Probing the Meaning of

Quantum Mechanics

Superpositions, Dynamics, Semantics and Identity

Editors

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Quantum Mechanics and Quantum Information: Physical, Philosophical and Logical Approaches

Cagliari, Italy 23 – 25 July 2014

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EDITORIAL PREFACE

Quantum mechanics is recognized today as one of the most successful physical theories ever. However, its interpretation is still problematic, or more specifically, an understanding of what quantum mechanics talks about does not exist yet. Although the features that are considered at the origin of these problems — Schrödinger's cat situation, entanglement, non-separability and contextuality — are well identified and nowadays resources of well-established technical applications, we continue to lack a semantic and a conceptual representation explaining the relation of quantum theory to physical reality.

Independently of the many developments in the 20th Century, during the last decades quantum mechanics has given rise to a new technological era, a revolution taking place today especially within the field of quantum information processing, reaching from quantum teleportation and cryptography to quantum computation. This development is related to the philosophy of quantum mechanics which addresses epistemic and ontic interpretations, the meaning of individuality and identity, of quantum possibility, objective probability and chance, free will and causality. Next to the philosophical foundations also the logical and mathematical structure of the theory is considered in the present book. The underlying mathematical structure related to quantum mechanics is based on a vectorial Hilbert space, which was employed in 1932 by John von Neumann in his famous book *Mathematical Foundations of Quantum Mechanics*. Few years later, in 1936, John von Neumann and Garret Birkhoff published a seminal paper titled "The logic of quantum mechanics". The subject of that paper was an investigation of the logical structure underlying the language of quantum mechanics, establishing a link between logic, language and quantum mechanics. From this standpoint several schools in the foundations of quantum mechanics and quantum logic with different approaches followed.

The differences in approach have still grown over time, up to the present day. The quantum logical approaches consider many different aspects of quantum mechanics such as: *truth notions*, the *logic related to superpositions and entanglement*, *modal approaches*, *logical systems associated to quantum computational gates* and *categorical approaches.* These topics define a new extensive area of investigation in modern quantum logic and, at the same time, provide a deep contribution to the understanding of the theory of quanta. All these different subjects related to quantum mechanics are discussed and analyzed in the present book.

This book attempts to address foundational questions in an interdisciplinary manner considering the physical, philosophical, mathematical and logical aspects of quantum mechanics. Going from philosophy of quantum mechanics to quantum logic, from categorical approaches to quantum information processing, the originality of this book resides in the multiplicity of approaches which focus on the common aim, to grasp the meaning of the theory of quanta. We believe that the advancement of such understanding will be possible, not through the strict separation of smaller and smaller subjects of research, but on the very contrary, through the common sharing and discussion of multiple perspectives that come from different fields.

The novelty of the book comes from the multiple perspectives put forward by top researchers in quantum mechanics, from Europe as well as North and South America, discussing the meaning and structure of the theory of quanta. The book comprises in a balanced manner physical, philosophical, logical and mathematical approaches to quantum mechanics and quantum information. From quantum superpositions and entanglement to dynamics and the problem of identity, from quantum logic, computation and quasi-set theory to the category approach and teleportation, from realism and empiricism to operationalism and instrumentalism, the book touches from different perspectives some of the most intriguing questions related to one of the most important physical theories of our time. From Buenos Aires to Brussels and Cagliari, from Florence to Florianópolis, the interaction between different groups is reflected in the many different articles. This book is interesting not only to the specialists but also to the general public attempting to get a grasp on some of the most fundamental questions of present quantum physics.

D. Aerts, C. de Ronde, H. Freytes & R. Giuntini.

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Cagliari, May 13th, 2016
Roberto Giuntini

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ON THE NOTION OF TRUTH IN QUANTUM MECHANICS: A CATEGORY-THEORETIC STANDPOINT

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The category-theoretic representation of quantum event structures provides a canonical setting for confronting the fundamental problem of truth valuation in quantum mechanics as exemplified, in particular, by Kochen-Specker's theorem. In the present study, this is realized on the basis of the existence of a categorical adjunction between the category of sheaves of variable local Boolean frames, constituting a topos, and the category of quantum event algebras. We show explicitly that the latter category is equipped with an object of truth values, or classifying object, which constitutes the appropriate tool for assigning truth values to propositions describing the behavior of quantum systems. Effectively, this category-theoretic representation scheme circumvents consistently the semantic ambiguity with respect to truth valuation that is inherent in conventional quantum mechanics by inducing an objective contextual account of truth in the quantum domain of discourse. The philosophical implications of the resulting account are analyzed. We argue that it subscribes neither to a pragmatic instrumental nor to a relative notion of truth. Such an account essentially denies that there can be a universal context of reference or an Archimedean standpoint from which to evaluate logically the totality of facts of nature. In this light, the transcendence condition of the usual conception of correspondence truth is superseded by a reflective-like transcendental reasoning of the proposed account of truth that is suitable to the quantum domain of discourse.

Keywords: Quantum mechanics, Kochen-Specker theorem, category theory, categorical adjunction, topos theory, subobject classifier, correspondence truth, contextual semantics.

1. On the Traditional Framework of Correspondence Truth in Physical Science

The logic of a physical theory reflects the structure of elementary propositions referring to the behavior of a physical system in the domain of the corresponding theory. Elementary propositions concern the values of physical quantities pertaining to a system and are of the form "the value of the physical quantity A lies in a measurable subset of the real numbers", for which we use the shorthand notation "$A \in \Delta$". Propositions of the type "$A \in \Delta$" refer to the world in the most direct conceivable sense. They can only be ascertained as true or false by corresponding, or not corresponding, to "facts" or, more generally, to "actual states of affairs" or, in contemporary physical terms, to "events" conceived in the physicist's understanding as actual measurement outcomes, as concrete elements of empirical reality [1]. In this respect, elementary propositions are posits in the process of probing the natural world. They can be used to indicate properties of physical systems in the following sense: a system S is characterized by a property a, namely a particular value of the physical quantity A, if and only if the corresponding proposition P_A is true, i.e., if and only if "$A = a$" or "$A \in \Delta$". For instance, "$q_x = 5$" is true of the system S if and only if the x component of S's position is 5 according to a proper system of coordinates. To say, therefore, that the elementary proposition "$A \in \Delta$" is *true of* the system S is equivalent to say that in S, A has a value in Δ. Physics is fundamentally concerned about the truth values of such propositions of the form "$A \in \Delta$", where Δ varies over the Borel subsets of the real line and A varies over the physical quantities of system S, when S is in a given state. Of course, the state may change in time, and hence the truth values also will change in time.

In addition to the syntactic aspect, dealing with the logico-mathematical operations and relations among the propositions pertaining to a system, the logic of a physical theory incorporates the task of providing a semantics, focusing particularly on the task of establishing a criterion of truth or, in a more complete sense, defining propositional truth. In relation to the latter, the correspondence theory of truth has frequently been regarded within physical science as the most eminent. Although the correspondence theory admits various different formulations, the core of any correspondence theory is the idea that a proposition is true if and only if it corresponds to or matches reality. The classical version of the theory describes this relationship as a correspondence to the facts about the world [2, pp. 70-72].

If so, then adopting a correspondence theory of truth amounts to endorsing instances of the following scheme:

> [CF] The proposition that P is true if and only if P corresponds to a fact.

Alternatively, if one construes the notion of a "fact" in terms of the weaker notion of an obtaining "state of affairs", as in an Austin-type theory [3, pp. 149-162], then, [CF] is re-expressed as follows:

> [CS] The proposition that P is true if and only if there is a state of affairs X such that P corresponds to X and X obtains.

The useful feature of states of affairs, especially in the philosophical discourse, is that they refer to something that can be said to obtain or fail to obtain, to be the case or not to be the case, to be a fact or fail to be a fact, that is, they exist even when they are not concretely manifested or realized.

It is worthy to note in this connection that the traditional theory of truth as correspondence, regardless of its exact formulation, has frequently been associated with the view that truth is *radically non-epistemic* [4, p. 606]. This means that the truth (or falsity) of a proposition is entirely independent of anyone's cognitive capacities, beliefs, theories, conceptual schemes, and so on. According to this non-epistemic conception, the truthmakers of propositions — namely, facts or actual states of affairs, forming the object-end of the truthmaking relation — are totally independent of human conceptualization and thus can be conceived of as being *completely autonomous* in themselves, or as residing in the world *purely extensionally*, that is, in a manner independent of our worldviews and particular discursive practices and contexts. In this sense, the view of a radically non-epistemic conception of truth incorporates the following transcendence condition:

> *Transcendence condition*: The truth of a proposition transcends our possible knowledge of it, or its evidential basis; it is empirically unconstrained.

As a consequence of the preceding condition, even if it is impossible to produce a framework on which we may ascertain the truth value of a proposition this does not imply that the proposition does not possess any such value. It always has one. It is either determinately true or determinately false independently of any empirical evidence or cognitive means by which we may establish which value it is. The possession of truth values is therefore entirely independent of our means of warranting their assignment.

The transcendence condition, no doubt, attempts to capture the realist intuition that a proposition cannot be claimed true or false in virtue of its knowability or justifiability. For, a proposition may be true without being justified, and vice versa. Agreed! But what does the transcendence thesis, in its totality, really presuppose, especially when viewed within the traditional framework of correspondence truth? It presupposes the existence of a 'platonic' universe of true propositions, entirely independent of our ability in having access to it, and, henceforth, elements of this ideal world, in this case, propositions, possess determinate truth values entirely independent of our capability in forming justified convictions about them. In other words, it is important to realize that this thesis does not simply aim to establish an objective basis or attribute a non-epistemic character to the notion of truth — that, for instance, the content of declarative propositions is rendered true (or false) on the basis of worldly conditions and not on some relevant beliefs of ours — but this particular conception tends to be so radically non-epistemic that at the end leads to a notion of truth with absolutely no epistemic features.

Be that as it may, it is particularly interesting that the semantics underlying the propositional structure of classical mechanics allows truth-value assignment in conformity with the usual traditional conception of a correspondence account of truth. In classical physics, the algebra of propositions of a classical system is isomorphic to the lattice of subsets of phase space, a Boolean lattice, L_B, that can be interpreted semantically by a two-valued truth-function. This means that to every proposition $P \in L_B$ one of the two possible truth values 1 (true) and 0 (false) can be assigned; accordingly, any proposition is either true or false (*tertium non datur*) [5, p. 21]. Specifically, each point in phase space, representing a classical state of a given system S, defines a truth-value assignment to the subsets representing the propositions. Each subset to which the point belongs represents a true proposition or a property that is instantiated by the system. Likewise, each subset to which the point does not belong represents a false proposition or a property that is not instantiated by the system. Thus, every possible property of S is selected as either occurring or not; equivalently, every corresponding proposition pertaining to S is either true or false. Henceforth, all propositions of a classical system are *semantically decidable*.

From a physical point of view, this is immediately linked to the fact that classical physics views objects-systems as bearers of determinate properties.

That is, properties of classical systems are considered as being *intrinsic* to the system; they are independent of whether or not any measurement is attempted on them and their definite values are independent of one another as far as measurement is concerned. All properties pertaining to a classical system are simultaneously determinate and potentially available to the system, independently of any perceptual evidence or cognitive means by which we may verify or falsify them. Accordingly, the propositions of a classical system are considered as possessing determinate truth values — they are either determinately true or determinately false — *prior to and independent* of any actual investigation of the states of affairs the propositions denote; that is, classical mechanical propositions possess investigation-independent truth values, thus capturing the radically non-epistemic character of a traditional correspondence account of truth [6]. Consequently, since in a classical universe of discourse all propositions are meant to have determinate truth-conditions, there is supposed to exist implicitly an Archimedean standpoint from which the totality of facts may be logically evaluated.

2. Propositional Truth in Standard Quantum Mechanics

In contradistinction to the Boolean propositional structure of classical mechanics, the logical structure of a quantum system is neither Boolean nor possible to be embedded into a Boolean lattice. On the standard (Dirac-von Neumann) codification of quantum theory, the elementary propositions pertaining to a system form a non-Boolean lattice, L_H, isomorphic to the lattice of closed linear subspaces or corresponding projection operators of a Hilbert space. Thus, a proposition pertaining to a quantum system is represented by a projection operator P on the system's Hilbert space H, or equivalently, it is represented by the linear subspace H_P of H upon which the projection operator P projects [7, Sect. 4.2]. Since each projection operator P on H acquires two eigenvalues 1 and 0, where the value 1 can be read as "true" and 0 as "false", the proposition asserting that "the value of the physical quantity A of system S lies in a certain range of values $\varDelta$", or equivalently, that "system S in state $|\psi\rangle$ acquires the property $P(A)$", is said to be true if and only if the corresponding projection operator P_A obtains the value 1, that is, if and only if $P_A|\psi\rangle = |\psi\rangle$. Accordingly, the state $|\psi\rangle$ of the system lies in the associated subspace H_A which is the range of the operator P_A, i.e., $|\psi\rangle \in H_A$. In such a circumstance, the property $P(A)$ pertains to the quantum system S. Otherwise, if $P_A|\psi\rangle = 0$ and, hence, $|\psi\rangle \in \perp H_A$ (subspace completely orthogonal to H_A), the counter property $\neg P(A)$ pertains to S, and the proposition is said to be false. It might appear, therefore, that propositions of this kind have a well-defined truth value in a sense analogous to the truth-value assignment in classical mechanics.

There is, however, a significant difference between the two situations. Unlike the case in classical mechanics, for a given quantum system, the propositions represented by projection operators or Hilbert space subspaces are not partitioned into two mutually exclusive and collectively exhaustive sets representing either true or false propositions. As already pointed out, only propositions represented by subspaces that contain the system's state are assigned the value "true" (propositions assigned probability 1 by $|\psi\rangle$), and only propositions represented by spaces orthogonal to the state are assigned the value "false" (propositions assigned probability 0 by $|\psi\rangle$) ([8, pp. 46-47], [9, pp. 213-217]). Hence, propositions represented by subspaces that are at some non-zero or non-orthogonal angle to the unit vector $|\psi\rangle$ or, more appropriately, to the ray representing the quantum state are not assigned any truth value in $|\psi\rangle$. These propositions are neither true nor false; they are assigned by $|\psi\rangle$ a probability value different from 1 and 0; thus, they are *undecidable* or *indeterminate* for the system in state $|\psi\rangle$ and the corresponding properties are taken as indefinite.

This kind of semantic indeterminacy imposes an inherent ambiguity with respect to the classical binary true/false value assignments rigorously expressed, for the first time, by Kochen-Specker's theorem [10]. According to this, for any quantum system associated to a Hilbert space of dimension higher than two, there does not exist a two-valued, truth-functional assignment $h : L_H \to \{0, 1\}$ on the set of closed linear subspaces, L_H, interpretable as quantum mechanical propositions, preserving the lattice operations and the orthocomplement. In other words, the gist of the theorem, when interpreted semantically, asserts the impossibility of assigning definite truth values to *all* propositions pertaining to a physical system at any one time, for any of its quantum states, without generating a contradiction.

3. Motivating the Application of the Categorical Framework to Quantum Semantics

It should be underlined, however, that although the preceding Kochen-Specker result forbids a global, absolute assignment of truth values to quantum mechanical propositions, it does not exclude ones that are contextual [11]. Here, "contextual" means that the truth value given to a proposition *depends* on which subset of mutually commuting projection operators (meaning "simultaneously measurable") one may consider it to be a member, i.e., it *depends* on which other compatible propositions are given truth values at the same time. To explicate this by means of a generalized example, let A, B and E denote observables of the same quantum system S, so that the corresponding projection operator A commutes with operators B and E ($[A, B] = 0 = [A, E]$), not however the operators B and E with each other ($[B, E] \neq 0$). Then, due to the incompatibility of the last pair of observables, the result of a measurement of A depends on whether the system had previously been subjected to a measurement of the observable B or a measurement of the observable E or in none of them. Thus, the value of the observable A depends upon the set of mutually commuting observables one may consider it with, that is, the value of A depends upon the selected set of measurements. In other words, the value of the observable A cannot be thought of as pre-fixed, as being independent of the experimental context actually chosen, as specified, in our example, by the $\{A, B\}$ or $\{A, E\}$ frame of mutually compatible observables. Accordingly, the truth value assigned to the associated proposition "$A \in \varDelta$" — i.e., "the value of the observable A of system S lies in a certain range of values $\varDelta$" — should be contextual as it depends on whether A is thought of in the context of simultaneously ascribing a truth value to propositions about B, or to propositions about E. Of course, the formalism of quantum theory does not imply how such a contextual valuation might be obtained, or what properties it should possess. To this end, we resort to the powerful methods of categorical topos theory, which directly captures the idea of structures varying over contexts, thus providing a natural setting for studying contextuality phenomena ([12],[13],[14]), providing consistently context-dependent operations in logic ([15], [16]) and representing physical processes ([17], [18]).

In this work, the research path we propose implements the intuitively clear idea of probing the global structure of a quantum algebra of events (or propositions) in terms of structured multitudes of interlocking local Boolean frames. It is probably one of the deepest insights of modern quantum theory that whereas the totality of all experimental/empirical facts can only be represented in a globally non-Boolean logical structure, the acquisition of every single fact depends on a locally Boolean context. Indeed, we view each preparatory Boolean environment of measurement as a context that offers a 'classical perspective' on a quantum system. A classical perspective or context is nothing but a set of commuting physical quantities, as in the preceding example, or, more precisely, a complete Boolean algebra of commuting projection operators generated by such a set. Physical quantities in any such algebra can be given consistent values, as in classical physics. Thus, each context functions as a Boolean frame providing a local 'classical' viewpoint on reality. No single context or perspective can deliver a complete picture of the quantum system, but, by applying category-theoretic reasoning, it is possible to use the collection of all of them in an overall structure that will capture the entire system. It is also of great importance how the various contexts relate to each other. Categorically speaking, this consideration is naturally incorporated into our scheme, since the category-theoretic representation of quantum event structures in terms of Boolean localization contexts can be described by means of a *topos*, which stands for a category of sheaves of variable local Boolean frames encoding the global logical information of these localization contexts.

In a well defined sense, topos theory provides us with the first natural examples of global multi-valued functional truth structures. By definition, a topos, conceived as a category of sheaves for a categorical topology, is equipped with an internal object of truth values, called a *subobject classifier*, which generalizes the classical binary object of truth values used for valuations of propositions [19, pp. 338-343]. As explained below, this generalized object of truth values in a topos is not ad hoc, but reflects genuine constraints of the surrounding universe of discourse. We will show, in particular, that the topos-theoretic representation scheme of quantum event algebras by means of variable local Boolean frames induces an object of truth values, or classifying object, which constitutes the appropriate tool for the definition of quantum truth-value assignments, corresponding to valuations of propositions describing the behavior of quantum systems. This, in effect, characterizes the novelty of our approach and its fruitfulness for a consistent contextual account of truth in the quantum domain in comparison to a multiplicity of various other approaches on the foundations of quantum physics.

4. Category-Theoretic Representation of Quantum Event Algebras

As indicated in the preceding section, the global semantic ambiguity of the non-Boolean logical structure of quantum mechanics, expressed formally by Kochen-Specker's theorem, does not exclude local two-valued truth-functional assignments with respect to complete Boolean algebras of projection operators on the Hilbert space of a quantum system. More precisely, each self-adjoint operator representing an observable has associated with it a Boolean subalgebra which is identified with the Boolean algebra of projection operators belonging to its spectral decomposition. Hence, given a set of observables of a quantum system, there always exists a complete Boolean algebra of projection operators, namely, a local Boolean subalgebra of the global non-Boolean event algebra of a quantum system with respect to which a local two-valued truth-functional assignment is meaningful, if and only if the given observables are simultaneously measurable. Consequently, the possibility of local two-valued truth-functional assignments of the global non-Boolean event algebra of a quantum system points to the assumption that complete Boolean algebras play the role of local Boolean logical frames for contextual true/false value assignments. The modeling scheme we propose in order to implement this idea in a universal way, so that the global structure of a quantum system to be modeled categorically in terms of a topos of sheaves of local Boolean frames, uses the technical apparatus of categorical sheaf theory (MacLane and Moerdijk [20], Awodey [21]). The basic conceptual steps involved in the realization of the suggested approach may be summarized as follows.

4.1. *Conceptual considerations*

Firstly, we introduce the notion of a topological covering scheme of a quantum event algebra [22] consisting of epimorphic families of local Boolean logical frames. From the physical point of view, this attempt amounts to analyze or 'co-ordinatize' the information contained in a quantum event algebra L by means of structure preserving morphisms $B \to L$, having as their domains locally defined Boolean event algebras B. Any single map from a Boolean domain to a quantum event algebra does not suffice for a complete determination of the latter's information content, and thus, it contains only a limited amount of information about it. Specifically, it includes only the amount of information related to a Boolean reference context and inevitably is constrained to represent exclusively the abstractions associated with it. This problem is confronted by employing a sufficient amount of maps, organized in terms of covering sieves, from the coordinatizing Boolean domains to a quantum event algebra so as to cover it completely. These maps play exactly the role of local Boolean covers for the filtration of the information associated with a quantum structure of events, in that, their domains B provide Boolean coefficients associated with typical measurement situations of quantum observables. The local Boolean covers capture individually complementary features of a quantum algebra of events and provide collectively its categorical local decomposition in the descriptive terms of Boolean reference frames. Technically, this is described by an action of a category of local Boolean frames on a global quantum event algebra, forming a presheaf.

Secondly, we define appropriate compatibility conditions between overlapping local Boolean covers. This is necessary since it enforces an efficient, uniquely defined pasting code between different local covers of a quantum algebra of events. Technically, this is described by the notion of a Boolean localization functor, or equivalently, by a structure sheaf of Boolean coefficients of a quantum event algebra. Category-theoretically, one may think of a Boolean localization functor as consisting of epimorphic families or diagrams of Boolean covers interconnected together targeting a quantum event algebra.

Thirdly, we establish the necessary and sufficient conditions for the isomorphic representation of quantum event algebras in terms of Boolean localization functors. The crucial technical and semantical method used in order to establish these conditions is based on the existence of a *categorical adjunction*, consisting of a pair of adjoint functors, between the category of (pre)sheaves of variable local Boolean frames and the category of quantum event algebras. In essence, this *Boolean frames-quantum adjunction* provides a bidirectional and functorial translational mechanism of encoding and decoding information between the Boolean and quantum kind of structures respecting their distinctive form. It is precisely the existence of this categorical adjunction that allows the isomorphic representation of quantum event algebras in terms of sheaves of variable local Boolean frames.

The notion of a sheaf incorporates the requirements of consistency under extension from the local Boolean to the global quantum level, and inversely, under restriction of the global quantum to the local Boolean level. The functional dependence implicated by a categorical sheaf relativizes the presupposed rigid relations between quantum events with respect to variable local Boolean frames conditioning the actualization of events (cf. [23]). The category of sheaves of variable local Boolean frames, encoding the global logical information of Boolean localization functors, constitutes a topos providing the possibility of applying the powerful logical classification methodology of topos theory with reference to the quantum universe of discourse.

Topos theoretical approaches to the logical foundations of quantum mechanics have also been considered, from a different viewpoint, by Isham and Butterfield [24], Döring and Isham ([13], [25]), Heunen, Landsmann and Spitters ([14], [26]), Abramsky and Brandenburger [12], van den Berg and Heunen [27]. In Zafiris and Karakostas [28] we compare each one of the preceding approaches with our topos-theoretic representation scheme of quantum event algebras. It should be noted in this connection that of particular relevance to the present work, regarding the specification of an object of truth values or classifying object, although not based on categorical methods, appears to be the approach to the foundations of quantum logic by Davis [29] and Takeuti [30], according to whom, 'quantization' of a proposition of classical physics is equivalent to interpreting it in a Boolean extension of a set theoretical universe, where B is a complete Boolean algebra of projection operators on a Hilbert space.

It is not possible to provide here a concise account of category theory. For a general introduction to this well-developed mathematical framework, topos theory and categorial logic, the reader may consult ([19], [31], [32]). For the convenience of the reader, we define in the following subsection the basic categorical structures involved in the functorial representation of a quantum event algebra as a topos of (pre)sheaves over the base category of Boolean event algebras ([33], [28]).

4.2. *Functorial modeling of quantum event algebras*

A *Boolean categorical event structure* is a small category, denoted by $\mathcal{B}$, which is called the category of Boolean event algebras. The objects of $\mathcal{B}$ are σ-Boolean algebras of events and the arrows are the corresponding Boolean algebraic homomorphisms.

A *quantum categorical event structure* is a locally small co-complete category, denoted by $\mathcal{L}$, which is called the category of quantum event algebras. The objects of $\mathcal{L}$ are quantum event algebras and the arrows are quantum algebraic homomorphisms. A quantum event algebra L in $\mathcal{L}$ is defined as an *orthomodular σ-orthoposet* [5], that is, as a partially ordered set of quantum events, endowed with a maximal element 1, and with an operation of orthocomplementation $[-]^* : L \to L$, which satisfy, for all $l \in L$, the following conditions: [i] $l \leq 1$, [ii] $l^{**} = l$, [iii] $l \vee l^* = 1$, [iv] $l \leq \acute{l} \Rightarrow \acute{l}^* \leq l^*$, [v] $l \perp \acute{l} \Rightarrow l \vee \acute{l} \in L$, [vi] for $l, \acute{l} \in L$, $l \leq \acute{l}$ implies that l and $\acute{l}$ are compatible, where $0 := 1^*$, $l \perp \acute{l} := l \leq \acute{l}^*$, and the operations of meet $\wedge$ and join $\vee$ are defined as usually. The σ-completeness condition, meaning that the join of countable families of pairwise orthogonal events exists, is required in order to have a well defined theory of quantum observables over L.

A *quantum observable* $\mathcal{O}$ is defined to be an algebraic morphism from the Borel algebra of the real line $Bor(\mathbb{R})$ to the quantum event algebra L, $\mathcal{O} : Bor(\mathbb{R}) \to L$, such that: [i] $\mathcal{O}(\emptyset) = 0$, $\mathcal{O}(R) = 1$, [ii] $E \bigcap F = \emptyset \Rightarrow \mathcal{O}(E) \perp \mathcal{O}(F)$, for $E, F \in Bor(\mathbb{R})$, [iii] $\mathcal{O}(\bigcup_n E_n) = \bigvee_n \mathcal{O}(E_n)$, where

$E_1, E_2, \ldots$ denotes a sequence of mutually disjoint Borel sets of the real line. Addition and multiplication over $\mathbb{R}$ induce on the set of quantum observables the structure of a partial commutative algebra over $\mathbb{R}$.

If L is isomorphic with the orthocomplemented lattice of orthogonal projections on a complex Hilbert space, then from von Neumann's spectral theorem the quantum observables are in bijective correspondence with the Hermitian operators on the Hilbert space. As noted in Section 2, the original quantum logical formulation of quantum theory depends in an essential way on the identification of propositions with projection operators on a complex Hilbert space. In the sequel, the measure-theoretic σ-completeness condition is not going to play any particular role in the exposition of the arguments, so one may drop it and consider complete Boolean algebras and complete orthomodular lattices instead.

The crucial observation pointing to the significance of the proposed functorial modeling approach to quantum event algebras is that despite the fact that each object L in $\mathcal{L}$ is not Boolean, there always exists an underlying categorical diagram of Boolean subalgebras of L, where each one of them is generated by the following well-known compatibility condition: any arbitrary pair of elements $l, \acute{l} \in L$ are compatible if the sublattice generated by $\{l, l^*, \acute{l}, \acute{l}^*\}$ is a Boolean algebra, namely, if it is a Boolean sublattice. Thus, if these Boolean sublattices of events can be appropriately identified in terms of local Boolean logical frames of a quantum event algebra, then the global information of the latter can be recovered in a structure-preserving way by a suitable sheaf-theoretic construction gluing together categorical diagrams of local Boolean subobjects.

A topological gluing construction of this form can only take place in an extension of $\mathcal{B}$ enunciating the free completion of $\mathcal{B}$ under colimits. Technically, this process requires initially the construction of the *functor category of presheaves of sets on Boolean event algebras*, denoted by $\mathbf{Sets}^{\mathcal{B}^{op}}$, where $\mathcal{B}^{op}$ is the opposite category of $\mathcal{B}$. The objects of $\mathbf{Sets}^{\mathcal{B}^{op}}$ are all functors $\mathbf{P} : \mathcal{B}^{op} \rightarrow \mathbf{Sets}$, and morphisms all natural transformations between such functors. Each object $\mathbf{P}$ in the category of presheaves $\mathbf{Sets}^{\mathcal{B}^{op}}$ is a contravariant set-valued functor on $\mathcal{B}$, called a *presheaf* on $\mathcal{B}$. The functor category of presheaves on Boolean event algebras, $\mathbf{Sets}^{\mathcal{B}^{op}}$, provides the instantiation of a structure known as *topos.* A topos exemplifies a well-defined notion of a universe of variable sets. It can be conceived as a local mathematical framework corresponding to a generalized model of set theory or as a generalized topological space.

For each Boolean algebra B of $\mathcal{B}$, a presheaf $\mathbf{P}$ on B, denoted by $\mathbf{P}(\mathrm{B})$, is a set, and for each Boolean homomorphism $f : C \to B$, $\mathbf{P}(f) : \mathbf{P}(B) \to \mathbf{P}(\mathrm{C})$ is a set-theoretic function such that, if $p \in \mathbf{P}(\mathrm{B})$, the value $\mathbf{P}(f)(p)$ for an arrow $f : C \to B$ in $\mathcal{B}$ is called the *restriction* of p along f and is denoted by $\mathbf{P}(f)(p) = p \cdot f$. From a physical viewpoint, the purpose of introducing the notion of a presheaf $\mathbf{P}$ on $\mathcal{B}$, in the environment of the functor category $\mathbf{Sets}^{\mathcal{B}^{op}}$, is to identify an element of $\mathbf{P}(\mathrm{B})$, that is, $p \in \mathbf{P}(\mathrm{B})$, with an event observed by means of a physical procedure over a Boolean domain cover for a quantum event algebra. This identification forces the interrelation of observed events, over all Boolean reference frames of the base category $\mathcal{B}$, to fulfil the requirements of a uniform and homologous fibred structure, further considered in Section 5.

We notice now that each Boolean algebra B of $\mathcal{B}$ gives naturally rise to a contravariant Hom-functor $\mathbf{y}[B] := Hom_{\mathcal{B}}(-, B)$. This functor defines a presheaf on $\mathcal{B}$ for each B in $\mathcal{B}$. Concomitantly, the functor $\mathbf{y}$ is a full and faithful functor from $\mathcal{B}$ to the contravariant functors on $\mathcal{B}$, i.e.,

$$\mathbf{y} : \mathcal{B} \longrightarrow \mathbf{Sets}^{\mathcal{B}^{op}},$$

defining an embedding $\mathcal{B} \hookrightarrow \mathbf{Sets}^{\mathcal{B}^{op}}$, which is called the Yoneda embedding ([20, p. 26], [21, pp. 187-189]). According to the Yoneda Lemma, there exists an injective correspondence between elements of the set $\mathbf{P}(B)$ and natural transformations in $\mathbf{Sets}^{\mathcal{B}^{op}}$ from $\mathbf{y}[B]$ to $\mathbf{P}$ and this correspondence is natural in both $\mathbf{P}$ and B, for every presheaf of sets $\mathbf{P}$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$ and Boolean algebra B in $\mathcal{B}$. The functor category of presheaves of sets on Boolean event algebras $\mathbf{Sets}^{\mathcal{B}^{op}}$ is a complete and co-complete category. Thus, the Yoneda embedding $\mathbf{y} : \mathcal{B} \longrightarrow \mathbf{Sets}^{\mathcal{B}^{op}}$ constitutes the sought free completion of $\mathcal{B}$ under colimits of diagrams of Boolean event algebras.

Then, the *Boolean realization functor of a quantum categorical event structure* $\mathcal{L}$ *in* $\mathbf{Sets}^{\mathcal{B}^{op}}$, namely, the functor of generalized elements of $\mathcal{L}$ in the environment of the category of presheaves on Boolean event algebras, is defined as

$$\mathbf{R} : \mathcal{L} \to \mathbf{Sets}^{\mathcal{B}^{op}},$$

where the action on a Boolean algebra B in $\mathcal{B}$ is given by

$$\mathbf{R}(L)(B) := \mathbf{R}_L(B) = Hom_{\mathcal{L}}(\mathbf{M}(B), L).$$

The functor $\mathbf{R}(L)(-) := \mathbf{R}_L(-) = Hom_{\mathcal{L}}(\mathbf{M}(-), L)$ is called the *functor of Boolean frames* of L, where $\mathbf{M} : \mathcal{B} \to \mathcal{L}$ is a *Boolean modeling functor* of $\mathcal{L}$, i.e., a forgetful functor assigning to each Boolean event algebra the underlying quantum event algebra and to each Boolean homomorphism the underlying quantum algebraic homomorphism. In this manner, a Boolean event algebra B, instantiated by the measurement of a quantum observable via the spectral decomposition of the corresponding self-adjoint operator, acts as a partial coordinatizing logical frame of a quantum event algebra L in terms of Boolean coefficients. Notice that the Boolean logical frames of L, denoted by

$$\psi_B : \mathbf{M}(B) \to L,$$

being instantiated by the evaluation of the functor $\mathbf{R}(L)(-)$ at each B in $\mathcal{B}$, are not ad hoc but they are interrelated by the operation of presheaf restriction. Explicitly, this means that for each Boolean homomorphism $f : C \to B$, $\mathbf{R}(L)(f) : \mathbf{R}(L)(B) \to \mathbf{R}(L)(C)$ is a function between sets of Boolean frames in the opposite direction such that, if $\psi_B \in \mathbf{R}(L)(B)$ is a Boolean frame, the value of $\mathbf{R}(L)(f)(\psi_B)$, or equivalently, the corresponding Boolean frame $\psi_C : \mathbf{M}(C) \to L$ is given by the restriction or pullback of ψ_B along f, denoted by $\mathbf{R}(L)(f)(\psi_B) = \psi_B \cdot f = \psi_C$.

In this setting, the problem of establishing a functorial representation of quantum event algebras is solved exactly by finding the left adjoint functor $\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \to \mathcal{L}$ to the Boolean realization functor $\mathbf{R} : \mathcal{L} \to \mathbf{Sets}^{\mathcal{B}^{op}}$. In other words, the existence of the left adjoint functor $\mathbf{L}$ paves the way for an explicit reconstruction of quantum event algebras by means of appropriate diagrams of Boolean logical frames in a structure-preserving manner. Additionally, as will be shown in the sequel, it can be used for addressing efficiently the problem of truth valuation in the quantum domain by circumventing the Kochen-Specker types of prohibition. This becomes possible by exploiting the topos logical structure of the category of diagrams of Boolean frames, and then using the left adjoint functor for the classification of quantum event algebras in terms of contextual truth valuations with respect to interconnected covering families of local Boolean frames.

In this context of reasoning, we can show that there exists a *categorical adjunction* between the categories $\mathbf{Sets}^{\mathcal{B}^{op}}$ and $\mathcal{L}$. More precisely, there exists *a pair of adjoint functors* $\mathbf{L} \dashv \mathbf{R}$ as follows [33]:

$$\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \underset{\longrightarrow}{\longleftarrow} \mathcal{L} : \mathbf{R}.$$

The *Boolean frames-quantum adjunction* consists of the functors $\mathbf{L}$ and $\mathbf{R}$, called left and right adjoints, as well as the natural bijection:

$$Nat(\mathbf{P}, \mathbf{R}(L)) \cong Hom_{\mathcal{L}}(\mathbf{LP}, L).$$

The established bijective correspondence, interpreted functorially, assures that the Boolean realization functor of $\mathcal{L}$, realized for each L in $\mathcal{L}$ by its functor of Boolean frames,

$$\mathbf{R}(L) : B \mapsto Hom_{\mathcal{L}}(\mathbf{M}(B), L),$$

has a left adjoint functor $\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \to \mathcal{L}$, which is defined for each presheaf $\mathbf{P}$ of Boolean algebras in $\mathbf{Sets}^{\mathcal{B}^{op}}$ as the colimit $\mathbf{L}(\mathbf{P})$, explicitly constructed for the case of interest $\mathbf{P} = \mathbf{R}(L)$ in the next section. Thus, the following diagram, where the Yoneda embedding is denoted by $\mathbf{y}$, commutes:

$$\begin{tikzcd}
\mathcal{B} \arrow[d, tail, "\mathbf{y}"'] \arrow[dr, "\mathbf{M}"] & \\
\mathbf{S}ets^{\mathcal{B}^{op}} \arrow[r, dotted, shift left, "\mathbf{L}"] & \mathcal{L} \arrow[l, dotted, shift left, "\mathbf{R}"]
\end{tikzcd}$$

The pair of adjoint functors $\mathbf{L} \dashv \mathbf{R}$ formalizes categorically the process of encoding and decoding information between diagrams of Boolean frames and quantum event algebras respecting their distinctive structural form. In general, the existence of an adjunction between two categories always gives rise to a family of universal morphisms, called *unit* and *counit* of the adjunction, one for each object in the first category and one for each object in the second. In this way, each object in the first category induces a certain property in the second category and the universal morphism carries the object to the universal for that property. Most importantly, every adjunction extends to an *adjoint equivalence* of certain subcategories of the initial functorially correlated categories. It is precisely this category-theoretic fact which determines the necessary and sufficient conditions for the isomorphic representation of quantum event algebras by means of suitably restricted functors of Boolean frames.

Every categorical adjunction is completely characterized by the unit and counit natural transformations, acquiring the status of universal mapping properties [21, pp. 208-215]. In relation to the Boolean frames-quantum adjunction, for any presheaf $\mathbf{P}$ of Boolean event algebras in the functor category $\mathbf{Sets}^{B^{op}}$, the unit of the adjunction is defined as

$$\delta_{\mathbf{P}} : \mathbf{P} \longrightarrow \mathbf{RLP}.$$

On the other side, for each quantum event algebra L in $\mathcal{L}$, the counit is defined as

$$\epsilon_L : \mathbf{LR}(L) \longrightarrow L.$$

The representation of a quantum event algebra L in $\mathcal{L}$, in terms of the functor of Boolean frames $\mathbf{R}(L)$ of L, is full and faithful if and only if the counit of the Boolean frames-quantum adjunction is a quantum algebraic isomorphism, that is structure-preserving, injective and surjective. In turn, the counit of the Boolean frames-quantum adjunction is a quantum algebraic isomorphism if and only if the right adjoint functor is full and faithful. In the latter case we characterize the Boolean modeling functor $\mathbf{M} : \mathcal{B} \to \mathcal{L}$ as a proper or dense modeling functor.

5. Equivalence Classes of Pointed Boolean Frames

The functorial representation of a quantum event algebra L in $\mathcal{L}$ through the category of presheaves of Boolean event algebras $\mathbf{Sets}^{\mathcal{B}^{op}}$ requires an explicit calculation of the colimit, $\mathbf{L}(\mathbf{P}) = \mathbf{LR}(L)$, when the functor on which it acts is the presheaf functor of Boolean frames of L. This task is simplified by the observation that there exists an underlying colimit-preserving faithful functor from the category $\mathcal{L}$ to the category **Sets**. Thus, we can calculate the colimit in **Sets** by means of equivalence classes and, then, show how the obtained set of equivalence classes carries the structure of a quantum event algebra.

In order to calculate, in general, the colimit $\mathbf{L}(\mathbf{P})$ for any $\mathbf{P}$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$, it is necessary to specify the index or parameterizing category corresponding to the functor $\mathbf{P}$, which is defined over the base category of Boolean event algebras $\mathcal{B}$. This index category is called the *category of elements of a presheaf* $\mathbf{P}$, denoted by $\int(\mathbf{P}, \mathcal{B})$, and defined as follows: it has objects all pairs (B, p) and morphisms $(\acute{B}, \acute{p}) \to (B, p)$ are those morphisms $u : \acute{B} \to B$ of $\mathcal{B}$ for which $p \cdot u = \acute{p}$, that is the restriction or pullback of p along u is $\acute{p}$. Projection on the second coordinate of $\int(\mathbf{P}, \mathcal{B})$ defines a functor $\int_{\mathbf{P}} : \int(\mathbf{P}, \mathcal{B}) \to \mathcal{B}$ called the *split discrete fibration* induced by $\mathbf{P}$, where $\mathcal{B}$ is the base category of the fibration, as in the diagram below. Notice that the fibers of this fibration are categories in which the only arrows are identity arrows, i.e., they are actually sets. Then, if B is an object of $\mathcal{B}$, the inverse image under $\int_{\mathbf{P}}$ of B is simply the set $\mathbf{P}(B)$, although its elements are written as pairs so as to form a disjoint union.

$$
\begin{tikzcd}
\int(\mathbf{P},\mathcal{B}) \arrow[d, "\int_{\mathbf{P}}"'] & \\
\mathcal{B} \arrow[r, "\mathbf{P}"] & \mathbf{Sets}
\end{tikzcd}
$$

Consequently, the index category corresponding to the functor $\mathbf{P}$ is the category of its elements $\mathcal{I} \equiv \int(\mathbf{P},\mathcal{B})$, whence the functor $[\mathbf{M} \circ \int_{\mathbf{P}}]$ defines the diagram $\mathbf{X} : \mathcal{I} \to \mathcal{L}$ over which the colimit should be calculated. Hence, we obtain:

$$\mathbf{L}(\mathbf{P}) = \mathbf{L_M}(\mathbf{P}) = Colim\{\int(\mathbf{P},\mathcal{B}) \xrightarrow{\int_{\mathbf{P}}} \mathcal{B} \xrightarrow{\mathbf{M}} \mathcal{L} \dashrightarrow \mathbf{Sets}\}.$$

Now we consider the case of our interest where $\mathbf{P} = \mathbf{R}(L)$, i.e., the presheaf functor $\mathbf{P}$ represents the functor of Boolean frames of a quantum event algebra L. In this case, the category of elements $\int(\mathbf{R}(\mathbf{L}),\mathcal{B})$ has objects all pairs (B,ψ_B), where B is a Boolean event algebra and $\psi_B : \mathbf{M}(B) \to L$ is a Boolean frame of L defined over B. The morphisms of $\int(\mathbf{R}(\mathbf{L}),\mathcal{B})$, denoted by $(\acute{B},\psi_{\acute{B}}) \to (B,\psi_B)$, are those Boolean event algebra homorphisms $u : \acute{B} \to B$ of the base category $\mathcal{B}$ for which $\psi_B \cdot u = \psi_{\acute{B}}$, that is the restriction or pullback of the Boolean frame ψ_B along u is $\psi_{\acute{B}}$.

Next, we define the *set of pointed Boolean frames* of a quantum event algebra L as

$$\mathbf{Y}(\mathbf{R}_L) = \{(\psi_B, q)/(\psi_B : \mathbf{M}(B) \to L),\, q \in \mathbf{M}(B)\}.$$

Note that a pointed Boolean frame $\zeta := (\psi_B, q)$ of L over B consists of a Boolean frame $\psi_B : \mathbf{M}(B) \to L$ together with a projection operator $q \in \mathbf{M}(B)$. The crucial observation is that the composition law in the category $\int(\mathbf{R}(\mathbf{L}),\mathcal{B})$ induces a transitive and reflexive relation $\Re$ on the set $\mathbf{Y}(\mathbf{R}_L)$, defined as follows,

$$(\psi_B \cdot u,\, \acute{q})\, \Re\, (\psi_B,\, u(\acute{q})),$$

for any Boolean homomorphism $u : \acute{B} \to B$ in the base category $\mathcal{B}$. The next step is to make this relation also symmetric by postulating that for pointed Boolean frames ζ, η in $\mathbf{Y}(\mathbf{R}_L)$, where ζ, η denote pairs in the above set, we have

$$\zeta \sim \eta,$$

if and only if $\zeta \Re \eta$ or $\eta \Re \zeta$. Finally, by considering a sequence ξ_1, ξ_2, ..., ξ_k of pointed Boolean frames in the set $\mathbf{Y}(\mathbf{R}_L)$ and also ζ, η such that

$$\zeta \sim \xi_1 \sim \xi_2 \sim \ldots \sim \xi_{k-1} \sim \xi_k \sim \eta,$$

we define an equivalence relation on the set $\mathbf{Y}(\mathbf{R}_L)$ as follows,

$$\zeta \bowtie \eta := \zeta \sim \xi_1 \sim \xi_2 \sim \ldots \sim \xi_{k-1} \sim \xi_k \sim \eta,$$

if and only if there exists a path of Boolean transition homomorphisms in $\mathcal{B}$ inducing the above equivalence. Then, for each pair $\zeta = (\psi_B, q) \in \mathbf{Y}(\mathbf{R}_L)$, we define the equivalence class at pointed Boolean frame ζ:

$$Q_\zeta = \{\iota \in \mathbf{Y}(\mathbf{R}_L) : \zeta \bowtie \iota\}.$$

We finally define the *quotient set of equivalence classes of pointed Boolean frames* as follows,

$$\mathbf{Y}(\mathbf{R}_L)/\bowtie := \{Q_\zeta : \zeta = (\psi_B, q) \in \mathbf{Y}(\mathbf{R}_L)\},$$

and use the notation $Q_\zeta = \|(\psi_B, q)\| = \psi_B \otimes q$ to denote the equivalence class at a single pointed Boolean frame $\zeta = (\psi_B, q)$. The quotient set $\mathbf{Y}(\mathbf{R}_L)/\bowtie$ completes the calculation of the colimit $\mathbf{LR}(L)$ in $\mathbf{Sets}$. The tensor notation is justified by the fact that the equivalence classes of the form $\psi_B \otimes q$ are canonically identified as elements of the tensor product of the functors $\mathbf{R}(L)$ and $\mathbf{M}$ over the base category $\mathcal{B}$. Thus, we finally obtain:

$$\mathbf{LR}(L) \cong \{\mathbf{Y}(\mathbf{R}_L)/\bowtie\} \equiv \mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}.$$

Importantly, the quotient set $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ is naturally endowed with a quantum event algebraic structure as follows. First, the orthocomplementation is defined by the assignment:

$$Q_\zeta^* = (\psi_B \otimes q)^* := \psi_B \otimes q^*.$$

Second, the unit element is defined by:

$$\mathbf{1} := \psi_B \otimes 1.$$

Third, two equivalence classes in the quotient set $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ can be ordered if and only if they have a common refinement. Consequently, the partial order structure is defined by the assignment,

$$(\psi_B \otimes q) \preceq (\psi_C \otimes r),$$

if and only if,

$$d_1 \preceq d_2,$$

where we have made the following identifications,

$$(\psi_B \otimes q) = (\psi_D \otimes d_1)$$

$$(\psi_C \otimes r) = (\psi_D \otimes d_2),$$

with d_1, $d_2 \in \mathbf{M}(D)$, according to the pullback diagram,

$$\begin{tikzcd}
\mathbf{M}(D) \arrow[r, "\beta"] \arrow[d, "\gamma"] & \mathbf{M}(B) \arrow[d, "\alpha"] \\
\mathbf{M}(C) \arrow[r, "\lambda"] & L
\end{tikzcd}$$

such that $\beta(d_1) = q$, $\gamma(d_2) = r$, and $\beta : \mathbf{M}(D) \to \mathbf{M}(B)$, $\gamma : \mathbf{M}(D) \to \mathbf{M}(C)$ denote the pullback of $\alpha : \mathbf{M}(B) \to L$ along $\lambda : \mathbf{M}(C) \to L$ in the category of quantum event algebras.

We conclude that the ordering relation between any two equivalence classes of pointed Boolean frames in the set $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ requires the existence of pullback compatibility conditions between the corresponding Boolean frames, established in the next section. This is a consideration of central significance for an isomorphic representation of quantum event algebras in terms of pullback compatible Boolean frames.

6. Boolean Localization Functors

In confronting this issue, let us recall that for each quantum event algebra L in $\mathcal{L}$ the counit of the Boolean frames-quantum adjunction is defined as

$$\epsilon_L : \mathbf{LR}(L) \longrightarrow L.$$

If we express the calculation of the colimit in terms of equivalence classes of pointed Boolean frames, viz. $\mathbf{LR}(L) \cong \mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$, we obtain:

$$\epsilon_L : \mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M} \longrightarrow L.$$

Thus, the counit ϵ_L fits into the following diagram:

$$
\begin{tikzcd}
\mathbf{R}(L)\otimes_{\mathcal{B}}\mathbf{M} \arrow[dr, "\epsilon_L"] & \\
\mathbf{M}(B) \arrow[u, "\psi_B\otimes(-)"] \arrow[r, "\psi_B"] & L
\end{tikzcd}
$$

Accordingly, for every Boolean frame $\psi_B : \mathbf{M}(B) \to L$ the projection operator $q \in \mathbf{M}(B)$ is mapped to an event in L only through its factorization via the equivalence class $\psi_B \otimes q$ of pointed Boolean frames, or equivalently,

$$\epsilon_L([\psi_B \otimes q]) = \psi_B(q), \qquad q \in \mathbf{M}(B).$$

We note that if the counit natural transformation ϵ_L at L can be appropriately restricted to an isomorphism $\mathbf{R}(L)\otimes_{\mathcal{B}}\mathbf{M} \cong L$, then the structural information of L can be completely encoded and classified logically through equivalence classes of pointed Boolean frames. For this purpose, we need to impose suitable conditions on families of Boolean frames in $\mathbf{R}(L)$, which are going to play the role of local Boolean covers of L. Intuitively, we require that local Boolean covers of L should constitute a minimal generating class of Boolean frames which jointly should form an epimorphic family covering L entirely on their overlaps, and moreover they should be compatible under refinement operations. We formulate these notions as follows.

A *functor of Boolean coverings* for a quantum event algebra L in $\mathcal{L}$ is defined as a subfunctor $\mathbf{T}$ of the functor of Boolean frames $\mathbf{R}(L)$ of L,

$$\mathbf{T} \hookrightarrow \mathbf{R}(L).$$

For each Boolean algebra B in $\mathcal{B}$, a subfunctor $\mathbf{T} \hookrightarrow \mathbf{R}(L)$ is equivalent to an algebraic right ideal or sieve of quantum homomorphisms $\mathbf{T} \triangleright \mathbf{R}(L)$, defined by the requirement that, for each B in $\mathcal{B}$, the set of elements of $\mathbf{T}(B) \subseteq [\mathbf{R}(L)](B)$ is a set of Boolean frames $\psi_B : \mathbf{M}(B) \to L$ of $\mathbf{R}(L)(B)$, called *Boolean covers of* L, satisfying the following property:

{If $[\psi_B : \mathbf{M}(B) \to L] \in \mathbf{T}(B)$, viz. it is a Boolean cover of L, and $\mathbf{M}(v) : \mathbf{M}(\acute{B}) \to \mathbf{M}(B)$ in $\mathcal{L}$ for $v : \acute{B} \to B$ in $\mathcal{B}$, then $[\psi_B \circ \mathbf{M}(v) : \mathbf{M}(\acute{B}) \to \mathcal{L}] \in \mathbf{T}(B)$, viz. it is also a Boolean cover of L}.

A family of Boolean covers $\psi_B : \mathbf{M}(B) \longrightarrow L$, B in $\mathcal{B}$, is the *generator of an ideal of Boolean coverings* $\mathbf{T}$, if and only if, this ideal is the smallest among all containing that family. The ideals of Boolean coverings for an L in $\mathcal{L}$ constitute a partially ordered set under inclusion of subobjects. The minimal ideal is the empty one, namely $\mathbf{T}(B) = \emptyset$ for all B in $\mathcal{B}$, whereas the maximal ideal is the set of all Boolean frames of L for all B in $\mathcal{B}$.

We remind that the ordering relation between any two equivalence classes of pointed Boolean frames in the set $\mathbf{R}(L)\otimes_{\mathcal{B}}\mathbf{M}$ requires the existence of pullback compatibility between the corresponding Boolean frames. Thus, if we consider a functor of Boolean coverings $\mathbf{T}$ for a quantum event algebra L, we require that the generating family of Boolean covers they belong to is compatible under pullbacks. Then, the *pairwise gluing isomorphism* of the Boolean covers $\psi_B : \mathbf{M}(B) \longrightarrow L$, B in $\mathcal{B}$, and $\psi_{\acute{B}} : \mathbf{M}(\acute{B}) \longrightarrow L$, $\acute{B}$ in $\mathcal{B}$, is defined as

$$\Omega_{B,\acute{B}} : \psi_{\acute{B}B}(\mathbf{M}(B)\times_L\mathbf{M}(\acute{B})) \longrightarrow \psi_{B\acute{B}}(\mathbf{M}(B)\times_L\mathbf{M}(\acute{B}))$$

$$\Omega_{B,\acute{B}} = \psi_{B\acute{B}} \circ {\psi_{\acute{B}B}}^{-1} \ ,$$

where $\mathbf{M}(B)\times_L\mathbf{M}(\acute{B})$, together with the two projections $\psi_{B\acute{B}}$ and $\psi_{\acute{B}B}$, is the *pullback or categorical overlap of the Boolean covers* $\psi_B : \mathbf{M}(B) \longrightarrow L$, B in $\mathcal{B}$, and $\psi_{\acute{B}} : \mathbf{M}(\acute{B}) \longrightarrow L$, $\acute{B}$ in $\mathcal{B}$, with common codomain a quantum event algebra L, as shown in the diagram:

$$\begin{tikzcd}
\mathbf{M}(B)\times_L\mathbf{M}(\acute{B}) \arrow[r, "\psi_{B,\acute{B}}"] \arrow[d, "\psi_{\acute{B},B}"] & \mathbf{M}(B) \arrow[d, "\psi_B"] \\
\mathbf{M}(\acute{B}) \arrow[r, "\psi_{\acute{B}}"] & L
\end{tikzcd}$$

An immediate consequence of the previous definition is the satisfaction of the following *Boolean coordinate cocycle conditions* for injective Boolean covers:

$$\begin{array}{rl}
\Omega_{B,B} = 1_B & \quad 1_B\text{: identity of } B \\
\Omega_{B,\acute{B}} \circ \Omega_{\acute{B},\acute{\acute{B}}} = \Omega_{B,\acute{\acute{B}}} & \quad \text{if } \mathbf{M}(B) \cap \mathbf{M}(\acute{B}) \cap \mathbf{M}(\acute{\acute{B}}) \neq 0 \\
\Omega_{B,\acute{B}} = {\Omega^{-1}}_{\acute{B},B} & \quad \text{if } \mathbf{M}(B) \cap \mathbf{M}(\acute{B}) \neq 0.
\end{array}$$

Thus, the gluing isomorphism assures that the Boolean covers $\psi_{\acute{B}B} : (\mathbf{M}(B)\times_L\mathbf{M}(\acute{B})) \to L$ and $\psi_{B\acute{B}} : (\mathbf{M}(B)\times_L\mathbf{M}(\acute{B})) \to L$ cover the same part of L compatibly.

Now, given a functor of Boolean coverings $\mathbf{T}$ for a quantum event algebra L, we call it *a functor of Boolean localizations of* L, or a *structure* *sheaf of Boolean coefficients of* L, if and only if the Boolean coordinate cocycle conditions are satisfied. Then, we can show that for a dense generating family of Boolean covers in a Boolean localization functor $\mathbf{T}$ of L, meaning that they generate an epimorphic ideal of Boolean coverings of L, the counit of the Boolean frames-quantum adjunction is restricted to a quantum algebraic isomorphism, that is structure-preserving, injective and surjective [33]. In turn, the right adjoint functor of the adjunction restricted to a Boolean localization functor is full and faithful. This argument can be formalized more precisely in topos theoretic terminology using the technical means of a subcanonical Grothendieck topology (called the topology of epimorphic families) on the base category of Boolean algebras. For details the interested reader should consult [34]. Consequently, $\mathcal{L}$ is a reflection of the topos of variable local Boolean frames $\mathbf{Sets}^{\mathcal{B}^{op}}$, and the structure of a quantum event algebra L in $\mathcal{L}$ is preserved by the action of a family of Boolean frames if and only if this family forms a Boolean localization functor of L. The next step is to show that the category of quantum event algebras $\mathcal{L}$ is equipped with a subobject classifier or classifying object acting as an object of truth values for the valuation of propositions describing the behavior of quantum systems.

7. Existence of the Subobject Classifier in $\mathcal{L}$

A proper understanding of the meaning of the existence of a subobject classifier in the category of quantum event algebras presupposes the clarification of the notion of subobjects in any categorical environment, since it is going to be the main conceptual tool in our argumentation. A subobject of an object X in any category $\mathcal{X}$ with pullbacks is an equivalence class of monic arrows targeting X, denoted by $\mu : M \hookrightarrow X$. The set of all subobjects of X in the category $\mathcal{X}$, denoted by $\boldsymbol{\Theta}_{\mathcal{X}}(X)$, is a partially ordered set under inclusion of subobjects. Now, $\boldsymbol{\Theta}_{\mathcal{X}}$ can be construed as a presheaf functor in $\mathbf{Sets}^{\mathcal{X}^{op}}$ by the operation of pulling back as follows. Given an arrow $g : Y \to X$ in $\mathcal{X}$, the pullback of any monic arrow $\mu : M \hookrightarrow X$ along the arrow g is a new monic arrow $\acute{\mu} : \acute{M} \hookrightarrow Y$, that is a subobject of Y, and obviously the assignment $\mu \mapsto \acute{\mu}$ defines a function $\boldsymbol{\Theta}_{\mathcal{X}}(g) : \boldsymbol{\Theta}_{\mathcal{X}}(X) \to \boldsymbol{\Theta}_{\mathcal{X}}(Y)$.

An immediate question that arises in this setting is related with the possibility of representing the subobject functor $\boldsymbol{\Theta}_{\mathcal{X}}$ in $\mathbf{Sets}^{\mathcal{X}^{op}}$ internally by an object Ω in the category $\mathcal{X}$ such that, for each X in $\mathcal{X}$, there exists a natural isomorphism:

$$\iota_X : \boldsymbol{\Theta}_{\mathcal{X}}(X) \cong Hom_{\mathcal{X}}(X, \Omega).$$

If the subobject functor becomes representable with representing object Ω in $\mathcal{X}$, then we say that the category $\mathcal{X}$ is equipped with a subobject classifier. By this term we mean a universal monic arrow

$$T := \mathit{True} : 1 \hookrightarrow \Omega$$

such that, to every monic arrow $\mu : M \hookrightarrow X$ in $\mathcal{X}$, there is a unique characteristic arrow ϕ_μ, which, with the given monic arrow μ, forms a pullback diagram in $\mathcal{X}$:

$$\begin{tikzcd} M \arrow[r, "!"] \arrow[d, "\mu"] & 1 \arrow[d, "T"] \\ X \arrow[r, "\phi_\mu"] & \Omega \end{tikzcd}$$

This is equivalent to saying that every subobject of X in $\mathcal{X}$ is uniquely a pullback of the universal monic T. Conversely, satisfaction of this property amounts to saying that the subobject functor $\mathbf{\Theta}_{\mathcal{X}}$ is representable by the object Ω, or equivalently, that it is isomorphic to $Hom_{\mathcal{X}}(-, \Omega)$. Notice that the bijection ι_X sends each subobject $\mu : M \hookrightarrow X$ of X to its unique characteristic arrow $\phi_\mu : X \to \Omega$, and conversely.

A particularly important example, where the subobject classifier of a category always exists, is the case when the category $\mathcal{X}$ is a topos. The paradigmatic instance of the function of the subobject classifier in a topos is provided by the classifying function of the two-valued object $\mathit{2} := \{false, true\} = \{0, 1\}$ in the topos **Sets** of normal sets. The terminal object in **Sets** is given by $\mathit{1} = \{*\}$, i.e., by the one-point set. A subobject $\mu : M \hookrightarrow X$ of a set X is just a subset $M \subset X$, which is classified by the characteristic function $\phi_\mu : X \to \mathit{2}$ on the set X, defined as follows: for every $x \in X$, $\phi_\mu(x) = 1 = true$ if $x \in M$, and $\phi_\mu(x) = 0 = false$ otherwise. Then, by using the injective map $True : \mathit{1} = \{*\} \to \{false, true\} = \{0, 1\}$ defined by $\{*\} \mapsto \mathit{1}$, we indeed obtain the subobject classifier in **Sets**, where $\Omega = \mathit{2}$ classifies subsets of X in **Sets** according to the pullback diagram:

$$\begin{tikzcd} M \arrow[r, "!"] \arrow[d, "\mu"] & 1 \arrow[d, "T"] \\ X \arrow[r, "\phi_\mu"] & 2 \end{tikzcd}$$

Thus, in the topos **Sets** the two-valued Boolean algebra $2 = \{false, true\} = \{0, 1\}$ is the classifying object or object of truth values, which means that the internal logic of the category **Sets** is the usual interpretation of classical logic. Equivalently, we may say that the category of Boolean event algebras $\mathcal{B}$ is classified internally by the two-valued Boolean event algebra 2.

In contradistinction with the preceding case, as already noted in Section 2, Kochen-Specker's theorem prohibits the existence of a global classifying two-valued homomorphism on a quantum event algebra, and thus, the latter cannot be embedded into a Boolean event algebra. Due to this, a pertinent issue regarding the logic of quantum events is to find out if there exists another classifying object Ω in $\mathcal{L}$ acting as an object of truth values in analogy with the classical case. The representation of quantum event algebras in terms of Boolean localization functors is particularly suited to address this issue. It is useful to recapitulate the role of Boolean localization functors by means of the following diagram displaying the function of the counit of the Boolean frames-quantum adjunction:

$$\begin{tikzcd} \mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M} \arrow[dr, "\epsilon_L"] & \\ \mathbf{M}(B) \arrow[u, "\psi_B \otimes (-)"] \arrow[r, "\psi_B"] & L \end{tikzcd}$$

If the functor of Boolean frames $\mathbf{R}(L)$ of L is restricted to a Boolean localization functor $\mathbf{T}$, then the counit of the adjunction localized to $\mathbf{T}$ becomes an isomorphism, and thus quantum events are represented isomorphically in terms of equivalence classes $\psi_B \otimes q$, $q \in \mathbf{M}(B)$, of pointed Boolean covers, namely pointed Boolean frames qualified as local covers of

L. Thus, the action of Boolean localization functors on quantum event algebras L makes the category $\mathcal{L}$ a reflection of $\mathbf{Sets}^{\mathcal{B}^{op}}$. Technically speaking, this means that $\mathcal{L}$ is a complete category and monic arrows are preserved by the right adjoint Boolean realization functor of quantum event algebras. In particular, there exist a terminal object and pullbacks of monic arrows [20]. Thus, there exists a subobject functor for a quantum categorical event structure $\mathcal{L}$ equipped with Boolean localization functors.

The *subobject functor* of $\mathcal{L}$ is defined as follows:

$$\mathbf{Sub} : \mathcal{L}^{op} \to \mathbf{Sets}.$$

The functor **Sub** is a contravariant functor by pulling back. Composition with a proper or dense Boolean modeling functor defines a presheaf in $\mathbf{Sets}^{\mathcal{B}^{op}}$, called the *Boolean frames modeled subobject functor* of $\mathcal{L}$, as follows:

$$\mathbf{Sub} \circ \mathbf{M} : \mathcal{B}^{op} \to \mathcal{L}^{op} \to \mathbf{Sets}.$$

In a compact notation we obtain,

$$\Upsilon_{\mathbf{M}} := \Upsilon(\mathbf{M}(-)) := \mathbf{Sub} \circ \mathbf{M} : \mathcal{B}^{op} \to \mathbf{Sets},$$

such that,

$$\mathcal{B}^{op} \ni B \mapsto \{[Dom(m) \hookrightarrow \mathbf{M}(B)]\} \in \mathbf{Sets},$$

where the range denotes the set of subobjects of $\mathbf{M}(B)$, i.e., the set of equivalence classes of monic quantum homomorphisms m from $Dom(m)$ to $\mathbf{M}(B)$.

The set $\Upsilon_{\mathbf{M}}(B) = \Upsilon(\mathbf{M}(B))$ is defined as the set of all *subobjects* of $\mathbf{M}(B)$, for every B in $\mathcal{B}$, in the category $\mathcal{L}$. Notice that the set $\Upsilon(\mathbf{M}(B))$, for every B in $\mathcal{B}$, is a partially ordered set under inclusion of subobjects of $\mathbf{M}(B)$.

We emphasize that $\Upsilon_{\mathbf{M}}$ is an object in $\mathbf{Sets}^{\mathcal{B}^{op}}$, so the pertinent question is if it can be represented internally in $\mathcal{L}$ by means of a concrete quantum event algebra Ω, which would play in this manner the role of a *truth-values object*, or equivalently, a *subobject classifier* in $\mathcal{L}$, defined as follows.

The *subobject classifier* of the category of quantum event algebras is a universal monic quantum homomorphism

$$T := True : 1 \hookrightarrow \Omega$$

such that, for every subobject K of L in $\mathcal{L}$, represented by the monic arrow $m : K \hookrightarrow L$, there is a unique characteristic arrow ϕ_m, which, with the given monic arrow m, forms a pullback diagram:

$$\begin{array}{ccc} K & \xrightarrow{!} & 1 \\ \Big\downarrow{m} & & \Big\downarrow{T} \\ L & \xrightarrow{\phi_m} & \Omega \end{array}$$

This is equivalent to saying that every subobject of L in $\mathcal{L}$ is uniquely a pullback of the universal monic T.

The significance of the internal representation of $\Upsilon_{\mathbf{M}}$ in $\mathcal{L}$ boils down to the fact that if this is the case, we could legitimately interpret the concrete classifying object Ω as a truth-values object in $\mathcal{L}$, and consequently use it to classify quantum propositions circumventing in this way the Kochen-Specker prohibition. Thus, in this case, subobjects of a quantum event algebra should be characterized in terms of characteristic functions, which take values not in *2*, but precisely in the truth-values object Ω in $\mathcal{L}$. Most important, in that case the category of quantum event algebras $\mathcal{L}$ would be endowed with a subobject classifier.

7.1. *The quantum object of truth values*

The issue of the existence of a classifying or truth-values object Ω in $\mathcal{L}$ can be fruitfully examined from the perspective of the Boolean-frames quantum adjunction. We have seen previously that the counit of the adjunction, $\epsilon_L : \mathbf{LR}(L) \longrightarrow L$, evaluated at any quantum event algebra L in $\mathcal{L}$, when restricted to a Boolean localization functor of L becomes an isomorphism. In particular, if Ω exists in $\mathcal{L}$, there must be a Boolean localization functor $\mathbf{T}$ of Ω such that,

$$\epsilon_\Omega : \mathbf{R}(\Omega) \otimes_{\mathcal{B}} \mathbf{M} \cong \Omega,$$

where $\mathbf{R}(\Omega)$ is localized to $\mathbf{T} \hookrightarrow \mathbf{R}(\Omega)$.

Henceforth, it is easily deduced that the subobject functor $\Upsilon_{\mathbf{M}}$ is representable internally in $\mathcal{L}$ by a classifying or truth-values object Ω if and only if there exists a natural isomorphism:

$$\Upsilon(\mathbf{M}(-)) \cong \mathbf{R}(\Omega) := Hom_{\mathcal{L}}(\mathbf{M}(-), \Omega).$$

The naturality condition means that the isomorphism holds at each Boolean algebra B in $\mathcal{B}$.

At the other side of the bidirectional adjunction, we have seen that for any presheaf $\mathbf{P} \in \mathbf{Sets}^{B^{op}}$, the unit is defined as $\delta_{\mathbf{P}} : \mathbf{P} \longrightarrow \mathbf{RLP}$. Thus, if we consider as $\mathbf{P} \in \mathbf{Sets}^{B^{op}}$ the subobject functor $\Upsilon(\mathbf{M}(-))$, we obtain the following natural transformation

$$\delta_{\Upsilon(\mathbf{M}(-))} : \Upsilon(\mathbf{M}(-)) \longrightarrow \mathbf{RL}\Upsilon(\mathbf{M}(-)),$$

or equivalently,

$$\delta_{\Upsilon(\mathbf{M}(-))} : \Upsilon(\mathbf{M}(-)) \longrightarrow Hom_{\mathcal{L}}(\mathbf{M}(-), \mathbf{L}\Upsilon(\mathbf{M}(-)).$$

Hence, by inspecting the unit $\delta_{\Upsilon(\mathbf{M}(-))}$ evaluated at $\Upsilon(\mathbf{M}(-))$ with respect to a Boolean localization functor of L, we arrive at the following conclusion. If the unit $\delta_{\Upsilon(\mathbf{M}(-))}$, restricted to a Boolean localization functor $\mathbf{T}$ of L for every L in $\mathcal{L}$, is an isomorphism, then the subobject functor $\Upsilon(\mathbf{M}(-))$ becomes representable in $\mathcal{L}$ by the quantum truth-values object Ω, defined as follows,

$$\Omega := \mathbf{L}\Upsilon(\mathbf{M}(-)),$$

and thus the category of quantum event algebras is endowed with a subobject classifier.

It is remarkable that the unit of the Boolean fames-quantum adjunction $\delta_{\Upsilon(\mathbf{M}(-))}$, localized at $\mathbf{T}$, depicts exactly the object of truth values Ω in $\mathcal{L}$, which is represented, in virtue of the counit isomorphism, as the colimit $\Omega := \mathbf{L}\Upsilon(\mathbf{M}(-))$ in the category of elements of the subobject functor $\Upsilon(\mathbf{M}(-))$. It is straightforward to verify the latter remark, in case the unit $\delta_{\Upsilon(\mathbf{M}(-))}$ is an isomorphism, by noticing that

$$\Omega := \mathbf{L}\Upsilon(\mathbf{M}(-)) \cong \mathbf{L}[\mathbf{RL}\Upsilon(\mathbf{M}(-))] \cong \mathbf{LR}\Omega$$

is precisely an expression of the counit isomorphism for the quantum event algebra Ω restricted to $\mathbf{T}$.

We emphasize that the above unit isomorphism $\delta_{\Upsilon(\mathbf{M}(-))}$, evaluated at each Boolean algebra B serving as the domain of a local Boolean cover of Ω, means that there exists a bijection $\delta_{\Upsilon(\mathbf{M}(B))} := \Delta$ sending each subobject $\lambda : Dom(\lambda) \hookrightarrow \mathbf{M}(B)$ of $\mathbf{M}(B)$ to its unique characteristic arrow ${\Delta^{\lambda}}_B : \mathbf{M}(B) \to \Omega$, and conversely. This is equivalent to saying that every subobject of $\mathbf{M}(B)$ in $\mathcal{L}$ is uniquely a pullback of the universal monic T. Thus, the following diagram is a classifying pullback diagram in $\mathcal{L}$ for each quantum algebraic homomorphism or classifying arrow

$${\Delta^{\lambda}}_B : \mathbf{M}(B) \to \Omega$$

from the domain of a Boolean cover $\mathbf{M}(B)$, such that λ is a subobject of $\mathbf{M}(B)$:

$$
\begin{array}{ccc}
Dom(\lambda) & \xrightarrow{\quad !\quad} & 1 \\
\Big\downarrow{\scriptstyle \lambda} & & \Big\downarrow{\scriptstyle T} \\
\mathbf{M}(B) & \xrightarrow{\quad \Delta^{\lambda}{}_{B} \quad} & \mathbf{L}\Upsilon(\mathbf{M}(-)) := \Omega
\end{array}
$$

Furthermore, the characteristic arrow $\Delta^{\lambda}{}_{B}$ is identified with a local Boolean cover of Ω with respect to the Boolean localization functor $\mathbf{T}$ for which the counit ϵ_Ω is an isomorphism. In other words, at each Boolean algebra B local subobjects of $\mathbf{M}(B)$ correspond bijectively with local Boolean covers of Ω.

8. Quantum Logical Structure of Truth Values and Criterion of Truth

It is now important to provide an explicit representation of the elements of the quantum truth-values object Ω by calculating the colimit $\mathbf{L}\Upsilon(\mathbf{M}(-))$. In order to calculate the latter, it is necessary to specify the index or parameterizing category corresponding to the subobject functor $\Upsilon(\mathbf{M}(-))$, which is defined over the base category of Boolean event algebras $\mathcal{B}$. Following the reasoning presented in Section 5, this index category, denoted by $\int(\boldsymbol{\Upsilon}(\mathbf{M}(-)), \mathcal{B})$, is called the *category of elements of the functor* $\Upsilon(\mathbf{M}(-))$. Its objects are all pairs (B, λ), where λ is a subobject of $\mathbf{M}(B)$, that is a monic quantum homomorphism in $\mathbf{M}(B)$. Its morphisms are given by the arrows $(\acute{B}, \acute{\lambda}) \longrightarrow (B, \lambda)$, namely, they are those morphisms $v : \acute{B} \longrightarrow B$ of $\mathcal{B}$ for which $\lambda * v = \acute{\lambda}$, where $\lambda * v$ denotes the pullback of the subobject λ of $\mathbf{M}(B)$ along v and $\acute{\lambda}$ is a subobject of $\mathbf{M}(\acute{B})$.

Consequently, the index category corresponding to the subobject functor $\Upsilon(\mathbf{M}(-))$ is the category of its elements $\int(\boldsymbol{\Upsilon}(\mathbf{M}(-)), \mathcal{B})$, whence the functor $[\mathbf{M} \circ \int_{\Upsilon(\mathbf{M}(-))}]$ defines the diagram $\mathbf{X} : \mathcal{I} \to \mathcal{L}$ over which the colimit should be calculated. Hence, we obtain:

$$
\begin{aligned}
\mathbf{L}(\Upsilon(\mathbf{M}(-))) &= \mathbf{L}_{\mathbf{M}}(\Upsilon(\mathbf{M}(-))) \\
&= Colim\{\int(\boldsymbol{\Upsilon}(\mathbf{M}(-)), \mathcal{B}) \longrightarrow \mathcal{B} \xrightarrow{\mathbf{M}} \mathcal{L} \dashrightarrow \mathbf{Sets}\}.
\end{aligned}
$$

Next, we consider the set of all pairs of the form,

$$
\boldsymbol{\Sigma} = \{(\lambda, q)/(\ \lambda : Dom(\lambda) \hookrightarrow \mathbf{M}(B)),\ q \in \mathbf{M}(B)\},
$$

for all Boolean algebras B in $\mathcal{B}$, where $\lambda : Dom(\lambda) \hookrightarrow \mathbf{M}(B)$ is a subobject of $\mathbf{M}(B)$ and q is a projection operator in $\mathbf{M}(B)$. Then, from the composition law in the category of elements $\int(\mathbf{\Upsilon}(\mathbf{M}(-)), \mathcal{B})$, we obtain the relation $\top$,

$$(\lambda * u, \acute{q}) \top (\lambda, u(\acute{q})),$$

for any Boolean homomorphism $u : \acute{B} \to B$ in the base category $\mathcal{B}$. On the basis of considerations in Section 5, it can be easily seen that the above relation induces an equivalence relation defined by the identification equations:

$$[\lambda * v] \otimes \acute{q} = \lambda \otimes v(\acute{q}), \quad \lambda \in \Upsilon(\mathbf{M}(B)),\ \acute{q} \in \mathbf{M}(\acute{B}),\ v : \acute{B} \longrightarrow B.$$

Furthermore, if we define $\lambda * v = \acute{\lambda}$, $v(\acute{q}) = q$, where $\acute{\lambda}$ is a subobject of $\mathbf{M}(\acute{B})$ and $q \in \mathbf{M}(B)$, we obtain the equations:

$$\acute{\lambda} \otimes \acute{q} = \lambda \otimes q.$$

Thus, the set $\Upsilon(\mathbf{M}(-)) \otimes_{\mathcal{B}} \mathbf{M}$ is identified as the quotient of the set $\mathbf{\Sigma}$ by the equivalence relation generated by the above equations.

We conclude that the quotient set of equivalence classes of $\mathbf{\Sigma}$, identified with $\Upsilon(\mathbf{M}(-)) \otimes_{\mathcal{B}} \mathbf{M}$, provides the calculation of the colimit $\mathbf{L}\Upsilon(\mathbf{M}(-))$ in **Sets**. The tensor notation indicates that the equivalence classes of the form $\lambda \otimes q$ are canonically identified as elements of the tensor product of the functors $\Upsilon(\mathbf{M}(-))$ and $\mathbf{M}$ over the base category of Boolean event algebras $\mathcal{B}$.

It remains to show that the quotient set $\Upsilon(\mathbf{M}(-)) \otimes_{\mathcal{B}} \mathbf{M}$ has actually the structure of a quantum event algebra Ω restricted to the Boolean localization functor $\mathbf{T}$, characterized by the property that all arrows of the form ${\Delta^{\lambda}}_B$ serve as local Boolean covers of $\Upsilon(\mathbf{M}(-)) \otimes_{\mathcal{B}} \mathbf{M} \equiv \Omega$ with respect to $\mathbf{T}$, for which the counit ϵ_{Ω} is an isomorphism.

For this purpose, since the property of pullback compatibility with respect to $\mathbf{T}$ exists in $\mathcal{L}$, we may consider the arrows $h : \mathbf{M}(D) \to \mathbf{M}(B)$ and $\acute{h} : \mathbf{M}(D) \to \mathbf{M}(\acute{B})$ and the following pullback diagram in $\mathcal{L}$

$$\begin{tikzcd}
\mathbf{M}(D) \arrow[r, "h"] \arrow[d, "\acute{h}"] & \mathbf{M}(B) \arrow[d, "{\Delta^{\lambda}}_B"] \\
\mathbf{M}(\acute{B}) \arrow[r, "{\Delta^{\lambda}}_{\acute{B}}"] & \Omega
\end{tikzcd}$$

such that the relations, $h(d) = q$, $\acute{h}(d) = \acute{q}$ and $\lambda * h = \acute{\lambda} * \acute{h}$, are satisfied. Then, from the identification equations of the equivalence relation, we obtain:

$$\lambda \otimes q = \lambda \otimes h(d) = [\lambda * h] \otimes d = [\acute{\lambda} * \acute{h}] \otimes d = \acute{\lambda} \otimes \acute{h}(d) = \acute{\lambda} \otimes \acute{q}.$$

We may further define,

$$\lambda * h = \acute{\lambda} * \acute{h} = \tau,$$

where τ is a subobject of $\mathbf{M}(D)$. Then, apparently, it holds that:

$$\lambda \otimes q = \tau \otimes d$$

$$\acute{\lambda} \otimes \acute{q} = \tau \otimes d.$$

Now, it can be easily deduced that the set $\Upsilon(\mathbf{M}(-)) \otimes_{\mathcal{B}} \mathbf{M} \equiv \Omega$ is endowed with a partial order relation with respect to $\mathbf{T}$, as follows,

$$\lambda \otimes b \leq \rho \otimes c,$$

if and only if there exist quantum algebraic homomorphisms $\beta : \mathbf{M}(D) \to \mathbf{M}(B)$ and $\gamma : \mathbf{M}(D) \to \mathbf{M}(C)$, and some d_1, d_2 in $\mathbf{M}(D)$, such that, $\beta(d_1) = b$, $\gamma(d_2) = c$, and $\lambda * \beta = \rho * \gamma = \tau$. Thus, we obtain:

$$\lambda \otimes b = \tau \otimes d_1$$

$$\rho \otimes c = \tau \otimes d_2.$$

We conclude that

$$\lambda \otimes b \leq \rho \otimes c,$$

if and only if,

$$\tau \otimes d_1 \leq \tau \otimes d_2 \Longleftrightarrow d_1 \leq d_2.$$

The set $\Upsilon(\mathbf{M}(-)) \otimes_{\mathcal{B}} \mathbf{M} \equiv \Omega$ may be further endowed with a maximal element which admits the following presentations:

$$\mathbf{1} = \tau \otimes 1 := \mathit{true} \qquad \forall \tau \in \Upsilon(\mathbf{M}(D))$$

$$\mathbf{1} = id_{\mathbf{M}(B)} \otimes b := \mathit{true} \qquad \forall b \in \mathbf{M}(B),$$

and an orthocomplementation operator,

$$[\tau \otimes d]^{\star} = \tau \otimes d^{\star}.$$

Then, it is easy to verify that the set $\Upsilon(\mathbf{M}(-)) \otimes_{\mathcal{B}} \mathbf{M} \equiv \Omega$ endowed with the prescribed operations is actually a quantum event algebra. Thus, Ω

constitutes the classifying or truth-values quantum event algebra, where the characteristic arrows $\Delta^{\lambda}{}_{B}$ serve as local Boolean covers of Ω with respect to the corresponding Boolean localization functor $\mathbf{T}$. We form, therefore, the following conclusions.

Quantum Truth-Values Algebra: The elements of the quantum truth-values algebra $\Omega \equiv \Upsilon(\mathbf{M}(-)) \otimes_{\mathcal{B}} \mathbf{M}$ are equivalence classes represented in tensor product form as follows:

$$[\delta_{\Upsilon(\mathbf{M}(B))}]^{\lambda}(b) := \Delta^{\lambda}{}_{B}(b) = \lambda \otimes b,$$

where,

$$[\lambda * v] \otimes \acute{b} = \lambda \otimes v(\acute{b}), \quad \lambda \in \Upsilon(\mathbf{M}(B)),\ \acute{b} \in \mathbf{M}(\acute{B}),\ v : \acute{B} \to B,\ v(\acute{b}) = b,$$

and $[\delta_{\Upsilon(\mathbf{M}(B))}]^{\lambda} := \Delta^{\lambda}{}_{B}$ denotes a local Boolean cover of Ω in the Boolean localization functor $[\delta_{\Upsilon(\mathbf{M}(-))}]^{(-)} := \mathbf{T}$ of Ω using the unit isomorphism. Thus, truth-value assignment in quantum mechanics is localized with respect to equivalence classes of compatible Boolean frames belonging to a Boolean localization functor of a quantum event algebra.

Criterion of Truth : The criterion of truth for the category of quantum event algebras $\mathcal{L}$ with respect to a Boolean localization functor of a quantum event algebra L in $\mathcal{L}$, given the preceding specification of the quantum truth-values object Ω, is the following:

$$\Delta^{\lambda}{}_{B}(b) = \lambda \otimes b = \mathit{true} \qquad \textbf{iff} \qquad b \in \mathit{Image}(\lambda), \qquad \forall \lambda \in \Upsilon(\mathbf{M}(B)),$$

where b is the projection operator that identifies a corresponding quantum event or proposition $p = \psi_B(b) \equiv \psi_B \otimes b$ with respect to the local Boolean cover $\psi_B : \mathbf{M}(B) \to L$, i.e., with respect to the context of $\mathbf{M}(B)$.

9. Semantical Aspects of Quantum Truth Valuation

The conceptual essence of the existence of a quantum truth-values object Ω in the category of quantum event algebras, as specified concretely in the previous section, is associated with the fact that Ω constitutes the appropriate quantum algebra for valuations of propositions describing the behavior of a quantum system, in analogy with the classical case, where the two-valued Boolean algebra *2* is properly used. Thus, in the quantum case, subobjects of a quantum event algebra L admit classifying truth-values assignments in terms of characteristic morphisms, which take values not in *2*, but in the truth-values object Ω with respect to a Boolean localization functor of L.

Intuitively, within the proposed category-theoretic scheme, the fundamental operation of the classifying object Ω is based on the following functorial aspect: a Boolean localization functor acting on a quantum event algebra L induces a corresponding Boolean localization functor in the subobject classifier Ω, such that the characteristic arrows of the subobjects of the Boolean domain covers in the former play the role of local Boolean covers in the latter. Due to the significance of this remark, let us explain the functionality of the quantum truth-values object Ω in more detail by considering the following pullback diagram

$$\begin{tikzcd} Dom(l * \psi_B) \arrow[r] \arrow[d, "l * \psi_B"] & K \arrow[d, "l"] \\ \mathbf{M}(B) \arrow[r, "\psi_B"] & L \end{tikzcd}$$

where the monic quantum homomorphism $l : K \hookrightarrow L$ denotes a subobject of a quantum event algebra L, $\psi_B : \mathbf{M}(B) \longrightarrow L$ signifies a local Boolean cover belonging to a Boolean localization functor of L, and $l * \psi_B \equiv \lambda$ expresses the pullback of the quantum subobject l along ψ_B, thus, denoting the subobject λ of $\mathbf{M}(B)$. According to the function of the subobject classifier Ω, the characteristic arrow of the subobject $l : K \hookrightarrow L$ of L is specified as an equivalence class of pullbacks of the subobject l along its restrictions on all Boolean covers in a Boolean localization functor of L. Thus, for each projection operator b in $\mathbf{M}(B)$, coordinatizing a quantum event or proposition $p = \psi_B(b)$ of L, with respect to a local Boolean cover $\psi_B : \mathbf{M}(B) \to L$, we obtain:

$$l \otimes \psi_B(b) = (l * \psi_B) \otimes b = \lambda \otimes b.$$

Consequently, by using the criterion of truth of Section 8, it is valid that $\lambda \otimes b = true$ if and only if b belongs to the $Image(\lambda)$. We stress that this criterion holds for all λ in the set of subobjects $\Upsilon(\mathbf{M}(B))$. Then, when can we say that a particular proposition p in L is in the $Image(l)$? Clearly, since each λ is the restriction of l with respect to a corresponding subobject of $\mathbf{M}(B)$, this will be the case if and only if $\lambda \otimes b = true$ for all and only those λ for which this restriction is non-empty. Thus, for each Boolean cover $\mathbf{M}(B)$ of L, the value $\mathbf{1} = true$ in Ω is assigned to all those b in $\mathbf{M}(B)$ belonging to the restriction of the subobject $l : K \hookrightarrow L$ of L with respect to the subobject λ of $\mathbf{M}(B)$, for all these λ. In particular, if the Boolean covers are monic morphisms, each pullback of this form is expressed as the intersection of the subobject l with the corresponding cover in the Boolean localization functor.

Conceptually, every quantum event or proposition of a quantum event algebra L is localized with respect to all Boolean frames $\mathbf{M}(B)$ belonging to a Boolean localization functor of L by means of pulling back or restricting. Accordingly, truth-value assignment in quantum mechanics is contextualized with respect to ideals of local Boolean covers, which are pullback compatible and cover completely a quantum event algebra. With respect to each such contextualization we obtain a contextual truth valuation of the restricted proposition associated with the corresponding frame $\mathbf{M}(B)$ specified by the truth rule $\lambda \otimes b = \mathit{true}$ if and only if b belongs to the $\mathit{Image}(\lambda)$, holding for every subobject λ of $\mathbf{M}(B)$, where b represents the restricted quantum proposition with respect to the Boolean frame $\mathbf{M}(B)$. In this sense, elementary propositions associated with the description of the behavior of a quantum system in various contexts of observation, identified by local frames in Boolean localization functors of a quantum event algebra, are naturally assigned truth values in Ω according to the preceding criterion of truth.

We emphasize in this respect that the quantum truth-values object Ω enables not only a determinate truth valuation in each fixed frame $\mathbf{M}(B)$, but in addition, it amalgamates internally all compatible truth valuations with respect to all Boolean frames belonging to a Boolean localization functor of L. This, in effect, is established by the formation of equivalence classes, which are represented in tensor product form, via the truth-value *true* in Ω. The maximally compatible equivalence class identified with the maximal element *true* in Ω specifies a complete description of states of affairs with respect to the considered Boolean localization functor. In this way, truth-value assignment in quantum mechanics is localized and consequently contextualized with respect to tensor product equivalence classes formed among compatible Boolean frames belonging to a Boolean localization functor of a quantum event algebra.

Importantly, the attribution of truth values to quantum mechanical propositions arising out of the preceding category-theoretic scheme circumvents consistently the semantic ambiguity with respect to binary truth-value assignments to propositions that is inherent in conventional quantum mechanics, in the following sense. All propositions that are certainly true or certainly false (assigned probability value 1 or 0) according to conventional quantum mechanics are also certainly true or certainly false according to the category-theoretic approach. The remaining propositions (assigned probability value different from 1 and 0) are semantically undecidable according to the former interpretation, they are neither true nor false, while they have determinate (albeit unknown) truth values, they are either true or false, according to the latter. These values, however, depend not only on the state of the physical system that is considered but also on the context through which the system is investigated, thus capturing the endemic feature of quantum contextuality. Indeed, as already explained, the existence of the subobject classifier Ω leads naturally to contextual truth-value assignments to quantum mechanical propositions, where each proposition pertaining to a physical system under investigation acquires a determinate truth value with respect to the context defined by the corresponding observable to be measured.

A particularly interesting application of the proposed scheme refers to the following case involving partially overlapping, incompatible Boolean frames, physically realized as experimental contexts for the measurement of quantum observables pertaining to a system. Thus, let A and E be two incompatible observables of a quantum system in a given state sharing one or more projection operators in their corresponding spectral decompositions. Let $\mathbf{M}(B_A)$ and $\mathbf{M}(B_E)$ be the corresponding Boolean subalgebras in the system's Hilbert space quantum event structure associated to the observables A and E, respectively. From a physical perspective, the quantum truth-values object Ω takes into account the whole set of possible ways of assigning truth values to the propositions associated with the projectors of the spectral decomposition of a given observable. Then, the subobject classifier Ω makes it possible to refer, at least partially, to the truth valuation of propositions represented by projectors pertaining to incompatible observables with respect to the initially chosen, without facing a Kochen-Specker contradiction, in the following sense: once an observable is selected to be measured, say A, and thus the associated context of measurement is fixed, we may consistently refer to Boolean truth valuations of observable E, as far as its common projectors with A are concerned, by taking into account the Boolean information that $\mathbf{M}(B_A) \cap \mathbf{M}(B_E)$ has about $\mathbf{M}(B_E)$. It is important to realize that in this framework no Kochen-Specker contradiction arises, since these truth valuations are considered from a fixed context. Furthermore, the sheaf theoretical representation of a quantum algebra of events, in terms of Boolean localization functors, takes precisely into account the compatibility conditions of these Boolean subalgebras with respect to their intersection in such a way as to leave invariant the amount of information contained in a quantum system. As indicated in Section 6, this underlying invariance property is satisfied if and only if the counit of the adjunction, restricted to those Boolean localization functors, is an isomorphism, that is, structure-preserving, $1-1$ and onto. Inevitably, this state of affairs allows one to formalize the extent to which we can consider as objective properties of a physical system, and hence, attribute well-defined truth values to their corresponding propositions, those properties represented by projectors pertaining to the overlaps of different Boolean covers without facing no-go theorems.

It is instructive to note at this point that considering a preparatory Boolean environment for a system to interact with a measuring arrangement does not determine which event will take place, but it does determine the *kind* of event that will take place. It forces the outcome, whatever it is, to belong to a certain definite Boolean subalgebra of events for which the standard measurement conditions are invariant. Such a set of standard conditions for a definite kind of measurement constitutes a set of necessary and sufficient constraints for the occurrence of an event of the selected kind. As already explained in Section 4, this equivalently means in the light of our approach that a Boolean algebra in the lattice of quantum events picked by an observable to be measured instantiates locally a physical context, which may serve as a logical Boolean reference frame relative to which results of measurement are being coordinatized. In this respect, Boolean frames or instances of concrete experimental arrangements in quantum mechanics play a role analogous to the reference frames of rods and clocks in relativity theory in establishing a perspectival aspect to reality. We may further observe that the variation of the base Boolean event algebras in the proposed category-theoretic scheme is actually arising from any experimental praxis aiming to fix or prepare the state of a quantum system and corresponds, in this sense, to the variation of all possible Boolean preparatory contexts pertaining to a quantum system for extracting information about it. The key philosophical meaning of this approach implies, therefore, the view that the quantum world is comprehended through overlapping Boolean frames, which interlock, in a category-theoretical environment, to form a coherent picture of the whole in a nontrivial way. Most importantly, this viewpoint is formalized categorically as an instance of the concept of the adjunction of our interpretative scheme between the category of presheaves of variable local Boolean frames and the category of quantum event algebras. In this way, the global structural information of a quantum event algebra can be captured homomorphically or, by restriction to a Boolean localization functor, can be completely constituted (up to isomorphism) by means of gluing together the information collected in all compatible Boolean frames in the form of appropriate equivalence classes. Moreover, the Boolean frames-quantum adjunction provides the conceptual and technical means to show that the category of quantum event algebras is equipped with a quantum truth-values object Ω, a classifying object, which generalizes the classical binary object by classifying information in terms of contextual truth valuations with respect to these distinct Boolean frames. We claim that this development is fundamental philosophically for a novel realist understanding of truth-semantics suited to the quantum domain.

10. Contextual Conception of Truth in Quantum Mechanics

In view of the preceding considerations, therefore, and in relation to philosophical matters, we propose a contextual account of truth that is compatible with the propositional structure of quantum theory by conforming to the following instance of the correspondence scheme:

> [CC] The proposition that P-in-C is true if and only if there is a state of affairs X such that (1) P expresses X in C and (2) X obtains,

where C denotes, in general, the context of discourse, and specifically, in relation to the aforementioned quantum mechanical considerations, the experimental context $C_A(\mathbf{M})$, linked to the proposition P under investigation, that is associated with a particular Boolean frame $\mathbf{M}(B)$ belonging to a Boolean localization functor of L, and A indicating the physical magnitude under investigation.

Let us initially note that the proposed contextual account of truth satisfies Tarski's [35, p. 188] criterion of material adequacy, known as "convention T" or "schema T", for a theory of truth:

> (T) The proposition that "P" is true if, and only if, P

where the symbol "P" in (T) represents the name of the proposition which P stands for. To this purpose, let us consider a particular proposition P: "system S has the property $P(A)$". Assume context-dependence with regard to $P(A)$, i.e., the latter property of S holds within context C. Then, proposition P is concisely translated as: "system S has the property $P(A)$-in-C".

Suppose now, harmlessly, that this proposition is true. If so, the following instance of the T-schema must be true a priori:

> The proposition that "system S has the property $P(A)$-in-C" is true if, and only if, system S has the property $P(A)$-in-C.

If, however, the property $P(A)$ of S is context-dependent upon C, then the proposition that system S has the property $P(A)$ must also be context-dependent upon C. Thus, in conformity with the propositional status entering into the scheme [CC], the preceding instance of the T-schema can be written equivalently in succinct form as:

> The proposition that "P-in-C" is true if, and only if, P-in-C.

Clearly, the logical operation of the bi-conditional in the preceding T-sentence is governed again by the T-schema, or the truism, that the content of a proposition determines the necessary and sufficient conditions under which it is true. Thus for any given true proposition which is context-dependent upon C, the fact that makes it true is the context-dependent fact (or state of affairs) upon C that the proposition expresses. Truth contextuality follows naturally from the contextuality of makers of propositional truths. If the latter are context-dependent, then whatever truths may be expressed about them must also be contextual.

The proposed account of truth, as encapsulated by the scheme [CC] of contextual correspondence, incorporates explicitly a context-dependence texture of the "world-word" relation, if the world, in its microphysical dimension, is to be correctly describable. The truth-making relationship is now established, not in terms of a raw un-conceptualized reality, as envisaged by the traditional scheme of correspondence truth, but between a well-defined portion of reality as carved out by the experimental context and the propositional content that refers to the selected context. Such interdependence of propositional content and referential context is not by virtue of some meta-scientific principle or philosophical predilection, but by virtue of the microphysical nature of physical reality displaying a context-dependence of facts. Truthmakers of quantum mechanical propositions, namely facts or actual states of affairs, are not in general pre-determined, pre-fixed; they are not 'out there' wholly unrestrictedly, regardless of the consideration of a well-defined context of discourse. On the other hand, the traditional conception of correspondence truth, as exemplified either by the alethic scheme [CF] or [CS], alluded to in the introduction, and involving a direct context-independent relation between singular terms of propositions and definite autonomous facts of an external reality, may be viewed as a *species* or as a *limit case* of the more generic alethic scheme of contextual correspondence [CC], when the latter is applied in straightforward unproblematic circumstances where the non-explicit specification of a context of discourse poses no further consequences.

If, however, ascriptions of truth values to propositions are context-dependent in some way as the scheme [CC] implies, it would appear, according to traditional thinking, that one is committed to antirealism about truth. In our opinion, this assumption is mistaken. The contextual account of truth suggested here conforms to a realist conception of truth, which, moreover, is compatible with contemporary physics; it subscribes neither to an epistemic nor to a relative notion of truth. Such an account essentially denies that there can be a "God's-eye view" or an absolute Archimedean standpoint from which to state the totality of facts of nature. For, in virtue of the Kochen-Specker theorem, there simply does not exist, within a quantum mechanical discourse, a consistent binary assignment of determinately true or determinately false propositions independent of the appeal to a context. Propositional content seems to be linked to a context. This connection between referential context and propositional content means that a descriptive elementary proposition in the domain of quantum mechanics is, in a sense, incomplete unless it is accompanied by the specified conditions of an experimental context under which the proposition becomes effectively truth-valued [36]. In view of our approach, the latter observation underlines the fact that the conceptual significance of a logic of propositions referring to the description of a quantum system lies, not at the level of non-contextual propositions forming the original axiomatized poset structure of quantum logic, but on the level of propositions holding in distinct Boolean frames. The proposed categorical framework reveals precisely that the logic of a quantum event structure is to be sought not in its internal constitution as a set-theoretical entity endowed with unrestricted primary qualities, but rather, in the form of its relationship with the Boolean kind of structure through the established network of adjoint functors between the topos of variable overlapping Boolean frames and the category of quantum event algebras. This conception enlightens yet further the connection between a quantum algebra of events and its underlying building blocks of Boolean algebras by clarifying the contextual character of quantum theory.

We note in this respect that the proposed account of truth of contextual correspondence [CC] ought to be disassociated from a pragmatic instrumental notion of truth. From the category theoretical perspective of the present paper, the reference to a Boolean preparatory experimental context should not be viewed primarily as offering the evidential or verificationist basis for the truth of a proposition; it does not aim to equate truth to verification. Nor should it be associated with practices of instrumentalism, operationalism and the like; it does not aim to reduce theoretical terms to products of operational procedures. It rather provides the appropriate *conditions* under which it is possible for a proposition to receive consistently a truth value. Whereas in classical mechanics the conditions under which elementary propositions are claimed to be true or false are determinate independently of the context in which they are expressed, in contradistinction, the truth-conditions of quantum mechanical propositions are determinate within a context. In other words, the specification of the context is *part and parcel* of the truth-conditions that should obtain for a proposition in order the latter to be invested with a determinate (albeit unknown) truth value. Otherwise, the proposition is, in general, semantically undecidable. In the quantum description, therefore, the introduction of the experimental context is to select at any time t a specific complete Boolean subalgebra $\mathbf{M}(B_A)$ in the global non-Boolean algebra of propositions of a quantum system as co-definite; that is, each proposition in $\mathbf{M}(B_A)$ is assigned at time t a definite truth value, "true" or "false", or equivalently, each corresponding property of the system either obtains or does not obtain [37]. Significantly, due to the functioning of the quantum classifying object Ω, as specified by the explicit criterion of Section 8, the truth of a proposition in a Boolean frame remains invariant with respect to truth valuations of all other compatible propositions in all Boolean frames $\mathbf{M}(B)$ belonging to a Boolean localization functor of a quantum event algebra. Thus, in our approach, the specification of a particular Boolean frame, or a concrete experimental context, provides in a consistent manner the necessary conditions whereby bivalent assignment of truth values to quantum mechanical propositions is *in principle* applicable. This marks the fundamental difference between conditions for well-defined attribution of truth values to propositions and mere verification conditions.

This element also signifies the transition from the transcendence condition of the conventional correspondence theory of truth of Section 1 to a reflective-like *transcendental* reasoning of the proposed account of truth. That is, it signifies the transition from the uncritical qualification of truth values to propositions beyond the limits of experience and acknowledging them as being true or false *simpliciter*, to the demarcation of the limits of possible experience or to the establishment of pre-conditions which make possible the attribution of truth values to propositions. In the quantum description, therefore, the specification of the experimental context forms a *pre-condition* of quantum physical experience, which is necessary if quantum mechanics is to grasp empirical reality at all. Any microphysical fact or event that 'happens' is raised at the empirical level only in conjunction with the specification of an experimental context that conforms to a set of observables co-measurable by that context. The introduction of the experimental context furnishes thus the status of a material presupposition of any empirical access to the quantum level of reality, of any possible empirical inquiry on the microscopic scale, and hence of any possible cognizance of microphysical objects as scientific objects of experience. In this respect, the specification of the context constitutes a methodological act preceding any empirical truth in the quantum domain and making it possible.

Nor the proposed contextual conception of truth is a relative notion itself; the propositions to which it applies are relative. They are relative to a specific Boolean subalgebra of propositions which are determinately true or false of a system at any particular time. For, as already argued, a quantum mechanical proposition is not true or false *simpliciter*, but acquires a determinate truth value with respect to a well-defined context of discourse as specified by the state of the quantum system concerned and a particular observable to be measured. Thus, the conditions under which a proposition is true are *jointly* determined by the context in which the proposition is expressed and the actual microphysical state of affairs as projected into the specified context. What makes a proposition true, therefore, is not that is relative to the context (as an alethic relativist must hold, see, for instance, [38]) but whether or not the conditions in question obtain. The obtainment of the conditions implies that it is possible for us to make, in an overall consistent manner, meaningful statements that the properties attributed to quantum objects are part of physical reality. In our approach, therefore, the reason that a proposition is true is because it designates an objectively existing state of affairs, albeit of a contextual nature. In relation to the latter, we wish to emphasize the fact that, in contrast to a panoptical "view from nowhere" of the classical paradigm, the general epistemological implication of quantum theory acknowledges in an essential way a contextual character of knowledge. The classical idea that one can reasonably talk about "all entities", as if the terms "entity" or "object" had a unique, fixed meaning, independently of the appeal to a particular context of reference, is inadequate in the microphysical level of discourse. Whereas in the old classical paradigm, reality was conceived as an absolute concept totally independent of the process of knowledge, in the new quantum paradigm, epistemology — namely, the understanding of the process of knowing — has to be explicitly included in the description of natural phenomena. Epistemology necessarily becomes now an integral part of the theory, an issue pointing at the same time at the meta-theoretical, philosophical level towards an interconnection among epistemological and ontological considerations.

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A LOGICAL ACCOUNT OF QUANTUM SUPERPOSITIONS

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In this paper we consider the phenomenon of superpositions in quantum mechanics and suggest a way to deal with the idea in a logical setting from a syntactical point of view, that is, as subsumed in the language of the formalism, and not semantically. We restrict the discussion to the propositional level only. Then, after presenting the motivations and a possible world semantics, the formalism is outlined and we also consider within this scheme the claim that superpositions may involve contradictions, as in the case of the Schrödinger's cat, which (it is usually said) is both alive and dead. We argue that this claim is a misreading of the quantum case. Finally, we sketch a new form of quantum logic that involves three kinds of negations and present the relationships among them. The paper is a first approach to the subject, introducing some main guidelines to be developed by a 'syntactical' logical approach to quantum superpositions.

Keywords: Superpositions; quantum logic; modal logic; Schrödinger's cat; contradictions; collapse; quantum deduction.

1. Introduction

Superposition is one of the most strange and difficult concepts of quantum mechanics. It is used in the most impressive applications of the theory, being essential, for instance, in quantum information theory.[a] But it is rather difficult to understand precisely what a state of superposition stands for, although this understanding is important for any attempt at meeting the challenge of providing a coherent interpretation of quantum theory. Surely

[a]Suffice to have a look at the SEP entry 'Quantum Entanglement and Information'.

superposition is one of the keys to the multiplicity of quantum mysteries, and must be dealt with in any attempt to understand quantum mechanics.

The traditional Copenhagen interpretation, so as other 'collapse interpretations' of quantum mechanics, assumes that a quantum superposition disappears when a measurement is made, and then only one of the involved states takes place as the state of the system after the measurement (with one of the eigenstates of the measured operator emerging as the state of the system). Common to all these theories is the fact that, if we think that each superposed state stands for a certain 'property' of the system, we never attribute to the superposed system all of the particular properties involved in the superposition: superposition means a different thing than 'having all the involving properties at once', and some no-go theorems grant that under plausible conditions, these 'properties' cannot have actual values at once. The differences reside in the interpretation about *what* (or *who*) causes the collapse or the change of state.

In this paper we shall be dealing with interpretations that assume some form of collapse, and in section 2 we provide a particular way to introduce syntactically the idea of superpositions in the vocabulary of a formal quantum language. To begin with, we advance a Kripke-style semantics for the system. We will not revise here all the history related to the phenomenon, so we assume that the reader is comfortable with the concepts to be introduced below, including those involving modal logic.

In particular, in section 3 we shall employ our formalism to discuss the claim that entanglement and superpositions should be understood as involving or representing contradictions (even if only *potential* contradictions; see [7] and [9] for a defense and development of such claims). We shall argue that this is not the most interesting understanding of what is going on in the formalism of quantum theory, and suggest that it involves presuppositions which are difficult to swallow. To advance a claim we had already defended before (see [1], [2], [3]), we argue that quantum superpositions are better understood not as involving contradictions, but rather a different kind of opposition, traditionally known (from the square of opposition) as *contrariety*.[b] As we hope to make clear, this opposition is in tune with

[b]We must also acknowledge that we are not the first to find such a claim; later we realized that it is present also in other authors, yet in a different perspective; see for instance [5, pp.220-1], [11], [13]. Indeed, it seems to be a well-known fact among quantum logicians that a 'quantum negation' would have these characteristics, so that some of them call it *choice negation* in contradistinction to *exclusion negation* (see [13, p.582]) which has the characteristics of 'classical' negation. We shall be back to this point later.

traditional approaches to the proper understanding of negation in quantum logics. We develop this in section 4 discussing aspects of negation involved in quantum mechanics. The final section with a conclusion then follows.

2. A modal logic of superpositions

In this section we present a modal logic in which we introduce the notions of superposition and measurement at the syntactical level. It is just an attempt to do it, for we are still very tentatively for instance about which should be the right modal system that would underly our system. Furthermore, it seems also that a kind of temporal logic could be profitably used in this context. But these are works to be done. In saying that, we hope the reader takes this paper as a first work in the subject, approaching it from the scratch.

2.1. *Syntax*

Let us assume that our basic logic is the standard propositional modal system S4 with $\neg$, $\wedge$, $\vee$ and $\Diamond$ as primitive, being $\rightarrow$ and $\leftrightarrow$ defined as below. Perhaps a weaker system will be enough, say T, but we wish that the Euclidean property does not hold,[c] so we shall not assume S5 to begin with. We enlarge the language of our system with two new connectives, a binary connective '$\star$', representing 'superposition' and a unary one, 'M', which will stand for 'a measurement is made on'. Furthermore, to facilitate the physical interpretation, we will denote the propositional variables by '$|\psi\rangle$', '$|\psi_1\rangle$', '$|\phi\rangle$', '$|\phi_1\rangle$', etc.[d] Intuitively, propositional variables will denote only pure states that are not superposed. Formulas are defined as usual, except that $\star$ and M apply only to both propositional variables and formulas of the type $|\psi_1\rangle \star |\psi_2\rangle$, but not to formulas in general. Intuitively speaking, $|\psi\rangle \star |\phi\rangle$ is read as indicating the superposition of $|\psi\rangle$ and $|\phi\rangle$, while $M\alpha$ is read as 'a measurement is made on the state α'.

We define formulas explicitly to avoid confusion. We begin with a simple class $\mathcal{B}$ of formulas called *basic formulas*:

i) $|\psi\rangle$, $|\phi\rangle$, $|\omega\rangle$, ... are basic formulas.

[c]A frame for the system is Euclidean if wRw' and wRw'' entails that $w'Rw''$, being R the accessibility relation and w and w' standing for worlds. Soon we will make it clear why the Euclidean property is not desired in our system.

[d]From now on we shall not make more distinctions betwen use and mention, leaving the details for the context.

ii) If α and β are basic formulas, so that α and β share no proper subformula, then $(\alpha \star \beta)$ is a basic formula.

iii) These are the only basic formulas.

Notice that basic formulas are of the form $|\psi\rangle$, or things like $(|\psi\rangle \star |\phi\rangle)$ and longer iterations of $\star$, such as $(|\psi\rangle \star |\phi\rangle) \star |\omega\rangle$ (already using the standard parentheses convention). We have a proviso in the second clause in order to avoid things like $|\psi\rangle \star |\psi\rangle)$ from being formulas. So one may formally write the superposition of many *diverse* states, but we shall not allow superposition of a state with itself.

The *molecular formulas* are now defined as follow:

i) If α and β are any formulas, then $\neg\alpha$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$ and $\Diamond\alpha$ are molecular formulas.

ii) If α is a *basic formula*, then $M\alpha$ is a molecular formula.

iii) These are the only molecular formulas.

Notice that it does not make sense to put an operator like M in front of molecular formulas in general, but only in front of basic formulas. As we shall make clear in the next table, the idea is that M formalizes that a measurement in a system in a given state is being made, so it would be strange or even senseless to measure a conjunction of states or negation of a statement that the system is in a given state, for instance.

In Table 1 below we exemplify the intuitive understanding of a superposed state $|\psi\rangle \star |\phi\rangle$ with a particular case of the entanglement of two states, typical of the Schrödinger's cat case, to be considered below. More explanations are given after the presentation of the axiomatics.

Table 1.

Logic	*Informal interpretation (with examples)*
$\|\psi\rangle$	state vector, wave function $\|\psi\rangle$
$M\|\psi\rangle$	a measurement is made on the system in the state $\|\psi\rangle$
$\|\psi\rangle \star \|\phi\rangle$	Example: $\underbrace{1/\sqrt{2}(\|\psi_A^1\rangle \otimes \|\psi_B^2\rangle)}_{\|\psi\rangle} \pm \underbrace{1/\sqrt{2}(\|\psi_A^2\rangle \otimes \|\psi_B^1\rangle)}_{\|\phi\rangle}$
$\Diamond\|\psi\rangle$	$\|\psi\rangle$ is a possible state the system may evolve to or $\|\psi\rangle$ is a state *in potentia* of the system.
$\neg\|\psi\rangle$	*not* the state $\|\psi\rangle$
$\|\psi\rangle \wedge \|\phi\rangle$	state $\|\psi\rangle$ *and* state $\|\phi\rangle$
etc.	etc.

The bi-conditional $\leftrightarrow$ is defined as usual, being the implication defined as follows (Sasaki hook). The reason for using this implication is that within the formalism of classical modal logic, it acts as the standard implication $(\neg|\psi\rangle \vee |\phi\rangle)$, but we can also associate our system to some axiomatization of an orthomodular quantum logic, and then the chosen implication would be adequate, as is well known [8, p.166].

Definition 2.1 (Implication).

$$|\psi_1\rangle \rightarrow |\psi_2\rangle := \neg|\psi_1\rangle \vee (|\psi_1\rangle \wedge |\psi_2\rangle)$$

Now, we advance a modal logic semantics to cope with the above intuitions about the workings of our apparatus.

2.2. *Semantics*

Here we sketch a way to introduce a Kripke-style semantics for our logic based on a possible world semantics for S4. We give here just a sketch of this formal semantics because we are more worried with the 'concrete' part of the logic, to employ the words of Hardegree [11, p.50], that is, in its connections with the quantum realm.

A frame $\mathcal{F}$ is a pair $\langle W, R\rangle$, being W a non-empty set of worlds and R being a binary relation on W, the accessibility relation. A valuation for basic formulas can be introduced as follow:.

Definition 2.2 (Valuation). *A valuation is a mapping* $\mu : \mathcal{B} \times W \mapsto \{0, 1\}$*, where* $\mathcal{B}$ *is the set of basic formulas.*

Now, obviously, we are not willing to take every valuation into account. One of our claims is that a superposition is a *sui generis* state a system may be in, which is not reducible to anything else except by a measurement. So, we shall discard valuations that could conduce to $\mu(|\psi_1\rangle, w) = \mu(|\psi_2\rangle, w) = \mu(|\psi_1\rangle \star |\psi_2\rangle, w) = 1$. That is, when a system is in a superposition ($\mu(|\psi_1\rangle \star |\psi_2\rangle, w) = 1$ obtains), the individual superposed states *should not* obtain (so, we must have $\mu(|\psi_1\rangle, w) = \mu(|\psi_2\rangle, w) = 0$ when $\mu(|\psi_1\rangle \star |\psi_2\rangle, w) = 1$).

In order to preserve this fundamental idea, as a further condition for our semantics we shall filter the valuations, dividing them between acceptable and unacceptable ones. The valuations which shall have "physical content" in our interpretation are those accepting the following consistency requirement:

Acceptability We shall only take into account valuations μ such that, for any world w, whenever $\mu(\alpha \star \beta, w) = 1$, then for any subformula γ of α or β, $\mu(\gamma, w) = 0$.

That is, if a superposition $(\alpha \star \beta)$ is the case in w, then α is not the case in w, β is not the case in w, and also none of their proper subformulas are the case in w.

Having selected those valuations, we now extend μ to more complex formulas as follows:

Definition 2.3.

i) The usual definitions for propositional connectives and modal operators;
ii) $\mu(M|\psi_1\rangle, w) = 1$ iff $\mu(|\psi_1\rangle, w) = 1$ and $\exists w' \neq w, wRw'$, such that $\mu(|\psi_1\rangle, w') = 1$.
iii) $\mu(M(\alpha \star \beta), w) = 1$ iff (i) there is $w' \neq w$ such that wRw' and $\mu(\alpha, w') \neq \mu(\beta, w')$, and (ii) $\forall w' \neq w$, if $wRw', \mu(\alpha, w') \neq \mu(\beta, w')$.

This condition says that the valuation attributes distinct values to α and β to all accessible words distinct from the actual world and that there exists at least one of such worlds (to which the superposed system evolves after the measurement).

Given those conditions, it is easy to show that some interesting formulas are valid.

To begin with, one may easily check that as a result of our acceptability constraint, $|\psi_1\rangle \star |\psi_2\rangle \rightarrow (\neg|\psi_1\rangle \wedge \neg|\psi_2\rangle)$ is valid. In fact, in any valuation μ satisfying the antecedent ($\mu(|\psi_1\rangle \star |\psi_2\rangle, w) = 1$ for some w), by our acceptability condition we must also have $\mu(|\psi_1\rangle, w) = \mu(|\psi_2\rangle, w) = 0$, so that $\mu(\neg|\psi_1\rangle, w) = 1$ and $\mu(\neg|\psi_2\rangle, w) = 1$.

Consider now the formula $M|\psi_1\rangle \rightarrow |\psi_1\rangle$. Suppose that there exist w and μ such that $\mu(M|\psi_1\rangle, w) = 1$ but $\mu(|\psi_1\rangle, w) = 0$. However, since $\mu(M|\psi_1\rangle, w) = 1$ we have by definition that $\mu(|\psi_1\rangle, w) = 1$, contradiction. So, $M|\psi_1\rangle \rightarrow |\psi_1\rangle$ is also valid.

Still taking into account a simple case, consider $M(|\psi_1\rangle \star |\psi_2\rangle) \rightarrow \neg\Diamond(|\psi_1\rangle \wedge |\psi_2\rangle)$. Suppose again that there exist w and μ such that $\mu(M(|\psi_1\rangle \star |\psi_2\rangle) \rightarrow \neg\Diamond(|\psi_1\rangle \wedge |\psi_2\rangle), w) = 0$. Then, we have

$$\mu(M(|\psi_1\rangle \star |\psi_2\rangle), w) = 1$$

and

$$\mu(\neg\Diamond(|\psi_1\rangle \wedge |\psi_2\rangle), w) = 0.$$

From this last line we have that there exists w' accessible to w such that both $\mu|\psi_1\rangle, w') = 1$ and $\mu(|\psi_2\rangle), w') = 1$. From the truth of the antecedent, however, we must have among other things that $\mu(|\psi_1\rangle, w') \neq \mu(|\psi_2\rangle), w')$. Anyway one chooses such values, we get a contradiction.

For a more sophisticated case, consider $|\psi_1\rangle \star |\psi_2\rangle \wedge M(|\psi_1\rangle \star |\psi_2\rangle) \rightarrow (\Diamond|\psi_1\rangle \vee \Diamond|\psi_2\rangle)$. Suppose, again for a proof by *reductio*, that there exist w and μ such that

$$\mu(|\psi_1\rangle \star |\psi_2\rangle \wedge M(|\psi_1\rangle \star |\psi_2\rangle), w) = 1$$

and

$$\mu((\Diamond|\psi_1\rangle \vee \Diamond|\psi_2\rangle), w) = 0.$$

By the fact that $\mu(|\psi_1\rangle \star |\psi_2\rangle, w) = 1$ we know (by the acceptability constraint) that $\mu(|\psi_1\rangle, w) = \mu(|\psi_2\rangle, w) = 0$. From $\mu(M(|\psi_1\rangle \star |\psi_2\rangle), w) = 1$, we know that there exists w' accessible to w such that $\mu(|\psi_1\rangle, w') \neq \mu(|\psi_2\rangle, w')$. Now, given that $\mu((\Diamond|\psi_1\rangle \vee \Diamond|\psi_2\rangle), w) = 0$, we know that both $\mu(\Diamond|\psi_1\rangle, w) = 0$ and $\mu(\Diamond|\psi_2\rangle, w) = 0$, so that $\mu(|\psi_1\rangle, w') = \mu(|\psi_2\rangle, w') = 0$, contradicting the fact that those formulas must have opposite truth values. So, the formula is valid. Notice that not even the system T was used here.

2.3. *Postulates*

The previous discussion helps us in providing some interesting formulas that are valid in our system, according to the semantics sketched above. Let us now take those formulas as a minimal axiomatic basis for our treatment of quantum superpositions and measurement.

The postulates of our system are those of S4 plus the following ones. We also add an intuitive explanation of the meaning of each postulate, following the informal suggestions given at Table 1:

(1) $|\psi_1\rangle\star|\psi_2\rangle \rightarrow \neg(|\psi_1\rangle\vee|\psi_2\rangle)$ — When a system is in a quantum superposition, it is not in both the superposed states. This will be relevant for the discussion on contradictions. In fact, using the standard 'quantum semantics' and taking the orthogonal complementation of a state for its negation, then if we suppose that the superposed states are orthogonal, that is, that something like $|\psi_1\rangle \star |\psi_1\rangle^\perp$ happens, then this axiom will avoid that the system has both properties associated with the states (to the extent that we can speak of the system and of its properties of course).[e]

[e]That is, we are purposely avoiding a purely instrumentalistic view. See below.

(2) $M|\psi\rangle \to |\psi\rangle$ — Being in a state that is not superposed, the system, if measured, evolves to the same state.
(3) $M(|\psi_1\rangle \star |\psi_2\rangle) \wedge (|\psi_1\rangle \star |\psi_2\rangle) \to (\Diamond|\psi_1\rangle \vee \Diamond|\psi_2\rangle)$ — After a measurement of a system in a quantum superposition, the system evolves to only one of the component states.
(4) $M(|\psi_1\rangle \star |\psi_2\rangle) \to \neg\Diamond(|\psi_1\rangle \wedge |\psi_2\rangle)$ — After a measurement, a system represented by a quantum superposition does not evolve to both superposed states at once (in a same world).

Of course the above schemata can be extended to involve more than two states. We think that the axioms capture the basic ideas concerning superpositions and measurements under the general view that collapse in quantum theory can be assumed. Furthermore, we shall depart from some views, in particular the standard Copenhague interpretation, in assuming that we *can* speak of quantum systems even before measurement. This is, we think, the main novelty of quantum mechanics on what concerns the interpretation of quantum states. Thus, we agree with Sunny Auyang in that “physical theories are about things” [4, p.152], so we shall assume a realistic point of view in saying that there are quantum systems which may be in certain states and that these states may be described by a superposition. Furthermore, we can measure the relevant observables for the systems in certain states. The observables are subsumed in the above axiomatics, for we are assuming that, in measuring a certain state, in reality we are measuring a certain observable in that state, and the axiomatics does not depend on the particular observable being measured, so that they need not be considered in our logical framework.

Furthermore, we also do not make reference to the specific mechanism of collapse, neither the observer (as in von Neumann’s original proposal) nor anything else. This detail does not matter to our schema, so that entering into these controversies would not be productive to our present study.

3. Schrödinger’s cat and contradictions

Schrödinger’s cat is a paradigmatic example of a quantum superposition. We think it is not necessary to revise the details of the description of the experimental situation here, for it is quite well known in the discussions on the philosophy of quantum mechanics. Here, as said before, we shall assume that we can speak of the cat even before a measurement of the entangled state between the cat and the radioactive material inside the cage. That is, the cat is an *element of reality* even when in a superposed state. Hence, there are three possible situations for the cat: no measurement is made and (1) she is in a superposed state; or else a measurement is made and (2) she is alive or (3) she is dead. But, of course, she cannot be alive and dead at once, for situation (1) *does not* say that. Recall that such a situation was ascribed by Schrödinger as being *the* characteristic trait of quantum mechanics [12]. In fact, the superposed state vector can be written as follows, if we consider the system composed by the cat plus the radioactive material that activates the deadly poison:

$$|\mathrm{cat}\rangle + |\mathrm{material}\rangle = \frac{1}{\sqrt{2}}(|\mathrm{cat\ alive}\rangle \otimes |\mathrm{no\ decay}\rangle + |\mathrm{cat\ dead}\rangle \otimes |\mathrm{decay}\rangle) \quad (1)$$

or, in a simpler way,

$$|\mathrm{cat}\rangle = \frac{1}{\sqrt{2}}(|\mathrm{cat\ alive}\rangle + |\mathrm{cat\ dead}\rangle) \quad (2)$$

The superposed state is a vector expressing a situation where the cat is neither definitively alive nor definitively dead, but in a *limbo*, expressed by the superposition. According to us, and following Schrödinger, this is the great novelty of quantum mechanics. As was already much discussed in the literature, superpositions cannot be understood or explained in terms of classical concepts; it is a *sui generis* idea.[f]

There is also a widely quoted passage by Schrödinger which deserves attention, for it is not usually mentioned and which enters quite well in the discussion. Just after the well known (and highly quoted!) passage where he presents his description of the situation of the cat, we can read that the situation

> "*[i]n itself it would not embody anything unclear or contradictory.* There is a difference between a shaky or out-of-focus photograph and a snapshot of clouds and fog banks." (our emphasis) [12]

It seems that he is suggesting that the superposed state acts as a snapshot of clouds, really a situation involving vagueness of some sort. It is not that the cat, when in the superposed state, is blurred by the cloud, but she *is* the cloud. And, as we see (and agree with Schrödinger), there is no contradiction here, if by a contradiction we understand, as in standard logic, a

[f] The following passage by Dirac is also usually quoted: "[t]he nature of the relationships which the superposition principle [that one which enables the formation of superposed states] requires to exist between the states of any system is of a kind that cannot be explained in terms of familiar physical concepts." [10, p.11]

conjunction (not a vector sum) of two propositions, one of them being the classical negation of the other.

This affirmative can be seen from a more 'technical' point of view. In the formalism of quantum mechanics, the situations 'cat alive' and 'cat dead' are represented by arrays in orthogonal subspaces, say S and $S^{\perp}$ so that $S \oplus S^{\perp} = \mathcal{H}$ (the whole Hilbert space, being $\oplus$ the direct sum of subspaces). Thus we should agree with Gary Hardegree when he says that

> "Since a given vector x may fail to be an element of either S or $S^{\perp}$, the quantum negation differs from classical exclusion negation, being instead a species of *choice negation*. A choice negation is characterized by the fact that a sentence A and its choice negation $A^{\perp}$ may *both* fail to be true at the same time. Common examples of choice negation include intuitionistic negation and the standard (diametrical) negation of three-valued logic." [11]

This 'choice negation' will be briefly discussed in the next section, where we will identify it with the operation that gives us the *contrary* of a certain proposition, in the sense of the square of opposition. Let us read by a moment $|\psi\rangle$ and $\neg|\psi\rangle$ as vectors in S and in $S^{\perp}$ respectively. Even if $S \oplus S^{\perp} = \mathcal{H}$, there may be vectors of $\mathcal{H}$ which are neither in S nor in $S^{\perp}$, so $|\psi\rangle$ and $\neg|\psi\rangle$ do not exhaust all possible situations. Thus, being *not* in state $|\psi\rangle$, this does not mean that the system is in state $|\psi\rangle^{\perp}$, for it can be in the superposed state (a sum of two non null vectors, one in S, another in $S^{\perp}$). This motivates the discussion about the meaning of the negation in the quantum context. Our claim is that it is not 'classical' negation, where α is true iff $\neg\alpha$ is false ('exclusion negation' to use Hardegree's and van Fraassen's term). According to us, the quantum negation is (in van Fraassen's terms, to whom Hardegree attributes the name),[g] a 'choice negation' or, as we prefer to say, *contrary negation*. In the next section we shall discuss this point in the context of an alternative interpretation of the above logic.

Taking into account our system, some results can be easily obtained, and their intuitive meaning are clear enough:

Theorem 3.1. $\vdash (|\psi_1\rangle \star |\psi_2\rangle) \to (\neg|\psi_1\rangle \wedge \neg|\psi_2\rangle)$
Proof: Immediate from our first axiom.∎

[g] van Fraassen introduces this terminology in [13]. However, in this paper van Fraassen attributes the terminology to other origins.

A similar result is the following one:

Theorem 3.2. $\vdash (|\psi_1\rangle \star |\psi_2\rangle) \rightarrow \neg(|\psi_1\rangle \wedge |\psi_2\rangle)$

Corollary 3.1. *Here and below sometimes we shall use the quantum mechanics notation for emphasis. The* $^{\perp}$ *operator may be understood as emphasizing the choice negation. A stronger situation than that one shown before (without the need of a measurement):* $|\psi\rangle \star |\psi\rangle^{\perp} \rightarrow \neg(|\psi\rangle \wedge |\psi\rangle^{\perp})$

Theorem 3.3. $\vdash M(|\psi_1\rangle \star |\psi_2\rangle) \rightarrow \neg(|\psi_1\rangle \wedge |\psi_2\rangle)$
Proof:
1. $(|\psi_1\rangle \star |\psi_2\rangle)$ *(hypothesis)*
2. $M(|\psi_1\rangle \star |\psi_2\rangle) \rightarrow \neg\Diamond(|\psi_1\rangle \wedge |\psi_2\rangle)$ *(Axiom 4)*
3. $\neg\Diamond(|\psi_1\rangle \wedge |\psi_2\rangle)$ *(1, 2 Modus Ponens)*
4. $\Box\neg(|\psi_1\rangle \wedge |\psi_2\rangle)$ *(basic modal logic)*
5. $\neg(|\psi_1\rangle \wedge |\psi_2\rangle)$ *(T principle)* ▮

That is, a measurement on a system in superposition never has both the superposed states as a result. Notice that here we have employed only the resources of T.

We could continue exploring our system here, mainly in trying to link it with quantum mechanics. But since our aim is just to introduce the logical system with a minimum of discussion, we leave this job for future works. Anyway, two further theorems follow, whose proofs are immediate:

Theorem 3.4. $|\psi_1\rangle \vdash \neg(|\psi_1\rangle \star |\psi_2\rangle)$

Theorem 3.5. $|\psi_1\rangle \wedge |\psi_2\rangle \vdash \neg(|\psi_1\rangle \star |\psi_2\rangle)$

Finally, the promised explanation about the preference for not using S5. If in the world ω_0 the system (say, the cat) is in a superposed state and in ω_1 she is alive and in ω_2 (both accessible from ω_0) she is dead, of course we don't want that these two last worlds are both accessible to each other, so the accessibility relation should not be Euclidean.

4. Many worlds and a new quantum logic

Another way of interpreting our system is by considering many worlds. In this case, we do not speak of collapse, but of bifurcation. Thus, in making a measurement we get two actual worlds and the considered system may be in both, but since the 'parallel' worlds do not access one another, we shall not have a contradiction here either; that is, the conjunction of two contradictory propositions. Indeed, let us consider once again the case of the cat. In one world, say w_1, the cat is alive, while in w_2 it is dead. Now if we read one of the states as the negation of the other, it seems that in this case we may have subcontrary situations, for the cat can be in *both* states in different worlds, that is, the propositions can be both true, although not both false. But even here there is no strict contradiction (one of them is true if and only if the other one is false; in particular, we don't have the conjunction of the two situations).

There is a logic that can express this situation, a non-adjunctive logic. In such logics, we can have propositions like $|\psi\rangle$ and $\neg|\psi\rangle$, but not their conjunction, that is, they can both be true, but not in the same world. But the involved negation must be treated with care. So, let us just discuss this concept a little.

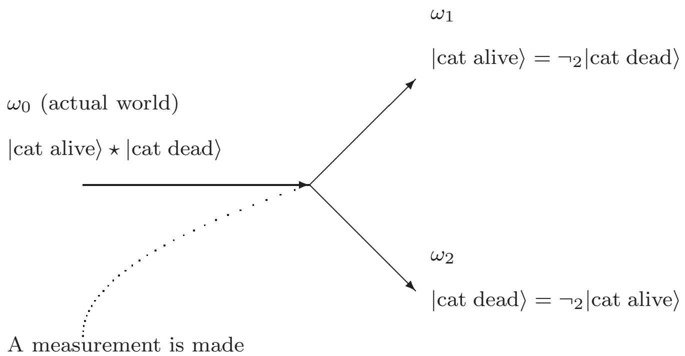

Figure 1. A measurement is made in a superposed entangled system and the world splits in two. Both states of the cat, alive and dead become actual, but not in the same world.

Inspired in the square of opposition (and in [6]), we can consider three kinds of negation, which we term *standard negation*, $\neg_1$ (characterized by 'exclusion negation' and delivering contradictories — the diagonals of the square), *contrary-negation*, $\neg_2$ (or 'choice-negation' in the sense of Hardegree's quotation of the previous section), given by the upper side of the square, and *sub-contrary-negation*, $\neg_3$, given by the bottom side of the square. Our previous discussion about the cat has suggested that $|\text{cat dead}\rangle$ should be read as $\neg_2|\text{cat alive}\rangle$ and vice-versa. A typical $\neg_3$ is paraconsistent negation, while classical logic formalizes $\neg_1$ and intuitionistic logic (for instance) also deals with $\neg_2$. Thus, in the case of many worlds, we can read 'cat in world 1' as '$\neg_2$(cat in world 2)' and vice-versa, so that they can be both false but not both true. In this case, they can be both true but neither in this case do we have a 'true' contradiction (involving $\neg_1$).

We sketch here a minimum of such a quantum logic. We can take our system from above and just change the notion of deduction as follows (we use the symbol $\Vdash$ for this new deduction):

Definition 4.1 (Quantum Deduction). *Let Γ be a set of formulas of the language of our system and let α be a formula. We say that α is quantum deduced from Γ, and write $\Gamma \Vdash \alpha$, if one of the following clauses hold:*

(1) $\alpha \in \Gamma$, or
(2) α is a thesis of the logical system, or
(3) There exists a subset $\Delta \subseteq \Gamma$ such that $\Delta \cup \{\alpha\}$ is non-trivial (according to classical logic), and $\Delta \vdash \alpha$, where $\vdash$ is the standard (classical) deduction symbol.

A set of formulas Δ is $\vdash -$*non-trivial* (according to classical logic) if there is a formula β such that $\Delta \nvdash \beta$. Analogously, we can define $\Vdash -$ non triviality. The most typical situation is to require that Δ be consistent according to classical logic, that is, there is no formula β such that $\Delta \vdash \beta$ and $\Delta \vdash \neg_1\beta$. A consistent set of formulas is of course non-trivial. But the most interesting case is that $\Delta \cup \{|\psi\rangle \wedge \neg_3|\psi\rangle\}$ is non-trivial. That is, our system enables $\Gamma \Vdash |\psi\rangle \wedge \neg_3|\psi\rangle$ without trivializing the system. In doing so, our system, which is standard logic plus the above notion of deduction plus $\neg_3$, is paraconsistent. Anyway, we have neither $\Gamma \Vdash |\psi\rangle \wedge \neg_2|\psi\rangle$ nor $\Gamma \Vdash |\psi\rangle \wedge \neg_1|\psi\rangle$, as it is easy to see, and that is what matters.

Of course you could say that once we have admitted the possibility of a paraconsistent quantum logic, *then* some form of contradiction is possible in the case of superpositions. Really, perhaps you can force the things this way, once you provide a reasonable interpretation about what does a paraconsistent negation mean, that is, what is the intuition behind $\neg_3|\psi\rangle$ (and not only a formal setting). Anyway, in this case, we should agree that we can speak of the cat having properties (contradictory separated properties) before measurement and that these properties do have true values, something questionable in the usual interpretations of the quantum realm.

An interesting theorem can be obtained in considering the above negations, at least if we consider the modal logic S5 as the underlying logic (the case of S4 involving $\neg_3$ must be further investigated).

Theorem 4.1. *We cannot derive a contradiction from a superposition even by using a paraconsistent negation $\neg_3$: that is, superposition does not entail 'contradictions'!*

Proof:

1. $\neg(|\psi\rangle \wedge |\psi\rangle^{\perp}) \rightarrow \Diamond\neg(|\psi\rangle \wedge |\psi\rangle^{\perp})$ *(modal logic – T system)*

2. $\Diamond\neg(|\psi\rangle \wedge |\psi\rangle^{\perp}) \rightarrow \neg\Box(|\psi\rangle \wedge |\psi\rangle^{\perp})$ *(in S5, $\neg\Box$ stands for a paraconsistent negation "$\neg_3$", that is, $p, \neg_3 p \nvdash q$ and $p, \neg_3 p \nvdash \neg_1 q$ [6]).*

3. $|\psi\rangle \star |\psi\rangle^{\perp} \rightarrow \sim(|\psi\rangle \wedge |\psi\rangle^{\perp})$ *(Our axiom plus propositional calculus) — being superposed, the system is not in both states even with the paraconsistent negation $\neg_3$.* ∎

Other arguments contrary to the reading that superpositions may involve contradictions can be seen in the papers by Arenhart and Krause mentioned in our references. The figure below shows the interrelations among the three negations we have considered.

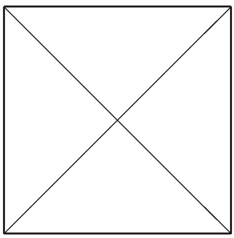

Figure 2. The interplay among the negations, inspired by the square.

5. Conclusion

In this paper we provided a first approach to a logical understanding of superpositions and their measurement. Obviously, superpositions will not be fully understood from a purely logical approach, but we feel that it is fair to put some things clearly by introducing an explicit talk about superposition and measurement in the object language. Perhaps this can help us in spotting the most thorny issues involved with superpositions so as in helping us to achieve a better understanding of the subject. We have approached the problem by first trying to set some intuitive properties of quantum superpositions and the result of measuring a physical system in a superposition, and only then sought to provide for some formal counterparts to those ideas.

Certainly much more is still required in order to achieve a better logical (and physical) understanding of quantum superpositions but, as it happens to almost every inquire into a great mystery, one must proceed with great care and a disposition to revise what was already settled. In particular, we hope to have convinced the reader that, given some fairly uncontroversial assumptions about superpositions and their measurements, there is no sensible sense to be made of the claim that superpositions involve contradictions, a very common claim in popular accounts to quantum mechanics. So, we agree that quantum mechanics produces oddities, but a dead and alive cat is not one of them.

We guess our approach hits correctly at some of the core features of superpositions and their measurements, at least for those willing to accept some form of collapse. We grant that this is a very big 'if', but one has to make a choice. Furthermore, still talking about choices, the modal logic underlying our approach is still very much open to further discussion. We have used most of the time the resources of T, but perhaps distinct systems of modal logic and even distinct fragments of a tense reading of the modal operators could be even better suited (given that 'measure' has a dynamical understanding in collapse interpretations). These are some paths we intend to investigate as a sequence to this first step.

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MIXING CATEGORIES AND MODAL LOGICS IN THE QUANTUM SETTING

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The study of the foundations of Quantum Mechanics, especially after the advent of Quantum Computation and Information, has benefited from the application of category-theoretic tools and modal logics to the analysis of Quantum processes: we witness a wealth of theoretical frameworks casted in either of the two languages. This paper explores the interplay of the two formalisms in the peculiar context of Quantum Theory.

After a review of some influential abstract frameworks, we show how different modal logic frames can be extracted from the category of finite dimensional Hilbert spaces, connecting the Categorical Quantum Mechanics approach to some modal logics that have been proposed for Quantum Computing. We then apply a general version of the same technique to two other categorical frameworks, the 'topos approach' of Doering and Isham and the sheaf-theoretic work on contextuality by Abramsky and Brandenburger, suggesting how some key features can be expressed with modal languages.

Keywords: Quantum mechanics, category theory, modal logic.

1. Introduction

The development of Quantum Computation and Information has caused a new wave of studies in Quantum Mechanics: the possibility of defining quantum algorithms, and the fact that some of them outperform their classical counterparts, has elicited novel theoretical issues and challenges. They range from the abstract study of general features of quantum phenomena, such as non-locality, to the design of logical languages to encode quantum protocols, to diagrammatic calculi for quantum-flavoured categories. Two sets of tools seem to be especially useful in these explorations, the framework of Category Theory and formal languages from Modal Logic. The aim of this paper is to suggest possible ways to connect these tools in a way that is meaningful for the quantum setting.

We will start with an overview of some categorical approaches. This is in no way a review of all the categorically-informed outlooks on the quantum world; we will merely cherry-pick those that are more amenable for our investigation, in the hope that the chosen examples will still carry enough importance in the eyes of the reader. The first research program we describe was pioneered by Abramsky and Coecke. Their work started from the analysis of the categorical structure of the category of finite-dimensional Hilbert spaces and linear maps [1,3] In the last decade this research project has produced many results and a renewed interest in symmetric monoidal categories, the categories used to model compound systems. We then turn our attention to the category of set-valued contravariant functors over the poset of abelian sub-algebras of $\mathcal{B}(H)$, the algebra of bounded operators of a given Hilbert space H. This topos and its associated logic are studied in a line of research carried on by Doering and Isham [9]. Last but not least, we will briefly describe a sheaf-theoretic analysis of non-locality and contextuality initiated by Abramsky and Brandenburger [2].

The second section of the paper will focus on modal languages that have been used to encode quantum phenomena and processes. Again, there are a wealth of modal logic proposed in this context; the main approach we will cover, proposed by Baltag and Smets, exploits the formalism of PDL to represent quantum algorithms and to design a proof system able to prove their correctness. This research group has proposed different logics for this task; here we focus on LQP, the Logic of Quantum Programs, and its compound version LQP^n. This approach is connected with both the traditional logical studies of the foundations of Quantum Mechanics, the so-called standard Quantum Logic, and the "Dynamic Turn" in Logic, that is, the use of modal logics to reason about processes and information. The semantics of these logics are relational structures called Dynamic Quantum Frames, namely relational versions of Hilbert spaces.[a]

Our first step in merging categorical frameworks and modal languages is to describe a procedure to extract modal languages and transition systems from certain functors, first proposed in a seminal paper by Joyal Winskel and Nielsen [11]. We then show how the frames of LQP and LQP^n are related to the categories studied by Abramsky and Coecke. The bridge between the category of finite-dimensional Hilbert spaces and the semantics of these is given by the choice of a specific functor from such category into $\mathbf{Set}_{part}$, the category of sets and partial functions. This construction

[a]This line of research is developed in multiple papers, we refer to the main ones [5,6].

suggests how to obtain a richer semantics, containing probabilistic information, with the choice of a different functor. The intended use for such semantics is the formalization of protocols where probabilities play an essential part.

A more general version of the same construction can be exploited to extract modal languages from the other categorical frameworks. In the case of the topos approach the modal language can be used to describe the relationships between partial states seen as elements of the image of the spectral presheaf. Moreover, we link the Kochen-Specker Theorem to the existence of transition-preserving functions between the transition systems arising from the terminal presheaf and the spectral presheaf, respectively. In the sheaf setting put forward by Abramsky and Brandenburger the modal logic obtained is shown to capture the important notion of strong contextuality.

We refer to the more detailed work [8] for a detailed explanation of the results in subsections 4.1 and 4.2, further discussion and examples. In what follows we employ concepts and notation from Category Theory, Modal Logic and Quantum Computing; we direct the reader to the textbooks in the references [7,13,15] for the necessary background in these areas.

2. Categorical frameworks

We begin by fleshing out the categorical approaches that we listed.

2.1. *Categorical quantum mechanics*

In their paper [3] Abramsky and Coecke outline a study of the foundations of Quantum Mechanics from a category-theoretic perspective. The target of this study is $\mathbf{FdHil}_{\mathbb{C}}$, the category having as objects finite-dimensional Hilbert spaces over the field of complex numbers and as morphisms linear maps.[b] This category can be thought of as the formal environment where Quantum Computing takes place.[c]

The key observation is the following:

Theorem 2.1 (3). *The category* **FdHil** *is a dagger compact closed category with biproducts.*

This in particular means that **FdHil** is:

(1) a symmetric monoidal category

[b]We will drop the subscript in what follows.
[c]The limitation to finite-dimensional Hilbert spaces is a rather standard one in Quantum Computation [15].

(2) a compact closed category
(3) a dagger category
(4) a category with biproducts

and furthermore that all these layers of structure coexist together, namely that the category satisfies some coherence conditions (in [13, pp. 158-9]).

Intuitively, a symmetric monoidal category is a category equipped with an operation to mold objects into compound objects. In **FdHil** this role is fulfilled by the tensor product. Monoidal categories have a special object I which is the unit of the operation, in **FdHil** this is the one-dimensional Hilbert space $\mathbb{C}$. This unit object can be used to characterize scalars in general as morphisms $I \to I$; this definition specializes well, since the linear maps of type $\mathbb{C} \to \mathbb{C}$ correspond indeed to the scalars in $\mathbb{C}$.

A compact closed category is a category having, for each object, a dual object with particular properties. In **FdHil** these are the conjugate spaces, that is, the spaces in which scalars and inner product are the complex conjugate with respect to the original space. Dagger categories have a contravariant, identity-on-objects and involutive endofunctor, namely an operation † that modifies only morphisms and switches domains and codomains. This corresponds to the conjugate-transpose of a linear map in **FdHil**. Via this additional structure we can characterize unitary maps as isomorphisms such that $f^{-1} = f^{\dagger}$ and self-adjoints maps as morphisms such that $f = f^{\dagger}$. This also suggests the abstract characterization of projectors as self-adjoint morphisms such that $f \circ f = f$.

Finally, a category with biproducts is a category with a distinguished object, called *zero object*, and an operation to merge objects together. Contrarily to the monoidal operation, biproducts stand for objects that are completely determined by their components. The zero object in **FdHil** is the 0-dimensional vector space, while the biproduct is the direct sum of Hilbert spaces.

The central ingredients of Quantum Mechanics can be recovered in this categorical framework, and we can give an abstract representation of quantum protocols. Furthermore, we can prove the correctness of a protocol via the commutation of the appropriate diagram.

2.2. *A topos for quantum mechanics*

The second research program that we sketch is part of a general enterprise aimed at reconstructing physical theories within the language of suitable topoi.[d] In the eyes of the authors Doering and Isham, a suitable topos for a given physical theory will have an object Σ, playing the role of the state space, and an object $\mathcal{R}$, representing the possible values of physical quantities. The latter are represented as arrows $\Sigma \to \mathcal{R}$, while propositions are encoded as sub-objects of Σ. We limit ourselves to this superficial description of the general picture and zoom in the case of quantum mechanics.

Given a Hilbert space H, consider the algebra of bounded operators $\mathcal{B}(H)$. In quantum mechanics, a physical quantity is usually represented by a self-adjoint operator in such algebra. Call $\mathcal{V}(H)$ the poset of all abelian sub-algebras of $\mathcal{B}(H)$, ordered by inclusion. We can think of each abelian sub-algebra V as a "classical snapshot" over the quantum system H, where classicality is given by the fact that all the physical quantities represented by operators in V commute. According to the authors the right topos for quantum mechanics is $\mathbf{Set}^{\mathcal{V}(H)}$.

In this setting the presheaf acting as the state space is the *spectral presheaf* $\underline{\Sigma}$, the functor assigning to each $V \in \mathcal{V}(H)$ the set of multiplicative linear functionals $\lambda V \to \mathbb{C}$ and defined on morphisms as restriction. We can think of these functionals as partial states, assigning a value to each bounded operator in V. Proposition are thus sub-objects of $\underline{\Sigma}$. Being functors from $\mathcal{V}(H)$, propositions will now have a different extension depending on the 'context' at which they are evaluated. While in the topos **Set** the set of truth values for proposition is the set $\{0, 1\}$, in the presheaf category $\mathbf{Set}^{\mathcal{V}(H)}$ the sub-object classifier is the presheaf of sieves, hence the truth value of a proposition is also contextual.

2.3. *Sheaf-theoretic analysis of contextuality*

As third and final case study, we turn to the sheaf-theoretic analysis of non-locality and contextuality proposed by Abramsky and Brandenburger [2]. This work shares some similarities with the topos approach, indeed they both deal with presheaves over categories of 'contexts', but nevertheless it is casted at a higher level of generality. We are concerned with the general architecture of the approach and will therefore only depict it in broad strokes.

The primitives of their setting are a set of measurements X and a set of outcomes O, both assumed to be finite. Given a collection of measurements

[d] For a comprehensive survey of the approach we point the reader to the reference [9]. A closely related line of work, that could be amenable for a similar analysis, stems from the reference [10]. For the standard reference in Topos Theory see the classic book [14].

$U \subseteq X$, a *section* is a function $s : U \to O$ associating an outcome to each measurements. We can thus define a presheaf $\mathcal{E} : \wp(X)^{op} \to \mathbf{Set}$, where the powerset is seen as a poset category, assigning $\mathcal{E}(U) = O^U$ and defined on morphisms by restriction: if $i : U \subseteq U'$ then $\mathcal{E}(i)(s) = s|_U$. Therefore such presheaf encodes all the possible events: for each collection of measurements it returns all the possible scenarios in the form of functions matching the measurements to a corresponding outcome. The condition on morphisms ensures that functions restrict coherently to the subsets of their domain.

This definition makes $\mathcal{E}$ into a presheaf; the finiteness of X and O ensures that partial functions can always be glued together by taking the union of their graphs, hence $\mathcal{E}$ is actually a sheaf. This mathematical object is called the *sheaf of events* for the given pair X and O. It is then possible to compose this functor with the endofunctor given by the distributions over a commutative semiring.

3. Dynamic quantum logic

The Logic of Quantum Programs LQP and it compound version LQP^n were designed by Baltag and Smets to express quantum algorithms and prove their correctness [5,6]. The core ideas behind this logics are two. First, we can see the states of a physical system as states of a Modal Logic frame. Second, the dynamics of the system can be captured by means of a PDL-style formalism, i.e., a modal logic formalism containing constructors for modalities. In particular, the intuition is that measurements can be seen as tests and the evolutions of the system as programs.

How do we prove the correctness of an algorithm in this setting? Essentially, by proving that it is a validity of the logic. More precisely, if we are able to represent the systems we want to study as Modal Logic frames, we can express the correctness of an algorithm by proving that the formula encoding the algorithm is true at all states in all systems, i.e., is a validity of the corresponding class of Modal Logic frames. The key result that we need to apply the above line of reasoning is Soundness: we need to show that if a formula is provable in the logic (from some premises) then it is true in all states in all Modal Logic frames (satisfying the premises).

3.1. *The logic LQP*

The logic LQP is an implementation of these ideas. Given a set of atomic propositions At and a set of atomic actions $AtAct$, the set of formulas $\mathcal{F}_{LQP}$

is built by mutual recursion as follows:

$$\psi ::= p \mid \neg\psi \mid \psi \wedge \phi \mid [\pi]\psi$$

where $p \in At$ and the action π is defined as

$$\pi ::= U \mid \pi^{\dagger} \mid \pi \cup \pi' \mid \pi; \pi' \mid \psi?$$

where $U \in AtAct$. The basic actions are meant to represent unitary transformations, while the tests $\psi?$ capture the measurement of a certain property. The composition of actions stands for the sequential composition of quantum gates or measurements, the dagger is the conjugate transpose and the nondeterministic union of actions is exploited to render the nondeterministic behaviour caused by the measurements. Denote with Act the collection of all actions.

The semantics of such a language is the following:

Definition 3.1. Given a Hilbert space H, call L_H its lattice of linear subspaces. A *concrete quantum dynamic frame* is a tuple $\langle \Sigma_H, \{\xrightarrow{P_a?}\}_{a \in L_H}, \{\xrightarrow{U}\}_{U \in \mathcal{U}} \rangle$ such that

(1) Σ_H is the set of all one-dimensional linear subspaces of H
(2) $\{\xrightarrow{P_a?}\}_{a \in L_H}$ is a family of *quantum tests*, partial maps from Σ_H into Σ_H associated to the projectors of the Hilbert space H. Given $\overline{v} \in \Sigma_H$, the partial map $\xrightarrow{P_a?}$ is defined as $\xrightarrow{P_a?} (\overline{v}) = \overline{P_a(v)}$. The map is undefined if $P_a(v)$ is the zero vector.
(3) $\{\xrightarrow{U}\}_{U \in \mathcal{U}}$ is a collection of partial maps from Σ_H into Σ_H associated to the unitary maps from H into H. As for projectors, the map $\xrightarrow{U}$ is defined as $\xrightarrow{U} (\overline{v}) = \overline{U(v)}$.

Call Γ_{CQDF} the class of all concrete quantum dynamic frames.

Definition 3.2. Given a concrete quantum dynamic frame, a set $T \subseteq \Sigma_H$ and a relation $R \subseteq \Sigma_H \times \Sigma_H$, define the following operations

- $T^{\perp} = \{s | \forall t \in T \; s \perp t\}$
- $[R]T = \{s | \forall t \in T \; (s,t) \in R \Rightarrow t \in T\}$
- $R^{\dagger} = \{(s,t) | t \in ([R]\{s\}^{\perp})^{\perp}\}$

where $s \perp t$ is the orthogonality relation between one-dimensional linear subspaces in Σ_H.

Definition 3.3. An *LQP-model* M consists of a concrete quantum dynamic frame $\langle \Sigma_H, \{\xrightarrow{P_a?}\}_{a \in L_H}, \{\xrightarrow{U}\}_{U \in \mathcal{U}} \rangle$ and a valuation function $V : At \to \wp(\Sigma_H)$.

Given an LQP-model, we define an interpretation of the actions and the satisfaction relation by mutual recursion.

Definition 3.4. Call $Cl(\{R_i\}_{i \in I})$ the closure of a set of relations with respect to the operations of relational composition, union and the dagger operation described in Definition [3.2]. An *interpretation of the actions* in an LQP-model is a function $i : Act \to Cl(\{\xrightarrow{P_a?}\}_{a \in L_H} \cup \{\xrightarrow{U}\}_{U \in \mathcal{U}})$ such that

- $i(U) \in \{\xrightarrow{U}\}_{U \in \mathcal{U}}$
- $i(\pi \cup \pi') = i(\pi) \cup i(\pi')$
- $i(\pi^{\dagger}) = i(\pi)^{\dagger}$
- $i(\pi; \pi') = i(\pi); i(\pi')$
- $i(\psi?) = \xrightarrow{P_a?}$ where a is the span of the set $\{s \in \Sigma_H | M, s \vDash_{LQP} \psi\}$

Where ; on the right-hand side is the composition of relations (partial functions in this case), $\cup$ is the union of relations and $\dagger$ is the previously defined operation on relations.

Definition 3.5. Given a model M, a state s in the model and a formula $\psi \in \mathcal{F}_{LQP}$, the satisfaction relation $\vDash_{LQP}$ is defined as

- $M, s \vDash_{LQP} p$ iff $s \in V(p)$
- $M, s \vDash_{LQP} \neg\psi$ iff $M, s \nvDash_{LQP} \psi$
- $M, s \vDash_{LQP} \psi \wedge \phi$ iff $M, s \vDash_{LQP} \psi$ and $M, s \vDash_{LQP} \phi$
- $M, s \vDash_{LQP} [\pi]\psi$ iff for all $(s, s') \in i(\pi)$ we have $M, s' \vDash_{LQP} \psi$

Theorem 3.1 (6 p. 21). *There exist a proof system of LQP which is sound with respect to the class* Γ_{CQDF}.

We refer to the literature [6] for the details of the proof system.

3.2. *The logic LQP^n*

Nevertheless, the formalism of LQP is not enough to express quantum protocols. We need to express *locality*, that is, we need to express the fact that some quantum gates or measurements are performed locally, on certain subsystems. For this reason an enhanced version of LQP, called LQP^n, was developed to capture locality in systems of n qubits.

Suppose given a natural number n. Set $N = \{1, \dots, n\}$ and $I \subseteq N$. The syntax of LQP^n is the same as that of LQP plus the propositional constants

$$\top_I \,|\, 1 \,|\, +$$

and the constant actions $triv_I$. The new propositional constants are used to characterize local properties, while $triv_I$ is used to define local actions.

Definition 3.6. Let H' be a Hilbert space of dimension 2 with basis $\{|1\rangle, |0\rangle\}$. Consider the Hilbert space $H := \otimes_{i=1}^{i=n} H'$ consisting of n copies of H' and call *n-partite quantum dynamic frame* the concrete quantum dynamic frame associated to H.

Set $N = \{1, \dots, n\}$. We write H_I to indicate the tensor product of the Hilbert spaces indexed by the indices in $I \subseteq N$. Thus in particular $H_N = H$.

Call Γ_{CQDF^n} the class of n-partite quantum dynamic frames.

Definition 3.7. The satisfaction relation $\models_{LQP^n}$ contains that of LQP and is defined on the new formulas as

- $M, s \models_{LQP^n} 1$ iff $s = \overline{\otimes_{i=1}^{i=n} |1\rangle_i}$
- $M, s \models_{LQP^n} +$ iff $s = \overline{\otimes_{i=1}^{i=n} |+\rangle_i}$
- $M, s \models_{LQP^n} \top_I$ iff $s \in \top_I^{\Sigma}$

Hence 1 and + are used to denote specific states, while the last condition means that $\top_I$ is true at a state iff that state is I-separated.

Theorem 3.2 (6 p.28). *There is a proof system of LQP^n which extends that of LQP and is sound with respect to the class Γ_{CQDF^n}.*

The correctness of a protocol can then be proved by encoding it in a formula of the logic and then showing that such formula is a validity.

Theorem 3.3 (4,6). *In the logic LQP^n we can give a formal correctness proof of the following algorithms: Teleportation, Quantum Secret Sharing, Superdense Coding, Entanglement Swapping and Logic Gate Teleportation.*

Unfortunately, with this logic we can only encode quantum protocols that succeed with probability 1.

4. Drawing the connection

We now address the question: is there a systematic way to relate these two formalisms that can bear meaningful fruits in the quantum setting?

Definition 4.1. A *small category* is a category such that the collection of objects and the collection of maps are both sets. For a small category $\mathbf{C}$ we indicate with $\mathbf{C}_0$ the set of objects and with $\mathbf{C}_1$ the set of morphisms.

Definition 4.2. Given a small category $\mathbf{C}$ and a functor $U : \mathbf{C} \to \mathbf{Rel}$, a $(\mathbf{C}, U)$-frame is a pair $\langle W, \{R_f\}_{f \in \mathbf{C}_1} \rangle$ such that:

- $W := \{(x, C) | x \in U(C), C \in \mathbf{C}_0\}$
- $R_f := \{((x, C)(U(f)(x), C')) | f : C \to C' \in \mathbf{C}_1\}$

Notice that if $\mathbf{C}$ is small then W is the union of set-many sets, and thus is a set. Similarly, as there are set-many morphisms in $\mathbf{C}$, the collection of all relations R_f will be a set.

This is essentially the construction known as 'category of elements', repackaged into a labelled transition system. An analogous procedure was studied in Theoretical Computer Science in connection with presheaves [16]. In that context, because of contravariance, the relations are defined as follows:

$$R_f := \{((x, C)(y, C')) | U(f)(y) = x, f : C \to C' \in \mathbf{C}_1\}$$

This construction can be turned into a full functor embedding presheaves into the category of transition systems and transition-preserving functions.[e] As a consequence we obtain the following Proposition.

Proposition 4.1. *Suppose $U, U' : \mathbf{C}^{op} \to \mathbf{Set}$. There exists a natural transformation $\theta : U \to U'$ iff there is a transition-preserving function $\overline{\theta}$ from the $(\mathbf{C}, U)$-frame to the $(\mathbf{C}, U')$-frame defined as $\overline{\theta}((x, C)) = (\theta_C(x), C)$.*

The obvious modal language to describe these transition systems is the one provided by the base category $\mathbf{C}$:

$$\phi ::= \quad p \,|\, \neg\phi \,|\, \psi \wedge \phi \,|\, \langle \alpha \rangle \phi$$

where $\alpha \in \mathbf{C}_1$ and $p \in At$, a set of atomic propositions. Notice that this is a fragment of the so-called *path logic* introduced in the presheaf approach' to concurrency [11].[f] Suppose given a model $\mathcal{M} = \langle W, \{R_i\}_{i \in \mathbf{C}_1}, V \rangle$, where

[e] If the transitions on transition system given from a presheaf are defined following the functions, instead of be defined as the inverses of such functions, then there is such a natural transformation if and only if there is a *functional bisimulation* between the two corresponding transition systems.

[f] In the original formulation path logic was conceived with infinitary conjunctions and disjunctions.

$\langle W, \{R_i\}_{i \in \mathbf{C}_1} \rangle$ is one of the aforementioned transition systems and V is a valuation. The satisfaction of the formulas is defined as usual for the propositional case, while for the modalities put:

$$\mathcal{M}, w \vDash \langle \alpha \rangle \phi \quad \text{iff} \quad \exists (w, w') \in R_\alpha \wedge \mathcal{M}, w' \vDash \phi$$

The additional structure of the base category $\mathbf{C}$ is then reflected in the modal language within the modalities, and can be interpreted provided that the functor giving the semantics matches this structure with a corresponding structure on the target category. More precisely, suppose there is an endofunctor $T : \mathbf{C} \to \mathbf{C}$; this will be reflected in the syntax with the existence of modalities of shape $\langle T\alpha \rangle$. Given a functor $U : \mathbf{C} \to \mathbf{Rel}$, if there is a functor $T' : \mathbf{Rel} \to \mathbf{Set}$ such that the diagram

$$\begin{tikzcd} \mathbf{C} \arrow[r, "T"] \arrow[d, "U"'] & \mathbf{C} \arrow[d, "U"] \\ \mathbf{Rel} \arrow[r, "T'"] & \mathbf{Rel} \end{tikzcd}$$

commutes then we can concretely define the satisfaction relation as [g]

$$\mathcal{M}, w \vDash \langle T\alpha \rangle \phi \quad \text{iff} \quad \exists (w, w') \in T'(R_\alpha) \wedge \mathcal{M}, w' \vDash \phi$$

This intuition will be exploited, for example, when dealing with the dagger operator in the next section.

4.1. *Recovering the semantics of LQP*

The procedure outlined in the previous section cannot be applied to non small categories, if we want to have a set-sized carrier and set-many relations. However, we can give up the idea of obtaining a single Modal Logic frame from a category and have instead a frame from every small subcategory. Note that this entails having class-many frames. Hence from a big category we can still recover a class of frames, and study such class with modal logics.

We want to apply this construction to **FdHil** to recover the semantics of LQP and LQP^n. In this particular case, we are interested in having one

[g] We choose **Rel** here just to remain general, but the same concept applies to subcategories such as **Set**. Also the same idea can be generalized to functors of type $\mathbf{C}^n \to \mathbf{C}$.

frame for each physical system, since this was the original idea underlying LQP, thus we will consider the frames generated by the subcategories of **FdHil** with only *one* object. Notice that the procedure depends on the choice of a functor $\mathbf{FdHil} \to \mathbf{Rel}$; each functor produces a different class of frames, and therefore a different modal logic.

Consider the functor $S : \mathbf{FdHil} \to \mathbf{Set}_{part}$ defined as:

$$\begin{aligned} H &\mapsto \Sigma_H \\ L : H \to V &\mapsto S(L) : \Sigma_H \to \Sigma_V \end{aligned}$$

where Σ_H is the set of one-dimensional linear subspaces of H and the functions $S(L)$ are the partial functions defined as $S(L)(\overline{v}) = \overline{L(v)}$. We now define a $(\mathbf{H}, S)$-frame, for $\mathbf{H}$ the full subcategory of **FdHil** containing only a single Hilbert space H.

Definition 4.3. An $(\mathbf{H}, S)$-frame is a pair $\langle W, \{R_L\}_{L \in \mathbf{H}_1} \rangle$ such that

- $W := \Sigma_H$
- $\{R_L\}_{L \in \mathbf{H}_1} := \{S(L) | L \in \mathbf{H}_1\}$

Hence in this case the carrier of the Modal Logic frame is the set of all one-dimensional subspaces of H. Alternatively, since such subspaces are in bijection with the unitary vectors that represent the states of a quantum systems, W is the set of all pure states of H.

Notice that the concrete quantum dynamic frame given by a Hilbert space H is a substructure of the corresponding $(\mathbf{H}, S)$-frame: the latter has all the partial functions corresponding to linear maps of type $H \to H$, the former only those "constructed from" unitary maps and projectors. So if we interpret the programs in the syntax of LQP in the right way, that is, we send tests to the partial functions corresponding to projections and basic actions to the functions corresponding to unitary transformations, we get the same validities of the class Γ_{CQDF} in the language of LQP. This happens simply because all the additional relations that are in the $(\mathbf{H}, S)$-frame but not in the concrete quantum dynamic frame are not expressible in the language.

Theorem 4.1 (8). *The logic of the class of $(\mathbf{H}, S)$-frames in the language $\mathcal{F}_{LQP}$ contains all the theorems of LQP. Similarly for LQP^n, when the class of frames is suitably restricted.*

4.2. *Capturing probabilistic information*

We hint now at how to use the same technique to obtain richer relational structures containing probabilistic information. Consider $F : \mathbf{FdHil} \to \mathbf{Set}_{part}$ defined as:

$$\begin{aligned} H &\mapsto A_H \\ L : H \to V &\mapsto F(L) : A_H \to A_V \end{aligned}$$

The set A_H is the set of functions $s_\rho : L_H \to [0,1]$, where L_H is the lattice of closed linear subspaces of H, defined as

$$s_\rho(a) = tr(P_a \rho)$$

where P_a is the projector associated to the subspace a and ρ is a density operator on H.

A linear map $L : H \to V$ is sent to the partial function $F(L) : A_H \to A_V$, where

$$F(L)(s_\rho) = s_{\frac{L\rho L^\dagger}{tr(L\rho L^\dagger)}}$$

$F(L)(s_\rho)$ is not defined if $tr(L\rho L^\dagger) = 0$. Recall that the density operators are exactly the positive linear maps of trace 1. The operator $L\rho L^\dagger$ is still positive, and the denominator $tr(L\rho L^\dagger)$ ensures that it is an operator of trace 1. Therefore $\frac{L\rho L^\dagger}{tr(L\rho L^\dagger)}$ is again a density operator, so the function $F(L)$ is well defined.

Definition 4.4. A $(\mathbf{H}, F)$-frame is a pair $\langle W, \{R_L\}_{L\in\mathbf{H}_1}\rangle$ defined as

- $W := A_H$
- $\{R_L\}_{L\in\mathbf{H}_1} := \{F(L) | L : H \to H\}$

Hence in this case the carrier of the Modal Logic frame is the set of functions $s_\rho : L_H \to [0,1]$ defined above. Such functions are associated to density operators on H, which represent *both pure and strictly mixed states* of the quantum system H. The set $\{R_L\}_{L\in\mathbf{H}_1}$ is the collection of maps generated by all linear maps of type $H \to H$.

We can design a language to express the features of these richer frames and show that, upon translation, all the theorems of LQP (LQP^n) are validities of the class of $(\mathbf{H}, F)$-frames (given by compound systems) in this enriched language [8].

4.3. *Path logic for the quantum topos*

The logic that is naturally associated to topoi is intuitionistic. The quantum topos $\mathbf{Set}^{\mathcal{V}(H)}$ is no exception, and the authors argue about how a higher-order typed language interpreted in $\mathbf{Set}^{\mathcal{V}(H)}$ can be useful in rendering the properties of a quantum system H. Here we want to remark that each presheaf in $\mathbf{Set}^{\mathcal{V}(H)}$ is amenable for the treatment described at the beginning of this section.

We have seen how from each presheaf we can obtain a transition system; in this environment the most interesting choice is the spectral presheaf $\underline{\Sigma}$: since the elements in $\underline{\Sigma}(V)$ are partial states $\lambda : V \to \mathbb{C}$, the elements of the transition system generated by $\underline{\Sigma}$ will be all pairs (λ, V). Because of the definition of $\underline{\Sigma}$ on morphisms, two such partial states are related if one is the restriction of the other. The path logic given by $\mathcal{V}(H)$ can thus be used to describe how these partial states are connected. More generally, it remains to be seen how the path logic for such a presheaf category interacts with the internal logic of the topos.

We can however offer a remark on the purely structural level. One of the insight of the topos-theoretic point of view on quantum mechanics is the observation that the Kochen-Specker Theorem [12] is equivalent to the fact that, for a Hilbert space H of dimension greater than 2, there is no global element for the spectral presheaf of $\mathbf{Set}^{\mathcal{V}(H)}$, that is to say, no natural transformation $\theta : \underline{1} \to \underline{\Sigma}$. Because of proposition 4.1 we know that the latter is the case if and only if there is a transition-preserving function from the transition system given by $\underline{1}$ to the transition system arising from $\underline{\Sigma}$.

4.4. *A language for sections in the sheaf of events*

Suppose given the sheaf of events $\mathcal{E} : \wp(X)^{op} \to \mathbf{Set}$ for given sets of measurements X and of outcomes O. Notice that because of the finiteness of X the category $\wp(X)^{op}$ will be finite and therefore small. Moreover, observe that the elements of the transition system obtained from $\mathcal{E}$ are the sections themselves, paired with their domain, that is, they are pairs $(s : U \to O, U)$ for $U \subseteq X$.[h]

The path logic resulting from this base category is a modal logic whose modalities are indexed by the inclusion of sets of measurements. In the

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[h] If we consider the functor $\mathcal{E}$ composed with a distribution functor $\mathcal{D}_R$ for a given commutative semiring R then the objects of the transition systems will be pairs (d, U), where d is an R-distribution over the sections over U.

case of $\mathcal{E}$ this concretely means that $(s : U \to O, U) \vDash \langle U \subseteq U' \rangle \phi$ iff there is a section $s' : U' \to O$ whose retraction is s, or in other words, a section *extending* s, and such that $(s', U') \vDash \phi$. We can therefore use this modal logic to describe how sections over different sets are related to each other, similarly to what we sketched for the topos approach. An important property of the systems considered in this approach is the existence of the so-called *global sections*, namely sections over the whole set X. This property can be expressed with the path logic formula $\langle \emptyset \subseteq X \rangle \top$: the unique section $r : \emptyset \to O$ satisfies this formula if it can be extended to a section over the whole X, i.e., if there exists a global section. We refer to the literature [2] and in particular to Theorem 8.1 in loc. cit. for a discussion on the existence of global sections.

5. Conclusions

We have seen how we can associate modal logic to some influential categorical frameworks employed in the study of quantum phenomena. In our first case study we showed how from the category **FdHil** and the functor S we can obtain the class of Modal Logic frames for the dynamics quantum logics LQP and LQP^n. This constitutes the link between these two research programs. Along the same lines we have sketched how, with the choice of the functor F, it is possible to obtain a richer semantics containing probabilistic information.

The same approach was used to associate a modal logic to the topos-theoretic analysis of Doering and Isham. In that case the path logic could be used to describe the relationships between partial states seen as elements of the image of the spectral presheaf. Moreover, via the topos representation of quantum mechanics we linked the Kochen-Specker Theorem to the existence of transition-preserving functions between the transition systems arising from the terminal presheaf and the spectral presheaf, respectively.

Finally, we addressed the work on contextuality by Abramsky and Brandenburger, casted in the language of Sheaf Theory. The modal logic obtained in that setting was shown to capture the important notion of strong contextuality.

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BINARY GATES IN THREE VALUED QUANTUM COMPUTATIONAL LOGICS

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The standard theory of quantum computation relies on the idea that the *qubit* — the basic information quantity — is represented by a superposition of elements of the standard quantum computational basis $\mathcal{B}^{(2)} = \{|0\rangle, |1\rangle\}$. In the present paper we focus on the case of qutrits where the standard quantum computational bases is replaced by a three-valued computational basis $\mathcal{B}^{(3)} = \{|0\rangle, |\frac{1}{2}\rangle, |1\rangle\}$. Recently in [13], unary gates on the Hilbert space $\mathbb{C}^3$ were considered. In this work we propose an extensive method that allows to extend binary gates to the framework of qutrits.

1. Introduction

The theory of quantum computation has suggested new forms of quantum logic — the so called *quantum computational logics* — that turn out to have some typical unsharp features [4]. The main difference between the quantum logic introduced by Birkhoff and von Neumann [3] and the quantum computational logics, could be summarized as follows: in the Birkhoff and von Neumann approach, the meaning of a sentence is interpreted as a closed subspace of the Hilbert space associated with a physical system [5]; on the other hand, in quantum computational logics the meaning of a sentence is identified with a quantum information quantity: a *qubit*, a *quregister* (that is a system of qubits) or, more generally, a mixture of quregisters (the so called *qumix*) [6–8,11,14].

In [2,13] authors proposed an extension of the standard quantum computational logic where the basic information quantity is a *qutrit* (a unit vector of the Hilbert space $\mathbb{C}^3$) instead of the usual qubit. Following this approach, the standard quantum computational basis is replaced by the three-valued quantum computational basis $\mathcal{B}^{(3)} = \{|0\rangle, |\frac{1}{2}\rangle, |1\rangle\}$, and in this context several unary quantum gates were discussed. In this paper we introduce an extensive method that allows to extend binary gates to the framework of qutrits.

The paper is organized as follows: in Section 2 we provide all the basic notions necessary to render the article self-contained; in Section 3 we summarize the behaviour of some unary quantum gate in the framework of the standard and the three-valued quantum computational logic, respectively; in Section 4 we introduce an extensive method to define some binary gate in three-valued quantum computational logics. Finally, we close the paper with a few comments on possible further developments.

2. Basic Notions

In this section we introduce all the notions necessary for a consistent reading of this paper.

Consider the n-fold tensor product Hilbert space $\otimes^n \mathbb{C}^d$, with $n \geq 1$ and $d \geq 2$.

The canonical orthonormal basis $\mathcal{B}^{(d^n)}$ of $\otimes^n \mathbb{C}^d$ is defined as follows:

$$\mathcal{B}^{(d^n)} = \{|x_1, ..., x_n\rangle : x_i \in \{0, \frac{1}{d-1}, \frac{2}{d-1}, ..., 1\}, \forall i \in \{1, ..., n\}\}$$

where

- $|x_1, ..., x_n\rangle$ is an abbreviation of the tensor product $|x_1\rangle \otimes ... \otimes |x_n\rangle$;
- the vector $|\frac{i}{d-1}\rangle \in \mathbb{C}^d$ (with $0 \leq i \leq d-1$) is a d-dimensional column vector with 1 in the $(i+1)^{th}$-entry and 0 in all the other $d-1$ entries.

In the special case where $n = 1$ and $d = 3$, we obtain the basis $\mathcal{B}^{(3)} = \{|0\rangle, |\frac{1}{2}\rangle, |1\rangle\}$.

Definition 2.1. Qudit
A unit vector in the Hilbert space $\mathbb{C}^d$ (with $d \geq 2$) is called *qudit*.

As a special case of the Definition above, if $d = 3$ the unit vector is the *qutrit*, whose extensive expression is $|\psi\rangle = a|0\rangle + b|\frac{1}{2}\rangle + c|1\rangle$ (with $|a|^2 + |b|^2 + |c|^2 = 1$).

Definition 2.2. Quregister and Qumix
A *quregister* is a unit vector in $\otimes^n \mathbb{C}^d$ and a *qumix* (or mixed state) is a density operator in $\otimes^n \mathbb{C}^d$.

So, a vector in $\otimes^n \mathbb{C}^d$ is a n-fold tensor product of d-dimensional vectors. Trivially, qudits are special cases of quregisters.

Definition 2.3. Truth-values of a quregister
We say that the *truth-value of a quregister* $|x_1, ..., x_{n-1}, \frac{i}{d-1}\rangle \in \otimes^n \mathbb{C}^d$ is $\frac{i}{d-1}$, with $0 \leq i \leq d-1$.

Indeed, the truth-value of a quregister depends on its last component only. In particular, if $i = 0$ we say that the register is *false* and if $i = d-1$ we say that the register is *true*.

Let us remark that the number of different truth-values over the Hilbert space $\otimes^n \mathbb{C}^d$ is d, for any value of n. Hence, in the Hilbert space $\otimes^n \mathbb{C}^3$, three possible truth-values only are available.

Definition 2.4. The truth-value projectors
A *truth-value projection* on $\otimes^n \mathbb{C}^d$ is a projector $P^{(d^n)}_{\frac{i}{d-1}}$ whose range is the closed subspace spanned by the set of all quregisters whose n-th component is $|\frac{i}{d-1}\rangle$, where $P^{(d^n)}_{\frac{i}{d-1}} = I^{(n-1)} \otimes P^{(d)}_{\frac{i}{d-1}}$ and $0 \leq i \leq d-1$.

In particular, the *truth-projection* on $\otimes^n \mathbb{C}^d$ is the projection operator $P^{(d^n)}_1$ whose range is the closed subspace spanned by the set of all true quregisters of $\otimes^n \mathbb{C}^d$.

According to the Born rule, for any qumix $\rho \in \otimes^n \mathbb{C}^d$, a notion of probability can be introduced as follows:

Definition 2.5. $\frac{i}{d-1}$-probability
Let ρ be a qumix in $\otimes^n \mathbb{C}^d$. The probability that ρ has the truth-value $\frac{i}{d-1}$ (with $0 \leq i \leq d-1$) is defined by:

$$\mathtt{p}^{(d)}_{\frac{i}{d-1}} = \mathtt{tr}(P^{(d^n)}_{\frac{i}{d-1}} \rho),$$

where $\mathtt{tr}$ is the trace functional.

From an intuitive point of view, $\mathtt{p}^{(d)}_{\frac{i}{d-1}}$ represents the probability that the information stored by the qumix ρ is the truth-value $\frac{i}{d-1}$.

The unitary evolution of quregisters and qumixes is dictated by quantum logical gates and the quantum operations they naturally induce: unitary transformations mapping quregisters and qumixes in $\otimes^n \mathbb{C}^d$ into quregisters and qumixes in $\otimes^n \mathbb{C}^d$, respectively. We may distinguish between:

- semiclassical quantum gates: unitary operators that transform basis elements into basis elements;

- genuinely quantum gates: unitary operators that transform basis elements into a superposition of basis states;
- genuinely entangled gates: unitary operators that transform basis elements into entangled states.

In this paper, we will be mostly interested in the Hilbert space $\mathbb{C}^3$. We say that a density operator (qumix) in $\mathbb{C}^3$ is a *qutrit-density operator* and a quantum logical gate on $\mathbb{C}^3$ is a *qutrit-gate*.[a]

3. Extending Unary Quantum Gates

In this section we discuss extensions of two known quantum gates to the case of qutrits. These constructions exploit the fact that in $\mathbb{C}^3$ — as well as in $\otimes n3$ — the new truth value widens the usual behavior of gates in $\mathbb{C}^2$ — as well as in $\otimes n2$ — along distinct degrees of freedom. In fact, we will see that single gates in $\mathbb{C}^2$ may admit several extensions in the case of qutrits.

3.1. *The negation*

Qubit case: For any $n \geq 1$, the *negation* on $\bigotimes^n \mathbb{C}^2$ is the unitary operator $Not^{(2^n)}$ such that, for every element $|x_1, ..., x_n\rangle$ of the computational basis $\mathcal{B}^{(2^n)}$,

$$Not^{(2^n)}(|x_1, ..., x_n\rangle) = |x_1, ..., x_{n-1}\rangle \otimes |1 - x_n\rangle .$$

We have that:

$$Not^{(2^n)} = \begin{cases} \sigma_x & \text{if } n = 1; \\ I^{(n-1)} \otimes \sigma_x, & \text{otherwise,} \end{cases}$$

where $\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the "first" Pauli matrix.

The negation is an example of *semiclassical* gate: its application to the vectors of the standard quantum computational basis reproduces the behavior of the classical negation: $Not|0\rangle = |1\rangle$, $Not|1\rangle = |0\rangle$.

Qutrit case: Given the usual logical basis $\mathcal{B}^{(3)} = \{|0\rangle, |\frac{1}{2}\rangle, |1\rangle\}$ on $\mathbb{C}^3$, we can define a negation $Not^{(3)}_{|\frac{1}{2}\rangle}$ as expected by

$$Not^{(3)}_{|\frac{1}{2}\rangle}|x\rangle = |1 - x\rangle,$$

[a] When a qutrit gate A is applied to a density operator ρ on $\mathbb{C}^3$, the evolution of ρ is given by: $A\rho A^\dagger$. Since no danger of confusion will be impending, for the sake of notational simplicity, from now on we write $A(\rho)$.

where $x \in \{0, \frac{1}{2}, 1\}$. We use the subscript $|\frac{1}{2}\rangle$ to emphasize the fact that $|\frac{1}{2}\rangle$ is a fixpoint of $Not^{(3)}_{|\frac{1}{2}\rangle}$, i.e. $Not^{(3)}_{|\frac{1}{2}\rangle}|\frac{1}{2}\rangle = |\frac{1}{2}\rangle$. We can immediately obtain a matrix form $Not^{(3)}{}_{|\frac{1}{2}\rangle} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ such that:

$$Not^{(3)}_{|\frac{1}{2}\rangle} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ b \\ a \end{pmatrix}.$$

This idea can be easily applied to the other basis states as follows:

$$Not^{(3)}{}_{|0\rangle} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } Not^{(3)}{}_{|1\rangle} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let us remark that, for any $i \in \{0, \frac{1}{2}, 1\}$, $Not^{(3)}_{|i\rangle} \cdot Not^{(3)}_{|i\rangle} = I^{(3)}$.

3.2. *The Hadamard gate*

Qubit case: For any $n \geq 1$, the *Hadamard gate* on $\bigotimes^{n} \mathbb{C}^2$ is the linear operator $H^{(2^n)}$ such that for every element $|x_1, ..., x_n\rangle$ of the computational basis $\mathcal{B}^{(2^n)}$:

$$H^{(2^n)}(|x_1, ..., x_n\rangle) = |x_1, ..., x_{n-1}\rangle \otimes \frac{1}{\sqrt{2}} \left((-1)^{x_n} |x_n\rangle + |1 - x_n\rangle\right).$$

We have that

$$H^{(2^n)} = \begin{cases} H & \text{if } n = 1; \\ I^{n-1} \otimes H, & \text{otherwise,} \end{cases}$$

where H is the *Hadamard* matrix:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The basic property of $H^{(2^n)}$ is that, for any $|\psi\rangle \in \bigotimes^{n} \mathbb{C}^2$,

$$H^{(2^n)}\left(H^{(2^n)}(|\psi\rangle)\right) = |\psi\rangle.$$

Hadamard gate is an example of *genuinely quantum* gate: its application to the vectors of the standard quantum computational basis produces as output a superposition. Clearly, this behavior has no classical counterpart.

Qutrit case: In [1], the following extension of the Hadamard gate to the case of qutrits is considered

$$H^{(3)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{6}(-1+i\sqrt{3}) & -\frac{1}{6}(1+i\sqrt{3}) \\ 1 & -\frac{1}{6}(1+i\sqrt{3}) & \frac{1}{6}(-1+i\sqrt{3}) \end{pmatrix}$$

as a tool in the framework of distillation protocols for fault tolerant quantum computation — precisely Magic State Distillation.

We state without proof the main properties of $H^{(3)}$:

Lemma 3.1.

(1) For any $|\psi\rangle \in \mathcal{B}^{(3)}$, $H^{(3)}|\psi\rangle = a|0\rangle + b|\frac{1}{2}\rangle + c|1\rangle$ *s.t* $|a|^2 = |b|^2 = |c|^2 = \frac{1}{3}$*;*
(2) $H^{(3)}$ *is a genuinely quantum gate;*
(3) $H^{(3)} \cdot H^{(3)} = Not^{(3)}_{|0\rangle} \neq I$*.*

Along different lines, other possible extensions of the Hadamard gate are the *square root of the identity gates* $\sqrt{I}^{(3)}_{|i\rangle}$ [10].

$$\sqrt{I}^{(3)}_{|0\rangle} = 1 \oplus H_{(\mathbb{C}^2)} = \frac{\sqrt{2}}{2} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix};$$

$$\sqrt{I}^{(3)}_{|\frac{1}{2}\rangle} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix};$$

$$\sqrt{I}^{(3)}_{|1\rangle} = H_{(\mathbb{C}^2)} \oplus 1 = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

where $\oplus$ indicates the matrix direct sum. The name “square root of the identity” derives from the fact that a double application of $\sqrt{I}^{(3)}_{|i\rangle}$ is indeed the identity operator.

Some properties of the gates above follow:

Lemma 3.2. *For any* $i \in \{0, \frac{1}{2}, 1\}$*:*

(1) $\sqrt{I}^{(3)}_{|i\rangle}$ *is a* genuinely *quantum qutrit-gate;*
(2) $\sqrt{I}^{(3)}_{|i\rangle} \cdot \sqrt{I}^{(3)}_{|i\rangle} = I$*;*

(3)

$$\sqrt{I}^{(3)}_{|0\rangle}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}a\\\frac{1}{\sqrt{2}}(b+c)\\\frac{1}{\sqrt{2}}(b-c)\end{pmatrix};\qquad \sqrt{I}^{(3)}_{|\frac{1}{2}\rangle}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}(a+c)\\b\\\frac{1}{\sqrt{2}}(a-c)\end{pmatrix};$$

$$\sqrt{I}^{(3)}_{|1\rangle}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}(a+b)\\\frac{1}{\sqrt{2}}(a-b)\\c\end{pmatrix}.$$

Let us notice that, in the Hilbert space $\mathbb{C}^3$, the Hadamard gate $H^{(3)}$ does not behave as a square root of the identity, since in general $\sqrt{I}^{(3)}_{|i\rangle}\sqrt{I}^{(3)}_{|i\rangle}|\psi\rangle \neq \psi$. The operator $\sqrt{I}^{(3)}_{|i\rangle}$, instead, is a square root of identity for any $i \in \{0, \frac{1}{2}, 1\}$.

4. Extending Binary Quantum Gates

In this section we discuss extensions of several well known binary quantum gates to the case of qutrits. These constructions exploit the fact that in $\mathbb{C}^3$ — as well as in $\otimes n3$ — the new truth value widens the usual behavior of gates in $\mathbb{C}^2$ — as well as in $\otimes n2$ — along distinct degrees of freedom. In fact, we will see that single gates in $\mathbb{C}^2$ may admit several extensions in the case of qutrits.

4.1. *Reversible binary gates*

4.1.1. *The Swap gate*

The *Swap* gate is the semiclassical unitary operator such that, for any $\rho, \sigma \in \mathbb{C}^2$, $Swap(\rho \otimes \sigma) = \sigma \otimes \rho$. This gate is extremely useful since it allows to "move" — to swap, in fact — the qubits during a computational protocol.
Qubit case: The explicit formulation of the $Swap^{(2)}$ gate is the following:

$$Swap^{(2)} = \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}.$$

Qutrit case:
It can be seen that the $Swap^{(2)}$ gate can be generalized as follows to the case of qutrit-density operators [9]:

$$Swap^{(3)} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}.$$

4.1.2. *The* $\sqrt{Swap}$ *gate*

The $\sqrt{Swap}$ is an example of genuinely quantum gate.
Qubit case: For any $n \geq 1$, the $\sqrt{Swap}^{(2)}$ gate on $\otimes^n\mathbb{C}^2$ can be defined by:

$$\sqrt{Swap}^{(2)}(|x_1, ..., x_n\rangle) =$$
$$\begin{cases} \frac{1}{2}((1+i)|x_{n-1}x_n\rangle + (1-i)|x_n x_{n-1}\rangle), & \text{if } n = 2;\\ |x_1, ..., x_{n-2} \otimes \frac{1}{2}\rangle((1+i)|x_{n-1}x_n\rangle + (1-i)|x_n x_{n-1}\rangle), & \text{if } n < 2. \end{cases}$$

The matrix formulation of $\sqrt{Swap}^{(2)}$ gate in $\mathbb{C}^2$ is:

$$\sqrt{Swap}^{(2)} = \begin{pmatrix} 1&0&0&0\\ 0&\frac{1+i}{2}&\frac{1-i}{2}&0\\ 0&\frac{1-i}{2}&\frac{1+i}{2}&0\\ 0&0&0&1 \end{pmatrix}.$$

Qutrit case
According with [15], a matrix expression of $\sqrt{Swap}$ gate in $\mathbb{C}^3$ is:

$$\sqrt{Swap}^{(3)} = \frac{1}{2}\begin{pmatrix} 2&0&0&0&0&0&0&0&0\\ 0&1+i&0&1-i&0&0&0&0&0\\ 0&0&1+i&0&0&0&1-i&0&0\\ 0&1-i&0&1+i&0&0&0&0&0\\ 0&0&0&0&2&0&0&0&0\\ 0&0&0&0&0&1+i&0&1-i&0\\ 0&0&1-i&0&0&0&1+i&0&0\\ 0&0&0&0&0&1-i&0&1+i&0\\ 0&0&0&0&0&0&0&0&2 \end{pmatrix}$$

4.2. *Irreversible binary gates*

One may immediately note that, in the case of $Swap$ and $\sqrt{Swap}$ gates, the dimensions of the input and the output coincide. Actually, these gates are unitary: represent a reversible transformation. It can be seen that this is not the case in general. For instance, consider the functions And and Xor from $\{0,1\}^2$ to $\{0,1\}$. In these cases different inputs may correspond to a single output; these functions are no longer reversible and therefore are not representable by unitary operators. In quantum computation, a standard method for addressing this issue involves the introduction of an auxiliary *ancilla*.

In this section we propose an application of this method to the framework of qutrits. First, it can be easily seen that, given an irreversible function, $\varphi : \{0, \frac{1}{2}, 1\}^2 \to \{0, \frac{1}{2}, 1\}$, the extension $f_\varphi : \{0, \frac{1}{2}, 1\}^3 \to \{0, \frac{1}{2}, 1\}^3$, defined by

$$f_\varphi(x, y, z) = (x, y, \varphi(x, y) \oplus_{\frac{3}{2}} z) \tag{1}$$

where x, y are the control outputs, $\varphi(x, y) \oplus_{\frac{3}{2}} z$ is the target output, and $\oplus_{\frac{3}{2}}$ is the sum modulo $\frac{3}{2}$, allows to represent φ in terms of a reversible transformation. The price to pay for this reversible extension is that the introduction of an extra-ancilla z increases the dimension of the original space.

Let us now discuss the And and Xor quantum gates.

4.3. *The Conjunction*

Qubit case

In $\mathbb{C}^2$ the conjunction is usually defined by:

$$And^{(2)}(\rho, \sigma) = T(\rho \otimes \sigma \otimes P_0^{(2)})T,$$

where the Toffoli gate T is $T(|x\rangle|y\rangle|z\rangle) = |x\rangle|y\rangle|xy \oplus z\rangle$, $\oplus$ is the sum modulo 2 and $P_0^{(2)}$ plays the role of an extra ancilla. As usual, in classical logic, for any $x, y \in \{0,1\}$ is $And(x, y) = x \cdot y$. In the quantum computational case it can be easily seen that, for any $\rho, \sigma \in \mathbb{C}^2$, $\mathtt{p}_1(And^{(2)}(\rho, \sigma)) = \mathtt{p}_1^{(2)}(\rho) \cdot \mathtt{p}_1^{(2)}(\sigma)$.

Qutrit case

In the case of three-valued logic, Gödel conjunction (i.e. $min(x, y)$) has the following irreversible truth-table:

And	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

Obviously, since $And(x, y)$ is irreversible, this function can't be represented by a unitary operator. However, Equation (1) yields: $f_{And}(x, y, z) = (x, y, \min(x, y) \oplus_{\frac{3}{2}} z)$ and, setting $z = 0$, we have:

$$f_{And}(x, y, 0) = (x, y, \min(x, y)).$$

The $f_{And}(x, y, z)$ function induces the following table:

In	f_{And}	**Out**
(0, 0, 0)		(0, 0, 0)
(0, 0, 1/2)		(0, 0, 1/2)
(0, 0, 1)		(0, 0, 1)
(0, 1/2, 0)		(0, 1/2, 0)
(0, 1/2, 1/2)		(0, 1/2, 1/2)
(0, 1/2, 1)		(0, 1/2, 1)
(0, 1, 0)		(0, 1, 0)
(0, 1, 1/2)		(0, 1, 1/2)
(0, 1, 1)		(0, 1, 1)
(1/2, 0, 0)		(1/2, 0, 0)
(1/2, 0, 1/2)		(1/2, 0, 1/2)
(1/2, 0, 1)		(1/2, 0, 1)
(1/2, 1/2, 0)		(1/2, 1/2, 1/2)
(1/2, 1/2, 1/2)		(1/2, 1/2, 1)
(1/2, 1/2, 1)		(1/2, 1/2, 0)
(1/2, 1, 0)		(1/2, 1, 1/2)
(1/2, 1, 1/2)		(1/2, 1, 1)
(1/2, 1, 1)		(1/2, 1, 0)
(1, 0, 0)		(1, 0, 0)
(1, 0, 1/2)		(1, 0, 1/2)
(1, 0, 1)		(1, 0, 1)
(1, 1/2, 0)		(1, 1/2, 1/2)
(1, 1/2, 1/2)		(1, 1/2, 1)
(1, 1/2, 1)		(1, 1/2, 0)
(1, 1, 0)		(1, 1, 1)
(1, 1, 1/2)		(1, 1, 0)
(1, 1, 1)		(1, 1, 1/2)

Thanks to the ancilla z, the function $f_{And}(x, y, z)$ is reversible and the target bit of the output of $f_{And}(x, y, 0)$ correctly represents the outputs of the three-valued Gödel conjunction. It can be seen that the truth table above is representable by the following $FAnd$ matrix:

$$FAnd = \begin{pmatrix} Q_1 &&&&&&&& \\ & Q_1 &&&&&&& \\ && Q_1 &&&&&& \\ &&& Q_1 &&&&& \\ &&&& Q_2 &&&& \\ &&&&& Q_2 &&& \\ &&&&&& Q_1 && \\ &&&&&&& Q_2 & \\ &&&&&&&& Q_3 \end{pmatrix}$$

where $Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $Q_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and $Q_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Consequently, we can define, for any qutrit-density operators $\rho, \sigma \in \mathbb{C}^3$, the *qutrit Gödel conjunction* by:

$$And(\rho, \sigma) = (FAnd(\rho \otimes \sigma \otimes P_0^{(3)})),$$

where $P_0^{(3)}$ plays the role of a 0-ancilla (similarly to $z = 0$ in $f_{And}(x, y, z)$). We list without proof some easy observations on the qutrit Gödel conjunction:

- $\mathtt{p}_1^{(3)}(And(\rho, \sigma)) = \mathtt{p}_{\frac{1}{2}}^{(3)}(\rho)\mathtt{p}_1^{(3)}(\sigma) + \mathtt{p}_1^{(3)}(\rho)\mathtt{p}_{\frac{1}{2}}^{(3)}(\sigma) + \mathtt{p}_{\frac{1}{2}}^{(3)}(\rho)\mathtt{p}_{\frac{1}{2}}^{(3)}(\sigma)$;
- $\mathtt{p}_{\frac{1}{2}}^{(3)}(And(\rho, \sigma)) = \frac{1}{2}\mathtt{p}_1^{(3)}(\rho)\mathtt{p}_1^{(3)}(\sigma)$.

4.4. *The Xor gate*

Qubit case: The standard quantum Xor gate is the unitary operator such that, for any basis vectors $|\psi\rangle$,$|\phi\rangle$ of a Hilbert space, $Xor^{(2)}(|\psi\rangle|\phi\rangle) = |\psi\rangle|\psi \oplus \phi\rangle$. The matrix form of the $Xor^{(2)}$ gate in $\mathbb{C}^2$ is the following:

$$Xor^{(2)} = CNot = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Qutrit case: In the case of $\{0, \frac{1}{2}, 1\}$, let us note that different versions of the Xor function have been considered in the literature. For instance,

$$|x - y| \text{ or } f_{Xor}(x, y) = \begin{cases} (x, x \oplus_2 y), \text{ if } x, y \in \{0, 1\} \\ (x, y), \text{ if } x = \frac{1}{2} \text{ or } y = \frac{1}{2} \end{cases} \tag{2}$$

where $\oplus_2$ is the sum modulus 2. In this last case, the matrix presentation is the following:

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&0&0 \end{pmatrix}.$$

It seems reasonable that we may require that an extension $Xor^{(3)}$ would satisfy the following natural properties:

(1) $Xor^{(3)}(1,1) = 0$ (boundary condition);
(2) $Xor^{(3)}(0,x) = x$ (neuter element);
(3) $Xor^{(3)}(x,y) = Xor^{(3)}(y,x)$ (commutativity);
(4) $Xor^{(3)}(x, Xor^{(3)}(y,z)) = Xor^{(3)}(Xor^{(3)}(x,y),z)$ (associativity);

since in the case of $\mathbb{C}^2$ it does. It can be seen that both functions in Display 2 fail associativity.

What's the most suitable 3-valued extension of the $Xor^{(2)}$ gate in qutrits' framework? We devote the remain of this section to propose a tentative answer. A first step towards this goal can be found in the following:

Lemma 4.1. *There exists only one* 3*-valued* Xor*:* $x + y - 2xy$*.*

Proof. It is easy to check that $x + y - 2xy$ satisfies the requirements (1)-(4) above. As regards uniqueness, one can see that $\frac{1}{2} = Xor(\frac{1}{2}, 0) = Xor(\frac{1}{2}, Xor(1,1)) = Xor(\frac{1}{2}, 1)$. Moreover, if $Xor(\frac{1}{2}, \frac{1}{2}) = 1$, then $Xor(1, Xor(\frac{1}{2}, \frac{1}{2})) = 0 \neq Xor(Xor(1, \frac{1}{2}), \frac{1}{2})) = \frac{1}{2}$; if $Xor(\frac{1}{2}, \frac{1}{2}) = 0$, then $Xor(1, Xor(\frac{1}{2}, \frac{1}{2})) = 1 \neq Xor(Xor(1, \frac{1}{2}), \frac{1}{2})) = 0$. $\square$

Let us remark that, according with [12, p. 2], this 3-valued Xor is the *least sensitive* (*most robust*) *exclusive or*, in the sense that if we slightly change the inputs, then "the result of the fuzzy operation does not change much" (see [12, Def. 3]).

One may readily realise that the function in Lemma 4.1 is not reversible if it takes values in $\{0, \frac{1}{2}, 1\}$. Call a function $Xor_f : [0,1]^2 \to [0,1]^2$ a *minimal* Xor *without ancilla* iff $\forall x, y \in \{0,1\}$, $Xor_f(x,y) = (x, x \oplus_2 y)$.

Consider a minimal Xor without ancilla Xor_f restricted to $\{0, \frac{1}{2}, 1\}$, and let $g : \{0, \frac{1}{2}, 1\}^2 \to \{0, \frac{1}{2}, 1\}$ be $g(x, y) = \pi_2(Xor_f(x, y))$, where π_2 is the second coordinate projection. We have the following:

Lemma 4.2. *Let Xor_f be a minimal Xor without ancilla in $\{0, \frac{1}{2}, 1\}$. If Xor_f is reversible, then g is neither commutative nor associative.*

Proof. Suppose Xor_f is reversible. Then, if $a \in \{0, 1\}$, $g(a, \frac{1}{2}) \notin \{0, 1\}$, since otherwise Xor_f wouldn't be reversible. Therefore, for $a \in \{0, 1\}$, $g(a, \frac{1}{2}) = \frac{1}{2}$. If g is commutative, then $g(0, \frac{1}{2}) = g(\frac{1}{2}, 0) = g(\frac{1}{2}, 1) = g(1, \frac{1}{2})$. Therefore, $Xor_f(\frac{1}{2}, 0) = Xor_f(\frac{1}{2}, 1)$, against the hypothesis. As regards associativity, note that for, $a \neq b$, $g(\frac{1}{2}, a) \neq g(\frac{1}{2}, b)$. By direct inspection, it can be seen that g is not associative. $\square$

We have seen in Lemma 4.1 that there exists a unique, irreversible, three-valued Xor satisfying Conditions (1)–(4). Since Lemma 4.2, any minimal Xor would induce a failure of commutativity and associativity. Therefore, any reversible gate preserving Conditions (1)–(4) will be in $\otimes^3\mathbb{C}^3$.

Resorting again to Equation (1), we obtain the reversible extension: $f_{Xor}(x, y, z) = (x, y, (x+y-2xy) \oplus_{\frac{3}{2}} z)$. Obviously, $f_{Xor}(x, y, 0) = (x, y, x+y-2xy)$. The truth table induced by the 3-valued Xor is:

In	f_{Xor}	**Out**
(0, 0, 0)		(0, 0, 0)
(0, 0, 1/2)		(0, 0, 1/2)
(0, 0, 1)		(0, 0, 1)
(0, 1/2, 0)		(0, 1/2, 1/2)
(0, 1/2, 1/2)		(0, 1/2, 1)
(0, 1/2, 1)		(0, 1/2, 0)
(0, 1, 0)		(0, 1, 1)
(0, 1, 1/2)		(0, 1, 0)
(0, 1, 1)		(0, 1, 1/2)
(1/2, 0, 0)		(1/2, 0, 1/2)
(1/2, 0, 1/2)		(1/2, 0, 1)
(1/2, 0, 1)		(1/2, 0, 0)
(1/2, 1/2, 0)		(1/2, 1/2, 1/2)
(1/2, 1/2, 1/2)		(1/2, 1/2, 1)
(1/2, 1, 1)		(1/2, 1/2, 0)
(1/2, 1, 0)		(1/2, 1, 1/2)
(1/2, 1, 1/2)		(1/2, 1, 1)
(1/2, 1, 1)		(1/2, 1, 0)
(1, 0, 0)		(1, 0, 1)
(1, 0, 1/2)		(1, 0, 0)
(1, 0, 1)		(1, 0, 1/2)
(1, 1/2, 0)		(1, 1/2, 1/2)
(1, 1/2, 1/2)		(1, 1/2, 1)
(1, 1/2, 1)		(1, 1/2, 0)
(1, 1, 0)		(1, 1, 0)
(1, 1, 1/2)		(1, 1, 1/2)
(1, 1, 1)		(1, 1, 1)

It can be seen that the truth table above is implemented by the following $FXor$ matrix:

$$FXor = \begin{pmatrix} Q_1 & & & & & & & & \\ & Q_2 & & & & & & & \\ & & Q_3 & & & & & & \\ & & & Q_2 & & & & & \\ & & & & Q_2 & & & & \\ & & & & & Q_2 & & & \\ & & & & & & Q_3 & & \\ & & & & & & & Q_2 & \\ & & & & & & & & Q_1 \end{pmatrix}.$$

Finally, we can reasonably define, for any $\rho, \sigma \in \mathbb{C}^3$, the *qutrit* $Xor^{(3)}$ *gate* by: $Xor^{(3)}(\rho, \sigma) = FXor(\rho \otimes \sigma \otimes P_0)$. Some easy properties of this gate follows:

- $\mathtt{p}_1^{(3)}(Xor(\rho,\sigma)) = \mathtt{p}_{\frac{1}{2}}^{(3)}(\rho) + \mathtt{p}_{\frac{1}{2}}^{(3)}(\sigma)\left(\mathtt{p}_0^{(3)}(\rho) + \mathtt{p}_1^{(3)}(\sigma)\right)$;
- $\mathtt{p}_{\frac{1}{2}}^{(3)}(Xor(\rho,\sigma)) = \frac{1}{2}\left(\mathtt{p}_0^{(3)}(\rho)\mathtt{p}_1^{(3)}(\sigma) + \mathtt{p}_1^{(3)}(\rho) + \mathtt{p}_0^{(3)}(\sigma)\right)$.

5. Conclusions

Further possible developments of this note will be:

- enquiring into the logical properties of 3-valued quantum computational logics through the notion of *effect probability* [13];
- investigating the algebraic structures naturally associated to 3-valued quantum computational logics;
- extending the present results to a n-valued quantum computational logics.

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THE GTR-MODEL: A UNIVERSAL FRAMEWORK FOR QUANTUM-LIKE MEASUREMENTS

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We present a very general geometrico-dynamical description of physical or more abstract entities, called the *general tension-reduction* (GTR) model, where not only states, but also measurement-interactions can be represented, and the associated outcome probabilities calculated. Underlying the model is the hypothesis that indeterminism manifests as a consequence of unavoidable fluctuations in the experimental context, in accordance with the *hidden-measurements interpretation* of quantum mechanics. When the structure of the state space is Hilbertian, and measurements are of the *universal* kind, i.e., are the result of an average over all possible ways of selecting an outcome, the GTR-model provides the same predictions of the Born rule, and therefore provides a natural completed version of quantum mechanics. However, when the structure of the state space is non-Hilbertian and/or not all possible ways of selecting an outcome are available to be actualized, the predictions of the model generally differ from the quantum ones, especially when sequential measurements are considered. Some paradigmatic examples will be discussed, taken from physics and human cognition. Particular attention will be given to some known psychological effects, like question order effects and response replicability, which we show are able to generate non-Hilbertian statistics. We also suggest a realistic interpretation of the GTR-model, when applied to human cognition and decision, which we think could become the generally adopted interpretative framework in quantum cognition research.

Keywords: Probability; hidden-measurements; degenerate measurements; hidden-variables; Born rule; Bloch sphere; order effects; response replicability; quantum cognition.

1. Introduction

Probability is the key notion used by scientists of different disciplines to quantify, in a meaningful and optimal way, their lack of knowledge regarding certain properties of the systems under study. Before the advent of quantum mechanics, only classical probabilities were taken into consideration, based on the structure of Boolean algebra and obeying the classical Kolmogorovian axioms [1]. On the other hand, quantum probabilities, which are based on a different structure of the experimental propositions, cannot be represented in a fixed probability space (if multiple measurements are considered), and therefore generally disobey the Kolmogorovian axioms.

However, the difference between classical and quantum probabilities is not in a simple correspondence with the difference between macroscopic and microscopic physical entities. Indeed, also macroscopic entities can behave in a quantum-like way, depending on the nature of the experimental actions that an experimenter is considering in relation to them. In other terms, quantum and classical probabilities can always emerge from our experimental investigations, depending on the *structure of possibilities* that are taken into consideration.

This means that classic and quantum probabilities should be considered as special cases of more general probability structures, which can simply be called (depending on the context) non-classical, non-quantum, or quantum-like. These more general structures are non-classical in the sense that they do not obey the Kolmogorovian axioms, and they are quantum-like in the sense that, similarly to quantum probabilities, they are based on logical connectives that are *dynamical*, i.e., describing the possible outcomes of *actions* (measurements) that can be performed on the different entities. However, they are also non-quantum, in the sense that they are not necessarily purely quantum, as the structure of the associated state space is not necessarily Hilbertian and the probability values are not necessarily those predicted by the Born rule.

From the viewpoint of physics, the interest of investigating more general probability models lies for instance in the possibility of shedding new light onto the problem of the semiclassical limit, i.e., in the understanding of the transition from pure quantum to pure classical regimes. Indeed, if it is true that classical and quantum probabilities are based on different, not commensurable structures, obtained as different limits of more general quantum-like situations, then it is also clear that we need more general models, of a "mixed" quantum-classical kind, if we want to describe the mesoscopic regions of our reality, which cannot be incorporated within the pure quantum or pure classical limit models.

A preliminary analysis of these intermediary regions of reality, described by non-classic and non-quantum probabilities, was carried out in some detail by one of us and his collaborators in the past decades, using a paradigmatic model called the ϵ-model [2,3]: a generalization of a two-level (qubit) system where an additional real parameter ϵ can be continuously varied, from 0 to 1, so as to produce a non-singular classic-to-quantum transition. Even though the initial motivation in studying these more general probability models came from physics, it became clear early on that their significance went beyond the field of physics, and was of great interest also in the description of human cognitive processes, as today studied in the new emerging field of theoretical and experimental investigation called *quantum cognition* [4,5,6,7,8,9,10,11].

Quantum cognition resulted, among other things, from the observation that human concepts, understood as abstract entities interacting with human minds that are sensitive to their meanings, can produce highly contextual dynamics, impossible to explain by only using traditional modelizations in terms of logico-rational thinking processes, and therefore classical probabilities. Similarly, quantum mechanics was historically created, as a mathematical theory, to offer a consistent description of entities whose behavior appeared also to be highly contextual. Therefore, it was quite natural at some point to assume that the quantum formalism could also play a role in the modelization of human cognition and decision-making.

This intuition was followed by an increasing number of successful applications of the quantum formalism, and today it is a well-established hypothesis that, in addition to our classical logical layer, describable by classical probability theory, there is an additional quantum conceptual layer, describable by standard quantum mechanics [12]. This quantum description of the human cognitive behavior, however, has nothing to do with the fact that our human brains would be quantum machines. Quantum cognition is not concerned with the modeling of human brains as quantum computers, but with the possibility of using quantum probabilities, and the structure of Hilbert spaces, to elucidate the working of our mental processes, particularly those mistakenly understood as irrational.

Clearly, the contextuality of human concepts mirrors that of elementary quantum entities. Consider for instance the problem of concept combination, i.e., the problem that not all concept combinations will have an intersective semantics, so that the meaning of a combination of concepts will not always be reducible to the meaning of the individual concepts forming the combination, as new meanings are constantly able to emerge. This emergence effect can be naturally described within the quantum formalism by means of the superposition principle and the related constructive and destructive interference effects, which can explain the observed overextension and underextension of the probabilities (with respect to the classical predictions). The superposition principle is in turn a key ingredient in the creation of entangled states that can be used to describe the situation of concepts connected through meaning, which are able to violate Bell's inequalities in a way similar to quantum microscopic entities [13,14]. Other fundamental aspects of the quantum formalism, like for instance quantum field many particles dynamics, can be also be exploited to describe typical situations where human judgments and decisions are at play [15].

This "unreasonable" success of quantum mathematics in the modelization of human cognitive and decision situations requires of course to be widely explained. At the same time, one needs to investigate what are its limits, i.e., to what extent the standard quantum formalism can be used to model all sorts of cognitive situations. These two issues are intimately related. Indeed, there are no a priori reasons for the contextuality built-in in the standard Hilbertian formalism to be exactly the same (in terms of structure) as that incorporated in our human mental processes, also because the latters describe a (non-physical, mental) layer of our reality which, evolutionarily speaking, is much younger than the layer of the fundamental physical processes. This means that, starting from a more general model, containing both the quantum and classical regimes as special situations, one should be able to explain why certain aspects of the quantum formalism, in particular the Born rule, are so effective in describing many empirical data, and at the same time insufficient to model many others, considering that a Hilbert space, equipped with the Born rule, necessarily imposes some specific constraints (like the QQ-equality introduced by [16]), that can be violated by our complex cognitive and decisional processes.

To provide a convincing explanation of both the success of quantum probabilities, and their lack of universality, the present article is organized as follows. In Sec. 2, we use the simple example of a coin flipping experiment to motivate a general description of an entity that gives rise to a general measurement model, called the *general tension-reduction model* (GTR-model). The model was recently derived in the ambit of quantum cognition studies [17,18], but is of great interest also for physics, as it offers a non-circular derivation of the Born rule and therefore constitutes a possible solution to the measurement problem [19,20].

Our strategy here is not that of repeating our previous more formal derivations of the model, but to motivate its construction starting from more qualitative and general considerations. More precisely, inspired by our analysis of the coin flipping measurement of Sec. 2, the GTR-model will be introduced in Sec. 3. In Sec. 4, it will be shown to allow for the description of experiments where some of the outcomes can be degenerate and, in Sec. 5, we show that it can naturally handle also the situation of composite entities. In Sec. 6, we explain how the Born rule can be deduced, when a huge (universal) average is performed over all possible kinds of measurements, showing that the Born rule can be interpreted as a first order approximation of a more general theory, thus explaining its great success also in the description of cognitive experiments. In Sec. 7, we explicitly show that the GTR-model can describe more general structures than the Kolmogorovian and Hilbertian ones and, in Sec. 8, we apply the model to human cognition, showing that one needs its full structural richness to described some of the experimental data, like question order effects and response replicability. Finally, in Sec. 9, we offer a few concluding remarks.

2. Measuring a coin

Consider the process that consists in flipping a coin onto the floor. If we are interested in knowing the final upper face of the coin, three possible outcomes have to be distinguished: “head,” “tail” and “edge,” and to these three outcomes three different probabilities can be associated: $P(\mathrm{h})$, $P(\mathrm{t})$ and $P(\mathrm{e})$. In the case of an *American nickel*, their typical experimental values are [21]:

$$P(\mathrm{h}) = \frac{2999.5}{6000}, \quad P(\mathrm{t}) = \frac{2999.5}{6000}, \quad P(\mathrm{e}) = \frac{1}{6000}. \tag{1}$$

Flipping a coin onto the floor is a simple action producing a non-predeterminable outcome, and the same is true when we toss a die, draw a ball from a urn, etc. All these simple experiments, called *chance games*, have been largely used in the past to study the logic of probabilities, and culminated in modern classical probability theory, axiomatized by Kolmogoroff [1]. However, the actions associated with chance games are of a very special kind, and one should expect a probability model derived from their analysis to also be a very special model, not necessarily able to describe all the probability structures that can emerge from our experiments.

To explain in which sense the random processes traditionally studied by classical probability theory are special, take the coin flipping example.

The associated probabilities will depend in part on how the coin is manufactured. For instance, if we have exactly three possible outcomes, this is because there are three distinct faces (two flat faces and a curved one), and the values of the probabilities certainly also depend, in part, on their relative surfaces, on the exact location of the center of mass of the coin, and so on. These are the so-called *intrinsic* properties of the coin, i.e., the attributes it always possesses, in a stable and permanent way (at least for as long as the coin exists).

But a coin, as a physical entity, is not only described by its intrinsic (always actual) properties: its condition is also determined by those properties that can contextually change over time, like its position and orientation in space, its linear and angular momentum, its temperature, etc. Now, a process like that of flipping a coin is special because it is usually so conceived that we cannot learn anything about its non-intrinsic properties from the obtained probabilities. In other terms, flipping a coin is an experiment which tells us nothing about the *state* of the coin prior to its execution: there is no *discovery aspect* involved, but only a *creation aspect*.

In [17,18], we have named these special — creation only — processes, *solipsistic measurements*, with the term "solipsistic" used in a metaphorical sense, to express the idea that the measurement tells us nothing about the pre-measurement state of the entity, but only about the measurement process itself. Note that in the following we shall use the terms "measurement" and "experiment" almost interchangeably, being clear that a measurement is an experiment aimed at the observation of a given quantity, and that also the flipping of a coin can be interpreted as a measurement process, whose outcomes are the values taken by a quantity called the "upper face" of the coin.

It is worth observing that solipsistic measurements have a truly remarkable property: being totally insensitive to the initial state of the entity,[a] they are necessarily all *mutually compatible experiments*. Not in the sense that they can be performed at the same time (almost no experiments have this remarkable property), but in the sense that the order with which they are carried out is irrelevant. This because the final state obtained by the first measurement cannot influence the outcome of the second one, and vice versa. It is then clear why the Kolmogorovian probability model, which is founded on the paradigm of the solipsistic measurements, is unable to ac-

[a]It is because solipsistic measurements are processes that are extremely sensitive to small fluctuations that, statistically speaking, they are totally insensitive to variations in the initial state of the entity.

count for the quantum probabilities: quantum measurements, from which quantum probabilities result, are non-solipsistic indeterministic processes, producing statistics of outcomes which strongly depend on the initial pre-measurement states, and therefore cannot in general be mutually compatible measurements.

Let us exploit a bit further the evocative example of the coin to see how we can go from solipsistic measurements to a more general class of measurements, to model more general (non-Kolmogorovian) probability situations. Clearly, an experimenter is free to conceive different measurements, by simply defining different observational protocols. In the case of the coin, one can for instance consider different ways to produce its flipping. Solipsistic measurements, as is known, correspond to a situation where the experimenter has to flip the coin with a vigorous momentum, on a sufficiently hard floor. On the other hand, if the experimenter decides to flip the coin in a less vigorous way, or onto a softer surface, or even a sticky one, it is easy to imagine that the statistics of outcomes will start depending on how the coin is initially positioned before the flipping, for instance on the bottom of the dice cup that is used to produce the shot.

To model this possibility, we need to find a representation that allows us to express a dependency of the probabilities on the pre-measurement state. A very simple idea would be to represent the state of the coin directly in terms of the associated outcome probabilities. Of course, by doing so we will not be able anymore to describe solipsistic measurements (as is clear that for them the different initial states are all associated with the same outcome probabilities), but let us explore anyway this idea, as it will show us the path for its natural generalization.

So, let us assume that we have chosen a given flipping protocol (i.e., a given measurement), and that by repeating the experiment many times, with the coin always in the same initial state inside the cup, we have obtained the three outcome probabilities $P(\mathrm{h})$, $P(\mathrm{t})$ and $P(\mathrm{e})$. The idea is to describe the initial state of the coin as an abstract point-particle in $\mathbb{R}^3$, with position $\mathbf{x} = (P(\mathrm{h}), P(\mathrm{t}), P(\mathrm{e}))$, i.e., as a point-particle whose coordinates are precisely the outcome probabilities. Since $P(\mathrm{h}) + P(\mathrm{t}) + P(\mathrm{e}) = 1$, it follows that $\mathbf{x}$ belongs to a two-dimensional regular simplex $\triangle_2$, i.e., to an equilateral triangle of side $\sqrt{2}$.

We obtain in this way a simple and natural geometric representation of the probabilities characterizing the measurement under investigation. However, as we said, this representation cannot be used to describe solipsistic measurements, as two states $\mathbf{x}$ and $\mathbf{x}'$, if different, will necessarily be associated with different outcome probabilities. In fact, this representation doesn't allow, neither, to describe *deterministic* measurements, such that for a given initial state the outcome would be predetermined. These are experiments such that the protocol allow the experimenter to flip the coin in a perfectly controlled (fluctuation free) way, so as to know in advance what will be the final state, given the initial one.

If we want to obtain a representation that can be used to also represent solipsistic and deterministic measurements, we thus need to find a way to describe the state of the entity independently from the outcome probabilities. For this, we need to introduce in our model some additional *elements of reality*, describing the *interactions* between the measured entity and the measuring system, i.e., between the coin and the cup-floor system, in our example. To do so, we start by observing that the point $\mathbf{x}$ exactly defines three disjoint triangular sub-regions in $\triangle_2$, which we will call A_{h}, A_{t} and A_{e} (see the first drawing of Fig. 1). A simple geometric calculation then shows that [17–19,22]:

$$P(\mathrm{h}) = \frac{\mu(A_{\mathrm{h}})}{\mu(\triangle_2)}, \quad P(\mathrm{t}) = \frac{\mu(A_{\mathrm{t}})}{\mu(\triangle_2)}, \quad P(\mathrm{e}) = \frac{\mu(A_{\mathrm{e}})}{\mu(\triangle_2)}, \tag{2}$$

where μ denotes the Lebesgue measure. In other terms, the relative areas of the three sub-regions defined by the state-vector $\mathbf{x}$ are exactly the outcome probabilities. To exploit this remarkable geometric property of simplexes, we are now going to describe a *tension-reduction process* that will allow us to represent, in an abstract way, the (possibly) indeterministic part of a measurement process.

Assume that $\triangle_2$ is an elastic and disintegrable membrane, stretched between its three vertex points — let us call them $\mathbf{x}_{\mathrm{h}}$, $\mathbf{x}_{\mathrm{t}}$ and $\mathbf{x}_{\mathrm{e}}$, respectively — and that the state of the entity is represented by a point-particle firmly attached to the membrane, at some point $\mathbf{x}$. Assume also that the line segments separating the three regions A_{h}, A_{t} and A_{e} are like "tension lines," along which the membrane can less easily disintegrate. Then, consider the following process: the membrane disintegrates, at some unpredictable point $\boldsymbol{\lambda}$. If $\boldsymbol{\lambda} \in A_{\mathrm{h}}$, the disintegrative process propagates inside the entire sub-region A_{h}, but not in the other two sub-regions A_{t} and A_{e}, because of the tension lines. This will cause the two anchor points of A_{h}, $\mathbf{x}_{\mathrm{t}}$ and $\mathbf{x}_{\mathrm{e}}$, to tear away, and being the membrane elastic, it will consequently collapse toward the remaining anchor point $\mathbf{x}_{\mathrm{h}}$, drawing in this way the point particle (which is attached to it) to the same final position, representative of the outcome "head."

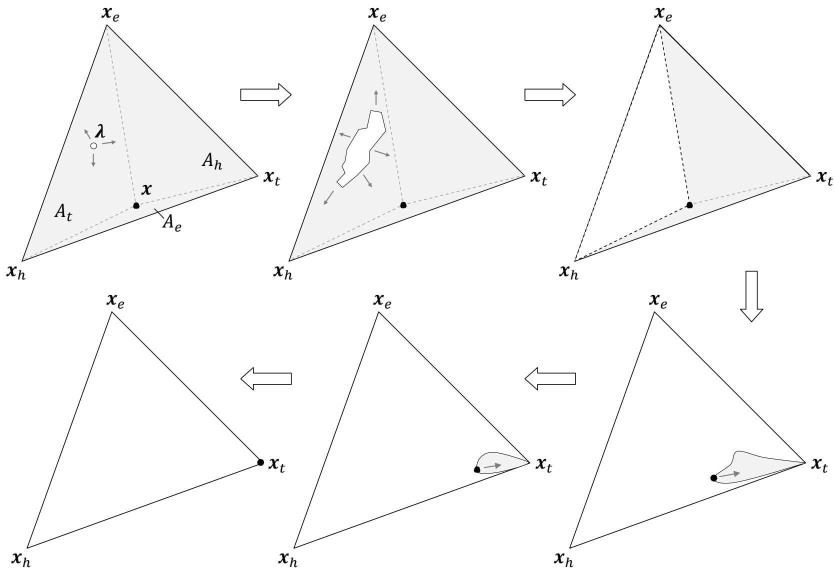

Fig. 1. A 2-dimensional triangular membrane stretched between the three vertex points $\mathbf{x}_\mathrm{h}$, $\mathbf{x}_\mathrm{t}$ and $\mathbf{x}_\mathrm{e}$, with the point particle attached to it at point $\mathbf{x}$, giving rise to three disjoint convex regions A_h, A_t, and A_e. The vector $\boldsymbol{\lambda}$, here assumed to belong to region A_t, indicates the initial point of disintegration of the membrane, which by collapsing brings the point particle to point $\mathbf{x}_\mathrm{t}$, corresponding to the final outcome of the measurement.

Similarly, if the initial disintegration point happens in A_t, the final outcome will be "tail," represented by the vector $\mathbf{x}_\mathrm{t}$ (see Fig. 1), and if the initial disintegration point happens in A_e, the final outcome will be "edge," represented by the vector $\mathbf{x}_\mathrm{e}$. If the membrane is uniform, it is clear that the probability to obtain outcome $\mathbf{x}_\mathrm{h}$ is just given by the relative area of sub-region A_h, which according to (2) corresponds to the first component of state vector $\mathbf{x}$, and similarly for the other two outcomes.

Of course, a uniformly disintegrable membrane is a very special situation, and a priori an (uncountable) infinity of different membranes, characterized by different *ways of disintegrating*, can be considered, like for instance those giving rise to solipsistic measurements, whose outcome probabilities are independent of the initial state $\mathbf{x}$. These can be understood as the limit of membranes that become less and less disintegrable in their interior points and more and more disintegrable in the points belonging only to their three edges, so that the position of the point particle on $\triangle_2$, representative of the initial state, becomes irrelevant in the determination of the outcome probabilities, which will only depend on the probabilities that the initial disintegration happens in one of the three edges of $\triangle_2$. More precisely, if the disintegration probability of the edge opposite to $\mathbf{x}_{\rm h}$ is $P(\rm h)$, then this will also be the probability for outcome $\mathbf{x}_{\rm t}$, for (almost) all initial states, and similarly for the other two edges.

The membranes describing deterministic measurements, on the other hand, can be understood as the limit of membranes becoming more and more disintegrable in a sub-region that becomes increasingly small and less and less disintegrable everywhere else. Indeed, in this limit we obtain a structure which almost surely will start disintegrating in a single predetermined point $\boldsymbol{\lambda}$. Then, for almost all initial states one can predict the outcome in advance, with certainty. Indeed, if the initial state $\mathbf{x}$ is such that $\boldsymbol{\lambda} \in A_{\rm h}$, the outcome will be $\mathbf{x}_{\rm h}$, with probability $P(\rm h) = 1$, and similarly for the other two outcomes. Note that if we have said that outcomes are predetermined for *almost* all, and not all initial states, this is because we cannot exclude the special situation $\mathbf{x} = \boldsymbol{\lambda}$, of classical unstable equilibrium, which remains clearly indeterminate (but do not contribute to the probability calculus, being this possibility of zero Lebesgue measure).

To describe the most general typology of disintegrable membrane, we only need to introduce a *probability density* $\rho : \triangle_2 \to [0, \infty[$, $\int_{\triangle_2} \rho(\mathbf{y}) d\mathbf{y} = 1$, characterizing the propensity of the membrane (which will be called ρ-*membrane*) to disintegrate in its different possible sub-regions. This means that if the initial state produces the three sub-regions $A_{\rm h}$, $A_{\rm t}$ and $A_{\rm e}$, the probabilities for obtaining the three outcomes $\mathbf{x}_{\rm h}$, $\mathbf{x}_{\rm t}$ and $\mathbf{x}_{\rm e}$, will be given by the integrals:

$$P(\mathbf{x} \to \mathbf{x}_i | \rho) = \int_{A_i} \rho(\mathbf{y}) d\mathbf{y}, \quad i \in \{\rm h, t, e\}, \tag{3}$$

and of course, in the special case of a uniform membrane, we simply have $\rho(\mathbf{y}) = \frac{1}{\mu(\triangle_2)} = \frac{2}{\sqrt{3}}$, for all $\mathbf{y}$, and we recover (2).

Before continuing in the construction of our model, a remark is in order. It is clear that the 'tension-reduction' mechanism associated with the disintegrable elastic membranes can only describe idealized measurements of the *first kind*, i.e., measurements such that, if repeated a second time, will produce exactly the same outcome, with probability 1. This is so because if $\mathbf{x}$ corresponds to one of the three vertices of $\triangle_2$, then, being already located in one of the end points of the elastic structure, its position cannot be altered by a new membrane's collapse, when the measurement process is repeated.

Of course, not all measurements are of the first kind, but certainly most of them can be made, at least ideally, of the first kind. For instance, to make the coin flipping a measurement of the first kind it is sufficient to specify in the protocol that if the coin is already located on the floor, i.e., if the initial state of the coin is an *on-floor state*, and not an *on-cup state*, then what the experimenter has to do is to simply observe if the upper face is "head," "tail" or "edge," and take the result of such observation as the outcome of the measurement.

Clearly, the flipping of the coin producing the three on-floor outcomes "head," "tail" and "edge," is not the only coin-measurement that we can perform. Imagine for a moment the following *coin shaking* measurement, operationally defined by the following protocol: If the coin is on the floor, then put it at the center of the bottom of the cup, with exactly the same upper face, then shake the cup, following a predetermined procedure (that we don't need to specify here), and finally look at the bottom of the cup, to see what is the obtained upper face. On the other hand, if the coin is already in the cup, just observe its upper face, which will then be the outcome of the measurement.

We now have two different measurements, the *coin-flipping* measurement, which can produce the outcome states "floor-head," "floor-tail" and "floor-edge," and the *coin-shaking* measurement, which can produce the outcome states "cup-head," "cup-tail" and "cup-edge," and of course we cannot associate the same membrane (i.e., the same measurement simplex) to both measurements. This not only because their outcome states are different, but also because the associated flipping and shaking procedures are different.

If different membranes can represent different measurements, belonging to a same state space, then in addition to the tension-reduction process describing the membrane's collapse we have to introduce a mechanism allowing the states belonging to one membrane to be measured with respect to another membrane. In other terms, we need to describe a process that can transform an off-membrane state into an on-membrane state, in order to be subjected to its (possibly) indeterministic collapse. A process of this kind has to be able, in particular, to bring the state "cup-head" in contact with the 'potentiality region' of the flipping-membrane, or the state "floor-head" in contact with the 'potentiality region' of the shaking-membrane.

Since we are here interested in obtaining a geometrical representation, and that we want the description to be as simple as possible, a very natural choice is to use a (deterministic) *orthogonal projection* process. In other terms, if $\mathbf{x}$ is an off-membrane state, with respect to the considered measurement, we can describe the latter as a *two-stage process*. The first stage, purely deterministic, would correspond to the point particle orthogonally "falling" onto the membrane, along a rectilinear path, until it reaches its on-membrane position; the second stage, which can either be deterministic or indeterministic (depending on the nature of the membrane), is then the tension-reduction process produced by the disintegration and subsequent contraction of the membrane that we have previously described.

3. The general tension-reduction (GTR) model

In the previous section, we have considered some measurements possibly performed on a coin, generalizing the solipsistic ones usually considered in the classical games of chance. Of course, it was not our intention to describe in a complete and self-consistent way all possible states and measurements that can be described in relation to a coin entity. Our example was just meant to fix ideas and allow introducing some of the basic concepts of a general geometrical description of an entity, which includes not only its states, but also its measurements, and this by means of an interaction mechanism, based on the disintegration of a membrane, which is able to produce the different outcomes and associated probabilities. Based on the intuition we have gained, we are now in a position to reason in more general and abstract terms to identify the fundamental ingredients of what we have called the *general tension-reduction* (GTR) model [17,18].

We begin by summarizing what we have obtained so far, using now a more formal language. Let Σ be the set of *all states* of a given entity S. By "all states" we don't necessarily mean all conceivable states, but more specifically the set of those states that are relevant for the description of the entity in the different measurement contexts it can be meaningfully associated with. If the entity is finite-dimensional, or can be conveniently approximated as such (this is generally the case in all practical experimental situations), which we will assume to be so in the following, then Σ can be taken to be a subset of the M-dimensional Euclidean space $\mathbb{R}^M$, for some given finite integer $2 \leq M < \infty$.

A measurement on S, producing N different outcomes, with $N \leq M$, is then described as a $(N-1)$-dimensional simplex $\triangle_{N-1}$, whose N vertices $\mathbf{x}_i$, $i = 1, \ldots, N$, are representative of the N possible outcomes. A measurement of the non-degenerate kind (degenerate measurements will be discussed in Sec. 4) is then a two-stage process, bringing the initial state $\mathbf{x} \in \Sigma$ to one of the outcome states $\mathbf{x}_i$, $i = 1, \ldots, N$. The first stage of the process corresponds to the point particle associated with the state $\mathbf{x}$ orthogonally "falling" onto the $(N-1)$-dimensional ρ-membrane, associated with $\triangle_{N-1}$, and firmly attaching to it. If we write $\mathbf{x} = \mathbf{x}^{\parallel} + \mathbf{x}^{\perp}$, with $\mathbf{x}^{\parallel}$ and $\mathbf{x}^{\perp}$ the two components of $\mathbf{x}$ parallel and orthogonal to $\triangle_{N-1}$, respectively, then this first deterministic stage of the measurement corresponds to the transition: $\mathbf{x} \to \mathbf{x}^{\parallel}$.

The second stage is a process that can either be deterministic or indeterministic, depending on the nature of the ρ-membrane. Its description is a straightforward generalization of what we have explained already in the $N = 3$ coin example (see Fig. 1). The presence of the point particle on the ρ-membrane, at point $\mathbf{x}^{\parallel}$, creates N disjoint sub-regions A_i, $i = 1, \dots N$ (A_i is the convex closure of $\{\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{x}^{\parallel}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_N\}$), such that $\triangle_{N-1} = \cup_{i=1}^{N} A_i$. These sub-regions are separated by $(N-2)$-dimensional "tension-surfaces," along which the elastic substance can less easily disintegrate. Then, as soon as the ρ-membrane starts disintegrating, at some point $\boldsymbol{\lambda}$, if $\boldsymbol{\lambda} \in A_i$, the disintegrative process will cause the $N-1$ anchor points $\mathbf{x}_j$, $j \neq i$, of A_i, to tear away, and consequently the ρ-membrane will contract toward the remaining anchor point $\mathbf{x}_i$, drawing the abstract point particle to that position. Thus, we have a two-stage process producing the possible transitions $\mathbf{x} \to \mathbf{x}^{\parallel} \to \mathbf{x}_i$, with associated probabilities:

$$P(\mathbf{x} \to \mathbf{x}_i|\rho) = \int_{A_i} \rho(\mathbf{y}) d\mathbf{y}, \quad i \in \{1, \dots, N\}. \tag{4}$$

Let us explore a little further the general structure of Σ. All points belonging to a measurement simplex $\triangle_{N-1}$ are also possible states of the entity under investigation, as is clear that they are all measurable with respect to a ρ-membrane associated with $\triangle_{N-1}$. But it is also clear that within the $(N-1)$-dimensional sub-space generated by $\triangle_{N-1}$, there cannot be other points representative of states in addition to those belonging to $\triangle_{N-1}$ itself. Indeed, *bona fide* states are those that, at least in principle, can participate in all well-defined measurements, but the points in that sub-space lying outside of the simplex cannot be orthogonally projected onto the latter, and therefore cannot be measured with respect to its ρ-membrane.

Of course, this doesn't mean that the points outside of a simplex, in each simplex sub-space, cannot also be used to describe some kind of states, i.e., some real conditions characterizing the entity under study. However, since these states would describe situations where the entity would not be available in producing any outcome (i.e., in providing an answer when subjected to a measurement's interrogative process), they have to be considered states of a non-ordinary kind. Let us call them *confined states*, to express the idea that they describe situations where the entity is confined in a "place of reality" which is out of reach for ordinary measurement contexts.

In the example of the coin, we can imagine a situation where the coin has been glued to the wall, with a very strong glue, so that it cannot be subjected anymore to the coin-flipping measurement, or the coin-shaking one. In a cognitive psychology experiment, we can consider the situation of a person who is asked to choose one among a set of predetermined responses to a given question, but with the question and responses expressed in a language that the person cannot understand, so that no meaningful answer can be obtained from her. In physics we can also mention the example of color confinement, the well-known difficulty in directly observing single color-charged entities, like quarks, in our Euclidean spatial theater. Having said that, in the following we will limit our discussion to ordinary (non-confined) states, participating to all possible measurements, which means that by the term "state" we will mean (if not mentioned otherwise) a condition in which the entity is available to take part in all well-defined measurements. Consequently, the $(N-1)$-dimensional section of the state space Σ that contains the measurement simplex $\triangle_{N-1}$, will be taken to be precisely $\triangle_{N-1}$.

As we said, different measurements are associated with different simplexes, i.e., with simplexes having different relative orientations. This means that the dimension M of the state space Σ needs to be large enough to accommodate all these different orientations, in a way that the points belonging to the different simplexes are all mutually orthogonally projectable, so that they can all participate to the different possible measurements. For this to be the case, it is easy to imagine that the dimension M of Σ will generally have to be considerably larger than N, taking into account the fact that in each subspace generated by a simplex no other simplexes can be present.

Another issue to be addressed is the center of the simplexes. Of course, they all have to be centered at the same point, say the origin of the system of coordinates considered, otherwise it would not be possible to ensure the overall functioning of the orthogonal projection mechanism. But there is another reason why all measurement simplexes need to share the same origin. The point at the origin corresponds to a state which has quite a remarkable property: it is the state which manifests the same availability in producing whatever outcome, in whatever measurement, and if measurements are described by uniform effective membranes (see Sec. 6, for the central role played by uniform probability distributions ρ_u), it is also the state producing the same probabilities $\frac{1}{N}$, for all outcomes in all measurements. In a sense, it is the most *neutral* state among all possible ones, and if we assume that such condition of *maximum neutrality* should exist, at least in principle, then the different simplexes will have to share the same origin.

What about the shape of the state space Σ in $\mathbb{R}^M$? First of all, what we know is that no states can be at a distance from the origin that is greater than that of the apex points of the different $(N-1)$-dimensional measurement simplexes. Since these points are by definition at distance 1 from the center of the simplex to which they belong, and that all simplexes share the same center, we have that all the M-dimensional vectors of Σ are necessarily contained in a M-dimensional ball of radius 1, although of course they will not generally fill such ball. We can also remark that each $(N-1)$-simplex contains an inscribed $(N-1)$-ball of radius $\frac{1}{N-1}$, and since all simplexes have the same origin, within Σ there is a M-ball of radius $\frac{1}{N-1}$ possessing a maximum density of states, as it contains all the inscribed $(N-1)$-balls associated with the different measurements. Thus, in case the entity under consideration would be associated with a continuity of measurement simplexes, we can expected such M-ball to be completely filled with states. Also, considering that for $N=2$ the inscribed 1-sphere (a line segment) has radius 1, if we have a continuity of two-outcome measurement simplexes $\triangle_1$, oriented along all possible directions in $\mathbb{R}^M$, the state space will be precisely a M-dimensional ball of radius 1, i.e., $\Sigma = B_1(\mathbb{R}^M)$.

Of course, apart being contained in a unit ball, and containing a smaller ball having a maximum density of states, nothing can be said a priori about the shape of Σ, i.e., about the envelope containing the extremal points of Σ. These extremal points can be of two kinds: either they are at a distance 1 from the origin, and thus correspond to one of the vertices of a measurement simplex, or they belong to one of the sub-simplexes of a measurement simplex, and then their distance from the origin will be smaller than 1. This means that, apart the above mentioned two-dimensional case, Σ will generally not have a spherical symmetry.

Strictly speaking, the question of the shape of Σ is meaningful only if we have a continuity of states, i.e., if Σ is a region of $\mathbb{R}^M$ completely filled with states. In this case, for consistency reasons, it is reasonable to assume that it will be a *convex* region. Indeed, by definition, a convex region is a set of points such that, given any two points, the line joining them lies entirely within it. This means that the region is connected, in the sense that it is possible to go from one point to another without leaving the region. If we assume, as we did, that a measurement is a process during which the state of the entity changes in a continuous way within Σ, and that both during the first deterministic stage and the second possibly indeterministic stage the abstract point particle representative of the state follows rectilinear trajectories, then the only way to guarantee that in all circumstances these trajectories are made of states, i.e., that they belong to Σ, is to require Σ to be a convex set.

4. Degenerate measurements in the GTR-model

To complete our description of the GTR-model, we need to consider the possibility of measurements such that different final states can be associated with a same outcome. These are called *degenerate* measurements in quantum mechanics and we will adopt here the same terminology. However, since our approach is more general, we will have to distinguish between two different possibilities, that we will call *submeasurements of the first type* and *submeasurements of the second type* (not to be confused with von Neumann's designation of measurements of the first and second kind). Submeasurements of the first type are degenerate measurements in which the experimenter can in principle distinguish between all the possible outcomes, but decides (for whatever reason) not to do so, thus identifying some of them.

For instance, in the example of the coin, we can imagine the situation where the "edge" outcome is conventionally identified with the "head" outcome, so that one obtains an effective "head" or "tail" two-outcome measurement, where "head" is now re-interpreted as either point $\mathbf{x}_{\mathrm{h}}$ or point $\mathbf{x}_{\mathrm{e}}$. But apart from this identification, the measurement protocol, and therefore the associated membrane's collapse mechanism, remains exactly the same. In other terms, a submeasurement of the first type, by only identifying some of the outcomes, produces a change of state of the entity that is identical to that produced by an experiment where such identification is not considered.

Submeasurements of the second type, on the other hand, correspond to experimental situations where the distinction between certain outcomes becomes impossible to realize in practice, even in principle. This means that the outcome states are different than those associated with the corresponding non-degenerate situations, as if they were not, the distinction between the different outcomes would always be possible. In other terms, a submeasurement of the second type is characterized by a different experimental context, and therefore the membrane's mechanism describing its unfolding will also be different.

In the example of the coin, we can consider the following modified flipping protocol. Once the coin has reached the floor, before taking knowledge of the value of the upper face, a colleague performs the following additional operations: if she finds that the upper face is "tail," she does nothing. If instead she finds that the upper face is "edge," or "head," she takes the coin and places it on a table, tail up, and then without saying a word leaves the room. Clearly, if the final location of the coin is on the floor, the outcome is "tail," and more precisely "floor-tail". On the other hand, if the final location of the coin is on the table, then the outcome is "table-tail," and evidently such new state contains no information that would allow the experimenter to associate it either with the "floor-head" state or the "floor-edge" state, of the associated non-degenerate measurement.

If we denote $P_{\mathrm{deg}}(\mathrm{t})$ the probability of obtaining "floor-tail," and $P_{\mathrm{deg}}(\bar{\mathrm{t}})$ the probability of obtaining "table-tail," in the degenerate measurement, we clearly have:

$$P_{\mathrm{deg}}(\mathrm{t}) = P(\mathrm{t}), \quad P_{\mathrm{deg}}(\bar{\mathrm{t}}) = P(\mathrm{h}) + P(\mathrm{e}). \tag{5}$$

Degenerate measurements of the second type which obey equalities of the above kind will be said to be *quantum-like*, as is clear that quantum measurements always obey them.

Let us now show how we can modify the membrane's mechanism to describe submeasurements of the second type. Starting from the non-degenerate situation, we must alter the functioning of the membrane in such a way that not only (5) will be satisfied, but also the collapse will have to produce the two states "floor-tail" ($\mathbf{x}_{\mathrm{t}}$) and "table-tail" ($\mathbf{x}_{\bar{\mathrm{t}}}$), instead of the three states "floor-tail" ($\mathbf{x}_{\mathrm{t}}$), "floor-head" ($\mathbf{x}_{\mathrm{h}}$) and "floor-edge" ($\mathbf{x}_{\mathrm{e}}$). In the following, we only analyze the $N = 3$ situation, the generalization to more general measurements being straightforward.

The first deterministic stage of the measurement is exactly the same as for the corresponding non-degenerate situation, with the point particle initially in state $\mathbf{x}$ orthogonally "falling" onto the membrane and firmly attaching to it. Instead, the second indeterministic stage is different, being that the two sub-regions A_{h} and A_{e} will now be fused together, so as to form a single larger subregion $A_{\bar{\mathrm{t}}} = A_{\mathrm{h}} \cup A_{\mathrm{e}}$. To fix ideas, we can think that a special reactive substance has been applied along the common boundary between A_{h} and A_{e}, the effect of this special substance being twofold: firstly, it produces the effective fusion of the two subregions into a single one, so that if the membrane breaks in a point belonging to, say, A_{h}, the tearing will now propagate also across the boundary with A_{e} (because of the presence of the reactive substance), causing the collapse of the entire subregion $A_{\bar{t}}$. Secondly, it produces the early detachment of the common anchor point $\mathbf{x}_t$, with the consequent contraction of the elastic membrane, drawing the point particle attached to it to a given position on the line segment (the 1-simplex) subtended by $\mathbf{x}_h$ and $\mathbf{x}_e$. Finally, also the last two anchor points $\mathbf{x}_h$ and $\mathbf{x}_e$ will detach, causing the membrane to shrink toward the particle, but without affecting its acquired position (see Fig. 2).

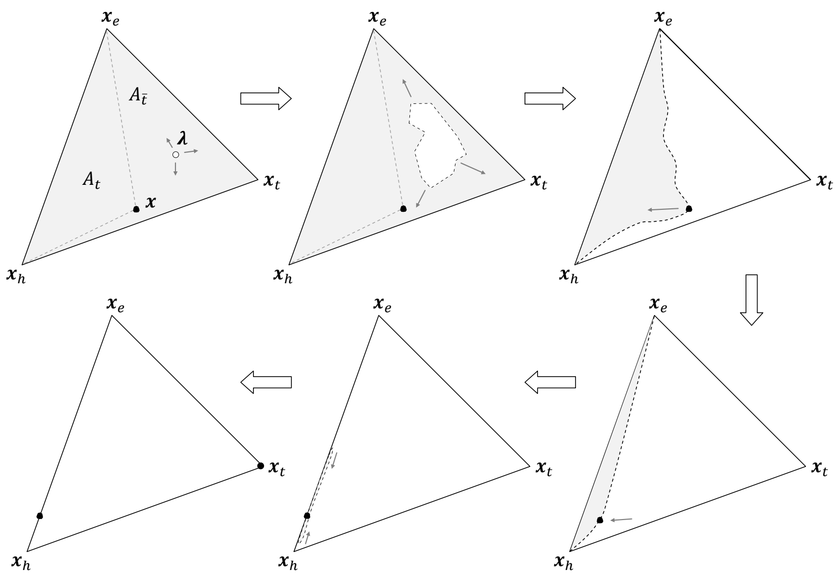

Fig. 2. The breaking of the elastic membrane (in grey color), when the two regions A_h and A_e have been fused into a single larger region $A_{\bar{t}}$. The process is here represented in the case where the initial breaking point is in A_h. At first, the membrane disintegrates inside $A_h \subset A_{\bar{t}}$, causing the anchor point $\mathbf{x}_t$ to detach and the particle to be drawn to a position somewhere on the line segment between $\mathbf{x}_e$ and $\mathbf{x}_h$. Then, also the two anchor points $\mathbf{x}_e$ and $\mathbf{x}_h$ simultaneously detach, causing the membrane to shrink toward the particle, without affecting its acquired position. Note that the on-membrane state has been denoted $\mathbf{x}$, instead of $\mathbf{x}^{\parallel}$, for notational consistency with Fig. 1.

On the other hand, if the membrane breaks in A_t, then only A_t will collapse, producing the final outcome $\mathbf{x}_t$, exactly as in the non-degenerate situation. So, when performing the degenerate measurement, only two transitions are now possibly produced by the membrane mechanism. If the membrane disintegrates in A_{t}, the outcome is $\mathbf{x}_{\mathrm{t}}$, which means that the abstract point particle representative of the final state will be at a maximal (unit) distance from the origin of the system of coordinates, that is, from the center of the M-dimensional sphere in which Σ is inscribed. On the other hand, if the membrane disintegrates in the composite subregion $A_{\bar{\mathrm{t}}}$, the membrane's collapse will not necessarily produce a vector of maximal length within Σ. Therefore, it is natural to assume in this case that a third deterministic process can possibly occur, to complete the measurement, bringing the point particle to a final position $\mathbf{x}_{\bar{\mathrm{t}}}$ of maximal distance from the origin, at the surface of Σ, representative of the final outcome-state of the measurement.

In view of (4), it is clear that the degenerate membrane's mechanism we have just described obeys (5), considering that the first and third stages of the measurement are purely deterministic. There is however another aspect of interest that needs to be discussed, in relation to this possible third stage of a degenerate measurement of the second type. More precisely, we need to distinguish the following two possibilities. Considering the outcome $\mathbf{x}_{\bar{\mathrm{t}}}$, either (1) its orthogonal projection onto the membrane falls onto some point of the line segment between $\mathbf{x}_{\mathrm{e}}$ and $\mathbf{x}_{\mathrm{h}}$, or (2) it falls onto whatever other point of the membrane, not lying on that line segment. Condition (1) corresponds to the situation where the corresponding non-degenerate measurement having $\mathbf{x}_{\bar{\mathrm{t}}}$ as the initial state, can only produce the two outcomes $\mathbf{x}_{\mathrm{e}}$ and $\mathbf{x}_{\mathrm{h}}$, and therefore cannot produce the outcome $\mathbf{x}_{\mathrm{t}}$. On the other hand, condition (2) corresponds to the situation where the probability of obtaining $\mathbf{x}_{\mathrm{t}}$ will be generally non-zero.

We shall say that a submeasurement of the second type, obeying condition (1) above, is a *projection-like submeasurement*, as we know that degenerate quantum measurements do obey this condition as a result of the projection formula. A simple way to automatically implement (1) in our model is to ask the third deterministic stage to be structurally similar to the first one, in the sense that the point particle, when reemerging from the membrane, it will always do so following a rectilinear path, orthogonal to the membrane's plane. Such final state, when projected back onto the membrane, in a repetition of the measurement, will then land onto the same edge of the triangular membrane, in accordance with condition (1). It is worth observing, however, that a degenerate measurement of the second type will generally not be projection-like, in the above sense. This is immediately clear in our coin example. Indeed, when flipping the coin in a "table state" onto the floor, if we don't use a very special experimental protocol, all three states “floor-head,” “floor-tail” and “floor-edge” will be easily obtained.

5. Composite entities in the GTR-model

Another important class of measurements we want to describe in our GTR-model is that of measurements performed on *composite entities* (also called *join entities*). To remain within the ambit of our example, consider the situation where instead of a single coin we now have two coins (not necessarily identical). Then, we can perform a first coin-flipping measurement with the first coin, observe the outcome, and do the same with the second coin, observing again the outcome. As each one-entity measurement can produce 3 different outcomes: “head,” “tail” and “edge,” the combination of the two measurements can produce $9 = 3 \times 3$ different outcomes: “head-head,” “head-tail,” “head-edge,” “tail-head,” “tail-tail,” “tail-edge,” “edge-head,” “edge-tail” and “edge-edge.”

It is important to observe that, to be able to perform separately the two measurements, it has to be possible to act separately on the two coins and obtain in a separate way the outcomes of the two experimental actions. We are not saying by this that the two coins must be *experimentally separated*, in the sense that their measurements cannot produce correlations, but that the measurements themselves have to be separable. In physics for instance, when considering measurements on composite entities, these are usually separable, but in more general situations this may not necessarily be the case [23,24].

An important characteristic of separable measurements is that they can be performed either sequentially or simultaneously, and that when they are performed in a sequential way the order of the sequence is irrelevant. This is evidently the case in the coin-flipping measurements. Indeed, we can either flip the two coins simultaneously, or one after the other, in whatever order, and we will always obtain the same statistics of outcomes. However, a separable measurement can either produce or not produce correlated outcomes, depending on the nature of the state of the join entity. If such state describes a condition of (experimental) separation of the sub-entities forming the join entity, then no correlations will be observed in the statistics of outcomes. We shall call states of this kind *product states*. The term “product” is here to be understood in the specific sense of a state that when subjected to a separable measurement, the obtained set of outcomes can be described as the *Cartesian product* of the set of outcomes obtained when individual measurements are performed separately, on the different sub-entities, in whatever order.

In the GTR-model, the situation of separable measurements performed on product states can be described by simply considering distinct membranes, one for each sub-entity forming the join entity, working independently from one another. In the case of the two-coin entity, its initial state is then described by a couple of initial states, one for each individual coin, and its measurement is described by a couple of membranes, acting separately on these two one-coin initial states. Of course, it is also possible to represent this double-membrane process as a single higher-dimensional membrane process. For this, instead of two three-dimensional triangular membranes, we will have a 8-dimensional hypermembrane (a 8-simplex), with 9 vertices. The initial state of the two coins will then be described by a vector whose first three components define the state of the first coin, the successive three components define the state of the second coin, and the remaining components (the number of which depends on the dimension $M \geq 8$ of the state space) are fully determined by them (as the join entity state, being a product state, it has to be fully determined by the state of its components). This means that the higher-dimensional single-membrane measurement process, when projected onto its first $3+3$ components, will describe two independent processes, in accordance with the fact that, for a separable measurement performed on a product state, "the whole has to be equivalent to the sum of its parts."

The situation is however different if either the state is a non-product state, or the measurement itself is a non-separable measurement, or both. For instance, we may decide to change the operational definition of the flipping measurement by introducing a small rigid rod whose two ends are glued to the two coins (in a way that we don't need to specify here), before flipping them. The presence of the connecting rod means that the two-coin measurement cannot anymore be conceived as two separate one-coin measurements, as by flipping one coin we will also, inevitably, flip the other one, and because of the connecting rod (which may break or not during the experiment), correlated outcomes can be observed (which one can show are able to violate Bell's inequalities; see for instance [13,25,26]). In other terms, the measurement becomes non-separable. The situation where the state of the two coins is non-product is similar. For instance, one can consider that the two coins are strongly magnetized, and therefore able to attract each other. This means that the description of their state also has to include a description of the magnetic field connecting them, which similarly to the above rigid rod example can produce correlations, under certain conditions.

Different from the situation of separable measurements performed on product states, in the case of non-separable measurements and/or non-product states, two separate membranes will clearly not be sufficient to obtain a full description not only of the state of the two-coin entity, but also of the statistics of (correlated) outcomes that the measurement is able to produce. Only a genuinely higher dimensional structure will be able in this case to account for all the experimental possibilities, and in particular the $3+3$ components describing the two initial individual coin states will generally not allow to deduce the value of the remaining components, in accordance with the fact that, for non-product (non-separable) entities, "the whole will be more than just the sum of its parts."

6. The quantum mechanical example

In the previous sections we have used the coin example as an heuristic to explain the basic ingredients and structure of the GTR-model. In this section, we consider an important implementation of the GTR-model: *quantum mechanics*. Clearly, the GTR-model is much more general than quantum mechanics, in the sense that it is able to account for a much wider class of measurements and states than those usually considered in the standard quantum formalism. Also, even when the model is reduced to quantum mechanics, by means of two assumptions that we are now going to enunciate, it still remains a more general framework than quantum mechanics, in the sense that it describes a completed version of the latter where the Born rule can be derived in a non-circular way. In that sense, the GTR-model also offers a possible solution to the longstanding measurement problem [19].

The two additional assumptions that are needed to derive the Born rule of probabilistic assignment from the GTR-model are the following:

Hypothesis 1: Metaignorance. *The experimenter does not control which specific measurement is actualized at each run of the measurement, among those that can actualize the given outcome-states. In other terms, the experimenter lacks knowledge not only about the (almost deterministic) measurement interaction that each time is actualized, but also about the way it is each time selected.*

Hypothesis 2: Hilbertian structure. *The state space Σ is a generalized Bloch sphere.*

Let us explain how the above two assumptions can be used to derive the quantum mechanical Born rule. We start by analyzing the consequences of the first one: *metaignorance* (or *metaindifference*, to use the terminology of [27]; see also [28]). This assumption is natural for the following reason. In a typical quantum measurement (like, say, a Stern-Gerlach spin-measurement), we are in a situation where the experimenter doesn't want to control in whatsoever way its development. This because a measurement is generally understood as a process of *observation*, and an observation is meant to alter in the least possible way the observed entity. Of course, if the process creates the very property that is observed, then there will be an "intrinsic invasiveness" to it, impossible to remove. This is precisely the situation of quantum measurements, where the entity transitions from an initial to a final state, with the latter being generally different from the former (if the pre-measurement state is not an eigenstate).

In other terms, in a measurement context the only controlled aspect is that relative to the definition of the possible outcome-states. But apart from that, the experimenter will avoid as much as possible to interfere with the natural process of actualization of these potential outcomes. Thus, underlying the metaignorance assumption there is the idea that if nothing limits the way the measuring system and the measured entity can interact, they will naturally explore all possible (available) ways of interacting. This means that the quantum statistics, and more generally the statistics of any observational process where the experimenter doesn't try to influence the outcomes, corresponds to what has been called a *universal average* [17–19,28], i.e., an average over all possible measurement processes associated with a predetermined set of outcomes. Measurements subtending a universal average are called *universal measurements*, and our point is that quantum measurements are just a special example of universal measurements.

More specifically, to each experimental situation characterized by N given outcomes, an uncountable infinity of measurements can be defined, each one characterized by a different ρ-membrane, i.e., by a $(N-1)$-simplex associated with a different probability density ρ. A universal measurement then corresponds to a two-stage process, where firstly a ρ-membrane is selected among the infinity of possible ones, and secondly, from that specific ρ-membrane, a $\boldsymbol{\lambda}$-measurement-interaction is also selected, so producing the (almost) deterministic collapse of the membrane.

So, to calculate the probabilities associated with a universal measurement one has to perform an average over the probabilities obtained from all these different possible measurements. Of course, if such a huge average is addressed directly, one will be confronted with technical problems related to the foundations of mathematics and probability theory. A good strategy, similar to that used in the definition of the *Wiener measure*, is to proceed as follows. First, one shows that any probability density ρ can be described as the limit of a suitably chosen sequence of *cellular probability densities* ρ_n, as the number of cells n tends to infinity, in the sense that for every initial state $\mathbf{x}$ and final state $\mathbf{y}$, one can always find a sequence of cellular ρ_n such that the transition probability $P(\mathbf{x} \to \mathbf{y}|\rho_n)$, associated with the ρ_n-membrane, tends to the transition probability $P(\mathbf{x} \to \mathbf{y}|\rho)$, associated with the ρ-membrane, as $n \to \infty$ [18,19].

By a cellular probability density we mean here a probability density describing a structure made of a finite number n of regular cells (of whatever shape), tessellating the hypersurface of the $(N-1)$-simplex, which can only be of two sorts: uniformly breakable, or uniformly unbreakable. Then, if one excludes the totally unbreakable case (as it would produce no outcomes), we have a total number $2^n - 1$ of possible ρ_n-membranes. This means that for each n, one can unambiguously define the average probability:

$$\langle P(\mathbf{x} \to \mathbf{y})\rangle_n \equiv \frac{1}{2^n - 1} \sum_{\rho_n} P(\mathbf{x} \to \mathbf{y}|\rho_n), \tag{6}$$

where the sum runs over all the possible $2^n - 1$ probability densities made of n cells. Clearly, $\langle P(\mathbf{x} \to \mathbf{y})\rangle_n$ is the probability for the transition $\mathbf{x} \to \mathbf{y}$, when a probability density ρ_n (a cellular ρ_n-membrane) is chosen at random, in a uniform way.

Then, to obtain the transition probabilities of the universal measurement, i.e., of the average over all possible ρ-measurements, one has to calculate the infinite cell limit of the above average:

$$\langle P(\mathbf{x} \to \mathbf{y})\rangle_{\text{univ}} = \lim_{n\to\infty} \langle P(\mathbf{x} \to \mathbf{y})\rangle_n, \tag{7}$$

and it is possible to demonstrate that [18,19]:

$$\langle P(\mathbf{x} \to \mathbf{y})\rangle_{\text{univ}} = P(\mathbf{x} \to \mathbf{y}|\rho_u), \tag{8}$$

where ρ_u denotes the uniform probability density, i.e., the probability density associated with a uniform membrane, for which all points have the same probability of disintegrating.

A model only contemplating uniform membranes has been called the *uniform tension-reduction* (UTR) model [17,18]. The result (8) then tells us that every time a statistics of outcomes is generated by a universal average, an effective UTR-model will naturally emerge. This is however not sufficient to derive the quantum mechanical Born rule. For this, the second assumption, about the structure of the state space, is also needed. Here again we will not go into any technical details of the proof and just explain in broad terms the gist of it, referring the interested reader to [17–19].

It is well known that the rays of a two-dimensional Hilbert space can be represented as points at the surface of a 3-dimensional unit sphere, called the *Bloch sphere* [29], with its internal points describing the so-called *density operators* (also called *density matrices*). What is less known is that a similar representation can be worked out for general N-dimensional Hilbert spaces [19,30–35]. The standard 3-d Bloch sphere is then replaced by a generalized (N^2-1)-dimensional Bloch sphere, with the only difference that for $N > 2$ only a convex portion of it will be filled with states. In other terms, the state space Σ is a M-dimensional convex set, with $M = N^2 - 1$, inscribed in a unit sphere of same dimension.

It is then possible to show that the set of eigenvectors associated with a given observable, i.e., with a self-adjoint operator, are precisely described, within Σ, by the N vertexes of a $(N-1)$-simplex inscribed in the generalized Bloch sphere [19]. And when the point particle representative of the initial ray-state, located at some point $\mathbf{x}$ at a unit distance from the origin, plunges into the sphere to reach its on-membrane position $\mathbf{x}^{\parallel}$, following a path orthogonal to the simplex, it can also be shown that this deterministic movement precisely causes the off-diagonal elements of the associated density operator (in the measurement's basis) to gradually vanish, which means that the on-membrane state is precisely a *decohered state*, described by a fully reduced density operator.

Different from the usual description of *decoherence theory* [36], the decohered on-membrane state $\mathbf{x}^{\parallel}$ does not correspond to the final state of the measurement, but to the state prior to the indeterministic collapse of the membrane. In probabilistic terms, the disintegrative/collapse process is governed by the Born rule, as is clear that the coordinates of the on-membrane (reduced density operator) state $\mathbf{x}^{\parallel}$ are precisely the transition probabilities and that, as we already remarked in Sec. 2, Eq. (2), the relative Lebesgue measure of the different subregions correspond to the values of the corresponding coordinates of the on-membrane vector. Also, in the situation where the measurement is degenerate, a third purely deterministic process can also happen, of the *purification* kind, through which the entity takes a maximal "distance" from the measurement context represented by the membrane, in accordance with the predictions of the Lüders-von Neumann projection formula [19].

To complete our description of the quantum mechanical example, we briefly mention the situation with multipartite systems, formed by multiple entities. For this, we recall that the existence of a generalized Bloch representation is based on the observation that one can always find a basis for the density operators acting in $\mathbb{C}^N$, made of N^2 orthogonal (in the Hilbert-Schmidt sense) operators. One of them is the identity matrix $\mathbb{I}$, and the other N^2-1 correspond to a determination of the generators of $SU(N)$, the *special unitary group of degree* N. These are traceless and orthogonal self-adjoint matrices Λ_i, $i=1,\ldots,N^2-1$, which can be chosen to be normalized as follows: $\mathrm{Tr}\,\Lambda_i\Lambda_j = 2\delta_{ij}$. Then, any density operator state D can be uniquely determined by a (N^2-1)-dimensional real vector $\mathbf{x}$, by writing [19]:

$$D(\mathbf{x}) = \frac{1}{N}\left(\mathbb{I} + c_N\,\mathbf{x}\cdot\mathbf{\Lambda}\right) = \frac{1}{N}\left(\mathbb{I} + c_N\sum_{i=1}^{N^2-1} x_i\Lambda_i\right), \tag{9}$$

where we have defined the constant: $c_N \equiv \sqrt{\frac{N(N-1)}{2}}$.

If the entity in question is a joint entity formed by two sub-entities, with Hibert spaces $\mathbb{C}^{N_A}$ and $\mathbb{C}^{N_B}$, respectively, so that $\mathbb{C}^N = \mathbb{C}^{N_A}\otimes\mathbb{C}^{N_B}$ and $N = N_A N_B$, it is possible to introduce a tensorial determination of the generators, such that the generators of $SU(N)$ are expressed as tensor products of the generators of $SU(N_A)$ and $SU(N_B)$. In this way, one can show that the vector $\mathbf{x}$ representative of the state of the joint entity can always be written as a direct sum of three vectors [37]:

$$\mathbf{x} = d_{N_A}\mathbf{x}^A \oplus d_{N_B}\mathbf{x}^B \oplus \mathbf{x}^{\mathrm{corr}}. \tag{10}$$

where $d_{N_A} = (\frac{N_A-1}{N-1})^{\frac{1}{2}}$, $d_{N_B} = (\frac{N_B-1}{N-1})^{\frac{1}{2}}$, $\mathbf{x}^A$ belongs to the one-entity Bloch sphere $B_1(\mathbb{R}^{N_A^2-1})$ and describes the state of the A-entity, $\mathbf{x}^B$ belongs to the one-entity Bloch sphere $B_1(\mathbb{R}^{N_B^2-1})$ and describes the state of the B-entity, and $\mathbf{r}^{\mathrm{corr}}$ is that component of the state describing the correlations between the two sub-entities.

In accordance with our analysis of Sec. 5, it is then possible to show that: (1) when the initial state $\mathbf{x}$ is representative of a *product state*, the components of the two individual vectors $\mathbf{x}^A$ and $\mathbf{x}^B$ are independent of one another, and the components of the correlation vector $\mathbf{x}^{\mathrm{corr}}$ are entirely determined by the components of the latter. This means that the indeterministic process described by the $(N-1)$-dimensional two-entity measurement's membrane is equivalent to that obtained by two sequential measurements, in whatever order, performed by two (N_A-1)- and (N_B-1)-dimensional membranes.

On the other hand, when the initial state is not of the product form $D = D_A\otimes D_B$, the two vectors $\mathbf{x}^A$ and $\mathbf{x}^B$ are not anymore independent and the components of the correlation vector $\mathbf{x}^{\mathrm{corr}}$ cannot anymore be deduced from the components of $\mathbf{x}^A$ and $\mathbf{x}^B$. This means that in a non-product situation the collapse mechanism of the full $(N-1)$-dimensional membrane is needed to describe the statistics of outcomes of the quantum measurement, which cannot be decomposed into two separate and independent collapse mechanisms [37].

7. Beyond Kolmogorov & Hilbert

In the previous section we have shown that quantum mechanics provides a specific realization of the GTR-model, when the membranes are uniform (a situation we have called the UTR-model) and the state space is a generalized Bloch sphere, which can be seen as the natural completion of the standard Hilbert space structure in which also density matrices are allowed to play the role of pure states. In this section, we provide a few examples of situations where both classical and quantum probabilities structures are violated, showing in this way that the GTR-model can describe more general probability models than those associated with purely classical and purely quantum experimental propositions.

7.1. *Beyond classical*

We start by considering the violation of the classical probability model. More precisely, we will show that the joint probabilities of sequential measurements cannot generally be fitted into a classical probability model equipped with a single sample space. For this, we consider the simplest possible situation: that of an entity whose measurements only have two possible outcomes. The first measurement is characterized by a one-dimensional ρ_A-membrane (an elastic band) stretched between the two opposite outcome-states $\mathbf{a}$ and $-\mathbf{a}$ (we will call it the A-measurement), and a second measurement is characterized by a one-dimensional ρ_B-membrane stretched between the two opposite outcome-states $\mathbf{b}$ and $-\mathbf{b}$ (we will call it the B-measurement).

We assume that the entity is initially in eigenstate $\mathbf{b}$ of the B-measurement. This means that the probability $P(\to \mathbf{b} \to \mathbf{a}|\mathbf{b})$ of having first the transition to state $\mathbf{b}$, then to state $\mathbf{a}$, knowing that the initial state is $\mathbf{b}$, is: $P(\to \mathbf{b} \to \mathbf{a}|\mathbf{b}) = P(\to \mathbf{b}|\mathbf{b})P(\to \mathbf{a}|\mathbf{b}) = P(\mathbf{b} \to \mathbf{a})$. Also, the probability $P(\to \mathbf{a} \to \mathbf{b}|\mathbf{b})$ of having first the transition to state $\mathbf{a}$, then to state $\mathbf{b}$, knowing that the initial state is $\mathbf{b}$, is: $P(\to \mathbf{a} \to \mathbf{b}|\mathbf{b}) = P(\to \mathbf{a}|\mathbf{b})P(\to \mathbf{b}|\mathbf{a}) = P(\mathbf{b} \to \mathbf{a})P(\mathbf{a} \to \mathbf{b})$. We thus find that:

$$P(\to \mathbf{b} \to \mathbf{a}|\mathbf{b}) - P(\to \mathbf{a} \to \mathbf{b}|\mathbf{b}) = P(\mathbf{b} \to \mathbf{a})[1 - P(\mathbf{a} \to \mathbf{b})]. \qquad (11)$$

This means that whenever $P(\mathbf{b} \to \mathbf{a}) \neq 0$ and $P(\mathbf{a} \to \mathbf{b}) \neq 1$, the right hand side of (11) is different from zero, i.e., $P(\to \mathbf{b} \to \mathbf{a}|\mathbf{b}) \neq P(\to \mathbf{a} \to \mathbf{b}|\mathbf{b})$, which is a violation of classical probability, as the (static) propositions of classical probability theory, based on Boolean algebra, always commute. More precisely, in classical theory the probability of the event "**b** then **a**" has to coincide with the probability of the event "**a** then **b**," i.e., $P_{\rm c}(\to \mathbf{b} \to \mathbf{a}) = P_{\rm c}(\to \mathbf{a} \to \mathbf{b})$, for whatever initial state of the entity under study. Thus, the GTR-model easily violates classical probability.

As a specific example, we can assume that ρ_A is uniform, whereas ρ_B describes an elastic band uniformly breakable only inside an interval of length 2ϵ, centered at the origin of the sphere, and such that $\epsilon < \cos\theta$, with θ the angle between the two elastic bands. Then, we have the transition probabilities: $P(\mathbf{a} \to \mathbf{b}) = 1$, and $P(\mathbf{b} \to \mathbf{a}) = \frac{1}{2}(1+\cos\theta)$, which are clearly different if $\cos\theta \neq 1$ (see Fig. 3).

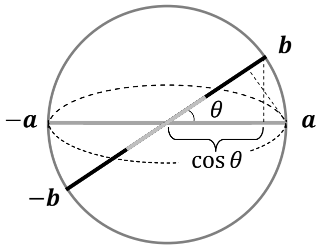

Fig. 3. Two measurements, characterized by different one-dimensional breakable elastic structures. The measurement having the two outcomes **a** and −**a** is characterized by a uniform ρ_A-membrane, whereas the measurement having the two outcomes **b** and −**b** is characterized by a ρ_B-membrane that can uniformly break only inside an interval centered at the origin, whose length is strictly less than $2\cos\theta$. If the initial state is **a**, all the breakable points of ρ_B will contribute to the transition $\mathbf{a} \to \mathbf{b}$. On the other hand, if the initial state is **b**, only the points belonging to the segment of length $1+\cos\theta$ will contribute to the transition $\mathbf{b} \to \mathbf{a}$.

7.2. *Beyond quantum*

To show that the GTR-model can also easily violate quantum probability, we can still use the example of Fig. 3. Indeed, $P(\mathbf{b} \to \mathbf{a}) \neq P(\mathbf{a} \to \mathbf{b})$ is already a manifest violation of the Born rule, as is clear that according to the latter we should always have the equality (sometimes called the *reciprocity law*): $P(\mathbf{b} \to \mathbf{a}) = P(\mathbf{a} \to \mathbf{b})$. Indeed, $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$, implying:

$P(|\phi\rangle \to |\psi\rangle) = |\langle\psi|\phi\rangle|^2 = |\langle\phi|\psi\rangle|^2 = P(|\psi\rangle \to |\phi\rangle)$. This tells us that the transition probability between two states only depends on their relative orientation in the Hilbert space, as measured by the modulus of their scalar product, and not on the specific direction taken by the transition.

One should not conclude, however, that when the reciprocity law is satisfied the probability model would be Hilbertian. Indeed, as soon as $\rho_A = \rho_B$, we have $P(\mathbf{b} \to \mathbf{a}) = P(\mathbf{a} \to \mathbf{b})$, but this doesn't mean that the probabilities produced by the elastic bands are necessarily given by the Born rule. For this, as we explained in Sec. 6, the probability densities have to be uniform. In other terms, testing the reciprocity law is not the same as testing the quantumness of the model, as the reciprocity law can also be satisfied by more general probability models than the Hilbert one. To make this point even more clear, let us introduce a quantity called the *q-test* [6,16,38]:

$$\begin{aligned} q \equiv\; &[P(\to \mathbf{a} \to -\mathbf{b}|\mathbf{x}) + P(\to -\mathbf{a} \to \mathbf{b}|\mathbf{x})] \\ &- [P(\to \mathbf{b} \to -\mathbf{a}|\mathbf{x}) + P(\to -\mathbf{b} \to \mathbf{a}|\mathbf{x})], \end{aligned} \tag{12}$$

where $\mathbf{x}$ is some given initial state. It can be shown that if the probability model is Hilbertian, then independently of the dimension of the Hilbert space we must have $q = 0$, which is usually called the "QQ-equality" [6,16,38,39].

We will give a simple proof of the "QQ-equality" in Sec. 8.2. Let us here calculate explicitly the value of q using the GTR-model, to show that the $q = 0$ condition can be easily violated. For this, we observe that: $P(\to \mathbf{a} \to -\mathbf{b}|\mathbf{x}) = P(\mathbf{x} \to \mathbf{a})P(\mathbf{a} \to -\mathbf{b})$, and similarly for the other terms in (12). Thus, we can write:

$$\begin{aligned} q = &[P(\mathbf{x} \to \mathbf{a})P(\mathbf{a} \to -\mathbf{b}) + P(\mathbf{x} \to -\mathbf{a})P(-\mathbf{a} \to \mathbf{b})] \\ &- [P(\mathbf{x} \to \mathbf{b})P(\mathbf{b} \to -\mathbf{a}) + P(\mathbf{x} \to -\mathbf{b})P(-\mathbf{b} \to \mathbf{a})]. \end{aligned} \tag{13}$$

Limiting our discussion to two-outcome situations, we can use $P(\mathbf{x} \to -\mathbf{b}) = 1 - P(\mathbf{x} \to \mathbf{b})$ and $P(\mathbf{x} \to -\mathbf{a}) = 1 - P(\mathbf{x} \to \mathbf{a})$, so that (13) becomes:

$$q = q_1 + q_2, \tag{14}$$

$$q_1 \equiv P(-\mathbf{a} \to \mathbf{b}) - P(-\mathbf{b} \to \mathbf{a}) \tag{15}$$

$$\begin{aligned} q_2 \equiv\; &P(\mathbf{x} \to \mathbf{a})[P(\mathbf{a} \to -\mathbf{b}) - P(-\mathbf{a} \to \mathbf{b})] \\ &+P(\mathbf{x} \to \mathbf{b})[P(-\mathbf{b} \to \mathbf{a}) - P(\mathbf{b} \to -\mathbf{a})]. \end{aligned} \tag{16}$$

The term q_1 is called the *relative indeterminism* contribution, and the term q_2 is called the *relative asymmetry* contribution [39]. To simplify the discussion, we assume that all measurements are described by symmetrical probability densities: $\rho_A(y) = \rho_A(-y)$ and $\rho_B(y) = \rho_B(-y)$, as it is the case in the example of Fig. 3. Then, we have: $P(\mathbf{a} \to -\mathbf{b}) = P(-\mathbf{a} \to \mathbf{b})$ and $P(-\mathbf{b} \to \mathbf{a}) = P(\mathbf{b} \to -\mathbf{a})$, so that $q_2 = 0$, but:

$$q_1 = \int_{\cos\theta}^{1} [\rho_B(y) - \rho_A(y)]dy. \tag{17}$$

Clearly, if $\rho_B \neq \rho_A$, then $q_1 \neq 0$, so that $q \neq 0$, showing again that the probability model described by the GTR-model can extend beyond quantum.

It is interesting to also observe that being the quantum mechanical situation characterized by uniform probability densities, i.e., $\rho_B = \rho_A = \rho_u$, this means that in a pure quantum model both q_1 (relative indeterminism) and q_2 (relative asymmetry) are zero. But this is not the only way to satisfy the QQ-equality, as also the condition $q_1 = -q_2$ can guarantee that $q = 0$. In other terms, the QQ-equality is a necessary but not sufficient condition to test the quantumness of a probability model.

In fact, even when the stronger condition $q_1 = q_2 = 0$ is satisfied, the model can still be non-Hilbertian. To see this, consider the situation where $\rho_B = \rho_A = \rho$, with ρ a symmetrical (but non-uniform) probability distribution. We then know that, similarly to quantum mechanics, the reciprocal law is satisfied and that $q_1 = q_2 = 0$. As we are now going to show, this doesn't mean however that the probability model is structurally equivalent to that described by the Born rule.

According to the general theorem we have stated in Sec. 6, only when ρ is a uniform distribution we recover the exact formulae predicted by the Born rule. For example, in the situation of a two-outcome measurement, the Born rule gives:

$$P_{\text{Born}}(\mathbf{a} \to \pm\mathbf{b}) = |\langle\psi_{\mathbf{b}}|\psi_{\mathbf{a}}\rangle|^2 = \frac{1}{2}(1 \pm \cos\theta), \tag{18}$$

which is very different form the GTR-model expression:

$$P(\mathbf{a} \to \pm\mathbf{b}) = \int_{-1}^{\pm\cos\theta} \rho(y)dy. \tag{19}$$

For instance, for the specific choice $\rho_\epsilon(y) = \frac{1}{2\epsilon}\chi_{[-\epsilon,\epsilon]}(y)$, with $\chi_{[-\epsilon,\epsilon]}$ the characteristic function of the interval $[-\epsilon, \epsilon]$, and $\epsilon \in [0,1]$ (the so-called ϵ-model; see [2,3]), we can write the more explicit expression:

$$\begin{aligned} P(\mathbf{a} \to \pm\mathbf{b}) = \delta_{\pm,-1}\Theta(-\cos\theta - \epsilon) + \delta_{\pm,+1}\Theta(\cos\theta - \epsilon) \\ + \frac{1}{2}\left(1 \pm \frac{\cos\theta}{\epsilon}\right)\chi_{[-\epsilon,\epsilon]}(\cos\theta), \end{aligned} \tag{20}$$

which clearly predicts different values than the Born rule (here Θ denotes the Heaviside step function, equal to 1 when the argument is positive and equal to 0 otherwise, and δ denotes the Kronecker delta, equal to 1 when the two indices are the same and equal to 0 otherwise).

We observe that in the limit $\epsilon \to 0$, $\rho_\epsilon(y) \to \delta(y)$, i.e., the elastic becomes only breakable in its middle point, which corresponds to a measurement with no fluctuations. Then the third term of (20) vanishes and one recovers an almost classical situation (almost because for $\cos\theta = \pm\epsilon$ we are in a situation of unstable equilibrium). On the other hand, in the opposite uniform limit $\epsilon \to 1$, $\rho_\epsilon(y) \to \frac{1}{2}$, the first two terms of (20) vanish and the third term tends to the pure quantum expression (18).

Now, albeit the values of the probabilities predicted by the Born rule (18) and by the GTR-model (19) (or more specifically the ϵ-model (20)) are manifestly different, one may nevertheless ask if the quantum model (i.e., the UTR-model in the Bloch sphere) would nevertheless describe the same experimental situations than a symmetric GTR-model, when all the elastic bands are the same, considering that both models satisfy (at least in the two-outcome situation) the equalities $q_1 = q_2 = 0$. As we are now going to show, the answer is negative. For this, we need to consider three distinct measurements. If they are purely quantum we have, with obvious notation [19]:

$$\begin{aligned}\langle\psi_{\mathbf{a}}|\psi_{-\mathbf{b}}\rangle &= \langle\psi_{\mathbf{a}}|\left(|\psi_{\mathbf{c}}\rangle\langle\psi_{\mathbf{c}}| + |\psi_{-\mathbf{c}}\rangle\langle\psi_{-\mathbf{c}}|\right)|\psi_{-\mathbf{b}}\rangle \\ &= \langle\psi_{\mathbf{a}}|\psi_{\mathbf{c}}\rangle\langle\psi_{\mathbf{c}}|\psi_{-\mathbf{b}}\rangle + \langle\psi_{\mathbf{a}}|\psi_{-\mathbf{c}}\rangle\langle\psi_{-\mathbf{c}}|\psi_{-\mathbf{b}}\rangle,\end{aligned} \tag{21}$$

where for the first equality we have used the resolution of the identity: $|\psi_{\mathbf{c}}\rangle\langle\psi_{\mathbf{c}}| + |\psi_{-\mathbf{c}}\rangle\langle\psi_{-\mathbf{c}}| = \mathbb{I}$. It immediately follows that if we can find an experimental situation in the GTR-model with identical and symmetrical elastic bands such that, say, $P(\mathbf{c} \to -\mathbf{b}) = 0$ (implying $\langle\psi_{\mathbf{c}}|\psi_{-\mathbf{b}}\rangle = 0$) and $P(\mathbf{a} \to -\mathbf{c}) = 0$ (implying $\langle\psi_{\mathbf{a}}|\psi_{-\mathbf{c}}\rangle = 0$), but also $P(\mathbf{a} \to -\mathbf{b}) \neq 0$ (implying $\langle\psi_{\mathbf{a}}|\psi_{-\mathbf{b}}\rangle \neq 0$), then such situation would clearly be incompatible with the quantum identity (21), and therefore would be modelizable by the symmetric GTR-model, but not by the Born rule of quantum mechanics. A simple situation of this kind is described in Figure 4.

8. The human cognition example

In Sec. 6 we have shown that the quantum mechanical (Hilbertian) model is a special case of the GTR-model, obtained by considering only uniform membranes and a Blochean state space. In this section we want to provide another important implementation of the GTR-model: *human cognition*. In

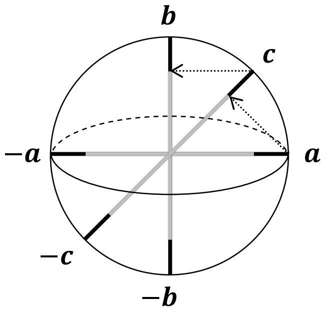

Fig. 4. Three different measurements represented by three identical symmetric ρ_ϵ-elastic bands oriented along the directions **a**, **b** and **c**, respectively. The angle between **a** and **b** is $\frac{\pi}{2}$, and the angle between **a** and **c**, and between **c** and **b**, is $\frac{\pi}{4}$. The unbreakable segments of the elastics are in black color, the uniformly breakable segments in gray color, and the picture corresponds to the choice $\epsilon = \frac{1}{\sqrt{2}}$ (the breakable segments are therefore of length $\sqrt{2}$).

that respect, we recall that as from the beginning of the present century the quantum formalism has been applied with success to model a large number of cognitive phenomena, like information processing, human judgment and decision making, perception and memory, as well as concept combinations and conceptual reasoning; see for instance: [4–11,17,18] and the references cited therein.

However, different from the situation of the elementary microscopic entities interacting with macroscopic measuring apparatuses, described by standard quantum mechanics, the general structure and the possible symmetries characterizing the human cognitive activity, when human minds interact with different conceptual entities, remain to be identified. Indeed, even though the quantum formalism has proven to work pretty well as a model, to fit many of the existing experimental data, it also fails to do so for many others (some examples will be given in the following). This means that the general probabilistic structures of the data generated by the human minds, in the different cognitive contexts, goes beyond that described by pure classical and pure quantum models. Therefore, to study these more general structures one needs an ampler theoretical framework than just classical or quantum mechanics, able to embrace from the beginning the full complexity of these non-physical (mental) processes. The GTR-model provides this more general framework, both in the Chatton and Occam sense: it is not more complex than necessary (Occam's razor), but also it is not less complex than necessary (Chatton's anti-razor).

Before explaining why the GTR-model provides a natural, coherent and unitary description of human cognition, at a quite fundamental level, it is useful to briefly address a possible objection, which consists in saying that the GTR-model would contain too many free parameters, allowing it to easily fit all sorts of empirical data, but because of this it would not make it of great interest in providing an explanation for the observed phenomena. In fact, considering that an infinity of different probability densities ρ can be used to describe each measurement (in addition to the choice of the orientations of the measurement simplexes), the model clearly allows for an uncountable infinity of free parameters!

To answer this objection (which by the way is usually also addressed in relation to pure Hilbertian models, obeying the Born rule; see for instance the recent discussion in [40]), one has to distinguish between phenomenological models, where the different parameters are just introduced *ad hoc*, to obtain a good (or exact) data fit, but only for a reduced set of isolated experimental situations, from more fundamental models, designed with the precise intent of describing *all possible situations* in a given domain of experimentation, possibly also deriving the observed phenomena from first principles, so providing for them convincing explanations (and whenever possible, predictions).

To give an example taken from physics, no one would ever have objected to Einstein that his general relativity theory was a bad explanation because it contained too many free parameters, as for instance the stress-energy tensor appearing in Einstein's field equations could take any functional dependence on the space-time coordinates. Of course, this was not a weak trait of Einstein's model, but its intrinsic richness, considering that his equations had to be applicable to all possible densities and fluxes of energy-momentum in spacetime. And this is precisely what made his theory a universal one. What was important in Einstein's equations is that the free parameters associated with the stress-energy tensor always remained in a clear and logical relation with respect to the other fundamental quantities of the theory, like Einstein's and Ricci's tensors and the metric. And as we are now going to explain, the same holds true, *mutatis mutandis*, for the GTR-model.

When the GTR-model is applied to interrogative contexts involving human minds (a measurement can always be understood as an interrogative context, with the different outcomes being the available answers), different from the standard quantum formalism it allows for a clear distinction not only between 'a question and its possible answers,' but also between 'the different *ways* an answer can be selected.' Indeed, different human subjects, when subjected to a same interrogation (or situation eliciting a decision), will have each, in general, a different 'way of choosing an answer.' Different 'ways of choosing' can be described within the GTR-model by means of different probability densities ρ (in the same way as different energy-momentum distributions can be described by means of different stress-energy tensors in general relativity). On the other hand, in standard quantum mechanics all measurements are described by the same uniform probability densities ρ_u, which means that a pure quantum model can only describe situations where all subjects participating in the experiment act as perfect "Bornian clones," all selecting an answer exactly in the same way (which is "the way of a uniform membrane").

Also, the disintegration-collapse of a membrane expresses in a very intuitive way what we humans typically perceive when facing a decisional context [17,18]. Indeed, when we are subjected to an interrogation, or a situation requiring a decision, we know that at the mental level a (neural) state of equilibrium will be built, expressing a sort of balancing of the tensions between the initial state of the conceptual entity we are subjected to, and the available (mutually excluding and competing) answer-states. The building of this equilibrium is described in the GTR-model by the abstract point particle entering the sphere and reaching an on-membrane position, so producing "tension lines" going from its position to the end points representative of the different outcomes. At some moment, always in accordance with what we can subjectively feel, some fluctuations will disturb this mental equilibrium, in a way that we cannot predict in advance, and trigger an irreversible and almost instantaneous process, drawing the abstract point particle to one of the vertices of the mental simplex, reducing in this way the previous tensional equilibrium (hence the "tension-reduction" name given to the model, which was suggested to us by Jerome Busemeyer).

So, the different membranes in the GTR-model are representative of aspects of the minds of the different subjects, understood as dynamic memory structures sensitive to meaning. On the other hand, and consequently, the abstract point particle interacting with a membrane is not to be interpreted (as is usually done) as a description of the subject's beliefs, but as an objective (intersubjective) element of a conceptual reality that is independent of the minds of the individuals that can possibly interact with it. This means that the different locations of the abstract point particle within the generalized Bloch sphere describe the different states a conceptual entity can be in, and that all these states have the same objective status for the different subjects participating in an experiment [41,42]; but different subjects, because of their different *forma mentis* (their different ρ), will extract a different meaning from them, in a given cognitive context, i.e., each subject will choose in a different way an outcome, i.e., an answer to the addressed question (and therefore each subject will be generally associated with a different statistics of outcomes).

To give an example, consider the concept *Food*. When it is not under the influence of a specific context, we can say that it is in its "ground" state, which can be understood as a sort of basic prototype of the concept. But as soon as *Food* is contextualized, for instance in the ambit of the phrase *This food is very juicy*, its state will change. This means that its previous ground state will stop playing the role of a prototype, which will be played then by its new state, in a sort of new 'contextualized prototype'. Now, the difference between the concept *Food* in a ground state and in an "excited" state, like the one associated with the above "juicy context," can be evaluated by subjecting the concept to an additional context: that of a human mind that is asked to select a good representative of the concept, among a number of possible predetermined choices. The difference between the ground state *Food* and the excited state *This food is very juicy* will then manifest in the fact that, assuming for example that *Fruit* and *Vegetable* are among the possible choices of representatives, the former will be chosen much more frequently (i.e., with a higher probability) than the latter, when the concept is in its "juicy" excited state, rather than in its ground state.

8.1. *Replicable measurements*

As we said, a ρ-membrane describes in the GTR-model an aspect of a participant's mind subjected to a given interrogative (or decisional) context. It is then natural to consider variations of the probability density ρ, when measurements are repeated, to account for the replicability effects that are easy to observe in experimental situations. In quantum mechanics, the replicability of an outcome is only predicted by the theory in relation to the repetition of the same measurement, according to von Neumann first kind condition. As we have seen in Sec. 2, the 'tension-reduction' mechanism associated with a disintegrable elastic membrane does automatically guarantee that if a measurement is repeated a second time, it will produce exactly the same result, with probability 1. This because the membrane's collapse cannot alter the position of the point particle when already located in one of the vertices of the measurement simplex.

However, if, following a measurement A, a second different measurement B is performed, and then, following the B-measurement, measurement A is performed once again, one will not generally obtain the same outcome obtained in the first A-measurement with probability 1. Indeed, the intermediary B-measurement will generally produce an outcome that is not an eigenstate of the A-measurement, so that when A is performed a second time the outcome will not be certain in advance and could be different from the first A-outcome (unless the two measurements are compatible, a situation described in quantum mechanics by two commuting self-adjoint operators).

When the A and B measurements are interrogative processes, and the measuring apparatus is a human mind, we know however that the situation is different. Indeed, it is to expected in this case that, in most situations, if we have given a certain answer to question A, then we will give the same answer to that same question if we are asked it a second time, even if in the meantime we have also answered to question B. This can happen for many reasons, like desire of coherence, learning, fear of being judged when we change opinion, etc.

In quantum mechanics replicability is easy to model if one assumes that the two measurements A and B are associated with compatible (commuting) observables. The problem is that response replicability is expected to be observed also when the observables A and B are non-compatibles, i.e., when they describe interrogative contexts that, for instance, can give rise to *question order effects*, as commonly observed in social and behavioral research [43–45]. In other terms, as [46] recently emphasized, an experimental situation where both question order effects and response replicability are present cannot be modelized by the standard quantum formalism.

What about the GTR-model? Is it able to jointly describe (i.e., jointly model) question order effects and response replicability? The answer is clearly affirmative, as for this it is sufficient to allow the probability density ρ, describing the mind aspect of a respondent in relation to a given interrogation, to change in such a way that if the measurement is repeated, following an intermediary measurement, the same answer will be obtained with certainty. In Fig. 5 we give an example of how the probability densities ρ_A and ρ_B characterizing two two-outcome measurements A and B have to change, to guarantee response replicability.

Note that the elastic bands represented in Fig. 5 are locally uniformly breakable, i.e., uniformly breakable on an interior segment (represented in grey color in the figure) and unbreakable everywhere else (black color in the figure). Note also that the specific structure of these elastics is precisely that required to model in an exact way the data obtained in experiments where subjects were asked to answer the following two incompatible "yes-no" questions, producing some typical question order effects [6,16,47]: "Do you generally think Bill Clinton is honest and trustworthy?" (A-measurement) and "Do you generally think Al Gore is honest and trustworthy?" (B-measurement). We shall not give here the details of this exact modelization, for which we refer the interested reader to [39].

8.2. *Non-Hilbertian order effects*

Another example of the insufficiency of the quantum formalism in the modeling of psychological experiments is question order effects. This statement may appear a bit surprising, as these effects are considered by many authors to be among the most successful quantitative predictions of quantum theory in social and behavioral sciences. But this depends on the perspective that is taken on the whole issue. To explain what we mean, we start by considering an equality originally derived by [48] and rediscovered by [16], which we have discussed already in Sec. 7.2. It is very simple to derive: for this, we denote P_a and P_b two orthogonal projection operators acting on some Hilbert space $\mathcal{H}$, representing two properties a and b of the entity under study. We also consider the complementary projection operators $P_{\bar{a}} = \mathbb{I} - P_a$ and $P_{\bar{b}} = \mathbb{I} - P_b$, describing the orthocomplementary properties $\bar{a}$ and $\bar{b}$, and define the following self-adjoint operator:

$$Q \equiv P_b P_a P_b - P_a P_b P_a + P_{\bar{b}} P_{\bar{a}} P_{\bar{b}} - P_{\bar{a}} P_{\bar{b}} P_{\bar{a}}. \tag{22}$$

Taking its average $q \equiv \mathrm{Tr}\, QD$, over an arbitrary state D (which can also be a density matrix), and considering that the probability $P(ba|D)$ of observing, in a sequence, first property b then property a, is described in quantum physics by the average [6]: $P(ba|D) = \mathrm{Tr}\, P_b P_a P_b D$, and similarly for the other sequential (i.e., conditional) probabilities, then using the additivity of the trace, we can write:

$$q = [P(ba|D) - P(ab|D)] + [P(\bar{b}\bar{a}|D) - P(\bar{a}\bar{b}|D)], \tag{23}$$

which apart from the different notation is precisely (12). The so-called QQ-equality consists in observing that, for whatever initial state D, $q = 0$ [6,16]. This can be easily proven by replacing $P_{\bar{a}} = \mathbb{I} - P_a$ and $P_{\bar{b}} = \mathbb{I} - P_b$ into (22), then developing the various terms and see that most of them simplify, so that one is left with the equality $Q = (P_b^2 - P_b) - (P_a^2 - P_a)$. Using the

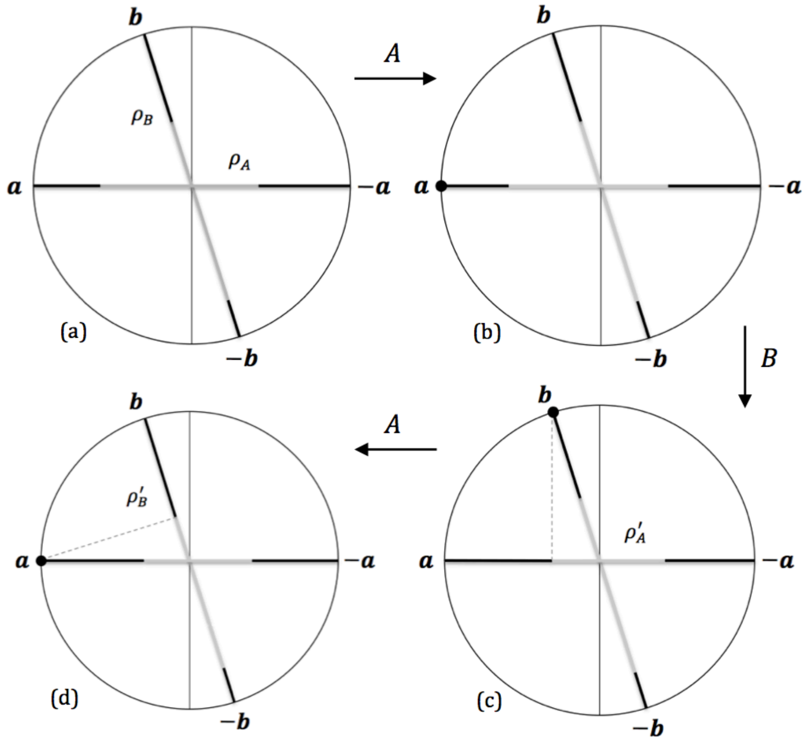

Fig. 5. A sequence of three two-outcome measurements, A, B and then again A, in the GTR-model. Figure (a) represents the situation before measurement A. The representative point particle is located somewhere at the surface of the sphere, outside of the plane of the two elastic bands describing the two (incompatible) measurements A and B, characterized by the non-uniform probability distributions ρ_A and ρ_B, respectively. The black parts of the elastics are unbreakable, whereas their grey parts are uniformly breakable. Following the first A-measurement, it is here assumed that the indeterministic breaking of the elastic has produced outcome $\mathbf{a}$, as indicated in Figure (b). Then, in the second B-measurement, it is assumed that the transition $\mathbf{a} \to \mathbf{b}$ has occurred, as well as the corresponding membrane's change $\rho_A \to \rho'_A$, as illustrated in Figure (c). The outcome $\mathbf{a}$ of the third A-measurement, described in Figure (d), is then certain in advance, in accordance with the hypothesis of replicability, and the $\mathbf{b} \to \mathbf{a}$ deterministic transition is also associated with the change $\rho_B \to \rho'_B$ of the B-elastic. Subsequent B-measurements and/or A-measurements will then not change anymore the nature of the elastic bands, and state transitions will become perfectly deterministic.

idempotency of the orthogonal projection operators, we thus have $Q = 0$, and therefore also $q = 0$, for whatever state D.

The condition $q = 0$ is generally considered to be a good test of the Hilbertian character of the probabilities involved in an experiment. However, as emphasized in Sec. 7.2, it can also be obeyed by non-Hilbertian models. This because what we have called the relative indeterminism contribution and the relative asymmetry contribution to q, can also mutually compensate, whereas these contributions need to be both zero in a Hilbertian model. But as we have seen, even when these contributions are both zero, the model can still be non-Hilbertian.

This departure from the Hilbertian model can be better appreciated by performing a detailed analysis of data like the Clinton/Gore ones we have mentioned in the previous section. Indeed, although the associated probabilities appear to almost obey the $q = 0$ condition, not only there are no reasons to expect that such condition would be exactly obeyed in the ideal limit of an infinite number of participants, but also, and more important, when these probabilities are described using the GTR-model, one immediately sees that their structure is highly non-Hilbertian. This because the breakable elastic bands that are needed to exactly model the data are very different from the globally uniformly breakable structures that are typical of quantum measurements. For example, the elastics depicted in Fig. 5 (a) are precisely those required to generate the Clinton/Gore probabilities, and they are manifestly non-uniform and non-symmetric, i.e., non-Bornian (see also the general discussion in [39], where additional quantum identities are derived and shown to be strongly violated by the Clinton/Gore data, and similar ones).

8.3. *Individual and collective minds*

To provide another argument as to why the standard quantum formalism cannot be considered to be sufficient to describe typical psychological measurements, let us come back for a moment to the results of Sec. 6. We have shown that when different measurements, all having the same outcomes, are averaged out, in what we have called a universal average, one recovers the Born rule, provided the state space is Hilbertian. A physicists may rightly ask to what exactly these different measurements do correspond. Indeed, when a same measurement is performed a number of times in a physics laboratory, in order to obtain a sufficiently rich statistics of data, and deduce some robust experimental probabilities, the quantum entity is simply prepared every time in the same state, measured by means of the same instrument, in a large number of equivalent runs of the experiment.

If the interpretation of a quantum measurement as a universal measurements is correct also for physics, this would mean that even though at each run of the measurement the same apparatus is used, the latter would nevertheless each time select an outcome *in a different way*, i.e., according to a different ρ-membrane, but since the experimenter would not know which ρ is each time actualized by the apparatus, all these outcomes generated by the different membranes would be averaged out in the final statistics, yielding an effective description in terms of a uniform membrane, which is the Born rule.

The existence of the hidden-membranes and the hidden-interactions associated with their possible breaking points remains of course hypothetical for the time being in physics, considering that we don't have any direct access to such non-spatial (or pre-spatial) layer of our physical reality, from our limited Euclidean theater [19]. This of course does not mean that cleverly designed experiments wouldn't be able in future to reveal these hidden and multidimensional dynamical structures, but for the time being they only remain a compelling theoretical explanation about how the quantum mechanical Born rule can emerge, as a first order approximation, from a substratum of more general probabilistic theories.

What is the situation in psychological measurements? The main difference is that in that ambit there is an aspect of the measurements that, contrary to quantum measurements in physics laboratories, is not at all hidden. Indeed, in a typical psychological measurement the data are obtained from a number of different participants, for instance about a thousand in the previously mentioned Clinton/Gore experiment. And since these participants are all subjected to the same questions, in relation to a conceptual entity presented to them in the same intersubjective initial state, each participant in the measurement is the manifestation of a different ρ-membrane, corresponding to that specific mind aspect characterizing the way each of them, different from all the others, will select one of the available answers.

In other terms, if the process of actualization of potential ρ remains totally hidden (and also hypothetical for the time being) in physics laboratories, it is instead a perfectly manifest element of reality in psychological measurements. Considering however that the number n of participants is necessarily finite, and in many measurements not necessarily large, it can be expected that in some situations the obtained average will not be well approximated by a universal one, and therefore the final probability model will not be Hilbertian. To put it in a different way, the abstract "collective mind" of the participants may not be representative of a "pure quantum mind," if some "ways of choosing an outcome" are not actualizable, because the statistical sample of the available ρ-membranes is too small.

It is however important to emphasize that when we limit our considerations to a single measurement situation, then a Hilbert space probabilistic model, or a Kolmogorovian model, will always be sufficient to fit the experimental data, as these two models are "universal probabilistic machines," capable of representing all possible probabilities appearing in nature, in a *single* measurement context [17,49]. But when we look for a consistent representation for different non-compatible measurements, this is where classical probabilities become totally inadequate, and pure quantum (Bornian) probabilities become too specific to describe all possible experimental situations.

The situation becomes even more problematic when we try not only to devise a consistent model for the description of a collection of different measurements (different questions) associated with different outcomes (different answers), but when we also consider the possibility of combining these different measurements and their outcomes in a sequential way. This introduces an additional difficulty, which is precisely the problem of distinguishing the individual level from the collective one. Indeed, when psychologists consider sequential measurements, to highlight possible question order effects, it is not the abstract collective mind that is subjected to the sequence of measurements, in different orders, but each one of the individual minds of the participants. In other terms, the sequence of measurements is first performed at the individual level (each participant is asked to answer two questions in a given succession), then an average of their obtained answer is considered.

Of course, the reason why an overall question order effect is observed is because the effect manifests at the individual level, and is then transferred from the individual to the collective level, in the final statistics of outcomes. If we ask the first question to an individual, then the second question to another individual, no order effects would be observed (and the same holds true for response replicability effects). It is of course essential that a same individual in the sample of respondents replies to the two questions in a given order, for the effect to manifest, being it generated at the level of the individual mind and not of the collective one.

All we are saying is of course perfectly evident, but it is important not to mix these two different levels: the individual and the collective. Let us consider the previously mentioned Clinton/Gore example to further clarify our point. Imagine for a moment that we have found a way to perfectly clone the i-th individual participating in the opinion pool. If we repeat many times the "A then B" and the "B then A" sequential measurements, using these i-clones (we can only use each clone once, because of response replicability), then calculate the relative frequencies of the observed outcomes and use the GTR-model to fit the data, we would find two elastic bands with a specific orientation in the Bloch sphere, characterized by some generally non-uniform probability densities $\rho_A^{(i)}$ and $\rho_B^{(i)}$. This is the description at the individual level.

When we consider the responses obtained from all the n different individuals participating in the pool, the probabilities we end up calculating are equivalent to a uniform average over the different sequential probabilities that would have been generated by these i-clones, for $i = 1, \ldots, n$, i.e., the probabilities we can deduce from the associated $\rho_A^{(i)}$- and $\rho_B^{(i)}$-elastics. These overall probabilities can in turn be modeled by using also two effective elastic bands, characterized by some probability densities ρ_A and ρ_B and a specific orientation. So, at the formal level, the abstract collective mind is described as if it was an individual mind, also performing the sequential measurements, according to its specific "forma mentis."

However, and this is the subtle point we want to clarify with the present discussion, the collective mind does not really perform a sequential measurement. To see this, consider the following sequential measurement performed "at the collective level." We first ask question A ("Do you generally think Bill Clinton is honest and trustworthy?"), and to obtain an answer we randomly select one of the participants, ask the question and collect the answer. Then, we ask question B ("Do you generally think Al Gore is honest and trustworthy?") and to obtain the answer we again randomly select one of the participants, ask the question and collect the answer. Similarly, we can perform the same sequential process in reversed order, by asking first B and then A. Now, apart the very special circumstance where the same participant would be selected in one or both of the above sequences (a possibility whose probability tends to zero as the number of participants increases), no order effects will be observed in this way. In other terms, at the collective level, if the participants are randomly chosen at each measurement, no order effects will be observed, and of course the same remains true for the response replicability effects.

This is because if we randomly chose a new participant every time that we ask a question, all memory effects will be destroyed, and all measurements will become compatible. At the individual level, measurements are generally incompatible because the answer given to a first measurement remains in the field of consciousness of the respondent, changing in this way the state of the conceptual entity when a second question is asked. For example, in the Clinton/Gore experiment, the entity which is measured by each individual mind is the conceptual entity *Honesty and trustworthiness* (which for brevity, we shall simply denote *Honesty*).

Prior to a measurement, we can consider that such entity is in its "ground" (most neutral) state, this being true for all the respondents. At the individual level, when performing measurement A, a subject is asked if s/he thinks Clinton is honest. Considering this as a measurement of the conceptual entity *Honesty*, the interrogation can be rephrased as follows: "What best represents *Honesty*, between the two possibilities: *Clinton is honest* and *Clinton is not honest*?" It is also worth observing that the outcomes *Clinton is honest* and *Clinton is not honest* are here to be considered as "excited states" of the conceptual entity *Honesty*.

Now, when a subject is submitted to the A-measurement, the outcome (i.e., the answer) will generally remain in her/his field of consciousness when the same subject is submitted to the subsequent B-measurement, corresponding to the question "Do you generally think Al Gore is honest?" This means that when the B-measurement is performed immediately after A, the measured conceptual entity will not anymore be in its ground state, but in the excited state corresponding to the outcome of the A-measurement. If, say, the answer to the A-interrogation is "yes," that is, "Clinton is honest," the effective subsequent B-measurement will be: "What best represents *Clinton is honest* between the two possibilities: '*Gore is honest* and *Gore is not honest*?"

In other terms, the short term memory of a participant is what allows her/him to keep track of the change of the state of the conceptual entity under consideration, in a sequence of different measurements, and this memory effect, manifesting at the individual level, is what in the end produces the order (and replicability) effects. This memory effect would of course be lost at the collective level, if respondents are selected in a random way at each measurement in a sequence. It could however be restored if the experimental protocol would be so designed to keep track of the obtained answer, and use them as a new input state when a successive measurement is performed, on a new randomly chosen subject.

Having elucidated this difference between the individual and collective levels, we want now explain how the averaging procedure, when performed on sequential measurements operated at the individual level, can generate a *symmetry breaking process* that can also be held in part responsible for the departure of the experimental probabilities from Hilbertian-like symmetries, like for instance that expressed by the QQ-equality. For this, we consider the simple situation of a collective mind formed by only two individuals and we assume that each of them, when subjected to a sequence of two two-outcome measurements A and B, will use the same locally uniform and symmetric probability density, in both measurements. In other terms, we assume that $\rho_A^{(i)} = \rho_B^{(i)} = \frac{1}{2\epsilon_i}\chi_{[-\epsilon_i,\epsilon_i]}$, where the index $i = 1, 2$, denotes the individual subject. Thus, we are here in a situation where both the relative indeterminism contribution and the relative asymmetry contribution are zero and the symmetry expressed by the QQ-equality is obeyed. In other terms, even though the measurements are not purely quantum, they still obey the QQ-equality that is also obeyed by pure quantum systems, and in that sense (but only in that sense) the situation can be considered to be, at the individual level, close to a pure quantum situation.

What happens then when we consider a uniform average over these two participants? To see this, we denote $\mathbf{a}$ and $-\mathbf{a}$ the two outcomes of measurement A, and $\mathbf{b}$ and $-\mathbf{b}$ those of measurement B, as represented in the Bloch sphere. According to (20), if we assume that the angle θ between the two elastic bands ($\cos\theta = \mathbf{a}\cdot\mathbf{b}$) is such that $\cos\theta < \epsilon_i$, $i = 1, 2$, and that the angle θ_A between the unit vector $\mathbf{x}$ describing the initial state and outcome $\mathbf{a}$ ($\cos\theta_A = \mathbf{x}\cdot\mathbf{a}$) is such that $\cos\theta_A < \epsilon_i$, $i = 1, 2$, then for the sequence of outcomes "$\mathbf{a}$ then $\mathbf{b}$" we have the i-individual probability:

$$\begin{aligned} P^{(i)}(\to \mathbf{a} \to -\mathbf{b}|\mathbf{x}) &= P^{(i)}(\mathbf{x} \to \mathbf{a})P^{(i)}(\mathbf{a} \to \mathbf{b}) \\ &= \frac{1}{2}(1 + \frac{1}{\epsilon_i}\cos\theta_A)\frac{1}{2}(1 + \frac{1}{\epsilon_i}\cos\theta) \\ &= \frac{1}{4}[1 + \frac{1}{\epsilon_i}(\cos\theta + \cos\theta_A) + \frac{1}{\epsilon_i^2}\cos\theta\cos\theta_A]. \end{aligned} \quad (24)$$

If now we consider the uniform average: $P(\to \mathbf{a} \to -\mathbf{b}|\mathbf{x}) \equiv \frac{1}{2}[P^{(1)}(\to \mathbf{a} \to -\mathbf{b}|\mathbf{x}) + P^{(2)}(\to \mathbf{a} \to -\mathbf{b}|\mathbf{x})]$, we obtain:

$$P(\to \mathbf{a} \to -\mathbf{b}|\mathbf{x}) = \frac{1}{4}\left[1 + \frac{\epsilon_1 + \epsilon_2}{2\epsilon_1\epsilon_2}(\cos\theta + \cos\theta_A) + \frac{\epsilon_1^2 + \epsilon_2^2}{2\epsilon_1^2\epsilon_2^2}\cos\theta\cos\theta_A\right]. \quad (25)$$

We observe that (25) can be written in the form (24) only if $(\epsilon_1 + \epsilon_2)^2 = 2(\epsilon_1^2 + \epsilon_2^2)$, or $(\epsilon_1 - \epsilon_2)^2 = 0$, i.e., if $\epsilon_1 = \epsilon_2$. But since by hypothesis $\epsilon_1 \neq \epsilon_2$, we immediately see that the effective elastic bands describing the two measurements from the viewpoint of the "collective mind" (here just formed by two individuals), not only they are not anymore the same, but also they are not anymore symmetric (for a specific calculation, see [39]). Also, the obtained averaged probabilities will generally violate the QQ-quality. In other terms, the averaging procedure induces a breaking of possible symmetries in the structure of the probabilities, and a clear departure from the Hilbertian model.

A remark is in order. In Sec. 6, we have explained that when all possible membranes (i.e., all possible ways of choosing an outcome) are allowed to be actualized in a single (non-sequential) measurement context (in what we have called a universal average, or universal measurement), the averaged probabilities are then described by an effective uniform membrane, equivalent to the Born rule, when the state space is considered to be Hilbertian. As we have just seen, the process of averaging over different membranes becomes much more involved when we deal with sequential measurements. Certainly, when considering a finite number of participants (the actual situation in real experiments), not all behaving as "Bornian clones," the final statistics will be non-Hilbertian. However, it remains an open question to determine what would be the probability model of a 'universal sequential measurement,' i.e., of an average over sequential measurements when all possible probability densities are included in the calculation. We plan to come back to this interesting question in future works.

9. Concluding remarks

In the present work we have reviewed and further illustrated some of the results we have recently obtained in [17–20,28,37,39,50], to emphasize the interest and role played by the GTR-model (and the associated hidden-measurement approach) in the description of very general measurement situations, extending beyond the pure classical and pure quantum ones. As we have explained, these more general situations, and the associated probability models, are certainly relevant in the description of both physical and psychological experimental situations.

Classical (Kolmogorovian) probabilities generally describe "static propositions," i.e., our lack of knowledge about the actual elements of reality that are present in the system under study. On the other hand, quantum probabilities generally describe "dynamic propositions," i.e., our lack of knowledge about processes of actualization of potential properties (if we exclude the special situations of measurements performed on eigenstates). Somehow in between these two descriptions, we can consider mixed measurements, where both static and dynamic logics, discovery and creation processes, actuality and potentiality, determinism and indeterminism, play an equivalent role. These more general, hybrid contexts, cannot be described using the too limited Hilbertian or Kolmogorovian models, but require more general structures.

In that respect, it is important to realize that our reality, because of its extreme complexity, is able to manifest all sorts of mixtures of creation and discovery processes, also at a fundamental level. Therefore, in our investigations we need to be equipped with probabilistic models that are able to cope with such complexity, beyond the very specific classical and quantum structures. This of course does not mean that, in certain ambits, one will not try to highlight some possible remarkable symmetries, but to correctly describe them we certainly need a general enough theoretical approach, as only in this way we can hope to understand the full logic behind them. The example of the QQ-equality is in that sense paradigmatic: we know that the equality is exactly obeyed by pure Hilbertian models, but we have seen it can also be obeyed by a class of non-Hilbertian models, which are precisely those describing the question order effect in "opinion pool" psychological measurements. This means that if we want to understand the reasons behind these observed regularities, we cannot do so from the limited Hilbertian viewpoint, but need a more general approach. One of the scope of the present article was to point out that the GTR-model precisely provides such needed more general approach.

Actually, we think that the GTR-model does more than this. Indeed, if it is correct to say that the Kolmogorovian model is a universal model for the description of situations governed by "static information," certainly we cannot say that the Hilbertian model, equipped with the Born rule, is a universal model for situations governed by (non-Boolean) "dynamic information," where also lack of knowledge about processes of actualization of potential properties is considered. This not only because the state space is Hilbertian, which may be a too severe constraint in certain situations, but also because very specific collapsing membranes are considered in quantum mechanics: the uniform ones. When all possible structures of membranes and state spaces are allowed, one certainly obtains the most general possible probabilistic description, i.e., a universal model for the description of both static and dynamic situations, which we have called the GTR-model.

As our simple coin example illustrates, the necessity of using the more general GTR-model already manifests when considering experimental situations involving macroscopic objects. This because classical properties, and the associated classical probabilities, are insufficient to describe all possible observations. In fact, macroscopic objects possess more physical properties than those usually accounted for by classical mechanics, like for instance the "upper face" property of a coin, or of a die [25]. And when these non-ordinary properties are considered, and tested in an operational way, not only the classical Kolmogorovian probabilities become inadequate, but the Hilbertian (Bornian) ones as well. On the other hand, the universality of the GTR-model allows to properly handle these non-ordinary measurement situations, and we cannot exclude that its structural richness will not also be instrumental in the description of anomalies manifesting in measurements with elementary physical entities (see for instance [51] for an example of possible anomalies in the ambit of coincidence measurements).

However, if the interest of the GTR-model for elementary (microscopic) physical systems remains to be evaluated (apart of course its theoretical interest in deriving and explaining the Born rule as a universal average), it is already a necessary tool for properly modeling human cognition, as we have illustrated in this work, considering that the data already in our possession cannot be exactly fitted by means of classical and Hilbertian models. We therefore hope that more scientists will decide to adopt it with advantage, both conceptually and as a mathematical instrument, to explore the ubiquitous quantum-like (but not necessarily pure quantum) structures.

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'PROBABILISTIC KNOWLEDGE' AS 'OBJECTIVE KNOWLEDGE' IN QUANTUM MECHANICS: POTENTIAL IMMANENT POWERS INSTEAD OF ACTUAL PROPERTIES

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In classical physics, probabilistic or statistical knowledge has been always related to ignorance or inaccurate subjective knowledge about an actual state of affairs. This idea has been extended to quantum mechanics through a completely incoherent interpretation of the Fermi-Dirac and Bose-Einstein statistics in terms of "strange" quantum particles. This interpretation, naturalized through a widespread "way of speaking" in the physics community, contradicts Born's physical account of Ψ as a "probability wave" which provides statistical information about outcomes that, in fact, cannot be interpreted in terms of 'ignorance about an actual state of affairs'. In the present paper we discuss how the metaphysics of actuality has played an essential role in limiting the possibilities of understating things differently. We propose instead a metaphysical scheme in terms of *immanent powers* with definite *potentia* which allows us to consider quantum probability in a new light, namely, as providing objective knowledge about a potential state of affairs.

Keywords: Quantum probability, objectivity, immanent powers, potentia.

1. Reality, Metaphysics and Knowledge

The term "metaphysics" comprises a series of many different definitions. The most known is the one proposed by Aristotle who defined it as a theory of "being *qua* being" [Met. 1003a20], a theory about what it means or implies to "be" in its different senses. Ever since, metaphysics and its problems have remained at the center of debates in western philosophical thought. But while for some, it is considered as a supreme form of knowledge, for others, it remains an occupation constituted by unfruitful discussions and pseudoproblems. One of the very first problems discussed in metaphysical terms is the famous problem of motion, which goes back to pre-socratic philosophy and the tension between the Heraclitean and Parmenidean schools of thought. In fact, it was Plato and Aristotle who created an opposition between Heraclitus, who embraced the doctrine of permanent motion and becoming in the world; and Parmenides, who taught the non-existence of motion and change in reality, reality being absolutely one and determined. As remarked by Verelst and Coecke:

> The contradicting conclusions deriving from pre-Socratic philosophy were of a major concern to Plato and Aristotle, because they challenged the existence of truth and certainty about the world and therefore about the actions of human beings in it. This uncertainty had given rise to a philosophical discipline, Sophism, that simply denied any relation between reality and what we say about it ([45], *Theaetetus*, 42, 152(d,e)). Its subjectivism stems from a radical empiricism, which holds that things are for me as I perceive them. But since reality as we perceive it is always in a process of permanent change this implies, as Plato points out in the *Theaetetus*, also the non-existence of stable, individual things in the world. [46, p. 44]

Metaphysics introduced a fundamental idea according to which it was possible to acquire knowledge of reality and existence through a system of principles, a theory. One of the first such metaphysical systems, which still today plays a major role in our understanding of the world around us is that proposed by Aristotle through his logical and ontological principles: the Principle of Existence (PE), the Principle of Non-Contradiction (PNC) and the of Principle Identity (PI). As Verelst and Coecke make the point:

> The three fundamental principles of classical (Aristotelian) logic: the existence of objects of knowledge, the principle of contradiction and the principle of identity, all correspond to a fundamental aspect of his ontology. This is exemplified in the three possible usages of the verb 'to be': existential, predicative, and identical. The Aristotelian syllogism always starts with the affirmation of existence: something is. The principle of contradiction then concerns the way one can speak (predicate) validly about this existing object, i.e. about the true and falsehood of its having properties, not about its being in existence. The principle of identity states that the entity is identical to itself at any moment (a=a), thus granting the stability necessary to name (identify) it. [46, p. 44]

Here we see the fundamental relation between logic and metaphysics. Aristotle had developed a metaphysical scheme in which, through the notions of *actuality* and *potentiality*, he was able to articulate both the Heraclitean and the Eleatic metaphysical schools of thought. On the one hand, potentiality constrained the undetermined, contradictory and non-individual realm of existence, on the other, the mode of being of actuality was determined through PE, PNC and PI. Through these principles the notion of entity was capable of unifying, of totalizing in terms of a "sameness", creating certain stability for knowledge to be possible. But even though Aristotle claimed that Being is said in many ways presenting at first both actual and potential realms as ontologically equivalent, from chapter 6 of book Θ of *Metaphysics*, he seems to place actuality in the central axis of his architectonic, relegating potentiality to a mere supplementary role.[a]

Both actuality and potentiality were part of a metaphysical representation and understood as characterizing modes of existence independent of observation. This is the way through which metaphysical thought was able to go beyond the *hic et nunc*, creating a world beyond the world, a world of concepts and representations. Such representation or transcendent description of the world is considered by many as the origin of metaphysical thought itself.[b] And this is the reason why, as noticed by Edwin Burtt [7, p. 224]: "[...] there is no escape from metaphysics, that is, from the final implications of any proposition or set of propositions. The only way to avoid becoming a metaphysician is to say nothing."

2. Classical Physics: Actual Properties and States of Affairs

The importance of potentiality, which was first placed by Aristotle in equal footing to actuality as a mode of existence, was soon diminished in the history of western thought. As we have seen above, it could be argued that the seed of this move was already present in the Aristotelian architectonic, whose focus was clearly placed in the actual realm. The realm of potentiality, as a different (ontological) mode of the being was neglected becoming

[a]Aristotle argues: "We have distinguished the various senses of 'prior', and it is clear that actuality is prior to potentiality. [...] For the action is the end, and the actuality is the action. Therefore even the word 'actuality' is derived from 'action', and points to the fulfillment." [1050a17-1050a23] Aristotle then continues to provide arguments in this line which show "[t]hat the good actuality is better and more valuable than the good potentiality." [1051a4-1051a17]

[b]The need of metaphysical principles in order to account for physical experience has been beautifully exposed by Borges in a story called *Funes the Memorious*.

not more than mere (logical) *possibility*, a teleological process of fulfillment. In relation to the development of physics, the focus and preeminence was also given to actuality. The 17th century division between *res cogitans* and *res extensa* played in this respect an important role separating very clearly the realms of actuality and potentiality. The philosophy which was developed after Descartes kept *res cogitans* (thought) and *res extensa* (entities as acquired by the senses) as separated realms.[c] As remarked by Heisenberg [29, p. 73]: "Descartes knew the undisputable necessity of the connection, but philosophy and natural science in the following period developed on the basis of the polarity between the 'res cogitans' and the 'res extensa', and natural science concentrated its interest on the 'res extensa'." This materialistic conception of science based itself on the main idea that extended things exist as being definite, that is, in the actual realm of existence. With modern science the actualist Megarian path was recovered and potentiality dismissed as a problematic and unwanted guest. The transformation from medieval to modern science coincides with the abolition of Aristotelian hilemorphic metaphysical scheme —in terms of potentiality and actuality— as the foundation of knowledge. However, the metaphysical scheme grounded on his logic still remained the basis for correct reasoning. As Verelst and Coecke remark:

> Dropping Aristotelian metaphysics, while at the same time continuing to use Aristotelian logic as an empty 'reasoning apparatus' implies therefore loosing the possibility to account for change and motion in whatever description of the world that is based on it. The fact that Aristotelian logic transformed during the twentieth century into different formal, axiomatic logical systems used in today's philosophy and science doesn't really matter, because the fundamental principle, and therefore the fundamental ontology, remained the same ([40], p. xix). This 'emptied' logic actually contains an Eleatic ontology, that allows only for static descriptions of the world. [46, p. 44]

It was Isaac Newton who was able to translate into a closed mathematical formalism both, the ontological presuppositions present in Aristotelian (Eleatic) logic and the materialistic ideal of *res extensa* —with actuality as its mode of existence. In classical mechanics the representation of the

[c]While *res cogitans*, the soul, was related to the *indefinite* realm of potentiality, *res extensa*, i.e. the entities as characterized by the principles of logic, related to the actual.

state of the physical system is given by a point in phase space Γ and the physical magnitudes are represented by real functions over Γ. These functions commute between each other and can be interpreted as possessing definite values independently of measurement, i.e. each function can be interpreted as being actual. The term 'actual' refers here to *preexistence* (within the transcendent representation) and not to *hic et nunc* observation. Every physical system may be described exclusively by means of its actual properties. The change of the system may be described by the change of its actual properties. Potential or possible properties are considered as the points to which the system might arrive in a future instant of time. As also noted by Dieks:

> In classical physics the most fundamental description of a physical system (a point in phase space) reflects only the actual, and nothing that is merely possible. It is true that sometimes states involving probabilities occur in classical physics: think of the probability distributions ρ in statistical mechanics. But the occurrence of possibilities in such cases merely reflects our ignorance about what is actual. The statistical states do not correspond to features of the actual system (unlike the case of the quantum mechanical superpositions), but quantify our lack of knowledge of those actual features. [24, p. 124]

Classical mechanics tells us via the equation of motion how the state of the system moves along the curve determined by the initial conditions in Γ and thus, as any mechanical property may be expressed in terms of Γ's variables, how all of them evolve. Moreover, the structure in which actual properties may be organized is the (Boolean) algebra of classical logic.

3. Physical Probability and Subjective Ignorance

We believe that a realist coherent interpretation of Quantum Mechanics (QM) should be capable of providing an understanding of its own physical concepts. Since Born's 1926 interpretation of the quantum wave function Ψ, probability has become one of the key notions in the description of quantum phenomena. But the difficulties to interpret quantum probability were already explicit in Born's original paper.

> Schrödinger's quantum mechanics [therefore] gives quite a definite answer to the question of the effect of the collision; but there is no question of any causal description. One gets no answer to the

> question, 'what is the state after the collision' but only to the question, 'how probable is a specified outcome of the collision'.
> Here the whole problem of determinism comes up. From the standpoint of our quantum mechanics there is no quantity which in any individual case causally fixes the consequence of the collision; but also experimentally we have so far no reason to believe that there are some inner properties of the atom which condition a definite outcome for the collision. [...] I myself am inclined to give up determinism in the world of the atoms. But that is a philosophical question for which physical arguments alone are not decisive. [47, p. 57]

In his paper Born formulated the now-standard interpretation of $\psi(x)$ as encoding a probability density function for a certain particle to be found at a given region. The wave function is a complex-valued function of a continuous variable. For a state ψ, the associated probability function is $\psi^*\psi$, which is equal to $|\psi(x)|^2$. If $|\psi(x)|^2$ has a finite integral over the whole of three-dimensional space, then it is possible to choose a normalizing constant. The probability that a particle is within a particular region V is the integral over V of $|\psi(x)|^2$. However, even though this interpretation worked fairly well, it soon became evident that the concept of probability in the new theory departed from the physical notion considered in classical statistical mechanics as *lack of knowledge* about a preexistent (actual) state of affairs described in terms of definite valued properties.

In the history of physics the development of probability took place through a concrete physical problem and has a long history which goes back to the 18th century. The physical problem with which probability dealt was the problem of characterizing a state of affairs even though the possessed knowledge of it was incomplete. Or in other words, "gambling". This physical problem was connected later on to a mathematical theory developed by Laplace and others. But it was only after Kolmogorov that this mathematical theory found a closed set of axioms [36]. Although there are still today many interpretational problems regarding the physical understanding of classical probability, when a realist physicist talks about probability in statistical mechanics he is discussing about the (average values of) properties of an uncertain —but existent— state of affairs.[d] This is

[d]In this respect it is important to remark that the orthodox interpretation of probability in terms of relative frequencies, although provides a conceptual framework to relate to measurement outcomes, refers to 'events' and to 'properties of a system'; in this sense

why the problem to determine a definite state of affairs in QM —the sets of definite valued properties which characterize the quantum system— poses also problems to the interpretation of probability and possibility within the theory itself. As noticed by Schrödinger in a letter to Einstein:

> It seems to me that the concept of probability [related to quantum theory] is terribly mishandled these days. Probability surely has as its substance a statement as to whether something *is* or *is not* the case —an uncertain statement, to be sure. But nevertheless it has meaning only if one is indeed convinced that the something in question quite definitely *is* or *is not* the case. A probabilistic assertion presupposes the full reality of its subject. [6, p. 115]

Schrödinger [43, p. 156] knew very well that in QM it is not possible to asigne a definite value to all properties of a quantum state. As he remarked: "[...] if I wish to ascribe to the [quantum mechanical] model at each moment a definite (merely not exactly known to me) state, or (which is the same) to all determining parts definite (merely not exactly known to me) numerical values, then there is no supposition as to these numerical values to be imagined that would not conflict with some portion of quantum theoretical assertions." This impossibility would be exposed three decades after in formal terms by Kochen and Specker [35].

We understand mathematics, contrary to physics, as a non-representational discipline which respects no metaphysical nor empirical limits whatsoever. The mathematician does not need to constrain himself to any set of metaphysical principles but only to the internal structure and coherency of the mathematical theory he is dealing with. 'Probability' is regarded by the mathematician as a 'theory of mathematics' and in this sense departs from any conceptual physical understanding which relates the formal structure to the world around us. A mathematician thinks of a probability model as the set of axioms which fit a mathematical structure and wonders about the internal consistency rather than about how this structure relates and can be interpreted in relation to experience and physical reality. As noticed by Hans Primas:

> Mathematical probability theory is just a branch of pure mathematics, based on some axioms devoid of any interpretation. In this framework, the concepts 'probability', 'independence', etc. are

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it is not necessarily linked to a realistic physical representation but rather supports an empiricist account of the observed measurement results.

> conceptually unexplained notions, they have a purely mathematical meaning. While there is a widespread agreement concerning the essential features of the calculus of probability, there are widely diverging opinions what the referent of mathematical probability theory is. [41, p. 582]

The important point is that when a mathematician and a physicist talk about 'probability' they need not refer to the *same* concept. While for the mathematician the question of the relation between the mathematical structure of probability and experience plays no significant role, for the physicist who assumes a realist stance the question of probability is *necessarily* related to experience and physical reality.

Luigi Accardi proved in 1981 that there is a direct relation between Bell inequalities and probability models [1]. The theorem of Accardi states that any theory which violates Boole-Bell inequalities[e] has a non-Kolmogorovian probability model. Since only Kolmogorovian models can be interpreted as referring to a degree of ignorance of a presupposed state of affairs described by a set of definite valued preexistent properties, this means that QM possesses a probability model which cannot be interpreted in terms of ignorance of such preexistent reality.

The fact that QM possesses a non-Kolmogorovian probability model is not such a big issue from a mathematical perspective: many mathematicians work with these probability structures and do not get astonished in any way by them. But from a representational realist perspective which understands that a physical theory must be capable of providing a conceptual representation of physical reality, the question which arises is very deep, namely, what is the meaning of a concept of probability which does not talk about the degree of knowledge of a definite state of affairs? From our perspective, if such a question is not properly acknowledged, the statement "QM is a theory about probabilities" looses all physical content. It might be regarded as either an obvious mathematical statement with no interest —it only states the well known fact that in QM there is a (non-Kolmogrovian) probability measure assigned via Gleason's theorem— or a meaningless physical statement, since we do not know what quantum probability is in terms of a physical concept. According to our stance, if

[e]As remarked by Itamar Pitowsky [40, p. 95]: "In the mid-nineteenth century George Boole formulated his 'conditions of possible experience'. These are equations and inequalities that the relative frequencies of (logically connected) events must satisfy. Some of Boole's conditions have been rediscovered in more recent years by physicists, including Bell inequalities, Clauser Horne inequalities, and many others."

we are to understand QM as a physical theory, and not merely as an algorithmic structure which predicts measurement outcomes, it is clear that we still need to provide a link between the mathematical structure and a set of physical concepts which are capable of providing a coherent account of quantum phenomena.

4. Empirical Terms vs Physical Concepts

In the first decades of the 20th century logical positivists fought strongly against dogmatic metaphysical thought, imposing a reconsideration of observability beyond the *a priori* categories of Kantian metaphysics. In their famous *Manifesto* [8] they argued that: "Everything is accessible to man; and man is the measure of all things. Here is an affinity with the Sophists, not with the Platonists; with the Epicureans, not with the Pythagoreans; with all those who stand for earthly being and the here and now." Their main attack to metaphysics was designed through the idea that one should focus in "statements as they are made by empirical science; their meaning can be determined by logical analysis or, more precisely, through reduction to the simplest statements about the empirically given." Their architectonic stood on the distinction between *empirical terms*, the empirically "given" in physical theories, and *theoretical terms*, their translation into simple statements. This separation and correspondence between theoretical statements and empirical observation would have deep consequences not only regarding the problems addressed in philosophy of science but also with respect to the limits of development of many different lines of research. The important point is that even though within the philosophy of science community this distinction has been strongly criticized and even characterized as "naive"; many of the problems discussed in the literature still presuppose it implicitly. Indeed, as remarked by Curd and Cover:

> Logical positivism is dead and logical empiricism is no longer an avowed school of philosophical thought. But despite our historical and philosophical distance from logical positivism and empiricism, their influence can be felt. An important part of their legacy is observational-theoretical distinction itself, which continues to play a central role in debates about scientific realism. [9, p. 1228]

One of the major consequences of this "naive" perspective towards observation is that physical concepts become supplementary elements in the analysis of physical theories. Indeed, when a physical phenomenon is understood as independent of physical concepts and metaphysical presuppositions, *empirical terms* configure an objective set of data which can be directly related —without any metaphysical constrain— to a formal scheme. Actual empirical observations become then the very fundament of physical theories which, following Mach, should be understood as providing an "economical" account of such observational data. As a consequence, metaphysics and physical concepts are completely out of the main picture.

Empirical Data ————— *Theoretical Terms*

(Supplementary Interpretation)

According to this scheme, physical concepts are not essentially needed since the analysis of a theory can be done by addressing the logical structure which accounts for the empirical data. The role of concepts becomes then accessory: adding metaphysics might help us to picture what is going on according to a theory. Like van Fraassen argues [45], it might be interesting to know what the world is like according to an interpretation of a formalism. However, one can perfectly do without interpretation when the question addressed is only related to empirical findings. Many realists within philosophy of physics while stress the need of an interpretation, accept the (empiricist) idea that the formalism already provides direct access to empirical data. Like a Troyan horse, these supposedly realist schemes hide within the main (metaphysical) presupposition of the enemy (empiricism). Indeed, within philosophy of physics, many who claim to be realists, agree with empiricists that metaphysical schemes are only necessary when attempting to "understand" —a term which remains dependent on the philosophical stance— a physical theory. The distance between realism and empiricism seems to be the strength with which they argue for or against the need of interpretation. This is the main reason why the "interpretation" of a theory has been understood in philosophy of physics as something "added" to an already formalized empirical theory. Thus, physical concepts are not directly related to the metaphysical foundation of phenomena and experience.[f]

Against these empiricist based perspectives —extensively widespread within philosophy of physics even in the context of supposedly realist approaches— we understand that each physical theory is a triad composed

[f] A clear expression of this situation is the so called "underdetermination problem" which implicitly assumes that a theory can account for phenomena independently of a metaphysical scheme.

by a *mathematical formalism*, a *conceptual network* and a limited specific *field of phenomena*. In this scheme physical notions play a fundamental role. *Physical concepts are defined through metaphysical principles which configure and determine physical experience itself.* Physical observation cannot be considered in terms of "common sense" realism for that would presuppose that the world is constrained by our "common sense" understanding of it. This idea breaks with the basic humble attitude of science, which accepts that we do not know how the world really is.

Physical observation is always both metaphysically and theory laden. We are always within a particular physical (and metaphysical) representation. A realist analysis which believes in the possibility of representing reality in terms of a formal and metaphysical theory must always begin, not by collecting a set of "naked" empirical data —as it is the case of empiricists—, but by considering and making explicit the metaphysical presuppositions related to the theory and its phenomena. Our philosophical post-Kantian realist position stresses the need to consider physical notions as fundamental elements of a theory, without which physical observation of phenomena cannot be defined. As obvious as it is, a 'field' cannot be observed without the notion of 'field', we simply cannot observe a 'particle' or a 'wave' without presupposing such physical concepts. These concepts are undoubtedly part of a metaphysical architectonic developed through centuries. Naturalizing such concepts in terms of "common sense" *givens* is turning a specific metaphysical scheme into dogma. As Einstein makes the point:

> Concepts that have proven useful in ordering things easily achieve such an authority over us that we forget their earthly origins and accept them as unalterable givens. Thus they come to be stamped as 'necessities of thought,' 'a priori givens,' etc. The path of scientific advance is often made impossible for a long time through such errors. [25, p. 102]

In particular, it is important to remark that all classical physical entities —as we discussed above— presuppose PE, PNC and PI. These are not principles that are found or observed in the world, but the very conditions that allow us to determine physical experience itself [16]. As remarked by Einstein in his famous recommendation, which led Heisenberg to the principle of indetermination: "It is only the theory which can tell you what can be observed." Einstein's philosophical position has been many times charichaturized in the literature as a scientific realist.[g] The fact that "he was not the friend of any simple realism" [32, p. 206] can be witnessed from the very interesting remark, recalled by Heisenberg, in which Einstein explained:

> I have no wish to appear as an advocate of a naive form of realism; I know that these are very difficult questions, but then I consider Mach's concept of observation also much too naive. He pretends that we know perfectly well what the word 'observe' means, and thinks this exempts him from having to discriminate between 'objective' and 'subjective' phenomena. No wonder his principle has so suspiciously commercial a name: 'thought economy.' His idea of simplicity is much too subjective for me. In reality, the simplicity of natural laws is an objective fact as well, and the correct conceptual scheme must balance the subjective side of this simplicity with the objective. But that is a very difficult task. [30, p. 66]

Einstein's position was orthodoxy at the time. Most of the founding fathers of QM —exception made of Dirac— were also part of this same neo-Kantian tradition which understood that the observation of physical phenomena was metaphysically constrained. This idea goes back to Hume himself who argued that the notion of *causation* is not something grounded empirically, it is never found in the observable world. Rather, as Kant would later on clearly expose, it is a metaphysical presupposition which allows us to make sense of physical phenomena. Following Einstein's dictum, Heisenberg went also against the positivist interpretation of empirical science as disconnected from metaphysical presuppositions:

> The history of physics is not only a sequence of experimental discoveries and observations, followed by their mathematical description; it is also a history of concepts. For an understanding of the phenomena the first condition is the introduction of adequate concepts. Only with the help of correct concepts can we really know what has been observed. [31, p. 264]

[g]As remarked by Howard [33, p. 73], Einstein was certainly part of the neo-Kantian tradition: "Einstein was dismayed by the Vienna Circle's ever more stridently anti-metaphysical doctrine. The group dismissed as metaphysical any element of theory whose connection to experience could not be demonstrated clearly enough. But Einstein's disagreement with the Vienna Circle went deeper. It involved fundamental questions about the empirical interpretation and testing of theories."

Going back to our scheme, the three elements that compose a physical theory form a perfect circle with no preeminence of one over the other. All three elements are interrelated in such a way that only through their mutual inter-definition we can access physical experience.

Conceptual Network *Mathematical Formalism*

Field of Representable Phenomena

In order to be clear about our perspective of analysis we would like to make explicit our philosophical stance which considers, not only the radical importance of conceptual representation, but also the metaphysical and theory ladenness of physical observation:

Representational Realism: *A representational realist account of a physical theory must be capable of providing a physical (and metaphysical) representation of reality in terms of a network of concepts which coherently relates to the mathematical formalism of the theory and allows us to make predictions of a definite field of phenomena. Observability in physics is always both metaphysically and theory laden, and thus, must be regarded as dependent of each particular physical representation.*

According to our stance, physical statements about phenomena must be necessarily related to the physical (and metaphysical) representation provided by the theory in terms of definite physical concepts. Crudely put, there is no "common sense" experience in physics. Every experience in physics is a restricted experience, constrained by physical concepts and metaphysical presuppositions. Physics is in essence a metaphysical enterprise. As Einstein remarked: "The problem is that physics is a kind of metaphysics; physics describes 'reality'. But we do not know what 'reality' is. We know it only through physical description..." From this perspective, the problem with the orthodox formalism of QM is that it provides predictions which have not yet been coherently related to a network of adequate physical concepts. To say it shortly, we do not know what QM is talking about. But if we do not know how to account for the 'clicks' in detectors in a typical quantum experiment, we simply do not understand either what a quantum phenomenon really is. Still today, the 'quantum clicks' that we find in the lab have no conceptual explanation. Boole-Bell type inequalities have proven that 'quantum clicks' lie outside the scope of classical local-realistic theories.

5. The EPR-Battle: Counterfactual Statements and (Actual) Elements of Physical Reality

The power of physics comes from its amazing predictive capacity; something that is exposed through the empirical confirmation of (operational) counterfactual statements. If a theory is empirically adequate then counterfactual statements of the type: "if we measure physical quantity A, the result will be x; but if we measure instead physical quantity B the result will be y" are always considered to be what the theory is really talking about. As clearly expressed by Griffiths [28, p. 361]: "If a theory makes a certain amount of sense and gives predictions which agree reasonably well with experimental or observational results, scientists are inclined to believe that its logical and mathematical structure reflects the structure of the real world in some way, even if philosophers will remain permanently skeptical." Indeed, operational counterfactual statements conform the core of the objective physical reality the theory talks about.

Counterfactual reasoning is a *necessary condition* not only for constructing a representation that provides an objective account of physical reality independent of the choices and actions of subjects but also for physical discursivity itself. We should remark, due to the ongoing debate about counterfactuals in QM, that these kind of physical statements need not be necessarily related to "possible worlds" or to "the reification of modalities" —a particular way of analyzing these subjects by logicians and analytic metaphysics which has also penetrated deeply philosophy of physics. In physics, operational counterfactual reasoning does not imply that every statement is actually real. Obviously, the fact that I can imagine an experience in the future does not imply its reality. Operational counterfactual reasoning has been assumed in every physical theory that we know and allows a theory to make predictions in terms of meaningful operational statements. Their main structure is very simple: "if I do this then that will happen."

Meaningful Operational Statements (MOS): *If given a specific situation a theory is capable of predicting in terms of definite physical statements the outcomes of possible measurements, then such physical statements are meaningful relative to the theory and must be considered as constitutive parts of the particular representation of physical reality that the theory* *provides. Measurement outcomes must be understood only as exposing the empirical adequacy (or not) of the theory.*

MOS are not necessarily statements about future events, such as for example "if I measure the spin in the x-direction, I will obtain spin-up with probability 0.4 and spin-down with probability 0.6." MOS can be also statements about the past or the present. For example, according to some physical theories I can claim that "the earth was formed about 4.54 billion years ago" (long before even physics was imagined!), or that if someone would perform a free fall experiment in the moon at this very moment, due to its gravity, "the object would be falling accelerated at 1.6 $\frac{m}{s^2}$." That is indeed the magic of both physics (and metaphysics), the possibility to represent, think and imagine beyond the here and now.

Physical statements that allow us to predict specific phenomena have been always intuitively related to physical reality. According to physicists, if we possess an empirically adequate physical description of a state of affairs we can predict what will happen in any particular experiment.[h] For instance, we also know what might have happened if I had performed an experiment in the past or in the present, in a different place to the one I am now. Experiments in classical physics allow us to learn about the preexistent properties of a system. The strong realist presupposition is that once we have an empirically adequate theory we don't even need to perform an experiment in order to know the result! Take for example a physical object as a small ball, one can imagine all the possible experiments that one could perform inside a lab with it. We know the acceleration of this ball in a free fall experiment on earth will be 9.8 $\frac{m}{s^2}$ and we can also predict the motion of the ball if we through it inside the room. There are indeed many experiments we could perform of which we know the answer beforehand by simply calculating their results using classical mechanics. There is no single physicist that would dare go against the predictions of Newtonian mechanics. And that is the whole beauty of physics, at least from a realist perspective: physical representations talk about physical reality independently of the here and now.

[h]It is interesting to notice that in such kind of statements we see the two main understandings of actuality coming together: the actuality *hic et nunc* of observations is an expression of the actual *preexistent* mode of existence of properties. Unfortunately, it is very frequent to find in the literature a mixture between these two different meanings of actuality.

The importance of counterfactual statements as related to physical reality was stressed, in the context of QM, by Einstein himself in 1935, in what would become on of the most famous articles in the foundational literature: the "EPR paper" [26]. In this paper, Einstein together with his students Podolsky and Rosen, used his famous definition of what was to be considered an *element of physical reality* in order to show that QM seemed not to be a *complete theory*. As remarked by them: "Whatever the meaning assigned to the term *complete*, the following requirement for a complete theory seems to be a necessary one: *every element of physical reality must have a counterpart in the physical theory*." [*Op. cit.*, p. 777] Indeed, this seems to be a necessary condition for a theory which attempts to provide an account of physical reality. However, Einstein's definition stressed only a limited set of the MOS predicted by quantum theory. His definition focused only on those MOS which could be related to an actualist metaphysical account of physical reality —leaving aside the more general probabilistic statements.

Einstein's (Actual) Element of Physical Reality: *If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of reality corresponding to that quantity.*

As remarked by Aerts and Sassoli [2, p. 20]: "the notion of 'element of reality' is exactly what was meant by Einstein, Podolsky and Rosen, in their famous 1935 article. An element of reality is a state of prediction: a property of an entity that we know is actual, in the sense that, should we decide to observe it (i.e., to test its actuality), the outcome of the observation would be certainly successful." Indeed, certainty and actuality were the restrictive constraints of what could be considered in terms of physical reality.

But Bohr, contrary to Einstein, had a very different standpoint regarding the meaning of physics in general, and of QM in particular. In his 1935 reply paper to EPR [4] which appeared in the following volume of *Physical Review*, he argued that in QM things were completely different to any other physical theory. Bohr wanted to presuppose classical discourse in terms of classical notions (e. g., 'waves' and 'particles') even at the price of restricting the multiple contexts of analysis provided by the formalism itself. Even though each basis was directly related to the correct predictions of statistical outcomes of observables, it was argued that in order to discuss about quantum properties the very precondition was the choice of a single context (interpreted in terms of an experimental arrangement). In this way, quantum physics had been restricted to the here and now experimental set-up. At the same time, Bohr [47, p. 7] had strongly argued about the impossibility of providing a physical representation of QM beyond classical notions, claiming that: "[...] the unambiguous interpretation of any measurement must be essentially framed in terms of classical physical theories, and we may say that in this sense the language of Newton and Maxwell will remain the language of physicists for all time." According to him [*Op. cit.*, p. 7], "it would be a misconception to believe that the difficulties of the atomic theory may be evaded by eventually replacing the concepts of classical physics by new conceptual forms." The choice of Bohr was to stick to classical discourse and give up counterfactual reasoning of MOS in QM —which was explicitly used within the EPR argument. Bohr was willing to develop a new complementarity scheme even at the price of abandoning the physical representation of quantum reality.

Bohr took as a standpoint the idea that observed measurement outcomes were perfectly defined in QM and added the necessity of choosing a particular context between the many possible ones[i] in order to recover a classical "*as-if* discourse" in terms of 'waves' and 'particles'.

> [...] the choice between the experimental procedures suited for the prediction of the position or the momentum of a single particle which has passed through a slit in a diaphragm, we are, in the 'freedom of choice' offered by the last arrangement, just concerned with the *discrimination between different experimental procedures which allow of the unambiguous use of complementarity classical concepts.* [4, p. 699] (emphasis added)

But this complementarity scheme designed by the Danish physicist precluded —since it denied operational counterfactual reasoning itself— the very possibility of relating MOS to an objective physical description of reality —independent of *subjective choices.* After Bohr's reply [4], unlike a classical object which preexists (in terms of definite valued properties) independently of the choice of any experiment, it was accepted that quantum systems and properties were explicitly dependent on the choice of an experimental arrangement or context.

Once and again it was repeated that Bohr had been "the true winner of the EPR battle" —as well as of the Solvay confrontation some years before. However, no one could really explain why. Bohr had designed a

[i] A problem known today in the literature as the infamous basis problem. One that has found no true solution until the present.

contradictory algorithmic language based on his complementarity principle according to which, it only made sense to talk about "waves" and "particles" once the choice of an experimental set up had been performed by the physicist in the lab. Subjectivity had been introduced for the first time *within* physical description, creating what is known today as "the quantum omelette" (see for discussion [19]). As most clearly stated by Jaynes:

> [O]ur present [quantum mechanical] formalism is not purely epistemological; it is a peculiar mixture describing in part realities of Nature, in part incomplete human information about Nature —all scrambled up by Heisenberg and Bohr into an omelette that nobody has seen how to unscramble. Yet we think that the unscrambling is a prerequisite for any further advance in basic physical theory. For, if we cannot separate the subjective and objective aspects of the formalism, we cannot know what we are talking about; it is just that simple. [34, p. 381]

But to be fare, it was not true that Bohr wanted to discuss about reality. His scheme was totally consistent and very difficult to tackle. It rested on an understanding of physics in highly pragmatic terms, as a "tool" to approach intersubjective agreement between experimentalists. In this respect, Bohr was indeed much closer to logical positivism than he himself would have admitted.

> Physics is to be regarded not so much as the study of something *a priori* given, but rather as the development of methods of ordering and surveying human experience. In this respect our task must be to account for such experience in a manner independent of individual subjective judgement and therefor objective in the sense that it can be unambiguously communicated in ordinary human language. [5]

Even though Bohr was a neo-Kantian and understood phenomena as related to categories and metaphysical principles, against metaphysical questions and problems, he was in line with the positivist appeal to Sophists and their epistemological perspective which placed the subject —and his here and now experience— as the fundament of knowledge itself. Bohr, contrary to the logical positivists who presupposed a "common sense" here and now observation, had placed his *a priori* in classical language —which had the purpose of constraining phenomena as classical space-time phenomena. Bohr had reintroduced Protagoras *dictum* within physics, adding to it the importance of classical language and connecting physics to the main philosophical debate of the 20th century: the linguistic turn. His new precept could be read in the following terms: *the subject and his (classical) language are the measure of all things*. Bohr was not interested in the problem of reality. Instead of getting into the riddle of an ontological analysis Bohr focused on epistemological concerns. Indeed, as remarked by A. Petersen, his long time assistant:

> Traditional philosophy has accustomed us to regard language as something secondary and reality as something primary. Bohr considered this attitude toward the relation between language and reality inappropriate. When one said to him that it cannot be language which is fundamental, but that it must be reality which, so to speak, lies beneath language, and of which language is a picture, he would reply, "We are suspended in language in such a way that we cannot say what is up and what is down. The word 'reality' is also a word, a word which we must learn to use correctly." Bohr was not puzzled by ontological problems or by questions as to how concepts are related to reality. Such questions seemed sterile to him. He saw the problem of knowledge in a different light. [39, p. 11]

Bohr wanted to develop a language that would allow us to account for phenomena in terms of classical physical concepts, even at the price of dissolving the relation between such concepts and physical reality. His complementarity approach was designed in order to support the inconsistencies of such incompatible relations. This had very important consequences for the development of physics. As Arthur Fine makes the point:

> [The] instrumentalist moves, away from a realist construal of the emerging quantum theory, were given particular force by Bohr's so-called 'philosophy of complementarity'; and this nonrealist position was consolidated at the time of the famous Solvay conference, in October of 1927, and is firmly in place today. Such quantum nonrealism is part of what every graduate physicist learns and practices. It is the conceptual backdrop to all the brilliant success in atomic, nuclear, and particle physics over the past fifty years. Physicists have learned to think about their theory in a highly nonrealist way, and doing just that has brought about the most marvelous predictive success in the history of science. [9, p. 1195]

Contrary to Fine we do not understand this as “the most marvelous” epoch of science but rather as a quite obscure period in which we have not advanced much in really understanding one of the main theories of the 20th Century. After more than one century after its creation we still don’t know what QM is talking about.

Our representational realist stance attempts to bring back metaphysical considerations within the analysis of QM by taking into account three main desiderata: the first is that physical observation is both metaphysically and theory laden; the second is that operational counterfactual reasoning about MOS is the kernel of physical discourse and, in consequence, cannot be abandoned if we seek to find an objective representation of physical reality; the third and final desideratum is that predictions must be necessarily related to the physical representation of reality provided by the theory in terms of adequate physical concepts. Against Bohr, (actual) experiments and measurements cannot be regarded as the point of departure, since it is only the theory which can tell you what can be observed. We need to do exactly the opposite, we need to read out an objective physical description from the formalism escaping at the same time dogmatic classical metaphysics —which is today still grounded in the (classical) metaphysics of actuality through PE, PNC and PI. Just like Einstein taught us to do in Relativity, we need to concentrate on what the theory predicts in operational terms, and be ready to come up with new physical concepts that match the formalism and explain phenomena. From our perspective —contrary to Bohr’s dogmatism with respect to classical language and physical experience—, every new physical theory determines a radically new field of experience which is necessarily related to a language constituted by new physical concepts.

6. Actual Properties and Observation in QM

The general metaphysical principle implied by the understanding of Newtonian mechanics, that ‘Actuality = Reality’, has become an unquestionable dogma within physics. As a silent fundament all of physics has been developed following the metaphysics of actuality. And even though QM was born from a deep positivist deconstruction of the *a priori* classical Newtonian notions —and in this sense the philosophy of Mach can be understood as the very precondition for the creation of both QM and relativity theory— it was very soon reestablished within the limits of classical metaphysics itself. The constrains of actuality have been unquestionably accepted by philosophers of physics either in terms of *hic et nunc* observation (empiricism and its variants) or as the mode of *preexistence* of properties (realism). Both positions have remained captive of actualism; trapped in the metaphysical net designed —through PE, PNC and PI— by Aristotle around the 5th century before Christ and imposed by Newton in the 18th Century of our time. Actual (preexistent) properties and actual (here and now) observations are two sides of the same (metaphysical) coin.

Today, both realists and anti-realists support an anti-metaphysical understanding of observation within philosophy of physics. It is accepted by both parties that QM is an empirically adequate theory and, consequently, that the problem is not related to the understanding of quantum phenomena. This can be directly linked not only to the positivist "common sense" or "naive" understanding of observation, but also —maybe more importantly— to Bohr's analysis of QM based on the idea that any physical phenomena is a classical phenomena, or in his own words, to the idea that "[...] the unambiguous interpretation of any measurement must be essentially framed in terms of classical physical theories".

Bas van Fraassen [44, pp. 202-203] has followed the Bohrian path in terms of his constructive empiricism. According to him: "To develop an empiricist account of science is to depict it as involving a search for truth only about the empirical world, about what is actual and observable." Making explicit the fact that observation should be understood in terms of "common sense" observation. The problem, according to van Fraassen, only appears with respect to the "non-observable" entities —such as e.g, an atom or an electron. Instrumentalists assume exactly the same ground (as empiricist) considering actual observation in terms of "common sense" observation of measurement outcomes. As made explicit by Fuchs and Peres: "[...] quantum theory does not describe physical reality. What it does is provide an algorithm for computing probabilities for the macroscopic events ('detector clicks') that are the consequences of experimental interventions."

Realists approaches to QM have focused on Einstein's implicit use of the *elements of physical reality* in terms of actuality and his recommendation to extend QM —which should be considered as "incomplete"— to a more general framework, one that goes back to a description in terms of actual properties restoring a classical way of thinking about *what there is*. This idea, presupposed by the Hidden Variable Program (HVP), has also permeated strongly most realist interpretations of QM, which in one way or the other have ended up always discussing in terms of actual properties, grounded as well on "common sense" observation. Such is the case of the modal interpretation of Dieks, Griffiths' consistent histories approach, and the many worlds interpretation. Even those interpretations that have argued in favor of considering a different realm to that of actuality —such as the ones proposed by Heisenberg, Popper, Margenau and Piron— were not able to advance in an ontological definition of such non-actual realm. Potentialities, propensities and dispositions have been repeatedly defined only in terms of *a process of becoming actual* (see for discussion [12]). These teleological schemes have betrayed, because of their standpoint and focus on the measurement problem, any true possibility of progress and development beyond the realm of actuality.

Actuality imposes a mode of existence (of both properties and observations) determined by PE, PNC and PI. Everything is reduced then either to: *yes-no properties* (realism) or *yes-no experimental observations* (empiricism). But what if QM cannot be subsumed under the metaphysical equation imposed by Newtonian physics according to which: Actuality = Reality?

7. Revisiting Quantum Physical Reality

We believe it would be no exaggeration to claim that the EPR paper together with Bohr's reply, have determined the fate of QM up to the present. The EPR paper ended with a recommendation to extend QM in order to recover a classical actualist understanding about *what there is*:

> While we thus have shown that the wave function does not provide a complete description of the [actual] physical reality, we left open the question of whether or not such a description [of actual properties] exists. We believe, however, that such theory is possible. [26, p. 780]

Bohr argued instead that:

> While [...] in classical physics the distinction between object and measuring agencies does not entail any difference in the character of the description of the phenomena concerned, it fundamental importance in quantum theory [...] has its root in the indispensable use of classical concepts in the interpretation of all proper measurements, even tough the classical theories do not suffice in accounting for the new types of regularities with which we are concerned in atomic physics. [4, p. 701]

EPR was the final battle of the two main figures in the physics of the 20th century, and even though Bohr was declared the only triumphant survivor, both lines of research were developed under the constraints and limits of the logical positivist "naive" or "common sense" understanding of observation. As a consequence of this development within philosophy of QM one of the main discussions of the founding fathers regarding the meaning of quantum phenomena and experience was completely abandoned. The measurement problem which accepted quantum observations as perfectly well defined givens of experience begun very soon to concentrate the attention of everyone.

After the clash of the two titans, physicists were confronted with a choice between two different paths. Either they could follow Bohr and be satisfied with an inconsistent intersubjective language ruled by complementarity with no direct reference to physical reality, or they could follow Einstein and try to find a new formalism that would allow them to recover a classical actualist type-description of physical (quantum) reality. But this crossroad, imposed by Einstein, Bohr and logical positivism, hides a "road sign" called *Wolfgang Pauli* which exposed the lines of a more radical resolution to the quantum riddle. Indeed, between the founding fathers, we regard Wolfgang Pauli as the most radical and revolutionary thinker of them all. Against Bohr, he stood always close to metaphysics and the problem of reality; beyond Einstein, he was ready to seriously reconsider the meaning and definition of physical reality itself.[j]

> When the layman says 'reality' he usually thinks that he is speaking about something which is self-evidently known; while to me it appears to be specifically the most important and extremely difficult task of our time to work on the elaboration of a new idea of reality. [37, p. 193]

What do we mean when we say that "particles are physically real according to classical mechanics"? Following Heisenberg and his closed theory approach, this question has a definite answer according to our representational realist stance. It means that classical mechanics is capable of expressing through the relation between the formalism of mathematical calculus and a network of concepts —such as, for example, space, time, particle, mass, position, velocity— a specific field of phenomena. It is in this sense that the notion of particle is a metaphysical machinery which allows us to express reality.

[j] Which for Einstein was determined through space-time separability.

In QM we have a sound formalism, with features such as contextuality, superposition and indetermination, which defy a realist classical scheme in terms of an ASA. However, all approaches until today have stood close to physical reality understood in an actualist fashion. Because of this, the features of the quantum formalism have been regarded as obstacles which we need to bypass or escape in order to restore our classical way of thinking about *what there is*. Our proposed line of research, following Pauli, is to turn this problem upside-down. We need to develop a representation of physical reality according to the quantum formalism, not the other way around.

Heisenberg's closed theory approach is the key to abandon another presupposed dogma —also imposed by Bohr— according to which QM must be related to classical physics in terms of a reductionistic *limit*. Once we accept the possibility of considering an independent metaphysical scheme to account for QM, the "problems" addressed in the literature become instead essential features of the metaphysical system we need to construct in order to coherently relate the formalism with physical reality and experience. We need to develop a new way of understanding reality beyond the ruling of actuality. To escape the ruling of actuality —both in terms of *hic et nunc* observation and *preexistent* properties— means to abandon, on the one hand, the idea that we have a clear definition of what is observed according to QM, and on the other hand, the idea that actuality is the only possible way to conceive and understand physical reality. Our strategy is the same strategy followed by Einstein in order to make sense of the Lorentz transformations in relativity theory. That is, to take as a standpoint the formalism and the predictive power of the theory in order to develop new physical concepts which relate coherently to the formalism and can allow us to represent and explain the phenomena in question.

8. Generalized Elements of Physical Reality

The quantum wave function Ψ provides definite physical statements regarding observables through the Born rule. The MOS provided by QM which have been used in order to develop outstanding experimental technological developments are of the following type:

Definition 8.1. MOS in QM: Given a vector in Hilbert space, Ψ, the Born rule allows us to predict the average value of (any) observable O.

$$\langle \Psi | O | \Psi \rangle = \langle O \rangle$$

This prediction is independent of the choice of any particular basis.

To take seriously the orthodox quantum formalism means for us to take into account all the predictions provided by the theory; i.e., both *certain predictions* (probability equal to unity) and *statistical predictions* (probability between zero and unity). This means we need to create a new understanding of probability in terms of objective knowledge and abandon its classical understanding of probability in terms of ignorance about an actual state of affairs. But how to do so in relation to physical reality? We believe that a good standpoint is the generalization of Einstein's famous definition of an *element of physical reality.* This redefinition must keep the relation imposed between operational predictive statements and reality, but leave aside both the actualist constraint imposed by certainty (probability equal to unity) and the intromission of the notion of measurement.

Generalized Element of Physical Reality: *If we can predict in any way (i.e., both probabilistically or with certainty) the value of a physical quantity, then there exists an element of reality corresponding to that quantity.*

By extending the limits of what can be considered as physically real, we have also opened the door to a new understanding of QM beyond classical physics. The problem is now set: we now need to find *a physical concept that is capable of being statistically defined in objective terms.* That means to find a notion that is not defined in terms of yes-no experiments (as it is the case of classical properties), but is defined instead in terms of a probabilistic measure. Of course, this first step must be accompanied by developing a network of physical notions that accounts for what QM is talking about, beyond measurement outcomes. In the end, our new non-classical physical scheme will also have to be capable of taking into account the main features brought in by the orthodox formalism.

i. *Our network of physical concepts must provide a deeper understanding of the principle of indetermination, the principle of superposition, the quantum postulate and quantum phenomena in general.*
ii. *Our network of physical concepts must also explain the physical meaning of non-locality, non-separability and quantum contextuality.*
iii. *Our metaphysical scheme must be capable of recovering an objective notion of measurement (one that exposes a preexistent state of affairs).*
iv. *The physical representation must account for all MOS in QM respecting (operational) counterfactual reasoning.*

We believe it is possible to come up with a physical network of concepts that takes into account these features. The price we are willing to pay is the abandonment of the metaphysics of actuality. Only then it will be possible to construct a new non-classical metaphysical scheme with physical concepts specifically designed in order to account for the formalism of QM.

9. Immanent Powers in QM

Our research has analyzed the idea of considering a mode of existence truly independent of actuality, namely, ontological potentiality. Elsewhere, we have introduced ontological potentiality as the realm of which QM talks about. This realm is defined by the principles of indetermination, superposition and difference. In order to advance we also need to introduce the notions of *Potential State of Affairs* (PSA), *potential effectuation* and *immanent cause* [13,17]. Indeed, by developing these new concepts, we expect that our formal analysis regarding *quantum possibility* [22] can find a coherent physical interpretation. But now the question arises: what are the "things" which exist and interact within this potential realm? Our answer is: *immanent powers* with definite *potentia*.[k] Indeed, while entities are composed by properties which exist in the actual mode of being, we have argued that an interesting candidate to consider what exists according to QM is the notion of *immanent power*. Elsewhere [12,13], we have put forward such an ontological interpretation of powers. In the following we summarize such ideas and provide an axiomatic characterization of QM in line with these concepts.

We should remark that our proposal differs substantially with respect to the hilemorphic schemes which define causal powers, propensities and dispositions always in terms of the actual realm —attempting to solve the infamous measurement problem. Such schemes still discuss in terms of properties and systems attempting to bridge the gap between "classical reality" and QM. The physical representation we propose, grounded on the closed theory scheme, escapes classical representation imposing an independent ontology for QM right from the start. The measurement and basis problems become, within our scheme, epistemic problems. These orthodox problems do not address what QM really talks about but rather, make reference to the way in which we humans acquire knowledge from the MOS implied by the theory.

[k]It will become clear in the following the radical distance of our approach with respect to many power ontologies developed within philosophy of QM.

The mode of being of a power is potentiality, not thought in terms of classical possibility (which relies on actuality) but rather as a mode of existence —i.e., in terms of ontological potentiality. To possess the power of *raising my hand*, does not mean that in the future 'I *will* raise my hand' *or* that in the future 'I *will not* raise my hand'; what it means is that, here and now, I possess a power which exists in the mode of being of potentiality, *independently of what happens or will happen in actuality.* Powers do not exist in the mode of being of actuality, they are not actual existents, they are undetermined potential existents. Powers, like classical properties, preexist to observation; unlike them preexistence is not defined in the actual mode of existence in terms of an Actual State of Affairs (ASA), instead we have a *potential preexistence* of powers which determines a Potential State of Affairs (PSA). While an ASA can be defined in terms of a set of actual properties, a PSA is defined as a set of powers with definite potentia.

Immanent powers are indetermined. The concept of 'immanent power' allow us to compress experience into a picture of the (quantum) world, just like entities such as 'particles', 'waves' and 'fields', allow us to do so in classical physics. We cannot "see" powers in the same way we see objects.[1] Powers are experienced in actuality through *elementary processes.* A power is sustained by a logic of actions which do not necessarily take place in the actual realm. Immanent powers interact in the potential realm through *potential effectuations.* While the actual realm reflects only an epistemic aspect of immanent powers, one that regards a specific type of knowledge we can acquire from them; the potential realm exposes the truly ontological relations between powers.

According to our representational realist stance, one of the most important problems regards the ontological meaning of a *quantum superposition* [20]. What does it mean to have a mathematical expression such as: $\alpha|\uparrow\rangle + \beta|\downarrow\rangle$, which allows us, through the Born rule, to predict (even though probabilistically) experimental outcomes? We believe that quantum superpositions are the central elements of QM. But what is the physical representation of these quantum superpositions? Our theory of powers has been explicitly developed in order to try to find an answer to this particular question.

Given a superposition in a particular basis, $\Sigma\, c_i|\alpha_i\rangle$, the powers are represented by the elements of the basis, $|\alpha_i\rangle$, while the coordinates in square

[1]It is important to notice there is no difference in this point with the case of entities: we cannot "see" entities —not in the sense of having a complete access to them. We only see perspectives which are unified through the notion of object.

modulus, $|c_i|^2$, are interpreted as the potentia of each respective power. *Powers can be superposed to different —even contradictory— powers* [17]. We understand a quantum superposition as encoding a set of powers each of which possesses a definite *potentia*. This we call a *Quantum Situation* (QS). For example, the quantum situation represented by the superposition $\alpha|\uparrow\rangle + \beta|\downarrow\rangle$, combines the contradictory powers, $|\uparrow\rangle$ and $|\downarrow\rangle$, with their respective potentia, $|\alpha|^2$ and $|\beta|^2$. Contrary to the orthodox interpretation of the quantum state, we do not assume the metaphysical identity of the multiple mathematical representations given by different bases [20]. Each superposition is basis dependent and must be considered as a distinct quantum situation. For example, the superpositions $c_{x1}|\uparrow_x\rangle + c_{x2}|\downarrow_x\rangle$ and $c_{y1}|\uparrow_y\rangle + c_{y2}|\downarrow_y\rangle$, which are representations of the same Ψ and can be derived from one another via a change in basis, are interpreted as two different quantum situations, QS_{Ψ,B_x} and QS_{Ψ,B_y}. Each one of them describes a particular quantum state of affairs. Notice that this interpretation departs radically from the orthodox understanding of superposed states as making reference to a quantum system. While orthodoxy relates superpositions to the state of a quantum system (e.g., photon, electron, etc.) our proposal relates quantum superpositions to quantum situations.

The logical structure of a superposition is such that a power and its opposite can exist at one and the same time, violating PNC [10]. Within a situation in which I can raise my hand, both powers (i.e., the power 'I am *able to* raise my hand' and the power 'I am *able not to* raise my hand') co-exist. A QS is *compressed activity*, something which *is* and *is not* the case, *hic et nunc*. Its expression cannot be thought in terms of identity but is expressed as a difference, as a *quantum of action*. This difference is expressed in the actual realm through actual effectuations.

Our understanding of QM can be condensed in the following eight postulates which contain the relation between our proposed concepts and the orthodox formalism of the theory.

I. Hilbert Space: QM is mathematically represented in a vector Hilbert space.

II. Potential State of Affairs (PSA): A specific vector Ψ with no given mathematical representation (basis) in Hilbert space represents a PSA; i.e., the definite existence of a multiplicity of *immanent powers*, each one of them with a specific *potentia*.

III. Actual State of Affairs (ASA): Given a PSA and the choice of a definite basis $B, B', B'', ...$, etc. —or equivalently a Complete Set of

Commuting Observables (C.S.C.O.)—, a context is defined in which a set of immanent powers, each one of them with a definite potentia, are univocally determined as related to a specific experimental arrangement (which in turn corresponds to a definite ASA). The context builds a bridge between the potential and the actual realms, between quantum powers and classical objects. The experimental arrangement (in the ASA) allows the powers (in the PSA) to express themselves in actuality through elementary processes which produce *actual effectuations*.

IV. Quantum Situations, Immanent Powers and Potentia: Given a PSA, Ψ, and the context or basis, we call a quantum situation to any superposition of one or more than one power. In general given the basis $B = \{|\alpha_i\rangle\}$ the quantum situation $QS_{\Psi,B}$ is represented by the following superposition of immanent powers:

$$c_1|\alpha_1\rangle + c_2|\alpha_2\rangle + ... + c_n|\alpha_n\rangle \tag{1}$$

We write the quantum situation of the PSA, Ψ, in the context B in terms of the order pair given by the elements of the basis and the coordinates in square modulus of the PSA in that basis:

$$QS_{\Psi,B} = (|\alpha_i\rangle, |c_i|^2) \tag{2}$$

The elements of the basis, $|\alpha_i\rangle$, are interpreted in terms of *powers*. The coordinates of the elements of the basis in square modulus, $|c_i|^2$, are interpreted as the *potentia* of the power $|\alpha_i\rangle$, respectively. Given the PSA and the context, the quantum situation, $QS_{\Psi,B}$, is univocally determined in terms of a set of powers and their respective potentia. (Notice that in contradistinction with the notion of *quantum state* the definition of a *quantum situation* is basis dependent and thus intrinsically contextual.)

V. Elementary Process: In QM we only observe discrete shifts of energy (quantum postulate). These discrete shifts are interpreted in terms of *elementary processes* which produce actual effectuations. An elementary process is the path which undertakes a power from the potential realm to its actual effectuation. This path is governed by the *immanent cause*[m] which allows the power to remain potentially preexistent within

[m]The *immanent cause* allows us to connect the power with its actual effectuation without destroying nor deteriorating the power itself. The immanent cause allows for the expression of effects remaining both in the effects and its cause. It does not only remain in itself in order to produce, but also, that which it produces stays within. Thus, in its production of effects the potential does not deteriorate by becoming actual —as in

the potential realm independently of its actual effectuation. Each power $|\alpha_i\rangle$ is univocally related to an elementary process represented by the projection operator $P_{\alpha_i} = |\alpha_i\rangle\langle\alpha_i|$.

VI. Actual Effectuation of an Immanent Power (Measurement): Immanent powers exist in the mode of being of ontological potentiality. An *actual effectuation* is the expression of a specific power within actuality. Different actual effectuations expose the different powers of a given QS. In order to learn about a specific PSA (constituted by a set of powers and their potentia) we must measure repeatedly the actual effectuations of each power exposed in the laboratory. (Notice that we consider a laboratory as constituted by the set of all possible experimental arrangements that can be related to the same Ψ.) An actual effectuation does not change in any way the PSA.

VII. Potentia (Born Rule): A *potentia* is the strength of an immanent power to exist in the potential realm and to express itself in the actual realm. Given a PSA, the potentia is represented via the Born rule. The potentia p_i of the immanent power $|\alpha_i\rangle$ in the specific PSA, Ψ, is given by:

$$Potentia\ (|\alpha_i\rangle, \Psi) = \langle\Psi|P_{\alpha_i}|\Psi\rangle = Tr[P_\Psi P_{\alpha_i}] \tag{3}$$

In order to learn about a QS we must observe not only its powers (which are exposed in actuality through actual effectuations) but we must also measure the potentia of each respective power. In order to measure the potentia of each power we need to expose the QS statistically through a repeated series of observations. The potentia, given by the Born rule, coincides with the probability frequency of repeated measurements when the number of observations goes to infinity.

VIII. Potential Effectuations of Immanent Powers (Schrödinger Evolution): Given a PSA, Ψ, powers and potentia evolve deterministically, independently of actual effectuations, producing *potential effectuations* according to the following unitary transformation:

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle \tag{4}$$

the case of the hylomorphic scheme. Actual results are single effectuations, singularities which expose the superposition in the actual mode of existence, while superpositions remain evolving deterministically according to the Schrödinger equation in the potential mode of existence, even interacting with other superpositions and producing new potential effectuations.

> While *potential effectuations* evolve according to the Schrödinger equation, *actual effectuations* are particular expressions of each power (that constitutes the PSA, Ψ) in the actual realm. The ratio of such expressions in actuality is determined by the potentia of each power.

According to our physical representation, just like classical physics talks about entities constituted by properties that preexist in the actual realm, QM talks about powers with definite potentia that preexist in an ontological potential realm, independently of the specific actual context of inquiry (QS) or the particular set of actualizations. QM talks about potential effectuations: the ontological evolution and interaction of immanent powers with definite potentia. Actual effectuations are only epistemic expressions of the immanent powers and their potentia. Our representation allows us to define powers objectively, independently of the epistemic context of inquiry. Through the immanent cause we are able to argue that a 'click' in a detector is the partial expression of an immanent power in the actual realm —instead of the complete actualization of a system (e.g. photon, electron, etc.). Only through the repetition of measurements we can find out (at the epistemic level) what is the specific potentia of this or that immanent power (in the ontic level). Both the basis and measurement problems are dissolved when acknowledging that the process of observation is restored as an epistemic display of an ontic level. Our representation allows us to regain an objective picture of physical reality independent of measurements and subjective choices. The price we are willing to pay is the abandonment of the Newtonian metaphysical equation presupposed in the analysis of QM, exposing the fact that: Quantum Reality $\neq$ Actuality.

The notion of *immanent power* is completely different from the notions of *causal power*, dispositions and propensities put forward by Heisenberg, Margenau, Popper and Piron —notions that have been redeveloped in the last years by Suarez, Dorato and Esfeld. Causal powers, propensities and dispositions are defined as non-actualized properties (see for a detailed discussion [12]). While these notions were developed in order to keep talking about systems and properties within QM, our notion of immanent power is designed to escape "classical reality". While our notion of immanent power exists and is defined in a manner completely independent of actuality; the notions of causal powers, dispositions and propensities are exclusively defined —going back to the Aristotelian hylemorphic scheme— in terms of their actualization. Propensity-type interpretations have focused in trying to solve the infamous measurement problem. Instead, rather than analyzing the quantum to classical limit, our scheme is the only one that proposes to discuss and focus on the interaction and evolution of that which exists according to QM in a manner completely independent of the representation provided by classical physics.

According to our physical representation QM is talking about potential effectuations of immanent powers and their potentia. In the mathematical level this is described by entangled quantum superpositions which evolve according to the Schrödinger equation of motion. Thus, the distance we take from the hylemorphic proposals is very radical. Firstly, our approach breaks with the presupposed existence of a reductionistic quantum to classical limit. Secondly, within our scheme the measurement and the basis problems are regarded only as epistemic problems. We have solved them by showing how they can be related to the objective measure —in the actual realm— of immanent powers and their potentia. Thirdly, the immanent power interpretation of QM that we put forward does not attempt to solve the "no-problems" discussed in the literature. These are considered as pseudoproblems which require the strong presupposition that QM is in fact talking about "classical reality". Our physical representation has been designed in order to solve two ontic problems: the contextuality problem and the superposition problem presented in [19,20]. Indeed, our (metaphysical) interpretation explains not only the physical meaning of the contextual character of QM in a natural manner through the introduction of the contextual notion of *quantum situation*, but also the physical meaning of quantum superpositions and more specifically, the objective nature of quantum probability.

10. Quantum Probability as an Objective Measure of the Potentia of Powers

An analogy can be useful in order to picture the physical representation we attempt to put forward for QM in terms of powers and potentia. Imagine two baseball players called Matthias and John. If we talk about baseball, everyone can understand that both Matthias and John possess a definite set of potentia withy respect to the powers of batting, running and pitching. What does it mean, for example, that Matthias possesses the power of batting with 0.9 potentia? It means obviously that he is a very good batter. That if I through 100 balls to him, he will be capable to batt approximately 90 balls. If I would like to learn (at the epistemic level) about the (ontic) power of Matthias to batt I obviously need to do statistics. The more statistics I perform the better I will learn about the potentia of his power to batt. Of course, exactly the same applies to John, if we would like to learn about his powers of batting, running and pitching, we would also need to perform a statistical measure. The statistics we could obtain from the performance of each player in many baseball games become in our scheme an (epistemic) measure of the (ontic) potentia of the powers in question. QM talks about the powers that exist in Nature, how they evolve and interact —it does not talk about subjects.

The analogy we are making with baseball is very important to understand the importance of considering immanent powers as notions which exist in quantum physical reality. In the movie Moneyball,[n] an American biographical sports drama film we find an account of the Oakland Athletics baseball team's 2002 season and their general manager Billy Beane's attempts to assemble a competitive team. In the film, Beane (Brad Pitt) and assistant GM Peter Brand (Jonah Hill), faced with the franchise's limited budget for players, build a team of undervalued talent by scouting and analyzing players only in terms of the statistical exposure of the potentia of each one of their powers (batting, running and pitching). The movie tells us how during a visit to the Cleveland Indians, Beane meets Peter Brand (Jonah Hill), a young Yale economics graduate with radical ideas about how to assess players' value. Rather than relying on the scouts' experience and intuition, Brand selects players based exclusively on their on-base percentage (OBP); i.e. the actual effectuation of the potentia of each player's powers. The movie tells us how, using these ideas, the Oakland Athletics 2002' campaign ranks among the most famous in franchise history winning 20 consecutive games between August 13 and September 4, 2002. Two years later the Red Sox won the 2004 World Series using the model pioneered by the Athletics. This model, which simply measures the potentia of the powers of each player has changed radically the way baseball is understood today. My strong claim is that the ontological understanding of what is going on in this model is related to the physical understanding of QM itself.

Our notion of immanent power recovers in a natural way the contextual character of QM. Indeed, in order to measure the potentia of a specific power (of John or Matthias), I will need to prepare a specific context. For example, if I want to measure the power of batting of John, then I will need to provide John with a batt, I will also need to have someone who throws balls to him in order to be able to do statistics. Notice that this is in no way different to a Stern-Gerlach situation. Obviously, if I through only one ball

[n]This movie is an adaptation of Michael Lewis' book *Moneyball: The Art of Winning an Unfair Game* [38].

I might not get enough information. If John misses this first throw I might not even learn if John can batt at all. It should become clear by now that the knowledge we might gain (or not) of a specific power changes in no way its ontological existence. It also becomes clear that the context of batting is epistemically incompatible to those of running or pitching. Exactly the same contextual aspect is found in Stern-Gerlach experiments. As we can see, immanent powers are contextual existents. However, it also beocmes clear that even though powers are *epistemically incompatible*, they are not *ontologically incompatible*. It is this aspect which is required in order to make sense of QM in objective terms and solve the contextuality problem prsented in [19].

Our notion of immanent power also provides a natural explanation to the measurement problem. In our terms, this amounts to the relation between the potentia of a power and its actual effectuation as described in terms of the immanent cause (see for a detailed discussion [14]). The fact that Matthias can batt 0.9 of the times does not imply in any way that if I throw a ball to Matthias he will batt the ball. This possibility is completely indetermined since the notion of immanent power is intrinsically statistical. That is the quantifying fuzzy nature of powers. Powers are notions which —unlike properties which are either true or false, 0 or 1— are quantified in a continuous manner; they posses an objective measure, namely, a potentia between 0 and 1. A power is *either* potentially true or false [21], but its potentia will range *between* 0 and 1.

We would like to remark the fact that our notion of immanent power is maybe the first physical notion to be characterized ontologically in terms of an objective probability measure. This concept escapes the ruling of actuality since it is founded on a different set of metaphysical principles to that of classical entities. Indeed, powers are indetermined, paraconsistent and contextual existents. Powers can be superposed and entangled with different —even contradictory— powers. A power, contrary to a property which can be only true or false possesses an intrinsic probabilistic measure, namely, its potentia. A potentia is intrinsically statistical, but this statistical aspect has nothing to do with ignorance. It is instead an objective feature of quantum physical reality itself.

Objective knowledge of properties is determined through yes-no experiments; one experiment is enough to completely characterize a property. Boolean classical logic and truth tables are suitable structures that allow us to define the elements of physical reality present in classical physics —namely, actual properties. Contrary to classical properties, objective knowledge of powers with definite potentia can be only approached by performing statistical experiments. One experiment is simply not enough in order to determine the potentia of a power. Boolean classical logic and correspondence truth tables are not suitable structures and notions that would allow us to characterize the physical elements of reality present in QM which are, as we have discussed above, intrinsically statistical. Indeed, we must remark that our scheme implies the need of developing a new potential notion of truth, one that is not understood as a one-to-one correspondence relation of propositions with an actual state of affairs. We leave this problem for a future work.

11. Conclusion

Our conceptual physical scheme allows us to provide a metaphysical ground to Pauli's intuition —also shared by Heisenberg, Popper, Margenau, Piron and many others— that quantum statistics and quantum probability expose objective features of a quantum situation, instead of "ignorance" or "inaccurate" knowledge of an ASA. Quantum probabilities of physical quantities calculated though the Born rule become in our scheme the objective gnoseological counterpart of an ontological potentia, an element of physical reality that provides an objective measure of ontologically existent quantum powers. We believe that the key to disentangle the quantum riddle and create a coherent representation of quantum physical reality is to abandon the metaphysical dogma, presupposed in classical physics, that Reality = Actuality. We should acknowledge in this respect that "common sense" is just a name for the naturalization of dogmatic metaphysics. Indeed, our time calls to work on the elaboration of a new idea of reality, but in order not to engage ourselves in pseudoproblems we should also acknowledge right from the start that this project is in itself a metaphysical enterprise.

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ON WHEN A SEMANTICS IS NOT A GOOD SEMANTICS: THE ALGEBRAISATION OF ORTHOMODULAR LOGIC

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It has been taken for granted for a long time that orthomodular lattices are the "algebraic counterpart" of orthomodular quantum logic. Pavičic and Megill have questioned this claim by pointing out that orthomodular quantum logic is sound and complete with respect to a proper supervariety of the variety $\mathcal{OML}$ of orthomodular lattices (the so-called weakly orthomodular lattices). The same authors conclude that "in the syntactical structure of quantum logic there is nothing orthomodular". After reviewing in a certain detail some concepts from Abstract Algebraic Logic, especially Blok and Pigozzi's theory of algebraisable logics, we argue that the weakly orthomodular semantics introduced by Pavičic and Megill is not a good semantics, and that the role of $\mathcal{OML}$ as an algebraic counterpart of quantum logic is unaffected by their allegations.

1. Introduction

This paper contains no new results; indeed, it scarcely contains anything that goes beyond folklore. However, different folks have different lore, and since quantum logic is an area to which mathematicians, physicists, and philosophers — let alone logicians — claim some rights, clashes are inevitable. Under some circumstances, then, it may not be out of place to review a few facts that could be far from obvious to a significant portion of this composite community, lest some claims you encounter in the literature lead more people than one would expect to believe that well-established results in the area are actually false. This is what we try to do in the present article.

2. A paradox to begin with

From a logic L one expects two things: syntax and semantics. Syntax is some way of manipulating strings of symbols, in order to separate the grain from the chaff. What counts as the grain could vary, depending on what we are interested in, but typically it will be *theorems*, i.e., *provable formulae*. Semantics is a way of associating meanings with strings of symbols, typically by specifying some class $\mathcal{C}$ of *models* of L, i.e., structures in which suitably interpreted formulae of L are either *true* of *false*. Now, it is sometimes said that a class $\mathcal{C}$ is a good semantics for L if L is at least weakly sound and complete with respect to $\mathcal{C}$, that is, if all theorems of L are true in $\mathcal{C}$ and all non-theorems are false.

This seems trivial. Let us see where it can lead, if taken literally. Take the lattice $\mathbf{N}_5$ with the universe $\{1, a, b, c, 0\}$ ordered as in Figure 1. Endow $\mathbf{N}_5$ with a unary operation $'$ defined by

$'$	
1	0
a	c
b	a
c	a
0	1

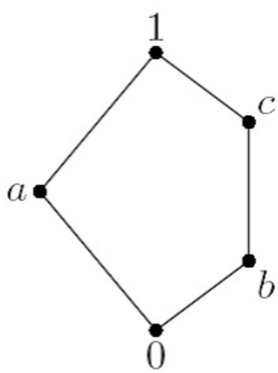


Fig. 1. The algebra $\mathbf{N}_5'$.

Just like you would do with Boolean algebras, define a formula α to be true in $\mathbf{N}_5'$ if the equation $\alpha \approx 1$ holds in $\mathbf{N}_5'$ in the usual algebraic sense expressed by $\mathbf{N}_5' \models \alpha \approx 1$. It is not difficult to verify that $\mathbf{N}_5'$ is a model of classical logic, the key fact being that if $p \to q \approx 1$ and $p \approx 1$ hold, then $q \approx 1$ holds as well, where $p \to q$ stands for $p' \vee q$. So classical logic is sound with respect to $\mathbf{N}_5'$. Conversely, observing that the two-element Boolean algebra $\mathbf{2}$ is a subalgebra of $\mathbf{N}_5'$, we get that any non-theorem β can be falsified on $\mathbf{N}_5'$, by any valuation falsifying β on $\mathbf{2}$. Thus, we have proved:

Prop 2.1. Classical logic is (weakly) sound and complete with respect to a non-modular lattice with a non-involutive complementation.

Strong soundness and completeness can also be established along similar, albeit slightly more convoluted, lines. Certainly, the theorem above does *not* show that classical logic is non-distributive, non-modular, and fails the double negation law. There would be no need to point this out, were it not for the fact that similar theorems have appeared in peer-reviewed outlets, making precisely such bizarre claims. Deferring references and a more detailed discussion thereof until Section 4, let us dwell for the time being on Proposition 2.1: what it *does* show is that classical logic has models that *viewed from the outside* are non-modular and do not satisfy double negation. But *seen with the eyes of classical logic* the model $\mathbf{N}_5'$ is distributive and satisfies double negation, that is $\mathbf{N}_5' \models (p \wedge (q \vee r)) \leftrightarrow ((p \wedge q) \vee (p \wedge r)) \approx 1$ and $\mathbf{N}_5' \models p \leftrightarrow p'' \approx 1$ both hold, with $\leftrightarrow$ being the classical equivalence connective. This is so because in $\mathbf{N}_5'$ we have that $b \leftrightarrow c = 1$, and since classical logic can only distinguish things up to classical equivalence, it cannot tell the difference between b and c in $\mathbf{N}_5'$.

To rule out such models, a Leibnizian principle of *identifying indiscernibles* may be employed, restricting the class of models of a given logic L to what we expect. And since what we expect (in the vast majority of cases) is the result of the algebraisation process, the next section will review it in some detail.

3. Algebraisability

This section is a concise primer of abstract algebraic logic, tailored for the dialectical purposes of the present paper. More detailed information on the subject can be found in the standard references for the area: [2,7,8]. The universal algebraic drudgery is the usual one, for which the reader can consult classical sources like [5].

3.1. *Logics and equational consequence*

Let ν be a similarity type, presented as a sequence of *connectives*, and let X be a set of variables. Furthermore, let $\mathbf{Fml}_L(X) = \langle Fml_L(X), \nu \rangle$ be the absolutely free algebra of type ν, freely generated by X. We disregard whatever differences there may be between logical and algebraic types, so that $Fml_L(X)$ can be alternatively viewed as a set of terms or as a set of well-formed formulae over a set of variables X, typically assumed to be countably infinite. The symbols $p, q, x, y...$ will be used as metavariables for members of X. Equations on ν, in this perspective, are nothing but ordered pairs of formulas, i.e. members of $(Fml_L(X))^2$. The traditional notation $\alpha \approx \beta$ will be used in place of the ordered pair notation $\langle \alpha, \beta \rangle$.

We will identify a logic L with a *deductive system*, i.e., a pair $\langle \mathbf{Fml}_L(X), \vdash_L \rangle$, where $\vdash_L \subseteq \wp(Fml_L(X)) \times Fml_L(X)$ is a *consequence relation* satisfying the following conditions:

(1) if $\alpha \in \Gamma$, then $\Gamma \vdash \alpha$
(2) if $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \alpha$
(3) if $\Gamma \vdash \alpha$ and $\Delta \vdash \gamma$ for every $\gamma \in \Gamma$, then $\Delta \vdash \alpha$
(4) if $\Gamma \vdash \alpha$, then $\Delta \vdash \alpha$ for some finite $\Delta \subseteq \Gamma$
(5) if $\Gamma \vdash \alpha$, then $\sigma(\Gamma) \vdash \sigma(\alpha)$ for any endomorphism σ of $\mathbf{Fml}_L$.

Notice that we dropped the subscript L from $\vdash_L$ in the process, and also the indication of the set of variables X from $\mathbf{Fml}_L(X)$. Whenever the logic L or the set X of variables are unimportant or clear from context, we will do it systematically. Since $\mathbf{Fml}_L$ is free, there is a natural bijection between its endomorphisms σ and maps $\sigma_0 \colon X \to Fml_L(X)$, namely, each σ is an extension of a σ_0 to a homomorphism, via

$$\sigma(\diamond (p_1, ..., p_n)) = \diamond (\sigma (p_1), ..., \sigma (p_n)),$$

for any connective $\diamond$ in ν. With no danger of confusion, both σ_0 and σ will be sometimes called *substitutions*. If Γ, Δ are sets of formulas, $\Gamma \vdash_L \Delta$ is short for: $\Gamma \vdash_L \alpha$ for all $\alpha \in \Delta$.

Sometimes, the term *consequence relation* is used in a wider sense: it is only required that the conditions (1)–(3) hold. Then, consequence relations satisfying (4) are called *finitary*, and those satisfying (5) *structural*. Every finitary and structural consequence relation is a consequence relation of some deductive system defined in a traditional way (cf. [12]), by axioms and inference rules. To see that it must be so, take as axioms all formulae α such that $\vdash \alpha$ (i.e., $\emptyset \vdash \alpha$) holds; then adopt as inference rules all pairs (Δ, α) such that Δ is finite and $\Delta \vdash \alpha$ holds.

A subset T of Fml_L that is closed under $\vdash$, i.e., $T \vdash \alpha$ implies $\alpha \in T$ for any $\alpha \in Fml_L$, is called a *theory* over L. From the structurality of $\vdash$ it follows that the set of theories is closed under inverse substitutions, that is, if T is a theory and σ a substitution, then $\sigma^{-1}(T)$ is also a theory. The smallest theory over L is the set $Th(L) = \{\alpha \in Fml_L \colon \vdash \alpha\}$, whose members are called *theorems* of L. Observe that $Th(L) = \bigcap\{T \colon T \text{ is a theory of } L\}$.

Now, let $\mathbf{A}$ be an algebra of the same type as $\mathbf{Fml}_L$, let $\alpha\,(p_1, ..., p_n)$ be a formula of the same type containing at most the indicated variables, and let $\overrightarrow{a} = a_1, ..., a_n$ be elements of A. By the freeness of $\mathbf{Fml}_L$, any homomorphism $h : \mathbf{Fml}_L \to \mathbf{A}$ is uniquely determined by its action on variables. This justifies the notation $\alpha^{\mathbf{A}}(\overrightarrow{a})$ to denote the element $h\,(\alpha)$, whenever h is such that $h\,(p_i) = a_i$, for all $i \leq n$. With this notation at hand, when $\mathcal{K}$ is a class of algebras, all of the same type as $\mathbf{Fml}_L$, we define the *equational consequence relation* of $\mathcal{K}$ as the relation $\vdash_{Eq(\mathcal{K})} \subseteq \wp((Fml_L)^2) \times (Fml_L)^2$ such that, for all sets of equations $E \subseteq (Fml_L)^2$ and for all equations $\alpha \approx \beta \in (Fml_L)^2$,

$$E \vdash_{Eq(\mathcal{K})} \alpha \approx \beta \text{ iff for every } \mathbf{A} \in \mathcal{K} \text{ and every } \overrightarrow{a} \in A^n, \\ \text{if } \gamma^{\mathbf{A}}(\overrightarrow{a}) = \delta^{\mathbf{A}}(\overrightarrow{a}) \text{ for all } \gamma \approx \delta \in E, \text{ then } \alpha^{\mathbf{A}}(\overrightarrow{a}) = \beta^{\mathbf{A}}(\overrightarrow{a}).$$

In the same vein, given $E \subseteq (Fml_L)^2$, $\mathbf{A}$ an algebra of the same type as $\mathbf{Fml}_L$, and $\overrightarrow{a} = a_1, ..., a_n$ elements of A, we will write $E^{\mathbf{A}}(\overrightarrow{a})$ to mean that for all $\gamma \approx \delta \in E$, $\gamma^{\mathbf{A}}(\overrightarrow{a}) = \delta^{\mathbf{A}}(\overrightarrow{a})$, omitting parentheses if E is a singleton[a]. Again, if E, E' are sets of equations, $E \vdash_{Eq(\mathcal{K})} E'$ means $E \vdash_{Eq(\mathcal{K})} \epsilon$ for all $\epsilon \in E'$.

3.2. *Algebraic semantics*

Let $L = \langle \mathbf{Fml}_L(X), \vdash_L \rangle$ be a logic in the type ν, and let $\tau = \{\gamma_i(p) \approx \delta_i(p)\}_{i \in I}$ be a set of equations in a single variable of ν. We may also view τ as a function which maps formulas in Fml_L to sets of equations of the same type. Thus, we let $\tau(\alpha)$ stand for the set

$$\{\gamma_i(p/\alpha) \approx \delta_i(p/\alpha)\}_{i \in I},$$

where $\gamma_i(p/\alpha)$ refers to the result of uniformly replacing any occurrences of p in γ_i by α, and similarly for $\delta_i(p/\alpha)$. For $\Gamma \subseteq Fml_L$, $\tau(\Gamma)$ is defined as $\bigcup\{\tau(\gamma) : \gamma \in \Gamma\}$.

[a]Consequence relations of deductive systems and equational consequence relations are both instances of a more general concept, devised by Blok and Jónsson [1], according to which the base set need not be a set of formulas or equations, but can be any old set. We will not enter into that.

Now, let $\mathcal{K}$ be a class of algebras also of the same type. We say that $\mathcal{K}$ is an *algebraic semantics* for L if, for some such τ, the following condition holds for all $\Gamma \cup \{\alpha\} \subseteq Fml_L$:

$$\Gamma \vdash_L \alpha \quad \text{iff} \quad \tau(\Gamma) \vdash_{Eq(\mathcal{K})} \tau(\alpha).$$

If $\mathcal{K}$ is an algebraic semantics for L via the mapping τ, we also say that L is the τ-*assertional logic* of $\mathcal{K}$ [3].

Here's the intuitive idea behind all this. Given an algebra $\mathbf{A} \in \mathcal{K}$, valuations on $\mathbf{A}$ — namely, homomorphisms $h : \mathbf{Fml}_L \rightarrow \mathbf{A}$, where $\mathbf{A} \in \mathcal{K}$ — are interpreted as "assignments of meanings" to elements of Fml_L, while elements of A can be seen as "meanings of propositions" or "truth values". The translation map τ equationally defines, for every $\mathbf{A} \in \mathcal{K}$, a "truth set" $T \subseteq A$ in the following sense: $\mathbf{A}$ satisfies $\tau(\alpha)$ just in case every valuation of α on $\mathbf{A}$ maps it to a member of T. Under such circumstances, intuitively speaking, the meaning of α in $\mathbf{A}$ belongs to the set of "true values" and hence is true. What is at stake here is an algebra-based concept of valid argument: α follows from the set of premises Γ just in case, whenever we assign a "true" value to all the premises in some algebra in $\mathcal{K}$, the conclusion α is also assigned a "true" value. In particular, if ν contains a nullary connective 1, it is possible to choose τ to be the singleton $\{p \approx 1\}$, which forces, for any $\mathbf{A} \in \mathcal{K}$, the truth set T to be the singleton $\left\{1^{\mathbf{A}}\right\}$. For example, the variety $\mathcal{BA}$ of Boolean algebras is an algebraic semantics for classical logic CL via $\tau = \{p \approx 1\}$; we also say (in this and in similar cases) that CL is the 1-*assertional logic* of $\mathcal{BA}$.

As natural as it may seem, the notion of algebraic semantics — which is, essentially, the brainchild of Lindenbaum and Tarski in the pioneering era of algebraic logic — has several shortcomings, hereafter reviewed. Metaphorically speaking, there are logics that cannot find an algebraic partner; moreover, the *ménage* between a logic and its algebraic semantics is an open couple relationship, and its partners do not enjoy equal rights.

- *There are logics with no algebraic semantics* [4]. An example is the deductive system $\langle \mathbf{Fml}_L, \vdash_{HI} \rangle$, where $\vdash_{HI}$ is the derivability relation of the Hilbert-style calculus whose sole axiom is $\alpha \rightarrow \alpha$ and whose sole inference rule is modus ponens.
- *The same logic can have more than one algebraic semantics* [4]. The concept of algebraic semantics depends on the chosen translation map, and different translations may lead to different places. By Glivenko's Theorem, for example, CL admits not only $\mathcal{BA}$ as an algebraic semantics, but also the variety $\mathcal{HA}$ of *Heyting algebras*, by choosing the nonstandard map $\tau = \{\neg\neg p \approx 1\}$.

- *There can be different logics with the same algebraic semantics.* Since Heyting algebras are an algebraic semantics for *intuitionistic logic* IL, the previous example shows that both CL and IL have $\mathcal{HA}$ as an algebraic semantics.
- *The relationship between a logic and its algebraic semantics can be asymmetric.* The property of belonging to the set of "true values" of an algebra $\mathbf{A} \in \mathcal{K}$ must be definable by means of the set of equations τ, whence the class $\mathcal{K}$ has the expressive resources to indicate when a given *formula* is valid in L. On the other hand, the logic L need not have the expressive resources to indicate when a given *equation* holds in $\mathcal{K}$.

The observation that such unwelcome states of affairs are all but infrequent spurred logicians to make the transition to a stronger concept, to be credited to Blok and Pigozzi [2]. This notion is the heart of the next subsection.

3.3. *Equivalent algebraic semantics*

Given an equation $\alpha \approx \beta$ and a set of formulas in two variables $\rho = \{\alpha_j(p,q)\}_{j\in J}$, we use the abbreviation

$$\rho(\alpha,\beta) = \{\alpha_j(p/\alpha, q/\beta)\}_{j\in J}.$$

ρ, in the same guise as τ, will be also regarded as a function, mapping this time equations to sets of formulas. Again, for $E \subseteq (Fml_L)^2$, $\rho(E)$ is defined as $\bigcup\{\rho(\alpha,\beta) : \alpha \approx \beta \in E\}$.

A logic $L = \langle \mathbf{Fml}_L(X), \vdash_L\rangle$ is said to be *algebraisable* with *equivalent algebraic semantics* $\mathcal{K}$ (where $\mathcal{K}$ is a class of algebras of the same type as $\mathbf{Fml}_L(X)$) iff there exist a map τ from formulas to sets of equations, and a map ρ from equations to sets of formulas such that the following conditions hold for any $\Gamma \cup \{\alpha,\beta\} \subseteq Fml_L$ and for all $E \subseteq (Fml_L)^2$:

AL1: $\Gamma \vdash_L \alpha$ iff $\tau(\Gamma) \vdash_{Eq(\mathcal{K})} \tau(\alpha)$;
AL2: $E \vdash_{Eq(\mathcal{K})} \alpha \approx \beta$ iff $\rho(E) \vdash_L \rho(\alpha,\beta)$;
AL3: $\alpha \dashv\vdash_L \rho(\tau(\alpha))$;
AL4: $\alpha \approx \beta \dashv\vdash_{Eq(\mathcal{K})} \tau(\rho(\alpha,\beta))$.

This definition can be drastically simplified:

Lemma 3.1. *A logic* L *is algebraisable with equivalent algebraic semantics* $\mathcal{K}$ *iff it satisfies either AL1 and AL4, or else AL2 and AL3.*

Clearly, the Blok-Pigozzi notion of equivalent algebraic semantics subsumes the Lindenbaum-Tarski notion of algebraic semantics. Every equivalent algebraic semantics for L is, in particular, an algebraic semantics for L in virtue of AL1. But the converse need not hold. Actually, the previous definition goes a long way towards meeting the criticisms of the previous subsection. First of all, it bans logical chauvinism: by AL2, logics must have the expressive resources to indicate which equations hold in the corresponding class of algebras. Second, it curbs (although it does not completely prohibit) promiscuity. If L is algebraisable with equivalent algebraic semantics $\mathcal{K}$, then $\mathcal{K}$ might still not be the unique equivalent algebraic semantics for L. A trivial example demonstrates this situation: both the variety $\mathcal{BA}$ of Boolean algebras and the singleton $\{\mathbf{2}\}$, where $\mathbf{2}$ is the 2-element Boolean algebra, are equivalent algebraic semantics for CL. However, we have the following:

Theorem 3.1. *[2, Thm. 2.15] In case L is finitary, any two equivalent algebraic semantics for L generate the same* quasivariety*.*

This quasivariety is in turn an equivalent algebraic semantics for the same logic, and so has every right to be called *the* equivalent quasivariety semantics for L.

On the other hand, it is possible to have different algebraisable logics with the same equivalent algebraic semantics (see e.g. [16] for an example of two different algebraisable logics whose equivalent variety semantics is the variety of Abelian ℓ-groups). None the less, if L and L' are algebraisable with equivalent quasivariety semantics $\mathcal{K}$ and with the *same set* of defining equations $\tau(p)$, then L and L' must coincide.

4. Algebraisability and quantum logic

Let us now focus on the issue alluded to in the very title of the present paper: the semantics of quantum logics, and in particular of the oldest and noblest member of this class: *orthomodular quantum logic* (OML). Before coming to the point, let us recall some relevant definitions.

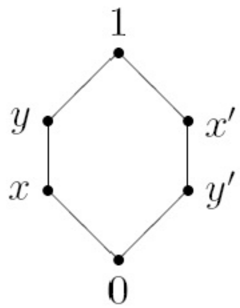
An *ortholattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, ', 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ s.t. the reduct $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice and the equations $x'' \approx x$, $x \wedge x' \approx 0$ and $x \vee x' \approx 1$ are satisfied. An ortholattice satisfying the identity

$$(x' \wedge (x \vee y)) \vee y' \vee (x \wedge y) \approx 1$$

is called a *weakly orthomodular lattice*, while an ortholattice satisfying the identity

$$x \vee y \approx ((x \vee y) \wedge y') \vee y$$

is an *orthomodular lattice*. As the names themselves suggest, every orthomodular lattice is weakly orthomodular, but the converse fails to hold — a counterexample is the lattice **O6** depicted below:



We make a note of a particular formula of type $\langle 2, 2, 1, 0, 0 \rangle$, of primary importance for the following considerations: the *Sasaki hook*, defined as

$$x \to y = x' \vee (x \wedge y) .$$

Let us now revert to OML. Here, the association between logic and algebra is somewhat obvious: actually, as it happened for many other nonclassical logics, in this case semantics predated syntax. Upon noting that the mathematical representatives of quantum events, viz. closed subspaces in a complex separable Hilbert space, formed a possibly nondistributive orthomodular lattice, Birkhoff and von Neumann *defined* OML in such a way as to obtain a match with the variety of orthomodular lattices (see e.g. [6]). Using terminology that was not available to them but that was briefly recalled above, Birkhoff and von Neumann introduced OML as the 1-assertional logic of the variety $\mathcal{OML}$ of orthomodular lattices, i.e. as the logic whose consequence relation is given by:

$$\Gamma \vdash_{OML} \alpha \text{ iff } \{\gamma \approx 1 : \gamma \in \Gamma\} \vdash_{Eq(\mathcal{OML})} \alpha \approx 1,$$

which, as we know, *ipso facto* implies that $\mathcal{OML}$ is an algebraic semantics for OML with set of defining equations $\tau(p) = \{p \approx 1\}$. The subsequent Hilbert-style, or Gentzen-style, axiomatisations of OML (see e.g. [10] or [13]) provided traditional, syntactically defined deductive systems that were sound and complete with respect to orthomodular lattices.

Actually, more than that is true: $\mathcal{OML}$ is an *equivalent algebraic semantics* for OML, as the next theorem shows.

Theorem 4.1. *OML is algebraisable, with set of equivalence formulae*

$$\rho(\alpha, \beta) = \{(\alpha \to \beta) \wedge (\beta \to \alpha)\},$$

and set of defining equations $\tau(p) = \{p \approx 1\}$. Its equivalent quasivariety semantics is the variety $\mathcal{OML}$.

Proof. We already observed that OML is strongly sound and complete w.r.t. $\mathcal{OML}$. This settles Condition (AL1) in Lemma 3.1. To round off our proof, it is sufficient to show that Condition (AL4) is also met, namely, that for an arbitrary equation $\alpha \approx \beta$

$$\alpha \approx \beta \dashv\models_{\mathcal{OML}} (\alpha \to \beta) \wedge (\beta \to \alpha) \approx 1.$$

However, it is known that the quasiequation $(\alpha \to \beta) \wedge (\beta \to \alpha) \approx 1 \Rightarrow \alpha \approx \beta$ holds in $\mathcal{OML}$, while the converse implication is obvious. □

In the light of Theorem 3.1, no other class of algebras which is at least a quasivariety can claim to play the same rôle: the unique position of $\mathcal{OML}$ as *the* algebraic match of orthomodular quantum logic is secure. Although the adequacy of OML for quantum mechanics can be, and has been, disputed — for one, the class of all lattices of closed subspaces of Hilbert spaces validates identities that are not universally satisfied throughout $\mathcal{OML}$ [9] — the least we can say is that, whether or not OML adequately represent a proper formalisation of the intrinsic logic of quantum mechanics, its algebraic counterpart can't be anything else than the variety of orthomodular lattices.

Surprisingly enough, however, this claim has been questioned by Pavičic and Megill in a series of papers [14,15]. They consider a Hilbert-style system for OML, taken from [11], and observe that this system is sound and complete w.r.t. a supervariety of $\mathcal{OML}$, the variety $\mathcal{WOML}$ of *weakly* orthomodular lattices. Furthermore, they observe a few additional facts:

- orthomodular lattices are exactly those members of $\mathcal{WOML}$ which satisfy the quasiequation $(\alpha \to \beta) \wedge (\beta \to \alpha) \approx 1 \Rightarrow \alpha \approx \beta$;
- $\mathcal{WOML}$ and $\mathcal{OML}$ have the same quasiequational theory, if we restrict ourselves to equations of the form $\alpha \approx 1$;
- The soundness and completeness proof also works if we replace $\mathcal{WOML}$ with the single algebra **O6**.

Having remarked the above, they venture on such bold claims as:

Deductive quantum logic is not orthomodular [15, p. 26].
The proofs of completeness [of orthomodular quantum logic and classical logic] introduce hidden axioms of orthomodularity and distributivity in the respective Lindenbaum algebras of the logic [15, p. 27].
In the syntactical structure of quantum logic there is nothing orthomodular. The orthomodularity appears through the definition of the equivalence relation [14, p. 14 of the preprint version].

If we grant this, a puzzling conclusion follows: orthomodularity does not lie in the structure of quantum logic, as all of us were taught, but “in the eye of the beholder”. Appealing to the framework we presented in the previous sections, however, it is now comparatively easy to detect the flaws in the argument by Pavičic and Megill. What they manage to show, indeed, is that deductive quantum logic (as they call it) has the variety of weakly orthomodular lattices as an *algebraic* semantics, period. They do not show that this variety is an *equivalent* algebraic semantics for the same logic — and of course they cannot do so, because there is a unique quasivariety with this property, and such a slot is already filled (as we have seen) by $\mathcal{OML}$. But there is a further instructive lesson to be drawn from Pavičic's and Megill's proof. $\mathcal{WOML}$ falls short of the requirements needed for an equivalent algebraic semantics for OML precisely because it includes algebras (like **O6**) containing as members distinct but OML-indiscernible elements. And this, in turns, happens because $\mathcal{WOML}$ does not universally satisfy the quasiequation

$$(\alpha \to \beta) \wedge (\beta \to \alpha) \approx 1 \Rightarrow \alpha \approx \beta,$$

which, true to form, characterises orthomodular lattices within the class of all ortholattices (as Pavičic and Megill correctly observe). Therefore it is not true that $\Sigma \dashv\models_{\mathcal{WOML}} \tau(\rho(\Sigma))$, for an arbitrary set of equations Σ: Pavičic's and Megill's semantics is not a good semantics in the sense we specified above.

Two additional remarks are in order, lest the issue be clouded by irrelevant details that could trigger unfounded counter-objections. First, let us assess more thoroughly the claim by Pavičic and Megill to the effect that ”in the syntactical structure of quantum logic there is nothing orthomodular” and ”the orthomodularity appears through the definition of the equivalence relation”. These authors maintain that orthomodularity is kind of smuggled into the standard completeness proof for orthomodular logic by defining the Lindenbaum congruence as

$$(\alpha, \beta) \in \Omega \text{ iff } \vdash_{OML} (\alpha \to \beta) \wedge (\beta \to \alpha).$$

According to them, if we modify this definition as follows:

$$\begin{aligned}(\alpha, \beta) \in \Omega \text{ iff } & \vdash_{OML} (\alpha \to \beta) \wedge (\beta \to \alpha) \\ & \text{and } v(\alpha) = v(\beta) \text{ for all homomorphisms of } \mathbf{Fml} \text{ to } \mathbf{O6},\end{aligned}$$

we get a soundness and completeness proof w.r.t. $\mathcal{WOML}$ and our models need no longer be orthomodular. That figures. The suggested refinement of the Lindenbaum congruence, in fact, has a twofold effect. First, it allows OML-indiscernible but distinct elements to sneak into the models for our logic. Second, it renders the new congruence undefinable by means of the symmetrisation of the Sasaki hook — or, indeed, by any other set of equivalence connectives, on pain of contradicting Theorems 3.1 and 4.1. The results surveyed above explain away all the doubts that have been cast on the correspondence between orthomodular quantum logic and orthomodular lattices.

In the second place, one should not be misled by our semantic definition of OML. Although this semantic approach was historically prior to any syntactic axiomatisation, if we proceeded as Pavičic and Megill do and considered, say, their Hilbert-style system H for OML defining

$$\Gamma \vdash^{*}_{OML} \alpha \quad \text{iff there is a finite sequence of formulas ending with } \alpha, \text{ each of which is either an axiom of } H \text{ or is obtained from its predecessors by a rule of } H,$$

it would have been an easy exercise to prove that $\Gamma \vdash_{OML} \alpha$ iff $\Gamma \vdash^{*}_{OML} \alpha$. This is, indeed, what these authors refer to when they mention the soundness and completeness theorem of OML.

To conclude, the allegations to be found in [14,15] rest on confusions that can be conveniently cleared up with the help of some tools from contemporary abstract algebraic logic. Although these misunderstandings may seem evident to a practicioner of the field, we thought it important to point them out in some detail for the benefit of the wider quantum structures community, whose members may not be wholly familiar with these concepts and methods. We hope that, by so doing, we will prevent once and for all such incorrect claims from spreading any further[b].

[b]We thank Matthew Spinks for helpful discussions on the themes of this paper.

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VON NEUMANN, EMPIRICISM AND THE FOUNDATIONS OF QUANTUM MECHANICS

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> The system of logics which one uses should be derived from aggregate experiences relative to the main application which one wishes to make — logics should be inspired by experience.
>
> John von Neumann

1. Introduction

The issue of the application of mathematics to natural science has received a great deal of consideration in the last few years (see, for instance, Field [1980] and [1989], Hellman [1989], Steiner [1989] and [1998], and Shapiro [1997]). In this paper, I am concerned with two tasks: (1) to provide a formal framework to accommodate one important aspect of this issue — the heuristic use of mathematics in theory construction — and (2) to illustrate how this framework works by focusing on a particular case-study: von Neumann's use of logic and set theory in his work in the foundations of quantum mechanics. The reason to consider von Neumann's work derives from the interplay of mathematical, logical and heuristic considerations that he wove together in the process of providing an axiomatization for quantum mechanics and exploring some of its consequences. Moreover, his work provides important insights into the nature of theory construction in physics.

As we shall see below, we cannot separate von Neumann's attitude towards mathematics from his attitude towards logic. He was searching for a unified approach to physics, where logic, mathematics and probability nicely hang together. This approach naturally raises the issue of the cognitive status of logic and mathematics in von Neumann's view. In addressing this issue, what I want to establish is that there are three major empiricist trends in von Neumann's work: (i) von Neumann was an empiricist in his view of mathematics (in the sense that mathematical theories are often created from empirical demands); (ii) von Neumann was also an empiricist with regard to logic (in the sense that logic should be inspired and modified by experience; moreover, in his view, there are as many logics as physical phenomena demand). Finally, (iii) in developing his approach to the foundations of quantum mechanics, von Neumann countenanced an empiricist version of the semantic approach (stressing the empirical adequacy of scientific theories, rather than their truth). In this way, we have a new understanding of von Neumann's accomplishments, and we can determine the cognitive status he assigned to them.

But to represent the moves made by von Neumann in the articulation of his approach, we need a particular formal framework. One of the contentions of the present work is that the framework developed by da Costa and French — the *partial structures approach* — is helpful here. This framework has two major components: an open-ended notion of structure (partial structure) and a weak concept of truth (quasi-truth). Both notions received a formal characterization in a number of interesting papers by da Costa (see da Costa [1986], and Mikenberg, da Costa and Chuaqui [1986]), and in later works, da Costa and French have provided extensive applications of them to various problems in the philosophy of science (see da Costa and French [1989], [1990], [1993a], [1993b], and [2003]). In my view, this framework helps us to understand von Neumann's work in the foundations of physics; so let me first say a few things about it.[a]

[a]The formal framework presented below is part of a broader program whose main task is to articulate a constructive empiricist philosophy of mathematics (see Bueno [2009]), thus extending van Fraassen's proposal to this domain (see van Fraassen [1980], [1985], [1989], and [1991]). Although I have no plans to elaborate on this program here, let me just make one point. As is well known, the aim of science for the constructive empiricist is not truth, but something weaker — empirical adequacy (van Fraassen [1980], p.12). Similarly, the aim of mathematics in the present proposal is accordingly weaker. It is not truth, but *quasi-truth* (roughly speaking, truth with regard to a limited domain).

2. Partial Structures and Quasi-Truth

The partial structures approach (see Mikenberg *et al.* [1986], and da Costa and French [1989], [1990], and [2003]) relies on three main notions: partial relation, partial structure and quasi-truth. One of the main motivations for introducing this proposal comes from the need for supplying a formal framework in which the "openness" and "incompleteness" of information dealt with in scientific practice can be accommodated. This is accomplished firstly by extending the usual notion of structure, in order to model the partialness of information we have about a certain domain (introducing then the notion of a partial structure). Secondly, the Tarskian characterization of the concept of truth is generalized for such "partial" contexts, advancing the corresponding concept of quasi-truth.

The first step, that paves the way to introduce partial structures, is to formulate an appropriate notion of partial relation. When investigating a certain domain of knowledge Δ (say, the physics of particles), we formulate a conceptual framework which helps us in systematizing the information we obtain about Δ. This domain is represented by a set D of objects (which includes *real* objects, such as configurations in a Wilson chamber and spectral lines, and *ideal* objects, such as quarks). D is studied by the examination of the relations holding among its elements. However, it often happens that, given a relation R defined over D, we do not know whether all the objects of D (or n-tuples thereof) are related by R. This is part and parcel of the "incompleteness" of our information about, and is formally accommodated by the concept of partial relation. The latter can be characterized as follows. Let D be a non-empty set. An n-place *partial relation* R over D is a triple $\langle R_1, R_2, R_3\rangle$, where R_1, R_2, and R_3 are mutually disjoint sets, with $R_1 \cup R_2 \cup R_3 = D^n$, and such that: R_1 is the set of n-tuples that (we know that) belong to R, R_2 is the set of n-tuples that (we know that) do not belong to R, and R_3 is the set of n-tuples for which it is not known whether they belong or not to R. (Note that if R_3 is empty, R is a usual n-place relation which can be identified with R_1.)

However, in order to represent appropriately the information about the domain under consideration, we need a notion of structure. The following characterization, spelled out in terms of partial relations and based on the standard concept of structure, is meant to supply a notion which is broad enough to accommodate the partiality usually found in scientific practice.

A *partial structure* A is an ordered pair $\langle D, R_i \rangle_{i \in I}$, where D is a non-empty set, and $(R_i)_{i \in I}$ is a family of partial relations defined over D.[b]

We have now defined two of the three basic notions of the partial structures approach. In order to spell out the last one (quasi-truth), we will need an auxiliary notion. The idea is to use the resources supplied by Tarski's definition of truth. But since the latter is only defined for full structures, we have to introduce an intermediary notion of structure to link it to the partial ones. This is the first role of those structures which extend a partial structure A into a full, total structure (which are called A-normal structures). Their second role is model-theoretic, namely to put forward an interpretation of a given language and to characterize semantic notions. Let $A = \langle D, R_i \rangle_{i \in I}$ be a partial structure. We say that the structure $B = \langle D', R'_i \rangle_{i \in I}$ is an *A-normal structure* if (i) $D = D'$, (ii) every constant of the language in question is interpreted by the same object both in A and in B, and (iii) R'_i extends the corresponding relation R_i (in the sense that, each R'_i, supposed to be an n-place relation, is not necessarily defined for all n-tuples of elements of D'). Note that, although each R_i is *defined* for all n-tuples over D', it holds for some of them (the R'_{i1}-component of R'_i), and it doesn't hold for others (the R'_{i2}-component).

As a result, given a partial structure A, there are several A-normal structures. Suppose that, for a given n-place partial relation R_i, we don't know whether $R_i a_1 \ldots a_n$ holds or not. One way of extending R_i into a full R'_i relation is to look for information to establish that it *does hold*, another way is to look for the contrary information. Both are *prima facie* possible ways of extending the partiality of R_i. But the same indeterminacy may be found with other objects of the domain, distinct from $a_1, \ldots, a_n$ (for instance, does $R_i b_1 \ldots b_n$ hold?), and with other relations distinct from R_i (for example, is $R_j b_1 \ldots b_n$ the case, with $j \neq i$?). In this sense, there are *too many* possible extensions of the partial relations that constitute A. Therefore, we need to provide constraints to restrict the acceptable extensions of A.

In order to do that, we need first to formulate a further auxiliary notion (see Mikenberg, da Costa and Chuaqui [1986]). A *pragmatic structure* is a partial structure to which a third component has been added: a set of accepted sentences P, which represents the accepted information about the structure's domain. (Depending on the interpretation of science which is

[b]The partiality of partial relations and structures is due to the "incompleteness" of our knowledge about the domain under investigation — with further information, a partial relation may become total. Thus, the partialness modeled here is not "ontological", but "epistemic".

adopted, different kinds of sentences are to be introduced in P: realists will typically include laws and theories, whereas empiricists will add mainly certain laws and observational statements about the domain in question.) A *pragmatic structure* is then a triple $A = \langle D, R_i, P\rangle_{i\in I}$, where D is a non-empty set, $(R_i)_{i\in I}$ is a family of partial relations defined over D, and P is a set of accepted sentences. The idea is that P introduces constraints on the ways that a partial structure can be extended (the sentences of P hold in the A-normal extensions of the partial structure A).

Our problem is, given a pragmatic structure A, what are the necessary and sufficient conditions for the existence of A-normal structures? Here is one of these conditions (Mikenberg *et al.* [1986]). Let $A = \langle D, R_i, P\rangle_{i\in I}$ be a *pragmatic* structure. For each partial relation R_i, we construct a set M_i of atomic sentences and negations of atomic sentences, such that the former correspond to the n-tuples which satisfy R_i, and the latter to those n-tuples which do not satisfy R_i. Let M be $\bigcup_{i\in I} M_i$. Therefore, a pragmatic structure A admits an A-normal structure if and only if the set $M \cup P$ is *consistent*.

Assuming that such conditions are met, we can now formulate the concept of quasi-truth. A sentence α is *quasi-true* in a pragmatic structure $A = \langle D, R_i, P\rangle_{i\in I}$ if there is an A-normal structure $B = \langle D', R'_i\rangle_{i\in I}$ such that α is true in B (in the Tarskian sense). If α is not quasi-true in A, we say that α is *quasi-false* in A. Moreover, we say that a sentence α is *quasi-true* if there is a pragmatic structure A and a corresponding A-normal structure B such that α is true in B (according to Tarski's account). Otherwise, α is *quasi-false*.

The idea, intuitively speaking, is that a quasi-true sentence α does not necessarily describe, in an appropriate way, the whole domain to which it refers, but only an aspect of it — the one modeled by the relevant partial structure A. For there are several different ways in which A can be extended to a full structure, and in some of these extensions α may not be true. Thus, the notion of quasi-truth is strictly weaker than truth: although every true sentence is (trivially) quasi-true, a quasi-true sentence is not necessarily true (since it may be false in certain extensions of A).[c]

It may be argued that because quasi-truth has been defined in terms of full structures and the standard notion of truth, there is no gain with

[c]The notions of full structure, partial structure and quasi-truth can all be characterized in second-order logic. Hence, using Boolos's [1985] plural interpretation of second-order quantifiers, they are nominalistically acceptable, assuming their first-order components also are. For alternative nominalization strategies, see Bueno [2009] and [2013].

its introduction. In my view, there are several reasons why this is *not* the case. Firstly, as we have just seen, despite the use of full structures, quasi-truth is weaker than truth: a sentence which is quasi-true in a particular domain — that is, with respect to a given partial structure A — may not be true if considered in an extended domain. Thus, we have here a sort of "underdetermination" — involving distinct ways of extending the same partial structure — that makes the notion of quasi-truth especially appropriate for the empiricist. Secondly, one of the points of introducing the notion of quasi-truth, as da Costa and French ([1989] and [1990]) have argued in detail, is that in terms of this notion, a formal framework can be advanced to accommodate the "openness" and "partialness" typically found in science. Bluntly put, the actual information at our disposal about a certain domain is modeled by a *partial* (but not full) structure A. Full, A-normal structures represent ways of extending the actual information which are possible according to A. In this respect, the use of full structures is a semantic expedient of the framework (in order to provide a definition of quasi-truth), but no epistemic import is assigned to them. Thirdly, it is possible to dispense with full structures in the formulation of quasi-truth, since the latter can be characterized in a different way, but still preserving all its features, independently of the standard Tarskian type account of truth (Bueno and de Souza [1996]). This provides, of course, the strongest argument for the dispensability of full structures (as well as of the Tarskian account) *vis--vis* quasi-truth. Therefore, full, A-normal structures are entirely inessential; their use here is only a convenient device.

To illustrate the use of quasi-truth, let us consider an example. As is well known, Newtonian mechanics is appropriate to explain the behavior of bodies under certain conditions (say, bodies which, roughly speaking, have "low" velocity, are not subject to strong gravitational fields, etc.). But with the formulation of special relativity, we know that if these conditions are not satisfied, Newtonian mechanics is false. In this sense, these conditions specify a family of partial relations, which delimit the context in which the theory holds. Although Newtonian mechanics is not true (and we know under what conditions it is false), it is *quasi-true*; that is, it is true in a given context, determined by a pragmatic structure and a corresponding A-normal one (see da Costa and French [1993a]).[d]

[d]Can the partial structures approach help the constructive empiricist to develop a philosophy of mathematics? I think it can. As I mentioned above, the main idea is that mathematics doesn't have to be true to be good, but only quasi-true; that is, true with regard to a *possible* domain — the one delimited by certain mathematical structures. It

3. Partial Isomorphism and Partial Homomorphism

In empirical research, we often lack complete information about the domain of inquiry. We know that certain relations definitely hold for objects in the domain, and others clearly don't; but for a number of relations we simply don't know (given our current information) whether they hold or not. Typically, we have at best partial structures to represent our information about the domain under investigation (and this includes not only empirical structures, but also theoretical ones). As a result, partial structures help us to accommodate the partiality of our information (see French [1997]).

But what is the *relationship* between the various partial structures articulated in a particular domain? Since we are dealing with partial structures, a second-level of partiality emerges: we can only establish partial relationships between the (*partial*) structures at our disposal. This means that the usual requirement of introducing an isomorphism between theoretical and empirical structures can hardly be met. Relationships weaker than full isomorphisms, full homomorphisms etc. have to be introduced, otherwise scientific practice — where partiality of information appears to be ubiquitous — cannot be properly accommodated (for details, see French [1997], French and Ladyman [1997], and Bueno [1997]).

Following a related move made elsewhere (see French and Ladyman [1999], Bueno [1997], French and Ladyman [1997], and Bueno, French and Ladyman [2002]), it is possible to characterize, in terms of the partial structures approach, appropriate notions of *partial isomorphism* and *partial homomorphism*. And because these notions are more open-ended than the standard ones, they accommodate better the partiality of structures found in science. Here they are:

Let $S = \langle D, R_i \rangle_{i \in I}$ and $S = \langle D', R'_i \rangle_{i \in I}$ be partial structures. So, each R_i is of the form $\langle R_1, R_2, R_3 \rangle$, and each $R'i$ of the form $\langle R'_1, R'_2, R'_3 \rangle$.

We say that a partial function $f : D \mapsto D'$ is a partial isomorphism between S and S' if (i) f is bijective, and (ii) for every x and $y \in D$, $R_1xy \leftrightarrow R'_1 f(x)f(y)$ and $R_2xy \leftrightarrow R'_2 f(x)f(y)$. So, when R_3 and R'_3 are empty (that is, when we are considering total structures), we have the standard notion of isomorphism.

turns out that quasi-truth also extends van Fraassen's own account of empirical adequacy, which is ultimately truth with regard to a limited domain — that of the observable (for details, see Bueno [1997]). Thus, a unified empiricist view (encompassing mathematics and empirical science) can be articulated using the partial structures approach.

Moreover, we say that a partial function $f : D \mapsto D'$ is a *partial homomorphism* from S to S' if for every x and every y in D, $R_1xy \rightarrow R'_1f(x)f(y)$ and $R_2xy \rightarrow R'_2f(x)f(y)$. Again, if R_3 and R'_3 are empty, we obtain the standard notion of homomorphism as a particular case.[e]

Using these notions, we can provide a framework for accommodating the application of mathematics to theory construction in science. The main idea, roughly speaking, is to bring structure from mathematics to an empirical domain. In other words, mathematics is applied by "bringing structure" from a mathematical domain (say, functional analysis) into a physical, but mathematized, domain (such as quantum mechanics). What we have, thus, is a structural perspective, which involves the establishment of relations between structures in different domains. (Note that, at the level we are considering here, the physics is already mathematized.) But, as Redhead [1975] has nicely spelled out, there is typically "surplus" structure at the mathematical level (see also French [1999]). And given this "surplus", only *some* structure is brought from mathematics to physics; in particular those relations which help us to find counterparts, at the empirical domain, of relations that hold at the mathematical domain. In this way, by "transferring structures" from a mathematical to a physical domain, empirical problems can be better represented and examined. This assumes, of course, that we have information about the relations between the two domains, although it may well not be complete. Only the information we know to hold is used; but there are other relations at the mathematical level that may not have any counterpart at the empirical level. (This will become clear in the discussion of von Neumann's work below.)

It is straightforward to accommodate this situation using partial structures. The partial homomorphism represents the situation in which only some structure is brought from mathematics to physics (via the R_1- and R_2-components, which represent our current information about the relevant domain), although "more structure" could be found at the mathematical domain (via the R_3-component, which is left "open"). Moreover, given the partiality of information, just *part* of the mathematical structures is preserved, namely that part about which we have enough information to match the empirical domain.

By bringing structure from one domain into another, the application of mathematics has a heuristic role in theory construction: it suggests a way

[e]Note that the notions of partial isomorphism and partial homomorphism can also be formulated in second-order logic. Hence, given Boolos's [1985] plural quantifier, they are acceptable by the nominalist.

of searching for relations at the empirical domain, given their counterparts at the mathematical level. However, since mathematical theories don't have to be true to be applicable, but only quasi-true, the empiricist is allowed to use these theories without incurring into unacceptable ontological commitments.[f]

Of course, far more could be said about these issues. But here I am only concerned with presenting the overall formal framework assumed by the present proposal (in terms of partial structures, quasi-truth, and partial isomorphisms and homomorphisms). However, I hope enough has been said to indicate that this framework can be taken by the empiricist to provide an account of the role of structure in the application of mathematics (see Bueno [2009] and [2013]). To illustrate these points, we shall see now how this framework can be used to consider von Neumann's application of mathematics to physics.

4. Von Neumann, Logic and the Application of Mathematics

4.1. *The 1925-1926 mess*

Von Neumann's work in the foundations of quantum mechanics emerged in a context where there was a demand for a proper mathematical formulation of the theory. The demand arouse out of what can be called the "1925-1926 mess". In these years, two entirely distinct formulations of quantum mechanics were devised: on the one hand, Heisenberg, Born, and Jordan formulated, in a series of papers in 1925, the so-called matrix mechanics; on the other hand, in 1926, also in a series of works, Schrödinger articulated wave mechanics.[g] The two formulations couldn't be more different. Matrix mechanics is expressed in terms of a system of matrices defined by algebraic equations, and the underlying space is discrete. Wave mechanics is articulated in a continuous space, which is used to describe a field-like process in a configuration space governed by a single differential equation. However, despite these differences, the two theories seemed to have the same empirical consequences. For example, they gave coinciding energy values for the hydrogen atom.

[f]Of course, in order to support this claim, the empiricist needs a nominalization strategy for mathematics. This can be done using the partial structures approach and adapting Hellman's modal-structural interpretation, or by invoking ontologically neutral quantifiers (see Hellman [1989], Azzouni [2004], Bueno [2009] and [2013]).
[g]For a detailed critical discussion, and references, see Muller [1997].

Schrödinger's explanation for this was to claim that the two theories were *equivalent*, and this was the main point of one of his papers of 1926. In the opening paragraph of this work (Schrödinger [1926]), in which he tried to establish the equivalence, Schrödinger notes:

> Considering the extraordinary differences between the starting-points and the concepts of Heisenberg's quantum mechanics [matrix mechanics] and of the theory which has been designated "ondulatory" or "physical" mechanics [wave mechanics] [...] it is very strange that these two new theories *agree with one another* with regard to the known facts where they differ from the old quantum theory. That is really very remarkable because starting-points, presentations, methods and in fact the whole mathematical apparatus, seem fundamentally different. Above all, however, the departure from classical mechanics in the two theories seems to occur in diametrically opposed directions. In Heisenberg's work the classical continuous variables are replaced by systems of *discrete* numerical quantities (matrices), which depend on a pair of integral indices, and are defined by *algebraic* equations. The authors themselves describe the theory as a "true theory of a discontinuum". On the other hand, wave mechanics shows just the reverse tendency; it is a step from classical point mechanics towards a *continuum-theory*. In place of a process described in terms of a finite number of dependent variables occurring in a finite number of differential equations, we have a continuous *field-like* process in configuration space, which is governed by a *single partial differential* equation, derived from a Principle of [Least] Action. (Schrödinger [1926], pp. 45-46; for a discussion, see Muller [1997], pp. 49-58)

According to Schrödinger, despite the conceptual and methodological differences, the two theories yielded the same results because they were equivalent. His strategy to prove the equivalence was clear enough: to establish an isomorphism between the canonical matrix- and wave-operator algebras. This was, of course, a straightforward strategy.

The problem, as Muller indicates ([1997], pp. 52-53), is that Schrödinger only established a mapping that assigns one matrix to each wave-operator, but he didn't establish the converse. More surprisingly, Schrödinger himself acknowledges this point. In a footnote to his paper, he remarks:

> In passing it may be noted that the converse of this theorem is also true, at least in the sense that certainly *not more* than *one*

> linear differential operator [wave-operator] can belong to a given *matrix*. [...] [However] we have not proved that a linear operator [wave-operator], corresponding to an arbitrary matrix, *always exists*. (Schrödinger [1926], p. 52)

In other words, the equivalence between matrix and wave mechanics hasn't been proved after all. The claim that Schrödinger established the result in 1926 is therefore a "myth" (Muller [1997]).

Given the importance of the two theories, and since they were thought of as having the same empirical consequences, it comes as no surprise that to establish the equivalence between them was taken as a substantial achievement. There were several attempts to do so. Dirac, for instance, provided a distinctive approach (see Dirac [1930]). As is well known, his main idea was to put each self-adjoint operator in diagonal form. And this approach in fact succeeds in putting matrix and wave mechanics in a uniform setting. But it faces an unexpected problem: it is inconsistent! In the case of those operators that cannot be put in diagonal form, Dirac's method requires the introduction of "improper" functions with self-contradictory properties (the so-called $\delta-$functions). However, from the viewpoint of classical mathematics, there are no such functions, since they require that a differential operator is also an integral operator; but this condition cannot be met.

4.2. *Von Neumann's equivalence proof*

Given the failure of these two attempts to establish the equivalence between matrix and wave mechanics, an entirely new approach was required to settle the issue. This was one of von Neumann's achievements in his [1932] book on the mathematical foundations of quantum mechanics. But von Neumann's involvement with foundational issues in quantum mechanics started earlier. In a paper written in 1927 with Hilbert and Nordheim, the problem of finding an appropriate way of introducing probability into quantum mechanics had been explicitly addressed (see Hilbert, Nordheim and von Neumann [1927]). The approach was articulated in terms of the notion of the amplitude of the density for relative probability (for a discussion, see Rédei [1997]). But it faced a serious technical difficulty (which was acknowledged by the authors): the assumption was made that every operator is an integral operator, and therefore Dirac's problematic function had to be assumed. As a result, an entirely distinct account was required to adequately introduce probability in quantum mechanics. And this led to von Neumann's 1932 work, using Hilbert spaces.

Von Neumann's book is the development of three papers that he wrote in 1927 (for a discussion and references, see Rédei [1997] and [1998]). It is distinctive not only for its clarity, but also for the fact that there is no use of Dirac's δ-function. Probabilities are introduced via convenient trace functions, and relevant operators (projection operators) are defined on Hilbert spaces. So, von Neumann is able to claim that there is indeed a way of introducing probabilities in quantum mechanics without inconsistency (see von Neumann [1932]).

What about the equivalence between matrix and wave mechanics? In von Neumann's hand, the problem becomes an issue of structural similarity. Bluntly put, von Neumann realized that there was a *similarity of structure* between the theory of Hilbert spaces and quantum mechanics, and that we could adequately *represent* claims about quantum systems by exploring the geometry of Hilbert spaces. But how can we accommodate this intuition? That is, how can we accommodate the *structure similarity* that von Neumann found between quantum mechanics and part of functional analysis?[h]

In a nutshell, structure similarity can be accommodated via an appropriate *morphism* between the structures under consideration; that is, by a transformation which preserves the relevant (features of those) structures. Of course, the strongest form of morphism is *isomorphism* — the full preservation of structure. However, the use of this notion would be inappropriate in this case, for the following reason. By 1927, quantum mechanics could be seen as a semi-coherent assemblage of principles and rules for applications. Weyl's 1931 book was an attempt to impose a degree of coherence via the introduction of group-theoretic techniques (the uncertainty principle, for example, could be obtained via group theory). Dirac's 1930 work represents a further attempt to lay out a coherent basis for the theory. However, as von Neumann perceived, neither of these offered a mathematical framework congenial for the introduction of probability at the most fundamental level, and this was one of the major motivations for the introduction of Hilbert spaces. (There is, of course, a great deal more to be said about this, but here is not the place to do so.) Given this, there is no way of spelling out the similarity of structure between quantum mechanics (as it was at the end of 1920's and the beginning of 1930's) and Hilbert spaces in terms of the existence of *full isomorphisms*. Moreover, since there is *more structure* in functional analysis than was actually used by von Neumann

[h]In fact, it was due to this structure similarity that claims about quantum systems could be *represented* in terms of Hilbert spaces.

in his axiomatization of quantum mechanics, we can say that the relationship between those "quantum" and mathematical structures is captured by appropriate *partial homomorphisms*. The partiality involved in the formulation of quantum mechanics is then captured by the partial nature of the homomorphism — only those components of quantum mechanics about which we have enough information are "preserved".

An important feature of von Neumann's axiomatization was his systematic search for *analogies* between mathematical structures and between the latter and "physical" structures (that is, structures employed in the description of physical phenomena). These analogies played an important role in von Neumann's equivalence proof of matrix and wave mechanics. What von Neumann established is a mathematical relation between two systems of *functions*. And, in this case, the relation is a full isomorphism. On the one hand, we have *functions* — defined on the "discrete" space of index values $Z = (1, 2, \dots)$ — which are sequences $x_1, x_2, \dots$, and are used in the formulation of matrix mechanics; on the other hand, we have *functions* defined on the continuous state-space Ω of a mechanical system (Ω is a k-dimensional space, with k being the number of classical mechanical degrees of freedom), and Ω's functions are *wave* functions $\varphi(q_1, \dots, q_k)$. Von Neumann explicitly points out that the spaces Z and Ω are quite different ([1932], p. 28). It is not surprising then that Dirac's method of unification faced so many mathematical difficulties, since this method assumed a *direct analogy* between Z and Ω (see von Neumann [1932], pp. 17-27).

> The method [...] resulted in an analogy between the "discrete" space of index values Z [...] and the continuous state space Ω [...]. That this cannot be achieved without some violence to the formalism and to mathematics is not surprising. The spaces Z and Ω are in reality very different, and every attempt to relate the two must run into great difficulties. (von Neumann [1932], p. 28)

However, the correct analogy is not between the spaces Z and Ω, but between the *functions defined on them*. Von Neumann calls the totality of functions on Z, satisfying certain conditions, F_Z, and the totality of function on Ω, also satisfying certain conditions, F_Ω ([1932], pp. 28-29). Once this point is clearly seen, von Neumann presents his equivalence proof. But note that what he proved is that F_Z and F_Ω are isomorphic, and therefore his theorem is restricted to the mathematical structures employed in matrix and in wave mechanics. Their *physical content*, as it were, is left untouched.

And it was here that finding the *right analogy* paid off. For it was the existence of the mathematical equivalence (between F_Z and F_Ω) that led von Neumann to search for a more basic mathematical formulation for quantum mechanics. However, since F_Z was nothing but a Hilbert space, it was natural to adopt a slightly more abstract formulation of it — not tied to the particular features of F_Z — as the basis for shaping the structure of quantum mechanics. In von Neumann's own words:

> Since the systems F_Z and F_Ω are isomorphic, and since the theories of quantum mechanics constructed on them are mathematically equivalent, it is to be expected that a unified theory, independent of the accidents of the formal framework selected at the time, and exhibiting only the really essential elements of quantum mechanics, will then be achieved, if we do this: *Investigate the intrinsic properties* (common to F_Z and F_Ω) *of these systems of functions, and choose these properties as a starting point.*
> The system F_Z is generally known as "Hilbert space". Therefore, our first problem is to investigate the fundamental properties of Hilbert space, independent of the special form of F_Z or F_Ω. The mathematical structure which is described by these properties (which in any specific special case are equivalently represented by calculations within F_Z or F_Ω, but for general purposes are easier to handle directly than by such calculations), is called "abstract Hilbert space". (von Neumann [1932], pp. 32-33; the italics are mine)

In other words, once the right analogy is found — that is, once the appropriate structural relationship is uncovered — the crucial step is taken. The rest is to explore the consequences.

4.3. *Quasi-truth, empiricism and quantum mechanics*

Let us reflect on what von Neumann is doing here. He is bringing some structure from functional analysis, by using the theory of abstract Hilbert spaces, and he is using this structure to provide the basis for quantum mechanics. Does this require that the theory of Hilbert spaces is true? By no means: the only requirement is that it has the adequate structure; that is, that it provides a system of functions which generalizes the mathematical properties satisfied by F_Z and F_Ω. We can think of the latter as partial structures A (in the sense that they would provide partial information about a given physical system). What von Neumann is doing is to overcome that partiality by finding a convenient "A-normal structure", which extends and generalizes the relations in A — and this is the abstract Hilbert space. In this sense, the only requirement is the quasi-truth (and *not* the truth) of Hilbert space theory. In other words, we need only the *possibility* that there is a structure that extends the partial information contained in A.[i]

These features the idea that a mathematical theory doesn't have to be true to be applicable and the role of analogies in theory construction in mathematics — suggest that von Neumann's mathematical practice might be captured by the framework developed here. As is well known, according to van Fraassen, a scientific theory doesn't have to be true to be good, but only empirically adequate (van Fraassen [1980], p. 12). Similarly, in this account of von Neumann's approach, a mathematical theory doesn't have to be true to be applicable, but only consistent with a particular physical description (in the present case, consistent with quantum mechanics). In this way, a mathematical theory has only to be *quasi-true* to be applicable (where the body of accepted sentences P provides the relevant accepted information from physics). Moreover, just as analogies are explored in the construction of physical theories (see French [1997], and French and Ladyman [1997]), the same goes in the case of mathematics, as the discussion of the equivalence between matrix and wave mechanics clearly indicates.

But there is a third feature worth considering in von Neumann's application of mathematics. It concerns the role of logic in this process — and as we shall see, von Neumann has a pluralist and empiricist view about logic. According to him, there is a plurality of logics, depending on the particular context we study, and such logics should be constrained by experience. As we shall also see, pluralism and empiricism about logic support the empiricist framework advanced here. They entail that logic is motivated by experience, and they strengthen the role of analogies between mathematics and physics as part of the development of physics and mathematics. In this sense, experience has a double role of generating logics and demanding further mathematical structures. In what follows, I shall elaborate on these points.

[i]A noted above, since we are dealing here with mathematical objects, the empiricist will have to adopt a nominalization strategy in order to avoid ontological commitments to them. This can be articulated by the introduction of convenient modal operators, defined in terms of quasi-truth, and adapting the modal-structural interpretation of mathematics, due to Hellman [1989]. Alternatively, ontologically neutral quantifiers can be invoked (see Azzouni [2004], Bueno [2009], and [2013]).

4.4. *Logic and empiricism*

As I have indicated, with the introduction of Hilbert spaces in quantum mechanics, von Neumann made a top-down move from mathematical to physical theories. Now what I intend to delineate is the corresponding bottom-up move in von Neumann's thought as one goes from experience through logic to highly abstract mathematical structures.

One of the outcomes of von Neumann's approach to quantum mechanics was the creation of quantum logic. He showed that we can formulate a family of propositions that describe the state of a physical system, in such a way that the geometrical structure underlying this family gives us information about the physical system. The idea is found in von Neumann's 1932 book, and is developed further in the celebrated 1936 paper by Birkhoff and von Neumann, where "quantum logic" as such was first introduced.

The study of the geometry associated with the relevant family of propositions give us its "logic". In the case of classical mechanics, the family of propositions generate a Boolean algebra, but in the context of quantum mechanics, we have "a sort of" projective geometry. The reason for the proviso is that in the 1936 paper, Birkhoff and von Neumann only consider physical systems with a finite number of degrees of freedom (that is, whose states can be characterized by finitely many parameters, in a state-space with a finite dimension). Now, with *this* finiteness assumption, the resulting "logic" (which is isomorphic to the projective geometry of all subspaces of a finite-dimensional Hilbert space) *is indeed* a projective geometry. But the question arises as to the logic of a physical system that has *infinitely* many degrees of freedom. Von Neumann explicitly addressed this problem in the work he produced after the 1936 paper with Birkhoff. And his solution was to provide a generalization of projective geometry, which led to the formulation of *continuous geometry* and to what is now called *von Neumann algebras* of Hilbert space operators (see von Neumann [1960] and [1981], Murray and von Neumann [1936], and for a discussion, Bub [1981a] and [1981b], and Rédei [1997], [1998], and [2000], and Rédei and Stöltzner (eds.) [2000]).

In other words, it was because of von Neumann's *pluralism* that he was concerned with determining the logics adequate to each particular domain — moving from *finite* physical systems (represented by projective geometry) to *infinite* ones (represented by continuous geometry). Moreover, it was because of his *empiricism* that he undertook the search for distinct logics based on experience, that is the various types of physical systems we have to accommodate. In other words, the case of the generalization of projective geometry (continuous geometry) is a clear example of a mathematical theory that is created from physical, *empirical* demands. I shall call this von Neumann's *first empiricist feature*. Let us consider some others.

4.5. *The 1937 manuscript: logics and experience*

In an unpublished manuscript, written about 1937, von Neumann discusses his view about the status of logic and alternative logical systems (see von Neumann [1937a]). This work was written shortly after the publication of Birkhoff and von Neumann [1936], and von Neumann's chief task is to explore some consequences of the approach to logic left unnoticed in the 1936 paper.

The manuscript begins with a remark making it clear that it was *not* part of what was then considered work in the "foundations" of mathematics:

> We propose to analyze in this note logics from a point of view which is fundamentally different from the one current in present-day "foundations" investigations. While we are not questioning at all the great importance and value of those investigations, we do believe that our present system of logics is open to criticism on other counts also. And it seems to us that these other "counts" are at least as fundamental and essential as those which form the subject-matter of the current "foundations" analysis. (von Neumann [1937a], p. 1)

Von Neumann had several reasons for not being satisfied with the then dominant foundational research. The most important of them was the entirely *a prioristic* way in which that research was typically conducted. Logic was essentially taken as *classical* logic, and there was hardly any room for the development of logics that were *specific* and *appropriate* to the particular domain of science under consideration. As opposed to this, the approach favored by von Neumann was

> much more *directly connected with and inspired by the connection of logics with the physical world*, and particularly also with probability. For this reason we even see some point in the proposition that [the problems this approach generates] deserve precedence over the generally considered problems of "foundations". (von Neumann [1937a], p. 1; the italics are mine)

According to von Neumann, it is crucial to have room for the articulation of logics that are faithful to the domain in question. In fact, he goes still further claiming that logics should be inspired by experience.

> The basic idea [of this approach to logic] is that *the system of logics which one uses should be derived from aggregate experiences relative to the main application which one wishes to make — logics should be inspired by experience.* (von Neumann [1937a], p. 2; the italics are mine)

In other words, not only the construction of mathematical theories, but also the construction of logical systems is *heuristically* motivated by experience — in order for the resulting logic to be *adequate* to the domain in question — and a logic should also reflect the main traits of the empirical domain. *Logic* should be inspired by experience (this is von Neumann's *second empiricist feature*).

However, as well as being heuristically inspired by experience, a logic can also play a *critical* role, being open to revision on empirical grounds:

> If any accepted physical theory which has a reasonable claim to "universality" permits to draw inferences concerning the structure of logics, *these inferences should be used to reform logics, and not to criticize the theory.* (von Neumann [1937a], pp. 2-3; the italics are mine)

In other words, a logic can be revised on empirical grounds (*third empiricist feature*). As a result, a logic is relative to a particular domain of inquiry. And relying on this revisability principle, von Neumann provides a further argument against the traditional approach to logic:

> We hope to show on the pages which follow that an *absolutely* consequent application of this principle leads to very plausible results, and particularly to more natural ones than the usual, rigidly dogmatic, attitude in logic. (von Neumann [1937a], p. 3)

All these considerations (especially the domain-dependence of logic) become completely clear with quantum logic, which as is well known, due to the demands of quantum phenomena, lacks the distributivity of classical logic. In von Neumann's own words:

> Quantum mechanics is a particularly striking example of how a physical theory may be used to modify logics — and *for this reason*

> it was first investigated in [Birkhoff and von Neumann [1936]]. (von Neumann [1937a], p. 3; the italics are mine)

However, von Neumann has something still more radical to advance:

> We propose to show in this note that even classical mechanics is incompatible with the usual system of "infinite" (or "transcendental") logics [that is, logics which are not related to experience]. We will determine the system of logics to which classical mechanics lead. We will see that this system lends itself much better to an extension to "probability logics" — which is an absolutely indispensable one if we bear in mind the modern developments of physical theory. (von Neumann [1937a], p. 3)

But once this is achieved, something more will be established:

> We will also see that this "classical mechanical" system of logics, when combined with the "finite" "quantum mechanical" system of logics given in [Birkhoff and von Neumann [1936]], leads to a satisfactory, general "infinite" system of "quantum mechanical" logics, which contains also probability theory. (von Neumann [1937a], pp. 3-4)

And *these* are the "plausible results" that von Neumann referred to above in his criticism of the "dogmatic" approach to logic.

Therefore, the picture that emerges from this paper involves, firstly, *pluralism* (there are as many logics as physical phenomena demand); secondly, *anti-apriorism* (the demand for new logics arises from experience); and thirdly, *empirically driven change* (logic is open to change via empirical considerations). These are, of course, three features that are cherished by the *empiricist*. In this sense, there is more to logic than the *a priori* exploration of the logical consequence relation, and the main features of a logic should be obtained from, and modified by, experience (the domain of *application*).

Given the empiricist features of von Neumann's account of logic (in particular, given the idea that logic is open to change via empirical considerations), it is interesting to note that von Neumann thought that his account was comparable to the criticism of classical logic made by intuitionist and relevant logicians. As he points out in his paper with Birkhoff on the logic of quantum mechanics:

> The models for propositional calculi which have been considered in the preceding sections are also interesting from the standpoint of pure logic. *Their nature is determined by quasi-physical and technical reasoning*, different from the introspective and philosophical considerations which have had to guide logicians hitherto. Hence it is interesting to compare the modifications which they introduce into Boolean algebra, with those which logicians on "intuitionist" and related grounds have tried introducing. (Birkhoff and von Neumann [1936], p. 119; the italics are mine.)

However, as Birkhoff and von Neumann point out, there is a crucial distinction between the criticisms of classical logic as articulated by the quantum and the intuitionist logicians:

> The main difference seems to be that whereas logicians have usually assumed that properties L71-L73 of negation [namely, $(A')' = A$; $A \wedge A' = f$; $A \vee A' = t$; $A \to B$ implies $B' \to A'$] were the ones least able to withstand a critical analysis, the study of mechanics points to the *distributive identities* L6 [$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$ and $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$] as the weakest link in the algebra of logic. (Birkhoff and von Neumann [1936], p. 119)

In their view, their own approach was closer to that of the relevant logician:

> Our conclusion agrees perhaps more with those critiques of logic, which find most objectionable the assumption that $A' \vee B = t$ implies $A \to B$. (Birkhoff and von Neumann [1936], p. 119)

All these principles (pluralism, anti-apriorism, and empiricism) are forcefully articulated in von Neumann's work on quantum logic, since he thinks this logic is fully adequate for the quantum domain, and it is importantly different from classical logic (even if one is a fragment of the other). In my view, not only has von Neumann adopted these methodological and epistemological principles but also he has fruitfully explored them to develop several of his most lasting contributions. In other words, these principles have informed von Neumann's research, providing him with heuristic guidelines for theory construction. Of course, all this can be clearly seen with the development of quantum logic.

Now, we have here a sophisticated interaction between mathematics, logic and physics. As von Neumann articulated them, logics (such as quantum logic) are inspired by experience, but once articulated they generate new mathematical structures (such as the continuous geometry demanded by physical systems with infinitely many degrees of freedom). Such structures are in turn applied to model physical phenomena. Of course, once mathematical structures are formulated, they can also be studied in purely mathematical terms, independently of any concern with physics — as von Neumann's own mathematical work with continuous geometry beautifully illustrates (see, for instance, von Neumann [1981]).

So we have here a movement from the bottom up, from experience through logic to highly abstract mathematical structures (from quantum logic to continuous geometry), in addition to the movement from the top down, from mathematical structures down to experience. The latter move was described earlier with von Neumann's use of the theory of Hilbert spaces in the formulation of quantum mechanics.

Returning now to von Neumann's 1937 manuscript, we can see another aspect of his *pluralism*. According to von Neumann, in order to have a proper understanding of *classical* mechanics we need an appropriate logic, and even in *this* case, we need logics that are non-classical ("probability logics"), in the sense that weaker, probabilistic "implication" relations among statements are invoked. The idea is not that the Kolmogorovian notion of probability is non-classical, but rather that there are probabilistic "implication" relations. This proposal is explored further in another unpublished manuscript, whose subject is quantum logic (von Neumann [1937b]; I present the details of the approach in the discussion of this manuscript in Section 4.7 below). In this article, von Neumann outlined four possibilities concerning the interplay between the "dimension" of a physical system and its logic, each alternative leading to an appropriate logical system. The four cases are: (i) a system "which behaves in the sense of classical physics (in particular mechanics) and which posses only a finite number of possible different states" (von Neumann [1937b], p. 11); (ii) a system similar to (i) but in which the number of states is discretely infinite; (iii) a system similar to (i) but in which the number of states is continuously infinite; and finally (iv) a system where the "finiteness of [the number of states] remains essentially untouched, but the 'classical' way of looking at things (as practiced in [(i)-(iii)]) is replaced by a 'quantum mechanical' one" (von Neumann [1937b], p. 16). Moreover, von Neumann also advanced a final synthesis, combining "these two kinds of extensions", namely from the finite to the infinite, and from the "classical" to the "quantum" approach (*ibid.*). Unfortunately, this is an unfinished work, and von Neumann only discussed case (i) (see von Neumann [1937b], pp. 11-15).

But note that the three methodological and epistemological principles mentioned above play a heuristic role at this point. What led von Neumann to consider these four possibilities concerning a physical system was: (a) his logical *pluralism* (since he was searching for logics appropriate to the domain under study, whether it is classical or quantum physics); (b) his *anti-apriorism* (by refusing to adopt a logic which was not properly motivated by experience), and (c) his *empiricism* with regard to logic (which required an empirical investigation to determine the adequacy of a logic to the physical domain in question). In this sense, these principles have informed and supported von Neumann's research.

4.6. *The status of mathematics*

It is important to note that I am *not* saying that von Neumann was an empiricist about logic and mathematics in the naive, and untenable, sense that these disciplines *are* empirical. Once *motivated* by empirical considerations, logic and mathematics are *developed* and *articulated* as deductive disciplines in the standard way. And the adoption of this standard practice is absolutely clear in von Neumann's own mathematical writings.

For instance, von Neumann criticized Dirac for his introduction of the δ-function on *mathematical* grounds, along the lines I have previously indicated (see von Neumann [1932], pp. 23-27). According to von Neumann, since this function lies "beyond the scope of mathematical methods generally used", and he desires to "describe quantum mechanics with the help of these latter methods" (*ibid.*, p. 27), he moves on to his own formulation of quantum mechanics in terms of Hilbert spaces. This surely illustrates the importance von Neumann saw in subscribing to standard methods of mathematical research. As he insists:

> The method of Dirac, mentioned above (and this is overlooked today in a great part of quantum mechanical literature, because of the clarity and elegance of the theory), in no way satisfies the requirements of mathematical rigor — not even if these are reduced in a natural and proper fashion to the extent common elsewhere in theoretical physics. For example, the method adheres to the fiction that each self-adjoint operator can be put in diagonal form. In the case of those operators for which this is not actually the case, this requires the introduction of "improper" functions with self-contradictory properties. The insertion of such a mathematical "fiction" is frequently necessary in Dirac's approach, even though the problem at hand is merely one of calculating numerically

> the result of a clearly defined experiment. (von Neumann [1932], pp. viii-ix)

However, von Neumann points out:

> There would be no objection here if these concepts [Dirac's "improper" functions], which cannot be incorporated into the present day framework of analysis, were intrinsically necessary for the physical theory. Thus, as Newtonian mechanics first brought about the development of the infinitesimal calculus, which, in its original form, was undoubtedly not self-consistent, so quantum mechanics might suggest a new structure for our "analysis of infinitely many variables" — i.e., the mathematical technique would have to be changed, and not the physical theory. (von Neumann [1932], p. ix)

This passage illustrates the extension of von Neumann's *empiricism*: he would be prepared to change the standard mathematical techniques, if this were required by our physical theories. And the precedent for this in the case of Newtonian mechanics is surely well taken. Nevertheless, von Neumann continues:

> But this is by no means the same case. It should rather be pointed out that the quantum mechanical "Transformation theory" can be established in a manner which is just as clear and unified, but which is also without mathematical objections. It should be emphasized that the correct structure need not consist in a mathematical refinement and explanation of the Dirac method, but rather that it requires a procedure differing from the very beginning, namely, the reliance on the Hilbert theory of operators. (von Neumann [1932], p. ix)

That is, once we have the appropriate mathematical framework, there is no need for introducing deviant techniques in mathematics in order for us to formulate quantum mechanics.

4.7. *The 1954 manuscript: a unified approach*

One of the striking features of von Neumann's work is his search for an approach in which logic and the mathematical and physical theories we employ to explain the phenomena hang nicely together. This means that we should search for an account that admits different kinds of generalization, such as the following: Suppose we are considering a given type of physical system. The approach von Neumann tried to provide was such that the same framework that is used to describe a physical system with a *finite* number of degrees of freedom could be extended to accommodate a system with an *infinite* number of degrees. Moreover, the geometric structures associated with such descriptions should be such that they are also preserved and generalized when the extension to the infinite case is performed. Furthermore, the logic derived from the geometric structure should also be amenable to generalization. Finally, given the crucial role played by probability theory in physics (especially in quantum mechanics), and given the close connection that von Neumann saw between probability and logic, he searched for an approach that allowed probability to be adequately introduced in quantum mechanics, and which also made clear the relationship between logic and probability. These are broad constraints and von Neumann adopted them as heuristic devices in theory construction. As we shall see, they played a crucial role in his program.

These points are clearly formulated in an unpublished paper that von Neumann presented in 1954 in the Congress of the Mathematicians in Amsterdam. Half a century after Hilbert's celebrated 1900 paper on mathematical problems, von Neumann was asked to consider further unsolved problems in mathematics. And he emphasized the importance of providing a unified approach to logic, probability and physics. Once again, he was keen on exploring analogies and structural similarities between different domains of mathematics and physics. He first considered classical physics, its underlying logic and probability, in a finite context:

> If you take a classical mechanism of logics, and [...] limit yourself to logics referred to a finite set, it is perfectly clear that logics in that range are equivalent to the theory of all subsets of that finite set, and that probability means that you have attributed weights to single points, that you can attribute a probability to each event, which means essentially that the logical treatment corresponds to set theory in that domain and that a probabilistic treatment corresponds to introducing measures. (von Neumann [1954], p. 21)

However, given the importance that von Neumann attached to generalizations, it would be important to extend these considerations to the infinite case. But this can be done without difficulties:

> But it is quite possible to extend this to the usual infinite sets. And one also has this parallelism that logics corresponds to set

> theory and probability theory corresponds to measure theory and that given a system of logics, so given a system of sets, if all is right, you can introduce measures, you can introduce probability [...]. (von Neumann [1954], p. 21)

Note, in particular, the structural analogies von Neumann introduces between logic and set theory, on the one hand, and between probability and measure theory, on the other. The idea is that there are structural relations between these domains, and he explores them in order to find a unified approach.

Things become more delicate when we try to extend this unified picture to quantum mechanics:

> In the quantum mechanical machinery the situation is quite different. Namely instead of the sets use the linear subsets of a suitable space, say of a Hilbert space. The set-theoretical situation of logics is replaced by the machinery of projective geometry, which in itself is quite simple. (von Neumann [1954], p. 21)

As it could be expected, the delicate part consists in the introduction of probability:

> However, all quantum mechanical probabilities are defined as inner products of vectors. Essentially if a state of a system is given by one vector, the transition probability in another state is the inner product of the two which is the square of cosine of the angle between them. In other words, probability corresponds precisely to introducing the angles geometrically. Furthermore, there is only one way to introduce it. The more so because in the quantum mechanical machinery the negation of a statement, which is represented by a linear set of vectors, corresponds to the orthogonal complement of this linear space. And therefore, as soon as you have introduced into the projective geometry the ordinary machinery of logics, you must have introduced the concept of orthogonality. [...] So in order to have logics you need in this sense a projective geometry with a concept of orthogonality in it. (von Neumann [1954], pp. 21-22)

In this way, logic and probability go hand to hand. As von Neumann stresses:

> In order to have probability all you need is a concept of all angles, I mean angles other than 90^o. Now it is perfectly quite true that in a

> geometry, as soon as you can define the right angle, you can define all angles. Another way to put it is that, if you take the case of an orthogonal space, those mappings of this space on itself, which leave orthogonality intact, leave all angles intact; in other words, in those systems which can be used as models of the logical background for quantum theory, it is true that as soon as all the ordinary concepts of logics are fixed under some isomorphic transformation, all of probability theory is already fixed. (von Neumann [1954], p. 22)

In conclusion, the unified approach could be achieved, in the sense that

> logics and probability theory arise simultaneously and are derived simultaneously. (von Neumann [1954], p. 22)

The importance of these considerations is that, by putting forward a unified approach, von Neumann could provide a better understanding of logic, probability and physics. The search for such understanding is something that can be traced back to von Neumann's work in the 1930's, and he has explicitly discussed these topics in the unpublished manuscript of 1937 that I mentioned earlier (see von Neumann [1937b]). As well as examining the relationship between the "dimension" of a physical system and its logic, which we considered above, in this work von Neumann also indicates how classical ("strict-") logic can be obtained as a particular case of probability logic. The main idea is again to provide an approach in which logic, probability and physics are closely tied together.

> Let S be the physical system, or rather the mathematical model of a physical system, to which we wish to apply logics. The system L of logics is then the set of all statements $a, b, c, \ldots$ which can be made concerning S. Such a statement is always one concerning the outcome of a certain measurement, which is to be performed on S. [...] The fundamental relations and operations for elements of L are these:
>
> (I) The *relation* of "*implication*": $a \leq b$.
> $a \leq b$ means this: If a measurement of a on S has shown a to be true, then an immediately subsequent measurement of b on S will certainly show b to be true.
>
> (II) The *operation* of "*negation*": $\neg a$.
> $\neg a$ obtains as follows: The same measurement which is used to decide about the validity of a is also used to decide about the validity of $\neg a$, but when the result concerning the validity of

> a is "yes", then the one concerning $\neg a$ is "no", and conversely. (von Neumann [1937b], pp. 3-4)

The operations of conjunction and disjunction can also be defined (von Neumann [1937b], p. 5), which leads von Neumann to call L the system of "strict logics" (*ibid.*, p. 6).

However, as he notes, when we apply L to "physical reality", a further structure emerges, which can only be expressed in terms of probability (*ibid.*).

> For any well defined state of our knowledge concerning the mathematical description of physical reality, that is for any reasonable model S, a probability function exists. So we have in L:
>
> (VI) The (*real number-valued) function* called "*probability function*": $P(a, b)$. $P(a, b) = \theta$ (θ a real number) means this: If a measurement of a on S has shown a to be true, then the probability of an immediately subsequent measurement of b on S showing b to be true, is equal to θ.
>
> If we consider the structure which L aquires by making use of the function $P(a, b)$ — that is, of all relations $P(a, b) = \theta$ (of course necessarily $0 \leq \theta \leq 1$) — then L appears as a new system, which we will call the system of "*probability logics*". (von Neumann [1937b], pp. 6-7)

But what is the relation between strict- and probability-logics? It is clear enough: the former is a particular case of the latter.

> It is easy to see that the system of strict logics is part of the system of probability logics, since $a \leq b, \neg a$ can be defined in terms of $P(a, b) = \theta (0 \leq \theta \leq 1)$. Indeed (VI) makes the following statements obvious:
>
> (F) $P(a, b) = 1$ is equivalent to $a \leq b$.
> (G) $P(a, b) = 0$ is equivalent to $a \leq \neg b$.
>
> Hence $a \leq b$ is directly defined by (F), that is by $P(a, b) = 1$; and $\neg a$ is indirectly defined by (G), as the (unique) c, for which $d \leq c$ is equivalent to $d \leq \neg a$, that is, for which $P(d, c) = 1$ is equivalent to $P(d, a) = 0$. (von Neumann [1937b], p. 7)

However, there is no "reduction" of probability-logic to strict-logic. And in this sense, since probability-logic naturally arises in the context of physical theories, it cannot be dismissed as irrelevant, and to some extent it is even irreducible.

> For a θ with $0 \leq \theta \leq 1$, however, no such "reduction" of $P(a, b) = \theta$ to strict logics seems possible. (von Neumann [1937b], p. 8)

In other words:

> Probability logics cannot be reduced to strict logics, but constitutes an essentially wider system than the latter, and statements of the form $P(a, b) = \theta (0 \leq \theta \leq 1)$ are perfectly new and sui generis aspects of physical reality. (von Neumann [1937b], pp. 9-10)

The point of these remarks is to indicate the importance that von Neumann assigned to the development of an integrated approach to physics, logic and probability, and how he used this as a heuristic constraint in theory construction. More importantly, he was clear in *not* claiming that a unified theory is more likely to be true than a less unified one (von Neumann [1947], p. 7), and in this sense, such unification was only a *pragmatic* feature, not an epistemic one.[j] This is again an empiricist (and anti-realist) component in his work.

But von Neumann's work in the foundations of probability and quantum theory led him to articulate elements of what we now recognize as an empiricist version of the semantic approach. Together with his empiricism with regard to logic and mathematics, this constitutes the third empiricist trend in von Neumann's thought. To this issue we shall turn now.

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[j] As von Neumann reminds us: "The attitude that theoretical physics does not explain the phenomena, but only classifies and correlates, is today accepted by most theoretical physicists. [This attitude is of course clearly spelled out by Duhem [1906], when he distinguished the explanation of the phenomena (i.e. the search for their underlying causes) from their classification (i.e. to provide structural representations for them). Von Neumann is clearly siding with Duhem's anti-realism about unobservable entities here.] This means that the criterion of success for such a theory is simply whether it can, by a simple and elegant classifying and correlating scheme, cover very many phenomena, which without this scheme would seem complicated and heterogeneous, and whether the scheme even covers phenomena which were not considered or even not known at the time when the scheme was evolved. (These two latter statements express, of course, the unifying and predicting power of a theory.) *Now this criterion, as set forth here, is clearly to a great extent of an aesthetical nature.* For this reason it is very closely akin to the mathematical criteria of success, which [...] are almost entirely aesthetical" (von Neumann [1947], p. 7; the italics are mine). This passage forcefully indicates that, for von Neumann (as for any good empiricist!), unification is a *pragmatic*, aesthetic feature of a theory, not an *epistemic* one (see also von Neumann [1935-36/2000] and [1954/2000]).

4.8. *Von Neumann and the semantic approach*

As is well known, the semantic approach to science supplies a new perspective to the issue of the structure of scientific theories. The main components of this approach, in the formulation provided by van Fraassen, are: the use of *models of data* to represent empirical information; the employment of *state-spaces* to represent theoretical information; the adoption of *laws of succession* to represent the evolution of a physical system; and the requirement of *empirical adequacy* as a major criterion of theory choice (see van Fraassen [1980], [1989] and [1991]). As we shall see now, each of these four components were clearly formulated by von Neumann — in a strictly empiricist way (see also Suppe [1989] and [1977]).

With regard to models of data (observation space), von Neumann and Birkhoff are explicit:

> It is clear that an "observation" of a physical system M can be described generally as a writing down of the readings from various compatible measurements. Thus if the measurements are denoted by the symbols $\mu_1, \ldots, \mu_n$, then an observation of M amounts to specifying numbers $x_1, \ldots, x_n$ corresponding to the different μ_k.
> It follows that the most general form of a prediction concerning M is that the point $(x_1, \ldots, x_n)$ determined by actually measuring $\mu_1, \ldots, \mu_n$, will lie in a subset S of $(x_1, \ldots, x_n)$-space. Hence if we call the $(x_1, \ldots, x_n)$-spaces associated with M, its "observation-spaces", we may call the subsets of the observation-spaces associated with any physical system M, the "experimental propositions" concerning M. (Birkhoff and von Neumann [1936], p. 106)

Furthermore, von Neumann also formulates the notion of a state-space (phase-space), which is crucial for the representation of theoretical information:

> According to this concept [phase-space], any physical system M is at each instant hypothetically associated with a "point" p in a fixed phase-space Σ; this point is supposed to represent mathematically the "state" of M, and the "state" of M is supposed to be ascertainable by "maximal" observations. (Birkhoff and von Neumann [1936], p. 106)[k]

[k]Of course, there is a history to be told about the evolution of the notion of phase-space in physics. I shall not embark on this here.

The third component of the semantic approach, the introduction of laws of succession (law of propagation), is then explicitly considered:

> Furthermore, the point p_0 associated with M at a time t_0, together with a prescribed mathematical "law of propagation", fix the point p_t associated with M at any later time t. (Birkhoff and von Neumann [1936], p. 106)

As examples to illustrate these concepts, von Neumann mentions a few cases (note the reference to function spaces, which are subsequently articulated by Redhead [1975]):

> Thus in classical mechanics, each point of Σ corresponds to a choice of n position and n conjugate momentum coordinates — and the law of propagation may be Newton's inverse-square law of attraction. Hence in this case Σ is a region of ordinary $2n$-dimensional space. In electrodynamics, the points of Σ can only be specified after certain *functions* — such as the electromagnetic and electrostatic potential — are known; hence Σ is a function-space of infinitely many dimensions. Similarly, in quantum theory the points of Σ correspond to so-called "wave-functions", and hence Σ is again a function-space — usually assumed to be Hilbert space. (Birkhoff and von Neumann [1936], p. 106)

As further examples, Birkhoff and von Neumann consider:

> In electrodynamics, the law of propagation is contained in Maxwell's equations, and in quantum theory, in equations due to Schrödinger. In any case, the law of propagation may be imagined as inducing a steady fluid motion in the phase-space. (Birkhoff and von Neumann [1936], p. 106)

The last component of the semantic approach, which is particularly important in the context of an empiricist view, is the formulation of a convenient notion of empirical adequacy. After spelling out the mathematical content of models of data and state-spaces, von Neumann indicates the importance of assigning empirical content to the models considered; otherwise, we would be considering only mathematical structures.

> Now before a phase-space can become imbued with reality [empirical content], its elements and subsets must be correlated in some way with "experimental propositions" (which are subsets of different observation-spaces). Moreover, this must be so done that

> set-theoretical inclusion (which is the analogue of logical implication) is preserved. (Birkhoff and von Neumann [1936], p. 107)

In other words, a physical theory is empirically adequate if there is a transformation between certain components of the state-space and the data models that preserves the relevant relations; that is, which assigns to the observable components of the state-space a counterpart in the models of data.

Note that, in this passage, when von Neumann talks about imbuing the state-space with "reality", he is not adding a realist gloss to the resulting formalism. He is actually talking about the *empirical content* of the theory under consideration. This is clear for the following reasons: Firstly, von Neumann was careful in *not* adding a realist gloss to his claims. He insisted that the unificatory and predictive components of a theory are only *aesthetic, pragmatic* features, and *not* epistemic (see von Neumann [1947], p. 7), and the fact that a theory is unified and empirically successful doesn't increase the probability that it is true. Of course, this move meshes nicely with an empiricist approach. Secondly, in a remark made immediately after the passage I've just quoted, von Neumann explicitly considers the *empirical substructures* of the state-spaces (the components which refer only to the observables of a physical system, and which are crucial for an empiricist view):

> There is an obvious way to do this in dynamical systems of the classical type. One can measure position and its first time-derivative velocity — and hence momentum — explicitly, and so establish a one-one correspondence which preserves inclusion between subsets of phase-space and subsets of a suitable observation space. (Birkhoff and von Neumann [1936], p. 107)

This passage clearly indicates that von Neumann is concerned with the observables of a physical theory (this is the focus of the empirical substructures) and the existence of a "one-one" correspondence that preserves the relevant structure between the state-space and the models of data. And this is, of course, the main feature of van Fraassen's notion of empirical adequacy (see van Fraassen [1980], p. 64, and Bueno [1997]).

However, Birkhoff and von Neumann are concerned not only with the empirical adequacy of deterministic theories, but also of statistical ones (a point which is discussed at length in van Fraassen [1989]):

> In quantum theory [...] the possibility of predicting in general the readings from measurements on a physical system M from a

> knowledge of its "state" is denied; only statistical predictions are always possible. (Birkhoff and von Neumann [1936], p. 107)

In this way, it becomes clear that all the main components of the semantic approach (especially in the empiricist gloss given by van Fraassen) can be found in von Neumann's work.

Von Neumann was also clear in his emphasis on the role of mathematical structures to accommodate physical phenomena, and also when talking about "physical reality", he always made it clear that he was considering the *models* we use to represent information about reality. In this sense, given the emphasis on structural considerations, and since von Neumann didn't add a realist gloss to them, his overall approach meshes nicely with an empiricist account (see von Neumann [1947]).[1]

But is this account of von Neumann's epistemic attitude towards science adequate? For given that von Neumann developed a quantum-logical approach to quantum mechanics, and since quantum logic is often used to support a *realist* interpretation of quantum mechanics, wasn't von Neumann a realist after all? The problem with this suggestion lies in the assumption that quantum logic supports realism. Surely there are quantum-logical interpretations that are meant to enhance the realist position; this is the case of Putnam's view (see Putnam [1968] and [1976], and for a discussion Bub [1981b], and da Costa [1997]). However, quantum logic, as such, is not necessarily tied to realism. When Putnam articulated his version of quantum logic, he *added* to the quantum-logical approach a realist gloss. But this is not necessary. It is interesting to note that van Fraassen himself, while not sharing the realist additions to quantum logic, has formulated his modal interpretation of quantum mechanics *quantum-logically* (see van Fraassen [1991]). In other words, quantum logic provides a broad conceptual framework in which both realists and anti-realists can work. The distinctions between these two kinds of interpretations (realist and anti-realist) arise from the different assumptions they add. The importance of this is that we can understand why von Neumann's formulation of quantum logic wasn't articulated as part of a realist enterprise. Given his concerns with structures and the lack of a realist gloss over them, it meshes more smoothly with an empiricist outlook.

[1]It is worth mentioning that Birkhoff and von Neumann also consider Hankel's principle of the "perseverance of formal laws" as one of the requirements to be met by their approach (Birkhoff and von Neumann [1936], p. 118). This perseverance is introduced as a heuristic component in the formulation of quantum logic.

4.9. *Logic and constructive empiricism*

But it is not only in the formulation of the semantic approach, and in the role assigned to quantum logic that von Neumann's and van Fraassen's approaches are related; they also share a similar kind of logical pluralism. And of course this pluralism makes a lot of sense in an empiricist setting. According to van Fraassen, the correctness of a logical system is always *relative*, and in this sense, the question about the determination of the *right logic* cannot be addressed in absolute terms:

> A logical system is considered correct for a language if it provides a catalogue of the valid inferences in that language. So the question "Which is the right logic?" may perhaps be rephrased as: Assuming that natural language is adequately represented by a certain formal language L, what logic is correct for L from a semantic point of view? (van Fraassen [1971], p. 3)

As a result, we cannot talk about the correct logic in absolute terms, but we have to consider the particular context (the particular domain or language) we are concerned with.

> From the semantic point of view, the correct logic is always derivative: It is found by examining semantic relations (defined in terms of truth, reference, and so on) among statements. Thus, if what the intuitionist means by his statements is understood, it can then be seen that intuitionistic logic is the correct logic for his language. (van Fraassen [1971], p. 4)

So, just as von Neumann stressed the context-dependence of logic (we have different logics depending on the empirical domain we are examining), van Fraassen emphasizes the language-dependence of logic. In both cases the outcome is clear enough:

> Since *we have now denied that there is a unique logic* we must face the charge of a self-defeating relativism. For what logical system shall govern the appraisal of our own reasoning in semantic inquiry? Our answer to this is fairly straightforward: In Metalogic we use a part of natural language commonly known as "mathematical English", in which we describe and discuss only mathematical

> (that is, set-theoretic) objects. (van Fraassen [1971], p. 4; the italics are mine)[m]

However, van Fraassen adds, despite his talk of set-theoretic objects, he is by no means a platonist:

> We are deliberately speaking of mathematical objects in the idiom of naive platonism; the reader is asked not to infer that this is our position in philosophy of mathematics. After all, any philosophy of mathematics must eventually make sense of the common language of mathematical mankind. (van Fraassen [1971], p. 5, note 6)

The idea is to make sense of the mathematical language without inflating the ontology, that is, without assuming platonism, and advocating a logical pluralism.

Given van Fraassen's rejection of the existence of a unique (correct) logic, he claims that a logic is correct only relative to a given domain. According to him, "we accept classical logic as correct *within a certain (perhaps rather limited) domain*" (van Fraassen [1971], p. 5; the italics are mine). This is, of course, the natural position for the empiricist, since it avoids the essentialism typical of a topic neutral account of logic. It is this domain dependence of logic that von Neumann explored so fruitfully in his work in the foundations of quantum mechanics. The logic he was then exploring was the logic of elementary statements, and for the reasons discussed in Birkhoff and von Neumann [1936], this logic is correct for the language of these statements.

The point of these considerations is simple. If we adopt a broad notion of empiricism (namely, constructive empiricism), we can understand how one can be an empiricist in mathematics. It doesn't require any form of reductionism of mathematics to the observable (just as van Fraassen's approach doesn't require any such reduction in science), but it allows mathematics and logic to be motivated and changed on empirical grounds. This form of empiricism is by no means foreign to mathematical practice, but it has informed, as I indicated here, von Neumann's own approach to mathematics.

[m] And van Fraassen notes: "When this language [mathematical English] is understood it can be seen that classical logic [. . .] is the correct logic for that language. To understand this language may involve understanding our beliefs concerning what sets are like and what sets exist — and some of these beliefs are rather audacious" (van Fraassen [1971], p. 4).

5. Conclusion

Von Neumann adopted a clear empiricist and pluralist view about logic, which played an important role in the development of his research both in the foundations of quantum mechanics (with the formulation of quantum logic) and in mathematics itself (with the creation of continuous geometry). However, these two features are by no means incompatible with the adoption of a classical attitude with regard to the *practice of mathematics*; in particular, with regard to the rejection of inconsistencies (such as those found in Dirac's use of the δ-function).

The answer to the question *Was von Neumann an empiricist*? should now be clear. As we saw above, a number of factors support a positive answer, since according to von Neumann: (1) mathematical theories are often created from physical, *empirical demands* (first empiricist feature); (2) logic should be inspired by experience (second empiricist feature); (3) logic can be revised on empirical grounds (third empiricist feature); (4) there are as many logics as physical phenomena demand (logical pluralism); (5) the *demand* for new logics arises from experience (logical anti-apriorism); and finally (6) von Neumann clearly formulated the semantic approach, with special emphasis on the *empirical adequacy* of the resulting theories, rather than their truth.

If these considerations are still not enough, I shall leave the last word to von Neumann himself, quoting a passage which leaves room for no doubt:

> As a mathematical discipline travels far from *its empirical source*, or still more, if it is a second and third generation *only indirectly inspired by ideas coming from "reality"*, it is beset with very grave dangers. It becomes more and more purely aestheticizing, more and more purely l'art pour l'art. This need not be bad, if the field is surrounded by correlated subjects, *which still have closer empirical connections* [...]. But there is a grave danger that the subject will develop along the line of least resistance, that the stream, so far from its source, will separate into *a multitude of insignificant complexities.* In other words, at a great distance from its *empirical source*, or after much "abstract" inbreeding, a mathematical subject is in *danger of degeneration.* (von Neumann [1947]; the italics are mine)

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THE BORN RULE AND FREE WILL: WHY LIBERTARIAN AGENT-CAUSAL FREE WILL IS NOT "ANTISCIENTIFIC"

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In the libertarian "agent causation" view of free will, free choices are attributable only to the choosing agent, as opposed to a specific cause or causes outside the agent. An often-repeated claim in the philosophical literature on free will is that agent causation necessarily implies lawlessness, and is therefore "antiscientific." That claim is critiqued and it is argued, on the contrary, that the volitional powers of a free agent need not be viewed as anomic, specifically with regard to the quantum statistical law (the Born Rule). Assumptions about the role and nature of causation, taken as bearing on volitional agency, are examined and found inadequate to the task. Finally, it is suggested that quantum theory may constitute precisely the sort of theory required for a nomic grounding of libertarian free will.

Keywords: Born Rule; free will; anomic action.

1. The Born Rule and free choices

The *agent causation* (AC) theory of free will holds that truly free human choices are attributable not to specific events or causes external to a choosing agent, nor to desires or other internal psychological influences, but only to the volitional power of the choosing agent. In effect, that is what "volition" means according to AC. But the latter is currently a minority view. The more "mainstream," conservative approach to the problem of free will is to assert that "free will" simply means acting in accordance with our desires in a way that is free of external constraints. This view is called *compatibilism*, because it was developed specifically to be compatible with deterministic laws. In effect, it defines the term "free will' in such a way that we can say we are making free choices as to how to behave even when all of our behaviors are fully determined by past causes and inexorable deterministic laws (or even fated in the sense of being elements of a "block world" in which the future exists in the same way as the past and present). In contrast, AC is a form of *incompatibilism*, which denies that free will is compatible with determinism. It holds that in order for us to have free will, the world must be fundamentally indeterministic. AC is the *libertarian* form of incompatibilism; it asserts that the world is in fact indeterministic, and that we do have free will. Our own volitional power is taken as the primary cause of our choices. The complementary form of incompatibilism is to assert that the world is deterministic and therefore to deny that we have free will.

It is often asserted that the agent causation view requires lawlessness, and is necessarily "antiscientific." For example, the entry on incompatibilism in the Stanford Encyclopedia of Philosophy states: "Libertarians who hold this view [agent causation] are committed, it seems, to the claim that free will is possible only at worlds that are at least partly lawless, and that our world is such a world." [1]

Sider [2] argues that this alleged incompatibility of agent causation with scientific laws extends beyond ordinary deterministic laws to the indeterministic probability law (i.e. the Born Rule) of quantum theory. He briefly entertains the idea that agent causation could 'peacefully coexist' with an indeterministic law such as the Born Rule, concluding in the negative, as follows:

"In the previous sections I was ignoring quantum mechanics. For instance, I was assuming that if a cause occurs, its effect must occur, even though quantum mechanics merely says that causes make their effects probable. Why did I ignore quantum mechanics? Because randomness is not freedom... A libertarian might concede that randomness is not sufficient for freedom, but nevertheless claim that quantum randomness makes room for freedom, because it makes room for agent causation. Imagine that it is 1939, and Hitler has not yet decided to invade Poland. He is trying to decide what to do among the following three options:

Invade Poland
Invade France
Stop being such an evil guy and become a ballet dancer

Quantum mechanics assigns probabilities to each of these possible decisions; it does not say which one Hitler will choose. Suppose, for the sake of argument, that the probabilities are as follows:

95% Invade Poland
4.9% Invade France
0.1% become a ballet dancer

If this picture [of quantum theory leaving room for free will] were correct, then my criticism of libertarianism as being anti-scientific would be rebutted: agent causation could peacefully coexist with quantum mechanics. In fact, though, the coexistence picture makes agent causation a slave to quantum-mechanical probabilities." (p. 124)

In other words, Sider assumes that Hitler, as modeled above, must 'mindlessly follow the probabilities' and therefore is not really free to choose.

On the next page, he concludes:

"Quantum mechanics does not help the agent-causation theorist. I will now go back to ignoring quantum mechanics." (p. 125)

However, Sider's formulation depends on the highly nontrivial but unsupported claim that a human agent can be represented as a quantum system with a well-defined state over the relevant time period, and that the agent's choices can be characterized by eigenvalues of an operator represented the set of options available to the agent. So, for example, if the agent (Hitler) is presented with a choice of actions, the measured operator is assumed to be one whose possible outcomes (eigenvalues) are "Invade Poland, Invade France, or Become Ballet Dancer." Then the Born probabilities for measurement outcomes are assumed to apply to those possible actions, considered as eigenstates, conditioned on the presumed stable-over-time initial quantum state of the choosing agent. It then follows, so the argument goes, that an agent would be a "slave" to those quantum statistics in order to be in compliance with the Born Rule. If not a slave, then the agent would presumably be able to violate the rule willy-nilly, and thus be in engaging in anti-scientific, "anomic action."[a]

[a]Technically, to say a set of events is 'anomic' is not a statement about law violation, but about the fact the events are not covered by any law. But clearly, if a set of events is not covered by any law whatsoever, than any given extant law must be violated by them; that is, the set of events must fail to conform to that law. Thus it follows that the claim that actions of free will are anomic is equivalent to the claim that events resulting from such actions would have to violate any extant law. For otherwise they would be consistent with that law, and would therefore fail to be anomic.

The main purpose of this paper is to argue that this argument fails because a human agent cannot be assumed to be modeled in this way.[b] First, however, I will note some objections of Clarke [4] to the claim that libertarian free will must be inconsistent not only with deterministic laws but also with statistical laws such as the Born Rule. Clarke argues against this claim by noting that:

"probabilistic laws of nature also do not require, for any finite number of trials, any precise distribution of outcomes. The probabilities involved in diachronic laws...are the chances that events of one type will cause, or will be followed by, events of another type. ... These probabilities, we may assume, determine single-case, objective probabilities, or propensities. Actual distributions can diverge from proportions matching these probabilities. As trials of some process governed by such a law increase, it becomes increasingly likely that the distribution of outcomes will match the probabilities given by the laws. But for any finite number of repetitions, any distribution at all is possible for outcomes neither determined to occur nor determined not to occur." (pp. 390-391)

Thus, a statistical law is not violated unless very large numbers of precisely repeated experimental runs yield statistically significant deviations from expected mean values, where even "statistically significant" can be a matter of context and degree. Highly unlikely strings of outcomes may occur, and yet a statistical law may still not be violated. The point here is that the demonstration of a violation of a statistical law requires a high hurdle of empirical evidence.

Clarke presents an example in which an agent may "freely choose to obey the laws" by choosing to obtain a distribution of choice outcomes that comport with a prescribed probabilistic law in a psychology experiment. He notes that each of the agent's individual choices are free, even in the face of such a law. This author concurs that the situation envisioned in this example bears against the idea that an agent with libertarian free will must necessarily violate statistical laws. However, one might still wonder whether an agent governed not by an arbitrary statistical law but by the laws of quantum mechanics might somehow be constrained by them in a

[b]Pereboom similarly claims that "... although our being undetermined agent-causes has not been ruled out as a coherent possibility, it is not credible given our best physical theories." [3, 422] By this, he presumably endorses Sider's argument.

way that would either void his causal agency or necessarily violate the laws of quantum mechanics; i.e., by resulting in observable deviations from the Born Rule. It is this situation that is considered herein.

Concerning the Born Rule, the first thing to note is that in order to predict physically relevant probabilities of outcomes with the Rule, one must have a clearly defined system and a clearly defined observable being measured on that system. A system definition must be able to state how many degrees of freedom (usually considered as 'particles') are in play, and exactly what the initial state of that system is. An observable definition must be able to state exactly what forces are acting on the system and what sort of detection process constitutes the outcome of the observable being measured. These requirements may be straightforwardly met for microscopic systems in the laboratory, but it is a highly nontrivial matter as to whether they may be met under conditions obtaining in the context of human behavior.

Now, consider Sider's apparent assumption that a human agent, if truly free, should be able to make choices that would deviate from the Born Rule (for otherwise he would be a 'slave' to the rule and thus not be making free choices). Such a claim assumes that one could set up repeatable experiments in which the agent could be precisely defined as a quantum system in an unambiguous quantum state, whose applicable observable was also tightly enough defined so as to be able to detect such deviations. It is only if such deviations were in principle detectable that there could be a violation from the statistical laws of quantum mechanics, as observed in Clarke's remark quoted above. However, there are very good reasons to think that this is not the case.

For one thing, as noted above, one has to be able to perform precisely repeatable experiments. Does exposing a given human agent to repeated opportunities to make a choice constitute a precisely repeatable experiment of this type? Almost certainly not. The human agent is an open system, continually exposed to variable influences from his or her environment: air currents, radiant energy, etc; as well to internal fluctuations (number of blood cells in the brain, number of activated neurons, etc.). Assuming the brain is the most relevant bodily system concerning the choice, the state(s) and the number of relevant degrees of freedom in the brain are in continual flux. There is no justification for assuming that the agent would be in the same quantum state over any extended period of time, in particular the time interval in which repeated choices would be presented. No matter how tightly one might attempt to control the agent's environment, one is dealing with an enormously complex and under-defined system, from a quantum-mechanical perspective.

It we want to be more careful in the application of physical law, the example of Hitler facing different choices of action is better modeled as a macroscopic system with several possible macrostates, where the macrostates correspond to the choices. Each such macrostate is instantiated by an enormous number of microstates (we will see just how enormous in what follows). It is only at the level of the microstates that quantum laws would become relevant, not at the level of the macroscopic system of "Hitler's body and brain." So the choice discussed by Sider would more accurately be described by probabilities dictated by the Gibbs ensembles of statistical mechanics, not the Born Rule. However, quantum effects could be relevant at the level of individual microstates, such that quantum indeterminism might enter at that point.

Thus, there is indeed an entry for fundamental indeterminism allowing in principle for free will, but not in such a way that it would lead to observable Born Rule or other law violations. One can see this by noting that the number of microstates corresponding to a sample of ice in a macrostate defined by a temperature of 273K and normal atmospheric pressure is roughly $10^{1,299,000,000,000,000,000,000,000}$. (This is calculated using $S = klogW$, where S is the system's entropy, k is Boltzmann's thermodynamic constant, and W is the number of microstates corresponding to a given macrostate; and the experimentally determined value of S for the given ice macrostate is input.[c]) This is the number of microscopic (quantum-level) different possible configurations of the atoms in the ice that give rise to exactly the same outward, classically observable properties for the sample. (Just for comparison, the number of atoms in the entire universe is estimated to be "only" about 10^{80}.) Given that the body and brain are largely composed of water, the number of microstates involved in describing macroscopic human behaviorial states is of the same astonishing order.

Now suppose a human being could exercise free will by volitionally altering some of the microstates in his brain, exploiting quantum indeterminacy. Granted, this would require that the brain's neural wiring be delicately balanced so as to be able to manifest such alterations as changes in the relevant macrostates. But as a human brain is a far-from equilibrium biological

[c]Details are given by Lambert [5] at http://entropysite.oxy.edu/microstate/, where it is noted concerning this enormous number of microstates that "Writing 5,000 zeroes per page, it would take not just reams of paper, not just reams piled miles high, but light years high of reams of paper to list all those microstates!"

system, this is certainly a conceivable brain function. Could one detect any Born Rule violation as a result of this process? In order to do so, one would have to have exactly repeatable input/measurement/output data; that is, one would have to have data demonstrating that specific neural atoms or molecules underwent state transitions at rates not in conformity with the rule. Even if this were in principle possible, the number of microstates that would have to be taken into account would, as above, be hyper-astronomical. At the rate of recording even one microstate's atomic transitions per second (way too optimistic to be realistic), this would take hugely longer than the age of the universe. Clearly, the model of Hitler and his putative quantum choice-eigenstates is grossly oversimplified.

The example of mental activity influencing a choice raises another important consideration: if we are going to discuss the relevance of quantum indeterminism for free will, we must take into account whether it should be understood as describing only physical/material systems, or whether in some way it also pertains to mental activity. Now, a physicalist would deny that the mind is anything different from the physical brain, and an idealist would assert that quantum mechanics describes mental substance in various guises, among them apparently solid matter. Meanwhile, a dualist would presumably say that physical theory describes only material substance and therefore mind is not addressed by quantum theory. The very fact that there are very different metaphysical views concerning the nature of matter and mind, and the role of either substance in explaining and understanding human behavior, dictates that we must tread with extreme caution when invoking physical theories (whether presumed deterministic or not) as a basis to argue either for or against the existence of free will.

Against this backdrop, it would seem reasonable to point out, with prudence, at least the possibility that quantum indeterminism might provide an opening for free will–if only as an avenue of possible escape from the alleged 'fatedeness' of future actions. To rule out that possibility based on a demonstrably oversimplified application of quantum laws to choices modeled as quantum observables and human beings ostensibly labeled by stable-over-time quantum states would seem to be precipitous.

Even assuming that one could model human choices directly by quantum states corresponding one-to-one with specific choices (as opposed to taking into account the macrostate/microstate relationship), the crucial point is that at the level of individual instances the Born Rule gives only propensities for outcomes.[d] A human agent described as a huge quantum system might instantaneously be subject to those propensities yet, given quantum indeterminism, still have room to make a free choice (in the sense that the choice is made by the agent as a volitional, primary substance). If another instance outwardly presenting the same choice came before the agent, it is in fact overwhelmingly unlikely that the agent is in exactly the same state that she was just prior to the previous choice, so that the Born Rule propensities are likely not the same as in the previous instance. Even if the experiment is repeated many times, a resulting set of outcomes in which so many parameters are ill-defined and subject to change has no bearing on whether any particular statistical law is being violated.

Thus, it is a highly nontrivial matter to try to apply the Born Rule to macroscopic biological systems and their macroscopically defined choices; yet Sider's argument for the failure of free choice presumes without argument that one can straightforwardly do so.[e] This may not be possible due to the quickly changing and therefore ill-defined nature of the physical systems constituting the choosing agent during any relevant time interval, and the similarly ill-defined-over-time status of the "choice observable." (And that is to disregard the distinction between microstates and macrostates, where it is probably the latter that correspond to human choices.) Thus it is far from established that there would be any necessary statistical violation of the Born Rule resulting from free choices even if such choices are made possible by fundamental quantum indeterminacy.

Finally, the claim that free will must be anomic and "anti-scientific" also encompasses psychological laws. But the latter are either empirically observed statistical regularities or fallible theoretical models. In either case, no large-scale, apparently deterministic regularity is necessarily inconsistent with fundamental indeterminacy and the attendant possible opening for free will. For example, the Ideal Gas Law, $PV = nRT$, is a large-scale, apparently deterministic statistical effect of microscopic processes and yet does not conflict with fundamental quantum indeterminacy.

[d] Here we also need to correct an inaccuracy in the quoted argument by Sider: "quantum mechanics merely says that causes make their effects probable." No, a 'cause' (considered as a quantum state subject to measurement) can give rise to equally possible measurement outcomes. Thus a quantum mechanical 'cause' can set up a number of equally possible effects, none of which is any more probable than the others.

[e] Indeed the idea that macroscopic objects like humans are describable by quantum states, while routinely assumed, is also debatable. In particular this is not the case in the Transactional Interpretation; [6, pp. 112-115].

Concerning the fallibility of theoretical models of human behavior, and generalizations made from empirical observation: to every rule formulated to 'cover' human behavior, there is an exception. For example, this author recalls the following introductory statement in a college sociology textbook: "Everybody loves a parade." But on the contrary, certainly there are people do not love parades. Does that make their behavior anti-scientific? Perhaps more to the point, is an arbitrary volitional choice–one in which the choosing agent provides no reason or cause for their choose– 'antiscientific'? An affirmative answer to this question seems to be an underlying assumption of the arguments against agent causation. Yet if quantum theory indicates that genuine indeterminism is a feature of the world, then one can point to no reason or cause for an electron to 'choose' spin up over spin down when both are equally likely and equally possible. Yet one such outcome always occurs.[f] And quantum theory is a (set of) well-corroborated scientific law(s). Moreover, since one such outcome does in fact occur, quantum theory might even be seen as *demanding* some sort of primitively volitional capacity on the part of quantum systems. In fact, this is not a new idea: physicist Freeman Dyson famously opined that

> "... I think our consciousness is not just a passive epiphenomenon carried along by the chemical events in our brains, but is an active agent forcing the molecular complexes to make choices between one quantum state and another. In other words, mind is already inherent in every electron, and the processes of human consciousness differ only in degree but not in kind from the processes of choice between quantum states which we call "chance" when they are made by electrons." [9, p. 249]

2. Critique of causal notions invoked in support of the alleged lawlessness of free choice

Let us now turn to Sider's claim that we must be somehow "detached" from our choices if they are not considered as caused by our beliefs and/or desires [2, p. 121]. The idea that choices must be attributable to beliefs or desires is questionable in itself, since it does not take into account situations, as alluded to just above, in which we are called upon to make a completely arbitrary choice that does not involve any necessary belief or desire. For

[f]Here I disregard "many worlds theories" which hold that all outcomes occur, because (as I have argued elsewhere) I think they face serious problems; cf. [7]. I also disregard the "Bohmian" theory because of weaknesses in accounting for the putative corpuscles in the relativistic domain and because of a cogent argument by Brown and Wallace [8] that it amounts to a many-worlds picture.

example, an experimenter needs to generate some statistics for an experiment on an entangled EPR pair. He stands at one measuring device (a Stern-Gerlach apparatus) and his colleague stands at another. In each run, they must choose an orientation for the S-G magnet, say 'z' or 'x'. They have no belief or desire relevant to making each choice; it is completely arbitrary. Yet there is no logical incompatibility with each of their choices being free, and also connected to themselves as choosing agents.

Thus, there is no necessary 'detachment' in making a completely arbitrary choice. The experimenters make an arbitrary choice because they need to do so in order to conduct the experiment. At most, one might say that they have a 'belief' or 'desire' that they should have roughly the same number of x's and z's, but that does not constitute any belief or desire applying to any individual choice. This is similar to the scenario considered by Clarke in which an agent is given a target distribution and he freely chooses how to approximate that distribution in each individual case.

More generally, a claim such as Sider's concerning the causal relationship between an agent and his or her choice must be based both on specific theories of (i) causation and (ii) the 'Self' that is, on some notion of who or what it is that is to be considered as characterized by beliefs and desires, in what aspect of the Self those beliefs and desires consist, and in what sense the choosing Self is 'detached' from them if they are not subject to the notion of causality invoked in (i). These kinds of details are not provided in support of the claim, and it is far from clear that (apart from the consideration of arbitrary choices as above) it would hold across the board, in the context of all possible theories of causation and its relation to the Self.

Let us first consider (i): causation. I will not attempt to present and defend any particular theory of causation for purposes of this paper, which has a narrow focus on the "anomic" claim. What must be remembered, however, is that causation (as Hume pointed out) is not something that can be grounded empirically. It is never found in the observable world. Rather, causation is a vexed theoretical construct invoked in an attempt to explain the regularities that we see in the world. In particular, it is invoked as a kind of 'missing link' whenever we see that a particular type of input seems always to be followed by a unique type of output.

Since causation is not externally observed, it is certainly possible that it could be an aspect of volition, which (if it exists) is a wholly internal sort of influence. (Does viewing causation as an aspect of volition seem to conflict with the concept of causation as explaining apparently deterministic regularities? Not if it is kept in mind that the *apparent* determinism of the classical realm arises from a fundamental quantum indeterminacy, in the limit of large quantum numbers and/or large numbers of interacting degrees of freedom.) In this case, it would be entirely conceivable to regard the agent as the primary cause of his or her free choice, through an irreducible act of volition. As above, that volition need not be 'caused by' desires or beliefs; it could be invoked to fulfill an arbitrary criterion. This brings us to an appropriately non-detached view of the Self, point (ii): that is, the action can be considered as causally connected to the Self through that primary volitional act.

In this picture, the volitional power is the essence of the *external intervention* whose causal efficacy is crucial to libertarian free will. As incompatibilists correctly note, under strict determinism there is no external intervention in the flow of events. However, quantum theory predicts only a set of outcomes without specifying which of the set will occur; yet one always does. Consider this puzzling fact in light of Curie's Principle, a version of the Principle of Sufficient Reason which says that a specific outcome must always be attributable to a specific cause which actualizes that outcome as opposed to all others. This principle (which may or may not apply to Nature) specifies that the specific cause is a necessary condition for something to happen; absent that specific cause, nothing will actually happen.

If Curie's Principle does in fact apply to Nature, it would appear that some external intervention is actually necessary in order for any of that set of possible events to occur. Were this the case, it would directly rebut the presumption of 'randomness' attributed by Sider to an agent subject at some level to the Born Rule: on the contrary, the choosing agent would use his or her freedom of choice to provide the asymmetric cause demanded by Curie's Principle. (But even if Curie's Principle does not strictly apply to Nature at the quantum level, there is certainly room, in-principle, for the agent to 'tip the scales' for one outcome as opposed to the others.)

3. Conclusion

This paper has focused on two specific critiques of libertarian, agent-causal free will: (i) the claim that it must be anomic or "antiscientific"; and (ii) that it must be causally detached from the choosing agent. The present author is aware that the topic of volitional agent causation is a deep and vexed issue with a very lengthy literature. This paper does not address the broader concerns; its primary intent is to point out that the grounds often adduced for claiming that agent causation is inconsistent with 'our best scientific theories' [10] are weak, based as they are on misunderstandings (or at best, unsupported assumptions) concerning the nature and applicability of the quantum probability law to human choices. For example, O'Connor [11] discusses Pereboom's criticism that agent causation is (in O'Connor's terms) "inconsistent with seeing human beings as part of the natural world of cause and effect." But this statement and its attendant critiques of agent-causal free will presuppose a particular metaphysical view of the 'natural world' that excludes quantum indeterminism. Or, if quantum processes are (however cursorily) considered as part of the natural world, the quantum probability law is presumed (without argument and likely erroneously, as argued herein) to apply to human beings and their choices as being describable by well-defined, stable-over-time quantum states and eigenvalues of observables, respectively. Thus, the treatment of quantum theory in connection with free will has been considerably less careful than would warrant sweeping negative conclusions about its compatibility with processes in the natural world.

Moreover, the puzzle of indeterministic, seemingly uncaused 'collapse' to one outcome from a set of eligible outcomes seems to beg for an external intervention of some sort, in order to satisfy Curie's Principle. It might seem farfetched to think of quantum objects such as electrons or photons as having volition, yet it is certainly conceivable that some very primitive and elementary form of volition might obtain at this level. While volition is a conscious mental function, some of the quantum pioneers thought of the quantum domain as mental, or at least idea-like, in nature (for example, Heisenberg's non-actual 'potentiae' [12, pp. 154-155]. Pauli remarked that the quantum process of actualization of events "acausally weaves meaning into the fabric of nature." [13]. Clearly, 'meaning' is something that arises from the mental realm, not from inanimate material systems.

Considering the elementary constituents of matter as imbued with even the minutest propensity for volition would, at least in principle, allow the possibility of a natural emergence of increasingly efficacious agent volition as the organisms composed by them became more complex, culminating in a human being. And allowing for volitional causal agency to enter, in principle, at the quantum level would resolve a very puzzling aspect of the indeterminacy of the quantum laws–the seeming violation of Curie's Principle in which an outcome occurs for no reason at all. This suggests that, rather than bearing against free will, the quantum laws could be the ideal nomic setting for agent-causal free will.

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FUZZY APPROACH FOR CNOT GATE IN QUANTUM COMPUTATION WITH MIXED STATES

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In the framework of quantum computation with mixed states, a fuzzy representation of CNOT gate is introduced. In this representation, the incidence of non-factorizability is specially investigated.

Keywords: CNOT quantum gate, quantum operations, non-factorizability.

0. Introduction

The concept of quantum computing, introduced at the beginning of 1980s by Richard Feynman, is animated by the fact that quantum systems make possible new interesting forms of computational and communication processes. In fact, quantum computation can be seen as an extension of classical computation where new primitive information resources are introduced. Especially, the concept of quantum bit (qubit for short) which is the quantum counterpart of the classical bit. Thus, new forms of computational processes are developed in order to operate with these new information resources. In classical computation, information is encoded by a series of bits represented by the binary values 0 and 1. Bits are manipulated via ensemble of logical gates such as *NOT, OR, AND* etc, that form a circuit giving out the result of a calculation.

Standard quantum computing is based on quantum systems described by finite dimensional Hilbert spaces, such as $\mathbb{C}^2$, that is the two-dimensional Hilbert space where a generic *qubit* lives. Hence, a qubit is represented by a unit vector in $\mathbb{C}^2$ and, generalizing to a positive integer n, n-*qubits* are represented by unit vectors in $\mathbb{C}^{2^n} = \otimes^n \mathbb{C}^2 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$ (n- times). Similarly to the classical case, we can introduce and study the behavior of a number of *quantum logical gates* (hereafter quantum gates for short) operating on qubits. As in the classical case, a quantum circuit is identified with an appropriate composition of quantum gates. They are mathematically represented by unitary operators acting on pure states. In this framework only reversible processes are considered. But for many reasons this restriction is unduly. On the one hand, it does not encompass realistic physical states described by mixtures. In fact, a quantum system rarely is in a pure state. This may be caused, for example, by the incomplete efficiency in the preparation procedure and also by manipulations on the system as measurements over pure states, both of which produce statistical mixtures. It motivated the study of a more general model of quantum computational processes, where pure states and unitary operators are replaced by density operators and quantum operations, respectively. This more general approach, where not only reversible transformations are considered, is called *quantum computation with mixed states* [1,6,7,9,10]. In this powerful model, fuzzy logic can play an important role to describe certain aspects of the combinational structures of quantum circuits.

Our work is motivated on a fuzzy behavior of the CNOT quantum gate that arises when the model of quantum computation with mixed states is considered.

The paper is organized as follows: Section 1 contains generalities about tensor product structures to describe bipartite quantum systems. In Section 2 we recall some basic notions about the model of quantum computation with mixed states. In Section 3 we study fuzzy aspects of the CNOT when factorized inputs are considered. Section 4 generalizes the precedent section by considering non-factorized states. Finally in Section 5 we establish necessary and sufficient condition on the input of CNOT for which the factorizability of density operators in $\otimes^2\mathbb{C}^2$ is preserved by CNOT.

1. Bipartite systems

The notion of *state of a physical system* is familiar from its use in classical mechanics, where it is linked to the initial conditions (the initial values of position and momenta) which determine the solutions of the equations of motion for the system. For any value of time, the state is represented by a point in the phase space. In classical physics, compound systems can be decomposed into their subsystems. Conversely, individual systems can be combined to give overall composite systems. In this way, a classical global system is completely described in terms of the states of its subsystems and their mutual dynamic interactions. This property about classical systems is known as *separability principle*. From a mathematical point of view, the separability condition of classical systems comes from the fact that states of compound systems are represented as direct sum of the states of their subsystems.

In quantum mechanics the description of the state becomes substantially modified. A quantum state can be either pure or mixed. A *pure state* is described by a unit vector in a Hilbert space and it is denoted by $|\varphi\rangle$ in Dirac notation. When a quantum system is not in a pure state, it is represented by a probability distibution of pure states, the so called *mixed state*. Mixed states are mathematically modelled by *density operators* on a Hilbert space, i.e. positive self-adjoint, trace class operators. In terms of density operators, a pure state $|\psi\rangle$ can be represented as the projector $\rho = |\psi\rangle\langle\psi|$. Thus, in quantum theory, the most general description of a quantum states is encoded by density operators.

In quantum mechanics a system consisting of many parts is represented by the tensor product of the Hilbert spaces associated with the individual parts. We restrict our investigation to compound systems living in the bipartite Hilbert space of the form $\mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional. But not all density operators on $\mathcal{H}_1 \otimes \mathcal{H}_2$ are expressible as $\rho = \rho_1 \otimes \rho_2$, where ρ_i is a density operator living in $\mathcal{H}_i$, for $i \in \{1, 2\}$. Thus, there exist properties of quantum systems that characterize the whole system but that are not reducible to the local properties of its parts. Unlike classical physics, compound quantum systems can violate the separability principle.

From a mathematical point of view, the origin of this difference between classical and quantum systems arises from the tensor product structure related to the Hilbert spaces. More precisely, the non-factorizability property of quantum states is related to the fact that the direct sum of $\mathcal{H}_1$ and $\mathcal{H}_2$ is a proper subset of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

In what follows we provide a formal description of this instance of non-factorizability of quantum states in compound systems of the form $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Due to the fact that the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and I are a basis for the set of operators over $\mathbb{C}^2$, an arbitrary density operator ρ over $\mathbb{C}^2$ may be represented as

$$\rho = \frac{1}{2}(I + s_1\sigma_1 + s_2\sigma_2 + s_3\sigma_3)$$

where s_1, s_2 and s_3 are three real numbers such $s_1^2 + s_2^2 + s_3^2 \leq 1$. The triple (s_1, s_2, s_3) represents the point of the Bloch sphere that is uniquely associated to ρ. A similar canonical representation can be obtained for any n-dimensional Hilbert space by using the notion of generalized Pauli-matrices.

Definition 1.1. Let $\mathcal{H}$ be a n-dimensional Hilbert space and $\{|\psi_1\rangle, \ldots, |\psi_n\rangle\}$ be the canonical othonormal basis of $\mathcal{H}$. Let k and j be two natural numbers such that: $1 \leq k < j \leq n$. Then, the *generalized Pauli-matrices* are defined as follows:

$$^{(n)}\sigma_1^{[k,j]} = |\psi_j\rangle\langle\psi_k| + |\psi_k\rangle\langle\psi_j|$$

$$^{(n)}\sigma_2^{[k,j]} = i(|\psi_j\rangle\langle\psi_k| - |\psi_k\rangle\langle\psi_j|)$$

and for $1 \leq k \leq n-1$

$$^{(n)}\sigma_3^{[k]} = \sqrt{\frac{2}{k(k+1)}}(|\psi_1\rangle\langle\psi_1| + \cdots + |\psi_k\rangle\langle\psi_k| - k|\psi_{k+1}\rangle\langle\psi_{k+1}|).$$

If $\mathcal{H} = \mathbb{C}^2$ one immediately obtains: $^{(2)}\sigma_1^{[1,2]} = \sigma_1$, $^{(2)}\sigma_2^{[1,2]} = \sigma_2$ and $^{(2)}\sigma_3^{[1]} = \sigma_3$.

Let ρ be a density operator of the n-dimensional Hilbert space $\mathcal{H}$. For any j, where $1 \leq j \leq n^2 - 1$, let

$$s_j(\rho) = tr(\rho\sigma_j).$$

The sequence $\langle s_1(\rho) \ldots s_{n^2-1}(\rho)\rangle$ is called the *generalized Bloch vector* associated to ρ, in view of the following well known result:

Theorem 1.1. *15 Let ρ be a density operator of the n-dimensional Hilbert space $\mathcal{H}$ and let $\sigma_j \in \mathfrak{P}_n$. Then ρ can be canonically represented as follows:*

$$\rho = \frac{1}{n}I^{(n)} + \frac{1}{2}\sum_{j=1}^{n^2-1} s_j(\rho)\sigma_j$$

where $I^{(n)}$ is the $n \times n$ identity matrix.

A kind of converse of Theorem 1.1 reads: a matrix ρ having the form $\rho = \frac{1}{n} I^{(n)} + \frac{1}{2}\sum_{j=1}^{n^2-1} s_j(\rho)\sigma_j$ is a density operator if an only if its eigenvalues are non-negative.

Let us consider the Hilbert space $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$. For any density operator ρ on $\mathcal{H}$, we denote by ρ_a the partial trace of ρ with respect to the system $\mathcal{H}_b$ (i.e. $\rho_a = tr_{\mathcal{H}_b}(\rho)$) and by ρ_b the partial trace of ρ with respect to the system $\mathcal{H}_a$ (i.e. $\rho_b = tr_{\mathcal{H}_a}(\rho)$). For the next developments it is useful to recall the following technical result:

Lemma 1.1. *Let ρ be a density operator in a n-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ where $dim(\mathcal{H}_a) = m$ and $dim(\mathcal{H}_b) = k$. If we divide ρ in $m \times m$ blocks $B_{i,j}$, each of them is a k-square matrix, then:*

$$\rho_a = tr_{\mathcal{H}_b}(\rho) = \begin{bmatrix} trB_{1,1} & trB_{1,2} & \dots & trB_{1,m} \\ trB_{2,1} & trB_{2,2} & \dots & trB_{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ trB_{m,1} & trB_{m,2} & \dots & trB_{m,m} \end{bmatrix}$$
$$\rho_b = tr_{\mathcal{H}_a}(\rho) = \sum_{i=1}^{m} B_{i,i}.$$

Definition 1.2. Let ρ be a density operator in a Hilbert space $\mathcal{H}_m \otimes \mathcal{H}_k$ such that $dim(\mathcal{H}_m) = m$ and $dim(\mathcal{H}_k) = k$. Then ρ is said to be (m,k)-*factorizable* iff $\rho = \rho_m \otimes \rho_k$ where ρ_m is a density operator in $\mathcal{H}_m$ and ρ_k is a density operator in $\mathcal{H}_k$.

It is well known that, if ρ is (m,k)-factorizable as $\rho = \rho_m \otimes \rho_k$, this factorization is unique and ρ_m and ρ_k correspond to the reduced states of ρ on $\mathcal{H}_m$ and $\mathcal{H}_k$, respectively.

Suppose that $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$, where $dim(\mathcal{H}_a) = m$ and $dim(\mathcal{H}_b) = k$. Let us consider the generalized Pauli matrices $\sigma_1^a, \dots, \sigma_{m^2-1}^a$ and $\sigma_1^b, \dots, \sigma_{k^2-1}^b$ arising from $\mathcal{H}_a$ and $\mathcal{H}_b$, respectively.

If we define the following coefficients:

$$M_{j,l}(\rho) = tr(\rho[\sigma_j^a \otimes \sigma_l^b]) - tr(\rho[\sigma_j^a \otimes I^{(k)}])tr(\rho[I^{(m)} \otimes \sigma_l^b])$$

and if we consider the matrix $\mathbf{M}(\rho)$ defined as

$$\mathbf{M}(\rho) = \frac{1}{4} \sum_{j=1}^{m^2-1} \sum_{l=1}^{k^2-1} M_{j,l}(\rho)(\sigma_j^a \otimes \sigma_l^b)$$

then $\mathbf{M}(\rho)$ represents the "additional component" of ρ when ρ is not a factorized state. More precisely, we can establish the following proposition:

Proposition 1.1. *15 Let ρ be a density operator in $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$. Then,*

$$\rho = \rho_a \otimes \rho_b + \mathbf{M}(\rho).$$

The above proposition gives a formal representation of the instance of holism mentioned at the beginning of the section. In fact, a state ρ in $\mathcal{H}_a \otimes \mathcal{H}_b$ does not only depend on its reduced states ρ_a and ρ_b, but also the summand $\mathbf{M}(\rho)$ is involved. Let us notice that $\mathbf{M}(\rho)$ is not a density operator and then it does not represent a physical state.

2. Quantum computation with mixed states

As already mentioned, a qubit is a pure state in the Hilbert space $\mathbb{C}^2$. The standard orthonormal basis $\{|0\rangle, |1\rangle\}$ of $\mathbb{C}^2$, where $|0\rangle = (1,0)^\dagger$ and $|1\rangle = (0,1)^\dagger$, is generally called *logical basis*. This name refers to the fact that the logical truth is related to $|1\rangle$ and the falsity to $|0\rangle$. Thus, pure states $|\psi\rangle$ in $\mathbb{C}^2$ are superpositions of the basis vectors $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, where c_0 and c_1 are complex numbers such that $|c_0|^2 + |c_1|^2 = 1$. Recalling the Born rule, any qubit $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ may be regarded as a piece of information, where the numbers $|c_0|^2$ and $|c_1|^2$ correspond to the probability-values associated to the information described by the basic states $|0\rangle$ and $|1\rangle$, respectively. Hence, we confine our interesting to the probability value $p(|\psi\rangle) = |c_1|^2$, that is related to the basis vector associated with the logical truth.

Arbitrary quantum computational states live in $\otimes^n \mathbb{C}^2$. A special basis, called the 2^n-*computational basis*, is chosen for $\otimes^n \mathbb{C}^2$. More precisely, it consists of the 2^n orthogonal states $|\iota\rangle$, $0 \leq \iota \leq 2^n$ where ι is in binary representation and $|\iota\rangle$ can be seen as tensor product of states (Kronecker product) $|\iota\rangle = |\iota_1\rangle \otimes |\iota_2\rangle \otimes \ldots \otimes |\iota_n\rangle$, with $\iota_j \in \{0,1\}$. Then, a pure state $|\psi\rangle \in \otimes^n \mathbb{C}^2$ is a superposition of the basis vectors $|\psi\rangle = \sum_{\iota=1}^{2^n} c_\iota |\iota\rangle$, with $\sum_{\iota=1}^{2^n} |c_\iota|^2 = 1$.

As already mentioned, in the usual representation of quantum computational processes, a quantum circuit is identified with an appropriate composition of *quantum gates*, mathematically represented by *unitary operators* acting on pure states of a convenient (n-fold tensor product) Hilbert space $\otimes^n \mathbb{C}^2$ [14].

In what follows we give a short description of the model of quantum computers with mixed states.

We associate to each vector of the logical basis of $\mathbb{C}^2$ two density operators $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ that represent, in this framework, the falsity-property and the truth-property, respectively. Let us consider the operator $P_1^{(n)} = \otimes^{n-1} I \otimes P_1$ on $\otimes^n \mathbb{C}^2$. By applying the Born rule, we consider the probability of a density operator ρ as follows:

$$p(\rho) = Tr(P_1^{(n)} \rho). \tag{1}$$

Note that, in the particular case in which $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, we obtain that $p(\rho) = |c_1|^2$. Thus, this probability value associated to ρ is the generalization of the probability value considered for qubits. In the model of quantum computation with mixed states, the role of quantum gates is replaced by quantum operations. A *quantum operation* [11] is a linear operator $\mathcal{E} : \mathcal{L}(H_1) \to \mathcal{L}(H_2)$ - where $\mathcal{L}(H_i)$ is the space of linear operators in the complex Hilbert space H_i ($i = 1, 2$) - representable (following the first Kraus representation theorem) as $\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$, where A_i are operators satisfying $\sum_i A_i^\dagger A_i = I$. It can be seen that a quantum operation maps density operators into density operators. Each unitary operator U has a natural correspondent quantum operation $\mathcal{O}_U$ such that, for each density operator ρ, $\mathcal{O}_U(\rho) = U\rho U^\dagger$. In this way, quantum operations are generalizations of unitary operators. It provides a powerful mathematical model where also irreversible processes can be considered.

3. CNOT quantum operation as fuzzy connective

As in classical case, also in quantum computation it is useful to implement some kind of “if-then-else”operations. More precisely, it means that we have to consider the evolution of a set of qubits depending upon the values of some other set of qubits. The gates that implement these kind of operations are called “controlled gates”. The controlled gates we are interested on, is the *controlled-NOT gate* (CNOT, for short). An usual application of the CNOT gate is to generate entangle states, starting from factorizable ones. This is a crucial step for quantum teleportation protocol and quantum cryptography.

The CNOT gate, takes two qbits as input, a control qbit and a target qbit, and performs the following operation:

- if the control qbit is $|0\rangle$, then CNOT behaves as the identity
- if the control bit is $|1\rangle$, then the target bit is flipped.

Thus, CNOT is given by the unitary transformation

$$|i\rangle|j\rangle \mapsto |i\rangle|i\widehat{+}j\rangle$$

where $i, j \in \{0, 1\}$ and $\widehat{+}$ is the sum modulo 2. Note that, confining in the computational basis only, the behaviour of CNOT replaces the classical XOR connective. The matrix representation of CNOT is given by:

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{2}$$

Since CNOT is a unitary matrix, it naturally admits an extension as quantum operation. Noting that $\text{CNOT}^\dagger = \text{CNOT}$, its extension as quantum operation is given by:

$$\mathbb{CNOT}(\rho \otimes \sigma) = \text{CNOT}\ (\rho \otimes \sigma)\ \text{CNOT}. \tag{3}$$

Theorem 3.1. *Let ρ, σ be two density operators in $\mathbb{C}^2$. Then:*

$$p(\mathbb{CNOT}(\rho \otimes \sigma)) = (1 - p(\rho))p(\sigma) + (1 - p(\sigma))p(\rho).$$

Proof. Let

$$\rho = \begin{bmatrix} 1-a & r \\ r^* & a \end{bmatrix} \quad \textit{and} \quad \sigma = \begin{bmatrix} 1-b & t \\ t^* & b \end{bmatrix}$$

be density operators in $\mathbb{C}^2$. It is easy to check that the diagonal elements of $\rho \otimes \sigma$ are $d_{11} = (1-a)(1-b)$, $d_{22} = (1-a)b$, $d_{33} = a(1-b)$ and $d_{44} = ab$. Similarly, the diagonal elements of $\mathbb{CNOT}(\rho \otimes \sigma)$ are: $d'_{11} = d_{11}$, $d'_{22} = d_{22}$ and $d'_{33} = d'_{44}$ and $d'_{44} = d_{33}$. Thus

$$\begin{aligned} p(\mathbb{CNOT}(\rho \otimes \sigma)) &= d'_{22} + d'_{44} \\ &= (1-a)b + b(1-a) \\ &= (1 - p(\rho))p(\sigma) + (1 - p(\sigma))p(\rho). \end{aligned}$$ □

The above theorem allows us to consider $\mathbb{CNOT}$ as a fuzzy connective in accord to the probability value $p(\mathbb{CNOT}(- \otimes -))$.

In fact: let $x, y \in [0, 1]$; the usual product operation $x \cdot y$ in the unitary real interval defines the conjunction in the fuzzy logical system called

Product Logic [3]. The operations $\neg_{\text{Ł}} x = 1 - x$ and $x \oplus y = \min\{x + y, 1\}$ define the negation and the disjunction of the infinite value Łukasiewicz calculus respectively [4]. The operations $\langle \cdot, \oplus, \neg_{\text{Ł}} \rangle$ endow the interval $[0, 1]$ of an algebraic structure known as *Product MV-algebra* (PMV-algebra, for short) [5,13]. In this case the PMV-algebra $\langle [0, 1]\cdot, \oplus, \neg_{\text{Ł}} \rangle$ is the standard model of the a fuzzy logic system, called *Product Many Valued Logic.*

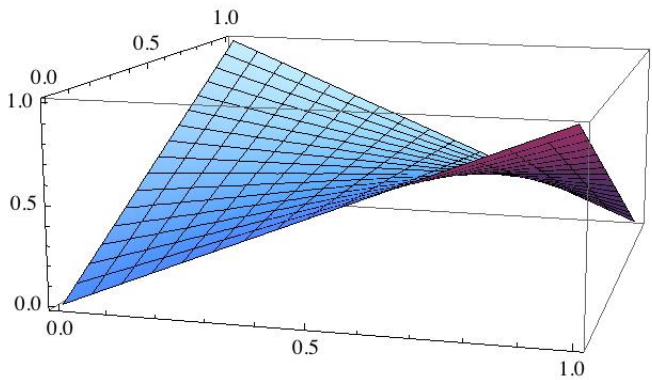
If ρ, σ are two density operators in $\mathbb{C}^2$, then $(1 - p(\rho))p(\sigma) + p(\rho)(1 - p(\sigma)) \leq 1$. Thus, $p(\mathbb{CNOT}(\rho \otimes \sigma))$ can be expressed in terms of PMV-operations. More precisely:

$$\begin{aligned} p(\mathbb{CNOT}(\rho \otimes \sigma)) &= (1 - p(\rho))p(\sigma) + (1 - p(\sigma))p(\rho) \\ &= (\neg_{\text{Ł}} p(\rho) \cdot p(\sigma)) \oplus (\neg_{\text{Ł}} p(\sigma) \cdot p(\rho)). \end{aligned}$$

In this way $\mathbb{CNOT}$ can be relate to the fuzzy connective given by the PMV-polynomial term $(\neg_{\text{Ł}} x \cdot y) \oplus (\neg_{\text{Ł}} y \cdot x)$, establishing a link between $\mathbb{CNOT}$ and a fuzzy logic system. Let us notice that there are other quantum gates admitting a similar fuzzy representation [8,9].

In Figure 1 we show the behavior of $p(\mathbb{CNOT}(- \otimes -)$ as a fuzzy connective.

Fig. 1. $p(\mathbb{CNOT}(\rho \otimes \sigma))$



4. $\mathbb{CNOT}$ on general density operators

In the precedent section we have introduced the behaviour of the $\mathbb{CNOT}$ gate on factorized states of the form $\rho \otimes \sigma$. For a more general approach, we now assume that the input state can be any arbitrary mixed state ρ in $\otimes^2\mathbb{C}^2$. We remark how this kind of studies suggests a holistic form of quantum logics [2,17]. Let ρ be a density operator in $\otimes^2\mathbb{C}^2$ and ρ_1, ρ_2 the reduced states of ρ. Since $\mathbb{CNOT}$ is linear, by Proposition 1.1, we have that

$$\mathbb{CNOT}(\rho) = \mathbb{CNOT}(\rho_1 \otimes \rho_2) + \mathrm{CNOT}(\mathbf{M}(\rho))\mathrm{CNOT}. \tag{4}$$

The summand $\mathbb{CNOT}(\rho_1 \otimes \rho_2)$ will be called the *fuzzy component* of $\mathbb{CNOT}(\rho)$ and we denote by $\mathcal{C}(\rho)$ the quantity $\mathrm{CNOT}(\mathbf{M}(\rho))\mathrm{CNOT}$.

Theorem 4.1. *Let ρ be a density operator in $\mathbb{C}^4$ such that*

$$\rho = (r_{ij})_{1\leq i,j\leq 2^2} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{bmatrix}.$$

Then:

(1) $p(\mathbb{CNOT}(\rho)) = r_{22} + r_{33}$,
(2) $p(\mathbb{CNOT}(\rho_1 \otimes \rho_2)) = (r_{11} + r_{22})(r_{22} + r_{44}) + (r_{11} + r_{33})(r_{33} + r_{44})$,
(3) $-1 \leq Tr(P_1\mathcal{C}(\rho)) = 2(r_{22}r_{33} - r_{11}r_{44}) \leq 1$,
(4) $Tr(P_1\mathcal{C}(\rho)) = \frac{1}{2}$ *iff,* $r_{11} = r_{44} = 0$ *and* $r_{22} = r_{33} = \frac{1}{2}$,
(5) $Tr(P_1\mathcal{C}(\rho)) = -\frac{1}{2}$ *iff,* $r_{11} = r_{44} = \frac{1}{2}$ *and* $r_{22} = r_{33} = 0$.

Proof. 1) It is immediate to see that $diag(\mathbb{CNOT}(\rho)) = (r_{11}, r_{22}, r_{44}, r_{33})$. Then $p(\mathbb{CNOT}(\rho)) = tr(P_1\mathbb{CNOT}(\rho)) = r_{22} + r_{33}$.

2) By Lemma 1.1 we have that

$$\rho_1 = \begin{bmatrix} r_{11} + r_{22} & r_{13} + r_{24} \\ r_{31} + r_{42} & r_{33} + r_{44} \end{bmatrix} \text{ and } \rho_2 = \begin{bmatrix} r_{11} + r_{33} & r_{12} + r_{34} \\ r_{21} + r_{43} & r_{22} + r_{44} \end{bmatrix}.$$

By Theorem 3.1 we have that

$$\begin{aligned} p(\mathbb{CNOT}(\rho_1 \otimes \rho_2)) &= (1 - p(\rho_1))p(\rho_2) + (1 - p(\rho_2))p(\rho_1) \\ &= (1 - (r_{33} + r_{44})(r_{22} + r_{44}) + (1 - r_{22} + r_{44})(r_{33} + r_{44}) \\ &= (r_{11} + r_{22})(r_{22} + r_{44}) + (r_{11} + r_{33})(r_{33} + r_{44}). \end{aligned}$$

3,4,5) By Proposition 1.1,

$$\begin{aligned} Tr(P_1\mathcal{C}(\rho)) &= p(\mathbb{CNOT}(\rho)) - p(\mathbb{CNOT}(\rho_1 \otimes \rho_2)) \\ &= r_{22} + r_{33} - (r_{11} + r_{22})(r_{22} + r_{44}) - (r_{11} + r_{33})(r_{33} + r_{44}) \\ &= 2(r_{22}r_{33} - r_{11}r_{44}). \end{aligned}$$

Note that $Tr(P_1\mathcal{C}(\rho))$ assumes the maximum value when $r_{11}r_{44} = 0$. If $r_{11} = 0$ then $1 = r_{2,2} + r_{3,3} + r_{4,4}$ and the maximum of $r_{2,2}r_{3,3}$ occurs when $r_{44} = 0$. It implies that $r_{2,2} + r_{3,3} = 1$. Thus $\max\{r_{2,2}r_{3,3}\}$ occurs when $r_{2,2} = r_{3,3} = \frac{1}{2}$. In this way, $\max\{Tr(P_1\mathcal{C}(\rho))\}$ occurs when $r_{11} = r_{44} = 0$ and $r_{22} = r_{33} = \frac{1}{2}$ and $\max\{Tr(P_1\mathcal{C}(\rho))\} = \frac{1}{2}$. With a similar argument we prove that $\min\{Tr(P_1\mathcal{C}(\rho))\} = -\frac{1}{2}$ and it occurs when $r_{11} = r_{44} = \frac{1}{2}$ and $r_{22} = r_{33} = 0$. □

Example 4.1. Werner states provide an interesting example to show the behavior of $\mathbb{CNOT}$ on a non-factorized states. Werner states, originally introduced in [18] for two particles to distinguish between classical correlation and the Bell inequality satisfaction, have many interests for their applications in quantum information theory. Examples of this, are entanglement teleportation via Werner states [12], the study of deterministic purification [16], etc. Werner states in $\otimes^2\mathbb{C}^2$ are generally represented by the following expression:

$$\rho_w(\alpha) = \frac{1}{4}\begin{bmatrix} 1-\alpha & 0 & 0 & 0 \\ 0 & 1+\alpha & -2\alpha & 0 \\ 0 & -2\alpha & 1+\alpha & 0 \\ 0 & 0 & 0 & 1-\alpha \end{bmatrix}.$$

where $\alpha \in [0,1]$. By Theorem 4.1 we have that:

$$p(\mathbb{CNOT}(\rho_w(\alpha))) = \tfrac{1+\alpha}{2},$$
$$p(\mathbb{CNOT}(\rho_w(\alpha)_1 \otimes \rho_w(\alpha)_2)) = \tfrac{1}{2},$$
$$Tr(P_1\mathcal{C}(\rho_w(\alpha)))) = \tfrac{\alpha}{2}.$$

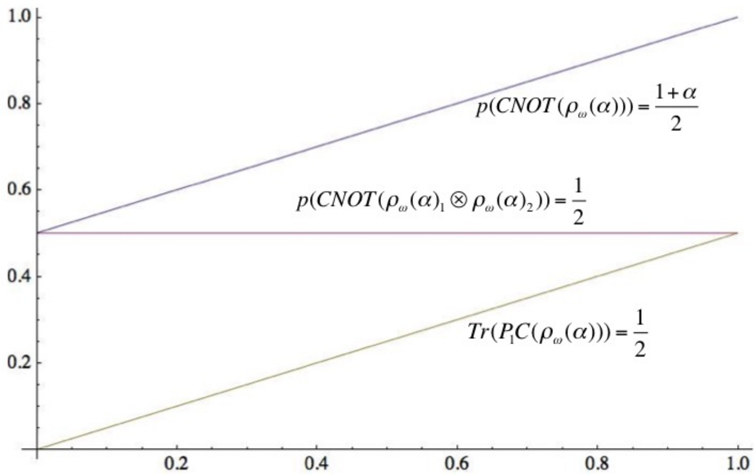
Note that, for each $\alpha \in [0,1]$ the probability value of the fuzzy component $p(\mathbb{CNOT}(\rho_w(\alpha)_1 \otimes \rho_w(\alpha)_2))$ does not change. Thus, the variation of probability value $p(\mathbb{CNOT}(\rho_w(\alpha)))$ is ruled by the variation of $Tr(P_1\mathcal{C}(\rho_w(\alpha))))$. The Figure 2 shows the incidence of $Tr(P_1\mathcal{C}(\rho_w(\alpha))))$ in the probability value $p(\mathbb{CNOT}(\rho_w(\alpha)))$ when the parameter w varies.

5. Preservation of factorizability by $\mathbb{CNOT}$

As we noted at the beginning of Section 3, CNOT allows us to entangled factorized states. As an example, if the control qbit is in a superposition state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ ($\alpha, \beta \neq 0$) and the target is $|0\rangle$, then CNOT generates the entangled state

$$(\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle \mapsto \alpha(|0\rangle \otimes |0\rangle) + \beta(|1\rangle \otimes |1\rangle).$$

Fig. 2. Incidence of $Tr(P_1\mathcal{C}(\rho_w(\alpha)))$



In this section we shall study a generalization of this situation. More precisely, we characterize the input $\rho \otimes \sigma$ for which $\mathbb{CNOT}$ generates a non-factorizable state.

Theorem 5.1. *Let ρ and σ be two density operators in $\mathbb{C}^2$. Then $\mathbb{CNOT}(\rho \otimes \sigma)$ is factorizable iff one of the following two conditions holds:*

(1) $\rho = \begin{bmatrix} a_1 & 0 \\ 0 & 1-a_1 \end{bmatrix}$ *and* $\sigma = \begin{bmatrix} \frac{1}{2} & b \\ b & \frac{1}{2} \end{bmatrix}$,
(2) $\rho = P_i$ *where* $i \in \{0,1\}$,
(3) $\sigma = \frac{1}{2}\begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{bmatrix}$.

Proof. Consider the following two generic density operators in $\mathbb{C}^2$

$$\rho = \begin{bmatrix} a_1 & a \\ a^* & 1-a_1 \end{bmatrix} \quad and \quad \sigma = \begin{bmatrix} b_1 & b \\ b^* & 1-b_1 \end{bmatrix}.$$

By Proposition 1.1, $\mathbb{CNOT}(\rho \otimes \sigma)$ has the following form:

$$\mathbb{CNOT}(\rho \otimes \sigma) = \mathbb{CNOT}(\rho \otimes \sigma)_1 \otimes \mathbb{CNOT}(\rho \otimes \sigma)_2 + \mathbf{M}(\mathbb{CNOT}(\rho \otimes \sigma)).$$

Thus we have to establish conditions on ρ, σ such that $\mathbf{M}(\mathbb{CNOT}(\rho \otimes \sigma)) = 0$. Since $\mathbf{M}(\mathbb{CNOT}(\rho\otimes\sigma)) = \mathbb{CNOT}(\rho\otimes\sigma) - \mathbb{CNOT}(\rho\otimes\sigma)_1 \otimes \mathbb{CNOT}(\rho\otimes\sigma)_2$, is straightforward to see that

$$\mathbf{M}(\mathbb{CNOT}(\rho \otimes \sigma)) = (x_{i,j})_{1 \le i,j \le 4}$$

where

11) $x_{11} = a_1(1-a_1)(1-2b_1) = -x_{22} = -x_{33} = x_{44}$,
12) $x_{12} = -2ia_1(a_1-1)Im(b)$,
13) $x_{13} = -a(b^* + 2Re(b)(a_1(2b_1-1)-b_1))$,
14) $x_{14} = a(b_1 - 2Re(b)(b^* + 2ia_1Im(b)))$.

23) $x_{23} = -a(b_1 - 1 + 2Re(b)(b - 2ia_1Im(b)))$,
24) $x_{24} = a(b^* - 2Re(b)(a_1 + b_1 - 2a_1b_1))$.

34) $x_{34} = 2ia_1Im(b)(a_1-1)$.

The other entries of $\mathbf{M}(\mathbb{CNOT}(\rho \otimes \sigma))$ are obtained by the conjugation of the above entries. Let us consider the system of equations

$$(x_{i,j} = 0)_{1 \le i,j \le 4}. \tag{5}$$

Note that $x_{11} = 0$ iff $b_1 = \frac{1}{2}$, $a_1 = 0$ or $a_1 = 1$. We shall study these cases

1 **Case** $b_1 = \frac{1}{2}$

By $x_{13} = 0$ we have that $-a(b^* - Re(b)) = 0$. Thus we have to consider two subcases, $a = 0$ or $b^* = Re(b)$ i.e. $b \in \mathbb{R}$.

1.1 Note that the conditions $b_1 = \frac{1}{2}$, $a = Im(b) = 0$ is a solution of the system (5) that characterize the input

$$\rho = \begin{bmatrix} a_1 & 0 \\ 0 & 1-a_1 \end{bmatrix} \quad and \quad \sigma = \begin{bmatrix} \frac{1}{2} & b \\ b & \frac{1}{2} \end{bmatrix}.$$

In this way $\mathbb{CNOT}(\rho \otimes \sigma) = \begin{bmatrix} \frac{a_1}{2} & a_1 b & 0 & 0 \\ a_1 b & \frac{a_1}{2} & 0 & 0 \\ 0 & 0 & \frac{1-a_1}{2} & (1-a_1)b \\ 0 & 0 & (1-a_1)b & \frac{1-a_1}{2} \end{bmatrix}$ which is factorizable as $\rho \otimes \sigma$.

1.2 $b \in \mathbb{R}$. By $x_{23} = 0$ we have that $-a(-\frac{1}{2} + b^2) = 0$, giving the following three possible cases:

- $a = 0$ which is the case 1.1.

– $b = \pm\frac{1}{2}$. It provides solutions to the system (5) that respectively characterizes the input

$$\sigma = \frac{1}{2}\begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{bmatrix} \text{ for an arbitrary } \rho.$$

In this way $\mathbb{CNOT}(\rho \otimes \sigma) = \frac{1}{2}\begin{bmatrix} a_1 & \pm a_1 & \pm a & a \\ \pm a_1 & a_1 & a & \pm a \\ \pm a^* & a^* & 1-a_1 & \pm(1-a_1) \\ a^* & \pm a^* & \pm(1-a_1) & 1-a_1 \end{bmatrix}$ which is factorizable as

$$\frac{1}{2}\begin{bmatrix} a_1 & \pm a \\ \pm a^* & 1-a_1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{bmatrix}.$$

2 **Case** $a_1 = 0$

By $x_{13} = 0$ $-a(b^* - Re(b)b_1) = 0$. Thus we have to consider two subcases, $a = 0$ or $b^* = 2Re(b)b_1$

2.1 Note that $a = a_1 = 0$ is a solution of the system 5 that characterizes the input: $\rho = P_1$ and arbitrary σ.
In this case $\mathbb{CNOT}(P_1 \otimes \sigma) = P_1 \otimes (\sigma_1 \sigma \sigma_1)$, where σ_1 is the Pauli matrix introduced above.

2.2 $b^* = 2Re(b)b_1$. Since $b_1 \in \mathbb{R}$, $b \in \mathbb{R}$ and $b(1 - 2b_1) = 0$. It provides two possibles subcases $b = 0$ or $b_1 = \frac{1}{2}$.

– $b = 0$. By $x_{14} = 0$ we have that $a = 0$ or $b_1 = 0$. The case $a = 0$ is an instance of the case 2.1. If $b_1 = 0$ then, the equation $x_{23} = 0$ forces $a = 0$ which is also an instance of the case 2.1.

– $b_1 = \frac{1}{2}$. It is an instance of the case 1.

3 **Case** $a_1 = 1$

By $x_{13} = 0$ we have that $-a(b^* + 2Re(b)(b_1 - 1)) = 0$. It provides two possibles subcases: $a = 0$ or $b(2b_1 - 1) = 0$ where $b \in \mathbb{R}$.

3.1 $a = 0$, $a_1 = 1$ is a solution of the system 5 that characterize the input: $\rho = P_0$ and arbitrary σ.
In this way $\mathbb{CNOT}(P_0 \otimes \sigma) = P_0 \otimes \sigma$.

3.2 $b(2b_1 - 1)$ gives two possibilities: $b = \frac{1}{2}$ or $b = 0$

– $b = \frac{1}{2}$ is an instance of the case 1.1.

– $b = 0$. By $x_{14} = 0$ we have that $ab_1 = 0$. The case $a = 0$ is an instance of the case 3.1. If $b_1 = 0$, by $x_{23} = 0$ follows that $a = 0$ which is also an instance of the case 3.1.

Thus we have analyzed all possible solution of the system 5 characterizing the preservation of factorizability for $\mathbb{CNOT}$. □

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A MODAL LOGIC OF INDISCERNIBILITY

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This paper is a continuation of the authors' attempts to deal with the notion of indistinguishability (or indiscernibility) from a logical point of view. Now we introduce a two-sorted first-order modal logic to enable us to deal with objects of two different species. The intended interpretation is that objects of one of the species obey the rules of standard S5, while the objects of the other species obey only the rules of a weaker notion of indiscernibility. Quantum mechanics motivates the development. The basic idea is that in the 'actual' world things may be indiscernible but in another accessible world they may be distinguished in some way. That is, indistinguishability needs not be seen as a necessary relation. Contrariwise, things might be distinguished in the 'actual' world, but they may be indiscernible in another world. So, while two quantum systems may be entangled in the actual world, in some accessible world, due to a measurement, they can be discerned, and on the other hand, two initially separated quantum systems may enter in a state of superposition, losing their individualities. Two semantics are sketched for our system. The first is constructed within a standard set theory (the ZFC system is assumed at the metamathematics). The second one is constructed within the theory of quasi-sets, which we believe suits better the purposes of our logic and the mathematical treatment of certain situations in quantum mechanics. Some further philosophically related topics are considered.

Keywords: Indiscernibility; non identity; modal logic; quasi-set theory; non-standard semantics.

1. Introduction

> "[P]hotons (...) are individuals without identity."
> (Herman Weyl [23, p.246])

Classical logic and standard mathematics (say, the mathematics that can be developed within a standard set theory such as ZFC, the Zermelo-Fraenkel set theory with the Axiom of Choice, with our without *Urelemente* —or simply *atoms*) are 'Leibnizian' in the sense that indiscernible things —sets having the same elements or atoms belonging to the same sets— turn out to be the very same thing. The standard approach to deal with indiscernible things within such frameworks is to consider some congruence relation other than identity to represent indiscernibility. In this case the elements belonging to the same equivalence class are taken as 'indiscernible'. So good for physics (for this may work), so bad for philosophy, because even if equivalent, the objects are in fact either distinct or identical, and in this last case they are the very same object. Leibniz reigns in classical logic. So, we address the following question: suppose we admit a metaphysics comprising truly indiscernible objects, objects that may form aggregates with more than one element but so that these elements are in no way discernible from one another. As Frank Ramsey and Ludwig Wittgenstein have noted when they criticized the (Leibnizian) definition of identity in *Principia Mathematica* (PM), this is a logical possibility [20, p.28], [24, 5.5302].[a] Nowadays, quantum physics seems to provide reasons to suppose that this may in fact be the case: bosons may share the same quantum state and in some cases nothing in the universe can distinguish them, and this of course would include logic. Fermions, on the contrary, must obey Pauli's exclusion principle, so they cannot partake all their quantum numbers. Thus, in principle, they always present a distinction. But the problem is that in certain situations (say in an entangled state), again, nothing in the universe can tell us which is which. So, their (alleged) individuality is lost. In both cases, it is doubtful whether they can be regarded as *individuals* in the standard (and informal) sense of the word: an individual, by definition, is something that *has an identity*, something that *retains its identity* even when mixed among others of similar kind. But quantum objects seem to be entities of another category: their indiscernibility and *non-individuality* seem to come from an ontological source!

[a]The fact that the definition of PM involves a hierarchy of types and not set theoretical notions does not matter here, for the very idea continues to be the same: indistinguishable objects are identical.

We have already introduced a first-order logic to cope with indiscernible but non-identical objects in [3]. Here we extend the discussion by arguing that indiscernibility, read as a binary relation, needs not be necessary (in the alethic modal sense), that is, two objects[b] may be indistinguishable in *this world* but may be discernible in another accessible world. A typical example is given by an entangled state. In this state (suppose involving two quantum objects for simplicity) they cannot be discerned, but after a measurement, they can (at least this is what we tend to agree with). Conversely, two discernible objects (note that we don't need to use the words 'identical' and 'distinct', for 'indiscernible' and 'discernible' suffice; see [17]) may enter in a superposed state, and in this case something completely new arises: their 'separate' wave functions don't exist anymore, but a new one has arisen, one for the composed system, which have no more 'parts' in the usual sense.

The system we shall present is a simple one, but perhaps its most striking features and novel facts concern semantics. We provide two distinct semantics to it, a move that shows that the consideration of the metamathematics used in discussing semantics may have important consequences. The first semantics is developed within a standard framework (which can be taken to be ZFC). We argue that this semantics has some shortcomings in the sense that it rules out the metaphysical assumption of indiscernible but non-identical things (in its strong sense that these objects are *really* indiscernible and not just 'epistemologically indiscernible'). Then we develop a semantics based on quasi-set theory and discuss it from the logical and philosophical points of view.

An important point to be mentioned concerns the content of the last footnote. We can certainly speak of *two* objects as being indiscernible and this does not entail that they are *different*. Let us explain this a little further. In order to say that they are different, we need to grant that there is a property (characteristics, attribute, whatever you wish) satisfied by just one of them. Of course (within classical mathematics) we don't need to explicitly present this property, but just show that it exists. For quantum objects, as it is well known (if the reader is skeptic with this, take a collection of bosons in the same quantum state) there may be not such a property, and even so they are not taken to be the same entity. But if we use the resources of standard mathematics, we can grant the existence of such a property, for standard mathematics is committed with Leibniz's laws. So, we beg the question at the start, for we are showing that two entities are

[b]Notice that our talk of two objects does not entail that they have an *identity*. See below.

distinct *because* this hypothesis is consonant with our metaphysics (and our mathematics), which already involves this idea in its core.

Let us go now to our logic and then to further discussions.

2. An elementary modal logic of indiscernibility

In this section we present a minimal nucleus of an elementary modal logic comprising a notion of indiscernibility which does not collapse into identity. Let $\mathscr{L}$ be a first-order two sorted language defined as follows:

Definition 2.1. The language $\mathscr{L}$ encompasses the following list of primitive symbols:

1. [Propositional connectives]: $\neg$ (negation) and $\rightarrow$ (material implication);
2. [Universal quantifier]: the universal quantifier $\forall$;
3. [Modal operator]: the unary operator $\Box$ (necessity),
4. [Relations]: two binary predicate symbols, $\approx$ (identity) and $\cong$ (indistinguishability, or indiscernibility);
5. [Variables]: two species of them: (i) variables of the first species $x_1, x_2, \ldots$ (which we represent by x, y, z) and (ii) variables of the second species $X_1, X_2, \ldots$, represented by X, Y, Z;
6. [Predicates]: for every $n \in \omega$, a denumerable collection of non-logical predicates of arity n.
7. [Auxiliary symbols]: parentheses and comma.

In this formulation, we shall make use of neither individual constants nor function symbols. We use t with eventual subscripts for terms in general (that is, variables) of any of the two species.

Before we define formulas, we define the type of a predicate. We call ι the *type of terms for the first species*, which intuitively represent entities without identity conditions, and we call η the *type of terms of the second species*, representing everyday objects, (supposedly) having identity conditions. Every predicate will have a type, in the sense that it relates entities of a certain type in a given order. For instance (see the definitions below), a predicate of type $\langle \eta, \iota, \eta \rangle$ is a relation of weight 3 and relates objects of types η, ι, η respectively.

Definition 2.2. A type τ is any finite sequence composed of η and ι.

For instance, $\langle \iota, \iota \rangle$, $\langle \eta, \iota, \eta \rangle$ are also types.

Definition 2.3. The type τ^P of a n-ary predicate symbol P is a type of length n.

There are three possible cases for the type of a predicate symbol. On the extremes, the type of a predicate may be composed only of η or only of ι; in those cases we call the predicate *classical* (intuitively, it relates only classical objects) and *non-classical* (it relates only entities without identity conditions), respectively. On the other hand, some types are mixed. For instance, the type $\langle \eta, \iota \rangle$ is the type of a binary relation that relates a classical object to a non-classical object. The same holds for other possible combinations. In principle, nothing should prevent a predicate from having more than one type. One instance could be the relation 'is a part of'. One could reasonably claim that some electrons are part of Angelina, and also that her lips are part of her. The former are non-classical objects according to our classification, the latter are classical, while Angelina is a classical object too. So a relation like this would have two types, $\langle \eta, \eta \rangle$ and $\langle \iota, \eta \rangle$. To simplify matters here, the only relation allowed to have more than one type will be $\cong$, which has two types, a classical and a non-classical, that is, it has types $\langle \eta, \eta \rangle$ and $\langle \iota, \iota \rangle$. In that sense, it relates both kinds of things, but never mixes them. The type of $\approx$ is $\tau^{\approx} = \langle \eta, \eta \rangle$, because as we shall require, identity holds only between classical things.

The atomic formulas are defined as comprising the expressions of the form $x \cong y$, $X \cong Y$, $X \approx Y$, and for predicate symbols P of type τ^P, $Pt_1, \ldots, t_n$ is a formula provided that each t_i is of the appropriate type as required by τ^P. The usual clauses for connectives, quantifiers and modal operators are also assumed. It follows from our definition that the binary predicate of identity does not apply to the objects denoted by the variables of the first species (objects of the first species). All the standard definitions of the other sentential connectives, the operator for possibility $\Diamond$ and the existential quantifier are defined as usual.

The postulates of our logic are the following ones (see also [3]). First, a standard complete axiom system for modal first-order classical logic with identity with respect to the objects of the second species. As for the objects of the first species, the corresponding axioms of the same logic, except those of identity. Furthermore, we add the following axioms:

(i) The relation of indiscernibility is reflexive, symmetric and transitive with respect to objects of the first species.
(ii) $\forall X \forall Y (X \cong Y \to X \approx Y)$
(iii) The axioms for the modal elementary system S5. We use S5 here

for convenience, but perhaps other systems could be used instead. For what we shall state below, the modal system K seems to be enough.

We remark that the substitution law (sometimes called Leibniz Law) is valid only with respect to identical objects of the second species. That is, we have that

$$\forall X \forall Y (X \approx Y \to (\alpha(X) \to \alpha(Y))$$

with the standard restrictions, but it is NOT the case that

$$\forall x \forall y (x \cong y \to (\alpha(x) \to \alpha(y)).$$

Thus, identity and indistinguishability are distinct relations. The reciprocal of the above axiom (ii) is a theorem, as we shall see below. So, our system requires of objects of second species that whenever indiscernibility holds, identity holds too (indiscernibility implies identity and vice-versa); objects of first species, on the other hand, may at best be indiscernible. As we remarked before, in our intended interpretation, we think of the objects of second species as those physical objects described by classical physics, that is, roughly speaking, the objects of our surroundings (in our scale). The objects of the first species will be thought of as representing elementary quantum objects. Here we follow Schrödinger in that the notion of identity lacks sense with respect to them [21, pp.17-8] (see [10, chap.3] for details).

Theorem 2.1 (The reciprocal of axiom (ii)). *For all X and Y, $X \approx Y \to X \cong Y$. In words, identical objects of the second species are indiscernible.*
Proof:

1. $X \approx Y$ *(premise)*
2. $X \approx Y \to (X \cong X \to X \cong Y)$ *(instance of the substitutivity axiom for objects of the second species)*
3. $X \cong X \to X \cong Y$ *(1,2, Modus Ponens)*
4. $\forall X(X \cong X)$ *(axiom)*
5. $\forall X(X \cong X) \to X \cong X$ *(axiom)*
6. $X \cong X$ *(4, 5, Modus Ponens)*
7. $X \cong Y$ *(3, 6, Modus Ponens)*
8. $X \approx Y \to X \cong Y$ *(1-7, Deduction Theorem)* ▮

The theorem and the axioms show that identity and indiscernibility coincide for objects of the second species (denoted by the variables of the second species). As for the Deduction Theorem, it can clearly be applied in this case. Having that result, other important theorems are easy to prove, such as that indiscernibility is reflexive, symmetric and transitive also for objects of the second species, and also that a version of the substitution of indiscernibles holds for objects of the second species too.

The assumption of classical logic for the underlying logic of our system is only a choice of ours. The idea of having a system to deal with indiscernible but non identical objects is a very general one, and it would surely be interesting to study it associated with other systems as well, such as some paraconsistent logics (see [6]).

3. Classical semantics

We now introduce a 'classical' Kripke-style semantics for our system; we call it classical in the sense that the underlying framework in which it is developed is ZFC. After presenting it, we shall be able to discuss the inadequacy of this semantics for encompassing the underlying motivations for the logic presented above, given that the semantics is formulated within a standard framework such as ZFC, which involves the notion of identity for all objects, while our system suggests that for the objects of the first species the concept of identity should not be applied.

The Kripke semantics we have in mind makes use of a domain of objects D composed of two disjoint sets D_1 and D_2 such that the variables of the first species have as their range the set D_1 and the variables of the second species range over the elements of D_2. Now, one of the guiding intuitions behind the modal behavior of our indiscernibility relation concerns the fact that some entities (e.g. a pair os electrons) may be indiscernible in our world, but could be discerned in some other world (for instance, as a result of a measurement). To grant that indiscernible objects in one world may be discernible in other world, and to preserve indiscernibility by properties in cases where the relation $\cong$ holds between two entities in one world, we begin by endowing worlds $w \in W$ with the structure of a *Weyl aggregate*, defined as follows:

Definition 3.1. A Weyl aggregate over a non-empty set D is an ordered pair $\langle D, \sim\rangle$, where $\sim$ is an equivalence relation over D.

Intuitively, a Weyl aggregate determines a partition of D into classes of 'indiscernible' elements, in the sense that the elements of the same equivalence classes are thought of as the indiscernible objects of the set D [23, App. B]. Our next step is to consider each world w as a Weyl aggregate over D_1; each world will therefore provide for a particular partition of D_1. To avoid ambiguities, we shall denote by $\sim_i$ the equivalence relation which interprets the indistinguishability relation in w_i. The relation of identity is interpreted in the diagonal of D_2, that is, the set $\Delta_{D_2} = \{\langle x_i, x_i \rangle : x_i \in D_2\}$. The indistinguishability relation $\cong$ stands just for the equivalence relation $\sim_i$ in each world w_i when dealing with objects of the first type, and it stands for the identity relation when dealing with objects of the second type. The identity sign means the identity relation over the domain D_2.

More formally, a frame $\mathcal{F} = \langle W, R \rangle$ is comprised by a set W of worlds and an accessibility relation R between worlds which is an equivalence relation between worlds (since we are working within S5). Given a domain $D = D_1 \cup D_2$, each $w_i \in W$ is endowed with the structure of a Weyl aggregate $w_i = \langle D_1, \sim_i \rangle$, as defined above. For ease in reading the next definition, we associate with each type τ a function f from types τ of predicates to the domains of interpretation as follows, defined by induction:

(i) $f(\eta) = \mathcal{P}(D_2)$.
(ii) $f(\iota) = \mathcal{P}(D_1)$
(iii) for $\tau = \langle p_1, \ldots, p_n \rangle$, then $f(\tau) = \mathcal{P}(f(p_1) \times \ldots \times f(p_n))$.

Notice that $f(\eta)$ is the set of all properties of objects of the second type, $f(\iota)$ is the set of all properties of objects of the first kind, and for eah type τ there is a set of all the relations of type τ. For instance, for $\tau = \langle \iota, \eta \rangle$, $f(\tau)$ is the set of all relations between objects of the first kind (elements of D_1) with objects of the second kind (that is, elements of D_2).

Now we generalize the indistinguishability relation $\sim_i$ of each world w_i to every object of the domain and to properties and relations of every type defined in the domain. Notice that the generalization is always relative to a world w_i.

Definition 3.2. Extending $\sim_i$. Given w_i, we define:

(i) for $a, b \in D_2$, $a \sim_i b$ if and only if $a = b$ (that is, indiscernibility for classical objects in the metalanguage is identity)
(ii) for two n-tuples $\langle a_1, \ldots a_n \rangle$ and $\langle b_n \ldots b_n \rangle$ of elements of D_1 and D_2 (that is, each element of the n-tuple has a type), we define $\langle a_1, \ldots a_n \rangle \sim_i \langle b_n \ldots b_n \rangle$ iff $a_1 \sim_i b_1$ and $\ldots$ and $a_n \sim_i b_n$.
(iii) given τ and X and Y elements of $f(\tau)$, we say that $X \sim_i Y$ iff for each $a \in X$ there is a $b \in Y$ and $a \sim_i b$ and, for each $b \in Y$ there is an $a \in X$ such that $b \sim_i a$ and $card(X) = card(Y)$.

So, we have an indiscernibility relation $\sim_i$ for every element of the range of the function f. That is, we can have a partition of $f(\tau)$ for each τ

relative to $\sim_i$. For instance, for ι, two subsets of D_1 may be indiscernible without being identical: it is enough that they have the same cardinal and indiscernible elements of the appropriate kind.

A Kripke model is a 4-tuple $\mathscr{K} = \langle W, R, D, V\rangle$ (in the sense of [13, p.243]), where D is as above and V is an interpretation mapping such that:

(i) $V(\approx, w_i) = \Delta_{D_2}$, for every $w_i \in W$
(ii) $V(\cong, w_i) = \sim_i \cup \Delta_{D_2}$; that is, the indiscernibility relation provided by the Weyl aggregate and the identity in D_2;
(iii) $V(P, w_i) \in f(\tau^P)$.

Remark: The reader should pay attention to the symbology: $=$ is the metatheoretical identity (that is, the identity in ZFC), while $\approx$ is the symbol of identity of our language. As usual, we leave to the context the distinction between use and mention.

For instance, we may have $D = D_1 \cup D_2 = \{a, b, c\} \cup \{1, 2\}$, so that $\sim_1$ may be the following relation, which extends the identity relation:

$$\sim_1 = \{\underbrace{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle 1, 1\rangle, \langle 2, 2\rangle}_{\text{identity}}, \langle b, c\rangle, \langle c, b\rangle\}.$$

Now, an assignment μ of the variables of $\mathscr{L}$ to elements of the domain so that $\mu(x) \in D_1$ and $\mu(X) \in D_2$. Each formula has a truth value V_μ which may be 1 (truth) or 0 (false) relative to an assignment, defined as follows:

(i) $V_\mu(x \cong y, w_i) = 1$ iff $\langle \mu(x), \mu(y)\rangle \in \sim_i$, and 0 otherwise.
(ii) $V_\mu(X \approx Y, w_i) = 1$ iff $\mu(X) = \mu(Y)$.
(iii) $V_\mu(Pt_1 \ldots t_n, w_i) = 1$ iff there is an X in $f(\tau^P)$ such that $X \sim_i V(P, w_i)$ and $\langle \mu(t_1) \ldots \mu(t_n)\rangle \in X$
(iv) $V_\mu(\neg\alpha, w_i) = 1$ iff $V_\mu(\alpha, w_i) = 0$, and 0 otherwise.
(v) $V_\mu(\alpha \to \beta, w_i) = 1$ iff $V_\mu(\alpha, w_i) = 0$ or $V_\mu(\beta, w_i) = 1$.
(vi) $V_\mu(\Box\alpha, w_i) = 1$ iff $V_\mu(\alpha, w') = 1$, for every $w' \in W$ such that $w_i R w'$, and 0 otherwise.
(vii) $V_\mu(\forall s\alpha, w_i) = 1$ (where s is a variable of either the first or of the second species) iff $V_\rho(\alpha, w_i) = 1$ for every assignment ρ which is a s-alternative of μ, that is, such that for every variable t except possibly s, (being t of the same species as s), $\rho(t) = \mu(t)$.

We emphasize that we can in fact assert the last affirmation, namely, that $\rho(t) = \mu(t)$ even in the case of t being a variable of the first species, for both $\rho(t)$ and $\mu(t)$ are elements of a 'standard' set, that is, a set in ZFC, so that the identity relation makes sense for them.

Now, as usual, α is valid in a model $\mathscr{K}$ iff $V_\mu(\alpha, w) = 1$ for every w and μ. Perhaps the most important thing is to discuss the nature of such a 'semantics' for our logical system. We believe that logic comprises semantics, at least an informal one (see also [5]). If this was not so, we wouldn't have contact, for instance, with the 'meaning' of intuitionistic logic. Indeed, as is well known, but sometimes misunderstood, from the formal point of view we can get intuitionistic logic (better, the so called Brouwer-Heyting system) by dropping the excluded middle law from classical logic (in Kleene's axiomatization, for instance; see [14]). But this is only from a formal point of view, for the meanings of these two logics are completely different. In the same vein, in order to keep our logic consonant with indiscernible objects but so that indiscernibility does not collapse into identity, we need to speak of a 'philosophically well-grounded' semantics for our system. To do so, we turn to a non-classical set theory to be used in the metamathematics (see also [1]). Before that, let us insist a little about the reason of this move.

The problem concerning the above semantics is that the relation of identity makes sense (in the metamathematics) also for the elements of D_1, because D_1 is a standard set (in our chosen metatheory, ZFC). But we have interpreted the indiscernibility relation $\cong$ as an equivalence relation $\sim$ only, due to the chosen axioms, and not necessarily in the identity of D_1. But such a semantics is not in agreement with the real spirit of our logic, for identity continues to hold (in the metamathematics) for the objects interpreted by the variables of the first species. Thus, we should require a semantics for our logic where the notion of identity would not apply to the elements in the domain of the variables of the first species. Of course this cannot be done within a standard set theory such as ZFC. For this reason, we sketch below a semantics for our logic constructed within *quasi-set theory.* But before that, let us comment on some peculiarities of our system.

4. Peculiarities of our system

Certain standard results that hold in the modal predicate system S5 don't have analogue when the considered relation is indistinguishability and not identity. In this section we shall present some of these results and try to interpret them within a context, namely, by supposing that the objects of the first species play the role of quantum objects according to a standard interpretation of quantum mechanics (that is, in most cases, we shall be thinking in some version of quantum mechanics that enables us to suppose indiscernible but not identical objects, thus we leave versions like Bohm's out of this discussion).[c]

The first result is that $x \cong y$ does not entail $\Box(x \cong y)$, that is, the non-necessity of indistinguishability. To show that, as usual, we need to consider a conter-example to the formula $x \cong y \to \Box(x \cong y)$, that is, a model of $x \cong y$ that is not a model of $\Box(x \cong y)$. For that, consider a set of worlds $W = \{w_1, w_2\}$, $D_1 = \{a, b\}$, D_2 is any non-empty set whatever. Now, given that each world is a Weyl aggregate, we put $w_1 = \langle D_1 \sim_1 \rangle$, with $\sim_1 = \{\langle a, a\rangle, \langle b, b\rangle, \langle a, b\rangle, \langle b, a\rangle\}$, while $w_2 = \langle D_1, \sim_2 \rangle$, with $\sim_2 = \{\langle a, a\rangle, \langle b, b\rangle\}$. That is, a and b are indiscernible in w_1, but are not indiscernible w_2. Also, R is such that w_1 accesses w_2 and vice versa. Consider an assignment μ such that $\mu(x) = a$ and $\mu(y) = b$. It is simple to see that $V_\mu(x \cong y, w_1) = 1$, $V_\mu(x \cong y, w_2) = 0$, so that $V_\mu(\Box(x \cong y), w_1) = 1$.

Thus, while indiscernibility is not necessary, the same doesn't happen with identity, as the next theorem shows. Really, we can easily show that $X \approx Y \to \Box(X \approx Y)$ (the necessity of identity) as follows:

1. $X \approx X$ (logical thesis)
2. $X \approx Y \to (\alpha(X) \to \alpha(Y))$ (axiom — the substitutivity of identity)
3. $X \approx Y \to (\Box(X \approx X) \to \Box(X \approx Y))$ (instance of the axiom, by taking $\alpha(X)$ as $\Box(X \approx X)$.
4. $\Box(X \approx X)$ (from 1, by Gödel's rule of necessitation)
5. $X \approx Y \to \Box(X \approx Y)$ (from 3 and 4, by the standard logical rules)

But, if indiscernibility doesn't entail necessary indiscernibility, what does this result mean?

We can interpret this result by saying that objects that are indiscernible in the actual world (termed w_0) may be not indiscernible in another accessible world (call it w_1). That is, there may be a world, accessible to us, where indiscernible objects from our point of view may not be indiscernible. In particular, suppose that a and b are indiscernible individuals in w_0, and R is the accessibility relation, so that $w_0 R w_1$, and in w_1 it is not the case that a and b are indiscernible. This fact could be taken to imply either that there are properties in w_1 which do not exist in w_0, which would introduce a kind of *possibilia* concerning properties, or the individuals a and b

[c]In fact, Bohm's metaphysics coincides with classical mechanics in that all its objects are *individuals*.

are simply discernible in ω_1. Leaving aside the discussion of possibilia in regarding properties (we haven't found in the literature any discussion on this topic), we prefer the other alternative. A possible situation may be as follows.

Suppose that in ω_0 we have two particles of the same species (two photons, say) correlated as in an EPR-like experiment. Before they have interacted, each of them may be described by its own wave function, which gives us their quantum states. After interaction, the quantum state is described by either a symmetric (for bosons) or by an anti-symmetric (for fermions) wave-function, so that their individual characteristics become blurred; if we accept to speak of quantum objects behind the formalism, they are now indiscernible by all means provided by quantum mechanics (or even by God, if God is not a logician —see below): they are in a state of superposition, and cannot be described in isolation. But now we perform the experiment, sending the particles to opposite directions to detectors A and B where we measure, say, the spin of particle 1 which arrived in A in the z–direction, getting an 'UP' result.[d] Thus, says quantum mechanics, we *know* by what Einstein called 'a spooky action at a distance' that the spin of particle 2 in B is down, even without measuring it. We shall not pursue the details here, which can be found in standard books on quantum mechanics. The important thing is that the particles (perhaps it would be better to say, their correspondents in the world w_1), become discernible! But, it is quite important to say, if the two particles were of the same species, we cannot identify *which one* (of the pair of particles) had its state measured at A. This is a typical quantum result that will be important for what follows.

Thus, we have a situation where two objects (the involved particles) are indiscernible in a "world" but discernible in another accessible "world". But, what matters here, indiscernibility needs not be necessary, as our logic shows. That is, it may be that in some other world w_2, no measurement was made, so that the particles continue indiscernible.

Another interesting fact, typical from modal logics, can be re-discussed here. For instance, can we name (label) the particles, say be calling 'John' and 'Peter' the two entangled particles mentioned above? If we can do it, do these names act as rigid designators? The quantum situation is a fantastic realm for questioning standard logical and semantical notions. If we accept that quantum objects do not have precise identity conditions (remember that this is just one of the possible metaphysics that can be associated to

[d] As is well known, the label '1' does not indicate individuation.

quantum entities), then names and labels in general will not work as usual, for there is no precise and unambiguous sense in naming 'John' a certain atom or particle. Of course sometimes the quantum object (an atom, say) can be isolated and in this situation the name 'John' seems to make sense. But the atom can be put together with others of similar species, and then the alleged identity of John is lost forever. We can never determine again, without ambiguity, that a particular atom is John.[e] In a certain in imprecise sense, all of them may be John, and (in a more precise talk) the measured results will do the same results. John has no identity. Some time ago, Dalla Chiara and Toraldo di Francia, have called our attention to the fact that the quantum realm is a *land of anonymity* (see [7]), so, no proper names make sense at all, and then neither rigid designators too!

5. Quasi-sets

Since this theory is still not in the reader's mind, we need to recall here its main ideas. Intuitively speaking, a quasi-set (qset for short) is a collection of objects such that some of them may be indistinguishable without turning out to be identical.[f] Of course this is not a strict "definition" of a quasi-set, but act more or less as Cantor's 'definition' of a set as "any collection into a whole M of definite and separate, that is, distinguishable objects m of our intuition or our thought", serving just to provide an intuitive account of the concept.[g] By 'indistinguishable' we mean agreement with respect to all properties, and in saying that a and b are 'identical' we mean intuitively that they are *the very same entity*. The definitions of these concepts depend on the employed language and logic, but here we consider only their informal meanings.

[e]There is an interesting film made by the IBM research team (available at the YouTube), termed "The boy and his atom". The scientists have magnified atoms over 100 million times and made a figure of a boy with them. In a certain moment, one atom is isolated from a group of similar atoms and the boy "plays" with it. But later the atom merges the atoms of the boy's hand, and after a few moment, it becomes isolated again. Can we say that the atom now isolated is "the same one" as before? The reader should be convinced that there is no sense in saying that. The atoms are exactly similar. 'Identity' (in the standard sense) seems in fact do not work here at all.

[f]Important to say that we have discussed some objections addressed to this theory for some people think that once we have 'more than one' object, they must be different, so identity applies to them. The reader who thinks this way needs to read again about the IBM film mentioned above, and a counter-argument in [16]. Really, quasi-set theory shows in particular that such a claim does not hold in general, for quasi-set comprising indiscernible objects may have a cardinal grater than 1.

[g]A more detailed discussion about this theory can be found in [10] and [11].

The quasi-set theory $\mathfrak{Q}$ has in its main motivations some considerations taken from quantum physics, mainly from Schrödinger's idea that the concept of identity does not make sense when applied to elementary particles in orthodox quantum mechanics (see [10]). Another motivation, in our opinion, is the need, stemming from philosophical difficulties of dealing with collections of absolutely indistinguishable items that would be not 'the same' entity.[h] Of course, from a formal point of view, $\mathfrak{Q}$ can also be formally developed independently of any intended interpretation, but here we shall always keep in mind this 'quantum' motivation since, after all, it is the intended interpretation that has motivated the development of the theory.

The first point is to guarantee that identity and indistinguishability (or indiscernibility) will not collapse into one another when the theory is formally developed. We of course could just take an equivalence relation within a standard set theory such as ZFC to mimic indiscernibility, but this is just what we don't want to do; instead, we wish to deal with 'legitimate' (in a sense of the word to be explicated below) indistinguishable objects. Thus, we assume that identity, that will be symbolized by '$=$', is not a primitive relation, but that the theory has a weaker concept of indistinguishability, symbolized by '$\equiv$' as primitive. This is just an equivalence relation and holds among all objects of the considered domain. The ur-elements of the domain are divided up into two classes of objects, the m-objects, that stand for 'micro-objects', and M-objects, for 'macro-objects'.[i] Quasi-sets are those objects of the domain which are not ur-elements. Identity is defined for M-objects and 'sets' (entities that obey the primitive predicate Z) only. Thus, if we take just the part of theory obtained by ruling out the m-objects and collections (quasi-sets) that have m-objects in their transitive closure, we obtain a copy of ZFU (ZFC with ur-elements); if we further eliminate the M-objects, we get just a copy of the 'pure' ZFC.

Indiscernible m-objects are termed non-individuals by historical reasons (see [10] for a wide discussion). From the axioms of the theory $\mathfrak{Q}$ we can form collections of m-objects which may have a cardinal, termed its

[h]This is of course a way of speech. Despite the fact that some interpretations (such as Bohm's) presuppose an ontology similar to that of classical physics, in the sense of dealing with individuals, we shall keep here with the mainstream account of assuming that quantum objects may be 'absolutely indiscernible' in certain situations.

[i]Within $\mathfrak{Q}$, there are no explicit relations among m and M atoms, but we guess that the theory could be supplemented by mereological axioms enabling us to say that the M objects can be 'formed' by m objects in some sense. A first attempt to consider such a 'quantum mereology' was done in [15].

quasi-cardinal, but not an associated ordinal. Thus, the concepts of ordinal and cardinal are independent, as in some formulations of ZFC proper, so, there are quasi-sets that cannot be ordered. Informally speaking, there may be quasi-sets of m-objects such that its elements cannot be identified by names, counted, ordered, although there is a sense in saying that these collections have a cardinal which cannot be defined in terms of ordinals. It is just by using quasi-cardinals that we can say (within $\mathfrak{Q}$) that a quasi-set has 'more than one' element. This discourse is of course dubious if we notice that the notion of identity does not hold in certain situations, but here the number (the quasi-cardinal) is what matters, and this resembles the Fock space quantum formalism (see [22]).

It is important to remark that, when $\mathfrak{Q}$ is used in connection with quantum physics, the m-objects are thought of as representing quantum entities (henceforth q-objects), but they are not necessarily "particles" in the standard sense. Generally speaking, whatever 'objects' sharing the property of being indistinguishable can also be values of the variables of $\mathfrak{Q}$. For a survey of the various different meanings that the word 'particle' has acquired in connection with quantum physics see [9, Chap.6].

Another important feature of $\mathfrak{Q}$ is that standard mathematics can be developed using its resources, because the theory is conceived in such a way that ZFU (and hence also ZFC) is a subtheory of $\mathfrak{Q}$. In other words, the theory is constructed so that it extends standard Zermelo-Fraenkel with ur-relements (ZFU); thus standard sets of ZFU must be viewed as particular qsets, that is, there are qsets that have all the properties of the sets of ZFU, and the objects of $\mathfrak{Q}$ that correspond to the ur-elements of ZFU are identified with the M-atoms of $\mathfrak{Q}$. To make the distinction, the language of $\mathfrak{Q}$ encompasses a unary predicate Z such that $Z(x)$ says that x is (a copy of) a set of ZFU.

It is also possible to show that there is a translation from the language of ZFU into the language of $\mathfrak{Q}$, so that the translations of the postulates of ZFU become theorems of $\mathfrak{Q}$; thus, there is a 'copy' of ZFU in $\mathfrak{Q}$, and we refer to it as the 'classical' part of $\mathfrak{Q}$. In this copy, all the standard mathematical concepts can be stated, as for instance, the concept of ordinal (for Z-sets). This 'classical part' of $\mathfrak{Q}$ plays an important role in the formal developments of the next sections.

Furthermore, it should be recalled that the theory is constructed so that the relation of indiscernibility, when applied to M-atoms or Z-sets, collapses into standard identity of ZFU. The Z-sets are qsets whose transitive closure, as usually defined, does not contain m-atoms or, in other words, they are constructed in the classical part of the theory.

In order to distinguish between Z-sets and qsets that may have m-atoms in their transitive closure, we write (in the metalanguage) $\{x : \varphi(x)\}$ for the former and $[x : \varphi(x)]$ for the latter. In $\mathfrak{Q}$, we term 'pure' those qsets that have only m-objects as elements (although these elements may be not always indistinguishable from one another, that is, the theory is consistent with the assumption of the existence of different kinds of m-atoms, i.e., not all of them must be indiscernible from one each other).

The concept of *extensional identity*, as said above, is a defined notion, and it has the properties of standard identity of ZFU. More precisely, we write $x = y$ (read 'x and y are extensionally identical') iff they are both qsets having the same elements (that is, $\forall z(z \in x \leftrightarrow z \in y)$) or they are both M-atoms and belong to the same qsets (that is, $\forall z(x \in z \leftrightarrow y \in z)$).

Since m-atoms may be indiscernible, in general is not possible to attribute an ordinal to collections of m-atoms. As a consequence, for these collections it is not possible to define the notion of cardinal number in the usual way, that is, through ordinals.[j] In the theory, to remedy this situation, we admit also a primitive concept of quasi-cardinal which intuitively stands for the 'quantity' of objects in a collection. The axioms for this notion grant that certain quasi-sets x, in particular, those whose elements are m-objects, may have a quasi-cardinal, written $qc(x)$, even when it is not possible to attribute an ordinal to them.

To link the relation of indistinguishability with qsets, the theory also encompasses an 'axiom of weak extensionality', which states, informally speaking, that those quasi-sets that have the same quantity of elements of the same kind, expressed by the quasi-cardinals (in the sense that they belong to the same equivalence class of indistinguishable objects) are themselves indistinguishable. One of the interesting consequences of this axiom is related to the quasi-set version of the non observability of permutations in quantum physics, which is one of the most basic facts regarding indistinguishable quanta (see [12]). In brief, remember that in standard set theories, if $w \in x$, then

$$(x - \{w\}\ ) \cup \{z\} = x \text{ iff } z = w.$$

We can 'exchange' (without modifying the original arrangement) two elements iff they are *the same* elements, by force of the axiom of extensionality. In contrast, in $\mathfrak{Q}$ we can prove the following theorem, where $[\![z]\!]$,

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[j] We just recall that an ordinal is a transitive set which is well-ordered by the membership relation, and that a cardinal is an ordinal α such that for no $\beta < \alpha$ there exists a bijection from β to α.

and similarly $[\![w]\!]$, stand for a quasi-set with quasi-cardinal 1 whose only element is indistinguishable from z and, respectively, from w (the reader should not think that this element *is identical to either* z or w, because the relation of equality does not apply to these items; the set theoretical operations can be understood according to their usual definitions):

Theorem 5.1 (Unobservability of Permutations). *Let x be a finite quasi-set such that x does not contain all indistinguishable from z, where z is an m-atom such that $z \in x$. If $w \equiv z$ and $w \notin x$, then there exists $[\![w]\!]$ such that:*

$$(x - [\![z]\!]) \cup [\![w]\!] \equiv x.$$

The theorem works to the effect that, supposing that x has n elements, then if we 'exchange' their elements z by corresponding indistinguishable elements w (set theoretically, this means performing the operation $(x - [\![z]\!]) \cup [\![w]\!]$), then the resulting quasi-set remains *indistinguishable* from the one we started with. In a certain sense, it does not matter whether we are dealing with x or with $(x - [\![z]\!]) \cup [\![w]\!]$. So, within $\mathfrak{Q}$, we can express that 'permutations are not observable', without necessarily introducing symmetry postulates, and in particular we derive 'in a natural way' the quantum statistics [10, Chap.7].

6. Quasi-set semantics for the logic of indiscernibility

In this section we shall be working within $\mathfrak{Q}$. Informally, we say that two objects are *of the same species* when they are indiscernible.

To define a quasi-set semantics for our language $\mathscr{L}$, we repeat almost all the previous definition with obvious adaptations, as follows. Let $\langle W, R\rangle$ be a frame where W is a set of worlds (that is, W is a qset satisfying the predicate Z), and R is an equivalence relation on W, the accessibility relation. A Kripke quasi-model for $\mathscr{L}$ is a 4-tuple $\mathscr{K} = \langle W, R, D, V\rangle$, where $D = D_1 \cup D_2 \neq \emptyset$, D_1 being a qset whose elements are m-objects (occasionally indiscernible), and D_2 is a set.

As before, there is a quasi-function f from types to the adequate qsets comprising the domain. We now define an interpretation V of the predicate symbols in each world w_i as follows:

(i) $V(\approx, w_i) = \Delta_{D_2}$, for every $w_i \in W$

(ii) $V(\cong, w_i) = \equiv \cup \Delta_{D_2}$; that is, the indiscernibility relation is provided by the indiscernibility relation of $\mathfrak{Q}$ between the elements of D_1 and the identity in D_2;

(iii) $V(P, w_i) \in f(\tau^P)$.

Now, as before, we define a valuation related to an assignment μ, V_μ. This is a quasi-function from $\{\text{formulas}\} \times W$ onto $\{0, 1\}$, defined as below, for each assignment μ of the variables of $\mathscr{L}$ to the domain, so that $\mu(x) \in D_1$ and $\mu(X) \in D_2$, for x and X variables of the first and of the second species respectively.

(i) $V_\mu(x \cong y, w) = 1$ iff $\mu(x) \equiv \mu(y)$, and 0 otherwise.
(ii) $V_\mu(X \approx Y, w) = 1$ iff $\mu(X) = \mu(Y)$, and 0 otherwise.
(iii) $V_\mu(Pt_1 \dots t_n, w_i) = 1$ iff there is an X in $f(\tau^P)$ such that $X \equiv V(P, w_i)$ and $\langle \mu(t_1) \dots \mu(t_n) \rangle \in X$
(iv) $V_\mu(\neg\alpha, w) = 1$ iff $V_\mu(\alpha, w) = 0$, and 0 otherwise.
(v) $V_\mu(\alpha \to \beta, w) = 1$ iff $V_\mu(\alpha, w) = 0$, or $V_\mu(\beta, w) = 1$.
(vi) $V_\mu(\Box\alpha, w) = 1$ iff $V_\mu(\alpha, w') = 1$, for every $w' \in W$ such that wRw', and 0 otherwise.
(vii) $V_\mu(\forall s\alpha, w) = 1$ (where s is a variable of either the first or of the second species) iff $V_\rho(\alpha, w) = 1$ for every assignment ρ which is a s-alternative of μ, that is, such that for every variable t except possibly s, (being t of the same species as s), $\rho(t) \equiv \mu(t)$.

The main items that deserve special attention are (i), (ii), and (iii). Firstly, we remark that according to (i), indiscernible objects in the language are taken as indiscernible m-atoms in the domain D. This makes our semantics in agreement with the ideas encoded in the logic. As for (ii), the definition gives the usual modal conditions: identical objects (from the point of view of $\mathscr{L}$) are *the very same object* in D, for we are here considering objects of the second species, to which the notion of identity makes sense.

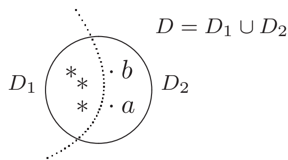

Figure 1. A typical qset taken as the domain of the individuals in our semantics, composed of individuals of both species. The elements of D_1 are indiscernible.

Concerning (iii), a main difference emerges from the classical semantics. Recall that in classical semantics each world is a Weyl aggregate, with an indiscernibility relation. Each world *has its own* indiscernibility relation, so that *objects indiscernible in one world could be discernible in another world.* However, in quasi-set theoretical semantics, in every world the indiscernibility relation is interpreted as $\equiv$ in $\mathfrak{Q}$. However, that relation is the same for every world. So, in this case the indiscernibility is indeed necessary. *This means that classical semantics generates a distinct system of logic than quasi-set theoretical semantics.* Of course, both systems may generate the same system if in the classical case we impose the restriction that the indiscernibility relation be the same in every world.

Another point also deserves explanation for the reader. This concerns item (vii), for quantifiers. Quantification over objects that don't have well defined identity conditions is thought by many not to make sense (see for instance [4]). Some may say that there is no way to quantify over them, but we can circumvent this criticism by showing that we can proceed in $\mathfrak{Q}$ exactly as we do in (say) ZFC (see [8]), as shown in [2]. In this paper, it is shown that quantification over non-individuals is perfectly well defined, but here is not the place to discuss such issues.

7. Conclusion

In conclusion, we consider again the suggestion that a logic of indiscernibility is something that should be investigated from several points of view. The first account, as we have said, is to work within a standard set theory such as ZFC. But in this approach, the best we can do is to consider indiscernible objects by means of an equivalence relation (or by a congruence), but by the inevitable fact that all objects represented within ZFC are individuals, this "mock" indiscernibility can be shown to be false by extending the considered structures (we are always working within a certain mathematical structure) can be extended to a rigid one. The "right way", so to say, is to work within a non-standard set theory such as $\mathfrak{Q}$. The modal account to indiscernibility serves to enlighten some other aspects not pursued till now. We hope that our paper has provided some such insights. But the discussion goes on.

There are in fact many distinct lines of investigation that deserve to be pursued. Our axiomatization, presented earlier, is certainly not complete with respect to both of our semantics. As we mentioned, in the classical semantics we could establish that $x \cong y \to \Box(x \cong y)$ is not a theorem. However, as a consequence of our quasi-set theoretical semantics, this formula is logically valid in this second semantics. So, the systems are indeed different, and distinct sets of axioms will be required. Furthermore, in classical semantics systems distinct from S5 could be profitably used. One could thing, for instance, of systems in which the accessibility relation R is such that a world w_i accesses only worlds w_j such that $\sim_j \subseteq \sim_i$, that is, where the indiscernibility relations are more 'fine grained'. In this sense, a world would access another world only if more things get discerned in the accessed world. On the other hand, alternative systems could operate the other way: w_i accesses w_j only if $\sim_i \subseteq \sim_j$. In this case, the indiscernibility relation involves more elements, items that previously were discernible now get indiscernible. These are interesting ways to further develop our system, and we hope to pursue such issues in a future paper.

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THE POSSIBILITY OF A NEW METAPHYSICS FOR QUANTUM MECHANICS FROM MEINONG'S THEORY OF OBJECTS

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According to de Ronde it was Bohr's interpretation of Quantum Mechanics (QM) which closed the possibility of understanding physical reality beyond the realm of the actual, so establishing the Orthodox Line of Research. In this sense, it is not the task of any physical theory to look beyond the language and metaphysics supposed by classical physics, in order to account for what QM describes. If one wishes to maintain a realist position (though not nave) regarding physical theories, one seems then to be trapped by an array of concepts that do not allow to understand the main principles involved in the most successful physical theory thus far, mainly: the quantum postulate, the principle of indetermination and the superposition principle. If de Ronde is right in proposing QM can only be completed as a physical theory by the introduction of 'new concepts' that admit as real a domain beyond actuality, then a new ontology that goes beyond Aristotelian and Newtonian actualism is needed. It was already in the early 20th century that misunderstood philosopher Alexius von Meinong proposed a Theory of Objects that admits a domain of being beyond existence-actuality. Member of the so called 'School of Brentano', Meinong's concerns were oriented to provide an ontology of everything that can be thought of, and at the same time an intentionality theory of how objects are thought of. I wish to argue that in Meinong's theory of objects we find the rudiments of the ontology and the intentionality theory we need to account for QM's basic principles: mainly the possibility of predicating properties of non-entities, or in other words, the possibility of objectively describing a domain of what is, that is different from the domain of actual existence.

0. Introduction

The wonder of Newtonian physics has many facets. It is not only a 'useful' theory we can apply to calculate velocities, accelerations, falls; it is not only a theory that allows us to perform experiments or to pose questions about how things are, given some conditions, or how things would be, given some others. It is also, and maybe mainly, a theory that allows us to understand the world in which we live in. Of course, not all physicists or philosophers are realists, who commit themselves to the thesis that holds that the terms of the theory actually refer to independent entities in the world. But, beyond the realism debate, I believe it to be without doubt that classical mechanics is a complete theory that allows us to understand reality, by offering certain concepts that seem to describe the macroscopic world we inhabit. Even if there is no independent reality where Newtonian particles collide and interact, that picture of the world seems to hold true when we inquire into certain domains of nature. So strong is said picture of the world, that it seems impossible to think beyond the concepts of Newtonian physics.

It is in this sense, I believe, that Quantum Mechanics (QM) is said to lack a consensus regarding its interpretation. What does it mean, for a theory that possesses a successful mathematical formalism, and unprecedented levels of empirical adequacy, to be without a unanimous interpretation? This means, basically, that "if we are to understand QM as a physical theory, and not merely as a mathematical or algorithmic structure, it is clear that we still need to provide a link between the mathematical structure and a set of physical concepts which are capable of providing a coherent account of quantum phenomena" (de Ronde, 2015A:8). In other words, we do not know what the theory is about. QM has a rigorous formalism, empirical adequacy, and outstanding technological applications, but it lacks still those concepts that allow us to form a picture of the world, to think about the 'reality' described by the theory, to do physics in the most complete sense. There is agreement in the literature, regarding the fact that QM possesses indeed a successful formalism and adequate experimental arrangements, *even though* we do not have the appropriate concepts to account for all of this:

> "(1) The only consensual part of the theory is a formal skeleton enabling one to calculate the probability of various experimental outcomes at any time, given the initial preparation (Peres, 1995; Schwinger, 2001). (2) This formal skeleton is often complemented with bits and pieces of former pictures of the world borrowed from classical physics, but connected to one another in an unfamiliar and unruly way. A recurring complaint is that, as long as we are left without any truly coherent representation of the world and of its 'ontological furniture' compatible with the quantum formalism, we cannot claim that we truly 'understand' quantum mechanics" (Bitbol, 2010:54-55)

"Quantum mechanics brilliantly succeeds as a mathematical formalism: the numbers it provides are always successfully compared with experimental results. But it is often said to fail as an explanatory theory allowing us to understand the laws of atomic processes" (Lurat, 2007:230).

"Scientific advances can significantly change our view of what the world is like, and one of the tasks of the philosophy of science is to take successful theories and tease out of them their broader implications for the nature of reality. Quantum mechanics, one of the most significant advances of twentieth century physics, is an obvious candidate for this task, but up till now efforts to understand its broader implications have been less successful than might have been hoped. The interpretation of quantum theory found in textbooks, which comes as close as anything to defining "standard" quantum mechanics, is widely regarded as quite unsatisfactory. Among philosophers of science this opinion is almost universal, and among practicing physicists it is widespread. It is but a slight exaggeration to say that the only physicists who are content with quantum theory as found in current textbooks are those who have never given the matter much thought, or at least have never had to teach the introductory course to questioning students who have not yet learned to 'shut up and calculate!'" (Griffiths, 2011:2).

"Regarding its formal structure we could say that quantum mechanics seems to be a 'finished theory'. In terms of empirical adequacy, it provides outstanding results, its mathematical structure —developed in the first three decades of the 20th century by people like Werner Heisenberg, Pascual Jordan, Max Born, Erwin Schrödinger and Paul Dirac— seems able to provide until now the adequate modeling to any experiment we can think of. However, apart from its fantastic accuracy, even today its physical interpretation remains an open problem. In the standard formulation, quantum mechanics assigns a quantum mechanical state to a system, but 'the state' has a meaning only in terms of the outcomes of the measurements performed and not in terms of 'something' which one can coherently relate to physical reality. It is not at all clear, apart from measurement outcomes, what is the referent of this quantum state, in particular, and of the formal structure, in general. If we are to ask too many questions, problems start to pop

> up and simple answers seem doomed to inconsistency" (de Ronde, 2011:9)

Regardless of this agreement concerning the lack of a proper conceptual scheme that would allow a comprehensive understanding of QM in terms of a 'physical reality' of some sort, one could very well argue that this is indeed a futile enterprise, one that should be abandoned in favor of a more pragmatic or instrumentalist approach. Of this opinion are, for instance, Fuchs & Peres (2000:1): "Contrary to those desires, quantum theory does *not* describe physical reality. What it does is provide an algorithm for computing *probabilities* for the macroscopic events ('detector clicks') that are the consequences of our experimental interventions. This strict definition of the scope of quantum theory is the only interpretation ever needed, whether by experimenters or theorists". So, we could ask ourselves, why bother with finding concepts or an interpretation for a theory that is already providing everything it should?

It is inevitable, in order to answer such a question, to make explicit what one believes should be a physical or even a scientific theory. The instrumentalist approach simply decides to ignore the fact that the theory —might— 'lacks' something, and wishes to pursue and insist in the already achieved successes of the theory. The interpretative 'problems' of the theory, such as the measurement problem, the basis problem, non-locality, non-separability… and the list goes on, are only set aside, swept under the rug, to allow for the wonderful computations to carry on. To argue with a position that does not acknowledge the existence of a theoretical problem is a hard enterprise, given how the desiderata concerning a physical theory are so different from one another. To them, I can only ask: is that all? Are we to satisfy ourselves by claiming that the most successful scientific theory produced by mankind is nothing more than an algorithm to compute probabilities, with no reference whatsoever to physical reality? Can we really settle with a theory that has no comprehensive concepts, but opens questions regarding nature and being we would choose never to answer?

To all of those who believe 'no' is the best answer for the posed questions, and believe physics is more than an algorithm that computes the results of the experiments we ourselves have designed, to all of those who yet believe in some kind of physical reality to be known by human scientific endeavor, a long road of problems lies ahead. I believe QM lacks concepts that would allow us to comprehend the reality which its formalism already describes, in the strong sense that implies that we need to find these concepts. For the unconvinced instrumentalist who remains happy computing, the remainder of this article will seem pointless. To the one who shares the desideratum of comprehending reality through physics, we need now to inquire into how this could be approached.

I will begin by presenting a map of possible interpretations for QM and I will argue in favor of the line of interpretations that states the need to find new concepts for QM. In section 3, I will offer a brief presentation of Meinong's theory of objects, which I will apply, in section 4, to some of the problematic issues of QM.

1. The interpretation of QM

The quest of conceptually comprehending QM until today can be presented, following de Ronde (2011), in two main lines of inquiry: first, the tradition that beginning with Bohr has tried to comprehend QM based on classical concepts and has tried to make the formalism compatible with basic classical metaphysical principles; and second, another line of inquiry which attempts to take the successful formalism as a starting point and so, tries to find the appropriate metaphysical principles that would account for it:

> "We believe that an interesting distinction that can help us to understand the huge interpretational map of quantum mechanics relates to the position one takes with respect to metaphysics. This controversial relation between physics and metaphysics displaces the problem of truth to a secondary stage and concentrates its analysis in the conditions of possibility to access and distinguish physical phenomena. Metaphysical schemes provide the coordinates through which the representational map of realistic stances can be developed. Among those who attempt to provide a metaphysical account of quantum mechanics there is a first group that tries, in different ways, to 'restore a classical way of thinking about what there is'. Staying close to at least some of the classical notions of physics (space-time, causality, objects, etc.) these approaches have no problem to give up the orthodox formulation of quantum mechanics. A second group also interested in the metaphysical question regarding quantum mechanics attempts to begin 'right from the start' with the successful mathematical formalism in its orthodox form, trying to learn about its structure and internal features in order to find a metaphysical scheme which is able to fit the formalism. We might consider the first group as going from metaphysics into the formal structure while the second group goes from the formal structure into the metaphysical scheme" (de Ronde, 2011:54).

The first path can be characterized, then, as that which attempts to comprehend the new theory, QM, with the old concepts, the classical Aristotelian-Newtonian ones. Bohr himself stated that “the unambiguous interpretation of any measurement must be essentially framed in terms of classical physical theories, and we may say that in this sense the language of Newton and Maxwell will remain the language of physicists for all time” (Bohr, quoted in de Ronde, 2011: 7). Probably one of the most famous examples in this line of inquiry is Bohm’s hidden variable program, which according to de Ronde, “is forced to change the formalism with seemingly ad hoc moves; moves which can be only justified in relation to the prior metaphysical commitments” (de Ronde, 2011:54-5).

The Bohr-inspired program that seeks to find an interpretation for QM in the concepts and language of classical mechanics finds, among other approaches, its philosophical grounds in an interpretation of Kantian transcendental philosophy. Following Pringe’s interpretation, the general idea is that the limits of possible experience require that a phenomenon be constituted both through sensibility and understanding, that is, through empirical intuitions and *a priori* concepts. Anything that falls out of these limits is considered by Kant to be metaphysical and not subject to scientific inquiry, inasmuch as it is beyond possible experience. According to Kant, certain *a priori* conditions must be met in order for something to be an object of possible experience: sensibility must provide the necessary empirical intuitions which are synthesized by the faculty of understanding according to the categories table. One of the most important elements in this table is the concept of causality. So, according to Pringe’s Kantian reading of Bohr’s interpretation of QM:

> “If the quantum postulate is assumed, all pretension of reaching a spatial-temporal representation, which is *at the same time* causal, of an object subject to the postulate, must be abandoned. That is, if an object is within the domain of validity of the postulate, it won’t be possible —as it is in classical physics— to synthesize the set of contingent data of a measurement, according to the concept of cause, as the effect of said object, representing this in space and time, in such a way that its states modify each other causally” (Pringe, 2012:183).

This means that the quantum postulate forbids ‘quantum objects’ to be objects of possible experience, inasmuch as it is not possible to synthesize the multiplicity of empirical data following the concept of causality. The main problem of this conclusion lies in the fact that objectivity is then lost for the quantum domain. So, how does QM remain a scientific theory, given that it violates Kantian transcendental conditions of validity?

The Bohrian answer to this question, as it is known, is contextuality. Within each experimental arrangement or measurement process, Bohr argues, we can synthesize the given objects, meeting thus the necessary conditions. The problem is, we cannot give a coherent account of the results of multiple experiments, since these immediately become incompatible in terms of the Kantian categories. Pringe goes on to argue that quantum phenomena must be described in classical terms, thus guaranteeing Kantian transcendental conditions.

> "In effect, quantum phenomena are contextual, given that their validity is restricted to a determined type of experimental arrangement, and they are complementary, inasmuch as they mutually exclude one another; but at the same time, they are all necessary to account for experimental evidence. We face, then, a multiplicity of phenomena, whose objective character is established, but they do not yet acquire *systematic unity*" (Pringe, 2012:188-9).

This systematic unity is, of course, a necessary condition for scientific knowledge. So, quantum phenomena are given in terms of classical concepts which are referred to specific and distinct experimental arrangements. Now, each quantum phenomenon, from each arrangement, is incompatible, in classical terms, with each other (in most of the cases). Though incompatible, these results are mutually complementary, in the sense that they are all necessary to account for the empirical data that the theory produces. So, how can all these mutually incompatible and complementary results be brought to systematic unity in order to guarantee the objective validity of scientific knowledge?

> "[...] Bohr distinguishes quantum *objects* from quantum *phenomena*, which are nothing more, than classical descriptions, whose totality exhausts the available information regarding the firsts. The systematic unity of quantum phenomena will only be reached when they are subsumed under the concept of quantum object. [...] The concept of a certain quantum object or system contains the representation of its state, and with it, the information about the different probabilities of the different results of the possible measurements that can be realized on the system. In this way, the

> multiple quantum phenomena are unified by a probabilistic law. Given a certain quantum phenomenon, the representation of the state of the quantum object establishes the *probability* of each and every phenomena of the object. So, the multiplicity of phenomena is synthesized through the concept of the object and subsumed under it. This synthesis allows, then, to carry out predictions such that, given a certain phenomenon, the probabilities of the different results of possible measurements are calculated based on the so called 'wave function' of the system" (Pringe, 2012:189).

The wave function then plays the role of the quantum object, which can never be directly given into intuition neither can it be synthesized according to the categories, but operates as the regulative systematic unity of the different quantum phenomena which *are* in fact, given to intuition and synthesized. The objective validity of QM is then grounded on the objective validity of classical physics, the reason for this being that classical concepts are the only ones that can attain objective validity. I shall quote Pringe *in extenso* one last time to appreciate the conclusion of such an analysis:

> "In the first place, the objective validity of a classical object consists in its synthetic function of an empirical multiple, thanks to which, the intuitive representation of an object is constituted. On the contrary, the objective validity of the concept of a quantum object is based rather in its regulative task to provide systematic unity to the complementary phenomena (whose objectivity is guaranteed by the use of classical concepts to interpret the experimental results).
>
> In second place, the concept of a classical object acquires objective reality when a given empirical multiplicity is subsumed under the concept thanks to the mediation of a scheme. So, the concept is exhibited directly in intuition. On the contrary, as we have seen, as a consequence of the quantum postulate, the conditions under which an empirical multiplicity is given, which should be synthesized by the concept of a quantum object, are incompatible with those conditions under which the concept can be applied. Therefore, a direct exhibition of such a concept in intuition is not possible. The concept of a quantum object acquires objective reality, rather, through an indirect exhibition in intuition, carried out through symbolic analogies" (Pringe, 2012:192-3)

We can now appreciate what it means, in philosophical terms, that quantum mechanics can only be interpreted in terms of classical physics. These Bohrian declarations can be grounded in Kantian transcendental philosophy. What it means to be able to constitute an object is incompatible with quantum theory. Thus, all that is left for QM is to settle with classical representations, mutually incompatible, but mutually complementary. There is no quantum object we can constitute, but this concept operates not in a constitutive manner, but a regulative one, providing systematic unity to the multiple phenomena, presented in classical terms, through symbolic analogies.

In other words, we could say that the Bohrian interpretation of QM that seeks to understand QM in terms of classical concepts is right, if Kantian transcendental philosophy is also right. That is, it makes no sense to pursue new constitutive concepts for QM if the limits established in Kant's *Critique of Pure Reason* are indeed the *a priori* limits of what can be constituted by human thinking. All that is left is the possibility of finding metaphysical concepts, which would perform a regulative role, but not a constitutive one. This is, clearly, one possibility: but it implies that we must always find objective validity for these concepts, as Pringe says, 'indirectly'. Now, of course, we must ask ourselves, why should we trust Kant? The most direct way to 'refute' Kant's limits to human experience would be to find new limits to experience that are compatible with the quantum principles. But, again, why seek them if one thinks Kant is right about them? It would seem we need further motivation to enter such an enterprise.

It can be argued that we find in Kant the philosophical grounding for classical mechanics[a]. The space, which is the empty form of sensibility according to Kant, is the space of Euclidean geometry, which is, at the same time, the absolute space of Newtonian physics. The pure concept of causality, under which we synthesize phenomena and constitute objects of experience according to Kant, is the concept of causality that is needed for classical mechanics' descriptions of macroscopic interactions between bodies. These *could* be the transcendental limits of human experience. But they could also very well be the transcendental limits of *classical experience*. Why should these be the limits of all human experience? One may

[a]Regardless of whether or not Kant took Newtonian mechanics and Euclidean geometry as starting points for his theory, and regardless of whether or not these principles are needed for Kantian philosophy, the truth is that they seem highly compatible and that history, specially neo-kantism, has taken Kantian philosophy as a transcendental fundament for physics.

argue that both the appearance of Relativity Theory and QM are sufficient reason to believe that we need new limits for human experience[b]. We need a new ontology that is not grounded on classical metaphysical principles.

Another approach to a Kantian interpretation of QM is found in Michael Bitbol's work. The French author wishes not to accept the fixed given limits of experience developed by Kant, but to embrace his 'reflective metaphysical program' in order to analyze the different problems that arise from quantum theory. In this sense, the task is not to limit experience to Kant's words, but to inquire once again into the limits of human experience, based on the new developments brought about by QM:

> "Kant's motto is that, despite its stemming from the "extravagant claims of speculative reason" (Kant, 1997, Introduction), metaphysics should not be rejected but disciplined. It should be given an epistemological rather than ontological status, so much so that ontology itself is seen as an epistemological tool. At the very end of Kant's work of reconstruction, metaphysical statements are then no longer seen as representations of something "out there", but as rules in a grammatical pre-ordering of experience. [...] Hence, metaphysics becomes nothing else than a reflective analysis of the powers and credence of reason" (Bitbol, 2010:59).

In this way, metaphysics is not seen as an objective description of an independently existing reality, but as a way to determine the possibility of knowledge. Given QM's 'new knowledge', one might argue, we need metaphysics to establish its conditions of possibility:

> "One can thus adopt a pragmatic definition of the a priori instead of a purely intellectual one (Pihlstrm, 2003). According to this definition, an a priori form is no longer a universally necessary intellectual condition for objective knowledge, but a pragmatic condition locally and provisionally necessary for the determination of some intersubjectively shared domain of experimental or technological intervention" (Bitbol, 2010:62).

So Kant's all-limiting *a priori* becomes, under this new perspective, a contextual limitation to specific cases of knowledge. The task of a reflective metaphysics is not any more, then, to establish the limits of possible

[b]Of course, no a posteriori theory or evidence could refute a priori arguments. The point is that now, there seems to be reason enough to believe that those limits set by Kant are indeed to narrow.

knowledge for all human cognitive activity, but rather, to describe the *a priori* elements that are at stake in each context.

> "This being granted, a solution (or rather dissolution) of the measurement problem boils down to finding a way to articulate the indefinite chain of relational statements of the quantum theory to the absolute statements that are used in experimental work. An articulation of this kind can easily be found, provided one realizes that the latter absolute statements are in fact indexical; provided one realizes that these statements are only 'absolute' relative to us, to our scale, to the open community of experimenters to which we belong (Rovelli, 1996 ; Bitbol, 2008). At this point, one is bound to realize the ineliminability of situatedness from the apparently neutral descriptions of quantum mechanics, and to accomplish thereby the reflective move typical of Kant's renewed definition of metaphysics" (Bitbol, 2010:75).

So the measurement problem is 'dissolved' because we come to the understanding that the 'absolute statements' of QM are in fact relative statements, the term of the relation being the community of scientists. The 'pragmatic *a priori*' means nothing else than the explicitation of the metaphysical principles that underlie each experimental arrangement. Since each of these is in fact produced by the community itself, all that remains is to acknowledge this fact and consider QM as interpreted in our own terms:

> "But in quantum physics, no event should be ascribed autonomy. In this case, every event is tantamount to an observable value ascription, and an observable is only defined relative to an effective instrumental possibility of assessing it. In quantum physics, the instrumental context is not only a way of getting access to an event; it is a way of generating it" (Bitbol, 2010:78).

So, according to this pragmatic *a priori*, we are to settle ourselves with no more than the conditions of possibility of a given situation, which coincides with the fact that we determine ourselves the conditions for a given experimental arrangement. There is nothing beyond that situation and the so called 'paradoxes' of QM are *dissolved* inasmuch as they no longer constitute a problem, if we accept that each measurement is *situated*:

> "This represents a major difference with classical physics. In classical physics, the simple truth that we act as situated subjects of knowledge could be bracketed, and a naturalized description of

> the world including ourselves taken as objects could pretend to be universal. Instead, quantum physics manifests the bounds of this attitude of all-pervasive naturalization. It makes one realize that the irreducible fact of situatedness is a necessary presupposition of objective knowledge and cannot thus be objectified itself. This, of course, was pointed out by many generations of transcendental philosophers, from Kant to Husserl and beyond; but quantum physics leaves little room for those who want to ignore their lesson" (Bitbol, 2008:212).

In my opinion, Bitbol's so called transcendental interpretation of QM boils down to a sophisticated defense of an instrumentalist position. To ascribe Husserl or Kant such a conception is as fair as believing we have come any closer to an understanding of the problems involved in QM, because we call "pragmatic *a priori*" the renounce of a realist program for QM. Kant believed that the *a priori* concepts of pure understanding referred to actual, existing, independent reality, by way of the empirical intuitions that are synthesized under such concepts. Husserl believed that the constitution of phenomena in natural attitude is guided by the world itself, and that in ultimate stance, the question about how phenomena are constituted is the question as to why the subjective constitution of phenomena is valid, in the sense that it corresponds with the reality 'out there'. The whole point of the transcendental question into the conditions of possible experience is to determine in a universal manner how it is that we know the world. There is nothing transcendental, in any relevant sense, in the claim that we generate an instrumental context each time we perform an experiment.

Pringe's reading of Bohr's interpretation is based on a solid understanding of Kantian philosophy. My only criticism to it is that, while Kant's philosophy successfully grounds in transcendental conditions classical mechanics, it fails to bring us any closer to an understanding of QM and closes the door for any project that seeks to really empower QM by acknowledging that the success of the theory should be taken seriously. We cannot understand QM, know what the theory is talking about, if we try to force it into old schemes, and settle with 'symbolic analogies'. The question of what does QM talk about, needs to be taken seriously, instead of trying to explain why the question cannot be answered.

On the other hand, Bitbol's position falls short of being transcendental or realist in any relevant sense. If anything, it is an elaborate account of the claim that QM needs no interpretation: because we are situated, we cannot escape our situation, and therefore must settle with a contextual reading of the results of QM, that denies them any kind of autonomy, validity and even reality.

Let us now move into the second group of interpretations of QM, that which wishes to find the proper metaphysical principles for QM taking as a starting point the successful formalism, and which wishes to do so not in instrumentalist terms, neither in nave realist terms, but in the sort of realism that takes into consideration the fact that scientific theories *represent* reality, a reality that exists out there, but that we can only access through a certain array of concepts. De Ronde calls this a "Representational Realist Stance", and defines it as follows:

> "A representational realist account of a physical theory must be capable of providing a physical (and metaphysical) representation of reality in terms of a network of concepts which coherently relates to the mathematical formalism of the theory and allows to make predictions of a definite field of phenomena (expressed through such concepts)" (de Ronde, 2015:12-3).

We can take, then, the realist stance and the search for a new ontology of QM as two fundamental desiderata in the quest of interpreting QM. If we do so, we can better see what the problem is with interpretations that still seek to keep the classical concepts for QM. The argument is simple: it is the theory which tells us how to understand reality and what is and is not out there. Physical theories are based on metaphysical principles which are adopted without question and, of course, without possible scientific justification, since they are the basic principles upon which the concepts of the scientific theory are developed. In the case of physical theory, de Ronde argues, we are still trapped by Aristotle's basic metaphysical principles: the Principle of Existence, which determines that an entity, that which exists, can only do so only in spatio-temporal way; the Principle of Non-Contradiction, which forbids the attribution of contradictory properties to anything that exists, since it assumes that reality is in itself of a non-contradictory nature; and the Principle of Identity, which asserts that an entity is identical to itself, and that its essential properties are maintained through time. The basic assumption in Aristotelian metaphysics and later in Newtonian, taken to the extreme, is that everything that is, all that exists, can only do so for real, in actuality. In other words, there is only one real existent mode of being: the mode of actuality:

> "The general metaphysical principle implied by the understanding of Newtonian mechanics, that 'Actuality = Reality', has become an

> unquestionable dogma within physics. As a silent fundament all of physics has been developed following the metaphysics of actuality. And even though QM was born from a deep positivist deconstruction of the a priori classical Newtonian notions -and in this sense the philosophy of Mach can be understood as the very precondition for the creation of both QM and relativity theory- it was very soon reestablished within the limits of classical metaphysics itself. The constrains of actuality have been unquestionably accepted by philosophers of physics either in terms of *hic et nunc* observation (empiricism and its variants) or as the mode of preexistence of properties (realism). Both positions have remained captive of actualism; trapped in the metaphysical net designed (through the PE, PNC and PI) by Aristotle around the 5th century before Christ and imposed by Newton in the 18th Century of our time. Actual (preexistent) properties and actual (here and now) observations are two sides of the same (metaphysical) coin" (de Ronde, 2015:20)

All attempts to understand QM have been precluded to do so, due to the limitations imposed by such a metaphysic, "But what if QM cannot be subsumed under the metaphysical equation imposed by Newtonian physics: Actuality = Reality?" (de Ronde, 2015:21-2). If that is the case, and it is the unquestioned presupposition that reality can only be in the mode of actuality which has prevented a successful interpretation of QM, then a new path is clear ahead:

> "We need to develop a new way of understanding reality beyond the ruling of actuality. To escape the ruling of actuality —both in terms of *hic et nunc* observation and pre-existent properties— means to abandon, on the one hand, the idea that we have a clear definition of what is observed according to QM, and on the other hand, the idea that actuality is the only possible way to conceive and understand physical reality. Our strategy is to take as a standpoint the formalism and its predictive power in order to develop new physical concepts which relate coherently to the formalism and can allow us to think about the physical meaning of quantum phenomena." (de Ronde, 2015:23-4).

The project then comes to light. To search for new concepts for QM means to develop a new ontology. A new ontology is not the same as new ontic categories. The task is not to expand the list of what there is, but to rethink the principles under which we claim that something is or can be.

The principle that underlies all previous metaphysical endeavors and, therefore, all attempts to find a proper conceptualization of QM, seems to be the principle that equates reality to actuality. Two things are then needed to carry on forward: a new ontology that does not reduce reality to what is actual, and a new theory of experience that allows us to understand how such a domain of reality, which is not actual, could be thought of, experienced. In other words, we need a new ontology and 'a new' phenomenology.

2. Meinong's Theory of Objects

In this section I wish to offer a schematic presentation of philosopher Alexius von Meinong's *Gegestandtheorie* or theory of objects. Based on the conclusion of the previous section regarding the need for a "new ontology" that would allow to fully grasp the principles of QM, I believe Meinong's ontology is a good place to start. Meinong is a disciple of Austrian philosopher Franz Brentano, who can be considered to have founded a philosophical school, the so called "School of Brentano", of which I wish to recover one main principle that appears clearly in Meinong's philosophy and is relevant for the present purposes[c]. The key point of Brentano, or at least of the 'Brentanian philosophers', is the correlation between the psychological and the ontological.

Brentano's reading of Aristotelian realism leads him to consider that there is a parallelism between mental acts and their objects, one the one hand, with objects 'in themselves', on the other. The crucial thesis is that those objects as they are, unlike Kant's *nomena*, are given to the mind and can be fully known. Thus, philosophy is the inquiry both into the mental acts and its correlates (psychology), and into objects and their nature as such (ontology). In the words of Barry Smith:

> "Descriptive psychology, as Brentano here understands it, seems to consist precisely in a psychology that will issue in an ontologically sophisticated theory of the different types of parts, of such a sort that the specification of parts will be at the same time a specification of the ways in which these parts are fitted together into wholes" (Smith, 1994:47)

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[c]For a comprehensive reading on the philosophy of Brentano and its disciples, see: Smith, Barry (1994), *Austrian Philosophy. The Legacy of Franz Brentano*, Open Court Publishing Company, Chicago and LaSalle, Illinois.

A psychological investigation, thus, would yield as results not only information regarding mental acts themselves and consciousness as such, but, inasmuch as investigating the mental correlates of mental acts is investigating objects, it would also yield the ontological information of the objects as they are.

Historically speaking, this parallelism found in Brentano's theory, was inherited by most (if not all) of his disciples, who later focused in different domains of inquiry. In the case of Meinong, the disciple who interests us here, this was translated into a theory of objects which is, as we will see in a moment, both a theory of objects as they can be thought of, as well as a theory of objects as they are or can be:

> "For where Brentano applied his descriptive realist method almost exclusively in the area of psychology, his students extended it in systematic ways to other domains of inquiry. We can in fact distinguish in their work three branches of what might be called 'descriptive ontology': the ontology of *things* (or objects in the narrow sense), the ontology of *states of affairs*, and the ontology of *values*, a tripartite division which flows in an obvious way from Brentano's tripartite division of acts.
>
> The ontology of things or objects arises when one turns from the psychology of presentation to an investigation of the non-psychological correlates of presenting acts. 'Object' is then understood as: 'possible correlate of presentation' " (Smith, 1994:52).

In order to make clear how this tripartite division follows obviously, let me very briefly present Brentano's tripartite division of acts: all acts of consciousness are of one of the following type: an act of presentation, in which the object is simply present to the mind; an act of judging in which the object's existence is either affirmed or negated; and an act of interest, in which the object (both presented and usually judged to exist) is loved or hated[d]. It is in the first simple sense in which Meinong's theory of objects can be first understood: an object is that which can be thought of. As such, it must have some sort of being. Which 'type' or mode of being it has, must be 'decided' once we know more about that object we are thinking of.

This common principle of the School of Brentano is of crucial importance in the task to 'find new concepts' for QM. As it has been shown in the previous section, especially considering Pringe's Kantian interpretation of Bohr, an ontology that does not allow us to comprehend how the ontological

[d]Cf. Smith (1994:42-4)

domain to which 'quantum objects' belong, can be experienced, would not be a very fruitful enterprise. In this sense, the typical psychological-ontological parallelism of the School of Brentano should prove interesting: it is not only a theory about what there is (ontology) what we seek, but also a theory about how what is can be thought of (psychology). The problem with the Kantian approach is that it limits experience only to the domain of what can be empirically intuited, thus closing both ontology and psychology to very limited possibilities, and leaving everything else outside of the 'scientific' knowable world and relegated to mere metaphysical speculation.

It is important to make a distinction between an object of possible experience and an object that plays a role in experience. From the Kantian perspective, we only have experience of spatio-temporal objects, yet, there are other objects, such as regulative objects or metaphysical objects, which cannot be experienced, but perform a function in guiding or regulating experience. Such objects are, for instance, God, the Soul or even Truth, understood as a regulative idea. The problem arises with the idea that certain objects that we claim, belong to nature, such as QM-objects would be, are not objects of possible experience, and all they can do is regulate or guide our experience of classical objects, which, in its turn, would tell us something about the domain of QM. The accusation against Kant-Bohr-Pringe is not that they find no role for these objects which cannot be experienced, the problem is the claim that there are, in physicial nature, objects which cannot be experienced and should perform a function similar to that which performs, for instance, the idea of God.

Let us now move into Meinong's theory. A good starting point for the present discussions is Meinong's 'prejudice in favor of the actual'. The idea is that the interest of inquiry has always been so focused on what exists in the sense in which spatio-temporal objects exists, that a whole domain of objects of knowledge has been left aside, objects which *are* in their own sense. A theory of objects, then, should focus not only in those objects that exist in such a manner, but in all objects that have some sort of being:

> "If we remember how metaphysics has always been conceived as including in its subject matter the farthest and the nearest, the greatest and the smallest alike, we may be surprised to be told that metaphysics cannot take on such a task. It may sound strange to hear that metaphysics is not universal enough for a science of Objects, and hence cannot take on the task just formulated. For the intentions of metaphysics have been universal (a fact which has so often been disastrous to its success). Without doubt, metaphysics

> has to do with everything that exists. However, the totality of what exists, including what has existed and will exist, is infinitely small in comparison with the totality of the Objects of knowledge. This fact easily goes unnoticed, probably because the lively interest in reality which is part of our nature tends to favor that exaggeration which finds the non-real a mere nothing — or, more precisely, which finds the non-real to be something for which science has no application at all or at least no application of any worth" (Meinong, 1981:77)

Meinong's concern, then, can be said to exceed the preoccupation for what is 'real' or what exists, or that which metaphysics encompasses. Meinong is concerned with providing a theory that can account for all objects, i.e., everything that can be thought of. In this sense, there are more objects than 'things'. We can think of more things than those that actually exist. And in a very relevant sense, we can have experience of more objects than those that exist only in the form of spatio-temporal entitites.

We can approach Meinong's theory of objects by comparing it with Russell's theory of definite descriptions. Historically speaking, Russell can be considered the victor in a dispute between the two authors, regarding how to consider the attribution of properties to non-entities:

> "Meinong was concerned with the problem of explaining the apparently correct attributions of properties to non-entities, especially in intensional contexts, and the closely related problem of intensionality. It is often said that Russell's theory of descriptions simply solved the problem of ascriptions of properties to non-entities; and this is usually supported by pointing to the Russellian analysis of non-existence claims as claims about entities" (Routley & Routley, 1973:225).

Let us take the example of Pegasus and the corresponding statement "Pegasus does not exist". According to Russell, since logic and hence all predication is always extensional, no statement can correctly be predicated of a logical subject that does not exist. So, the statement about Pegasus should be correctly paraphrased into "The class of existing items does not include Pegasus". We can see here how the second statement does not have Pegasus as its logical subject, but the set of existing things. Given that Pegasus does not exist, I cannot correctly attribute it with any property at all, not even non-existence. Now, the consequence of the Russellian approach is that I cannot predicate *anything* at all of a non-existing entity. Thus, the statement "Pegasus is a winged horse", given how there is no Pegasus, must be considered false, the same as the statement "Pegasus is identical to Pegasus". This 'solution' offered by Russell and accepted by the analytic tradition of philosophy of language has as a consequence the impossibility of predicating of anything that does not exist. Now, this logical maneuver carried out by Russell can, of course, be considered legitimate and suiting very specific purposes. It goes along great, for example, with a positivist ontology as the one assumed nowadays, but it falls short for pretty much any other purpose:

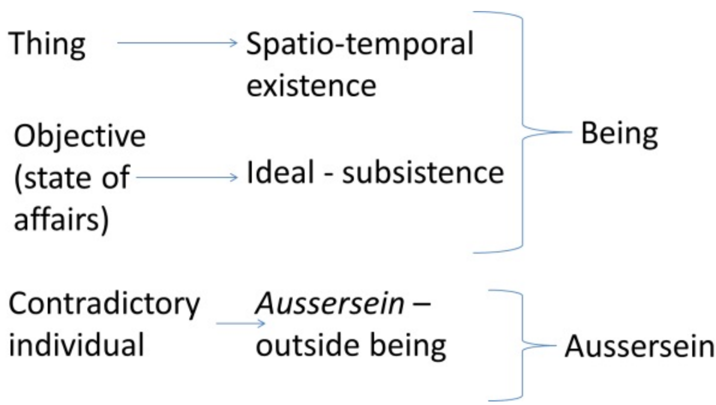
> "While such a 'solution' to the problem of the attribution of properties to non-entities might be satisfactory for a few limited purposes, for many purposes it is not. Russell's theory does not even begin to provide a solution to the main problems which concerned Meinong, viz. that of obtaining a satisfactory account and explanation of truth (or factuality) in intensional discourse, and of the logical behaviour of subjects, descriptions and quantified expressions in intensional discourse, of explaining the apparent truth of some statements about non-entities and the peculiarity or falsity of others, and of obtaining a non-Platonistic account of mathematics." (Routley & Routley, 1973:226)

Meinong's solution to this problem runs in a very different line. Rather than denying the possibility of attributing properties to non-entities, Meinong proposes quite a different principle to start from, the so called "principle of independence of being-so (*Sosein*) from being (*Sein*)" and it states as follows: "The Object is by nature indifferent to being, although at least one of its two Objectives of being, the Object's being or non-being, subsist" (Meinong, 1960: 82 and 86 respectively). We can know an object, we can think about it, predicate of it, without yet knowing whether or not the object exists. Things are said to exist when they do so in space-time. In this sense, all of mathematics is able to predicate from objects that do not exist.

An object is a simple unit; it can be a part of an *objective*, which is Meinong's term for what is usually known as a state of affairs, a complex 'situation'. For example, the golden mountain is an object, "the golden mountain is made of gold" is an objective. The golden mountain does not exist, yet the just stated objective *subsists*. "Why doesn't the golden mountain exist?", one might ask. And Meinong's answer would be "because it's made of gold and there are no such things as golden mountains". Yet we only know that golden mountains do not exist, because we understand what a golden mountain is, we can grasp the object 'golden mountain': "If I should be able to judge that a certain Object is not, then I appear to have had to grasp the Object in some way beforehand, in order to say anything about its not-being, or more precisely, in order to affirm or deny the ascription of non being to the Object" (Meinong, 1960:84).

Now, regarding certain other objects, such as the round square, we cannot attribute them with any kind of being. They are, in fact, *Aussersein* or outside being. Yet, for Meinong, unlike for Russell, round squares are indeed round and square, hence we know they are impossible objects. The objective "round squares are impossible figures" subsists. This objective has a being so, inasmuch as it subsists, even if the object, the round square, must be attributed with not-being. Compare this with Russell, for whom round squares are neither round nor square, since the class of existing items does not comprise such entities, nothing can be predicated of them with truth.

We can see the elements of Meinong's ontology in the following chart:



The most important consequence of Meinong's theory of objects, at least for our present purposes, is that it allows to know and describe objects without a prior commitment to its existence. The domain of being is expanded beyond existence, allowing for science to inquire into objects that do not exist in the way of spatio-temporal things, but rather have a different kind of being, they subsist:

> "The first distinctive thesis of Meinong's theory is that very many objects do not exist in any way at all. Nevertheless we can make

> true statements in which such objects occur as proper subjects. Since the term 'object' carries no existential commitment, the standard attempt to represent Meinong's theory as a Platonistic theory or as a levels-of-existence theory is, on the face of it, seriously mistaken in the case of ground-floor objects at least. Thus one very important feature which Meinong's theory shares with any thoroughgoing and genuinely non-existential logic is the rejection of the Ontological Assumption (OA), the view embodied in all standard modern logical theories and most empiricist theories (e.g. Hume), that one cannot make true statements about what does not exist. Alternatively, the OA is the thesis that a non-entity cannot be the proper subject of a true statement (where the proper subject contrasts with the apparent subject which is eliminated under analysis into canonical form). The OA was explicitly rejected in Meinong's Independence Thesis (IT) stating the independence of Sosein from Sein: according to the thesis an item's having properties does not imply its existence" (Routley & Routley, 1973:227).

I'd like to offer a brief presentation of two more concepts that are central to Meinong's theory of object and could prove useful in the following section. The first one is the concept of incomplete object. A *complete object* is, paradigmatically, a thing, a spatio-temporal entity. They are complete both in the ontological and gnoseological senses. An object is complete when it is completely determined in all its properties regarding all its relations with all other objects. In other words, a complete object is the one that has a determined answered for all questions posed in terms of the law of the excluded middle. From the gnoseological side, a complete object is that of which I can know all is properties, all its determinations. This glass of water next to me, for instance, is a complete object. When I ask "is it transparent?" I answer "yes", "is it 10cm tall?", yes... and so on. The object is determined in every sense, even if I do not know all of its properties, in principle I could, or if I don't it is due to empirical reasons and not due to a natural impediment of the object.

An *incomplete object*, on the other hand, is that which is not determined regarding all of its properties. This means, not only that one does not know these attributes, but that indeed the object is, in an ontological sense, undetermined. Excluded middle does not apply to it. For example, the abstract circle described by an Euclidean geometry is not determined as to its size, its color, its texture, its location, etc. It is not that these properties are unknown, they are simply not in the object. I could not know, for example, that the circle's area is $\pi.r^2$, yet the object *is* determined in this respect, whereas the color of the circle is not something unknown, but undetermined. It is just a property that the abstract geometrical circle does not possess.

An object that is either determined or undetermined will also be (un)determined in its mode of being:

> "An object which is completely determined in its Being-So (*Sosein*), is also completely determined in its mode of Being (*Sein*). Correspondingly that object which is incompletely determined in its Being-So, is also incompletely determined in its Being. Of the incomplete object A, which is not determined as to its Being, it cannot be stated 'A is' or 'A is not'. It is still possible that it is and that it is not. Here again, possibility is a third alternative to the two contradictory factuality determinations and, moreover, something definitely positive. The indeterminateness of incomplete objects with respect to factuality of Being and Being-So is wholly compatible with their determinateness as to the possibility of Being and Being-So. Thus, though factuality cannot be attributed to incomplete objects, possibility may. The freedom of the incomplete objects from the law of the excluded middle enables them to be the 'carriers' (*Trger*) of 'pure possibilities"' (Michaelis, 1942:401).

Which brings us to the second and last concept I wish to introduce, that is, the concept of possibility. "Possibility is a quantitative property which can be intensified up to the limit of factuality" (Michaelis, 1942:397), meaning that possibility can be thought of as a line that goes from impossibility to factuality/actuality. Possibility is predicated not upon objects themselves, but upon objectives; one should not say "A is (not) possible", but "it is (not) possible that A". Then, in one extreme of the possibility line, we would find all the objectives that have as components completely undetermined objects, contradictory objects such as the round square. In 'the middle' of the line, we would find incomplete objects, such as mathematical entities or objects that are probabilistic in their own nature; until the other extreme of the line, where determined, complete objects, that is, spatio-temporal things, are found.

To conclude this section, the aim of which was no other than to offer a schematic presentation of some of the rudiments of Meinong's ontology in the hope they can be applied to some of the issues of QM in the following section, we can state that Meinong's theory of objects takes as a starting point the concept of object, that what can be thought of, and frees objects from the restraint of existence in the task of knowledge; meaning that there is a realm of being that goes beyond existence/actuality that can be scientifically known. Objectives can be composed both by real existing objects or inexistent objects. In both cases we say of objectives that they subsist. Finally, objects can be either complete or incomplete in the strong ontological sense, and depending on this, we will find them in one place or another of the possibility line, which again, is an ontological possibility and not an epistemic one.

3. A Meinongian ontology for QM

In this section I would like to apply some of the Meinongian concepts presented in the previous section, to some of the main interpretative problems in QM. In particular, I will address quantum superpositions.

Quantum superpositions, also known in the literature as "Schrödinger's cats" raise a series of difficulties that concern not only the discussions in foundations of QM, but also the theory itself. The problem regarding superpositions, according to (da Costa & de Ronde, 2013), is that they seem to violate the principle of non-contradiction when establishing mutually exclusive terms as in for example $\alpha|\uparrow\rangle + \beta|\downarrow\rangle$. Moreover, a further problem appears at the time of determining what objective physical process is responsible for the measurement outcome of only one of the terms; this is known as the "measurement problem". These problems are yet without an accepted solution and, according to de Ronde, the way to solve them lies not within the insistence on the 'measurement problem', on trying to find a way to classically justify the measurement outcomes, which are always taken as a starting, legitimizing point, but rather, by looking into the superpositions themselves. All attempts are guided by the will to understand QM in classical terms, by ignoring that there are, in fact, superpositions as described by the quantum formalism, but also being used for the most diverse technological applications[e].

This misguided approach rests, among others, in one metaphysical supposition that is operating in every attempt to interpret QM, that is, that reality equals actuality. In other words, that something can only be considered as real, as 'truly existent', with ontological density, if and only if that something is in the mode of the actual. Actual here must be understood as

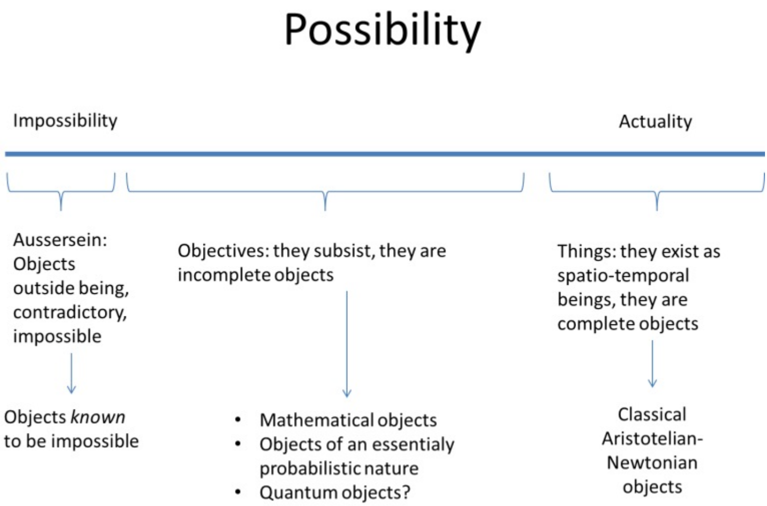
[e]See de Ronde 2015B, specially 25-6.

a metaphysical mode of being, that in which the object in question is completely determined in every respect, as it is the case of physical bodies in classical Newtonian physics. But, "if we are willing to discuss the possibility that 'Quantum Physical Reality Actuality', then there is plenty of space to interpret and represent quantum superpositions in terms of (non-actual) physical reality" (de Ronde, 2015B:3). What would it mean for 'quantum physical reality' to be different from actuality?

An answer can be sought in Meinong's ontology, as I have presented it in the previous section. To begin with, we can consider any definition of a quantum superposition, as the one quoted above, to be a subsisting objective in the Meinongian sense. We mean by this two things. First, that the state of superposition as such should not be thought of as an object, but as an objective. Remember that objectives are, for Meinong, states of affairs, complex objects, made of objects. Hence, in $\alpha| \uparrow \rangle + \beta| \downarrow \rangle$ we find the terms $| \uparrow \rangle$ and $| \downarrow \rangle$ as being the objects that conform the whole objective. Second, by saying that the entire objective subsists, we are saying that it has a specific mode of being, that is not the mode of being of existent actual things, and we are also saying, that we need not worry (yet) about whether or not the objects that compose this objective, the terms in the equation, exist or not; because, as explained in the previous section, judgments about existence are independent of other judgments we can make about objectives.

Superpositions are contradictory only when thought from the perspective of what is actual. But we could understand the wave function, a complex objective which has several bases, some of which are states of superpositions, as being an *incomplete object*. Now, incomplete objects are undetermined not only in a subjective sense, in the sense that we don't know certain properties it has, but also in an ontological sense, that is, incomplete objects are indeed undetermined regarding many of its properties. This indetermination is only a problem, again, if we want the wave function to be a complete object and more precisely a spatio-temporal thing that exists in the same sense as Newtonian particles do. But are we not, in this way, calling for unnecessary problems? Why insist in a metaphysical principle, which as such cannot be demonstrated, if it turns out to make any coherent interpretation impossible?

According to Meinong possibility is a line that goes from impossibility to actuality. What if the domain that QM describes does not belong to the realm of actuality, but to that of pure possibility?:



A first consequence of this way of interpreting the problem would be that: the superpositions problems should be investigated separately from the measurement problem. By this I mean, one issue is understanding measurement in the sense of 'actualization' of quantum states, that is, one issue is the relation there is between the domain of reality described by QM and how that domain becomes actual in a measurement. But the wave function, as described by QM, should constitute an independent and legitimate problem for quantum theory, one that should be understood in terms of possibility, rather than actuality. Thinking of as an incomplete object allows to understand that there is nothing contradictory in the expression of a superposition state. At the same time, if 'real', in the sense of strong being, in the sense of 'what really is', is separated from what is actual, allowing to enlarge the domain of relevant being beyond actuality, then, the wave function, understood as an incomplete object, can be said to be in its own right, even if it doesn't exist in the sense of the actual. This being of the wave function, now in the mode of possibility rather than actuality, is from a Meinongian perspective a legitimate domain of being, in the relevant sense that it is a domain susceptible of scientific inquiry.

Following this line of thought, the primordial question should no longer be "how does nature decide the result of a measurement given a superposition?", but, rather, "what is a superposition and what can it do besides being actualized in a measurement?" It is not that the process of actualization through measurement were irrelevant or unimportant for QM, but it seems that some previous knowledge is necessary in order to tackle it. We do not yet fully understand what a superposition is, but we do have the formalism, and also the possibility of grounding this understanding in a different ontology from the one of classical Newtonian mechanics. By taking the realm of possibility seriously, and the independence of being-so from being, I believe, we can start seeking for new physical concepts that allow the proper comprehension.

We can seek for these concepts now, not in terms of complete determined actual classical objects, but with a different ontology. We can think of superpositions as incomplete objects. This change allows superpositions to bear "mutually contradictory properties" without collapsing understanding. The relevant question would be now, not how one term of the equation appears instead of the other in a measurement outcome, but, for instance, how do different superposition states, none of them actualized, become 'entangled' and interact with one another? If this occurs independently of a measurement process, as it seems to be the case, then we need to begin to understand that these curious states have objective physical properties that are entirely independent of actuality.

Superpositions and the wave function must be understood in their own right. This means we need to find the metaphysical principles that allow for their comprehension regardless of the domain of actuality. In this sense, de Ronde proposes to replace the classical Aristotelian metaphysical principles upon which classical physics are based, for the principles that spring from the quantum theory: instead of the Aristotelian principles of Existence (that being is being is the mode of the Aristotelian entity, as a space-time unity, a complete object in the Meinongian sense); of non-contradiction, and of identity (that is, the identity of an entity through time); instead of said principles, "This realm [the realm of QM] is defined by the principles of indetermination, superposition and difference" (de Ronde, 2015:26). This means that to understand what constitutes a quantum object; we need to define it according to the principles proper to the domain it belongs, the domain of QM, the domain of possibility; and not in terms of actuality.

4. Conclusions

I have argued in favor of a realist approach to the problem of the interpretation of QM and, following that line, I have offered some arguments as to why such problem should be sought to be solved from a perspective that prioritizes the quantum formalism and the elements proper of the theory that have led to so much experimental and technological success, over those orthodox interpretations that seek to maintain all physical understanding within the limits of classical physics. Having accepted then the need for new ontological concepts that could allow for such an interpretation, I have offered a brief presentation of Meinong's theory of objects. In particular, the concepts of subsistence, possibility and incomplete objects, open, I believe, the opportunity to consider that the wave function, as a sort of 'quantum object', belongs to a different mode of being than that of actuality. I do not claim to have found here any revolutionary results, but only to have offered some basic ontological considerations in order to guide the discussion in the foundations of QM. In this sense, I believe that Meinong's ontology allows for a scientific comprehension of objects that do not exist in the mode of the actual, and that these objects could be the objects of QM. If this is true, the only result I have to offer is an indication as to how to proceed in the quest to comprehend quantum reality: that is, not in its constant reference to measurements and actuality, but as a legitimate domain in itself.

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ENTANGLEMENT OF FORMATION FOR WERNER STATES AND ISOTROPIC STATES VIA LOGICAL GATES

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To what extent is a logical characterization of entanglement possible? We investigate some correlations that hold between the concept of entanglement of formation for Werner states and for isotropic states and the probabilistic behavior of some quantum logical gates.

Keywords: Entanglement; quantum logics.

1. Introduction

Entanglement represents a characteristic feature of quantum theory, often described as "puzzling and mysterious". The first explicit analysis of this concept is due to Schrödinger, who proposed the German term "Verschärfung", later translated into English as "entanglement".

How can we describe "entangled quantum objects" from an intuitive point of view? Let us refer to a simple paradigmatic case. We are concerned with a composite physical system S consisting of two subsystems S_1 and S_2 (say, a two-electron system). The observer has a *maximal information* about S: a *pure state*, which is identified with a unit vector $|\psi\rangle$ of a Hilbert space $\mathcal{H}_S$ (representing the "mathematical environment" for S). What can be said about the states of the two subsystems? Due to the form of $|\psi\rangle$ and to the quantum-theoretic rules that concern the mathematical description of composite systems, such states cannot be pure: they are represented by two identical *mixed states* (or *mixtures*), which codify a "maximal degree of uncertainty". A typical possible form of $|\psi\rangle$ is the following *Bell-state*:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle),$$

which lives in the space $\mathcal{H}_S = \mathbb{C}^2 \otimes \mathbb{C}^2$ (the tensor product of the space $\mathbb{C}^2$ with itself), whose canonical basis consists of the four vectors $|0,0\rangle, |0,1\rangle, |1,0\rangle, |1,1\rangle$.

This gives rise to the following physical interpretation: the global system S might satisfy the properties that are *certain* either for the state $|0,0\rangle$ or for the state $|1,1\rangle$ with probability-value $\frac{1}{2}$. At the same time, $|\psi\rangle$ determines that the state of both subsystems (S_1 and S_2) is the mixed state $\frac{1}{2}\mathtt{I}$ (where $\mathtt{I}$ is the identity operator of $\mathbb{C}^2$.) Although it is not determined whether the state of the global system S is $|0,0\rangle$ or $|1,1\rangle$, the two subsystems S_1 and S_2 can be described as "entangled", because in both possible cases they would satisfy the same properties, turning out to be "indistinguishable". As a consequence, any measurement performed by an observer either on system S_1 or on system S_2 would instantaneously transform the *potential* properties of both subsystems into actual properties (by the *collapse of the wave function-principle*.)

The celebrated "Einstein Podolsky Rosen paradox" (*EPR*) is based on a similar physical situation. As is well known, what mainly worried Einstein was the possibility of "non-local effects": the subjective decision of an observer (who may choose among different *incompatible observables* to be measured on the system S_1) seems to determine the instantaneous emergence of an actual property for the system S_2, which might be very "far" from S_1 (possibly inaccessible by means of a light-signal).

Strangely enough, the critical concept of entanglement did not play a relevant role in the foundational debate inspired by the "quantum logical approaches to quantum theory" (after the pioneering work of Birkhoff and von Neumann). The interest for abstract investigations about entanglement emerged again during the last three decades in the framework of the new quantum-computation theories, where the theoretic analysis has often interacted with the experimental research about quantum non-locality. In this context, the characteristic *EPR*-situations (no longer described as mysterious and potentially paradoxical) have been used as a powerful resource, even from a technological point of view (for instance, in the applications to teleportation-phenomena and to quantum cryptography).

The question “to what extent is a logical characterization of entanglement possible?” can be naturally discussed in the framework of new forms of quantum logic that have been suggested by the theory of quantum computation ([1], [2]). The basic semantic idea of these logics (called *quantum computational logics*) can be sketched as follows. Linguistic formulas are supposed to describe pieces of quantum information, mathematically represented as possible states of quantum objects that can store a given information; while the logical connectives are interpreted as *quantum-logical gates* that process quantum information. Accordingly, any formula of the quantum computational language can be regarded as a synthetic logical description of a quantum circuit.

In [4] we have shown how a logical characterization of entanglement (for bipartite systems) is possible by using a somewhat sophisticated language that is an extension of the standard quantum computational language (whose logical connectives are interpreted as special examples of unitary operators). The use of unitary gates is, instead, sufficient for a logical characterization of entanglement in the case of some special classes of quantum states (in particular, the *Werner states* and the *isotropic states*). In this article we will generalize some results proved in [3].

2. States, entanglement and entanglement-measures

Let us first recall some basic definitions. The general mathematical environment for quantum computation is the Hilbert space

$$\mathcal{H}^{(n)} := \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{n-times}$$

(the n-fold tensor product of the space $\mathbb{C}^2$). Any piece of quantum information is represented by a density operator ρ of a space $\mathcal{H}^{(n)}$. A *quregister* (or *quregister-state*) is represented by a unit-vector $|\psi\rangle$ (which is a pure state) of a space $\mathcal{H}^{(n)}$ or, equivalently, by the corresponding density operator $\mathtt{P}_{|\psi\rangle}$ (the projection-operator that projects over the closed subspace determined by $|\psi\rangle$). A *register* (which represents a *certain* piece of information) is an element $|x_1, \ldots, x_n\rangle$ of the canonical orthonormal basis of a space $\mathcal{H}^{(n)}$ (where $x_i \in \{0, 1\}$). The registers $|0\rangle$ and $|1\rangle$ represent the two classical bits.

Any density operator ρ can be represented as a mixture of pure states

$$\rho = \sum_i w_i P_{|\psi\rangle_i} \text{ (with } \sum_i w_i = 1).$$

Generally such representation is not unique. A density operator ρ that cannot be represented as a projection $P_{|\psi\rangle}$ is called a *proper mixture*.

Following a standard convention, we assume that the bit $|1\rangle$ represents the truth-value *Truth*, while bit $|0\rangle$ represents the truth-value *Falsity*. On this basis, we can identify, in each space $\mathcal{H}^{(n)}$, two special projection-operators $\mathtt{P}_1^{(n)}$ and $\mathtt{P}_0^{(n)}$ that represent, respectively, the *truth-property* and the *falsity-property*. The truth-property $\mathtt{P}_1^{(n)}$ is the projection-operator that projects over the closed subspace spanned by the set of all registers $|x_1, \ldots, x_{n-1}, 1\rangle$; while the falsity-property $\mathtt{P}_0^{(n)}$ is the projection-operator that projects over the closed subspace spanned by the set of all registers $|x_1, \ldots, x_{n-1}, 0\rangle$. In this way, truth and falsity are dealt with as mathematical representatives of possible physical properties. Accordingly, by applying the Born-rule, one can naturally define the probability-value $\mathtt{p}(\rho)$ of any density operator ρ of $\mathcal{H}^{(n)}$ as follows:

$$\mathtt{p}(\rho) := \mathrm{tr}(\mathtt{P}_1^{(n)}\rho), \text{ where tr is the trace-functional.}$$

Hence, $\mathtt{p}(\rho)$ represents the probability that the information ρ is true.

The concept of entanglement can be defined both for pure and for mixed states. Consider the product-space

$$\mathcal{H}^{(m+n)} = \mathcal{H}^{(m)} \otimes \mathcal{H}^{(n)}.$$

Any density operator ρ of $\mathcal{H}^{(m+n)}$ represents a possible state for a composite physical system $S = S_1 + S_2$ (consisting of two subsystems). According to the quantum formalism, ρ determines the two *reduced states* $Red^{(1)}_{[m,n]}(\rho)$ and $Red^{(2)}_{[m,n]}(\rho)$ that represent the states of S_1 and of S_2 (in the context ρ). In such a case, we say that ρ is a *bipartite state* with respect to the decomposition (m, n). It may happen that ρ is a pure state, while $Red^{(1)}_{[m,n]}(\rho)$ and $Red^{(2)}_{[m,n]}(\rho)$ are proper mixtures. In this case the information about the whole system is more precise than the pieces of information about its parts. As an example, consider the following density operator:

$$\rho = \mathtt{P}_{\frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)}$$

(the projection that projects over the closed subspace spanned by the vector $\frac{1}{\sqrt{2}}(|0, 0\rangle + |1, 1\rangle)$).

We have:

$$Red^{(1)}_{[1,1]}(\rho) = Red^{(2)}_{[1,1]}(\rho) = \frac{1}{2}\mathtt{I}.$$

Definition 2.1. *(Factorizability, separability and entanglement)*
Let ρ be a bipartite state of $\mathcal{H}^{(m+n)}$ (with respect to the decomposition (m,n)).

1) ρ is called a *(bipartite) factorized state* of $\mathcal{H}^{(m+n)}$ iff $\rho = \rho_1 \otimes \rho_2$, where ρ_1 and ρ_2 are density operators of $\mathcal{H}^{(m)}$ and $\mathcal{H}^{(n)}$, respectively;
2) ρ is called a *(bipartite) separable state* of $\mathcal{H}^{(m+n)}$ iff $\rho = \sum_i w_i \rho_i$, where each ρ_i is a bipartite factorized state of $\mathcal{H}^{(m+n)}$, $w_i \in [0,1]$ and $\sum_i w_i = 1$;
3) ρ is called a *(bipartite) entangled state* of $\mathcal{H}^{(m+n)}$ iff ρ is not separable.

Accordingly, a pure state is entangled iff it is non-factorizable. Proper mixtures, instead, may be non-factorizable, separable (and, hence, non-entangled). An example is represented by the following proper mixture:

$$\rho = \frac{1}{2}\mathsf{P}_{|0,0\rangle} + \frac{1}{2}\mathsf{P}_{|1,1\rangle}$$

Definition 2.2. *(Maximally entangled states and maximally mixed states)*

1) A pure bipartite state ρ of $\mathcal{H}^{(m+n)}$ is called *maximally entangled* iff $Red^{(1)}_{[m,n]} = \frac{1}{2^m}\mathtt{I}^{(m)}$ or $Red^{(2)}_{[m,n]} = \frac{1}{2^n}\mathtt{I}^{(n)}$, where $\mathtt{I}^{(m)}$ and $\mathtt{I}^{(n)}$ are the identity operators of the spaces $\mathcal{H}^{(m)}$ and $\mathcal{H}^{(n)}$, respectively.
2) A state ρ of $\mathcal{H}^{(m+n)}$ is called a *maximally mixed state* of $\mathcal{H}^{(m+n)}$ iff $\rho = \frac{1}{2^{(m+n)}}\mathtt{I}^{(m+n)}$.

How to measure the "entanglement-degree" of a given state? Different definitions for the concept of *entanglement-measure*, which quantify different aspects of entanglement, have been proposed in the literature ([5,6]). Generally it is required that any notion of *normalized entanglement-measure* (EM) satisfies the following minimal conditions for any density operator ρ:

(1) $EM(\rho) \in [0,1]$;
(2) $EM(\rho) = 0$, if ρ is separable;
(3) $EM(\rho) = 1$, if ρ is a maximally entangled pure state;
(4) $EM(\rho)$ is *invariant* under *locally unitary maps*. This means that: $EM(\rho) = EM((U_1^{(m)} \otimes U_2^{(n)})\rho(U_1^{(m)} \otimes U_2^{(n)})^\dagger)$, for any unitary operators $U_1^{(m)}$ of $\mathcal{H}^{(m)}$ and $U_2^{(n)}$ of $\mathcal{H}^{(n)}$ (where $(U_1^{(m)} \otimes U_2^{(n)})^\dagger$ is the adjoint of $U_1^{(m)} \otimes U_2^{(n)}$).

One of the most interesting notions of entanglement-measure is the concept of *entanglement of formation*, which is defined in terms of the notion of *von Neumann-entropy.*

Definition 2.3. *(The von Neumann-entropy)*
Let ρ be a density operator of the space $\mathcal{H}^{(n)}$. The *von Neumann-entropy* of ρ is defined as follows:

$$E_S(\rho) = -\sum_i \lambda_i \ln \lambda_i,$$

where λ_i are the eigenvalues of ρ.

Definition 2.4. *(The entanglement of formation)*
Let ρ be a bipartite state of the space $\mathcal{H}^{(m+n)}$). The *entanglement of formation* of ρ is defined as follows:

$$E_F(\rho) = \inf \left\{ \sum_i w_i E_S(Red^{(j)}_{[m,n]}(\mathtt{P}_{|\psi_i\rangle})) : \rho = \sum_i w_i \mathtt{P}_{|\psi_i\rangle} \right\},$$

where $j \in \{1, 2\}$.

Apparently, the number $E_F(\rho)$ is determined by the set of all values of the von Neumann-entropy of the two pure reduced states that correspond to all possible representations of ρ as a mixture of pure states.

One can easily show that $E_F(\rho)$ satisfies the minimal conditions that are required for all concepts of normalized entanglement-measure.

3. Quantum logical gates and entanglement-measures

The question "to what extent is a logical characterization of entanglement possible?" is deeply connected with the probabilistic behavior of *quantum logical gates* (briefly, *gates*), the quantum operations that process quantum information. It seems reasonable to assume that any logical characterization of entanglement should provide a definition for an accepted notion of entanglement-measure (say, the concept of entanglement of formation) in terms of the probabilistic behavior of some gates.

In this perspective, we will first recall the definition of some basic gates that play an important role both from the computational and from the logical point of view. One is dealing with special examples of unitary operators that transform quregisters into quregisters (in a reversible way).

Definition 3.1. *(The negation)*
For any $n \geq 1$, the *negation* (defined on $\mathcal{H}^{(n)}$) is the linear operator $\mathtt{NOT}^{(n)}$ such that, for every element $|x_1, \ldots, x_n\rangle$ of the canonical basis,

$$\mathtt{NOT}^{(n)}|x_1, \ldots, x_n\rangle = |x_1, \ldots, x_{n-1}\rangle \otimes |1 - x_n\rangle.$$

In particular, we obtain:

$$\mathtt{NOT}^{(1)}|0\rangle = |1\rangle; \quad \mathtt{NOT}^{(1)}|1\rangle = |0\rangle,$$

according to the classical truth-table of negation.

Definition 3.2. *(The Toffoli-gate)*
For any $m, n, p \geq 1$, the *Toffoli-gate* (defined on $\mathcal{H}^{(m+n+p)}$) is the linear operator $\mathtt{T}^{(m,n,p)}$ such that, for every element $|x_1, \ldots, x_m\rangle \otimes |y_1, \ldots, y_n\rangle \otimes |z_1, \ldots, z_p\rangle$ of the canonical basis,

$$\begin{aligned}\mathtt{T}^{(m,n,p)}&|x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_p\rangle \\ &= |x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_{p-1}\rangle \otimes |x_m y_n \widehat{+} z_p\rangle,\end{aligned}$$

where $\widehat{+}$ represents the addition modulo 2.

Definition 3.3. *(The* $\mathtt{XOR}$*-gate)*
For any $m, n \geq 1$, the $\mathtt{XOR}$-*gate* (defined on $\mathcal{H}^{(m+n)}$) is the linear operator $\mathtt{XOR}^{(m,n)}$ such that, for every element $|x_1, \ldots, x_m\rangle \otimes |y_1, \ldots, y_n\rangle$ of the canonical basis,

$$\mathtt{XOR}^{(m,n)}|x_1, \ldots, x_m, y_1, \ldots, y_n\rangle = |x_1, \ldots, x_m, y_1, \ldots, y_{n-1}\rangle \otimes |x_m \widehat{+} y_n\rangle.$$

Definition 3.4. *(The Hadamard-gate)*
For any $n \geq 1$, the *Hadamard-gate* (defined on $\mathcal{H}^{(n)}$) is the linear operator $\sqrt{\mathtt{I}}^{(n)}$ such that for every element $|x_1, \ldots, x_n\rangle$ of the canonical basis:

$$\sqrt{\mathtt{I}}^{(n)}|x_1, \ldots, x_n\rangle = |x_1, \ldots, x_{n-1}\rangle \otimes \frac{1}{\sqrt{2}}\left((-1)^{x_n}|x_n\rangle + |1 - x_n\rangle\right).$$

In particular we obtain:

$$\sqrt{\mathtt{I}}^{(1)}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle); \sqrt{\mathtt{I}}^{(1)}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Hence, $\sqrt{\mathtt{I}}^{(1)}$ transforms bits into genuine qubits.

Definition 3.5. *(The square root of* NOT*)*
For any $n \geq 1$, the *square root of* NOT (defined on $\mathcal{H}^{(n)}$) is the linear operator $\sqrt{\mathtt{NOT}}^{(n)}$ such that for every element $|x_1,\ldots,x_n\rangle$ of the canonical basis:

$$\sqrt{\mathtt{NOT}}^{(n)}|x_1,\ldots,x_n\rangle = |x_1,\ldots,x_{n-1}\rangle \otimes \left(\frac{1-i}{2}|x_n\rangle + \frac{1+i}{2}|1-x_n\rangle\right),$$

where $i = \sqrt{-1}$.

Definition 3.6. *(The swap-gate)*
For any $m,n \geq 1$, the *swap-gate* (defined on $\mathcal{H}^{(m,n)}$) is the linear operator $\mathtt{SW}^{(m,n)}$ such that, for every element $|x_1,\ldots,x_m\rangle \otimes |y_1,\ldots,y_n\rangle$ of the canonical basis,

$$\mathtt{SW}^{(m,n)}|x_1,\ldots,x_m,y_1,\ldots,y_n\rangle = |y_1,\ldots,y_n,x_1,\ldots,x_m\rangle.$$

All gates can be canonically extended to the set of all density operators. Let $\mathtt{G}$ be any gate defined on $\mathcal{H}^{(n)}$. The corresponding *density-operator gate* (also called *unitary quantum operation*) ${}^{\mathfrak{D}}\mathtt{G}$ is defined as follows for any density operator ρ of $\mathcal{H}^{(n)}$:

$${}^{\mathfrak{D}}\mathtt{G}\rho = \mathtt{G}\,\rho\,\mathtt{G}^\dagger \ \text{ (where } \mathtt{G}^\dagger \text{ is the adjoint of } \mathtt{G}).$$

For the sake of simplicity, also the operations ${}^{\mathfrak{D}}\mathtt{G}$ will be briefly called *gates*.

The Toffoli-gate has a special logical interest, since it allows us to define a conjunction that behaves according to classical logic, whenever it is applied to certain pieces of information (bits and registers). At the same time this conjunction may have some deeply anti-classical features when the inputs are uncertain pieces of quantum information. The most peculiar property is represented by a *holistic* behavior: generally the conjunction defined on a global piece of quantum information (represented by a given density operator) cannot be described as a function of its separate parts. This is the reason why the quantum computational conjunction is called "holistic".

Definition 3.7. (*The holistic conjunction*)
For any $m,n \geq 1$ and for any density operator ρ of $H^{(m+n)}$), the *holistic conjunction* $\mathtt{AND}^{(m,n)}$ is defined as follows:

$$\mathtt{AND}^{(m,n)}(\rho) = {}^{\mathfrak{D}}\mathtt{T}^{(m,n,1)}(\rho \otimes P_0^{(1)}).$$

Clearly, the falsity-property $P_0^{(1)}$ plays, in this definition, the role of an *ancilla*. Generally we have:

$$\mathtt{AND}^{(m,n)}(\rho) \neq \mathtt{AND}^{(m,n)}(Red^{(1)}_{[m,n]}(\rho) \otimes Red^{(2)}_{[m,n]}(\rho)).$$

Hence, the holistic conjunction defined on a global information consisting of two parts does not generally coincide with the conjunction of the two separate parts. As an example, consider the following density operator (which corresponds to a maximally entangled pure state):

$$\rho = P_{\frac{1}{\sqrt{2}}(|0,0\rangle+|1,1\rangle)}.$$

We have:

$$\mathtt{AND}^{(1,1)}(\rho) = P_{\frac{1}{\sqrt{2}}(|0,0,0\rangle+|1,1,1\rangle)} ,$$

which also represents a maximally entangled pure state. At the same time we have:

$$\mathtt{AND}^{(1,1)}(Red^{(1)}_{[1,1]}(\rho) \otimes Red^{(2)}_{[1,1]}(\rho)) = \mathtt{AND}^{(1,1)}\left(\frac{1}{2}\mathtt{I}^{(1)} \otimes \frac{1}{2}\mathtt{I}^{(1)}\right),$$

which is a proper mixture.

Now we introduce a highly representative class of gates, called *connective-gates*.

Definition 3.8. (*Connective-gate*)
Consider a Hilbert space $\mathcal{H}^{(m+n+p)}$ with $m, n \geq 0$ and $p \geq 1$.
A *connective-gate* of $\mathcal{H}^{(m+n+p)}$ is a unitary operator $\mathtt{G}^{(m,n,p)}$ that can be represented in the following form:
$\mathtt{G}^{(m,n,p)} = [P_0^{(m)} \otimes P_0^{(n)} \otimes \mathtt{I}^{(p-1)} \otimes U_{00}] + [P_0^{(m)} \otimes P_1^{(n)} \otimes \mathtt{I}^{(p-1)} \otimes U_{01}] + [P_1^{(m)} \otimes P_0^{(n)} \otimes \mathtt{I}^{(p-1)} \otimes U_{10}] + [P_1^{(m)} \otimes P_1^{(n)} \otimes \mathtt{I}^{(p-1)} \otimes U_{11}]$,
where U_{ij} are unitary operators of $\mathcal{H}^{(1)}$, $\mathtt{I}^{(0)} = 1, P_0^{(0)} = P_1^{(0)} = \frac{1}{2}$.

Apparently, any connective-gate $\mathtt{G}^{(n)}$ applied to a register $|x_1, \ldots, x_n\rangle$ transforms the target-bit $|x_n\rangle$ into a qubit that determines the probability-value of $\mathtt{G}^{(n)}(|x_1, \ldots, x_n\rangle)$.

One can easily check that the negation $\mathtt{NOT}^{(1)}$ (of $\mathcal{H}^{(1)}$) and the Toffoli-gate $\mathtt{T}^{(1,1,1)}$ (of $\mathcal{H}^{(1+1+1)}$) are connective-gates, since they can be represented in the following form:

$$\mathtt{NOT}^{(1)} = (\frac{1}{2}\cdot\frac{1}{2}\cdot 1)\mathtt{NOT}^{(1)} + (\frac{1}{2}\cdot\frac{1}{2}\cdot 1)\mathtt{NOT}^{(1)} + (\tfrac{1}{2}\cdot\tfrac{1}{2}\cdot 1)\mathtt{NOT}^{(1)} + (\tfrac{1}{2}\cdot\tfrac{1}{2}\cdot 1)\mathtt{NOT}^{(1)};$$

$$\begin{aligned}\mathtt{T}^{(1,1,1)} &= [P_0^{(1)} \otimes P_0^{(1)} \otimes 1 \cdot \mathtt{I}^{(1)}] + [P_0^{(1)} \otimes P_1^{(1)} \otimes 1 \cdot \mathtt{I}^{(1)}] \\ &\quad + [P_1^{(1)} \otimes P_0^{(1)} \otimes 1 \cdot \mathtt{I}^{(1)}] + [P_1^{(1)} \otimes P_1^{(1)} \otimes 1 \cdot \mathtt{NOT}^{(1)}].\end{aligned}$$

Similar representations can be given for the gates $\mathtt{NOT}^{(p)}$, $\mathtt{T}^{(m,n,p)}$, $\mathtt{XOR}^{(n,p)}$, $\sqrt{\mathtt{NOT}}^{(p)}$, $\sqrt{\mathtt{I}}^{(p)}$. An example of gate that is not a connective-gate is the swap-gate .

The following theorem shows that entanglement cannot be generally characterized by the probabilistic behavior of connective-gates ([4]).

Theorem 3.1.
There is no connective-gate $\mathtt{G}^{(m,n,1)}$ *that satisfies the following condition: there exists a function* $\mu : [0,1] \to [0,1]$ *such that for any density operator* ρ *of* $\mathcal{H}^{(m)} \otimes \mathcal{H}^{(n)}$, $\mu(\mathtt{p}({}^{\mathfrak{D}}\mathtt{G}^{(m,n,1)}(\rho \otimes P_0^{(1)})))$ *is an entanglement-measure for* ρ.

4. Werner states and isotropic states

Although (by Theorem 3.1) entanglement-measures cannot be generally characterized in terms of the probabilistic behavior of connective-gates, the situation changes if we restrict our attention to some special classes of states. An interesting example is represented by the class of all *Werner states*, introduced in [9] in order to show that entangled bipartite states do not necessarily exhibit non-local correlations.

Definition 4.1. (Werner state)
A *Werner state* is a bipartite state ρ of a space $\mathcal{H}^{(2n)}$ that satisfies the following condition for any unitary operator U of $\mathcal{H}^{(n)}$:

$$^{\mathfrak{D}}(U \otimes U)\, \rho = \rho.$$

Hence, any Werner state is invariant under local unitary transformations.

One can prove that the class of all Werner states of $\mathcal{H}^{(2n)}$ can be represented as a one-parameter manifold of states.

Lemma 4.1. *Any Werner state of the space* $\mathcal{H}^{(2n)}$ *can be represented as follows:*

$$\rho_w^{(2n)} = \frac{1}{2^{2n}-1}\Big[\Big(1 - \frac{w}{2^n}\Big)\, \mathtt{I}^{(2n)} + \Big(w - \frac{1}{2^n}\Big)\, \mathtt{SW}^{(n,n)}\Big],$$

where $-1 \le w \le 1$ *(while* $\mathtt{I}^{(2n)}$ *and* $\mathtt{SW}^{(n,n)}$ *are the identity operator and the swap-gate of the space* $\mathcal{H}^{(2n)}$*).*

Notice that the number w represents the expectation value of $\mathtt{SW}^{(n,n)}$ for the state $\rho_w^{(2n)}$ (i.e. $w = \mathrm{tr}(\mathtt{SW}^{(n,n)}\rho_w^{(2n)})$).

A simple correlation connects the entanglement of formation for a Werner state $\rho_w^{(2n)}$ with the parameter w ([8]).

Lemma 4.2. *The entanglement of formation of any Werner space* $\rho_w^{(2n)}$ *is*

$$E_F(\rho_w^{(2n)}) = \begin{cases} s\left(\frac{1-\sqrt{1-w^2}}{2}\right) & \textit{if } w \in [-1,0]; \\ 0, & \textit{otherwise.} \end{cases}$$

(where $s(x) = -x\log_2 x - (1-x)\log_2(1-x)$*).*

On this basis one can prove that the entanglement of formation of Werner states can be represented in terms of the probabilistic behavior of the holistic conjunction.

Lemma 4.3. *[3]*
Let $\rho_w^{(2n)}$ be a Werner state of $\mathcal{H}^{(2n)}$.

1) $\mathrm{p}(\rho_w^{(2n)}) = \frac{1}{2}$;
2) $\mathrm{p}(\mathtt{AND}^{(n,n)}(\rho_w^{(2n)})) = \frac{2^{2n-1}+2^{n-1}w-1}{2(2^{2n}-1)}$;
3) $\mathrm{p}(Red_{[n,n]}^{(1)}(\rho_w^{(2n)})) = \mathrm{p}(Red_{[n,n]}^{(2)}(\rho_w^{(2n)})) = \frac{1}{2}$.

Theorem 4.1. *Let $\rho_w^{(2n)}$ be a Werner state of $\mathcal{H}^{(2n)}$.*

$$E_F(\rho_w^{(2n)}) = \begin{cases} s\left(\frac{1-\sqrt{1-\left(\frac{2(2^{2n}-1)\mathrm{p}(\mathtt{AND}^{(n,n)}(\rho_w^{(2n)}))-2^{2n-1}+1}{2^{n-1}}\right)^2}}{2}\right), \\ \quad \textit{if } \frac{1}{4}+\frac{1}{4(1-2^n)} \leq \mathrm{p}(\mathtt{AND}^{(n,n)}(\rho_w^{(2n)})) < \frac{1}{4}+\frac{1}{4(1-2^{2n})}; \\ 0, \ \textit{otherwise.} \end{cases}$$

Proof. By Lemma 4.2 and Lemma 4.3. □

Similar results can be found for another interesting class of states that contains all *isotropic states.*

Definition 4.2. *(Isotropic state)*
An *isotropic state* is a bipartite state ρ of a space $\mathcal{H}^{(2n)}$ that satisfies the following condition for any unitary operator U of $\mathcal{H}^{(n)}$:

$${}^{\mathfrak{D}}(U \otimes U^*)\rho = \rho.$$

Hence, any isotropic state ρ of $\mathcal{H}^{(2n)}$) is invariant under the whole $U \otimes U^*$-group of transformations (where U is any unitary operator of $\mathcal{H}^{(n)}$ and U^* is the complex conjugate of U).

As happens in the case of Werner states, one can prove that also the class of all isotropic states of $\mathcal{H}^{(2n)}$) can be represented as a one-parameter manifold of states.

Lemma 4.4.
Any isotropic state of trhe space $\mathcal{H}^{(2n)}$) can be represented as follows:

$$\rho_\iota^{(2n)} = \frac{1}{2^{2n}-1}\Big[\left(1-\frac{\iota}{2^n}\right)\mathtt{I}^{(2n)} + \left(\iota-\frac{1}{2^n}\right)\mathtt{P}^{(n,n)}\Big],$$

where $0 \leq \iota \leq 2^n$ *and*
$\mathtt{P}^{(n,n)} = \sum_{x_1,\ldots,x_n,y_1,\ldots,y_n=0}^{1} |x_1,\ldots,x_n,x_1,\ldots,x_n\rangle\langle y_1,\ldots,y_n,y_1,\ldots,y_n|$.

Notice that the number ι represents the expectation value of $\mathtt{P}^{(n,n)}$ for the state $\rho_\iota^{(2n)}$ (i.e. $\iota = \mathrm{tr}(P^{(n,n)}\rho_\iota^{(2n)})$).

Lemma 4.5. *([7])*
Let $\rho_\iota^{(2n)}$ *be an isotropic state of* $\mathcal{H}^{(2n)}$.

$$E_F(\rho_\iota^{(2n)}) = \begin{cases} h(s(\gamma(\iota)) + (1-\gamma(\iota))\log_2(2^n-1)) \ \textit{if}\ \iota \in (1, 2^n] \\ 0 \ \textit{otherwise} \end{cases},$$

where $\gamma(\iota) = \frac{1}{2^{2n}}(\sqrt{\iota}+\sqrt{(2^n-1)(2^n-\iota)})^2$, *s is the binary Shannon entropy (i.e.* $s(x) = -x\log_2 x - (1-x)\log_2(1-x)$*) and h is the convex-hull of the inner expression (i.e. the largest convex curve nowhere larger than the given one).*

On this basis one can prove that the entanglement of formation of isotropic states also can be represented in terms of the probabilistic behavior of the holistic conjunction.

Lemma 4.6. *Let* $\rho_\iota^{(2n)}$ *be an isotropic state of* $\mathcal{H}^{(2n)})$.

1) $\mathrm{p}(\rho_\iota^{(2n)}) = \frac{1}{2}$
2) $\mathrm{p}(\mathtt{AND}^{(n,n)}(\rho_\iota^{(2n)})) = \frac{2^{2n-1}+2^{n-1}\iota-1}{2(2^{2n}-1)}$

Proof. 1) $\mathrm{p}(\rho_\iota) = \mathrm{tr}(P^{(2n)}\rho_\iota) = \frac{1}{2}\Big[\sum_{i=1}^{2^{2n-1}}\left(1-\frac{\iota}{2^n}\right) + \sum_{i=1}^{2^{n-1}}\left(\iota - \frac{1}{2^n}\right)\Big] = \frac{1}{2}$.
2) $\mathrm{p}(\mathtt{AND}^{(n,n)}(\rho_\iota)) = \mathrm{tr}(P^{(2n+1)}\,{}^{\mathfrak{D}}\mathtt{T}^{(n,n,1)}(\rho_\iota \otimes P_0^{(1)})) = \mathrm{tr}((P_1^{(n)} \otimes P_1^{(n)})\rho_\iota) = \frac{1}{2^{2n}-1}\Big[\sum_{i=1}^{2^{2(n-1)}}\left(1-\frac{w}{2^n}\right) + \sum_{i=1}^{2^{n-1}}\left(\iota-\frac{1}{2^n}\right)\Big] = \frac{2^{2n-1}+2^{n-1}\iota-1}{2(2^{2n}-1)}$. In particular, for any entangled state $\rho_\iota^{(2n)}$ (i.e. $\iota \in (1, 2^n]$), we have: $\frac{1}{4} + \frac{1}{4(1+2^n)} < \mathrm{p}(\mathtt{AND}^{(n,n)}(\rho_\iota^{(2n)})) \leq \frac{1}{2}$. For any separable state $\rho_\iota^{(2n)}$ (i.e. $\iota \in [0,1]$), we have: $\frac{1}{4} + \frac{1}{4(1-2^{2n})} \leq \mathrm{p}(\mathtt{AND}^{(n,n)}(\rho_\iota^{(2n)})) \leq \frac{1}{4} + \frac{1}{4(1+2^n)}$. For the factorized state $\rho_\iota^{(2n)} = \frac{1}{2^n}\mathtt{I}^{(n)} \otimes \frac{1}{2^n}\mathtt{I}^{(n)}$ (i.e. $\iota = \frac{1}{2^n}$), we have: $\mathrm{p}(\mathtt{AND}^{(n,n)}(\rho_\iota^{(2n)})) = \frac{1}{4}$.
□

Theorem 4.2. *Let $\rho_\iota^{(2n)}$ be an isotropic state of $\mathcal{H}^{(2n)}$).*

$$E_F(\rho_\iota^{(2n)}) = \begin{cases} h\Big[s(\gamma\Big(\frac{2(2^{2n}-1)\mathrm{p}(\mathtt{AND}^{(n,n)}(\rho_\iota^{(2n)}))-2^{2n-1}+1}{2^{n-1}}\Big)) \\ +(1-\gamma\Big(\frac{2(2^{2n}-1)\mathrm{p}(\mathtt{AND}^{(n,n)}(\rho_\iota^{(2n)}))-2^{2n-1}+1}{2^{n-1}}\Big))\log_2(2^n-1)\Big], \\ \quad \textit{if } \frac{1}{4}+\frac{1}{4(1+2^n)} < \mathrm{p}(\mathtt{AND}^{(n,n)}(\rho_\iota^{(2n)})) \leq \frac{1}{2}; \\ 0, \ \textit{otherwise.} \end{cases}$$

Proof. By Lemma 4.5 and Lemma 4.6. □

On this basis one can conclude that (unlike the general case) the probabilistic behavior of the holistic conjunction allows us to characterize entanglement both for Werner states and for isotropic states.

One might wonder whether the capacity of characterizing entanglement may depend on the specific features of the holistic conjunction. The answer to this question is negative. In fact, similar results can be obtained by using (instead of $\mathtt{AND}^{(m,n)}$) the gate $\mathtt{XOR}^{(m,n)}$ or other gates that represent a binary Boolean function (in the Hilbert-space environment).

Theorem 4.3. *Let $\rho_w^{(2n)}$ be a Werner state $\mathcal{H}^{(2n)}$.*

1) $\mathrm{p}(\mathtt{XOR}^{(n,n)}(\rho_w^{(2n)})) = \frac{2^{2n-1}}{2^{2n}-1}\Big(1-\frac{w}{2^n}\Big)$

2) $$E_F(\rho_w^{(2n)}) = \begin{cases} s\left(\frac{1-\sqrt{1-\left(\frac{2^{2n-1}-(2^{2n}-1)\mathrm{p}(\mathtt{XOR}^{(n,n)}(\rho_w^{(2n)}))}{2^{n-1}}\right)^2}}{2}\right), \\ \quad \textit{if } \frac{2^{2n-1}}{2^{2n}-1} < \mathrm{p}(\mathtt{XOR}^{(n,n)}(\rho_w^{(2n)})) \leq \frac{2^{n-1}}{2^n-1}; \\ 0, \ \textit{otherwise.} \end{cases}$$

Proof. Similar to Theorem 4.1. In particular, for any entangled state $\rho_w^{(2n)}$ (i.e. $w \in [-1, 0)$), we have: $\frac{2^{2n-1}}{2^{2n}-1} < \mathrm{p}(\mathtt{XOR}^{(n,n)}(\rho_w^{(2n)})) \leq \frac{2^{n-1}}{2^n-1}$. For any separable state $\rho_w^{(2n)}$ (i.e. $w \in [0, 1]$), we have: $\frac{2^{n-1}}{2^n+1} \leq \mathrm{p}(\mathtt{XOR}^{(n,n)}(\rho_w^{(2n)})) \leq \frac{2^{2n-1}}{2^{2n}-1}$. For the factorized state $\rho_w^{(2n)} = \frac{1}{2^n}\mathtt{I}^{(n)} \otimes \frac{1}{2^n}\mathtt{I}^{(n)}$ (i.e. $w = \frac{1}{2^n}$), we have: $\mathrm{p}(\mathtt{XOR}^{(n,n)}(\rho_w^{(2n)})) = \frac{1}{2}$. □

Theorem 4.4. *Let $\rho_\iota^{(2n)}$ be an isotropic state of $\mathcal{H}^{(2n)}$.*

1) $\mathrm{p}(\mathtt{XOR}^{(n,n)}(\rho_\iota^{(2n)})) = \frac{2^{2n-1}}{2^{2n}-1}\Big(1-\frac{\iota}{2^n}\Big)$

2) $E_F(\rho_\iota^{(2n)}) =$

$$\begin{cases} h\Big[s(\gamma\Big(\frac{2^{2n-1}-(2^{2n}-1)\mathrm{p}(\,\mathtt{XOR}^{(n,n)}(\rho_\iota^{(2n)}))}{2^{n-1}}\Big)) \\ +(1-\gamma\Big(\frac{2^{2n-1}-(2^{2n}-1)\mathrm{p}(\,^{\mathfrak{D}}\mathtt{XOR}^{(n,n)}(\rho_\iota^{(2n)}))}{2^{n-1}}\Big))\log_2(2^n-1)\Big], \\ \quad \textit{if}\ \mathrm{p}(\,\mathtt{XOR}^{(n,n)}(\rho_\iota^{(2n)})) < \frac{2^{n-1}}{2^n+1}; \\ 0, \ \textit{otherwise.} \end{cases}$$

Proof. Similar to Theorem 4.2. In particular, for any entangled state $\rho_\iota^{(2n)}$ (i.e. $\iota \in (1, 2^n]$), we have: $\mathrm{p}(\,\mathtt{XOR}^{(n,n)}(\rho_\iota^{(2n)})) < \frac{2^{n-1}}{2^n+1}$. For any separable state $\rho_\iota^{(2n)}$ (i.e. $\iota \in [0,1]$), we have: $\frac{2^{n-1}}{2^n+1} \le \mathrm{p}(\,\mathtt{XOR}^{(n,n)}(\rho_\iota^{(2n)})) \le \frac{2^{2n-1}}{2^{2n}-1}$. For the factorized state $\rho_\iota^{(2n)} = \frac{1}{2^n}I^{(n)} \otimes \frac{1}{2^n}\mathtt{I}^{(n)}$ (i.e. $\iota = \frac{1}{2^n}$), we have: $\mathrm{p}(\,\mathtt{XOR}^{(n,n)}(\rho_\iota^{(2n)})) = \frac{1}{2}$. $\square$

Both $\mathtt{AND}^{(m,n)}$ and $\mathtt{XOR}^{(m,n)}$ belong to a special class of gates that we call *binary Boolean gates.*

Definition 4.3. *(Binary Boolean gate)*
Consider the 16 binary Boolean functions g_j (with $1 \le j \le 16$) defined on the set $\{0,1\}$. A unitary operator $G_j^{(m,n,p)}$ is called a *binary Boolean gate* representing the Boolean function g_j in the space $\mathcal{H}^{(m+n+p)}$ iff $G_j^{(m,n,p)}$ satisfies the following condition, for any element $|x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_p\rangle$ of the canonical basis of $\mathcal{H}^{(m+n+p)}$:

$$\begin{aligned} &\mathtt{G_j}^{(m,n,p)}|x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_p\rangle \\ &\qquad = |x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_{p-1}, g_j(x_m, y_n)\widehat{+}z_p\rangle), \end{aligned}$$

where $\widehat{+}$ is the addition modulo 2.

One can easily show that all binary Boolean gates are connective-gates, but generally not the other way around. For instance, the Hadamard-gate defined on the space $\mathcal{H}^{(3)}$ is a connective-gate that is not a binary Boolean gate.

Consider now the following proper subset of the set **BBF** of all binary Boolean functions:

$$\mathbf{BBF}^* = \mathbf{BBF} - \{g_1, g_2, g_3\},$$

where: $g_1(x,y) = 0,\ g_2(x,y) = 1 - x(1-y),\ g_3(x,y) = 1 - y(1-x)$.

Theorem 4.5.
Let ρ be either a Werner state or an isotropic state. For any choice of a binary Boolean function $g_j \in \mathbf{BBF}^*$, there is a corresponding binary Boolean gate $\mathtt{G_j}^{(m,n,p)}$ such that the entanglement of formation of ρ can be determined as a function of the probabilistic behavior of $\mathtt{G_j}^{(m,n,p)}$.

Proof. Similar to Theorem 4.1 and 4.2. □

In conclusion, the question "to what extent is a logical characterization of entanglement possible?" gives rise to different answers that depend on the class of gates and on the class of states that is taken into consideration. In [4] we have proved that a logical characterization of entanglement for bipartite states is possible by using some elements of a "non-standard" class of gates that contains both unitary and anti-unitary operators. An interesting open question is the following: is it possible to characterize entanglement by using *only* unitary gates that are not necessarily connective-gates?

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TIME, CHANCE AND QUANTUM THEORY

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I propose an understanding of Everett and Wheeler's relative-state interpretation of quantum mechanics, which restores the feature of indeterminism to the theory. This incorporates a theory of probability as truth values in a many-valued logic for future statements, and a contextual theory of truth which gives objective and subjective perspectives equal validity.

Keywords: Relative-state interpretation; probability; many-valued logic.

1. Introduction

When quantum mechanics emerged in the 1920s as the basic format in which all subsequent theories of physics would be couched, it exhibited two distinct features which represented its radical conceptual departure from classical physics. These two features have been given [21] the confusingly similar labels[a] of *indeterminacy* and *indeterminism*.

Indeterminacy is the more startling and harder to grasp of these two new concepts, yet it has the clearer mathematical formulation in the theory. Physically, it is the idea that physical quantities need not have definite values. Mathematically, this is expressed by the representation of such quantities by hermitian operators instead of real numbers, and the mathemtical fact that not every vector is an eigenvector of such an operator. This yields the physical notion of *superposition*, well known as a stumbling block in understanding the theory.

Indeterminism, on the other hand, is simply the opposite of determinism, as canonically expressed by Laplace [12]. It is the denial of the idea

[a]The confusing similarity of these words reflects the confusion of the two concepts in the conceptual fog from which the theory was slowly clarifying. It is still tempting to think of the two ideas as related; but it is well to distinguish them.

that the future of the universe is rigidly determined by its present state. It can be seen as not so much a rejection, more a reluctant abandonment: if Laplace's determinism is seen, not as a statement of faith, but as a declaration of intent — "We will look for an exact cause in the past of everything that happens in the physical world" — then quantum theory is an admission of failure: "We will not try to find the reason why a radioactive nucleus decays at one time rather than another; we do not believe that there is any physical reason for its decay at a particular time."

Indeterminism, unlike indeterminacy, does not present any conceptual difficulty. In the history of human thought, the determinism of classical physics is a fairly recent and, to many, a rather troubling idea. To the pre-scientific mind, whether in historical times or in our own childhood, the apparent fact of experience is surely that there are things which might happen but they might not: you can never be sure of the future. As it is put in the Bible, "time and chance happeneth to them all".

On simply being told that the new theory of physics is to be indeterministic, one might think one could see the general form that theory would take: given a mathematical description of the state of a physical system at time t_0, there would be a mathematical procedure which, instead of yielding a single state at each subsequent time $t > t_0$, would yield a set of possible states and a probability distribution over this set. This would give a precise expression of the intuition that some things are more likely to happen than others; the probabilities would be objective facts about the world.

But this is not how indeterminism is manifested in quantum mechanics. As it was conceived by Bohr and formalised by Dirac [6] and von Neumann [22] and most textbooks ever since (the "Copenhagen interpretation"), the indeterminism does not enter into the way the world develops by itself, but only in the action of human beings doing experiments. In the basic laws governing physical evolution, probabilities occur only in the projection postulate, which refers to the results of experiments. The idea that experiments should play a basic part in the theory has been trenchantly criticised by John Bell [4]. This unsatisfactory, poorly defined and, frankly, ugly aspect of the theory consititutes the *measurement problem* of quantum mechanics.

It could be argued that the indeterminism of quantum theory is played down in the Copenhagen interpretation, in the sense that the role of time in physical probability is minimised. This is not a formal feature of the theory, but an aspect of its presentation. Experiments are presented as if they took no time to be completed; probabilities are properties of the instantaneous state. Heisenberg referred to the contents of the state vector as representing potentiality, and this aspect of the state vector has been emphasised by de Ronde [5], but it seems to me that the idea of something being potential looks to the future for its actualisation.

The theory remained in this state (and some say it still does) until the papers of Everett and Wheeler in 1957. Everett, as endorsed by Wheeler, argued that there was no need for the projection postulate and no need for experiments to play a special role in the theory. But this did not give a satisfactory place for indeterminism in the theory: on the contrary, quantum mechanics according to Everett is essentially deterministic and probability is generally regarded as a *problem* for the Everett-Wheeler interpretation.

There are at least four different concepts that go under the name of "probability" (or, if you think that it is a single concept, four theories of probability) [10]. Probability can be a strength of logical entailment, a frequency, a degree of belief, or a propensity. Physicists tend to favour what they see as a no-nonsense definition of probability in terms of frequencies, but in the context of physics this is ultimately incoherent. Classical deterministic physics has a place for probability in situations of incomplete knowledge; this is how Laplace considers probability. The concept being used here is degree of belief. Some quantum physicists [8] would like to retain this concept in the indeterministic realm. Many, however, would reject this subjectivist approach as abandoning the ideal of an objective theory. The most appropriate concept for an indeterministic theory, as outlined above, would seem to be that of propensity; if there is a set of possible future states for a system, the system has various propensities to fall into these states. This gives an objective meaning of "probability". This concept is also appropriate to the Copenhagen interpretation of quantum mechanics: an experimental setup has propensities to give the different possible results of the experiment. The Everett-Wheeler picture, however, seems to have no place for propensities.

Everett liked to describe the universal state vector, which occupies centre stage in his theory, as giving a picture of *many worlds*; Wheeler, I think rightly, rejected this language. Both views, however, attribute simultaneous existence to the different components of the universal state vector which are the seat of probabilities in the Copenhagen interpretation. It is hard to see how propensities can attach to them. Everett, consequently, pursued the idea of probabilities as frequencies, and even claimed to derive the quantum-mechanical formula for probability (the Born rule) rather than simply postulating it. It is generally agreed, however, that his argument was circular (see Wallace in [23, p. 127]).

In this article I will explore the Everett-Wheeler formulation of quantum mechanics from the point of view of a sentient physical system inside the world, such as each of us. I argue that it is necessary to recognise the validity of two contexts for physical propositions: an external context, in which the universal state vector provides the truth about the world and its development is deterministic; and an internal context in which the world is seen from a component of the universal state vector. In the latter context the development in time is indeterministic; probabilities apply, in a particular component at a particular time, to statements referring to the future. This leads to a fifth concept of probability: the probability of a statement in the future tense is the *degree of truth* of that statement. I conclude with a formulation of the predictions of quantum mechanics from the internal perspective which fits the template for an indeterministic theory described earlier in this section: for each state at time t_0, the universal state vector provides a set of possible states at subsequent times $t > t_0$ and a probabiilty distribution over this set.

I start by arguing for the relative-state interpretation of Everett and Wheeler as the best approach to the problem of understanding quantum theory.

2. The relative-state understanding of quantum mechanics

How can we approach the problem of understanding quantum mechanics?

First, some principles: what understanding a scientific theory means to me. I want to take our best science seriously; whatever a successful theory supposes, I am disposed to believe that that is the truth about the world. But secondly, I want to take my own experience seriously. Too often, expositors of science tell us that what we experience is an illusion. Of course, illusions do exist: when I see a stick bending as it is put into water at less than a right angle to the surface of the water, that is an illusion, and it can be demonstrated to be an illusion by immediate appeal to other experiences.[b] But there are other experiences that are too basic to be illusory. Such fundamental aspects of our experience as time, and consciousness, and free will, and the solidity of solids [18], cannot be illusions. It may be that

[b]Such illusions are less common than is often thought. We are told that Copernicus discovered that it was an illusion that the sun goes round the earth. Not so: it is *true* that the sun goes round the earth. Einstein taught us that any such statement is relative to a frame of reference, and in my rest frame the sun does indeed go round me once a day. This reflection is relevant to the understanding of quantum mechanics suggested here, with "frame of reference" replaced by "perspective".

we have a false theory of these phenomena — for example, that free will consists of an interruption to the laws of nature, or that the solid state consists of a mathematical continuum — it may, indeed, be difficult to define them precisely — but that does not mean that the phenomena themselves are not real.

Now here's the problem. It seems to be impossible to apply both of these principles in understanding quantum mechanics. One of the basic facts of our experience which I cannot disbelieve is that (properly conducted) experiments have well-defined, unique results. But there are no such unique results of experiment in pure quantum theory.

What do I mean by "pure quantum theory"? The mathematical machinery is not in question: Hilbert spaces to describe the possible states of systems, tensor products to combine them, Hamiltonians to describe interactions, the Schrödinger equation to govern evolution. That is clear-cut. But if the theory is to include our unique experience, it seems that further, more vague, elements have to be added to this clear theory. What is traditionally taught as quantum mechanics therefore includes, in addition to the Schrödinger equation, the "collapse postulate": a stochastic evolution which delivers a state vector incorporating the uniqueness that we experience. For some reason, this only happens after a "measurement", though nobody has ever made precise what exactly a measurement is; so it does not describe our experience when we are not making measurements [19]. I therefore follow Everett and Wheeler in not including this part of the text-book account in what I mean by "quantum theory".

The preceding criticisms do not apply to theories of spontaneous collapse like the GRW theory [9], which was designed to overcome these objections, but such theories go beyond orthodox quantum theory and, in principle, are empirically distinguishable from it. This response to the challenge of understanding quantum theory essentially consists of saying "It's hopeless — quantum mechanics can't be understood; we need to replace it with a different theory". That may be correct, but in this article I want to persevere in the attempt to understand the pure theory of quantum mechanics.

It then seems that my two basic principles are incompatible. If the science of quantum theory is true, then experiments have no unique results and our experience of such uniquenesss cannot be trusted. We cannot take quantum mechanics seriously and at the same time take our basic experience seriously.

This dilemma has the same form as some perennial philosophical problems. For example,

(1) The existence of space-time *vs* the passage of time;
(2) Determinism *vs* free will (or, indeed, *in*determinism *vs* free will);
(3) The physical description of brain states *vs* conscious experience;
(4) Duty *vs* self-interest

This general class of philosophical problems has been discussed by Thomas Nagel in his book *The View from Nowhere* [15]. Each of the above examples is, or seems to be, a contradiction between two statements or principles, both of which we seem to have good reason to believe. In each case one of the statements is a general universal statement — what Nagel calls "a view from nowhere" — to which assent seems to be compelled by scientific investigation or moral reflection; the other is a matter of immediate experience, seen from inside the universe (a view from "now here").

Nagel suggests that we can resolve these contradictions by recognising that the apparently opposed statements can both be true, but in different contexts (or from different perspectives). Statements in the view from nowhere do not contradict statements in the view from now here, they just do not engage with them; the two types of statement are *incommensurable*. But, provided we are careful to specify the context of each statement, we will see that in each pair, the apparently opposed statements are *compatible*.

In order to see how these ideas apply to quantum mechanics, let us look at the famous example of Schrödinger's cat.

Schrödinger's sad story [17] is often presented as a challenge to quantum mechanics. When the unfortunate cat has been in Schrödinger's diabolical device for a time t, the crude argument (not Schrödinger's!) goes, its state is

$$|\psi_{\text{cat}}(t)\rangle = \mathrm{e}^{-\gamma t}|\text{alive}\rangle + \sqrt{1 - \mathrm{e}^{-2\gamma t}}|\text{dead}\rangle$$

So why don't we see such superpositions of live and dead cats?

The answer is simple. If we are watching the cat, hoping to see a superposition like the above, the interaction by which we see it actually produces the entangled state

$$|\Psi(t)\rangle = \mathrm{e}^{-\gamma t}|\text{alive}\rangle_{\text{cat}}|\ddot{\smile}\rangle_{\text{observer}} + \sqrt{1 - \mathrm{e}^{-2\gamma t}}|\text{dead}\rangle_{\text{cat}}|\ddot{\frown}\rangle_{\text{observer}}$$

in which $|\ddot{\smile}\rangle$ is the observer state of seeing a live cat and $|\ddot{\frown}\rangle$ is the state of seeing a dead cat. Nowhere in this total state is there an observer seeing a superposition of a live and a dead cat.

But that doesn't tell us what there actually *is* in the state $|\Psi(t)\rangle$. In order to understand the meaning of this superposition, let us look at it more carefully.

If the observer is watching the cat continuously over the period from time 0 to time t, they will be able to note the time, if any, at which they see the cat die. Then the joint state of the cat and the observer is something like

$$|\Psi(t)\rangle = \mathrm{e}^{-\gamma t}|\text{alive}\rangle_{\text{cat}}|\text{“The cat is alive”}\rangle_{\text{observer}} \\ + \sqrt{2\gamma}\int_0^{t'} \mathrm{e}^{-\gamma t'}|\text{dead}\rangle_{\text{cat}}|\text{“I saw the cat die at time } t'\text{”}\rangle_{\text{observer}} dt' \quad (1)$$

in which the observer states contain propositions which are physically encoded in the brain of the observer. But what is their status as propositions; are they true or false? Each is believed by a brain which has observed the fact it describes, and that fact belongs to reality. As a human belief, each statement could not be more true. Yet they cannot all be true, for they contradict each other.

This conflict shows the necessity of considering the perspective from which a statement is made when discussing its truth value. When this is done, it becomes possible for contradictory statements to be simultaneously true, each in its own context.

In general, the state of the universe can be expanded in terms of the states of any observer inside the universe as

$$|\Psi(t)\rangle = \sum_n |\eta_n\rangle|\Phi_n(t)\rangle$$

where the $|\eta_n\rangle$ form an orthonormal basis of observer states, which we can take to be eigenstates of definite experience; the $|\Phi_n(t)\rangle$ are the corresponding states of the rest of the universe at time t. The latter are not normalised; indeed, most of them will be zero. It is only possible for the observer to experience being in one of the states $|\eta_n\rangle$, and in this state it is true for the observer that the only experience they have is η_n; the observer is justified, at time t, in deducing that the rest of the universe is in the unique state $|\Phi_n(t)\rangle$. This is the *internal* truth relative to the experience state $|\eta_n\rangle$.

But there is also the *external* truth that the state of the whole universe is $|\Psi(t)\rangle$. From this standpoint all the experiences η_n truly occur. In Everett's terms, $|\Phi_n(t)\rangle$ is the *relative* state of the rest of the universe relative to the observer's state $|\eta_n\rangle$.

Thus there are the following two types of truth involved.

External truth: The truth about the universe is given by a state vector $|\Psi(t)\rangle$ in a Hilbert space $\mathcal{H}_U$, evolving according to the Schrödinger equation. If the Hilbert space can be factorised as

$$\mathcal{H}_U = \mathcal{H}_S \otimes \mathcal{H}_E$$

where $\mathcal{H}_S$ contains states of an experiencing observer, then

$$|\Psi(t)\rangle = \sum_n |\eta_n\rangle|\Phi_n(t)\rangle$$

and all the states $|\eta_n\rangle$ for which $|\Phi_n(t)\rangle \neq 0$ describe experiences which actually occur at time t.

Internal truth from the perspective $|\eta_n\rangle$: I actually have experience η_n, which tells me that the rest of the universe is in the state $|\Phi_n(t)\rangle$. This is an objective fact; everybody I have talked to agrees with me.

This distinction between internal and external truth is an example of Nagel's distinction between the view from nowhere and the view from "now here". Scientists might be tempted to exalt the external statement as the objective truth, downgrading internal statements as merely subjective. Indeed, Nagel himself uses the terminology of "objective" and "subjective" [14]. But he does not use a dismissive qualifier like "merely" to denigrate the subjective: he is at pains to emphasise that the truth of an internal statement has a vividness and immediacy, resulting from the fact that it is actually experienced, compared to which external truth is "bleached-out". This applies most obviously in contexts like ethics and aesthetics, but we would do well to remember it in our scientific context; as I have pointed out above, it is the internal statement which has the scientific justification of being supported by evidence, and is objective in the usual sense that it is empirical and is agreed by all observers who can communicate with each other.

But the situation is more complicated than this might suggest. It is not that there is a God-like being who can survey the whole universe and make statements about the universal state vector, distinct from us physical beings who are trapped in one component of $|\Psi\rangle$. We physical beings are the ones who make statements about $|\Psi\rangle$, for good theoretical reasons, from our situation in which we experience just the one component $|\eta_n\rangle|\Phi_n\rangle$. From that perspective, what are we to make of the other components $|\eta_m\rangle|\Phi_m\rangle$?

Consider a measurement process, in which an initial state $|\Psi(0)\rangle = |\eta_0\rangle|\Phi_0(0)\rangle$, containing only one experience η_0, develops in time t to an entangled state $|\Psi(t)\rangle = \sum |\eta_n\rangle|\Phi_n(t)\rangle$. The external statement is:

> $|\Psi(t)\rangle$ represents a *true* statement about the universe, and all its components are *real*.

The observer who experiences only $|\eta_n\rangle$ must say:

> I know that only $|\eta_n\rangle$ is **real** (because I experience only that), and therefore $|\Phi_n(t)\rangle$ represents a **true** statement about the rest of the universe. But I also know that $|\Psi(t)\rangle$ is *true* (because I've calculated it). The other $|\eta_m\rangle|\Phi_m\rangle$ represent things that **might have happened** but **didn't**.

These statements are font-coded, using bold type for internal (vivid, experienced) judgements, and italic for external (pale, theoretical) ones, even though these are made by an internal observer.

It is a constant temptation in physics, on finding a quantum system in a superposition $|\phi\rangle + |\psi\rangle$, to think that it is either in the state $|\phi\rangle$ or in the state $|\psi\rangle$. This, after all, is the upshot when we look at the result of an experiment. Despite the stern warnings of our lecturers when we are learning the subject, and the proof from the two-slit experiment that "+" cannot mean "or" [21, pp. 188-189], we all probably slip into this way of thinking at times; and the common-sense view of Schrödinger's cat seems to justify it. On the other hand, the many-worlds view insists that both terms in the superposition are real, and therefore "+" means "and". What I am suggesting here is that both "and" and "or" are valid interpretations of "+" in different contexts: "and" in the external view, "or" in the internal view.

How Many Worlds? The foregoing needs some refinement. Quantum superposition is not just a single binary operation "+" on states, but is modified by coefficients: $a|\phi\rangle + b|\psi\rangle$ is a weighted superposition of the (normalised) states $|\phi\rangle$ and $|\psi\rangle$. In interpreting "+" as "or", it is easy to incorporate this weighting of the disjuncts by interpreting it in terms of probability. But if we interpret "+" as "and", as in the many-worlds interpretation, what can it mean to weight the conjuncts? Let us look back to the paradigmatic geometrical meaning of vector addition. "Going north-east" is the vector sum of "going north" and "going east", and does indeed mean going north *and* going east at the same time. But "going NNE" also means going north and going east at the same time; only there is more going north than going east. So if a superposition $a|\phi\rangle + b|\psi\rangle$ of macroscopic states $|\phi\rangle$ and $|\psi\rangle$ means that both $|\phi\rangle$ and $\psi\rangle$ are real in different worlds, we must accept that they are not "equally real", as is often carelessly stated by Everettians (including Everett himself [7] — in a footnote — but not Wheeler [24]), but that they are real to different extents $|a|^2$ and $|b|^2$. Adding up these degrees of reality, we then find that there are not many worlds but $(|a|^2 + |b|^2 =)$ one.

Another argument for this conclusion uses the physical observable of particle number. The many-worlds view regards the state vector $|\Psi(t)\rangle = \sum_n |\eta_n\rangle|\Phi_n(t)\rangle$ as describing many (say N) worlds, with N different copies of the observer having the different experiences $|\eta_n\rangle$. Suppose the observer's name is Alice. There is an observable called Alice number, of which each of the states $|\eta_n\rangle|\Phi_n(t)\rangle$ is an eigenstate with eigenvalue 1. Then $|\Psi(t)\rangle$ is also an eigenstate of Alice number with eigenvalue 1 (not N). There is only one Alice.

3. Probability and the future

3.1. *Probability*

The observer in this measurement process might go on to say:

> I **saw** a transition from $|\eta_1\rangle$ to $|\eta_n\rangle$ at some time $t' < t$. But I *know* that $|\Psi(t')\rangle$ didn't collapse. The other $|\eta_m\rangle|\Phi_m\rangle$ **might** come back and interfere with me in the future; but this has very low **probability**.

This shows how the conflicting statements about time development in the Everett-Wheeler and Copenhagen interpretations can after all be compatible. The continuous Schrödinger-equation evolution postulated by the Everett-Wheeler interpretation refers to the external view; the collapses postulated by the Copenhagen interpretation refer to the internal view, i.e. to what we actually see. The occurrence of collapse, in the form of perceived quantum jumps in the memory of an observer, becomes a theorem rather than a postulate (see [19] for some indications, but work remains to be done on this). We also see that it is only the internal statement that mentions probability. But what does it mean?

The meaning of probability is a long-standing philosophical problem (see, for example, [10]). There are in fact several distinct concepts which go by the name of probability, sharing only the fact that they obey the same mathematical axioms. The clearest of these, perhaps, is degree of belief, which has the advantage that it can be defined operationally: someone's degree of belief in a proposition is equal to the odds that they are prepared to offer in a bet that the proposition is true. The subjective nature of this concept seems to chime with the fact that it belongs in internal statements, as we have just seen, and indeed similar views of probability are often adopted by Everettians even though their general stance is objectivist.

However, we have also seen that "internal" should not be equated with "subjective", and our experience in a quantum-mechanical world seems to require a description in terms of objective chance. Things happen randomly, but with definite probabilities that cannot be reduced to our beliefs. The value of the half-life of uranium 238 is a fact about the world, not a mere consequence of someone's belief.

Such objective probability (or "chances") can only refer to future events.

3.2. *The future*

What kinds of statements can be made at time t_0 about some future time $t > t_0$, if the universal state vector is known to be $|\Psi(t_0)\rangle$ and its decomposition with respect to experience states of a particular observer is $\sum_n |\eta_n\rangle|\Phi_n(t_0)\rangle$? From the external perspective, the future state $|\Psi(t)\rangle$ is determined by the Schrödinger equation and there is no question of any probability. From the internal perspective relative to an experience state $|\eta_n\rangle$, there is a range of possible future states $|\eta_m\rangle$, and probabilities must enter into the statement of what the future state will be. But here is a fundamental problem: there is *no such thing* as what the future state will be. As Bell pointed out [3], quantum mechanics gives no connection between a component of $|\Psi\rangle$ at one time and any component at another time; so what is it that we can assign probabilities to? How can "the probability that my state will be $|\eta_m\rangle$ tomorrow" mean anything when "my state will be $|\eta_m\rangle$ tomorrow" has no meaning?

This puzzle takes us back to ways of thinking that are much older than quantum mechanics, indeed older than all of modern science. The success of Newtonian deterministic physics has led us to assume that there always is a definite future, and even when we drop determinism we tend to continue in the same assumption. There is a future, even if we do not and cannot know what it will be. But this was not what Aristotle believed, and maybe it is not what we believed when we were children.

Aristotle, in a famous passage [1], considered the proposition "There will be a sea-battle tomorrow". He argued that this proposition is neither true nor false (otherwise we are forced into fatalism). Thus he rejected the law of excluded middle for future-tense statements, implying that they obey a many-valued logic. Modern logicians [16] have considered the possibility of a third truth-value in addition to "true" or "false", namely u for "undetermined", for future-tense statements. But, interestingly, Aristotle admitted that the sea-battle might be more or less likely to take place. This suggests that the additional truth values needed for future-tense statements are not limited to one, u, but can be any real number between 0 and 1 and should be identified with the probability that the statement will come true.[c] Turning this round gives us an objective form of probability which applies to future events, or to propositions in the future tense; in a slogan,

$$\text{Probability} = \text{degree of future truth.}$$

A form of temporal logic incorporating this idea is developed in [20]. It contains a lattice of propositions in the context of a particular observer and a particular time t_0 ("now"). The lattice is generated by propositions corresponding to the observer's experience eigenstates, labelled by time t. The observer is assumed to have a memory of experiences occurring before t_0: thus propositions in the present tense ($t = t_0$) and in the past tense ($t < t_0$) have truth values 0 or 1, as in classical logic, but those in the future tense ($t > t_0$) can take any truth value in the real interval $[0, 1]$. General propositions are formed from these dated experience propositions by means of conjunction, disjunction and negation ("and", "or" and "not"). A conjunction of experience propositions is a history in the sense of the consistent-histories formulation of quantum mechanics; then a conjunction of histories is another history, but the general proposition is a disjunction of histories — something which is not usually considered in this formulation. It is shown in [20] that, given a weak form of the assumption that the histories are consistent, the usual formula for the probability of a history can be extended to truth values for all propositions, with logical properties that are to be expected from identifying truth values with probabilities.

The Open Future. We find it hard, in a scientific theory, to accommodate the idea that there is no definite future. To be sure, we have indeterministic theories in which the future is not uniquely determined by the past, but such stochastic theories deal with complete histories encompassing past, present and future; probabilities refer to which of these histories is actual. Indeterminism, in the usual stochastic formulation, consists of the fact that there are many such histories containing a given past up to a certain time, so the future extension is not unique; but the underlying assumption is that only one of these future histories is real, so that the future is fixed even though it is not determined. In contrast, the formulation of quantum mechanics outlined here — or what Bell [3] called the "Everett (?) theory" — is, I think, the only form of scientific theory in which the future is genuinely open. Unlike Bell, I do not regard this as a problem for

[c]This idea motivated Łukasiewicz [13] in formulating modern many-valued logic, and has been applied to quantum mechanics by Pykacz [2].

the theory; it tells a truth which we should be happy to acknowledge. The function of the theory is to provide a catalogue of possibilities and specify how these change (deterministically) with time; it does not and cannot say which of the possibilities is actualised at any time. The "measurement problem" of quantum theory is no more than the difficulty of accepting this format for a scientific theory; with a change of gestalt, we can see it as a natural way to formulate indeterminism.

However, I must emphasise the roles that entanglement and the concept of internal truth play in this resolution of the measurement problem. Without these, there would be a "preferred basis" problem: if the universal wave function is a catalogue of possibilities, what basis defines the components which are to be regarded as possibilities? But there is no preferred-basis problem in the relative-state interpretation, as understood here. The possibilities are given by experience states, which only exist if the universal Hilbert state has a tensor product structure in which one of the factors describes a system capable of experience, i.e. which has a basis of states exhibiting the structure of propositions describing experience. It is not required that this structure should be unique; in principle, it is possible that the universal Hilbert space has more than one tensor product structure with the required properties. If this should be so, statements about these different experiences would also be (internally) true, relative to these different structures; this would not detract from the truth of the original experience propositions. Both kinds of internal proposition would be compatible with the external truth of the same universal state vector.

This potential ambiguity is not peculiar to quantum mechanics: a theory of conscious observers in classical physics would also admit the logical possibility that a single physical structure could admit two different interpretations in terms of conscious beings. It is hard to imagine that this could actually occur with systems complex enough to record experiences; nevertheless, it has been shown that in simpler systems such ambiguous factorisation can arise in quantum mechanics, and that decoherence can be exhibited in both factorisations [11]. This does not show that an understanding based on such factorisation, like that outlined here, is untenable.

4. Summary

Here is how I understand nonrelativistic quantum mechanics.

1. From the external perspective there is, at each time t, a true description of the physical world given by a state vector $|\Psi(t)\rangle$ in a Hilbert space $\mathcal{H}_U$.

2. The sequence of states $|\Psi(t)\rangle$ satisfies the Schrödinger equation with a universal Hamiltonian H_U.

3. In general, a description of the physical world from the external perspective is given by a closed subspace of the Hilbert space $\mathcal{H}_U$. Such a description has, at time t, the degree of truth $\langle\Psi(t)|\Pi|\Psi(t)\rangle$ where Π is the orthogonal projection onto the relevant closed subspace.

4. Suppose the universe has a subsystem S which has sufficient structure to experience and record propositions about the physical world. Then $\mathcal{H}_U = \mathcal{H}_S \otimes \mathcal{H}'$ where $\mathcal{H}_S$ is the state space of the experiencing subsystem S. Let $\{|\eta_n\rangle\}$ be an orthonormal basis of $\mathcal{H}_S$ which includes all possible experiences of S. Then the true state of the universe (from the external perspective) can be expanded as

$$|\Psi(t)\rangle = \sum_n |\eta_n\rangle|\Phi_n(t)\rangle$$

with $|\Phi_n(t)\rangle \in \mathcal{H}'$. From the external perspective, an experience $|\eta_n\rangle$ is real at time t if $|\Phi_n(t)\rangle \neq 0$, and the component $|\eta_n\rangle|\Phi_n(t)\rangle$ describes a world which has a degree of reality $\langle\Phi(t)|\Phi(t)\rangle$ at time t.

5. From the internal perspective of the experiencing system S in the state $|\eta_N\rangle$ at time t_0, there is, for each time $t \leq t_0$, one true experience (recorded in memory) described by a basis vector $|\eta_{n(t)}\rangle$ with $n(t_0) = N$.

6. From the perspective of the experiencing system S in state $|\eta_N\rangle$ at time t_0, the statement that its experience state at a future time $t > t_0$ will be $|\eta_m\rangle$ has truth value

$$\frac{\left|\langle\Psi(t)|\,(\Pi_m \otimes I)\,\mathrm{e}^{-iH(t-t_0)/\hbar}\,(\Pi_N \otimes I)\,|\Psi(t_0)\rangle\right|^2}{\langle\Psi(t)|\,(\Pi_m \otimes I)\,|\Psi(t)\rangle\langle\Psi(t_0)|\,(\Pi_N \otimes I)\,|\Psi(t_0)\rangle}$$

where Π_m is the orthogonal projector onto $|\eta_m\rangle$ in $\mathcal{H}_S$. The experiencing subject S refers to this as the probability that they will experience η_m at time t.

7. From the perspective of the experiencing system S in state $|\eta_N\rangle$ at time t_0, the significance of the universal state vector $|\Psi(t)\rangle$ is as follows.

For $t \leq t_0$, $|\Psi(t)\rangle$ describes what might have happened but didn't, as well as what actually did happen.

For $t > t_0$, $|\Psi(t)\rangle$ describes what might be going to happen at time t.

Thus, from the internal perspective, the universal state vector $|\Psi(t)\rangle$ is not a description of reality but an influence governing changes in reality. Things that didn't actually happen still, in principle, have an effect on what is going to happen.

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A TOPOS THEORETIC FRAMEWORK FOR PARACONSISTENT QUANTUM THEORY

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In this paper, we show how a particular form of paraconsistent logic arises naturally from a topos-theoretic formulation of quantum theory. We argue that paraconsistent approaches have the potential to shed new light on some conceptual problems in the interpretation of quantum theory, and suggest some potential directions for related mathematical research.

1. Introduction

Over the last century, a vast variety of different forms of non-classical logic have been brought to bear on the conceptual problems of quantum theory. Apart from the canonical orthomodular quantum logic of Birkhoff and Von Neumann [1], approaches from fuzzy logic [2], supervaluationism [3], linear logic [4] and topos theoretic intuitionistic logic [5], to name but a few, have all been used to provide new perspectives on the quantum world. However, the idea of applying paraconsistent logic to quantum mechanics (henceforth QM) has, thus far, only be tentatively courted (the most comprehensive treatment so far seems to be in [6]. In this paper, we attempt to contribute to this exciting new project by showing how the topos theoretic formulation of QM provides a natural semantics for a popular and well known form of paraconsistent logic in the same way that the orthodox Hilbert space formalism provides a natural semantics for non-distributive quantum logic.

In section 2, we will provide a quick overview of the relevant aspects of the topos theoretic approach to quantum logic. We will then go on to show how paraconsistency naturally arises out of this formalism. Section 3 concludes by making some general remarks concerning the application of paraconsistent logic to QM and sketching some arguments concerning the advantages of the approach developed in section 2.

2. Paraconsistent Topos Quantum Logic

The primary formal setting for this paper will be the topos theoretic formulation of QM (henceforth TQT), as developed by Isham, Butterfield, Döring and others (e.g [7–9]). Of course, this is an extremely rich subject, and we can only give the most superficial of introductions here. Specifically, we will focus exclusively on one aspect of the logical structure of the formalism.

The starting point for TQT's approach to the logic of QM is the infamous Kochen Specker theorem (KST). Specifically, TQT is concerned with overcoming the following immediate corollary of KST,

VAL: There is no non-contextual valuation assignment on the lattice of projections on a Hilbert space with dimension greater than 2 which takes values in a Boolean algebra and respects the functional relationships between the projections.

In one sense, **VAL** can be seen as a consequence of the non-Boolean (or, more pertinently, non-distributive) structure of projections lattices. Thus, we read:

> while a classical property state is selected by a two-valued homomorphism on the Boolean lattice of classical properties, a quantum property state ... is not selected by a two-valued homomorphism on the non-Boolean lattice of subspaces representing the properties of a quantum mechanical system. There are no such two-valued homomorphisms. [10]

Given the usual interpretational assumption that we should represent the physical propositions that can be made about a quantum system Q with associated Hilbert space H by elements of the orthomodular lattice $P(H)$ of projections onto H, **VAL** immediately licenses the conclusion that is is impossible to simultaneously assign classical truth values to all of the physical propositions associated with Q. There will always be some physical propositions about Q that have no determinate classical truth value. So Q, as a quantum system, is somehow *underdetermined* in a fundamental way.

Philosophically, we can take this as the starting point of TQT, which attempts to strip **VAL** of its conceptually problematic implications by rejecting many of the basic interpretational assumptions of the Hilbert space formalism of QM. In particular, TQT rejects the formalisation of physical propositions as projection operators, and provides a new formal representation of physical propositions.

2.1. *The Spectral Presheaf*

As we noted above, **VAL** guarantees that it is generally impossible to assign classical truth values to all of the projection operators onto a fixed Hilbert space H in a non-contextual way that preserves functional relationships. However, if we take any given Boolean subalgebra B of the lattice $P(H)$ of projections onto H, it will generally be possible to assign classical truth values to all of the projections in B in a straightforward and unproblematic way. From an algebraic perspective, this is a consequence of the fact that there always exist Boolean algebra homomorphisms from any Boolean algebra B into the two element Boolean algebra $\mathbf{2} = \{0, 1\}$. We call the set of all such homomorphisms the 'Stone space of B', denoted Ω_B, i.e. $\Omega_B = \{\lambda : B \to \mathbf{2} | \lambda \text{ is a Boolean algebra homomorphism}\}$. Now, we can turn Ω_B into a topological space if we take the basis sets to be all those sets of the form $U_b = \{\lambda \in \Omega_B | \lambda(b) = 1\}$ (where $b \in B$). This topology turns Ω_B into a compact totally disconnected Hausdorff space. By Stone's theorem, we know that any Boolean algebra B is isomorphic to the set $cl(\Omega_B)$ of all clopen subsets of Ω_B. So we can represent B by the clopen subsets of its Stone space. Now, one of the main ideas behind TQT is to find a way to 'paste together' the Stone spaces of the individual Boolean subalgebras of $P(H)$ into a single unified space that acts like a Stone space for the whole projection lattice. Of course, **VAL** tells us that we can't just take the set of homomorphisms from $P(H)$ into $\mathbf{2}$, since there are no such homomorphisms. This is where topos theory becomes an important tool.

Given an arbitrary complete orthomodular lattice L (for our purposes, we can generally think of this as the projection lattice of some given Hilbert space), let $B(L)$ denote the set of complete Boolean subalgebras of L, partially ordered by inclusion. Then, following Döring and Cannon [11], we can define the spectral presheaf of L in the following way,

Def 2.1.1: The 'spectral presheaf', $\underline{\Omega}$ of a complete orthomodular lattice L is the presheaf over $B(L)$ defined by

Objects: Given $B \in B(L)$, $\underline{\Omega}_B = \Omega_B$

Arrows: Given $B' \subseteq B$, $\underline{\Omega}_{B,B'} : \Omega_B \to \Omega_{B'}$

$$\lambda \mapsto \lambda|_{B'}$$

So the component of $\underline{\Omega}$ at an element $B \in B(L)$ is just the Stone space of B, and the component of $\underline{\Omega}$ at an inclusion morphism $\lambda : B' \to B$ is just the restriction morphism that takes $\lambda \in B$ and returns its restriction to B', $\lambda|_{B'}$. It was shown by Döring and Cannon [11] that two orthomodular lattices are isomorphic (as orthomodular lattices) if and only if their spectral presheaves, as defined above, are isomorphic as presheaves. This tells us that the spectral presheaf $\underline{\Omega}$ is a complete invariant of the orthomodular lattice L.

Intuitively, the idea is that $\underline{\Omega}$ acts like a generalised Stone space for the corresponding orthomodular lattice L (which we know is unique). This interpretation is justified by the fact that $\underline{\Omega}$ is 'composed' of the individual Stone spaces of L's Boolean subalgebras. Intuitively, we would like to consider the 'elements' of $\underline{\Omega}$, since, given the interpretation of $\underline{\Omega}$ as the Stone space of L, its elements should act as truth value assignments on the elements of L. But, in the case where $L = P(H)$ for some H with $dim(h) > 2$, we know that such valuation functions don't exist. Indeed, it was actually shown [12] that the non-existence of global elements of $\underline{\Omega}$ in the case where $L = P(H)$, $dim(H) > 2$, is equivalent to KST. So the interpretation of $\underline{\Omega}$ as the generalised Stone space for $P(H)$ is justified, since its global elements, if they existed, would correspond to non-contextual truth value assignments on $P(H)$.

Now, in light of this interpretation, together with Stone's theorem, it makes sense to consider the 'clopen subsets' of $\underline{\Omega}$. However, since $\underline{\Omega}$ is a presheaf, not a set, it does not have subsets. However, it does have *subobjects* (sub prehseaves). The following map (one of the central formal constructions of TQT) allows us to generate an interesting class of such subobjects.

Def 2.1.2: Given $a \in L$, and $B \in B(L)$, we define the daseinisation of a at B to be $\delta(a)_B = \bigwedge\{b \in B | b \geq a\}$. We then define the 'outer daseinisation presheaf' $\underline{\delta(a)}$ over $B(L)$ by

Objects: Given $B \in B(L)$, $\underline{\delta(a)}_B = \{\lambda \in \Omega_B | \lambda(\delta(a)) = 1\}$

Arrows: Given $B' \subseteq B$, $\underline{\delta(a)}_{B,B'} : \delta(a)_B \to \delta(a)_{B'}$

$$\lambda \mapsto \lambda|_{B'}$$

So, given $a \in L$, and $B \in B(L)$, the component of the outer daseinisation presheaf, $\underline{\delta(a)}$, of a at B is the part of B's Stone space that makes $\delta(a)_B$ true (assigns it the value 1). We can think of $\delta(a)_B$ as B's 'best approximation' to a. In the case where $L = P(H)$, it might be that a is some projection that is incompatible with some projections in B. So, in the measurement context represented by B, we won't be able to meaningfully assert the proposition a. Then $\delta(a)_B$ is the closest thing we can say to a that can be meaningfully asserted from the classical context B.

By Stone duality, $\underline{\delta(a)}_B \in cl(\Omega_B)$, i.e. $\underline{\delta(a)}_B$ is a clopen subset of Ω_B. Since this holds for all $B \in B(L)$, we call $\underline{\delta(a)}$ a 'clopen subobject' of $\underline{\Omega}$. It is a basic theorem of TQT that the lattice $Sub_{cl}(\underline{\Omega})$ of clopen subobjects of the spectral presheaf is a complete Heyting algebra (under natural component wise lattice operations).

So, 2.1.2 defines a map $\underline{\delta}$ from L into $Sub_{cl}(\underline{\Omega})$. It is well known [see 11] that this map satisfies the following properties,

Theorem 2.1.3:

(i) $\underline{\delta}$ is injective.

(ii) $\underline{\delta}$ preserves all joins, i.e. $\bigvee_{i\in I} \underline{\delta(a_i)} = \underline{\delta(\bigvee_{i\in I} a_i)}$, for any family $\{a_i | i \in I\} \subseteq L$.

(iii) $\underline{\delta(a)}$ is order preserving (monotone), i.e. $a \leq b$ in L implies $\underline{\delta(a)} \leq \underline{\delta(b)}$ in $Sub_{cl}(\underline{\Omega})$.

(iv) $\underline{\delta(0)} = \bot$, $\underline{\delta(1)} = \top$, where $0, 1$ are the minimal and maximal elements of L, respectively, and $\bot$ and $\top$ are the minimal and maximal elements of $Sub_{cl}(\underline{\Omega})$ ($\bot$ is the presheaf that takes each $B \in B(L)$ to the empty set and $\top$ is just $\underline{\Omega}$).

(v) $\underline{\delta(a \wedge b)} \leq \underline{\delta(a)} \wedge \underline{\delta(b)}$.

So we can think of $\underline{\delta}$ as an order preserving injection of L into the complete Heyting algebra of clopen subobjects of the spectral presheaf of L. Since this map is monotone and join preserving, it has a monotone meet preserving upper adjoint, defined as follows,

Def 2.1.4: Given $\underline{S} \in Sub_{cl}(\underline{\Omega})$, define $\varepsilon(\underline{S}) = \bigvee\{a \in L | \underline{\delta(a)} \leq \underline{S}\}$.

Döring and Cannon showed that ε has the following properties,

Theorem 2.1.5:

(i) ε preserves all meets.

(ii) ε is order preserving (monotone).

(iii) $\varepsilon(\underline{\delta(a)}) = a$, for any $a \in L$.

(iv) $\underline{\delta(\varepsilon(S))} \leq \underline{S}$, for any $\underline{S} \in Sub_{cl}(\underline{\Omega})$.

(v) $\varepsilon(\underline{S} \vee \underline{T}) \geq \varepsilon(\underline{S}) \vee \varepsilon(\underline{T})$.

ε can be used to define an equivalence relation on $Sub_{cl}(\underline{\Omega})$, defined by $\underline{S} \sim \underline{T}$ iff $\varepsilon(\underline{S}) = \varepsilon(\underline{T})$. We let E denote the set of all equivalence classes of $Sub_{cl}(\underline{\Omega})$ under this equivalence relation. E can be turned into a complete lattice by defining $\bigwedge_{i\in I}[\underline{S}_i] = [\bigwedge_{i\in I} \underline{S}_i]$, $[\underline{S}] \leq [\underline{T}]$ iff $[\underline{S}] \wedge [\underline{T}] = [\underline{S}]$ and $\bigvee_{i\in I}[\underline{S}_i] = \bigwedge\{[\underline{T}] | [\underline{S}_i] \leq [\underline{T}]\ \forall i \in I\}$. Döring and Cannon also showed the following important result.

Theorem 2.1.6: E and L are isomorphic as complete lattices. In particular, the maps $g : E \to L$ and $f : L \to E$ defined by $g([\underline{S}]) = \varepsilon(\underline{S})$ and $f(a) = [\underline{\delta(a)}]$ are an inverse pair of complete lattice isomorphisms.

Now, the Heyting algebraic structure of $Sub_{cl}(\underline{\Omega})$ makes the clopen subobjects a natural model for first order intuitionistic logic. This is the logical structure that is usually considered in the context of TQT. However, the ε map defined above allows for the definition of a new type of logical structure on $Sub_{cl}(\underline{\Omega})$. For the remainder of the paper, we will assume that L is the projection lattice of a fixed Hilbert space H.

Def 2.1.7: Given $\underline{S} \in Sub_{cl}(\underline{\Omega})$, define $\underline{S}^* = \underline{\delta(\varepsilon(S)^{\perp})}$, i.e. $\underline{S}^*$ is the daseinisation of the orthocomplement of $\varepsilon(\underline{S})$ ($\perp$ denotes the orthocomplement of $P(H)$. Recall that $\perp$ satisfies (a) $a \vee a^{\perp} = 1$, (b) $a \wedge a^{\perp} = 0$, (c) $a \leq b$ implies $a^{\perp} \geq b^{\perp}$, (d) $a^{\perp\perp} = a$, (e) $(a \wedge b)^{\perp} = a^{\perp} \vee b^{\perp}$, (f) $(a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp}$).

The $*$ operation can be thought of as the 'translation' of the orthocomplement operation into TQT. It represents a new form of negation on $Sub_{cl}(\underline{\Omega})$, the properties of which are outlined below.

Theorem 2.1.8: The $*$ operation has the following properties,

(i) $\underline{S} \vee \underline{S^*} = \top$,

(ii) $\underline{S}^{**} \leq \underline{S}$,

(iii) $\underline{S}^{***} = \underline{S}^*$,

(iv) $\underline{S} \wedge \underline{S}^* \geq \bot$,

(v) $(\underline{S} \wedge \underline{T})^* = \underline{S}^* \vee \underline{T}^*$,

(vi) $(\underline{S} \vee \underline{T})^* \leq \underline{S}^* \wedge \underline{T}^*$,

(vii) $\varepsilon(\underline{S}) \vee \varepsilon(\underline{S}^*) = 1$,

(viii) $\varepsilon(\underline{S}) \wedge \varepsilon(\underline{S}^*) = 0$,

(ix) $\underline{S} \leq \underline{T}$ implies $\underline{S}^* \geq \underline{T}^*$.

Proof: (i) Since $\underline{S} \geq \underline{\delta(\varepsilon(S))}$, we have

$\underline{S} \vee \underline{S}^* = \underline{S} \vee \underline{\delta(\varepsilon(S)^{\perp})} \geq \underline{\delta(\varepsilon(S))} \vee \underline{\delta(\varepsilon(S)^{\perp})} = \underline{\delta(\varepsilon(S) \vee \varepsilon(S)^{\perp})} = \underline{\delta(1)}$
$= \top$.

(ii) $S^{**} = \underline{\delta(\varepsilon(S)^{\perp})}^* = \underline{\delta(\varepsilon(\delta(\varepsilon(S)^{\perp}))^{\perp})} = \underline{\delta(\varepsilon(S)^{\perp\perp})} = \underline{\delta(\varepsilon(S))} \leq \underline{S}$.

(iii) $S^{***} = \underline{\delta(\varepsilon(S))}^* = \underline{\delta(\varepsilon(\delta(\varepsilon(S)))^{\perp})} = \underline{\delta(\varepsilon(S)^{\perp})} = \underline{S}^*$.

(iv) This is obvious, but the reason that we don't have equality can be seen below,

$\underline{S} \wedge \underline{S}^* = \underline{S} \wedge \underline{\delta(\varepsilon(S)^{\perp})} \geq \underline{\delta(\varepsilon(S))} \wedge \underline{\delta(\varepsilon(S)^{\perp})} \geq \underline{\delta(\varepsilon(S) \wedge \varepsilon(S)^{\perp})} = \underline{\delta(0)}$
$= \bot$.

(v) $(\underline{S} \wedge \underline{T})^* = (\underline{S \wedge T})^* = \underline{\delta(\varepsilon(S \wedge T)^\perp)} = \underline{\delta((\varepsilon(S) \wedge \varepsilon(T))^\perp)} = \underline{\delta(\varepsilon(S)^\perp \vee \varepsilon(T)^\perp)} = \underline{\delta(\varepsilon(S)^\perp)} \vee \underline{\delta(\varepsilon(T)^\perp)} = \underline{S}^* \vee \underline{T}^*$.

(vi) $(\underline{S} \vee \underline{T})^* = (\underline{S \vee T})^* = \underline{\delta(\varepsilon(S \vee T)^\perp)} \leq \underline{\delta((\varepsilon(S) \vee \varepsilon(T))^\perp)} = \underline{\delta(\varepsilon(S)^\perp \wedge \varepsilon(T)^\perp)} \leq \underline{\delta(\varepsilon(S)^\perp)} \wedge \underline{\delta(\varepsilon(T)^\perp)} = \underline{S}^* \wedge \underline{T}^*$.

(vii) $\varepsilon(\underline{S}) \vee \varepsilon(\underline{S}^*) = \varepsilon(\underline{S}) \vee \varepsilon(\underline{\delta(\varepsilon(S)^\perp)}) = \varepsilon(\underline{S}) \vee \varepsilon(\underline{S})^\perp = 1$.

(viii) $\varepsilon(\underline{S}) \wedge \varepsilon(\underline{S}^*) = \varepsilon(\underline{S}) \wedge \varepsilon(\underline{\delta(\varepsilon(S)^\perp)}) = \varepsilon(\underline{S}) \wedge \varepsilon(\underline{S})^\perp = 0$.

(ix) Let $\underline{S} \leq \underline{T}$. Then $\varepsilon(\underline{S}) \leq \varepsilon(\underline{T})$. So $\varepsilon(\underline{S})^\perp \geq \varepsilon(\underline{T})^\perp$. So $\underline{\delta(\varepsilon(S)^\perp)} \geq \underline{\delta(\varepsilon(T)^\perp)}$, i.e. $\underline{S}^* \geq \underline{T}^*$.

Crucially, property (iv) is only an inequality. $*$ is a *paraconsistent* negation that violates the law of 'explosion' (whereby a contradiction implies every other proposition). Indeed, this theorem guarantees that $*$ models a very particular type of paraconsistent negation.

Corollary 2.1.9: Equipped with $*$ and the implication operation $\Rightarrow$ defined by $\underline{S} \Rightarrow \underline{T} = \underline{S}^* \vee \underline{T}$, $Sub_{cl}(\underline{\Omega})$ is a model of 'dialectical logic with quantifiers' (DLQ), a well known form of paraconsistent logic [13].

So, we have seen that, by translating the orthocomplement of traditional orthomodular quantum logic into TQT, a well known form of paraconsistent logic arises in a natural and straightforward way.

3. Paraconsistency and Quantum Theory

The results of the preceding section demonstrate a natural and straightforward connection between quantum theory (specifically, TQT) and paraconsistent logic. The author is inclined to argue that this is probably the strongest and most natural connection between the two subjects that has been observed so far. In [8], another form of paraconsistent logic was derived from the algebraic structure of $Sub_{cl}(\underline{\Omega})$, but it only arose as a dual form of the already well known intuitionistic structure. DLQ is a well known and important form of paraconsistent logic that has a number of important applications. For example, DLQ appears to have an important role to play as the underlying logic of paraconsistent set theory (see [14] and [15] for details). Indeed, in [15], $*$ is used to make initial steps in the project of unifying TQT with quantum set theory, as developed in [16,17].

In [6], some general arguments were made in support of the idea that paraconsistent logic could have an important role to play in aiding our understanding of quantum superpositions. Although this is not the setting for a detailed analysis of this argument, there is a straightforward sense in which the conclusion can be seen as very intuitive. If we consider a spin 1/2 particle in the superposition $1/\sqrt{2}\,|\uparrow\rangle + 1\sqrt{2}\,|\downarrow\rangle$ (for some fixed direction), it seems very natural to say that the proposition 'the spin of the particle is up and it is not the case that the spin of the particle is up' is at least partially true. Paraconsistent logic allows us to respect this intuition without trivialising our whole logical system. It should be noted that this is directly analogous to some of the arguments that have been made to justify the adoption of intuitionistic logic for quantum theory. In that case, the argument rests on the intuition that the proposition 'the spin of the particle is up or the spin of the particle is not up' is at least partially false. It seems clear that the intuition supporting the argument for the adoption of paraconsistent logic is at least as strong as the argument for the adoption of intuitionistic logic in this case.

In [6], the authors suggest a particular form of paraconsistent logic that might be useful for thinking about quantum superpositions. Again, there is no space to discuss their proposals in any detail, but it should be noted that the present approach has the advantage that the paraconsistent logical structure arises naturally, *as part of the formalism of the theory.* The fact that this form of paraconsistent logic arises naturally from the formalism is both conceptually encouraging and technically convenient. Indeed, the paraconsistent structure of $Sub_{cl}(\underline{\Omega})$ has already been used to extend the TQT formalism [15].

Another recently discovered connection between paraconsistency and quantum theory was explored in [20], where the authors showed that paraconsistent logic can play an important role in deriving quantum probability rules in the context of Bayesian probability theory. The underlying logic used in this approach is closely related to that used in [6], so this could provide an independent justification for that approach, that strengthens its links with the operational and formal structure of quantum theory.

At any rate, the author hopes that this paper can serve as a humble invitation to the study of the application of paraconsistent logic to quantum theory. It seems clear that a number of closely related ideas, both conceptual and mathematical, are emerging in this area, and the stage is set for an exciting new wave of research.

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A POSSIBLE SOLUTION TO THE SECOND ENTANGLEMENT PARADOX

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Entangled states are in conflict with a general physical principle which expresses that a composite entity exists if and only if its components also exist, and the hypothesis that pure states represent the actuality of a physical entity, i.e., its 'existence'. A possible way to solve this paradox consists in completing the standard formulation of quantum mechanics, by adding more pure states. We show that this can be done, in a consistent way, by using the extended Bloch representation of quantum mechanics, recently introduced to provide a possible solution to the measurement problem. Hence, with the solution proposed by the extended Bloch representation of quantum mechanics, the situation of entangled states regains full intelligibility.

Keywords: Entanglement; density operators; $SU(N)$; extended Bloch representation.

Since their discovery, entangled states were considered to describe enigmatic situations. More precisely, we can distinguish two entanglement paradoxes. The first one is well-known, as it attracted most of the attention of physicists throughout the years: it is related to the experimental fact that two entangled entities are able to produce perfect correlations, even when separated by large spatial distances. In other terms, the first entanglement paradox is about the existence of *non-local* effects, allowing entities to become spatially separated, by arbitrarily large distances, without becoming experimentally separated.

The second entanglement paradox is in our opinion even more puzzling, although lesser known. Schrœdinger, the discoverer of entanglement, was perfectly aware of this difficulty, for instance when he emphasized that for two quantum entities in an entangled state only the properties of the pair appeared to be defined, whereas the individual properties of each one of the two entities forming the pair remained totally undefined [1] (see also [2], Sec. 7.3, and the references therein).

Let us explain why this observation is able to generate a paradox. For this, we start by enouncing two very general physical principles (GPP) [3]:

GPP1 *A physical entity S is said to exists at a given moment, if and only if it is in a pure state at that moment.*

GPP2 *A composite physical entity S, formed by two sub-entities S^A and S^B, is said to exist at a given moment if and only if S^A and S^B exist at that moment.*

On the other hand, according to standard quantum mechanics (SQM), the following two principles are also assumed to hold:

SQMP1 *If S is a physical entity with Hilbert space $\mathcal{H}$, each ray-state of $\mathcal{H}$ is a pure state of S, and all the pure states of S are of this kind.*

SQMP2 *The pure states of a composite entity S, formed by two sub-entities S^A and S^B, with Hilbert spaces $\mathcal{H}^A$ and $\mathcal{H}^B$, are the rays of $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$. S^A and S^B are in the ray-states $|\psi^A\rangle$ and $|\phi^B\rangle$ if and only if S is in the product ray-state $|\psi^A\rangle \otimes |\phi^B\rangle$.*

The paradox results from the observation that the above four principles are not compatible with each other, precisely because of the existence of entangled states. Indeed, if the composite entity S is in the entangled state:

$$|\psi\rangle = a_1\, e^{i\alpha_1}|\psi^A\rangle \otimes |\phi^B\rangle + a_2\, e^{i\alpha_2}|\phi^A\rangle \otimes |\psi^B\rangle, \tag{1}$$

with $a_1, a_2, \alpha_1, \alpha_2 \in \mathbb{R}$, $0 \leq a_1, a_2 \leq 1$, $a_1^2 + a_2^2 = 1$, $|\psi^A\rangle, |\phi^A\rangle \in \mathcal{H}^A$, $|\psi^B\rangle, |\phi^B\rangle \in \mathcal{H}^B$, $\langle\psi^A|\phi^A\rangle = \langle\psi^B|\phi^B\rangle = 0$, considering that $|\psi\rangle$ is a ray-state, by the SQMP1 it describes a pure state of S. By the GPP1, we know that S exists, and by the GPP2, the two sub-entities S^A and S^B also exist. But then, by the SQMP1, S^A and S^B are in ray-states, and by the SQMP2, S is in a product state, which is a contradiction.

Facing this conflict between the above four principles, a possible strategy is that of considering that the GPP2 does not have general validity, in the sense that when a composite entity is in an entangled state, its sub-entities would simply cease to exist, in the same way that two water droplets cease to exist when fused into a single larger droplet. However, this strategy is not fully consistent, as two quantum entities do not completely disappear when entangled, considering that there are properties associated with the pair that remain always actual.

For instance, we are still in the presence of two masses, which can be separated by a large spatial distance. So, entanglement is neither a situation where two masses are completely fused together, nor a situation where a spatial connection would bond them together, making it difficult to spatially separate them (as in chemical bonds). Also, entangled entities can remain perfectly correlated, and the property of "being perfectly correlated" is clearly a property which is meaningful only when we are in the presence of two *existing* entities.

In other terms, we cannot affirm that in an entangled state the composing entities cease to exist, and it is very difficult to even conceive a situation where the GPP2 would cease to apply. Even the above example of two droplets of water fused together cannot be considered as a counterexample, as in this case we are not really allowed to describe the larger droplet, once formed, to be the combination of two *actual* sub-droplets. So, it doesn't seem reasonable to abandon the GPP2, and of course it is neither reasonable to abandon the GPP1, which is almost tautological if we consider that a pure state, by definition, describes the objective condition in which an entity *is* in a given moment, precisely because of its *actual* existence.

Regarding the GPP1, it could be objected that also an entity which is not in a pure state can exist. This is correct, but we have here to understand that the term 'pure state' is a sort of pleonasm, as if we consider the notion of state at its most fundamental level, i.e., as a description of the *reality of an entity*, in a given moment, then all states are by definition pure states. Indeed, the non-pure states (i.e., the statistical mixtures of states), only describe our subjective lack of knowledge regarding the actual condition of the physical entity under consideration. Even when we are ignorant about the objective condition of an entity, i.e., its pure state, we can of course still say that the entity exists. However, and this is how the GPP1 should be understood, if the entity exists, in a given moment, then it must be possible (at least in principle, and independently of our subjective 'state of knowledge') to characterize its potential behavior under all possible interactions, and such characterization is precisely what a pure state is all about.

So, the only way to resolve the paradox seems to be that of revisiting the SQM1. To do so, we start by observing that there is a well-defined procedure in SQM that allows to associate individual states to entangled sub-entities. If, say, we are only interested in the description of S^A, irrespective of its correlations with S^B, all we have to do is to take a *partial trace*. For this, one has to rewrite the ray-state (1) in operatorial form. Defining: $D_\psi = |\psi\rangle\langle\psi|$, $D_\psi^A = |\psi^A\rangle\langle\psi^A|$, $D_\phi^A = |\phi^A\rangle\langle\phi^A|$, $D_\psi^B = |\psi^B\rangle\langle\psi^B|$ and $D_\phi^B = |\phi^B\rangle\langle\phi^B|$, we have:

$$D_\psi = a_1^2\, D_\psi^A \otimes D_\phi^B + a_2^2\, D_\phi^A \otimes D_\psi^B + I^{\mathrm{int}}, \tag{2}$$

where the interference contribution is given by:

$$I^{\mathrm{int}} = a_1 a_2\, e^{-i\alpha} |\psi^A\rangle\langle\phi^A| \otimes |\phi^B\rangle\langle\psi^B| + \mathrm{c.c.}, \tag{3}$$

with $\alpha = \alpha_2 - \alpha_1$. The state of S^A, irrespective of its correlations with S^B, can then be naturally defined by taking the partial trace: $D^A = \mathrm{Tr}_B\, D_\psi$, and similarly for S^B: $D^B = \mathrm{Tr}_A\, D_\psi$. A simple calculation yields:

$$D^A = a_1^2\, D_\psi^A + a_2^2\, D_\phi^A, \quad D^B = a_1^2\, D_\phi^B + a_2^2\, D_\psi^B. \tag{4}$$

However, (4) cannot be considered to solve the second entanglement paradox, as is clear that the *reduced* one-entity states D^A and D^B will not in general be ray-states, but *density operators*, and by the SQM1 we cannot interpret them as pure states. Therefore, we cannot use the GPP1 to decree the existence of S^A and S^B.

The following questions then arise: To save the intelligibility of the entangled states, shouldn't we complete the SQM by also allowing density operators to describe pure states? And more importantly: Do we have sufficient physical arguments to consider such a *completed quantum mechanics* (CQM), and can it be formulated in a sufficiently general and consistent way? It is the purpose of this article to provide positive answers the above questions, showing that the second entanglement paradox can be solved.

For this, we start by observing that the first reason to consider that density operators should also describe pure states is precisely the existence of the above mentioned partial trace procedure. Indeed, there is no logical reason why by focusing on a component of a system in a well-defined pure state, taking a partial trace, we would suddenly become ignorant about the condition of such component.

Another important reason is that a same density operator admits infinitely many representations as a mixture of one-dimensional projection operators [4]. This immediately suggests that the mixture interpretation is generally inappropriate, not only because it remains ambiguous, but also, and especially, because it fails to capture the dimension of potentiality that a density operator is able to describe.

Another relevant observation is that composite entities in ray-states can undergo unitary evolutions such that the evolution inherited by their sub-entities will make them continuously go from ray-states to density operator states, and return; a situation hardly compatible with the statistical ignorance interpretation of the density operators (see [5], sect. 7.5).

But we think there is an even more important reason to consider that the density operators can also describe pure states: if we do so, it becomes possible to derive the *Born rule* and provide a solution to the measurement problem, as recently demonstrated in what we have called the *extended Bloch representation of quantum mechanics* [6].

Let us briefly explain how this works. As is known, the ray-states of two-dimensional systems (like spin-$\frac{1}{2}$ entities) can be represented as points at the surface of a 3-dimensional unit sphere, called the *Bloch sphere* [7], with the density operators being located inside of it. What is less known is that a similar representation can be worked out for general N-dimensional systems [8]. The 3-d Bloch sphere is then replaced by a (N^2-1)-dimensional unit sphere, with the difference that, for $N > 2$, only a convex portion of it is filled with states.

When this generalized Bloch sphere representation is adopted, as an alternative way to represent the quantum states, it can be further extended by also including the measurements. These are geometrically described as $(N-1)$-simplexes inscribed in the sphere, whose vertices are the eigenvectors of the measured observables. These measurement simplexes, in turn, can be viewed as abstract structures made of an unstable and elastic substance, and it can be shown that an ideal quantum measurement is a process where the abstract point particle representative of the state first plunges into the sphere, in a deterministic way, along a path orthogonal to the simplex, then attaches to it, and following its indeterministic disintegration, and consequent collapse, is brought to one of its vertices, thus producing the outcome of the measurement, in a way that is perfectly consistent with the Born rule and the projection postulate [6].

We will not describe here the details of this 'hidden-measurement mechanism', as this is not the scope of this article. We only emphasize that its functioning requires the point particle representative of the state to move from the surface to the interior of the sphere, then back to the surface, thus implicitly ascribing the status of pure states also to the density operators.

In other terms, if we take seriously the extended Bloch representation, we can say in retrospect that a key obstacle in our understanding of quantum measurements is that SQM was not considering all the possible pure states that can describe the condition of a physical entity, and that the missing ones were precisely those located inside of the generalized Bloch sphere, i.e., the density operators.

Our final and in a sense most important argument in favour of the 'density operators are pure states' interpretation is about showing that within the extended Bloch formalism a composite entity in an entangled state is naturally described as a system formed by two components that always remain in well defined states, precisely corresponding to the reduced states (4), plus a third 'element of reality' describing their non-local correlation. For this, we start by observing that (2) can be written as [6]:

$$D_{\psi} = \frac{1}{N}\left(\mathbb{I} + c_N\, \mathbf{r}\cdot\mathbf{\Lambda}\right), \tag{5}$$

where the real unit vector $\mathbf{r}$ is the representative of the ray-state D_{ψ} within the generalized Bloch sphere $B_1(\mathbb{R}^{N^2-1})$, $c_N = (\frac{N(N-1)}{2})^{\frac{1}{2}}$, and the components of the operator-vector $\mathbf{\Lambda}$ are (a determination of) the generators of $SU(N)$, the *special unitary group of degree* N, which are self-adjoint, traceless matrices obeying $\mathrm{Tr}\,\Lambda_i\Lambda_j = 2\delta_{ij}$, $i,j\in\{1,\ldots,N^2-1\}$, forming a basis, together with the identity operator $\mathbb{I}$, for all the linear operators on $\mathcal{H}=\mathbb{C}^N$.

In the same way, with $\mathcal{H}^A = \mathbb{C}^{N_A}$, $\mathcal{H}^B = \mathbb{C}^{N_B}$, $N = N_A N_B$, we can define the Bloch vectors: $\mathbf{r}^A, \mathbf{s}^A, \overline{\mathbf{r}}^A \in B_1(\mathbb{R}^{N_A^2-1})$, $\mathbf{r}^B, \mathbf{s}^B, \overline{\mathbf{r}}^B \in B_1(\mathbb{R}^{N_B^2-1})$, representative of the states D^A_{ψ}, D^A_{ϕ}, D^A, D^B_{ψ}, D^B_{ϕ}, D^B, respectively, by: $D^A_{\psi} = \frac{1}{N_A}(\mathbb{I}^A + c_{N_A}\,\mathbf{r}^A\cdot\mathbf{\Lambda}^A)$, $D^A_{\phi} = \frac{1}{N_A}(\mathbb{I}^A + c_{N_A}\,\mathbf{s}^A\cdot\mathbf{\Lambda}^A)$, $D^A = \frac{1}{N_A}(\mathbb{I}^A + c_{N_A}\,\overline{\mathbf{r}}^A\cdot\mathbf{\Lambda}^A)$, $D^B_{\psi} = \frac{1}{N_B}(\mathbb{I}^B + c_{N_B}\,\mathbf{r}^B\cdot\mathbf{\Lambda}^B)$, $D^B_{\phi} = \frac{1}{N_B}(\mathbb{I}^B + c_{N_B}\,\mathbf{s}^B\cdot\mathbf{\Lambda}^B)$, $D^B = \frac{1}{N_B}(\mathbb{I}^B + c_{N_B}\,\overline{\mathbf{r}}^B\cdot\mathbf{\Lambda}^B)$, where the Λ^A_i are the N_A^2-1 generators of $SU(N_A)$, the Λ^B_j are the N_B^2-1 generators of $SU(N_B)$, and $\mathbb{I}^A$ and $\mathbb{I}^B$ are the identity operators on $\mathcal{H}^A$ and $\mathcal{H}^B$, respectively. Note that for $N=2$, the components Λ_i in (5) are simply the Pauli matrices, and $\mathbf{r}$ is a vector in the usual three-dimensional Bloch sphere.

At this point, we observe that it is possible to use the remarkable property that the trace of a tensor product is the product of the traces, to construct a determination of the $SU(N)$ generators in terms of tensor products of the generators of $SU(N_A)$ and $SU(N_B)$. More precisely, defining the N^2 self-adjoint $N\times N$ matrices:

$$\Lambda_{(i,j)} = \frac{1}{\sqrt{2}}\,\Lambda^A_i\otimes\Lambda^B_j,$$

where $i=0,\ldots,N_A^2-1$, $j=0,\ldots,N_B^2-1$, and we have defined $\Lambda^A_0 = (\frac{2}{N_A})^{\frac{1}{2}}\mathbb{I}^A$ and $\Lambda^B_0 = (\frac{2}{N_B})^{\frac{1}{2}}\mathbb{I}^B$, it is easy to check that, apart $\Lambda_{(0,0)} = (\frac{2}{N})^{\frac{1}{2}}\mathbb{I}$, the remaining N^2-1 matrices are all traceless, mutually orthogonal and properly normalized, and therefore constitute a bona fide determination of the generators of $SU(N)$ that can be used in (5), to express the components of the vector $\mathbf{r}$, representative of the composite entity's state.

Using the orthogonality of the generators, the components of $\mathbf{r}$ are given by: $r_i = e_N \mathrm{Tr}\, D_\psi \Lambda_i, \quad i=1,\ldots,N^2-1$, with $e_N = \frac{N}{2c_N}$, and similarly for the components of the Bloch vectors representing the sub-entities' states. With a direct calculation, one can then show that the entangled state $\mathbf{r}$ is of the tripartite direct sum form:

$$\mathbf{r} = d_{N_A}\overline{\mathbf{r}}^A \oplus d_{N_B}\overline{\mathbf{r}}^B \oplus \mathbf{r}^{\mathrm{corr}}. \tag{6}$$

where we have defined $d_{N_A} = (\frac{N_A-1}{N-1})^{\frac{1}{2}}$ and $d_{N_B} = (\frac{N_B-1}{N-1})^{\frac{1}{2}}$. In (6), the vector $\overline{\mathbf{r}}^A = a_1^2\mathbf{r}^A + a_2^2\mathbf{s}^A$ belongs to the one-entity Bloch sphere $B_1(\mathbb{R}^{N_A^2-1})$, and describes the state of S^A, whereas the vector $\overline{\mathbf{r}}^B = a_1^2\mathbf{s}^B + a_2^2\mathbf{r}^B$ belongs to the one-entity Bloch sphere $B_1(\mathbb{R}^{N_B^2-1})$, and describes the state of S^B. On the other hand, $\mathbf{r}^{\mathrm{corr}}$ is the component of the state which describes the correlation between the two sub-entities, and is of the form:

$$\mathbf{r}^{\mathrm{corr}} = d_{N_A,N_B}\overline{\mathbf{r}}^{AB} + \mathbf{r}^{\mathrm{int}}, \tag{7}$$

where $\overline{\mathbf{r}}^{AB} = a_1^2\mathbf{r}_1^{AB} + a_2^2\mathbf{r}_2^{AB} \in B_1(\mathbb{R}^{(N_A^2-1)(N_B^2-1)})$ is a vector with components $[\overline{\mathbf{r}}^{AB}]_{(i,j)} = a_1^2[\mathbf{r}_1^{AB}]_{(i,j)} + a_2^2[\mathbf{r}_2^{AB}]_{(i,j)}$, with $[\mathbf{r}_1^{AB}]_{(i,j)} = \mathbf{r}_i^A\mathbf{s}_j^B$, $[\mathbf{r}_2^{AB}]_{(i,j)} = \mathbf{s}_i^A\mathbf{r}_j^B$, $i=1,\ldots N_A^2-1$, $j=1,\ldots N_B^2-1$, $d_{N_A,N_B} = (\frac{(N_A-1)(N_B-1)}{N-1})^{\frac{1}{2}}$, and the vector $\mathbf{r}^{\mathrm{int}} \in \mathbb{R}^{(N_A^2-1)(N_B^2-1)}$ describes the interference contribution (3).

If the first two one-entity generators are chosen to be: $\Lambda_1^A = |\psi^A\rangle\langle\phi^A| + |\phi^A\rangle\langle\psi^A|$, $\Lambda_2^A = -i(|\psi^A\rangle\langle\phi^A| - |\phi^A\rangle\langle\psi^A|)$, $\Lambda_1^B = |\psi^B\rangle\langle\phi^B| + |\phi^B\rangle\langle\psi^B|$, $\Lambda_2^B = -i(|\psi^B\rangle\langle\phi^B| - |\phi^B\rangle\langle\psi^B|)$, $\mathbf{r}^{\mathrm{int}}$ only has four non-zero components, which for a suitably chosen order for the joint-entity generators are:

$$\mathbf{r}^{\mathrm{int}} = e_N\sqrt{2}a_1a_2(\cos\alpha, \cos\alpha, -\sin\alpha, \sin\alpha, 0, \ldots, 0). \tag{8}$$

According to (6), and different from the SQM formalism, we see that the extended Bloch representation allows to describe an entangled state as a "less tangled" condition in which the two sub-entities are always in the well-defined states $\overline{\mathbf{r}}^A$ and $\overline{\mathbf{r}}^B$, belonging to their respective one-entity Bloch spheres, which are clearly distinguished from their correlation, described by the vector (7), which cannot be deduced from the states of the two sub-entities, in accordance with the general principle that the whole is greater than the sum of its parts (so that the states of the parts cannot generally determine the state of the whole).

We can observe that the interference contribution $\mathbf{r}^{\rm int}$ is what distinguishes the entangled state (1)-(2) from the *separable state*: $D_\psi^{\rm sep} = a_1^2\, D_\psi^A \otimes D_\phi^B + a_2^2\, D_\phi^A \otimes D_\psi^B$. However, even when the interference contribution is zero, the separable (but non-product) state $D_\psi^{\rm sep}$ does not describe a situation of two experimentally separated entities, as is clear that the state vector $\overline{\mathbf{r}}^A$ is not independent from the state vector $\overline{\mathbf{r}}^B$, since their components both contain the parameters a_1^2 and a_2^2. Also, the components of $\overline{\mathbf{r}}^{AB}$ cannot be deduced from the knowledge of the components of $\overline{\mathbf{r}}^A$ and $\overline{\mathbf{r}}^B$, which means that a separable state is not a separated state, but a state that still describes a situation where the whole is greater than the sum of its parts.

Of course, when $a_2 = 0$ (or $a_1 = 0$), we are back to the situation of a so-called *product state*. This manifests at the level of the Bloch representation in the fact that $\mathbf{r} = d_{N_A}\mathbf{r}^A \oplus d_{N_B}\mathbf{s}^B \oplus d_{N_A,N_B}\mathbf{r}_1^{AB}$, with the two sub-entity states $\mathbf{r}^A$ and $\mathbf{s}^B$ now totally independent from one another, and able to fully determine the joint-entity contribution $\mathbf{r}_1^{AB}$, so that there are no genuine emergent properties in this case.

Therefore, considering the general description (6), and the previously mentioned arguments in favor of the 'density operators are pure states' interpretation, we are now in a position to formulate a completed quantum mechanical principle, in replacement of the SQMP1, which can restore the full intelligibility of entangled states:

CQMP1 *If S is a physical entity with Hilbert space $\mathcal{H}$, then each density operator of $\mathcal{H}$ is a pure state of S and all the pure states of S are of this kind.*

Of course, the CQMP1 does not imply that a density operator cannot be used to also describe a situation of subjective ignorance of the experimenter, regarding the pure state of an entity. It simply means that within the quantum formalism a same mathematical object can be used to model different situations, which are not easy to experimentally distinguish, because of the linearity of the trace used to calculate the transition probabilities.

However, a distinction between 'pure state-density operators' and 'mixed state-density operators' is in principle possible, for instance if one can set up an experimental context producing a non-linear evolution of the states, as in this case mixtures and pure states will evolve in a different way, and their ontological difference can become observable (see [6] for some additional considerations regarding the distinguishability of pure and mixed states in a measurement context).

It is worth mentioning that a completed quantum mechanics retaining all the principles of SQM, apart the SQM1 which is to be replaced by the CQMP1, was already proposed by one of us many years ago [3]. At the time the proposal was motivated by the existence of a mechanistic classical laboratory situations able to violate Bell's inequalities exactly as quantum entities in EPR-experiments can do [9]. Today, considering that the 'density operators are pure states' interpretation is an integral part of the extended Bloch representation, which provides a possible solution to the measurement problem [6] and, as we have shown in this article, also allows to obtain a partitioning of the entangled states where their correlative aspects remain clearly and naturally "disentangled" from the description of the sub-entities' states, we believe that the proposal has reached the status of a firmly founded scientific hypothesis, only waiting for an experimental confirmation.

A last remark is in order. Even in the simplest case of two entangled qubits ($N_A = N_B = 2$), where the two one-entity states $\overline{\mathbf{r}}^A$ and $\overline{\mathbf{r}}^B$ can be represented within our 3-dimensional Euclidean space (for instance as directions, in the case of spins), the correlation vector $\mathbf{r}^{\mathrm{corr}}$ is already 9-dimensional, and therefore is no longer describable within our Euclidean theatre. This is in accordance with the observed non-local effects that are produced by entangled entities, which are insensitive to spatial separation, and which therefore should be understood as effects resulting from the existence of genuinely *non-spatial* correlations.

In other terms, the solution we have proposed to the second entanglement paradox, via the extended Bloch representation, also suggests that non-locality should be understood as a manifestation of the non-spatial nature of quantum entities. This means that our approach also offers a possible solution to the first entanglement paradox, as is clear that if quantum entities are non-spatial entities, then their interconnections, when in entangled states, need not to happen through space, and therefore can remain perfectly insensitive to spatial separation.

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