Ultra-Violet Behavior of Bosonic Quantum Membranes

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ABSTRACT

We treat the action for a bosonic membrane as a sigma model, and then compute quantum corrections by integrating out higher membrane modes. As in string theory, where the equations of motion of Einstein's theory emerges by setting $\beta = 0$, we find that, with certain assumptions, we can recover the equations of motion for the background fields, i.e. $R_{\mu\nu} + (1/4)F_{\mu\alpha\beta\gamma\mu}F_{\nu}^{\alpha\beta\gamma} = 0$ and $D^{\alpha}F_{\alpha\beta\gamma} = 0$. for the membrane case. Although the membrane theory is non-renormalizable on the world volume by power counting, the investigation of the ultra-violet behavior of membranes may give us insight into the supersymmetric case, where we hope to obtain higher order M-theory corrections to 11 dimensional supergravity.

1   Introduction

At present, little is known about the action for M-theory [1,2], other than that it contains 11 dimensional supergravity in the low energy limit. Higher order corrections to 11D supergravity [3] are unknown. In this series of papers, we hope to compute these higher order corrections.

In string theory, the usual 10D supergravity action is derived by treating the original string action as a sigma model and them integrating the higher

modes. By setting $\beta = 0$ via conformal invariance, we then obtain the equations of motion of 10D supergravity, plus higher order corrections to any order [4].

We would like to apply this same general technique to M-theory, treating the 11D supermembrane action [5] as a sigma model in order to compute the higher order corrections to M-theory. There are, of course, several obstacles to performing such a procedure.

First, by power counting, the membrane theory in higher dimensions is non-renormalizable on the world volume. We find that the degree of divergence for any N-point function can be made arbitrarily high by adding higher vertex corrections, thereby rending invalid the standard renormalization technique.

Second, there are problems with the quantization of supermembranes, i.e. they are quantum mechanically unstable [6]. (This instability was the original reason why many abandoned supermembrane theory soon after it was formulated. However, it may be possible to re-interpret this instability in terms of 0-branes in matrix models [7].)

Third, the precise relationship between membranes and M-theory is also not known. In particular, M-theory also contains five-branes, and perhaps higher order corrections to membranes as well.

Our philosophy, however, will be to investigate the first problem. Although the model is superficially non-renormalizable by power counting methods, it may possess enough symmetry to eliminate large classes of diverges. For example, there is no counterpart to the $\beta = 0$ equation for membranes, because there is no conformal symmetry on the world volume. However, in a later paper we will show that supersymmetry will in fact set the analogous supermembrane term to zero because of the super Bianchi identities. Thus, supersymmetry (which demands compatibility with the 11D supergravity background equations) is sufficient to render the theory one-loop renormalizable.

Our ultimate goal is to show whether or not supersymmetry is sufficient to kill the divergences of the supermembrane theory to all orders. This, in turn, would allow us to compute the higher order corrections to 11D supergravity in the M-theory action. If a recursion relation can be written for the higher order corrections, then we may be able to make statements concerning the entire theory, to all orders.

However, even if this ultimate goal is not realized, we expect to find in-

teresting surprises. For example, we will show that, unlike the string case, one needs both one-loop and two-loop graphs in order to derive the standard equations of motion for the graviton and anti-symmetric tensor field. In the same way that the non-renormalizable four-fermion theory or the massive vector meson theory proved to be interesting laboratories for particle physics, it may turn out that supermembrane actions, even if they are inherently non-renormalizable, may be an interesting laboratory for M-theory.

2   Riemann Normal Co-ordinates

Our starting point is the bosonic membrane action:

$$L_1 = \frac{1}{2\alpha}\sqrt{\gamma}\gamma^{ij}g_{\mu\nu}\partial_i\phi^\mu\partial_j\phi^\nu \tag{1}$$

where $g_{\mu\nu}$ is the space-time metric, where Greek letters $\mu, \nu, \alpha = 0, 1, 2...10$, where γ^{ij} is the metric on the three-dimensional world volume, where Roman letters $i, j, k = 1, 2, 3$, and where ϕ^μ is the membrane co-ordinate.

To this action, we add a contribution from the anti-symmetric field:

$$L_2 = \beta\epsilon^{ijk}A_{\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda \tag{2}$$

which is found in the bosonic part of the supermembrane action. The total action is then $L_T = L_1 + L_2$, with α and β being two coupling constants.

Notice that the action of this theory is gauge invariant under the transformation:

$$\delta A_{\mu\nu\lambda} = \partial_\mu\Lambda_{\nu\lambda} + ... \tag{3}$$

Notice that the action also contains a world volume metric γ^{ij}. In the usual string action, this metric can be eliminated entirely via a gauge choice and a conformal transformation. However, in the covariant membrane case, we cannot eliminate all the degrees of freedom of the non-propagating world volume metric. Instead, we will simply treat the metric γ^{ij} as a classical background field. This means that we will have to keep γ^{ij} arbitrary and quantize the theory on a classical curved world volume.

Next, we wish to power expand this action using the background field method applied to sigma models, using Riemann normal co-ordinates [6]. Let the space-time variable $\phi^\mu(\tau)$ obey a standard geodesic equation:

$$\frac{d^2\phi^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{d\phi^\rho}{d\tau}\frac{d\phi^\sigma}{d\tau} = 0 \tag{4}$$

Now expand the membrane co-ordinate ϕ^μ around a classical configuration ϕ^μ_{cl}:

$$\phi^\mu = \phi^\mu_{\text{cl}} + \pi^\mu \tag{5}$$

where π^μ is the quantum correction to the classical configuration. Now power expand π^μ in terms of ξ^μ:

$$\pi^\mu = \xi^\mu - \frac{1}{2}\Gamma^\mu_{\rho\sigma}\xi^\rho\xi^\sigma - \frac{1}{3!}\Gamma^\mu_{\rho\sigma\lambda}\xi^\rho\xi^\sigma\xi^\lambda... \tag{6}$$

The various co-efficients in this power expansion can be laboriously computed by inserting the expression back into the geodesic equation. For example, we find that $\Gamma^\lambda_{\mu\nu}$ is the usual Christoffel symbol, and:

$$\Gamma^\mu_{\rho\sigma\lambda} = \partial_\rho\Gamma^\mu_{\sigma\lambda} - \Gamma^\alpha_{\rho\sigma}\Gamma^\mu_{\alpha\lambda} - \Gamma^\alpha_{\rho\lambda}\Gamma^\mu_{\alpha\sigma} \tag{7}$$

In general, the higher coefficients are equal to:

$$\Gamma^\nu_{\mu_1\mu_2...\mu_n\alpha\beta} = D_{\mu_n}\cdots D_{\mu_1}\Gamma^\nu_{\alpha\beta} \tag{8}$$

where we take the covariant derivatives only with respect to the lower indices.

Our goal is now to power expand the Lagrangian L_1+L_2 in terms of ξ^μ, and then integrate out ξ^μ from the action. This will give us a series of potentially divergent graphs, whose structure we wish to examine.

In general, this power expansion becomes prohibitively difficult as we progress to higher and higher orders, so we will instead use the formalism introduced by Mukhi [9].

One reason why this expansion is unwieldy is because the standard Taylor expansion is non-covariant. If we have a function I and power expand it, we find:

$$I = \sum_{n=0}^{\infty} I^{(n)} \tag{9}$$

where:

$$I^{(n)} = \frac{1}{n!} \int dx_1 \xi^{\mu_1} \partial_{\mu_1}^{x_1} \int dx_2 \xi^{\mu_2} \partial_{\mu_2}^{x_2} \cdots \int dx_n \xi^{\mu_n} \partial_{\mu_n}^{x_n} I \tag{10}$$

where the co-ordinates on the three dimensional world volume are given by x_i, and where ∂_i^x is a functional derivative:

$$\partial_\mu^x = \frac{\delta}{\delta \phi^\mu(x)} \tag{11}$$

Clearly, iterating the operator:

$$\int dx \, \xi^\mu(x) \partial_\mu^x \tag{12}$$

yields non-covariant results.

Let us define instead the operator Δ:

$$\Delta = \int dx \, \xi^\mu(x) D_\mu \tag{13}$$

where D_μ is a functional covariant derivative. For example:

$$D_\mu A^\nu[\phi(y)] = \left[\partial_\mu A^\nu(\phi(x)) + \Gamma_{\mu\lambda}^\nu(\phi(x)) A^\lambda(\phi(x)) \right] \delta^3(x - y) \tag{14}$$

Then we can power expand the Lagrangian as follows:

$$L = \sum_{n=0}^{\infty} L^{(n)} \tag{15}$$

where:

$$L^{(n)} = \frac{1}{n!} \Delta^n L \tag{16}$$

To perform the power expansion, we derive the following identities:

$$\Delta \xi^\mu = 0$$

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$$
\begin{aligned}
\Delta(\partial_i \phi^\mu) &= D_i \xi^\mu \\
\Delta(D_i \xi^\mu) &= R^\mu_{\nu\rho\sigma} \xi^\nu \xi^\rho \partial_i \phi^\sigma \\
\Delta T_{\mu_1\mu_2...} &= \xi^\rho D_\rho T_{\mu_1\mu_2...}
\end{aligned}
\tag{17}
$$

where $T_{\mu_1\mu_2...}$ is an arbitrary tensor, and:

$$
D_i \xi^\mu = \partial_i \xi^\mu + \Gamma^\mu_{\rho\sigma} \xi^\rho \partial_i \phi^\sigma
\tag{18}
$$

and:

$$
R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\lambda_{\nu\sigma} \Gamma^\mu_{\lambda\rho} - \Gamma^\lambda_{\nu\rho} \Gamma^\mu_{\lambda\sigma}
\tag{19}
$$

Now let us power expand the original action in terms of ξ. Let us replace $\phi^\mu_{\rm cl}$ with the symbol ϕ^μ.

We find:

$$
\begin{aligned}
L_1^{(0)} =&\ \frac{1}{2\alpha} \sqrt{\gamma} \gamma^{ij} g_{\mu\nu} \partial_i \phi^\mu \partial_j \phi^\nu \\[4pt]
L_1^{(1)} =&\ \frac{1}{\alpha} \sqrt{\gamma} \gamma^{ij} g_{\mu\nu} \partial_i \phi^\mu D_j \xi^\nu \\[4pt]
L_1^{(2)} =&\ \frac{1}{2\alpha} \sqrt{\gamma} \gamma^{ij} R_{\mu\nu\sigma\rho} \partial_i \phi^\mu \partial_j \phi^\sigma \xi^\nu \xi^\rho + \frac{1}{2\alpha} \sqrt{\gamma} \gamma^{ij} g_{\mu\nu} D_i \xi^\mu D_j \xi^\nu \\[4pt]
L_1^{(3)} =&\ \frac{1}{6\alpha} \sqrt{\gamma} \gamma^{ij} R_{\mu\nu\rho\sigma;\lambda} \partial_i \phi^\mu \partial_j \phi^\sigma \xi^\nu \xi^\rho \xi^\lambda + \frac{2}{3\alpha} \sqrt{\gamma} \gamma^{ij} R_{\mu\nu\rho\sigma} \partial_i \phi^\mu D_j \xi^\sigma \xi^\nu \xi^\rho \\[4pt]
L_1^{(4)} =&\ \frac{1}{24\alpha} \sqrt{\gamma} \gamma^{ij} \left[R_{\mu\nu\rho\sigma;\pi\kappa} + 4 R^\lambda_{\nu\rho\mu} R_{\lambda\pi\kappa\sigma} \right] \partial_i \phi^\mu \partial_j \phi^\sigma \xi^\nu \xi^\sigma \xi^\pi \xi^\kappa \\[2pt]
&+ \frac{1}{4\alpha} \sqrt{\gamma} \gamma^{ij} R_{\mu\rho\sigma\nu;\pi} \partial_i \phi^\mu D_j \xi^\nu \xi^\rho \xi^\sigma \xi^\pi + \frac{1}{6\alpha} \sqrt{\gamma} \gamma^{ij} R_{\mu\sigma\rho\nu} D_i \xi^\mu D_i \xi^\nu \xi^\rho \xi^\sigma \\[4pt]
L_1^{(5)} =&\ \frac{1}{120\alpha} \sqrt{\gamma} \gamma^{ij} \left(R_{\mu\nu\alpha\beta;\gamma\delta\lambda} + 14 R^\epsilon_{\alpha\beta\mu;\gamma} R_{\epsilon\delta\lambda\nu} \right) \partial_i \phi^\mu \partial_j \phi^\nu \xi^\alpha \xi^\beta \xi^\gamma \xi^\delta \xi^\lambda \\[2pt]
&+ \frac{1}{15\alpha} \sqrt{\gamma} \gamma^{ij} \left(R_{\mu\alpha\beta\nu;\gamma\delta} + 2 R^\pi_{\alpha\beta\mu} R_{\pi\gamma\delta\nu} \right) \partial_i \phi^\mu D_j \xi^\nu \xi^\alpha \xi^\beta \xi^\gamma \xi^\delta \\[2pt]
&+ \frac{1}{12\alpha} \sqrt{\gamma} \gamma^{ij} R_{\mu\alpha\beta\nu;\gamma} D_i \xi^\mu D_j \xi^\nu \xi^\alpha \xi^\beta \xi^\gamma \\[4pt]
L_1^{(6)} =&\ \frac{1}{720\alpha} \sqrt{\gamma} \gamma^{ij} \left(R_{\mu\alpha\beta\nu;\gamma\delta\epsilon\lambda} + 22 R_{\mu\alpha\beta\pi;\gamma\delta} R^\pi_{\epsilon\lambda\nu} + 14 R^\pi_{\alpha\beta\mu;\gamma} R_{\pi\delta\epsilon\nu;\lambda} \right. \\[2pt]
&+ \left. 16 R^\pi_{\alpha\beta\mu} R_{\pi\gamma\delta\rho} R^\rho_{\epsilon\lambda\nu} \right) \partial_i \phi^\mu \partial_j \phi^\nu \xi^\alpha \xi^\beta \xi^\gamma \xi^\delta \xi^\epsilon \xi^\lambda \\[2pt]
&+ \frac{1}{72\alpha} \sqrt{\gamma} \gamma^{ij} \left(R_{\mu\alpha\beta\nu;\gamma\delta\epsilon} + 4 R^\pi_{\alpha\beta\mu;\gamma} R_{\pi\delta\epsilon\nu} + R^\pi_{\alpha\beta\nu;\gamma} R_{\pi\delta\epsilon\mu} \right) \partial_i \phi^\mu D_i \xi^\nu \xi^\alpha \xi^\beta \xi^\gamma \xi^\delta \xi^\epsilon \\[2pt]
&+ \frac{1}{40\alpha} \sqrt{\gamma} \gamma^{ij} \left(R_{\mu\alpha\beta\nu;\gamma\delta} + \frac{8}{9} R^\pi_{\alpha\beta\mu} R_{\pi\gamma\delta\nu} \right) D_i \xi^\mu D_j \xi^\nu \xi^\alpha \xi^\beta \xi^\gamma \xi^\delta
\end{aligned}
\tag{20}
$$

The power expansion of the Lagrangian involving the anti-symmetric field is given by:

$$
\begin{aligned}
L_2^{(0)} &= \beta\sqrt{\gamma}\,\epsilon^{ijk}A_{\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda \\
L_2^{(1)} &= 3\beta\epsilon^{ijk}A_{\mu\nu\lambda}D_i\xi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda + \beta\epsilon^{ijk}D_\rho A_{\mu\nu\lambda}\xi^\rho\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda \quad (21)
\end{aligned}
$$

Since the original action was gauge invariant in the anti-symmetric field, we wish to preserve this symmetry, so let us re-write $L_2^{(1)}$ by introducing the tensor:

$$
F_{\sigma\mu\nu\lambda} = D_\sigma A_{\mu\nu\lambda} - D_\mu A_{\nu\lambda\sigma} + D_\nu A_{\lambda\sigma\mu} - D_\lambda A_{\sigma\mu\nu} \tag{22}
$$

Because of gauge invariant, all subsequent terms will involve this covariant tensor.

Then we can write:

$$
L_2^{(1)} = \beta\epsilon^{ijk}F_{\sigma\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\sigma \tag{23}
$$

$$
L_2^{(2)} = \frac{\beta}{2}\epsilon^{ijk}\left(3F_{\sigma\mu\nu\lambda}D_i\xi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\sigma + D_\rho F_{\sigma\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\rho\xi^\sigma\right) \tag{24}
$$

$$
\begin{aligned}
L_2^{(3)} = \;&\frac{\beta}{6}\epsilon^{ijk}\Big(6F_{\sigma\mu\nu\lambda}D_i\xi^\mu D_j\xi^\nu\partial_k\phi^\lambda\xi^\sigma + 6D_\rho F_{\sigma\mu\nu\lambda}D_i\xi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\rho\xi^\sigma \\
&+ 3R^\mu_{\alpha\beta\gamma}\partial_i\phi^\gamma F_{\sigma\mu\nu\lambda}\xi^\alpha\xi^\beta\xi^\sigma\partial_j\phi^\nu\partial_k\phi^\lambda \\
&+ D_\pi D_\rho F_{\sigma\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\rho\xi^\sigma\xi^\pi\Big)
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
L_2^{(4)} = \;&\frac{\beta}{4!}\epsilon^{ijk}\Big\{6F_{\sigma\mu\nu\lambda}D_i\xi^\mu D_j\xi^\nu D_k\xi^\lambda\xi^\sigma \\
&+ 18R^\mu_{\alpha\beta\gamma}F_{\sigma\mu\nu\lambda}\partial_i\phi^\gamma D_j\xi^\nu\partial_k\phi^\lambda\xi^\alpha\xi^\beta\xi^\sigma \\
&+ 18D_\rho F_{\sigma\mu\nu\lambda}D_i\xi^\mu D_j\xi^\nu\partial_k\phi^\lambda\xi^\sigma\xi^\rho \\
&+ 9D_\pi D_\rho F_{\sigma\mu\nu\lambda}D_i\xi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\sigma\xi^\rho\xi^\pi \\
&+ 9R^\mu_{\alpha\beta\gamma}D_\rho F_{\sigma\mu\nu\lambda}\partial_i\phi^\gamma\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\alpha\xi^\beta\xi^\rho\xi^\sigma
\end{aligned}
$$

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$$+ \quad D_\kappa D_\pi D_\rho F_{\sigma\mu\nu\lambda} \partial_i \phi^\mu \partial_j \phi^\nu \partial_k \phi^\lambda \xi^\rho \xi^\rho \xi^\pi \xi^\kappa$$
$$+ \quad 3R^\mu_{\alpha\beta\gamma} F_{\sigma\mu\nu\lambda} D_i \xi^\gamma \partial_j \phi^\nu \partial_k \phi^\lambda \xi^\alpha \xi^\beta \xi^\sigma$$
$$+ \quad 3R^\mu_{\alpha\beta\gamma;\pi} \partial_i \phi^\gamma F_{\sigma\mu\nu\lambda} \partial_j \phi^\nu \partial_k \phi^\lambda \xi^\alpha \xi^\beta \xi^\sigma \xi^\pi \Big\} \qquad (26)$$

Now we wish to simply the action a bit. We first wish to eliminate terms linear in ξ. If we add the contribution from $L_1^{(1)}$ and $L_2^{(1)}$, we find:

$$
\begin{aligned}
L_T^{(1)} &= \frac{1}{\alpha} \sqrt{\gamma} \gamma^{ij} g_{\mu\nu} \partial_i \phi^\mu D_j \xi^\nu + \beta \epsilon^{ijk} F_{\sigma\mu\nu\lambda} \partial_i \phi^\mu \partial_j \phi^\nu \partial_k \phi^\lambda \xi^\sigma \\
&= \partial_i \phi^\mu \left(\frac{1}{\alpha} \sqrt{\gamma} \gamma^{ij} g_{\mu\nu} D_j \xi^\nu + \beta \epsilon^{ijk} \partial_j \phi^\nu \partial_k \phi^\lambda \xi^\sigma F_{\sigma\mu\nu\lambda} \right) \\
&= \partial_i \phi^\mu \nabla^i \xi_\mu \qquad (27)
\end{aligned}
$$

where we define ∇ by:

$$\nabla^i \xi_\mu = \frac{1}{\alpha} \sqrt{\gamma} \gamma^{ij} g_{\mu\nu} D_j \xi^\nu + \beta \epsilon^{ijk} \partial_j \phi^\nu \partial_k \phi^\lambda \xi^\sigma F_{\sigma\mu\nu\lambda} \qquad (28)$$

The linear term $L^{(1)}$ may be set to zero, if we impose:

$$\nabla^i \partial_i \phi^\mu = 0 \qquad (29)$$

which then defines ϕ^μ_{cl}.

Now let us add the two contributions together from the two parts and organize them by the ξ co-ordinates.

We find:

$$
\begin{aligned}
L_T^{(2)} &= \xi^\alpha \xi^\beta L_{\alpha\beta}^{(2)} + D_i \xi^\alpha \xi^\beta L_{\alpha\beta}^{i(2)} + D_i \xi^\alpha D_j \xi^\beta L_{\alpha\beta}^{ij(2)} \\
L_T^{(3)} &= \xi^\alpha \xi^\beta \xi^\sigma L_{\alpha\beta\gamma}^{(3)} + D_i \xi^\alpha \xi^\beta \xi^\sigma L_{\alpha\beta\gamma}^{i(3)} + D_i \xi^\alpha D_j \xi^\beta \xi^\sigma L_{\alpha\beta\gamma}^{ij(3)} \\
L_T^{(4)} &= \xi^\alpha \xi^\beta \xi^\gamma \xi^\delta L_{\alpha\beta\gamma\delta}^{(4)} + D_i \xi^\alpha \xi^\beta \xi^\gamma \xi^\delta L_{\alpha\beta\gamma\delta}^{i(4)} \\
&+ D_i \xi^\alpha D_j \xi^\beta \xi^\gamma \xi^\delta L_{\alpha\beta\gamma\delta}^{ij(4)} + D_i \xi^\alpha D_j \xi^\beta D_k \xi^\gamma \xi^\delta L_{\alpha\beta\gamma\delta}^{ijk(4)} \qquad (30)
\end{aligned}
$$

where:

$$L_{\alpha\beta}^{(2)} = \frac{1}{2\alpha} \sqrt{\gamma} \gamma^{ij} R_{\mu\alpha\beta\sigma} \partial_i \phi^\mu \partial_j \phi^\sigma + \frac{\beta}{2} \epsilon^{ijk} D_\alpha F_{\beta\mu\nu\lambda} \partial_i \phi^\mu \partial_j \phi^\nu \partial_k \phi^\lambda$$

$$
\begin{aligned}
L^{i(2)}_{\alpha\beta} &= \frac{3\beta}{2}\epsilon^{ijk}F_{\beta\alpha\nu\lambda}\partial_j\phi^\nu\partial_k\phi^\lambda \\
L^{ij(2)}_{\alpha\beta} &= \frac{1}{2\alpha}\sqrt{\gamma}\gamma^{ij}g_{\alpha\beta}
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
L^{(3)}_{\alpha\beta\gamma} &= \frac{1}{6\alpha}\sqrt{\gamma}\gamma^{ij}R_{\mu\alpha\beta\sigma;\gamma}\partial_i\phi^\mu\partial_j\phi^\sigma \\
&+ \frac{\beta}{6}\epsilon^{ijk}D_\gamma D_\beta F_{\alpha\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda \\
&+ \frac{\beta}{2}\epsilon^{ijk}R^\mu_{\alpha\beta\pi}F_{\gamma\mu\nu\lambda}\partial_i\phi^\pi\partial_j\phi^\nu\partial_k\phi^\lambda \\
L^{i(3)}_{\alpha\beta\gamma} &= \frac{2}{3\alpha}\sqrt{\gamma}\gamma^{ij}R_{\mu\beta\gamma\alpha}\partial_j\phi^\mu \\
&+ \beta\epsilon^{ijk}D_\beta F_{\gamma\alpha\nu\lambda}\partial_j\phi^\nu\partial_k\phi^\lambda \\
L^{ij(3)}_{\alpha\beta\gamma} &= \beta\epsilon^{ijk}F_{\gamma\alpha\beta\lambda}\partial_k\phi^\lambda
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
L^{(4)}_{\alpha\beta\gamma\delta} &= \frac{1}{24\alpha}\sqrt{\gamma}\gamma^{ij}\partial_i\phi^\mu\partial_j\phi^\sigma\left[R_{\mu\alpha\beta\sigma;\gamma\delta}+R^\lambda_{\alpha\beta\mu}R_{\lambda\gamma\delta\sigma}\right] \\
&+ \frac{\beta}{4!}\epsilon^{ijk}D_\delta D_\gamma D_\alpha F_{\beta\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda \\
&+ \frac{9\beta}{4!}\epsilon^{ijk}R^\mu_{\alpha\beta\sigma}D_\gamma F_{\delta\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda \\
&+ \frac{3\beta}{4!}\epsilon^{ijk}R^\mu_{\alpha\beta\sigma;\delta}\partial_i\phi^\sigma F_{\gamma\mu\nu\lambda}\partial_j\phi^\nu\partial_k\phi^\lambda \\
L^{i(4)}_{\alpha\beta\gamma\delta} &= \frac{1}{4\alpha}\sqrt{\gamma}\gamma^{ij}R_{\mu\alpha\beta\gamma;\delta}\partial_i\phi^\mu + \frac{18\beta}{4!}\epsilon^{ijk}R^\mu_{\beta\delta\pi}F_{\delta\mu\alpha\lambda}\partial_j\phi^\pi\partial_k\phi^\lambda \\
&+ \frac{9\beta}{4!}\epsilon^{ijk}D_\delta D_\beta F_{\gamma\alpha\nu\lambda}\partial_j\phi^\nu\partial_k\phi^\lambda + \frac{3\beta}{4!}\epsilon^{ijk}R^\mu_{\beta\gamma\alpha}F_{\delta\mu\nu\lambda}\partial_j\phi^\nu\partial_k\phi^\lambda \\
L^{ij(4)}_{\alpha\beta\gamma\delta} &= \frac{1}{6\alpha}\sqrt{\gamma}\gamma^{ij}R_{\alpha\beta\gamma\delta} + \frac{18\beta}{4!}\epsilon^{ijk}D_\delta F_{\gamma\alpha\beta\lambda}\partial_k\phi^\lambda \\
L^{ijk(4)}_{\alpha\beta\gamma\delta} &= \frac{6\beta}{4!}\epsilon^{ijk}F_{\delta\alpha\beta\gamma}
\end{aligned}
\tag{33}
$$

Before we can begin to set up the perturbation series, we must first diagonalize the quadratic term $L^{(2)}_T$. The space-time matrix $g_{\mu\nu}$ can be eliminated in favor of the usual vierbein e^a_μ, where the Roman index a represents tangent space indices.

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Now let us replace $g_{\mu\nu}$ with $e^a_\mu e^a_\nu$. With a little bit of algebra, we can move the vierbein past the derivative and prove the identity:

$$
\begin{aligned}
e^a_\mu D_i \xi^\mu &= e^a_\mu \left(\partial_i \xi^\mu + \partial_i \phi^\lambda \Gamma^\mu_{\lambda\nu} \xi^\nu \right) \\
&= \partial_i \xi^a + \partial_i \phi^\mu \omega^{ab}_\mu \xi^a = D_i \xi^a
\end{aligned}
\tag{34}
$$

where $\xi^a = e^a_\mu \xi^\mu$ and where:

$$
\omega^{ab}_\mu = -(\partial_\mu e^a_\nu) e^{b\nu} + e^a_\nu \Gamma^\nu_{\mu\beta} e^{b\beta}
\tag{35}
$$

which is self-consistent with the equation $D_\mu e^a_\nu = 0$, as desired.

In this fashion, we can now write everything with tangent space indices. We find that the only change is that we must replace Green letters $\alpha, \beta, \gamma, \delta$ with Roman letters a, b, c, d.

For the general case, we find:

$$
L_T^{(n)} = \sum_{k=0}^{n-1} D_{i_1} \xi^{a_1} D_{i_2} \xi^{a_2} \cdots D_{i_k} \xi^{a_k} \cdots \xi^{a_n} L^{i_1 i_2 \cdots i_k (n)}_{a_1 \cdots a_n}
\tag{36}
$$

where a_i are defined in the tangent space.

3 Regularization

Now that we have power expanded the original action in terms of ξ^a, where a is the tangent space index on curved space-time, we must now integrate over the quantum field ξ^a defined on the tangent space, which will leave us with divergent terms whose structure we wish to analyze.

We will power expand around the term:

$$
L_1^{(2)} = \frac{1}{2\alpha} \sqrt{\gamma} \gamma^{ij} D_i \xi^a D_j \xi^a
\tag{37}
$$

where the space-time metric $g_{\mu\nu}$ has been absorbed into the vierbeins.

If we perform the integration over ξ^a, then (subject to a regularization scheme):

$$\langle T\xi^a(x)\xi^b(x')\rangle \sim \alpha\Lambda\delta^{ab} + ... \tag{38}$$

where the right hand side is linearly divergent via some large momentum scale Λ, and there are important corrections to this equation crucially dependent on the regularization scheme.

Then the first term in $L_T^{(2)}$ contributes the following term:

$$L_{ab}^{(2)}\langle T\xi^a\xi^b\rangle \tag{39}$$

which in turn yields the two equations:

$$\langle T\xi^a\xi^b\rangle\frac{1}{\alpha}\sqrt{\gamma}\gamma^{ij}R_{\mu ab\sigma}\partial_i\phi^\mu\partial_j\phi^\sigma \tag{40}$$

and:

$$\beta\langle T\xi^a\xi^b\rangle\epsilon^{ijk}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda D_a F_{b\mu\nu\lambda} \tag{41}$$

Unlike the superstring, there is no conformal symmetry by which we can set this divergent term to zero. In this paper, we will simply set the lowest order divergent term to zero by fiat. This is a weakness in this approach. This, in some sense, defines the model, i.e. the theory can only propagate on certain background fields which set the lowest order divergent term to zero.

However, for the supermembrane, we will show in a later paper that there is enough supersymmetry to allow us to set this divergent term to zero, so we have:

$$R_{\mu\nu} = 0; \quad D^a F_{abcd} = 0 \tag{42}$$

The second equation is just the equation of motion for the anti-symmetric field, as expected. However, the first equation is rather troubling, since there should be a term proportional to F^2. The fact that this term is missing means that the equations of motion are actually inconsistent. There exists no action involving $g_{\mu\nu}$ and $F_{\alpha\beta\gamma\delta}$ which yields these equations of motion. Thus, we must carefully analyze our regularization scheme and go to higher interactions. This is different from the superstring case, where the one-loop results are sufficient to yield self-consistent equations of motion. For the membrane, we find that we must go to two loops in order to obtain self-consistent results.

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Now let us generalize this result to higher orders by carefully introducing a regularization scheme. There is a problem with dimensional regularization, however. If we analytically continue the integral:

$$\int \frac{d^d p}{(p^2 + m^2)} \sim \Gamma(1 - \frac{d}{2}) \tag{43}$$

we find that it is finite for $d = 3$. It diverges with a pole at $d = 4$, but is formally finite for odd dimensions. This strange result does not change for higher loops, since multiple integrals over the momenta yield factors of $\Gamma(k)$, where k is half-integral, which is again finite for $d = 3$. Furthermore, when we introduce supersymmetry, we find that dimensional regularization does not respect this symmetry, which only holds at $d = 3$ for supermembranes. Hence, dimensional regularization poses some problems. In fact, supersymmetry is so stringent, it appears that finding a suitable regularization method is problematic for any method.

We will, instead, use standard point-splitting and proper time methods, separating the points on the world volume at which the various $\xi^\mu(x)$ meet at a vertex. This, of course, will violate general covariance and supersymmetry by point-splitting. However, point-splitting methods are convenient since the divergence within a Feynman integral occur when fields are defined at the same world volume point, i.e. $x \to y$ on the world volume.

Then the two-point Green's function can be written as:

$$\langle T\xi^a(x)\xi^b(x')\rangle = -iG^{ab}(x - x') \tag{44}$$

where x and x' represents points on the three dimensional world volume, and:

$$\frac{1}{\alpha}\left[-D_i^{ab}\sqrt{\gamma}\gamma^{ij}D_j^{bc} + \sqrt{\gamma}\left(\zeta R + m^2\right)\right]G^{ac}(x, x') = \delta^3(x - x')\delta^{ab} \tag{45}$$

(Although the theory is massless, notice that we added in a small mass m^2 in order to handle infrared divergences. In non-linear sigma models of this type, it can be shown that this mass regulartor cancels against other terms in the perturbation theory. Notice that we introduce a parameter ζ which takes into account the curvature on the world volume. This term will be of interest when

we introduce fermions into our formalism. However, here we can set this term ζ to zero for our case.)

The solution of this Green's function is complicated by two facts. First, the Green's function is defined over both a curved three dimensional world volume manifold and 11 dimensional space-time manifold, and hence we have to use the formalism developed for general relativity.

Second, the Green's function will in general introduce unwanted curvatures on the world volume. This is because the covariant derivative D_i contains the connection field $\partial_i \phi^\mu \omega_\mu^{ab}$. If we set this equal to A_i^{ab}, then $D_i^{ab} = \partial_i \delta^{ab} + A_i^{ab}$, which is the familiar covariant derivative in $O(D)$ gauge theory. Thus, when we invert the opertor $D_i \sqrt{\gamma} \gamma^{ij} D_j$, we encounter gauge invariant terms like the square of R_{ij}^{ab}, where

$$[D_i, D_j]^{ab} = R_{ij}^{ab} \tag{46}$$

In two dimensions, curvature terms of this sort do not contribute to the perturbation expansion to lowest order when $d = 2$ [8]. However, these terms do in fact contribute to the perturbation expansion when $d = 4$ [10]. In fact, the presence of these terms renders the quantum theory of the $d = 4$ non-linear sigma model non-renormalizable, since they introduce new counter-terms not present in the original action.

Given the potential problems with this term, let us introduce the proper time formalism [11]. Let s be the Schwinger proper time variable:

$$\tilde{G}^{ab}(x, x') = (\gamma(x))^{1/4} G^{ab}(x, x')(\gamma(x'))^{1/4} = i\alpha \int_0^\infty ds \langle x, s | x', 0 \rangle^{ab} \tag{47}$$

where $\gamma = \det \gamma^{ij}$ and we choose a positive metric on the world volume.

We impose the boundary condition:

$$\langle x, 0 | x', 0 \rangle^{ab} = \delta^{ab} \delta^3(x - x') \tag{48}$$

Now let us assume the ansatz for the Green's function:

$$
\begin{aligned}
\langle x, s | x', 0 \rangle &= \frac{i}{(4\pi i s)^{d/2}} \gamma(x)^{1/4} \Delta^{1/2}(x, x') \gamma(x')^{1/4} F^{ab}(x, x'; is) \\
&\times \exp\left(-\frac{\sigma(x, x')}{2is} - ism^2\right)
\end{aligned}
\tag{49}
$$

where $\sigma(x, x')$ is one-half the square of the distance along the geodesic between x and x', where Δ is given by:

$$\gamma(x)^{1/4}\Delta(x, x')\gamma(x')^{1/4} = -\det\left(-\sigma_{,i,j}(x, x')\right) \tag{50}$$

and:

$$F(x, x')^{ab} = \sum_{n=0}^{\infty} (is)^n a_n^{ab}(x, x') \tag{51}$$

The object of this section is to power expand the Green's function in terms of $\sigma(x, x')$. In particular, σ obeys a number of useful identities, among them:

$$\sigma_i \sigma^i = 2\sigma \tag{52}$$

where $\sigma_i = \partial_i \sigma$, and we raise and lower indices via γ^{ij}. From this identity, we can establish a large number of identities for various derivatives of σ.

When calculating Feynman diagrams, we will find that they diverge according to inverse powers of σ. Hence, we can compare the large momentum cut-off Λ to σ, i.e.

$$\Lambda \sim \sigma^{-1/2} \tag{53}$$

This new Green's function satisfies the "Schrodinger" equation:

$$-\frac{\partial}{\partial is}\langle x, s | x', 0\rangle = H\langle x, s | x', 0\rangle \tag{54}$$

where the "Hamiltonian" is given by:

$$H = -\gamma^{-1/4} D_i \gamma^{1/2} \gamma^{ij} D_j \gamma^{-1/4} + \zeta R + m^2 \tag{55}$$

If we insert the expansion of the Green's function into the defining equation for the Green's function, we are left with a constraint on the undetermined function F:

$$-\frac{\partial F}{\partial is} = \zeta R F + \frac{1}{is}\sigma^{,i} F_{,i} - \frac{1}{\Delta^{1/2}}\left(\Delta^{1/2} F\right)^{;i}_{,i} \tag{56}$$

Inserting the power expansion for F into this expression, we now have a recursive relation among the a_n coefficients appearing within F:

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$$\sigma^i a_{n+1,i} + (n+1)a_{n+1} = \frac{1}{\Delta^{1/2}} \left(\Delta^{1/2} \right)^{;i}_{\ i} - \zeta R a_n \tag{57}$$

Now let us solve the a_n iteratively. The equation for a_0^{ab} give us:

$$\sigma^i D_i a_0 = 0 \tag{58}$$

The goal of this exercise is to extract out the divergent terms within the Green's function. Let us, therefore, slowly let x approach x'. Repeated differentiation of the previous formula yields:

$$
\begin{aligned}
\lim_{x \to x'} D_i a_0 &= 0 \\
\lim_{x \to x'} D_i \gamma^{ij} D_j a_0 &= -\frac{1}{2} R_{ij}^{ab} \\
\lim_{x \to x'} a_{0,;i}{}^i{}_{;j}{}^j &= \frac{1}{2} \operatorname{Tr} R_{ij} R^{ij}
\end{aligned}
\tag{59}
$$

where $R_{ij}^{ab} = [D_i, D_j]^{ab}$ and:

$$
\begin{aligned}
\sigma &\to \frac{1}{2}(x - x')^2 \\
\sigma_{i;j} &\to \gamma_{ij} \\
\sigma_{i;j;k;l} &\to \frac{1}{2}\left(R_{lijk} + R_{kijl} \right) \\
\Delta &\to 1
\end{aligned}
\tag{60}
$$

After a certain amount of algebra, we find the desired result for the coefficients a_n^{ab}:

$$
\begin{aligned}
a_0 &\to \delta^{ab} \\
a_1 &\to \frac{1}{6}R - \zeta R \\
a_2 &\to \frac{1}{2}\left[\left(\frac{1}{6} - \zeta\right)R \right]^2 + \frac{1}{6}\left(\frac{1}{5} - \zeta\right)R_i^{;i} + \frac{1}{12}\operatorname{Tr} R_{ij}R^{ij} \\
&\quad - \frac{1}{180}R_{ij}R^{ij} + \frac{1}{180}R_{ijkl}R^{ijkl}
\end{aligned}
\tag{61}
$$

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Now we wish to insert these values for a_n^{ab} into the expression for the Green's function, in order to see how badly it diverges as a function of σ. In the limit as $x \to x'$, many terms drop out, and we are left with:

$$
\begin{aligned}
\tilde{G}(x, x')^{ab} &= i\alpha \int_0^\infty ds \, \langle x, s | x; 0 \rangle^{ab} \\
&= i\alpha \int_0^\infty ds \, \frac{i\sqrt{\gamma}}{(4\pi is)^{d/2}} \exp\left(-\frac{\sigma}{2is} - ism^2\right) \sum_{n=0}^\infty (is)^n a_n^{ab} \quad (62)
\end{aligned}
$$

We now use the integral:

$$
\int_0^\infty dx \, x^{\nu-1} \exp\left(-\frac{i\mu}{2}[x + (\beta^2/x)]\right) = -i\pi\beta^\nu e^{i\nu\pi/2} H_{-\nu}^{(2)}(\beta\mu) \quad (63)
$$

where $H^{(2)}$ is a Bessel function of the third kind, or a Hankel function.

For the case of interest, $d = 3$, we have:

$$
\tilde{G}(x, x')^{ab} = \alpha \frac{\pi\sqrt{\gamma}}{(4\pi)^{3/2}} \sum_{n=0}^\infty a_n^{ab} \left(\frac{-\sigma}{2m^2}\right)^{(1/2)(n-1/2)} H_{-n+1/2}^{(2)}\left(\sqrt{-2m^2\sigma}\right) \quad (64)
$$

We now make the definitions $\mu = 2m^2$, $\beta = (-\sigma/2m^2)^{1/2}$, $\nu = n+1-(d/2)$. To find the power expansion in σ, we use the fact that:

$$
H_{-n+1/2}^{(2)} = -i(-1)^{n-1} \left(\frac{2z}{\pi}\right)^{1/2} (j_{n-1} - iy_{n-1}) \quad (65)
$$

(where we set $z = \sqrt{-2m^2\sigma}$) and the fact that:

$$
\begin{aligned}
j_n(z) &= z^n \left(-\frac{1}{z}\frac{d}{dz}\right)^n \frac{\sin z}{z} \\
y_n(z) &= -z^n \left(-\frac{1}{z}\frac{d}{dz}\right)^n \frac{\cos z}{z}
\end{aligned} \quad (66)
$$

In particular, we find that the propagator in curved space is linearly divergent in momentum, as expected. The troublesome terms, a_1 and a_2, we

see, are finite in three dimensions, and so can be dropped from our discussion (which is not the case for $d = 4$). Although they ruin the renormalization program for four dimensions, we find that they drop out in three dimensions.

For completeness, we present the entire series:

$$
\begin{aligned}
\tilde{G}(x, x')^{ab} &= \alpha i \frac{\sqrt{2\pi\gamma}}{(4\pi)^{3/2}} \left\{ \frac{e^{-iz}}{\sqrt{\sigma}} a_0^{ab} \right. \\
&+ \left. \sum_{n=1}^{\infty} (-1)^n a_n^{ab} \left(\frac{z}{-2m^2} \right)^{2n-1} \left(-\frac{1}{z} \frac{d}{dz} \right)^{n-1} \frac{e^{-iz}}{z} \right\}
\end{aligned}
\tag{67}
$$

Notice that, as $x \to x'$, we find that the integral diverges linearly with the momentum, but the troublesome curvature term involving R_{ij}^{ab} do not contribute (as they do in four dimensions). Hence, from now on, we can simply use the fact that the propagator diverges linearly with momentum.

Lastly, we can also use this formalism to compute two point functions involving derivatives. If we have two point functions like:

$$
\langle T \partial_i \xi^a(x) \partial_j \xi^b(y) \rangle = \partial_i^x \partial_j^y \langle T \xi^a(x) \xi^b(y) \rangle = \partial_i^x \partial_j^y \tilde{G}^{ab}(\sigma)
\tag{68}
$$

where $\tilde{G}^{ab}(\sigma)$ is the propagator, then we can, for small distance separations, use the fact that $\sigma(x, y) \sim (1/2)(x - y)^i \gamma_{ij} (x - y)^j$, so that:

$$
\partial_i^x \partial_j^y \sigma(x, y) \sim -\gamma_{ij}
\tag{69}
$$

By taking repeated derivatives of the propagator as a function of the separation σ, one can therefore contruct the contraction of an arbitrary number of ξ fields.

4   Two Loop Order

We saw earlier that the one loop result was inconsistent. An action of the form $R + F_{\mu\nu\alpha\beta}^2$ cannot have equations of motion given by $R_{\mu\nu} = 0$ and $D^\mu F_{\mu\nu\alpha\beta} = 0$ We must therefore probe the two loop result to see if we can re-establish the consistency of the model.

Consider first the case of two external lines $N = 2$.

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The contraction of the term $L_{2,\alpha\beta}^{(3)}$ yields:

$$L_{2,aa}^{i(3)} \sim \epsilon^{ijk} F_{aa\nu\lambda} \partial_j \phi^\nu \partial_k \phi^\lambda = 0 \tag{70}$$

which vanishes by the anti-symmetry of the F tensor.

The most interesting two loop graph is given by the contraction of $L_{2\alpha\beta\gamma}^{ij(3)}$ with itself. This gives us the contraction:

$$\epsilon^{ijk} F_{\gamma\alpha\beta\lambda} \partial_k \phi^\lambda \langle T D_i \xi^\alpha(x) D_j \xi^\beta(x) \xi^\gamma(x) D_{\bar{i}} \xi^{\bar{\alpha}}(y) D_{\bar{j}} \xi^{\bar{\beta}}(y) \xi^{\bar{\gamma}}(y) \rangle \epsilon^{\bar{i}\bar{j}\bar{k}} F_{\bar{\gamma}\bar{\alpha}\bar{\beta}\bar{\lambda}} \partial_{\bar{k}} \phi^{\bar{\lambda}} \tag{71}$$

when $x \to y$.

If we perform the contractions over ξ, we find, with a little bit of work, the following result:

$$\beta^2 \alpha^3 \Lambda^4 \sqrt{\gamma} \gamma^{ij} \partial_i \phi^\lambda \partial_j \phi^{\bar{\lambda}} F_{cab\lambda} F_{\bar{\lambda}}^{cab} \tag{72}$$

Notice that the divergence can be absorbed into a rescaling: $\alpha = \alpha_R/\Lambda$. (In the next section, we will see that the leading divergences can in fact be absorbed by this rescaling to all orders.)

After rescaling, we find that the equation of motion of the graviton is given by:

$$R_{\mu\nu} + \frac{1}{4} F_{\mu\alpha\beta\gamma} F_\nu^{\alpha\beta\gamma} = 0 \tag{73}$$

(Unfortunately, the term proportional to $g_{\mu\nu}$ does not appear in the equations of motions, signalling a possible inconsistency. This is normally solved for the superstring case by adding an another field, the dilaton. We will see that this possible inconsistency vanishes for the supermembrane case.)

There is also a self-consistency between the equations of motion for the metric and the anti-symmetric field which must be re-established at every loop order, and hence this provides a powerful check on the correctness of any model of membranes.

Higher order graphs are easy to construct but more tedious to evaluate. We will present the contractions necessary to perform two and three loop calculations, but will not explicitly compute the graphs.

For example, the R^2 and DR two loop terms are contained in the contraction of $L_{1,\alpha\beta\gamma\delta}^{(4)}$, so we have the two loop contribution:

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$$\frac{1}{24\alpha}\sqrt{\gamma}\gamma^{ij}\partial_i\phi^\mu\partial_j\phi^\sigma\left[R_{\mu\alpha\beta\sigma;\gamma\delta}+R^\lambda_{\alpha\beta\mu}R_{\lambda\gamma\delta\sigma}\right]\langle T\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta\rangle \qquad (74)$$

Two loop curvature terms are also contained in the contraction of the square of $L^{(3)}_{1\alpha\beta\gamma}$. This contraction yields:

$$\begin{aligned}
\langle L^{(3)}_1 L^{(3)}_1\rangle &=& \left(\frac{2}{3\alpha}\right)^2\sqrt{\gamma}\gamma^{ij}R_{\mu\beta\gamma\alpha}\partial_j\phi^\mu\langle TD_i\xi^\alpha(x)\xi^\beta(x)\xi^\gamma(x) \\
&& \times\; D_{\bar{i}}\xi^{\bar\alpha}(y)\xi^{\bar\beta}(y)\xi^{\bar\gamma}(y)\rangle\sqrt{\gamma}\gamma^{\overline{ij}}R_{\bar\mu\bar\beta\bar\gamma\bar\alpha}\partial_{\bar{j}}\phi^{\bar\mu}D_{\bar{i}}\xi^{\bar\alpha}\xi^{\bar\beta}\xi^{\bar\gamma}+... \quad (75)
\end{aligned}$$

These two terms give us two loop correction terms to the curvature tensor, yielding complicated combinations of R^2 and DR terms.

Lastly, we can also calculate the two and three loop contribution for the anti-symmetric field. For example, two loop corrections to the equations of motion are given by contracting $L^{(4)}_{2\alpha\beta\gamma\delta}$ with two propagators. This term is contained within:

$$\begin{aligned}
\langle L^{(4)}_2\rangle &=& \beta\epsilon^{ijk}\left\{\frac{1}{4!}D_\delta D_\gamma D_\alpha F_{\beta\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda \right. \\
&& \left. +\; \frac{9}{4!}R^\mu_{\alpha\beta\sigma}D_\gamma F_{\delta\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda\right\}\langle T\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta\rangle+... \qquad (76)
\end{aligned}$$

This gives us terms like RF and $DDDF$.

Similarly, we can also contract over the square of $L^{ij(4)}_{2\alpha\beta\gamma\delta}$, which will give us a $FDDF$ term. It is contained within:

$$\begin{aligned}
\langle (L^{(4)}_{2\alpha\beta\gamma})^2\rangle &=& \left(\frac{18\beta}{4!}\right)^2\epsilon^{ijk}D_\delta F_{\gamma\alpha\beta\lambda}\partial_k\phi^\lambda\langle TD_i\xi^\alpha(x)D_j\xi^\beta(x)\xi^\gamma(x)\xi^\delta(x) \\
&& \times\; D_{\bar{i}}\xi^{\bar\alpha}(y)D_{\bar{j}}\xi^{\bar\beta}(y)\xi^{\bar\gamma}(y)\xi^{\bar\delta}(y)\rangle\epsilon^{\overline{ijk}}D_{\bar\delta}F_{\bar\gamma\bar\alpha\bar\beta\bar\lambda}\partial_{\bar{k}}\phi^{\bar\lambda}+... \qquad (77)
\end{aligned}$$

5 Power Counting

Now let us analyze the divergence of graphs to all orders in perturbation theory. Because the coupling constant has negative dimension, we can always increase

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the degree of divergence of any multi-loop graph by adding more insertions. In this sense, the theory is not renormalizable. But we will see in this section how many divergences we can absorb via the coupling constant α and β.

Consider first the Lagrangian L_1 with only the metric tensor, without the anti-symmetric field. Let L be the number of loops in an arbitrarily complicated Feynman graph. Then its contribution to the over-all divergence is $3L$, due to d^3p. Let I be the number of internal lines in the graph. So its contribution is $-2L$ due to $1/p^2$. Let V_n be the number of n-point vertices in the graph. Since each n-point graph in the action has two momenta associated with it, it can contribute at most $2V_n$. Let E equal the number of external lines in the graph. Since each external line subtracts off a line which could have become an internal line, it contributes $-N$. Then the superficial divergence of any graph D is given by:

$$D = 3L - 2I + 2\sum_{n=3}^{\infty} V_n - N \tag{78}$$

Now calculate the number of momentum integrations. Each internal line contributed d^3p. Each n-point vertex contributes a momentum-conserving delta function $\delta^3(\sum p_i)$, which deletes three momentum integrations per vertex. And then there is one over-all conservation of momentum factor. The sum of these integrations, in turn, contributes an over-all $(d^3p_i)^L$ momentum integration for the loops. Thus, we have:

$$L = I - \sum_{n=3}^{\infty} V_n + 1 \tag{79}$$

Now insert the second equation into the first, and we obtain:

$$D = L - N + 2 \tag{80}$$

Notice that the degree of divergence D is just a function of the number of loops L and the number of external lines E.

Now let us see if we can re-absorb this divergence into the coupling constant α. Let Λ be the momentum cut-off for the graph. Recall that the perturbation expansion parameter is α. Then the leading divergences of the N-point amplitude A_N, symbolically speaking, diverge as:

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$$A_N = \sum_{L=1}^{\infty} \alpha^{L-1} A_{N,L} \tag{81}$$

where we only compute the loop corrections.

We have just shown that $A_{N,L}$ diverges as:

$$A_{N,L} \sim \Lambda^{L-N+2} \tilde{A}_{N,L} \tag{82}$$

Now let us re-define the coupling constant as:

$$\alpha \sim \frac{\alpha_R}{\Lambda} \tag{83}$$

which we performed in the last section for the single loop.

Rescaling the graph, we now have:

$$A_N = \Lambda^{3-N} \left(\sum_{L=1}^{\infty} \alpha^{L-1} \tilde{A}_{N,L} \right) \tag{84}$$

Thus, the leading superficial divergence can be absorbed into α by a rescaling. The larger N, the faster the graph converges. In particular, we see that the amplitude is formally finite for $N = 3$ and beyond, but diverges still for $N = 2$. We can eliminate the $N = 2$ divergence by simply declaring that the background fields obey the standard equations of motion, thereby defining the model.

Now let us generalize the simple power counting to the general case, including the anti-symmetric field. The power counting is much worse, since we now have 3 momenta attached to each vertex function, rather than 2. The leading divergences all come from this sector.

So the degree of divergence is now given by:

$$D = 3L - 2I + 3 \sum_{n=3}^{\infty} V_n + X_N \tag{85}$$

(If some of the lines on the vertex are external lines, this reduces the degree of divergence of the graph, so we have to compensate this by adding in X_N. For example, $X_2 = -2, X_3 = -3$.)

The number of momentum integrations is given by:

$$L = I - \sum_{n=3}^{\infty} V_n + 1 \qquad (86)$$

Notice that we can no longer cancel both I and $\sum V_n$ to arrive at a simple relationship involving just L and N. Thus, we need one more constraint to eliminate the vertex factors.

Let us count the number of lines in a graph. Each of the V_n vertices contributes n lines to the graph. Thus, they collectively contribute $\sum nV_n$ lines to the graph. When two of these vertices V_n are joined, they form an internal line, which is therefore counted twice. This means that the sum $\sum nV_n$ counts each internal line twice, and each external line once (since external lines are not paired off). Thus, we have:

$$\sum_{n=3}^{\infty} nV_n = 2I + N \qquad (87)$$

By examining these three sets of equations, we see that, in general, it is not possible to eliminate all the V_n in a graph. Therefore, we will only concentrate on the leading divergence within a graph and ignore lower order divergences.

Let us see which vertices contribute the most to the over-all divergence. A vertex V_n contains 3 momenta. Let us say that we replace it with two small vertices V_{n_1} and V_{n+2} which are joined by an internal line. The over-all contribution from these two attached vertices is given by $3 + 3 - 2 = 4$, where the -2 comes from $1/p^2$. Thus, we can always increase the over-all divergence of a graph by replacing V_n with pairs of smaller n-point vertices. This process can be continued, until we are left with a graph with only V_4 and V_5 vertices left. Thus, the leading divergence is now given with only V_4 and V_5. If we eliminate V_4, we are left with:

$$D = 2L - \frac{1}{2}V_5 + \frac{N}{2} + X_N + 1 \qquad (88)$$

(Notice that this equation depends on whether the overall number of vertex lines is even or odd. If it is even, then $V_5 = 0$.)

In this way, we can compute the over-all divergence of a graph. However, there is simple short-cut we can use. If we examine the perturbation expansion

of L_1 and L_2, we see that the primary difference is that the internal vertices of L_1 contain two derivatives, while the internal vertices of L_2 contain three derivatives. Because the coupling constant $1/\alpha$ appears in front of each term in L_1, we see that each internal vertex function diverges, at most, like Λ^3. But if we let β remain a finite constant, we see that each internal vertex in L_2 diverges as Λ^3 as well. Thus, by only rescaling α but keeping β finite, we see that the divergence of the purely metric theory is identical to the theory coupled to anti-symmetric tensor fields. (β, although it is finite, will ultimately be fixed by requiring consistency in the equations of motion of the background fields).

6 Conclusion

In field theory, the study of non-renormalizable Lagrangians, such as the four-fermion model, or massive vector theories, has given us insight in deep physical processes. Likewise, the bosonic membrane action, by naive power counting, is non-renormalizable on the world volume, but may give us insight into M-theory. Although the bosonic membrane theory is ultimately probably not a consistent quantum theory, the techniques we have used here will generalize to the supermembrane case.

In this paper, we have expanded the bosonic membrane action around Riemann normal co-ordinates, treating the theory as a non-linear sigma model, and calculated the regularized propagator and higher loop graphs.

In particular, we found:

a) The standard dimensional regularization method apparently breaks down at $d = 3$, where the Gamma function no longer has a pole. Instead, we developed the proper time and point-splitting formalism in curved space for the $d = 3$ membrane action. Although we lost general covariance, this gave us an intuitive way in which to isolate all the divergences of higher graphs, since the singularities emerge when two fields touch on the world volume. It is then a simple matter to analyze complicated graphs visually and isolate their divergences.

b) The renormalization program in four dimensions for the non-linear sigma model is ruined by the presence of terms like $\mathrm{Tr}\,[R_{ij}R^{ij}]$. However, we have

shown that these terms are not a problem in three dimensions.

c) We found that the single loop graph was insufficient to generate self-consistent equations of motion for the background fields. This was surprising, since setting $\beta = 0$ in the usual string formalism yields self-consistent equations of motions at the first loop level.

d) Since the formalism we have developed works for arbitrary loop level, we can calculate higher order corrections to the equations of motion. We find, at two loop level, new terms which have the form: R^2, DR, $FDDF$, FRF, etc.

e) We found that, by naive counting arguments on arbitrary loop graphs, we could absorb the leading divergences into a rescaling of the coupling constant α. By setting $\alpha \to \alpha_R/\Lambda$, where Λ is a large momentum cut-off parameter, we could absorb the leading divergences. The amplitudes then diverge as Λ^{3-N}, so the leading divergences actually vanish if $N = 3, 4, \ldots$ For the case $N = 2$, we set the divergence to zero, thereby yielding the equations of motions. We found that the counting of divergences remains the same if the coupling constant β for the anti-symmetric fields is finite. This doesn't mean that the action is renormalizable, of course, since we still have to analyze non-leading graphs and many other subtle problems.

One weakness of this formalism is that the equations of motion emerge only after setting one divergences to zero by hand. Hence, we have to define the theory by placing in the background fields on-shell.

In the superstring case, conformal symmetry allows us to set $\beta = 0$. However, the entire motivation of our approach is to analyze the D = 11 supermembrane, where supersymmetry is sufficient to set these lower order divergences to zero. Our ultimate goal, therefore, is to see whether supersymmetry is strong enough to control the divergences found in the supermembrane theory, and whether we can obtain the the M-theory action by expanding around higher order corrections to the standard D=11 supergravity action, and then use recursion relations to probe the entire action. The supersymmetric case will be discussed in a forthcoming paper.

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