A NOTE ON THE STABILITY OF QUANTUM SUPERMEMBRANES

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We re-examine the question of the stability of quantum supermembranes. In the past, the instability of supermembranes was established by using a regulator, i.e. approximating the membrane by $SU(N)$ super Yang-Mills theory and letting $N \rightarrow \infty$. In this paper, we (a) show that the instability persists even if we directly examine the continuum theory, which then opens the door to other types of regularizations. (b) give heuristic arguments that even a theory of unstable membranes at the Planck length may still be compatible with experiment. (c) resolve a certain puzzling discrepancy between earlier works on the stability of supermembranes.

1 Quantum Supermembranes

To analyze the quantum stability of supermembranes, we start with the Hamiltonian in the light cone gauge for the supermembrane[1,2]:

$$H = \int d^2\sigma \left[\frac{1}{2}(P^I)^2 + \frac{1}{4}(\{X^I, X^J\})^2 - \frac{i}{2}\bar{\theta}\Gamma^I\{X^I, \theta\} \right] \quad (1)$$

where $I = 1, 2, ..., 9$ and $\{A, B\} = \partial_1 A\, \partial_2 B - (1 \leftrightarrow 2)$. To demonstrate the instability of this Hamiltonian without going to a $SU(N)$ super Yang-Mills theory[3], we let $f(\sigma_1, \sigma_2)$ represent a function of the membrane world-variables and consider $X_\mu(f)$, which describes a string-like configuration. The potential function in the Hamiltonian vanishes if we substitute this string-like configuration. In other words, along these string-like configurations, the wave function can "leak" out to infinity, and hence the system is unstable.

We first choose variables. Let $a = 1, 2, ...8$, and $I = 1, 2, ...9$. Then choose co-ordinates $X^I = (x_9, Y_9, x_a, Y_a)$. where we have split off the string co-ordinate by setting $x = x(f)$, while Y cannot be written as a string. We can fix the gauge by choosing $Y_9 = 0$.

Then the Hamiltonian can be split up into several pieces[4]:
$H = H_1 + H_2 + H_3 + H_4$ where:

$$H_1 = -\frac{1}{2}\int d^2\sigma \left[\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial x_a}\right)^2 \right]$$

$$H_2 = -\frac{1}{2}\int d^2\sigma \left(\frac{\partial}{\partial Y_a}\right)^2 + \frac{1}{2}\int d^2\sigma d^2\bar{\sigma}d^2\sigma' \left[Y_a(\bar{\sigma})z^T(\bar{\sigma},\sigma')z(\sigma',\sigma)Y_a(\sigma)\right]$$

$$H_3 = -\frac{i}{2}\int d^2\sigma d^2\bar{\sigma}\,\bar{\theta}(\bar{\sigma})\left[z(\bar{\sigma},\sigma)\gamma_9 + z_a(\bar{\sigma},\sigma)\gamma_a\right]\theta(\sigma)$$

$$z(\bar{\sigma},\sigma) = \delta^2(\bar{\sigma},\sigma)\partial_{\sigma_1}x\partial_{\bar{\sigma}_2} - (1 \leftrightarrow 2)$$

$$z_a(\bar{\sigma},\sigma) = \delta^2(\bar{\sigma},\sigma)\partial_{\sigma_1}x_a\partial_{\bar{\sigma}_2} - (1 \leftrightarrow 2) \qquad (2)$$

and the index σ is shorthand for $\{\sigma_1,\sigma_2\}$. Notice that $z(\bar{\sigma},\sigma)$ is an anti-symmetric function. H_4 contains other terms in Y, which will vanish at the end of the calculation.

Now consider the term H_2. The eigenfunction for H_2 is:

$$\Phi_0 = A(\det\Omega)^2 \exp\left(-\frac{1}{2}\int d^2\bar{\sigma}d^2\sigma\, Y_a(\bar{\sigma})\Omega(\bar{\sigma},\sigma)Y_a(\sigma)\right)$$

$$H_2\Phi_0 = 4\int d^2\sigma \int d^2\sigma'\, z^T(\bar{\sigma},\sigma')z(\sigma',\sigma)\,\Phi_0 \qquad (3)$$

Let us introduce eigenvectors E^σ_{MN}, where $M \neq N$, as follows:

$$\int d^2\sigma\, z(\bar{\sigma},\sigma)\, E^\sigma_{MN} = i\lambda_{MN}E^\sigma_{MN} \qquad (4)$$

where M, N label a complete set of orthonormal functions, which can be either continuous or discrete, and λ_{MN} are the anti-symmetric eigenvalues of z, so the eigenvalue of H_2 becomes: $\sum_{M,N}|\lambda_{MN}|$.

Now change fermionic variables to:

$$\theta(\sigma) = \sum_{M\neq N}\theta^{MN}E^\sigma_{MN} \qquad (5)$$

The original fermionic variables are real, so $\theta^{MN\dagger} = \theta^{NM}$.

Then H_3 reduces to:

$$H_3 = \sum_{M<N}\theta^{MN\dagger}\left(\lambda_{MN}\gamma_9 + \lambda^a_{MN}\gamma_a\right)\theta^{MN} \qquad (6)$$

After a bit of work, we find that:

$$H_3\xi = -8\sum_{M<N}\omega_{MN}\xi \qquad (7)$$

where: $\omega_{MN} = \{(\lambda^2_{MN} + (\lambda^a_{MN})^2\}^{1/2}$. To find total energy, we now sum the contributions of H_2 and H_3:

$$(H_2 + H_3)\Psi = 8 \sum_{M<N} \left(|\lambda_{MN}| - \omega_{MN} \right) \Psi \tag{8}$$

Then we see that ω_{MN} asymptotically approaches $|\lambda_{MN}|$ in this limit, so that the ground state energy of H_2 and H_3 vanishes. In conclusion[4], we see that the principal contributions to the ground state energy comes from H_2 and H_3, which in turn cancel if we are far from the membrane. Furthermore, we see that the energy eigenvalue of the operator is continuous for the ground state, which means that the system is unstable.

Although the system is unstable, we speculate how this may still be compatible with known phenomenology. We note that because the decay time of such a quantum membrane is on the order of the Planck time, it is possible that unstable membranes decay too rapidly to be detected by our instruments. Consider the standard decay of the quark bound state:

$$\Gamma = \frac{16\pi\alpha^2}{3} \frac{|\psi(0)|^2}{M^2} \tag{9}$$

where $\phi(0)$ is the wave function of the bound state at the origin, and M is the mass of the decay product.

We expect that $|\psi(0)|$ to be on the order of a fermi^{-3}, the rough size of the quark-anti-quark bound state. For our purposes, we assume that the membrane is on the order of the Planck length. On dimensional grounds, we therefore expect that the lifetime of the membrane to be on the order of $T \sim M^2 L^{-3}$ where L is the Planck length.

For relatively light-weight membranes, we find that the lifetime is much smaller than Planck times, so we will, as expected, never see these particles.

For very massive membranes, we find that the lifetime becomes arbitrarily long, which seems to violate experiment. However, the coupling of very massive membranes, much heavier than the Planck mass, is very small, and hence they barely couple to the particles we see in nature. Again, we see that unstable membranes cannot be measured in the laboratory.

2 Resolving a Discrepancy in the Stability of Membranes

In this section, we try to resolve a certain discrepancy with regard to the instability of supermembranes. In ref. 5, it was pointed out that it is possible to choose a gauge where the Hamiltonian becomes quadratic. Thus, it appears that the non-linearity of supermembranes, and hence their instability, is an illusion.

To resolve this puzzle, go to light cone co-ordinates: $\{X^+, X^-, X_{d-2}, X_i\}$, where $i = 1, 2, 3, ...d - 3$. Our goal is to re-write everything in terms of X_i.

Choose the gauge $P^+ = p^+$; $X^+ = p^+ \tau$. We still have one more gauge degree of freedom left. We choose (Δ is the volume term):

$$P_{d-2}^2 + \Delta = \partial_a X_i \partial_a X_i \tag{10}$$

This determines P_{d-2} such that the left-hand side is quartic, but the right hand side is quadratic.

Now let us solve the constraints. The $P^2 + \Delta$ constraint can be solved for P^-, which now becomes the new light cone Hamiltonian. We find:

$$P^- = \frac{1}{2p^+} \left(P_i^2 + w^{ab} \partial_a X_i \partial_b X_i \right) \tag{11}$$

Notice that the light cone Hamiltonian has now become quadratic! And lastly, X_{d-2} is fixed by the other constraints $\partial_a X_\mu P^\mu = 0$.

At this point, we now have a Hamiltonian P^- defined entirely in terms of quadratic functions, defined in terms of the canonical variables X_i, P_i. The action seems to be stable. But this contradicts all our previous results.

There is, however, a loophole to this proof. Notice that the $\dot{X}_\mu P^\mu$ term in the Lagrangian $L(X, P)$ decomposes as $\dot{X}_\mu P^\mu = p^+ P^- - \dot{X}_{d-2} P_{d-2} - \dot{X}_i P_i$.

The key point in all of this is that $\dot{X}_{d-2} P_{d-2}$ does not vanish. If we take the τ derivative of X_{d-2}, we find that this term contains non-linear functions of $\dot{X}_i$, and hence changes the canonical commutation relations between the transverse variables. Thus, the non-linearity of the theory creeps back in and is now hidden in the commutation relations. This resolves the apparent discrepancy.

References

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