	

Because $38.97 = 30 + 8 + \frac{9}{10} + \frac{7}{100}$, when we divide, we have to divide each part separately:

$38.97 \div 10 = (30 + 8 + \frac{9}{10} + \frac{7}{100}) \div 10$

$= 30 \div 10 + 8 \div 10 + \frac{9}{10} \div 10 + \frac{7}{100} \div 10$

$= 3 + \frac{8}{10} + \frac{9}{100} + \frac{7}{1000} = 3.897$

ten-thousands	thousands	hundreds	tens	ones	.	tenths	hundredths	thousandths
				3	.	8	9	7

We can either imagine that each digit has moved to the right one column or that the decimal point has moved to the left one space:

$38.97 \div 10$

$= 38.97$

$= 3.897$

As before, we can run into some trouble if the decimal point runs out past the digits. For example, consider 8.125 ÷ 1000. If we try to move the decimal point to the left three places, we will run out of number:

$$8.125 \div 1000 =$$

$$.\ \ 8.125$$

The trick, as before, is to put another zero above the crook of each step of the decimal place's movement:

$$8.125 \div 1000 =$$

$$.\,0\,0\,8.125$$

$$= .008125$$

This can be verified by distributing the division of the 1000. To evaluate 8.125 ÷ 1000, we split up the 8.125 into $8 + \frac{1}{10} + \frac{2}{100} + \frac{5}{1000}$ and divide each part by 1000:

$$8 \div 1000 + \frac{1}{10} \div 1000 + \frac{2}{100} \div 1000 + \frac{5}{1000} \div 1000$$

$$= \frac{8}{1000} + \frac{1}{10000} + \frac{2}{100000} + \frac{5}{1000000} = .008125.$$

We could also write 8.125 with base-ten columns, then move each digit to the left three spaces:

tens	ones	.	tenths	hundredths	thousandths	ten-thousandths	hundred-thousandths	millionths
	8	.	1	2	5			
		.	0	0	8	1	2	5

Questions:

(1) Show with factoring and cancellation why $\frac{1}{100} \times 10 = \frac{1}{10}$.

(2) Show that $\frac{3}{100} \div 10 = \frac{3}{1000}$ by treating the problem as one of dividing fractions.

(3) Show 98.164 × 10 using

(a) the distribution of the 10.

(b) base-ten column notation.

(c) the "move the decimal point" short-cut.

(4) Show 18.7 ÷ 1000 using

(a) the distribution of the ÷ 1000

(b) base-ten column notation

(c) the "move the decimal place" short-cut

Chapter 35: Multiplying Decimals

The procedure to multiply decimals is not very difficult to teach children. As with everything else, however, it is best if it is first explained. This is easy when the decimals are first converted into fractions.

In finding the proper names for decimals (chapter 32), we have already half-way converted them into decimals. For example, we have seen how $381.14 = 381 + \frac{14}{100}$. Basically, the digits after the decimal point form the numerator, and the denominator is the smallest one used in decimal (in this instance, hundredths). The only remaining step is to convert the whole number into a fraction with the same denominator. In this example, we proceed by putting a 100 in both the numerator and denominator of the 381:

$$381 + \frac{14}{100} = \frac{381 \times 100}{100} + \frac{14}{100} = \frac{38100}{100} = \frac{38114}{100}$$

As another example, the decimal $7.914 = 7 + \frac{914}{1000}$ becomes:

$$7 + \frac{914}{1000} = \frac{7 \times 1000}{1000} + \frac{914}{1000} = \frac{7000}{1000} + \frac{914}{1000} = \frac{7914}{1000}$$

As a final example, $3{,}070.5 = 3{,}070 + \frac{5}{10}$ becomes:

$$3070 + \frac{5}{10} = \frac{3070 \times 10}{10} + \frac{5}{10} = \frac{30700}{10} + \frac{5}{10} = \frac{30705}{10}$$

By looking at a series of examples, a pattern can emerge:

$381.14 = \frac{38114}{100}$ $\quad$ $7.914 = \frac{7914}{1000}$ $\quad$ $3070.5 = \frac{30705}{10}$

In each case, the numerator consists of all the digits of the number with the decimal point removed. The denominator is the smallest denominator used in the expanded form of the decimal.

This is remarkably like the "slang number notation" which was discussed earlier in this book. The number 1,500 can be called "fifteen hundred" because it can be viewed as a 15 in the hundreds place:

thousands	hundreds	tens	ones	.	tenths
1	5	0	0	.	

=

thousands	hundreds	tens	ones	.	tenths
	15	0	0	.	

Similarly, the number 3.9 could be called "thirty-nine tenths" because it can be viewed as a 39 in the tenths place:

thousands	hundreds	tens	ones	.	tenths
			3	.	9

=

thousands	hundreds	tens	ones	.	tenths
				.	39

Normally, of course, if we have a two-digit number in a base-ten column, we "carry" the ten's digit up and into the next column to the left. However, those who use slang like to bend the rules a little bit, and sometimes this can help to make a point more clearly. When a person asks "how many hundreds of dollars did your new brakes cost?" you might say "fifteen hundred dollars." Similarly, if someone asks "how many tenths of an inch was our estimate off?" you might answer "thirty-nine tenths."

Hopefully, your students will quickly notice and adopt the short-cut for converting numbers from decimal to fraction form. The fraction $17.19 = \frac{1719}{100}$, for example, and $0.394 = \frac{394}{1000}$.

Your students should also be able to convert quickly from fractions to decimals, at least when the denominator is a power of 10. For example, $\frac{29704}{100} = 297.04$ and $\frac{398}{10} = 39.8$. This is just a matter of dividing by a power of 10, because $\frac{29704}{100} = 29704 \div 100$, which has been discussed in detail in chapter 34.

In any case, when your students are able to convert numbers from decimal form to fraction form and back easily, they are ready to see the short-cut for multiplying decimals. Let us begin with an example like 37.4×2.91. In order to multiply this number, we first convert the multiplicand and multiplier into fractional form: $37.4 \times 2.91 = \frac{374}{10} \times \frac{291}{100}$. To multiply these fractions, we multiply the numerators together and the denominators together separately:

$\frac{374}{10}\times\frac{291}{100} = \frac{374\times 291}{10\times 100}$. In other words, we will multiply the two numbers together as if they were whole numbers (374 × 291) and then divide the result by 1000. To work this out, we use long multiplication:

$$\begin{array}{rrrrrr}
 & & & {}^{1} & & {}^{3} \\
 & & & 3 & 7 & 4 \\
\times & & & 2 & 9 & 1 \\
\hline
 & & & 3 & 7 & 4 \\
 & 3 & 3 & 6 & 6 & \\
+\,7 & 4 & 8 & & & \\
\hline
1\,0 & 8 & 8 & 3 & 4 &
\end{array}$$

In order to divide 108,834 by 1000, we merely move the decimal place over three places:

$$108.834$$

This is our answer: 37.4 × 2.91 = 108.834.

A good way to double-check this answer is with a little mental math. The number 37.4 is a little less than 40 and 2.91 is a little less than 3. Thus, we expect that 37.4 × 2.91 should be a little bit less than 40 × 3 = 120. Indeed, 108.834 is a little less than 120.

For another example, let us multiply 0.081 × 3.5. First, we convert the two numbers into fractions, then we multiply them as fractions: 0.081 × 3.5 = $\frac{81}{1000}\times\frac{35}{10} = \frac{81\times 35}{1000\times 10} = \frac{81\times 35}{10000}$. In order to multiply 81 × 35, we resort to long-multiplication. In order to divide by 10000, we move the decimal place to the left 4 times (as many times as there are zeros in 10000):

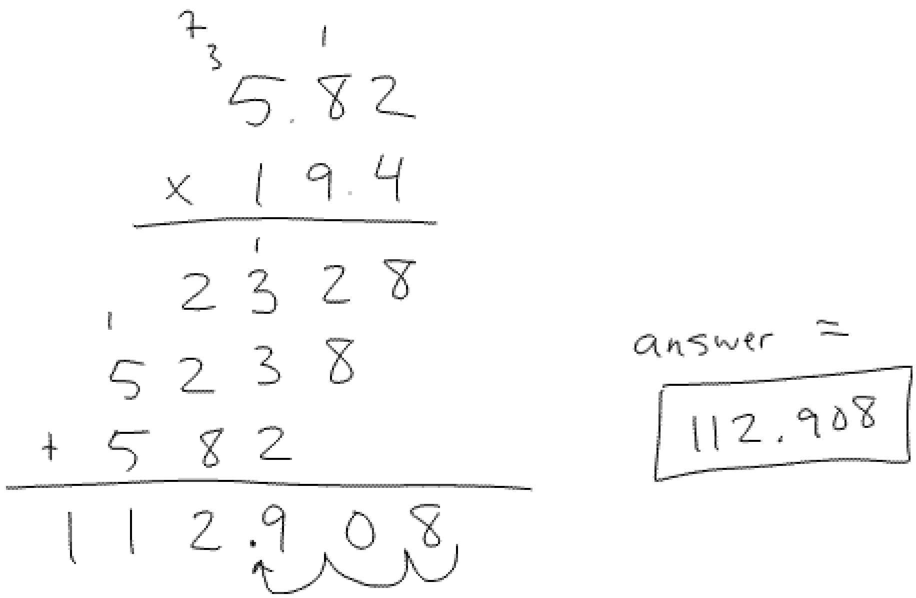
$$\begin{array}{rrrr}
 & & 8 & 1 \\
 & \times & 3 & 5 \\
\hline
 & 4 & 0 & 5 \\
+\,2 & 4 & 3 & \\
\hline
2 & 8 & 3 & 5
\end{array}
\qquad
.2835$$

0.2835 = answer

We conclude that 0.081 × 3.5 = 0.2835.

The traditional short-cut for multiplying decimals works in the exact same manner, although it skips the step of converting the two decimals into fractions. First, we multiply the two numbers together as if they were whole numbers. Next, we divide by a power of ten by "moving the decimal point to the left." The number of times we move the decimal point will be the total number of places after the decimal points of the two numbers. For example, in 37.4 × 2.91, there are 3 places used after decimal places – one for the 4 in 37.4 and two for the 91 in 2.91. Thus, we multiply 374 × 291 and then move the decimal point to the left 3 places. This is exactly what we did in the earlier example. The "one decimal place" in 37.4 corresponds to the "divided by 10" in $37.4 = \frac{374}{10}$ and the "two decimal places" in 2.91 corresponds to the "divided by 100" in $2.91 = \frac{291}{100}$.

The traditional way to illustrate this short-cut method is to first write the numbers in long multiplication form (with their decimal points), then multiply the numbers together (ignoring the decimal points), and then move the decimal point in the answer as many spaces to the left as there are places after the decimal points. For example, the work to compute 5.82 × 19.4 will look like:



To explain why this works, we add the additional step of converting the numbers to fractions: $5.82 \times 19.4 = \frac{582}{100} \times \frac{194}{10} = \frac{582 \times 194}{1000}$. This is why we need to multiply 582 × 194 as whole numbers and then divide the result by 1000.

When we skip the explanation, students can become confused. A popular complaint is "why don't we line up the decimal places?" This is reasonable, because in all additions and subtractions, we emphasize the importance of making sure the decimal points of all numbers are lined up correctly. This is the only way to ensure that ones are added to ones, tens are added to tens, and so on. However, in this case, we are actually multiplying whole numbers together. The only reason why we write:

$$\begin{array}{r} 5.82 \\ \times\ 19.4 \\ \hline \end{array} \qquad \text{instead of} \qquad \begin{array}{r} 582 \\ \times\ 194 \\ \hline \end{array}$$

is to save space and time with a short-cut. Indeed, we are really performing the multiplication on the right. However, it is useful to write out the problem as on the left, so that we can see the numbers that we really want to multiply, and keep track of the number of decimal places involved. This information is necessary for us to know how many places to move the decimal place when we are done.

Students can also become confused when multiplying decimals that end in zeros. For example, to multiply 37 × 4.30 we can follow the usual process exactly:

$$\begin{array}{r} {}^{2}_{2} \\ 37 \\ \times\ 4.30 \\ \hline 00 \\ 111 \\ +\ 148 \\ \hline 159.10 \end{array} \qquad \text{answer: } \boxed{159.10}$$

We multiply 37 × 430 and then move the decimal place to the left twice, once for each digit after the decimal point (the 3 and 0 in 4.30).

However, the number 4.30, when expanded out, is $4 + \frac{3}{10} + \frac{0}{100}$. Because $\frac{0}{100} = 0 \div 100 = 0$ (zero things shared among 100 gives zero to each), we do not need this fraction, and thus $4.30 = 4 + \frac{3}{10} = 4.3$ (which can also be easily illustrated with base-ten columns).

Thus, it works just as well to multiply 37 × 4.3, which looks like:

$$
\begin{array}{r}
2 \\
2 \\
37 \\
\times\ 4.3 \\
\hline
111 \\
148 \\
\hline
159.1
\end{array}
\qquad\qquad
\text{answer: } \boxed{159.1}
$$

The answer is the same (159.1 = 159.10 for the same reasons that 4.3 = 4.30), but the work is shorter and less confusing. Remember, we want our students to be always on the look-out for short-cuts and ways to make the math clearer and easier to compute and understand!

Questions:

(1) Convert each of the following decimals into a single fraction:

(a) 39.297 (b) 0.0081 (c) 995.62 (d) 29.6

(2) Compute 3.41 × 2.05 by converting the numbers into fractions and multiplying them as fractions. Show all your work.

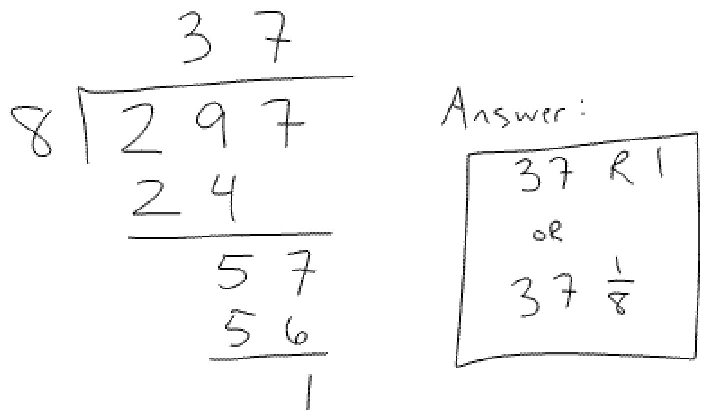
(3) Compute 1.4 × 2.57 using the short-cut method. Then briefly explain what happened using fractions.

(4) Explain in detail why 3.510 = 3.51.

Chapter 36: Long Division with Decimals

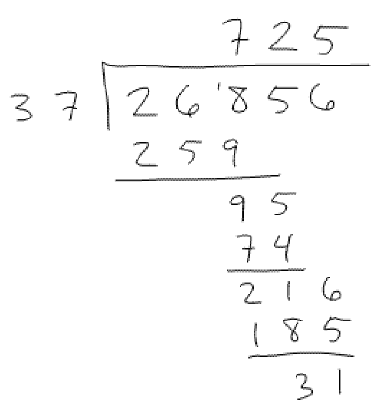
Just as with the multiplication of decimals, the division of decimals is much like that of whole numbers. Similarly, the key to explaining the usual short-cut process also involves fractions.

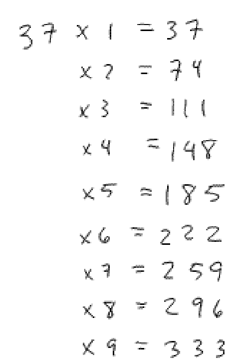
We have seen how to use long division to divide whole numbers, for example 297 ÷ 8 is calculated as:



We divide the hundreds by the divisor, then convert the leftovers into tens. We divide the tens by the divisor, then convert the leftovers into ones. We then divide the ones by the divisor and then either leave the leftovers as a remainder, or else write this as a fraction. In the above example, we had a remainder of 1, which becomes 1/8 when divided into 8 equal parts.

The hardest part about decimal long division is that the divisors tend to have 2 or 3 digits, and thus their multiples are not basic math facts that the students have memorized. When we divide by 8 as above, for example, we know automatically that 8 × 3 = 24 and 8 × 7 = 56. However, when we divide by 37, it is not immediately clear what 37 × 7 and 37 × 8 are exactly. We can estimate that 37 × 7 < 40 × 7 = 280 and 37 × 8 < 40 × 8 = 320, but approximations like these are not always enough to guess the multiple needed.

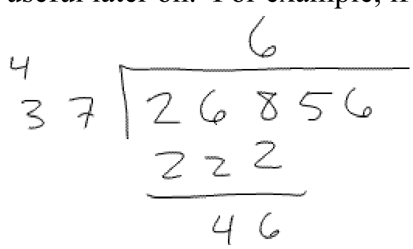




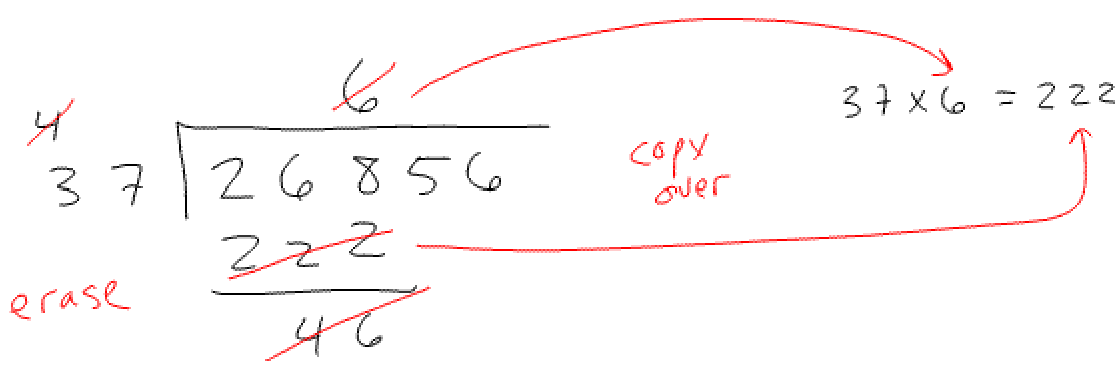
Some teachers encourage their students to start with a list of all the multiples of the divisor. For example, when dividing 26,856 ÷ 37, the students will start by listing out 37 × 1 = 37, 37 × 2 = 74, 37 × 3 etc. These can be computed fairly quickly by starting with 37 × 1= 37 and then adding 37's to get the subsequent products (37 × 2 = 37 + 37 = 74 and 37 × 3 = 74 + 37 = 111, etc.). For a very long problem, this can definitely save time. In this example, we can immediately see from the list that 37 × 7 = 259 is the largest multiple that fits into 268. Unfortunately, this process can easily lead to mistakes. A carrying error, for example saying that 37 × 3 = 74 + 37 = 101, will snowball into a great number of other mistakes and cause the whole problem to be wrong. Also, when students feel the need to always follow this procedure, they will generally take a long time to compute division problems.

The best solution is to teach your students to use a combination of mental math and estimation. With 26,856 ÷ 37, for example, there really is no need to compute all the multiples of 37. When we first confront "how many times does 37 go into 268?" it is clear that 5 is too small, because if we over-estimate 37 with 40, we get 5 × 40 = 200, which is already 68 too small. Similarly, the if we under-estimate 37 with 30, we get 9 × 30 = 270, which is too big. Thus 5 is too small and 9 is too big. At worst, we will need to figure out 6 × 37, 7 × 37, and 8 × 37. If we start by calculating 7 × 37 we will luck out (in this example), but otherwise we will then immediately know to go to 6 × 37 (if we go over with 7 × 37) or 8 × 37 (if we are at least 37 under with 7 × 37).

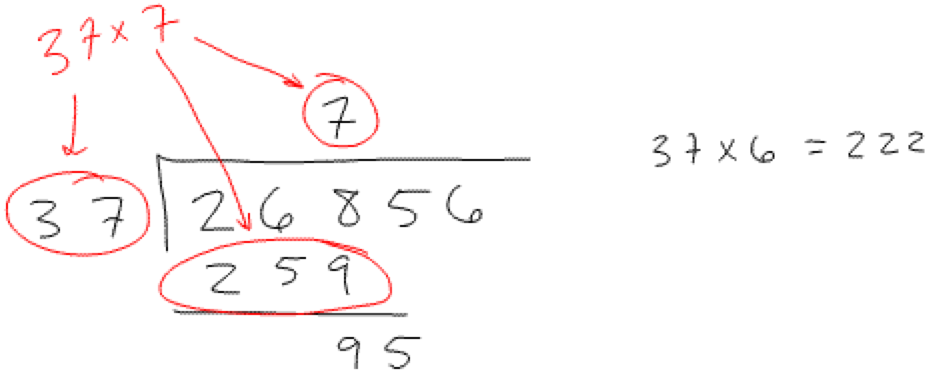
One very important thing to teach your students is to keep track of all the multiples that don't work out. If you calculate 6 × 37 = 222 and it turns out to be too small, write this fact out on the side of the paper. If we just erase everything and start over, we will feel as though we "haven't gotten anywhere" whereas we really have learned something valuable that might be very useful later on. For example, if we began by trying 6, our work would look like:



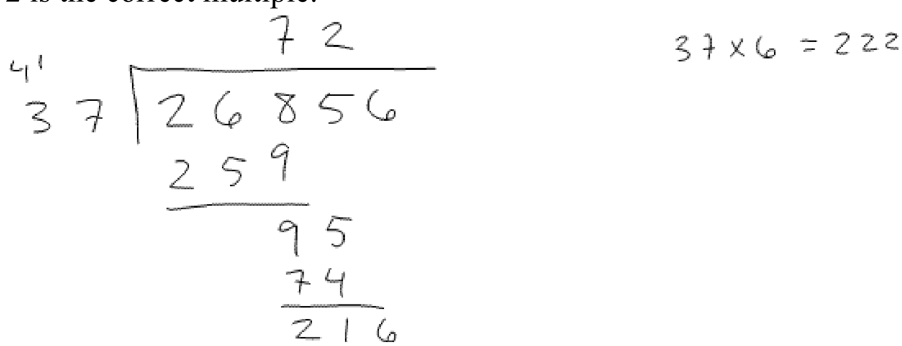
We see that our remainder of 46 is bigger than our divisor of 37, and thus that the choice of 6 was too small. Rather than erase everything and start from scratch, we have learned some very valuable things here. First of all, 6 × 37 = 222, which we should copy over to the side:



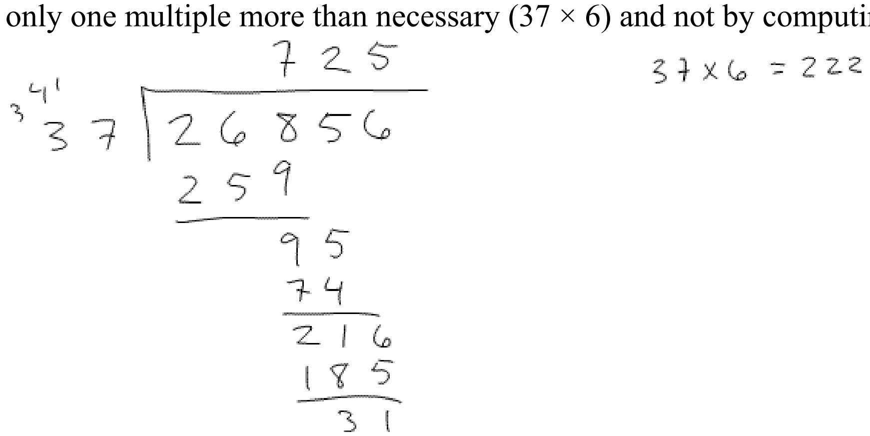
Second, we can only fit one 37 into 46, and thus 7 will be the correct digit to use. Third, we can easily calculate 37 × 7 = 222 + 37 = 259. We do not need to copy this over to the side, because it is written right into the problem. This might come in handy later:



Next, we figure that 37 will go into 95 only a small number of times. We estimate that 2 × 40 = 80 is close and 3 × 40 = 120 is too much, but these are over-estimates. If we try 3, we can mentally calculate that 37 × 3 = (30 + 7) × 3 = 30 × 3 + 7 × 3 = 90 + 21 will be too much. Thus, 2 is the correct multiple:



Normally, the question "how many times does 37 go into 216?" would be a hard one. However, we already know that 37 × 6 = 222 is a tiny bit too much (less than 37 over). This tells us immediately that 5 is the multiple to use. Thus, we can complete the problem by computing only one multiple more than necessary (37 × 6) and not by computing all of them:



As was said earlier, it is this process of estimating, multiplying, correcting, and re-trying that makes long division difficult. The rest is rather easy to figure out and explain.

The first thing to do when dividing decimals is to change the problem so that the divisor is a whole number. This is easiest to see by converting the problem into a fraction. For example, if we want to divide $39.75 \div 2.7$, we write it as $\frac{39.75}{2.7}$ and then multiply top and bottom by as many 10's as is necessary to turn the 2.7 into a whole number. In this case, there is only one digit after the decimal place in 2.7, and so we multiply the top and bottom by a single 10, making it $39.75 \div 2.7 = \frac{39.75}{2.7} = \frac{39.75\times 10}{2.7\times 10} = \frac{397.5}{27}$. Similarly, if we wanted to divide $492 \div 0.19$, we would multiply top and bottom by 100 because $492 \div 0.19 = \frac{492}{0.19} = \frac{492\times 100}{0.19\times 100} = \frac{49200}{19}$ is necessary to convert the denominator into a whole number.

When this trick is illustrated with the usual long-division notation, it looks like:

$$2.7\overline{)39.75} \Longrightarrow 27\overline{)397.5}$$

$$0.19\overline{)492} \Longrightarrow 19\overline{)49200}$$

The short-cut trick, which your students might (or might not) notice is called "move the decimal points the same number of spaces." In order to turn 2.7 into a whole number, we must move the decimal point to the right one space. Thus, we move the decimal point of 39.75 to the right one space as well:

$$2.7\overline{)39.75} \Longrightarrow 27\overline{)397.5}$$

Similarly, to make 0.19 a whole number, we need to move the decimal place to the right two spaces, and thus we do the same to the dividend 492:

$$0.19\overline{)492} \Longrightarrow 19\overline{)49200}$$

The reason why this works is because we are "doing the same thing to the top and bottom of a fraction." In the second instance, we are taking the fraction $\frac{492}{0.19}$ and chopping everything into 100 equal-sized pieces. This makes our number of pieces 100 times more (from 492 to 49200) and 100 times smaller (from 0.19 to make a whole to 19 to make a whole).

After this trick is accomplished, the long division proceeds just as before. The only difference is that we do not stop with a remainder of ones, but convert them into tenths and continue the process into the extended base-ten columns. A good way to introduce this is to begin by dividing by a small number. This avoids all the confusion of estimation and enables the class to focus on the new concept. For example, let us walk through the problem of 95 ÷ 4.

To begin, we take the first digit, the 9, which represents 9 tens. We can share out 8 of these by giving 2 to each of the 4, leaving 1 ten behind:

$$
\begin{array}{r|l}
 & 2 \text{ tens} \\ \hline
4 & 9 \text{ tens} + 5 \text{ ones} \\
 & -\,8 \text{ tens} \\ \cline{2-2}
 & 1 \text{ ten}
\end{array}
$$

We then convert 1 left-over ten into 10 ones and add them to the 5 ones for a total of 15 ones. It might be a good idea to act this out with the fake 1 and 10 dollar bills:

$$
\begin{array}{r|l}
 & 2 \text{ tens} \\ \hline
4 & 9 \text{ tens} + 5 \text{ ones} \\
 & -\,8 \text{ tens} \\ \cline{2-2}
 & \cancel{1 \text{ ten}} \longrightarrow 10 \text{ ones} \\ \cline{2-2}
 & \qquad\qquad\quad 15 \text{ ones}
\end{array}
$$

We can give 3 ones to each of the 4 students sharing the fake \$95. This uses up \$12, leaving \$3 left-over:

$$
\begin{array}{r|l}
 & 2 \text{ tens} + 3 \text{ ones} \\ \hline
4 & 9 \text{ tens} + 5 \text{ ones} \\
 & -\,8 \text{ tens} \\ \cline{2-2}
 & \cancel{1 \text{ ten}} \longrightarrow 10 \text{ ones} \\ \cline{2-2}
 & \qquad\qquad\quad 15 \text{ ones} \\
 & \qquad\qquad\; -\,12 \text{ ones} \\ \cline{2-2}
 & \qquad\qquad\qquad 3 \text{ ones}
\end{array}
$$

We can continue at this point by following the same process. Just as the 1 leftover ten became 10 ones, we can convert our 3 leftover ones into 30 tenths. With fake money, this can be easily explained. Because we cannot share 3 one-dollar bills among 4 people, we need to use change. Thus, we break our 3 one-dollar bills into 30 dimes:

2 tens + 3 ones
4 | 9 tens + 5 ones
− 8 tens
~~1 ten~~ → 10 ones
15 ones
− 12 ones
~~3 ones~~ → 30 tenths

We can take our 30 dimes (or tenths-of-dollars) and share out 28 of them by giving 7 to each of the 4 students. This will leave us with a remainder of 2 dimes:

2 tens + 3 ones + 7 tenths
4 | 9 tens + 5 ones
− 8 tens
~~1 ten~~ → 10 ones
15 ones
− 12 ones
~~3 ones~~ → 30 tenths
− 28 tenths
2 tenths

We can now convert our 2 tenths into 20 hundredths, because $\frac{2}{10} = \frac{20}{100}$. With fake money, this is equivalent to cashing the 2 dimes in for 20 pennies (hundredths-of-dollars). At this point, we can evenly divide the 20 pennies by 4 evenly, giving each of the 4 people 5 hundredths:

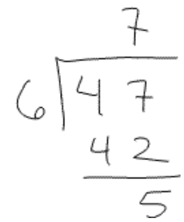
```
      2 tens + 3 ones + 7 tenths + 5 hundredths
4  |  9 tens + 5 ones
    - 8 tens
      1 ten  -> 10 ones
                15 ones
              - 12 ones
                3 ones -> 30 tenths
                        - 28 tenths
                          2 tenths -> 20 hundredths
                                    - 20 hundredths
                                       0 hundredths
```

As before, we can save ourselves a whole lot of writing by doing this with base-ten number notation. We could represent the above calculation with number columns as:

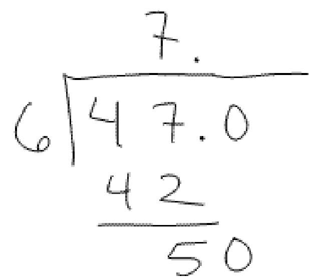
	tens	ones	.	tenths	hundredths
	2	3	.	7	5
4	9	5			
	-8				
	1	5			
	-1	2			
		3	.	0	
		-2		8	
				2	0
				-2	0
					0

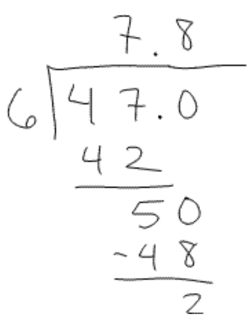
Of course, as long as your students know that the digit to the left of the decimal point represents ones, the digit to the right represents tenths, etc. there isn't even a need for the labels.

For example, we can divide 47 ÷ 6. We begin by trying to divide 4 tens by 6, but not even 1 ten can be given to each of the 6 groups. When we convert the 4 tens into 40 ones and add it to the 7 ones we have already, we next ask "how many times does 6 go into 47?" We know from basic multiplication facts that 6 × 7 = 42 and use this:



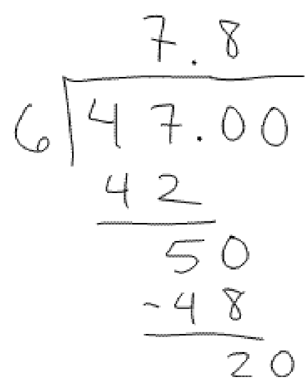
Next, in order to convert the remaining 5 ones into 50 tenths, all we need to do is intoduce a decimal point to the dividend. We can then write a zero after the decimal point, to make a tenths column, and "bring down" this zero to turn the 5 into a 50. Technically, there ought to be a decimal point in every number in this column. It is a good idea to put one up above, after the 7, to make sure the answer comes out correctly. However, most people just write the 5.0 as 50.

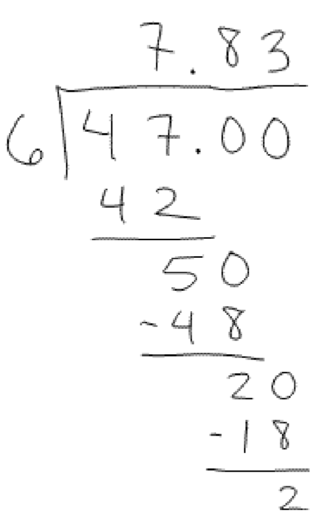




We know that 6 × 8 = 48 is the largest multiple of 6 that is not bigger than 50, and so we use this to compute the next step of the problem. Our remainder of 2 represents 2 tenths because it is in the tenths column.

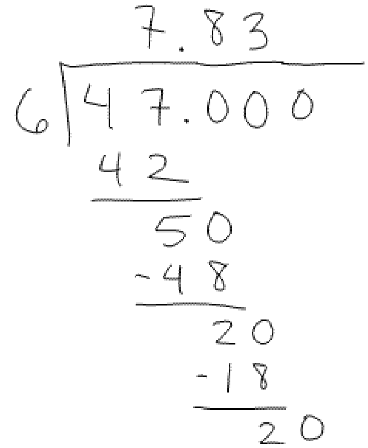
We can convert these 2 tenths into 20 hundredths merely by adding another zero. Traditionally, we put another zero at the end of the dividend, just to establish the next column.

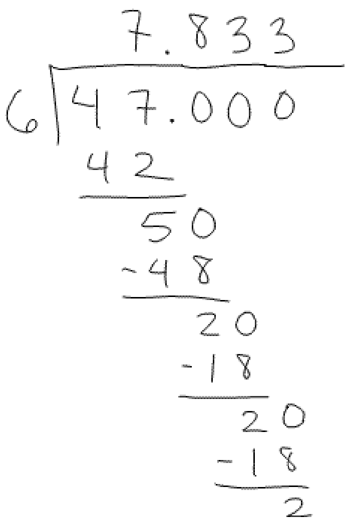




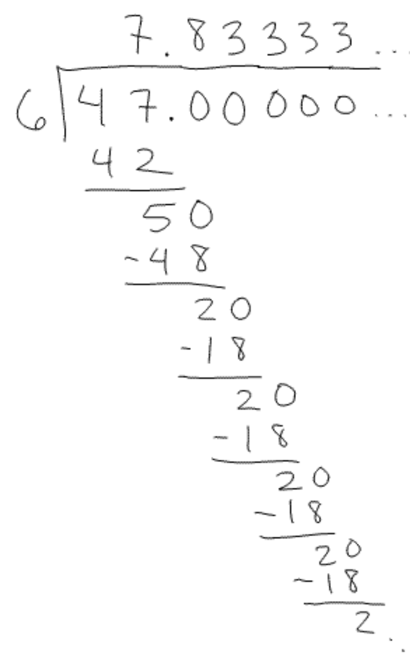
We can next divide the 20 hundredths by 6, resulting in 3 with a remainder of 2 hundredths.

We can convert the 2 hundredths into 20 thousandths just by adding more zeros to establish a thousandths column.





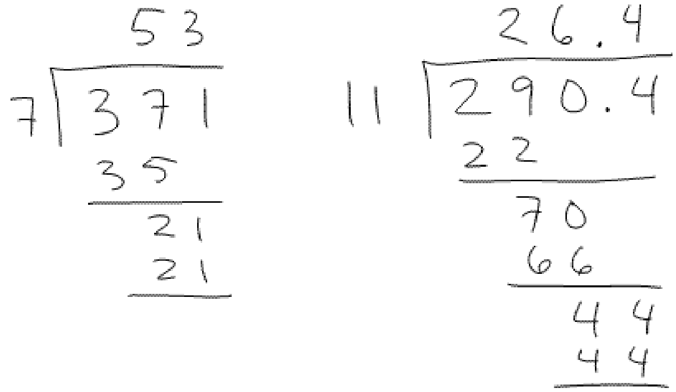
We can share out the 20 thousandths by giving 3 thousandths to each of the 6, using up 18 thousandths and leaving behind 2 thousandths.



Carry this process out for several more steps. Hopefully, someone in your class will begin to notice that it will never stop, but will repeat forever. From this point on, we will always turn the leftover 2 into 20 in the next column, divide this by 6 with a 3 and get another remainder of 2.

Rather than write out digits for the rest of our lives, there is a short-hand notation for this: put a bar over the part that repeats. In this example, $47 \div 6 = 7.8\overline{3}$, which is an abbreviated way of saying $47 \div 6 = 7.833333333333333...$

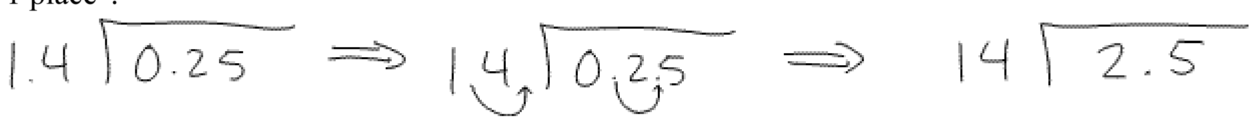
When we calculate a long-division problem, there are three potential results. One is that the number comes out evenly without needing to create new decimal places. Examples of this include 371 ÷ 7 and 290.4 ÷ 11, as illustrated below:



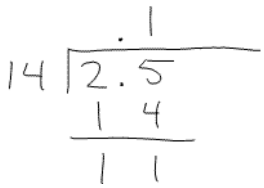
The second possibility is that we might need to add a few decimal places before the problem comes out evenly. We saw this earlier with 95 ÷ 4 = 23.75. This will only happen when the divisor factors entirely into 2's and 5's, the factors of the number 10 which underlies our base-ten number system. This means that whenever we divide by 2, 5, 4, 8, 10, 16, 20, 25, 32, or any other number whose prime factors are all 2's and 5's, the process of long division will eventually stop, though some additional decimal places might be needed.

The third and final possibility is that the problem will go on forever. This will happen every time we divide by a number that contains a factor other than 2 or 5 (unless the division comes out evenly, without the need for adding new decimal places). For example, when we divide by 3, 6, 7, 9, 11, 12, 13, 14, 15, 17, 18, or 19, the long division will either come out evenly or will go on forever. Furthermore, the number of decimal places we will need to add (before the answer begins to repeat) will never be more than the divisor. When we divided 47 ÷ 6, for example, we only had to add two digits (2 < 6) before reaching the repeating 3 in our answer $7.8\overline{3}$. If we divided by 17, however, we might have to add 17 more digits before our answer would begin to repeat.

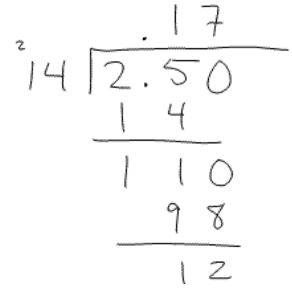
As a culminating example for this chapter, let us divide 0.25 ÷ 1.4. We begin by multiplying both the dividend and the divisor by 10 and "moving both decimal points to the right 1 place":



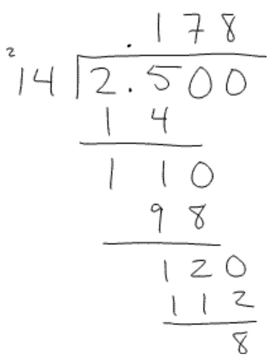
We will thus divide 2.5 ÷ 14 and get the same answer as 0.25 ÷ 1.4.



There is no way that 14 will go into 2 ones, and so we look instead at "how many times does 14 go into 25 tenths?" It helps to put a decimal point up in our answer at this point. Because 14 goes into 25 once, we put a 1 in the tenths place of our answer and compute the remainder of 11.

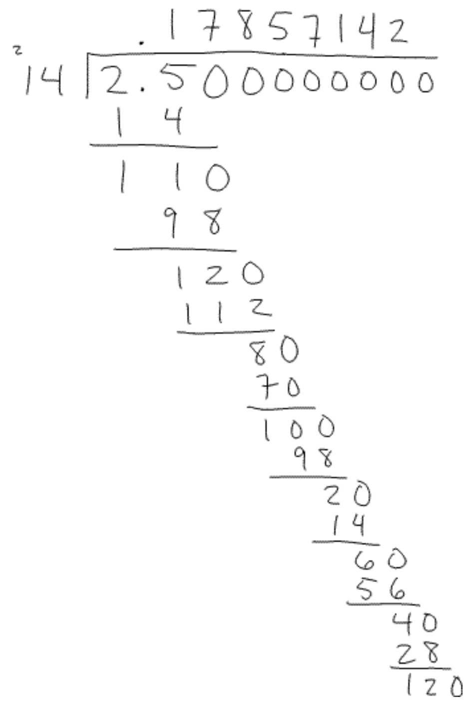


Next, we add a new decimal place and contemplate "how many times does 14 go into 110?" We can easily calculate that 14 × 5 = 70 (think 14 × 5 = 7 × 2 × 5 = 7 × 10) and 14 × 10 = 140. This means that 14 × 6 = 70 + 14 = 84, 14 × 7 = 84 + 14 = 98, and 14 × 8 = 98 + 14 = 112. We are thus able to fit 7 fourteens into 110, but not 8.



After this, we turn the remaining 12 hundredths into 120 thousandths and continue. We use our earlier calculation to see that 14 × 8 = 112 is as close as we can get without going over.

Already at this point, we know that 14 × 1 = 14, 14 × 5 = 70, 14 × 7 = 84, 14 × 7 = 98, and 14 × 8 = 112. Though this is only a partial list of multiples of 14, it can be very handy. For example, the next remainder is 8 thousandths, which will become 80 ten-thousandths. We can see that 14 × 5 = 70 is only 10 below, and thus the right choice.

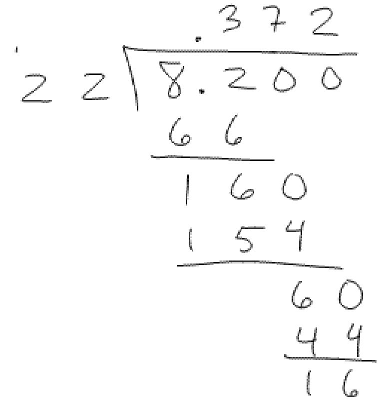


As we carry on, we need to figure out how many times 14 goes into 100, then 20, then 60, then 40, and then 120. We have had this remainder before, and had to earlier ask "how many times does 14 go into 120?" Furthermore, this was after we began extending the dividend by adding zeros to the end. When we subtract 14 × 8 = 112 as we did before, we will get a remainder of 8, as we did before.

Thus, the next digit will be an 8, and then another 5. Just as before, this will be followed by another 7, then another 1, and so on.

We have thus figured out the complete and exact answer: $0.25 \div 1.4 = 0.17\overline{857142}$.

This is the complete answer to the problem, finding digits in the millionths and beyond (even before it repeats). For most practical purposes, this is unnecessary. It is often sufficient to *round* the answer. To do this, the students should calculate to one more decimal place than the one desired. If this next digit is a 5 or more, then the student should raise the digit to the left by 1 and then leave it at that. If the last digit is 4 or less, the student should discard it.



For example, to calculate 8.2 ÷ 22 to the nearest hundredth, we use long division up to the thousandth place. The digit in the thousandths place is a 2, which is less than 5, and thus we discard it and say that 8.2 ÷ 22 $\approx$ 0.37. The "squiggly equals sign," $\approx$, is pronounced as "is approximately equal to." The exact answer (notice that we have divided 22 into 160 before?) is that 8.2 ÷ 22 = $0.3\overline{72}$ = 0.372727272... However, for most practical purposes, it is enough to say that the number, rounded to the nearest hundredth, is 0.37.

Teaching students to round numbers is a useful way to test and reinforce their knowledge of the base-ten number system. For example, a student asked to round 12,859.4 to the nearest thousand will need to know that the 2 is the digit in the thousands place. The digit to the right is an 8, bigger than 5, and thus we round the 2 up to a 3 and say that the number is approximately 13,000 (when rounded to the nearest thousand).

The most confusing aspect of rounding is what to do when rounding up a 9. For example, what do we do when rounding 37.952 to the nearest tenth? The digit to the right of the tenth's place is a 5, so we should round up the 9. However, there is no room for a 10 in the tenths place, so we replace it with a 0 and add 1 to the 7 in the ones place. The result is 38.0, which is the correct answer to "round 37.952 to the nearest tenth." We keep the "point zero" in this case, even though we would normally drop it, because it emphasizes that our answer is accurate into the tenth's place.

Rather than teach your students that rounding is yet another task and process for them to memorize, introduce it in the context of long division. In this manner, it can be viewed as a relief, an easy way to finish up a problem. It is astonishing how much you can get your students to learn when they think they are getting out of work!

Questions:

(1) Use fractions to explain the "move the decimal places the same distance" trick for dividing 39.87 ÷ 2.914. (Don't calculate the division, just show how to convert the divisor into a whole number.)

(2) Use long division to compute 127 ÷ 8 completely. Make a side note of each conversion necessary (from hundreds to tens, from tens to ones, from ones to tenths, etc.)

(3) Use long division to compute 99 ÷ 37 completely.

(4) Use long division to compute 7.85 ÷ 2.1 to the nearest hundredth.

Chapter 37: Large Numbers and Scientific Notation

Many people have difficulty grasping and comparing very large numbers. This is a shame, for this is not too hard a topic to introduce and explain. Students need only learn some vocabulary in order to name and write many large numbers. For even bigger numbers, there is scientific notation, which offers an easy introduction to exponents.

When a number has a large number of digits, it is traditional to use commas to separate the digits into groups of three, starting with the decimal point. For example, the number 2900459821.64 will usually be written as 2,900,459,821.64. This is the proper notation for fancy occasions like final answers, but is not necessary in the middle of working out a problem. For example, if the above number were multiplied by 10,000, it is much easier to move the decimal point to the right four spaces with the commas taken out:

2 9 0 0 4 5 9 8 2 1 . 6 4 0 0

before putting the commas back in for the final answer: 29,004,598,216,400.

If a student attempted to multiply 2,900,459,821.64 × 10,000 with the commas, the result can be quite confusing:

2 , 9 0 0 , 4 5 9 , 8 2 1 . 6 4 0 0

because some students will want to keep the commas where they are, for a final answer of 2,900,459,821,6400 or something like that.

The importance of the commas is not mathematical but rather linguistic – they help us to pronounce the number. Each group of three digits (to the left of the decimal point) names a different category of number, starting with the three closest to the decimal point and moving to the left. The first group are the ones, measured in hundreds, tens, and ones (although in both cases, we do not pronounce the "ones"). The next group are the thousands, also measured in hundreds, tens, and ones. The next group are the millions, also measured in hundreds, tens, and ones. The next groups are the billions, trillions, and quadrillions, each measured in hundreds, tens, and ones. The next group is usually called the quintillions, but some suggest that pentillion might be a better name. The group names go on, but this is quite enough for any practical use. A student who feels empowered by a knowledge of names can be encouraged to research this topic and present the results to the class, but the mathematical benefits of such work are negligible.

We can write out the group names with commas in the following way:

quadrillions , *trillions* , *billions* , *millions* , *thousands* , (*ones*)

The ones are written in parentheses as a reminder that these are not pronounced. The names of cycles can help to teach students the order of these. A "bi"-cycle has two wheels, just as "billion" is the second sort of "illion". A "tri"-cycle has three wheels, just as "trillion" is the third sort of "illion". A "quad" bike has four wheels, just as "quadrillion" is the fourth sort of "illion". Explaining the "m" in "million" can be a little more difficult for students because a one-wheeled cycle is called a "uni-cycle" and not a "mono-cycle." Similarly, monocles (a single eye-lens, famously used in caricatures of rich people), monorails (trains using a single rail), and monoliths (single-stone monuments) are not in the common vocabulary of most school children. Fortunately, the word "million" is found in everyday speech, so a teacher need only show the class what it means.

If we expand our base-ten number columns to include large numbers, it will look something like this:

hundred-quadrillions	ten-quadrillions	quadrillions	hundred-trillions	ten-trillions	trillions	hundred-billions	ten-billions	billions	hundred-millions	ten-millions	millions	hundred-thousands	ten-thousands	thousands	hundreds	tens	ones

However, it might be more useful to write it as:

hundred	ten		hundred	ten		hundred	ten		hundred	ten		hundred	ten		hundred	ten	
quadrillions			trillions			billions			millions			thousands					

For example, the number 29,004,598,216,400 will fit in the columns as:

hundred	ten		hundred	ten		hundred	ten		hundred	ten		hundred	ten		hundred	ten	
quadrillions			trillions			billions			millions			thousands					
				2	9	0	0	4	5	9	8	2	1	6	4	0	0

This number is pronounced "twenty-nine trillion, four billion, five-hundred ninety-eight million, two-hundred sixteen thousand, and four-hundred." If we write this half with numbers, the pattern becomes even easier to see. The number 29,004,598,216,400 is pronounced "29 trillion, 4 billion, 598 million, 216 thousand, and 400." In other words, we read each set of numbers between the parentheses as if they were less than a thousand, and then say the order of that set (trillions, billions, millions, thousands, etc.).

Students should know how to read out numbers as big as quadrillions. They should also be able to write out a number that is read aloud to them. This process is even easier – write out the numbers as they are read and separate them with parentheses. For example, a student who hears "sixty-four quadrillion..." should write "64,". When the number continues "two-hundred and seventeen trillion," the student should add to the number to make "64,217,". If the number continues "fifty billion," the student should know to put in a zero before the "50" (because the digits are being set down in threes) so that the number currently stands at "64,217,050,". The only tricky part to this process is to know to put in three zeros when an entire order of number (or more) is skipped. For example, if our number then finishes "and eight thousand," our student ought to know that there are no millions and no ones, so that the number is 64,217,050,000,008,000. The students should be encouraged to read their numbers back, to make sure that nothing has been skipped. A student who accidentally writes "64,217,050,008,000" should be able to catch that this is approximately 64 trillion and not the 64 quadrillion requested. This example, do note, is about as difficult as possible. It is best to start of with smaller numbers (in the thousands and millions) before moving up to larger numbers. Also, let the class have some experience writing out numbers like "seventy-five trillion" before more complicated ones like "seventy-five trillion, two-hundred thousand."

It is very important for people to have an appreciation for the relative sizes of large numbers. In politics, for example, a debate might involve the numbers 1.4 trillion, 500 million, and 70 billion. One of the best ways to compare numbers is either to set them out in number columns or otherwise line up their digits so that the places correspond. The three numbers in our current example will look like:

hundred	ten		hundred	ten		hundred	ten		hundred	ten		hundred	ten	
trillions			billions			millions			thousands					
		1	4	0	0	0	0	0	0	0	0	0	0	0
						5	0	0	0	0	0	0	0	0
				7	0	0	0	0	0	0	0	0	0	0

Notice that the number 1.4 trillion has a 1 in the trillion's place and the 4 in the hundred-billions place. This is because the 1 in 1.4 trillion represents 1 trillion. The .4 represents a "tenth of a trillion," which is what you get when you divide a trillion by ten – a number in the next column to the right. Another way to look at this is to say that 1.4 trillion = 1.4 × 1 trillion = 1.4 × 1,000,000,000,000 = 1,400,000,000,000.

As another example, 2.75 billion is written with a 2 in the billions place, then the 7 in the next place to the right, and the 5 in the place after that, resulting in: 2,750,000,000. It helps to think about it like this: 2.75 is more than 2 and less than 3. Thus 2.75 billion ought to be between 2 billion and 3 billion, as is 2,750,000,000.

In any case, to compare large numbers, it helps to set them in common terms. For example, to compare 500 million and 70 billion, we look at them on the number-line chart:

hundred	ten		hundred	ten		hundred	ten		hundred	ten	
billions			millions			thousands					
			5	0	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0	0	0	0

Rather than compare 500,000,000 and 70,000,000,000 directly, we can look at them as numbers with the same units. If we set our units to millions, then these two numbers are 500 million and 70,000 million:

hundred	ten		hundred	ten		hundred	ten		hundred	ten	
billions			millions			thousands					
			5	0	0	million					
	7	0	0	0	0	million					

The relationship between 500,000,000 and 70,000,000,000 is the same as that between 500 and 70,000. This can be seen as viewing the two numbers as a fraction: $\frac{500{,}000{,}000}{70{,}000{,}000{,}000} = \frac{500 \times 1{,}000{,}000}{70{,}000 \times 1{,}000{,}000} = \frac{500}{70{,}000}$. In fact, we can reduce this fraction even more by: $\frac{500}{70{,}000} = \frac{5 \times 100}{700 \times 100} = \frac{5}{700}$. We could also see this on the number chart, by using hundred-millions as our units (instead of ones or millions):

hundred	ten		hundred	ten		hundred	ten		hundred	ten	
billions			millions			thousands					
			5	hundred million							
	7	0	0	hundred million							

It certainly is weird to change "70 billion" to "700 hundred million," but mathematically these numbers are the same.

Now we know that the relationship between 500 million and 70 billion is the same as that between 5 and 700. If we divide 700 ÷ 5, we get 140, which means that 70 billion is 140 times bigger than 500 million. In other words, a government budget of $70 billion could be used to fund 140 projects that each cost $500 million.

As another example, let us compare 1.4 trillion to 70 billion. First, we write out the numbers as 1,400,000,000,000 and 70,000,000,000. Next, let's skip the base-ten number columns and just write the numbers so that their digits line up:

1,4 0 0,0 0 0,0 0 0,0 0 0

7 0,0 0 0,0 0 0,0 0 0

If we cut off the same number of zeros from both numbers (dividing by the same power of ten), the remainders of the numbers will have the same relationship as 1.4 trillion and 70 billion. We can make the numbers we need to compare as small as possible by cutting them off right after the 7, as illustrated below:

1,4 0 | 0,0 0 0,0 0 0,0 0 0

7 | 0,0 0 0,0 0 0,0 0 0

This puts the numbers both in terms of "ten-billions": 1.4 trillion is 140 ten-billion and 70 billion is 7 ten-billion. However, this is not important for our calculations. All we need to do is line up the numbers like this to see that the relationship between 1.4 trillion and 70 billion is the same as that between 140 and 7. If we divide 140 ÷ 7 we get 20, thus 1.4 trillion is 20 times as big as 70 billion. A sum of $1.4 trillion, for example, would be able to fund 20 projects that each cost $70 billion.

Already, we can see that writing large numbers can require a large number of zeros. This can be troublesome. For one thing, if we accidentally forget a zero or put an extra one in, this changes the number by a factor of ten. While a lazy student might feel that there is no substantial difference between 5000000 and 50000000, a factor of ten is a very large amount. A yearly salary of $50000 is respectible, for example, whereas $500000 is enormous and $5000 is well below the poverty line.

In scientific calculations, even larger numbers come into play. The distance light travels in a year, for example, is approximately 9,460,000,000,000,000 meters. To avoid having to write out all of these zeros, we can use *scientific notation*. To write a number into scientific notation, we first break it into two parts. The first part will have a non-zero digit in the one's place and the rest of its digits after a decimal point. The second part will be a power of ten. For example, the number 9,460,000,000,000,000 can be broken into 9.46 × 1,000,000,000,000,000. In other words, this number is 9.46 quadrillion. Next, we write the power of ten as a ten raised to an exponent. The exponent gives the number of times that 10 must be multiplied by itself to form the number. For whole-number powers of 10, this will be the number of zeros after the 1. Thus, in our example, 9.46 × 1,000,000,000,000, 000 = 9.46×10^{15} (there are 15 zeros after the 1).

Another way to do this is to put a decimal place after the first digit and then count the number of spaces we must shift the decimal place to arrive at this spot. For example:

insert decimal point here

9 460 000 000 000 000

this is 15 spaces from the original decimal point

This means we need to divide 9,460,000,000,000,000 by 10 fifteen times to turn it into 9.46. Thus, to turn 9.46 back into our number, we need to multiply by 10 fifteen times, which is represented by 9.46×10^{15}.

As another example, 35,980,000 can be converted into scientific notation as:

decimal point

3 5 980 000

7 spaces

Thus 35,980,000 = 3.598×10^{7}.

Students should also know how to convert from scientific notation back into whole numbers. The trick is to move the decimal place as many times to the right as given in the exponent of the number. For example, 2.7×10^{4} really means 2.7 × 10 × 10 × 10 × 10 because 10^4 means 10 × 10 × 10 × 10 (ten multiplied by itself 4 times). To multiply by 10 four times is to move the decimal point to the right four times, thus 2.7×10^{4} = 27000 = 27,000.

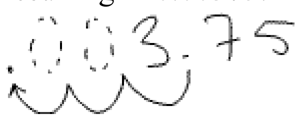
2.7 0 0 0

Similarly, the famous Avogadro's number 6.022 × 10^{23} is the number of carbon atoms in 12 grams of carbon. To convert this into a whole number, we need to move the decimal point 23 times:

6.0220000000000000000000.

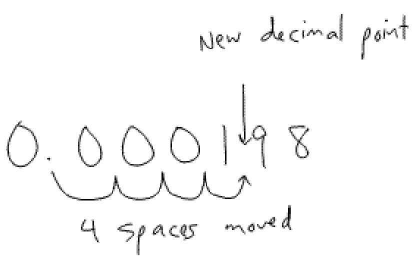
Thus, Avogadro's number is 602,200,000,000,000,000,000,000. This should show the advantage of scientific notation: 6.022×10^{23} tells us that the number is basically a 6 followed by 23 zeros, with no need to actually write them all out.

Scientific notation can also represent very small numbers. This is done by using a negative number as the exponent of 10. Just as subtracting 3 is the opposite of adding 3, for example, multiplying by 10^{-3} is the opposite of multiplying by 10^{3}. Thus, the number 3.75×10^{-3} represents 3.75 *divided* by 10 three times. This moves the decimal point to the left three times, resulting in 0.00375:



Similarly, the number 1.4×10^{-6} is formed by moving the decimal place to the left 6 times, resulting in 0.0000014:

To convert a very small number into scientific notation, we do the same as before. We put the new decimal place after the first non-zero digit (the non-zero digits are called the *significant digits*) and then figure out how many places we need to move the decimal point to reach the new position. For example, the number 0.000198 is converted as:

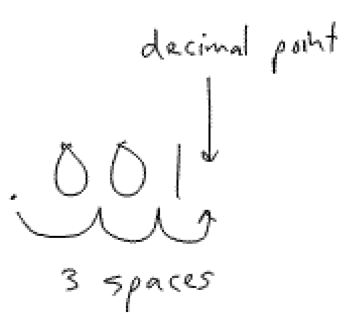


This means that $0.000198 = 1.98 \times 10^{-4}$.

Similarly, the number "one thousandth" is 0.001, as can be seen with base-ten columns:

tens	ones	.	tenths	hundredths	thousandths
		.	0	0	1

To convert this into scientific notation, we move the decimal point to right after the 1 by shifting it to the right three times:



Thus one-thousandth = 0.001 = 1×10^{-3}.

It can help to see the base-ten number columns written out with powers of ten:

millions	hundred-thousands	ten-thousands	thousands	hundreds	tens	ones	.	tenths	hundredths	thousandths	ten-thousandths	hundred-thousands
10^6	10^5	10^4	10^3	10^2	10^1	10^0		10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}

Questions:

(1) Make a chart of base-ten number columns that goes up to the hundred-quadrillions and use it to represent each of the following numbers:

(a) 35 million
(b) two-hundred and sixteen billion, four thousand and twelve
(c) 16 quadrillion, 75 trillion, 256 billion, 749 million, 218 thousand, and 15
(d) 8.4 million
(e) 27.91 billion
(f) 0.3 trillion
(g) 38,000 million
(h) 208 ten-thousand

(2) Write out the formal name of each of the following numbers:

(a) 29081
(b) 3100000
(c) 77460241
(d) 3504821222
(e) 991000000435
(f) 391498329785966843

(3) For each pair of numbers, state which one is larger, then say exactly how many times larger:

(a) 4 thousand 25 hundred
(b) 16 million 32 billion
(c) 0.2 trillion 10 billion
(d) 750 million 3 trillion

(4) Suppose a small college can be run with an annual budget of $50 million. How many colleges of this size could be funded with a yearly sum of $20 billion?

(5) Write out the following numbers without scientific notation:

(a) 6.49×10^{6} (b) 2.07×10^{4} (c) 3.9×10^{-3} (d) 8.197×10^{-2}

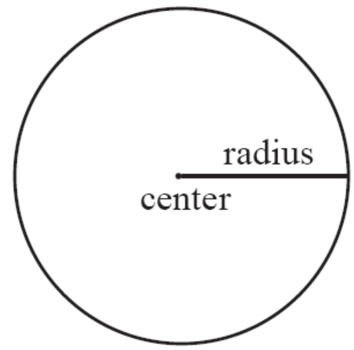
(6) Write the following numbers in scientific notation:

(a) 480,000
(b) 97 billion
(c) 1 quadrillion
(d) 0.00914
(e) one millionth

Chapter 38: Circles

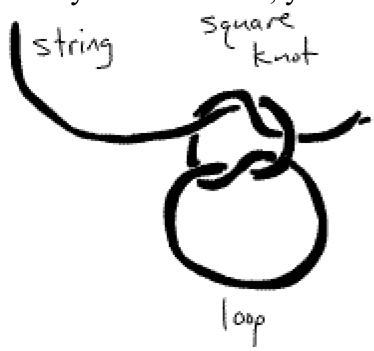
At an early age, children should be taught to recognize circles. A little later on, they can learn the names of the parts of a circle. In order to measure the circumference and area of a circle, however, it is necessary to understand decimals.

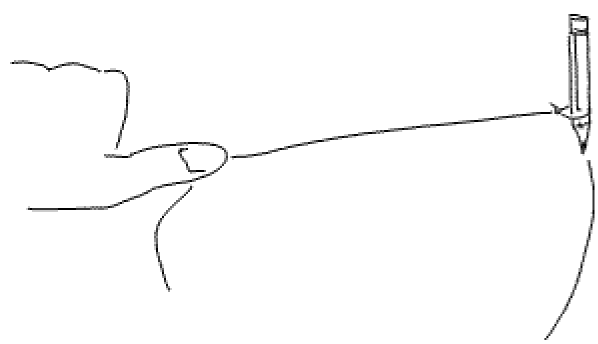
A circle is defined to be all the points on a plane (a flat surface, like a piece of paper) which are the same distance away from a single point. The single point is called the *center of the circle* and the distance is called the *radius*.



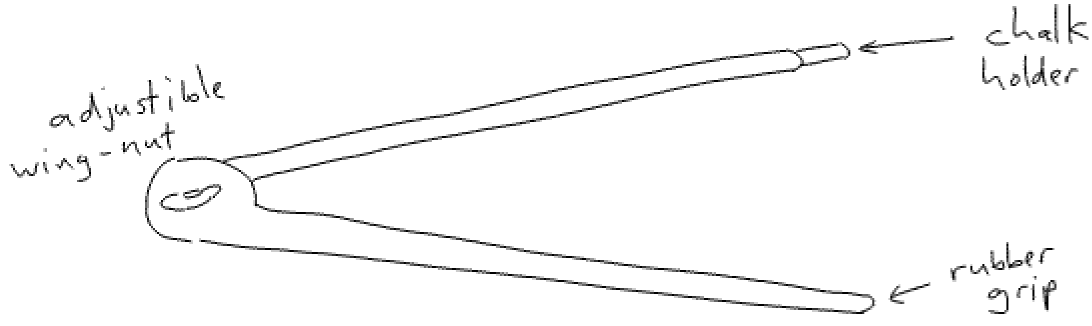
There are many ways to teach and emphasize this technical definition of a circle. The most dramatic would be to tie a loose loop of clothesline around a pole (like a flagpole or a basketball goal post). A child who held the line stretched taut and walked would walk around in a perfect circle. If this was done on pavement, sidewalk chalk could be used to draw a big circle on the ground. If this was done in snow, then a circle could be tromped into the ground. A pole is not necessary, but it is difficult to keep the center point fixed by other means (like having a person stand and hold the other end).

A less dramatic, but equally educational method is to make a loop at the end of a piece of string and fit a pencil, pen, or piece of chalk through it. If you push firmly down upon the string with your left thumb, you can trace a nice circle with the pen or chalk in your right hand:



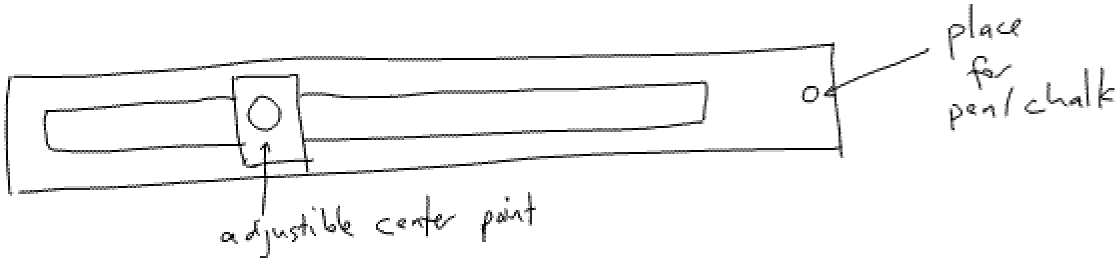


It is a good idea for you to practice drawing circles on the board in this fashion. It takes a bit of work to keep your thumb from slipping, but in the long run this method is much faster and versatile than using the blackboard compasses sold by school supply stores:



Every time you want to draw a circle of a different size, you have to loosen the wing nut, adjust the distance between the chalk and grip, and then tighten it back up again. Despite all of your best efforts, these sorts of compasses often slip, resulting in lousy circles.

There are some compasses that look a lot like rulers, either adjusting the place where the pen/chalk goes or else adjusting the center point. These work rather well. However, a piece of string is at least $50 cheaper (for the blackboard version) and works much faster in the hands of a skilled practitioner.



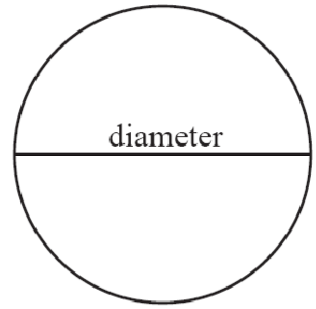
Whatever materials you use, it is a good idea to show how a circle can be drawn. Emphasize the center point (where your thumb or the grip touches the board) and the radius (the distance from that point to the chalk). Show how a small radius results in a small circle and a large radius results in a large circle.

Have your students try their hands at drawing circles on their papers with strings in the same fashion. First, have them draw the center point. Next, have them put a pencil through the loop at the end of the string. Next, have them hold the string down at the center point. Finally, they should slowly bring the pencil around the center, drawing a circle. It helps to trace the upper half of the circle first, then go back to the starting point and draw the bottom half. If you try to go all the way around, the string will begin to wind around your thumb and grow shorter, resulting in a spiral and not a circle.

If you have the funds available, have your students use ruler compasses. These are not much more expensive than the traditional metal compasses, but they work much better and avoid all the issues of handing out sharp metal spikes to small children.

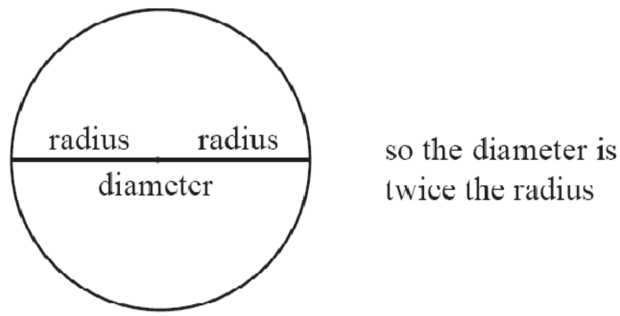
Work hard to make sure that each student has a set of the necessary tools for class work. Math class, especially the hands-on construction and drawing parts, becomes a lot less fun when kids have to wait for their turn to use a compass or a pair of scissors. Also, we want the kids to continue this work at home and elsewhere, drawing circles and straight lines not just for homework assignments, but also for crafts and whatever projects catch their fancy. A child who makes fairly nice circles frequently with a piece of string will be more comfortable with geometry than a child who makes excellent circles three or four times with a tool that stays in math class.

Another key circle term is the *diameter*, the longest possible distance from one end of the circle to the other.

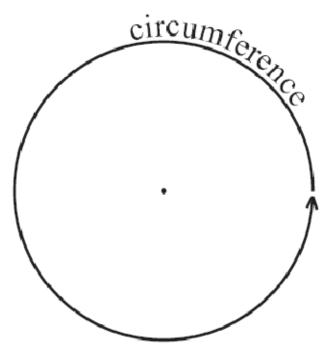


Have each student construct a circle on paper and then use a rule to draw the diameter. See if the class, in an open discussion, can figure out the two most interesting and important aspects of the diameter. First, the diameter always runs through the center of the circle. Second, the diameter is formed by two radii (the plural of radius is radii). This gives us the first circle formula: the diameter is twice the length of the radius. Similarly, the radius of a circle measures half the diameter. If your students do not notice this, have them draw a circle and then measure both the radius and the diameter. When you begin to write up all the results on the board (radius and diameter for each circle), hopefully someone will notice the relationship.

Have your class practice by giving them the radius of a circle (with or without a picture) and have them tell you the diameter (and vice-versa). For example, ask them to find the diameter of a circle with a radius of 6 feet.



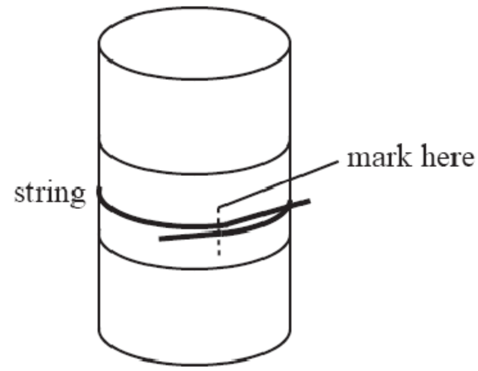
Another key concept of a circle is the *circumference*, the distance around the circle. This probably ought to be called the perimeter of the circle, but instead it has a special name. For a kid holding a rope looped around a pole, this is the distance walked in one complete lap around the pole.



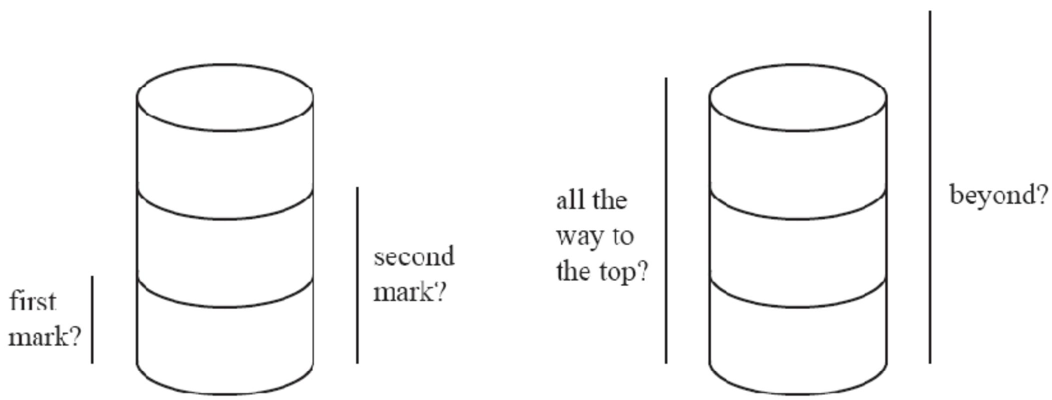
Measuring the circumference of a circle is tricky, mostly because we have only measured straight lines up until this point. We could try to measure the circumference by moving a straight ruler around a circle (a natural first trick to try), but if this is repeated several times, the answers will generally come out different each time.

A better way to measure the circumference of a circle is to not use a circle drawn on a board or a piece of paper, but instead to use a cylindrical object like a cup, cardboard tube, or coffee can. The object can be traced onto a piece of paper, so that the diameter and radius can be measured. A piece of string can then be tightly wrapped around the object and marked with a pen, so that it can be unwound, laid straight, and measured with a ruler.

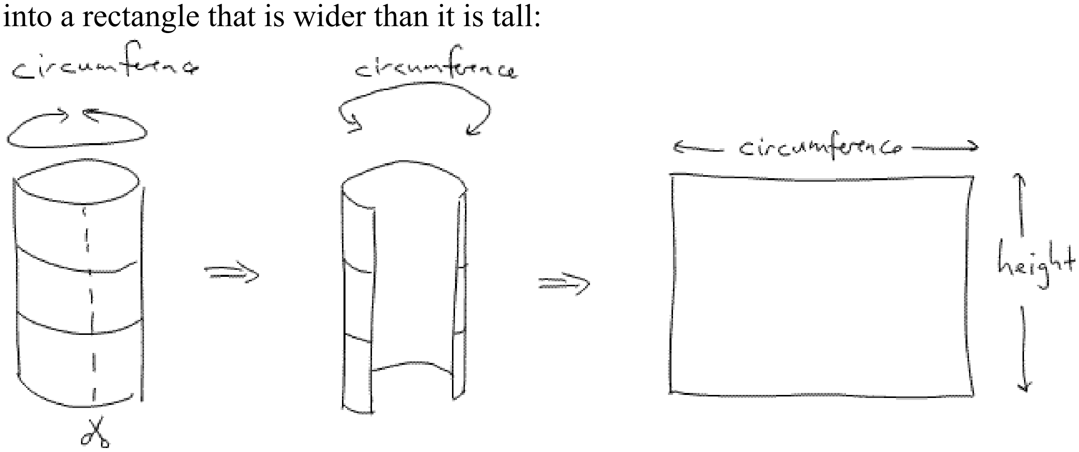
An excellent introductory exercise for this is to do this with a cardboard toilet-paper tube. First, divide the height of the tube into three equal parts and carefully trace circles around the tube at these heights. Next, wrap a piece of string around the tube and draw across the string to mark one complete circumference of the tube. Prepare all of this in advance.



When you are ready to introduce circumference to your class, show them the tube (it should be familiar to all) and the piece of string. Tell them that you are going to wrap the string around the tube, then unwind it and compare it to the overall length of the tube. At this point, you can wrap the string around the tube and show the students that the two pen marks line up. Next, challege the class to guess whether the string, when unwound, will go from the bottom to the first mark, the second mark, the third mark, or beyond:



Insist that each student in the class venture a guess, and mark their votes with tally marks on the board. The most common guess, even among adults, is that the string will reach the second mark. Thus, most people are quite surprised to see that the string goes a bit beyond the third mark. You can emphasize this by slicing open a toilet-paper tube vertically and unrolling it into a rectangle that is wider than it is tall:



A most wonderful aspect of toilet-paper tubes is that they are almost three times as tall (11cm) as they are across (4.5cm). This means that, when evenly divided into three parts, the space between the marks is approximately a diameter. To be more accurate, take a paper-towel tube (also about 4.5cm in diameter) and cut it down to a height of 13.5cm. This way, the question is not "how many marks up the tube will a wrapped-around string reach when unwound," but "how many diameters is the distance around the tube?" The answer, in both cases, is "a little more than 3." This is a lovely way to first introduce the amazing number π (pronounced *pi*), which is also a little bit more than 3 (and for the same reason).

The next step is to do essentially the same thing, but with more precise measurement. Have your class break up into groups of three or four and give each one a round object to trace and measure. The best objects are cylinders (like cups and coffee cans) and not flat (like a CD) or tapered (like a cone or vase). It is also better to avoid objects that can be easily bent and distorted (like a thin cardboard tube). It may be tricky to find just the right objects, but you only need to find a few because they can be re-used year after year. If you can, have the circles be of a variety of sizes. Tin cans work really well if you can make sure they have no sharp edges.

Have each group begin by tracing the circle onto a piece of paper. In the spirit of teamwork, one kid can hold the can in place and another can trace. Another kid can then use a ruler to draw the diameter of the circle (move the ruler around until the distance from end-to-end is largest) and measure the diameter's length. It is a good idea to have the students measure everything in millimeters, because this avoids the confusion of fractions.

Next, a child from each group should take a piece of string and wrap it tightly around their object, just as you demonstrated with the toilet paper tube. Another student can then take a pen and mark a line across the overlapping bits of string, to clearly mark the circumference. This is definitely a job best done by two people – one to hold the string taut and another to mark it. If the object has a big lip, then take care that the string does not slip down from it. The goal is for the string to go around the same distance which was traced onto the paper. Finally, the string should be unwound and measured carefully. For circles which are medium-sized or bigger, the circumference will generally be bigger than a standard 12-inch ruler. Thus, take care to have full metersticks available.

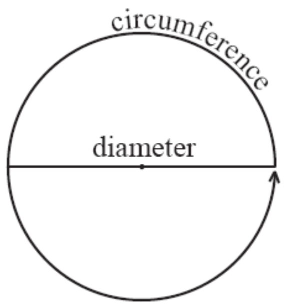
If you think that this will be too difficult for your class, then join one of the teams and do a demonstration first. Otherwise, wander about the room and keep each team on track.

As each team finishes, they should report up their data (diameter and circumference) to be recorded on the board. Have the class discuss the data briefly, hopefully noting that the larger diameters correspond with larger radii and vice-versa.

Finally, have each team divide the circumference of their circle by the diameter of their circle. If everything was done correctly and measured accurately, each team should end up with a number that is a little bit more than 3. Tell the class that they have each approximated a very special number called π. If the measurements are done with more precise equipment (or better yet, with abstract geometry), then π can be rounded to the nearest millionth as $\pi \approx 3.141593$. More frequently, however, π is rounded to the nearest hundredth as $\pi \approx 3.14$.

You can tell them that mathematicians have calculated this number to millions of digits, and yet the digits have never begun to repeat. Any time we divide two whole numbers with long division, the long division will either stop or repeat eventually. This means that π cannot be written as a fraction, a property called *irrational* (not a ratio of whole numbers).

In any case, we have now arrived at another key circle formula:

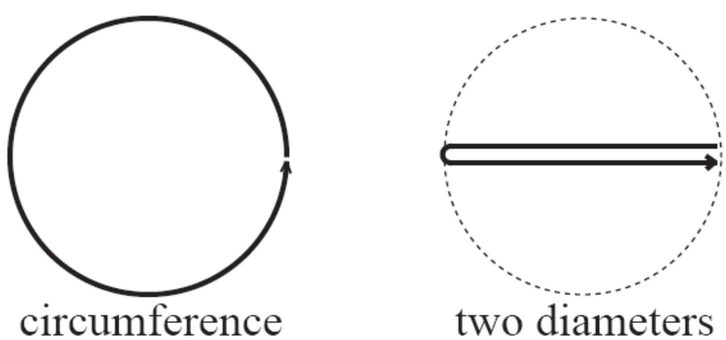


$$\frac{\text{circumference}}{\text{diameter}} = \pi \approx 3.14$$

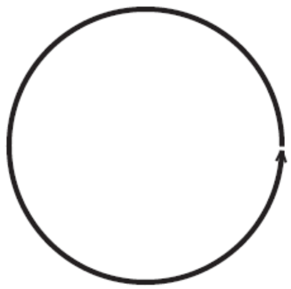
Because *circumference* ÷ *diameter* = π, we also know the equivalent multiplication equation: *diameter* × π = *circumference*. More traditionally, the letter *C* is used to represent the circumference, the letter *D* is used to represent the diameter, and the letter *r* is used to represent the radius. Thus, we know that $C = D \times \pi$. If we use the other circle formula, $D = 2 \times r$, then we end up with $C = 2 \times r \times \pi$. If we rearrange this with commutative property of multiplication and drop the multiplication signs, we get the usual formula for the circumference of a circle: $C = 2\pi r$.

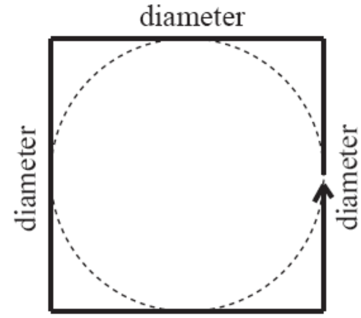
We are now begining to tread in the world of algebra. It really is just so much easier to write $C = 2\pi r$ than to write "the circumference of a circle is found by multiplying 2 times π times the radius of the circle." For elementary school children, it should be enough for them to know that the distance around a circle is a little bit more than three times the distance across, and that the exact number is called π, approximately 3.14.

One way to see that the distance around a circle is more than 2 diameters is to contemplate the following situation. If you had to fetch your things from the other side of a circular track field, what would be faster – to cut across the middle of the field or to walk around the circle? Clearly, it would be fastest to walk straight across the field to your things and then straight back. Thus, walking around the circumference of a circle is longer than walking the diameter twice:



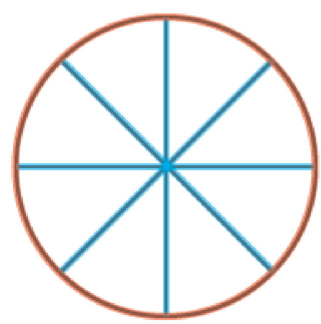
Similarly, if we walked in a big square around the circle, we would end up walking the length of 4 diameters. This is clearly a bigger distance than just walking around the circle:





We can conclude that the distance around a circle is somewhere between 2 and 4 diameters long. In face, as our exercise shows, this distance is a tiny bit more than 3.14 times the length of the diameter.

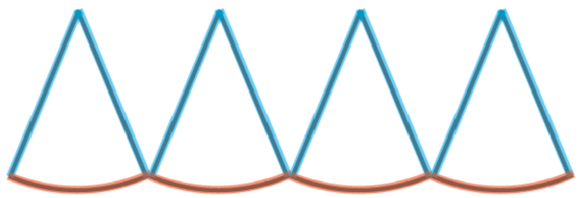
As a final exercise, we can help children discover the formula for the area of a circle. To begin, each child should mark a center point on a piece of paper and then draw a circle around it (either with a compass or with a piece of string). It does not work as well for the children to trace a round object here, because it is important for the center point to be accurate. Next, each child should draw a diameter across the circle with a rule. Next, the ruler should be used to draw three more lines, to divide the circle up into eighths (like a pizza). It is not very critical that the eighths be precise, but it works best if all the wedges (the technical term for a circle-wedge is a *sector*) are all pretty much the same size.



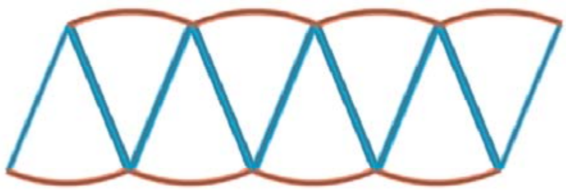
Next, have each student trace the circumference with a colored crayon or marker (red, for example). With a different color (for example, blue), each child should trace the other lines of the figure. This is an excellent time to reinforce the vocabulary words: circle, center, circumference, radius, diameter, and sector. Ask the students to count the diameters and the radii. Double check that each kid is able to explain why the length of a diameter is twice the length of a radius.

When everything has been colored, have each kid cut out the circle and then cut along the lines. The end result should be 8 sectors, each like a little wedge of pie, with the curved part one color and the two straight parts another. Point out to the class that the area of the whole circle will be the same as the areas of the 8 pieces all put together. We have cut apart and rearranged areas before!

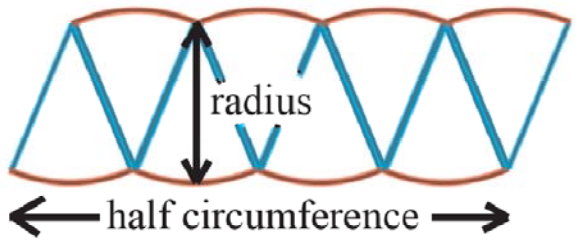
Next, have the students put 4 sectors in a row with their points facing upward, as illustrated below:



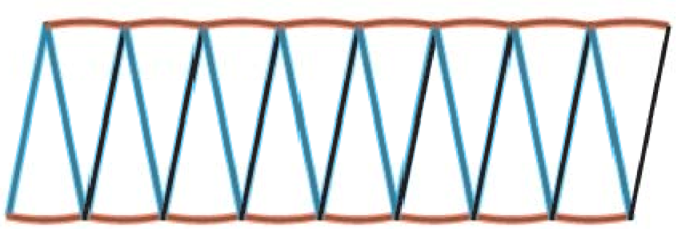
When this is done, have them put fit three of the sectors in the little gaps, facing downward, and then the eighth sector at the very end, as illustrated below:



Have your class discuss the resulting shape. Other than the fact that the top and bottom edges are a bit bumpy, they will hopefully recognize this as roughly a parallelogram. Also notice that the red coloring (or whatever color was used for the circumference) runs all the way across the top and bottom edges of this "almost parallelogram." This means that the length of the base is half of the circle's circumference. Also, the height of this "almost parallelogram" is the distance from the tip of a sector (the center of the circle) to the curved red edge (the circumference) and is thus the length of the circle's radius:

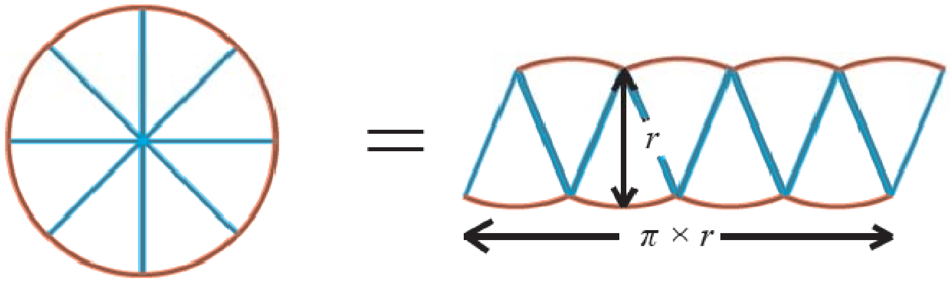


If any of the students in the class are skeptical and say that this is not very much like a parallelogram, have them cut each sector in half (into two smaller sectors) and make a new figure, with 8 wedges pointing upward and 8 more pointing downward:



The result will have the exact same base (half the circumference) and same height (the radius), but will look a lot more like a parallelogram.

In any case, because the area of a parallelogram is *base* × *height*, this means that the area of a circle can be rearranged and calculated as *half-circumference* × *radius*. If we use the circumference formula $C = 2 \times \pi \times r$, then half of the circumference is just $\pi \times r$. Thus, we obtain the following formula for the area of a circle: $Area = \pi \times r \times r$. If your students are comfortable with exponents, then this can be written most briefly as $A = \pi r^2$.



$$Area = base \times height = \pi \times r \times r = \pi r^2$$

When your students understand the formulas for circumference and area, you can have them run a variety of examples. Give them the radius or the diameter of a circle, and then have them approximate the circumference or the area (using $\pi \approx 3.14$). It is very likely, however, that they will work with these same formulas again in middle and high school, so it is much more important that they see where they come from, than that they can memorize and use them.

Questions:

(1) Find two circles of different sizes and use them to estimate π via the exercise detailed in this chapter.

(2) Draw a circle, color it, cut it into sectors, and rearrange the sectors into an "almost parallelogram" to verify the formula for the area of a circle, following the exercise detailed in this chapter.

(3) Find the radius, diameter, circumference, and area of a circle with:

- (a) radius = 3 in
- (b) diameter = 10 cm
- (c) circumference = 25 ft

Chapter 39: Word Problems

Word problems are famous for being the most difficult problems in mathematics. This reputation is well deserved because absolutely anything in mathematics can be phrased as a word problem. Many word problems require the reader to recall a specific formula, often from geometry or statistics. Other word problems require the reader to bring in outside information, for example conversion rates or constants from physics. Some word problems can only be solved by performing a large number of steps in order, each one providing information necessary for the subsequent steps. There is no way that one chapter could explain how to prepare students for all of these possibilities. However, there is a useful trick that helps with nearly all problems that require students to add, subtract, multiply, or divide information given in the problem: pay attention to units.

Teach your students to read over the whole word problem carefully. Have them try to envision what is going on in the problem. Encourage them to draw a sketch of what is going on. When you first introduce problem solving to the class, have them discuss the situation among themselves. Nothing helps to solve a problem like common sense. Also, we want children to view word problems as if they were a game at which there are many winning strategies, not as a task for which there is only one correct procedure. There can be a lot of fun in rising to the challenge of a word problem and figuring out different ways to take it on.

As your kids read through the word problem, have them write down all the numbers that appear. As well, have them write down the units for each problem. Tell them that it is not important whether units are written as singular or plural. Also teach them that the word "per" means "divided by." For example, a phrase like "55 miles per hour" means "55 miles divided by hours" and can be written as either $55\,\dfrac{miles}{hours}$ or else $\dfrac{55\,miles}{hour}$. The latter form will prove to be most useful.

The first trick to word problems has already been covered: you can only add and subtract numbers with the same units. For example, a word problem that reads "Blah blah blah 40 peanuts blah blah 15 peanuts..." has two numbers in it: 40 and 15. When these numbers are written with their units, they are 40 peanuts and 15 peanuts. Because the units are the same, it is very likely that the word problem is asking for 40 and 15 to be either added together or subtracted. The answer will either be 55 peanuts or else 25 peanuts. Only a close reading of the problem and a little common sense will be able to determine which of these two answers is correct. The problem could turn out to ask "How much more is 40 peanuts than 15 peanuts?" or else "If Kim has 40 peanuts and then receives 15 peanuts more, how much will she have?"

As a further tool for figuring out problems, have your students find the question word in the problem (what, how many, etc.) and scan for corresponding units. If the problem asks "how many feet?" then the answer should have feet as units. If the problem asks "how long will it take?" then the units of the answer will be hours, seconds, days, or some other unit of time. This can take a bit of thinking, because the units are not always so clearly stated. In the two earlier examples, the question asks only "how much." In these problems, the reader is expected to know that the problem is really asking "how many peanuts," thus the answer must be given in peanuts.

The one exception to this rule is when the numbers must both be plugged into a formula. For example, if the base and height of a triangle are both given in inches, the answer might involve calculating the area of the triangle, which will require multiplying and dividing instead of adding and subtracting. However, this can be detected by keeping an eye on the units of the answer, as area is measured in square units and square inches can only be obtained by multiplying inches by inches to get $in \times in = in^2$.

This trick can save students a whole lot of worrying. As soon as they notice that the numbers and answer in the problem are all measured with the same units, there is generally only one thing left to decide: should the numbers be added or subtracted? Of course, there could be three or more numbers given in the problem, but the general rule still holds – if the numbers all have the same units, then the answer will be found by adding and subtracting.

This rule extends further: numbers with different units cannot be added or subtracted, and thus must be either multiplied or divided. Even better, if we allow ourselves to flip fractions upside-down (replace them with their reciprocals), then these problems can be solved entirely with multiplication.

For example, suppose we want to solve the following problem: "A car will drive for 200 miles at an average speed of 50 miles per hour. How much time will this take?" We begin by identifying the numbers, with units: 200 *miles* and $\frac{50\,miles}{hour}$. The units are different, so this problem cannot be solved by adding or subtracting these numbers. Instead, the problem can be solved by multiplication, provided that we are allowed to flip $\frac{50\,miles}{hour}$ to make $\frac{hour}{50\,miles}$ and flip 200 *miles* (which is the same as $\frac{200\,miles}{1}$) into $\frac{1}{200\,miles}$. As the key to what we need to do, we look at the units of the desired answer. The question "how much time..." is looking for an answer with units of time, in this case, hours. We now look at the pieces before us, and try to figure out how they can be multiplied to produce an answer in hours: $\frac{50\,miles}{hour}$ or $\frac{hour}{50\,miles}$ and $\frac{200\,miles}{1}$ or $\frac{1}{200\,miles}$. The only fraction with hours up top is $\frac{hour}{50\,miles}$, so we'll use that one. In order to get rid of the miles down below, we'll multiply this by $\frac{200\,miles}{1}$ so that the miles up top and the miles down below will cancel. The end result will be:

$$\frac{hour}{50\ miles} \times \frac{200\ miles}{1} = \frac{200 \times hour \times \cancel{miles}}{50 \times \cancel{miles}} = \frac{4\ hour}{1} = 4\ hours$$

This is the answer to our problem: the car will drive for 4 hours.

Many children are taught to solve this particular sort of problem by using the $D = RT$ "dirt" formula, that "distance equals rate times time." In this particular instance, we need to use a variation of this formula, that $T = \frac{D}{R}$ ("time equals distance divided by rate"). This certainly works, and can be less confusing than the method described earlier in this chapter. However, students without algebra skills will need to memorize three formulas ($D = RT$,$T = \frac{D}{R}$, and $R = \frac{D}{T}$). Also, these formula only work for these limited types of problems.

As another example, take this problem: "A car gets 25 miles per gallon. How much gasoline will the car need to drive 150 miles?" This problem is nearly identical to the last example, and yet none of the dirt formulas can explain it. Instead, we write out the numbers with units, $\frac{25\,miles}{gallon}$ and 150 *miles*. The question asks "how much gasoline?" so the units of the answer must be a measure of gasoline, thus *gallons*. In order to end up with gallons, we will need to flip the fraction, to put *gallons* in the numerator. When we then look at $\frac{gallon}{25\,miles}$ and 150 *miles* and think about multiplication, we see right away that the unwanted miles will cancel out when we multiply:

$$\frac{\text{gallons}}{25 \text{ miles}} \times 150 \text{ miles} = \frac{150 \times \text{gallons} \times \cancel{\text{miles}}}{25 \times \cancel{\text{miles}}} = 6 \text{ gallons}$$

This is the answer to the question: the car will require 6 gallons of gasoline to drive 150 miles.

As another example, consider: "Joey rakes leaves for 4 hours and makes \$28. How many dollars per hour is this?" Here, our two numbers are 4 *hours* and 28 *dollars*. The answer we want should be measured in $\frac{dollars}{hour}$. The only way we can obtain this is by keeping the 28 *dollars* the way it is and flipping the 4 *hours* into $\frac{1}{4\,hours}$. When we multiply, we get:

$$28 \text{ dollars} \times \frac{1}{4 \text{ hours}} = \frac{28 \text{ dollars}}{4 \text{ hours}} = \frac{7 \text{ dollars}}{\text{hours}}$$

If we change the "hours" to "hour," we see that Joey makes 7 dollars per hour, the desired answer.

As a more complicated problem, consider: "Suppose gasoline costs \$2.89 per gallon and Max's car gets 32 miles per gallon. How far can Max drive on \$20 worth of gas?" Here, we have three numbers: $\frac{2.89\,dollars}{gallon}$, $\frac{32\,miles}{gallon}$, and 20 *dollars*. The question "how far" is looking for an answer in *miles*. To solve this problem, we keep the $\frac{32\,miles}{gallon}$ the way it is because it has the *miles* that we want in the numerator. We will have to flip the first fraction into $\frac{gallon}{2.89\,dollars}$ because otherwise the gallons will not cancel out ($\frac{2.89\,dollars}{gallon} \times \frac{32\,miles}{gallon} = \frac{2.89 \times dollars \times miles}{gallons \times gallons}$). When we multiply these two together, we get

$$\frac{\cancel{gallon}}{2.89\ dollars} \times \frac{32\ miles}{\cancel{gallon}} = \frac{32\ miles}{2.89\ dollars}$$

All we now need to do is multiply by our last number, 20 *dollars*, and all the units but the *miles* will cancel out:

$$\frac{32\ miles}{2.89\ \cancel{dollars}} \times 20\ \cancel{dollars} = \frac{32 \times 20}{2.89}\ miles$$

It is easy to mentally multiply 32 × 20 = 640, but dividing this by 2.89 will take a little bit of long division. However, because we end up with *miles* as units, we know that we have obtained the right answer:

$$\begin{array}{r} 221.4 \\ 2.89 \overline{)\,64000.0} \\ \underline{578} \\ 620 \\ \underline{578} \\ 420 \\ \underline{289} \\ 1310 \\ \underline{1156} \\ 154 \end{array}$$

If we round to the nearest whole number, we see that Max will be able to drive a tiny bit more than 221 miles on \$20 worth of gas.

As another example, consider: "Electricity costs 15 cents per kilowatt hour. At the beginning of March, Omar's electric meter read 29,624 kilowatt hours. At the end of March, Omar's meter read 30,104 kilowatt hours. How much will Omar have to pay for the electricity he used in March?" Here we have three numbers: $\frac{15\,cents}{kWh}$, 29624 *kWh*, and 30104 *kWh*. Two of these numbers have the same units, so we figure that they will need to be either added or subtracted. A little common sense will tell us that we need to subtract: 30104 *kWh* – 29624 *kWh* = 480 *kWh*. This can then be multiplied by the remaining number to obtain our answer:

$$\frac{15\ \text{cents}}{\cancel{\text{kWh}}} \times 480\ \cancel{\text{kWh}} = 15 \times 480\ \text{cents}$$

A little long multiplication tells us that the answer is 7200 cents, although it probably will make more sense to convert this into $72. Conversions will be covered in the next chapter. If a child added the two numbers instead of subtracting, the answer would have come out to be $8,952.20 which is a very unreasonable amount to pay for a month's worth of electricity.

Hopefully these examples have illustrated how much help keeping track of units can be in solving word problems. Numbers with the same units are either added or subtracted and numbers with different units are flipped (if necessary) and multiplied. All that a child needs more is a comfort with fractions and to know that "per" means "divided by."

There are, of course, many exceptions. Sometimes numbers need to be converted. For example, a problem containing the numbers 35 *inches* and 2 *feet* will probably not be a multiplication problem, but rather a conversion problem – the 2 *feet* will probably need to be converted into 24 *inches* and then the two numbers in inches will either be added or subtracted. Unless, of course, this is a geometry problem asking for the area of a shape with these dimensions. No single trick could possibly cover the multitudes of situations that people could set into words. However, this "keep track of your units" trick will take children very far and give them an excellent place to begin considering a word problem.

One way to reinforce this technique is with nonsense examples where only the question words, numbers, and "per" are in English. For example, consider this problem: "How many tweetles snortle blee 30 knopps voovie 5 knopps per tweetle?" This problem is completely ridiculous, but we can still figure that the answer must be in *tweetles*, and that our two numbers are: 30 *knopps* and $\frac{5\,knopps}{tweetle}$. We cannot add or subtract these numbers. Instead, we flip the fraction to put the desired *tweetles* in the numerator, and multiply:

$$\frac{\text{tweetle}}{5\ \cancel{\text{knopps}}} \times 30\ \cancel{\text{knopps}} = \frac{30\ \text{tweetle}}{5} = 6\ \text{tweetles}$$

The only reasonable answer is 6 tweetles, whatever that might mean.

When students pay attention to units, many problems in physics become easy. For example: "If the acceleration due to gravity is 9.8 meters per second squared, then what is the potential energy of a 20 kilogram mass suspended 2 meters off the floor?" The question here is asking for an amount of energy. If the student knows that energy is measured in *joules* and that a *joule* is the same as a $\frac{kg \times meter \times meter}{\sec \times \sec}$, then the three numbers given can be easily combined:

$$\frac{9.8 \text{ meters}}{\text{sec} \times \text{sec}} \times 20 \text{ kg} \times 2 \text{ meters} = \frac{9.8 \times 20 \times 2 \text{ kg} \times \text{m}^2}{\text{sec}^2}$$

A little bit of multiplication tells us that this object has $\frac{392\, kg \times m^2}{\sec^2}$ = 392 *joules* of potential energy. To make the problem a bit more difficult, students are usually expected to know the acceleration due to gravity on the surface of the earth. However, this is a problem from a high school physics course, and not generally something that elementary school students would need to know. However, it illustrates how even advanced problems can be solved quite immediately when students keep track of units.

One thing that can confuse children are red herrings – extra numbers that are not needed for computing the answer. As an example, take the problem: "Shawna bought 9 pencils for $2 at 10 in the morning. In the afternoon, she bought 5 more pencils for $1. How many pencils has she bought this day?" Many students would be overwhelmed by the fact that there are 5 different numbers in this problem, and feel the need to use all of them somehow. However, if we list out all the numbers with units, we get: 9 *pencils*, 2 *dollars*, 5 *pencils*, and 1 *dollar*. The units for the 10 are unclear; *o'clock* is not a sort of unit, so it would have to be 10 *hours* (past midnight). The problem specifically asks "how many pencils," so we want our answer to be in *pencils*. With this in mind, the clear answer is to add the two numbers with *pencils* as their units, ending up with an answer of 9 *pencils* + 5 *pencils* = 14 *pencils*.

For the students who find it ridiculous or confusing that the *dollars* and *hour* are unused, there is a useful exercise: have the students make up their own word problems. To begin, give the class a calculation, for example, 8 + 3. Next, have each child come up with a word problem whose solution requires making this particular calculation. This can help students to see the manner in which a calculation can be buried in a paragraph. Invite the students to come up with several different problems for the same calculation, and to share their problems with one another. Give them other problems that involve subtraction, multiplication, and division.

When your students have made several word problems for different calculations, have them pick their favorite and then add more information to it. Perhaps they could mention the prices of things, the number of people in the room, the age of the main character, or anything else of the sort. Then have them explain how the problem is solved. A few exercises like this can help students to realize that not all information in a problem is necessary. Real-life problems tend to have a lot of excess information (and sometimes not enough information). However, word problems found in textbooks and exams tend to avoid red-herring numbers.

A final trick to solving word problems is this: when the numbers in a problem are messy (decimals, fractions, numbers that do not come out evenly) then first try a version of the problem with nicer numbers. For example, suppose your students are confronted with a problem like: "To make 1.5 batches of cookies, $\frac{4}{3}$ of a cup of sugar is necessary. How much sugar would be needed to make 2.7 batches of cookies?" The numbers of this problem make it quite intimidating. So, let us first make a version of the problem where all of the numbers are replaced with nicer numbers: "To make 2 batches of cookies, 3 cups of sugar are needed. How much sugar would be needed to make 8 batches?" Notice how this problem is much easier, even though it is exactly the same (but for the numbers). Perhaps a student would be able to immediately figure this easier version out; we will need 4 times as much sugar, thus $4 \times 3 = 12$ cups of sugar will be necessary. With a few questions, you will probably get the student to explain that the 4 is found by dividing $8 \div 2$. This, then gives us the map with which to solve the harder version. Rather than dividing $8 \div 2$, we divide the numbers that the 8 and 2 replaced: $2.7 \div 1.2 = 2.25$. Next we multiply, not by 3 cups, but by $\frac{4}{3}$ of a cup (the number the 3 replaced). Thus, our final answer will be $2.25 \times \frac{4}{3} = \frac{2.25\times 4}{3} = \frac{9}{3} = 3$ cups of sugar.

As another example, suppose we need to solve: "James can wash a car in $\frac{2}{5}$ of an hour. How many cars can he wash in 3.6 hours?" To make the problem easier, we can switch the numbers to easier ones: "James can wash a car in 2 hours. How many cars can he wash in 10 hours?" Here, it becomes clear that we need to divide $10 \div 2$ to get the answer of 5. Similarly, the answer to the original problem will be obtained by dividing the corresponding numbers: $3.6 \div \frac{2}{5}$. This can be solved using the "flip and multiply" technique: $3.6 \div \frac{2}{5} = 3.6 \times \frac{5}{2} = \frac{3.6\times 5}{2} = \frac{18}{2} = 9$. James can wash exactly 9 cars in 3.6 hours.

These last two problems could also be solved using the "keep track of units" method. However, these both describe rates without using the clue word "per." In the first problem, the rate is "$\frac{4}{3}$ of a cup of sugar per 1.5 batches," written $\frac{\frac{4}{3}cup}{1.5batches}$. In the second problem, the rate is "$\frac{2}{5}$ of a car per hour," written $\frac{\frac{2}{5}car}{hour}$. Your students will have to stay on the look-out for disguised rates like these.

The only thing to watch out for with the "make the numbers nicer" trick is to avoid using the number 1. This is because multiplying by 1 and dividing by 1 result in the same answer. Thus, when you convert back to the unpleasant numbers, you might not be sure whether you need to multiply or divide. If you do not replace any of the numbers with 1, however, this problem will never arise.

All in all, word problems will probably always provide challenges for your students. However, if they think things through carefully, keep track of units, and temporarily replace complicated numbers with simple ones, the problems that arise in elementary school ought not to be too difficult. Begin with easy problems, work up to harder ones gradually, and give your students lots of exposure to a wide variety of problems. Also, having them write their own word problems is a great way to exercise their creativity and show them how the math and words connect.

Questions:

(1) What are the most likely possible answers to: "Blah blah... 20 dollars... blah blah... 90 dollars...?"

(2) Use the "keep track of units" method to solve: "How much will it cost to drive 200 miles in a car that gets 15 miles per gallon when gas costs 3 dollars per gallon?"

(3) Solve: "How many snoots burtle feebe-feebe gromp 60 hompers per snoot noog 150 hompers?"

(4) Make up a word problem that requires the calculation: $35 \div 7$. Include at least 4 different numbers, with units, in your problem.

(5) Show how the "make the numbers easier" method can be used on the following problem: "The instructions on a bottle of floor cleaner insists that 2.5 gallons of water be mixed with $\frac{3}{4}$ of a cup of the concentrated cleaner. However, there is only $\frac{1}{3}$ of a cup of the concentrated cleaner left in the bottle. How much water should this be mixed with?"

Chapter 40: Conversions

Conversions are just a special kind of word problems. The only additional difficulty is that students are expected to know the various conversion rates. The key is to view the conversion first as an equality, and then as a rate in the two possible ways (one flipped).

For example, the conversion between inches and feet, written as an equation, is 12 *inches* = 1 *foot*. The two ways to write this as a rate are $\frac{12\,inches}{1\,foot}$ and $\frac{1\,foot}{12\,inches}$. Any time we want to convert between inches and feet, we only need to multiply by one of these rates, specifically the one that has the desired unit in the numerator. For example, to convert 7 feet into inches, we multiply by $\frac{12\,inches}{1\,foot}$ (the rate with the desired *inches* up top) to get:

$$7 \text{ feet} \times \frac{12 \text{ inches}}{1 \text{ foot}} = \frac{7 \times 12 \times \cancel{\text{feet}} \times \text{inches}}{\cancel{\text{foot}}} = 84 \text{ inches}$$

Remember, as before, that there is no difference between writing units as singular or plural, so we are able to cancel out *feet* and *foot* because they represent the same unit.

On the other hand, to convert 54 inches into feet, we use the $\frac{1\,foot}{12\,inches}$ rate because it has the *feet* we desire up top. Thus we multiply:

$$54 \;\cancel{\text{inches}} \times \frac{1 \text{ foot}}{12 \;\cancel{\text{inches}}} = \frac{54 \text{ feet}}{12} = \frac{9}{2} \text{ feet}$$

By cancelling out the *inches* and reducing the fraction, we see that 54 inches is the same as $\frac{9}{2}$ *feet* (also known as 4.5 feet or $4\frac{1}{2}$ feet).

If students are confused about why rates are allowed to be flipped upside-down, there are a variety of ways to explain it. First of all, this stems from the "to divide by a fraction, flip and multiply" rule. If we allow ourselves to flip fractions, to put the desired units in the right place, then we can look at everything as multiplication, even though multiplying by a flipped fraction is really a division. Secondly, rates make sense both ways, flipped and otherwise. For example, the rate $\frac{1\,foot}{12\,inches}$ says "there is 1 foot for every 12 inches" or "1 foot per 12 inches" while the rate $\frac{12\,inches}{1\,foot}$ says "there are 12 inches in every foot" or "12 inches per foot." These describe the same relationship, just in different ways.

Similarly, if you were driving a car and were concerned about how far you had to travel, you might describe your speed as $\frac{60\,miles}{hour}$. This, pronounced "60 miles per hour" assures you that you will be 60 miles closer to where you want to be at the end of every hour. On the other hand, if you were more concerned about how much time was going by, you might look at things as $\frac{1\,hour}{60\,miles}$. In other words, you are using up "1 hour per 60 miles," meaning that an hour was going by for every 60 miles you traveled.

As a final example, suppose that lemons are on sale, 3 lemons for $2. If you were buying the lemons, you might think of this as $\frac{3\,lemons}{2\,dollars}$, indicating how many lemons you could get for your money. The person selling the lemons, however, probably thinks of this as $\frac{2\,dollars}{3\,lemons}$, focusing more on how much money the lemons will generate. These two rates describe the same relationship, and either one might be useful in solving a problem. The one to use can almost always be figured out by keeping track of units.

The following is a fairly comprehensive list of all the common conversions that elementary students ought to know. It is best to introduce these only one or two at a time, to avoid overwhelming and confusing the class:

12 *inches*	=	1 *foot*
3 *feet*	=	1 *yard*
5,280 *feet*	=	1 *mile*
1,000 *milimeters*	=	1 *meter*
100 *centimeters*	=	1 *meter*
1,000 *meters*	=	1 *kilometer*
60 *seconds*	=	1 *minute*
60 *minutes*	=	1 *hour*
24 *hours*	=	1 *day*
7 *days*	=	1 *week*
365 *days*	=	1 *year*
12 *months*	=	1 *year*
10 *years*	=	1 *decade*
100 *years*	=	1 *century*
1000 *years*	=	1 *millennium*
2 *cups*	=	1 *pint*
2 *pints*	=	1 *quart*
4 *quarts*	=	1 *gallon*
1,000 *mililiters*	=	1 *liter*

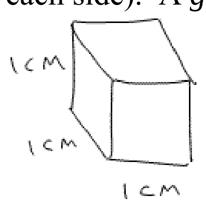
16 *ounces*	=	1 *pound*
2,000 *pounds*	=	1 *ton*
1,000 *miligrams*	=	1 *gram*
1,000 *grams*	=	1 *kilogram*
1,000 *kilograms*	=	1 *metric ton*

Less commonly-known (and expected of students) are the conversions between the English and metric units, each rounded to the nearest hundredth:

1 *inch*	≈	2.54 *centimeters*
1 *liter*	≈	1.06 *quarts*
1 *kilogram*	≈	2.20 *pounds*

Point out to your students that the prefix "cent" found in *centimeter*, *century*, and even "dollars and *cents*" refers to the number 100. Similarly, the "mil" in *milimeter*, *miligram*, *mililiter*, and *millennium* all refer to the number 1,000. Also, the prefix "dec" refers to the number 10, as in *decade*. There are many other metric units which use these prefixes, but they are rarely used and thus unimportant for students to memorize. However, it would be a wonderful class activity of pattern recognition and detective work for them to guess the conversions. Ask them to guess what the conversion between *centiliter* and *liter* for example, or between *centigram* and *gram*. Tell them that 10 *decimeters* = 1*meter*, then see if they can guess the relationship between *deciliters* and *liters*.

It might also be useful to briefly explain the deeper relationships between the metric units. A *mililiter* is the volume found in a cubic *centimeter* (a cube measuring a *centimeter* on each side). A *gram* is the weight of a *mililiter* of water.



= 1 mL = 1 g (when filled with water)

One area of concern with conversions are some commonly-held misconceptions about weeks, months, and years. Many people think that there are 4 weeks in a month. In truth, only February is exactly four weeks long (except in leap years), while all the other months have 2 or 3 more days. These days add up. If each month had only 4 weeks, then we could convert 1 *year* into $1\,year \times \frac{12\,months}{1\,year}$ = 12 *months*, and then into $12\,months \times \frac{4\,weeks}{1\,month}$ = 48 *weeks*, and finally into $48\,weeks \times \frac{7\,days}{1\,week}$ = 336 *days*. This ends up to be 29 days short of a full year of 365 days.

Because of that problem, avoid direct comparisons between weeks and months. Similarly, it is hard to make a direct comparison between days and months, because months can have anywhere between 28 and 31 days.

In any case, when your students know these common rates, conversions become quite easy. For example, suppose you want to know how many hours are in 1.4 days. We know that the relationship between days and hours is 24 *hours* = 1 *day*. As a rate, this is either $\frac{24\,hours}{1\,day}$ or else $\frac{1\,day}{24\,hours}$. Because we want a final answer in hours, we will use the first of these rates:

$$1.4\ \cancel{days} \times \frac{24\ hours}{1\ \cancel{day}} = 1.4 \times 24\ hours$$
$$= 33.6\ hours$$

$$\begin{array}{r} 24 \\ \underline{1.4} \\ 96 \\ \underline{24} \\ 33.6 \end{array}$$

A little long multiplication tells us that 1.4 *days* is the same as 33.6 *hours*.

Some conversion problems require a number of steps. For example, suppose we want to know how many *milimeters* are in 1 *centimeter*. The only conversion (listed in this chapter) we have for centimeters is 100 *centimeters* = 1 *meter*. Thus, we multiply:

$$1\ cm = 1\ \cancel{cm} \times \frac{1\ m}{100\ \cancel{cm}} = \frac{1\ m}{100}$$

Next, to convert these meters into milimeters, we use 1000 *milimeters* = 1 *meter* and multiply:

$$1\ cm = \frac{1\ m}{100} = \frac{1\ \cancel{m}}{100} \times \frac{1000\ mm}{1\ \cancel{m}} = \frac{1000\ mm}{100} = 10\ mm$$

Thus, we conclude that 1 *centimeter* = 10 *milimeters*. This is a fairly common conversion, thus can be useful to memorize. However, as illustrated above, it can be deduced from 100 *centimeters* = 1 *meter* and 1000 *milimeters* = 1 *meter*.

As another example, suppose we want to know how many cups are in 2.5 gallons. There is no direct relationship (listed in this chapter) between cups and gallons. However, we can convert gallons into quarts, then quarts into pints, and then pints into cups. If we line up all of these rates in a row, we could make the conversion in a single huge multiplication:

$$2.5\ gallons \times \frac{4\ quarts}{1\ gallon} \times \frac{2\ pints}{1\ quart} \times \frac{2\ cups}{1\ pint}$$

Notice how we arrange the rates so that the denominator of each has the unit from the numerator to the immediate left. This ensures that our old units will cancel, leaving only the last:

$$2.5\ \cancel{gallons} \times \frac{4\ \cancel{quarts}}{1\ \cancel{gallon}} \times \frac{2\ \cancel{pints}}{1\ \cancel{quart}} \times \frac{2\ cups}{1\ \cancel{pint}} = 2.5 \times 4 \times 2 \times 2\ cups$$

When we multiply 2.5 × 4 × 2 × 2 = 40, we get our answer: 2.5 gallons contain 40 cups.

As another example, how many *miles* is 1 *kilometer*? The only conversion between imperial and metric distance is 1 *inch* = 2.54 *centimeters*. Thus, we will need to convert the 1 *kilometer* first into centimeters, and then into inches, and then finally into miles. In fact, the only connection we have between *kilometers* and *centimeters* goes first through *meters*. Thus, we use 1 *kilometer* = 1,000 *meters* and 1 *meter* = 100 *centimeters*:

$$1\text{ Km} = 1\,\cancel{\text{Km}} \times \frac{1000\,\cancel{\text{m}}}{1\,\cancel{\text{Km}}} \times \frac{100\text{ cm}}{1\,\cancel{\text{m}}} = 100{,}000\text{ cm}$$

Next, we convert from *centimeters* into *inches*:

$$100{,}000\text{ cm} = 100{,}000\,\cancel{\text{cm}} \times \frac{1\text{ inch}}{2.54\,\cancel{\text{cm}}} = \frac{100{,}000}{2.54}\text{ inches}$$

$$\begin{array}{r} 39370.0 \\ 2.54\overline{)100000.00.0} \\ \underline{762} \\ 2380 \\ \underline{2286} \\ 940 \\ \underline{762} \\ 1780 \\ \underline{1778} \\ 200 \end{array}$$

This division will require a little long division, telling us that:
1 *kilometer* = 100,000 *centimeters* ≈ 39,370 *inches* (rounded to the nearest whole inch)

Next, we use 12 *inches* = 1 *foot* to convert from *inches* into *feet*:

$$39370\text{ in} = 39370\,\cancel{\text{in}} \times \frac{1\text{ ft}}{12\,\cancel{\text{in}}} = \frac{39370}{12}\text{ ft}$$

$$\begin{array}{r} 3280.8 \\ 12\overline{)39370.0} \\ \underline{36} \\ 33 \\ \underline{24} \\ 97 \\ \underline{96} \\ 100 \\ \underline{96} \\ 4 \end{array}$$

Rounded to the nearest foot, we now know that 1 *kilometer* ≈ 3,281 *feet*. Finally, we use the relationship 5,280 *feet* = 1 *mile* to finish the problem:

$$3281\text{ ft} = 3281\,\cancel{\text{ft}} \times \frac{1\text{ mi}}{5280\,\cancel{\text{ft}}} = \frac{3281}{5280}\text{ mi}$$

$$\begin{array}{r} .621 \\ 5280\overline{)3281.00} \\ \underline{31680} \\ 11300 \\ \underline{10560} \\ 7400 \end{array}$$

Rounded to the nearest hundredth, we see that 1 *kilometer* ≈ 0.62 *miles*.

A child with a good memory can be encouraged to remember additional relationships between the units, for example that 1 *gallon* = 16 *cups* or 1 *yard* = 36 *inches*, or 1 *kilometer* ≈ 0.62 *miles*. These can speed up calculations immensely. However, make the student work each one out at least once, and not just pull them out of the cover of a book or from a web site. It is far better for a student to know only the basic conversions and how to use them, than to memorize a string of numbers without knowing where they came from.

A slightly increased challenge comes from converting the units in rates. This works in just the same way – flip the rates, as necessary, to make the unwanted units cancel out. For example, suppose we want to convert 14 feet per second into miles per hour. To begin with, we write our rate as a fraction: $\frac{14\ feet}{\text{sec}}$. We do not want *feet* in the numerator, we want *miles* instead. This calls for the 5,280 *feet* = 1 *mile* conversion, written as a fraction with the *feet* on the bottom, so that it cancels out with our *feet* up top when we multiply:

$$\frac{14\ \text{ft}}{\text{sec}} = \frac{14\ \cancel{\text{ft}}}{\text{sec}} \times \frac{1\ \text{mi}}{5280\ \cancel{\text{ft}}} = \frac{14\ \text{mi}}{5280\ \text{sec}}$$

We'll worry about dividing later. Next, we want the *seconds* in the denominator to be converted into *hours*. We'll need to convert into *minutes* as an intermediate step, although the whole conversion can be done in one go:

$$\frac{14\ \text{mi}}{5280\ \text{sec}} = \frac{14\ \text{mi}}{5280\ \cancel{\text{sec}}} \times \frac{60\ \cancel{\text{sec}}}{1\ \cancel{\text{min}}} \times \frac{60\ \cancel{\text{min}}}{1\ \text{hr}} = \frac{14 \times 60 \times 60\ \text{mi}}{5280\ \text{hr}}$$

Now we have the units we desire, *miles per hour*, so all we need to do is multiply and divide:

$$\begin{array}{r} {}^{2} \\ 3600 \\ \underline{14} \\ 14400 \\ \underline{3600} \\ 50400 \end{array} \qquad \begin{array}{r} 9.54 \\ 5280\overline{)50400.00} \\ \underline{47520} \\ 28800 \\ \underline{26400} \\ 24000 \\ \underline{21124} \end{array}$$

Rounded to the nearest tenth, 14 feet per second is approximately 9.5 miles per hour.

Questions:

(1) Multiply with rates to make the following conversions:

(a) 3.8 *miles* into *inches*

(b) 250 *ounces* into *kilograms*

(c) 5 *gallons per hour* into *cups per minute*

Chapter 41: Ratios and Percents

Ratios and rates are essentially the same thing, written in different ways. Both compare two numbers with different units. A rate is written as a fraction, however, while a ratio is written with a colon. An example of a rate is $\frac{40\,miles}{hour}$. To write this as a ratio, we merely replace the horizontal bar with a colon and write 40 *miles* : 1 *hour*. Notice that the unspoken 1 in the denominator must be written in. Other examples of ratios are 2 *cups of water* : 1 *cup of rice* and 5 *women* : 3 *men*.

While ratios and rates are mathematically equivalent, there are some philosophical differences between the two. Rates tend to compare very different units, for example *dollars* and *lemons* or *gallons* and *minutes*. Ratios tend to compare rather similar units, for example *gallons of gasoline* and *gallons of oil*. Even in the example of 5 *women* : 3 *men*, this ratio is really comparing *people* to *people*.

Furthermore, rates tend to imply an on-going relationship. For example, in a problem where a car is traveling at the rate of $\frac{60\,miles}{hour}$, it will usually be supposed that the car will be continuing at this rate consistantly for the entire time. If the car drives for 10 hours, we will suppose that it travels at a steady $\frac{60\,miles}{hour}$ all the way through, thus traveling 600 miles. Similarly, if lemons are sold at the rate of $\frac{3\,lemons}{2\,dollars}$, we will suppose that this price will remain the same. If we want to buy 150 lemons, then it will cost us $100.

Ratios can also be used to represent on-going relationships, but not usually. If a classroom has 14 girls and 7 boys, we will say that the "boy-girl ratio" is 7 *boys* : 14 *girls*. This is a single relationship, pertaining to a single classroom, and will not mean anything if we increase the number of boys or girls.

Have your students practice writing ratios and rates in all the ways possible. For example, suppose a machine uses 3 pints of oil every 5 days. As a ratio, this is 3 *pints* : 5 *days*. As a rate, this becomes the fraction $\frac{3\,pts}{5\,days}$. This can also be flipped into $\frac{5\,days}{3\,pts}$, which converts back into the ratio 3 *pints* : 5 *days*.

When the units are clear, they are often dropped in ratios. For example, a school with 140 girls and 125 boys can be said to have a girl-boy ratio of 140 : 125. However, in formal speech and writing, the units are always mentioned, either in the ratio (for example, 3 *trucks* : 4 *cars*) or else beforehand (for example, a win-lose ratio of 7 : 5).

Note that when ratios are written without units, we cannot flip them around. The rates $\frac{15\,apples}{4\,dollars}$ and $\frac{4\,dollars}{15\,apples}$ are the same. However, the fractions $\frac{15}{4}$ and $\frac{4}{15}$ are not the same. For the exact same reason, 5 *cows* : 9 *people* is the same ratio as 9 *people* : 5 *cows*. However the ratios 5 : 9 and 9 : 5 are different.

Because ratios are essentially the same as rates, which are fractions, they can be reduced. For example, take the ratio 25 : 15. This is the same as the rate $\frac{25}{15}$ which reduces to $\frac{5}{3}$. When we convert back into a ratio, we finish with 5 : 3. Thus, we will say that 25 : 15 and 5 : 3 are *equivalent ratios*.

If you work out a number of equivalent ratios, hopefully your students will notice a short-cut: if we multiply or divide the two numbers by the same thing, we will get an equivalent ratio. For example, if we multiply both numbers of 4 : 10 by 6 we will get 24 : 60, an equivalent ratio. Similarly, we can divide both by 2 to get 2 : 5, another equivalent ratio.

When ratios involve fractions, it is very convenient to multiply both sides by the common denominator. For example, the ratio $\frac{7}{10} : \frac{3}{4}$ is rather intimidating. The two denominators are $10 = 2 \times 5$ and $4 = 2 \times 2$, so the common denominator is $2 \times 5 \times 2 = 20$. When we multiply both parts of the ratio by 20, we get $\frac{7}{10} \times 20 : \frac{3}{4} \times 20$, which simplifies to 14 : 15. This is equivalent to $\frac{7}{10} : \frac{3}{4}$, but much easier to work with.

Students should also be able to do the reverse and figure out what number is missing in a pair of equivalent ratios. For example, suppose we are told that 3 : 5 and ? : 20 are equivalent ratios. Because they are equivalent, we must be able to obtain the second one by multiplying or dividing each part of 3 : 5 by the same number. If we factor $20 = 2 \times 2 \times 5$, we can see that 20 is obtained by multiplying 5 by 4. Thus, the other part of the ratio, the question mark, must be obtained by multiplying 3 by 4. Thus, the ? = 12. We could illustrate this as:

$$\times 4 \left(\begin{array}{c} 3 : 5 \\ ? : 20 \end{array} \right) \times 4$$

As another example, suppose we want to fill in the square that makes the following ratios equivalent:

8 : 12 and □ : 15

This is trickier than the last example because there is no whole number which we can multiply by 12 to get 15. We could multiply by $\frac{15}{12}$, because $12 \times \frac{15}{12} = 15$, and thus the number in the square must be $8 \times \frac{15}{12} = 10$. An easier method, however, is to begin by reducing the first ratio by dividing both numbers by 4, to get 2 : 3. Now we can easily see that the 15 is formed by 3 × 5, and so the square must be 2 × 5 = 10.

÷4 8 : 12 ÷4
2 : 3
×5 □ : 15 ×5

Sometimes there are extended ratios, like 4 *cats* : 9 *dogs* : 3 *birds*. This is short-hand notation for three different ratios. The first, in this example, is 4 *cats* : 9 *dogs*. The next is 9 *dogs* : 3 *birds*. The last is 4 *cats* : 3 *birds*. To find an extended ratio equivalent to this, we need to multiply or divide all three parts by the same number. For example, if we multiply by 2, we will get 8 *cats* : 18 *dogs* : 6 *birds*, an equivalent extended ratio.

Elementary school children should be able to understand ratio notation and be able to tell if two ratios are equivalent. To do that, all they need to do is convert them into fractions, reduce, and see if they are the same.

Percents are another sort of fraction. If you have explained to your class that "per" means "divided by" and "cent" means "one hundred," then they should have a very easy time accepting the fact that "percent" means "divided by one hundred." Dividing by 100 is so useful in mathematics that the word "percent" has its own symbol: %. You can even show the class how the symbol might have evolved:

÷ 100 ⇒ / 100 ⇒ /00 ⇒ 0/0 ⇒ %

In any case, the symbol "%" can be replaced by "÷ 100" at any time. For example, 40% is the same as 40 ÷ 100, which is the same as $\frac{40}{100}$ or 0.4, when written as decimal. Similarly, $5\% = 5 \div 100 = \frac{5}{100} = 0.05$ and $175\% = 175 \div 100 = \frac{175}{100} = 1.75$.

To divide by 100, all we need to do is move the decimal point to the left two spaces. However, it is far more educational to teach your students that "percent" means "divided by 100" and have them remember the short-cut to this division, than to teach your students that "percent" means "move the decimal point to the left two spaces."

If we first multiply by 100 and then divide by 100, our number will end up unchanged. This is because multiplication and division are inverse operations. This means that we can multiply a number by 100 and then attach a percent sign to the end. This is called "representing a number as a percent" and is usually done to decimals. For example, the number 0.15 is the same as $\frac{0.15 \times 100}{100}$ = 15 ÷ 100 = 15%. Similarly, 0.007 = (0.007 × 100) ÷ 100 = 0.7%.

Again, we could teach students to "move the decimal point to the right two spaces and write a percent sign," but it would be much better to show them in detail what is happening. After a few examples, your students ought to notice the pattern and adopt the short-cut on their own. In this way, your students should not have to worry about "which way does the decimal point move?" when attaching or removing a percent sign. Instead, all they need to do is remember that "% means ÷ 100" and then figure it out on their own. Each time, it should take them less time to figure it out, until eventually they can recall the process instantly. At the same time, this reinforces the "per" trick for solving word problems and the importance of inverse operations.

While it is easy to convert between decimals and percents, it is also quite useful for students to know the percents which are equivalent to basic fractions. One way to do this is to first convert the fraction into a decimal. For example, $\frac{1}{2}$ works out to be 0.5 by long division:

$$\frac{1}{2} = \begin{array}{r} .5 \\ 2\overline{)1.0} \\ \underline{10} \end{array}$$

Thus, $\frac{1}{2}$ = 0.5 = 50%.

Another way to do this is to make an equivalent fraction with a denominator of 100. The trick here is to factor 100 = 2 × 2 × 5 × 5. Thus, we convert:

$$\frac{1}{2} = \frac{1 \times 2 \times 5 \times 5}{2 \times 2 \times 5 \times 5} = \frac{50}{100} = 50\%$$

This works nicely for all fractions whose denominators evenly divide 100.

For example $\frac{1}{4} = \frac{25}{100}$ = 25 % = 0.25, as can be shown either with long division or:

$$\frac{1}{4} = \frac{1 \times 5 \times 5}{2 \times 2 \times 5 \times 5} = \frac{25}{100}$$

Similarly, $\frac{3}{4} = \frac{75}{100}$ = 75% = 0.75. This can be figured either with long division, with equivalent fractions, or by multiplication:

$$\frac{3}{4} = 3 \times \frac{1}{4} = 3 \times \frac{25}{100} = \frac{75}{100}$$

We could just as easily have multiplied the decimal (3 × $\frac{1}{4}$ = 3 × 0.25 = 0.75) or multiplied the percent (3 × $\frac{1}{4}$ = 3 × 25% = 75%). It is an excellent exercise to have your students work this problem out in all the possible ways. Not only does this reinforce a large variety of skills, but it should give them a number of different ways to approach problems.

Similarly, have your students work out $\frac{1}{5} = \frac{20}{100} = 20\% = 0.2$ and then conclude that:

$\frac{2}{5} = 40\%$ $\quad$ $\frac{3}{5} = 60\%$ $\quad$ and $\quad$ $\frac{4}{5} = 80\%$

Other useful percents to know are:

$\frac{1}{10} = 10\%$ $\quad$ $\frac{1}{20} = 5\%$ $\quad$ $\frac{1}{25} = 4\%$ $\quad$ $\frac{1}{50} = 2\%$ $\quad$ and $\quad$ $\frac{1}{100} = 1\%$

along with all their multiples. For example, $\frac{7}{10} = 70\%$ and $\frac{3}{20} = 15\%$.

Not only are these very useful exercises, strengthening long division and fraction skills, but these are extremely valuable for quick computations.

Fractions whose denominators do not evenly divide 100 will not come out as nicely. The fraction 1/3, for example, comes out to $0.\overline{3}$ with long division:

$$\frac{1}{3} = \begin{array}{r} .3\overline{3}\ldots \\ 3\overline{)1.000\ldots} \\ \underline{9} \\ 10 \\ \underline{9} \\ 10 \\ \ddots \end{array}$$

If we round to the nearest hundredth, we get that $\frac{1}{3} \approx 0.33$ and thus $\frac{1}{3} \approx 33\%$. However, this is only an approximation. To be precise, we would have to write $\frac{1}{3} = 33.\overline{3}\%$ or else $\frac{1}{3} = 33\tfrac{1}{3}\%$. Similarly, $\frac{2}{3} = 66.\overline{6}\% = 66\tfrac{2}{3}\%$.

Try to teach the class that "=" should only be used for actual equalities, and that "≈" should be used when numbers are rounded or approximated.

Percentages are most often used to express the relationship between a part and a whole. For example, suppose a class contains 14 girls and 11 boys. The whole class consists of 14 + 11 = 25 children. The 11 boys constitute only part of this class. Specifically, the boys are $\frac{11}{25} = 11 \times \frac{1}{25} = 11 \times 4\% = 44\%$ of the class. The other part are girls, which can either be calculated as $\frac{14}{25} = 14 \times \frac{1}{25} = 14 \times 4\% = 56\%$, or else by taking the whole 100% of the class and subtracting the 44% boys to get 100% - 44% = 56%.

Notice that the ratio of boys to girls in that last example would be 11 : 14. We have to be very careful of the various fractions at play here. The ratio 11 *boys* : 14 *girls* is the same as the rate $\frac{11\,boys}{14\,girls}$. Because *boys* and *girls* are different units, they will not cancel out. Thus, this rate cannot be converted into a percent without units. However, if we convert *boys* into *children* and *girls* into *children*, we can then add 11 *boys* + 14 *girls* = 11 *children* + 14 *children* = 25 *children*. In this case, the 11 *children* who happen to be boys are the following fraction of the whole class: $\frac{11\,children}{25\,children}$. In this fraction, the *children* do cancel out, leaving behind 11/25 = 44%.

As another example, suppose an engine requires a fuel mix of 3 *cups oil* : 5 *cups gasoline*. If we convert the two parts into simply *cups*, we can see that the recipe makes a full 3 *cups* + 5 *cups* = 8 *cups* of fuel. Of this, the gasoline comprises $\frac{5\,cups}{8\,cups} = \frac{5}{8} = 62.5\%$ of the mixture.

This can certainly be confusing to students, so try the following exercise. Have the class break up into teams, and have each team chose their best basketball player. If you can, take them to a court and have the best players shoot free-throws. If you cannot, then get smaller balls and have the players try to throw them into waste baskets from a standarized distance. This could be played with toy balls, wadded-up pieces of paper, or anything else at hand. The other players on the team must keep track of the total number of shots attempted, the total number of successful baskets, and the total number of misses. In addition, they should help to return the balls back to the "pro" who is shooting. This can be a rather loud and excited activity, so you might want to save it for a Friday afternoon.

After a certain amount of time, have each team sit back down and share the data on their pro. Next, have them work out the hit-miss ratio, as well as the free-throw percent and the percent missed. For example, if the student threw 20 balls and got 7 in, then 13 must have missed. The hit-miss ratio will thus be 7 : 13. The free-throw percent will be $\frac{7}{20} = 35\%$. The precent missed will be 65%.

This sort of exercise can be used with just about anything. Have your class calculate the percent of attendance at the beginning of each class. Have them figure out the ratio of kids wearing pants to kids wearing shorts on a hot day. You want them to realize that you could never have more than 100% of the class do anything (attend, wear shorts, etc.) although you could have a ratio of 14 *pants* : 5 *shorts* which forms a rate that looks like more than 1, namely $\frac{14\,pants}{5\,shorts}$.

Remember, we can only add, subtract, and compare numbers with the same units. There really is no way to compare 20 *dollars* and 15 *feet*. We can only say that 20 *dollars* is more than 15 *dollars*. Similarly, the rate $\frac{14\,pants}{5\,shorts}$ is neither bigger nor less than 100%, because the first number has units (*pants per shorts*) and the second number has no units at all. This can seem tricky at first, but with a little experience, your students will hopefully catch on.

Much easier are problems where students must use percentages. All that is necessary is to remove the percent sign (by dividing by 100) and then multiply. For example, if a problem asks "what is 20% of $140?" then we turn 20% = 0.2 and then multiply 140 × 0.2 = 28. The answer is $28.

Many problems require students to subtract a percent, as when a store discounts items. For example: "how much does a $40 shirt cost when on sale for 15% off?" The long way to solve this problem is to first calculate 15% of the $40 by multiplying 40 × 0.15 = 6 and then subtracting 40 – 6 to get the answer: $34. A shorter way to do this is imagine that we have to pay 100% - 15% = 85% of the shirt's cost. When we multiply 40 × 0.85, we get the same answer: 34.

A very important lesson for children is that adding and subtracting the same percent are *not* inverse operations and will *not* end back at the same number. A very useful example is to start with $100, then add and subtract 10%. Ten-percent of $100 is $10, so when we add 10%, the sum goes up to $110. Ten-percent of $110 is $11, however, so when we subtract we end up at $99, not at $100. If you catch any students supposing that adding and subtracting the same percent will not change the original number, begin by praising them for being on the look-out for inverse operations. Use this as an opportunity to refresh the concept of inverse operations. Then, however, show them how this reasonable idea does not work out. We call it "adding 10%" and "subtracting 10%," but these are both computed with multiplications and thus not really opposite.

Questions:

(1) Write the ratio 8 *cups water* : 3 *cups detergent* as a rate.

(2) Name 5 ratios which are all equivalent to 21 : 28.

(3) Convert into percents:

(a) 0.19 (b) 1.3 (c) 0.015 (d) $\frac{9}{10}$ (e) $\frac{4}{25}$

(4) Write as a number without a percent:

(a) 29% (b) 9% (c) 3.2% (d) 285%

(5) Suppose a team's win-loss ratio is 11 : 9. What percent of the games have they won?

(6) Suppose 64% of the school is female. What is the male-female ratio for the school (reduced as much as possible)?

(7) What is 30% of 180?

(8) Show the two different ways to solve this problem:

"How much will a $80 pair of pants cost when discounted 30%?"

Chapter 42: Probabilities

Another useful application of fractions and percents are probabilities. Not only are there lots of fun games involving dice and other randomness for children to play, but a little exposure to the concept early on will help them greatly in the higher grades.

A *probability* is a number which represents how likely something is to happen. Things that always happen have probability 1. Things that never happen have probability 0. Things that sometimes happen have a probability between 0 and 1, usually written as a fraction or a percent.

The easiest probabilities to calculate occur when there are a certain number of things that can happen, all of them equally likely. For example, when a die is rolled there are six numbers than can land on top (1, 2, 3, 4, 5, and 6), all of them equally likely (assuming the die has not been "loaded" by putting a weight in one side). Similarly, when a coin is flipped, there are two ways it can land, *heads* and *tails*, both equally likely (unless there is something weird about the coin). A deck of cards has 52 cards, and when one is pulled out without looking, they are all equally likely to be drawn. If we want to calculate the probability of accomplishing something in a situation like this, we divide the number of ways to win by the number of possible outcomes.

For example, suppose we want to calculate the probability of rolling a 5 on a die. There is only 1 way to win (roll a 5) and 6 possible outcomes (1, 2, 3, 4, 5, and 6). Thus, the probability is $\frac{1}{6}$.

If we want to calculate the probability of rolling a 3 or higher, we need to count up all the ways to win. There are 4 different numbers on a die that are 3 or higher (3, 4, 5, and 6). There are still 6 possible outcomes. This means that the probability is $\frac{4}{6}$.

Now suppose we want to draw an ace out of a deck of cards. There are 4 aces in the deck and 52 cards total, thus the probability of success is $\frac{4}{52}$. This fraction can be reduced to $\frac{1}{13}$, but it might be better to leave it as $\frac{4}{52}$ for a while, so that your students see the 4 different aces and the 52 different cards written in the answer.

Popular probability problems for elementary children involve drawing socks out of drawers and balls out of urns, always randomly (in the dark or with eyes closed). For example, suppose a sock drawer contains 6 blue socks, 10 white socks, and 4 brown socks. If a child reaches in and randomly pulls out a sock, what is the probability that the sock is blue? To answer this, we need to count up the number of ways to succeed (6, because there are six socks that count as blue) and then the number of ways things can happen (6 + 10 + 4 = 20 different socks can be drawn). Thus, the probability of drawing out a blue sock is $\frac{6}{20} = 30\%$.

Here is an exercise for warming children up to probabilities. Divide the class into two teams and have each team choose a name. Write the name of the first team on three index cards, then fold them up. Write the name of the second team on one index card and fold it up, too. Then put all the folded-up cards in a hat or a box. To play the game, draw out a folded paper randomly, read the teams's name, and give that team a point. After each point, the card drawn should be folded back up and returned to the box.

It will probably not be long into the game before the players of the second team complain that the game is "not fair." Have them explain their complaint in detail, and have the class discuss why this game may or may not be fair. It is very possible that students will say that one team has "more chances" than the other. A discussion of chances might very well lead to the ratio 3 : 1. Don't explain the concept of probability, just try to channel the second team's outrage into a proper discussion of why the game is not fair. Press them to come up with numbers to explain their complaint. If the best you can get is "they have 3 chances and we only get 1," then leave it at that. That proves that the game is unfair.

Next, change the game. Each round of the game will involve drawing two different cards, one from the box of folded-up index cards and the other from a deck of cards. The second team will only score from the box and the first team will only score from the deck. The first team will get a point every time an ace is drawn out of the deck. The second team will only score when their name is drawn from the box of index cards (if the first team's name is drawn, no one gets a point).

It might take a few dozen rounds, but hopefully your class will realize that the game is now unfair in the other direction; the second team will now score more often. This time, the first team will need to explain why the game is unfair. Do not accept that their complaint without some sort of number evidence. This time, the "chances" argument will no longer work because the first team has 4 chances to win (there are 4 aces in the deck) while the second team has only 1 chance to win (1 card with their name). Hopefully, a full class discussion will point out the difference in the total number of cards in each situation. The first team has to deal with a deck of 52 while the second team has to deal with only 4 cards. The dream is that someone will say something along the lines of "1 out of 4" to describe the second team's chances. This, of course, is the probability. If something like this comes up, then have the class figure out that the probability for the first team is "4 out of 52."

When that discussion is finished, offer each team the ability to switch their way of scoring points for a new one: roll a die and score when a 6 is rolled. Hopefully, the second team will realize that this would be a bad trade, switching a 1 in 4 chance to score for a 1 in 6. The first team, however, would do well to take the offer, as 1 in 6 is better than 4 in 52. At this point, if the class has not struck upon the concept of probability, you can coax them along to the concept, to calculate how likely it is to score a point with each of the different methods.

Once this game has been taught to the class, it ought to be easy to extend in all sorts of different ways. Put a large list of options on the board (roll a 5 or a 6 on a die, draw a face card out of a deck, flip a coin to "heads," etc.) and have each team pick the way they want to score. You could even have each option be able to be chosen only once, so that the first team tries to score by one method, then crosses it off the list, and then the other time gets to choose one of the remaining methods. The goal, of course, is to get the students to calculate the probability of each method, and then compare the fractions to determine the highest-probability remaining method. By putting the kids in teams, you can have them each work out the probability of some of the scoring methods, then compare them all at once as a team. This also enables those students who have a better grasp on the concepts to explain them to the weaker students.

There is a great deal that can be done with probabilities, but much is beyond the scope of elementary mathematics. Probabilities can be combined, usually by multiplication but sometimes by addition, depending on the circumstances.

One very common situation which is slightly beyond what has been covered in this chapter so far is the rolling of two dice and summing the total. In this case, there are 36 equally-likely outcomes, which can be illustrated on a table like:

first die

second die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

sums

This table makes it clear that the most likely sum to show up is a combined total of 7. The probability of rolling a sum of 7 on two dice is $\frac{6}{36}$ because there are 6 different ways to get a 7 (six squares on the table with that sum) and 36 total ways the dice can come out (36 squares on the table). Similarly, the probability of rolling an 8 is $\frac{5}{36}$ because there are 5 squares on the table with a sum of 8. The least likely outcomes are 2 and 12, which each occur with probability $\frac{1}{36}$. Beyond this, however, is beyond the scope of elementary mathematics.

Questions:

(1) An urn has 7 red balls, 6 black balls, and 3 blue balls. What is the probability that a ball randomly pulled out will be red?

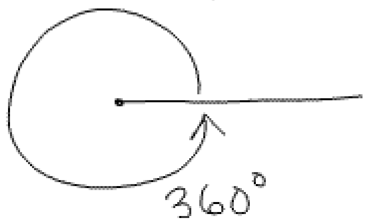
(2) Calculate the probability of succeeding with each method, then put them in order from most likely to least:

(a) randomly draw a face card or an ace out of a deck of 52
(b) roll a 1 or a 6 on a die
(c) flip a coin and have it land "tails"
(d) pull a red ball out of a bag with 5 red balls and 9 yellow balls
(e) randomly draw a heart out of a deck of 52 cards
(f) roll two dice and get a sum of 5

Chapter 43: Angles

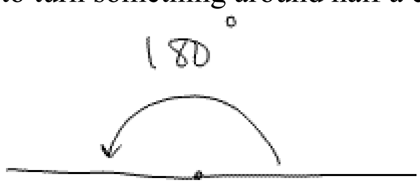
It is a good idea to introduce children to the concept of angle early on. For one thing, kids of a certain age enjoy precision and have lots of fun working out exercises with rulers and protractors. For another, there are not many concepts required for understanding angles, and yet many aspects of mathematics require some understanding of angles. In fact, we already had to talk about right angles when explaining what squares and rectangles were. Finally, angles provide wonderful opportunities for children to explore deductive reasoning in a tangible and reasonable fashion.

A clever child could deduce nearly everything about angles from one single concept: there are 360 *degrees* in a full circle:

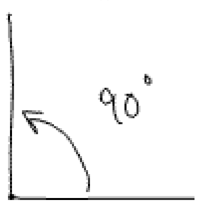


This means that if one end of a stick is held in place and the stick is turned once around, then the stick will have turned 360 *degree*, usually written as 360°.

A turn that goes only part-way around a circle will have fewer than 360°. For example, to turn something around half a circle is to put it through 180°:



Similarly, a quarter of a turn is 90°:

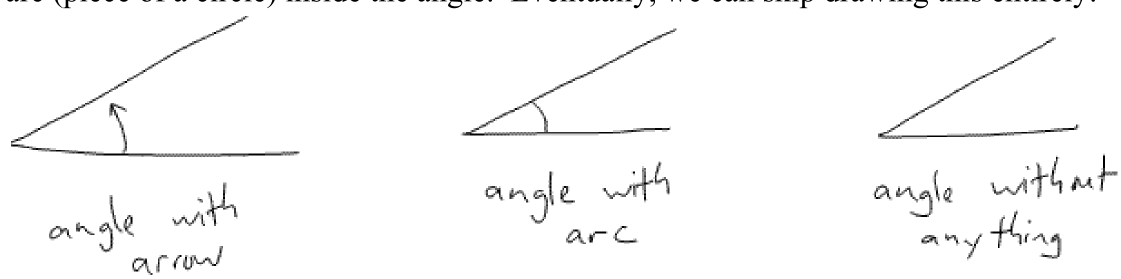


To practice and reinforce this concept, you can have children turn toy cars, arrows cut out of paper, pencils, or anything else that points in a particular direction. Have everyone begin with their object pointed toward the front of the classroom. Then, tell everyone to "turn 180°" and check to see that everyone's object is now pointing at the back of the classroom. If the kids want to have fun, they can drive a toy car along the ground and then have it "pull a 180," which means to skid out and end up facing the opposite direction. Similarly, to "pull a 360" is to turn the car entirely around so that it ends up facing the same direction in which it began. With young children, you can encourage them to make the usual screeching noises that accompany skids like this.

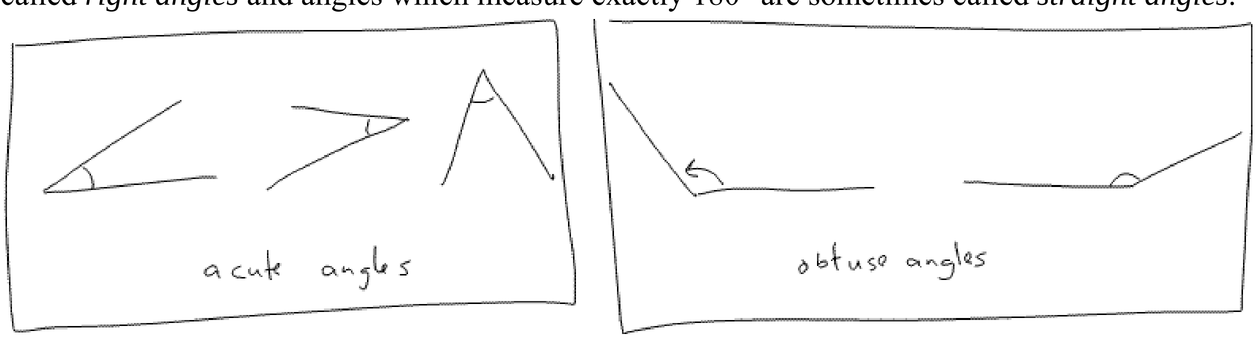
For turns of 90°, you'll have to differentiate between clockwise and counter-clockwise turns:

You can also have the class stand up and take commands like "everyone turn clockwise 90°" or "everyone turn 360° counter-clockwise." It should only take a few rounds of this game for everyone to get an understanding of how to measure rotations.

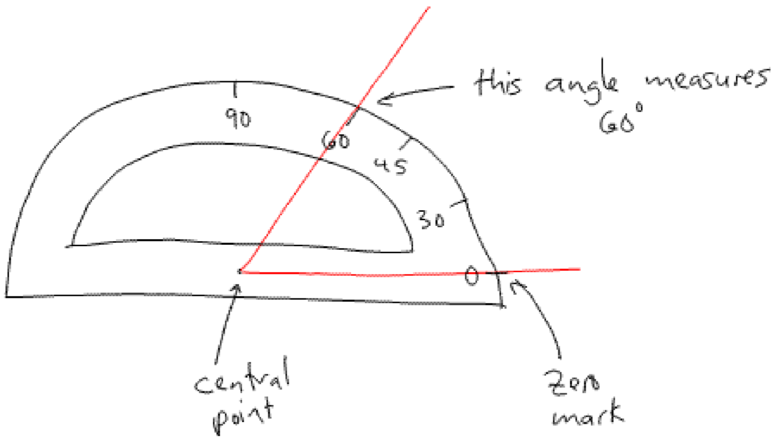
An *angle* consists of two line segments (called *edges*) which meet at a point (called the *vertex* of the angle). Draw some angles on the board and have the class guess the number of degrees that one of the lines would have to rotate around the vertex in order to line up with the other line. To begin with, indicate this movement with a little arrow. Later on, draw just a little arc (piece of a circle) inside the angle. Eventually, we can skip drawing this entirely:



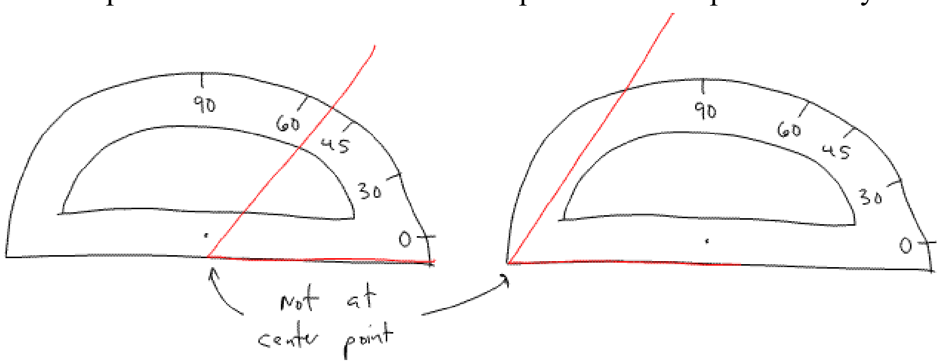
At first, have the students just identify angles as either 90°, less than 90°, 180°, or somewhere between 90° and 180°. The angles which are less than 90° are called *acute angles* and those between 90° and 180° are called *obtuse angles*. Angles which are exactly 90° are called *right angles* and angles which measure exactly 180° are sometimes called *straight angles*.



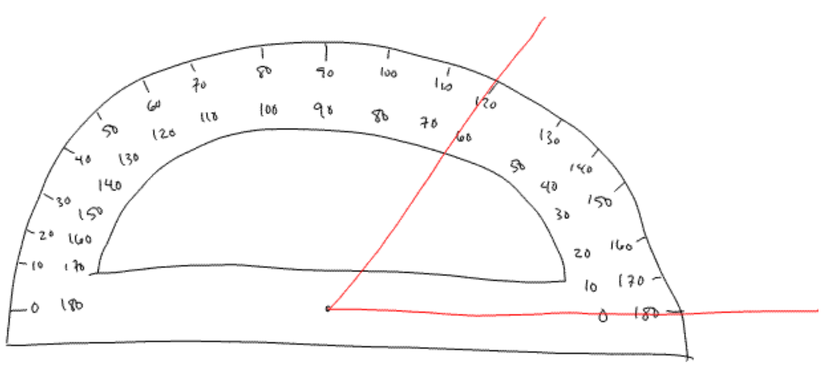
For our students to measure angles more exactly, we will need to provide them with *protractors*. A protractor is a bit like a ruler, except that it is curved like a semi-circle and measures angles instead of lengths. In order to measure an angle, the central point of the protractor must be placed over the vertex of the angle and one edge of the angle must line up with the zero at one end of the protractor. If the other edge of the angle is drawn out far enough, it will meet the protractor at the number which measures the angle:



There are two common mistakes made by students who are first beginning to learn how to use a protractor. The first is to fail to put the central point exactly over the vertex:



The other common mistake is to read the wrong number, confusing acute and obtuse angles. This happens because most protractors have a zero at both ends:

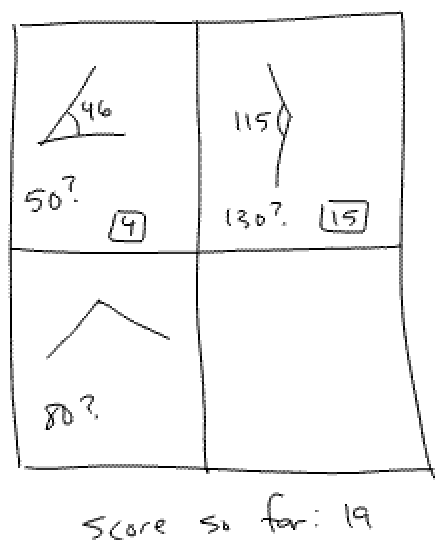


This angle crosses the protractor at both 60° and 120°. Because the angle is acute, however, the correct measurement is 60°.

It would be nice to walk the students through the use of a protractor and warn them about these mistakes in advance. However, when they make these mistakes, tell them that these are very common and understandable as you correct them.

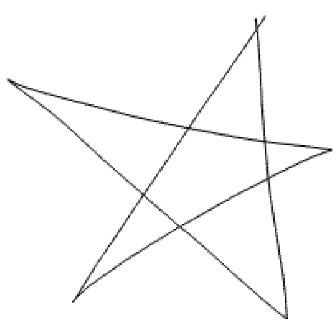
For practice, give the students photocopied worksheets with various angles on them. Have the children measure them with protractors and write in the angle measurements.

It would also benefit them to practice guessing angles, to develop a sense of proportion. To do this, pair up the students, each with a ruler and a protractor. First, each student draws an angle with the ruler. Next, they trade papers with their partner. Without using the protractor, each student must then guess the measure of the angle and write the guess down (with either the word "guess" or followed by a question mark). Next, each child should measure the angle to find the exact measurement, and then calculate the difference between the guess and the actual measurement. The difference between the guess and the actual angle should be calculated (subtract). The papers are traded back, and the students each verify that the angle was measured correctly. Then, another angle is drawn and the game is repeated. To make things more neat, have them first divide each side of the paper into quarters by drawing lines. The game will then consist of drawing, guessing, and measuring 8 angles (enough to fill both sides of each paper). Encourage the class to try angles which are both acute and obtuse. When the papers are all filled, each student should add up the amounts of error – the total is the score of the student who drew the angles.



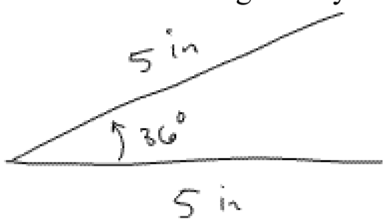
On the game board illustrated to the right, the student who has drawn these angles has thus-far earned a score of 19 points. Now it is the turn of the two players to measure this angle and calculate how far off is the guess of 80°.

For another angle exercise, have each student take a piece of paper and ruler, and try to draw as nice a five-pointed star as possible:

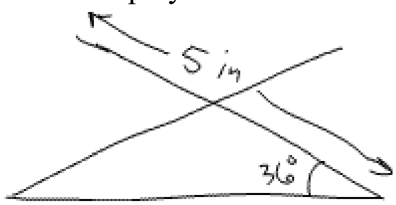


Most likely each student will draw a star like the one illustrated to the left, with some of the points bigger than others.

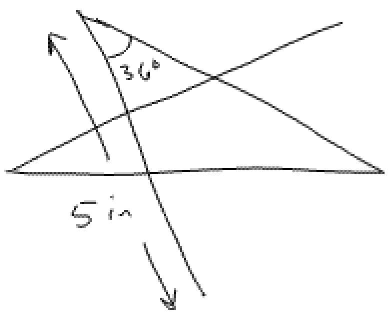
Next, give each student a protractor and another piece of paper. Have them measure a five-inch line across the middle of the paper with the ruler and then a 36° angle up from this line. Then, have them extend this second line out until it is also 5 inches long. Have them erase any bit of the line that goes beyond the vertex or beyond the 5 inches:



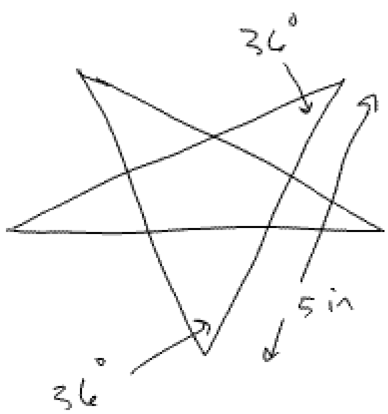
Next, have each student measure 36° up from the other end of the first line, and extend that line up by 5 inches:



Next, have them turn the page around so that they can measure another 36° angle from the end of this latest line. As always, this line should be made 5 inches long:

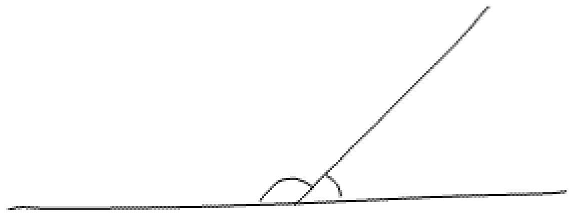


If the last two ends are connected together with a straight line, this line should be 5 inches long and make two more 36° angles:

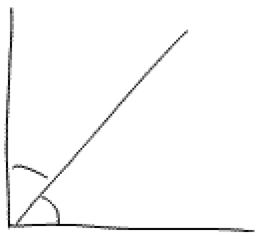


The result will be a perfect five-pointed star, but only if the angles and lines were measured accurately. Have the students cut them out, write their names on them, and decorate them as they wish. Not only does this practice measuring lines and angles, but it will often make students in the class quite happy. Many children have tried many times to draw a perfect five-pointed star and failed. The secret, you can tell them, is that each point must measure exactly 36°, which is hard to do without a protractor.

When two angles together make a straight line, their measures must add up to 180°. Such angles are called *supplementary*:

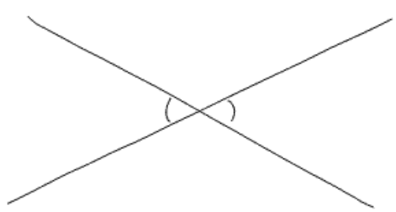


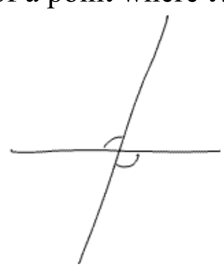
When two angles together make a right angle, they are called *complementary*:



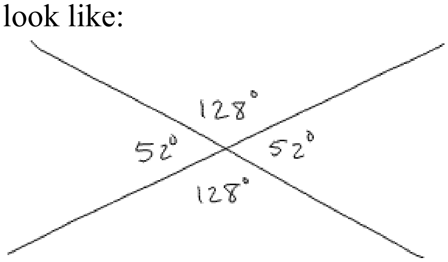
As a mnemonic for remembering these, you can tell your students that "c" comes before "s" in the alphabet, and thus "complementary" angles add up to a smaller amount (90°) than "supplementary" angles (180°).

Complementary angles really don't show up in mathematics until trigonometry. Supplementary angles, however, can be used to prove an important fact about *vertical angles*, angles which appear on opposite sides of a point where two straight lines intersect:





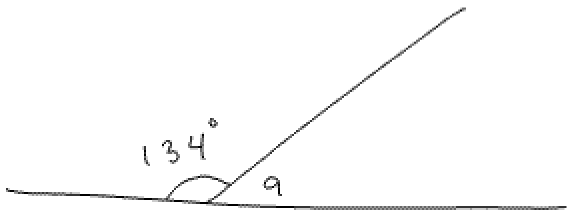
Have your students use a ruler to draw two straight, intersecting lines on a paper. Then, have them measure all four of the angles which occur. You might need to encourage them to rotate their papers around to make it easier to measure these angles. For example, this might look like:

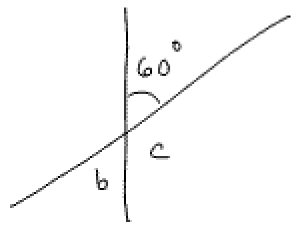


Ask the class to discuss their results. Hopefully, at least someone will notice that the measurements repeat, as in the example above. Cheer this student significantly, then explain to the class that angles like these are called *vertical*. Also, when angles have the same measure, they are called *congruent*. This, this student has discovered that "vertical angles are congruent." This is so important in mathematics, that it is called a *theorem*, an interesting result worth noting.

If the students have measured correctly, they might also notice that the two different angle measurements add up to 180°. If no one notices this, have each student add the two different measurements. This is a good time to explain the concept of supplementary angles.

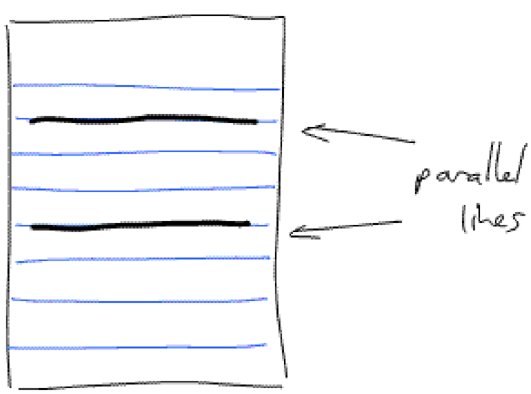
After this exercise, your students should be able to figure the exact measures of the angles marked *a*, *b*, and *c* in the figure below, without using a protractor:





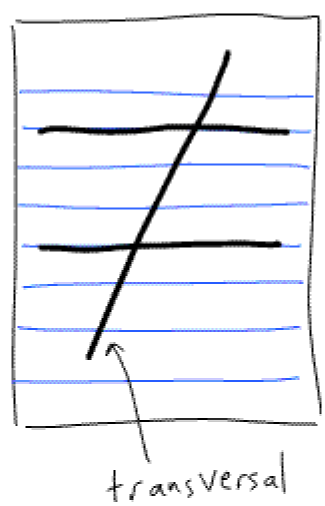
Angle *a* is supplementary to the 134° angle, and thus measures 180° - 134° = 46°. Angle *b* is vertical to the 60° angle and thus also measures 60°. Angle *c* is supplementary to both *b* and the 60° angle, thus measures 180° - 60° = 120°.

As a next exercise, have each student use a ruler to trace two of the lines on a sheet of lined paper, preferably lines which are several spaces apart. See if your students remember that these sorts of lines are called parallel lines:

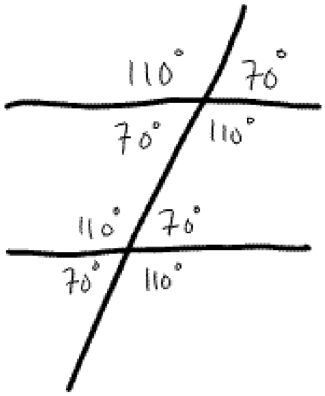


Next, have each student use the ruler to draw a line that crosses both of these lines (and not through any of the endpoints). Tell your students that the technical name for a going-across line like this is a *transversal*.

Have your students to count the number of angles which have been formed (eight) and then measure all of them with a protractor.

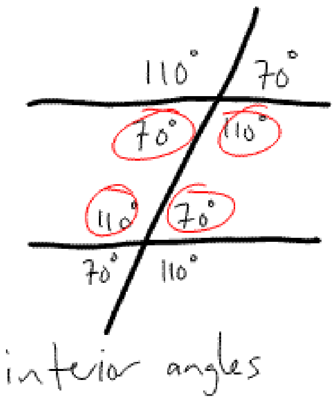


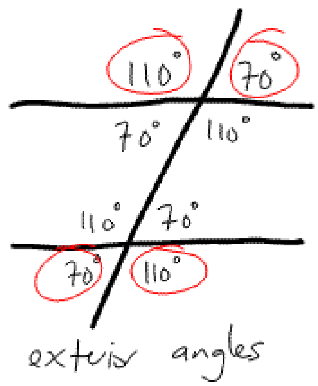
The end result ought to look something like:



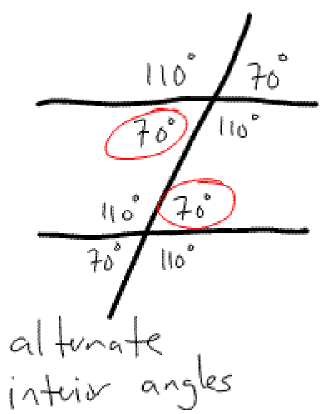
Again, have your class compare and discuss their results, to see if they notice any patterns or similarities. Even though they will probably have gotten different measurements, they will hopefully have gotten the same two measurements over and over, in the pattern illustrated at the left.

This is another important theorem in mathematics. Unfortunately, there is a great amount of vocabulary required to describe this pattern. The four angles betwen the two parallel lines are called *interior*, as if inside the two lines, and the other four angles are called *exterior*:



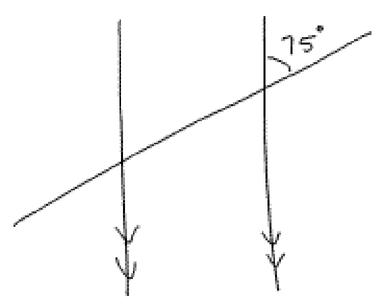


When two angles are on opposite sides of the transversal, they are called *alternate*:

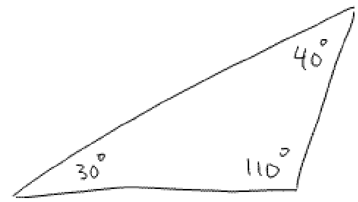


One of the ways to describe this pattern, thus, is to say: "when two parallel lines are crossed by a transversal, the alternate interior angles are congruent." Now that is a mouthful that no elementary school child should be forced to memorize. However, it would not be a bad idea to introduce these vocabulary words and have the students play around with them. Still, it is much more important that the students recognize the pattern than it is for them to remember the vocabulary.

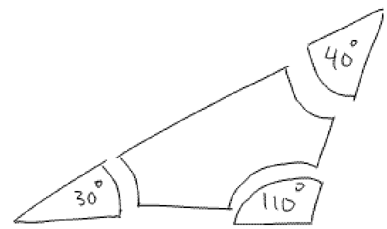
At this point, you should be able to show the class a figure like the one to the right and have them find the measures of all the missing angles, without using a protractor. Remember that parallel lines do not have to always be horizontal, they need only go in the same direction.

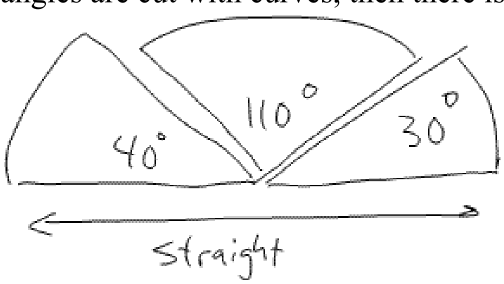


As another exercise, have each student draw a big triangle with a ruler on a piece of paper. Next, have each student measure the three angles with a protractor. Finally, have each student add up the sum of their three angles. As always, the class should then discuss their results. Hopefully, they will realize that the sum of the angles of each triangle came out to 180°, or perhaps nearly so (it is sometimes hard to measure angles exactly). Let them talk it over for a while, and challenge them to state their result exactly. Ideally, they will come up with something like: "the angles of a triangle add up to 180°."



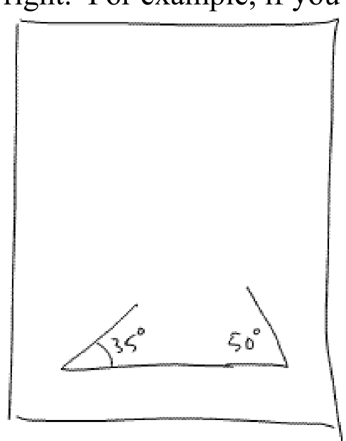
To reinforce this, have each student then cut out their triangle, and then cut out the three angles. It is very important that the angles be cut with arcs, and not with straight lines. Perhaps you can have them draw in the arcs first, and then trace them. The arcs are important, because the students should then put the points of the three angles together. If the angles are cut with curves, then there is only one clear way to do this:



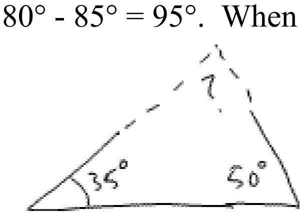


The three angles should all together form a straight line. This further verifies that the angles of a triangle add up to 180° (the measure of a straight angle).

As a final verification, have each student draw a straight line across the bottom of a page. Then, give each student an angle to measure up from the left and another to measure up from the right. For example, if you choose 35° and 50°, the students should draw:

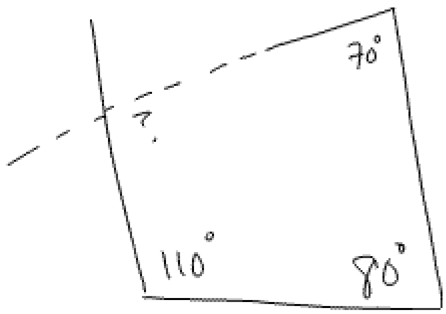


Now challenge the class. Tell them that in a few minutes, everyone will extend these two last lines until they meet and form a triangle. Ask them to guess the measure of the angle where the two lines meet. Hopefully, your students will now be comfortable enough with the idea of the angles of a triangle summing to 180° that they will calculate that the triangle has 35° + 50° = 85° so far, and thus the last angle should measure 180° - 85° = 95°. When they have convinced themselves (and each other) of this in class discussion, have everyone complete the triangle and see that they were correct.



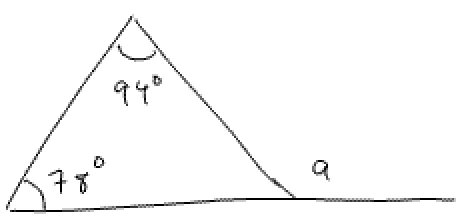
Hopefully, your students will find it exciting and empowering to watch a prediction like this come true.

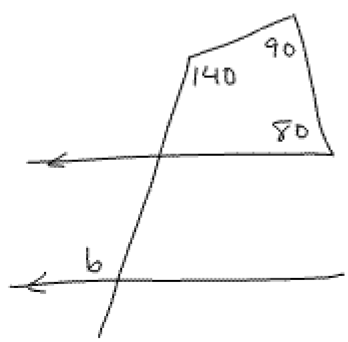
Everything that we have just discussed about triangles can be repeated with quadrilaterals. That is, you can have each student draw a four-sided figure on a paper with a ruler (encourage them to be wacky and not draw rectangles), then measure and sum the angles. Hopefully, they will be able to conclude and convince each other in class discussion that the angles will sum to 360°. If they cut out the four angles and put them together, they ought to make a full circle. Finally, you can instruct them to measure out an angle between two lines (say, 80°) and then two more angles off the ends (say, 110° and 70°), and have them predict the angle which will occur where the two ends meet:



To challenge your strongest students, invite them to repeat the process for five-sided figures, six-sided figures, and so on. This pattern is fairly tricky to notice and figure out, but that is all right. Anyone who puts in a hour or two of effort on the problem will certainly attain a certain level of mastery with angles and measurement.

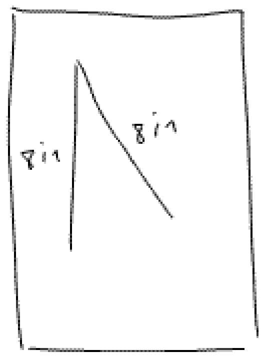
At this point, you should be able to draw more complicated figures for your students and have them figure out the missing angles. For example, your students ought to be able to find the angles *a* and *b* in the following:



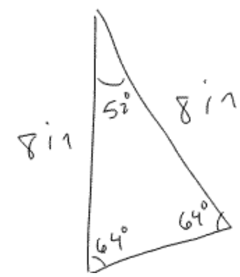


Work them up gradually to these sorts of problems. Teach them to work around the problem, figuring out the measure of any angle possible. If posed in the right way (and not too difficult), problems like these can be quite fun for students to figure out. Start out with one of these drawn on the board, and have the class discuss it. Encourage students to come up to the board and write in the measures of angles, taking care to explain their reasoning to the others. After some work like this, you can break the class into small groups to work on more problems. Finally, you can have the students work individually. In this manner, the stronger students can help the others to understand until everyone is able to solve these sorts of problems.

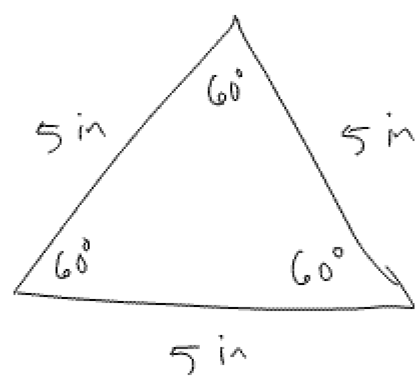
Another exercise is to have the students figure out the angles of isosceles triangles. Each student should draw a line, measure it, then draw an angle from one end that has the same length:



Next, have the students connect the two endpoints and then measure all three angles of the triangle. As always, see if the class can collectively detect the pattern: isosceles triangles always have two angles with the same measure. Furthermore, these angles will not include the angle between the two sides of the same length.



As a final exercise angle exercise, have the class try to figure out how to draw a perfect equilateral triangle. First, have them try to draw an equilateral triangle with each side 5 inches long using only a ruler. They will be able to make two sides measure 5 inches, but when the ends are connected, it is very unlikely that the last side will also measure 5 inches. After a little bit of trying, give them each a protractor, and see if this will help. Have them work either in teams (to encourage competition) or as a class (to encourage group discussion and brainstorming). Ideally, someone will realize that the angles of an equilateral triangle will all need to be the same, and thus will each need to be 60°, so that they can all add up to 180°. There are a number of ways they can come to this conclusion, so encourage them to discuss the matter at length. In any event, this is an excellent opportunity to introduce your children to some honest-to-goodness real mathematics – guessing, trying, reasoning, and then coming to deep conclusions about the way the universe of lines and numbers fits together.

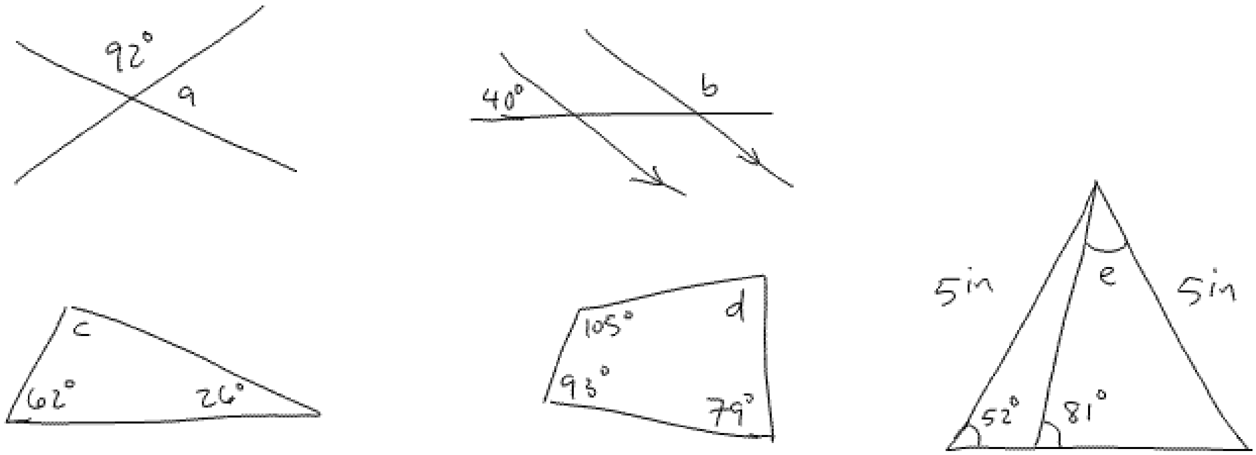


Questions:

(1) Draw a perfect five-pointed star by following the directions in this chapter.

(2) Draw a five-sided, a six-sided, and a 7-sided shape on pieces of paper. Use a protractor to measure all of the angles involved. Try to make a generalization about the sum of the angles of each of these figures.

(3) Find the measures of the angles marked *a*, *b*, *c*, *d*, and *e* in the following figure:

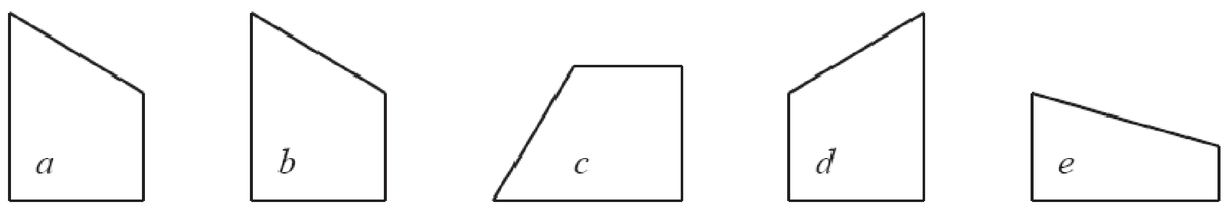


Chapter 44: Similar Triangles

Ratios and angles come together in the area of similar triangles. Similar triangles are key to understanding trigonometry, but are also valuable in solving a number of problems.

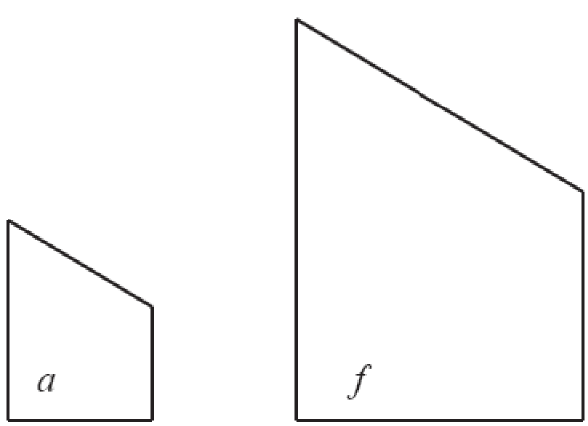
Two geometric objects are called *congruent* if they have the exact same shape and size. This means that if we slide one of them around (a movement or transformation called a translation), rotate it, and/or take its mirror image (called a reflection), then we can get the first object to exactly line up with the second one. This is easiest to do with shapes cut out of paper – if one shape can be placed on top of the other so that they match up exactly, then they are called *congruent*. Having children move shapes around like this is an excellent way to prepare them for the advanced concepts of functions and groups. This is also a nice way to emphasize the geometric concept of congruence. Basically, you put a number of shapes on a table, then ask the kids to identify the ones that can be matched up exactly.

For example, in the figures below, shapes *a*, *b*, *c*, and *d* are all congruent, but *e* is not.

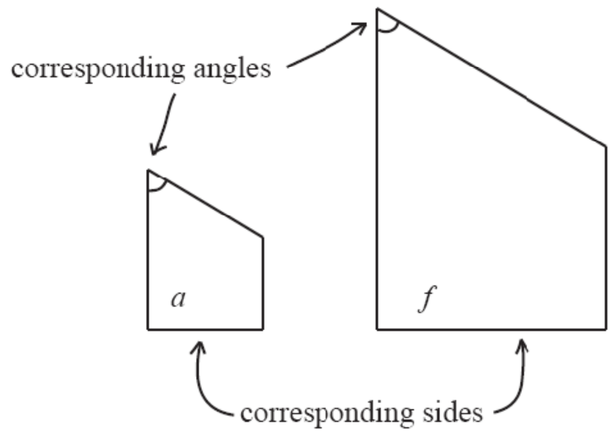


We could line up *a* and *b* by sliding one of them horizontally with a translation. We can rotate *a* counter-clockwise 90° and then translate it over to line up with *c*. We will need to flip *a* over with a reflection to match it up with *d*.

All of these tranformations preserve the size and shape of the original object. There is another transformation, called a *dihilation*, which proportionally shrinks or enlarges shapes. For example, if we double the height and width of the shape *a* above, it will look like *f* below:



The quadrilaterals *a* and *f* are called similar. Two figures are *similar* if they have the same shape, but not necessarily the same size. When shapes are similar, we can talk about their *corresponding parts*. If we were to shrink or enlarge one of the shapes and then move it to line up with the other, the corresponding parts are the ones which would line up. For example, the acute angles at the tops of *a* and *f* correspond and the sides between their right angles correspond:

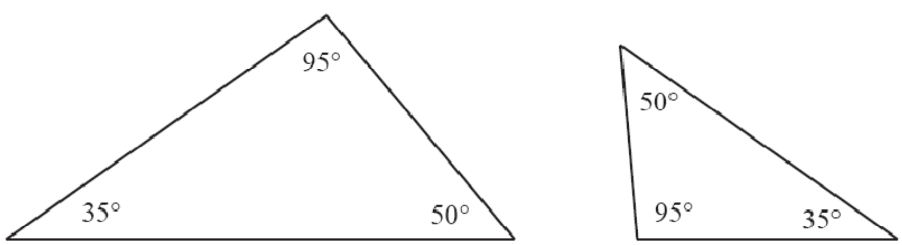


No matter how we might shrink or enlarge quadrilateral *a*, for example, the left-hand side will always be the longest. Similarly, the left-hand side of quadrilateral *f* will always be the longest. Thus, since these two shapes are similar, their left-hand sides must correspond.

The amount by which one shape must be shrunk or enlarged to become the size of a similar shape is called the *scale factor*. In the above example, the scale factor from *a* to *f* is 2 because each side was doubled in size to make shape *f*. To go from *f* to *a*, on the other hand, we would need a scale factor of $\frac{1}{2}$ in order to make each side half as long.

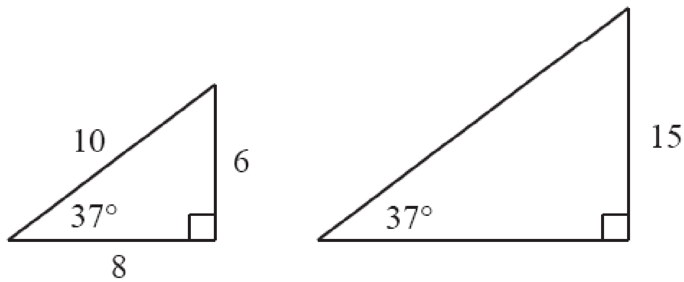
When a shape is scaled up or down with a dihilation, its size can change considerably. The angles, however, do not change. Thus, the corresponding angles of similar shapes are congruent.

With triangles, there is a very simple test for similarity. If two triangles have the same angles, then they are similar. For example, the two triangles below have the same three angles, and thus are similar.



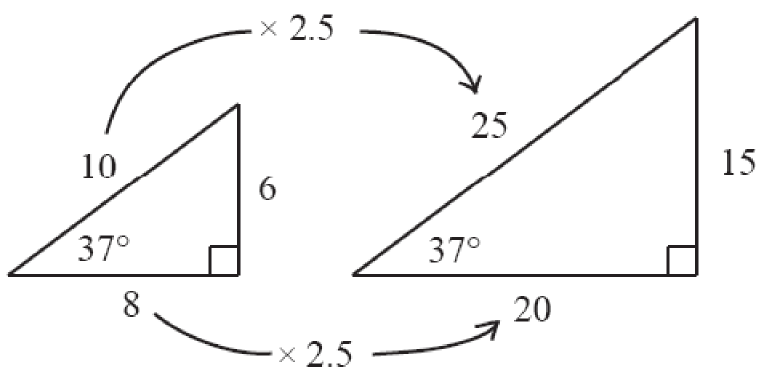
We have to take care to line up the corresponding sides correctly. The angles with the same measure correspond, thus the sides between them correspond. For example, in the above figures, the sides between the 35° and 50° correspond (and happen to be the longest sides of each triangle). Similarly, the right side of the first triangle and the left side of the second correspond, because these sides are between the corresponding 95° and 50° angles.

Once we know that two triangles are similar, it is only a matter of calculating the scale factor. For example, look at the following triangles:



We know that the angles of a triangle add up to 180°. The first triangle has angles of 37°, 90°, and 180° - 37° - 90° = 53°. The second triangle has the same three angles (for the same reason), and is thus similar to the first. Because the angles of any triangle add up to 180°, if triangles have two angles in common like this, then they are similar.

Next, we see that the side of length 6 in the first triangle corresponds with the side of length 15 in the second. We can use this information to calculate the scale factor which is used to multiply the sides of the first triangle to get those of the second. We start with 6 × ? = 15 and then work with the corresponding division problem: 15 ÷ 6 = ? A quick long division will conclude that the scale factor is 2.5. This means that the other two sides of the larger triangle can be found by multiplying 2.5 with the lengths of the corresponding sides:

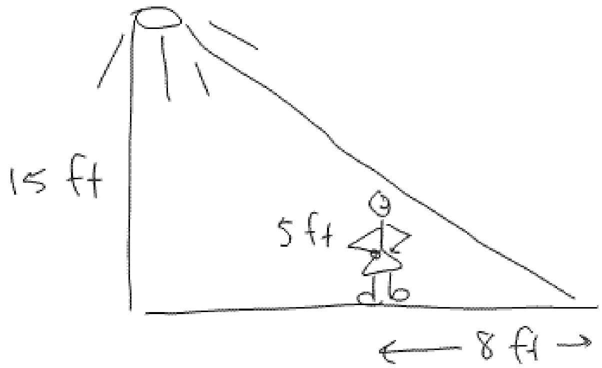


With a pair of similar triangles, the lengths of corresponding sides all form the same ratio. Using the above example, we have 25 : 10 is equivalent to 15 : 6 and 20 : 8. When these are written as fractions, they will all reduce to the scale factor of 2.5.

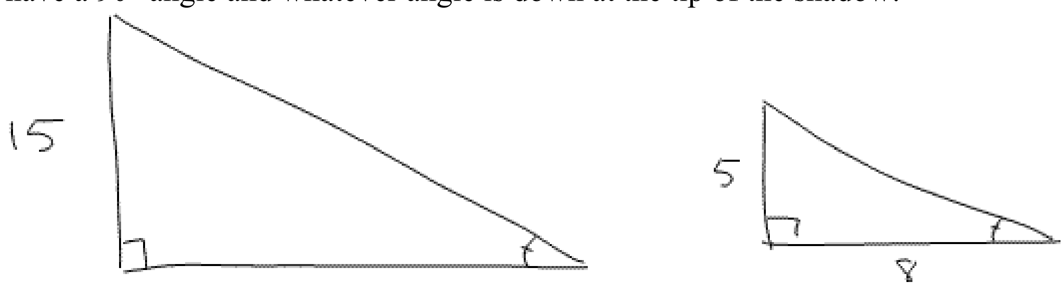
Furthermore, any two sides of one triangle will form the same ratio as the corresponding sides of the second triangle. If we take the height and base of the first triangle above, for example, we get the ratio 6 : 8. The corresponding height and base of the second triangle form an equivalent ratio of 15 : 20. Similarly, the ratio between the height and the longest side of the two triangles are 6 : 10 and 15 : 25, which are equivalent ratios.

Any of these ratios can be used to calculate lengths on similar triangles.

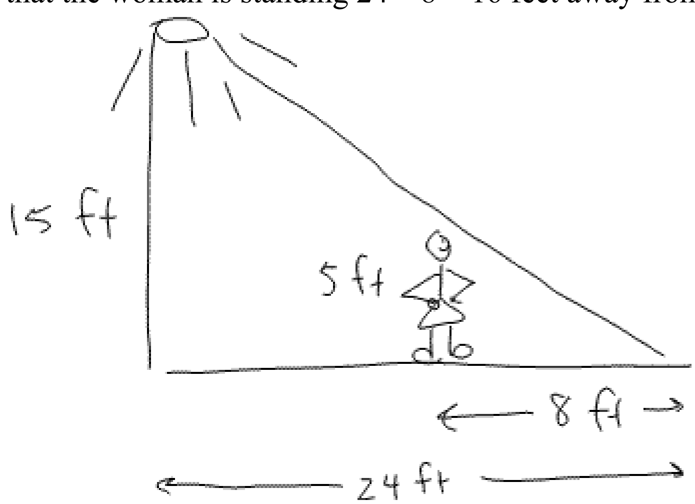
A classic similar-triangles problem runs like this: "A 15-foot street light casts a 8-foot shadow from a 5-foot tall woman. How far from the light is she standing?" It helps to draw the following picture of the situation:



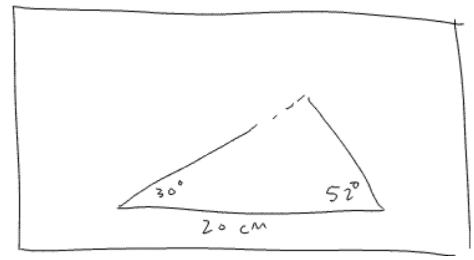
If we suppose that both the woman and the light are standing straight up and down at 90° angles, then we can break the picture up into two similar triangles. This is because both triangles have a 90° angle and whatever angle is down at the tip of the shadow:



The side of length 5 on the smaller triangle corresponds with the length of 15 on the bigger triangle, and thus the scale factor must be 15 ÷ 5 = 3. We can find the base of the larger triangle by multiplying 8 × 3 = 24. If we go back to the original picture, we can thus conclude that the woman is standing 24 – 8 = 16 feet away from the street light:



You can have your students figure out the properties of similar triangles with protractors, rulers, and paper. Break the class up into four groups. Have one group draw a 10-centimeter line across the bottom of a page. Another group can draw a 15-centimeter line on their papers. The other two groups can draw 20 and 25-centimeter lines. Next, each student draws a 30° angle from one end of the line and a 52° angle from the other. These are then extended until they connect, forming a triangle. As in the last chapter, you can quiz your students to see if they can guess the measure of the third angle (180° - 30° - 52° = 98°).



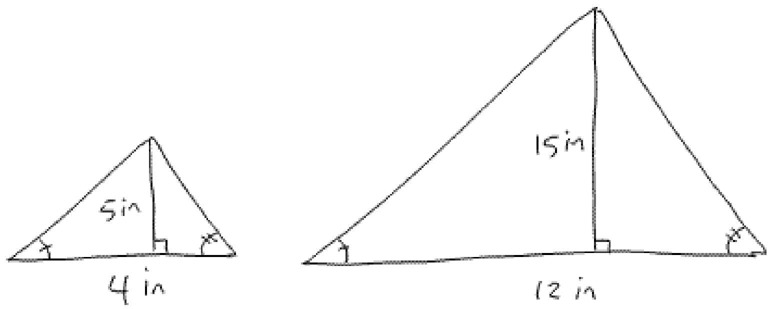
By doing this, all of the triangles will be similar, but of four different sizes. Next, have each student measure the other two lengths of the triangle (to the nearest half-centimeter), then report all these lengths up to the front of the room. Have each group report the lengths of their (ideally) identical triangles as a team, to help eliminate measuring errors. The measurements ought to come out roughly as follows:

longest side	medium side	shortest side
10 cm	8 cm	5 cm
15 cm	12 cm	7.5 cm
20 cm	16 cm	10 cm
25 cm	20 cm	12.5 cm

You can then discuss the data as a class, looking for patterns. Hopefully someone notices that all three sides of the 20 cm triangle are double those of the 10 cm triangle. If someone hypothesizes that a new triangle, drawn with a 5cm side and the same 30° and 52° angles, will have exactly half the lengths, then let them make one and try. Similarly, find some large paper for anyone who wants to try to start with a 30 cm line. Ideally, the patterns will gradually be discovered. We can either divide one side by another for each triangle and get the same ratio (for example, the shortest side of each is half the length of the largest). We could also compare the lengths of one side to another. For example, we could compare the longest and shortest sides of the smallest two triangles. The longest side of the bigger triangle is 15 cm, which is 1.5 times longer than the 10 cm longest side of the smallest triangle. Similarly, the shortest side of the larger triangle, 7.5 cm, is also 1.5 times the size of the 5 cm smallest triangle.

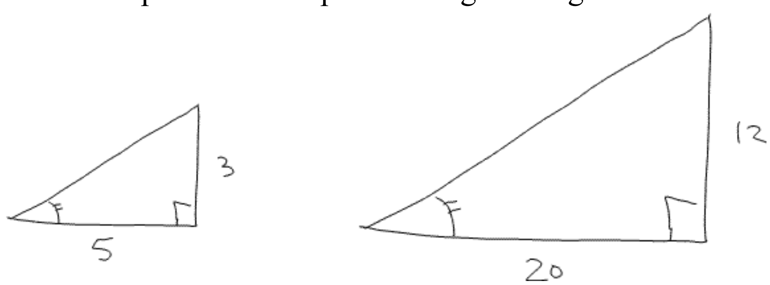
This, of course, leads to a discussion of similar triangles. From here, you can point out all the ratios and proportions that your students have overlooked, and go on to explain the definition, properties, and uses of similar triangles.

One interesting aspect of similar triangles we have not yet covered is how the area of similar triangles relate. If our scale factor is 3, for example, then both the base and height will be multiplied by 3, as in the example shown below:



The triangle on the left has an area of $\frac{1}{2}\times 4\times 5 = 10$ in^2. The triangle on the right has an area of $\frac{1}{2}\times 12\times 15 = 90$ in^2. This area is 9 times bigger. It is no coincidence that 9 = 3 × 3, because our scale factor increased the area of the triangle in two different dimensions.

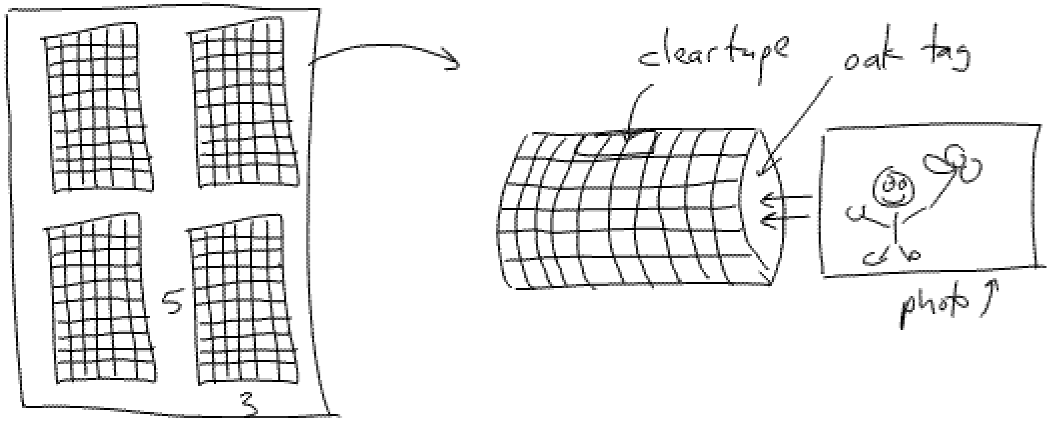
It is worth computing the areas of some similar triangles, to see if your students can detect this pattern. It helps to use right triangles because then the height can be one of the sides:



In this example, the scale factor is 4, which means that the area of the second triangle is 4 × 4 = 16 times bigger.

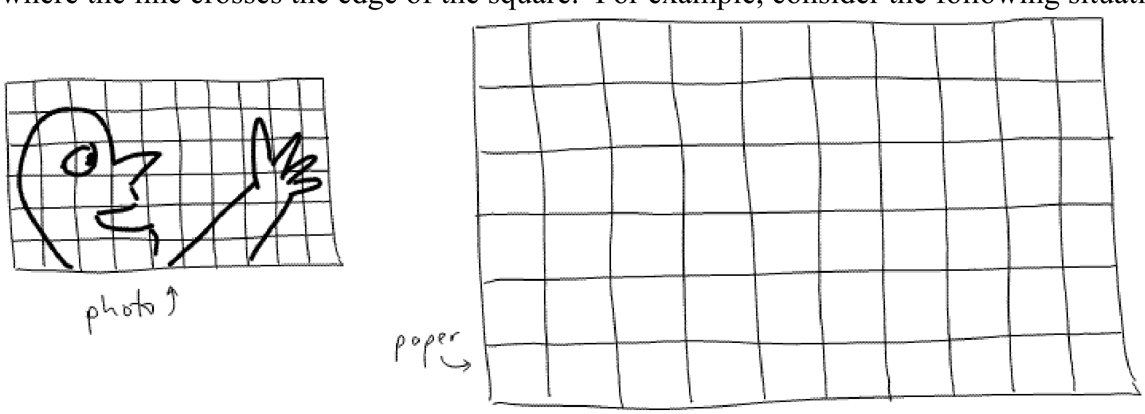
A fun and educational exercise for a class of children is to teach them the grid method for copying a picture or photograph. There are differing levels of set-up which can be put into this project. For the most extensive (and fun), each child should be encouraged to bring in a 3 × 5 photograph that they would like to copy. To avoid all the hassles of lost and forgotten photos, you could provide photos instead or in addition to these and let the class choose them. For a less fun (but far easier) project, you can photocopy all of the necessary materials and skip all student choice and selection. Instead of photos, you can have students pick images from magazines. However, it will help greatly if all the images used are the same size.

You should prepare in advance with a number of transparency sheets that are 3 × 5 in dimension and have lines drawn horizontally and vertically every half inch. If you carefully draw four of these onto a sheet of paper, you can photocopy it onto transparency sheets and then cut them out. To be extra clever, you can tape the top and bottom to a piece of oak tag the same size. Each student should get one of these and slide their photograph in behind the sheet:



Each student should also be provided with a piece of paper with a 6-inch by 10-inch grid, divided up with lines every 1-inch. These can also be photocopied easily onto paper. Of course, without a copier, you can have your students measure half-inch marks on their photos and draw lines, then draw 1-inch marks on a piece of paper and draw lines, but this could take a very long time.

At this point, have your students copy the photograph onto the piece of paper. They begin with the square in the upper-left corner of the photo and copy just that square into the corresponding square on the paper. The key is to look for lines in each square, keeping track of where the line crosses the edge of the square. For example, consider the following situation:



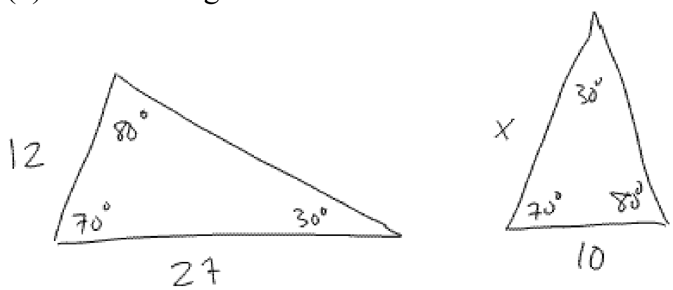
The top row has just about nothing to draw, except maybe some lines along the very bottom of the second and third squares. The first square of the second row, however, has a slightly-curved line that runs from the middle of the bottom up to about a third-of-the-way down from the right-hand edge. Thus, we draw a line as similar as possible in the first square of the second row from the top of the big piece of paper.

In this fashion, the task of copying a picture can be broken down into copying a large number of small, simple squares. Not only does this make a complicated job easier, but it also guarantees to preserve the shape, perspective, and proportions of the original image. It will take some time and patience to complete the picture, but artwork always takes time and patience. If your students want to complete the picture in the style of the great artists, they should then color in the picture (and thereby try to cover up the grid lines).

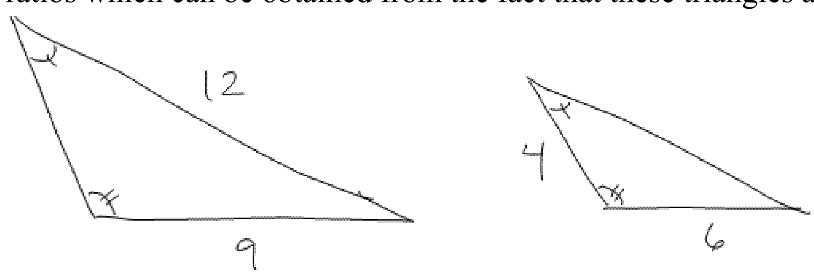
Hopefully, your students will be able to appreciate the fact that this exercise demonstrates the scaling and proportion concepts which are key to the idea of similarity.

Questions:

(1) Find the length of the side marked *x* in the following figure:



(2) Find all the missing lengths of the following triangles. Next, name all the sets of equivalent ratios which can be obtained from the fact that these triangles are similar.



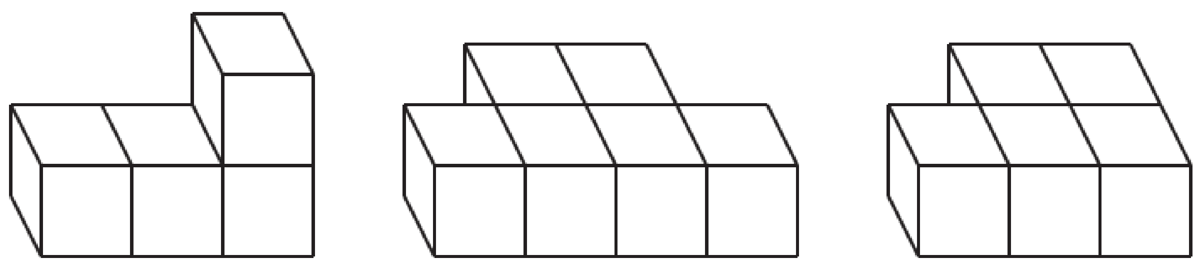
(3) Two triangles are similar. One triangle has 25 times the area of the first triangle. The base of the small triangle is 9 cm long. What is the length of the corresponding base of the big triangle?

Chapter 45: Volumes

It can be fun to introduce students to the computation of volumes. It will work best if we have enough supplies on hand for all the kids, but the supplies are all quite simple: cube blocks, cans, rice, paper, tape, rulers, protractors, and scissors.

The best way to begin the study of volumes is with a very large quantity of solid cubes. The wooden alphabet blocks that are frequently found in playrooms will work excellently – the more, the merrier. Try to make sure that all the cubes are the same size, however, so separate them by size and just use the size of which you have the most blocks.

Explain to the class that *volume* is the amount of space that a solid object takes up. Just as we used squares to measure the area, we use *cubes* to measure volume. To reinforce this idea, you can build several small objects with some of the blocks, then challenge the class to put them in order from largest volume to smallest volume. As a very simple example, consider the following three arrangements of blocks:

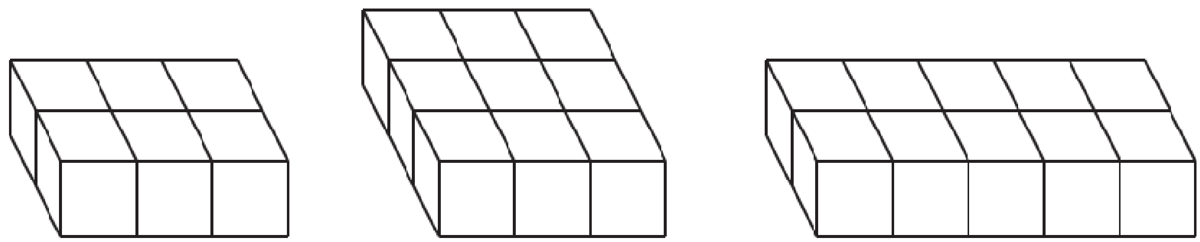


The middle arrangement is the largest, with a volume of 6 cubes, the last arrangement has a volume of 5 blocks, and the first arrangement is the smallest, with a volume of 4 blocks.

It should not take too many exercises like these before your class is easily able to count the volume of a building made of blocks. If they really want to play, you can divide the class up into groups and give each one a set of 20 blocks. Tell each group to hide some number of their blocks, and then use the rest to build a structure. When everything is built, each kid takes a piece of paper and then walks around the room, looking at each structure and trying to guess its volume. When everyone has written down a guess for each structure, then the class sits down. Each group, in turn, then explains the volume of their structure. When the kids walk around, they are not allowed to touch the buildings. However, for the final demonstration, the groups can take apart the buildings to make the number of blocks clear. If you want to make the game competative, you can have each team score a point for every student in the class (other than themselves) who guesses wrong on their volume. This will encourage the groups to be creative and tricky in putting together their buildings.

It should not take too much time with this game for your students to understand that volumes are measured in cubes. If they really enjoy the game, you can do it as a warm-up at the beginning of a lesson on volume.

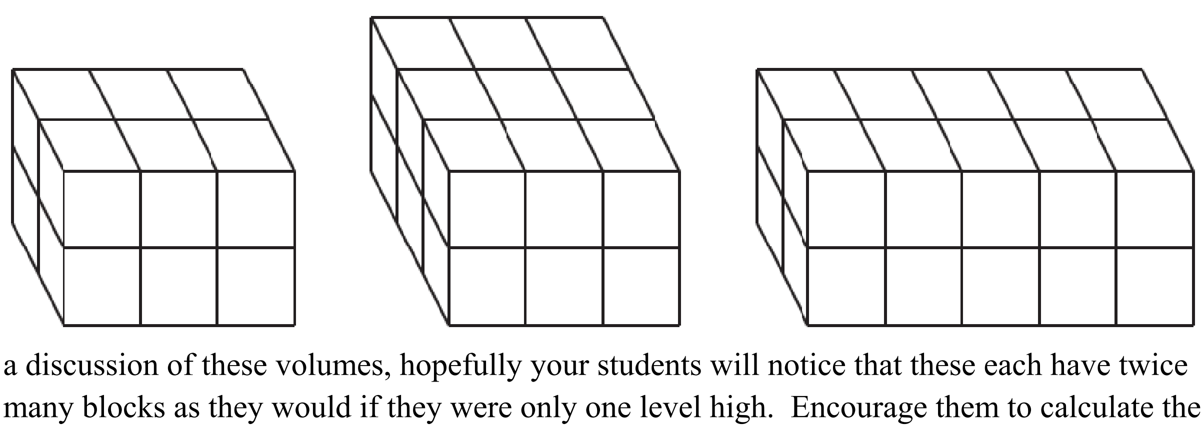
When the students have this basic grasp on volume, we can move on to calculate the volumes of boxes. To begin with, have the students build a one-level arrangement of blocks in the shape of a rectangle. This can be done individually, in small groups, or as a class – largely dictated by the amount of blocks and table surface available. Some of the possible arrangements might be:



Have your students discuss the volume of each structure. Hopefully, at least one student will notice that the blocks need not be counted individually, but that it is enough to multiply the length and the width, just as we would to find the area of the rectangle. For example, the first arrangement above has 2 × 3 = 6 blocks in volume, the second has 3 × 3 = 9 blocks of volume, and the third has 2 × 5 = 10 blocks in volume. If no student points this out so clearly, then make a really big rectangle (something like a 6 by 9) and challenge the students to see who can find the volume first. It is very likely that the first student will have multiplied to find the answer. Ask the student to explain his or her method, and use leading questions until the answer comes out.

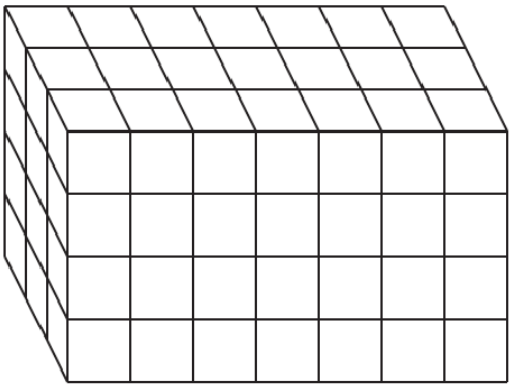
You could play the small-groups game again, and have each group build a big rectangle of blocks. This time, however, you might find that every single student guesses all the volumes correctly. There are no opportunities for hiding blocks and leaving gaps when the arrangements have to be solid rectangles like the above. While the game is less fun, it should indicate that your students are well along the path to understanding volume!

Next, have students build arrangements of blocks that are rectangles when viewed from the top, but two levels high. For example:



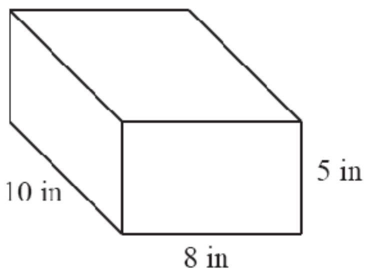
In a discussion of these volumes, hopefully your students will notice that these each have twice as many blocks as they would if they were only one level high. Encourage them to calculate the volume of the first as 2 × 3 × 2 = 12 cubes, the area of the second as 3 × 3 × 2 = 18 cubes, etc.

The next step is to encourage the class to build structures like these, but with any number of levels. This is where we are likely to run out of blocks, so it will probably need to be done by only 2 or 3 groups of children. If you are very limited with blocks, perhaps you will have to make one big structure for all the class to see, for example:



Hopefully, your students will be able to work out that the volume of the overall structure is $3 \times 4 \times 7 = 84$ blocks. Ideally, they will go straight to multiplication, but they might begin by calculating that each layer uses $3 \times 7 = 21$ blocks and then multiply this by 4. If they got really good at the two-layer stacks, they might notice that this is two of them on top of one another.

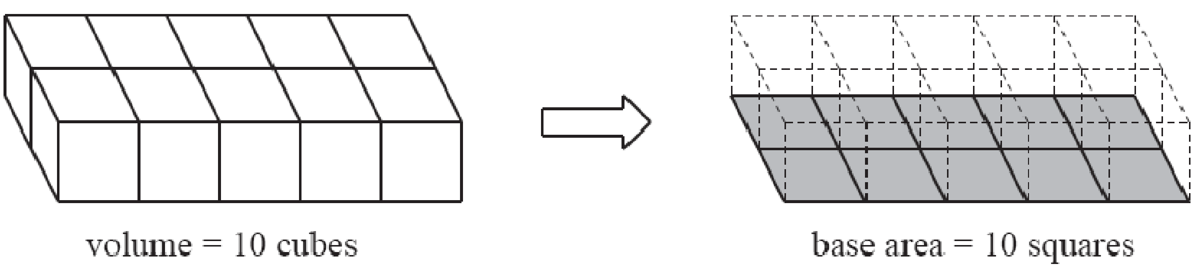
In any case, the next level of abstraction is to introduce some boxes on paper where the lengths are written out, but the individual blocks are not illustrated. For example:



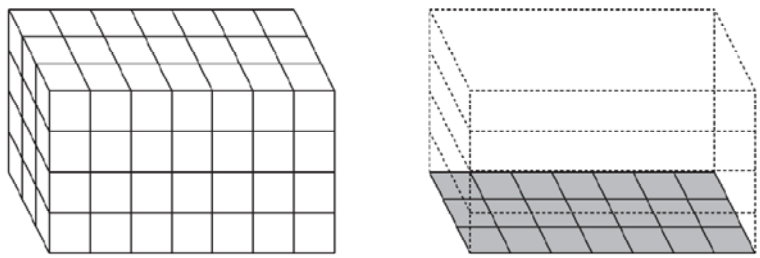
The volume of this block can be found by multiplying $10 \times 8 \times 5$ to get 400 cubes. If we include the units, then we get $10\,in \times 8\,in \times 5\,in = 400\,in \times in \times in = 400\,in^3$. A block that measures 1 inch on each side is called a *cubic inch*, thus this box has a volume of 400 *cubic inches*.

It might take a little longer to run through these exercises than to merely tell the class that the volume formula for a box is: *Volume* = *length* × *width* × *height*. However, it will benefit your class greatly for them to see the origins of this formula on their own.

It will help to explain to the class that an arrangement with one layer will have the same number for its bottom area as it does for its volume. This is because each square of area on its base will correspond with a cube of volume:

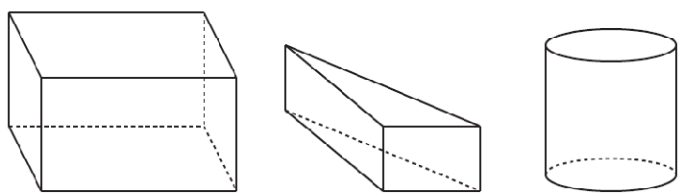


Rather than look at the volume of a box as *Volume* = *length* × *width* × *height*, we could instead look at it as *Volume* = *Area of base* × *height*. In fact, because the base of a box is a rectangle, the *Area of base* = *length* × *width* anyway. For example, returning to an earlier example:



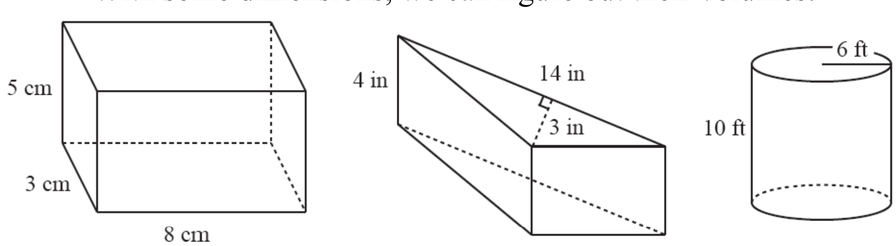
The base of this box is a rectangle that measures 3 by 7, thus has an area of 21 squares. The box has a height of 4, and thus the volume is 21 × 4 = 84 cubes.

The advantage of this method is that it works for all prisms. A *prism* is a solid object that starts with a flat base and is made of identical layers like this. For example:



The first object above has a top and a bottom that are both rectangles. The technical term for this shape is a *rectangular prism*, but most people call them *boxes*. The second object above has identical triangles for its top and bottom. This is called a *triangular prism*. The last shape has a circle for its top and bottom (they don't look like circles only because they are drawn as if viewed from the side). This ought to be called a *circular prism*, but instead is called a *cylinder*.

With some dimensions, we can figure out their volumes:



The formula for all three is: *Volume* = *Area of base* × *height*.

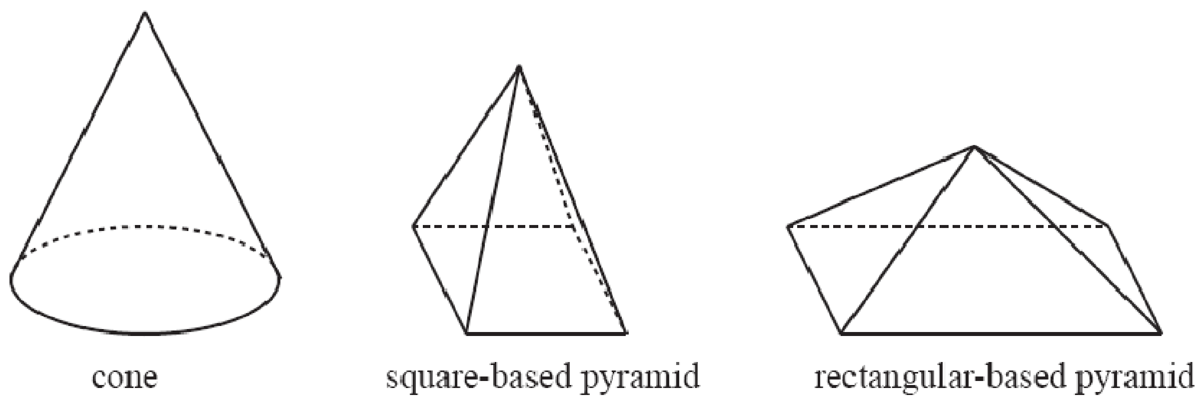
For the box, the area of the base is $3\,cm \times 8\,cm = 24\,cm^2$ and the height is 5 *cm*, so the volume is $24\,cm^2 \times 5\,cm = 120\,cm^3$.

For the triangular prism, the base is a triangle with a height of 3 *in* and a base of 14 *in* (you might have to turn the paper to see this), thus the area of the base is $\frac{1}{2} \times 3\,in \times 14\,in = 21\,in^2$. Because the prism is 4 inches tall, we can imagine that it is made of 4 layers, each with 21 cubic inches of volume, for a total volume of $21\,in^2 \times 4\,in = 84\,in^3$.

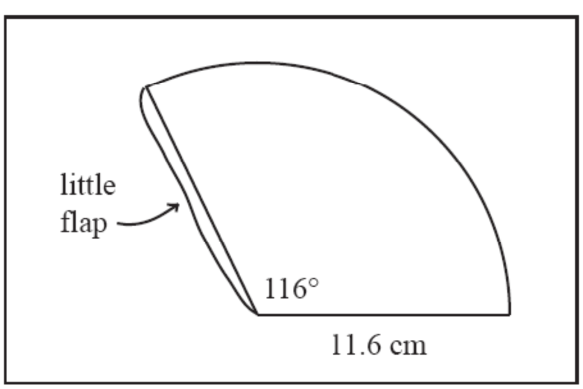
The base of the cylinder is a circle with radius 6 *ft*, thus the area is *Area of a circle* = $\pi \times$ *radius* $\times$ *radius* = $\pi \times 6\ ft \times 6\ ft = 36 \times \pi\ ft^2 \approx 36 \times 3.14\ ft^2 = 113.04\ ft^2$. The volume of the cylinder is thus *Volume of prism* = *Area of base* $\times$ *height* $\approx 113.04\ ft^2 \times 10\ ft = 1{,}130.4\ ft^3$.

Many books use the formula: *Volume of cylinder* = $\pi \times$ *radius* $\times$ *radius* $\times$ *height*. However, it is better to teach your children the more general formula for the volume of a prism instead (or in addition). This requires less memorization, can be used in lots of different situations, and makes some sense, especially if you start with building blocks of various layers.

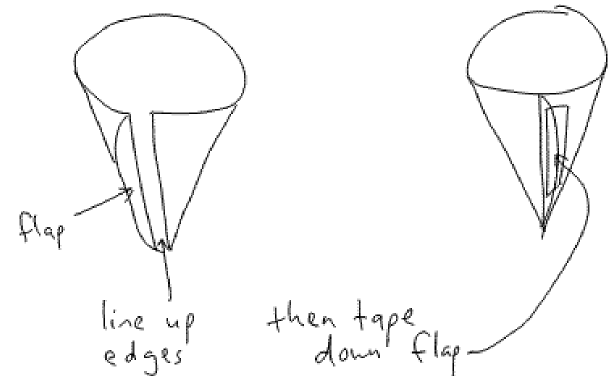
The next sort of volumes we can calculate are those which start from a flat base and then taper up to a point. Mathematicians call all of these *cones*, but most people use that term only for the ones with circular bases. When the base is a square or a rectangle, they are often called *pyramids*:



To explore shapes like these, we will compare them to prisms of the same height and base. The easiest way to do this is to begin comparing cones and cylinders. If you start with an empty 15-ounce can (a standard size for canned vegetables), it will be about 11 *cm* tall and 7.5 *cm* in diameter. In order to make a cone with the same dimensions, start by drawing a line 11.6 *cm* along the bottom of a piece of paper (oak tag would be even better). Next, use a protractor to measure a 116° angle from one end of this line. Finally, use a protractor, ruler-compass, or piece of string to draw a sector with this angle and radius. It wouldn't hurt, also, to add a little flap along one edge:



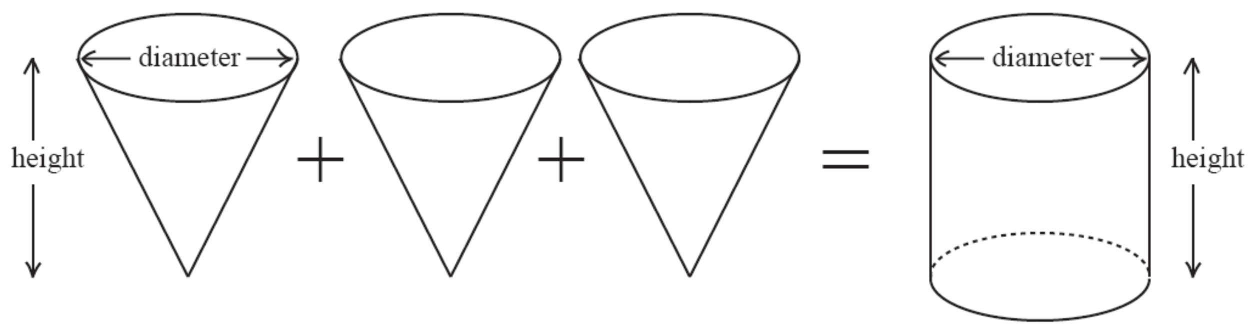
Cut this shape out. Fold along the line that separates the sector from the flap. Roll up the sector into a cone, lining the straight edge up with the fold. Tape the fold down onto the outside of the cone. It is all right to lightly fold the cone flat in order to line up the two edges precisely – you can always squeeze it into a more proper cone shape later.



This requires some precision (and maybe even some practice and failure), so you might want to make up a number of cones in advance, rather than leave the task to your students.

For the activity, break the class up into groups. Each group gets one cone, one empty can, a spoon, a plastic tub, and some sort of small, dry material, for example uncooked rice or oatmeal. One member of the group will hold the cone (trying to make it as rounded as possible), while another member slowly spoons the oatmeal (or whatever) into the cone. If they hold everything over the tub, then clean-up ought to be easy. When the cone is completely full (level across the top and not heaping over), the students should carefully pour all of its contents into the can. The group should then repeat this process as many times as possible (perhaps taking turns with the roles of holding the cone, spooning the oatmeal, and pouring into the can).

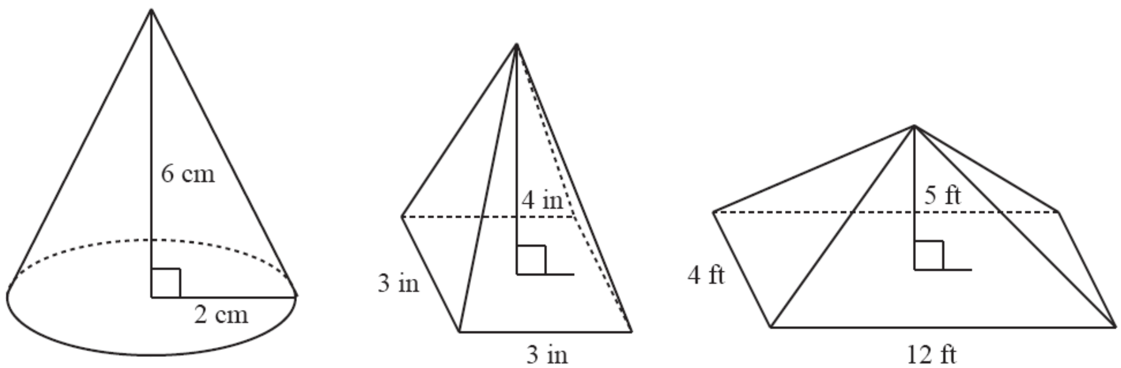
If everything is done correctly (the cones are the right size and filled correctly, nothing is spilled, etc.) then three cones of material will exactly fill the can (if the group is using oatmeal, they might need to pack it down a little). Two cones will certainly fall short and four cones will definitely overflow the can:



We can conclude that the volume of a cone is one-third that of a cylinder with the same height and diameter. Thus *Volume of a cone* = $\frac{1}{3} \times \pi \times$ *radius* $\times$ *radius* $\times$ *height*.

In fact, anything that tapers to a point like this will be exactly one-third the volume of a prism with the same height and base-area. For example, *Volume of a cone* = $\frac{1}{3}$ × *area of circular base* × *height*. Similarly, the volume of a pyramid is *Volume of pyramid* = $\frac{1}{3}$ × *area of base* × *height*.

For example, using the following objects:



The area of the base of the cone is π × 2 *cm* × 2 *cm* ≈ 12.56 cm^2 and the height is 6 *cm*, thus the volume is approximately $\frac{1}{3} \times 12.56\,cm^2 \times 6\,cm = 25.12\,cm^3$.

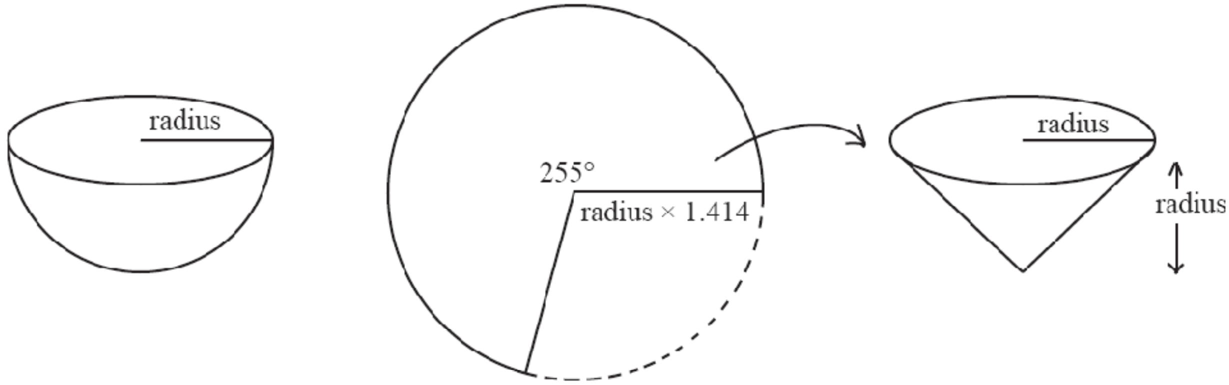
The square-bottomed pyramid has a base area of 3 *in* × 3 *in* = 9 in^2 and a height of 4 *in*, thus it has a volume of exactly $\frac{1}{3} \times 9\,in^2 \times 4\,in = 12\,in^3$.

The rectangular-bottomed pyramid on the right has a base area of 4 *ft* × 12 *ft* = 48 ft^2 and a height of 5 *ft*, so it has a volume of $\frac{1}{3} \times 48\ ft^2 \times 5\,ft = 60\ ft^3$.

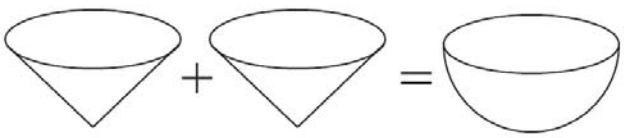
Most books cover these formula separately. However, the formula *Volume of a cone or pyramid* = $\frac{1}{3}$ × *area of base* × *height* is easier to remember and use.

The final volume formula which is often taught to elementary school children is the volume of a sphere. This is a bit more difficult to illustrate, mostly because it is not easy to make or find a sphere which can be filled with material. If you could find a coconut that looked very round, you could cut it in half with a hacksaw and an hour or so of hard labor. You could then scoop out the coconut meat, clean it up, and use that. Unfortunately, coconuts are not precisely spherical. However, your students might have so much fun playing with the shell that they'd forgive whatever slight errors might occur in the process. For a little bit more money than a coconut (but less than $20), plastic hemispheres can be purchased online.

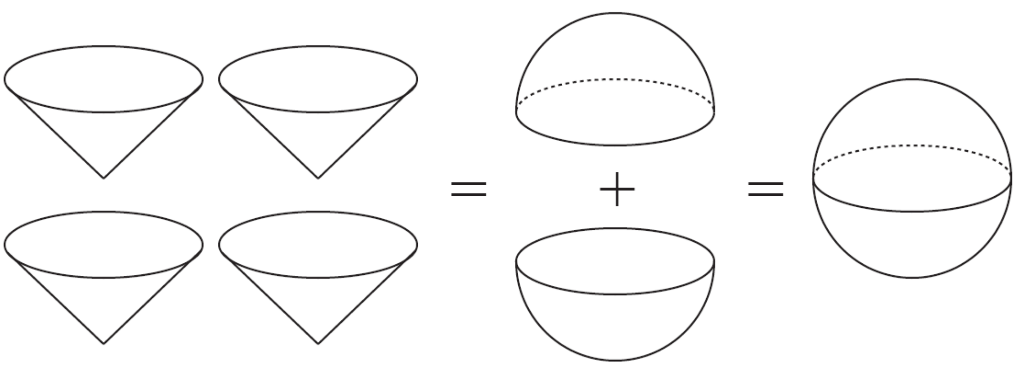
In any case, to make the demonstration you will need to have a hemisphere, but two would be even better. Measure the diameter of the inside of the sphere (this is important, especially if your sphere has a significant thickness, as will be the case with a coconut. Divide this number by 2 to get the radius, and then multiply by 1.414. Draw a circle with this radius on a piece of oak tag. Next, use a protractor to measure a sector of 255° out of this larger circle. If you cut out this and tape it together into a cone, it should have the same height and radius as your hemisphere:



As soon as you have a hemisphere and a cone with the same height and radius, you can conduct the demonstration. Just as with the cone and cylinder, you fill the cone with dry oatmeal or rice, and then pour it into the hemisphere. This time, if everything goes well, two cones will fill the hemisphere exactly:



This means that four cones will fill a full sphere:



The volume of each cone can be calculated using the radius and the height (which is the same as the radius). Each cone has volume $\frac{1}{3} \times \pi \times \mathit{radius} \times \mathit{radius} \times \mathit{radius} = \frac{1}{3} \times \pi \times \mathit{radius}^3$. The volume of the sphere is four times bigger.

We conclude: *Volume of a sphere* $= \dfrac{4}{3} \times \pi \times \mathit{radius} \times \mathit{radius} \times \mathit{radius} = \dfrac{4}{3} \times \pi \times \mathit{radius}^3$.

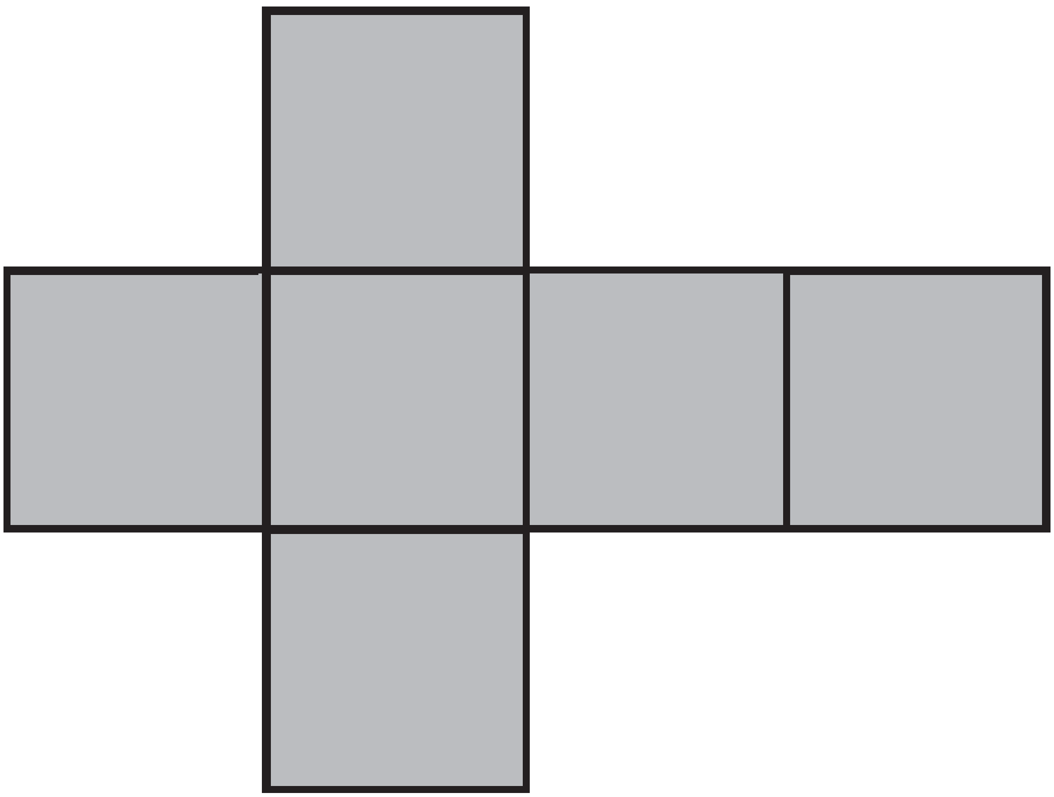
Questions:

(1) Build a cone with the same height and volume as a standard 15-ounce can. Use it to demonstrate the formula for the volume of a cone.

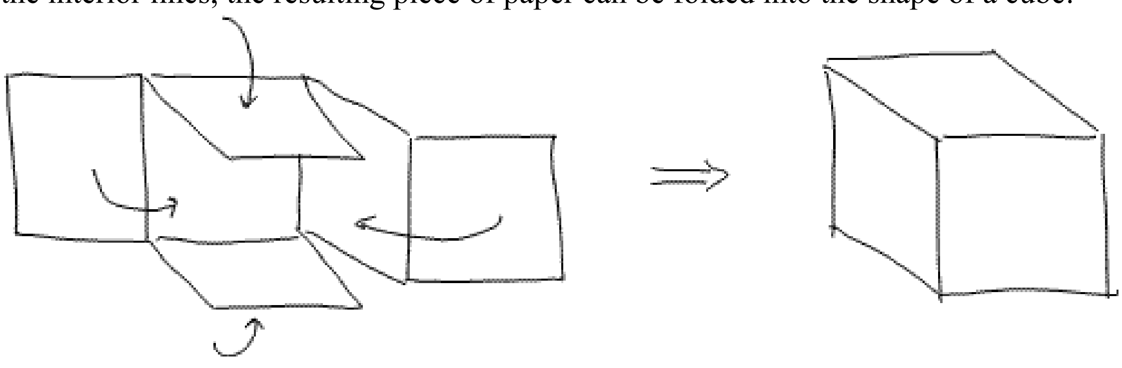
Chapter 46: Planar Nets

The last math exercise for this book is for your students to play with planar nets. This is a fun and empowering way for children to make three-dimensional objects out of paper.

A *planar net* is a flat plan for the surface of a solid object. The most common one is the planar net for a cube, which generally looks like:

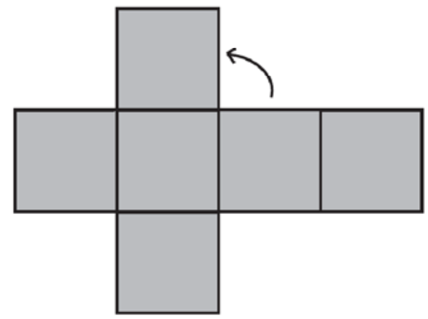


If your students cut this shape out (cutting only along the outer edge) and fold along all the interior lines, the resulting piece of paper can be folded into the shape of a cube:



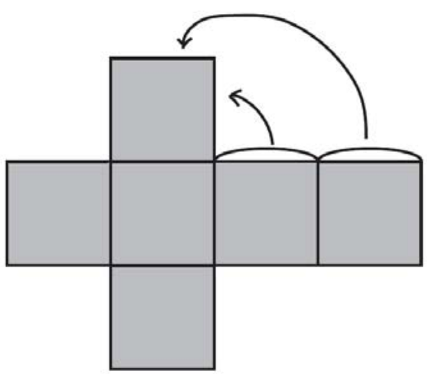
Not only is this fun, but it demonstrates the concept of the *surface area* of a cube. The area on the surface of the cube is the sum of the areas of the six squares of the planar net, the full area of the surface of the cube.

To make this more practical, it helps to put little tabs on the outside edge, to help hold the cube together when it is taped. It is only necessary to have one tab at any point where two edges come together. It is a very useful mental exercise for your students to figure out which edges will come together, before they actually cut out the net. For example, on the net of a cube, it is clear that the two edges indicated will line up when the cube is being formed:

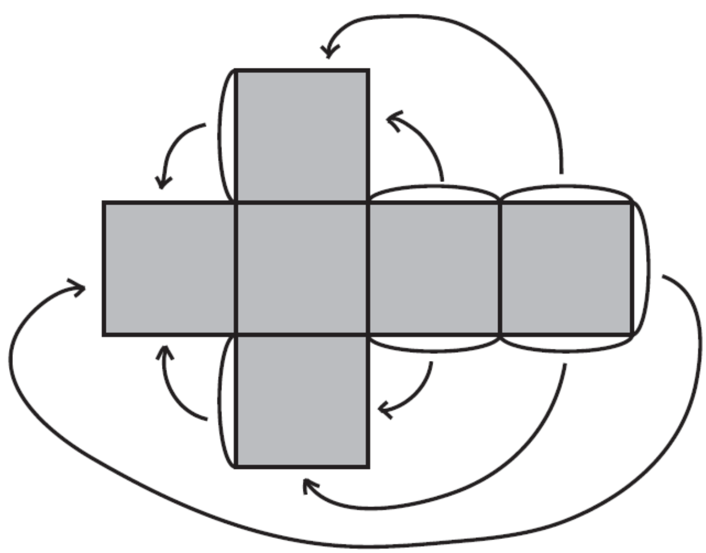


Thus, it will be a good idea to make a little flap or a tab on one of these, to fold down and help to tape the cube together.

When these edges are taped together, the edges next to them will also line up, and thus need a tab as well:

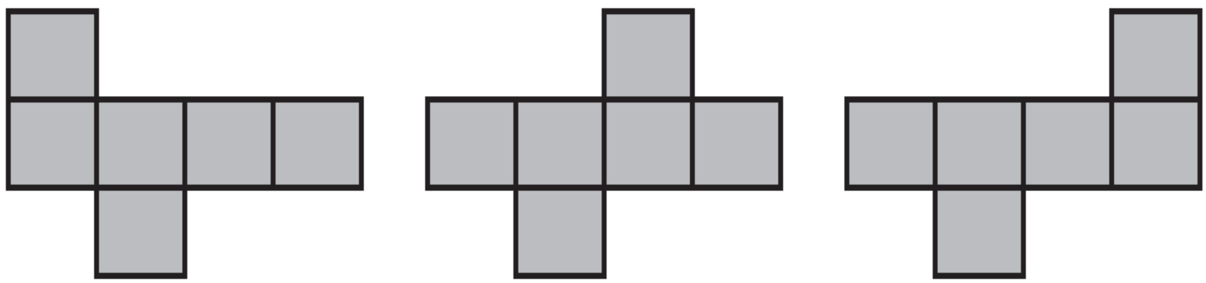


The full picture, lining up all the edges and putting in a tab for each pair might look like. It is all right for students to draw arrows like this on a photocopied planar net, because they will all be on the bits of paper which are cut off and thrown away.

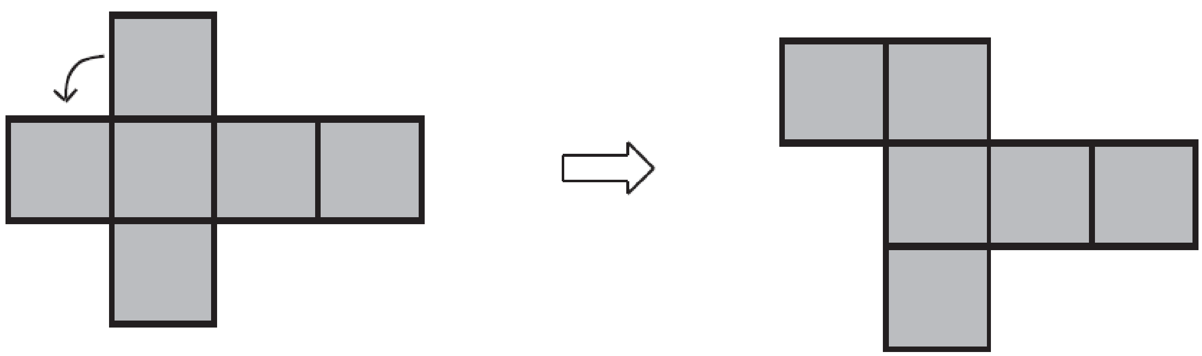


This exercise is good for students, because it forces them to mentally manipulate an object in three-dimensions. They can then cut out the net and then verify that the edges they imagined do line up. These sorts of mental skills are very useful, and should be encouraged.

Also, this sort of mental exercise can be used to make different planar nets which still form the same three-dimensional shape. If we cut the top square completely off of our cube net, for example, and attach it instead along any of the places where one of its edges is attached (as indicated by those arrows), we will obtain another planar net for a cube. The three possibilities are as follows:

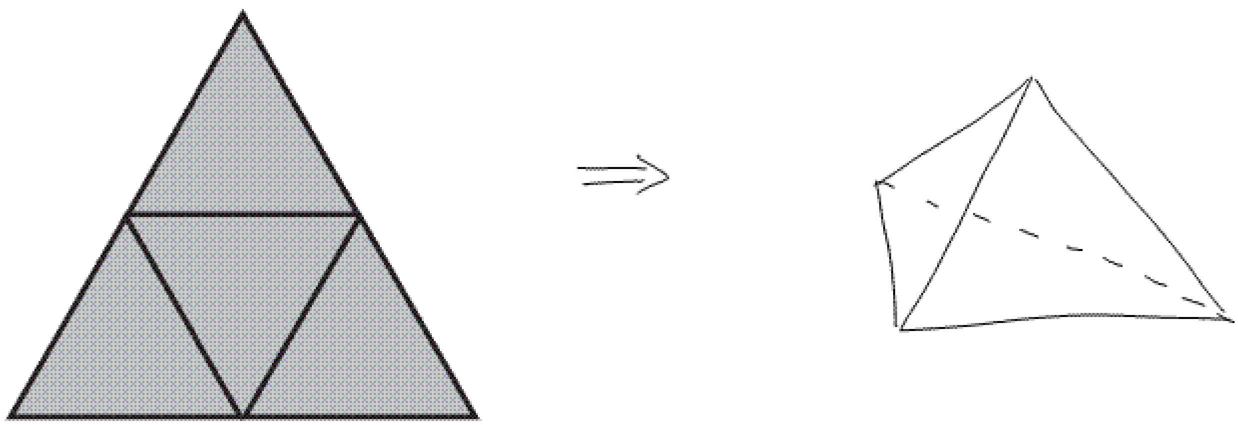


Similarly, if we go back to the first planar net, we could cut off the square on the far left and attach it up one square instead:

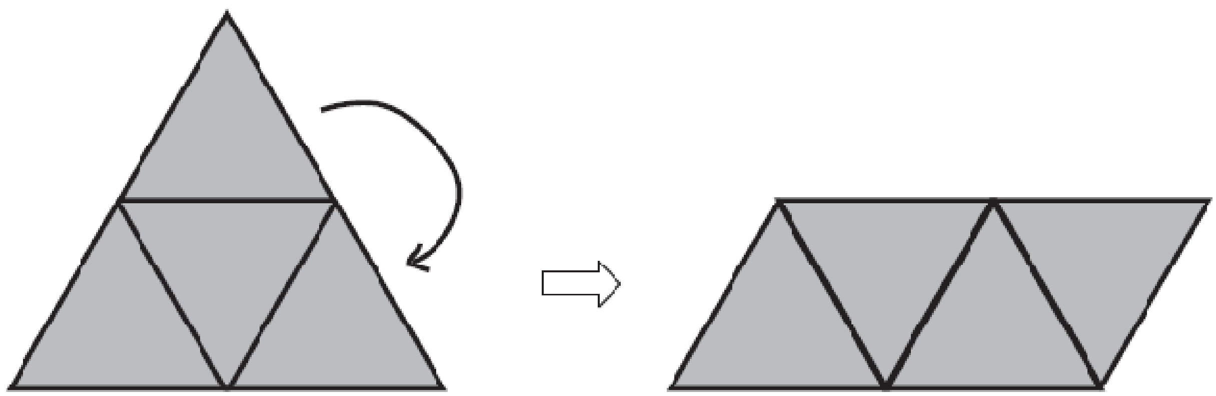


You could challenge your students to find all the possible different planar nets for a cube in this fashion. Be careful, however, for there are many answers!

A planar net for a *tetrahedron*, a pyramid made of four equilateral triangles, looks like:

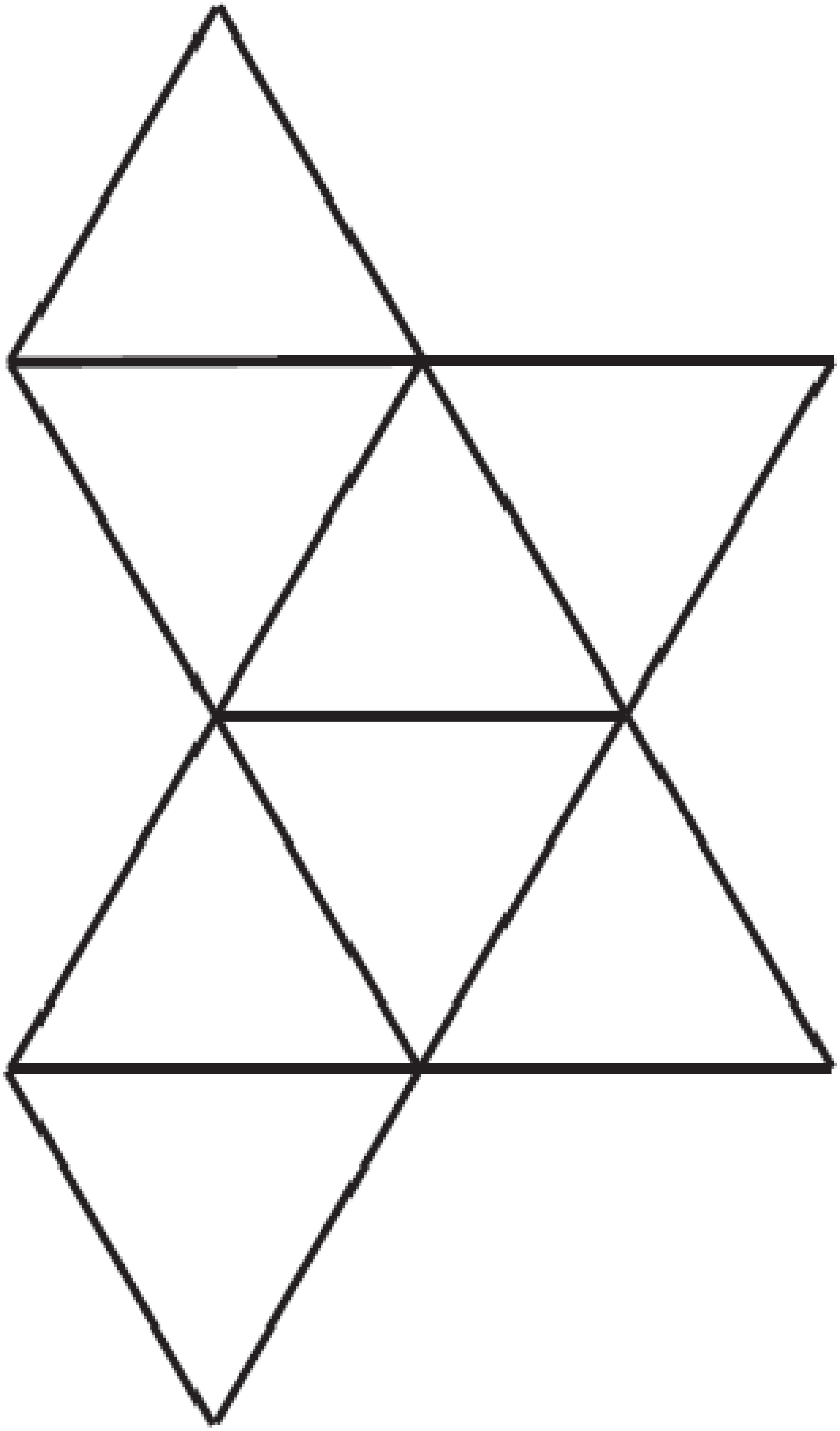


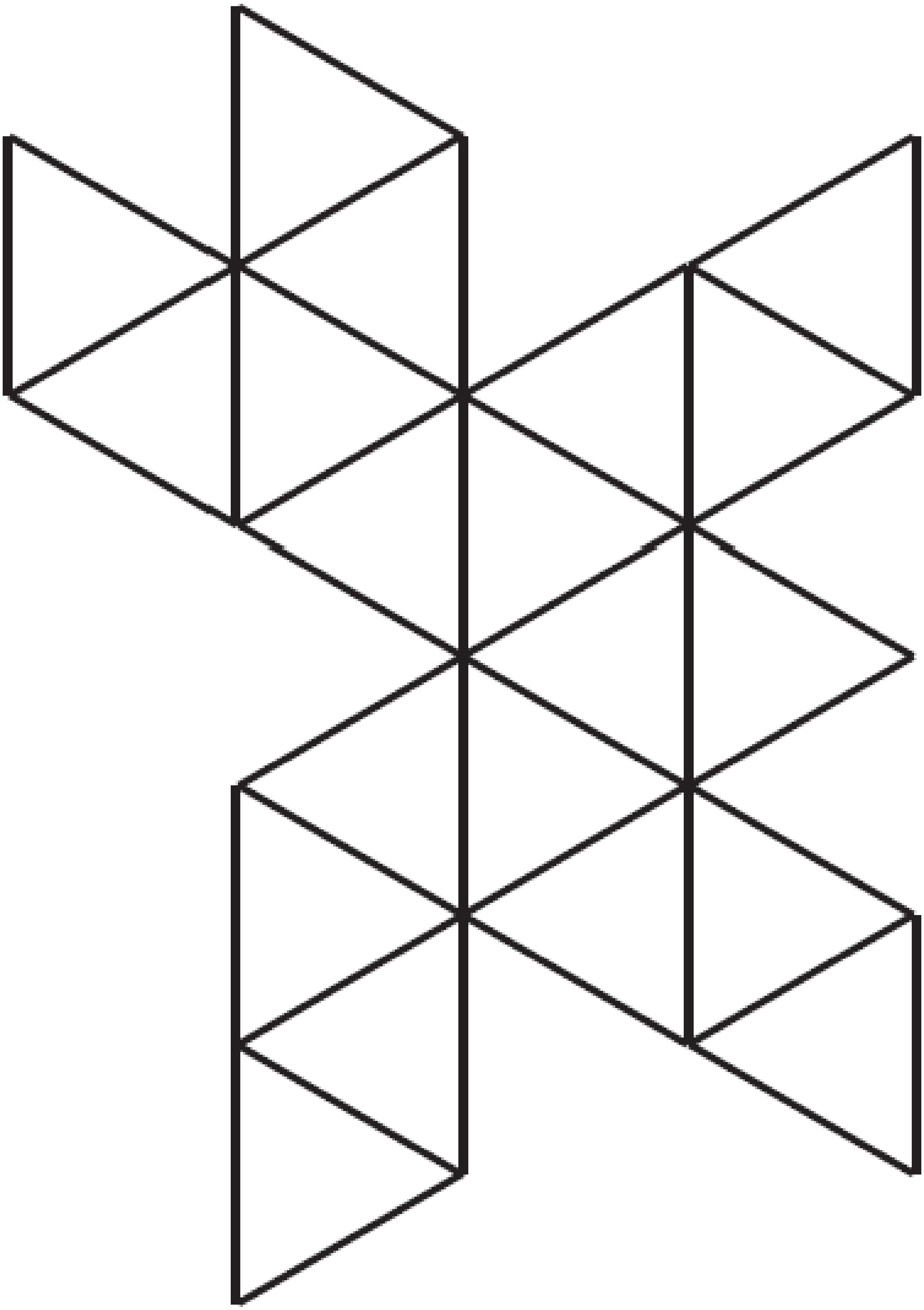
The two edges on the right-hand side of the planar net will be taped together, so we can use this to move the top triangle down and make a different planar net:

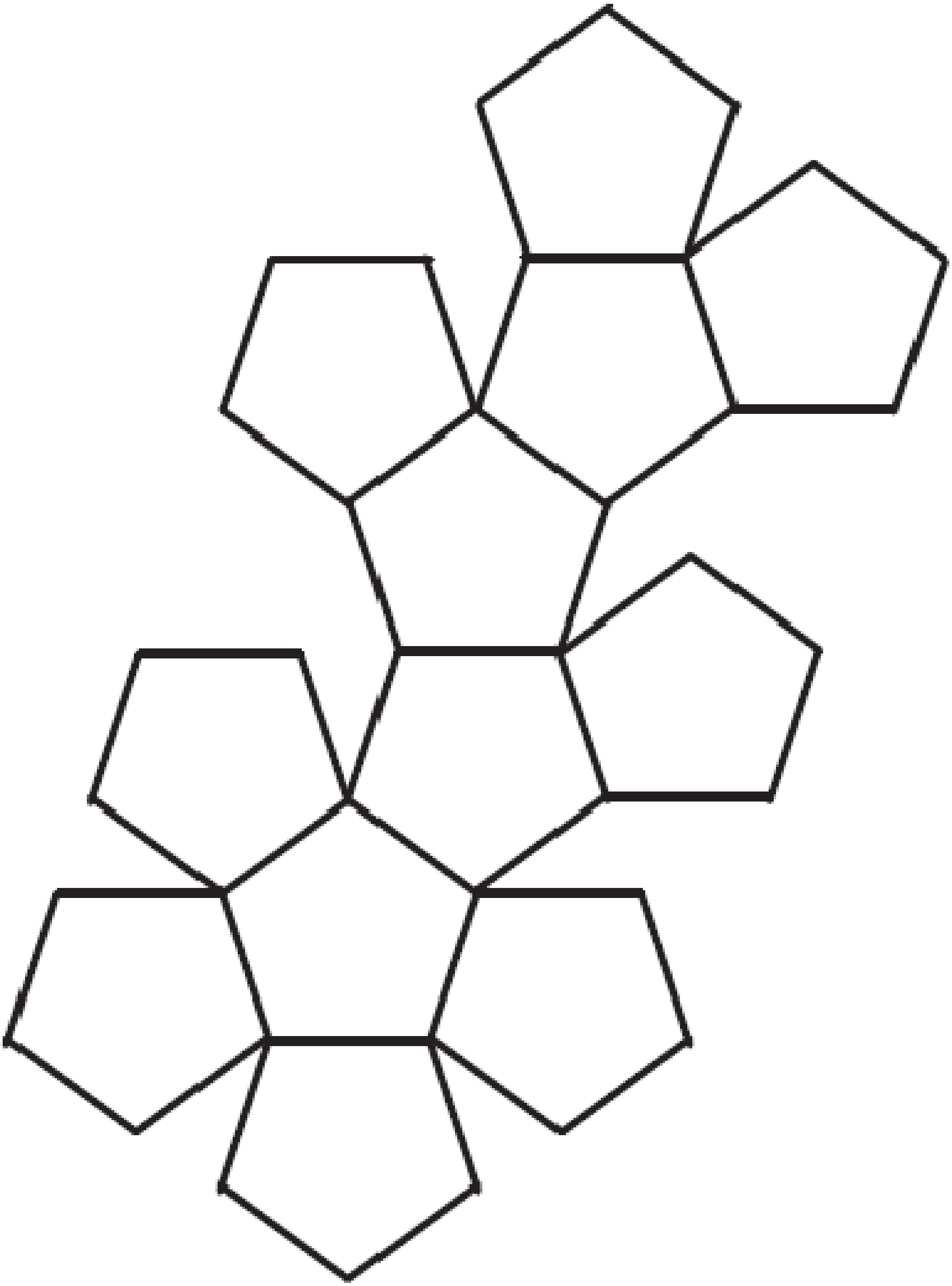


It really isn't all that necessary to worry your students about finding all the different planar nets that make the same shape. This is more an exercise in cutting, folding, taping, and enjoying the beauty of mathematical objects.

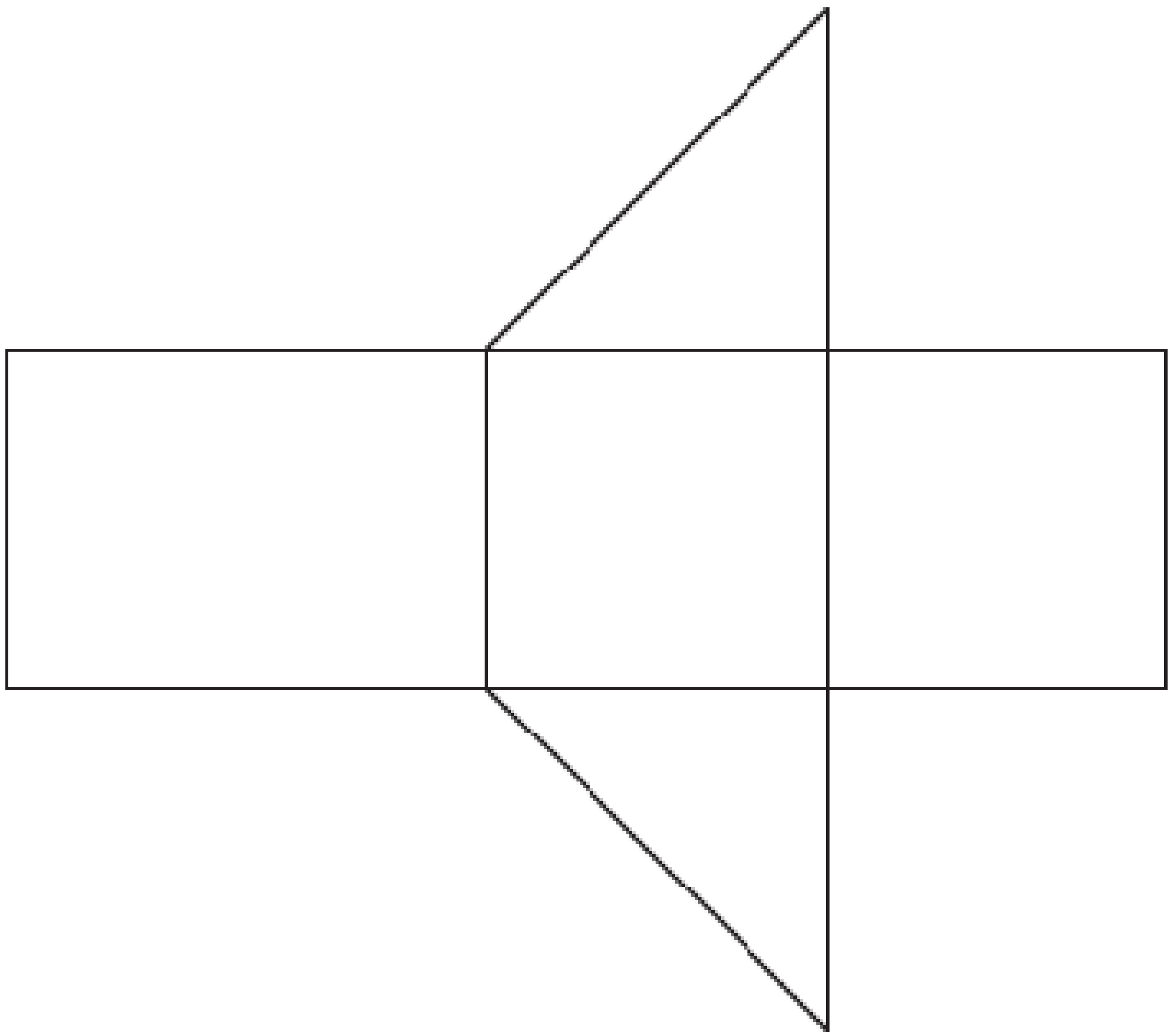
The tetrahedron and the cube are two examples of the *Platonic solids*. These are the three-dimensional shapes whose sides are all regular polygons (squares and equilateral triangles, in these cases) of the same size, whose corners all look the same. The other Platonic solids are the *octahedron* (formed by 8 equilateral triangles), the *icosahedron* (formed by 20 equilateral triangles), and the *dodecadedron* (formed by 12 regular pentagons). Planar nets for these shapes can be found on the next three pages. There are many other ways these nets could be drawn, but these versions each fit nicely on a single page.

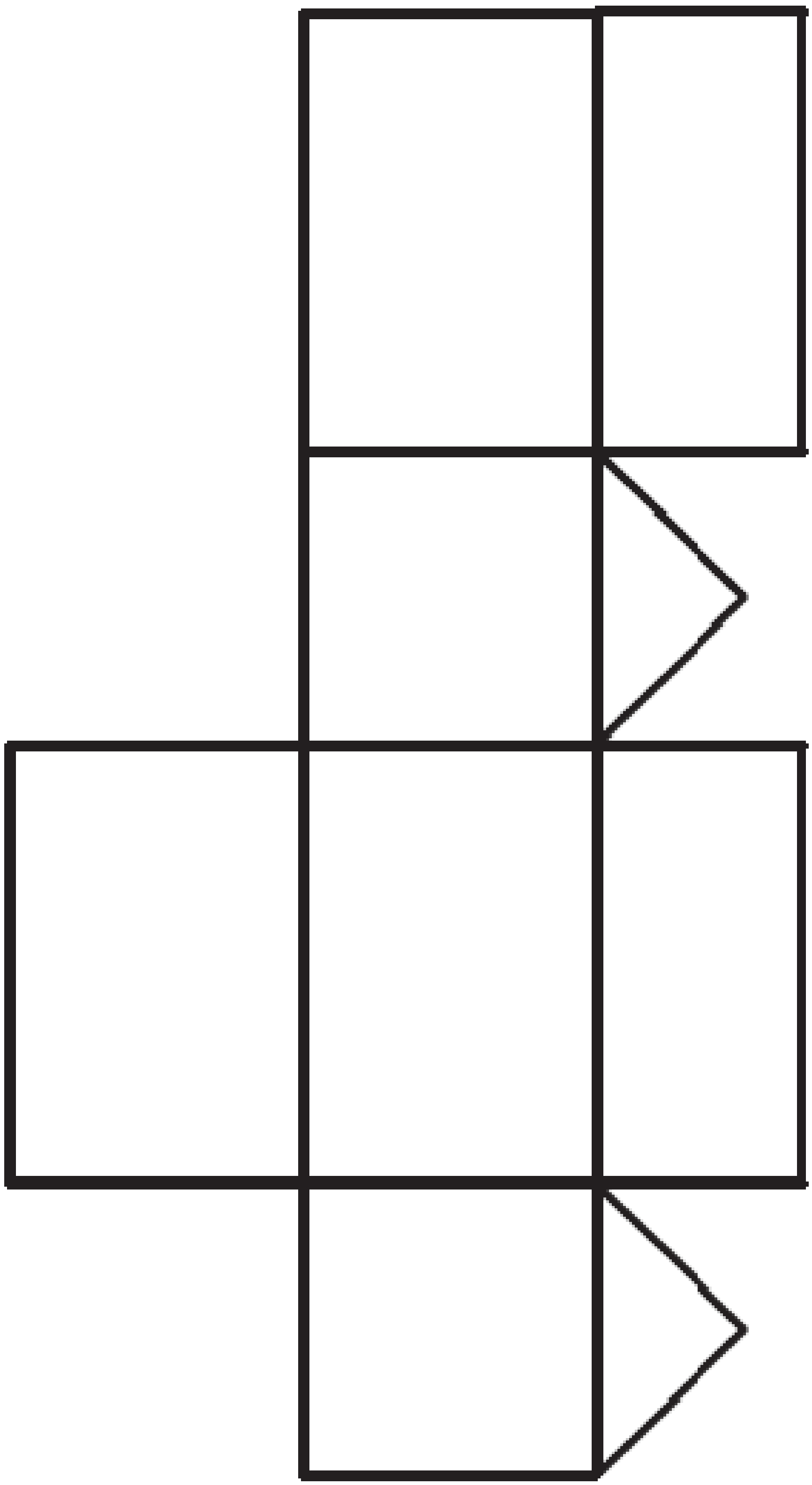


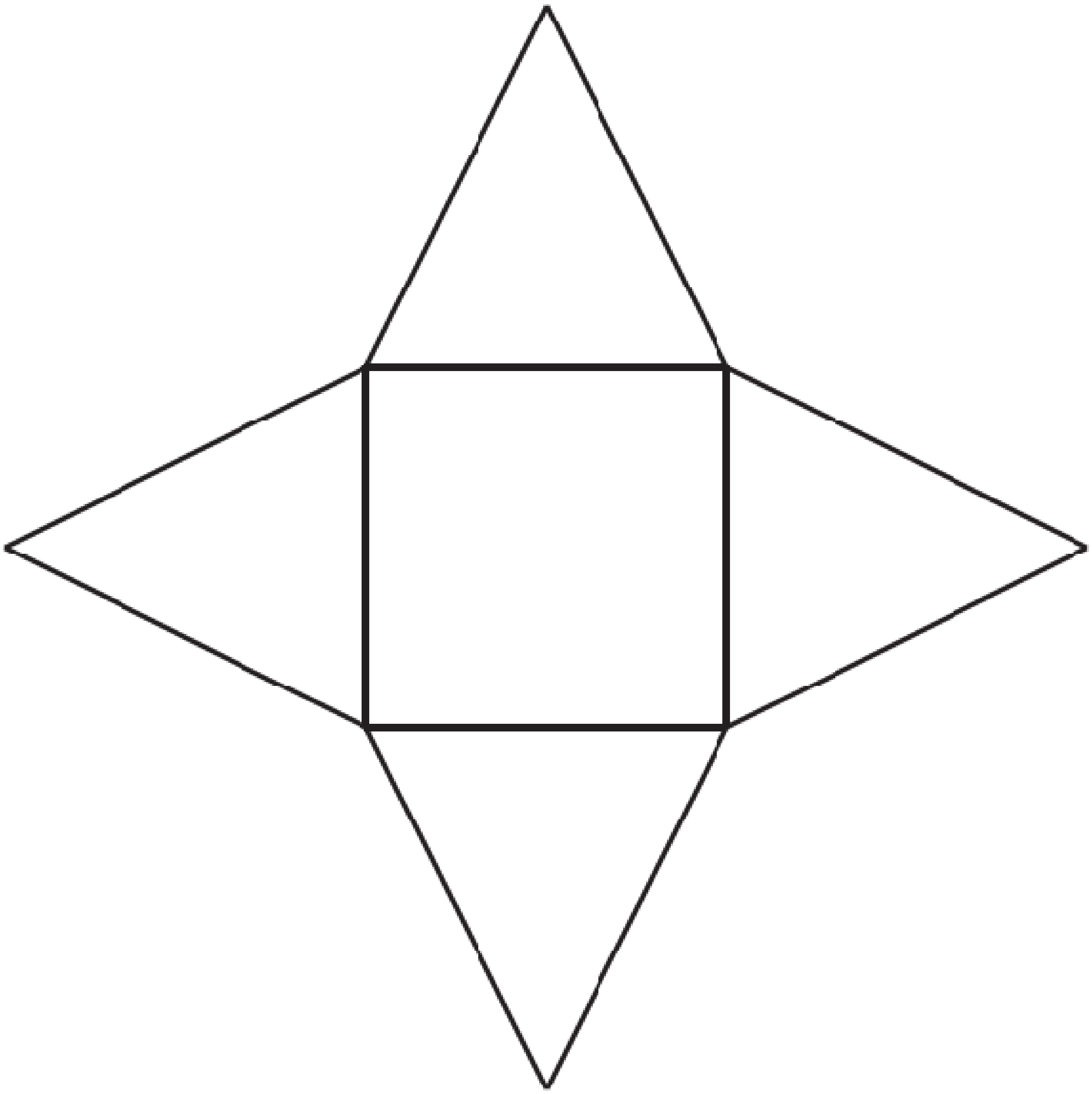




A fun exercise for students would be to give them a new planar net and have them try to guess the shape it will form before they cut it out and put it together. The Platonic solids are quite unusual (except for the cube), and thus will probably be difficult for your students to guess. However, after they have some experience with putting together planar nets, challenge them to guess what the following nets will make. Remember to encourage them to add tabs, to make taping the shapes together more easy (and to exercise their spatial reasoning).







Questions:

(1) Sketch two planar nets for a cube which are completely different from any of the ones illustrated in this chapter.

(2) Cut out and tape together one of each of the Platonic solids, using the nets in this chapter.

(3) Identify the shapes formed by the last three planar nets in this chapter (make them if necessary).