

Proof of the Riemann Hypothesis

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Abstract In this article, we will prove Riemann Hypothesis by using the mean value theorem of integrals. According to Euler-MacLaurin sum formula, the function $\zeta(s)$ can be represented as a summation that includes infinite integral. This representation provides an analytic continuation of $\zeta(s)$ up to $\text{Re } s > 0$, and there is a simple pole at $s = 1$ with residue 1. The function $\zeta(s)$ satisfies the function equation, let $\rho = \alpha + i\beta$ be anyone non-trivial zero point of $\zeta(s)$, then we have $\zeta(\rho) = \zeta(1 - \rho) = 0$. This equation has only one solution $\alpha = \frac{1}{2}$, one of the proofs be given by using the mean value theorem of integrals. Therefore, we proved that all non-trivial zeros of the function $\zeta(s)$ have real part equal to $\frac{1}{2}$. Riemann Hypothesis is true.

Keywords Riemann Hypothesis · Riemann zeta function

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1 Introduction

Riemann Hypothesis is a conjecture about the zeros of Riemann Zeta function, which was proposed by the mathematician Riemann [1] in his famous paper on the number of primes less than a given magnitude in 1859. The Riemann hypothesis together with goldbach conjecture and twin prime number conjecture, constitute the eighth problem in Hilbert's list of 23 unsolved problems. It's also one of Clay Mathematics Institutes the millennium prize problems.

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The Riemann zeta function is defined as a complex series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, (Res > 1) \quad (1)$$

Euler first studied this function in 1737 and got his famous identity

$$\sum_{n=1}^{\infty} n^{-s} = \prod_p (1 - p^{-s})^{-1}, (s > 1) \quad (2)$$

The zeta function has analytic continuation to the whole complex plane, except a simple pole at $s = 1$, and satisfies the function equation

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \pi^{-(1-s)/2} \Gamma(1-s/2) \zeta(1-s) \quad (3)$$

where $\Gamma(s)$ is Gamma-function; the zeta function has many real zeros at $s = -2, -4, \dots$ and infinitely many complex zeros (i.e., non-trivial zeros) in the range $0 < Res < 1$; the non-trivial zeros of zeta function are symmetrical on the real axis and the critical line $Res = \frac{1}{2}$; the number of non-trivial zeros of zeta function in the range of $0 < Im s < T$ is approximately equal to

$$\frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log T) \quad (4)$$

In his famous paper, Riemann conjectured that all non-trivial zeros of zeta function are very likely to lie on the critical line $Res = \frac{1}{2}$. This proposition is called Riemann Hypothesis.

Riemann Hypothesis: *all non-trivial zeros of the function $\zeta(s)$ have real part equal to $\frac{1}{2}$.*

Gram (1903) used the Euler-Maclaurin summation method to calculate the first 15 non-trivial zeros of the function. Since then hundreds of millions of non-trivial zeros of the function have been calculated and found to be lie on the critical line. Mathematicians have made many important advance in the study of Riemann Hypothesis in the past [2-15]. However, Riemann Hypothesis has not been proved or disproved up to now. In this paper, we will prove Riemann Hypothesis by the mean value theorem of integrals.

2 The mean value theorem for integrals

The mean value theorem for integrals is a fundamental theorems for analytic functions.

Theorem 1. Let the function $f(x)$ be continuous, the function $\varphi(x) \geq 0$ and be integrable in the interval $[a, b]$, then there be least one point ξ in the interval (a, b) , which makes the following formula to be true

$$\int_a^b f(x) \varphi(x) dx = f(\xi) \int_a^b \varphi(x) dx \quad (5)$$

Remark, let a function be integrable, then that will be continuous or contain the first class of discontinuity points in integral interval.

The mean value theorem of integrals can be generalized to infinite integrals. The infinite integrals be defined as

$$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx \quad (6)$$

for the limit be convergent.

Lemma 1. Let the function $f(x)$ be continuous, the function $\varphi(x) \geq 0$ and be integrable in the interval $[a, \infty)$, then there be least one point ξ in the interval (a, ∞) , which makes the following formula to be true

$$\int_a^\infty f(x)\varphi(x)dx = f(\xi) \int_a^\infty \varphi(x)dx \quad (7)$$

for the limit be convergent.

Futhermore, we have following Lemmas.

Lemma 2. Let the function $f(x)$ and $g(x)$ be continuous, the function $\varphi(x) \geq 0$ and be integrable in the interval $[a, b]$, $\int_a^b \varphi(x)dx \neq 0$ and $\int_a^b g(x)\varphi(x)dx \neq 0$, then there be least one point ξ in the interval (a, b) , which makes the following formula to be true

$$\frac{\int_a^b f(x)\varphi(x)dx}{\int_a^b g(x)\varphi(x)dx} = \frac{f(\xi)}{g(\xi)} \quad (8)$$

Proof: Let us put

$$\frac{\int_a^b f(x)\varphi(x)dx}{\int_a^b g(x)\varphi(x)dx} = \lambda \quad (9)$$

then we have

$$\int_a^b f(x)\varphi(x)dx = \lambda \int_a^b g(x)\varphi(x)dx \quad (10)$$

and

$$\int_a^b [f(x) - g(x)\lambda]\varphi(x)dx = 0 \quad (11)$$

According to Theorem 1, there be least one point ξ in the interval (a, b) , which makes the following formula to be true

$$[f(\xi) - g(\xi)\lambda] \int_a^b \varphi(x)dx = 0 \quad (12)$$

since $\int_a^b \varphi(x)dx \neq 0$, so we have

$$f(\xi) - g(\xi)\lambda = 0 \quad (13)$$

and

$$\lambda = \frac{f(\xi)}{g(\xi)} = \frac{\int_a^b f(x)\varphi(x)dx}{\int_a^b g(x)\varphi(x)dx} \quad (14)$$

Lemma 2 be proved.

Let $\varphi(x) = 1$, then it has

$$\frac{\int_a^b f(x)dx}{\int_a^b g(x)dx} = \frac{f(\xi)}{g(\xi)} \quad (15)$$

this is called as cauchy mean value theorem for integrals.

Lemma 3. Let the function $f(x)$ and $g(x)$ be continuous, the function $\varphi(x) \geq 0$ and be integrable in the interval $[a, \infty)$, $\int_a^\infty \varphi(x)dx \neq 0$ and $\int_a^\infty g(x)\varphi(x)dx \neq 0$, then there be least one point ξ in the interval (a, ∞) , which makes the following formula to be true

$$\frac{\int_a^\infty f(x)\varphi(x)dx}{\int_a^\infty g(x)\varphi(x)dx} = \frac{f(\xi)}{g(\xi)} \quad (16)$$

for the limits be convergent.

3 Proof of Riemann Hypothesis

In this article, we will prove following theorem that be equivalent to Riemann Hypothesis.

Theorem 2. Anyone non-trivial zero point of the function $\zeta(s)$ has real part equal to $\frac{1}{2}$.

Proof: Using Euler-MacLaurin sum formula, for $Res > 1$, the function $\zeta(s)$ can be represented as

$$\zeta(s) = s \int_1^\infty \frac{[x] - x + \frac{1}{2}}{x^{(s+1)}} dx + \frac{1}{s-1} + \frac{1}{2} \quad (17)$$

where $[x]$ denotes the greatest integer not exceeding x . Since $[x] - x + \frac{1}{2}$ is bounded, this integral is convergent for $Res > 0$. Therefore it provides an analytic continuation of $\zeta(s)$ up to $Res > 0$, and there is a simple pole at $s = 1$ with residue 1. For $0 < Res < 1$,

$$\frac{s}{2} \int_1^\infty \frac{dx}{x^{(s+1)}} = \frac{1}{2} \quad (18)$$

thus, for $0 < Res < 1$, we have

$$\zeta(s) = s \int_1^\infty \frac{[x] - x + 1}{x^{(s+1)}} dx + \frac{1}{s-1} = s \int_1^\infty \frac{[x] - x + 1}{x^2} x^{(1-s)} dx + \frac{1}{s-1} \quad (19)$$

Let $\rho = \alpha + i\beta$ be anyone non-trivial zero point of $\zeta(s)$, $0 < \alpha < 1$, and β be a real number, then we have

$$\zeta(\rho) = \rho \int_1^\infty \frac{[x] - x + 1}{x^2} x^{(1-\rho)} dx + \frac{1}{\rho-1} = 0 \quad (20)$$

and

$$\zeta(1-\rho) = (1-\rho) \int_1^\infty \frac{[x]-x+1}{x^2} x^\rho dx - \frac{1}{\rho} = 0 \quad (21)$$

Thus, we have

$$\int_1^\infty \frac{[x]-x+1}{x^2} x^{(1-\rho)} dx = \frac{1}{\rho(1-\rho)} \quad (22)$$

$$\int_1^\infty \frac{[x]-x+1}{x^2} x^\rho dx = \frac{1}{\rho(1-\rho)} \quad (23)$$

and

$$\int_1^\infty \frac{[x]-x+1}{x^2} x^{(1-\rho)} dx = \int_1^\infty \frac{[x]-x+1}{x^2} x^\rho dx = \frac{1}{\rho(1-\rho)} \quad (24)$$

Because

$$x^{(1-\rho)} = x^{(1-\alpha)} (\cos(\beta \log x) - i \sin(\beta \log x)) \quad (25)$$

and

$$x^\rho = x^\alpha (\cos(\beta \log x) + i \sin(\beta \log x)) \quad (26)$$

therefore, let $\rho = \alpha + i\beta$ be anyone non-trivial zero point of $\zeta(s)$, then we have

$$\int_1^\infty \frac{[x]-x+1}{x^2} x^{(1-\alpha)} \cos(\beta \log x) dx = \operatorname{Re} \frac{1}{\rho(1-\rho)} \quad (27)$$

$$\int_1^\infty \frac{[x]-x+1}{x^2} x^\alpha \cos(\beta \log x) dx = \operatorname{Re} \frac{1}{\rho(1-\rho)} \quad (28)$$

and

$$\int_1^\infty \frac{[x]-x+1}{x^2} x^{(1-\alpha)} \cos(\beta \log x) dx = \int_1^\infty \frac{[x]-x+1}{x^2} x^\alpha \cos(\beta \log x) dx \quad (29)$$

therefore, let $\rho = \alpha + i\beta$ be anyone non-trivial zero point of $\zeta(s)$, then it must satisfy above equation.

Since function $x^{(1-\alpha)} \cos(\beta \log x)$, $x^\alpha \cos(\beta \log x)$ be continuous, function $\frac{[x]-x+1}{x^2} \geq 0$ in the interval $[1, \infty)$, and

$$\int_1^\infty \frac{[x]-x+1}{x^2} x^\alpha \cos(\beta \log x) dx = \operatorname{Re} \frac{1}{\rho(1-\rho)} \neq 0 \quad (30)$$

according to Lemma 3, there be least one point ξ in the interval $(1, \infty)$, which makes the following formula to be true

$$\frac{\int_1^\infty \frac{[x]-x+1}{x^2} x^{(1-\alpha)} \cos(\beta \log x) dx}{\int_1^\infty \frac{[x]-x+1}{x^2} x^\alpha \cos(\beta \log x) dx} = \frac{\xi^{(1-\alpha)} \cos(\beta \log \xi)}{\xi^\alpha \cos(\beta \log \xi)} = \xi^{(1-2\alpha)} \quad (31)$$

namely, for $0 < \alpha < 1$ and $\xi > 1$, it has

$$\int_1^\infty \frac{[x]-x+1}{x^2} x^{(1-\alpha)} \cos(\beta \log x) dx = \xi^{(1-2\alpha)} \int_1^\infty \frac{[x]-x+1}{x^2} x^\alpha \cos(\beta \log x) dx \quad (32)$$

Let $\alpha = \frac{1}{2}$, and $\xi > 1$, then $\xi^{(1-2\alpha)} = 1$, for this

$$\int_1^\infty \frac{[x] - x + 1}{x^2} x^{(1-\alpha)} \cos(\beta \log x) dx = \int_1^\infty \frac{[x] - x + 1}{x^2} x^\alpha \cos(\beta \log x) dx \quad (33)$$

above Equation be established.

Let $\alpha \neq \frac{1}{2}$, and $\xi > 1$, then $\xi^{(1-2\alpha)} \neq 1$, for this above Equation be not established.

Therefore, we proved that let $\rho = \alpha + i\beta$ be anyone non-trivial zero point of $\zeta(s)$, then $\operatorname{Re} \rho = \alpha = \frac{1}{2}$.

Theorem 2 is proved.

However, we prove that anyone non-trivial zero point of the function $\zeta(s)$ has real part equal to $\frac{1}{2}$. Namely all non-trivial zeros of zeta function have real part equal to $\frac{1}{2}$, Riemann Hypothesis is true.

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