Warm-starting Contextual Bandits:
Robustly Combining Supervised and Bandit Feedback

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Abstract

We investigate the feasibility of learning from a mix of both fully-labeled supervised data and contextual bandit data. We specifically consider settings in which the underlying learning signal may be different between these two data sources. Theoretically, we state and prove no-regret algorithms for learning that is robust to misaligned cost distributions between the two sources. Empirically, we evaluate some of these algorithms on a large selection of datasets, showing that our approach is both feasible, and helpful in practice.

1. Introduction

In many real-world settings, a system must learn from multiple types of feedback; we consider the specific setting of learning jointly from fully labeled "supervised" examples and from online feedback "contextual bandit" (abbrev. CB) examples. For instance, in a system that chooses personalized content to display on a webpage, an expert may be able to provide an initial set of fully labeled examples to get a system started. After deployment, however, the system can only measure its performance (e.g., dwell time) on the content it displays and not other (counterfactual) options. In an automated translation system, professional translators can provide initial translations to seed a system, but the system may be able to further improve its performance based on, e.g., user satisfaction measures (Sokolov et al., 2015; Nguyen et al., 2017).

In both these settings (content display and translation), we desire an approach that is able to use the fully supervised expert data to "warm-start" a system, which later learns from CB feedback (Auer et al., 2002b; Langford & Zhang, 2007; Chu et al., 2011; Dudik et al., 2011; Agrawal & Goyal, 2013; Agarwal et al., 2014). Doing so has the added advantage of

ensuring that such a system does not need to suffer too much error in an initial exploration phase, which may be necessary in user-facing systems or in error- or safety-critical settings (Tewari & Murphy, 2017). However, it is generally unreasonable to assume that the expert supervision and the CB feedback in such settings are perfectly aligned: the "best" decision according to an expert may not necessarily match a user's choice. We need algorithms that operate well even in the case of unknown degrees of misalignment; we introduce a hypothesis class-specific notion of cost similarity used in our analysis, but not our algorithms (§2). We also highlight how simple strategies for combining the two sources without robustness to misalignment can perform significantly worse than learning from the ground truth source alone (§2.1).

Furthermore, different applications can differ in terms of which source—supervised or CB—is considered "ground truth". For example, while the CB feedback from users is the better signal about their preferences in content personalization (§3), the expert translations provide the ground truth in the translation setting for which user satisfaction is an imperfect proxy (§4). We develop algorithms for *both* settings, which effectively "search" for a good balance between fitting the CB feedback and supervised labels. In both cases, we provide regret bounds showing the value of the complementary data sources, dependent on their cost discrepancy and respective sample sizes. Importantly, our theory shows that our methods perform close to an oracle that knows the similarity of the two sources beforehand and uses it to optimally weight their examples, with a small additional penalty from searching for this weighting.

Empirically, we perform experiments based on fully-labeled examples from which CB feedback is simulated. We focus on the setting when CB data is ground truth and the supervised warm-start might have differing levels of bias. In an experimental study over hundreds of datasets (§5) we demonstrate the efficacy of our algorithm. As a snapshot, Figure 1 shows the empirical cumulative distribution functions (CDF) of algorithms across a number of experimental conditions, where each (x, y) value on the curve indicates that there is a y fraction of experimental conditions where the normalized error[1] of a method is below x. The plot ag-

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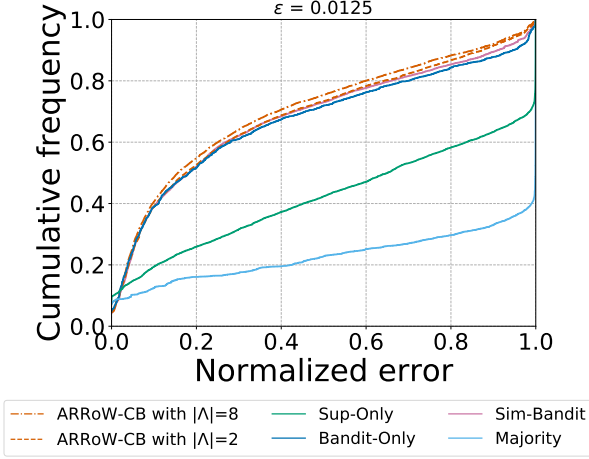[1]See §5 for a formal definition of normalized error.

Figure 1: Empirical CDFs of the performance of different methods across a number of datasets and experimental conditions (See §5 for descriptions of all algorithms, settings, and aggregation method). Our method ARRoW-CB has a parameter $|\Lambda|$, and we evaluate it with $|\Lambda|$ set to 8 and 2; SIM-BANDIT is a baseline also leveraging the warm-start; SUP-ONLY, MAJORITY[1] and BANDIT-ONLY learn using only the supervised and CB sources respectively. All CB methods use the ϵ-greedy strategy with $\epsilon = 0.0125$.

gregates across settings where the CB and supervised signals are perfectly aligned as well as where they are not. Overall, our main algorithm, namely ARRoW-CB with $|\Lambda| = 8$, outperforms all baselines in this aggregated summary, in particular beating the two algorithms (ARRoW-CB with $|\Lambda| = 2$ and SIM-BANDIT) that leverage both CB and supervised sources. More detailed results are presented in §5.

Relation to prior work. A theoretical study of domain adaptation (Ben-David et al., 2010; Mansour et al., 2009) and learning from multiple sources (Crammer et al., 2008) are the closest prior works. In these works, all data sources provide the same supervised feedback rather than the supervised/CB modality we investigate here, with the two sources having very different information per sample. Another related line of work is on "safe" CB learning (Kazerouni et al., 2017; Sun et al., 2017) to maintain performance better than a baseline policy at all times, somewhat related to our supervised ground truth setting. However, they do not study the distributional mismatch concerns central to our work.

Finally, there is a substantial literature on active learning from different data sources (Donmez & Carbonell, 2008; Urner et al., 2012; Yan et al., 2011; Malago et al., 2014; Zhang & Chaudhuri, 2015; Yan et al., 2018), combining multiple labeling oracles of varying quality. The CB setting

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[1] SUP-ONLY and MAJORITY do not explore or update on CB examples and we plot the average costs of their policies over all CB examples.

studied has important differences from active learning and the techniques do not carry over directly.

2. Notation and Problem Specification

We begin with some notation. For an event A, $I(A) = 1$ if A is true, and 0 otherwise. Denote by $[K]$ the set $\{1, 2, \ldots, K\}$. We use $\mathbb{1}_K$ to denote the all 1's vector in $\mathbb{R}^K$ and Δ^{K-1} for the K dimensional probability simplex.

In this paper, we study the problem of cost-sensitive interactive learning from multiple data sources. Specifically, we consider distributions over cost-sensitive examples (x, c), where $x \in \mathcal{X}$ is a context and c is a cost vector in $[0, 1]^K$; K being the number of actions (or "classes"). There are two distributions D^s (supervised) and D^b (CB), which have identical marginals over the context x, but different conditional distributions over cost vectors given x. We use the notation c^b (resp. c^s) to denote the cost vector c drawn from D^b (resp. D^s) to avoid writing D^b and D^s as subscripts in expectations. The interaction between the learner and the environment is described as follows:

Warm-start: The learner receives S, a dataset of n^s fully supervised examples drawn i.i.d. from D^s.

Interaction: For $t = 1, 2, \ldots, n^b$, the environment draws $(x_t, c_t^b) \sim D^b$ and reveals x_t to the learner, based on which the learner chooses a (possibly random) action $a_t \in [K]$ and observes $c_t^b(a_t)$, but not the cost of any other action.

In this paper, we focus on two learning settings: *CB ground truth setting* and *supervised ground truth setting*. In the CB ground truth setting (resp. supervised ground truth setting), the goal of the learner is to optimize the costs drawn from distribution D^b (resp. D^s).

To help make decisions, the learner is given a finite policy class Π that contains policies $\pi : \mathcal{X} \to [K]$[3]. The performance of the algorithm is measured by its *regret* to the retrospective-best policy in Π. We consider two notions of regret over the sequence $\langle x_t \rangle_{t=1}^{n^b}$, based on whether we consider the CB costs (c^b) or the supervised costs (c^s) as the ground truth:

$$\text{CB: } \mathbf{R}^b(\langle x_t, a_t \rangle_{t=1}^{n^b}) = \sum_{t=1}^{n^b} \mathbb{E}\big[c^b(a_t) \mid x_t\big] -$$
$$\min_{\pi \in \Pi} \sum_{t=1}^{n^b} \mathbb{E}\big[c^b(\pi(x_t)) \mid x_t\big], \quad (1)$$

$$\text{supervised: } \mathbf{R}^s(\langle x_t, a_t \rangle_{t=1}^{n^b}) = \sum_{t=1}^{n^b} \mathbb{E}\big[c^s(a_t) \mid x_t\big] -$$
$$\min_{\pi \in \Pi} \sum_{t=1}^{n^b} \mathbb{E}\big[c^s(\pi(x_t)) \mid x_t\big]. \quad (2)$$

In the content recommendation example (CB ground truth), x_t encodes a user profile and the system predicts which articles (a_t) to display. Here, c^b can be the negative dwell-

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[3] More generally, $\pi : \mathcal{X} \to \Delta^{K-1}$ and $\pi(x)$ is the distribution over actions given x.

time of users and c^s is the annotation of editors, which can have disagreements with c^b. The learner aims to optimize the dwell time over all displayed articles.

In the translation example (supervised ground truth), x_t encodes the text to be translated and a_t encodes its translation. Here, the learner aims to minimize errors against the expert translation (c^s) on x_t's, *despite the fact that the* system never sees these costs in its interaction phase. Note that the learner only observes the user feedback costs (c^b) in this interaction phase, which are imperfect proxies for c^s, and the only direct observations of c^s are on the warm-start examples. Nevertheless, we seek to optimize the accuracy of our translations given to the users, and hence regret is still measured over the interaction phase.

The utility of non ground truth examples are different in the two learning settings. In the CB ground truth setting, relying on the CB examples alone is sufficient to ensure vanishing regret asymptotically. The supervised warm-start primarily helps with a smaller regret in the initial phases of learning. On the other hand, in the supervised ground truth setting, the CB examples can have an asymptotically meaningful effect on the regret: for instance, if $D^s = D^b$, then utilizing CB examples can lead to a vanishing regret, whereas using supervised examples alone cannot.

We can leverage examples from a different source only when the cost structures are at least somewhat related. Therefore, we introduce a measure of similarity of two distributions over cost-sensitive examples.

Definition 1. *D_2 is said to be (α, Δ)-similar to D_1 with respect to Π, if for any policy π, $\mathbb{E}_{D_2} c(\pi(x)) - \mathbb{E}_{D_2} c(\pi^*(x)) \geq \alpha \left(\mathbb{E}_{D_1} c(\pi(x)) - \mathbb{E}_{D_1} c(\pi^*(x)) \right) - \Delta$, where $\pi^* = \operatorname{argmin}_{\pi \in \Pi} \mathbb{E}_{D_1} c(\pi(x))$.*

If we have a larger α and smaller Δ, examples from D_2 are more useful for learning under D_1. Prior similarity notions, such as in Ben-David et al. (2010), roughly assume a bound on $\max_{\pi \in \Pi} |\mathbb{E}_{D_1}[c(\pi(x))] - \mathbb{E}_{D_2}[c(\pi(x))]|$. The one-sided bound in our definition (instead of absolute value bound) on regret and an additional scaling factor α yield additional flexibility. Note that Definition 1 is only used in our analysis; our algorithms do not require knowledge of α and Δ. We give a more general condition which implies (α, Δ)-similarity, along with several examples in Appendix B.

Finally, we define some additional notation. In the t-th interaction round, our algorithms compute $\hat{c}_t$, an estimate of the unobserved vector c_t^b. We use $\mathbb{E}_S$ to denote sample averages on S and abbreviate $\mathbb{E}_{S_t}$ by $\mathbb{E}_t$ where $S_t = \{(x_\tau, \hat{c}_\tau)\}_{\tau=1}^t$ is the log of the CB examples up to time t.

2.1. Failure of Simple Strategies

The settings we have described so far might appear deceptively simple. It should be easy to include some additional supervised examples, which contain more feedback, into a CB algorithm. We now illustrate the difficulty of this task when the two distributions D^s and D^b are misaligned.

Consider the special case of 2-armed bandits (CB with a dummy context), where the CB source is the ground truth. D^s and D^b are deterministic with costs $(0.5, 0.5 + \frac{\Delta}{2})$ and $(0.5, 0.5 - \frac{\Delta}{2})$ for the two arms respectively, so that they are $(1, \Delta)$-similar. Suppose we see $n^s = \Omega(1/\Delta^3)$ examples in warmstart, and use them to initialize the means and confidence intervals on each arm to run the UCB algorithm (Auer et al., 2002a). Proposition 1 in Appendix C show that the optimal arm according to D^b, which is arm 2, is never played for the first $O(\exp(1/\Delta))$ rounds, incurring regret $\Omega(\Delta \exp(1/\Delta))$. So for any $\Delta < 0.5$, the regret is strictly larger than that of a UCB algorithm which ignores the warm-start and incurs at most $\tilde{O}(1/\Delta)$ regret. On the other hand, if $D^b = D^s$, then the UCB strategy described above incurs no regret.

What we observe here is a failure in competing simultaneously with two baselines: naively warmstarting by weighting examples from the two sources equally, or just ignoring the supervised source entirely. We will next describe an algorithm to compete not just with these two, but many possible weightings of the two sources. This extreme failure case shows that an arbitrary low-regret CB algorithm cannot handle biased warm-start data without extra care. Using additional randomization can help, but is not adequate by itself as we will see in our theory and experiments.

3. Contextual Bandit Ground Truth Setting

In this section, we study the setting where D^b, the distribution over CB examples, is considered the ground truth, as in the content recommendation example. Recall that in this setting, one could ignore the supervised warm-start examples and still achieve vanishing regret; the main goal here is to show that using the warm start data can help further reduce the regret, especially in early stages of learning.

3.1. Algorithm

Intuition of our approach. The key challenge in designing an algorithm for the CB ground truth setting is understanding how to effectively combine two data sources which might have unknown differences in their distributions. For the simpler supervised learning setting, Proposition 4 in Appendix I shows that it suffices to always use one of the two sources depending on the bias and relative number of examples. This has two caveats though: the bias is not known in practice, and completely ignoring one data source is ob-

Algorithm 1 Adaptive Reweighting for Robustly Warm-starting Contextual Bandits (ARRoW-CB)

Require: Supervised dataset S from D^s of size n^s, number of interaction rounds n^b, exploration probability ϵ, weighted combination parameters Λ, policy class Π.

1: **for** $t = 1, 2, \ldots, n^b$ **do**
2:    Observe instance x_t from D^b.
3:    Define $p_t := \frac{1-\epsilon}{t-1} \sum_{\tau=1}^{t-1} \pi_\tau^{\lambda_t}(x_t) + \frac{\epsilon}{K} \mathbb{1}_K$ for $t \geq 2$ and $p_t := \frac{1}{K} \mathbb{1}_K$ for $t = 1$.
4:    Predict $a_t \sim p_t$, and receive feedback $c_t^b(a_t)$.
5:    Define the inverse propensity score (IPS) cost vector $\hat{c}_t(a) := \frac{c_t^b(a_t)}{p_{t,a_t}} I(a = a_t)$, for $a \in [K]$.
6:    For every $\lambda \in \Lambda$, train π_t^λ by minimizing over $\pi \in \Pi$:

$$\lambda \sum_{\tau=1}^{t-1} \hat{c}_\tau(\pi(x_\tau)) + (1-\lambda) \sum_{(x,c^s) \in S} c^s(\pi(x)). \quad (3)$$

7:    Set $\lambda_{t+1} \leftarrow \operatorname{argmin}_{\lambda \in \Lambda} \sum_{\tau=1}^{t} \hat{c}_\tau(\pi_\tau^\lambda(x_\tau))$.
8: **end for**

viously wasteful when the two sources are identical. We choose to instead consider cost minimization on a dataset where the two sources are combined with different weights, and seek to learn these weights adaptively.

With these insights, we return to the actual problem setting of warm-starting a CB learner with supervised examples. Our algorithm for this setting is presented in Algorithm 1. The main idea is to minimize the empirical risk on a weighted dataset containing examples from the two sources. Our algorithm picks the mixture weighting by online model selection over a set of weighting parameters Λ, where we use the ground truth CB data at each time step to evaluate which $\lambda \in \Lambda$ has the best performance so far. For each $\lambda \in \Lambda$, we estimate a $\pi^\lambda \in \Pi$ as the empirical risk minimizer (ERM) for the λ-mixture between CB and supervised examples. We focus on the simplest ϵ-greedy algorithm for CBs, leaving similar modifications in more advanced CB algorithms for future work.

So long as $\{0, 1\} \subseteq \Lambda$, Algorithm 1 allows for relying on one source alone, while using a larger set of Λ significantly improves its empirical performance (see §5).[4]

We need some additional notation to present our regret bound. We define $V_t(\lambda)$ that governs the deviation of λ-weighted empirical costs for all policies in Π as

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[4]If we approximate the computation of the best policy in Step 6 using an online oracle as in prior works (Agarwal et al., 2014; Langford & Zhang, 2007), then the entire algorithm can be implemented in a streaming fashion since 7 for selecting the best λ also uses an online estimate a la Blum et al. (1999) for each λ as opposed to a holdout estimate for the current policy π_t^λ.

$V_t(\lambda) := 2\sqrt{(\frac{\lambda^2 K t}{\epsilon} + (1-\lambda)^2 n^s) \ln \frac{8n^b|\Pi|}{\delta}} + (\frac{\lambda K}{\epsilon} + (1-\lambda)) \ln \frac{8n^b|\Pi|}{\delta}$, and G_t that bounds the excess cost of the ERM solution using weighted combination parameter λ as

$$G_t(\lambda, \alpha, \Delta) := \frac{(1-\lambda)n^s \Delta + 2V_t(\lambda)}{\lambda t + (1-\lambda)n^s \alpha}.$$

We prove the following theorem in Appendix E.

Theorem 1. *Suppose D^s is (α, Δ)-similar to D^b. Then for any $\delta < 1/e$, with probability $1 - \delta$, the average CB regret of Algorithm 1 can be bounded as:*

$$\frac{1}{n^b} \mathbf{R}^b(\langle x_t, a_t \rangle_{t=1}^{n^b}) \leq \epsilon + 3\sqrt{\frac{\ln \frac{8n^b|\Pi|}{\delta}}{n^b}} + 32\sqrt{\frac{K \ln \frac{8n^b|\Lambda|}{\delta}}{n^b \epsilon}} +$$
$$\min_{\lambda \in \Lambda} \frac{\ln(en^b)}{n^b} \sum_{t=1}^{n^b} G_t(\lambda, \alpha, \Delta) \quad (4)$$

The bound (4) consists of many intuitive terms. The first ϵ term comes from uniform exploration; the second term is from the deviation of costs under D^b. The next term is the average regret incurred in performing model selection for λ; in our experiments $|\Lambda| = 8$ so that it can be thought of as $\widetilde{O}(\sqrt{K/(n^b \epsilon)})$. The final term involving a minimum over λ's is effectively finding the weighted combination which minimizes a bias-variance tradeoff in combining the two sources. Here the bias is controlled by Δ and in place of variance we use $V_t(\lambda)$ for high-probability results. Contrasting with learning with CB examples alone, we replace a $\sqrt{\frac{K \ln(|\Pi|/\delta)}{n^b \epsilon}}$ term with the middle term independent of $\ln |\Pi|$ and the average of G_t's which can be much smaller in favorable cases as we discuss below.

Identical distributions: A very friendly setting has $D^s = D^b$, corresponding to $(1, 0)$-similarity. Since the theorem holds with a minimum over all λ's in the set Λ, we can pick specific values λ_0 of our choice. One choice of λ_0 motivated from prior work (Ben-David et al., 2010) is to pick it such that $\lambda_0/(1-\lambda_0) = \epsilon/K$ to equalize the variance of the two sources, meaning each supervised example is worth K/ϵ CB examples. This setting of $\lambda_0 = \frac{\epsilon}{K+\epsilon}$ yields

$$G_t(\lambda_0, 1, 0) = O\left(\sqrt{\frac{K \ln \frac{n^b|\Pi|}{\delta}}{\epsilon t + K n^s}} + \frac{K \ln \frac{n^b|\Pi|}{\delta}}{\epsilon t + K n^s}\right). \quad \text{That is,}$$

after t CB samples, the effective sample size is $n^s + K t/\epsilon$.

Comparison with no warmstart: Whenever $1 \in \Lambda$, the minimum over $\lambda \in \Lambda$ in Theorem 1 can be bounded by its value at $\lambda = 1$, which corresponds to ignoring the warmstart examples and using bandit examples alone. For this special case, we have the following corollary.

Corollary 1. *Under conditions of Theorem 1, suppose that $1 \in \Lambda$. Then for any $\delta < 1/e$, with probability $1 - \delta$,*

$$\frac{1}{n^b} \mathbf{R}^b(\langle x_t, a_t \rangle_{t=1}^{n^b}) \leq \epsilon + O\left(\sqrt{\frac{\ln \frac{n^b|\Pi||\Lambda|}{\delta}}{n^b \epsilon}}\right).$$

The corollary follows from using the value of $G_t(1, \alpha, \Delta)$ along with some algebra, and shows that the regret of ARROW-CB is never worse than using the bandit source alone, up to a term scaling as $\ln |\Lambda|$. In particular, the usual choice of $\epsilon = O((n^b)^{-2/3})$ implies a $O((n^b)^{2/3})$ regret bound. Since a small value of $|\Lambda|$ suffices in our experiments, this is a negligible cost for robustness to arbitrary bias in the warmstart examples. Similarly comparing to $\lambda = 0$ lets us obtain a comparison against using the warmstart alone up to a model selection penalty, when $0 \in \Lambda$. The minimization over a richer set of λ leaves room for further improvements as shown in the case of $D^s = D^b$ above (which used a different setting of λ_0). Further improvements are also possible in the algorithm by using different λ values after reach round, which is not captured in the theory here.

4. Supervised Ground Truth Setting

In §3, we developed an algorithm and proved regret bounds for combining supervised and CB feedback, in the case where the CB cost is considered the ground truth. In this section, we consider the reverse setting where the supervised source constitutes the ground truth, recalling the motivating example in an automated translation setting from the introduction. Here, we wish to leverage the CB examples for learning the best policy relative to the distribution D^s.

Note that this setting is qualitatively different, since we only have a fixed number n^s of ground-truth examples while the number of CB examples grows over time. If we assign relative weights to individual supervised and CB examples as in Algorithm 1, the CB examples will eventually outweigh the supervised ones for any $\lambda > 0$, which is not desirable when the supervised source is the ground truth. In Algorithm 2, we address this problem by first computing the average costs of every policy on the supervised and CB examples separately, and then choosing a policy that minimizes a weighted combination of these averages. As a consequence, the relative weight of each CB example diminishes as their number grows, with the overall bias incurred from the CB source staying bounded.

Another difference between Algorithm 2 and Algorithm 1 is that, as opposed to using the CB examples collected online, we use subsets of warm start examples to guide the selection of weighted combination parameter λ. To this end, we introduce an epoch structure in the algorithm. In particular, at each epoch e, λ_e and π_e^λ's are updated exactly once, where a separate validation set is used to pick λ. In addition, we play with uniform randomization around the most recent policy as opposed to a running average of all policies trained so far, an outcome of using a separate validation set (line 12 of Algorithm 2) instead of progressive validation (7 of Algorithm 1). Since the exploration policy at the next epoch depends on the previous validation set, we must use a "fresh"

Algorithm 2 Combining contextual bandit and supervised data when supervised source is the ground truth

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Require: Supervised dataset S from D^s of size n^s, number of interaction rounds n^b, exploration probability ϵ, weighted combination parameters Λ, policy set Π.
1: Let $E = \lceil \log n^b \rceil$ be the number of epochs.
2: Define $t_e = \min(2^e, n^b)$ for $e \geq 1$, and $t_0 = 0$.
3: Partition S to $E+1$ equally sized sets $S^{\text{tr}}, S_1^{\text{val}}, \ldots, S_E^{\text{val}}$.
4: **for** $e = 1, 2, \ldots, E$ **do**
5:     **for** $t = t_{e-1}+1, t_{e-1}+2, \ldots, t_e$ **do**
6:         Observe instance x_t from D^b.
7:         Define $p_t := (1-\epsilon)\pi_{e-1}^{\lambda_{e-1}}(x_t) + \frac{\epsilon}{K}\mathbb{1}_K$ for $e \geq 2$, and $p_t := \frac{1}{K}\mathbb{1}_K$ for $e = 1$.
8:         Predict $a_t \sim p_t$ and receive feedback $c_t^b(a_t)$.
9:         Define the IPS cost vector $\hat{c}_t(a) := \frac{c_t^b(a_t)}{p_{t,a_t}}I(a = a_t)$, for $a \in [K]$.
10:     **end for**
11:     For each $\lambda \in \Lambda$, train π_e^λ as:
        $\arg\min_{\pi \in \Pi} \lambda\mathbb{E}_{t_e}\hat{c}(\pi(x)) + (1-\lambda)\mathbb{E}_{S^{\text{tr}}}c^s(\pi(x))$.
12:     Set $\lambda_e \leftarrow \arg\min_{\lambda \in \Lambda} \mathbb{E}_{S_e^{\text{val}}}c^s(\pi_e^\lambda(x))$.
13: **end for**

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validation set at each epoch. Avoiding this splitting is an interesting question for future work.

For the main result, we need the following notation for the deviation of λ-weighted empirical costs, where $E = \lceil \log n^b \rceil$ is the total number of epochs:

$$W_t(\lambda) := 2\sqrt{\left(\frac{\lambda^2 K}{t\epsilon} + \frac{(1-\lambda)^2(E+1)}{n^s}\right)\ln\frac{8E|\Pi|}{\delta}} + \left(\frac{\lambda K}{t\epsilon} + \frac{(1-\lambda)(E+1)}{n^s}\right)\ln\frac{8E|\Pi|}{\delta}.$$

Theorem 2. *Suppose that D^b is (α, Δ)-similar to D^s. Then for any $\delta < 1/e$, with probability $1 - \delta$, the average supervised regret of Algorithm 2 can be bounded as:*

$$\frac{1}{n^b}\mathbf{R}^s(\langle x_t, a_t\rangle_{t=1}^{n^b}) \leq \epsilon + 3\sqrt{\frac{\ln\frac{8|\Pi|}{\delta}}{n^b}} + \sqrt{\frac{2(E+1)\ln\frac{8E|\Lambda|}{\delta}}{n^s}}$$

$$+ \min_{\lambda \in \Lambda}\frac{2}{n^b}\sum_{t=1}^{n^b}\frac{\lambda\Delta + 2W_t(\lambda)}{(1-\lambda) + \lambda\alpha}. \quad (5)$$

The first term is the cost of exploration, while the second is the gap between the conditional and unconditional expectations over costs in defining the regret. The third term captures the complexity of model selection while the final is the performance upper bound for the best λ in our weighted combination set Λ. As before, this significantly improves upon the $O(\sqrt{\frac{\ln|\Pi|/\delta}{n^s}})$ bound from using supervised examples alone whenever the two sources have sufficient similarity. The proof can be found in Appendix F.

Identical distributions: When $D^s = D^b$, which implies that D^s is $(1, 0)$-similar to D^b, a single choice of $\lambda =$

$\frac{n^b \epsilon}{n^s K + n^b \epsilon}$ will ensure that the last term in Equation (5) is

at most $\tilde{O}\left(\sqrt{\frac{K \ln \frac{8E|\Pi|}{\delta}}{Kn^s + n^b \epsilon}} + \frac{K \ln \frac{8E|\Pi|}{\delta}}{Kn^s + n^b \epsilon}\right)$ (See Proposition 2

in Appendix G). That is, after n^b CB samples, the effective sample size is at most $n^s + n^b \epsilon / K$.

5. Experiments

Experimentally, we focus on the question of learning with the CB costs as the ground truth (§3). Our experiments seek to address the following questions: **a)** How much benefit does a small amount of supervised warm-start provide? **b)** How much benefit does the bandit feedback provide? **c)** How robust is our algorithm under a realistic mismatch in cost structures? **d)** How robust is our algorithm under adversarial cost structures (the "safety" question)?

We consider the following set of approaches:

BANDIT-ONLY: a baseline that only uses CB examples.

MAJORITY: always predicts $a \in$ $\text{argmin}_{a \in [K]} \mathbb{E}_{(x,c) \sim D^b}[c(a)]$ independent of the context, without exploration.

SUP-ONLY: a baseline that uses the best policy on supervised examples, without exploration.

SIM-BANDIT: a baseline that runs the CB algorithm on warm-start examples as well, providing cost for the chosen action only (from the supervised set) and then continues on the remaining CB examples.

ARROW-CB with $\Lambda = \{0, \frac{1}{8}\zeta, \frac{1}{4}\zeta, \frac{1}{2}\zeta, \zeta, \frac{1}{2} + \frac{1}{2}\zeta, \frac{3}{4} + \frac{1}{4}\zeta, 1\}$ (abbrev. **ARROW-CB with** $|\Lambda| = 8$), where $\zeta = \epsilon/(K + \epsilon)$; this is chosen because ζ is an approximate minimizer of $G_t(\lambda, 1, 0)$, and the $|\Lambda|$ used aims to ensure that $\min_{\lambda \in \Lambda} G_t(\lambda, \alpha, \Delta)$ is close to $\min_{\lambda \in [0,1]} G_t(\lambda, \alpha, \Delta)$ (see Prop. 3). For computational considerations, we use the last policy $\pi_t^{\lambda_t}$ rather than the averaged policy $\frac{1}{t-1} \sum_{\tau=1}^{t-1} \pi_\tau^{\lambda_t}$ in line 3 of Algorithm 1.

ARROW-CB with $\Lambda = \{0, 1\}$ (abbrev. **ARROW-CB with** $|\Lambda| = 2$): as argued in Proposition 3, choosing λ in $\{0, 1\}$ also approximately minimizes $G_t(\lambda, \alpha, \Delta)$.

In subsequent discussions, if not explicitly mentioned, ARROW-CB refers to ARROW-CB with $|\Lambda| = 8$.

All the algorithms (other than SUP-ONLY and MAJORITY, which do not explore) use ϵ-greedy exploration, with most of the results presented using $\epsilon = 0.0125$. We additionally present the results for $\epsilon = 0.1$ and $\epsilon = 0.0625$ in Appendix J. In general, the increased uniform exploration for larger ϵ leads to some performance penalty in the CB algorithms relative to SUP-ONLY, when the bias is small. However, the added exploration gives robustness to large bias as it is readily detected in more adversarial noise settings.

Datasets. We compare these approaches on 524 binary and multiclass classification datasets from Bietti et al.

(2018), which in turn are from `openml.org`. For each dataset, we use the multiclass label in the dataset to generate cost vectors c^b and c^s respectively. That is, given an example $(x, y) \in \mathcal{X} \times [K]$, $c^b(a) = I(a \neq y)$. We vary the number of warm-start examples and CB examples as follows: for a dataset of size n, we vary the number of warm-start examples n^s in $\{0.005n, 0.01n, 0.02n, 0.04n\}$, and the number of CB examples n^b in $\{0.92n, 0.46n, 0.23n, 0.115n\}$. Define the *warm-start ratio* as the ratio n^b/n^s. We group different settings of (dataset, n^s, n^b) by n^b/n^s, so that a separate plot is generated for each ratio in $R = \{2.875, 5.75, 11.5, 23, 46, 92, 184\}$. We filter out the settings where n^s is below 100.

Evaluation Criteria. For each (dataset, n^s, n^b) combination c, we can compute $e_{c,a}$ to be the average cost of algorithm a on the CB examples. Because the range of $e_{c,a}$ can vary significantly over different settings c, we normalize these to yield the *normalized error* of an algorithm on a dataset: $\texttt{err}_{c,a} := \frac{e_{c,a} - e_c^*}{\max_b e_{c,b} - e_c^*}$, where e_c^* is the error achieved by a fully supervised one-versus-all learning algorithm trained on all the examples with original labels in this dataset. Lower normalized error indicates better performance. We plot the cumulative distribution function (CDF) of the normalized errors for each algorithm. That is, for an algorithm a, at each point x, the y value is the fraction of c's such that $\texttt{err}_{c,a} \leq x$. In general, a high CDF value at a small x indicates that the algorithm is performing well over a large number of (dataset, n^s, n^b) combinations.

In some of the plots when investigating the effect of a particular type or level of noise, we find it useful to aggregate the plots further over all warm-start ratios in creating the CDF and this aggregation is done by a pointwise averaging of the individual CDFs.

Comparison with baselines using both sources. We present the CDFs of all algorithms under various noise models in Figure 2, with detailed results for individual noise levels, warm-start ratios and different ϵ values in Appendix J. In Figure 2, we aggregate over warm-start ratios as described earlier. We can see from the figures that ARROW-CB's CDFs (approximately) dominate those of SIM-BANDIT and ARROW-CB with $|\Lambda| = 2$, which use weightings of 0.5, and the best of $\{0, 1\}$ respectively. These gains highlight the importance of being more careful about selecting a good weighting, despite the earlier intuition from Proposition 4. We see that there is a potentially added benefit of using different λ's in different phases of learning which might even outperform the best setting in hindsight.

Results for aligned cost structures. In Fig. 2a, we consider the setting $c^s = c^b$. Here, ARROW-CB's CDF dominates all other algorithms other than SUP-ONLY. For SUP-ONLY, the warm-start policy is used greedily with no explo-
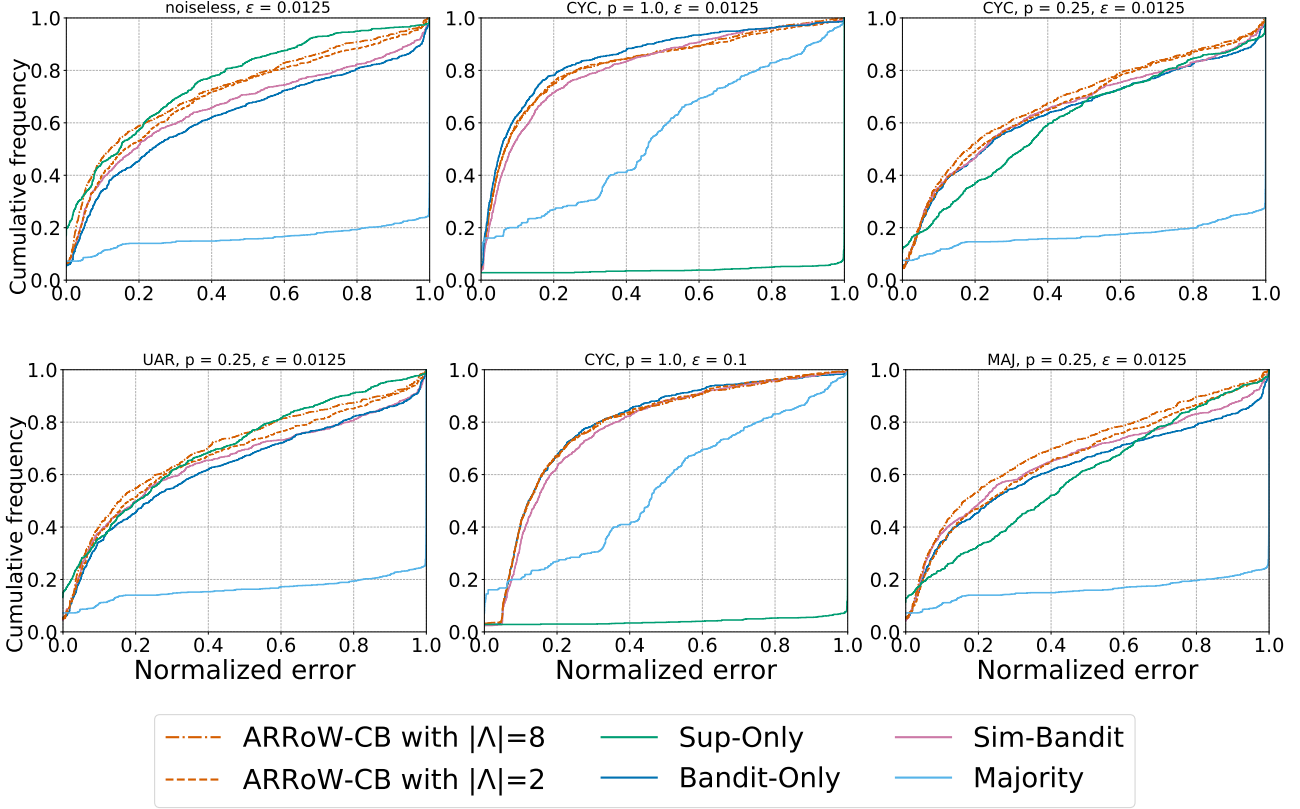
Figure 2: Comparison of all algorithms in the CB ground truth setting using the empirical CDF of the normalized performance scores. Left: unbiased warm-start examples with noiseless (top) and UAR with probability 0.5 (down) costs on warm-start examples. Middle: extreme noise rate using CYC noise type with probability 1.0. All CB algorithms use $\epsilon = 0.0125$ for exploration (top) and $\epsilon = 0.1$ (bottom). Right: moderate and potentially helpful noise rates. The corruption added to the warm-start examples are of types CYC (top) and MAJ (down) respectively, with probability 0.25.
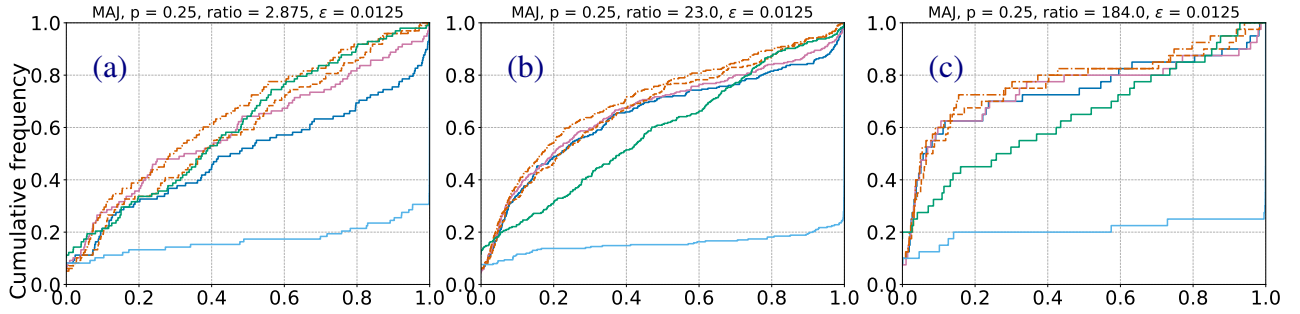
Figure 3: Effect of varying warm-start ratios for MAJ noise with $p = 0.25$. The warm-start ratios vary from 2.875 (left), 23 (middle) to 184 (right). Each CDF aggregates over all conditions of this noise type with the same warm-start ratio.

ration, making it a very strong baseline when there is no bias. We observe that ARRoW-CB uses the warm-start much more effectively than both the SIM-BANDIT and ARRoW-CB with $|\Lambda| = 2$ baselines. Our next experiments consider a uniform at random (UAR) noise setting, where the supervised data is unbiased (with respect to c^b) but has higher

variance. In particular, for every example (x, c^b), with probability $1 - p$ we set $c^s = c^b$ and with probability p we set c^s as the classification error against a uniform random label. From Claim 2 in the appendix, D^s is $(1 - p, 0)$-similar to D^b. We plot the CDFs of the algorithms in the case where $p = 0.25$ in Fig. 2d. The ordering of the CDFs stays the

same, with SUP-ONLY less dominant (unsurprisingly), and with the gaps between methods reduced with the reduced utility of the warm-start data.

Results with adversarial noise. We next conduct an experiment where c^s and c^b are highly misaligned in order to understand how robust ARROW-CB is to adversarial conditions. We consider the cycling noise model (CYC), where we set the supervised costs to be "off-by-one" from the CB costs. Specifically, if c^b declares action a to be the zero-cost action, then, with probability p, c^s corrupts the costs so that action $(a + 1) \bmod K$ becomes the zero-cost action. From Claim 3 in the appendix, D^s is $(1, 2p)$-similar to D^b. The CDF results for this experiment are in Figs. 2b, 2e ($p = 1$) for $\epsilon = 0.0125$ and $\epsilon = 0.1$, and Fig. 2c ($p = 0.25$) for $\epsilon = 0.0125$ respectively. Again, ARROW-CB is dominant amongst methods which use both the sources. In the case of $p = 1.0$, BANDIT-ONLY performs the best as the warm start examples are misleading. ARROW-CB performs slightly worse (Fig. 2b) due to the model selection overhead, as discussed following Theorem 1. This gap is reduced when we increase the ϵ value in ϵ-greedy to 0.1 (Fig. 2e). In the case of $p = 0.25$ (Fig. 2c), ARROW-CB outperforms all the methods, showing that it can utilize warm start examples even if they are moderately biased.

Results with majority noise. Finally, we consider the case of a noise model that replaces the ground truth label with the majority label, roughly modeling a "lazy annotator" who occasionally defaults to the most frequent class. For the majority noise model (MAJ), with probability $1 - p$, we set $c^s = c^b$ and with probability p we set c^s to a cost vector that has a zero for the most frequent label in this dataset and one elsewhere. From Claim 3 in the appendix, D^s is $(1, 2p)$-similar to D^b. The CDFs for this setting are shown in Figure 2f, where we again see ARROW-CB dominating all the baselines (similar to Figure 2c).

In sum, we observe that ARROW-CB is the *only* method which is the best or close across all the noise regimes; no other approach is consistently strong. In practical scenarios, where the extent of bias in the warm-start is difficult or costly to ascertain, this robust performance of ARROW-CB is extremely desirable. If we have some prior information about the noise level, it is prudent to prefer smaller ϵ when we expect a low noise (to compete well with SUP-ONLY), while a larger ϵ is preferred in high noise situations (to quickly detect the extent of bias).

While we present aggregates over warm-start ratios here, plots for each combination of noise type, level and warm-start ratio for three values of ϵ are shown in Appendix J.

Effect of warm-start ratio. In Fig. 3, we pick a moderate noise setting and study the ordering of the different methods as the number of warm-start examples n^s increases relative to n^b. We see ARROW-CB outperforming all methods. SUP-ONLY is strong on the left for a small ratio (2.875), while BANDIT-ONLY does well on the other extreme (184), and ARROW-CB consistently outperform both the baselines combining the two sources.

Overall. Overall, we see that effectively using warm-start examples can certainly improve the performance of CB approaches. ARROW-CB provides a way to do this in a robust manner, consistently outperforming most baselines. This is best evidenced in Figure 1, which further aggregates performance across the following 10 noise conditions on the warm start examples: noiseless and {UAR, CYC, MAJ} corruptions with probability p in $\{0.25, 0.5, 1.0\}$.

6. Discussion and Future Work

In this paper, we study the question of incorporating multiple data sources in CB settings. We see that even in simple cases, obvious techniques do not work robustly, and some care is required to handle biases from the non-ground-truth source.

Building on our results, there are several natural avenues for future work. Doing a similar modification to more advanced exploration algorithms (e.g. (Agrawal & Goyal, 2013; Agarwal et al., 2014)) is significantly more challenging. This falls into the general category of selecting the best from an ensemble of CB algorithms (where the ensemble corresponds to different weightings of the supervised and the CB examples). In ϵ-greedy, the policy training corresponds to training the CB algorithm on reweighted data, while the model selection over λ induces the action distribution. While the first step is typically straightforward even for other CB algorithms, finding an action distribution which looks good at this round, while allows the CB algorithms for different λ values to make subsequent updates is significantly harder (for instance, when using a UCB style strategy, each λ value might suggest a completely different action and expect reward feedback about it). A possible approach is to employ ideas from the CORRAL algorithm (Agarwal et al., 2017), but the cost of model selection is linear instead of logarithmic in $|\Lambda|$, and the approach is somewhat data inefficient due to restarts. More ambitiously, it is desirable for the schedule of supervised and CB examples to not be fixed in a warm-start fashion but based on active querying, such as by sending uncertain examples to a labeler for full supervision. Studying this and considering broader sources of feedback are both interesting future research.

Acknowledgments

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A. Additional Experimental Details

We run our experiments using Vowpal Wabbit (VW)[5]. In all our algorithms, we consider a scorer function class $\mathcal{F}$ that contains functions f that map (x, a) to estimated cost values. The policy class Π induced by $\mathcal{F}$ is defined as:

$$\Pi := \left\{ \pi_f : f \in \mathcal{F} \right\}, \text{ where } \pi_f(x) = \operatorname*{argmin}_{a \in [K]} f(x, a).$$

In general, our algorithms do not learn with respect to Π directly; instead, at each time t, they find some scorer function f_t, and use its induced policy π_{f_t} in Π to perform exploration and exploitation.

In all CB learning algorithms (BANDIT-ONLY and SIM-BANDIT), we use the ϵ-greedy exploration strategy with $\epsilon = 0.0125$ in most of the results for the main text, while the results for two other values (0.1 and 0.0625) in Appendix J. We use the importance weighted regression algorithm (IWR) (Bietti et al., 2018) to compute cost regressors f_t in $\mathcal{F}$. The function class we use consists of linear functions: $f_w(x, a) = \langle w_a, x \rangle$. In the supervised learning algorithm (SUP-ONLY), we use the cost-sensitive one against all algorithm (Beygelzimer et al., 2005) to train a cost regressor f_0 using all the warm start examples, and make no updates in the interaction stage. In the MAJORITY algorithm, we simply predict using the majority class in the dataset and make no updates in the interaction stage. *Note that there is no exploration in* SUP-ONLY *and* MAJORITY. For ARROW-CB, we also use the same set of ϵ values as BANDIT-ONLY and SIM-BANDIT. At time t and for λ in Λ, we approximate π_t^λ as follows. Instead of optimizing the objective function in Equation (3), we start with finding an approximate optimizer of the following objective function:

$$f_t^\lambda = \operatorname*{argmin}_{f \in \mathcal{F}} \left\{ (1 - \lambda) \sum_{(x,c) \in S} \sum_{a=1}^{K} (f(x, a) - c(a))^2 + \lambda \sum_{\tau=1}^{t} \frac{1}{p_{\tau, a_\tau}} (f(x_\tau, a_\tau) - c_\tau(a_\tau))^2 \right\},$$

and then take $\pi_t^\lambda = \pi_{f_t^\lambda}$. For computational efficiency, we choose not to find the exact empirical cost minimizer f_t^λ at each time. Instead, we use a variant of online gradient descent in VW with adaptive (Duchi et al., 2011), normalized (Ross et al., 2013) and importance-weight aware updates (Karampatziakis & Langford, 2011) on the objective function.

For computational efficiency, in ARROW-CB, at time $t \geq 2$, instead of mixing the average policy of $\left\{ \pi_\tau^\lambda \right\}_{\tau=1}^{t-1}$ with ϵ-uniform exploration, we predict by mixing the most recent policy π_t^λ with ϵ-uniform exploration.

We vary the learning rates of all algorithms from $\{0.1, 0.03, 0.3, 0.01, 1.0, 0.003, 3.0, 0.001, 10.0\}$. For each algorithm and dataset/setting combination, we first compute the average cost in the interaction stage. For different algorithm/dataset/setting combinations, we select different learning rates that minimize the corresponding average cost.

The source codes for running the experiments are available at `https://github.com/zcc1307/warmcb_scripts`.

B. Similiarity Conditions between Cost-sensitive Distributions

In this section, we introduce an alternative, more intuitive definition of similarity between two distributions of cost-sensitive examples. Then we show that this notion is stronger than Definition 1.

Definition 2. *Distribution D_2 is strongly (α, Δ)-similar to D_1 wrt policy class Π, if there exists a joint distribution D over $(x, c^+, c^-) \in \mathcal{X} \times [0, 1]^K \times [-1, 1]^K$ such that*

$$\mathbb{E}_D[c^+(a) + c^-(a) | x] = \mathbb{E}_{D_2}[c(a) | x], \quad \forall a \in [K] \text{ and } x \in \mathcal{X},$$

and there exists a policy $\pi^* \in \operatorname{argmin}_{\pi \in \Pi} \mathbb{E}_{D_1}[c(\pi(x))]$, such that both of the following hold:

1. *For any policy π in Π, $\mathbb{E}_D[c^+(\pi(x))] - \mathbb{E}_D[c^+(\pi^*(x))] \geq \alpha(\mathbb{E}_{D_1}[c(\pi(x))] - \mathbb{E}_{D_1}[c(\pi^*(x))])$.*
2. *The expected magnitude of c^- is bounded: for any policy π, $|\mathbb{E}_D[c^-(\pi(x))]| \leq \Delta/2$.*

In the definition above, we use $+/-$ superscripts to indicate the utility of c^+, c^-, which give a decomposition of the cost structure generated from D_2. Intuitively, c^+ is the component useful for learning under D_1, as low excess cost under c^+ implies low excess cost under D_1. In contrast, c^- may or may not have such property.

We have the following lemma upper bounding a policy's excess cost on D_1 in terms of its excess cost on D_2:

Lemma 1. *Suppose D_1 and D_2 are two distributions over cost-sensitive examples. If D_2 is strongly (α, Δ)-similar to D_1 with respect to policy class Π, then D_2 is (α, Δ)-similar to D_1 with respect to policy class Π.*

---

[5] `http://hunch.net/~vw/`

Proof. Suppose (x, c^+, c^-) is the decomposition of (x, c) that satisfies Definition 2. We have that

$$
\begin{aligned}
\mathbb{E}_{D_2}[c(\pi(x))] - \mathbb{E}_{D_2}[c(\pi^*(x))] &= (\mathbb{E}_D[c^+(\pi(x))] - \mathbb{E}_D[c^+(\pi^*(x))]) + (\mathbb{E}_D[c^-(\pi(x))] - \mathbb{E}_D[c^-(\pi^*(x))]) \\
&\geq \alpha(\mathbb{E}_{D_1}[c(\pi(x))] - \mathbb{E}_{D_1}[c(\pi^*(x))]) - \Delta,
\end{aligned}
$$

where the inequality follows from using the respective items in Definition 2 to lower bound each term (and observing that $\mathbb{E}_D[c^-(\pi(x))] - \mathbb{E}_D[c^-(\pi^*(x))] \leq \Delta/2 + \Delta/2 = \Delta$). The lemma follows. □

We now present a few examples that satisfy strong similarity.

Example 1: labeling the lowest-cost action. Consider a setting where eliciting the costs from an expert for warm-start examples is prohibitively expensive, but they can label the least costly action. Let D_1 be the CB source and let π^* be the best policy under D_1. Define D_2 by first sampling $(x, c_1) \sim D_1$ and returning $(x, \tilde{c})$ where $\tilde{c}(a) = I(a \neq a^*)$, for some $a^* \in \operatorname{argmin}_a c_1(a)$. Define $c^+(a) = I(a \neq \pi^*(x))$ and $c^- = \tilde{c} - c^+$.

Claim 1. D_2 is $(1, \Delta)$-*similar to* D_1, *with* $\Delta = 2\mathbb{P}(c^b(\pi^*(x)) \geq \min_{a \neq \pi^*(x)} c^b(a))$.

Proof. On one hand, for any policy π,

$$
\mathbb{E}_D c^+(\pi(x)) - \mathbb{E}_D c^+(\pi^*(x)) = \mathbb{P}(\pi(x) \neq \pi^*(x)) \geq \mathbb{E}_{D_1} c_1(\pi(x)) - \mathbb{E}_{D_1} c_1(\pi^*(x)).
$$

On the other hand, observe that given a cost sensitive example (x, c_1) and policy π^*, if $c_1(\pi^*(x)) < \min_{a \neq \pi^*(x)} c_1(a)$, then $\pi^*(x) = a^*$, in which case $\tilde{c} = c^+$ and $c^- = 0$. This implies that

$$
\mathbb{P}(c^- \neq 0) \leq \mathbb{P}(c_1(\pi^*(x)) \geq \min_{a \neq \pi^*(x)} c_1(a)) = \Delta/2.
$$

Therefore, for all policy π,

$$
\mathbb{E}|c^-(\pi(x))| \leq \mathbb{P}(c^- \neq 0) = \Delta/2. \qquad □
$$

Given a multiclass label y, defined its induced zero-one cost vector $c_y \in [0, 1]^K$ as follows: $c_y(a) = 0$ if $a = y$ and $c_y(a) = 1$ otherwise. From the definition of c_y, it can be seen that for any policy π, $\mathbb{E}c_y(\pi(x)) = \mathbb{P}(\pi(x) \neq y)$.

Example 2: Uniform-at-random corruption. Suppose D_0 is a joint distribution over multiclass examples (x, y), and the corrupted label $\tilde{y}$ has the following conditional distribution given (x, y): for all a in $[K]$, $\mathbb{P}(\tilde{y} = a|(x, y)) = (1 - p)I(a = y) + \frac{p}{K}$. Define D_1 and D_2 as the joint distribution of (x, c_y) and $(x, c_{\tilde{y}})$, respectively.

Claim 2. *Suppose D_1 and D_2 are defined as above. Then D_2 is $(1 - p, 0)$-strongly similar to D_1.*

Proof. Suppose $\pi^* \in \operatorname{argmin}_{\pi \in \Pi} \mathbb{E}_{D_1} c(\pi(x))$ is an optimal policy with respect to D_1.

For any policy π, by the definition of $\tilde{y}$ and $c_{\tilde{y}}$, we have that

$$
\mathbb{E}c_{\tilde{y}}(\pi(x)) = \mathbb{P}(\pi(x) \neq \tilde{y}) = (1 - p)\mathbb{P}(\pi(x) \neq y) + p\frac{K - 1}{K}.
$$

Therefore, for any policy π, the below identity holds:

$$
\mathbb{E}c_{\tilde{y}}(\pi(x)) - \mathbb{E}c_{\tilde{y}}(\pi^*(x)) = (1 - p)(\mathbb{E}c_y(\pi(x)) - \mathbb{E}c_y(\pi^*(x))).
$$

Therefore, taking $c^+ = c_{\tilde{y}}$, $c^- = 0$, $\alpha = 1 - p$ and $\Delta = 0$, it can be easily checked that the conditions of Definition 1 are satisfied. □

Example 3: general corruption. Suppose D_0 is a joint distribution over multiclass examples (x, y), and the corrupted label $\tilde{y}$'s conditional distribution given (x, y) has the following property: $\mathbb{P}(\tilde{y} = y|(x, y)) \geq 1 - p$. Define D_1 and D_2 as the joint distribution of (x, c_y) and $(x, c_{\tilde{y}})$, respectively.

Claim 3. *Suppose D_1 and D_2 are defined as above. Then D_2 is $(1, 2p)$-strongly similar to D_1.*

Proof. Suppose $\pi^* \in \operatorname{argmin}_{\pi \in \Pi} \mathbb{E}_{D_1} c(\pi(x))$ is an optimal policy with respect to D_1.

For every x, define deterministically that $c^+ = \mathbb{E}[c_y|x]$, and $c^- = \mathbb{E}[c_{\tilde{y}}|x] - \mathbb{E}[c_y|x]$. We have that $\mathbb{E}[c^+ + c^-|x] = \mathbb{E}[c_{\tilde{y}}|x]$ by the definitions of c^+ and c^-. In addition, by the construction of c^+, we immediately have that

$$
\mathbb{E}c^+(\pi(x)) - \mathbb{E}c^+(\pi^*(x)) = \mathbb{E}c_y(\pi(x)) - \mathbb{E}c_y(\pi^*(x)).
$$

What remains is to bound $\mathbb{E}c^-(\pi(x))$. By the definitions of c_y and $c_{\tilde{y}}$, we have that

$$|\mathbb{E}c^-(\pi(x))| = |\mathbb{P}(\pi(x) \neq \tilde{y}) - \mathbb{P}(\pi(x) \neq y)| \leq \mathbb{P}(y \neq \tilde{y}).$$

By the assumption on the conditional distribution of $\tilde{y}$ given (x, y), we have that the right hand side is at most p. The claim follows. $\square$

C. Proof Showing the Failure of Equal Data Weighting

In this section we formalize the example presented in §2.1. To recall, this is a 2-armed bandit setting (i.e. contextual bandits with a dummy context), where policy class $\Pi := \{\pi_1, \pi_2\}$, where π_i maps any context to action i, $i = 1, 2$. In addition, D^s (resp. D^b) is the Dirac measure on (x_0, c_0^s) (resp. (x_0, c_0^b)), and the respective cost vectors are:

$$c_0^s = (0.5, 0.5 + \frac{\Delta}{2}), \quad \text{and} \quad c_0^b = (0.5, 0.5 - \frac{\Delta}{2}).$$

We consider the following algorithm, which directly extends the UCB1 algorithm (Auer et al., 2002a) by additionally using the warm start examples to estimate the mean costs of the two actions. Note that as it is minimizing its cumulative cost, the algorithm computes lower confidence bounds of the costs and selects the minimum, which is equivalent to computing upper confidence bounds of the rewards and selecting the maximum.

Algorithm 3 A variant of the UCB1 algorithm that accounts for warm start examples

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Require: Supervised examples $S = \{(x_0, c^s)\}$ of size n^s, number of interaction rounds n^b.
1: **for** $t = 1, 2, \ldots, n^b$ **do**
2:      For $i = 1, 2$, define $n_{i,t-1} = \sum_{s=1}^{t-1} I(a_t = i)$.
3:      For $i = 1, 2$, compute empirical mean cost of action i: $\hat{\mu}_{i,t} = \frac{\sum_{(x_0, c^s) \in S} c^s(i) + \sum_{s=1}^{t-1} I(a_t=i) c_s^b(i)}{n^s + n_{i,t-1}}$.
4:      For $i = 1, 2$, compute $\text{lcb}_{i,t} = \hat{\mu}_{i,t} - 2\sqrt{\frac{\ln t}{n^s + n_{i,t-1}}}$.
5:      Take action $a_t = \operatorname{argmin}_{i \in \{1,2\}} \text{lcb}_{i,t}$.
6:      Observe $c_t^b(a_t)$.
7: **end for**

We have the following proposition on a lower bound of the regret of Algorithm 3, under the above settings of D^s and D^b.

Proposition 1. *Suppose D^s and D^b are defined as above. Additionally, Algorithm 3 is run with input n^s warm start examples drawn from D^s and number of interaction rounds $n^b \geq \exp(\Delta^2 n^s/16)$. Then, Algorithm 3 incurs a regret of $\Omega(\Delta \exp(\Delta^2 n_s/16))$.*

Proof. Suppose after $t - 1$ rounds of the interaction phase, Algorithm 3 has taken action i t_i times for $i = 1, 2$. As the cost vectors are deterministic, we can calculate the lower confidence bound estimates for the two actions in closed form:

$$\text{lcb}_{t,1} = 0.5 - 2\sqrt{\frac{\ln(t_1 + t_2 + 1)}{n^s + t_1}} \quad \text{and} \quad \text{lcb}_{t,2} = 0.5 + \frac{\Delta}{2}\frac{n^s - t_2}{n^s + t_2} - 2\sqrt{\frac{\ln(t_1 + t_2 + 1)}{n^s + t_2}}$$

Now, let us consider the first time we play action 2 in the interaction phase. At this point, t_2 is still 0 and t_1 should satisfy

$$\text{lcb}_{t,2} \leq \text{lcb}_{t,1}, \quad \text{that is} \quad \frac{\Delta}{2} - 2\sqrt{\frac{\ln(t_1 + 1)}{n^s}} \leq -2\sqrt{\frac{\ln(t_1 + 1)}{n^s + t_1}}.$$

The above condition implies that

$$2\sqrt{\frac{\ln(t_1 + 1)}{n^s}} \geq \frac{\Delta}{2}, \quad \text{equivalently} \quad t_1 \geq \exp(\Delta^2 n^s/16) - 1.$$

Denote by $T_0 := \exp(\Delta^2 n^s/16) - 1$. Therefore, after $n^b \geq \exp(\Delta^2 n^s/16)$ rounds of interaction, the regret of Algorithm 3 can be lower bounded by:

$$\sum_{t=1}^{n^b} c_0^b(a_t) - c_0^b(2) \geq \sum_{t=1}^{T_0 - 1} c_0^b(1) - c_0^b(2) = \frac{\Delta}{2} \cdot (T_0 - 1) = \Omega(\Delta \exp(\Delta^2 n^s/16)). \quad \square$$

In Algorithm 3, we used only $\ln(t+1)$ in the numerator, when an alternative might be to use $\ln(t+n^s+1)$. However, it is easily checked that after this simple modification, a similar exponential regret lower bound of Algorithm 3 can be proved (with the definition of T_0 changed to $T_0 := \exp(\Delta^2 n^s/16) - n^s - 1$).

D. Concentration Inequalities

We use a version of Freedman's inequality from (Beygelzimer et al., 2011).

Lemma 2 (Freedman's inequality). *Let $X_1, \ldots, X_n$ be a martingale difference sequence adapted to filtration $\{\mathcal{B}_i\}_{i=0}^n$, and $|X_i| \leq M$ almost surely for all i. Let $V = \sum_{i=1}^n \mathbb{E}[X_i^2 | \mathcal{B}_{i-1}]$ be the cumulative conditional variance. Then, with probability $1 - \delta$,*

$$\left| \sum_{i=1}^n X_i \right| \leq 2\sqrt{V \ln \frac{2}{\delta}} + M \ln \frac{2}{\delta}.$$

E. Proof of Theorem 1

We begin with some additional notation used in the analysis. Throughout this section, we let $\pi^* = \operatorname{argmin}_{\pi \in \Pi} \mathbb{E}c^b(\pi(x))$, the optimal policy in Π with respect to D^b.

Recall that for policy π and λ in $[0, 1]$, we define $\mathbb{E}_t[\hat{c}(\pi(x))] := \frac{1}{t} \sum_{s=1}^t \hat{c}_s(\pi(x))$ and $\mathbb{E}_S[c^s(\pi(x))] := \frac{1}{n^s} \sum_{(x,c) \in S} c(\pi(x))$. We additionally define the λ-weighted empirical cost of π as

$$\hat{L}_{\lambda,t}(\pi) = \frac{\lambda t \mathbb{E}_t[\hat{c}(\pi(x))] + (1-\lambda)n^s \mathbb{E}_S[c^s(\pi(x))]}{\lambda t + (1-\lambda)n^s},$$

and its expectation

$$L_{\lambda,t}(\pi) = \frac{\lambda t \mathbb{E}[c^b(\pi(x))] + (1-\lambda)n^s \mathbb{E}[c^s(\pi(x))]}{\lambda t + (1-\lambda)n^s}.$$

Observe that $\hat{L}_{1,t}(\pi) = \mathbb{E}_t[\hat{c}(\pi(x))]$ is π's empirical cost on the first t CB examples, $L_{1,t}(\pi) = \mathbb{E}c^b(\pi(x))$, $\hat{L}_{0,t}(\pi) = \mathbb{E}_S[c^s(\pi(x))]$ is π's empirical cost on the n^s supervised examples, and $L_{0,t}(\pi) = \mathbb{E}c^s(\pi(x))$. Denote by $\hat{\Gamma}_{\lambda,t}(\pi) = (\lambda t + (1-\lambda)n^s)\hat{L}_{\lambda,t}(\pi)$ the unnormalized λ-weighted empirical cost of π, and $\Gamma_{\lambda,t}(\pi) = (\lambda t + (1-\lambda)n^s)L_{\lambda,t}(\pi)$ its expectation.

Denote by $(x_{n^b+1}, c_{n^b+1}^s), \ldots, (x_{n^b+n^s}, c_{n^b+n^s}^s)$ an enumeration of the elements in S. Define filtration $\{\mathcal{B}_t\}_{t=0}^{n^b+n^s}$ as follows: $\mathcal{B}_0$ is the trivial σ-algebra, and

$$\mathcal{B}_t = \begin{cases} \sigma((x_{n^b+1}, c_{n^b+1}^s), \ldots, (x_{n^b+1}, c_{n^b+t}^s)), & t \in \{1, \ldots, n^s\}, \\ \sigma(S, (x_1, \hat{c}_1), \ldots, (x_{t-n^s}, \hat{c}_{t-n^s})), & t \in \{n^s+1, \ldots, n^s+n^b\}. \end{cases}$$

For reader's convenience, we also recall our earlier notation:

$$V_t(\lambda) = 2\sqrt{\left(\lambda^2 \frac{Kt}{\epsilon} + (1-\lambda)^2 n^s\right) \ln \frac{8n^b|\Pi|}{\delta}} + \left(\frac{\lambda K}{\epsilon} + (1-\lambda)\right) \ln \frac{8n^b|\Pi|}{\delta},$$

$$G_t(\lambda, \alpha, \Delta) = \frac{(1-\lambda)n^s \Delta + 2V_t(\lambda)}{\lambda t + (1-\lambda)n^s \alpha}.$$

In addition, denote by

$$\bar{G}_t(\lambda, \alpha, \Delta) := \min(1, G_t(\lambda, \alpha, \Delta)).$$

Proof of Theorem 1. Define event I as: for all π in Π,

$$\left| \frac{1}{n^b} \sum_{t=1}^{n^b} (\mathbb{E}[c_t^b(\pi(x_t))|x_t] - \mathbb{E}c^b(\pi(x))) \right| \leq \sqrt{\frac{\ln \frac{8|\Pi|}{\delta}}{2n^b}}.$$

By Hoeffding's inequality and union bound, I happens with probability $1 - \frac{\delta}{4}$. Specifically, on event I, for every policy π in

Π, as π^* is the policy in Π that minimizes $\mathbb{E}c^b(\pi(x))$, we have

$$\mathbb{E}c^b(\pi^*(x)) - \frac{1}{n^b}\sum_{t=1}^{n^b}\mathbb{E}[c^b(\pi(x_t))|x_t] \leq \mathbb{E}c^b(\pi(x)) - \frac{1}{n^b}\sum_{t=1}^{n^b}\mathbb{E}[c^b(\pi(x_t))|x_t] \leq \sqrt{\frac{\ln\frac{8|\Pi|}{\delta}}{2n^b}}.$$

Therefore,

$$\mathbb{E}c^b(\pi^*(x)) - \min_{\pi\in\Pi}\frac{1}{n^b}\sum_{t=1}^{n^b}\mathbb{E}[c^b(\pi(x_t))|x_t] \leq \sqrt{\frac{\ln\frac{8|\Pi|}{\delta}}{2n^b}}. \tag{6}$$

Recall that randomized policy $\pi_t : \mathcal{X} \to \Delta^{K-1}$ is defined as $\pi_t(x) = \frac{1-\epsilon}{t-1}\sum_{\tau=1}^{t-1}\pi_\tau^{\lambda_t}(x) + \frac{\epsilon}{K}\mathbb{1}_K$ for $t \geq 2$, and $\pi_1(x) = \frac{1}{K}\mathbb{1}_K$ for all x. With a slight abuse of notation, denote by $\mathbb{E}c^b(\pi_t(x)) := \mathbb{E}_{(x,c^b),a\sim\pi_t(x)}c^b(a)$. Observe that $\mathbb{E}[\mathbb{E}[c^b(a_t)|x_t]|\mathcal{B}_{n^s+t-1}] = \mathbb{E}c^b(\pi_t(x))$. Define event J as:

$$\left|\frac{1}{n^b}\sum_{t=1}^{n^b}\mathbb{E}[c^b(a_t)|x_t] - \frac{1}{n^b}\sum_{t=1}^{n^b}\mathbb{E}c^b(\pi_t(x))\right| \leq \sqrt{\frac{\ln\frac{8}{\delta}}{2n^b}}. \tag{7}$$

By Azuma's inequality, J happens with probability $1 - \frac{\delta}{4}$.

Denote by E the event that the events E_t, F_t defined in Lemmas 3 and 5 (both given below) and I, J hold simultaneously for all t. By a union bound over all E_t, F_t's and I, J, event E happens with probability $1 - \delta$. We henceforth condition on E happening.

Consider t in $\{2,\ldots,n^b\}$. We now give an upper bound on the expected excess cost of using randomized prediction π_t. Observe that by the definition of π_t,

$$\mathbb{E}c^b(\pi_t(x)) = \frac{1-\epsilon}{t-1}\sum_{\tau=1}^{t-1}\mathbb{E}c^b(\pi_\tau^{\lambda_t}(x)) + \epsilon\frac{1}{K}\sum_{a=1}^{K}\mathbb{E}c^b(a) \leq \frac{1}{t-1}\sum_{\tau=1}^{t-1}\mathbb{E}c^b(\pi_\tau^{\lambda_t}(x)) + \epsilon, \tag{8}$$

it suffices to upper bound $\frac{1}{t-1}\sum_{\tau=1}^{t-1}\mathbb{E}c^b(\pi_\tau^{\lambda_t}(x))$.

By Lemma 5 below, we have that

$$\frac{1}{t-1}\sum_{\tau=1}^{t-1}\mathbb{E}c^b(\pi_\tau^{\lambda_t}(x)) - \min_{\lambda\in\Lambda}\frac{1}{t-1}\sum_{\tau=1}^{t-1}\mathbb{E}c^b(\pi_\tau^\lambda(x)) \leq 16\sqrt{\frac{K\ln\frac{8n^b|\Lambda|}{\delta}}{(t-1)\epsilon}}. \tag{9}$$

In the above inequality, we can bound the individual summands of the second term by Lemma 3:

$$\mathbb{E}c^b(\pi_\tau^\lambda(x)) - \mathbb{E}c^b(\pi^*(x)) \leq \bar{G}_{\tau-1}(\lambda,\alpha,\Delta).$$

Therefore, rewriting Equation (9), we get that

$$\frac{1}{t-1}\sum_{\tau=1}^{t-1}\mathbb{E}c^b(\pi_\tau^{\lambda_t}(x)) - \min_{\lambda\in\Lambda}\left[\frac{1}{t-1}\sum_{\tau=1}^{t-1}(\mathbb{E}c^b(\pi^*(x)) + \bar{G}_{\tau-1}(\lambda,\alpha,\Delta))\right] \leq 16\sqrt{\frac{K\ln\frac{8n^b|\Lambda|}{\delta}}{(t-1)\epsilon}}.$$

Combining the above with Equation (8), we get that

$$\mathbb{E}c^b(\pi_t(x)) - \mathbb{E}c^b(\pi^*(x)) \leq \epsilon + \min_{\lambda\in\Lambda}\left[\frac{1}{t-1}\sum_{\tau=1}^{t-1}\bar{G}_{\tau-1}(\lambda,\alpha,\Delta)\right] + 16\sqrt{\frac{K\ln\frac{8n^b|\Lambda|}{\delta}}{(t-1)\epsilon}}.$$

Summing the above inequality over t in $\{2,\ldots,n^b\}$ and using that $\mathbb{E}c^b(\pi_1(x)) - \mathbb{E}c^b(\pi^*(x)) \leq 1$, we get,

$$\sum_{t=1}^{n^b}[\mathbb{E}c^b(\pi_t(x)) - \mathbb{E}c^b(\pi^*(x))] \leq n^b\epsilon + \sum_{t=2}^{n^b}\min_{\lambda\in\Lambda}\left[\frac{1}{t-1}\sum_{\tau=1}^{t-1}\bar{G}_{\tau-1}(\lambda,\alpha,\Delta)\right] + 16\sum_{t=2}^{n^b}\sqrt{\frac{K\ln\frac{8n^b|\Lambda|}{\delta}}{(t-1)\epsilon}} + 1.$$

Observe that for any function $\{N_t(\cdot)\}_{t=1}^{n^b}$, $\sum_{t=1}^{n^b}\min_{\lambda\in\Lambda}N_t(\lambda) \leq \min_{\lambda\in\Lambda}[\sum_{t=1}^{n^b}N_t(\lambda)]$. We further collect the coeffi-

cients on the $\bar{G}_\tau(\lambda, \alpha, \Delta)$ terms and use the upper bound $\sum_{t=\tau+1}^{n^b} \frac{1}{t-1} \leq \ln(en^b)$ for all $\tau \geq 1$, getting

$$\sum_{t=1}^{n^b} [\mathbb{E}c^b(\pi_t(x)) - \mathbb{E}c^b(\pi^*(x))] \leq n^b\epsilon + \ln(en^b) \cdot \min_{\lambda \in \Lambda} \sum_{\tau=0}^{n^b-2} \bar{G}_\tau(\lambda, \alpha, \Delta) + 16\sum_{t=2}^{n^b} \sqrt{\frac{K \ln \frac{8n^b|\Lambda|}{\delta}}{(t-1)\epsilon}} + 1. \qquad (10)$$

Therefore, we get:

$$\begin{aligned}
&\mathbf{R}^b(\langle x_t, a_t \rangle_{t=1}^{n^b}) \\
&\leq \quad \frac{1}{n^b} \sum_{t=1}^{n^b} [\mathbb{E}c^b(\pi_t(x)) - \mathbb{E}c^b(\pi^*(x))] + \sqrt{\frac{2\ln \frac{8|\Pi|}{\delta}}{n^b}} \\
&\leq \quad \epsilon + \frac{\ln(en^b)}{n^b} \min_{\lambda \in \Lambda} \sum_{t=0}^{n^b-2} \bar{G}_t(\lambda, \alpha, \Delta) + \frac{16}{n^b} \sum_{t=2}^{n^b} \sqrt{\frac{K \ln \frac{8n^b|\Lambda|}{\delta}}{(t-1)\epsilon}} + \sqrt{\frac{2\ln \frac{8|\Pi|}{\delta}}{n^b}} + \frac{1}{n^b} \\
&\leq \quad \epsilon + \frac{\ln(en^b)}{n^b} \min_{\lambda \in \Lambda} \sum_{t=1}^{n^b} \bar{G}_t(\lambda, \alpha, \Delta) + \frac{16}{n^b} \sum_{t=2}^{n^b} \sqrt{\frac{K \ln \frac{8n^b|\Lambda|}{\delta}}{(t-1)\epsilon}} + \sqrt{\frac{2\ln \frac{8|\Pi|}{\delta}}{n^b}} + \frac{\ln(e^2 n^b)}{n^b} \\
&\leq \quad \epsilon + 3\sqrt{\frac{\ln \frac{8n^b|\Pi|}{\delta}}{n^b}} + 32\sqrt{\frac{K \ln \frac{8n^b|\Lambda|}{\delta}}{n^b\epsilon}} + \frac{\ln(en^b)}{n^b} \min_{\lambda \in \Lambda} \sum_{t=1}^{n^b} \bar{G}_t(\lambda, \alpha, \Delta).
\end{aligned}$$

where the first inequality is from Equations (6) and (7) and dividing both sides by n^b; the second inequality is from Equation (10); the third inequality is from that $\bar{G}_0(\lambda, \alpha, \Delta) \leq 1$; the fourth inequality is from algebra, and our assumption that $\delta < 1/e$. The theorem follows. $\qquad \square$

The following lemma upper bounds the excess cost of $\pi_t^\lambda = \operatorname{argmin}_{\pi \in \Pi} \hat{L}_{\lambda, t-1}(\pi)$.

Lemma 3. *For every $t \in \{1, \ldots, n^b\}$, there exists an event E_t with probability $1 - \frac{\delta}{4n^b}$, such that the following holds for all λ in Λ:*

$$\mathbb{E}c^b(\pi_t^\lambda(x)) - \mathbb{E}c^b(\pi^*(x)) \leq \bar{G}_{t-1}(\lambda, \alpha, \Delta).$$

Proof. Define event E_t as: for all π in Π,

$$\left| [\lambda(t-1)\mathbb{E}c^b(\pi(x)) + (1-\lambda)n^s\mathbb{E}c^s(\pi(x))] - [\lambda(t-1)\mathbb{E}_{t-1}\hat{c}(\pi(x)) + (1-\lambda)n^s\mathbb{E}_S c^s(\pi(x))] \right|$$

$$\leq \quad 2\sqrt{(\lambda^2 \frac{K(t-1)}{\epsilon} + (1-\lambda)^2 n^s) \ln \frac{8n^b|\Pi|}{\delta}} + (\frac{\lambda K}{\epsilon} + (1-\lambda)) \ln \frac{8n^b|\Pi|}{\delta} \qquad (11)$$

In other words,

$$\left| \Gamma_{\lambda, t-1}(\pi) - \hat{\Gamma}_{\lambda, t-1}(\pi) \right| \leq V_{t-1}(\lambda). \qquad (12)$$

For a fixed π, applying Lemma 2 with $X_i = (1-\lambda)(c_{n^b+i}^s(\pi(x_{n^b+i}) - \mathbb{E}c^s(\pi(x)))$ for i in $\{1, \ldots, n^s\}$, $X_i = \lambda(\hat{c}_{i-n^s}(\pi(x_{i-n^s})) - \mathbb{E}c^b(\pi(x)))$ for i in $\{n^s + 1, \ldots, n^s + t\}$, and $M = (1-\lambda) + \frac{\lambda K}{\epsilon}$, and noting that $|X_i| \leq M$ almost surely for all i, $\mathbb{E}[X_i^2|\mathcal{B}_{i-1}] \leq (1-\lambda)^2$ for i in $\{1, \ldots, n^s\}$, and $\mathbb{E}[X_i^2|\mathcal{B}_{i-1}] \leq \lambda^2 \mathbb{E}[\frac{1}{p_{i-n^s, \pi(x_{i-n^s})}}|\mathcal{B}_{i-1}] \leq \frac{\lambda^2 K}{\epsilon}$ for i in $\{n^s + 1, \ldots, n^s + t - 1\}$, we get that Equation (11) holds for π with probability $1 - \frac{\delta}{4n^b|\Pi|}$. Therefore, by an union bound over all π in Π, E_t happens with probability $1 - \frac{\delta}{4n^b}$. We henceforth condition on E_t happening.

By the optimality of π_t^λ,

$$\hat{L}_{\lambda, t-1}(\pi_t^\lambda) \leq \hat{L}_{\lambda, t-1}(\pi^*).$$

Equivalently,

$$\hat{\Gamma}_{\lambda, t-1}(\pi_t^\lambda) \leq \hat{\Gamma}_{\lambda, t-1}(\pi^*).$$

Combining with Equation (12) applied to π_t^λ and π^*, we get that

$$\Gamma_{\lambda, t-1}(\pi_t^\lambda) - \Gamma_{\lambda, t-1}(\pi^*) \leq 2V_{t-1}(\lambda).$$

Using the (α, Δ)-similarity of D^s to D^b, and Lemma 4 below, we get that

$$(\mathbb{E}c^b(\pi_t^\lambda(x)) - \mathbb{E}c^b(\pi^*(x)))(\lambda(t-1) + (1-\lambda)n^s\alpha) \leq 2V_{t-1}(\lambda, \alpha) + (1-\lambda)n^s\Delta.$$

Therefore, by the definition of $G_t(\lambda, \alpha, \Delta)$, we have that $\mathbb{E}c^b(\pi_t^\lambda(x)) - \mathbb{E}c^b(\pi^*(x))) \leq G_{t-1}(\lambda, \alpha, \Delta)$. Combining the above with the fact that $\mathbb{E}c^b(\pi_t^\lambda(x)) - \mathbb{E}c^b(\pi^*(x)) \leq 1$, the lemma follows. $\qquad\square$

Lemma 4. *If D^s is (α, Δ)-similar to D^b, then for any policy π,*
$$(\mathbb{E}c^b(\pi(x)) - \mathbb{E}c^b(\pi^*(x)))(\lambda t + (1-\lambda)n^s\alpha) \leq (\Gamma_{\lambda,t}(\pi) - \Gamma_{\lambda,t}(\pi^*)) + (1-\lambda)n^s\Delta.$$

Proof. Using the definition of $\Gamma_{\lambda,t}$, we have that
$$\Gamma_{\lambda,t}(\pi) - \Gamma_{\lambda,t}(\pi^*) = \lambda t(\mathbb{E}c^b(\pi(x)) - \mathbb{E}c^b(\pi^*(x))) + (1-\lambda)n^s(\mathbb{E}c^s(\pi(x)) - \mathbb{E}c^s(\pi^*(x))).$$
Applying Lemma 1, the right hand side is at least
$$\lambda t(\mathbb{E}c^b(\pi(x)) - \mathbb{E}c^b(\pi^*(x))) + (1-\lambda)n^s(\alpha(\mathbb{E}c^b(\pi(x)) - \mathbb{E}c^b(\pi^*(x))) - \Delta).$$
The lemma follows immediately by algebra. $\qquad\square$

We also bound the cost overhead for selecting λ_t from Λ compared to using the best λ in hindsight. Define the progressive validation error using $\{\pi_\tau^\lambda\}_{\tau \leq t}$ as: $\hat{C}_{\lambda,t} = \sum_{\tau=1}^t \hat{c}_\tau(\pi_\tau^\lambda(x_\tau))$, and its expectation as: $C_{\lambda,t} = \sum_{\tau=1}^t \mathbb{E}c^b(\pi_\tau^\lambda(x))$.

Lemma 5. *For every $t \in \{2, \ldots, n^b\}$, there exists an event F_t with probability $1 - \frac{\delta}{4n^b}$, such that λ_t has the following property:*
$$\sum_{\tau=1}^{t-1} \mathbb{E}c^b(\pi_\tau^{\lambda_t}(x)) - \min_{\lambda\in\Lambda} \sum_{\tau=1}^{t-1} \mathbb{E}c^b(\pi_\tau^\lambda(x)) \leq 16\sqrt{\frac{K(t-1)}{\epsilon} \ln \frac{8n^b|\Lambda|}{\delta}}.$$

Proof. Define event F_t as: for all λ in Λ,
$$\left|C_{\lambda,t-1} - \hat{C}_{\lambda,t-1}\right| \leq 2\sqrt{\frac{K(t-1)}{\epsilon} \ln \frac{8n^b|\Lambda|}{\delta}} + \frac{K}{\epsilon} \ln \frac{8n^b|\Lambda|}{\delta}.$$
In other words,
$$\left|\sum_{\tau=1}^{t-1} \mathbb{E}c^b(\pi_\tau^\lambda(x)) - \sum_{\tau=1}^{t-1} \hat{c}_\tau(\pi_\tau^\lambda(x_\tau))\right| \leq 2\sqrt{\frac{K(t-1)}{\epsilon} \ln \frac{8n^b|\Lambda|}{\delta}} + \frac{K}{\epsilon} \ln \frac{8n^b|\Lambda|}{\delta}. \qquad (13)$$

For a fixed λ, by Lemma 2, taking $X_i = 0$ for i in $[n^s]$, $X_i = \hat{c}_{i-n^s}(\pi_{i-n^s}^\lambda(x_{i-n^s})) - \mathbb{E}c^b(\pi_{i-n^s}^\lambda(x))$ for i in $\{n^s + 1, \ldots, n^s + t - 1\}$, $M = \frac{K}{\epsilon}$, and noting that $|X_i| \leq M$ almost surely for all i, $\mathbb{E}[X_i^2|\mathcal{B}_{i-1}] = 0$ for i in $[n^s]$, and $\mathbb{E}[X_i^2|\mathcal{B}_{i-1}] \leq \mathbb{E}[\frac{1}{p_{i-n^s,\pi(x_{i-n^s})}}|\mathcal{B}_{i-1}] \leq \frac{K}{\epsilon}$ for i in $\{n^s + 1, \ldots, n^s + t - 1\}$, we get that Equation (13) holds with probability $1 - \frac{\delta}{4n^b|\Lambda|}$. By an union bound over all λ in Λ, the probability of F_t is at least $1 - \frac{\delta}{4n^b}$. We henceforth condition on F_t happening.

By the optimality of λ_t, we know that for all λ in Λ,
$$\hat{C}_{\lambda_t,t-1} \leq \hat{C}_{\lambda,t-1}.$$
Combining the above inequality with Equation (13) applied on λ_t and λ, we get that
$$C_{\lambda_t,t-1} - C_{\lambda,t-1} \leq 4\sqrt{\frac{K(t-1)}{\epsilon} \ln \frac{8n^b|\Lambda|}{\delta}} + 2\frac{K}{\epsilon} \ln \frac{8n^b|\Lambda|}{\delta}.$$
In addition, observe that $C_{\lambda_t,t-1} - C_{\lambda,t-1} \leq t - 1$ as $c^b \in [0,1]^K$ with probability 1. Combining the above facts with Lemma 6 below, we have that
$$C_{\lambda_t,t-1} - C_{\lambda,t-1} \leq \min\left(4\sqrt{\frac{K(t-1)}{\epsilon} \ln \frac{8n^b|\Lambda|}{\delta}} + 2\frac{K}{\epsilon} \ln \frac{8n^b|\Lambda|}{\delta}, t-1\right) \leq 16\sqrt{\frac{K(t-1)}{\epsilon} \ln \frac{8n^b|\Lambda|}{\delta}}.$$
In other words,
$$\sum_{\tau=1}^{t-1} \mathbb{E}c^b(\pi_\tau^{\lambda_t}(x)) - \sum_{\tau=1}^{t-1} \mathbb{E}c^b(\pi_\tau^\lambda(x)) \leq 16\sqrt{\frac{K(t-1)}{\epsilon} \ln \frac{8n^b|\Lambda|}{\delta}}.$$

As the above holds for any λ in Λ, the lemma follows. $\qquad\square$

Lemma 6. *For any positive real numbers $a, b > 0$, we have*
$$\min(\sqrt{ab} + b, a) \leq 2\sqrt{ab}.$$

Proof. The lemma follows from the straightforward calculations below:
$$\min(\sqrt{ab} + b, a) \leq \min(\sqrt{ab}, a) + \min(b, a) \leq \sqrt{ab} + \sqrt{ab} = 2\sqrt{ab}. \qquad \square$$

F. Proof of Theorem 2

We begin with some notation used in our analysis. Throughout this section, we let $\pi^* = \operatorname{argmin}_{\pi \in \Pi} \mathbb{E}c^s(\pi(x))$, the optimal policy in Π with respect to D^s. Define
$$H_t(\lambda, \alpha, \Delta) = \frac{\lambda \Delta + 2W_t(\lambda)}{(1 - \lambda) + \lambda \alpha}.$$

For policy π, λ in $[0, 1]$ and t_e for $e \in \{1, 2, \ldots, E\}$, define the λ-weighted empirical cost[6] of π as
$$\hat{M}_{\lambda, t_e}(\pi) = \lambda \mathbb{E}_{t_e} \hat{c}(\pi(x)) + (1 - \lambda)\mathbb{E}_{S^{\text{tr}}} c^s(\pi(x))$$

and its expectation
$$M_{\lambda, t_e}(\pi) = \lambda \mathbb{E}c^b(\pi(x)) + (1 - \lambda)\mathbb{E}c^s(\pi(x))$$

For convenience, we define $\tilde{n}^s = n^s/(E + 1)$.

Denote by $(x_{n^b+1}, c^s_{n^b+1}), \ldots, (x_{n^b+\tilde{n}^s}, c^s_{n^b+\tilde{n}^s})$ an enumeration of the elements in S^{tr}. Define filtration $\{\mathcal{B}_t\}_{t=0}^{n^b+\tilde{n}^s}$ as follows: $\mathcal{B}_0$ is the trivial σ-algebra, and

$$\mathcal{B}_t = \begin{cases} \sigma((x_{n^b+1}, c^s_{n^b+1}), \ldots, (x_{n^b+1}, c^s_{n^b+t})), & t \in \{0, \ldots, \tilde{n}^s\}, \\ \sigma(S^{\text{tr}}, (x_1, \hat{c}_1), (x_{t-\tilde{n}^s}, \hat{c}_{t-\tilde{n}^s})), & t \in \{\tilde{n}^s + 1, \ldots, \tilde{n}^s + n^b\}. \end{cases}$$

For reader's convenience, we also recall our earlier notation:
$$W_t(\lambda) = 2\sqrt{\left(\frac{\lambda^2 K}{t\epsilon} + \frac{(1-\lambda)^2(E+1)}{n^s}\right) \ln \frac{8E|\Pi|}{\delta}} + \left(\frac{\lambda K}{t\epsilon} + \frac{(1-\lambda)(E+1)}{n^s}\right) \ln \frac{8E|\Pi|}{\delta}.$$

Proof of Theorem 2. Define event I as: for all π in Π,
$$\left| \frac{1}{n^b} \sum_{t=1}^{n^b} \mathbb{E}[c^s(\pi(x_t))|x_t] - \mathbb{E}c^s(\pi(x)) \right| \leq \sqrt{\frac{\ln \frac{8|\Pi|}{\delta}}{2n^b}}. \tag{14}$$

Intuitively, under this event our regret measure for a policy is close to its expected value. By Hoeffding's inequality and union bound, I happens with probability $1 - \frac{\delta}{4}$. Specifically, on event I, as π^* is the policy in Π that minimizes $\mathbb{E}c^s(\pi(x))$, we have that for all π in Π,

$$\mathbb{E}c^s(\pi^*(x)) - \frac{1}{n^b} \sum_{t=1}^{n^b} \mathbb{E}[c^s(\pi(x_t))|x_t] \leq \mathbb{E}c^s(\pi(x)) - \frac{1}{n^b} \sum_{t=1}^{n^b} \mathbb{E}[c^s(\pi(x_t))|x_t] \leq \sqrt{\frac{\ln \frac{8|\Pi|}{\delta}}{2n^b}}.$$

Therefore,
$$\mathbb{E}c^s(\pi^*(x)) - \min_{\pi \in \Pi} \frac{1}{n^b} \sum_{t=1}^{n^b} \mathbb{E}[c^s(\pi(x_t))|x_t] \leq \sqrt{\frac{\ln \frac{8|\Pi|}{\delta}}{2n^b}}. \tag{15}$$

Recall that randomized policy $\pi_t : \mathcal{X} \to \Delta^{K-1}$ is defined as
$$\pi_t(x) = (1 - \epsilon)\pi_e^{\lambda_e}(x) + \frac{\epsilon}{K} \mathbb{1}_K, \tag{16}$$

for e in $\{1, 2, \ldots, E - 1\}$, $t \in (t_e, t_{e+1}]$, and $\pi_t(x) = \frac{1}{K} \mathbb{1}_K$ for all t in $(t_0, t_1]$. With a slight abuse of notation, denote by $\mathbb{E}c^s(\pi_t(x)) := \mathbb{E}_{(x,c^s), a \sim \pi_t(x)} c^s(a)$. Observe that for t in $(t_e, t_{e+1}]$, $\mathbb{E}[\mathbb{E}[c^s(a_t)|x_t]|\mathcal{B}_{\tilde{n}^s+t-1}] = \mathbb{E}c^s(\pi_t(x))$. Define event

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[6]Note that previously we used a cost of λ per example, whereas now it is λ per source, implying that the per example costs are λ/t_e and $(1 - \lambda)/n^s$ after t_e CB examples.

J as:

$$\left| \frac{1}{n^b} \sum_{t=1}^{n^b} \mathbb{E}[c^s(a_t)|x_t] - \frac{1}{n^b} \sum_{t=1}^{n^b} \mathbb{E}c^s(\pi_t(x)) \right| \leq \sqrt{\frac{\ln \frac{8}{\delta}}{2n^b}}. \tag{17}$$

By Azuma's inequality, J happens with probability $1 - \frac{\delta}{4}$.

Denote by E the event that the events E_e, F_e defined in Lemmas 7 and 8 (both given below) and I, J hold simultaneously for all t. By union bound over all E_e, F_e's and I, J, event E happens with probability $1 - \delta$. We henceforth condition on E happening.

Consider e in $\{1, 2, \ldots, E - 1\}$ and t in $(t_e, t_{e+1}]$. We now upper bound the expected excess cost of using randomized prediction π_t as defined in Equation (16). By the definition of π_t, we get

$$\mathbb{E}c^s(\pi_t(x)) = (1 - \epsilon)\mathbb{E}c^s(\pi_e(x)) + \epsilon \frac{1}{K} \sum_{a=1}^{K} \mathbb{E}c^s(a) \leq \mathbb{E}c^s(\pi_e(x)) + \epsilon. \tag{18}$$

Now, combining the above inequality with Lemmas 7 and 8, we have that for all t in $(t_e, t_{e+1}]$,

$$\mathbb{E}c^s(\pi_t(x)) \leq \min_{\lambda \in \Lambda} \mathbb{E}c^s(\pi_e^\lambda(x)) + \epsilon + \sqrt{\frac{2 \ln \frac{8E|\Lambda|}{\delta}}{\tilde{n}^s}}$$

$$\leq \mathbb{E}c^s(\pi^*(x)) + \epsilon + \sqrt{\frac{2 \ln \frac{8E|\Lambda|}{\delta}}{\tilde{n}^s}} + \min_{\lambda \in \Lambda} H_{t_e}(\lambda, \alpha, \Delta).$$

Summing the above inequality over all $t = t_1 + 1, \ldots, n^b$, grouping by epoch e, and using the fact that $0 \leq \mathbb{E}c^s(\pi_t(x)) \leq 1$ for $t \leq t_1 = 2$, we have

$$\sum_{t=t_1+1}^{n^b} \mathbb{E}c^s(\pi_t(x)) \leq n^b \mathbb{E}c^s(\pi^*(x)) + 2 + n^b \epsilon + (n^b - 2)\sqrt{\frac{2 \ln \frac{8E|\Lambda|}{\delta}}{\tilde{n}^s}} + \sum_{e=1}^{E-1} \sum_{t=t_e+1}^{t_{e+1}} \min_{\lambda \in \Lambda} H_{t_e}(\lambda, \alpha, \Delta).$$

Dividing both sides by n^b and some algebra yields

$$\frac{1}{n^b} \sum_{t=1}^{n^b} \mathbb{E}c^s(\pi_t(x)) \leq \frac{2}{n^b} + \mathbb{E}c^s(\pi^*(x)) + \epsilon + \sqrt{\frac{2 \ln \frac{8E|\Lambda|}{\delta}}{\tilde{n}^s}} + \frac{1}{n^b} \sum_{e=1}^{E-1} \sum_{t=t_e+1}^{t_{e+1}} \min_{\lambda \in \Lambda} H_{t_e}(\lambda, \alpha, \Delta).$$

Note that for every t in $[t_e + 1, t_{e+1}]$, as $t \leq t_{e+1} = 2t_e$, $W_t(\lambda) \geq \frac{1}{2}W_{t_e}(\lambda)$, hence $H_t(\lambda, \alpha, \Delta) \geq \frac{1}{2}H_{t_e}(\lambda, \alpha, \Delta)$. Therefore, the right hand side can be further upper bounded by

$$\frac{2}{n^b} + \mathbb{E}c^s(\pi^*(x)) + \epsilon + \sqrt{\frac{2 \ln \frac{8E|\Lambda|}{\delta}}{\tilde{n}^s}} + \frac{2}{n^b} \sum_{e=1}^{E-1} \sum_{t=t_e+1}^{t_{e+1}} \min_{\lambda \in \Lambda} H_t(\lambda, \alpha, \Delta)$$

$$\leq \frac{2}{n^b} + \mathbb{E}c^s(\pi^*(x)) + \epsilon + \sqrt{\frac{2 \ln \frac{8E|\Lambda|}{\delta}}{\tilde{n}^s}} + \min_{\lambda \in \Lambda} \frac{2}{n^b} \sum_{t=1}^{n^b} H_t(\lambda, \alpha, \Delta).$$

where the inequality is from that for any set of functions $\{N_t(\cdot)\}_{t=1}^{n^b}$, $\sum_{t=1}^{n^b} \min_{\lambda \in \Lambda} N_t(\lambda) \leq \min_{\lambda \in \Lambda}[\sum_{t=1}^{n^b} N_t(\lambda)]$.

To summarize, we have that

$$\frac{1}{n^b} \sum_{t=1}^{n^b} \mathbb{E}c^s(\pi_t(x)) - \mathbb{E}c^s(\pi^*(x)) \leq \frac{2}{n^b} + \epsilon + \sqrt{\frac{2 \ln \frac{8E|\Lambda|}{\delta}}{\tilde{n}^s}} + \min_{\lambda \in \Lambda} \frac{2}{n^b} \sum_{t=1}^{n^b} H_t(\lambda, \alpha, \Delta).$$

Combining with Equations (15) and (17) and using some algebra, we have

$$\frac{1}{n^b} \sum_{t=1}^{n^b} \mathbb{E}[c_t^s(\pi_t(x_t))|x_t] - \min_{\pi \in \Pi} \frac{1}{n^b} \sum_{t=1}^{n^b} \mathbb{E}[c^s(\pi(x_t))|x_t] \leq \epsilon + 3\sqrt{\frac{\ln \frac{8|\Pi|}{\delta}}{n^b}} + \sqrt{\frac{2 \ln \frac{8E|\Lambda|}{\delta}}{\tilde{n}^s}} + \min_{\lambda \in \Lambda} \frac{2}{n^b} \sum_{t=1}^{n^b} H_t(\lambda, \alpha, \Delta).$$

The theorem follows from the definition of $\tilde{n}^s$. $\qquad \square$

Lemma 7. *For every e, there exists an event E_e with probability $1 - \frac{\delta}{4E}$, on which for all λ in Λ, the excess cost of π_e^λ can*

be bounded as:

$$\mathbb{E}c^s(\pi_e^\lambda(x)) - \mathbb{E}c^s(\pi^*(x)) \le H_{t_e}(\lambda, \alpha, \Delta).$$

Proof. We first show the following concentration inequality: with probability $1 - \frac{\delta}{4E}$, for all π in Π,

$$\left| \hat{M}_{\lambda,t_e}(\pi) - M_{\lambda,t_e}(\pi) \right| \le W_{t_e}(\lambda). \tag{19}$$

To show the above statement, in light of the definitions of $\hat{M}, M$ and W, it suffices to show that for every π in Π, with probability $1 - \frac{\delta}{4E|\Pi|}$, we have

$$\left| \sum_{t=1}^{t_e} \frac{\lambda}{t_e} [\hat{c}_t(\pi(x_t)) - \mathbb{E}c(\pi(x))] + \sum_{(x,c^s)\in S} \frac{1-\lambda}{\tilde{n}^s} [c^s(\pi(x)) - \mathbb{E}c^s(\pi(x))] \right|$$

$$\le 2\sqrt{(\frac{\lambda^2 K}{t_e\epsilon} + \frac{(1-\lambda)^2}{\tilde{n}^s})\ln\frac{8E|\Pi|}{\delta}} + (\frac{\lambda K}{t_e\epsilon} + \frac{1-\lambda}{\tilde{n}^s})\ln\frac{8E|\Pi|}{\delta}. \tag{20}$$

For a fixed π, applying Lemma 2 with $X_i = \frac{1-\lambda}{\tilde{n}^s}(c_{n^b+i}^s(\pi(x_{n^b+i})) - \mathbb{E}c^s(\pi(x)))$ for i in $\{1, \ldots, \tilde{n}^s\}$, $X_i = \frac{\lambda}{t_e}(\hat{c}_{i-\tilde{n}^s}(\pi(x_{i-\tilde{n}^s})) - \mathbb{E}c^b(\pi(x)))$ for i in $\{\tilde{n}^s + 1, \ldots, \tilde{n}^s + t_e\}$, and $M = \frac{1-\lambda}{\tilde{n}^s} + \frac{\lambda K}{t_e\epsilon}$, and note that $|X_i| \le M$ almost surely for all i, $\mathbb{E}[X_i^2|\mathcal{B}_{i-1}] \le (\frac{1-\lambda}{\tilde{n}^s})^2$ for i in $\{1, \ldots, \tilde{n}^s\}$, and $\mathbb{E}[X_i^2|\mathcal{B}_{i-1}] \le (\frac{\lambda}{t_e})^2 \mathbb{E}[\frac{1}{p_{i-\tilde{n}^s,\pi(x_{i-\tilde{n}^s})}}|\mathcal{B}_{i-1}] \le \frac{\lambda^2 K}{t_e^2\epsilon}$ for i in $\{\tilde{n}^s + 1, \ldots, \tilde{n}^s + t_e\}$, we get that Equation (20) holds with probability $1 - \frac{\delta}{4E|\Pi|}$.

Therefore, by union bound over all π in Π Equation (19) holds for all λ simultaneously with probability $1 - \frac{\delta}{4E}$. As π_e^λ minimizes $\hat{M}_{\lambda,t}(\pi)$ over Π, we have that $\hat{M}_{\lambda,t}(\pi_e^\lambda) \le \hat{M}_{\lambda,t}(\pi^*)$. Combining this fact with Equation (19) on π_e^λ and π^*, we get

$$M_{\lambda,t_e}(\pi_e^\lambda) - M_{\lambda,t_e}(\pi^*) \le 2W_{t_e}(\lambda).$$

Observe that Lemma 1 and the definition of M implies that the left hand side of the above equation is at least $(\lambda\alpha + (1-\lambda))(\mathbb{E}c^s(\pi_e^\lambda(x)) - \mathbb{E}c^s(\pi^*(x))) - \lambda\Delta$. The lemma statement follows by straightforward algebra and the definition of $H_t(\cdot, \cdot, \cdot)$. □

Lemma 8. *For every e, there exists an event F_e with probability $1 - \frac{\delta}{4E}$, on which the policy $\pi_e^{\lambda_e}$ satisfies that*

$$\mathbb{E}c^s(\pi_e^{\lambda_e}) - \min_{\lambda\in\Lambda}\mathbb{E}c^s(\pi_e^\lambda(x)) \le \sqrt{\frac{2\ln\frac{8E|\Lambda|}{\delta}}{\tilde{n}^s}}.$$

Proof. Given any λ in Λ, as S_e^{val} is a sample independent of the π_e^λ, we have by Hoeffding's inequality that with probability $1 - \frac{\delta}{4E|\Lambda|}$,

$$\left| \mathbb{E}_{S_e^{\text{val}}}c^s(\pi_e^\lambda(x)) - \mathbb{E}c^s(\pi_e^\lambda(x)) \right| \le \sqrt{\frac{\ln\frac{8E|\Lambda|}{\delta}}{2\tilde{n}^s}}. \tag{21}$$

By union bound, with probability $1 - \frac{\delta}{4}$, Equation (21) holds for all λ in Λ simultaneously.

Observe that by the optimality of λ_e, for all λ in Λ, we have that

$$\mathbb{E}_{S_e^{\text{val}}}c^s(\pi_e^{\lambda_e}(x)) \le \mathbb{E}_{S_e^{\text{val}}}c^s(\pi_e^\lambda(x)).$$

Combining with Equation (21) on λ and λ_e, we get

$$\mathbb{E}c^s(\pi_e^{\lambda_e}(x)) \le \mathbb{E}c^s(\pi_e^\lambda(x)) + \sqrt{\frac{2\ln\frac{8E|\Lambda|}{\delta}}{\tilde{n}^s}}.$$

The lemma follows as the above inequality holds for every λ in Λ. □

G. Fixed choice of λ in Algorithm 2

Recall that

$$H_t(\lambda, \alpha, \Delta) = \frac{\lambda\Delta + 2W_t(\lambda)}{(1-\lambda) + \lambda\alpha},$$

and

$$W_t(\lambda) = 2\sqrt{\left(\frac{\lambda^2 K}{t\epsilon} + \frac{(1-\lambda)^2(E+1)}{n^s}\right)\ln\frac{8E|\Pi|}{\delta} + \left(\frac{\lambda K}{t\epsilon} + \frac{(1-\lambda)(E+1)}{n^s}\right)\ln\frac{8E|\Pi|}{\delta}}.$$

In addition, recall that $\frac{2}{n^b}\sum_{t=1}^{n^b} H_t(\lambda, \alpha, \Delta)$ is the last term of the regret bound Equation (5). We have the following proposition, showing that a single choice of λ ensures a small upper bound on this term when $\alpha = 1$ and $\Delta = 0$.

Proposition 2. *Suppose* $\lambda_0 = \frac{n^b\epsilon}{n^s K + n^b\epsilon}$. *Then,*

$$\frac{2}{n^b}\sum_{t=1}^{n^b} H_t(\lambda_0, 1, 0) = \tilde{O}\left(\sqrt{\frac{K\ln\frac{8E|\Pi|}{\delta}}{Kn^s + n^b\epsilon}} + \frac{K\ln\frac{8E|\Pi|}{\delta}}{Kn^s + n^b\epsilon}\right).$$

Proof. We have the following:

$$\frac{2}{n^b}\sum_{t=1}^{n^b} H_t(\lambda_0, 1, 0) \quad = \quad \frac{4}{n^b}\sum_{t=1}^{n^b} W_t(\lambda_0)$$

$$= \quad 4 \cdot \frac{1}{n^b}\sum_{t=1}^{n^b}\left[2\sqrt{\left(\frac{\lambda_0^2 K}{t\epsilon} + \frac{(1-\lambda_0)^2(E+1)}{n^s}\right)\ln\frac{8E|\Pi|}{\delta} + \left(\frac{\lambda_0 K}{t\epsilon} + \frac{(1-\lambda_0)(E+1)}{n^s}\right)\ln\frac{8E|\Pi|}{\delta}}.\right]$$

$$\leq \quad 4 \cdot \left[2\sqrt{\left(\frac{\lambda_0^2 K(\ln n^b + 1)}{n^b\epsilon} + \frac{(1-\lambda_0)^2(E+1)}{n^s}\right)\ln\frac{8E|\Pi|}{\delta} + \left(\frac{\lambda_0 K(\ln n^b + 1)}{n^b\epsilon} + \frac{(1-\lambda_0)(E+1)}{n^s}\right)\ln\frac{8E|\Pi|}{\delta}}.\right]$$

$$= \quad \tilde{O}\left(\sqrt{\left(\frac{\lambda_0^2 K}{n^b\epsilon} + \frac{(1-\lambda_0)^2}{n^s}\right)\ln\frac{8E|\Pi|}{\delta} + \left(\frac{\lambda_0 K}{n^b\epsilon} + \frac{(1-\lambda_0)}{n^s}\right)\ln\frac{8E|\Pi|}{\delta}}\right)$$

$$= \quad \tilde{O}\left(\sqrt{\frac{K\ln\frac{8E|\Pi|}{\delta}}{Kn^s + n^b\epsilon}} + \frac{K\ln\frac{8E|\Pi|}{\delta}}{Kn^s + n^b\epsilon}\right).$$

Where the first inequality is from that $H_t(\lambda_0, 1, 0) = 2W_t(\lambda_0)$; the second equality is from the definition of $W_t(\lambda)$; the inequality is by Jensen's inequality; the third equality is by dropping the logarithmic terms; the last inequality is by plugging in the choice of λ_0. □

H. Approximate optimality of $\Lambda = \{0, 1\}$

In this section, we show Proposition 3, which justifies that using $\Lambda = \{0, 1\}$ achieves a near-optimal regret bound in Theorem 1. Recall that

$$V_t(\lambda) = 2\sqrt{\left(\lambda^2\frac{Kt}{\epsilon} + (1-\lambda)^2 n^s\right)\ln\frac{8n^b|\Pi|}{\delta} + \left(\frac{\lambda K}{\epsilon} + (1-\lambda)\right)\ln\frac{8n^b|\Pi|}{\delta}},$$

$$G_t(\lambda, \alpha, \Delta) = \frac{(1-\lambda)n^s\Delta + 2V_t(\lambda)}{\lambda t + (1-\lambda)n^s\alpha}.$$

Proposition 3.

$$\min_{\lambda\in\{0,1\}} G_t(\lambda, \alpha, \Delta) \leq \sqrt{2}\min_{\lambda\in[0,1]} G_t(\lambda, \alpha, \Delta).$$

Proof. For any λ in $[0, 1]$, we give a lower bound on $V_t(\lambda)$. Note that by the fact that $\sqrt{a+b} \geq \sqrt{\frac{1}{2}}(\sqrt{a} + \sqrt{b})$, we have that

$$V_t(\lambda) \geq \frac{1}{\sqrt{2}}[(1-\lambda)V_t(0) + \lambda V_t(1)]. \tag{22}$$

Therefore,

$$G_t(\lambda, \alpha, \Delta)$$

$$\geq \frac{(1-\lambda)n^s\Delta + 2 \cdot \sqrt{\frac{1}{2}[(1-\lambda)V_t(0) + \lambda V_t(1)]}}{\lambda t + (1-\lambda)n^s\alpha}$$

$$\geq \frac{1}{\sqrt{2}} \frac{(1-\lambda)n^s\Delta + 2[(1-\lambda)V_t(0) + \lambda V_t(1)]}{\lambda t + (1-\lambda)n^s\alpha}$$

$$\geq \frac{1}{\sqrt{2}} \min\left(\frac{n^s\Delta + 2V_t(0)}{n^s\alpha}, \frac{2V_t(1)}{t}\right)$$

$$= \frac{1}{\sqrt{2}} \min_{\lambda \in \{0,1\}} G_t(\lambda, \alpha, \Delta),$$

where the first inequality is from Equation (22), the second inequality is by algebra, the third inequality is by the quasi-concavity of the LHS with respect to λ, and the fact that the minimum of a quasi-concave function over a convex set is attained at its boundary. As the above holds for any λ in $[0, 1]$, the proposition follows. $\square$

I. Combining Two Sources in Supervised Learning

Proposition 4. *For every policy class Π of VC dimension d and $\Delta \in [0, \frac{1}{4}]$, $m, n \geq 32d$, for any algorithm that outputs a policy $\hat{\pi}$ based on n examples from D^1 and m examples from D^2, there exists a pair of distributions (D^1, D^2) such that D^2 is $(1, \Delta)$-similar to D^1, and*

$$\mathbb{E}[\mathbb{E}_{D^1}c(\hat{\pi}(x)) - \min_{\pi \in \Pi}\mathbb{E}_{D^1}c(\pi(x))] \geq \tfrac{1}{16} \min\left(\sqrt{d/m} + 8\Delta, \sqrt{d/n}\right)$$

where the outer expectation is over draws from D^1 and D^2, and the algorithm's randomness.

The lower bound proof follows a similar strategy as in the classical classification setting with slight modifications. In order to prove the bound we make use of Assouad's Lemma. The statement below follows exactly from Yu (Yu, 1997).

Theorem 3 (Assouad's Lemma). *Let $d \geq 1$ be an integer and let $\mathcal{F}_d = \{P_\tau \mid \tau \in \{-1, +1\}^d\}$ be a class of 2^d probability measures indexed by binary strings of length d. Write $\tau \sim \tau'$ if τ and τ' differ in only one coordinate, and write $\tau \sim_j \tau'$ when that coordinate is the jth. Suppose that there are d pseudo-distances on $\mathcal{D}$ such that for any $x, y \in \mathcal{D}$*

$$\rho(x, y) = \sum_{j=1}^d \rho_j(x, y),$$

and further that, if $\tau \sim_j \tau'$,

$$\rho_j(\theta(P_\tau), \theta(P_{\tau'})) \geq \alpha.$$

Then for any estimator $\widehat{\theta}$,

$$\max_\tau \mathbb{E}_\tau \rho(\widehat{\theta}, \theta(P_\tau)) \geq d \cdot \frac{\alpha}{2} \min(\|P_\tau \wedge P_{\tau'}\| \mid \tau \sim \tau'\}.$$

Proof of Proposition 4. Since the VC dimension of Π is d, there exists a set $A = \{x_1, \ldots, x_d\}$ such that for any binary sequence $\tau \in \{-1, +1\}^d$, there exists some function π in Π such that for all $i \in \{1, \ldots, d\}$, $\pi(x_i) = \tau_i$. We now define two distributions for a binary classification problem, but express them as joint distributions over (x, c) in order to be consistent with the rest of the paper with the understanding that c will have a zero-one cost structure (where $c = (c(-1), c(+1))$ represent the costs of predicting labels -1 and $+1$). The two distributions D_τ^1 and D_τ^2 are each uniform over x_i, and the conditional distributions over costs are given by

$$D_\tau^1((1,0)|x = x_i) = \frac{1}{2} + \tau_i\epsilon$$

$$D_\tau^1((0,1)|x = x_i) = \frac{1}{2} - \tau_i\epsilon, \quad \text{and}$$

$$D_\tau^2((1,0)|x = x_i) = \frac{1}{2} + \tau_i\epsilon - \tau_i\beta$$

$$D_\tau^2((0,1)|x = x_i) = \frac{1}{2} - \tau_i\epsilon + \tau_i\beta.$$

Where $\epsilon \in [0, \frac{1}{2}]$, $\beta \in [0, \frac{\Delta}{2}]$ are free parameters to be determined later. Clearly D_τ^2 is $(1, \Delta)$-strongly similar to D_τ^1, therefore D_τ^2 is $(1, \Delta)$-similar to D_τ^1. This can be seen by taking $c^+ = \mathbb{E}_{D^1}[c|x]$ and $c^- = \mathbb{E}_{D^2}[c|x] - \mathbb{E}_{D^1}[c|x]$, and observing that $|\mathbb{E}c^-(\pi(x))| \leq \beta \leq \Delta/2$. Define P_τ as the product distribution of n copies of D_τ^1 and m copies of D_τ^2, which is the joint distribution of the input examples to the algorithm. In addition, define $\theta(P_\tau) = \tau$, and $\rho_j(\tau, \tau') = I(\tau_j \neq \tau'_j)$. Therefore, $\rho(\tau, \tau') = \sum_{j=1}^d \rho_j(\tau, \tau') = \sum_{j=1}^d I(\tau_j \neq \tau'_j)$ is the Hamming distance between τ and τ'. Suppose the algorithm returns a policy $\hat{\pi}$, there exists a binary sequence $\hat{\tau}$ such that $\hat{\pi}(x_i) = \hat{\tau}_i$. Observe that for any τ, by the definition of D_τ^1,

$$\mathbb{E}_{D_\tau^1} c(\hat{\pi}(x)) - \min_{\pi \in \Pi} \mathbb{E}_{D_\tau^1} c(\pi(x)) = \frac{2\epsilon}{d} \rho(\hat{\tau}, \tau).$$

By Assouad's Lemma (Theorem 3) with the ρ defined above and $\alpha = 1$, we have that there exists some τ in $\{-1, +1\}^d$, such that

$$\mathbb{E}\rho(\hat{\tau}, \tau) \geq \epsilon \min(\|P_\tau \wedge P_{\tau'}\|_1 \mid \tau \sim \tau')$$

This immediately implies that, for any algorithm that returns $\hat{\pi}$, there exists some τ such that

$$\mathbb{E}_{P_\tau}[c(\hat{\pi}(x)) - \min_{\pi \in \Pi} \mathbb{E}_{D_\tau^1} c(\pi(x))] \geq \epsilon \min(\|P_\tau \wedge P_{\tau'}\|_1 \mid \tau \sim \tau').$$

What remains is to bound $\|P_\tau \wedge P_{\tau'}\|_1$ for any binary sequences differing in one coordinate. Recall that $\|P_\tau \wedge P_{\tau'}\| = 1 - \frac{1}{2}\|P_\tau - P_{\tau'}\|_1$. We will bound $\frac{1}{2}\|P_\tau - P_{\tau'}\|_1 \leq H(P_\tau, P_{\tau'})$ using the Hellinger distance $H^2(P_\tau, P_{\tau'}) = \frac{1}{2}\sum_z (\sqrt{P_\tau(z)} - \sqrt{P_{\tau'}(z)})^2$. We recall that P_τ is in fact the product distribution over n copies of D_τ^1 and m copies of D_τ^2. For product measures

$$H^2(P_\tau, P_{\tau'}) \leq \sum_{i=1}^n (D_\tau^1, D_{\tau'}^1) + \sum_{i=1}^m (D_\tau^2, D_{\tau'}^2)$$

we need to bound the Hellinger distance for the biased and unbiased distributions.

Bounding the Hellinger Distance   We have

$$H^2(D_\tau^1, D_{\tau'}^1) = \frac{1}{2}\sum_{i=1}^d \sum_{c \in \{(0,1),(1,0)\}} (\sqrt{D_\tau^1(x_i, c)} - \sqrt{D_{\tau'}^1(x_i, c)})^2$$

$$= \frac{1}{2d}\sum_{c \in \{(0,1),(1,0)\}} (\sqrt{D_\tau^1(c|x_i)} - \sqrt{D_{\tau'}^1(c|x_i)})^2$$

$$= \frac{1}{d} H^2(B(\frac{1}{2} + \epsilon), B(\frac{1}{2} - \epsilon))$$

$$\leq \frac{8}{d}\epsilon^2,$$

where the first equality is by the definition of H^2; the second inequality is from that there is exactly one i such that $\sum_{c \in \{(0,1),(1,0)\}} (\sqrt{D_\tau^1(x_i, c)} - \sqrt{D_{\tau'}^1(x_i, c)})^2$ is nonzero, as $\tau \sim \tau'$; in the right hand side of third equality, $B(p)$ is the Bernoulli distribution with mean parameter p. Similarly,

$$H^2(D_\tau^2, D_{\tau'}^2) \leq \frac{8}{d}(\epsilon - \beta)^2$$

Hence,

$$H^2(P_\tau, P_{\tau'}) \leq \frac{8}{d}\left(n\epsilon^2 + m(\epsilon - \beta)^2\right)$$

Therefore, we have

$$\max_\tau \mathbb{E}_{P_\tau}[c(\hat{\pi}(x)) - \min_{\pi \in \Pi} \mathbb{E}_{D_\tau^1} c(\pi(x))] \geq \epsilon \left[1 - \sqrt{\frac{8}{d}\left(n\epsilon^2 + m(\epsilon - \beta)^2\right)}\right] \tag{23}$$

We now consider two separate cases regarding the settings of m, n, d and Δ.

Case 1: $\sqrt{\frac{d}{n}} \leq \sqrt{\frac{d}{m}} + 4\Delta$.   In this case, we let $\epsilon = \frac{1}{8}\sqrt{\frac{d}{n}}$ and $\beta = \frac{1}{8}\max(0, \sqrt{\frac{d}{n}} - \sqrt{\frac{d}{m}}) \in [0, \frac{\Delta}{2}]$. This gives that the right hand side of Equation (23) is at least $\frac{1}{8}\sqrt{\frac{d}{n}} \cdot \frac{1}{2} = \frac{1}{16}\sqrt{\frac{d}{n}}$.

Case 2: $\sqrt{\frac{d}{n}} > \sqrt{\frac{d}{m}} + 4\Delta$. In this case, we let $\epsilon = \frac{1}{8}\sqrt{\frac{d}{m}} + \frac{\Delta}{2}$ and $\beta = \frac{\Delta}{2}$. This gives that the right hand side of Equation (23) is at least $(\frac{1}{8}\sqrt{\frac{d}{m}} + \frac{\Delta}{2}) \cdot \frac{1}{2} = \frac{1}{16}(\sqrt{\frac{d}{m}} + 4\Delta)$.

In summary, for any choice of m, n, d, Δ with $m, n \geq 32d$ and $\Delta \in [0, \frac{1}{4}]$, we can find a pair of distributions (D_τ^1, D_τ^2) such that

$$\mathbb{E}_{P_\tau}[c(\hat{\pi}(x)) - \min_{\pi \in \Pi} \mathbb{E}_{D_\tau^1} c(\pi(x))] \geq \frac{1}{16} \min(\sqrt{\frac{d}{n}}, \sqrt{\frac{d}{m}} + 4\Delta).$$

The proposition follows. $\qquad\square$

J. Additional Experimental Results

We give a collection of cumulative distribution functions (CDFs) of our algorithms evaluated against different exploration parameters ϵ, noise settings and warm start ratios below:

1. In Figures 4 to 9, we present CDFs where all CB algorithms use ϵ-greedy with parameter $\epsilon = 0.0125$.

2. In Figures 10 to 15, we present CDFs where all CB algorithms use ϵ-greedy with parameter $\epsilon = 0.00625$.

3. In Figures 16 to 21, we present CDFs where all CB algorithms use ϵ-greedy with parameter $\epsilon = 0.1$.

The general trends in this more detailed comparison are similar to those observed in Section 5. For less noisy and small warm-start ratios, SUP-ONLY is particularly difficult as a baseline since it performs no exploration. With extreme noise, BANDIT-ONLY is the best as the supervised examples are misleading. ARROW-CB competes well on both the extremes, while outperforming all methods in several regimes. Importantly, ARROW-CB always beats the other methods that attempt to leverage both sources of data, and prevents the significant performance hit from relying on the wrong data source in either of the two extreme cases.
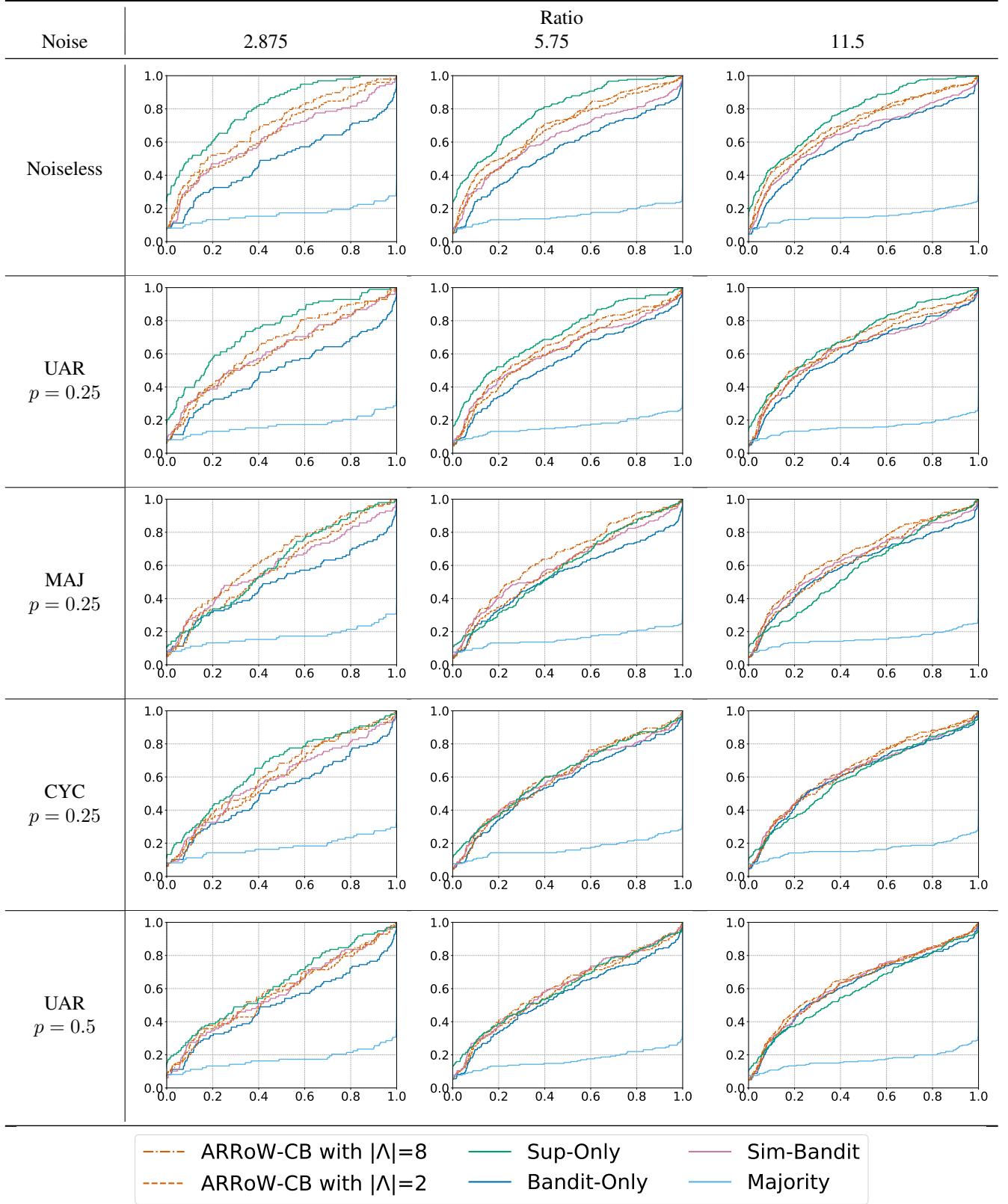
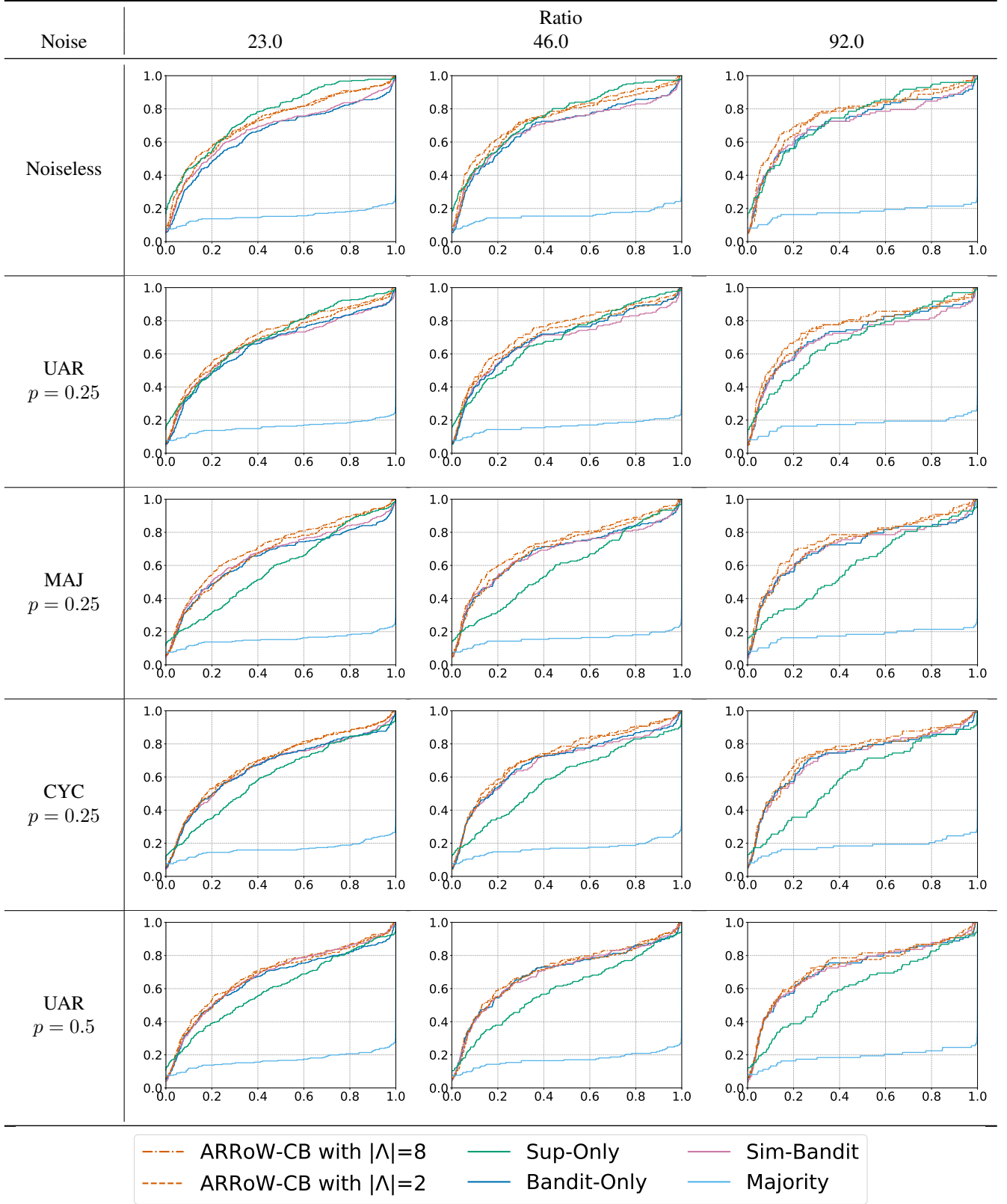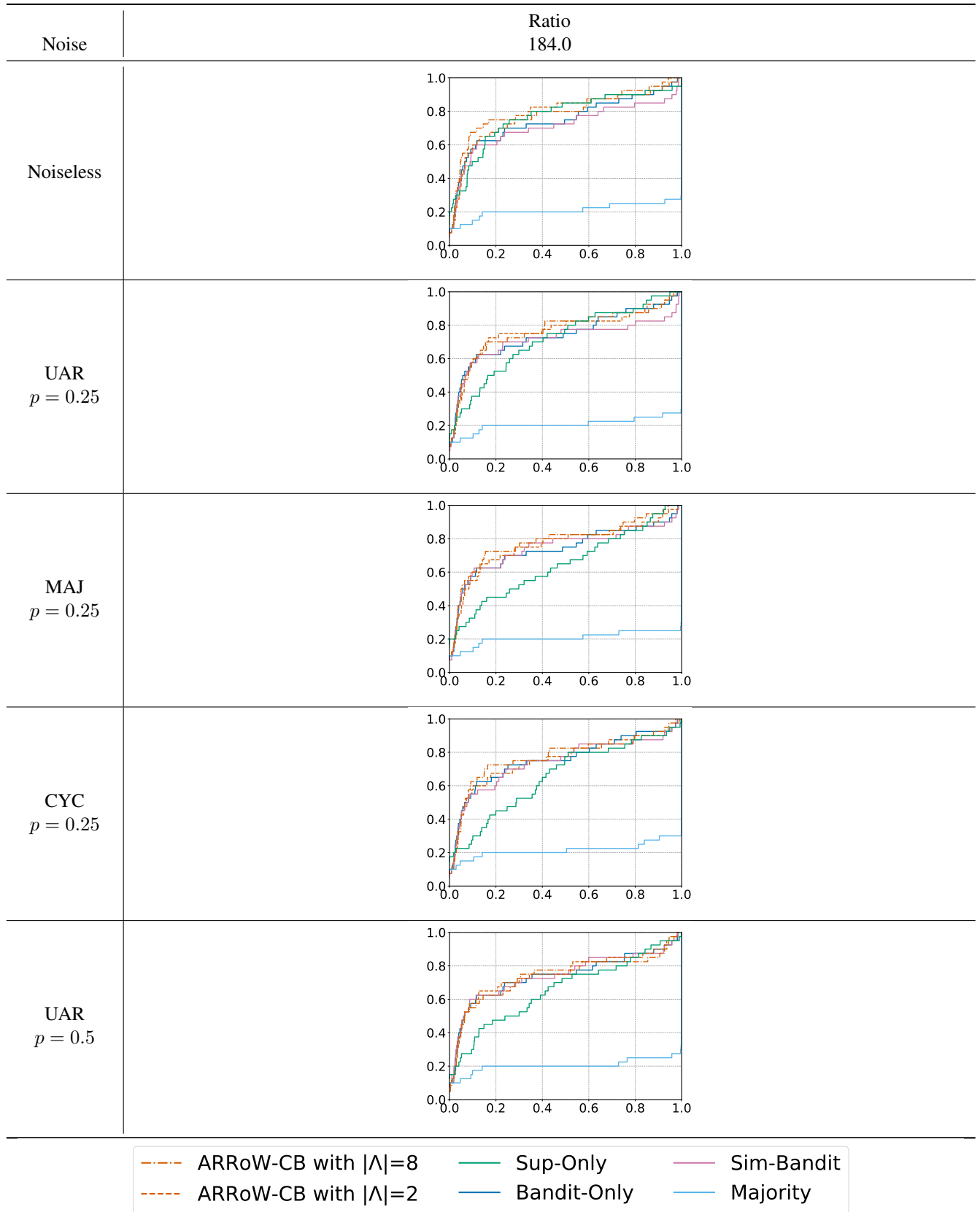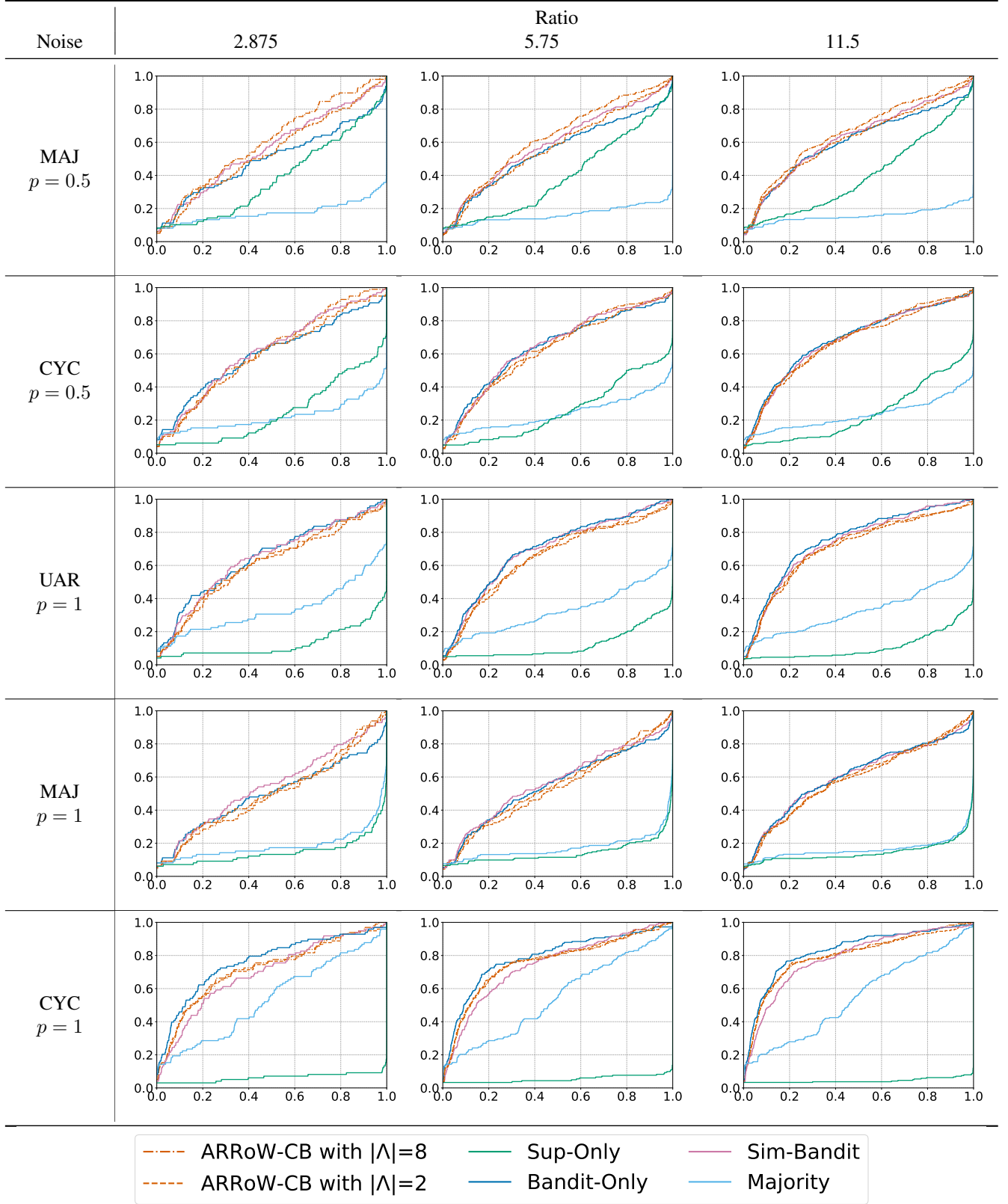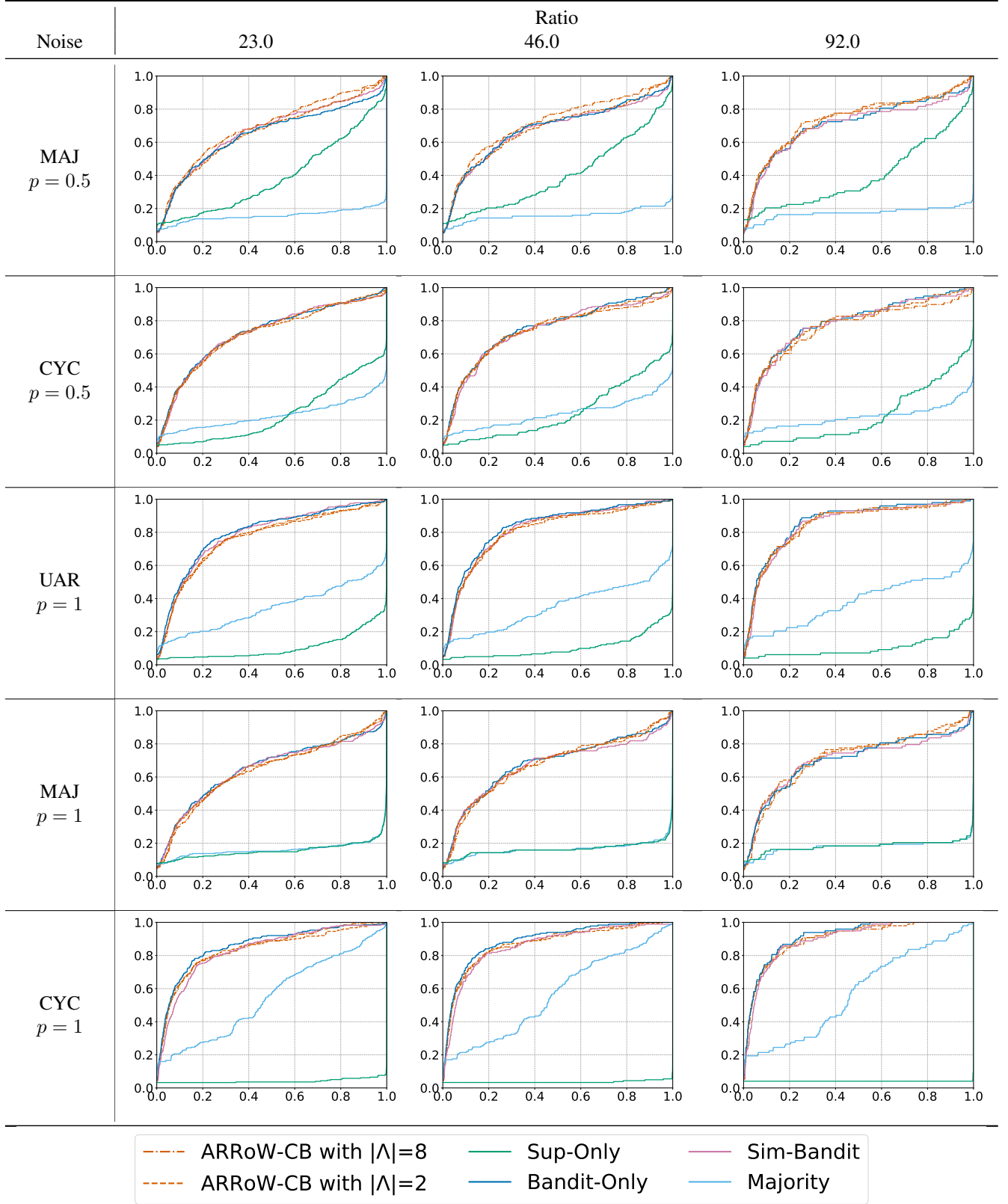Figure 4: Cumulative distribution functions (CDFs) for all evaluated algorithms, against different noise settings and warm start ratios in $\{2.875, 5.75, 11.5\}$. All CB algorithms use ϵ-greedy with $\epsilon = 0.0125$. In each of the above plots, the x axis represents scores, while the y axis represents the CDF values.
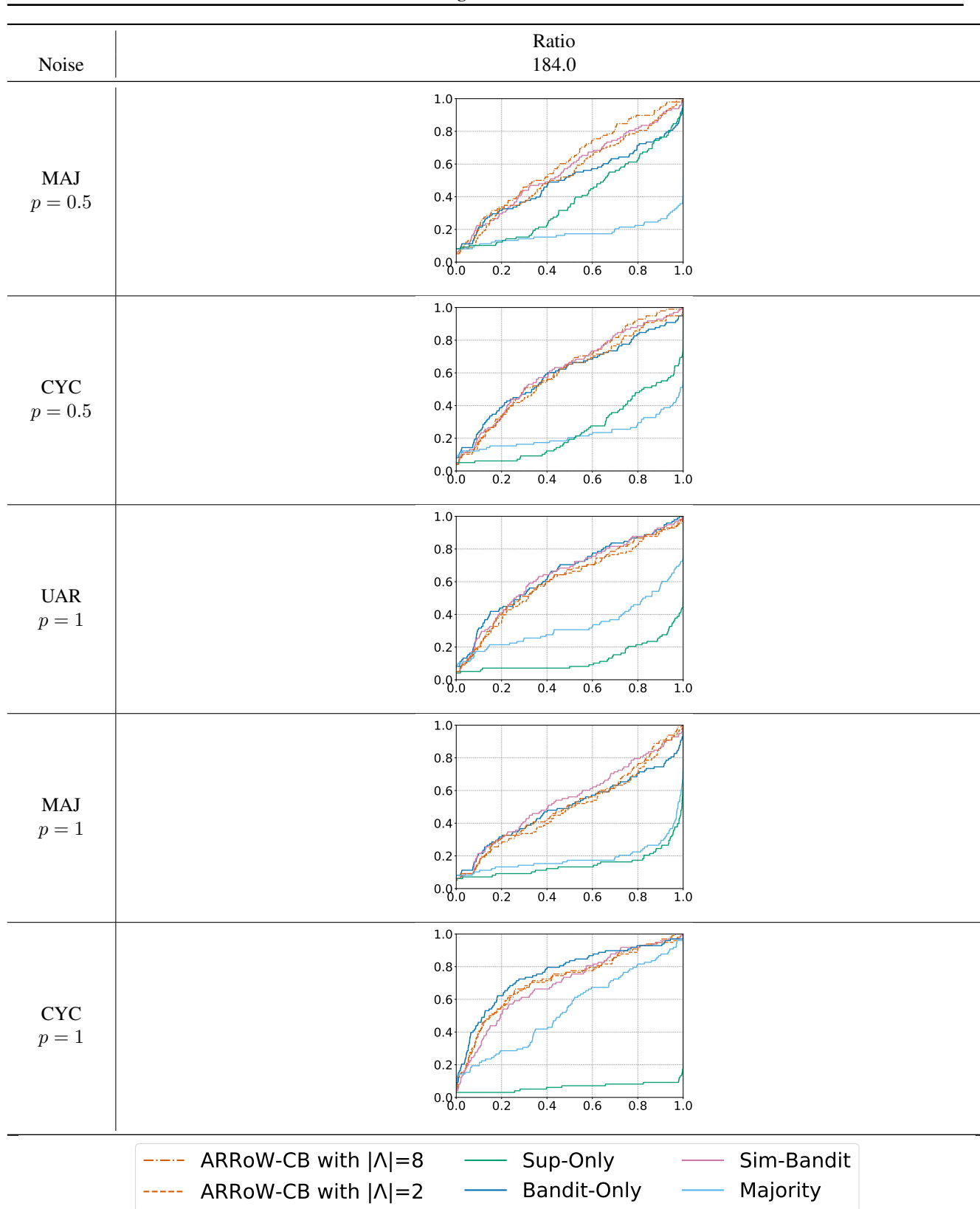
Figure 5: Cumulative distribution functions (CDFs) for all evaluated algorithms, against different noise settings and warm start ratios in $\{23.0, 46.0, 92.0\}$. All CB algorithms use ϵ-greedy with $\epsilon = 0.0125$. In each of the above plots, the x axis represents scores, while the y axis represents the CDF values.
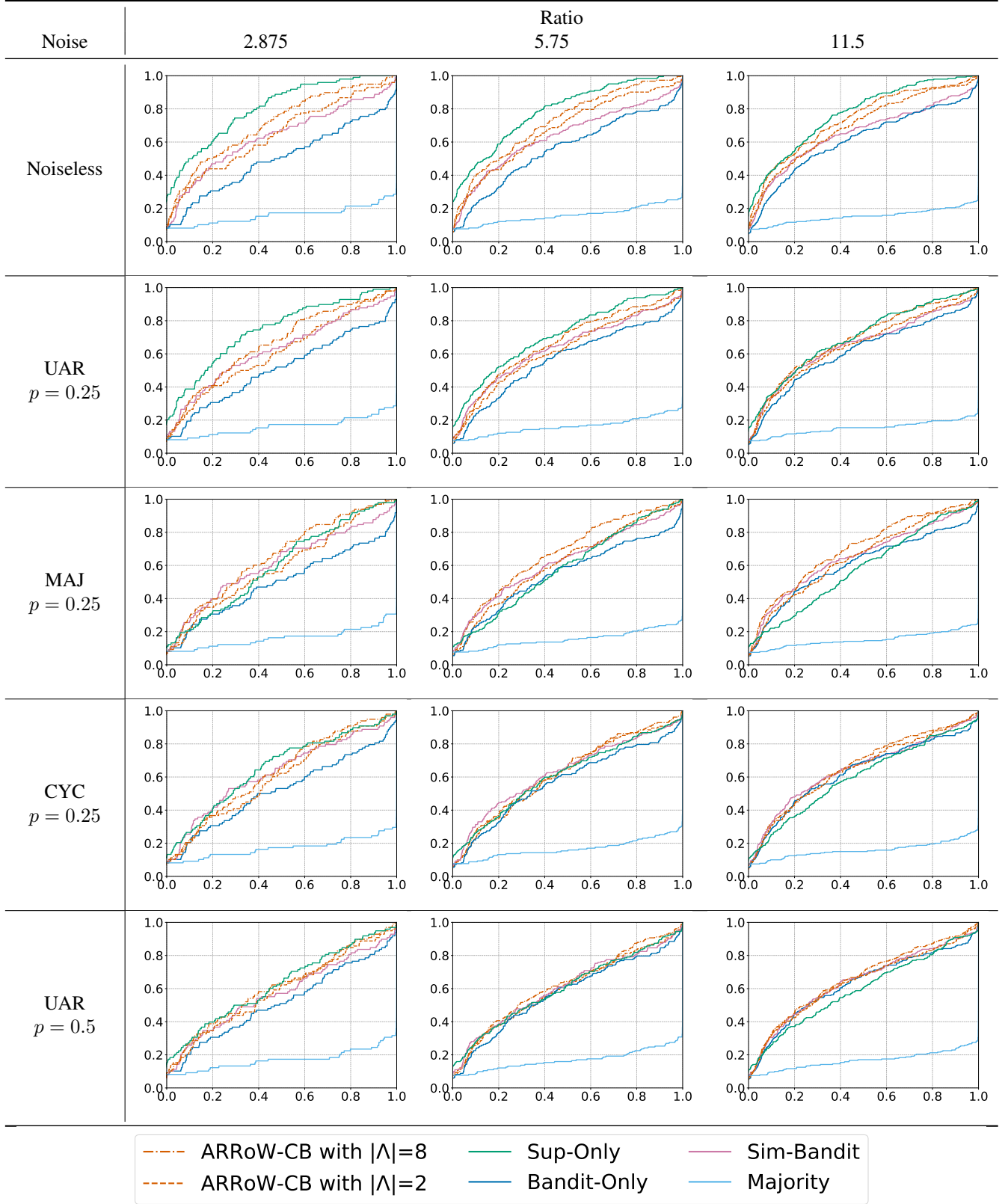
Figure 6: Cumulative distribution functions (CDFs) for all evaluated algorithms, against different noise settings and warm start ratios in $\{184.0\}$. All CB algorithms use ϵ-greedy with $\epsilon = 0.0125$. In each of the above plots, the x axis represents scores, while the y axis represents the CDF values.

Figure 7: Cumulative distribution functions (CDFs) for all evaluated algorithms, against different noise settings and warm start ratios in $\{2.875, 5.75, 11.5\}$. All CB algorithms use ϵ-greedy with $\epsilon = 0.0125$. In each of the above plots, the x axis represents scores, while the y axis represents the CDF values.
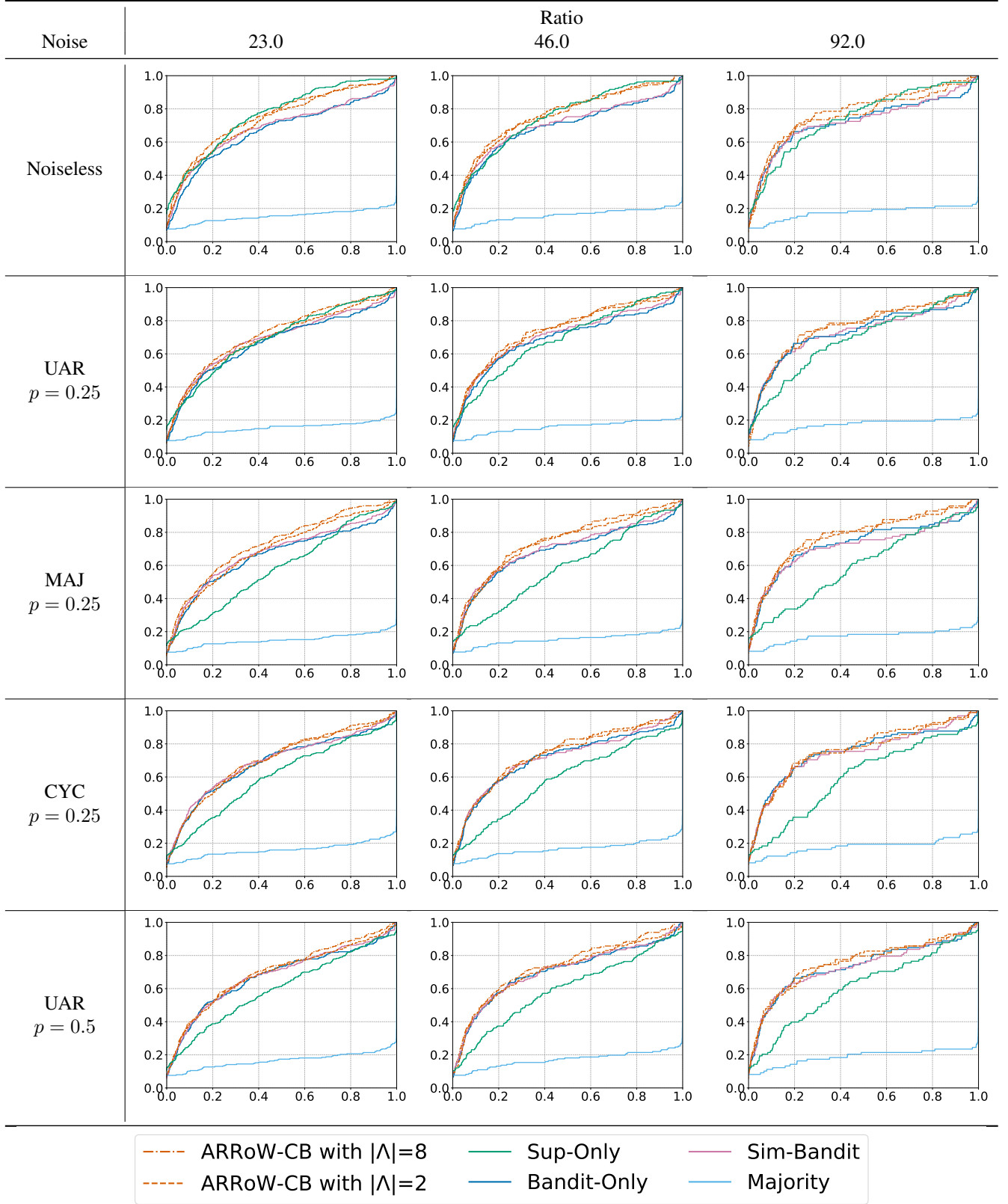
Figure 8: Cumulative distribution functions (CDFs) for all evaluated algorithms, against different noise settings and warm start ratios in $\{23.0, 46.0, 92.0\}$. All CB algorithms use ϵ-greedy with $\epsilon = 0.0125$. In each of the above plots, the x axis represents scores, while the y axis represents the CDF values.
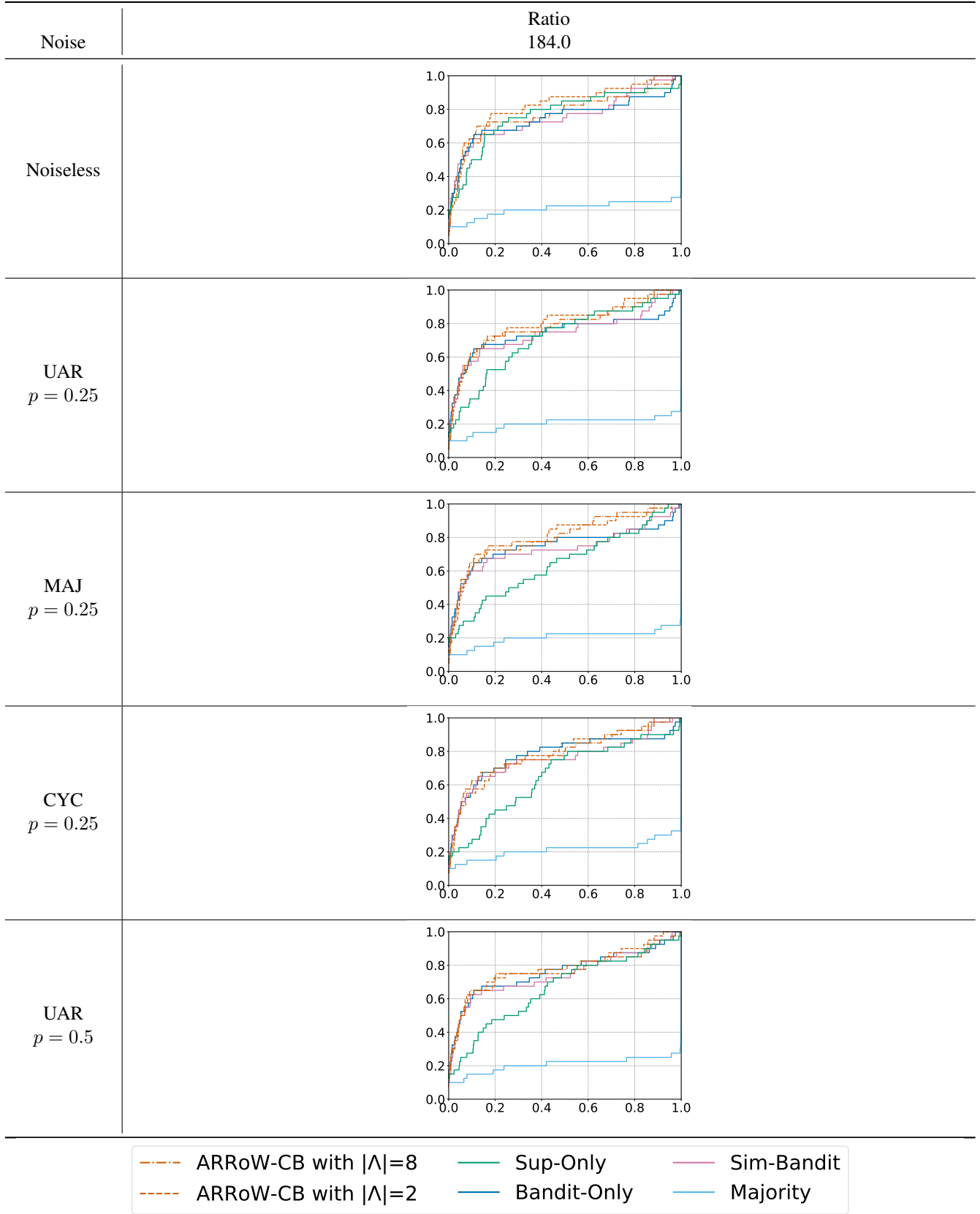
Figure 9: Cumulative distribution functions (CDFs) for all evaluated algorithms, against different noise settings and warm start ratios in $\{184.0\}$. All CB algorithms use ϵ-greedy with $\epsilon = 0.0125$. In each of the above plots, the x axis represents scores, while the y axis represents the CDF values.

Figure 10: Cumulative distribution functions (CDFs) for all evaluated algorithms, against different noise settings and warm start ratios in $\{2.875, 5.75, 11.5\}$. All CB algorithms use ϵ-greedy with $\epsilon = 0.00625$. In each of the above plots, the x axis represents scores, while the y axis represents the CDF values.
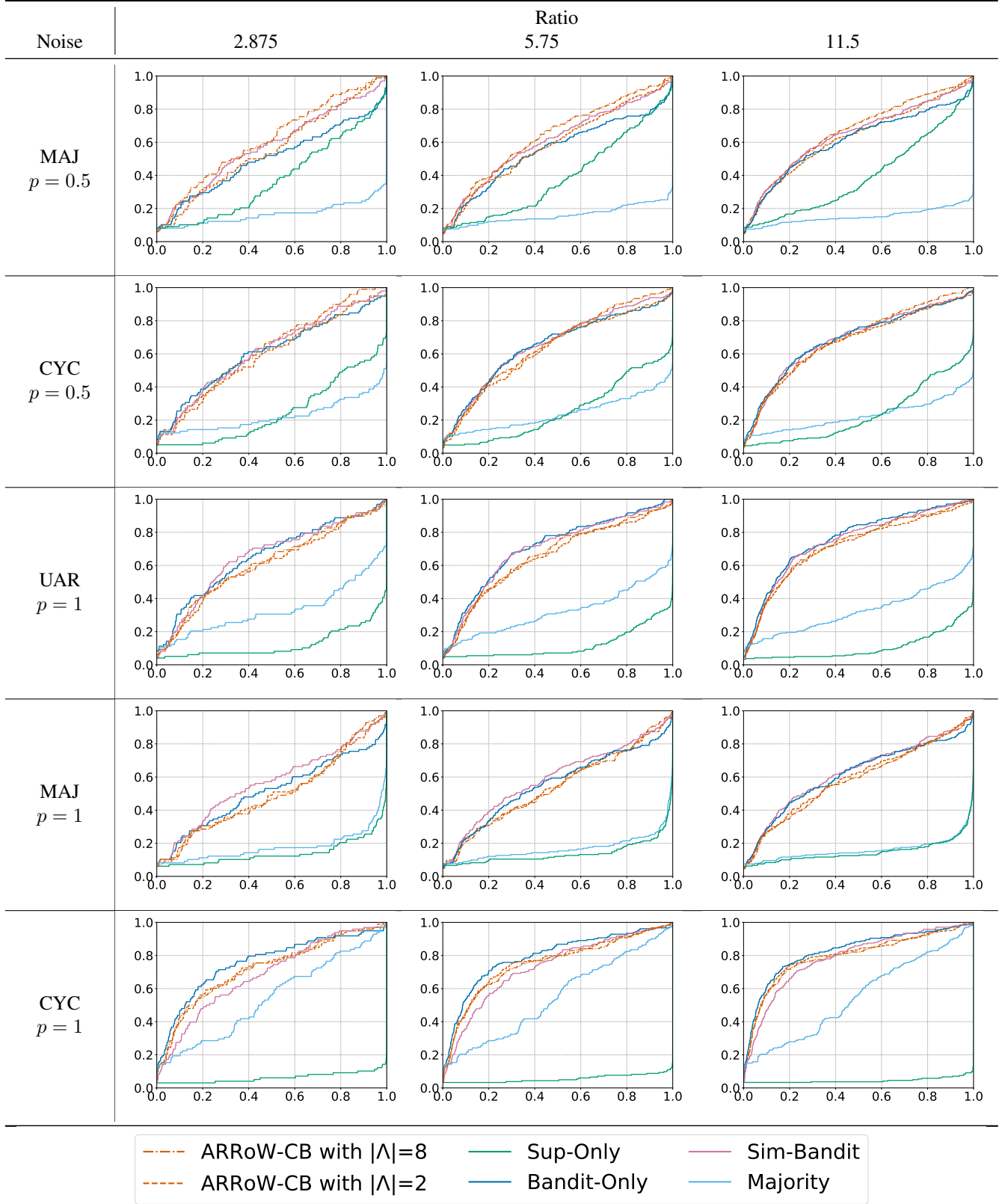
Figure 11: Cumulative distribution functions (CDFs) for all evaluated algorithms, against different noise settings and warm start ratios in $\{23.0, 46.0, 92.0\}$. All CB algorithms use ϵ-greedy with $\epsilon = 0.00625$. In each of the above plots, the x axis represents scores, while the y axis represents the CDF values.
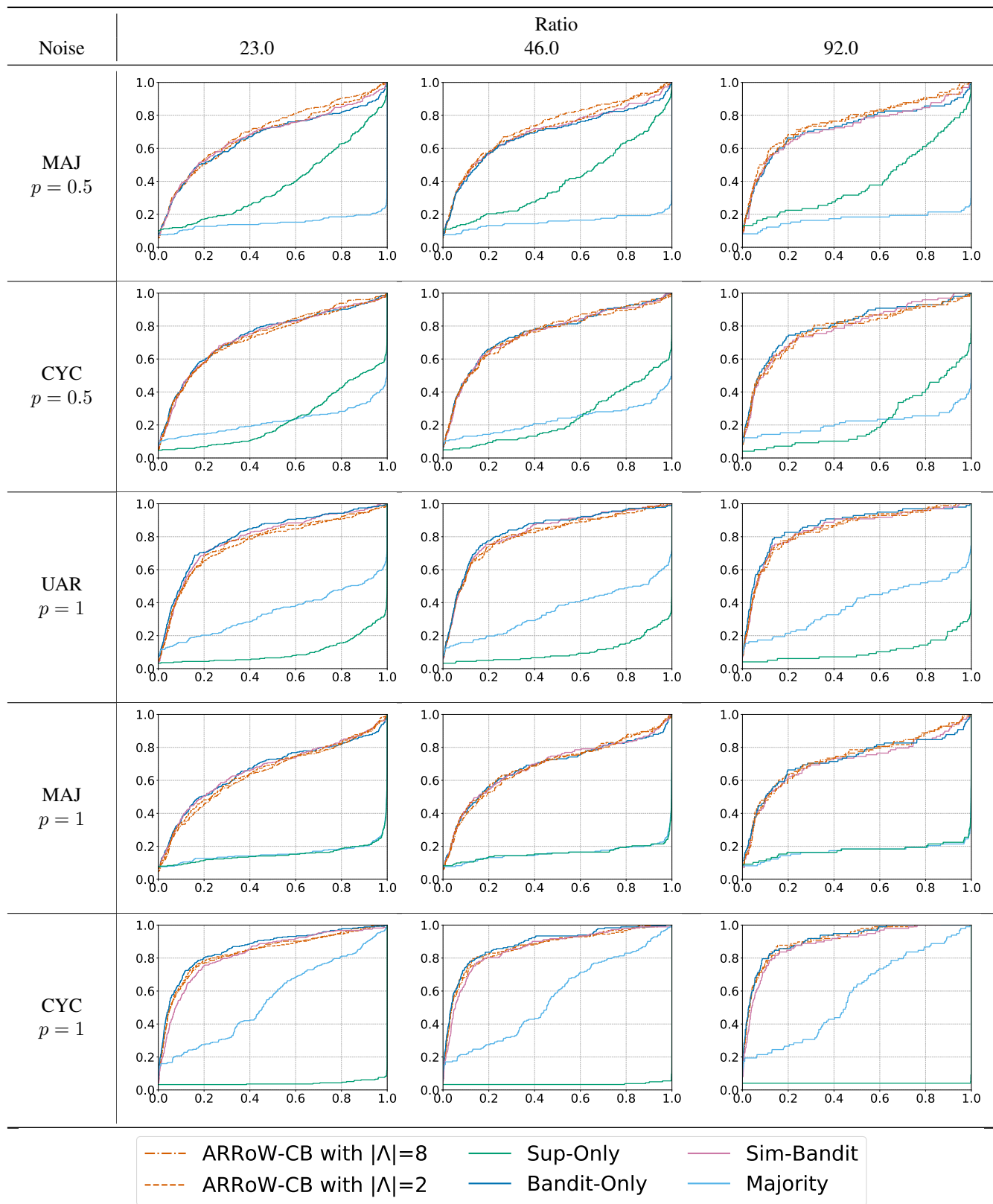
Figure 12: Cumulative distribution functions (CDFs) for all evaluated algorithms, against different noise settings and warm start ratios in $\{184.0\}$. All CB algorithms use ϵ-greedy with $\epsilon = 0.00625$. In each of the above plots, the x axis represents scores, while the y axis represents the CDF values.

Figure 13: Cumulative distribution functions (CDFs) for all evaluated algorithms, against different noise settings and warm start ratios in $\{2.875, 5.75, 11.5\}$. All CB algorithms use ϵ-greedy with $\epsilon = 0.00625$. In each of the above plots, the x axis represents scores, while the y axis represents the CDF values.
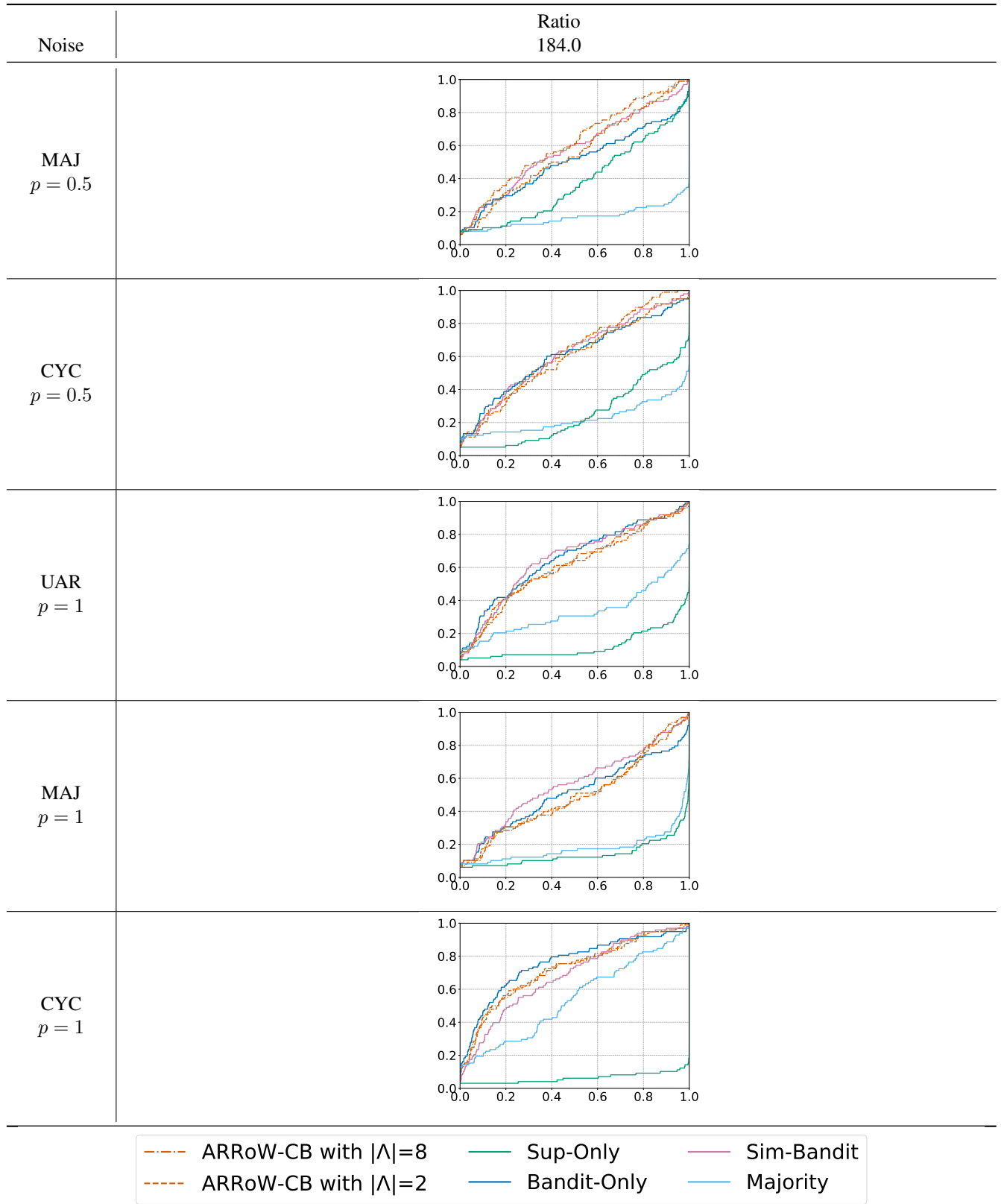
Figure 14: Cumulative distribution functions (CDFs) for all evaluated algorithms, against different noise settings and warm start ratios in $\{23.0, 46.0, 92.0\}$. All CB algorithms use ϵ-greedy with $\epsilon = 0.00625$. In each of the above plots, the x axis represents scores, while the y axis represents the CDF values.

Noise	Ratio 184.0
MAJ $p = 0.5$	
CYC $p = 0.5$	
UAR $p = 1$	
MAJ $p = 1$	
CYC $p = 1$	

- - · - ARRoW-CB with $|\Lambda|=8$ —— Sup-Only   —— Sim-Bandit
- - - ARRoW-CB with $|\Lambda|=2$ —— Bandit-Only   —— Majority

Figure 15: Cumulative distribution functions (CDFs) for all evaluated algorithms, against different noise settings and warm start ratios in $\{184.0\}$. All CB algorithms use ϵ-greedy with $\epsilon = 0.00625$. In each of the above plots, the x axis represents scores, while the y axis represents the CDF values.
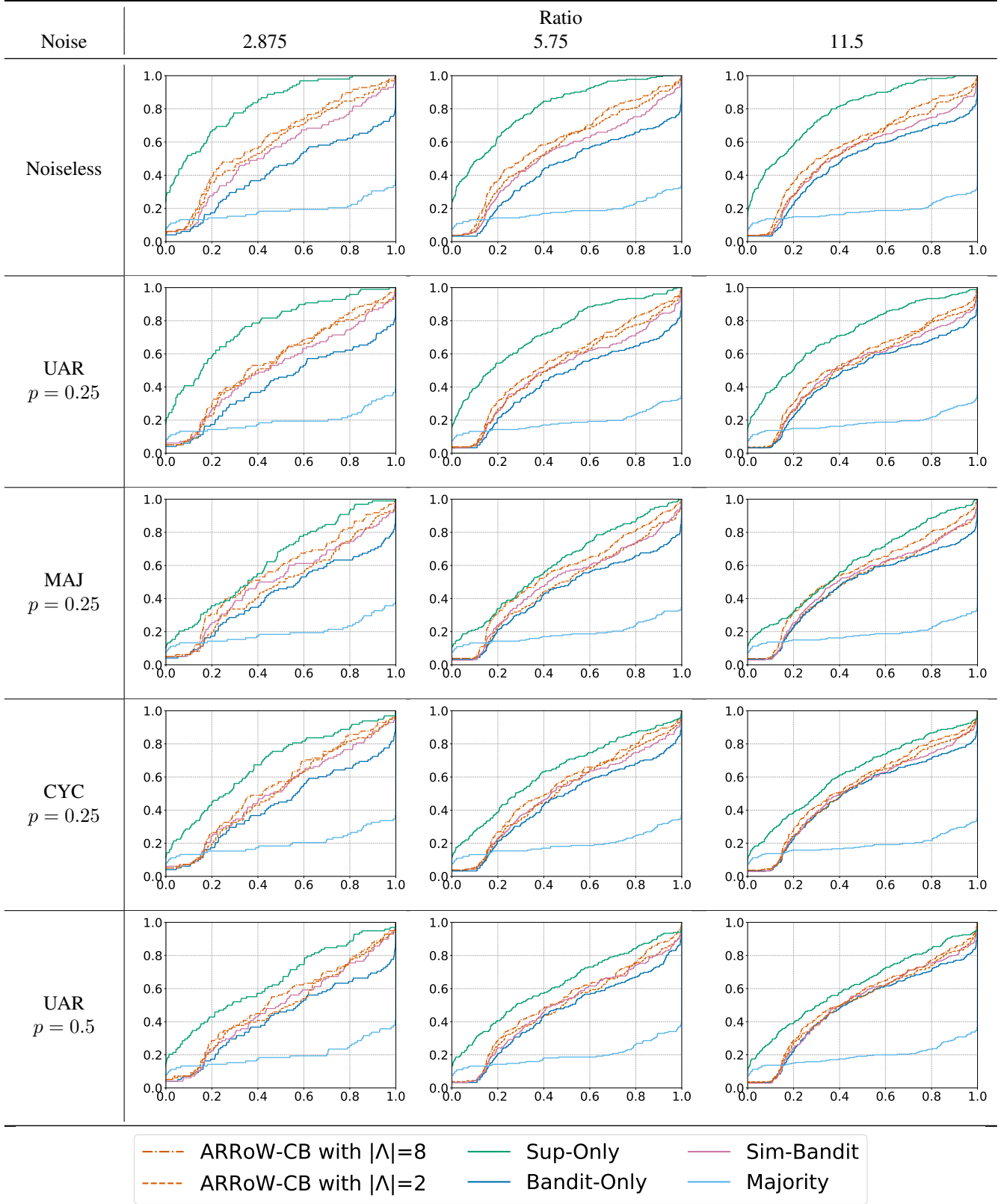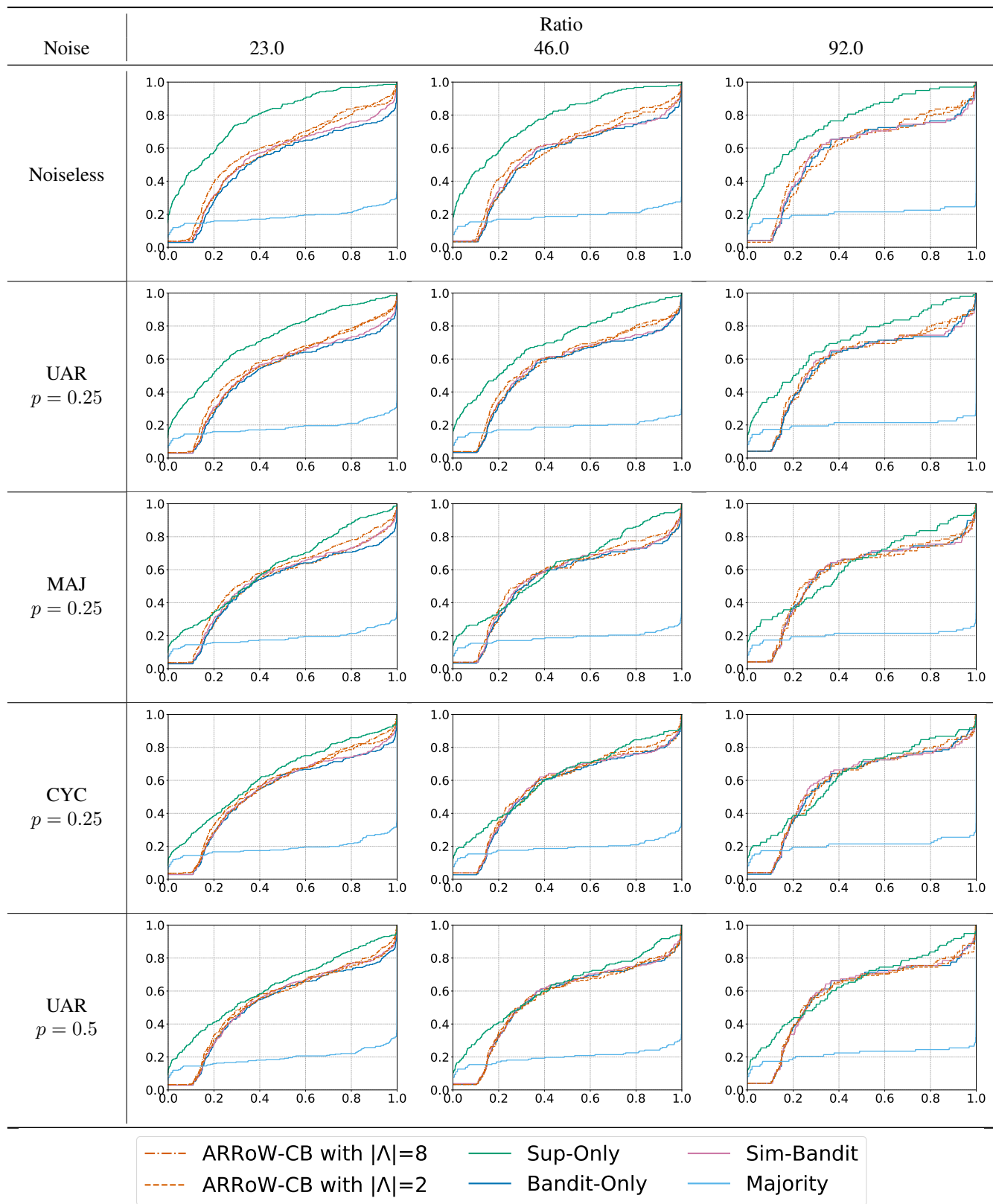
Figure 16: Cumulative distribution functions (CDFs) for all evaluated algorithms, against different noise settings and warm start ratios in $\{2.875, 5.75, 11.5\}$. All CB algorithms use ϵ-greedy with $\epsilon = 0.1$. In each of the above plots, the x axis represents scores, while the y axis represents the CDF values.
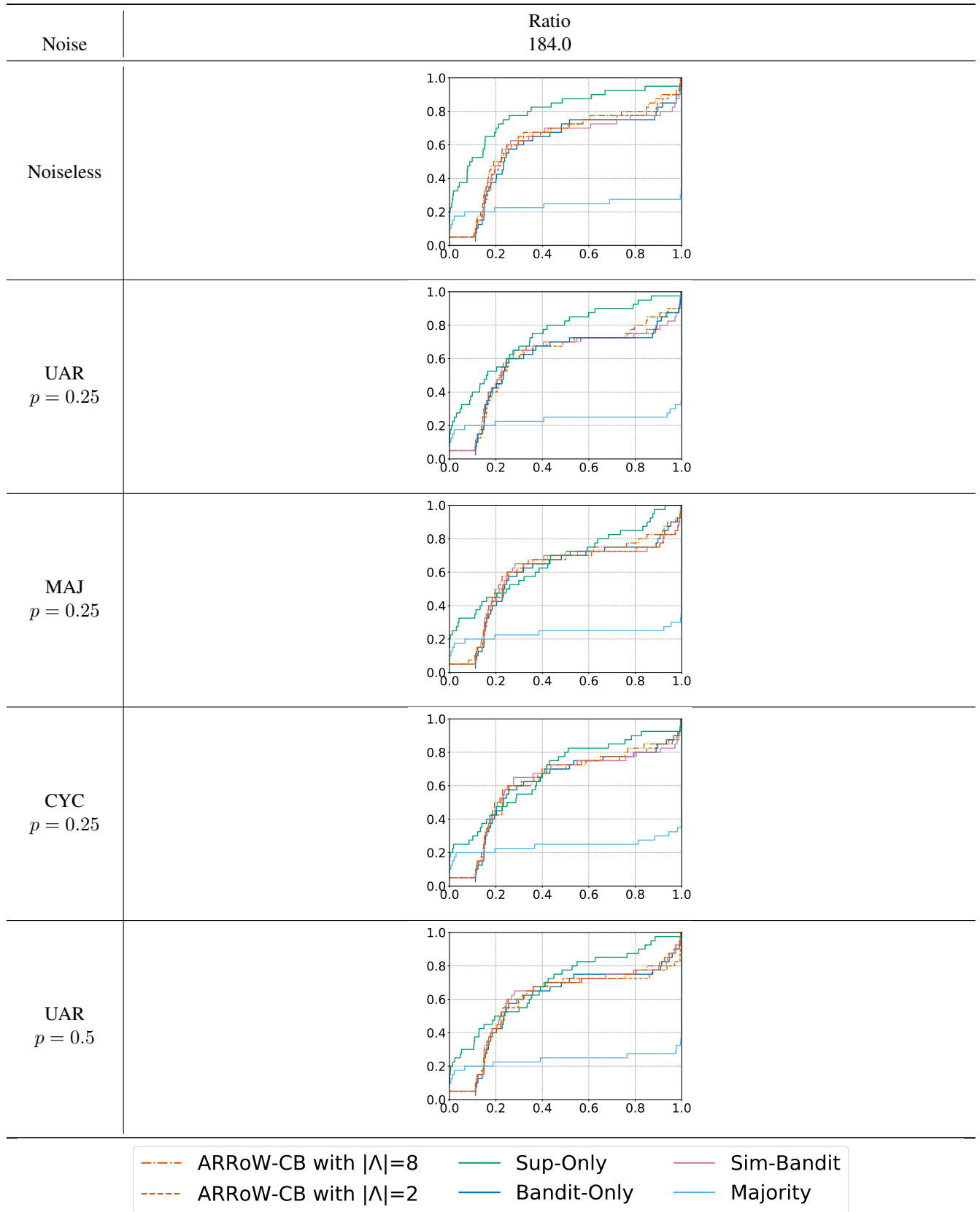
Figure 17: Cumulative distribution functions (CDFs) for all evaluated algorithms, against different noise settings and warm start ratios in $\{23.0, 46.0, 92.0\}$. All CB algorithms use ϵ-greedy with $\epsilon = 0.1$. In each of the above plots, the x axis represents scores, while the y axis represents the CDF values.

Noise	Ratio 184.0
Noiseless	
UAR $p = 0.25$	
MAJ $p = 0.25$	
CYC $p = 0.25$	
UAR $p = 0.5$	

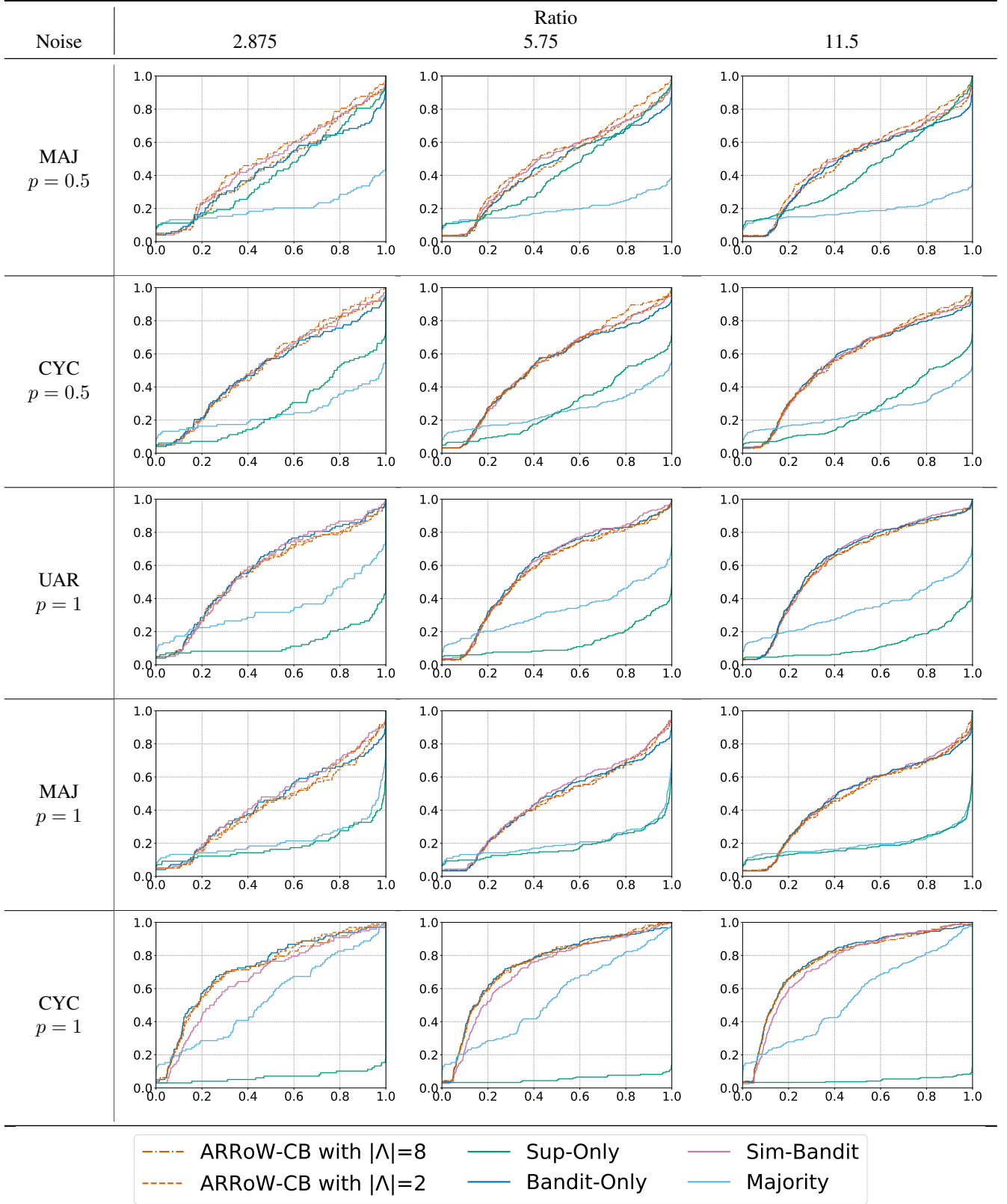Legend: ARRoW-CB with |Λ|=8, ARRoW-CB with |Λ|=2, Sup-Only, Bandit-Only, Sim-Bandit, Majority

Figure 18: Cumulative distribution functions (CDFs) for all evaluated algorithms, against different noise settings and warm start ratios in $\{184.0\}$. All CB algorithms use ϵ-greedy with $\epsilon = 0.1$. In each of the above plots, the x axis represents scores, while the y axis represents the CDF values.

Figure 19: Cumulative distribution functions (CDFs) for all evaluated algorithms, against different noise settings and warm start ratios in $\{2.875, 5.75, 11.5\}$. All CB algorithms use ϵ-greedy with $\epsilon = 0.1$. In each of the above plots, the x axis represents scores, while the y axis represents the CDF values.
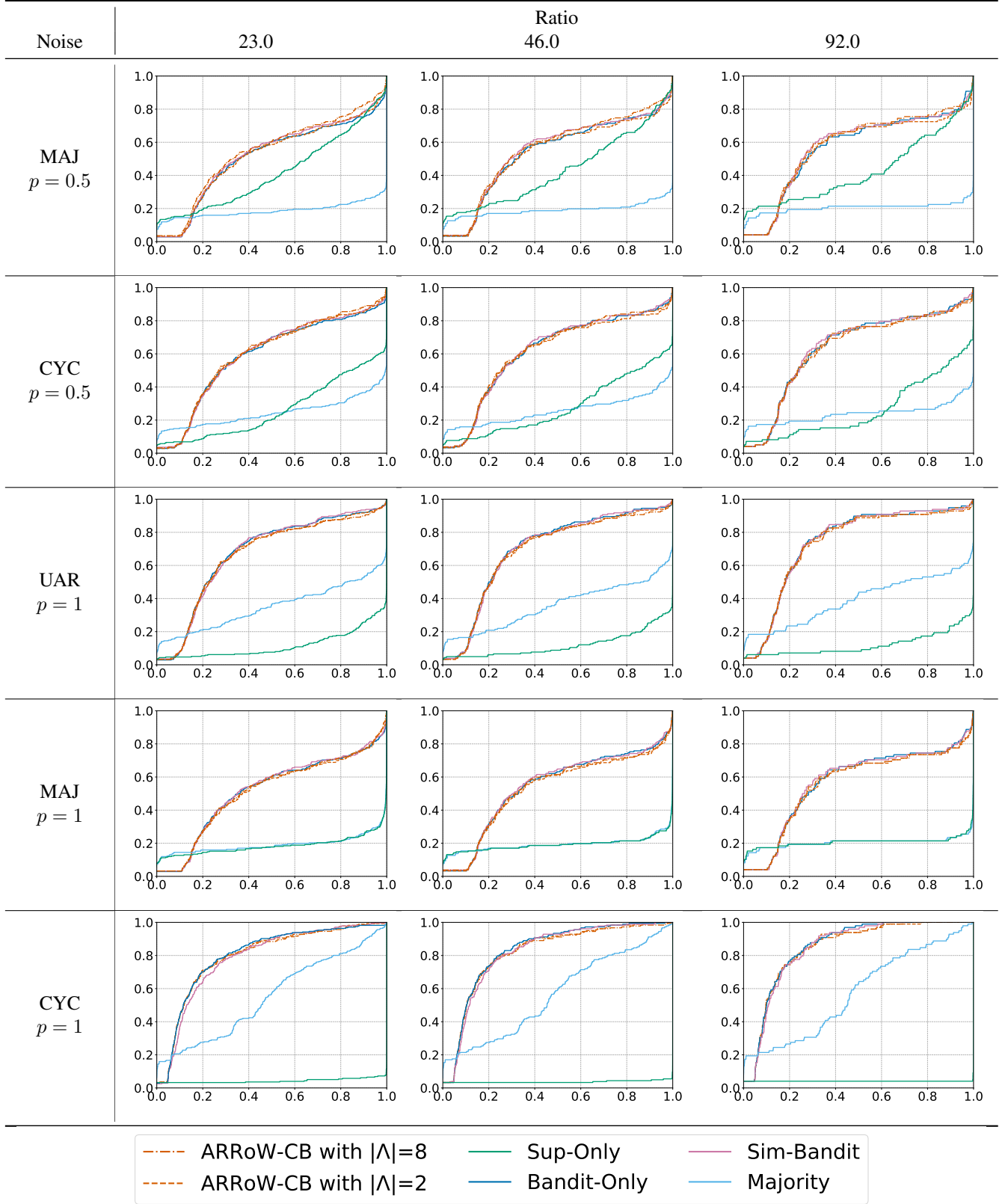
Figure 20: Cumulative distribution functions (CDFs) for all evaluated algorithms, against different noise settings and warm start ratios in $\{23.0, 46.0, 92.0\}$. All CB algorithms use ϵ-greedy with $\epsilon = 0.1$. In each of the above plots, the x axis represents scores, while the y axis represents the CDF values.

Noise	Ratio 184.0
MAJ $p = 0.5$	
CYC $p = 0.5$	
UAR $p = 1$	
MAJ $p = 1$	
CYC $p = 1$	

Legend: ARRoW-CB with $|\Lambda|=8$ · ARRoW-CB with $|\Lambda|=2$ · Sup-Only · Bandit-Only · Sim-Bandit · Majority
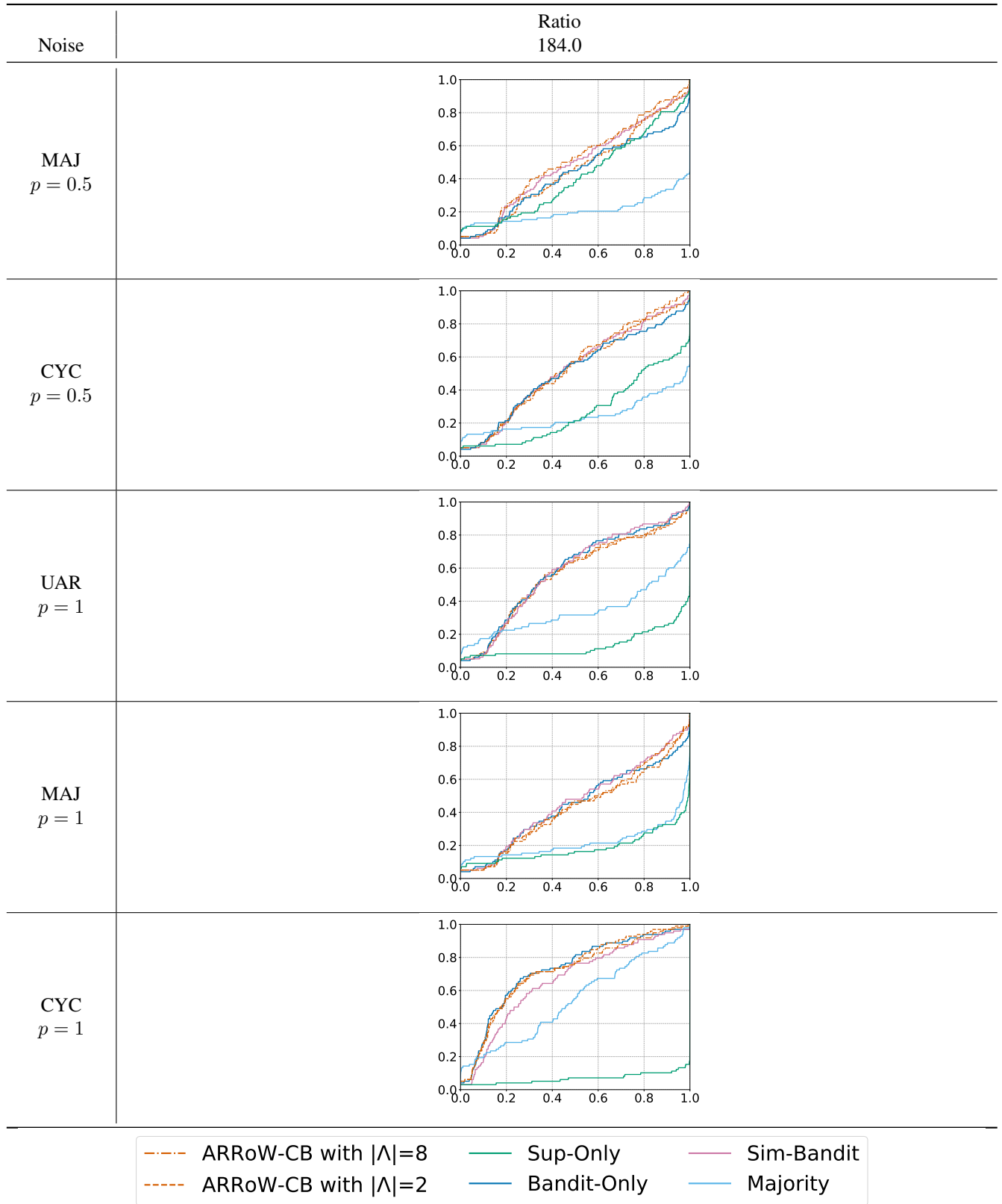
Figure 21: Cumulative distribution functions (CDFs) for all evaluated algorithms, against different noise settings and warm start ratios in $\{184.0\}$. All CB algorithms use ϵ-greedy with $\epsilon = 0.1$. In each of the above plots, the x axis represents scores, while the y axis represents the CDF values.