GENERALIZATION OF RAMANUJAN METHOD OF APPROXIMATING ROOT OF AN EQUATION

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ABSTRACT. We generalize Ramanujan method of approximating the smallest root of an equation which is found in Ramanujan Note books, Part-I. We provide simple analytical proof to study convergence of this method. Moreover, we study iterative approach of this method on approximating a root with arbitrary order of convergence.

1. INTRODUCTION.

Ramanujan method of approximating the smallest root z_0 of an equation of the form,

$$(1.1) \qquad \sum_{k=1}^{\infty} A_k z^k = 1$$

is found in Chapter 2 of Ramanujan note books, part-I without proof. It is assumed that all other roots of (1.1) have moduli strictly greater than $|z_0|$. For z sufficiently small, write

$$(1.2) \qquad \frac{1}{1 - \sum_{k=1}^{\infty} A_k z^k} = \sum_{k=1}^{\infty} P_k z^{k-1}$$

It follows easily that $P_1 = 1$ and

$$(1.3) \qquad P_n = \sum_{j=1}^{n-1} A_j P_{n-j}, \qquad n \geq 1$$

Ramanujan gives (1.3) and claims, with no hypothesis, that P_n/P_{n+1} approaches a root of (1.1)[1, 5]. In this present study, we give analytic proof of his method and generalize this to approximate a root of nonlinear equation with the arbitrary order of convergence.

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2. Generalization of Ramanujan method.

Suppose that f be a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line that satisfies $f(a)f(b) < 0$. Assume, further, that f is continuously differentiable on (a, b). We wish to find real number $\alpha \in [a, b]$ such that $f(\alpha) = 0$. Let z be an approximate of α that satisfies $|z - \alpha| < 1$ and $f'(z) \neq 0$. Define

$$(2.1) \qquad\qquad P(z)f(z) = 1, \qquad \text{If } z \neq \alpha$$

Now applying Leibniz's rule[3] of nth derivative on (2.1) and after simplification, we find that

$$(2.2) \qquad P_n(z) = -\frac{1}{f(z)} \sum_{k=0}^{n-1} \binom{n}{k} P_k(z) f^{(n-k)}(z), \qquad n > 1$$

where $P_k(z)$ is k^{th} derivative of $1/f(z)$, $(k = 1, 2, \ldots, n)$ and $P_0(z) = 1/f(z)$. Using (2.2), the next three values of $P(z)$'s are listed as follow as

$$P_1(z) = -\frac{f'(z)}{f(z)^2}$$

$$P_2(z) = -\frac{f^{(2)}(z)}{f(z)^2} + 2\frac{f'(z)^2}{f(z)^3}$$

$$P_3(z) = -\frac{f^{(3)}(z)}{f(z)^2} + 6\frac{f'(z)f^{(2)}(z)}{f(z)^3} - 6\frac{f'(z)^3}{f(z)^4}$$

and so on. Now define for $n \geq 1$

$$(2.3) \qquad\qquad H_n(z) = \frac{D^n\left(\frac{z-\alpha}{f(z)}\right)}{D^n\left(\frac{1}{f(z)}\right)}$$

The operator D^n in (2.3) denotes n^{th} order differentiation. Assume that, α is a zero of order one for $f(z)$ then $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Set

$$(2.4) \qquad\qquad Q(z) = \frac{1}{\sum_{j=1}^{\infty} \frac{f^{(j)}(\alpha)}{j!}(z-\alpha)^{j-1}}$$

It follows easily from Taylor's series [3] that $Q(z) = \frac{z-\alpha}{f(z)}$ for $z \neq \alpha$ and $Q(\alpha) = 1/f'(\alpha)$. Since α is zero of order one of $f(z)$, $Q(z)$ is differentiable at α. Subtituting (2.4) in (2.3), we find that

$$H_n(z) = \frac{D^n Q(z)}{D^n\left(\frac{Q(z)}{z-\alpha}\right)}$$

Expanding $Q(z)$ by Taylor's series at $z = \alpha$ on both numerator and denominator and applying the operator D^n and after simplification, we have

$$= \frac{(z-\alpha)^{n+1} \sum_{j=0}^{\infty} \frac{Q^{(n+j)}(\alpha)}{j!}(z-\alpha)^j}{n!(-1)^n Q(\alpha) + \sum_{j=1}^{\infty} \frac{Q^{(n+j)}(\alpha)}{n+j} \frac{(z-\alpha)^{n+j}}{(j-1)!}}$$

Thus, we obtain

$$(2.5) \qquad H_n(z) = O\left((z-\alpha)^{n+1}\right)$$

Since $|z - \alpha| < 1$ and letting $n \to \infty$, we obtain

$$(2.6) \qquad \lim_{n\to 0} H_n(z) = 0$$

On the other hand, using Leibnitz rule on (2.3), we find that

$$H_n(z) = \frac{(z-\alpha)D^n\left(\frac{1}{f(z)}\right) + nD^{n-1}\left(\frac{1}{f(z)}\right)}{D^n\left(\frac{1}{f(z)}\right)}$$

Since $D^n(1/f(z)) = P_n(z)$ and after simplfiiction, we obtain

$$(2.7) \qquad H_n(z) = z - \alpha + n\frac{P_{n-1}(z)}{P_n(z)}$$

Letting $n \to \infty$ and using (2.7), we obtain

$$(2.8) \qquad \alpha = z + \lim_{n\to\infty} n\frac{P_{n-1}(z)}{P_n(z)}$$

Remark 2.1. *If we replace $P_n(z)$ by $n!P_n$, $P_{n-1}(z)$ by $(n-1)!P_{n-1}$ and assuming $z = 0$ and $f(0) = -1$ in (2.1), (2.2) and (2.8), we get Ramanujan method of approximating the smallest root of an equation of the form (1.1)* [3].

The recursion formula given in (2.2) may lead numerical instability in machine computation due to divisibility of $f(z)$ when $f(z) \to 0$. To avoid such instability, multiply $f(z)^n$ on both sides of (2.1), (2.2) and set $T_k(z) = P_k(z)f(z)^{k+1}$, we obtain $T_0(z) = 1$ and

$$(2.9) \qquad T_n(z) = -\sum_{k=0}^{n-1}\binom{n}{k}T_k(z)f(z)^{n-k-1}f^{(n-k)}(z), \quad n > 1$$

Also, the equation (2.8) becomes

$$(2.10) \qquad \alpha = z + \lim_{n\to\infty} nf(z)\frac{T_{n-1}(z)}{T_n(z)}, \quad n = 1, 2, \ldots$$

Thus, n^{th} convergent of (2.10) gives

$$\alpha \approx z + nf(z)\frac{T_{n-1}(z)}{T_n(z)}, \qquad n = 1, 2, \ldots$$

3. Some rational approximations.

Ramanujan gives six examples to illustrate performance of his method to approximate the smallest root of equations. The approximations are given in the rational form. In particular, he provides an interesting rational approximation that $\log 2 = 375/541$. In this section, we present such rational approximations to m^{th} root of some integers and logarithm of some rational numbers.

3.1. m$^{\text{th}}$ root of a rational number.

Let us consider $f(z) = z^m - a$ and c be an integer which is very near to m^{th} root of a. Now, using (2.2), we obtain the following recursion

$$(3.1) \qquad T_n(c) = -\sum_{k=0}^{r}\binom{n}{k}\frac{m!}{(m-n+k)!}T_k(c)\,(c^m - a)^{n-k-1}\,c^{m-n+k}$$

and

$$(3.2) \qquad \sqrt[m]{a} = c + n(c^m - a)\frac{T_{n-1}(c)}{T_n(c)}$$

where $r = n - 1$ if $n \le m$ and $r = m$ if $n > m$. The sequence of rational approximations of $\sqrt{2}$ are

$$\frac{99}{70}, \frac{239}{169}, \frac{577}{408}, \frac{1393}{985}, \frac{3363}{2378}, \frac{8119}{5741}, \frac{19601}{13860}, \frac{47321}{33461}$$

which is corected up to 3, 4, 4, 5, 7, 7, 8, 9 decimal places respectively. Table 1 lists rational approximations of some irrational numbers using MATLAB.

Table 1. Rational approximations of Irrational numbers.

S.N0	Irrational No.	Rational Approx.	significant digits
1	$\sqrt[3]{9}$	$\frac{50623}{24337}$	10-digits
2	$\sqrt[9]{511}$	$\frac{4603}{2302}$	9-digits
3	$\sqrt[3]{2}$	$\frac{6064}{4813}$	8-digits
4	$\sqrt[5]{3100}$	$\frac{3110}{623}$	7-digits

3.2. Some Logarithmic values.

Let us consider $f(z) = \log(z+1) - a$ and $z = 0$ be an initial approximation (where a is a rational number). Table 2 lists rational approximations of logarithmic numbers using MATLAB.

TABLE 2. Rational approximations of logarithmic values.

S.No	Logarithm Nos.	Rational Approx.	significance digits
1	$\log_e 1.5$	$\frac{3858}{9515}$	7-digits
2	$\log_e 2.0$	$\frac{32781}{47293}$	6-digits
3	$\log_e 3.0$	$\frac{12667}{11530}$	7-digits
4	$\log_e 1.2$	$\frac{724}{3971}$	6-digits

3.3. **Some other examples.** Ramanujan lists the first ten convergents to the real root of $x + x^3 = 1$, with the last convergent being $13/19 = 0.684210\dots$. By Newton's method, the root is $0.682327804\dots$. Here, we provide three examples to approximate root of $x^3 - 2x - 5 = 0$, $e^x - 3 = 0$ and $x = \sin x + \frac{1}{2}$ by taking the initial approximations $x_0 = 2$, $x_0 = 1$ and $x_0 = 1$ respectively. Table 3 lists first ten convergents of roots all the three equations. The roots of first two equations converge to 14 decimal places at 9th and 8th convergent respectively whereas the root of $x = \sin x + \frac{1}{2}$ converges slowly to 1.49730038909589 at 23th convergent. We see from these examples, this method of approximating a real root is very slow process. However, we get good approximation when n is large.

TABLE 3. Examples for Generalized Ramanujan method

It	$x^3 - 2x - 5 = 0$	$e^x - 3 = 0$	$x = \sin x + \frac{1}{2}$
0	2.00000000000000	1.00000000000000	1.00000000000000
1	2.10000000000000	1.10363832351433	1.58288042035629
2	2.09433962264151	1.09853245432531	1.51838510578857
3	2.09455842997324	1.09861223692174	1.50077867834371
4	2.09455128205128	1.09861230157476	1.49783013943789
5	2.09455148653822	1.09861228868606	1.49735888023541
6	2.09455148143875	1.09861228866513	1.49730334991792
7	2.09455148154375	1.09861228866810	1.49729959647640
8	2.09455148154234	1.09861228866811	1.49730005778495
9	2.09455148154232	1.09861228866811	1.49730030987454

4. ITERARTIVE APPROACH ON GENERALIZED RAMANUJAN METHOD.

In this section, we study iterative approach on generalized Ramanujan's method to improve the accuracy of the root of nonlinear equations. If z_0 be an initial approximate root of $f(z)$ then using (2.10),

$$(4.1) \qquad z_{m+1} = z_m + nf(z_m)\frac{T_{n-1}(z_m)}{T_n(z_m)} \qquad m = 0, 1, 2, \dots$$

where

$$(4.2) \qquad T_n(z_m) = -\sum_{k=0}^{n-1} \binom{n}{k} T_k(z_m) f(z_m)^{n-k-1} f^{(n-k)}(z_m), \quad n > 1$$

Set $z_{m+1} - \alpha = \epsilon_{m+1}$ and $z_m - \alpha = \epsilon_m$. Then by using (2.5) and (2.7), we find that the order of convergence of (4.1) is $n+1$. (i.e)

$$(4.3) \qquad \epsilon_{m+1} = O\left(\epsilon_m^{n+1}\right)$$

Also, the condition of convergence is

$$(4.4) \qquad \left| n+1 - n\frac{T_{n-1}(z)T_{n+1}(z)}{T_n(z)^2} \right| < 1$$

Setting $n = 1$ in (4.1) and after simplification, we obtain Newton-Raphson method

$$z_{m+1} = z_m - \frac{f(z_m)}{f'(z_m)}, \qquad m = 0, 1, 2, \dots$$

Setting $n = 2$ in (4.1) and after simplification, we obtain Halley's method

$$z_{m+1} = z_m - \frac{f(z_m)/f'(z_m)}{1 - \frac{1}{2}f(z_m)f''(z_m)/f'(z_m)^2} \qquad m = 0, 1, 2, \dots$$

By varying the values of n, we can find more formulas with the higher order of accuracy. Thus, this new iterative method is more generalization of Newton-Raphson method [3] and Halley's method[4].

Example 4.1. *The equation $f(z) = z - \cos z$ has exactly one root $\alpha = 0.73908513321516$ between 0 and 1 and starting with $z_0 = 0$, generate the sequence $z_1, z_2, \dots$ by taking $m = 1 : 5$ and $n = 1 : 4$ in (4.5) are listed in the Table 4.*

TABLE 4. Computation of root of $z - \cos z = 0$

m	$n = 1$	$n = 2$	$n = 3$	$n = 4$
1	1.00000000000000	0.66666666666667	0.75000000000000	0.73846153846154
2	0.75036386784024	0.73903926244631	0.73908513352403	0.73908513321516
3	0.73911289091136	0.73908513321515	0.73908513321516	
4	0.73908513338528	0.73908513321516		
5	0.73908513321516			

We observed that from Table 4, when $m = 1$ and $n = 1 : 4$ (i.e. first row) the sequence corresponds to generalized Ramaujan method studied in section 2, which converges very slowly towards the root. Simliarly, when $n = 1$ and $m = 1 : 5$ (i.e first column) the sequence corresponds to Newton-Raphson method, which converges to root at 5th iteration, second column

correspond to Halley method, which converges to root at 4th iteration and so on.

5. Conclusion.

In conclusion we note that the generalization of Ramanujan method of approximating the real root of nonlinear equations has been developed in this article. Firstly, we have generalized Ramanujan's method of approximating the smallest root of equations, which is found in Ramanujan Note books - I, to any real root with simple analytic proof. Secondly, we have proved that iterative approch of this method has arbitrary order of convergence. Moreover, we have shown that Newton-Raphson and Halley's method are special cases of this generalized Ramanujan method.

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