

Ramanujan type $1/\pi$ Approximation Formulas

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Abstract

In this article we use theoretical and numerical methods to evaluate in a closed-exact form the parameters of Ramanujan type $1/\pi$ formulas.

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1 Introduction

We give the definitions of the Elliptic Integrals of the first and second kind respectively (see [9],[4]):

$$K(x) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-x^2 \sin^2(t)}} \text{ and } E(x) = \int_0^{\pi/2} \sqrt{1-x^2 \sin^2(t)} dt. \quad (1)$$

In the notation of Mathematica we have

$$K(x) = \text{EllipticK}[x^2] \text{ and } E(x) = \text{EllipticE}[x^2]. \quad (2)$$

Also we have (see [9],[7]):

$$\dot{K}(k) = \frac{dK(k)}{dk} = \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k}. \quad (3)$$

The elliptic singular moduli is defined to be the solution of the equation:

$$\frac{K(\sqrt{1-w^2})}{K(w)} = \sqrt{r}. \quad (4)$$

In Mathematica is stated as

$$w = k = k_r = k[r] = \text{InverseEllipticNomeQ}[e^{-\pi\sqrt{r}}]^{1/2}. \quad (5)$$

The complementary modulus is given by $k_r'^2 = 1 - k_r^2$.

Also we will need the following relation of the elliptic alpha function (see [7]):

$$a(r) = \frac{\pi}{4K(k_r)^2} - \sqrt{r} \left(\frac{E(k_r)}{K(k_r)} - 1 \right). \quad (6)$$

The Hypergeometric functions are defined by

$$\begin{aligned} {}_{m+1}F_m(a_1, a_2, \dots, a_{m+1}; b_1, b_2, \dots, b_m; z) &:= \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_{m+1})_n}{(b_1)_n (b_2)_n \dots (b_m)_n} \frac{z^n}{n!}, \text{ for } |z| < 1, \end{aligned} \quad (7)$$

and $(a)_0 := 1$, $(a)_n := a(a+1)(a+2) \dots (a+n-1)$, for each positive integer n .

2 The construction of some $1/\pi$ and $1/\pi^2$ formulas

It holds

$$\phi_1(z) = {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; z\right) = \frac{4K^2\left(\frac{1}{2}(1-\sqrt{1-z})\right)}{\pi^2}. \quad (8)$$

Consider the following equation with respect to the function $\phi_1(z)$:

$$\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(n!)^3} z^n (an + b) = \frac{g}{\pi} \Leftrightarrow b\phi_1(z) + az\phi_1'(z) = \frac{g}{\pi}.$$

Set $w = 1/2(1 - \sqrt{1-k^2})$, $1 - 2w = \sqrt{1-z} = k'_r$.

But

$$b\phi_1(z) + az\phi_1'(z) = \frac{g}{\pi} \Leftrightarrow g = \frac{4K(w)(aE(w) + (b + a(w-1) - 2bw)k(w))}{\pi(1-2w)},$$

hence

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(n!)^3} 4^n (w - w^2)^n (an + b) = \\ &= \frac{4K(\sqrt{w})(aE(\sqrt{w}) + (b - a + aw - 2bw - 2bw)K(\sqrt{w}))}{\pi^2(1-2w)}. \end{aligned}$$

For $w = k_r$ we get

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(n!)^3} 4^n (k_r k'_r)^{2n} (an + b) = \\ &= \frac{4K(k_r)(aE(k_r) + (b - a + ak_r^2 - 2bw - 2bk_r^2)K(k_r))}{\pi^2(1-2k_r^2)}. \end{aligned} \quad (9)$$

Now using the formula for $a(r)$, in the sense that

$$E(k_r) = K(k_r) - \frac{a(r)K(k_r)}{\sqrt{r}} + \frac{\pi}{4K(k_r)\sqrt{r}}, \quad (10)$$

for suitable values for a, b, c we get the following theorem:

Theorem 2.1

$$\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(n!)^3} 4^n (k_r k'_r)^{2n} (\sqrt{r}(1 - 2k_r^2)n + a(r) - \sqrt{r}k_r^2) = \frac{1}{\pi} \quad (11)$$

Example.

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(n!)^3} (40\sqrt{2} - 56)^n (an + b) = \frac{4a}{7\pi} + \frac{5a}{7\sqrt{2}\pi} + 4(-4a + \sqrt{2}a + 14b) \frac{\Gamma^2\left(\frac{9}{8}\right)}{7\pi\Gamma^2\left(\frac{5}{8}\right)}.$$

From which a special case is

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(n!)^3} (40\sqrt{2} - 56)^n \left(n + \frac{2}{7} - \frac{1}{7\sqrt{2}}\right) = \frac{8 + 5\sqrt{2}}{14\pi}.$$

Theorem 2.2

$$\sum_{n=0}^{\infty} \frac{B_n^{(2)}}{(n!)^2} (k_r)^{2n} (\sqrt{r}k_r'^2 n + a(r) - \sqrt{r}k_r^2) = \frac{1}{\pi}. \quad (12)$$

Proof.

We use the function

$$\phi_2(z) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) = \frac{2K(\sqrt{z})}{\pi}. \quad (13)$$

Then if

$$B_n^{(2)} := \sum_{j=0}^n \left[\binom{n}{j} \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{n-j} \right]^2$$

$$\phi_2^2(z) = \dots = \sum_{n=1}^{\infty} \frac{z^n}{(n!)^2} \sum_{j=0}^n \left[\binom{n}{j} \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{n-j} \right]^2, \quad (14)$$

where

$$c\phi_2(z) + bz\phi_2'(z) + az^2\phi_2''(z) = \sum_{n=0}^{\infty} \frac{B_n^{(2)}}{(n!)^2} z^n (an^2 + (b-a)n + c)$$

Hence we get

$$\sum_{n=0}^{\infty} \frac{B_n^{(2)}}{(n!)^2} k_r^{2n} (an^2 + (b-a)n + c) = \frac{2(aE(k_r) + (2b - 2bk_r^2 - 4a + 6ak_r^2)E(k_r)K(k_r))}{\pi^2(1 - k_r^2)^2} +$$

$$+ \frac{2(3a - 2b + 2c + (-4a + 2b - 2c)k_r^2)K(k_r)}{\pi^2(1 - k_r^2)}.$$

For $a = 0$, $b = 1$, $c = (-k_r^2 + a(r)r^{-1/2})k_r'^{-2}$, we get

Theorem 2.3 Set

$$B_n^{(3)} := \sum_{j=0}^n \left[\binom{n}{j} \left(\frac{1}{2} \right)_n \left(\frac{1}{2} \right)_{n-j} \right]^3, \quad (15)$$

then an $1/\pi^2$ formula is the following

$$\sum_{n=0}^{\infty} \frac{B_n^{(3)}}{(n!)^3} (2k_r k_r')^{2n} (n^2 + (b(r) - 1)n + c(r)) = \frac{3}{(1 - 2k_r^2)^2 r \pi^2} \quad (16)$$

where

$$b(r) = \frac{3a_r + \sqrt{r} - 6a(r)k_r^2 - 9\sqrt{r}k_r^2 + 12\sqrt{r}k_r^4}{\sqrt{r}(1 - 2k_r^2)^2}$$

and

$$c(r) = \frac{3a(r)^2 - 6a(r)\sqrt{r}k_r^2 - rk_r^2 + 4rk_r^4}{r(1 - 2k_r^2)^2}$$

Proof.

Set

$$\phi_3(z) = {}_3F_2 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; z \right)^2 = \left(\frac{16K^2 \left(\frac{1}{2}(1 - \sqrt{1-z}) \right)}{\pi^2} \right)^2,$$

then

$$c\phi_3(z) + bz\phi_3'(z) + az^2\phi_3''(z) = \sum_{n=0}^{\infty} \frac{B_n^{(3)}}{(n!)^3} z^n (an^2 + (b - a)n + c)$$

The left hand of the above equation is a function of $E(x)$, $K(x)$, and can evaluated when we set certain values to the parameters a , b , c .

Examples.

1)

$$\frac{1}{1200(161\sqrt{5} - 360)\pi^2} = \sum_{n=0}^{\infty} \frac{B_n^{(3)}}{(n!)^3} \left(51841 - 23184\sqrt{5} \right)^n \left(n^2 + \left(1 - \frac{521}{288\sqrt{5}} \right) n + \frac{5}{12} - \frac{521}{576\sqrt{5}} \right) \quad (17)$$

2)

$$b(163) = \frac{191211325848427}{151931373056001} - \frac{1010784962625383717350772720 \cdot 2^{2/3}}{151931373056001 (B_1 - \sqrt{489}B_2)^{1/3}} - \frac{4 \cdot 2^{1/3} ((B_1 - \sqrt{489}B_2)^{1/3})}{151931373056001}$$

$$B_1 = 5680848001702137216093843898647314524189$$

$$B_2 = 76896989960589381643149203281167$$

$$-5839006481108705728 + 9529627071955041072 \cdot b(163) - 4530513053635162884 \cdot b(163)^2 + 668649972819460401 \cdot b(163)^3 = 0$$

$$c(163) = \frac{14178679829869760}{24764813808128163} - \frac{4 (C_1 - \sqrt{489}C_2)^{1/3}}{24764813808128163} - \frac{6241484569597616793758909818952 \cdot 2^{2/3}}{24764813808128163 (C_3 - \sqrt{489}C_4)^{1/3}}$$

$$C_1 = 5512985602111283751597893407219881834715037026$$

$$C_2 = 101526256966667546381077303112958296550$$

$$C_3 = 2756492801055641875798946703609940917357518513$$

$$C_4 = 5076312848333773190538651556479148275$$

$$-24380823840878077184 + 13131020889593608594752 \cdot c(163) - 30513780896384581928640 \cdot c(163)^2 + 17765361127840243394169 \cdot c(163)^3 = 0$$

$$\sum_{n=0}^{\infty} \frac{4^n B_n^{(3)}}{(n!)^3} (k_{163} k'_{163})^{2n} (n^2 + (b(163) - 1)n + c(163)) = \frac{A}{\pi^2} \quad (18)$$

$$A = \frac{4 \left(12660947754667 + 26680 (A_1 - \sqrt{489}A_2)^{1/3} + 26680 (A_1 + \sqrt{489}A_2)^{1/3} \right)}{8254937936042721}$$

$$A_1 = 106866398697613339845357037$$

$$A_2 = 3136555671686449089$$

$$y_{163} = (k_{163} k'_{163})^2 = \frac{1}{16} - \frac{266933400}{(-1 + 557403\sqrt{489})^{1/3}} + \frac{10005}{2} (-1 + 557403\sqrt{489})^{1/3}$$

$$-1 + 16408588290048048 \cdot y_{163} - 768 \cdot y_{163}^2 + 4096 \cdot y_{163}^3 = 0$$

Formula (18) gives about 17 digits per term and is a formula for $1/\pi^2$. For $r = 253$ we have another such formula which gives 21 digits per term constructed in the same way as (18).

3 The study of a non usual $1/\pi$ formula

The j invariant is given by (see [17]):

$$j(z) = \left(\left(\frac{\eta(z/2)}{\eta(z)} \right)^{16} + 16 \left(\frac{\eta(z)}{\eta(z/2)} \right)^8 \right)^3, \quad (19)$$

where $z = \sqrt{-r}$, r -positive real and

$$\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})$$

is the Dedekind eta function.

Also

$$\frac{\eta(z)}{\eta(z/2)} = \frac{k_r^{1/12}}{2^{1/6} k_r^{1/6}}. \quad (20)$$

From [24] section 7, Theorem 7.4 and from [11] formula (5.8), when $q = e^{2\pi iz}$, $z = \sqrt{-r}$, r positive real, the modular j -invariant is also given by

$$j(z) = 1728 \frac{Q^3(q)}{Q^3(q) - R^2(q)}. \quad (21)$$

where

$$P(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n}, \quad Q(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}$$

and

$$R(q) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n}.$$

The function t_r is given from

$$t_r = \frac{Q_r}{R_r} \left(P_r - \frac{6}{\pi \sqrt{r}} \right), \quad (22)$$

where

$$P_r = P(-e^{-\pi \sqrt{r}}), \quad Q_r = Q(-e^{-\pi \sqrt{r}}) \text{ and } R_r = R(-e^{-\pi \sqrt{r}}).$$

i) Using Theorems 3 and 4 of [25], relation (21) equivalently can be transformed to

$$j(z) = \frac{432}{\beta_r(1 - \beta_r)}. \quad (23)$$

Also note that we have

$$j(\sqrt{-r}) = j_r = \frac{256(1 - k_r^2 + k_r^4)^3}{(k_r k'_r)^4} = \frac{432}{\beta_r(1 - \beta_r)}. \quad (24)$$

Hence with our method in [25] we can simplify the known results of [24] and [11] using the function β_r , which defined as the root of the equation:

$$\frac{{}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; 1-w\right)}{{}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; w\right)} = \sqrt{r}. \quad (25)$$

ii) Set now $m_r := k_r^2$ and let $a(r)$, $E(x)$ be the elliptic alpha function and the complete elliptic integral of the second kind respectively (see [7],[4]), then:

$$\begin{aligned} t_r &= \frac{1}{(1-2\beta_{r/4})u_{r/4}^2} \left(P(q) - \frac{6}{\sqrt{r}\pi} \right) = \\ &= \frac{1}{(1-2\beta_{r/4})u_{r/4}^2} \left(3 \frac{E(m_{r/4})}{K(m_{r/4})} - 2 + m_{r/4} - \frac{3\pi}{4\sqrt{r/4}K(m_{r/4})^2} \right) F_{r/4}^2 \end{aligned}$$

or

$$t_r = \frac{1 + m_{r/4} - \frac{6}{\sqrt{r}}a\left(\frac{r}{4}\right)}{\sqrt{1 - m_{r/4} + m_{r/4}^2(1 - 2\beta_{r/4})}}. \quad (26)$$

Hence from the above evaluations and the $1/\pi$ series in [6] and [11] we get the next reformulation:

Theorem 3.1 If we define

$$J_r := 1728j_r^{-1} = 4\beta_r(1 - \beta_r) \quad (27)$$

$$T_r := \frac{1 + k_r^2 - \frac{3}{\sqrt{r}}a(r)}{\sqrt{1 - k_r^2 + k_r^4(1 - 2\beta_r)}} = \frac{2j_r^{1/3}\sigma(r)G_r^8}{\sqrt{r}\sqrt{j_r - 1728}} \quad (28)$$

then

$$\frac{3}{\pi\sqrt{r}\sqrt{1 - J_r}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n \left(\frac{1}{2}\right)_n}{(n!)^3} (J_r)^n (6n + 1 - T_r) \quad (29)$$

Note. The function G_r is the Weber invariant and

$$\sigma(r) = 2\sqrt{r}(1 + k_r^2) - 6a(r)$$

(see [7],[5] chapter 5).

The above formulas (27), (28) and (29) can be used for numerical and theoretical evaluations.

Similarities of formula (29) and a fifth order base formula

From the identity

$${}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; \frac{1 - \sqrt{1-z}}{2}\right)^2 = {}_3F_2\left(\frac{1}{6}, \frac{5}{6}, \frac{1}{2}; 1, 1; z\right), \quad (30)$$

and using the following relations found in [7]:

$$K_s(x) = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2} - s, \frac{1}{2} + s; 1; x^2\right) \text{ and } E_s(x) = \frac{\pi}{2} {}_2F_1\left(-\frac{1}{2} - s, \frac{1}{2} + s; 1; x^2\right) \quad (31)$$

$$E_s = (1 - k^2)K_s + \frac{k(1 - k^2)}{1 + 2s} \dot{K}_s, \quad \dot{K}_s(t) = \frac{dK_s(t)}{dt} \quad (32)$$

$$a_s(x_r) := \frac{\pi}{4K_s(x_r)} \frac{\cos(\pi s)}{1 + 2s} - \sqrt{r} \left(\frac{E_s(x_r)}{K_s(x_r)} - 1 \right), \quad (33)$$

with $s = 1/3$ one can get, (working as in Theorem 2.1) the following Ramanujan-type $1/\pi$ formula:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n \left(\frac{1}{2}\right)_n}{(n!)^3} (4\beta_r(1 - \beta_r))^n \left(3n - 5 \frac{\beta_r - \frac{a_5(r)}{\sqrt{r}}}{1 - 2\beta_r} \right) = \frac{3}{2\pi\sqrt{r}(1 - 2\beta_r)}, \quad (34)$$

where the function $\alpha_5(r) = a_{1/3}(\sqrt{\beta_r})$ is algebraic for $r \in \mathbf{Q}_+^*$.

The parameters and the corresponding function $\alpha_5(r)$ of (34) are those of fifth singular moduli base theory. Also (34) in comparison with (29) gives the following theorem.

Theorem 3.2

$$10\alpha_5(r)r^{-1/2} = 10a_{1/3}(\sqrt{\beta_r})r^{-1/2} = 1 + 8\beta_r - \frac{1 + k_r^2 - 3a(r)r^{-1/2}}{\sqrt{1 - k_r^2 + k_r^4}} \quad (35)$$

The above formula is for general evaluation of elliptic alpha function in the fifth elliptic base.

Also from the cubic theory as in fifth, we have

$${}_3F_2\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{2}; 1, 1; w\right) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{1 - \sqrt{1-w}}{2}\right)^2 \quad (36)$$

we get

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{2}\right)_n}{(n!)^3} [4\alpha_3(r) - 4\alpha_3^2(r)]^n (n - b) = \frac{\sqrt{3}}{2\pi\sqrt{r}(1 - 2\alpha_3(r))} \quad (37)$$

$$b = \frac{4 \left(\alpha_3(r) - a_{1/6}[\alpha_3^{1/3}(r)]r^{-1/2} \right)}{3(1 - 2\alpha_3(r))} \quad (38)$$

4 Examples and Evaluations

1) For $r = 2$

$$J_2 = \frac{27}{125}$$

$$T_2 = \frac{5}{14}$$

and

$$\frac{15\sqrt{5}}{14\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n \left(\frac{1}{2}\right)_n}{(n!)^3} \left(\frac{27}{125}\right)^n \left(6n + \frac{9}{14}\right) \quad (39)$$

2) For $r = 4$ we have

$$\frac{11\sqrt{\frac{11}{3}}}{14\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n \left(\frac{1}{2}\right)_n}{(n!)^3} \left(\frac{8}{1331}\right)^n \left(6n + \frac{10}{21}\right) \quad (40)$$

3) For $r = 5$ we have

$$T_5 = \frac{1}{418} (139 + 45\sqrt{5})$$

$$J_5 = \frac{27(-1975 + 884\sqrt{5})}{33275}$$

Hence

$$\frac{\sqrt{21650 + 5967\sqrt{5}}}{\pi} =$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n \left(\frac{1}{2}\right)_n}{(n!)^3} \left(\frac{-53325 + 23868\sqrt{5}}{33275}\right)^n (836n + 93 - 15\sqrt{5}) \quad (41)$$

4) For $r = 8$ we have

$$k_8^2 = 113 + 80\sqrt{2} - 4\sqrt{2(799 + 565\sqrt{2})}$$

$$a(8) = 2(10 + 7\sqrt{2}) \left(1 - \sqrt{-2 + 2\sqrt{2}}\right)^2$$

Then

$$\frac{15\sqrt{\frac{5}{2}(84125 + 81432\sqrt{2})}}{9982\pi} =$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n \left(\frac{1}{2}\right)_n}{(n!)^3} \left(\frac{5643000 - 3990168\sqrt{2}}{1520875}\right)^n \left(\frac{3276 - 1125\sqrt{2} + 29946n}{4991}\right) \quad (42)$$

5) For $r = 18$ we have

$$k_{18} = (-7 + 5\sqrt{2})(7 - 4\sqrt{3})$$

$$a(18) = -3057 + 2163\sqrt{2} + 1764\sqrt{3} - 1248\sqrt{6}$$

$$\alpha_6 = \frac{1}{500}(68 - 27\sqrt{6})$$

$$\beta_{18} = \frac{1}{2} - \frac{7(49982 + 4077\sqrt{6})}{10\sqrt{5}(989 + 54\sqrt{6})^{3/2}}$$

$$J_{18} = \frac{637326171 - 260186472\sqrt{6}}{453870144125} \quad (43)$$

$$T_{18} = \frac{712075 + 49230\sqrt{6}}{1074514} \quad (44)$$

Hence we get the formula giving 8 digits per term:

(Note that the number of digits per term is determined by the value of J_r , approximately.)

$$\begin{aligned} & \frac{5\sqrt{23124123365 - 13274820\sqrt{6}}}{1074514\pi} = \\ & \sum_{n=0}^{\infty} \frac{(\frac{1}{6})_n (\frac{5}{6})_n (\frac{1}{2})_n}{(n!)^3} \left(\frac{637326171 - 260186472\sqrt{6}}{453870144125} \right)^n \times \\ & \times \left(6n + \frac{9(40271 - 5470\sqrt{6})}{1074514} \right) \end{aligned} \quad (45)$$

6) For $r = 27$

$$k_{27} = \frac{1}{2} \sqrt{\frac{1 + 100 \cdot 2^{1/3} - 80 \cdot 2^{2/3}}{2 + \sqrt{3} - 100 \cdot 2^{1/3} + 80 \cdot 2^{2/3}}}$$

$$a(27) = 3 \left[\frac{1}{2} (\sqrt{3} + 1) - 2^{1/3} \right]$$

$a(27)$ is obtained from [7] page 172.

$$J_{27} = \frac{56143116 + 157058640 \cdot 2^{1/3} - 160025472 \cdot 2^{2/3}}{817400375}$$

$$T_{27} = \frac{58871825 + 22512960 \cdot 2^{1/3} + 13208820 \cdot 2^{2/3}}{132566687}$$

Hence we get the 11 digits per term formula:

$$\begin{aligned} & \frac{935}{\pi} \sqrt{\frac{935}{3(761257259 - 157058640\sqrt[3]{2} + 160025472\sqrt[3]{4})}} = \\ & = \sum_{n=0}^{\infty} \frac{(\frac{1}{6})_n (\frac{5}{6})_n (\frac{1}{2})_n}{(n!)^3} \left(\frac{56143116 + 157058640\sqrt[3]{2} - 160025472\sqrt[3]{4}}{817400375} \right)^n \times \end{aligned}$$

$$\times \left(6n + \frac{6 (12282477 - 3752160\sqrt[3]{2} - 2201470\sqrt[3]{4})}{132566687} \right)$$

7) From the Wolfram pages 'Elliptic Lambda Function' and 'Elliptic Singular Value' we have:

$$k_{58} = (-1 + \sqrt{2})^6 (-99 + 13\sqrt{58})$$

and

$$a(58) = \frac{1}{64} (-70 + 99\sqrt{2} - 13\sqrt{29}) (5 + \sqrt{29})^6 (-444 + 99\sqrt{29})$$

Also using the cubic theta identities, (see [25] relations (2),(3),(4),(30)) we evaluate α_{174} numerically to 1500 digits and then β_{58} to 1500 digits accuracy. We then apply the 'Recognize' routine of Mathematica. The result is the minimum polynomial of β_{58} (this can be done also from (19) and (23)):

$$1 - 1399837865393267000x + 79684665286353732299517000x^2 - 159369327773031733812500000x^3 + 79684663886515866906250000x^4 = 0.$$

Solving this equation with respect to x we get the value of β_{58} in radicals. Thus

$$J_{58} = \frac{1399837865393267 - 259943365786104\sqrt{29}}{39842331943257933453125} \quad (46)$$

$$T_{58} = \frac{5 (1684967251 + 24160612\sqrt{29})}{10376469642} \quad (47)$$

The result is the formula

$$\frac{5\sqrt{\frac{5}{87} (13826969809210107 - 90211316\sqrt{29})}}{357809298\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n \left(\frac{1}{2}\right)_n}{(n!)^3} \left(\frac{1399837865393267 - 259943365786104\sqrt{29}}{39842331943257933453125} \right)^n \times \left(\frac{6117973}{32528118} - \frac{8628790}{25557807\sqrt{29}} + 6n \right) \quad (48)$$

which gives 18 digits per term.

8) For $r = 93$ (see [7] pg.158), we have

$$\sigma(93) = 6G_{93}^{-6} \left(\frac{\sqrt{3} + 1}{2} \right)^3 (15\sqrt{93} + 13\sqrt{31} + 201\sqrt{3} + 217).$$

From [5] chapter 34 we have

$$G_{93} = \frac{(3\sqrt{3} + \sqrt{31})^{1/4} (39 + 7\sqrt{31})^{1/6}}{2^{1/3}}$$

also

$$a(r) = \sqrt{r} \frac{1 + k_r^2}{3} - \frac{\sigma(r)}{6}$$

$$G_{93}^{-24} = 4k_{93}^2(1 - k_{93}^2)$$

Hence

$$(k_{93}k'_{93})^2 = \frac{1}{224589314596 + 129666700800\sqrt{3} + 40337431680\sqrt{31} + 23288826960\sqrt{93}}$$

$$J_{93}^{-1} = 119562334956358303022500 + 21474029280866147440000\sqrt{31} +$$

$$+ 470106000\sqrt{129368095019778762513344107725 + 23235195778655878514048710848\sqrt{31}}$$

$$T_{93} = \frac{10559116299575 + 1317692448000\sqrt{3} + 275805228680\sqrt{31} - 81807235875\sqrt{93}}{15081520900138},$$

where

$$j_{93} = 1728J_{93}^{-1}$$

This result is a very flexible formula that gives about 24 digits per term.

5 Neat Examples with Mathematica and Simplicity

The class number $h(-d)$, $d \in \mathbf{N}$ of the equivalent quadratic forms is given by

$$h(-d) = -\frac{w(d)}{2d} \sum_{n=1}^{d-1} \left(\frac{-d}{n} \right) n, \quad (49)$$

where $w(3) = 6$, $w(4) = 4$ else $w(d) = 2$. $\left(\frac{n}{m} \right)$, is the Jacobi symbol. Observe that $h(-163) = 1$ (see [17]). For small values of $h(-d)$ we have greater possibility to evaluate J_d and T_d in radicals.

The simplest way to evaluate the parameters J_{163} and T_{163} is again with Mathematica.

The general algorithm is:

i) Set $r = d$ and $k[r] = \text{InverseEllipticNomeQ}[e^{-\pi\sqrt{r}}]^{1/2}$, then we can evaluate β_r and j_r from relations (19) and (23). Hence we get the value of J_r as in section 4 example 7.

ii) For the evaluation of T_r we will need the value of $a(r)$ which is given from (see [7]):

$$a(r) = \frac{\pi}{4K^2} - \sqrt{r} \left(\frac{E}{K} - 1 \right). \quad (50)$$

This in Mathematica is given from

$$a(r) = \frac{\pi}{4\text{EllipticK}[k[r]^2]} - \sqrt{r} \left(\frac{\text{EllipticE}[k[r]^2]}{\text{EllipticK}[k[r]^2]} - 1 \right) \quad (51)$$

Hence taking the package

<< NumberTheory`Recognize`

and

Recognize[N[J₁₆₃, 1500], 16, x]

Recognize[N[T₁₆₃, 1500], 16, x]

we get two equations. After solving them we get if $r \in \mathbf{N}$ (here $r = 163$), the values of the parameters J_r and T_r in algebraic-closed forms. The results are the π formulas.

1) We have that J_{163} is root of

$$-64 + 2552810853189232588558727380998000x - 2198253790246041723377943360187500x^2 + 224451422498574115473590775022822688001953125x^3 = 0$$

hence

$$J_{163} = 4 \frac{C_1 - C_2 \left(-A_1 + \sqrt{489}B_1 \right)^{-1/3} + 30591288 \left(-A_1 + \sqrt{489}B_1 \right)^{1/3}}{10792555251621895860488211571345343375}$$

$$A_1 = 12737965652562547164590026038483234248161827096523072256574968383$$

$$B_1 = 229038073182066825378006485964950394558349727761749294205546402325349$$

$$C_1 = 8808429913332498766352891$$

$$C_2 = 902206261147132595923169636910570558029813352485594880$$

From $J_r = 4\beta_r(1 - \beta_r)$, we get the value of β_r and hence

$$T_{163} = 5 \frac{12948195754365757115 + 8 \left(A_2 - B_2\sqrt{489} \right)^{1/3} + 8 \left(A_2 + B_2\sqrt{489} \right)^{1/3}}{83470787671093501833}$$

where

$$A_2 = 3802386862487392962897493239274992371253057854289262$$

$$B_2 = 3865464212119923579732688315287754932290919450$$

The above parameters give 32 digits per term

2) Another evaluation is taking $d = r = 253$:

$$J_{253} = \frac{A_1 - A_2\sqrt{11} + 31990140\sqrt{A_3 - A_4\sqrt{11}}}{A_5}$$

$$A_1 = 2804365789259959094417576921792857440357087269234369$$

$$A_2 = 845548099807651569627713349319558464492321957799872$$

$$A_3 = 1433462642401972199773341051748172965440271797713951$$

$$6818782945906676740858207407330990565$$

$$A_4 = 43220524871261259540733172862370537466134334936322822$$

$$33926553935879770457716659641968088$$

$$A_5 = 1066755353338783886372226117351012749877681799897625$$

and

$$T_{253} = \frac{1875\sqrt{B_1 - B_2\sqrt{11}} + 3847208393012364625 + 752271279708923520\sqrt{11}}{6969874104047710086}$$

$$B_1 = 213216899528167866600672118125$$

$$B_2 = 60533150139616794053500831192$$

The above parameters give 41 digits per term.

Conclusion

We have given a way of how we can construct a very large number of Ramanujan's type $1/\pi$ formulas. It is true that in most cases, from $r = 1$ to 100 (or higher), using Mathematica program, such formulas are very simple, as long as $h(-d)$ remains small and the parameters are solutions of solvable polynomial equations.

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