Foundations of Physics, Vol. 19, No. 1, 1989

Is Life Improbable?

Received April 6, 1987

E. P. Wigner's argument that the probability of the existence of self-reproducing units, e.g., organisms, is zero according to standard quantum theory is stated and analyzed. Theorems are presented which indicate that Wigner's mathematical result in fact should not be interpreted as asserting the improbability of self-reproducing units.

Given the nature of its conclusions, E. P. Wigner's "On the Probability of the Existence of a Self-Reproducing Unit"(1) has received rather little attention. In this paper, Wigner presents a mathematical argument which "purports to show that, according to standard quantum mechanical theory, the probability is zero for the existence of self-reproducing states, e.g., organisms." Though he describes the argument as "not truly conclusive," some seem to have taken it as such. H. P. Yockey, for example, cites it with the remark that "for all physics has to offer, life should never have appeared and if it ever did it would soon die out";(2) D. Bohm cites it with similar approval.(3) J. Rothstein(4) treats it as an indication that standard quantum theory needs to be supplemented by a theory of preparation and measurement of quantum systems. Here we argue that, correctly interpreted, Wigner's mathematical result does not mean that organisms are unlikely in the context of standard quantum theory.

Let us first state Wigner's mathematical result and then discuss his interpretation of it. Wigner makes use of a fixed basis for the Hilbert space involved and describes vectors in terms of their components; for reasons given below, we will not do so. With this caveat, the following is a faithful description of Wigner's result. Let $U(H)$ denote the group of unitary

[1] Department of Mathematics, Yale University, New Haven, Connecticut 06520.

0015-9018/89/0100-0091$06.00/0

operators on the Hilbert space H and let $\langle v \rangle$ denote the linear span of a vector v in a Hilbert space.

Result (Wigner). Let H, H_1, and H_2 be finite-dimensional Hilbert spaces with $H = H_1 \otimes H_1 \otimes H_2$. Given $1 \leqslant n \leqslant \dim H_1$, let

$$\mathbf{S} = \{S \in U(H)\colon \exists V \subseteq H_1 \; \exists w \in H_1 \otimes H_2 \; \dim V = n,$$

$$w \neq 0 \text{ and } S\colon V \otimes \langle w \rangle \to V \otimes V \otimes H_2\}$$

If $\dim H_1$ is sufficiently greater than n, the probability that an element of $U(H)$ is in $\mathbf{S}$ is zero. ∎

Since the statement and Wigner's proof of this result are not quite rigorous, we do not label it a "theorem." Presumably the precise conclusion is that S is a set of measure zero with respect to the natural measure (Haar measure) on $U(H)$. Under this interpretation, Wigner's proof is convincing and could probably be made rigorous.

Wigner interprets this result as follows. H is the Hilbert space associated with a quantum system, and the decomposition $H = H_1 \otimes H_1 \otimes H_2$ corresponds to the fact that the matter in the system is thought of as consisting of three parts. The Hilbert space H_1 corresponds to the states of some matter in the system, part of which can form an organism; $V \subseteq H_1$ corresponds to the "living" states in which there actually is an organism present. The question is then: given a unitary operator $S\colon H \to H$ describing a certain amount of time evolution, is there a subspace $V \subseteq H_1$ and a "nutrient" state $w \in H_1 \otimes H_2$ such that S maps $V \otimes \langle w \rangle$ into $V \otimes V \otimes H_2$, i.e., such that after a time one organism and the nutrient will become two organisms (and some leftover material)? Assuming $\dim V \ll \dim H_1$ is natural because most states of the matter that may constitute an organism are in fact not "living." By the result, in this case for very few $S \in U(H)$ is the answer to the question, "yes." Wigner concludes that "the chances are nil for the existence of a set of 'living' states for which one can find a nutrient of such nature that the interaction *always* leads to multiplication."

Note, however, that in Wigner's result a decomposition of H as $H_1 \otimes H_1 \otimes H_2$ is fixed at the outset. (In his presentation, this is done implicitly by choosing a basis for H whereby vectors are described by their components $\Psi_{\kappa\lambda\mu}$. This why we have avoided using a basis.) There is no particular reason for fixing this. The factors H_1, H_1, and H_2 correspond to different parts of the matter composing the system, and surely what "matter" is, hence which decompositions of H as a tensor product are sensible, depends on the time evolution law S. Until dynamics or other structure is given, no one decomposition of an abstract Hilbert space as a tensor product is to be preferred over any other. If we let the decomposition depend on the choice of $S \in U(H)$, a different picture emerges.

Theorem 1. Let H, H_1, and H_2 be finite-dimensional Hilbert spaces with $\dim H = (\dim H_1)^2 \cdot \dim H_2$. Let $S \in U(H)$, let V be a subspace of H_1, and let w be a nonzero vector in $H_1 \otimes H_2$. Then there is a unitary isomorphism $U: H \to H_1 \otimes H_1 \otimes H_2$ such that, identifying H with $H_1 \otimes H_1 \otimes H_2$ by U, we have

$$S: V \otimes \langle w \rangle \to V \otimes V \otimes H_2$$

Proof. If $\dim V = 0$, the conclusion is trivial, so assume otherwise. By diagonalization, it is clear that S has invariant subspaces of all dimensions $\leqslant \dim H$. Choose an invariant subspace $W \subseteq H$ with $\dim W = \dim V \otimes V \otimes H_2$. Let U_0: $W \to V \otimes V \otimes H_2$ be a unitary isomorphism, and extend U_0 arbitrarily to a unitary isomorphism $U: H \to H_1 \otimes H_1 \otimes H_2$. Let w be any nonzero vector in $V \otimes H_2$. Then $V \otimes \langle w \rangle \subseteq V \otimes V \otimes H_2$, so U^{-1} maps $V \otimes \langle w \rangle$ into W. S preserves W and U maps W into $V \otimes V \otimes H_2$, so USU^{-1} maps $V \otimes \langle w \rangle$ into $V \otimes V \otimes H_2$. Thus, identifying H with $H_1 \otimes H_1 \otimes H_2$ by U, we have $S: V \otimes \langle w \rangle \to V \otimes V \otimes H_2$ as required. ∎

This means that, if we allow the decomposition of H as a tensor product to depend on the law S, we can always find *many* choices of a "living" subspace V and a nutrient state w that guarantee reproduction according to Wigner's criteria. Does this mean that self-reproducing units are plentiful in any quantum system? A look at the proof of the Theorem shows that this interpretation would be unwarranted. We have chosen $w \in V \otimes H_2$, that is, the nutrient actually consists of another "organism" and some extra material. This clearly violates the spirit, though not the letter, of Wigner's approach. [Note that, given a subspace $V \subseteq H_1$ of dimension n, every $S \in U(H)$ preserving $V \otimes V \otimes H_2 \subseteq H$ is in the set **S** as it is defined in the above result, so the same form of "cheating" is possible in his original formulation.]

Theorem 1 does, however, make some general features of the problem apparent. First, to argue that life is improbable, one needs *necessary* conditions for something to be "life," which can be very general. Refuting such an argument by no means indicates that life is probable, because to do so would require *sufficient* conditions that something be "life." The latter seem much harder to obtain. Second, one should carefully examine the proof of any theorem that attempts to deal with arbitrary self-reproducing entities. If the proof uses a "cheap trick," the theorem is likely to be less significant than it might at first seem.

It is worth noting that we can modify Theorem 1 to show for most $S \in U(H)$, we can find a one-dimensional "living" space V such that the nutrient state is in $V^{\perp} \otimes H_2$. This rules out nutrient states that already include the organism to be produced.

Theorem 2. Let H, H_1, and H_2 be finite-dimensional Hilbert spaces with $\dim H = (\dim H_1)^2 \cdot \dim H_2$. Let

$$\begin{aligned} \mathbf{S}' = \{ S \in U(H) \colon & \exists v \in H_1\ \exists w \in \langle v \rangle^{\perp} \otimes H_2, \\ & v \neq 0,\ w \neq 0, \text{ and } \exists \text{ unitary } U \colon H \to H_1 \otimes H_1 \otimes H_2, \\ & USU^{-1}(v \otimes w) \in \langle v \otimes v \rangle \otimes H_2 \} \end{aligned}$$

Then, as $\dim H \to \infty$, the probability that $S \in U(H)$ is in $\mathbf{S}'$ approaches 1. [Here, probability is with respect to Haar measure on $U(H)$.]

Proof. We will show that $\mathbf{S}'$ contains all $S \in U(H)$ having 3 eigenvalues α_i such that $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0$ for some positive real numbers c_i. Thus, U will be in $\mathbf{S}'$ unless all of its eigenvalues lie in some half of the unit circle,

$$\{ e^{i\theta} \colon \theta_0 < \theta < \theta_0 + \pi \}$$

This implies that, as $\dim H \to \infty$, the probability that $S \in U(H)$ in $\mathbf{S}'$ approaches 1. [Here, we are making use of the Weyl integration formula relating Haar measure on $U(H)$ and Haar measure on the maximal torus consisting of all diagonal elements of $U(H)$; see, e.g., Theorem 6.1 of Adams[5].]

Suppose that $u_i \in H$, $i = 1, 2, 3$, are nonzero, $Su_i = \alpha_i u_i$, and there exist $c_i > 0$ with $\sum c_i \alpha_i = 0$. Let $u = \sum \sqrt{c_i}\, u_i$. Since the u_i are orthogonal, $u \neq 0$ and

$$(u, Su) = \sum c_i \alpha_i = 0$$

Define U as follows. Choose $v \in H_1$ and $w \in \langle v \rangle^{\perp} \otimes H_2$ with $\|v \otimes w\| = \|u\|$. Let $U(u) = v \otimes w$ and let $U(Su)$ be any vector in $\langle v \otimes v \rangle \otimes H_2$ with norm equal to $\|Su\|$. Since $\|Uu\| = \|u\|$, $\|USu\| = \|Su\|$, and $(Uu, USu) = (u, Su) = 0$, U can be extended to a unitary isomorphism from H to $H_1 \otimes H_1 \otimes H_2$; do so arbitrarily. Then,

$$USU^{-1}(v \otimes w) = USu \in \langle v \otimes v \rangle \otimes H_2$$

as required, so $S \in \mathbf{S}'$. ∎

Again, while this proof sets limits on what any "proof of the improbability of self-reproducing units" could achieve, it should not be thought of as proving that self-reproducing units are probable. At present, mathematical arguments concerning the question of the probability of life can serve to refine our notion of such concepts as "entity" and "reproduction"; one probably should not expect these arguments to settle the question. A deeper mathematical investigation of the question is likely to require consideration of objects other than a Hilbert space and single unitary operator. First, life as we know it makes use of a continuous group of unitary time evolution operators that, far from being random, satisfy locality and symmetry constraints. Second, the likelihood of life seems to depend crucially not just on the time evolution laws of physics, but on boundary conditions, that is, constraints on the *state* of the system. For example, a source of free energy is required.

Perhaps these considerations can eventually be incorporated in a mathematical theory of self-reproducing systems. In the meantime, the empirical evidence for the existence of life and the validity of quantum theory strongly indicates that, in some sense, life is not of probability zero according to quantum theory, and that the task is to see what is this sense.

ACKNOWLEDGMENT

Some specifics of this paper, and my continued interest in the question as a whole, are largely due to conversations with Bruce Smith.

REFERENCES

1. E. P. Wigner, "The probability of the existence of a self-reproducing unit," in *Symmetries and Reflections* (Indiana University Press, Bloomington, 1967), pp. 200–208.
2. H. P. Yockey, *J. Theor. Biol.* **46**, 369 (1974).
3. D. Bohm, "Some remarks on the notion of order," in *Towards a Theoretical Biology: II Sketches*, C. H. Waddington, ed. (Aldine, Chicago, 1969), pp. 18–40.
4. J. Rothstein, *Bull. Am. Phys. Soc.* **14**, 504 (1969).
5. J. F. Adams, *Lectures on Lie Groups* (W. A. Benjamin, New York, 1969).