LECTURES ON SET THEORY

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Edition of August 9, 2017: chapter on proper forcing rewritten.

Edition of November 14, 2016: chapter on proper forcing changed; the proof of Theorem 28.5 was in error, and a new proof using a game is given (Theorem 28.33).

Edition of August 30, 2016: Proposition 27.21 corrected.

Some background on these notes:

0. The exercise solutions have not been carefully checked.

1. The axioms for first-order logic are due to Tarski.

2. The treatment of forcing follows Kunen, except for using Boolean values in the definition.

3. The proof of Hausdorff's theorem in Chapter 17 follows Hausdorff's original proof closely.

4. The treatment of proper forcing in Chapter 28 follows Jech to a large extent.

5. For PCF in chapters 30–32 we follow Abraham and Magidor.

6. Chapter 33 is based on Blass.

7. The proof that $\mathfrak{p} = \mathfrak{t}$ in Chapter 34 is based upon notes of Fremlin and a thesis of Roccasalvo.

8. The consistency proofs in Chapter 35 are partly from Kunen and partly from the author.

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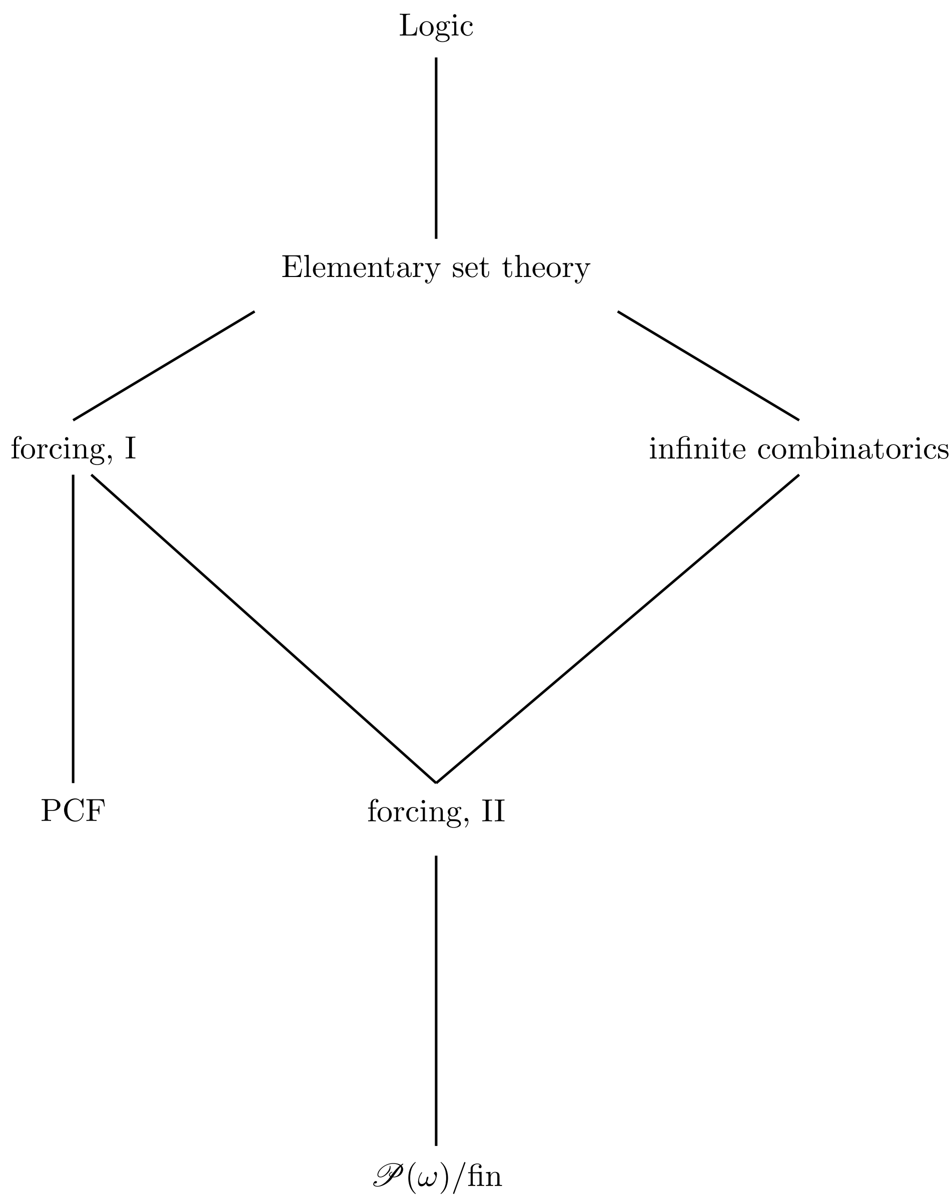
APPENDICES

A rough dependency diagram



1. Sentential logic

We go into the mathematical theory of the simplest logical notions: the meaning of "and", "or", "implies", "if and only if" and related notions. The basic idea here is to describe a formal language for these notions, and say precisely what it means for statements in this language to be true. The first step is to describe the language, without saying anything mathematical about meanings. We need very little background to carry out this development. ω is the set of all natural numbers $0, 1, 2, \ldots$. Let ω^+ be the set of all positive integers. For each positive integer m let $m' = \{0, \ldots, m-1\}$. A *finite sequence* is a function whose domain is m' for some positive integer m; the values of the function can be arbitary.

To keep the treatment strictly mathematical, we will define the basic "symbols" of the language to just be certain positive integers, as follows:

Negation symbol: the integer 1.
Implication symbol: the integer 2.
Sentential variables: all integers ≥ 3.

Let Expr be the collection of all finite sequences of positive integers; we think of these sequences as *expressions*. Thus an expression is a function mapping m' into ω^+, for some positive integer m. Such sequences are frequently indicated by $\langle \varphi_0, \ldots, \varphi_{m-1} \rangle$. The case $m = 1$ is important; here the notation is $\langle \varphi \rangle$.

The one-place function $\neg$ mapping Expr into Expr is defined by $\neg\varphi = \langle 1 \rangle^\frown \varphi$ for any expression φ. Here in general $\varphi^\frown \psi$ is the sequence φ followed by the sequence ψ.

The two-place function $\rightarrow$ mapping Expr$\times$Expr into Expr is defined by $\varphi \rightarrow \psi = \langle 2 \rangle^\frown \varphi^\frown \psi$ for any expressions φ, ψ. (For any sets A, B, $A \times B$ is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. So Expr $\times$ Expr is the set of all ordered pairs (φ, ψ) with φ, ψ expressions.)

For any natural number n, let $S_n = \langle n + 3 \rangle$.

Now we define the notion of a sentential formula—an expression which, suitably interpreted, makes sense. We do this definition by defining a *sentential formula construction*, which by definition is a sequence $\langle \varphi_0, \ldots, \varphi_{m-1} \rangle$ with the following property: for each $i < m$, one of the following holds:

$\varphi_i = S_j$ for some natural number j.

There is a $k < i$ such that $\varphi_i = \neg\varphi_k$.

There exist $k, l < i$ such that $\varphi_i = (\varphi_k \rightarrow \varphi_l)$.

Then a *sentential formula* is an expression which appears in some sentential formula construction.

The following proposition formulates the principle of *induction on sentential formulas.*

Proposition 1.1. *Suppose that M is a collection of sentential formulas, satisfying the following conditions.*

(i) S_i is in M, for every natural number i.
(ii) If φ is in M, then so is $\neg\varphi$.
(iii) If φ and ψ are in M, then so is $\varphi \to \psi$.

Then M consists of all sentential formulas.

Proof. Suppose that θ is a sentential formula; we want to show that $\theta \in M$. Let $\langle \tau_0, \ldots, \tau_m \rangle$ be a sentential formula construction with $\tau_t = \theta$, where $0 \le t \le m$. We prove by complete induction on i that for every $i \le m$, $\tau_i \in M$. Hence by applying this to $i = t$ we get $\theta \in M$.

So assume that for every $j < i$, the sentential formula τ_j is in M.

Case 1. τ_i is S_s for some s. By (i), $\tau_i \in M$.

Case 2. τ_i is $\neg\tau_j$ for some $j < i$. By the inductive hypothesis, $\tau_j \in M$, so $\tau_i \in M$ by (ii).

Case 3. τ_i is $\tau_j \to \tau_k$ for some $j, k < i$. By the inductive hypothesis, $\tau_j \in M$ and $\tau_k \in M$, so $\tau_i \in M$ by (iii). $\square$

Proposition 1.2. *(i) Any sentential formula is a nonempty sequence.*

(ii) For any sentential formula φ, exactly one of the following conditions holds:

(a) φ is S_i for some $i \in \omega$.

(b) φ begins with 1, and there is a sentential formula ψ such that $\varphi = \neg\psi$.

(c) φ begins with 2, and there are sentential formulas ψ, χ such that $\varphi = \psi \to \chi$.

(iii) No proper initial segment of a sentential formula is a sentential formula.

(iv) If φ and ψ are sentential formulas and $\neg\varphi = \neg\psi$, then $\varphi = \psi$.

(v) If $\varphi, \psi, \varphi', \psi'$ are sentential formulas and $\varphi \to \psi = \varphi' \to \psi'$, then $\varphi = \varphi'$ and $\psi = \psi'$.

Proof. (i): Clearly every entry in a sentential formula construction is nonempty, so (i) holds.

(ii): First we prove by induction that one of (a)–(c) holds. This is true of sentential variables—in this case, (a) holds. If it is true of a sentential formula φ, it is obviously true of $\neg\varphi$; so (b) holds. Similarly for $\to$, where (c) holds.

Second, the first entry of a formula differs in cases (a),(b),(c), so exactly one of them holds.

(iii): We prove this by complete induction on the length of the formula. So, suppose that φ is a sentential formula and we know for any formula ψ shorter than φ that no proper initial segment of ψ is a formula. We consider cases according to (ii).

Case 1. φ is S_i for some i. Only the empty sequence is a proper initial segment of φ in this case, and the empty sequence is not a sentential formula, by (i).

Case 2. φ is $\neg\psi$ for some formula ψ. If χ is a proper initial segment of φ and it is a formula, then χ begins with 1 and so by (ii), χ has the form $\neg\theta$ for some formula θ. But then θ is a proper initial segment of ψ and ψ is shorter than φ, so the inductive hypothesis is contradicted.

Case 3. φ is $\psi \to \chi$ for some formulas ψ and χ. That is, φ is $\langle 2 \rangle^\frown \psi^\frown \chi$. If θ is a proper initial segment of φ which is a formula, then by (ii), θ has the form $\langle 2 \rangle^\frown \xi^\frown \eta$ for some formulas ξ, η. Now $\psi^\frown \chi = \xi^\frown \eta$, so ψ is an initial segment of ξ or ξ is an initial segment of ψ. Since ψ and ξ are both shorter than φ, it follows from the inductive hypothesis that $\psi = \xi$. Hence $\chi = \eta$, and $\varphi = \theta$, contradiction.

(iv) is rather obvious; if $\neg\varphi = \neg\psi$, then φ and ψ are both the sequence obtained by deleting the first entry.

(v): Assume the hypothesis. Then $\varphi \to \psi$ is the sequence $\langle 2\rangle^\frown\varphi^\frown\psi$, and $\varphi' \to \psi'$ is the sequence $\langle 2\rangle^\frown\varphi'^\frown\psi'$. Since these are equal, φ and φ' start at the same place in the sequence. By (iii) it follows that $\varphi = \varphi'$. Deleting the initial segment $\langle 2\rangle^\frown\varphi$ from the sequence, we then get $\psi = \psi'$. $\square$

Parts (iv) and (v) of this proposition enable us to define values of sentential formulas, which supplies a mathematical meaning for the truth of formulas. A *sentential assignment* is a function mapping the set $\{0, 1, \ldots\}$ of natural numbers into the set $\{0, 1\}$. Intuitively we think of 0 as "false" and 1 as "true". The definition of values of sentential formulas is a special case of definition by recursion:

Proposition 1.3. *For any sentential assignment f there is a function F mapping the set of sentential formulas into $\{0, 1\}$ such that the following conditions hold:*

(i) $F(S_n) = f(n)$ for every natural number n.

(ii) $F(\neg\varphi) = 1 - F(\varphi)$ for any sentential formula φ.

(iii) $F(\varphi \to \psi) = 0$ iff $F(\varphi) = 1$ and $F(\psi) = 0$.

Proof. An *f-sequence* is a finite sequence $\langle(\varphi_0, \varepsilon_0), \ldots, (\varphi_{m-1}, \varepsilon_{m-1})\rangle$ such that each ε_i is 0 or 1, and such that for each $i < m$ one of the following holds:

(1) φ_i is S_n for some $n \in \omega$, and $\varepsilon_i = f(n)$.

(2) There is a $k < i$ such that $\varphi_i = \neg\varphi_k$ and $\varepsilon_i = 1 - \varepsilon_k$.

(3) There are $k, l < i$ such that $\varphi_i = \varphi_k \to \varphi_l$, and $\varepsilon_i = 0$ iff $\varepsilon_k = 1$ and $\varepsilon_l = 0$.

Now we claim:

(4) For any sentential formula ψ and any f-sequences $\langle(\varphi_0.\varepsilon_0), \ldots, (\varphi_{m-1}, \varepsilon_{m-1})\rangle$ and $\langle(\varphi'_0.\varepsilon'_0), \ldots, (\varphi'_{n-1}, \varepsilon'_{n-1})\rangle$ such that $\varphi_{m-1} = \varphi'_{n-1} = \psi$ we have $\varepsilon_{m-1} = \varepsilon'_{n-1}$.

We prove (4) by induction on ψ, thus using Proposition 1.1. If $\psi = S_n$, then $\varepsilon_{m-1} = f(n) = \varepsilon'_{n-1}$. Assume that the condition holds for ψ, and consider $\neg\psi$. There is a $k < m - 1$ such that $\neg\psi = \varphi_{m-1} = \neg\varphi_k$. By Proposition 1.2(iv) we have $\varphi_k = \psi$. Similarly, there is an $l < n - 1$ such that $\neg\psi = \varphi'_{n-1} = \neg\varphi'_l$ and so $\varphi'_l = \psi$. Applying the inductive hypothesis to ψ and the sequences $\langle\varphi_0, \ldots, \varphi_k\rangle$ and $\langle\varphi'_0, \ldots, \varphi'_l\rangle$ we get $\varepsilon_k = \varepsilon'_l$. Hence $\varepsilon_{m-1} = 1 - \varepsilon_k = 1 - \varepsilon'_l = \varepsilon'_{n-1}$.

Now suppose that the condition holds for ψ and χ, and consider $\psi \to \chi$. There are $k, l < m - 1$ such that $(\psi \to \chi) = (\varphi_k \to \varphi_l)$. By Proposition 1.2(v) we have $\varphi_k = \psi$ and $\varphi_l = \chi$. Similarly there are $s, t < n - 1$ such that $(\psi \to \chi) = (\varphi'_s \to \varphi'_t)$. By Proposition 1.2(v) we have $\varphi'_s = \psi$ and $\varphi'_t = \chi$. Applying the inductive hypotheis to ψ and the sequences $\langle\varphi_0, \ldots, \varphi_k\rangle$ and $\langle\varphi'_0, \ldots, \varphi'_s\rangle$ we get $\varepsilon_k = \varepsilon'_s$. Similarly, we get $\varepsilon_l = \varepsilon'_t$. Hence

$$\varepsilon_{m-1} = 0 \quad \text{iff} \quad \varepsilon_k = 1 \text{ and } \varepsilon_l = 0$$

$$\begin{aligned} &\text{iff} \quad \varepsilon'_s = 1 \text{ and } \varepsilon'_t = 0 \\ &\text{iff} \quad \varepsilon'_{n-1} = 0. \end{aligned}$$

This finishes the proof of (4).

(5) If $\langle \varphi_0, \ldots, \varphi_{m-1} \rangle$ is a sentential formula construction, then there is an f-sequence of the form $(\varphi_0, \varepsilon_0), \ldots, (\varphi_{m-1}, \varepsilon_{m-1})\rangle$.

We prove this by induction on m. First suppose that $m = 1$. Then φ_0 must equal S_n for some n, and $\langle (\varphi_0, f(n)) \rangle$ is as desired. Now suppose that $m > 1$ and the statement is true for $m - 1$. So let $\theta \stackrel{\text{def}}{=} \langle (\varphi_0, \varepsilon_0), \ldots .(\varphi_{m-2}, \varepsilon_{m-2}) \rangle$ be an f-sequence.

Case 1. $\varphi_{m-1} = S_p$. Then $\theta \frown \langle (\varphi_{m-1}, f(p)) \rangle$ is as desired.

Case 2. There is a $k < m$ such that $\varphi_{m-1} = \neg\varphi_k$. Then $\theta \frown \langle (\varphi_{m-1}, 1 - \varepsilon_k) \rangle$ is as desired.

Case 3. There are $k, l < m$ such that $\varphi_{m-1} = \varphi_k \rightarrow \varphi_l$. Then $\theta \frown \langle (\varphi_{m-1}, \delta) \rangle$ is as desired, where $\delta = 0$ iff $\varepsilon_k = 1$ and $\varepsilon_l = 0$.

Thus (5) holds. Now we can define the function F required in the Proposition. Let ψ be any sentential formula. Let $\langle \varphi_0, \ldots, \varphi_{m-1} \rangle$ be a sentential formula construction such that $\varphi_{m-1} = \psi$. By (5), let $\langle (\varphi_0, \varepsilon_0), \ldots, (\varphi_{m-1}, \varepsilon_{m-1}) \rangle$ be an f-sequence. We define $F(\psi) = \varepsilon_{m-1}$. This is unambiguous by (4).

Case 1. $\psi = S_n$ for some n. Then by the definition of f-sequence we have $F(\psi) = f(n)$.

Case 2. There is a $k < m$ such that $\psi = \varphi_{m-1} = \neg\varphi_k$. Then $\langle (\varphi_0, \varepsilon_0), \ldots, (\varphi_k, \varepsilon_k) \rangle$ is an f-sequence, so $F(\varphi_k) = \varepsilon_k$. So

$$F(\psi) = F(\varphi_{m-1}) = \varepsilon_{m-1} = 1 - \varepsilon_k = 1 - F(\varphi_k).$$

Case 3. There are $k, l < m$ such that $\psi = \varphi_{m-1} = \varphi_k \rightarrow \varphi_l$. Then $\langle (\varphi_0, \varepsilon_0), \ldots, (\varphi_k, \varepsilon_k) \rangle$ is an f-sequence, so $F(\varphi_k) = \varepsilon_k$; and $\langle (\varphi_0, \varepsilon_0), \ldots, (\varphi_l, \varepsilon_k) \rangle$ is an f-sequence, so $F(\varphi_l) = \varepsilon_l$. So

$$\begin{aligned} F(\psi) = 0 \quad \text{iff} \quad F(\varphi_{m-1}) = 0 \quad \text{iff} \quad e_{m-1} = 0 \quad \text{iff} \\ \varepsilon_k = 1 \text{ and } \varepsilon_l = 0 \quad \text{iff} \quad F(\varphi_k) = 1 \text{ and } F(\varphi_l) = 0. \end{aligned}$$ □

With f a sentential assignment, and with F as in this proposition, for any sentential formula φ we let $\varphi[f] = F(\varphi)$. Thus:

$$\begin{aligned} S_i[f] &= f(i); \\ (\neg\varphi)[f] &= 1 - \varphi[f]; \\ (\varphi \rightarrow \psi)[f] &= \begin{cases} 0 & \text{if } \varphi[f] = 1 \text{ and } \psi[f] = 0, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

The definition can be recalled by using *truth tables*:

φ	$\neg\varphi$
1	0
0	1

φ	ψ	$\varphi \to \psi$
1	1	1
1	0	0
0	1	1
0	0	1

Other logical notions can be defined in terms of $\neg$ and $\to$. We define

$\varphi \wedge \psi = \neg(\varphi \to \neg\psi)$.
$\varphi \vee \psi = \neg\varphi \to \psi$.
$\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$.

Working out the truth tables for these new notions shows that they mean approximately what you would expect:

φ	ψ	$\neg\psi$	$\varphi \to \neg\psi$	$\varphi \wedge \psi$	$\neg\varphi$	$\varphi \vee \psi$	$\varphi \to \psi$	$\psi \to \varphi$	$\varphi \leftrightarrow \psi$
1	1	0	0	1	0	1	1	1	1
1	0	1	1	0	0	1	0	1	0
0	1	0	1	0	1	1	1	0	0
0	0	1	1	0	1	0	1	1	1

(Note that $\vee$ corresponds to *non-exclusive or*: φ or ψ, or both.)

The following simple proposition is frequently useful.

Proposition 1.4. *If f and g map $\{0,1,\ldots\}$ into $\{0,1\}$ and $f(m) = g(m)$ for every m such that S_m occurs in φ, then $\varphi[f] = \varphi[g]$.*

Proof. Induction on φ. If φ is S_i for some i, then the hypothesis says that $f(i) = g(i)$; hence $S_i[f] = f(i) = g(i) = S_i[g]$. Assume that it is true for φ. Now S_m occurs in φ iff it occurs in $\neg\varphi$. Hence if we assume that $f(m) = g(m)$ for every m such that S_m occurs in $\neg\varphi$, then also $f(m) = g(m)$ for every m such that S_m occurs in φ, so $(\neg\varphi)[f] = 1 - \varphi[f] = 1 - \varphi[g] = (\neg\varphi)[g]$. Assume that it is true for both φ and ψ, and $f(m) = g(m)$ for every m such that S_m occurs in $\varphi \to \psi$. Now if S_m occurs in φ, then it also occurs in $\varphi \to \psi$, and hence $f(m) = g(m)$. Similarly for ψ. It follows that

$$(\varphi \to \psi)[f] = 0 \text{ iff } (\varphi[f] = 1 \text{ and } \psi[f] = 0) \text{ iff } (\varphi[g] = 1 \text{ and } \psi[g] = 0) \text{ iff } (\varphi \to \psi)[g] = 0.$$

$\square$

This proposition justifies writing $\varphi[f]$ for a finite sequence f, provided that f is long enough so that m is in its domain for every m for which S_m occurs in φ.

A sentential formula φ is a *tautology* iff it is true under every assignment, i.e., $\varphi[f] = 1$ for every assignment f.

Here is a list of common tautologies:

(T1) $\varphi \to \varphi$.
(T2) $\varphi \leftrightarrow \neg\neg\varphi$.
(T3) $(\varphi \to \neg\varphi) \to \neg\varphi$.
(T4) $(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$.
(T5) $\varphi \to (\neg\varphi \to \psi)$.
(T6) $(\varphi \to \psi) \to [(\psi \to \chi) \to (\varphi \to \chi)]$.
(T7) $[\varphi \to (\psi \to \chi)] \to [(\varphi \to \psi) \to (\varphi \to \chi)]$.
(T8) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$.
(T9) $(\varphi \wedge \psi) \to \varphi$.
(T10) $(\varphi \wedge \psi) \to \psi$.
(T11) $\varphi \to [\psi \to (\varphi \wedge \psi)]$.
(T12) $\varphi \to (\varphi \vee \psi)$.
(T13) $\psi \to (\varphi \vee \psi)$.
(T14) $(\varphi \to \chi) \to [(\psi \to \chi) \to ((\varphi \vee \psi) \to \chi)]$.
(T15) $\neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$.
(T16) $\neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$.
(T17) $[\varphi \vee (\psi \vee \chi)] \leftrightarrow [(\varphi \vee \psi) \vee \chi]$.
(T18) $[\varphi \wedge (\psi \wedge \chi)] \leftrightarrow [(\varphi \wedge \psi) \wedge \chi]$.
(T19) $[\varphi \wedge (\psi \vee \chi)] \leftrightarrow [(\varphi \wedge \psi) \vee (\varphi \wedge \chi)]$.
(T20) $[\varphi \vee (\psi \wedge \chi)] \leftrightarrow [(\varphi \vee \psi) \wedge (\varphi \vee \chi)]$.
(T21) $(\varphi \to \psi) \leftrightarrow (\neg\varphi \vee \psi)$.
(T22) $\varphi \wedge \psi \leftrightarrow \neg(\neg\varphi \vee \neg\psi)$.
(T23) $\varphi \vee \psi \leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$.

Now we describe a proof system for sentential logic. Formulas of the following form are *sentential axioms*; φ, ψ, χ are arbitrary sentential formulas.

(1) $\varphi \to (\psi \to \varphi)$.

(2) $[\varphi \to (\psi \to \chi)] \to [(\varphi \to \psi) \to (\varphi \to \chi)]$.

(3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$.

Proposition 1.5. *Every sentential axiom is a tautology.*

Proof. For (1):

φ	ψ	$\psi \to \varphi$	$\varphi \to (\psi \to \varphi)$
1	1	1	1
1	0	1	1
0	1	0	1
0	0	1	1

For (2): Let ρ denote this formula:

φ	ψ	χ	$\psi \to \chi$	$\varphi \to (\psi \to \chi)$	$\varphi \to \psi$	$\varphi \to \chi$	$(\varphi \to \psi) \to (\varphi \to \chi)$	ρ
1	1	1	1	1	1	1	1	1
1	1	0	0	0	1	0	0	1
1	0	1	1	1	0	1	1	1
1	0	0	1	1	0	0	1	1
0	1	1	1	1	1	1	1	1
0	1	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1

For (3):

φ	ψ	$\neg\varphi$	$\neg\psi$	$\neg\varphi \to \neg\psi$	$\psi \to \varphi$	(3)
1	1	0	0	1	1	1
1	0	0	1	1	1	1
0	1	1	0	0	0	1
0	0	1	1	1	1	1

$\square$

If Γ is a collection of sentential formulas, then a Γ-*proof* is a finite sequence $\langle \psi_0, \ldots, \psi_m \rangle$ such that for each $i \leq m$ one of the following conditions holds:

(a) ψ_i is a sentential axiom.

(b) $\psi_i \in \Gamma$.

(c) There exist $j, k < i$ such that ψ_k is $\psi_j \to \psi_i$. (Rule of *modus ponens*, abbreviated MP).

We write $\Gamma \vdash \varphi$ if there is a Γ-proof with last entry φ. We also write $\vdash \varphi$ in place of $\emptyset \vdash \varphi$.

Proposition 1.6. *(i) If $\Gamma \vdash \varphi$, f is a sentential assignment, and $\psi[f] = 1$ for all $\psi \in \Gamma$, then $\varphi[f] = 1$.*

(ii) If $\vdash \varphi$, then φ is a tautology.

Proof. For (i), let $\langle \psi_0, \ldots, \psi_m \rangle$ be a Γ-proof. Suppose that f is a sentential assignment and $\chi[f] = 1$ for all $\chi \in \Gamma$. We show by complete induction that $\psi_i[f] = 1$ for all $i \leq m$. Suppose that this is true for all $j < i$.

Case 1. ψ_i is a sentential axiom. Then $\psi_i[f] = 1$ by Proposition 1.5.

Case 2. $\psi_i \in \Gamma$. Then $\psi_i[f] = 1$ by assumption.

Case 3. There exist $j, k < i$ such that ψ_k is $\psi_j \to \psi_i$. By the inductive assumption, $\psi_k[f] = \psi_j[f] = 1$. Hence $\psi_i[f] = 1$.

(ii) clearly follows from (i), □

Now we are going to show that, conversely, if φ is a tautology then $\vdash \varphi$. This is a kind of completeness theorem, and the proof is a highly simplified version of the proof of the completeness theorem for first-order logic which will be given later.

Lemma 1.7. $\vdash \varphi \to \varphi$.

Proof.

(a)	$[\varphi \to [(\varphi \to \varphi) \to \varphi]] \to [[\varphi \to (\varphi \to \varphi)] \to (\varphi \to \varphi)]$	(2)
(b)	$\varphi \to [(\varphi \to \varphi) \to \varphi]$	(1)
(c)	$[\varphi \to (\varphi \to \varphi)] \to (\varphi \to \varphi)$	(a), (b), MP
(d)	$\varphi \to (\varphi \to \varphi)$	(1)
(e)	$\varphi \to \varphi$	(c), (d), MP

□

Theorem 1.8. (The deduction theorem) *If* $\Gamma \cup \{\varphi\} \vdash \psi$, *then* $\Gamma \vdash \varphi \to \psi$.

Proof. Let $\langle \chi_0, \ldots, \chi_m \rangle$ be a $(\Gamma \cup \{\varphi\})$-proof with last entry ψ. We replace each χ_i by several formulas so that the result is a Γ-proof with last entry $\varphi \to \psi$.

If χ_i is a logical axiom or a member of Γ, we replace it by the two formulas $\chi_i \to (\varphi \to \chi_i)$, $\varphi \to \chi_i$.

If χ_i is φ, we replace it by the five formulas in the proof of Lemma 1.7; the last entry is $\varphi \to \varphi$.

If χ_i is obtained from χ_j and χ_k by modus ponens, so that χ_k is $\chi_j \to \chi_i$, we replace χ_i by the formulas

$$[\varphi \to (\chi_j \to \chi_i)] \to [(\varphi \to \chi_j) \to (\varphi \to \chi_i)]$$
$$(\varphi \to \chi_j) \to (\varphi \to \chi_i)$$
$$\varphi \to \chi_i$$

Clearly this is as desired. □

Lemma 1.9. $\vdash \psi \to (\neg\psi \to \varphi)$.

Proof. By axiom (1) we have $\{\psi, \neg\psi\} \vdash \neg\varphi \to \neg\psi$. Hence axiom (3) gives $\{\psi, \neg\psi\} \vdash \psi \to \varphi$, and hence $\{\psi, \neg\psi\} \vdash \varphi$. Now two applications of Theorem 1.8 give the desired result. □

Lemma 1.10. $\vdash (\varphi \to \psi) \to [(\psi \to \chi) \to (\varphi \to \chi)]$.

Proof. Clearly $\{\varphi \to \psi, \psi \to \chi, \varphi\} \vdash \chi$, so three applications of Theorem 1.8 give the desired result. □

Lemma 1.11. $\vdash (\neg\varphi \to \varphi) \to \varphi$.

Proof. Clearly $\{\neg\varphi \to \varphi, \neg\varphi\} \vdash \varphi$ and also $\{\neg\varphi \to \varphi, \neg\varphi\} \vdash \neg\varphi$, so by Lemma 1.9, $\{(\neg\varphi \to \varphi, \neg\varphi\} \vdash \neg(\varphi \to \varphi)$. Then Theorem 1.8 gives $\{\neg\varphi \to \varphi\} \vdash \neg\varphi \to \neg(\varphi \to \varphi)$, and so using axiom (3), $\{\neg\varphi \to \varphi\} \vdash (\varphi \to \varphi) \to \varphi$. Hence by Lemma 1.7, $\{\neg\varphi \to \varphi\} \vdash \varphi$, and so Theorem 1.8 gives the desired result. □

Lemma 1.12. $\vdash (\varphi \to \psi) \to [(\neg\varphi \to \psi) \to \psi]$.

Proof.

$$\begin{aligned}
&\{\varphi \to \psi, \neg\varphi \to \psi, \neg\psi\} \vdash \neg\varphi \to \neg\psi \quad \text{using axiom (1)}\\
&\{\varphi \to \psi, \neg\varphi \to \psi, \neg\psi\} \vdash \psi \to \varphi \quad \text{using axiom (3)}\\
&\{\varphi \to \psi, \neg\varphi \to \psi, \neg\psi\} \vdash \neg\varphi \to \varphi \quad \text{using Lemma 1.10}\\
&\{\varphi \to \psi, \neg\varphi \to \psi, \neg\psi\} \vdash \varphi \quad \text{by Lemma 1.11}\\
&\{\varphi \to \psi, \neg\varphi \to \psi, \neg\psi\} \vdash \psi\\
&\qquad\{\varphi \to \psi, \neg\varphi \to \psi\} \vdash \neg\psi \to \psi \quad \text{by Theorem 1.8}\\
&\qquad\{\varphi \to \psi, \neg\varphi \to \psi\} \vdash \psi \quad \text{by Lemma 1.11}
\end{aligned}$$

Now two applications of Theorem 1.8 give the desired result. □

Theorem 1.13. *There is a sequence $\langle\varphi_0, \varphi_1, \ldots\rangle$ containing all sentential formulas.*

Proof. One can obtain such a sequence by the following procedure.

(1) Start with S_0.

(2) List all sentential formulas of length at most two which involve only S_0 or S_1; they are S_0, S_1, $\neg S_0$, and $\neg S_1$.

(3) List all sentential formulas of length at most three which involve only S_0, S_1, or S_2; they are S_0, S_1, S_2, $\neg S_0$, $\neg S_1$, $\neg S_2$, $\neg\neg S_0$, $\neg\neg S_1$, $\neg\neg S_2$, $S_0 \to S_0$, $S_0 \to S_1$, $S_0 \to S_2$, $S_1 \to S_0$, $S_1 \to S_1$, $S_1 \to S_2$, $S_2 \to S_0$, $S_2 \to S_1$, $S_2 \to S_2$.

(4) Etc. □

Theorem 1.14. *If not($\Gamma \vdash \varphi$), then there is a sentential assignment f such that $\psi[f] = 1$ for all $\psi \in \Gamma$, while $\varphi[f] = 0$.*

Proof. Let $\langle\chi_0, \chi_1, \ldots\rangle$ list all the sentential formulas. We now define $\Delta_0, \Delta_1, \ldots$ by recursion. Let $\Delta_0 = \Gamma$. Suppose that Δ_i has been defined. If not($\Delta_i \cup \{\chi_i\} \vdash \varphi$) then we set $\Delta_{i+1} = \Delta_i \cup \{\chi_i\}$. Otherwise we set $\Delta_{i+1} = \Delta_i$.

Here is a detailed proof that Δ exists. Let $M = \{\Omega : \Omega$ is a function with domain m' for some positive integer m, $\Omega_1 = \Gamma$, and for every positive integer i with $i + 1 \leq m$ we have

$$\Omega_{i+1} = \begin{cases} \Omega_i \cup \{\chi_i\} & \text{if not}(\Omega_i \cup \{\chi_i\} \vdash \varphi), \\ \Omega_i & \text{otherwise.}\} \end{cases}$$

(1) If $\Omega, \Omega' \in M$ with domains m', n' respectively, with $m \leq n$, then $\forall i \leq m[\Omega_i = \Omega'_i]$.

This is easily proved by induction on i.

(2) For every positive integer m there is a $\Omega \in M$ with domain m'.

Again this is easily proved by induction on m.

Now we define $\Delta_i = \Omega_i$, where $\Omega \in M$ and $i < \mathrm{dmn}(\Omega)$. This is justified by (1) and (2).

Now it is easily verified that the defining conditions for Δ hold.

Let $\Theta = \bigcup_{i\in\omega} \Delta_i$. By induction we have not$(\Delta_i \vdash \varphi)$ for each $i \in \omega$. In fact, we have $\Delta_0 = \Gamma$, so not$(\Delta_0 \vdash \varphi)$ by assumption. If not$(\Delta_i \vdash \varphi)$, then not$(\Delta_{i+1} \vdash \varphi)$ by construction.

Hence also not$(\Theta \vdash \varphi)$, since $\Theta \vdash \varphi$ means that there is a Θ-proof with last entry φ, and any Θ-proof involves only finitely many formulas χ_i, and they all appear in some Δ_j, giving $\Delta_j \vdash \varphi$, contradiction.

($*$) For any formula χ_i, either $\chi_i \in \Theta$ or $\neg\chi_i \in \Theta$.

In fact, suppose that $\chi_i \notin \Theta$ and $\neg\chi_i \notin \Theta$. Say $\neg\chi_i = \chi_j$. Then by construction, $\Delta_i \cup \{\chi_i\} \vdash \varphi$ and $\Delta_j \cup \{\neg\chi_i\} \vdash \varphi$. So $\Theta \cup \{\chi_i\} \vdash \varphi$ and $\Theta \cup \{\neg\chi_i\} \vdash \varphi$. Hence by Theorem 1.8, $\Theta \vdash \chi_i \to \varphi$ and $\Theta \vdash \neg\chi_i \to \varphi$. So by Lemma 1.12 we get $\Theta \vdash \varphi$, contradiction.

($**$) If $\Theta \vdash \psi$, then $\psi \in \Theta$.

In fact, clearly not$(\Theta \cup \{\psi\} \vdash \varphi)$ by Theorem 1.8, so ($**$) follows.

Now let f be the sentential assignment such that $f(i) = 1$ iff $S_i \in \Theta$. Now we claim

($***$) For every sentential formula ψ, $\psi[f] = 1$ iff $\psi \in \Theta$.

We prove this by induction on ψ. It is true for $\psi = S_i$ by definition. Now suppose that it holds for ψ. Suppose that $(\neg\psi)[f] = 1$. Thus $\psi[f] = 0$, so by the inductive assumption, $\psi \notin \Theta$, and hence by ($*$), $\neg\psi \in \Theta$. Conversely, suppose that $\neg\psi \in \Theta$. If $(\neg\psi)[f] = 0$, then $\psi[f] = 1$, hence $\psi \in \Theta$ by the inductive hypothesis. Hence by Lemma 1.9, $\Theta \vdash \varphi$, contradiction. So $(\neg\psi)[f] = 1$.

Next suppose that ($***$) holds for ψ and χ; we show that it holds for $\psi \to \chi$. Suppose that $(\psi \to \chi)[f] = 1$. If $\chi[f] = 1$, then $\chi \in \Theta$ by the inductive hypothesis. By axiom (1), $\Theta \vdash \psi \to \chi$. Hence by ($**$), $(\psi \to \chi) \in \Theta$. Suppose that $\chi[f] = 0$. Then $\psi[f] = 0$ also, since $(\psi \to \chi)[f] = 1$. By the inductive hypothesis and ($*$) we have $\neg\psi \in \Theta$. Hence $\Theta \vdash \neg\chi \to \neg\psi$ by axiom (1), so $\Theta \vdash \psi \to \chi$ by axiom (3). So $(\psi \to \chi) \in \Theta$ by ($**$).

Conversely, suppose that $(\psi \to \chi) \in \Theta$. Working for a contradiction, suppose that $(\psi \to \chi)[f] = 0$. Thus $\psi[f] = 1$ and $\chi[f] = 0$. So $\psi \in \Theta$ and $\neg\chi \in \Theta$ by the inductive hypothesis and ($*$). Since $(\psi \to \chi) \in \Theta$ and $\psi \in \Theta$, we get $\Theta \vdash \chi$. Since also $\neg\chi \in \Theta$, we get $\Theta \vdash \varphi$ by Lemma 1.9, contradiction.

This finishes the proof of ($***$).

Since $\Gamma \subseteq \Theta$, ($***$) implies that $\psi[f] = 1$ for all $\psi \in \Gamma$. Also $\varphi[f] = 0$ since $\varphi \notin \Theta$. □

Corollary 1.15. *If $\varphi[f] = 1$ whenever $\psi[f] = 1$ for all $\psi \in \Gamma$, then $\Gamma \vdash \varphi$.*

Proof. This is the contrapositive of Theorem 1.14. □

Theorem 1.16. *$\vdash \varphi$ iff φ is a tautology.*

Proof. $\Rightarrow$ is given by Proposition 1.6(ii). $\Leftarrow$ follows from Corollary 1.15 by taking $\Gamma = \emptyset$. □

EXERCISES

E1.1. Verify that

$$S_0 \to \neg S_1 = \langle 2, 3, 1, 4 \rangle$$

and

$$(S_0 \to S_1) \to (\neg S_1 \to \neg S_0) = \langle 2, 2, 3, 4, 2, 1, 4, 1, 3 \rangle.$$

E1.2. Prove that there is a sentential formula of each positive integer length.

E1.3. Prove that m is the length of a sentential formula not involving $\neg$ iff m is odd.

E1.4. Prove that a truth table for a sentential formula involving n basic formulas has 2^n rows.

E1.5. Use the truth table method to show that the formula

$$(\varphi \to \psi) \leftrightarrow (\neg \varphi \vee \psi)$$

is a tautology.

E1.6. Use the truth table method to show that the formula

$$[\varphi \vee (\psi \wedge \chi)] \leftrightarrow [(\varphi \vee \psi) \wedge (\varphi \vee \chi)]$$

is a tautology.

E1.7. Use the truth table method to show that the formula

$$(\varphi \to \psi) \to (\varphi \to \neg \psi)$$

is not a tautology. It is not necessary to work out the full truth table.

E1.8. Determine whether or not the following is a tautology:

$$S_0 \to (S_1 \to (S_2 \to (S_3 \to S_1))).$$

E1.9. Determine whether or not the following is a tautology; an informal method is better than a truth table:

$$(\{[(\varphi \to \psi) \to (\neg \chi \to \neg \theta)] \to \chi\} \to \tau) \to [(\tau \to \varphi) \to (\theta \to \varphi)].$$

E1.10. Determine whether the following statements are logically consistent. If the contract is valid, then Horatio is liable. If Horatio is liable, he will go bankrupt. Either Horatio will go bankrupt or the bank will lend him money. However, the bank will definitely not lend him money.

E1.11. Write out an actual proof for $\{\psi\} \vdash \neg \psi \to \varphi$. This can be done by following the proof of Lemma 1.9, expanding it using the proof of the deduction theorem.

2. First-order logic

Although set theory can be considered within a single first-order language, with only non-logical constant $\in$, it is convenient to have more complicated languages, corresponding to the many definitions introduced in mathematics.

All first-order languages have the following *symbols* in common. Again, as for sentential logic, we take these to be certain natural numbers.

1 (negation)
2 (implication)
3 (the equality symbol)
4 (the universal quantifier)
$5m$ for each positive integer m (variables ranging over elements, but not subsets, of a given structure) We denote $5m$ by v_{m-1}. Thus v_0 is 5, v_1 is 10, and in general v_i is $5(i+1)$.

Special first-order languages have additional symbols for the functions and relations and special elements involved. These will always be taken to be some positive integers not among the above; thus they are positive integers greater than 4 but not divisible by 5. So we have in addition to the above logical symbols some non-logical symbols:

Relation symbols, each of a certain positive rank.
Function symbols, also each having a specified positive rank.
Individual constants.

Formally, a first-order language is a quadruple (Rel, Fcn, Cn, rnk) such that Rel, Fcn, Cn are pairwise disjoint subsets of M (the sets of relation symbols, function symbols, and individual constants), and rnk is a function mapping $Rel \cup Fcn$ into the positive integers; $rnk(\mathbf{S})$ gives the *rank* of the symbol $\mathbf{S}$.

Now we will define the notions of terms and formulas, which give a precise formulation of meaningful expressions. Terms are certain finite sequences of symbols. A *term construction sequence* is a sequence $\langle \tau_0, \ldots, \tau_{m-1} \rangle$, $m > 0$, with the following properties: for each $i < m$ one of the following holds:

τ_i is $\langle v_j \rangle$ for some natural number j.

τ_i is $\langle \mathbf{c} \rangle$ for some individual constant $\mathbf{c}$.

τ_i is $\langle \mathbf{F} \rangle^\frown \sigma_0^\frown \sigma_1^\frown \cdots ^\frown \sigma_{n-1}$ for some n-place function symbol $\mathbf{F}$, with each σ_j equal to τ_k for some $k < j$, depending upon j.

A *term* is a sequence appearing in some term construction sequence. Note the similarity of this definition with that of sentential formula given in Chapter 1.

Frequently we will slightly simplify the notation for terms. Thus we might write simply v_j, or $\mathbf{c}$, or $\mathbf{F}\sigma_0 \ldots \sigma_{n-1}$ for the above.

The following two propositons are very similar, in statement and proof, to Propositions 1.1 and 1.2. The first one is the principle of *induction on terms*.

Proposition 2.1. *Let T be a collection of terms satisfying the following conditions:*

(i) Each variable is in T.

(ii) Each individual constant is in T.

(iii) If $\mathbf{F}$ is a function symbol of rank m and $\tau_0, \ldots, \tau_{m-1} \in T$, then also $\mathbf{F}\tau_0 \ldots \tau_{m-1} \in T$.

Then T consists of all terms.

Proof. Let τ be a term. Say that $\langle \sigma_0, \ldots, \sigma_{m-1} \rangle$ is a term construction sequence and $\sigma_i = \tau$. We prove by complete induction on j that $\sigma_j \in T$ for all $j < m$; hence $\tau \in T$. Suppose that $j < m$ and $\sigma_k \in T$ for all $k < j$. If $\sigma_j = \langle v_s \rangle$ for some s, then $\sigma_j \in T$. If $\sigma_j = \langle \mathbf{c} \rangle$ for some individual constant $\mathbf{c}$, then $s_j \in T$. Finally, suppose that σ_j is $\mathbf{F}\sigma_{k_0} \ldots \sigma_{k_{n-1}}$ with each $k_t < j$. Then $\sigma_{k_t} \in T$ for each $t < n$ by the inductive hypothesis, and it follows that $\sigma_j \in T$. This completes the inductive proof. □

Proposition 2.2. *(i) Every term is a nonempty sequence.*

(ii) If τ is a term, then exactly one of the following conditions holds:

(a) τ is an individual constant.

(b) τ is a variable.

(c) There exist a function symbol $\mathbf{F}$, say of rank m, and terms $\sigma_0, \ldots, \sigma_{m-1}$ such that τ is $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$.

(iii) No proper initial segment of a term is a term.

(iv) If $\mathbf{F}$ and $\mathbf{G}$ are function symbols, say of ranks m and n respectively, and if $\sigma_0, \ldots, \sigma_{m-1}, \tau_0, \ldots, \tau_{n-1}$ are terms, and if $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$ is equal to $\mathbf{G}\tau_0, \ldots \tau_{n-1}$, then $\mathbf{F} = \mathbf{G}$, $m = n$, and $\sigma_i = \tau_i$ for all $i < m$.

Proof. (i): This is clear since any entry in a term construction sequence is nonempty.

(ii): Also clear.

(iii): We prove this by complete induction on the length of a term. So suppose that τ is a term, and for any term σ shorter than τ, no proper initial segment of σ is a term. We consider cases according to (ii).

Case 1. τ is an individual constant. Then τ has length 1, and any proper initial segment of τ is empty; by (i) the empty sequence is not a term.

Case 2. τ is a variable. Similarly.

Case 3. There exist an m-ary function symbol $\mathbf{F}$ and terms $\sigma_0, \ldots, \sigma_{m-1}$ such that τ is $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$. Suppose that ρ is a term which is a proper initial segment of τ. By (i), ρ is nonempty, and the first entry of ρ is $\mathbf{F}$. By (ii), ρ has the form $\mathbf{F}\xi_0 \ldots \xi_{m-1}$ for certain terms $\xi_0, \ldots, \xi_{m-1}$. Since both σ_0 and ξ_0 are shorter terms than τ, and one of them is an initial segment of the other, the induction hypothesis gives $\sigma_0 = \xi_0$. Let $i < m$ be maximum such that $\sigma_i = \xi_i$. Since ρ is a proper initial segment of τ, we must have $i < m - 1$. But σ_{i+1} and ξ_{i+1} are shorter terms than τ and one is a segment of the other, so by the inductive hypthesis $\sigma_{i+1} = \xi_{i+1}$, contradicting the choice of i.

(iv): $\mathbf{F}$ is the first entry of $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$ and $\mathbf{G}$ is the first entry of $\mathbf{G}\tau_0, \ldots \tau_{n-1}$, so $\mathbf{F} = \mathbf{G}$. Then by (ii) we get $m = n$. By induction using (iii), each $\sigma_i = \tau_i$. □

We now give the general notion of a structure. This will be modified and extended for set theory later. For a given first-order language $\mathscr{L} = (Rel, Fcn, Cn, rnk)$, an *$\mathscr{L}$-structure* is a quadruple $\overline{A} = (A, Rel', Fcn', Cn')$ such that A is a nonempty set (the *universe* of the structure), Rel' is a function assigning to each relation symbol $\mathbf{R}$ a $rnk(\mathbf{R})$-ary relation on A, i.e., a collection of $rnk(\mathbf{R})$-tuples of elements of A, Fcn' is a function assigning to each function symbol $\mathbf{F}$ a $rnk(\mathbf{F})$-ary opeation on A, i.e., a function assigning a value in A to each $rnk(\mathbf{F})$-tuple of elements of A, and Cn' is a function assigning to each individual constant $\mathbf{c}$ an element of A. Usually instead of $Rel'(\mathbf{R})$, $Fcn'(\mathbf{F})$ and $Cn'(\mathbf{c})$ we write $\mathbf{R}^{\overline{A}}$, $\mathbf{F}^{\overline{A}}$, and $\mathbf{c}^{\overline{A}}$.

Now we define the "meaning" of terms. This is a recursive definition, similar to the definition of the values of sentential formulas under assignments:

Proposition 2.3. *Let $\overline{A}$ be a structure, and a a function mapping ω into A. (A is the universe of $\overline{A}$.) Then there is a function F mapping the set of terms into A with the following properties:*

(i) $F(v_i) = a_i$ for each $i \in \omega$.

(ii) $F(\mathbf{c}) = \mathbf{c}^{\overline{A}}$ for each individual constant $\mathbf{c}$.

(iii) $F(\mathbf{F}\sigma_0 \ldots \sigma_{m-1}) = \mathbf{F}^{\overline{A}}(F(\sigma_0), \ldots, F(\sigma_{m-1}))$ for every m-ary function symbol $\mathbf{F}$ and all terms $\sigma_0, \ldots, \sigma_{m-1}$

Proof. An $(\overline{A}, a)$-*term sequence* is a sequence $\langle(\tau_0, b_0), \ldots, (\tau_{m-1}, b_{m-1})\rangle$ such that each $b_i \in A$ and for each $i < m$ one of the following conditions holds:

(1) τ_i is $\langle v_j\rangle$ and $b_i = a_j$.

(2) τ_i is $\langle\mathbf{c}\rangle$ for some individual constant $\mathbf{c}$, and $b_i = \mathbf{c}^{\overline{A}}$.

(3) $\tau_i = \langle\mathbf{F}\rangle^\frown\tau_{k(0)}^\frown\cdots^\frown\tau_{k(n-1)}$ and $b_i = \mathbf{F}^{\overline{A}}(b_{k(0)}, \ldots, b_{k(n-1)})$ for some n-ary function symbol $\mathbf{F}$ and some $k(0), \ldots, k(n-1) < i$.

Now we claim

(4) For any term σ and any $(\overline{A}, a)$-term sequences

$$\langle(\tau_0, b_0), \ldots, (\tau_{m-1}, b_{m-1})\rangle \quad \text{and} \quad \langle(\tau'_0, b'_0), \ldots, (\tau'_{n-1}, b'_{n-1})\rangle$$

such that $\tau_{m-1} = \tau'_{n-1} = \sigma$ we have $b_{m-1} = b'_{n-1}$.

We prove (4) by induction on σ, thus using Proposition 2.1. If $\sigma = v_i$, then $b_{m-1} = a_i = b_{n-1}$. If σ is an individual constant $\mathbf{c}$, then $b_{m-1} = \mathbf{c}^{\overline{A}} = b'_{n-1}$. Finally, if $\sigma = \langle\mathbf{F}\rangle^\frown\rho_0\cdots^\frown\rho_{p-1}$, then we have:

$$\tau_{m-1} = \langle\mathbf{F}\rangle^\frown\tau_{k(0)}^\frown\cdots^\frown\tau_{k(p-1)} \quad \text{and} \quad b_{m-1} = \mathbf{F}^{\overline{A}}(b_{k(0)}, \ldots, b_{k(p-1)});$$
$$\tau'_{n-1} = \langle\mathbf{F}\rangle^\frown\tau'^\frown_{l(0)}\cdots^\frown\tau'_{l(p-1)} \quad \text{and} \quad b'_{m-1} = \mathbf{F}^{\overline{A}}(b'_{l(0)}, \ldots, b'_{l(p-1)})$$

with each $k(s)$ and $l(t)$ less than i. By Proposition 2.2(iv) we have $\tau_{k(s)} = \tau'_{l(s)}$ for every $s < p$. Now for every $s < p$ we can apply the inductive hypothesis to $\tau_{k(s)}$ and the sequences

$$\langle(\tau_0, b_0), \ldots, (\tau_{k(s)}, b_{k(s)}) \quad \text{and} \quad \langle(\tau'_0, b'_0), \ldots, (\tau'_{l(s)}, b'_{l(s)})$$

to obtain $b_{k(s)} = b'_{l(s)}$. Hence

$$b_{m-1} = \mathbf{F}^{\overline{A}}(b_{k(0)}, \ldots, b_{k(p-1)}) = \mathbf{F}^{\overline{A}}(b'_{l(0)}, \ldots, b'_{l(p-1)}) = b'_{n-1},$$

completing the inductive proof of (4).

(5) If $\langle \tau_0, \ldots, \tau_{m-1} \rangle$ is a term construction sequence, then there is an $(\overline{A}, a)$-term sequence of the form $\langle (\tau_0, b_0), \ldots, (\tau_{m-1}, b_{m-1} \rangle$.

We prove this by induction on m. For $m = 1$ we have two possibilities.

Case 1. τ_0 is v_j for some $j \in \omega$. Then $\langle (\tau_0, b_j) \rangle$ is as desired.

Case 2. τ_0 is $\mathbf{c}$, an individual constant. Then $\langle (\tau_0, \mathbf{c}^{\overline{A}}) \rangle$ is as desired.

Now assume the statement for $m - 1 \geq 1$. By the induction hypothesis there is an $(\overline{A}, a)$-term sequence of the form $\sigma \stackrel{\text{def}}{=} \langle (\tau_0, b_0), \ldots, (\tau_{m-2}, b_{m-2}) \rangle$. Then we have three possibilities:

Case 1. τ_{m-1} is v_j for some $j \in \omega$. Then $\sigma ^\frown \langle (\tau_{m-1}, b_j) \rangle$ is as desired.

Case 2. τ_{m-1} is $\mathbf{c}$, an individual constant. Then $\sigma ^\frown \langle (\tau_{m-1}, \mathbf{c}^{\overline{A}}) \rangle$ is as desired.

Case 3. τ_{m-1} is $\langle \mathbf{F} \rangle ^\frown \tau_{k(0)} ^\frown \cdots ^\frown \tau_{k(p-1)}$ for some p-ary function symbol $\mathbf{F}$ with each $k(s) < i$. Then $\sigma ^\frown \langle (\tau_{m-1}, \mathbf{F}^{\overline{A}}(b_{k(0)}, \ldots, b_{k(p-1)})) \rangle$ is as desired.

So (5) holds.

Now we can define F as needed in the Proposition. Let σ be a term. Let $\langle \tau_0, \ldots, \tau_{m-1} \rangle$ be a term construction sequence with $\tau_{m-1} = \sigma$. By (5), let $\langle (\tau_0, b_0), \ldots, (\tau_{m-1} b_{m-1}) \rangle$ be an $(\overline{A}, a)$-term sequence. Then we define $F(\sigma) = b_{m-1}$. This definition is unambiguous by (4). Now we check the conditions of the Proposition. Let σ be a term, and let $\langle (\tau_0, b_0), \ldots, (\tau_{m-1}, b_{m-1}) \rangle$ be an $(\overline{A}, a)$-term sequence with $\tau_{m-1} = \sigma$.

Case 1. $\sigma = v_j$ for some $j \in \omega$. Then $F(\sigma) = b_{m-1} = a_j$.

Case 2. $\sigma = \mathbf{c}$ for some individual constant $\mathbf{c}$. Then $F(\sigma) = b_{m-1} = \mathbf{c}^{\overline{A}}$.

Case 3. $\sigma = \langle \mathbf{F} \rangle ^\frown \rho_0 ^\frown \cdots ^\frown \rho_{p-1}$ with $\mathbf{F}$ a p-ary function symbol and each ρ_s a term. Then there exist $c(0), \ldots, c(p-1) < m - 1$ such that $\rho_s = \tau_{c(s)}$ for every $s < p$. Then $F(\tau_{c(s)}) = b_{c(s)} = \tau_{c(s)}^{\overline{A}}$ for each $s < p$, and hence

$$F(\sigma) = b_{m-1} = \mathbf{F}^{\overline{A}}(b_{s(0)}, \ldots, b_{s(p-1)}) = \mathbf{F}^{\overline{A}}(\tau_{s(0)}^{\overline{A}}, \ldots, \tau_{s(p-1)}^{\overline{A}}) = \mathbf{F}^{\overline{A}}(\rho_0^{\overline{A}}, \ldots, \rho_{p-1}^{\overline{A}}). \quad \square$$

With F as in Proposition 2.3, we denote $F(\sigma)$ by $\sigma^{\overline{A}}(a)$. Thus

$$\begin{aligned} v_i^{\overline{A}}(a) &= a_i; \\ \mathbf{c}^{\overline{A}}(a) &= \mathbf{c}^{\overline{A}}; \\ (\mathbf{F}\tau_0 \ldots \tau_{m-1})^{\overline{A}}(a) &= \mathbf{F}^{\overline{A}}(\tau_0^{\overline{A}}(a), \ldots, \tau_{m-1}^{\overline{A}}(a)). \end{aligned}$$

Here v_i is any variable, $\mathbf{c}$ any individual constant, and $\mathbf{F}$ any function symbol (of some rank, say m).

What $\sigma^{\overline{A}}(a)$ means intuitively is: replace the individual constants and function symbols by the actual members of A and functions on A given by the structure $\overline{A}$, and replace the variables v_i by coresponding elements a_i of A; calculate the result, giving an element of A.

Proposition 2.4. *Suppose that τ is a term, $\overline{A}$ is a structure, a, b assignments, and $a(i) = b(i)$ for all i such that v_i occurs in τ. Then $\tau^{\overline{A}}(a) = \tau^{\overline{A}}(b)$.*

Proof. By induction on τ:

$$\begin{aligned}
\mathbf{c}^{\overline{A}}(a) &= \mathbf{c}^{\overline{A}} = \mathbf{c}^{\overline{A}}(b);\\
v_i^{\overline{A}}(a) &= a(i) = b(i) = v_i^{\overline{A}}(b);\\
(\mathbf{F}\sigma_0 \ldots \sigma_{m-1})^{\overline{A}}(a) &= \mathbf{F}^{\overline{A}}(\sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a))\\
&= \mathbf{F}^{\overline{A}}(\sigma_0^{\overline{A}}(b), \ldots, \sigma_{m-1}^{\overline{A}}(b))\\
&= (\mathbf{F}\sigma_0 \ldots \sigma_{m-1})^{\overline{A}}(b).
\end{aligned}$$

The last step here is the induction step (many of them, one for each function symbol and associated terms). The inductive assumption is that $a(i) = b(i)$ for all i for which v_i occurs in $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$; hence also for each $j < m$, $a(i) = b(i)$ for all i for which v_i occurs in σ_j, so that the inductive hypothesis can be applied. $\square$

This proposition enables us to simplify our notation a little bit. If n is such that each variable occurring in τ has index less than n, then in the notation $\varphi^{\overline{A}}(a)$ we can just use the first n entries of a rather than the entire infinite sequence.

We turn to the definition of formulas. For any terms σ, τ we define $\sigma = \tau$ to be the sequence $\langle 3\rangle^\frown\sigma^\frown\tau$. Such a sequence is called an *atomic equality formula.* An *atomic non-equality formula* is a sequence of the form $\langle \mathbf{R}\rangle^\frown\sigma_0^\frown \cdots ^\frown \sigma_{m-1}$ where $\mathbf{R}$ is an m-ary relation symbol and $\sigma_0, \ldots \sigma_{m-1}$ are terms. An *atomic formula* is either an atomic equality formula or an atomic non-equality formula.

We define $\neg$, a function assigning to each sequence φ of symbols of a first-order language the sequence $\neg\varphi \stackrel{\text{def}}{=} \langle 1\rangle^\frown\varphi$. $\rightarrow$ is the function assigning to each pair (φ, ψ) of sequences of symbols the sequence $\varphi \rightarrow \psi \stackrel{\text{def}}{=} \langle 2\rangle^\frown\varphi^\frown\psi$. $\forall$ is the function assigning to each pair (i, φ) with $i \in \omega$ and φ a sequence of symbols the sequence $\forall v_i\varphi \stackrel{\text{def}}{=} \langle 4, 5i+5\rangle^\frown\varphi$.

A *formula construction sequence* is a sequence $\langle \varphi_0, \ldots, \varphi_{m-1}\rangle$ such that for each $i < m$ one of the following holds:

(1) φ_i is an atomic formula.

(2) There is a $j < i$ such that φ_i is $\neg\varphi_j$

(3) There are $j, k < i$ such that φ_i is $\varphi_j \rightarrow \varphi_k$.

(4) There exist $j < i$ and $k \in \omega$ such that φ_i is $\forall v_k\varphi_j$.

A *formula* is an expression which appears as an entry in some formula construction sequence.

The following is the principle of *induction on formulas.*

Proposition 2.5. *Suppose that Γ is a set of formulas satisfying the following conditions:*

(i) Every atomic formula is in Γ.

(ii) If $\varphi \in \Gamma$, then $\neg\varphi \in \Gamma$.

(iii) If $\varphi, \psi \in \Gamma$, then $(\varphi \to \psi) \in \Gamma$.

(iv) If $\varphi \in \Gamma$ and $i \in \omega$, then $\forall v_i \varphi \in \Gamma$.

Then Γ is the set of all formulas.

Proof. It suffices to take any formula construction sequence $\langle \varphi_0, \ldots, \varphi_{m-1} \rangle$ and show by complete induction on i that $\varphi_i \in \Gamma$ for all $i \in \omega$. We leave this as an exercise. $\square$

Proposition 2.6. *(i) Every formula is a nonempty sequence.*

(ii) If φ is a formula, then exactly one of the following conditions holds:

(a) φ is an atomic equality formula, and there are terms σ, τ such that φ is $\sigma = \tau$.

(b) φ is an atomic non-equality formula, and there exist a positive integer m, a relation symbol $\mathbf{R}$ of rank m, and terms $\sigma_0, \ldots, \sigma_{m-1}$, such that φ is $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$.

(c) There is a formula ψ such that φ is $\neg\psi$.

(d) There are formulas ψ, χ such that φ is $\psi \to \chi$.

(e) There exist a formula ψ and a natural number i such that φ is $\forall v_i \psi$.

(iii) No proper initial segment of a formula is a formula.

(iv) (a) If φ is an atomic equality formula, then there are unique terms σ, τ such that φ is $\sigma = \tau$.

(b) If φ is an atomic non-equality formula, then there exist a unique positive integer m, a unique relation symbol $\mathbf{R}$ of rank m, and unique terms $\sigma_0, \ldots, \sigma_{m-1}$, such that φ is $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$.

(c) If φ is a formula and the first symbol of φ is 1, then there is a unique formula ψ such that φ is $\neg\psi$.

(d) If φ is a formula and the first symbol of φ is 2, then there are unique formulas ψ, χ such that φ is $\psi \to \chi$.

(e) If φ is a formula and the first symbol of φ is 4, then there exist a unique natural number i and a unique formula ψ such that φ is $\forall v_i \psi$.

Proof. (i): First note that this is true of atomic formulas, since an atomic formula must have at least a first symbol 3 or some relation symbol. Knowing this about atomic formulas, any entry in a formula construction sequence is nonempty, since the entry is either an atomic formula or else begins with 1,2, or 4.

(ii): This is true on looking at any entry in a formula construction sequence: either the entry begins with 3 or a relation symbol and hence (a) or (b) holds, or it begins with 1, 2, or 4, giving (c), (d) or (e). Only one of (a)–(e) holds because of the first symbol in the entry.

(iii): We prove this by complete induction on the length of the formula. Thus suppose that φ is a formula of length m, and for any formula ψ of length less than m, no proper initial segment of ψ is a formula. Suppose that χ is a proper initial segment of φ and χ is a formula; we want to get a contradiction. By (ii) we have several cases.

Case 1. φ is an atomic equality formula $\sigma = \tau$ for certain terms σ, τ. Thus φ is $\langle 3\rangle^\frown\sigma^\frown\tau$. Since χ is a formula which begins with 3 (since χ is an initial segment of φ and is nonempty by (i)), (ii) yields that χ is $\langle 3\rangle^\frown\rho^\frown\xi$ for some terms ρ, ξ. Hence $\sigma^\frown\psi = \rho^\frown\xi$. Thus σ is an initial segment of ρ or ρ is an initial segment of σ. By Proposition 2.2(iii) it follows that $\sigma = \rho$. Then also $\tau = \xi$, so $\varphi = \chi$, contradiction.

Case 2. φ is an atomic non-equality formula $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$ for some m-ary relation symbol $\mathbf{R}$ and some terms $\sigma_0, \ldots, \sigma_{m-1}$. Then χ is a formula which begins with $\mathbf{R}$, and so there exist terms $\tau_0, \ldots, \tau_{m-1}$ such that χ is $\mathbf{R}\tau_0 \ldots \tau_{m-1}$. By induction using Proposition 2.2(iii), $\sigma_i = \tau_i$ for all $i < m$, so $\varphi = \chi$, contradiction.

Case 3. φ is $\neg\psi$ for some formula ψ. Then 1 is the first entry of χ, so by (ii) χ has the form $\neg\rho$ for some formula ρ. Thus ρ is a proper initial segment of ψ, contradicting the inductive hypothesis, since ψ is shorter than φ.

Case 4. φ is $\psi \to \theta$ for some formulas ψ, θ, i.e., it is $\langle 2\rangle^\frown\psi^\frown\theta$. Then χ starts with 2, so by (ii) χ has the form $\langle 2\rangle^\frown\sigma^\frown\tau$ for some formulas σ, τ. Now both ψ and σ are shorter than φ, and one is an initial segment of the other. So $\psi = \sigma$ by the inductive assumption. Then τ is a proper initial segment of θ, contradicting the inductive assumption.

Case 5. φ is $\langle 4, 5(i+1)\rangle^\frown\psi$ for some $i \in \omega$ and some formula ψ. Then by (ii), χ is $\langle 4, 5(i+1)\rangle^\frown\theta$ for some formula θ. So θ is a proper initial segment of ψ, contradiction.

(iv): These conditions follow from Proposition 2.2(iii) and (iii). □

Now we come to a fundamental definition connecting language with structures. Again this is a definition by recursion; it is given in the following proposition. First a bit of notation. If $a : \omega \to A$, $i \in \omega$, and $s \in A$, then by a^i_s we mean the sequence which is just like a except that $a^i_s(i) = s$.

Proposition 2.7. *Suppose that $\overline{A}$ is an $\mathscr{L}$-structure. Then there is a function G assigning to each formula φ and each sequence $a : \omega \to A$ a value $G(\varphi, a) \in \{0, 1\}$, such that*

(i) For any terms σ, τ, $G(\sigma = \tau, a) = 1$ iff $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$.

(ii) For each m-ary relation symbol $\mathbf{R}$ and terms $\sigma_0, \ldots, \sigma_{m-1}$, $G(\mathbf{R}\sigma_0 \ldots \sigma_{m-1}, a) = 1$ iff $\langle \sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a)\rangle \in \mathbf{R}^{\overline{A}}$.

(iii) For every formula φ, $G(\neg\varphi, a) = 1 - G(\varphi, a)$.

(iv) For all formulas φ, ψ, $G(\varphi \to \psi, a) = 0$ iff $G(\varphi, a) = 1$ and $G(\psi, a) = 0$.

(v) For all formulas φ and any $i \in \omega$, $G(\forall v_i\varphi, a) = 1$ iff for every $s \in A$, $G(\varphi, a^i_s) = 1$.

Proof. An $(\overline{A}, a)$-formula sequence is a sequence $\langle(\varphi_0, b_0), \ldots, (\varphi_{m-1}, b_{m-1})\rangle$ such that each b_s is a function mapping $M \stackrel{\text{def}}{=} \{a : a : \omega \to A\}$ into $\{0, 1\}$ and for each $i < m$ one of the following holds:

(1) φ_i is an atomic equality formula $\sigma = \tau$, and $\forall a \in M[b_i(a) = 1$ iff $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)]$.

(2) φ_i is an atomic nonequality formula $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$, and

$$\forall a \in M[b_i(a) = 1 \quad \text{iff} \quad \langle \sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a)\rangle \in \mathbf{R}^{\overline{A}}].$$

(3) There is a $j < i$ such that $\varphi_i = \neg\varphi_j$, and $\forall a \in M[b_i(a) = 1 - b_j(a)]$.

(4) There are $j, k < i$ such that $\varphi_i = \varphi_j \to \varphi_k$, and $\forall a \in M[b_i(a) = 0$ iff $b_j(a) = 1$ and $b_k(a) = 0]$.

(5) There are $j < i$ and $k \in \omega$ such that $\varphi_i = \forall v_k \varphi_j$, and $\forall a \in M[b_i(a) = 1$ iff $\forall u \in A[b_j(a^k_u) = 1]]$.

Now we claim

(6) For any formula ψ and any $(\overline{A}, a)$-formula sequences

$$\langle(\varphi_0, b_0), \ldots, (\varphi_{m-1}, b_{m-1})\rangle \quad \text{and} \quad \langle(\varphi'_0, b'_0), \ldots, (\varphi'_{n-1}, b'_{n-1})\rangle$$

such that $\varphi_{m-1} = \varphi'_{n-1} = \psi$ we have $b_{m-1} = b'_{n-1}$.

We prove (6) by induction on ψ, thus using Proposition 2.5. First suppose that ψ is an atomic equality formula $\sigma = \tau$. Then the desired conclusion is clear. Similarly for atomic nonequality formulas. Now suppose that ψ is $\neg\chi$. Then by Proposition 2.6(c) there are $j < m$ and $k < n$ such that $\chi = \varphi_j = \varphi'_k$. By the inductive hypothesis we have $b_j = b'_k$, and hence $\forall a \in M[b_{m-1}(a) = 1 - b_j(a) = 1 - b'_k(a) = b_{n-1}(a)]$, so that $b_{m-1} = b'_{n-1}$. Next suppose that ψ is $\chi \to \theta$. Then by Proposition 2.6(d) there are $j, k < m-1$ such that $\chi = \varphi_j$ and $\theta = \varphi_k$, and there are $s, t < n-1$ such that $\chi = \varphi'_s$ and $\theta = \varphi'_t$. Then $b_j = b'_s$ and $b_k = b'_t$ by the inductive hypothesis. Hence for any $a \in M$,

$$b_{m-1}(a) = 0 \quad \text{iff} \quad b_j(a) = 1 \text{ and } b_k = 0 \quad \text{iff} \quad b'_s = 1 \text{ and } b'_t = 0 \quad \text{iff} \quad b'_{n-1}(a) = 0.$$

Thus $b_{m-1} = b'_{n-1}$. Finally, suppose that ψ is $\forall v_k \theta$. Then by Proposition 2.6(e) there are $j, s < i$ such that $\varphi_j = \theta$ and $\varphi'_s = \theta$. So by the inductive hypothesis $b_j = b'_s$. Hence for any $a \in M$ we have

$$\begin{aligned} b_{m-1}(a) = 1 \quad & \text{iff} \quad \text{for every } u \in A[b_j(a^k_u) = 1] \\ & \text{iff} \quad \text{for every } u \in A[b'_s(a^k_u) = 1] \\ & \text{iff} \quad b'_{n-1}(a) = 1. \end{aligned}$$

Thus $b_{m-1} = b'_{n-1}$, finishing the proof of (6).

(7) If $\langle\varphi_0, \ldots, \varphi_{m-1}\rangle$ is a formula construction sequence, then there is an $(\overline{A}, a)$-formula sequence of the form $\langle(\varphi_0, b_0), \ldots, (\varphi_{m-1}, b_{m-1})\rangle$.

We prove (7) by induction on m. For $m = 1$ we have two possibilities.

Case 1. φ_0 is an atomic equality formula $\sigma = \tau$. Let $b_0(a) = 1$ iff $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$.

Case 2. φ_0 is an atomic nonequality formula $\mathbf{R}\sigma_0 \ldots, x_{m-1}$. Let $b_0(a) = 1$ iff $\langle\sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}\rangle \in \mathbf{R}^{\overline{A}}$.

Now assume the statement in (7) for $m - 1 \geq 1$. By the inductive hypothesis there is an $(\overline{A}, a)$-formula sequence of the form $\psi \stackrel{\text{def}}{=} \langle(\varphi_0, b_0), \ldots, (\varphi_{m-2}, b_{m-2})\rangle$. Then we have these possibilities for φ_{m-1}.

Case 1. φ_{m-1} is $\sigma = \tau$ for some terms σ, τ. Define $b_{m-1}(a) = 1$ iff $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$. Then $\psi^\frown\langle(\varphi_{m-1}, b_{m-1})\rangle$ is as desired.

Case 2. φ_{m-1} is $\mathbf{R}\sigma_0 \ldots \sigma_{p-1}$ for some terms $\sigma_0, \ldots, \sigma_{p-1}$. Define $b_{m-1}(a) = 1$ iff $\langle \sigma_0^{\overline{A}}(a), \ldots, \sigma_{p-1}^{\overline{A}}(a)\rangle \in \mathbf{R}^{\overline{A}}$. Then $\psi \frown \langle(\varphi_{m-1}, b_{m-1})\rangle$ is as desired.

Case 3. φ_{m-1} is $\neg\varphi_i$ with $i < m-1$. Define $b_{m-1}(a) = 1 - b_i(a)$ for any a. Then $\psi \frown \langle(\varphi_{m-1}, b_{m-1})\rangle$ is as desired.

Case 4. φ_{m-1} is $\varphi_i \to \varphi_j$ with $i, j < m-1$. Define $b_{m-1}(a) = 0$ iff $b_i(a) = 1$ and $b_j(a) = 0$. Then $\psi \frown \langle(\varphi_{m-1}, b_{m-1})\rangle$ is as desired.

Case 5. φ_{m-1} is $\forall v_k \varphi_i$ with $i < m-1$. Define $b_{m-1}(a)$ iff for all $u \in A$, $b_i(a_u^k) = 1$. Then $\psi \frown \langle(\varphi_{m-1}, b_{m-1})\rangle$ is as desired.

This completes the proof of (7).

Now we can define the function G needed in the Proposition. Let ψ be a formula and $a : \omega \to A$. Let $\langle \varphi_0, \ldots, \varphi_{m-1}\rangle$ be a formula construction sequence with $\varphi_{m-1} = \psi$. By (7) let $\langle(\varphi_0, b_0), \ldots, (\varphi_{m-1}, b_{m-1})\rangle$ be an $(\overline{A}, a)$-formula sequence. Then we define $G(\psi, a) = b_{m-1}(a)$. The conditions in the Proposition are clear. □

With G as in Proposition 2.7, we write $\overline{A} \models \varphi[a]$ iff $G(\varphi, a) = 1$. $\overline{A} \models \varphi[a]$ is read: "$\overline{A}$ is a model of φ under a" or "$\overline{A}$ models φ under a" or "φ is satisfied by a in $\overline{A}$" or "φ holds in $\overline{A}$ under the assignment a". In summary:

$\overline{A} \models (\sigma = \tau)[a]$ iff $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(b)$. Here σ and τ are terms.

$\overline{A} \models (\mathbf{R}\sigma_0 \ldots \sigma_{m-1})[a]$ iff the m-tuple $\langle \sigma_0^{\overline{A}}, \ldots, \sigma_{m-1}^{\overline{A}}\rangle$ is in the relation $R^{\overline{A}}$. Here $\mathbf{R}$ is an m-ary relation symbol, and $\sigma_0, \ldots, \sigma_{m-1}$ are terms.

$\overline{A} \models (\neg\varphi)[a]$ iff it is not the case that $\overline{A} \models \varphi[a]$.

$\overline{A} \models (\varphi \to \psi)[a]$ iff either it is not true that $\overline{A} \models \varphi[a]$, or it is true that $\overline{A} \models \psi[a]$. (Equivalently, iff ($\overline{A} \models \varphi[a]$ implies that $\overline{A} \models \psi[a]$).

$\overline{A} \models (\forall v_i \varphi)[a]$ iff $\overline{A} \models \varphi[a_s^i]$ for every $s \in A$.

We define some additional logical notions:

$\varphi \vee \psi$ is the formula $\neg\varphi \to \psi$; $\varphi \vee \psi$ is called the *disjunction* of φ and ψ.

$\varphi \wedge \psi$ is the formula $\neg(\varphi \to \neg\psi)$; $\varphi \wedge \psi$ is called the *conjunction* of φ and ψ.

$\varphi \leftrightarrow \psi$ is the formula $(\varphi \to \psi) \wedge (\psi \to \varphi)$; $\varphi \leftrightarrow \psi$ is called the *equivalence* between φ and ψ.

$\exists v_i \varphi$ is the formula $\neg\forall v_i \neg\varphi$; $\exists$ is the *existential quantifier*.

These notions mean the following.

Proposition 2.8. *Let $\overline{A}$ be a structure and $a : \omega \to A$.*
(i) $\overline{A} \models (\varphi \vee \psi)[a]$ iff $\overline{A} \models \varphi[a]$ or $\overline{A} \models \psi[a]$ (or both).
(ii) $\overline{A} \models (\varphi \wedge \psi)[a]$ iff $\overline{A} \models \varphi[a]$ and $\overline{A} \models \psi[a]$.
(iii) $\overline{A} \models (\varphi \leftrightarrow \psi)[a]$ iff ($\overline{A} \models \varphi[a]$ iff $\overline{A} \models \psi[a]$).
(iv) $\overline{A} \models \exists v_i \varphi[a]$ iff there is a $b \in A$ such that $\overline{A} \models \varphi[a_b^i]$.

Proof. The proof consists in reducing the statements to ordinary mathematical usage. (i):

$$\begin{aligned}
\overline{A} \models (\varphi \vee \psi)[a] \quad & \text{iff} \quad \overline{A} \models (\neg\varphi \to \psi)[a] \\
& \text{iff} \quad \text{either it is not true that } \overline{A} \models (\neg\varphi)[a] \text{ or it is true that } \overline{A} \models \psi[a] \\
& \text{iff} \quad \text{not}(\text{not}(\overline{A} \models \varphi[a])) \text{ or } \overline{A} \models \psi[a] \\
& \text{iff} \quad \overline{A} \models \varphi[a] \text{ or } \overline{A} \models \psi[a].
\end{aligned}$$

(ii):

$$\begin{aligned}
\overline{A} \models (\varphi \wedge \psi)[a] \quad & \text{iff} \quad \text{not}(\overline{A} \models (\varphi \to \neg\psi)[a]) \\
& \text{iff} \quad \text{not}(\text{not}(\overline{A} \models \varphi[a]) \text{ or } \overline{A} \models \neg\psi[a]) \\
& \text{iff} \quad \text{not}(\text{not}(\overline{A} \models \varphi[a]) \text{ or not}(\overline{A} \models \psi[a])) \\
& \text{iff} \quad \overline{A} \models \varphi[a] \text{ and } \overline{A} \models \psi[a].
\end{aligned}$$

(iii):

$$\begin{aligned}
\overline{A} \models (\varphi \leftrightarrow \psi)[a] \quad & \text{iff} \quad \overline{A} \models ((\varphi \to \psi) \wedge (\psi \to \psi))[a] \\
& \text{iff} \quad \overline{A} \models ((\varphi \to \psi)[a] \text{ and } \overline{A} \models (\psi \to \psi))[a] \\
& \text{iff} \quad (\overline{A} \models \varphi[a] \text{ implies that } \overline{A} \models \psi[a]) \text{ and} \\
& \qquad\ (\overline{A} \models \psi[a] \text{ implies that } \overline{A} \models \varphi[a]) \\
& \text{iff} \quad (\overline{A} \models \varphi[a] \text{ iff } \overline{A} \models \psi[a]).
\end{aligned}$$

(iv):

$$\begin{aligned}
\overline{A} \models \exists v_i \varphi[a] \quad & \text{iff} \quad \overline{A} \models \neg\forall v_i \neg\varphi[a] \\
& \text{iff} \quad \text{not(for all } b \in A(\overline{A} \models \neg\varphi[a^i_b])) \\
& \text{iff} \quad \text{not(for all } b \in A(\text{not}(\overline{A} \models \varphi[a^i_b])) \\
& \text{iff} \quad \text{there is a } b \in A \text{ such that } \overline{A} \models \varphi[a^i_b]. \qquad \square
\end{aligned}$$

We say that $\overline{A}$ is a *model* of φ iff $\overline{A} \models \varphi[a]$ for every $a : \omega \to A$. If Γ is a set of formulas, we write $\Gamma \models \varphi$ iff every structure which models each member of Γ also models φ. $\models \varphi$ means that every structure models φ. φ is then called *universally valid.*

Now we want to apply the material of Chapter 1 concerning sentential logic. By definition, a tautology in a first-order language is a formula ψ such that there exist formulas $\varphi_0, \varphi_1, \ldots$ and a sentential tautology χ such that ψ is obtained from χ by replacing each symbol S_i occurring in χ by φ_i, for each $i < \omega$.

Theorem 2.9. *If ψ is a tautology in a first-order language, then ψ holds in every structure for that language.*

Proof. Let $\overline{A}$ be any structure, and $b : \omega \to A$ any assignment. We want to show that $\overline{A} \models \psi[b]$. Let formulas $\varphi_0, \varphi_1, \ldots, \chi$ be given as in the above definition. For each sentential formula θ, let θ' be the first-order formula obtained from θ by replacing each sentential variable S_i by φ_i. Thus χ' is ψ. We define a sentential assignment f by setting, for each $i \in \omega$,

$$f(i) = \begin{cases} 1 & \text{if } \overline{A} \models \varphi_i[b], \\ 0 & \text{otherwise.} \end{cases}$$

Then we claim:

(*) For any sentential formula θ, $\overline{A} \models \theta'[b]$ iff $\theta[f] = 1$.

We prove this by induction on θ:

If θ is S_i, then θ' is φ_i, and our condition holds by definition. If inductively θ is $\neg\tau$, then θ' is $\neg\tau'$, and

$$\begin{aligned} \overline{A} \models \theta'[b] \quad &\text{iff} \quad \text{not}(\overline{A} \models \tau'[b]) \\ &\text{iff} \quad \text{not}(\tau[f] = 1) \\ &\text{iff} \quad \tau[f] = 0 \\ &\text{iff} \quad \theta[f] = 1. \end{aligned}$$

Finally if inductively θ is $\tau \to \xi$, then θ' is $\tau' \to \xi'$, and

$$\begin{aligned} \overline{A} \models \theta'[b] \quad &\text{iff} \quad (\overline{A} \models \tau'[b] \text{ implies that } \overline{A} \models \xi'[b] \\ &\text{iff} \quad \tau[f] = 1 \text{ implies that } \xi[f] = 1 \\ &\text{iff} \quad \theta[f] = 1. \end{aligned}$$

This finishes the proof of (*).

Applying (*) to χ, we get $\overline{A} \models \chi'[b]$, i.e., $\overline{A} \models \psi[b]$. □

EXERCISES

E2.1. Give an exact definition of a language for the structure $(\omega, <)$.

E2.2. Give an exact definition of a language for the set A (no individual constants, function symbols, or relation symbols).

E2.3. Describe a term construction sequence which shows that $+\bullet v_0v_0v_1$ is a term in the language for $(\mathbb{R}, +, \cdot, 0, 1, <)$.

E2.4. In any first-order language, show that the sequence $\langle v_0, v_0\rangle$ is not a term. Hint: use Proposition 2.2.

E2.5. In the language for $(\omega, S, 0, +, \cdot)$, show that the sequence $\langle +, v_0, v_1, v_2\rangle$ is not a term. Here $S(i) = i + 1$ for any $i \in \omega$. Hint: use Proposition 2.2.

E2.6. Prove Proposition 2.5.

E2.7. Show how the structure $(\omega, S, 0, +, \cdot)$ can be put in the general framework of structures.

E2.8. Prove that in the language for the structure $(\omega, +)$, a term has length m iff m is odd.

E2.9. Give a formula φ in the language for $(\mathbb{Q}, +, \cdot)$ such that for any $a : \omega \to \mathbb{Q}$, $(\mathbb{Q}, +, \cdot) \models \varphi[a]$ iff $a_0 = 1$.

E2.10. Give a formula φ which holds in a structure, under any assignment, iff the structure has at least 3 elements.

E2.11. Give a formula φ which holds in a structure, under any assignment, iff the structure has exactly 4 elements.

E2.12. Write a formula φ in the language for $(\omega, <)$ such that for any assignment a, $(\omega, <) \models \varphi[a]$ iff $a_0 < a_1$ and there are exactly two integers between a_0 and a_1.

E2.13. Prove that the formula

$$v_0 = v_1 \to (\mathbf{R}v_0v_2 \to \mathbf{R}v_1v_2)$$

is universally valid, where $\mathbf{R}$ is a binary relation symbol.

E2.14. Give an example showing that the formula

$$v_0 = v_1 \to \forall v_0(v_0 = v_1)$$

is not universally valid.

E2.15. Prove that $\exists v_0 \forall v_1 \varphi \to \forall v_1 \exists v_0 \varphi$ is universally valid.

3. Proofs

The purpose of this chapter is give the definition of a mathematical proof, and give the simplest proofs which will be needed in proving the completeness theorem in the next chapter. Given a set Γ of formulas in a first-order language, and a formula φ in that language, we explain what it means to have a proof of φ from Γ.

The following formulas are the *logical axioms*. Here φ, ψ, χ are arbitrary formulas unless otherwise indicated.

(L1a) $\varphi \to (\psi \to \varphi)$.
(L1b) $[\varphi \to (\psi \to \chi)] \to [(\varphi \to \psi) \to (\varphi \to \chi)]$.
(L1c) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$.
(L2) $\forall v_i(\varphi \to \psi) \to (\forall v_i\varphi \to \forall v_i\psi)$, for any $i \in \omega$.
(L3) $\varphi \to \forall v_i\varphi$ for any $i \in \omega$ such that v_i does not occur in φ.
(L4) $\exists v_i(v_i = \sigma)$ if σ is a term and v_i does not occur in σ.
(L5) $\sigma = \tau \to (\sigma = \rho \to \tau = \rho)$, where σ, τ, ρ are terms.
(L6) $\sigma = \tau \to (\rho = \sigma \to \rho = \tau)$, where σ, τ, ρ are terms.
(L7) $\sigma = \tau \to \mathbf{F}\xi_0 \ldots \xi_{i-1}\sigma\xi_{i+1} \ldots \xi_{m-1} = \mathbf{F}\xi_0 \ldots \xi_{i-1}\tau\xi_{i+1} \ldots \xi_{m-1}$, where $\mathbf{F}$ is an m-ary function symbol, $i < m$, and $\sigma, \tau, \xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots \xi_{m-1}$ are terms.
(L8) $\sigma = \tau \to (\mathbf{R}\xi_0 \ldots \xi_{i-1}\sigma\xi_{i+1} \ldots \xi_{m-1} \to \mathbf{R}\xi_0 \ldots \xi_{i-1}\tau\xi_{i+1} \ldots \xi_{m-1})$, where $\mathbf{R}$ is an m-ary relation symbol, $i < m$, and $\sigma, \tau, \xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots \xi_{m-1}$ are terms.

Theorem 3.1. *Every logical axiom is universally valid.*

Proof. (L1a–c): Universally valid by Theorem 2.9.

(L2): Assume that

(1) $\overline{A} \models \forall v_i(\varphi \to \psi)[a]$ and
(2) $\overline{A} \models \forall v_i\varphi[a]$;

We want to show that $\overline{A} \models \forall v_i\psi[a]$. To this end, take any $b \in A$; we want to show that $\overline{A} \models \varphi[a^i_b]$. Now by (1) we have $\overline{A} \models (\varphi \to \psi)[a^i_b]$, hence $\overline{A} \models \varphi[a^i_b]$ implies that $\overline{A} \models \psi[a^i_b]$. Now by (2) we have $\overline{A} \models \varphi[a^i_b]$, so $\overline{A} \models \psi[a^i_b]$.

(L3): We prove by induction on φ that if v_i does not occur in φ, and if $a, b : \omega \to A$ are such that $a(j) = b(j)$ for all $j \neq i$, then $\overline{A} \models \varphi[a]$ iff $\overline{A} \models \varphi[b]$. This will imply that (L3) is universally valid.

• φ is $\sigma = \tau$. Thus v_i does not occur in σ or in τ. Then

$$\begin{aligned} \overline{A} \models (\sigma = \tau)[a] \quad &\text{iff} \quad \sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a) \\ &\text{iff} \quad \sigma^{\overline{A}}(b) = \tau^{\overline{A}}(b) \quad \text{by Proposition 2.4} \\ &\text{iff} \quad \overline{A} \models (\sigma = \tau)[b]. \end{aligned}$$

• φ is $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$ for some m-ary relation symbol and some terms $\sigma_0, \ldots, \sigma_{m-1}$. We leave this case to an exercise.

• φ is $\neg\psi$ (inductively).

$$\begin{aligned}\overline{A} \models \varphi[a] \quad &\text{iff} \quad \text{not}(\overline{A} \models \psi[a]) \\ &\text{iff} \quad \text{not}(\overline{A} \models \psi[b]) \quad \text{(inductive hypothesis)} \\ &\text{iff} \quad \overline{A} \models \varphi[b].\end{aligned}$$

• φ is $\psi \to \chi$ (inductively).

$$\begin{aligned}\overline{A} \models \varphi[a] \quad &\text{iff} \quad (\overline{A} \models \psi[a] \text{ implies that } \overline{A} \models \chi[a]) \\ &\text{iff} \quad (\overline{A} \models \psi[b] \text{ implies that } \overline{A} \models \chi[b]) \\ &\qquad \text{(inductive hypothesis)} \\ &\text{iff} \quad \overline{A} \models \varphi[b].\end{aligned}$$

• φ is $\forall v_k\psi$ (inductively). By symmetry it suffices to prove just one direction. Suppose that $\overline{A} \models \varphi[a]$; we want to show that $\overline{A} \models \varphi[b]$. To this end, suppose that $u \in A$; we want to show that $\overline{A} \models \psi[b^k_u]$. Since $\overline{A} \models \varphi[a]$, we have $\overline{A} \models \psi[a^k_u]$. Now $k \neq i$, since v_i does not occur in φ. Hence $(a^k_u)(j) = (b^k_u)(j)$ for all $j \neq i$. Hence $\overline{A} \models \psi[b^k_u]$ by the inductive hypothesis, as desired.

This finishes our proof by induction of the statement made above. Now assume that $\overline{A} \models \varphi[a]$ and $u \in A$; we want to show that $\overline{A} \models \varphi[a^i_u]$. This holds by the statement above.

This finishes the proof of (L3).

(L4): Suppose that σ is a term and v_i does not occur in σ. To prove that $\overline{A} \models (\exists v_i(v_i = \sigma))[a]$, we want to find $u \in A$ such that $\overline{A} \models (v_i = \sigma)[a^i_u]$. Let $u = \sigma^{\overline{A}}(a)$. Then

$$(v_i)^{\overline{A}}[a^i_u] = u = \sigma^{\overline{A}}(a) = \sigma^{\overline{A}}(a^i_u)$$

by Proposition 2.4 (since v_i does not occur in σ, hence $a(j) = a^i_u(j)$ for all j such that v_j occurs in σ). Hence $\overline{A} \models (v_i = \sigma)[a^i_u]$.

(L5): Assume that $\overline{A} \models (\sigma = \tau)[a]$ and $\overline{A} \models (\sigma = \rho)[a]$. Then $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$ and $\sigma^{\overline{A}}(a) = \rho^{\overline{A}}(a)$, so $\tau^{\overline{A}}(a) = \rho^{\overline{A}}(a)$, hence $\overline{A} \models (\tau = \rho)[a]$.

(L6): Left as an exercise.

(L7): Assume that $\overline{A} \models (\sigma = \tau)[a]$. Then $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$, and so

$$\begin{aligned}(\mathbf{F}\xi_0 \ldots \xi_{i-1}\sigma\xi_{i+1} \ldots \xi_{m-1})^{\overline{A}}(a) &= \mathbf{F}^{\overline{A}}(\xi_0^{\overline{A}}(a), \ldots, \xi_{i-1}^{\overline{A}}(a), \sigma^{\overline{A}}(a), \xi_{i+1}^{\overline{A}}(a), \ldots, \xi_{m-1}^{\overline{A}}(a)) \\ &= \mathbf{F}^{\overline{A}}(\xi_0^{\overline{A}}(a), \ldots, \xi_{i-1}^{\overline{A}}(a), \tau^{\overline{A}}(a), \xi_{i+1}^{\overline{A}}(a), \ldots, \xi_{m-1}^{\overline{A}}(a)) \\ &= (\mathbf{F}\xi_0 \ldots \xi_{i-1}\tau\xi_{i+1} \ldots \xi_{m-1})^{\overline{A}}(a);\end{aligned}$$

it follows that $\overline{A} \models (\mathbf{F}\xi_0 \ldots \xi_{i-1}\sigma\xi_{i+1} \ldots \xi_{m-1} = \mathbf{F}\xi_0 \ldots \xi_{i-1}\tau\xi_{i+1} \ldots \xi_{m-1})[a]$, hence (L7) is universally valid.

(L8): Left as an exercise. $\square$

Now let Γ be a set of formulas. A Γ-*proof* is a finite sequence $\langle\varphi_0, \ldots, \varphi_{m-1}\rangle$ of formulas such that for each $i < m$ one of the following conditions holds:

(I1) φ_i is a logical axiom
(I2) $\varphi_i \in \Gamma$.
(I3) (modus ponens) There are $j, k < i$ such that φ_j is the formula $\varphi_k \to \varphi_i$.
(I4) (generalization) There exist $j < i$ and $k \in \omega$ such that φ_i is the formula $\forall v_k \varphi_j$.

Then we say that Γ *proves* φ, in symbols $\Gamma \vdash \varphi$, provided that φ is an entry in some Γ-proof. We write $\vdash \varphi$ in place of $\emptyset \vdash \varphi$.

Theorem 3.2. *If* $\Gamma \vdash \varphi$, *then* $\Gamma \models \varphi$.

Proof. Recall the notion $\Gamma \models \varphi$ from Chapter 2: it says that for every structure $\overline{A}$ for the implicit language we are dealing with, if $\overline{A} \models \psi[a]$ for all $\psi \in \Gamma$ and all $a : \omega \to A$, then $\overline{A} \models \varphi[a]$ for every $a : \omega \to A$. Now it suffices to take a Γ-proof $\langle \psi_0, \ldots, \psi_{m-1} \rangle$ and prove by complete induction on i that $\Gamma \models \psi_i$ for each $i < m$.

Case 1. ψ_i is a logical axiom. Then the result follows by Theorem 3.1.

Case 2. $\psi_i \in \Gamma$. Obviously then $\Gamma \models \psi_i$.

Case 3. There are $j, k < i$ such that φ_j is $\varphi_k \to \varphi_i$. Suppose that $\overline{A}$ is a model of Γ and $a : \omega \to A$. Then $\overline{A} \models \varphi_k[a]$ by the inductive hypothesis, and also $\overline{A} \models (\varphi_k \to \varphi_i)[a]$ by the inductive hypothesis. Thus $\overline{A} \models \varphi_k[a]$ implies that $\overline{A} \models \varphi_i[a]$, so $\overline{A} \models \varphi_i[a]$.

Case 3. There exist $j < i$ and $k \in \omega$ such that φ_i is $\forall v_k \varphi_j$. Given $u \in A$, we want to show that $\overline{A} \models \varphi_j[a^k_u]$; but this follows from the inductive hypothesis. □

One form of the completeness theorem, proved in the next chapter, is that, conversely, $\Gamma \models \varphi$ implies that $\Gamma \vdash \varphi$.

In this chapter we will show that many definite formulas φ are such that $\vdash \varphi$. We begin with tautologies.

Lemma 3.3. $\vdash \varphi$ *for any first-order tautology* φ.

Proof. Let χ be a sentential tautology, and let $\langle \psi_0, \psi_1, \ldots \rangle$ be a sequence of first-order formulas such that φ is obtained from χ by replacing each sentential variable S_i by ψ_i. For each sentential formula θ, let θ' be obtained from θ by replacing each sentential variable S_i by ψ_i. By Theorem 1.16, $\vdash \chi$ (in the sentential sense). Hence there is a sentential proof $\langle \theta_0, \ldots, \theta_m \rangle$ with $\theta_m = \chi$. We claim that $\langle \theta'_0, \ldots, \theta'_m \rangle$ is a first-order proof. Since $\theta'_m = \chi' = \varphi$, this will prove the lemma. If $i \le m$ and θ_i is a (sentential) axiom, then θ'_i is the corresponding first-order axiom:

$$\begin{aligned}
&[\rho \to (\sigma \to \rho)]' = [\rho' \to (\sigma' \to \rho')];\\
&[[\rho \to (\sigma \to \tau] \to [(\rho \to \sigma) \to (\rho \to \tau)]]' =\\
&\qquad [[\rho' \to (\sigma' \to \tau')] \to [(\rho' \to \sigma') \to (\rho' \to \tau')]];\\
&[(\neg\rho \to \neg\sigma) \to (\sigma \to \rho)]' = [(\neg\rho' \to \neg\sigma') \to (\sigma' \to \rho')].
\end{aligned}$$

If $j, k < i$ and θ_k is $\theta_j \to \theta_i$, then θ'_k is $\theta'_j \to \theta'_i$. □

We proceed with simple theorems concerning equality.

Proposition 3.4. $\vdash \sigma = \sigma$ *for any term* σ.

Proof. The following is a $\emptyset$-proof; on the left is the entry number, and on the right a justification. Let v_i be a variable not occurring in σ.

(1)	$v_i = \sigma \to (v_i = \sigma \to \sigma = \sigma)$	(L5)
(2)	$[v_i = \sigma \to (v_i = \sigma \to \sigma = \sigma)] \to [\neg(\sigma = \sigma) \to \neg(v_i = \sigma)]$	(taut.)
(3)	$\neg(\sigma = \sigma) \to \neg(v_i = \sigma)$	((1), (2), MP)
(4)	$\forall v_i[\neg(\sigma = \sigma) \to \neg(v_i = \sigma)]$	((3), gen.)
(5)	$\forall v_i[\neg(\sigma = \sigma) \to \neg(v_i = \sigma)] \to [\forall v_i \neg(\sigma = \sigma) \to \forall v_i \neg(v_i = \sigma)]$	(L2)
(6)	$\forall v_i \neg(\sigma = \sigma) \to \forall v_i \neg(v_i = \sigma)$	(4), (5), MP
(7)	$\neg(\sigma = \sigma) \to \forall v_i \neg(\sigma = \sigma)$	(L3)
(8)	$(7) \to [(6) \to [\neg(\sigma = \sigma) \to \forall v_i \neg(v_i = \sigma)]$	(taut.)
(9)	$(6) \to [\neg(\sigma = \sigma) \to \forall v_i \neg(v_i = \sigma)]$	(7), (8), MP
(10)	$\neg(\sigma = \sigma) \to \forall v_i \neg(v_i = \sigma)$	(6), (9), MP
(11)	$(10) \to [\exists v_i(v_i = \sigma) \to \sigma = \sigma]$	(taut.)
(12)	$\exists v_i(v_i = \sigma) \to \sigma = \sigma$	(10), (11), MP
(13)	$\exists v_i(v_i = \sigma)$	(L4)
(14)	$(13) \to [(12) \to \sigma = \sigma]$	(L1)
(15)	$(12) \to \sigma = \sigma$	((13), (14), MP)
(16)	$\sigma = \sigma$	((12), (15), MP)

□

Proposition 3.5. $\vdash \sigma = \tau \to \tau = \sigma$ *for any terms* σ, τ.

Proof. By (L5) we have

$$\vdash \sigma = \tau \to (\sigma = \sigma \to \tau = \sigma);$$

and by Proposition 3.4 we have $\vdash \sigma = \sigma$. Now

$$\sigma = \sigma \to ([\sigma = \tau \to (\sigma = \sigma \to \tau = \sigma)] \to (\sigma = \tau \to \tau = \sigma))$$

is a tautology, so $\vdash \sigma = \tau \to \tau = \sigma$. □

Proposition 3.6. $\vdash \sigma = \tau \to (\tau = \rho \to \sigma = \rho)$ *for any terms* σ, τ, ρ.

Proof. By (L5), $\vdash \tau = \sigma \to (\tau = \rho \to \sigma = \rho)$. By Proposition 3.5, $\vdash \sigma = \tau \to \tau = \sigma$. Now

$$(\sigma = \tau \to \tau = \sigma) \to ([\tau = \sigma \to (\tau = \rho \to \sigma = \rho)] \to [\sigma = \tau \to (\tau = \rho \to \sigma = \rho)])$$

is a tautology, so $\vdash \sigma = \tau \to (\tau = \rho \to \sigma = \rho)$. □

We now give several results expressing the principle of substitution of equals for equals. The main fact is expressed in Theorem 3.16, which says that under certain conditions the formula $\sigma = \tau \to (\varphi \leftrightarrow \psi)$ is provable, where ψ is obtained from φ by replacing some occurrences of σ by τ.

Lemma 3.7. *If σ and τ are terms, φ and ψ are formulas, v_i is a variable not occurring in σ or τ, and $\vdash \sigma = \tau \rightarrow (\varphi \rightarrow \psi)$, then $\vdash \sigma = \tau \rightarrow (\forall v_i \varphi \rightarrow \forall v_i \psi)$.*

Proof.

(1)	$\vdash \forall v_i[\sigma = \tau \rightarrow (\varphi \rightarrow \psi)]$	(hypothesis, gen.)
(2)	$\vdash \forall v_i(\sigma = \tau) \rightarrow \forall v_i(\varphi \rightarrow \psi)]$	(from (1), using (L2))
(3)	$\vdash \forall v_i(\varphi \rightarrow \psi) \rightarrow (\forall v_i \varphi \rightarrow \forall v_i \psi)$	((L2))
(4)	$\vdash \sigma = \tau \rightarrow \forall v_i(\sigma = \tau)$.	((L3))

Now putting (2)–(4) together with a tautology gives the lemma. □

To proceed further we need to discuss the notion of free and bound occurrences of variables and terms. This depends on the notion of a subformula. Recall that a formula is just a finite sequence of positive integers, subject to certain conditions. Atomic equality formulas have the form $\sigma = \tau$ for some terms σ, τ, and $\sigma = \tau$ is defined to be $\langle 3 \rangle^\frown \sigma^\frown \tau$. Atomic non-equality formulas have the form $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$ for some m, some m-ary relation symbol $\mathbf{R}$, and some terms $\sigma_0, \ldots, \sigma_{m-1}$. $\mathbf{R}$ is actually some positive integer k greater than 5 and not divisible by 5, and $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$ is the sequence $\langle k \rangle^\frown \sigma_0^\frown \cdots ^\frown \sigma_{m-1}$. Non-atomic formulas have the form

$\neg\varphi = \langle 1 \rangle^\frown \varphi$,
$\varphi \rightarrow \psi = \langle 2 \rangle^\frown \varphi^\frown \psi$, or
$\forall v_s \varphi = \langle 4, 5(s+1) \rangle^\frown \varphi$.

Thus every formula begins with one of the integers 1,2,3,4 or some positive integer greater than 5 not divisible by 5 which is a relation symbol. This helps motivate the following propositions.

Proposition 3.8. *If $\sigma = \langle \sigma_0, \ldots, \sigma_{k-1} \rangle$ is a term, then each σ_i is either of the form $5m$ with m a positive integer, or it is an odd integer greater than 5 which is a function symbol or individual constant.*

Proof. We prove this by induction on σ, thus using Proposition 2.1. The proposition is obvious if σ is a variable or individual constant. Suppose that $\mathbf{F}$ is a function symbol of rank m, $\tau_0, \ldots, \tau_{m-1}$ are terms, and σ is $\mathbf{F}\tau_0 \ldots \tau_{m-1}$, where we assume the truth of the proposition for $\tau_0, \ldots, \tau_{m-1}$. Suppose that $i < k$. If $i = 0$, then σ_i is $\mathbf{F}$, a function symbol. If $i > 0$, then σ_i is an entry in some τ_j, and the desired conclusion follows by the inductive hypothesis. □

Proposition 3.9. *Let $\varphi = \langle \varphi_0, \ldots \varphi_{k-1} \rangle$ be a formula, suppose that $i < k$, and φ_i is one of the integers 1,2,3,4 or a positive integer greater than 5 which is a relation symbol. Then there is a unique segment $\langle \varphi_i, \varphi_{i+1}, \ldots, \varphi_j \rangle$ of φ which is a formula.*

Proof. We prove this by induction on φ, thus using Proposition 2.5. We assume the hypothesis of the proposition. First suppose that φ is an atomic equality formula $\sigma = \tau$ with σ and τ terms. Thus $\sigma = \tau$ is the sequence $\langle 1 \rangle^\frown \sigma^\frown \tau$. Now by Proposition 2.2(ii), no entry of a term is among the integers $1, 2, 3, 4$ or is a positive integer greater than 5 which is a relation symbol. It follows from the assumption about i that $i = 0$, and hence the desired segment of φ is φ itself. It is unique by Proposition 2.6(iii). Second suppose that φ is an atomic non-equality formula $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$ with $\mathbf{R}$ an m-ary relation symbol and $\sigma_0, \ldots, \sigma_{m-1}$ terms. This is very similar to the first case. $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$ is the sequence $\langle \mathbf{R} \rangle ^\frown \sigma_0 ^\frown \cdots ^\frown \sigma_{m-1}$. By Proposition 2.2(ii) i must be 0, and hence the desired segment of φ is φ itself. It is unique by Proposition 2.6(iii).

Now assume inductively that φ is $\neg\psi$; so φ is $\langle 1 \rangle ^\frown \psi$. If $i = 0$, then φ itself is the desired segment, unique by Proposition 2.6(iii). If $i > 0$, then $\varphi_i = \psi_{i-1}$, where $\psi = \langle \psi_0, \ldots, \psi_{k-1} \rangle$. By the inductive hypothesis there is a segment $\langle \psi_{i-1}, \psi_i, \ldots, \psi_j \rangle$ of ψ which is a formula. This gives a segment $\langle \varphi_i, \varphi_{i+1}, \ldots, \varphi_{j+1} \rangle$ of φ which is a formula; it is unique by Proposition 2.6(iii).

Assume inductively that φ is $\psi \to \chi$ for some formulas ψ, χ. So φ is $\langle 2 \rangle ^\frown \psi ^\frown \chi$. If $i = 0$, then φ itself is the required segment, unique by Proposition 2.6(iii). Now suppose that $i > 0$. Now we have $\psi = \langle \varphi_1, \ldots, \varphi_m \rangle$ and $\chi = \langle \varphi_{m+1}, \ldots, \varphi_{k-1} \rangle$ for some m. If $1 \leq i \leq m$, then by the inductive assumption there is a segment $\langle \varphi_i, \varphi_{i+1}, \ldots, \varphi_n \rangle$ of ψ which is a formula. This is also a segment of φ, and it is unique by Proposition 2.6(iii). If $m+1 \leq i \leq k-1$, a similar argument with χ gives the desired result.

Finally, assume inductively that φ is $\forall v_s \psi$ with ψ some formula and $s \in \omega$. We leave this case to an exercise. □

The segment of φ asserted to exist in Proposition 3.9 is called the *subformula of* φ beginning at i. For example, consider the formula $\varphi \stackrel{\text{def}}{=} \forall v_0[v_0 = v_2 \to v_0 = v_2]$. The formula $v_0 = v_2$ occurs in two places in φ. In detail, φ is the sequence $\langle 4, 5, 2, 3, 5, 15, 3, 5, 15 \rangle$. Thus

$\varphi_0 = 4;$
$\varphi_1 = 5;$
$\varphi_2 = 2;$
$\varphi_3 = 3;$
$\varphi_4 = 5;$
$\varphi_5 = 15;$
$\varphi_6 = 3;$
$\varphi_7 = 5;$
$\varphi_8 = 15;$

On the other hand, $v_0 = v_2$ is the formula $\langle 3, 5, 15 \rangle$. It occurs in φ beginning at 3, and also beginning at 6.

Now a variable v_s is said to *occur bound* in φ at the j-th position iff with $\varphi = \langle \varphi_0, \ldots, \varphi_{m-1} \rangle$, we have $\varphi_j = v_s$ and there is a subformula of φ of the form $\forall v_s \psi = \langle \varphi_i, \varphi_{i+1}, \ldots, \varphi_m \rangle$ with $i + 1 \leq j \leq m$. If a variable v_s occurs at the j-th position of φ but does not occur bound there, then that occurrence is said to be *free*. We give some examples. Let φ be the formula $v_0 = v_1 \to v_1 = v_2$. All the occurrences of v_0, v_1, v_2 are free occurrences in φ. Note that as a sequence φ is $\langle 2, 3, 5, 10, 3, 10, 15 \rangle$; so $\varphi_0 = 2$, $\varphi_1 = 3$, $\varphi_2 = 5$, $\varphi_3 = 10$, $\varphi_4 = 3$, $\varphi_5 = 10$, and $\varphi_6 = 15$. The variable v_0, which is the integer 5, occurs free at the 2-nd position. The variable v_1, which is the integer 10, occurs free at the 3rd and 5th positions. The variable v_2, which is the integer 15, occurs free at the 6th position.

Now let ψ be the formula $v_0 = v_1 \to \forall v_1(v_1 = v_2)$. Then the first occurence of v_1 is free, but the other two occurrences are bound. As a sequence, ψ is $\langle 2, 3, 5, 10, 4, 10, 3, 10, 15\rangle$. The variable v_1 occurs free at the 3rd position, and bound at the 5th and 7th positions.

We also need the notion of a term occurring in another term, or in a formula. The following two propositions are proved much like 3.9.

Proposition 3.10. *If $\sigma = \langle\sigma_0, \ldots, \sigma_{m-1}\rangle$ is a term and $i < m$, then there is a unique term τ which is a segment of σ beginning at i.*

Proof. We prove this by induction on σ. For σ a variable or individual constant, we have $m = 1$ and so $i = 0$, and σ itself is the only possibility for τ. Now suppose that the proposition is true for terms $\tau_0, \ldots \tau_{n-1}$, $\mathbf{F}$ is an n-ary function symbol, and σ is $\mathbf{F}\tau_0 \ldots \tau_{n-1}$. If $i = 0$, then σ itself begins at i, and it is the only term beginning at i by Proposition 2.2(iii). If $i > 0$, then i is inside some term τ_k, and so by the inductive assumption there is a term which is a segment of τ_k beginning there; this term is a segment of σ too, and it is unique by Proposition 2.2(iii). □

Under the assumptions of Proposition 3.10, we say that τ occurs in σ beginning at i.

Proposition 3.11. *If $\varphi = \langle\varphi_0, \ldots, \varphi_{m-1}\rangle$ is a formula, $i < m$, and φ_i is a variable, an individual constant, or a function symbol, then there is a unique segment of φ beginning at i which is a term.*

Proof. We prove this by induction on φ. First suppose that φ is an atomic equality formula $\sigma = \tau$ for some terms σ, τ. Thus φ is $\langle 3\rangle^\frown\sigma^\frown\tau$. So $i > 0$, and hence i is inside σ or τ. If i is inside σ, then by Proposition 3.10, there is a term which is a segment of σ beginning at i; it is also a segment of φ, and it is unique by Proposition 2.2(iii). Similarly for τ.

We leave the other parts of the proof to an exercise. □

Under the assumptions of Proposition 3.11, we say that the indicated segment occurs in φ beginning at i.

We now extend the notions of free and bound occurrences to terms. Let σ be a term which occurs as a segment in a formula φ. Say that $\varphi = \langle\varphi_0, \ldots, \varphi_{m-1}\rangle$ and $\sigma = \langle\varphi_i, \ldots \varphi_k\rangle$. We say that this occurrence of σ in φ is *bound* iff there is a variable v_s which occurs bound in φ at some place t with $i \le t \le k$; the occurrence of σ is *free* iff there is no such variable.

We give some examples. The term $v_0 + v_1$ is bound in its only occurrence in the formula $\forall v_0(v_0 + v_1 = v_2)$. The same term is bound in its first occurrence and free in its second occurrence in the formula $\forall v_0(v_0 + v_1 = v_2) \wedge v_0 + v_1 = v_0$.

Suppose that σ, τ, ρ are terms, and τ occurs in σ beginning at i. By the *result of replacing that occurrence of τ by ρ* we mean the following sequence ξ. Say σ, τ, ρ have domains (lengths) m, n, p respectively. Then ξ is the sequence

$$\langle\sigma_0, \ldots, \sigma_{i-1}, \rho_0, \ldots, \rho_{p-1}, \sigma_{i+n}, \ldots, \sigma_{m-1}\rangle.$$

Put another way, if σ is $\theta^\frown\tau^\frown\eta$ with θ of length i, then ξ is $\theta^\frown\rho^\frown\eta$.

Proposition 3.12. *Suppose that σ, τ, ρ are terms, and the sequence ξ is obtained from ρ by replacing one occurrence of σ by τ. Then ξ is a term.*

Proof. We prove this by induction on ρ, thus by using Proposition 2.1. If ρ is a variable or an individual constant, then σ must be ρ itself, and ξ is τ, which is a term. Now suppose that ρ is $\mathbf{F}\eta_0 \ldots \eta_{m-1}$ for some m-ary function symbol $\mathbf{F}$ and some terms $\eta_0, \ldots, \eta_{m-1}$, and the proposition holds for $\eta_0, \ldots, \eta_{m-1}$. Say the occurrence of σ in ρ begins at i. If $i = 0$, then σ equals ρ, and hence ξ equals τ, which is a term. If $i > 0$, then i is inside some η_j, and hence the occurrence of σ is actually an occurrence in η_j by Proposition 2.2(iii). Replacing this occurrence of σ in η_j by τ we obtain a term by the inductive hypothesis; call this term η'_j. It follows that ξ is $\mathbf{F}\eta_0 \ldots \eta_{j-1}\eta'_j, \eta_{j+1} \ldots \eta_{m-1}$, which is a term. □

As an example, consider the term $v_0 \bullet (v_1 + v_2)$ in the language for $(\mathbb{Q}, +, \cdot)$. Replacing the occurrence of v_1 by $v_0 \bullet v_1$ we obtain the term $v_0 \bullet ((v_0 \bullet v_1) + v_2)$. Writing this out in detail, assuming that $\bullet$ corresponds to 9 and $+$ corresponds to 7, we start with the sequence $\langle 9, 5, 7, 10, 15\rangle$ and end with the sequence $\langle 9, 5, 7, 9, 5, 10, 15\rangle$.

Our first form of subsitution of equals for equals only involves terms:

Theorem 3.13. *If σ, τ, ρ are terms, and ξ is a sequence obtained from ρ by replacing an occurrence of σ in ρ by τ, then ξ is a term and $\vdash \sigma = \tau \to \rho = \xi$.*

Proof. ξ is a term by Proposition 3.12. Now we proceed by induction on ρ. If ρ is a variable or an individual constant, then σ must be the same as ρ, since ρ has length 1 and σ occurs in ρ. Then ξ is τ, and $\sigma = \tau \to \rho = \xi$ is $\sigma = \tau \to \sigma = \tau$, a tautology. So the proposition is true in this case.

Now assume inductively that ρ is $\mathbf{F}\eta_0 \ldots \eta_{m-1}$ with $\mathbf{F}$ an m-ary function symbol and $\eta_0, \ldots, \eta_{m-1}$ terms. There are two possibilities for the occurrence of σ. First, possibly σ is the same as ρ. Then ξ is τ, and again we have the tautology $\sigma = \tau \to \sigma = \tau$, Second, the occurrence of σ is within some η_i. Then by the inductive hypothesis, $\vdash \sigma = \tau \to \eta_i = \eta'_i$, where η'_i is obtained from η_i by replacing the indicated occurrence of σ by τ. Now an instance of (L7) is

$$\eta_i = \eta'_i \to \mathbf{F}\eta_0 \ldots \eta_{i-1} \ldots \eta_i\eta_{i+1} \ldots \eta_{m-1} = \mathbf{F}\eta_0 \ldots \eta_{i-1} \ldots \eta'_i\eta_{i+1} \ldots \eta_{m-1}.$$

Putting this together with $\vdash \sigma = \tau \to \eta_i = \eta'_i$ and a tautology gives $\vdash \sigma = \tau \to \rho = \xi$. □

Proposition 3.14. *Suppose that φ is a formula and σ, τ are terms. Suppose that σ occurs at the i-th place in φ, and if $i > 0$ and $\varphi_{i-1} = \forall$, then τ is a variable. Let the sequence ψ be obtained from φ by replacing that occurrence of σ by τ. Then ψ is a formula.*

Proof. Exercise. □

For the exact definition of ψ see the description before Proposition 3.12.

Lemma 3.15. *Suppose that σ and τ are terms, φ is a formula, and ψ is obtained from φ by replacing one free occurrence of σ in φ by τ, such that the occurrence of τ that results is free in ψ. Then $\vdash \sigma = \tau \to (\varphi \leftrightarrow \psi)$.*

Proof. We proceed by induction on φ. First suppose that φ is an atomic equality formula $\rho = \xi$. If the occurrence of σ that is replaced is in ρ, let ρ' be the resulting term. Then by Proposition 3.13, $\vdash \sigma = \tau \to \rho = \rho'$. Now (L5) gives $\vdash \rho = \rho' \to (\rho = \xi \to \rho' = \xi)$. Putting these two together with a tautology gives $\vdash \sigma = \tau \to (\rho = \xi \to \rho' = \xi)$. By symmetry, $\vdash \sigma = \tau \to (\rho' = \xi \to \rho = \xi)$. Hence $\vdash \sigma = \tau \to (\rho = \xi \leftrightarrow \rho' = \xi)$.

If the occurrence of σ that is replaced is in ξ, a similar argument using (L6) works.

Second, suppose that φ is an atomic non-equality formula $\mathbf{R}\rho_0 \ldots \rho_{m-1}$, with $\mathbf{R}$ an m-ary relation symbol and $\rho_0, \ldots, \rho_{m-1}$ terms. Say that the occurrence of σ that is replaced by τ is in ρ_i, the resulting term being ρ_i'. Then by Proposition 3.13, $\vdash \sigma = \tau \to \rho_i = \rho_i'$. By (L8) we have

$$\vdash \rho_i = \rho_i' \to (\mathbf{R}\rho_0 \ldots \rho_{m-1} \to \mathbf{R}\rho_0 \ldots \rho_{i-1}\rho_i'\rho_{i+1} \ldots \rho_{m-1}),$$

so by a tautology we get from these two facts

$$\vdash \sigma = \tau \to (\mathbf{R}\rho_0 \ldots \rho_{m-1} \to \mathbf{R}\rho_0 \ldots \rho_{i-1}\rho_i'\rho_{i+1} \ldots \rho_{m-1}),$$

and by symmetry

$$\vdash \sigma = \tau \to (\mathbf{R}\rho_0 \ldots \rho_{i-1}\rho_i'\rho_{i+1} \ldots \rho_{m-1} \to \mathbf{R}\rho_0 \ldots \rho_{m-1}),$$

and then another tautology gives

$$\vdash \sigma = \tau \to (\mathbf{R}\rho_0 \ldots \rho_{m-1} \leftrightarrow \mathbf{R}\rho_0 \ldots \rho_{i-1}\rho_i'\rho_{i+1} \ldots \rho_{m-1}),$$

This finishes the atomic cases. Now suppose inductively that φ is $\neg\chi$. The occurrence of σ in φ that is replaced actually occurs in χ; let χ' be the result of replacing that occurrence of σ by τ. Now the occurrence of σ in χ is free in χ. In fact, suppose that $\forall v_i\theta$ is a subformula of χ which has as a segment the indicated occurrence of σ, and v_i occurs in σ. Then $\forall v_i\theta$ is also a subformula of φ, contradicting the assumption that the occurrence of σ is free in φ. Similarly the occurrence of τ in χ' which replaced the occurrence of σ is free. So by the inductive hypothesis, $\vdash \sigma = \tau \to (\chi \leftrightarrow \chi')$, and hence a tautology gives $\vdash \sigma = \tau \to (\neg\chi \leftrightarrow \neg\chi')$, i.e., $\vdash \sigma = \tau \to (\varphi \leftrightarrow \psi)$.

We leave the case of an implication to an exercise.

Finally, suppose that φ is $\forall v_i\rho$. Then the occurrence of σ in φ that is replaced is in ρ. Let ρ' be obtained from ρ by replacing that occurrence of σ by τ. The occurrence of σ in ρ must be free since it is free in φ, as in the treatment of $\neg$ above; similarly for τ and ρ'. Hence by the inductive hypothesis, $\vdash \sigma = \tau \to (\rho \leftrightarrow \rho')$. Now since the occurrence of σ in φ is free, the variable v_i does not occur in σ. Similarly, it does not occur in τ. Hence by Proposition 3.7 and tautologies we get $\vdash \sigma = \tau \to (\forall v_i\rho \leftrightarrow \forall v_i\rho')$, i.e., $\vdash \sigma = \tau \to (\varphi \leftrightarrow \psi)$. □

The hypothesis that the term τ is still free in the result of the replacement in this proposition is necessary for the truth of the proposition. This hypothesis is equivalent to saying that the occurrence of σ which is replaced is not inside a subformula of φ of the form $\forall v_i\chi$ with v_i a variable occurring in τ.

Theorem 3.16. (Substitution of equals for equals) *Suppose that φ is a formula, σ is a term, and σ occurs freely in φ starting at indices $i(0) < \cdots < i(m-1)$. Also suppose that τ is a term. Let ψ be obtained from φ by replacing each of these occurrences of σ by τ, and each such occurrence of τ is free in ψ. Then $\vdash \sigma = \tau \to (\varphi \leftrightarrow \psi)$.*

Proof. We prove this by induction on m. If $m = 0$, then φ is the same as ψ, and the conclusion is clear. Now assume the result for m, for any φ. Now assume that σ occurs freely in φ starting at indices $i(0) < \cdots < i(m)$, and no such occurrence is inside a subformula of φ of the form $\forall v_j\chi$ with v_j a variable occurring in τ. Let θ be obtained from φ by replacing the last occurrence of σ, the one beginning at $i(m)$, by τ. By Proposition 3.15, $\vdash \sigma = \tau \to (\varphi \leftrightarrow \theta)$. Now we apply the inductive hypothesis to θ and the occurrences of σ starting at $i(0), \ldots, i(m-1)$; this gives $\vdash \sigma = \tau \to (\theta \leftrightarrow \psi)$. Hence a tautology gives $\vdash \sigma = \tau \to (\varphi \leftrightarrow \psi)$, finishing the inductive proof. □

Proposition 3.17. *Suppose that φ, ψ, χ are formulas, and the sequence θ is obtained from φ by replacing an occurrence of ψ in φ by χ. Then θ is a formula.*

Proof. Exercise. □

For the exact meaning of θ see the description before Proposition 3.12.

Another form of the substitution of equals by equals principle is as follows:

Theorem 3.18. *Let φ, χ, ρ be formulas, and let ψ be obtained from φ by replacing an occurrence of χ in φ by ρ. Suppose that $\vdash \chi \leftrightarrow \rho$. Then $\vdash \varphi \leftrightarrow \psi$.*

Proof. Induction on φ. If φ is atomic, then ψ is the same as ρ, and the conclusion is clear. Suppose inductively that φ is $\neg\varphi'$. If χ is equal to φ, then ψ is equal to ρ and the conclusion is clear. Suppose that χ occurs within φ', and let ψ' be obtained from φ' by replacing that occurrence by ρ. Assume that $\vdash \chi \leftrightarrow \rho$. Then by the inductive hypothesis $\vdash \varphi' \leftrightarrow \psi'$, so $\vdash \neg\varphi' \leftrightarrow \neg\psi'$, as desired.

The case in which φ is $\varphi' \to \varphi''$ is similar. Finally, suppose that φ is $\forall v_i\varphi'$, and χ occurs within φ'. Let ψ' be obtained from φ' by replacing that occurrence by ρ. Assume that $\vdash \chi \leftrightarrow \rho$. Then $\vdash \varphi' \leftrightarrow \psi'$ by the inductive assumption. So by a tautology, $\vdash \varphi' \to \psi'$, and then by generalization $\vdash \forall v_i(\varphi' \to \psi')$. Using (L2) we then get $\vdash \forall v_i\varphi' \to \forall v_i\psi'$. Similarly, $\vdash \forall v_i\psi' \to \forall v_i\varphi'$. Hence using a tautology, $\vdash \forall v_i\varphi' \leftrightarrow \forall v_i\psi'$. □

Now we work to prove two important logical principles: changing bound variables, and dropping a universal quantifier in favor of a term.

For any formula φ, $i \in \omega$, and term σ by $\mathrm{Subf}^{v_i}_{\sigma}\varphi$ we mean the result of replacing each free occurrence of v_i in φ by σ. We now work towards showing that under suitable conditions, the formula $\forall v_i\varphi \to \mathrm{Subf}^{v_i}_{\sigma}\varphi$ is provable. The supposition expressed in the first sentence of the following lemma will be eliminated later on.

Lemma 3.19. *Suppose that v_i does not occur bound in φ, and does not occur in the term σ.*

Assume that no free occurrence of v_i in φ is within a subformula of φ of the form $\forall v_j\chi$ with v_j a variable occurring in σ. Then $\vdash \forall v_i\varphi \rightarrow \mathrm{Subf}^{v_i}_{\sigma}\varphi$.

Proof.

(1)	$\vdash v_i = \sigma \rightarrow (\varphi \rightarrow \mathrm{Subf}^{v_i}_{\sigma}\varphi)$	(by Proposition 3.16 and a tautology)
(2)	$\vdash \varphi \rightarrow (\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi \rightarrow \neg(v_i = \sigma))$	(using a tautology)
(3)	$\vdash \forall v_i[\varphi \rightarrow (\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi \rightarrow \neg(v_i = \sigma))]$	(generalization)
(4)	$\vdash \forall v_i\varphi \rightarrow \forall v_i(\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi \rightarrow \neg(v_i = \sigma))$	(using (L2))
(5)	$\vdash \forall v_i(\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi \rightarrow \neg(v_i = \sigma)) \rightarrow (\forall v_i\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi \rightarrow \forall v_i\neg(v_i = \sigma))$	((L2))
(6)	$\vdash \forall v_i\varphi \rightarrow (\forall v_i\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi \rightarrow \forall v_i\neg(v_i = \sigma))$	((4), (5), a tautology)
(7)	$\vdash \neg\forall v_i\neg(v_i = \sigma) \rightarrow (\forall v_i\varphi \rightarrow \neg\forall v_i\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi)$	((6), a tautology)
(8)	$\vdash \neg\forall v_i\neg(v_i = \sigma)$	((L4))
(9)	$\vdash \forall v_i\varphi \rightarrow \neg\forall v_i\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi)$	((7), (8), modus ponens)
(10)	$\vdash \neg\mathrm{Subf}^{v_i}_{\sigma}\varphi \rightarrow \forall v_i\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi$	((L3))
(11)	$\vdash \forall v_i\varphi \rightarrow \mathrm{Subf}^{v_i}_{\sigma}\varphi$	((9), (10), a tautology)

□

Lemma 3.20. *If $i \neq j$, φ is a formula, v_i does not occur bound in φ, and v_j does not occur in φ at all, then $\vdash \forall v_i\varphi \rightarrow \forall v_j\mathrm{Subf}^{v_i}_{v_j}\varphi$.*

Proof.

$\vdash \forall v_i\varphi \rightarrow \mathrm{Subf}^{v_i}_{v_j}\varphi$	(by Lemma 3.19)
$\vdash \forall v_j\forall v_i\varphi \rightarrow \forall v_j\mathrm{Subf}^{v_i}_{v_j}\varphi$	(using (L2) and a tautology)
$\vdash \forall v_i\varphi \rightarrow \forall v_j\forall v_i\varphi$ (by (L3))	
$\vdash \forall v_i\varphi \rightarrow \forall v_j\mathrm{Subf}^{v_i}_{v_j}\varphi$	

□

Lemma 3.21. *If $i \neq j$, φ is a formula, v_i does not occur bound in φ, and v_j does not occur in φ at all, then $\vdash \forall v_i\varphi \leftrightarrow \forall v_j\mathrm{Subf}^{v_i}_{v_j}\varphi$.*

Proof. By Proposition 3.20 we have $\vdash \forall v_i\varphi \rightarrow \forall v_j\mathrm{Subf}^{v_i}_{v_j}\varphi$. Now v_j does not occur bound in $\mathrm{Subf}^{v_i}_{v_j}\varphi$ and v_i does not occur in $\mathrm{Subf}^{v_i}_{v_j}\varphi$ at all. Hence by Proposition 3.20 again, $\vdash \forall v_j\mathrm{Subf}^{v_i}_{v_j}\varphi \rightarrow \forall v_i\mathrm{Subf}^{v_j}_{v_i}\mathrm{Subf}^{v_i}_{v_j}\varphi$. Now $\mathrm{Subf}^{v_j}_{v_i}\mathrm{Subf}^{v_i}_{v_j}\varphi$ is actually just φ itself; so $\vdash \forall v_j\mathrm{Subf}^{v_i}_{v_j}\varphi \rightarrow \forall v_i\varphi$. Hence the proposition follows. □

For $i, j \in \omega$ and φ a formula, by $\mathrm{Subb}^{v_i}_{v_j}\varphi$ we mean the result of replacing all bound occurrences of v_i in φ by v_j. By Proposition 3.14 this gives another formula.

Proposition 3.22. *If v_i occurs bound in a formula φ, then there is a subformula $\forall v_i\psi$ of φ such that v_i does not occur bound in ψ.*

Proof. Induction on φ. Note that the statement to be proved is an implication. If φ is atomic, then v_i cannot occur bound in φ; thus the hypothesis of the implication is false, and so the implication itself is true. Now suppose inductively that φ is $\neg\chi$, and v_i

occurs bound in φ. Then it occurs bound in χ, and so by the inductive hypothesis, χ has a subformula $\forall v_i\psi$ such that v_i does not occur bound in ψ. This is also a subformula of φ. The implication case is similar. Finally, suppose that φ is $\forall v_k\chi$, and v_i occurs bound in φ. If it occurs bound in χ, then by the inductive hypothesis χ has a subformula $\forall v_i\psi$ such that v_i does not occur bound in ψ; this is also a subformula of φ. If v_i does not occur bound in χ, then we must have $i = k$ since v_i occurs bound in φ, and then φ itself is the desired subformula. □

Theorem 3.23. (Change of bound variables) *If ψ_j does not occur in φ, then* $\vdash \varphi \leftrightarrow \mathrm{Subb}^{v_i}_{v_j}\varphi$.

Proof. We proceed by induction on the number m of bound occurrences of v_i in φ. If $m = 0$, then $\mathrm{Subb}^{v_i}_{v_j}\varphi$ is just φ itself, and the conclusion is clear. Now assume that $m > 0$ and the conclusion is known for all formulas with fewer than m bound occurrences of v_i. By Proposition 3.22, let $\forall v_i\psi$ be a formula occurring in φ such that v_i does not occur bound in ψ. Let k be such that v_k does not occur in φ, and hence also does not occur in ψ, and with $k \neq j$. Note that $k \neq i$ since v_k does not occur in φ while v_i does. Then by Proposition 3.21 we have

(1) $$\vdash \forall v_i\psi \leftrightarrow \forall v_k \mathrm{Subf}^{v_i}_{v_k}\psi.$$

Let φ' be obtained from φ by replacing an occurrence of $\forall v_i\psi$ by $\forall v_k\mathrm{Subf}^{v_i}_{v_k}\psi$. By Theorem 3.18,

(2) $$\vdash \varphi \leftrightarrow \varphi'.$$

Now v_j does not occur in φ', and φ' has fewer than m bound occurrences of v_i. Hence by the inductive hypothesis,

(3) $$\vdash \varphi' \leftrightarrow \mathrm{Subb}^{v_i}_{v_j}\varphi'.$$

Now $k \neq i, j$ and v_k does not occur bound in $\mathrm{Subf}^{v_i}_{v_k}\psi$. Moreover, v_j does not occur in $\mathrm{Subf}^{v_i}_{v_k}\psi$ at all. Hence by Proposition 3.20,

$$\vdash \forall v_k\mathrm{Subf}^{v_i}_{v_k}\psi \leftrightarrow \forall v_j\mathrm{Subf}^{v_k}_{v_j}\mathrm{Subf}^{v_i}_{v_k}\psi.$$

Now clearly $\mathrm{Subf}^{v_k}_{v_j}\mathrm{Subf}^{v_i}_{v_k}\psi = \mathrm{Subf}^{v_i}_{v_j}\psi$; so

(4) $$\vdash \forall v_k\mathrm{Subf}^{v_i}_{v_k}\psi \leftrightarrow \forall v_j\mathrm{Subf}^{v_i}_{v_j}\psi.$$

Now $\mathrm{Subb}^{v_i}_{v_j}\varphi$ can be obtained from $\mathrm{Subb}^{v_i}_{v_j}\varphi'$ by replacing an occurrence of $\forall v_k\mathrm{Subf}^{v_i}_{v_k}\psi$ by $\forall v_j\mathrm{Subf}^{v_i}_{v_j}\psi$. Hence by (4) and Theorem 3.18 we get

(5) $$\vdash \mathrm{Subb}^{v_i}_{v_j}\varphi \leftrightarrow \mathrm{Subb}^{v_i}_{v_j}\varphi'.$$

(2), (3), and (5) now give the desired result, finishing the inductive proof. □

We can now strengthen Lemma 3.19 by eliminating one of its hypotheses; the remaining inessential hypothesis will be eliminated next.

Lemma 3.24. *Suppose that v_i does not occur in the term σ.*

Assume that no free occurrence of v_i in a formula φ is within a subformula of φ of the form $\forall v_j \chi$ with v_j a variable occurring in σ. Then $\vdash \forall v_i \varphi \to \mathrm{Subf}^{v_i}_{\sigma}\varphi$.

Proof. Choose j so that v_j does not occur in φ or in σ, with $i \neq j$. Then by the change of bound variables theorem 3.23, $\vdash \varphi \leftrightarrow \mathrm{Subb}^{v_i}_{v_j}\varphi$. From this, using generalization and (L2) we obtain

$$(1) \qquad \vdash \forall v_i \varphi \leftrightarrow \forall v_i \mathrm{Subb}^{v_i}_{v_j}\varphi.$$

Now v_i does not occur bound in $\mathrm{Subb}^{v_i}_{v_j}\varphi$, and no free occurrence of v_i in $\mathrm{Subb}^{v_i}_{v_j}\varphi$ is in a subformula of $\mathrm{Subb}^{v_i}_{v_j}\varphi$ of the form $\forall v_k \psi$, with v_k a variable occurring in σ. This is true since it is true of φ, and v_j does not occur in σ. Hence by Lemma 3.19 we get

$$(2) \qquad \vdash \forall v_i \mathrm{Subb}^{v_i}_{v_j}\varphi \to \mathrm{Subf}^{v_i}_{\sigma}\mathrm{Subb}^{v_i}_{v_j}\varphi.$$

Now v_i does not occur at all in $\mathrm{Subf}^{v_i}_{\sigma}\mathrm{Subb}^{v_i}_{v_j}\varphi$, so by change of bound variable,

$$(3) \qquad \vdash \mathrm{Subf}^{v_i}_{\sigma}\mathrm{Subb}^{v_i}_{v_j}\varphi \leftrightarrow \mathrm{Subb}^{v_j}_{v_i}\mathrm{Subf}^{v_i}_{\sigma}\mathrm{Subb}^{v_i}_{v_j}\varphi.$$

But clearly $\mathrm{Subb}^{v_j}_{v_i}\mathrm{Subf}^{v_i}_{\sigma}\mathrm{Subb}^{v_i}_{v_j}\varphi = \mathrm{Subf}^{v_i}_{\sigma}\varphi$. Hence from (1)–(3) and tautologies we get the result of the lemma. □

Theorem 3.25. (Universal specification) *Assume that no free occurrence of v_i in a formula φ is within a subformula of φ of the form $\forall v_j \chi$ with v_j a variable occurring in a term σ. Then $\vdash \forall v_i \varphi \to \mathrm{Subf}^{v_i}_{\sigma}\varphi$.*

Proof. Choose j so that v_j does not occur in φ or in σ, with $j \neq i$. Then by Lemma 3.24, $\vdash \forall v_i \varphi \to \mathrm{Subf}^{v_i}_{v_j}\varphi$. Hence using (L2) we easily get

$$(1) \qquad \vdash \forall v_j \forall v_i \varphi \to \forall v_j \mathrm{Subf}^{v_i}_{v_j}\varphi.$$

By (L3) we have

$$(2) \qquad \vdash \forall v_i \varphi \to \forall v_j \forall v_i \varphi.$$

Now no free occurrence of v_j in $\mathrm{Subf}^{v_i}_{v_j}\varphi$ is within a subformula of $\mathrm{Subf}^{v_i}_{v_j}\varphi$ of the form $\forall v_k \psi$ with v_k occurring in σ; this is true because it holds for φ. Also, v_j does not occur in σ. Hence by Lemma 3.24 we have

$$(3) \qquad \vdash \forall v_j \mathrm{Subf}^{v_i}_{v_j}\varphi \to \mathrm{Subf}^{v_j}_{\sigma}\mathrm{Subf}^{v_i}_{v_j}\varphi.$$

Clearly $\mathrm{Subf}^{v_j}_{\sigma}\mathrm{Subf}^{v_i}_{v_j}\varphi = \mathrm{Subf}^{v_i}_{\sigma}\varphi$, so from (1)–(3) the desired result follows. □

This finishes the fundamental things that can be proved. We now give various corollaries.

Corollary 3.26. $\vdash \forall v_i\varphi \to \varphi$. □

Proposition 3.27. *If v_i does not occur free in φ, then* $\vdash \varphi \leftrightarrow \forall v_i\varphi$.

Proof. By Corollary 3.26 we have

$$\vdash \forall v_i\varphi \to \varphi. \tag{1}$$

Now let v_j be a variable not occurring in φ. Then by a change of bound variable,

$$\vdash \varphi \leftrightarrow \mathrm{Subb}^{v_i}_{v_j}\varphi. \tag{2}$$

Hence using (L2) we easily get

$$\vdash \forall v_i \mathrm{Subb}^{v_i}_{v_j}\varphi \to \forall v_i\varphi. \tag{3}$$

Now note that v_i does not occur in $\mathrm{Subb}^{v_i}_{v_j}\varphi$. Hence by (L3) we get

$$\vdash \mathrm{Subb}^{v_i}_{v_j}\varphi \to \forall v_i \mathrm{Subb}^{v_i}_{v_j}\varphi. \tag{4}$$

Now from (1)–(4) the desired result easily follows. □

Proposition 3.28. $\vdash \forall v_i\forall v_j\varphi \leftrightarrow \forall v_j\forall v_i\varphi$, *for any formula φ and any $i, j \in \omega$.*

Proof.

$$\begin{array}{ll}
\vdash \forall v_i\forall v_j\varphi \to \varphi & \text{by Corollary 3.26 twice} \\
\vdash \forall v_i\forall v_i\forall v_j\varphi \to \forall v_i\varphi & \text{by (L2)} \\
\vdash \forall v_i\forall v_j\varphi \to \forall v_i\forall v_i\forall v_j\varphi & \text{using Prop. 3.27} \\
\vdash \forall v_i\forall v_j\varphi \to \forall v_i\varphi & \\
\vdash \forall v_j\forall v_i\forall v_j\varphi \to \forall v_j\forall v_i\varphi & \text{by (L2)} \\
\vdash \forall v_i\forall v_j\varphi \to \forall v_j\forall v_i\forall v_j\varphi & \text{using Prop. 3.27} \\
\vdash \forall v_i\forall v_j\varphi \to \forall v_j\forall v_i\varphi & \\
\vdash \forall v_j\forall v_i\varphi \to \forall v_i\forall v_j\varphi & \text{similarly} \\
\vdash \forall v_i\forall v_j\varphi \leftrightarrow \forall v_j\forall v_i\varphi & \square
\end{array}$$

Recall that $\exists v_i\varphi$ is defined to be the formula $\neg\forall v_i\neg\varphi$. The following simple propositions expand on this.

Proposition 3.29. $\vdash \neg\forall v_i\varphi \leftrightarrow \exists v_i\neg\varphi$ *for any formula φ and any $i \in \omega$.*

Proof. Exercise. □

Proposition 3.30. $\vdash \neg\exists v_i\varphi \leftrightarrow \forall v_i\neg\varphi$ *for any formula φ and any $i \in \omega$.*

Proof. Exercise. □

Some important results concerning $\exists$ are as follows.

Theorem 3.31. *If no free occurrence of v_i in a formula φ is within a subformula of the form $\forall v_k \psi$ with v_k occurring in a term σ, then* $\vdash \mathrm{Subf}_{\sigma}^{v_i} \varphi \to \exists v_i \varphi$.

Proof. Exercise. $\square$

Corollary 3.32. $\vdash \varphi \to \exists v_i \varphi$ *for any formula* φ. $\square$

Corollary 3.33. $\vdash \forall v_i \varphi \to \exists v_i \varphi$.

Proof. Exercise. $\square$

Proposition 3.34. *If v_i does not occur free in φ, then* $\vdash \varphi \leftrightarrow \exists v_i \varphi$.

Proof. Exercise. $\square$

Theorem 3.35. $\vdash \exists v_i \forall v_j \varphi \to \forall v_j \exists v_i \varphi$ *for any formula* φ.

Proof.

$$\begin{array}{lr}
\vdash \varphi \to \exists v_i \varphi & \text{by Corollary 3.32} \\
\vdash \forall v_j \varphi \to \forall v_j \exists v_i \varphi & \text{generalization, (L2)} \\
\vdash \neg \forall v_j \exists v_i \varphi \to \neg \forall v_j \varphi & \text{tautology} \\
\vdash \forall v_i [\neg \forall v_j \exists v_i \varphi \to \neg \forall v_j \varphi] & \text{generalization} \\
\vdash \forall v_i [\neg \forall v_j \exists v_i \varphi \to \neg \forall v_j \varphi] \to [\forall v_i \neg \forall v_j \exists v_i \varphi \to \forall v_i \neg \forall v_j \varphi] & \text{(L2)} \\
\vdash \forall v_i \neg \forall v_j \exists v_i \varphi \to \forall v_i \neg \forall v_j \varphi & \\
\vdash \neg \forall v_j \exists v_i \varphi \to \forall v_i \neg \forall v_j \varphi & \text{by Proposition 3.27} \\
\vdash \exists v_i \forall v_j \varphi \to \forall v_j \exists v_i \varphi & \text{tautology}
\end{array}$$

$\square$

Now we prove several results involving two formulas φ and ψ, and some variable v_i which is not free in one of them.

Proposition 3.36. *If v_i does not occur free in the formula φ, and ψ is any formula, then* $\vdash \forall v_i(\varphi \to \psi) \to (\varphi \to \forall v_i \psi)$.

Proof. By Proposition 3.27,

$$\vdash \varphi \to \forall v_i \varphi. \tag{1}$$

By (L2) we have $\vdash \forall v_i(\varphi \to \psi) \to (\forall v_i \varphi \to \forall v_i \psi)$, and hence by a tautology

$$\vdash \forall v_i \varphi \to [\forall v_i(\varphi \to \psi) \to \forall v_i \psi] \tag{2}$$

By a tautology, from (1) and (2) we get

$$\vdash \varphi \to [\forall v_i(\varphi \to \psi) \to \forall v_i\psi],$$

and then another tautology gives the desired result. □

Proposition 3.37. *If v_i does not occur free in the formula ψ, then* $\vdash \forall v_i(\varphi \to \psi) \to (\exists v_i\varphi \to \psi)$.

Proof.

(1)	$\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$	(taut.)
(2)	$\vdash \forall v_i(\varphi \to \psi) \to \forall v_i(\neg\psi \to \neg\varphi)$	((1), gen., (L2))
(3)	$\vdash \forall v_i(\neg\psi \to \neg\varphi) \to (\neg\psi \to \forall v_i\neg\varphi)$	(Prop. 3.36)
(4)	$\vdash (\neg\psi \to \forall v_i\neg\varphi) \to (\exists v_i\varphi \to \psi)$	(taut.)
	$\vdash \forall v_i(\varphi \to \psi) \to (\exists v_i\varphi \to \psi)$	((2)–(4), taut.)

□

Lemma 3.38. *If φ and ψ are formulas and v_i does not occur free in ψ, then* $\vdash \forall v_i\varphi \vee \psi \leftrightarrow \forall v_i(\varphi \vee \psi)$.

Proof.

(1)	$\vdash \forall v_i\varphi \vee \psi \leftrightarrow (\neg\psi \to \forall v_i\varphi)$	taut.
(2)	$\vdash \forall v_i\varphi \to \varphi$	Cor. 3.26
(3)	$\vdash (\neg\psi \to \forall v_i\varphi) \to (\neg\psi \to \varphi)$	(2), taut.
(4)	$\vdash (\neg\psi \to \varphi) \to (\varphi \vee \psi)$	taut.
(5)	$\vdash \forall v_i(\neg\psi \to \varphi) \to \forall v_i(\varphi \vee \psi)$	(4), gen., (L2)
(6)	$\vdash \forall v_i(\neg\psi \to \forall v_i\varphi) \to \forall v_i(\neg\psi \to \varphi)$	(3), gen., (L2)
(7)	$(\neg\psi \to \forall v_i\varphi) \to \forall v_i(\neg\psi \to \forall v_i\varphi)$	Prop. 3.27
(8)	$\vdash \forall v_i\varphi \vee \psi \to \forall v_i(\varphi \vee \psi)$	(1), (7), (6), (5)
(9)	$\vdash \varphi \vee \psi \to (\neg\psi \to \varphi)$	taut.
(10)	$\vdash \forall v_i(\varphi \vee \psi) \to \forall v_i(\neg\psi \to \varphi)$	(9), gen., (L2)
(11)	$\forall v_i(\neg\psi \to \varphi) \to (\neg\psi \to \forall v_i\varphi)$	Prop. 3.36
(12)	$\vdash (\neg\psi \to \forall v_i\varphi) \to \forall v_i\varphi \vee \psi$	taut.

The desired conclusion now follows from (8) and (10)–(12). □

Proposition 3.39. $\vdash \forall v_i(\varphi \wedge \psi) \leftrightarrow \forall v_i\varphi \wedge \forall v_i\psi$, *for any formulas φ, ψ.*

Proof.

$\vdash \forall v_i(\varphi \wedge \psi) \to \varphi \wedge \psi$	by Corollary 3.26
$\vdash \forall v_i(\varphi \wedge \psi) \to \varphi$	using a tautology
$\vdash \forall v_i\forall v_i(\varphi \wedge \psi) \to \forall v_i\varphi$	using (L2)
$\vdash \forall v_i(\varphi \wedge \psi) \to \forall v_i\varphi$	using Proposition 3.27
$\vdash \forall v_i(\varphi \wedge \psi) \to \forall v_i\psi$	similarly

$$
\begin{array}{lll}
(1) & \vdash \forall v_i(\varphi \wedge \psi) \to \forall v_i\varphi \wedge \forall v_i\psi & \text{a tautology} \\
& \vdash \forall v_i\varphi \to \varphi & \text{by Corollary 3.26} \\
& \vdash \forall v_i\psi \to \psi & \text{by Corollary 3.26} \\
& \vdash \forall v_i\varphi \wedge \forall v_i\psi \to \varphi \wedge \psi & \text{by a tautology} \\
& \vdash \forall v_i(\forall v_i\varphi \wedge \forall v_i\psi) \to \forall v_i(\varphi \wedge \psi) & \text{using (L2)} \\
& \vdash \forall v_i\varphi \wedge \forall v_i\psi \to \forall v_i(\varphi \wedge \psi). & \text{using Proposition 3.27}
\end{array}
$$

Now the desired result follows using (1) and a tautology. □

Proposition 3.40. *If φ and ψ are formulas and v_i does not occur free in ψ, then $\vdash \exists v_i\varphi \wedge \psi \leftrightarrow \exists v_i(\varphi \wedge \psi)$.*

Proof.

$$
\begin{array}{ll}
\vdash \neg\exists v_i\varphi \vee \neg\psi \leftrightarrow \forall v_i\neg\varphi \vee \neg\psi & \text{by Prop. 3.30} \\
\vdash \forall v_i\neg\varphi \vee \neg\psi \leftrightarrow \forall v_i(\neg\varphi \vee \neg\psi) & \text{by Prop. 3.38} \\
\vdash (\neg\varphi \vee \neg\psi) \leftrightarrow \neg(\varphi \wedge \psi) & \text{taut.} \\
\vdash \forall v_i(\neg\varphi \vee \neg\psi) \leftrightarrow \forall v_i\neg(\varphi \wedge \psi) & \text{gen., (L2)} \\
\vdash \forall v_i\neg(\varphi \wedge \psi) \leftrightarrow \neg\exists v_i(\varphi \wedge \psi). & \text{taut.}
\end{array}
$$

From these facts we get $\vdash \neg\exists v_i\varphi \vee \neg\psi \leftrightarrow \neg\exists v_i(\varphi \wedge \psi)$. The proposition follows by a tautology. □

Proposition 3.41. *If $\vdash \varphi \leftrightarrow \psi$, then $\vdash \forall v_i\varphi \leftrightarrow \forall v_i\psi$.*

Proof. Exercise. □

Proposition 3.42. *If $\vdash \varphi \leftrightarrow \psi$, then $\vdash \exists v_i\varphi \leftrightarrow \exists v_i\psi$.*

Proof. Exercise. □

Proposition 3.43. $\vdash \exists v_i(\varphi \vee \psi) \leftrightarrow \exists v_i\varphi \vee \exists v_i\psi$ *for any formulas φ, ψ.*

Proof.

$$
\begin{array}{ll}
\vdash \neg(\varphi \vee \psi) \leftrightarrow \neg\varphi \wedge \neg\psi & \text{a tautology} \\
\vdash \forall v_i\neg(\varphi \vee \psi) \leftrightarrow \forall v_i(\neg\varphi \wedge \neg\psi) & \text{by Proposition 3.41} \\
\vdash \forall v_i(\neg\varphi \wedge \neg\psi) \leftrightarrow \forall v_i\neg\varphi \wedge \forall v_i\neg\psi & \text{by Proposition 3.39} \\
\vdash \neg\forall v_i\neg(\varphi \vee \psi) \leftrightarrow \neg\forall v_i\neg\varphi \vee \neg\forall v_i\neg\psi; & \text{a tautology}
\end{array}
$$

this gives the desired result. □

E3.1. Do the case $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$ for some m-ary relation symbol and terms $\sigma_0, \ldots, \sigma_{m-1}$ in the proof of Theorem 3.1, (L3).

E3.2. Prove that (L6) is universally valid, in the proof of Theorem 3.1.

E3.3. Prove that (L8) is universally valid, in the proof of Theorem 3.1.

E3.4. Finish the proof of Proposition 3.9.

E3.5. Finish the proof of Proposition 3.11.

E3.6. Indicate which occurrences of the variables are bound and which ones free for the following formulas.

$\exists v_0(v_0 < v_1) \wedge \forall v_1(v_0 = v_1)$.
$v_4 + v_2 = v_0 \wedge \forall v_3(v_0 = v_1)$.
$\exists v_2(v_4 + v_2 = v_0)$.

E3.7. Prove Proposition 3.14.

E3.8. Indicate all free and bound occurrences of terms in the formula $v_0 = v_1 + v_1 \rightarrow \exists v_2(v_0 + v_2 = v_1)$.

E3.9. Prove Proposition 3.17.

E3.10. Show that the condition in Lemma 3.15 that the resulting occurrence of τ is free is necessary. Hint: use Theorem 3.2; describe a specific formula of the type in Proposition 3.15, but with τ not free, such that the formula is not universally valid.

E3.11. Do the case of implication in the proof of Lemma 3.15.

E3.12. Prove that the hypothesis of Theorem 3.25 is necessary.

E3.13. Prove Proposition 3.29.

E3.14. Prove Proposition 3.30.

E3.15. Prove Proposition 3.31.

E3.16. Prove Proposition 3.33.

E3.17. Prove Proposition 3.34.

E3.18. Prove Proposition 3.41.

E3.19. Prove Proposition 3.42.

E3.20. Prove that
$$\vdash \forall v_0 \forall v_1(v_0 = v_1) \rightarrow \forall v_0(v_0 = v_1 \vee v_0 = v_2).$$

E3.21. Prove that

$$\vdash \exists v_0(\neg v_0 = v_1 \wedge \neg v_0 = v_2) \rightarrow \exists v_0 \exists v_1(\neg v_0 = v_1).$$

4. The completeness theorem

The completeness theorem, in its simplest form, says that for any formula φ, $\vdash \varphi$ iff $\models \varphi$. We already know the direction $\Rightarrow$, in Theorem 3.2.

A more general form of the completeness theorem is that $\Gamma \vdash \varphi$ iff $\Gamma \models \varphi$, for any set $\Gamma \cup \{\varphi\}$ of formulas. Again the direction $\Rightarrow$ is given in Theorem 3.2.

Basic for the proof of the completeness theorem is the notion of consistency. A set Γ of formulas is *consistent* iff there is a formula φ such that $\Gamma \nvdash \varphi$.

Lemma 4.1. *For any set Γ of formulas the following conditions are equivalent:*

(i) Γ is inconsistent.

(ii) There is a formula φ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$.

(iii) $\Gamma \vdash \neg(v_0 = v_0)$.

Proof. (i)⇒(ii): Assume (i). Since $\Gamma \vdash \psi$ for every formula ψ, (ii) is clear.

(ii)⇒(iii): Assume (ii). Then the following is a Γ-proof:

A Γ-proof of φ.
A Γ-proof of $\neg\varphi$.
A $\emptyset$-proof of $\varphi \to (\neg\varphi \to \neg(v_0 = v_0)$. (This is a tautology; see Lemma 3.3.)
$\neg\varphi \to \neg(v_0 = v_0)$.
$\neg(v_0 = v_0)$.

(iii)⇒(i): By (iii) we have $\Gamma \vdash \neg(v_0 = v_0)$, while by Proposition 3.4 we have $\Gamma \vdash v_0 = v_0$. Then for any formula φ, the following is a Γ-proof of φ:

A $\emptyset$-proof of $v_0 = v_0$
A Γ-proof of $\neg(v_0 = v_0)$
A $\emptyset$-proof of $v_0 = v_0 \to (\neg(v_0 = v_0) \to \varphi)$. (This is a tautology; see Lemma 3.3.)
$\neg(v_0 = v_0) \to \varphi$
φ. □

A *sentence* is a formula which has no variable occurring free in it. A set Γ of sentences *has a model* iff there is a structure $\overline{A}$ for the language in question such that $\overline{A} \models \varphi[a]$ for every $\varphi \in \Gamma$ and every $a : \omega \to A$.

The following first-order version of the deduction theorem, Theorem 1.8, will be useful.

Theorem 4.2. (First-order deduction theorem) *If $\Gamma \cup \{\psi\}$ is a set of formulas, φ is a sentence, and $\Gamma \cup \{\varphi\} \vdash \psi$, then $\Gamma \vdash \varphi \to \psi$.*

Proof. Let $\langle \chi_0, \ldots, \chi_{m-1} \rangle$ be a $(\Gamma \cup \{\varphi\})$-proof with $\chi_i = \psi$ for some $i < m$. We modify this proof, replacing each χ_j by one or more formulas, converting the proof to a Γ-proof, in such a way that $\varphi \to \chi_j$ is in the new proof for every $j < m$. If χ_j is a logical axiom or a member of Γ, we replace it by the three formulas

$$\chi_j \to (\varphi \to \chi_j)$$
$$\chi_j$$
$$\varphi \to \chi_j.$$

If χ_j is φ, we replace it by the five formulas giving a little proof of $\varphi \to \varphi$; see Lemma 1.7. If there exist $k, l < j$ such that χ_k is $\chi_l \to \chi_j$, we replace χ_j by the formulas

$$\begin{aligned}
&(\varphi \to \chi_k) \to [(\varphi \to \chi_l) \to (\varphi \to \chi_j)] \\
&(\varphi \to \chi_l) \to (\varphi \to \chi_j) \\
&\varphi \to \chi_j.
\end{aligned}$$

If there exist $k < j$ and $l \in \omega$ such that χ_j is $\forall v_l \chi_k$, we replace χ_j by the formulas

$\forall v_l(\varphi \to \chi_k)$

a proof of $\forall v_l(\varphi \to \chi_k) \to (\varphi \to \forall v_l \chi_k)$ see Proposition 3.36

$\varphi \to \chi_j$. □

Theorem 4.3. *Suppose that every consistent set of sentences has a model. Then* $\Gamma \vdash \varphi$ *iff* $\Gamma \models \varphi$, *for every set* $\Gamma \cup \{\varphi\}$ *of formulas.*

Proof. Assume that every consistent set of sentences has a model. Note again that $\Gamma \vdash \varphi$ implies that $\Gamma \models \varphi$, by Theorem 3.2. We prove the converse by proving its contrapositive. Thus suppose that $\Gamma \cup \{\varphi\}$ is a set of formulas such that $\Gamma \nvdash \varphi$. We want to show that $\Gamma \not\models \varphi$, i.e., there is a model of Γ which is not a model of φ. For any formula ψ, let $[\![\psi]\!]$ be the *closure* of ψ, i.e., the sentence

$$\forall v_{i(0)} \ldots \forall v_{i(m-1)} \psi,$$

where $i(0) < \cdots < i(m-1)$ lists all the integers j such that v_j occurs free in ψ. Let $\Gamma' = \{[\![\psi]\!] : \psi \in \Gamma\}$. We claim that $\Gamma' \cup \{\neg[\![\varphi]\!]\}$ is consistent. Suppose not. Then $\Gamma' \cup \{\neg[\![\varphi]\!]\} \vdash \neg(v_0 = v_0)$. Hence by the deduction theorem, $\Gamma' \vdash \neg[\![\varphi]\!] \to \neg(v_0 = v_0)$, so $\Gamma' \vdash v_0 = v_0 \to [\![\varphi]\!]$. Hence, using Proposition 3.4, $\Gamma' \vdash [\![\varphi]\!]$. Now in a Γ'-proof that has $[\![\varphi]\!]$ as a member, replace each formula

$$\forall v_{i(0)} \ldots \forall v_{i(m-1)} \psi,$$

with $\psi \in \Gamma$, by the sequence

$$\begin{aligned}
&\psi \\
&\forall v_{i(m-1)} \psi \\
&\ldots\ldots\ldots \\
&\forall v_{i(0)} \ldots \forall v_{i(m-1)} \psi.
\end{aligned}$$

This converts the proof into a Γ-proof one of whose members is $[\![\varphi]\!]$. Thus $\Gamma \vdash [\![\varphi]\!]$. Using Corollary 3.26, it follows that $\Gamma \vdash \varphi$, contradiction.

Hence $\Gamma' \cup \{\neg[\![\varphi]\!]\}$ is consistent. Since this is a set of sentences, by supposition it has a model $\overline{M}$. Clearly $\overline{M}$ is a model of Γ. Since $\overline{M}$ is a model of $\neg[\![\varphi]\!]$, clearly there is an $a \in {}^{\omega}M$ such that $\overline{M} \models \neg\varphi[a]$. Thus $\overline{M}$ is not a model of φ. This shows that $\Gamma \not\models \varphi$. □

To prove that every consistent set of sentences has a model, we need several lemmas, starting with some additional facts about structures and satisfaction.

Lemma 4.4. *Suppose that $\overline{A}$ is a structure, a and b map ω into A, φ is a formula, and $a_i = b_i$ for every i such that v_i occurs free in φ. Then $\overline{A} \models \varphi[a]$ iff $\overline{A} \models \varphi[b]$.*

Proof. Induction on φ. For φ an atomic equality formula $\sigma = \tau$, the hypothesis means that $a_i = b_i$ for all i such that v_i occurs in σ or τ. Hence, using Proposition 2.4,

$$\overline{A} \models \varphi[a] \text{ iff } \sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a) \text{ iff } \sigma^{\overline{A}}(b) = \tau^{\overline{A}}(b) \text{ iff } \overline{A} \models \varphi[b].$$

For φ an atomic non-equality formula $\mathbf{R}\eta_0 \ldots \eta_{m-1}$, the hypothesis means that $a_i = b_i$ for all i such that v_i occurs in one of the terms η_j. Hence, again using Proposition 2.4,

$$\begin{aligned}
\overline{A} \models \varphi[a] \quad &\text{iff} \quad \langle \eta_0^{\overline{A}}(a), \ldots, \eta_{m-1}^{\overline{A}}(a) \rangle \in \mathbf{R}^{\overline{A}} \\
&\text{iff} \quad \langle \eta_0^{\overline{A}}(b), \ldots, \eta_{m-1}^{\overline{A}}(b) \rangle \in \mathbf{R}^{\overline{A}} \\
&\text{iff} \quad \overline{A} \models \varphi[b].
\end{aligned}$$

Assume inductively that φ is $\neg\psi$. The hypothesis implies that $a_i = b_i$ for all i such that v_i occurs free in ψ. Hence

$$\begin{aligned}
\overline{A} \models \varphi[a] \quad &\text{iff} \quad \text{not}(\overline{A} \models \psi[a]) \\
&\text{iff} \quad \text{not}(\overline{A} \models \psi[b]) \quad \text{(induction hypothesis)} \\
&\text{iff} \quad \overline{A} \models \varphi[b].
\end{aligned}$$

Assume inductively that φ is $\psi \to \chi$. The hypothesis implies that $a_i = b_i$ for all i such that v_i occurs free in ψ or in χ. Hence

$$\begin{aligned}
\overline{A} \models \varphi[a] \quad &\text{iff} \quad \text{not}(\overline{A} \models \psi[a]) \text{ or } \overline{A} \models \chi[a] \\
&\text{iff} \quad \text{not}(\overline{A} \models \psi[b]) \text{ or } \overline{A} \models \chi[b] \quad \text{(induction hypothesis)} \\
&\text{iff} \quad \overline{A} \models \varphi[b].
\end{aligned}$$

Now assume inductively that φ is $\forall v_k \psi$. By symmetry it suffices to show that $\overline{A} \models \varphi[a]$ implies that $\overline{A} \models \varphi[b]$. So, assume that $\overline{A} \models \varphi[a]$. Take any $u \in A$. Then $\overline{A} \models \psi[a^k_u]$. We claim that $(a^k_u)_i = (b^k_u)_i$ for every i such that v_i occurs free in ψ. If $i \neq k$ this is true since v_i also occurs free in φ, so that $a_i = b_i$; and $(a^k_u)_i = a_i = b_i = (b^k_u)_i$. If $i = k$, then $(a^k_u)_i = u = (b^k_u)_i$. It follows now by the inductive hypothesis that $\overline{A} \models \psi[b^k_u]$. Since u is arbitrary, $\overline{A} \models \varphi[b]$. □

As in the case of terms (see Proposition 2.4 and the comments after it), Lemma 4.4 enables us to simplify the notation $\overline{A} \models \varphi[a]$. Instead of a full assignment $a : \omega \to A$, it suffices to take a function $a : \{0, \ldots, m\} \to A$ such that every variable v_i occurring free in φ is such that $i \leq m$. Then $\overline{A} \models \varphi[a]$ means that $\overline{A} \models \varphi[b]$ for any b (or some b) such that b extends a. If φ is a sentence, thus with no free variables, then $\overline{A} \models \varphi$ means that $\overline{A} \models \varphi[b]$ for any, or some, $b : \omega \to A$.

Lemma 4.5. *Suppose that τ, ρ, and ν are terms, and ρ is obtained from τ by replacing all occurrences of v_i in τ by ν. Then for any structure $\overline{A}$ and any assignment $a : \omega \to A$, $\rho^{\overline{A}}(a) = \tau^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right)$.*

Proof. By induction on τ. If τ is v_k with $k \neq i$, then ρ is the same as τ, and both sides of the above equation are equal to a_k. If τ is v_i, then ρ is ν, and $\rho^{\overline{A}}(a) = \nu^{\overline{A}}(a) = v_i^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right) = \tau^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right)$. If τ is an individual constant $\mathbf{k}$, then ρ is equal to τ, and both sides of the equation in the lemma are equal to $\mathbf{k}^{\overline{A}}$.

Now suppose inductively that τ is $\mathbf{F}\eta_0 \ldots \eta_{m-1}$. Let μ_i be obtained from η_i by replacing all occurrences of v_i by ν. Then

$$\begin{aligned}
\rho^{\overline{A}}(a) &= (\mathbf{F}\mu_0 \ldots \mu_{m-1})^{\overline{A}}(a) \\
&= \mathbf{F}^{\overline{A}}(\mu_0^{\overline{A}}(a), \ldots, \mu_{m-1}^{\overline{A}}(a)) \\
&= \mathbf{F}^{\overline{A}}\left(\eta_0\left(a^i_{\nu^{\overline{A}}(a)}\right), \ldots, \eta_{m-1}\left(a^i_{\nu^{\overline{A}}(a)}\right)\right) \\
&= (\mathbf{F}\eta_0 \ldots \eta_{m-1})\left[a^i_{\nu^{\overline{A}}(a)}\right] \\
&= \tau^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right).
\end{aligned}$$

□

Lemma 4.6. *Suppose that φ is a formula, ν is a term, no free occurrence of v_i in φ is within a subformula of the form $\forall v_k \mu$ with v_k a variable occurring in ν, and $\overline{A}$ is a structure. Then $\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\varphi[a]$ iff $\overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right]$.*

Proof. By induction on φ. For φ a formula $\sigma = \tau$, let ρ and η be obtained from σ and τ by replacing all occurrences of v_i by ν. Then by Lemma 4.5,

$$\begin{aligned}
\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\varphi[a] \quad &\text{iff} \quad \overline{A} \models (\rho = \eta)[a] \\
&\text{iff} \quad \rho^{\overline{A}}(a) = \eta^{\overline{A}}(a) \\
&\text{iff} \quad \sigma^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right) = \tau^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right) \\
&\text{iff} \quad \overline{A} \models (\sigma = \tau)\left(a^i_{\nu^{\overline{A}}(a)}\right) \\
&\text{iff} \quad \overline{A} \models \varphi\left(a^i_{\nu^{\overline{A}}(a)}\right).
\end{aligned}$$

For φ a formula $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$, let η_i be obtained from σ_i by replacing all occurrences of v_i by ν. Then

$$\begin{aligned}
\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\varphi[a] \quad &\text{iff} \quad \overline{A} \models (\mathbf{R}\eta_0 \ldots \eta_{m-1})[a] \\
&\text{iff} \quad \langle \eta_0^{\overline{A}}(a), \ldots, \eta_{m-1}^{\overline{A}}(a) \in \mathbf{R}^{\overline{A}} \\
&\text{iff} \quad \left\langle \sigma_0^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right), \ldots \sigma_{m-1}^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right)\right\rangle \in \mathbf{R}^{\overline{A}} \\
&\text{iff} \quad \overline{A} \models (\mathbf{R}\sigma_0 \ldots \sigma_{m-1})\left[a^i_{\nu^{\overline{A}}(a)}\right] \\
&\text{iff} \quad \overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right].
\end{aligned}$$

Now suppose inductively that φ is $\neg\psi$. Then

$$\begin{aligned}\overline{A} \models \mathrm{Subf}^{v_1}_{\nu}\varphi[a] \quad &\text{iff} \quad \overline{A} \models (\neg\mathrm{Subf}^{v_1}_{\nu}\psi)\,[a]\\ &\text{iff} \quad \text{not}\left(\overline{A} \models (\mathrm{Subf}^{v_1}_{\nu}\psi)\right)[a]\\ &\text{iff} \quad \text{not}\left(\overline{A} \models \psi\left[a^i_{\nu^{\overline{A}}(a)}\right]\right)\\ &\text{iff} \quad \overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right].\end{aligned}$$

Suppose inductively that φ is $\psi \to \chi$. Then

$$\begin{aligned}\overline{A} \models \mathrm{Subf}^{v_1}_{\nu}\varphi[a] \quad &\text{iff} \quad \text{not}\left(\overline{A} \models \mathrm{Subf}^{v_1}_{\nu}\psi[a]\right) \text{ or } \overline{A} \models \mathrm{Subf}^{v_1}_{\nu}\chi[a]\\ &\text{iff} \quad \text{not}\left(\overline{A} \models \psi\left[a^i_{\nu^{\overline{A}}(a)}\right]\right) \text{ or } \overline{A} \models \chi\left[a^i_{\nu^{\overline{A}}(a)}\right]\\ &\text{iff} \quad \overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right].\end{aligned}$$

Finally, suppose inductively that φ is $\forall v_k\psi$. Now if v_i does not occur free in φ, then $\mathrm{Subf}^{v_i}_{\nu}\varphi$ is just φ itself, and $\overline{A} \models \varphi[a]$ iff $\overline{A} \models \varphi[a^i_{\nu^{\overline{A}}(a)}]$ by Lemma 4.4. Hence we may assume that v_i occurs free in φ.

If $k = i$, then $\mathrm{Subf}^{v_i}_{\nu}\varphi$ is φ, and by Lemma 4.4, $\overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right]$ iff $\overline{A} \models \varphi[a]$; so the theorem holds in this case. Now suppose that $k \neq i$. Then $\mathrm{Subf}^{v_i}_{\nu}\varphi$ is $\forall v_k\mathrm{Subf}^{v_i}_{\nu}\psi$. Suppose that $\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\varphi[a]$. Take any $u \in A$. Then $\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\psi[a^k_u]$. Now no free occurrence of v_i in ψ is within a subformula of the form $\forall v_s\mu$ with v_s occurring in ν. Hence by the inductive hypothesis $\overline{A} \models \psi\left[(a^k_u)^i_{\nu^{\overline{A}}(a^k_u)}\right]$. Now since φ is $\forall v_k\psi$ and v_i occurs free in φ, the assumption of the lemma says that v_k does not occur in ν. Hence $\nu^{\overline{A}}(a) = \nu^{\overline{A}}(a^k_u)$ by Proposition 2.4. Hence $\overline{A} \models \psi\left[(a^k_u)^i_{\nu^{\overline{A}}(a)}\right]$. Since $(a^k_u)^i_{\nu^{\overline{A}}(a)} = \left(a^i_{\nu^{\overline{A}}(a)}\right)^k_u$, it follows that $\overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right]$.

Conversely, suppose that $\overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right]$. Take any $u \in A$. Then $\overline{A} \models \psi\left[\left(a^i_{\nu^{\overline{A}}(a)}\right)^k_u\right]$. Since $\left(a^i_{\nu^{\overline{A}}(a)}\right)^k_u = (a^k_u)^i_{\nu^{\overline{A}}(a)}$, and $\nu^{\overline{A}}(a) = \nu^{\overline{A}}(a^k_u)$ (see above), by the inductive hypothesis we get $\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\psi[a^k_u]$. It follows that $\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\varphi[a]$. □

A set Γ of sentences is *complete* iff for every sentence φ, $\Gamma \vdash \varphi$ or $\Gamma \vdash \neg\varphi$. Γ is *rich* iff for every sentence of the form $\exists v_i\varphi$ there is an individual constant $\mathbf{c}$ such that $\Gamma \vdash \exists v_i\varphi \to \mathrm{Subf}^{v_i}_{\mathbf{c}}(\varphi)$.

The main lemma for the completeness proof is as follows.

Lemma 4.7. *If Γ is a complete, rich, consistent set of sentences, then Γ has a model.*

Proof. Let $B = \{\sigma : \sigma \text{ is a term in which no variable occurs}\}$. We define $\equiv$ to be the set

$$\{(\sigma, \tau) : \sigma, \tau \in B \text{ and } \Gamma \vdash \sigma = \tau\}.$$

By Propositions 3.4–3.6, $\equiv$ is an equivalence relation on B. Let π be the function which assigns to each $\sigma \in B$ the equivalence class $[\sigma]_\equiv$, and let A be the set of all equivalence classes.

We recall some basic facts about equivalence relations. An *equivalence relation* on a set M is a set R of ordered pairs (a,b) with $a,b \in M$ satisfying the following conditions:

(reflexivity) $(a,a) \in R$ for all $a \in M$.

(symmetry) For all $(a,b) \in R$ we have $(b,a) \in R$.

(transitivity) For all a,b,c, if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$.

Given an equivalence relation R on a set M, for each $a \in M$ we let $[a]_R = \{b \in M : (a,b) \in R\}$; this is the *equivalence class* of a. Some basic facts:

(a) For any $a,b \in M$, $(a,b) \in R$ iff $[a]_R = [b]_R$.

Proof. $\Rightarrow$: suppose that $(a,b) \in R$. Suppose also that $x \in [a]_R$. Thus $(a,x) \in R$. Since R is symmetric, $(b,a) \in R$. Since R is transitive, $(b,x) \in R$. Hence $x \in [b]_R$. This proves that $[a]_R \subseteq [b]_R$. Suppose that $x \in [b]_R$. Thus $(b,x) \in R$. Since also $(a,b) \in R$, by transitivity we get $(a,x) \in R$. So $x \in [a]_R$. This proves that $[b]_R \subseteq [a]_R$, and completes the proof that $[a]_R = [b]_R$.

$\Leftarrow$: Assume that $[a]_R = [b]_R$. Since R is reflexive on M, we have $(b,b) \in R$, and hence $b \in [b]_R$. Now $[a]_R = [b]_R$, so $b \in [a]_R$. Hence $(a,b) \in R$. $\square$

(b) For any $a,b \in M$, $[a]_R = [b]_R$ or $[a]_R \cap [b]_R = \emptyset$.

Proof. Suppose that $[a]_R \cap [b]_R \neq \emptyset$; say $x \in [a]_R \cap [b]_R$. Thus $(a,x) \in R$ and $(b,x) \in R$. By symmetry, $(x,b) \in R$. By transitivity, $(a,b) \in R$. By (a), $[a]_R = [b]_R$. $\square$

We are now going to define a structure with universe A. If $\mathbf{k}$ is an individual constant, let $\mathbf{k}^{\overline{A}} = [\mathbf{k}]_\equiv$.

(1) If $\mathbf{F}$ is an m-ary function symbol and $\sigma_0, \ldots, \sigma_{m-1}, \tau_0, \ldots, \tau_{m-1}$ are members of B such that $\sigma_i \equiv \tau_i$ for all $i < m$, then $\mathbf{F}\sigma_0 \ldots \sigma_{m-1} \equiv \mathbf{F}\tau_0 \ldots \tau_{m-1}$.

In fact, the hypothesis implies that $\Gamma \vdash \sigma_i = \tau_i$ for all $i < m$. Now we claim

(2) $\mathbf{F}\sigma_0 \ldots \sigma_{m-1} \equiv \mathbf{F}\sigma_0 \ldots \sigma_{m-i}\tau_{m-i+1} \ldots \tau_{m-1}$ for every positive integer $i \leq m+1$.

We prove (2) by induction on i. For $i = 1$ the statement is $\mathbf{F}\sigma_0 \ldots \sigma_{m-1} \equiv \mathbf{F}\sigma_0 \ldots \sigma_{m-1}$, which holds by Proposition 3.4. Now assume that $1 \leq i \leq m$ and $\mathbf{F}\sigma_0 \ldots \sigma_{m-1} \equiv \mathbf{F}\sigma_0 \ldots \sigma_{m-i}\tau_{m-i+1} \ldots \tau_{m-1}$. By logical axiom (L7) we also have

$$\mathbf{F}\sigma_0 \ldots \sigma_{m-i}\tau_{m-i+1} \ldots \tau_{m-1} \equiv \mathbf{F}\sigma_0 \ldots \sigma_{m-i-1}\tau_{m-i} \ldots \tau_{m-1},$$

so Proposition 3.6 yields

$$\mathbf{F}\sigma_0 \ldots \sigma_{m-1} \equiv \mathbf{F}\sigma_0 \ldots \sigma_{m-i-1}\tau_{m-i} \ldots \tau_{m-1}.$$

This finishes the inductive proof of (2). The case $i = m+1$ in (2) gives (1).

(3) If $\mathbf{F}$ is an m-ary function symbol, then there is a function $\mathbf{F}^{\overline{A}}$ mapping m-tuples of members of A into A, such that for any $\sigma_0, \ldots, \sigma_{m-1} \in B$, $\mathbf{F}^{\overline{A}}([\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv) = [\mathbf{F}\sigma_0 \ldots \sigma_{m-1}]_\equiv$.

In fact, we can define $\mathbf{F}^{\overline{A}}$ as a set of ordered pairs:

$$\begin{aligned}\mathbf{F}^{\overline{A}} = \{(x,y) :&\text{there are } \sigma_0, \ldots, \sigma_{m-1} \in B \text{ such that}\\ &x = \langle [\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv\rangle \text{ and } y = [\mathbf{F}\sigma_0 \ldots \sigma_{m-1}]_\equiv\}\end{aligned}$$

Then $\mathbf{F}^{\overline{A}}$ is a function. For, suppose that $(x,y),(x,z) \in \mathbf{F}^{\overline{A}}$. Accordingly choose elements $\sigma_0, \ldots \sigma_{m-1} \in B$ and $\tau_0, \ldots \tau_{m-1} \in B$ such that $x = \langle [\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv\rangle = \langle [\tau_0]_\equiv, \ldots, [\tau_{m-1}]_\equiv$, $y = [\mathbf{F}\sigma_0 \ldots \sigma_{m-1}]_\equiv$, and $z = [\mathbf{F}\tau_0 \ldots \varphi_{m-1}]_\equiv$. Thus for any $i < m$ we have $[\sigma_i]_\equiv = [\tau_i]_\equiv$, hence $\sigma_i \equiv \tau_i$. From (1) it then follows that $\mathbf{F}\sigma_0 \ldots \sigma_{m-1} \equiv \mathbf{F}\tau_0 \ldots \varphi_{m-1}$, hence $y = z$. So $\mathbf{F}^{\overline{A}}$ is a function. Clearly then (3) holds.

For $\mathbf{R}$ an m-ary relation symbol we define

$$\mathbf{R}^{\overline{A}} = \{x : \exists\sigma_0, \ldots \sigma_{m-1} \in B[x = \langle [\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv\rangle \text{ and } \Gamma \vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-1}]\}.$$

(4) If $\mathbf{R}$ is an m-ary relation symbol and $\sigma_i \equiv \tau_i$ for all $i < m$, then for any positive integer $i < m+1$, $\vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-1} \leftrightarrow \mathbf{R}\sigma_0 \ldots \sigma_{m-i}\tau_{m-i+1} \ldots \tau_{m-1}$.

We prove (4) by induction on i. For $i = 1$ the conclusion is $\vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-1} \leftrightarrow \mathbf{R}\sigma_0 \ldots \sigma_{m-1}$, so this holds by a tautology. Now assume our statement for $i < m$. Then by logical axiom (L8),

$$\vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-i}\tau_{m-i+1} \ldots \tau_{m-1} \rightarrow \mathbf{R}\sigma_0 \ldots \sigma_{m-i-1}\tau_{m-i} \ldots \tau_{m-1};$$

using Proposition 3.5 we can easily get

$$\vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-i}\tau_{m-i+1} \ldots \tau_{m-1} \leftrightarrow \mathbf{R}\sigma_0 \ldots \sigma_{m-i-1}\tau_{m-i} \ldots \tau_{m-1}.$$

This finishes the inductive proof of (4). Now we have

(5) If $\mathbf{R}$ is an m-ary relation symbol and $\sigma_0, \ldots, \sigma_{m-1} \in B$, then $\langle [\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv\rangle \in \mathbf{R}^{\overline{A}}$ iff $\Gamma \vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-1}$.

In fact, $\Leftarrow$ follows from the definition. Now suppose that $\langle [\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv\rangle \in \mathbf{R}^{\overline{A}}$. Then by definition there exist $\tau_0, \ldots, \tau_{m-1} \in B$ such that

$$\langle [\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv\rangle = \langle [\tau_0]_\equiv, \ldots, [\tau_{m-1}]_\equiv\rangle \text{ and } \Gamma \vdash \mathbf{R}\tau_0 \ldots \tau_{m-1}.$$

Thus $[\sigma_i]_\equiv = [\tau_i]_\equiv$, hence $\sigma_i \equiv \tau_i$, hence $\Gamma \vdash \sigma_i = \tau_i$, for each $i < m$. Now by (4), $\vdash \bigwedge_{i<m}(\sigma_i = \tau_i) \rightarrow (\mathbf{R}\sigma_0 \ldots \sigma_{m-1} \leftrightarrow \mathbf{R}\tau_0 \ldots \tau_{m-1})$. It follows that $\Gamma \vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-1}$, as desired; so (5) holds.

(6) For any $\sigma \in B$ we have $\sigma^{\overline{A}} = [\sigma]_\equiv$.

We prove (6) by induction on σ. If σ is an individual constant $\mathbf{k}$, then by definition $\mathbf{k}^{\overline{A}} = [\mathbf{k}]_{\equiv}$. Now suppose that (6) is true for $\tau_0, \ldots, \tau_{m-1} \in B$ and σ is $\mathbf{F}\tau_0 \ldots \tau_{m-1}$. Then

$$\sigma^{\overline{A}} = \mathbf{F}^{\overline{A}}([\tau_0]_{\equiv}, \ldots, [\tau_{m-1}]_{\equiv}) = [\mathbf{F}\tau_0 \ldots \tau_{m-1}]_{\equiv} = [\sigma]_{\equiv},$$

proving (6).

The following claim is the heart of the proof.

(7) For any sentence φ, $\Gamma \vdash \varphi$ iff $\overline{A} \models \varphi$.

We prove (7) by induction on the number m of the symbols $=$, relation symbols, $\neg$, $\rightarrow$, and $\forall$ in φ. For $m = 1$, φ is atomic, and we have

$$\begin{array}{rll}
\Gamma \vdash \sigma = \tau & \text{iff} & \sigma \equiv \tau \\
& \text{iff} & [\sigma]_{\equiv} = [\tau]_{\equiv} \\
& \text{iff} & \sigma^{\overline{A}} = \tau^{\overline{A}} \quad \text{by (6)} \\
& \text{iff} & \overline{A} \models \sigma = \tau; \\
\Gamma \vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-1} & \text{iff} & \langle [\sigma_0]_{\equiv}, \ldots, [\sigma_{m-1}]_{\equiv} \rangle \in \mathbf{R}^{\overline{A}} \quad \text{by (5)} \\
& \text{iff} & \langle \sigma_0^{\overline{A}}, \ldots, \sigma_{m-1}^{\overline{A}} \rangle \in \mathbf{R}^{\overline{A}} \quad \text{by (6)} \\
& \text{iff} & \overline{A} \models \mathbf{R}\sigma_0 \ldots \sigma_{m-1}.
\end{array}$$

Now we take the inductive steps.

$$\begin{array}{rll}
\Gamma \vdash \neg\psi & \text{iff} & \text{not}(\Gamma \vdash \psi) \\
& \text{iff} & \text{not}(\overline{A} \models \psi) \\
& \text{iff} & \overline{A} \models \neg\psi; \\
\Gamma \vdash \psi \rightarrow \chi & \text{iff} & \text{not}(\Gamma \vdash \psi) \text{ or } \Gamma \vdash \chi \\
& \text{iff} & \text{not}(\overline{A} \models \psi) \text{ or } \overline{A} \models \chi \\
& \text{iff} & \overline{A} \models \psi \rightarrow \chi.
\end{array}$$

Finally, suppose that φ is $\forall v_i \psi$. First suppose that $\Gamma \vdash \varphi$. We want to show that $\overline{A} \models \varphi$, so take any $\sigma \in B$ and let $u = [\sigma]_{\equiv}$; we want to show that $\overline{A} \models \psi[a^i_u]$, where $a : \omega \rightarrow A$. Let χ be the sentence $\mathrm{Subf}^{v_i}_{\sigma}\psi$. Then by Theorem 3.25 we have $\Gamma \vdash \chi$, and hence by the inductive assumption $\overline{A} \models \chi$. By (6) we have $\sigma^{\overline{A}} = [\sigma]_{\equiv}$. Hence by Lemma 4.6 we get $\overline{A} \models \psi[a^i_u]$.

Second suppose that $\Gamma \not\vdash \varphi$. Then by completeness $\Gamma \vdash \neg\varphi$, and hence $\Gamma \vdash \exists v_i \neg\psi$. Hence by richness there is an individual constant $\mathbf{c}$ such that $\Gamma \vdash \exists v_i \neg\psi \rightarrow \mathrm{Subf}^{v_i}_{\mathbf{c}}(\neg\psi)$, hence $\Gamma \vdash \neg\mathrm{Subf}^{v_i}_{\mathbf{c}}\psi$, and so $\Gamma \not\vdash \mathrm{Subf}^{v_i}_{\mathbf{c}}\psi$. By the inductive assumption, $\overline{A} \not\models \mathrm{Subf}^{v_i}_{\mathbf{c}}\psi$, and so by (6) and Lemma 4.6, $\overline{A} \not\models \psi[a^i_u]$, where $a : \omega \rightarrow A$ and $u = [\mathbf{c}]_{\equiv}$. So $\overline{A} \not\models \varphi$.

This finishes the proof of (7). Applying (7) to members φ of Γ we see that $\overline{A}$ is a model of Γ. $\square$

The following rather technical lemma will be used in a few places below.

Lemma 4.8. *Suppose that Γ is a set of formulas in $\mathscr{L}$, and $\langle\psi_0,\ldots,\psi_{m-1}\rangle$ is a Γ-proof in $\mathscr{L}$. Suppose that C is a set of individual constants such that no member of C occurs in any member of Γ. Let v_j be a variable not occurring in any formula ψ_k, and for each k let ψ'_k be obtained from ψ_k by replacing each member of C by v_j. Similarly, for each term σ let σ' be obtained from σ by replacing each member of C by v_j. Then $\langle\psi'_0,\psi'_1,\ldots,\psi'_{m-1}\rangle$ is a Γ-proof in $\mathscr{L}$.*

Proof. Assume the hypotheses. We need to show that if ψ_k is a logical axiom, then so is ψ'_k. We consider the possibilities one by one:

$$\begin{array}{ll}
(\varphi\to(\psi\to\varphi))' \text{ is } \varphi'\to(\psi'\to\varphi'); & \\
((\varphi\to(\psi\to\chi)\to((\varphi\to\psi)\to(\varphi\to\chi)))' \text{ is} & \\
\quad (\varphi'\to(\psi'\to\chi')\to((\varphi'\to\psi')\to(\varphi'\to\chi')); & \\
((\neg\varphi\to\neg\psi)\to(\psi\to\varphi))' \text{ is } (\neg\varphi'\to\neg\psi')\to(\psi'\to\varphi'); & \\
(\forall v_k(\varphi\to\psi)\to(\forall v_k\varphi\to\forall v_k\psi))' \text{ is } \forall v_k(\varphi'\to\psi')\to(\forall v_k\varphi'\to\forall v_k\psi'); & \\
(\varphi\to\forall v_k\varphi)' \text{ is } \varphi'\to\forall v_k\varphi' & \text{if } v_k \text{ does not occur in } \varphi; \\
(\exists v_k(v_k=\sigma))' \text{ is } \exists v_k(v_k=\sigma') & \text{if } v_k \text{ does not occur in } \sigma; \\
(\sigma=\tau\to(\sigma=\rho\to\tau=\rho))' \text{ is } (\sigma'=\tau'\to(\sigma'=\rho'\to\tau'=\rho'); & \\
(\sigma=\tau\to(\rho=\sigma\to\rho=\tau))' \text{ is } (\sigma'=\tau'\to(\rho'=\sigma'\to\rho'=\tau'); & \\
(\sigma=\tau\to\mathbf{F}\xi_0\ldots\xi_{i-1}\sigma\xi_{i+1}\ldots\xi_{m-1}=\mathbf{F}\xi_0\ldots\xi_{i-1}\tau\xi_{i+1}\ldots\xi_{m-1})' \text{ is} & \\
\quad \sigma'=\tau'\to\mathbf{F}\xi'_0\ldots\xi'_{i-1}\sigma'\xi'_{i+1}\ldots\xi'_{m-1}=\mathbf{F}\xi'_0\ldots\xi'_{i-1}\tau'\xi'_{i+1}\ldots\xi'_{m-1}; & \\
(\sigma=\tau\to(\mathbf{R}\xi_0\ldots\xi_{i-1}\sigma\xi_{i+1}\ldots\xi_{m-1}\to\mathbf{R}\xi_0\ldots\xi_{i-1}\tau\xi_{i+1}\ldots\xi_{m-1}))' \text{ is} & \\
\quad \sigma'=\tau'\to(\mathbf{R}\xi'_0\ldots\xi'_{i-1}\sigma'\xi'_{i+1}\ldots\xi'_{m-1}\to\mathbf{R}\xi'_0\ldots\xi'_{i-1}\tau'\xi'_{i+1}\ldots\xi'_{m-1}). &
\end{array}$$

Now back to our claim that $\langle\psi'_0,\ldots,\psi'_{m-1}\rangle$ is a Γ-proof. If ψ_k is a logical axiom, then by the above, ψ'_k is a logical axiom. If $\psi_k\in\Gamma$, then no member of C occurs in ψ_k, and hence $\psi'_k=\psi_k$. Suppose that $s,t<k$ and ψ_s is $\psi_t\to\psi_k$. Then ψ'_s is $\psi'_t\to\psi'_k$. If $s<k$ and $t\in\omega$, and ψ_k is $\forall v_t\psi_s$, then ψ'_k is $\forall v_t\psi'_s$. Thus our claim holds. □

Lemma 4.9. *Suppose that* $\mathbf{c}$ *is an individual constant not occurring in any formula in $\Gamma\cup\{\varphi\}$. Suppose that $\Gamma\vdash \mathrm{Subf}^{v_i}_{\mathrm{c}}\varphi$. Then $\Gamma\vdash\varphi$.*

Proof. Let $\langle\psi_0,\ldots,\psi_{m-1}\rangle$ be a Γ-proof with $\psi_j=\mathrm{Subf}^{v_i}_{\mathbf{c}}\varphi$. Let v_j and the sequence $\langle\psi'_0,\ldots,\psi'_{m-1}\rangle$ be as in Lemma 4.8, with $C=\{\mathbf{c}\}$. Then by Lemma 4.8, $\langle\psi'_0,\ldots,\psi'_{m-1}\rangle$ is a Γ-proof. Note that ψ'_j is $\mathrm{Subf}^{v_i}_{v_j}\varphi$. Thus $\Gamma\vdash\mathrm{Subf}^{v_i}_{v_j}\varphi$. Hence $\Gamma\vdash\forall v_j\mathrm{Subf}^{v_i}_{v_j}\varphi$, and so by Theorem 3.25, $\Gamma\vdash\varphi$. □

A first-order language $\mathscr{L}$ is *finite* iff $\mathscr{L}$ has only finitely many non-logical symbols. Note that in a finite language there are infinitely many integers which are not symbols of the language. We prove the main completeness theorem only for finite languages. This is not an essential restriction. With an expanded notion of first-order language the present proof still goes through.

Lemma 4.10. *Let $\mathscr{L}$ be a finite first-order language. Let $\mathscr{L}'$ extend $\mathscr{L}$ by adding individual constants $\mathbf{c}_0, \mathbf{c}_1, \ldots$. Suppose that Γ is a consistent set of formulas in $\mathscr{L}$. Then it is also consistent as a set of formulas in $\mathscr{L}'$.*

Suppose not. Let $\langle \psi_0, \ldots, \psi_{m-1} \rangle$ be a Γ-proof in the $\mathscr{L}'$ sense with ψ_i the formula $\neg(v_0 = v_0)$. Let C be the set of all constants $\mathbf{c}_i$ which appear in some formula ψ_k. Let v_j and $\langle \psi'_0, \psi'_1, \ldots, \psi'_{m-1} \rangle$ be as in Lemma 4.8. Then by Lemma 4.8, $\langle \psi'_0, \psi'_1, \ldots, \psi'_{m-1} \rangle$ is a Γ-proof. Clearly each ψ'_k is a $\mathscr{L}$ formula. Note that $\psi'_i = \psi_i = \neg(v_0 = v_0)$. So Γ is inconsistent in $\mathscr{L}$, contradiction. $\square$

Lemma 4.11. *Let $\mathscr{L}$ be a finite first-order language. Let $\mathscr{L}'$ extend $\mathscr{L}$ by adding individual constants $\mathbf{c}_0, \mathbf{c}_1, \ldots$.*

Then there is an enumeration $\langle \varphi_0, \varphi_1, \ldots \rangle$ of all of the sentences of $\mathscr{L}'$, and also an enumeration $\langle \psi_0, \psi_1, \ldots \rangle$ of all the sentences of $\mathscr{L}'$ of the form $\exists v_i \chi$.

Proof. Recall that a formula is a certain finite sequence of positive integers. First we describe how to list all finite sequences of positive integers. Given positive integers m and n, we can list all sequences of members of $\{1, \ldots, m\}$ of length n by just listing them in dictionary order. For example, with $m = 3$ and $n = 2$ our list is

$$\langle 1,1 \rangle$$
$$\langle 1,2 \rangle$$
$$\langle 1,3 \rangle$$
$$\langle 2,1 \rangle$$
$$\langle 2,2 \rangle$$
$$\langle 2,3 \rangle$$
$$\langle 3,1 \rangle$$
$$\langle 3,2 \rangle$$
$$\langle 3,3 \rangle$$

To list all finite sequences, we then do the following:

(1) List all sequences of members of $\{1\}$ of length 1. (There is only one such, namely $\langle 1 \rangle$.)

(2) List all sequences of members of $\{1, 2\}$ of length 1 or 2. Here they are:

$$\langle 1 \rangle$$
$$\langle 2 \rangle$$
$$\langle 1,1 \rangle$$
$$\langle 1,2 \rangle$$
$$\langle 2,1 \rangle$$
$$\langle 2,2 \rangle$$

(3) List all sequences of members of $\{1, 2, 3\}$ of length 1,2, or 3.

(4) General step: list all members of $\{1, \ldots, m\}$ of length $1, 2, \ldots m$.

Let $\langle \psi_0, \psi_1, \ldots \rangle$ be the listing described. Now we go through this list and select the ones which are sentences of $\mathscr{L}'$, giving the desired list $\langle \varphi_0, \varphi_1, \ldots \rangle$. Similarly for the list $\langle \psi_0, \psi_1, \ldots \rangle$ of all sentences of the form $\exists v_i \chi$. $\square$

Lemma 4.12. *Let $\mathscr{L}$ be a finite first-order language. Let $\mathscr{L}'$ extend $\mathscr{L}$ by adding individual constants $\mathbf{c}_0, \mathbf{c}_1, \ldots$.*

Suppose that Γ is a consistent set of sentences of $\mathscr{L}'$. Then there is a rich consistent set Δ of sentences with $\Gamma \subseteq \Delta$.

Proof. By Lemma 4.11, let $\langle \psi_0, \psi_1, \ldots \rangle$ enumerate all the sentences of $\mathscr{L}'$ of the form $\exists v_i \chi$; say that ψ_k is $\exists v_{t(k)} \psi'_k$ for all $k \in \omega$. Now we define an increasing sequence $\langle j(k) : k \in \omega \rangle$ by recursion. Suppose that $j(k)$ has been defined for all $k < l$. Let $j(l)$ be the smallest natural number not in the set

$$\{j(k) : k < l\} \cup \{s : \mathbf{c}_s \text{ occurs in some formula } \psi_k \text{ with } k \leq l\}.$$

Again we justify this definition. Let M be the set of all functions f defined on some set $m' = \{i \in \omega : i < m\}$ with $m \in \omega$ such that for all $l < m$, $f(l)$ is the smallest number not in the set

$$\{f(k) : k < l\} \cup \{s : \mathbf{c}_s \text{ occurs in some formula } \psi_k \text{ with } k \leq l\}.$$

(1) If $f, g \in M$, say with domains s', t' respectively, with $s \leq t$, then $f(k) = g(k)$ for all $k < s$.

We prove this by complete induction on k. Assume that it is true for all $k' < k$. Then $f(k)$ is the smallest number not in the set

$$\begin{aligned} &\{f(k'') : k'' < k\} \cup \{u : \mathbf{c}_u \text{ occurs in some formula } \psi''_k \text{ with } k'' \leq k\} = \\ &\{g(k'') : k'' < k\} \cup \{u : \mathbf{c}_u \text{ occurs in some formula } \psi''_k \text{ with } k'' \leq k\}, \end{aligned}$$

and this is the same as $g(k)$. So (1) holds.

(2) For each $m \in \omega$ there is a member of M with domain m'.

We prove this by induction on m. For $m = 0$ we take the empty function. Assume that $f \in M$ has domain m'. Define the extension g of f with domain $(m+1)'$ by letting $g(m)$ be the smallest number not in the set

$$\{f(k) : k < m\} \cup \{s : \mathbf{c}_s \text{ occurs in some formula } \psi_k \text{ with } k \leq m\}.$$

This proves (2).

Now we define $f(l)$ to be $g(l)$ for any $g \in M$ with l in the domain of g.

For each $l \in \omega$ let

$$\Theta_l = \Gamma \cup \{\exists v_{t(k)} \psi'_k \to \mathrm{Subf}^{v_{t(k)}}_{\mathbf{c}_{j(k)}} \psi'_k : k < l\}.$$

We claim that each set Θ_l is consistent. We prove this by induction on l. Note that $\Theta_0 = \Gamma$, which is given as consistent. Now suppose that we have shown that Θ_l is consistent. Now $\Theta_{l+1} = \Theta_l \cup \{\exists v_{t(l)}\psi'_l \to \mathrm{Subf}^{v_{t(l)}}_{\mathbf{c}_{j(l)}}\psi'_l\}$. Assume that Θ_{l+1} is inconsistent. Then $\Theta_{l+1} \vdash \neg(v_0 = v_0)$. By the deduction theorem 4.2, it follows that

$$\Theta_l \vdash (\exists v_{t(l)}\psi'_l \to \mathrm{Subf}^{v_{t(l)}}_{\mathbf{c}_{j(l)}}\psi'_l) \to \neg(v_0 = v_0),$$

hence easily

$$\Theta_l \vdash \neg(\exists v_{t(l)}\psi'_l \to \mathrm{Subf}^{v_{t(l)}}_{\mathbf{c}_{j(l)}}\psi'_l),$$

so that using tautologies

$$\begin{aligned} &\Theta_l \vdash \exists v_{t(l)}\psi'_l \quad \text{and} \\ &\Theta_l \vdash \neg\mathrm{Subf}^{v_{t(l)}}_{\mathbf{c}_{j(l)}}\psi'_l. \end{aligned}$$

Now by the definition of the sequence $\langle j(k) : k \in \omega\rangle$, it follows that $\mathbf{c}_{j(l)}$ does not occur in any formula in $\Theta_l \cup \{\psi'_l\}$. Hence by Lemma 4.9 we get $\Theta_l \vdash \neg\psi'_l$, and so $\Theta_l \vdash \forall v_{t(l)}\neg\psi'_l$. But we also have $\Theta_l \vdash \exists v_{t(l)}\psi'_l$, so that Θ_l is inconsistent, contradiction.

Now let $\Delta = \bigcup_{l\in\omega}\Theta_l$. We claim that Δ is consistent. Suppose not. Then $\Delta \vdash \neg(v_0 = v_0)$. Let $\langle\varphi_0, \ldots \varphi_{m-1}\rangle$ be a Δ-proof with $\varphi_i = \neg(v_0 = v_0)$. For each $k < m$ such that $\varphi_k \in \Delta$, choose $s(k) \in \omega$ such that $\varphi_k \in \Theta_{s(k)}$. Let l be such that $s(l)$ is largest among all $k < m$ such that $\varphi_k \in \Theta_{s(k)}$. Then $\langle\varphi_0, \ldots \varphi_{m-1}\rangle$ is a $\Theta_{s(l)}$-proof, and hence $\Theta_{s(l)}$ is inconsistent, contradiction.

Now clearly $\Gamma \subseteq \Delta$, since $\Theta_0 = \Gamma$. We claim that Δ is rich. For, let $\exists v_l\chi$ be a sentence. Say $\exists v_l\chi$ is ψ_m. Then $\exists v_l\chi$ is $\exists v_{t(m)}\psi'_m$, so that $l = t(m)$ and $c = \psi'_m$. Now the formula

$$\exists v_{t(m)}\psi'_m \to \mathrm{Subf}^{v_{t(m)}}_{\mathbf{c}_{j(m)}}\psi'_m$$

is a member of Θ_{m+1}, and hence is a member of Δ. This formula is $\exists v_l\chi \to \mathrm{Subf}^{v_l}_{\mathbf{c}_{j(m)}}\chi$. Hence Δ is rich. $\square$

Lemma 4.13. *Let $\mathscr{L}$ be a finite first-order language. Let $\mathscr{L}'$ extend $\mathscr{L}$ by adding individual constants $\mathbf{c}_0, \mathbf{c}_1, \ldots$.*

Suppose that Γ is a consistent set of sentences of $\mathscr{L}'$. Then there is a consistent complete set Δ of sentences with $\Gamma \subseteq \Delta$.

Proof. By Lemma 4.11, let $\langle\varphi_0, \varphi_1, \ldots\rangle$ be an enumeration of all the sentences of $\mathscr{L}'$. We now define by recursion sets Θ_i of sentences. Let $\Theta_0 = \Gamma$. Suppose that Θ_i has been defined so that it is consistent. If $\Theta_i \cup \{\varphi_i\}$ is consistent, let $\Theta_{i+1} = \Theta_i \cup \{\varphi_i\}$. Otherwise let $\Theta_{i+1} = \Theta_i \cup \{\neg\varphi_i\}$. We claim that in this otherwise case, still Θ_{i+1} is consistent. Suppose not. Then $\Theta_{i+1} \vdash \neg(v_0 = v_0)$, i.e., $\Theta_i \cup \{\neg\varphi_i\} \vdash \neg(v_0 = v_0)$. By the deduction theorem, $\Theta_i \vdash \neg\varphi_i \to \neg(v_0 = v_0)$, and then by Proposition 3.4 and a tautology $\Theta_i \vdash \varphi_i$. It follows that $\Theta_i \cup \{\varphi_i\}$ is consistent; otherwise $\Theta_i \cup \{\varphi_i\} \vdash \neg(v_0 = v_0)$, hence by the deduction theorem $\Theta_i \vdash \varphi_i \to \neg(v_0 = v_0)$, so by Proposition 4.3 and a tautology $\Theta_i \vdash \neg\varphi_i$. Together with $\Theta_i \vdash \varphi_i$, this shows that Θ_i is inconsistent, contradiction. So, $\Theta_i \cup \{\varphi_i\}$ is consistent. But this contradicts our "otherwise" condition. So, Θ_{i+1} is consistent. So the recursion continues.

Once more we give details on the recursion. Let M be the set of all functions f such that the domain of f is $m' = \{0, \ldots, m-1\}$ for some $m \in \omega$, and for all $i < m$ one of the following holds:

(1) $i = 0$ and $f(0) = \Gamma$.

(2) $i = j + 1$ for some $j \in \omega$, $f(j)$ is a set of sentences, $f(j) \cup \{\varphi_j\}$ is consistent, and $f(i) = f(j) \cup \{\varphi_j\}$.

(3) $i = j + 1$ for some $j \in \omega$, $f(j)$ is a set of sentences, $f(j) \cup \{\varphi_j\}$ is not consistent, and $f(i) = f(j) \cup \{\neg\varphi_j\}$.

We claim:

(4) If $f, g \in M$, say with domains m', n' respectively, with $m \leq n$, then $f(i) = g(i)$ for all $i < m$.

We prove this by induction on i. For $i = 0$ we have $f(0) = \Gamma = g(0)$. Suppose it is true for i, with $i + 1 < m$. Then by the definition of M we have two cases.

Case 1. $f(i)$ is a set of sentences, $f(i) \cup \{\varphi_i\}$ is consistent, and $f(i+1) = f(i) \cup \{\varphi_i\}$. Since $f(i) = g(i)$ by the inductive assumption, the definition of M gives $g(i+1) = g(i) \cup \{\varphi_i\} = f(i) \cup \{\varphi_i\} = f(i+1)$.

Case 2. $f(i)$ is a set of sentences, $f(i) \cup \{\varphi_i\}$ is not consistent, and $f(i+1) = f(i) \cup \{\neg\varphi_i\}$. Since $f(i) = g(i)$ by the inductive assumption, the definition of M gives $g(i+1) = g(i) \cup \{\neg\varphi_i\} = f(i) \cup \{\neg\varphi_i\} = f(i+1)$.

This finishes the inductive proof of (4).

(5) For all $f \in M$ and all i in the domain of f, $f(i)$ is a set of sentences.

This is easily proven by induction on i.

(6) For each $m \in \omega$ there is an $f \in M$ with domain m'.

We prove (5) by inducation on m. For $m = 0$ we can let f be the empty function. Suppose $f \in M$ with the domain of f equal to m'. If $m = 0$ we can let g be the function with domain $\{0\}$ and $g(0) = \Gamma$. Assume that $m > 0$. By (5), $f(m-1)$ is a set of sentences. Then we define g to be the extension of f such that

$$g(m) = \begin{cases} f(m-1) \cup \{\varphi_{m-1}\} & \text{if this set is consistent,} \\ f(m-1) \cup \{\neg\varphi_{m-1}\} & \text{otherwise.} \end{cases}$$

Thus (6) holds.

Now we define $\Theta_i = f(i)$ for any $f \in M$ which has i in its domain. Then by (5), each Θ_i is a set of sentences, $\Theta_0 = \Gamma$, and

$$\Theta_{i+1} = \begin{cases} \Theta_i \cup \{\varphi_i\} & \text{if this set is consistent,} \\ \Theta_i \cup \{\neg\varphi_i\} & \text{otherwise.} \end{cases}$$

Now we show by induction that each Θ_i is consistent. Since $\Theta_0 = \Gamma$, Θ_0 is consistent by assumption. Now suppose that Θ_i is consistent. If $\Theta_i \cup \{\varphi_i\}$ is consistent, then $\Theta_{i+1} = \Theta_i \cup \{\varphi_i\}$ and hence Θ_{i+1} is consistent. Suppose that $\Theta_i \cup \{\varphi_i\}$ is not consistent. Then $\Theta_i \cup \{\varphi_i\} \vdash \neg(v_0 = v_0)$, and hence an easy argument which we have used before gives $\Theta_i \vdash \neg\varphi_i$. Now $\Theta_{i+1} = \Theta_i \cup \{\neg\varphi_i\}$, so if Θ_{i+1} is not consistent we easily get $\Theta_i \vdash \varphi_i$. Hence Θ_i is inconsistent, contradiction. This completes the inductive proof.

Now let $\Delta = \bigcup_{i\in\omega} \Theta_i$. Then Δ is consistent. In fact, suppose not. Then $\Delta \vdash \neg(v_0 = v_0)$. Let $\langle \psi_0, \ldots, \psi_{m-1}\rangle$ be a Δ-proof with $\psi_i = \neg(v_0 = v_0)$. Let $\langle \chi_0, \ldots, \chi_{n-1}\rangle$ enumerate all of the members of Δ which are in the proof. Say $\chi_j \in \Theta_{s(j)}$ for each $j < n$. Let t be maximum among all the $s(j)$ for $j < n$. Then each χ_k is in Θ_t, so that $\langle \psi_0, \ldots, \psi_{m-1}\rangle$ is a Θ_t-proof. It follows that Θ_t is inconsistent, contradiction.

So Δ is consistent. Since $\Theta_0 = \Gamma$, we have $\Gamma \subseteq \Delta$. Finally, Δ is complete, since every sentence is equal to some φ_i, and our construction assures that $\varphi_i \in \Delta$ or $\neg\varphi_i \in \Delta$. $\square$

Lemma 4.14. *Let $\mathscr{L}$ be a first-order language. Let $\mathscr{L}'$ extend $\mathscr{L}$ by adding new non-logical symbols Suppose that $\overline{M}$ is an $\mathscr{L}'$-structure, and $\overline{N}$ is the $\mathscr{L}$-structure obtained from $\overline{M}$ by removing the denotations of the new non-logical symbols. Suppose that φ is a formula of $\mathscr{L}$, and $a : \omega \to M$. Then $\overline{M} \models \varphi[a]$ iff $\overline{N} \models \varphi[a]$.*

Proof. First we prove the following similar statement for terms:

(1) If σ is a term of $\mathscr{L}$, then $\sigma^{\overline{M}}(a) = \sigma^{\overline{N}}(a)$.

We prove this by induction on σ:

$$\begin{aligned}
v_i^{\overline{M}}(a) &= a_i = v_i^{\overline{N}}(a);\\
\mathbf{k}^{\overline{M}}(a) &= \mathbf{k}^{\overline{M}} = \mathbf{k}^{\overline{N}} = \mathbf{k}^{\overline{N}}(a) \quad \text{for } \mathbf{k} \text{ an individual constant of } \mathscr{L}\\
(\mathbf{F}\sigma_0 \ldots \sigma_{m-1})^{\overline{M}}(a) &= \mathbf{F}^{\overline{M}}(\sigma_0^{\overline{M}}(a), \ldots \sigma_{m-1}^{\overline{M}}(a))\\
&= \mathbf{F}^{\overline{N}}(\sigma_0^{\overline{N}}(a), \ldots \sigma_{m-1}^{\overline{N}}(a))\\
&= (\mathbf{F}\sigma_0 \ldots \sigma_{m-1})^{\overline{N}}(a).
\end{aligned}$$

Here $\mathbf{F}$ is a function symbol of $\mathscr{L}$. Thus (1) holds.

Now we prove the lemma itself by induction on φ:

$$\begin{aligned}
\overline{M} \models (\sigma = \tau)[a] \quad &\text{iff} \quad \sigma^{\overline{M}}(a) = \tau^{\overline{M}}(a)\\
&\text{iff} \quad \sigma^{\overline{N}}(a) = \tau^{\overline{N}}(a)\\
&\text{iff} \quad \overline{N} \models (\sigma = \tau)[a];\\
\overline{M} \models (\mathbf{R}\sigma_0 \ldots \sigma_{m-1})[a] \quad &\text{iff} \quad \langle \sigma_0^{\overline{M}}(a), \ldots, \sigma_{m-1}^{\overline{M}}(a)\rangle \in \mathbf{R}^{\overline{M}}\\
&\text{iff} \quad \langle \sigma_0^{\overline{N}}(a), \ldots, \sigma_{m-1}^{\overline{N}}(a)\rangle \in \mathbf{R}^{\overline{N}}\\
&\text{iff} \quad \overline{N} \models (\mathbf{R}\sigma_0 \ldots \sigma_{m-1})[a];\\
\overline{M} \models (\neg\varphi)[a] \quad &\text{iff} \quad \text{not}(\overline{M} \models \varphi[a])\\
&\text{iff} \quad \text{not}(\overline{N} \models \varphi[a])
\end{aligned}$$

$$
\begin{aligned}
& \text{iff} \quad \overline{N} \models (\neg\varphi)[a]; \\
\overline{M} \models (\varphi \to \psi)[a] \quad & \text{iff} \quad \text{not}(\overline{M} \models \varphi[a]) \text{ or } \overline{M} \models \psi[a] \\
& \text{iff} \quad \text{not}(\overline{N} \models \varphi[a]) \text{ or } \overline{N} \models \psi[a] \\
& \text{iff} \quad \overline{N} \models (\varphi \to \psi)[a]; \\
\overline{M} \models (\forall v_i\varphi)[a] \quad & \text{iff} \quad \text{for all } u \in M(\overline{M} \models \varphi[a^i_u]) \\
& \text{iff} \quad \text{for all } u \in N(\overline{N} \models \varphi[a^i_u]) \\
& \text{iff} \quad \overline{N} \models (\forall v_i\varphi)[a]. \qquad \square
\end{aligned}
$$

Theorem 4.15. (Completeness Theorem 1) *Every consistent set of sentences in a finite language has a model.*

Proof. Let Γ be a consistent set of sentences in the finite language $\mathscr{L}$. Let $\mathscr{L}'$ be obtained from $\mathscr{L}$ by adjoining individual constants $\mathbf{c}_i$ for each $i \in \omega$. By Lemmas 4.12 and 4.13 let Δ be a consistent rich complete set of sentences in $\mathscr{L}'$ such that $\Gamma \subseteq \Delta$. By Lemma 4.7, let $\overline{M}$ be a model of Δ. Let $\overline{N}$ be the $\mathscr{L}$-structure obtained from $\overline{M}$ by removing the denotations of the constants $\mathbf{c}_i$ for $i \in \omega$. By Lemma 4.14, $\overline{N}$ is a model of Γ. $\square$

Theorem 4.16. (Completeness Theorem 2) *Let $\Gamma \cup \{\varphi\}$ be a set of formulas in a finite language. Then $\Gamma \vdash \varphi$ iff $\Gamma \models \varphi$.*

Proof. By Theorems 4.3 and 4.15. $\square$

Theorem 4.17. (Completeness Theorem 3) *For any formula φ, $\vdash \varphi$ iff $\models \varphi$.*

Proof. Note that the implicit language $\mathscr{L}$ here is arbitrary, not necessarily finite. $\Rightarrow$ holds by Theorem 4.3. Now suppose that $\models \varphi$ in the sense of $\mathscr{L}$: for every $\mathscr{L}$-structure $\overline{M}$ and every $a : \omega \to M$ we have $\overline{M} \models \varphi[a]$. Let $\mathscr{L}'$ be the language whose non-logical symbols are those occurring in φ. There are finitely many such symbols, so $\mathscr{L}'$ is a finite language. By Lemma 4.14 we have $\models \varphi$ in the sense of $\mathscr{L}'$. Hence by Theorem 4.16, $\vdash \varphi$ in the sense of $\mathscr{L}'$. But every $\mathscr{L}'$-proof is also an $\mathscr{L}$-proof; so $\vdash \varphi$ in the sense of $\mathscr{L}$. $\square$

As the final topic of this chapter we consider the role of definitions. To formulate the results we need another elementary logical notion. We define $\exists! v_i\varphi$ to be the formula $\exists v_i[\varphi \wedge \forall v_j[\mathrm{Subf}^{v_i}_{v_j}\varphi \to v_i = v_j]]$, where j is minimum such that $j \neq i$ and v_j does not occur in φ.

Theorem 4.18. $\overline{A} \models \exists! v_i\varphi[a]$ *iff there is a unique $u \in A$ such that $\overline{A} \models \varphi[a^i_u]$.*

Proof. $\Rightarrow$: Assume that $\overline{A} \models \exists! v_i\varphi[a]$. Choose $u \in A$ such that

$$
\overline{A} \models (\varphi \wedge \forall v_j[\mathrm{Subf}^{v_i}_{v_j}\varphi \to v_i = v_j])[a^i_u]. \tag{1}
$$

In particular, $\overline{A} \models \varphi[a^i_u]$. Suppose that also $\overline{A} \models \varphi[a^i_w]$. By Lemma 4.4, $\overline{A} \models \varphi[(a^j_w)^i_w]$, i.e.,

$$
\overline{A} \models \varphi[(a^j_w)^i_{v_j(a^j_w)}].
$$

Now we apply Lemma 4.6, with a replaced by a^j_w and obtain

$$\overline{A} \models \mathrm{Subf}^{v_i}_{v_j}\varphi[a^j_w].$$

Since v_i does not occur free in $\mathrm{Subf}^{v_i}_{v_j}\varphi$, this implies that

$$\overline{A} \models \mathrm{Subf}^{v_i}_{v_j}\varphi[(a^i_u)^j_w]. \tag{2}$$

Now by (1) we have

$$\overline{A} \models (\mathrm{Subf}^{v_i}_{v_j}\varphi \to v_i = v_j)[(a^i_u)^j_w],$$

so by (2) we have $u = w$.

$\Leftarrow$: Suppose that $u \in A$ is unique such that $\overline{A} \models \varphi[a^i_u]$. To check the other part of $\exists! v_i\varphi$, suppose that $w \in A$ and $\overline{A} \models \mathrm{Subf}^{v_i}_{v_j}\varphi[(a^i_u)^j_w]$. Since v_i does not occur free in $\mathrm{Subf}^{v_i}_{v_j}\varphi$, it follows by Proposition 4.4 that $\overline{A} \models \mathrm{Subf}^{v_i}_{v_j}\varphi[a^j_w]$. Applying Lemma 4.6 with a replaced by a^j_w we obtain $\overline{A} \models \varphi[(a^j_w)^i_{v_j(a^j_w)}]$, i.e., $\overline{A} \models \varphi[(a^j_w)^i_w]$. By Proposition 4.4 again this yields $\overline{A} \models \varphi[a^i_w]$. Hence by supposition $u = w$, as desired. □

By a *theory* we mean a pair $(\mathscr{L}, \Gamma)$ such that $\mathscr{L}$ is a first-order language and Γ is a set of formulas in $\mathscr{L}$. A theory $(\mathscr{L}', \Gamma')$ is a *simple definitional expansion* of a theory $(\mathscr{L}, \Gamma)$ provided that the following conditions hold:

(1) $\mathscr{L}'$ is obtained from $\mathscr{L}$ by adding one new non-logical symbol.

(2) If the new symbol of $\mathscr{L}'$ is an m-ary relation symbol $\mathbf{R}$, then there is a formula φ of $\mathscr{L}$ with free variables among $v_0, \ldots, v_{m-1}$ such that

$$\Gamma' = \Gamma \cup \{\mathbf{R}v_0 \ldots v_{m-1} \leftrightarrow \varphi\}.$$

(3) If the new symbol of $\mathscr{L}'$ is an individual constant $\mathbf{c}$, then there is a formula φ of $\mathscr{L}$ with free variables among v_0 such that $\Gamma \vdash \exists! v_0\varphi$ and

$$\Gamma' = \Gamma \cup \{\mathbf{c} = v_0 \leftrightarrow \varphi\}.$$

(4) If the new symbol of $\mathscr{L}'$ is an m-ary function symbol $\mathbf{F}$, then there is a formula φ of $\mathscr{L}$ with free variables among $v_0, \ldots, v_m$ such that $\Gamma \vdash \forall v_0 \ldots \forall v_{m-1} \exists! v_m\varphi$ and

$$\Gamma' = \Gamma \cup \{\mathbf{F}v_0 \ldots v_{m-1} = v_m \leftrightarrow \varphi\}.$$

The basic facts about definitions are that the defined terms can always be eliminated, and adding a definition does not change what is is provable in the original language. In order to prove these two facts, we first show that any formula can be put in a certain standard form, which is interesting in its own right. A formula φ is *standard* provided that every atomic subformula of φ has one of the following forms:

$v_i = v_j$ for some $i, j \in \omega$.
$\mathbf{c} = v_0$ for some individual constant $\mathbf{c}$.

$\mathbf{R}v_0 \ldots v_{m-1}$ for some m-ary relation symbol $\mathbf{R}$.
$\mathbf{F}v_0 \ldots v_{m-1} = v_m$ for some m-ary function symbol $\mathbf{F}$.

Lemma 4.19. *If $\mathbf{c}$ is an individual constant and $i \neq 0$, then $\vdash \mathbf{c} = v_i \leftrightarrow \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)$.*

Proof. We argue model-theoretically. Suppose that $\overline{A}$ is a structure and $a : \omega \to A$. If $\overline{A} \models (\mathbf{c} = v_i)[a]$, then $\mathbf{c}^{\overline{A}} = a_i$. Then $v_0^{\overline{A}}(a^0_{a_i}) = a_i$ and $v_i^{\overline{A}}(a^0_{a_i}) = a_i$. Hence $\overline{A} \models (v_0 = v_i \wedge \mathbf{c} = v_0)[a^0_{a_i}]$, and so $\overline{A} \models \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)[a]$. Thus $\overline{A} \models (\vdash \mathbf{c} = v_i)[a]$ implies that $\overline{A} \models \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)[a]$.

Conversely, suppose that $\overline{A} \models \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)[a]$. Choose $x \in A$ such that $\overline{A} \models (v_0 = v_i \wedge \mathbf{c} = v_0)[a^0_x]$. Then $x = v_0^{\overline{A}}(a^0_x) = v_i^{\overline{A}}(a^0_x) = a_i$ and $\mathbf{c}^{\overline{A}} = v_0^{\overline{A}}(a^0_x) = a_i = v_i^{\overline{A}}(a)$. Hence $\overline{A} \models (\vdash \mathbf{c} = v_i)[a]$.

So we have shown that $\overline{A} \models (\vdash \mathbf{c} = v_i)[a]$ iff $\overline{A} \models \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)[a]$. It follows that $\models \mathbf{c} = v_i \leftrightarrow \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)$. Hence by the completeness theorem, $\vdash \mathbf{c} = v_i \leftrightarrow \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)$. $\square$

Lemma 4.20. *Suppose that $\mathbf{R}$ is an m-ary relation symbol and $\langle i(0), \ldots, i(m-1)\rangle$ is a sequence of natural numbers such that $m \leq i(j)$ for all $j < m$. Also assume that $k < m$. Then*

$$\vdash \mathbf{R}v_0 \ldots v_{k-1}v_{i(k)} \ldots v_{i(m-1)} \leftrightarrow \exists v_k[v_k = v_{i(k)} \wedge \mathbf{R}v_0 \ldots v_k v_{i(k+1)} \ldots v_{i(m-1)}].$$

Proof. Again we argue model-theoretically. Suppose that $\overline{A}$ is a structure and $a : \omega \to A$. First suppose that

$$\overline{A} \models \mathbf{R}v_0 \ldots v_{k-1}v_{i(k)} \ldots v_{i(m-1)}[a]$$

Thus

$$\begin{array}{ll}
& \langle a_0, \ldots, a_{k-1}, a_{i(k)}, \ldots, a_{i(m-1)}\rangle \in \mathbf{R}^{\overline{A}} \quad \text{hence} \\
& \langle (a^k_{a_i(k)})_0, \ldots, (a^k_{a_i(k)})_{k-1}, (a^k_{a_i(k)})_{i(k)}, \ldots, (a^k_{a_i(k)})_{i(m-1)}\rangle \in \mathbf{R}^{\overline{A}} \quad \text{hence} \\
& \langle (a^k_{a_i(k)})_0, \ldots, (a^k_{a_i(k)})_{k-1}, (a^k_{a_i(k)})_k, \ldots, (a^k_{a_i(k)})_{i(m-1)}\rangle \in \mathbf{R}^{\overline{A}} \quad \text{hence} \\
& \overline{A} \models [v_k = v_{i(k)} \wedge \mathbf{R}v_0 \ldots v_k v_{i(k+1)} \ldots v_{i(m-1)}][a^k_{a_i(k)}] \quad \text{hence} \\
(*) & \overline{A} \models \exists v_k[v_k = v_{i(k)} \wedge \mathbf{R}v_0 \ldots v_k v_{i(k+1)} \ldots v_{i(m-1)}][a].
\end{array}$$

Second, suppose that $(*)$ holds. Choose $s \in A$ such that

$$\overline{A} \models [v_k = v_{i(k)} \wedge \mathbf{R}v_0 \ldots v_k v_{i(k+1)} \ldots v_{i(m-1)}][a^k_s]$$

Then $s = a_{i(k)}$ and $\langle a_0, \ldots, a_{k-1}, s, a_{i(k+1)}, \ldots, a_{i(m-1)}\rangle \in \mathbf{R}^{\overline{A}}$, so

$$\langle a_0, \ldots, a_{k-1}, a_{i(k)}, a_{i(k+1)}, \ldots, a_{i(m-1)}\rangle \in \mathbf{R}^{\overline{A}},$$

so $\overline{A} \models \mathbf{R}v_0 \ldots v_{k-1}v_{i(k)} \ldots v_{i(m-1)}[a]$.

Now the Lemma follows by the completeness theorem. $\square$

Lemma 4.21. *Suppose that* $\mathbf{R}$ *is an* m*-ary relation symbol and* $\langle i(0), \ldots, i(m-1)\rangle$ *is a sequence of natural numbers such that* $m \le i(j)$ *for all* $j < m$*. Then there is a standard formula* φ *with free variables* $v_{i(j)}$ *for* $j < m$ *such that* $\vdash \mathbf{R}v_{i(0)} \ldots v_{i(m-1)} \leftrightarrow \varphi$*.*

Proof. This follows by an easy induction from Lemma 4.20. $\square$

The proof of the following lemma is very similar to the proof of Lemma 4.20.

Lemma 4.22. *Suppose that* $\mathbf{F}$ *is an* m*-ary function symbol and* $\langle i(0), \ldots, i(m)\rangle$ *is a sequence of natural numbers such that* $m+1 \le i(j)$ *for all* $j \le m$*. Also assume that* $k < m$ *Then*

$$\begin{aligned}&\vdash \mathbf{F}v_0 \ldots v_{k-1}v_{i(k)} \ldots v_{i(m-1)} = v_{i(m)} \leftrightarrow \\ &\quad \exists v_k[v_k = v_{i(k)} \wedge \mathbf{F}v_0 \ldots v_k v_{i(k+1)} \ldots v_{i(m-1)} = v_{i(m)}].\end{aligned}$$

Proof. Again we argue model-theoretically. Suppose that $\overline{A}$ is a structure and $a : \omega \to A$. First suppose that $\overline{A} \models (\mathbf{F}v_0 \ldots v_{k-1}v_{i(k)} \ldots v_{i(m-1)} = v_{i(m)})[a]$. Then

$$\begin{aligned}
&\mathbf{F}^{\overline{A}}(a_0, \ldots, a_{k-1}, a_{i(k)}, \ldots, a_{i(m-1)} = a_{i(m)}, \quad \text{hence}\\
&\mathbf{F}^{\overline{A}}((a^k_{i(k)})_0, \ldots, (a^k_{i(k)})_{k-1}, (a^k_{i(k)})_{i(k)}, \ldots, (a^k_{i(k)})_{i(m-1)} = (a^k_{i(k)})_{i(m)}, \quad \text{hence}\\
&\mathbf{F}^{\overline{A}}((a^k_{i(k)})_0, \ldots, (a^k_{i(k)})_{k-1}, (a^k_{i(k)})_{k}, \ldots, (a^k_{i(k)})_{i(m-1)} = (a^k_{i(k)})_{i(m)}, \quad \text{hence}\\
&\overline{A} \models (v_k = v_{i(k)} \wedge \mathbf{F}v_0 \ldots v_k v_{i(k+1)} \ldots v_{i(m-1)} = v_{i(m)})[a^k_{i(k)}], \quad \text{hence}\\
(*) \quad &\overline{A} \models \exists v_k[v_k = v_{i(k)} \wedge \mathbf{F}v_0 \ldots v_k v_{i(k+1)} \ldots v_{i(m-1)} = v_{i(m)}][a].
\end{aligned}$$

Second, suppose that $(*)$ holds. Choose s so that

$$\overline{A} \models [v_k = v_{i(k)} \wedge \mathbf{F}v_0 \ldots v_k v_{i(k+1)} \ldots v_{i(m-1)} = v_{i(m)}][a^k_s].$$

Then $s = a_{i(k)}$ and $\mathbf{F}^{\overline{A}}(a_0, \ldots, s, a_{i(k+1)} \ldots a_{i(m-1)}) = a_{i(m)}$. Hence

$$\mathbf{F}^{\overline{A}}(a_0, \ldots, a_{k-1}, a_{i(k)}, a_{i(k+1)} \ldots a_{i(m-1)}) = a_{i(m)},$$

and so $\overline{A} \models (\mathbf{F}v_0 \ldots v_{k-1}v_{i(k)} \ldots v_{i(m-1)} = v_{i(m)})[a]$.

The Lemma now follows by the completeness theorem. $\square$

Lemma 4.23. *Suppose that* $\mathbf{F}$ *is an* m*-ary function symbol and* $\langle i(0), \ldots, i(m)\rangle$ *is a sequence of natural numbers such that* $m+1 \le i(j)$ *for all* $j \le m$*. Then there is a standard formula* φ *with free variables* $v_{i(j)}$ *for* $j \le M$ *such that* $\vdash \mathbf{F}v_{i(0)} \ldots v_{i(m-1)} = v_{i(m)} \leftrightarrow \varphi$*.*

Proof. By an easy induction using Lemma 4.22 there is a formula ψ with free variables $v_{i(j)}$ for $j \le m$ such that the only nonlogical atomic formula which is a segment of ψ is

$\mathbf{F}v_0 \ldots v_{m-1} = v_{i(m)}$ and $\vdash \mathbf{F}v_{i(0)} \ldots v_{i(m-1)} = v_{i(m)} \leftrightarrow \varphi$. Now for any structure $\overline{A}$ and any $a : \omega \to A$ we have

$$(*) \qquad \overline{A} \models (\mathbf{F}v_0 \ldots v_{m-1} = v_{i(m)} \leftrightarrow \exists v_m[v_m = v_{i(m)} \wedge \mathbf{F}v_0 \ldots v_{m-1} = v_m)][a].$$

To prove $(*)$, first suppose that $\overline{A} \models (\mathbf{F}v_0 \ldots v_{m-1} = v_{i(m)})[a]$. Thus $\mathbf{F}(a_0, \ldots, a_{m-1}) = a_{i(m)}$. Hence

$$\begin{aligned}
&\mathbf{F}(a_0, \ldots, a_{m-1}) = a_{i(m)} \quad \text{hence}\\
&\mathbf{F}((a^m_{a_{i(m)}})_0, \ldots, (a^m_{a_{i(m)}})_{m-1}) = (a^m_{a_{i(m)}})_{i(m)} \quad \text{hence}\\
&\overline{A} \models (v_m = v_{i(m)} \wedge \mathbf{F}v_0 \ldots v_{m-1} = v_m)[a^m_{a_{i(m)}}] \quad \text{hence}\\
(**) \qquad &\overline{A} \models \exists v_m[v_m = v_{i(m)} \wedge \mathbf{F}v_0 \ldots v_{m-1} = v_m)][a]
\end{aligned}$$

Second, assume $(**)$. Choose $s \in A$ such that $\overline{A} \models [v_m = v_{i(m)} \wedge \mathbf{F}v_0 \ldots v_{m-1} = v_m)][a^m_s]$. It follows that $s = a_{i(m)}$ and $\mathbf{F}^{\overline{A}}(a_0, \ldots, a_{m-1} = s$, so $\mathbf{F}^{\overline{A}}(a_0, \ldots, a_{m-1} = a_{i(m)}$, hence $\overline{A} \models (\mathbf{F}v_0 \ldots v_{m-1} = v_{i(m)})[a]$.

This proves $(*)$. From $(*)$ the Lemma is clear. □

Lemma 4.24. *Suppose that* $\mathbf{F}$ *is an* m*-ary function symbol,* $\sigma_0, \ldots, \sigma_{m-1}$ *are terms, the integers* $i(0), \ldots, i(m)$ *are all greater than* m *and do not appear in any of the terms* σ_j*, and* $k < m$*. Then*

$$\begin{aligned}
&\vdash \mathbf{F}v_{i(0)} \ldots v_{i(k-1)}\sigma_k \ldots \sigma_{m-1} = v_{i(m)}\\
&\quad \leftrightarrow \exists v_{i(k)}[\sigma_k = v_{i(k)} \wedge \mathbf{F}v_{i(0)} \ldots v_{i(k)}\sigma_{k+1} \ldots \sigma_{m-1} = v_{i(m)}].
\end{aligned}$$

Proof. Arguing model-theoretically, let $\overline{A}$ be a structure and $a : \omega \to A$. Let $b = a^{i(k)}_{\sigma_k^{\overline{A}}(a)}$. First suppose that $\overline{A} \models (\mathbf{F}v_{i(0)} \ldots v_{i(k-1)}\sigma_k \ldots \sigma_{m-1} = v_{i(m)})[a]$. Thus

$$\begin{aligned}
&\mathbf{F}^{\overline{A}}(a_{i(0)}, \ldots, a_{i(k-1)}, \sigma_k^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a)) = v_{i(m)}^{\overline{A}}(a) \quad \text{hence}\\
&\mathbf{F}^{\overline{A}}(b_{i(0)}, \ldots, b_{i(k-1)}, b_{i(k)}, \sigma_{k+1}^{\overline{A}}(b), \ldots, \sigma_{m-1}^{\overline{A}}(b)) = v_{i(m)}^{\overline{A}}(b) \quad \text{hence}\\
&\overline{A} \models (\sigma_k = v_{i(k)} \wedge \mathbf{F}v_{i(0)} \ldots v_{i(k)}\sigma_{k+1} \ldots \sigma_{m-1} = v_{i(m)})[b] \quad \text{hence}\\
(*) \qquad &\overline{A} \models \exists v_{i(k)}(\sigma_k = v_{i(k)} \wedge \mathbf{F}v_{i(0)} \ldots v_{i(k)}\sigma_{k+1} \ldots \sigma_{m-1} = v_{i(m)})[a].
\end{aligned}$$

Second, suppose that $(*)$ holds. Choose $s \in A$ so that

$$\overline{A} \models (\sigma_k = v_{i(k)} \wedge \mathbf{F}v_{i(0)} \ldots v_{i(k)}\sigma_{k+1} \ldots \sigma_{m-1} = v_{i(m)})[a^{i(k)}_s].$$

Hence $\sigma_k^{\overline{A}}(a^{i(k)}_s) = s$ and

$$\begin{aligned}
&\mathbf{F}^{\overline{A}}((a^{i(k)}_s)_{i(0)}, \ldots, (a^{i(k)}_s)_{i(k)}, \sigma_{k+1}^{\overline{A}}(a^{i(k)}_s), \ldots \sigma^{\overline{A}}(a^{i(k)}_s) = (a^{i(k)}_s)_{i(m)}, \quad \text{hence}\\
&\mathbf{F}^{\overline{A}}((a^{i(k)}_s)_{i(0)}, \ldots, \sigma_k^{\overline{A}}(a^{i(k)}_s), \sigma_{k+1}^{\overline{A}}(a^{i(k)}_s), \ldots \sigma^{\overline{A}}(a^{i(k)}_s) = (a^{i(k)}_s)_{i(m)}, \quad \text{hence}\\
&\mathbf{F}^{\overline{A}}(a_{i(0)}, \ldots, a_{i(k-1)}, \sigma_k^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a)) = v_{i(m)}^{\overline{A}}(a) \quad \text{hence}\\
&\overline{A} \models (\mathbf{F}v_{i(0)} \ldots v_{i(k-1)}\sigma_k \ldots \sigma_{m-1} = v_{i(m)})[a]
\end{aligned}$$
□

Lemma 4.25. *Suppose that $\mathbf{F}$ is an m-ary function symbol, $\sigma_0, \ldots, \sigma_{m-1}$ are terms, the integers $i(0), \ldots, i(m)$ are all greater than m and do not appear in any of the terms σ_j.*

Then there is a formula φ with free variables among $v_{i(0)}, \ldots, v_{i(m)}$ such that the atomic subformulas of φ are the formulas $\sigma_k = v_{i(k)}$ for $k < m$ along with the formula $\mathbf{F}v_{i(0)} \ldots v_{i(m-1)} = v_{i(m)}$, and $\vdash \mathbf{F}\sigma_0 \ldots \sigma_{m-1} = v_{i(m)} \leftrightarrow \varphi$.

Lemma 4.26. *Suppose that τ is a term and $i \in \omega$ is greater than m for each m such that a function symbol of rank m occurs in τ, and such that v_m does not occur in τ.*

Then there is a standard formula φ with the same free variables occurring in $\tau = v_m$, such that $\vdash \tau = v_m \leftrightarrow \varphi$.

Proof. We go by induction on τ. If τ is v_i, then we can take φ to be $v_i = v_m$. If τ is an individual constant $\mathbf{c}$, then Lemma 4.19 gives the desired result. Finally, suppose inductively that τ is $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$. Then the desired result follows by Lemma 4.26, the inductive hypothesis, and Lemma 4.23. □

Lemma 4.27. *For any terms σ, τ there is a standard formula φ with the same free variables as $\sigma = \tau$ such that $\vdash \sigma = \tau \leftrightarrow \varphi$.*

Proof. Let i be greater than each m such that there is a function symbol of rank m appearing in $\sigma = \tau$, and also such that v_i does not occur in $\sigma = \tau$. Then

$$(1) \qquad \vdash \sigma = \tau \leftrightarrow \exists v_i(\sigma = v_i \wedge \tau = v_i).$$

We prove (1) model-theoretically. First suppose that $\overline{A} \models (\sigma = \tau)[a]$. Thus $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$. By Proposition 2.4 we then have

$$\begin{aligned} &\sigma^{\overline{A}}(a^i_{\sigma^{\overline{A}}(a)}) = \tau^{\overline{A}}(a^i_{\alpha^{\overline{A}}(a)}) \quad \text{hence} \\ &\overline{A} \models (\sigma = v_i \wedge \tau = v_i)[a^i_{\sigma^{\overline{A}}(a)}] \quad \text{hence} \\ (*) \qquad &\overline{A} \models \exists v_i(\sigma = v_i \wedge \tau = v_i)[a]. \end{aligned}$$

Second, suppose that $(*)$ holds. Choose $s \in A$ such that $\overline{A} \models (\sigma = v_i \wedge \tau = v_i)[a^i_s]$. Thus $\sigma^{\overline{A}}(a^i_s) = s = \tau^{\overline{A}}(a^i_s)$, hence $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$ by Proposition 2.4. That is, $\overline{A} \models (\sigma = \tau)[a]$. This finishes the proof of (1).

Now by (1) and Lemma 4.26 our lemma follows. □

The proof of the following lemma is very similar to that of Lemma 4.24.

Lemma 4.28. *Suppose that $\mathbf{R}$ is an m-ary relation symbol, $\sigma_0, \ldots, \sigma_{m-1}$ are terms, the integers $i(0), \ldots, i(m)$ are all greater than m and do not appear in any of the terms σ_j, and $k < m$. Then*

$$\begin{aligned} \vdash &\mathbf{R}v_{i(0)} \ldots v_{i(k-1)}\sigma_k \ldots \sigma_{m-1} \\ &\leftrightarrow \exists v_{i(k)}[\sigma_k = v_{i(k)} \wedge \mathbf{R}v_{i(0)} \ldots v_{i(k)}\sigma_{k+1} \ldots \sigma_{m-1}]. \end{aligned}$$

Proof. Arguing model-theoretically, let $\overline{A}$ be a structure and $a : \omega \to A$. Let $b = a^{i(k)}_{\sigma_k^{\overline{A}}(a)}$. First suppose that $\overline{A} \models (\mathbf{R}v_{i(0)} \ldots v_{i(k-1)}\sigma_k \ldots \sigma_{m-1}[a]$. Thus

$$\begin{aligned}
&\langle a_{i(0)}, \ldots, a_{i(k-1)}, \sigma_k^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a) \in \mathbf{R}^{\overline{A}}, \quad \text{hence}\\
&\langle b_{i(0)}, \ldots, b_{i(k-1)}, b_{i(k)}, \sigma_{k+1}^{\overline{A}}(b), \ldots, \sigma_{m-1}^{\overline{A}}(b) \in \mathbf{R}^{\overline{A}}, \quad \text{hence}\\
&\overline{A} \models (\sigma_k = v_{i(k)} \wedge \mathbf{R}v_{i(0)} \ldots v_{i(k)}\sigma_{k+1} \ldots \sigma_{m-1})[b] \quad \text{hence}\\
(*) \qquad &\overline{A} \models \exists v_{i(k)}(\sigma_k = v_{i(k)} \wedge \mathbf{R}v_{i(0)} \ldots v_{i(k)}\sigma_{k+1} \ldots \sigma_{m-1})[a].
\end{aligned}$$

Second, suppose that $(*)$ holds. Choose $s \in A$ so that

$$\overline{A} \models (\sigma_k = v_{i(k)} \wedge \mathbf{R}v_{i(0)} \ldots v_{i(k)}\sigma_{k+1} \ldots \sigma_{m-1})[a^{i(k)}_s].$$

Hence $\sigma_k^{\overline{A}}(a^{i(k)}_s) = s$ and

$$\begin{aligned}
&\langle (a^{i(k)}_s)_{i(0)}, \ldots, (a^{i(k)}_s)_{i(k)}, \sigma_{k+1}^{\overline{A}}(a^{i(k)}_s), \ldots \sigma^{\overline{A}}(a^{i(k)}_s)\rangle \in \mathbf{R}^{\overline{A}}, \quad \text{hence}\\
&\langle (a^{i(k)}_s)_{i(0)}, \ldots, \sigma_k^{\overline{A}}(a^{i(k)}_s), \sigma_{k+1}^{\overline{A}}(a^{i(k)}_s), \ldots \sigma^{\overline{A}}(a^{i(k)}_s)\rangle \in \mathbf{R}^{\overline{A}}, \quad \text{hence}\\
&\langle a_{i(0)}, \ldots, a_{i(k-1)}, \sigma_k^{\overline{A}}(a), \ldots \sigma_{m-1}^{\overline{A}}(a)\rangle \in \mathbf{R}^{\overline{A}}, \quad \text{hence}\\
&\overline{A} \models (\mathbf{R}v_{i(0)} \ldots v_{i(k-1)}\sigma_k \ldots \sigma_{m-1})[a]
\end{aligned}$$

□

Theorem 4.29. *For any formula φ there is a standard formula ψ with the same free variables as φ such that $\vdash \varphi \leftrightarrow \psi$.*

Proof. We proceed by induction on φ. For φ an atomic equality formula $\sigma = \tau$ the desired result is given by Lemma 4.27. Now suppose that φ is an atomic nonequality formul $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$. Using induction we see from Lemma 4.28 that there is a formula φ whose atomic parts are of the form $\sigma_k = v_{i(k)}$ and $\mathbf{R}v_{i(0} \ldots v_{i(m-1)}$ such that $\vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-1} \leftrightarrow \varphi$, and each $i(k)$ is greater than each n such that a function symbol of rank n occurs in some σ_l, and also is such that no $v_{i(k)}$ occurs in any σ_s, and eack $i(k) > m$. Now by Lemmas 4.21 and 4.26, $\vdash \varphi \leftrightarrow \psi$ for some standard formula ψ. The condition on free variables holds in each of these steps. Thus the atomic cases of the induction hold.

The induction steps are easy:

Suppose that $\vdash \varphi \leftrightarrow \psi$ with ψ standard. Then $\vdash \neg\varphi \leftrightarrow \neg\psi$ and $\neg\psi$ is standard.

Suppose that $\vdash \varphi \leftrightarrow \psi$ with ψ standard and $\vdash \varphi' \leftrightarrow \psi'$ with ψ' standard. Then $\vdash (\varphi \to \psi) \leftrightarrow (\psi \to \psi')$ and $\psi \to \psi'$ is standard.

Suppose that $\vdash \varphi \leftrightarrow \psi$ with ψ standard. Then $\vdash \forall v_i\varphi \leftrightarrow \forall v_i\psi$ with $\forall v_i\psi$ standard.

□

The following theorem expresses that defined notions can be eliminated.

Theorem 4.30. *Let $(\mathscr{L}', \Gamma')$ be a simple definitional expansion of $(\mathscr{L}, \Gamma)$, and let φ be a formula of $\mathscr{L}'$. Then there is a formula ψ of $\mathscr{L}$ with the same free variables as φ such that $\Gamma' \vdash \varphi \leftrightarrow \psi$.*

(Note here that $\vdash$ is in the sense of $\mathscr{L}'$.)

Proof. Let χ be a standard formula (of $\mathscr{L}'$) such that $\vdash \varphi \leftrightarrow \chi$, such that χ has the same free variables as φ. Now we consider cases depending on what the new symbol s of $\mathscr{L}'$ is. Let θ be as in the definition of simple definitional expansion, with θ instead of φ.

Case 1. s is an individual constant $\mathbf{c}$. Then we let ψ be obtained from χ by replacing every subformula $\mathbf{c} = v_0$ of χ by θ.

Case 2. s is an m-ary relation symbol $\mathbf{R}$. Then we let ψ be obtained from χ by replacing every subformula $\mathbf{R}v_0 \ldots v_{m-1}$ of χ by θ.

Case 3. s is an m-ary function symbol $\mathbf{F}$. Then we let ψ be obtained from χ by replacing every subformula $\mathbf{F}v_0 \ldots v_{m-1} = v_m$ of χ by θ. $\square$

The following theorem expresses that a simple definitional expansion does not increase the set of old formulas which are provable.

Theorem 4.31. *Let $(\mathscr{L}', \Gamma')$ be a simple definitional expansion of $(\mathscr{L}, \Gamma)$ with $\mathscr{L}$ finite, and let φ be a formula of $\mathscr{L}$. Suppose that $\Gamma' \vdash \varphi$. Then $\Gamma \vdash \varphi$.*

Proof. By the completeness theorem we have $\Gamma' \models \varphi$, and it suffices to show that $\Gamma \models \varphi$. So, suppose that $\overline{A} \models \psi$ for each $\psi \in \Gamma$. In order to show that $\overline{A} \models \varphi$, suppose that $a : \omega \to A$; we want to show that $\overline{A} \models \varphi[a]$. We define an $\mathscr{L}'$-structure $\overline{A}'$ by defining the denotation of the new symbol s of $\mathscr{L}'$. The three cases are treated similarly, but we give full details for each of them.

Case 1. s is $\mathbf{c}$, an individual constant. By the definition of simple definitional expansion, there is a formula χ of $\mathscr{L}$ with free variables among v_0 such that $\Gamma \vdash \exists! v_0 \chi$, and $\Gamma' = \Gamma \cup \{\mathbf{c} = v_0 \leftrightarrow \chi\}$. Then $\Gamma \models \exists! v_0 \chi$. Since $\overline{A} \models \Gamma$, it follows that $\overline{A} \models \chi[a^0_x]$ for a unique $x \in A$. Let $\mathbf{c}^{\overline{A}'} = x$. We claim that $\overline{A}' \models (\mathbf{c} = v_0 \leftrightarrow \chi)$. In fact, suppose that $b : \omega \to A$. If $\overline{A}' \models (\mathbf{c} = v_0)[b]$, then $b_0 = \mathbf{c}^{\overline{A}'} = x$. Then a^0_x and b agree at 0, so by Lemma 4.4, since the free variables of χ are among v_0, we have $\overline{A} \models \chi[b]$. By Lemma 4.14, $\overline{A}' \models \chi[b]$. Conversely, suppose that $\overline{A}' \models \chi[b]$. Then b and $a^0_{b(0)}$ agree on 0, so $\overline{A}' \models \chi[a^0_{b(0)}]$. Hence $\overline{A} \models \chi[a^0_{b(0)}]$ by Lemma 4.14. Since also $\overline{A} \models \chi[a^0_x]$ and $\overline{A} \models \exists! v_0 \chi$, it follows that $b(0) = x$. Hence $\overline{A}' \models (\mathbf{c} = v_0)[b]$. This proves the claim.

By the claim, $\overline{A}'$ is a model of Γ'. Hence it is a model of φ. By Lemma 4.14, $\overline{A}$ is a model of φ, as desired.

Case 2. s is $\mathbf{F}$, an m-ary function symbol. By the definition of simple definitional expansion, there is a formula χ of $\mathscr{L}$ with free variables among $v_0, \ldots, v_m$ such that $\Gamma \vdash \forall v_0 \ldots \forall v_{m-1} \exists! v_m \chi$, and $\Gamma' = \Gamma \cup \{\mathbf{F}v_0 \ldots v_{m-1} = v_m \leftrightarrow \chi\}$. Then $\Gamma \models \forall v_0 \ldots \forall v_{m-1} \exists! v_m \chi$. Let $x(0), \ldots, x(m-1) \in A$. Since $\overline{A} \models \Gamma$, it follows that $\overline{A} \models \chi[(\cdots(a^0_{x(0)})^1_{x(1)})\cdots)^{m-1}_{x(m-1)})^m_y]$ for a unique $y \in A$. Let $\mathbf{F}^{\overline{A}'}(x(0), \ldots, x((m-1)) = y$. We claim that $\overline{A}' \models (\mathbf{F}v_0 \ldots v_{m-1} = v_m \leftrightarrow \chi)$. In fact, suppose that $b : \omega \to A$. If $\overline{A}' \models (\mathbf{F}v_0 \ldots v_{m-1} = v_m)[b]$, then $\mathbf{F}^{\overline{A}'}(b_0, \ldots, b_{m-1}) = b_m$. Now b and $(\cdots(a^0_{b_0})^1_{b_1})\cdots)^m_{b_m}$ and b agree on $\{0, \ldots, m\}$, so by the definition of $\mathbf{F}^{\overline{A}'}$ we get $\overline{A} \models \chi[(\cdots(a^0_{b_0})^1_{b_1})\cdots)^m_{b_m}]$, and hence also $\overline{A} \models \chi[b]$, and by Lemma 4.14 $\overline{A}' \models \chi[b]$.

Conversely, suppose that $\overline{A}' \models \chi[b]$. Then $\overline{A} \models [(\cdots(a^0_{b_0})^1_{b_1})\cdots)^m_{b_m}]$, and therefore $\mathbf{F}^{\overline{A}'}(b_0, \ldots, b_{m-1}) = b_m$. This proves the claim.

By the claim, $\overline{A}'$ is a model of Γ'. Hence it is a model of φ. By Lemma 4.14, $\overline{A}$ is a model of φ, as desired.

Case 3. s is $\mathbf{R}$, an m-ary relation symbol. By the definition of simple definitional expansion, there is a formula χ of $\mathscr{L}$ with free variables among $v_0, \ldots, v_{m-1}$ such that $\Gamma' = \Gamma \cup \{\mathbf{R}v_0 \ldots v_{m-1} \leftrightarrow \chi\}$. Let

$$\begin{aligned}\mathbf{R}^{\overline{A}'} = \{\langle a_0, \ldots, a_{m-1}\rangle : \overline{A} \models \chi[a] \\ \text{for some } a : \omega \to A \text{ which extends } \langle a_0, \ldots, a_{m-1}\rangle\}.\end{aligned}$$

We claim that $\overline{A}' \models (\mathbf{R}v_0 \ldots v_{m-1} \leftrightarrow \chi)$. In fact, suppose that $b : \omega \to A$. If $\overline{A}' \models (\mathbf{R}v_0 \ldots v_{m-1}[b]$, then $\langle b_0, \ldots, b_{m-1}\rangle \in \mathbf{R}^{\overline{A}'}$, and so there is an extension $a : \omega \to A$ of $\langle b_0, \ldots, b_{m-1}\rangle$ such that $\overline{A} \models \chi[a]$. Since a and b agree on all k such that v_k occurs in χ, it follows that $\overline{A} \models \chi[b]$, and hence $\overline{A}' \models \chi[b]$.

Conversely, suppose that $\overline{A}' \models \chi[b]$. Then $\overline{A} \models \chi[b]$ by Lemma 4.14, and it follows that $\langle b_0, \ldots, b_{m-1}\rangle \in \mathbf{R}^{\overline{A}'}$. This proves the claim.

By the claim, $\overline{A}'$ is a model of Γ'. Hence it is a model of φ. By Lemma 4.14, $\overline{A}$ is a model of φ, as desired. □

Theorem 4.32. *Let m be an integer ≥ 2, and suppose that $(\mathscr{L}_{i+1}, \Gamma_{i+1})$ is a simple definitional expansion of $(\mathscr{L}_i, \Gamma_i)$ for each $i < m$. Suppose that φ is an $\mathscr{L}_m$ formula. Then there is an $\mathscr{L}_0$ formula ψ with the same free variables as φ such that $\Gamma_m \vdash \varphi \leftrightarrow \psi$.*

Proof. By induction on m. If $m = 2$, the conclusion follows from Theorem 4.30. now assume the result for m and suppose that $(\mathscr{L}_{i+1}, \Gamma_{i+1})$ is a simple definitional expansion of $(\mathscr{L}_i, \Gamma_i)$ for each $i \leq m$. Let φ be a formula of $\mathscr{L}_{m+1}$. Then by Theorem 4.30 there is a formula ψ of $\mathscr{L}$ with the same free variables as φ such that $\Gamma_{m+1} \vdash \varphi \leftrightarrow \psi$. By the inductive hypothesis, there is a formula χ with the same free variables as ψ such that $\Gamma_m \vdash \psi \leftrightarrow \chi$. Then $\Gamma_{m+1} \vdash \varphi \leftrightarrow \chi$. □

Theorem 4.33. *Let m be an integer ≥ 2, and suppose that $(\mathscr{L}_{i+1}, \Gamma_{i+1})$ is a simple definitional expansion of $(\mathscr{L}_i, \Gamma_i)$ for each $i < m$. Also assume that $\mathscr{L}_0$ is finite. Suppose that φ is an $\mathscr{L}_0$ formula and $\Gamma_m \vdash \varphi$. Then $\Gamma_0 \vdash \varphi$.*

Proof. By induction on m. If $m = 2$, the conclusion follows from Theorem 4.31. now assume the result for m and suppose that $(\mathscr{L}_{i+1}, \Gamma_{i+1})$ is a simple definitional expansion of $(\mathscr{L}_i, \Gamma_i)$ for each $i \leq m$. Suppose that φ is an $\mathscr{L}_0$ formula and $\Gamma_{m+1} \vdash \varphi$. Then by Theorem 4.31, $\Gamma_m \vdash \varphi$, and so by the inductive assumption, $\Gamma_0 \vdash \varphi$. □

EXERCISES

E4.1. Suppose that $\Gamma \vdash \varphi \to \psi$, $\Gamma \vdash \varphi \to \neg\psi$, and $\Gamma \vdash \neg\varphi \to \varphi$. Prove that Γ is inconsistent.

E4.2. Let $\mathscr{L}$ be a language with just one non-logical constant, a binary relation symbol $\mathbf{R}$. Let Γ consist of all sentences of the form $\exists v_1 \forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \varphi]$ with φ a formula with only v_0 free. Show that Γ is inconsistent. Hint: take φ to be $\neg\mathbf{R}v_0v_0$.

E4.3. Show that the first-order deduction theorem fails if the condition that φ is a sentence is omitted. Hint: take $\Gamma = \emptyset$, let φ be the formula $v_0 = v_1$, and let ψ be the formula $v_0 = v_2$.

E4.4. In the language for $\overline{A} \stackrel{\text{def}}{=} (\omega, S, 0, +, \cdot)$, let τ be the term $v_0 + v_1 \cdot v_2$ and ν the term $v_0 + v_2$. Let a be the sequence $\langle 0, 1, 2, \ldots \rangle$. Let ρ be obtained from τ by replacing the occurrence of v_1 by ν.

(a) Describe ρ as a sequence of integers.
(b) What is $\rho^{\overline{A}}(a)$?
(c) What is $\nu^{\overline{A}}(a)$?
(d) Describe the sequence $a^1_{\nu^{\overline{A}}(a)}$ as a sequence of integers.
(e) Verify that $\rho^{\overline{A}}(a) = \tau^{\overline{A}}(a^1_{\nu^{\overline{A}}(a)})$ (cf. Lemma 4.4.)

E4.5. In the language for $\overline{A} \stackrel{\text{def}}{=} (\omega, S, 0, +, \cdot)$, let φ be the formula $\forall v_0(v_0 \cdot v_1 = v_1)$, let ν be the formula $v_1 + v_1$, and let $a = \langle 1, 0, 1, 0, \ldots \rangle$.

(a) Describe $\mathrm{Subf}^{v_1}_{\nu}\varphi$ as a sequence of integers
(b) What is $\nu^{\overline{A}}(a)$?
(c) Describe $a^1_{\nu^{\overline{A}}(a)}$ as a sequence of integers.
(d) Determine whether $\overline{A} \models \mathrm{Subf}^{v_1}_{\nu}\varphi[a]$ or not.
(e) Determine whether $\overline{A} \models \varphi[a^1_{\nu^{\overline{A}}(a)}]$ or not.

E4.6. Show that the condition in Lemma 4.6 that

no free occurrence of v_i in φ is within a subformula of the form $\forall v_k \mu$ with v_k a variable occurring in ν

is necessary for the conclusion of the lemma.

E4.7. Let $\overline{A}$ be an $\mathscr{L}$-structure, with $\mathscr{L}$ arbitrary. Define $\Gamma = \{\varphi : \varphi$ is a sentence and $\overline{A} \models \varphi[a]$ for any $a : \omega \to A\}$. Prove that Γ is complete and consistent.

E4.8. Call a set Γ *strongly complete* iff for every formula φ, $\Gamma \vdash \varphi$ or $\Gamma \vdash \neg\varphi$. Prove that if Γ is strongly complete, then $\Gamma \vdash \forall v_0 \forall v_1(v_0 = v_1)$.

E4.9. Prove that if Γ is rich, then for every term σ with no variables occurring in σ there is an individual constant $\mathbf{c}$ such that $\Gamma \vdash \sigma = \mathbf{c}$.

E4.10. Prove that if Γ is rich, then for every sentence φ there is a sentence ψ with no quantifiers in it such that $\Gamma \vdash \varphi \leftrightarrow \psi$.

E4.11. Describe sentences in a language for ordering which say that $<$ is a linear ordering and there are infinitely many elements. Prove that the resulting set Γ of sentences is not complete.

E4.12. Prove that if a sentence φ holds in every infinite model of a set Γ of sentences, then there is an $m \in \omega$ such that it holds in every model of Γ with at least m elements.

E4.13. Let $\mathscr{L}$ be the language of ordering. Prove that there is no set Γ of sentences whose models are exactly the well-ordering structures.

E4.14. Suppose that Γ is a set of sentences, and φ is a sentence. Prove that if $\Gamma \models \varphi$, then $\Delta \models \varphi$ for some finite $\Delta \subseteq \Gamma$.

E4.15. Suppose that f is a function mapping a set M into a set N. Let $R = \{(a, b) : a, b \in M \text{ and } f(a) = f(b)\}$. Prove that R is an equivalence relation on M.

E4.16. Suppose that R is an equivalence relation on a set M. Prove that there is a function f mapping M into some set N such that $R = \{(a, b) : a, b \in M \text{ and } f(a) = f(b)\}$.

E4.17. Let Γ be a set of sentences in a first-order language, and let Δ be the collection of all sentences holding in every model of Γ. Prove that $\Delta = \{\varphi : \varphi \text{ is a sentence and } \Gamma \vdash \varphi\}$.

5. The axioms of set theory

ZFC, the axioms of set theory, are formulated in a language which has just one nonlogical constant, a binary relation symbol $\in$. The development of set theory can be considered as taking place entirely within this language, or in various finite definitional extensions of it.

Before introducing any set-theoretic axioms at all, we can introduce some more abbreviations.

$x \subseteq y$ abbreviates $\forall z(z \in x \to z \in y)$.
$x \subset y$ abbreviates $x \subseteq y \wedge x \neq y$.

For $x \subseteq y$ we say that x is *included* or *contained* in y, or that x is a subset of y. Then $x \subset y$ means *proper* inclusion, containment, or subset.

Now we introduce the axioms of ZFC set theory. We give both a formal and informal description of them. The informal versions will suffice for much of these notes.

Axiom 1. (Extensionality) If two sets have the same members, then they are equal. Formally:

$$\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \to x = y].$$

Note that the other implication here holds on the basis of logic.

Axiom 2. (Comprehension) Given any set z and any property φ, there is a subset of z consisting of those elements of z with the property φ.

Formally, for any formula φ with free variables among $x, z, w_1, \ldots, w_n$ we have an axiom

$$\forall z \forall w_1 \ldots \forall w_n \exists y \forall x(x \in y \leftrightarrow x \in z \wedge \varphi).$$

Note that the variable y is *not* free in φ.

From these first two axioms the existence of a set with no members, the *empty set* $\emptyset$, follows:

Proposition 5.1. *There is a unique set with no members.*

Proof. On the basis of logic, there is at least one set z. By the comprehension axiom, let y be a set such that $\forall x(x \in y \leftrightarrow x \in z \wedge x \neq x)$. Thus y does not have any elements. By the extensionality axiom, such a set y is unique. $\square$

Proposition 5.1 is written in usual mathematical fashion. More formally we would write

$$\text{ZFC} \vdash \exists v_0[\forall v_1[\neg(v_1 \in v_0)] \wedge \forall v_2[\forall v_1[\neg(v_1 \in v_2)] \to v_0 = v_2]].$$

The same applies to most of the results which we will state. But some results are metatheorems, describing a whole collection of results of this sort.

In general, the set asserted to exist in the comprehension axiom is unique; we denote it by $\{x \in z : \varphi\}$. We sometimes write $\{x : \varphi\}$ if a suitable z is evident. Note that this notation cannot be put into the framework of definitional extensions. But it is clear that uses of it can be eliminated, if necessary.

Axiom 3. (Pairing) For any sets x, y there is a set which has them as members (possibly along with other sets). Formally:

$$\forall x \forall y \exists z (x \in z \wedge y \in z).$$

The *unordered pair* $\{x, y\}$ is by definition the set $\{u \in z : u = x \text{ or } u = y\}$, where z is as in the pairing axiom. The definition does not depend on the particular such z that is chosen. This same remark can be made for several other definitions below. We define the *singleton* $\{x\}$ to be $\{x, x\}$.

Axiom 4. (Union) For any family $\mathscr{A}$ of sets, we can form a new set A which has as elements all elements which are in at least one member of $\mathscr{A}$ (maybe A has even more elements). Formally:

$$\forall \mathscr{A} \exists A \forall Y \forall x (x \in Y \wedge Y \in \mathscr{A} \rightarrow x \in A).$$

With A as in this axiom, we define $\bigcup \mathscr{A} = \{x \in A : \exists Y \in \mathscr{A} (x \in Y)\}$. We call $\bigcup \mathscr{A}$ the *union* of $\mathscr{A}$. Also, let $x \cup y = \bigcup \{x, y\}$. This is the *union* of x and y.

Axiom 5. (Power set) For any set x, there is a set which has as elements all subsets of x, and again possibly has more elements. Formally:

$$\forall x \exists y \forall z (z \subseteq x \rightarrow z \in y).$$

Axiom 6. (Infinity) There is a set which intuitively has infinitely many elements:

$$\exists x [\emptyset \in x \wedge \forall y \in x (y \cup \{y\} \in x)].$$

If we take the smallest set x with these properties we get the natural numbers, as we will see later.

Axiom 7. (Replacement) If a function has domain a set, then its range is also a set. Here we use the intuitive notion of a function. Later we define the rigorous notion of a function. The present intuitive notion is more general, however; it is expressed rigorously as a formula with a function-like property. The rigorous version of this axiom runs as follows.

For each formula with free variables among $x, y, A, w_1, \ldots, w_n$, the following is an axiom.

$$\forall A \forall w_1 \ldots \forall w_n [\forall x \in A \exists! y \varphi \rightarrow \exists Y \forall x \in A \exists y \in Y \varphi].$$

For the next axiom, we need another definition. For any sets x, y, let $x \cap y = \{z \in x : z \in y\}$. This is the *intersection* of x and y.

Axiom 8. (Foundation) Every nonempty set x has a member y which has no elements in common with x. This is a somewhat mysterious axiom which rules out such anti-intuitive situations as $a \in a$ or $a \in b \in a$.

$$\forall x [x \neq \emptyset \rightarrow \exists y \in x (x \cap y = \emptyset)].$$

Axiom 9. (Choice) This axiom will be discussed carefully later; it allows one to pick out elements from each of an infinite family of sets. A convenient form of the axiom to start with is as follows. For any family $\mathscr{A}$ of nonempty sets such that no two members of $\mathscr{A}$ have an element in common, there is a set B having exactly one element in common with each member of $\mathscr{A}$.

$$\forall\mathscr{A}[\forall x \in \mathscr{A}(x \neq \emptyset) \wedge \forall x \in \mathscr{A}\forall y \in \mathscr{A}(x \neq y \to x \cap y = \emptyset) \to \exists\mathscr{B}\forall x \in \mathscr{A}\exists! y(y \in x \wedge y \in \mathscr{B}).$$

The axiom of choice will not be used until later, where we will give several equivalent forms of it.

6. Elementary set theory

Here we will see how the axioms are used to develop very elementary set theory. The axiom of choice is not used in this chapter. To some extent the main purpose of this chapter is to establish common notation.

The proof of the following theorem shows what happens to Russell's paradox in our axiomatic development. Russell's paradox runs as follows, working in naive, non-axiomatic set theory. Let $x = \{y : y \notin y\}$. If $x \in x$, then $x \notin x$; but also if $x \notin x$, then $x \in x$. Contradiction.

Theorem 6.1. $\neg\exists z\forall x(x \in z)$.

Proof. Suppose to the contrary that $\forall x(x \in z)$. Let $y = \{x \in z : x \notin x\}$. Then $(y \in y \leftrightarrow y \notin y)$, contradiction. $\square$

Lemma 6.2. *If $\{x, y\} = \{u, v\}$, then one of the following conditions holds:*
(i) $x = u$ and $y = v$;
(ii) $x = v$ and $y = u$.

Proof. Since $x \in \{x, y\} = \{u, v\}$, we have $x = u$ or $x = v$.

Case 1. $x = u$. Since $y \in \{x, y\} = \{u, v\}$, we have $y = u$ or $y = v$. If $y = v$, that is as desired. If $y = u$, then $x = y$ too, and $v \in \{u, v\} = \{x, y\}$, so $v = x = y$. In any case, $y = v$.

Case 2. $x = v$. By symmetry to case 1, $y = u$. $\square$

Now we can define the notion of an *ordered pair*: $(x, y) = \{\{x\}, \{x, y\}\}$.

Lemma 6.3. *If $(x, y) = (u, v)$, then $x = u$ and $y = v$.*

Proof. Assume that $(x, y) = (u, v)$. Thus $\{\{x\}, \{x, y\}\} = \{\{u\}, \{u, v\}\}$. By Lemma 6.1, this gives two cases.

Case 1. $\{x\} = \{u\}$ and $\{x, y\} = \{u, v\}$. Then $x \in \{x\} = \{u\}$, so $x = u$. By Lemma 6.1 again, $\{x, y\} = \{u, v\}$ implies that either $y = v$, or else $x = v$ and $y = u$; in the latter case, $y = u = x = v$. So $y = v$ in any case.

Case 2. $\{x\} = \{u, v\}$ and $\{x, y\} = \{u\}$. Then $u \in \{u, v\} = \{x\}$, so $u = x$. Similarly $v = x$. Now $y \in \{x, y\} = \{u\}$, so $y = u = x = v$. $\square$

This lemma justifies the following definition:

$$1^{\text{st}}(a, b) = a \quad \text{and} \quad 2^{\text{nd}}(a, b) = b.$$

These are the *first and second coordinates* of the ordered pair.

The notion of intersection is similar to that of union, but there is a minor problem concerning what to define the intersection of the empty set to be. We have decided to let it be the empty set.

Theorem 6.4. *For any set $\mathscr{F}$ there is a set y such that if $\mathscr{F} \neq \emptyset$ then $\forall x[x \in y \leftrightarrow \forall z \in \mathscr{F}[x \in z]]$, while $y = \emptyset$ if $\mathscr{F} = \emptyset$.*

Proof. Let $\mathscr{F}$ be given. If $\mathscr{F} = \emptyset$, let $y = \emptyset$. Otherwise, choose $w \in \mathscr{F}$ and let $y = \{x \in w : \forall z \in \mathscr{F}[x \in z]\}$. $\square$

The set y in Theorem 6.4 is clearly unique, and we denote it by $\bigcap \mathscr{F}$. This is the *intersection* of $\mathscr{F}$. We already introduced in Chapter 2 the notations $\bigcup$, $\cup$, and $\cap$. To round out the simple Boolean operations we define

$$A \backslash B = \{x \in A : x \notin B\}.$$

This is the *relative complement* of B in A.

Sets a, b are *disjoint* iff $a \cap b = \emptyset$.

The replacement schema will almost always be used in connection with the comprehension schema. Namely, under the assumption $\forall x \in A \exists! y \varphi(x, y)$, we choose Y by the replacement axiom, so that $\forall x \in A \exists y \in Y \varphi(x, y)$; then we form

$$\{y \in Y : \exists x \in A \varphi(x, y)\}.$$

Lemma 6.5. $\forall A \forall B \exists Z \forall z(z \in Z \leftrightarrow \exists x \in A \exists y \in B(z = (x, y)))$.

Proof. Define

$$Z = \{z \in \mathscr{P}(\mathscr{P}(A \cup B)) : \exists a \in A \exists b \in B[z = (a, b)]\}.$$

Thus if $z \in Z$ then $\exists a \in A \exists b \in B[z = (a, b)]$. Now suppose that $a \in A$, $b \in B$, and $z = (a, b)$. Then $a, b \in A \cup B$, so $\{a\}, \{a, b\} \in \mathscr{P}(A \cup B)$, and so $z = (a, b) = \{\{a\}, \{a, b\}\} \in \mathscr{P}(\mathscr{P}(A \cup B))$, and hence $z \in Z$. $\square$

We now define $A \times B$ to be the unique Z of Lemma 6.5; this is the *cartesian product* of A and B. Normally we would define $A \times B$ as follows:

$$A \times B = \{(x, y) : x \in A \wedge y \in B\}.$$

This notation means

$$\{u : \exists x, y(u = (x, y) \wedge x \in A \wedge y \in B)\},$$

which is justified by the lemma.

An important informal notation is

$$(*) \qquad \{\tau(x, y) \in A : \varphi(x, y)\},$$

where $\tau(x, y)$ is some set determined by x and y. That is, there is a formula $\psi(w, x, y)$ in our set theoretic language such that $\forall x, y \exists! w \psi(w, x, y)$, and $\tau(x, y)$ is this w. For example $\tau(x, y)$ might be $x \cup y$, or (x, y). Then $(*)$ really means

$$(**) \qquad \{w \in A : \exists x, y[\psi(w, x, y) \wedge \varphi(x, y)]\}.$$

A *relation* is a set of ordered pairs.

Lemma 6.6. *If $(x, y) \in R$ then $x, y \in \bigcup\bigcup R$.*

Proof. $x \in \{x\} \in \{\{x\}, \{x, y\}\} = (x, y) \in R$, so $x \in \bigcup\bigcup R$. Similarly $y \in \bigcup\bigcup R$. □

This lemma justifies the following definitions of the *domain* and *range* of a set R (we think of R as a relation, but the definitions apply to any set):

$$\begin{aligned} \mathrm{dmn}(R) &= \{x : \exists y((x, y) \in R)\}; \\ \mathrm{rng}(R) &= \{y : \exists x((x, y) \in R)\}. \end{aligned}$$

Now we define, using the notation above,

$$R^{-1} = \{(x, y) \in \mathrm{rng}(R) \times \mathrm{dmn}(R) : (y, x) \in R\}.$$

This is the *inverse* or *converse* of R. Note that R^{-1} is a relation, even if R is not. Clearly $(x, y) \in R^{-1}$ iff $(y, x) \in R$, for any x, y, R. Usually we use this notation only when R is a function (defined shortly as a special kind of relation), and even then it is more general than one might expect, since the function in question does not have to be 1-1 (another notion defined shortly).

A *function* is a relation f such that

$$\forall x \in \mathrm{dmn}(f) \exists! y \in \mathrm{rng}(f)[(x, y) \in f].$$

Some common notation and terminology is as follows. $f : A \to B$ means that f is a function, $\mathrm{dmn}(f) = A$, and $\mathrm{rng}(f) \subseteq B$. We say then that f *maps* A *into* B. If $f : A \to B$ and $x \in A$, then $f(x)$ is the unique y such that $(x, y) \in f$. This is the *value of* f *with the argument* x. We may write other things like f_x, f^x in place of $f(x)$. Note that if $f, g : A \to B$, then $f = g$ iff $\forall a \in A[f(a) = g(a)]$. If $f : A \to B$ and $C \subseteq A$, the *restriction* of f to C is $f \cap (C \times B)$; it is denoted by $f \restriction C$. The *image* of a subset C of A is $f[C] \stackrel{\mathrm{def}}{=} \mathrm{rng}(f \restriction C)$. Note that $f[C] = \{f(c) : c \in C\}$. If $D \subseteq B$ then the *preimage* of D under f is $f^{-1}[D] \stackrel{\mathrm{def}}{=} \{x \in A : f(x) \in D\}$.

For any sets f, g we define

$$f \circ g = \{(a, b) : \exists c[(a, c) \in g \text{ and } (c, b) \in f]\}.$$

This is the *composition* of f and g. We usually apply this notation when there are sets A, B, C such that $g : A \to B$ and $f : B \to C$.

Lemma 6.7. *(i) If $g : A \to B$ and $f : B \to C$, then $(f \circ g) : A \to C$ and $\forall a \in A[(f \circ g)(a) = f(g(a))]$.*

(ii) If $g : A \to B$, $f : B \to C$, and $h : C \to D$, then $h \circ (f \circ g) = (h \circ f) \circ g$.

Proof. (i): First we show that $f \circ g$ is a function. Suppose that $(a, b), (a, b') \in (f \circ g)$. Accordingly choose c, c' so that $(a, c) \in g$, $(c, b) \in f$, $(a, c') \in g$, and $(c', b') \in f$. Then $g(a) = c$, $f(c) = b$, $g(a) = c'$, and $f(c') = b'$. So $c = c'$ and hence $b = b'$. This shows that $f \circ g$ is a function. Clearly $\text{dmn}(f \circ g) = A$ and $\text{rng}(f \circ g) \subseteq C$. For any $a \in A$ we have $(a, g(a)) \in g$ and $(g(a), f(g(a))) \in f$, and hence $(a, f(g(a))) \in (f \circ g)$, so that $(f \circ g)(a) = f(g(a))$.

(ii): By (i), both functions map A into D. For any $a \in A$ we have

$$(h \circ (f \circ g))(a) = h((f \circ g)(a)) = h(f(g(a))) = (h \circ f)(g(a)) = ((h \circ f) \circ g)(a).$$

Hence the equality holds. □

Given $f : A \to B$, we call f *injective*, or 1-1, if f^{-1} is a function; we call f *surjective*, or *onto*, if $\text{rng}(f) = B$; and we call f *bijective* if it is both injective and surjective.

A function f will sometimes be written in the form $\langle f(i) : i \in I \rangle$, where $I = \text{dmn}(f)$. As an informal usage, we will even define functions in the form $\langle \ldots x \ldots : x \in I \rangle$, meaning the function f with domain I such that $f(x) = \ldots x \ldots$ for all $x \in I$.

If A is a function with domain I, we define

$$\bigcup_{i \in I} A_i = \bigcup \text{rng}(A) \quad \text{and} \quad \bigcap_{i \in I} A_i = \bigcap \text{rng}(A).$$

EXERCISES

In the exercises that ask for counterexamples, it is reasonable to use any prior knowledge rather that restricting to the material in these notes.

E6.1. Prove that if $f : A \to B$ and $\langle C_i : i \in I \rangle$ is a system of subsets of A, then $f\left[\bigcup_{i \in I} C_i\right] = \bigcup_{i \in I} f[C_i]$.

E6.2. Prove that if $f : A \to B$ and $C, D \subseteq A$, then $f[C \cap D] \subseteq f[C] \cap f[D]$. Give an example showing that equality does not hold in general.

E6.3. Given $f : A \to B$ and $C, D \subseteq A$, compare $f[C \backslash D]$ and $f[C] \backslash f[D]$: prove the inclusions (if any) which hold, and give counterexamples for the inclusions that fail to hold.

E6.4. Prove that if $f : A \to B$ and $\langle C_i : i \in I \rangle$ is a system of subsets of B, then $f^{-1}\left[\bigcup_{i \in I} C_i\right] = \bigcup_{i \in I} f^{-1}[C_i]$.

E6.5. Prove that if $f : A \to B$ and $\langle C_i : i \in I \rangle$ is a system of subsets of B, then $f^{-1}\left[\bigcap_{i \in I} C_i\right] = \bigcap_{i \in I} f^{-1}[C_i]$.

E6.6. Prove that if $f : A \to B$ and $C, D \subseteq B$, then $f^{-1}[C \backslash D] = f^{-1}[C] \backslash f^{-1}[D]$.

E6.7. Prove that if $f : A \to B$ and $C \subseteq A$, then

$$\{b \in B : f^{-1}[\{b\}] \subseteq C\} = B \backslash f[A \backslash C].$$

E6.8. For any sets A, B define $A \triangle B = (A \backslash B) \cup (B \backslash A)$; this is called the *symmetric difference* of A and B. Prove that if A, B, C are given sets, then $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.

E6.9. For any set A let

$$\text{Id}_A = \{\langle x, x \rangle : x \in A\}.$$

This is the *identity function* on the set A. Justify this definition on the basis of the axioms.

E6.10. Suppose that $f : A \to B$. Prove that f is surjective iff there is a $g : B \to A$ such that $f \circ g = \mathrm{Id}_B$. Note: the axiom of choice might be needed.

E6.11. Let A be a nonempty set. Suppose that $f : A \to B$. Prove that f is injective iff there is a $g : B \to A$ such that $g \circ f = \mathrm{Id}_A$.

E6.12. Suppose that $f : A \to B$. Prove that f is a bijection iff there is a $g : B \to A$ such that $f \circ g = \mathrm{Id}_B$ and $g \circ f = \mathrm{Id}_A$. Prove this without using the axiom of choice.

E6.16. For any sets R, S define

$$R|S = \{\langle x, z\rangle : \exists y(\langle x, y\rangle \in R \wedge \langle y, z\rangle \in S)\}.$$

This is the *relative product* of R and S. Justify this definition on the basis of the axioms.

E6.14. Suppose that $f, g : A \to A$. Prove that

$$(A \times A)\backslash[((A \times A)\backslash f)|((A \times A)\backslash g)]$$

is a function.

E6.15. Suppose that $f : A \to B$ is a surjection, $g : A \to C$, and $\forall x, y \in A[f(x) = f(y) \to g(x) = g(y)]$. Prove that there is a function $h : B \to C$ such that $h \circ f = g$. Define h as a set of ordered pairs.

E6.16. The statement

$\forall A \in \mathscr{A} \forall B \in \mathscr{B}(A \subseteq B)$ implies that $\bigcup \mathscr{A} \subseteq \bigcap \mathscr{B}$

is slightly wrong. Fix it, and prove the result.

E6.17. Suppose that $\forall A \in \mathscr{A} \exists B \in \mathscr{B}(A \subseteq B)$. Prove that $\bigcup \mathscr{A} \subseteq \bigcup \mathscr{B}$.

E6.18. The statement

$\forall A \in \mathscr{A} \exists B \in \mathscr{B}(B \subseteq A)$ implies that $\bigcap \mathscr{B} \subseteq \bigcap \mathscr{A}$.

is slightly wrong. Fix it, and prove the result.

7. Ordinals, I

In this chapter we introduce the ordinals and give basic facts about them.

A set A is *transitive* iff $\forall x \in A \forall y \in x(y \in A)$; in other words, iff every element of A is a subset of A. This is a very important notion in the foundations of set theory, and it is essential in our definition of ordinals. An *ordinal number*, or simply an *ordinal*, is a transitive set of transitive sets. We use the first few Greek letters to denote ordinals. If α, β, γ are ordinals and $\alpha \in \beta \in \gamma$, then $\alpha \in \gamma$ since γ is transitive. This partially justifies writing $\alpha < \beta$ instead of $\alpha \in \beta$ when α and β are ordinals. This helps the intuition in thinking of the ordinals as kinds of numbers. We also define $\alpha \leq \beta$ iff $\alpha < \beta$ or $\alpha = \beta$.

By a vacuous implication we have:

Proposition 7.1. *$\emptyset$ is an ordinal.* $\square$

Because of this proposition, the empty set is a number; it will turn out to be the first nonnegative integer, the first ordinal, and the first cardinal number. For this reason, we will use 0 and $\emptyset$ interchangably, trying to use 0 when numbers are involved, and $\emptyset$ when they are not.

Proposition 7.2. *If α is an ordinal, then so is $\alpha \cup \{\alpha\}$.*

Proof. If $x \in y \in \alpha \cup \{\alpha\}$, then $x \in y \in \alpha$ or $x \in y = \alpha$. Since α is transitive, $x \in \alpha$ in either case. So $\alpha \cup \{\alpha\}$ is transitive. Clearly every member of $\alpha \cup \{\alpha\}$ is transitive. $\square$

We denote $\alpha \cup \{\alpha\}$ by $\alpha +' 1$. After introducing addition of ordinals, it will turn out that $\alpha + 1 = \alpha +' 1$ for every ordinal α, so that the prime can be dropped. This ordinal $\alpha +' 1$ is the *successor* of α. We define $1 = 0 +' 1$, $2 = 1 +' 1$, etc. (up through 9; no further since we do not want to try to justify decimal notation).

Proposition 7.3. *If A is a set of ordinals, then $\bigcup A$ is an ordinal.*

Proof. Suppose that $x \in y \in \bigcup A$. Choose $z \in A$ such that $y \in z$. Then z is an ordinal, and $x \in y \in z$, so $x \in z$; hence $x \in \bigcup A$. Thus $\bigcup A$ is transitive.

If $u \in \bigcup A$, choose $v \in A$ such that $u \in v$. then v is an ordinal, so u is transitive. $\square$

We sometimes write $\sup(A)$ for $\bigcup A$. In fact, $\bigcup A$ is the least ordinal $\geq$ each member of A. We prove this shortly.

Proposition 7.4. *Every member of an ordinal is an ordinal.*

Proof. Let α be an ordinal, and let $x \in \alpha$. Then x is transitive since all members of α are transitive. Suppose that $y \in x$. Then $y \in \alpha$ since α is transitive. So y is transitive, since all members of α are transitive. $\square$

Theorem 7.5. *$\forall x(x \notin x)$.*

Proof. Suppose that x is a set such that $x \in x$. Let $y = \{x\}$. By the foundation axiom, choose $z \in y$ such that $z \cap y = \emptyset$. But $z = x$, so $x \in z \cap y$, contradiction. $\square$

Theorem 7.6. *There does not exist a set which has every ordinal as a member.*

Proof. Suppose to the contrary that A is such a set. Let $B = \{x \in A : x \text{ is an ordinal}\}$. Then B is a set of transitive sets and B itself is transitive. Hence B is an ordinal. So $B \in A$. It follows that $B \in B$. contradicting Theorem 7.5. $\square$

Theorem 7.6 is what happens in our axiomatic framework to the Burali-Forti paradox.

Theorem 7.7. *If α and β are ordinals, then $\alpha = \beta$, $\alpha \in \beta$, or $\beta \in \alpha$.*

Proof. Suppose that this is not true, and let α and β be ordinals such that $\alpha \neq \beta$, $\alpha \notin \beta$, and $\beta \notin \alpha$. Let $A = (\alpha +' 1) \cup (\beta +' 1)$. Define $B = \{\gamma \in A : \exists \delta \in A[\gamma \neq \delta, \gamma \notin \delta, \text{ and } \delta \notin \gamma]\}$. Thus $\alpha \in B$, since we can take $\delta = \beta$. So $B \neq \emptyset$. By the foundation axiom, choose $\gamma \in B$ such that $\gamma \cap B = \emptyset$. Let $C = \{\delta \in A : \gamma \neq \delta, \gamma \notin \delta, \text{ and } \delta \notin \gamma\}$. So $C \neq \emptyset$ since $\gamma \in B$. By the foundation axiom choose $\delta \in C$ such that $\delta \cap C = \emptyset$. We will now show that $\gamma = \delta$, which is a contradiction.

Suppose that $\varepsilon \in \gamma$. Then $\varepsilon \notin B$. Clearly $\varepsilon \in A$, so it follows that $\forall \varphi \in A[\varepsilon = \varphi$ or $\varepsilon \in \varphi$ or $\varphi \in \varepsilon]$. Since $\delta \in A$ we thus have $\varepsilon = \delta$ or $\varepsilon \in \delta$ or $\delta \in \varepsilon$. If $\varepsilon = \delta$ then $\delta \in \gamma$, contradiction. If $\delta \in \varepsilon$, then $\delta \in \gamma$ since γ is transitive, contradiction. So $\varepsilon \in \delta$. This proves that $\gamma \subseteq \delta$.

Suppose that $\varepsilon \in \delta$. Then $\varepsilon \notin C$. It follows that $\gamma = \varepsilon$ or $\gamma \in \varepsilon$ or $\varepsilon \in \gamma$. If $\gamma = \varepsilon$ then $\gamma \in \delta$, contradiction. If $\gamma \in \varepsilon$ then $\gamma \in \delta$ since δ is transitive, contradiction. So $\varepsilon \in \gamma$. This proves that $\delta \subseteq \gamma$.

Hence $\delta = \gamma$, contradiction. $\square$

Proposition 7.8. *$\alpha \leq \beta$ iff $\alpha \subseteq \beta$.*

Proof. $\Rightarrow$: Assume that $\alpha \leq \beta$ and $x \in \alpha$. Then $x < \alpha \leq \beta$, so $x < \beta$ since β is transitive. Hence $x \in \beta$. Thus $\alpha \subseteq \beta$.

$\Leftarrow$: Assume that $\alpha \subseteq \beta$. If $\beta < \alpha$, then $\beta < \beta$, hence $\beta \in \beta$, contradicting Theorem 7.5. Hence $\alpha \leq \beta$ by Theorem 7.7. $\square$

Proposition 7.9. *$\alpha < \beta$ iff $\alpha \subset \beta$.*

Proof. $\alpha < \beta$ iff ($\alpha \leq \beta$ and $\alpha \neq \beta$) iff ($\alpha \subseteq \beta$ and $\alpha \neq \beta$) (by Proposition 7.8) iff $\alpha \subset \beta$. $\square$

Proposition 7.10. *$\alpha < \beta$ iff $\alpha +' 1 \leq \beta$.*

Proof. $\Rightarrow$: Assume that $\alpha < \beta$. If $\beta < \alpha +' 1$, then $\beta \in \alpha \cup \{\alpha\}$, so $\beta \in \alpha$ or $\beta = \alpha$. Since $\alpha \in \beta$, this implies that $\beta \in \beta$, contradicting Theorem 7.5. Hence by Theorem 7.7, $\alpha +' 1 \leq \beta$.

$\Leftarrow$: Assume that $\alpha +' 1 \leq \beta$. Then $\alpha < \alpha +' 1 \leq \beta$, so $\alpha < \beta$. $\square$

Proposition 7.11. *There do not exist ordinals α, β such that $\alpha < \beta < \alpha +' 1$.* $\square$

Theorem 7.12. *If A is a set of ordinals, then $\alpha \leq \bigcup A$ for each $\alpha \in A$, and if β is an ordinal such that $\alpha \leq \beta$ for all $\alpha \in A$ then $\bigcup A \leq \beta$.*

Proof. Suppose that A is a set of ordinals. If $\alpha \in A$, then $\alpha \subseteq \bigcup A$, and so $\alpha \leq \bigcup A$ by Proposition 7.8.

Now suppose that β is an ordinal such that $\alpha \leq \beta$ for all $\alpha \in A$. Take any $x \in \bigcup A$. Choose $y \in A$ such that $x \in y$. Then $y \leq \beta$. Also $x < y$, so $x < \beta$. Hence $x \in \beta$. This proves that $\bigcup A \subseteq \beta$. Hence $\bigcup A \leq \beta$ by Proposition 7.8. $\square$

Theorem 7.13. *If Γ is a nonempty set of ordinals, then $\bigcap \Gamma$ is an ordinal, $\bigcap \Gamma \in \Gamma$, and $\bigcap \Gamma \leq \alpha$ for every $\alpha \in \Gamma$.*

Proof. The members of $\bigcap \Gamma$ are clearly ordinals, so for the first statement it suffices to show that $\bigcap \Gamma$ is transitive. Suppose that $\alpha \in \beta \in \bigcap \Gamma$; and suppose that $\gamma \in \Gamma$. Then $\beta \in \gamma$, and hence $\alpha \in \gamma$ since γ is transitive. This argument shows that $\alpha \in \bigcap \Gamma$. Since α is arbitrary, it follows that $\bigcap \Gamma$ is transitive, and hence is an ordinal.

For every $\alpha \in \Gamma$ we have $\bigcap \Gamma \subseteq \alpha$, and hence $\bigcap \Gamma \leq \alpha$ by Proposition 7.8.

Suppose that $\bigcap \Gamma \notin \Gamma$. For any $\alpha \in \Gamma$ we have $\bigcap \Gamma \subseteq \alpha$, hence $\bigcap \Gamma \leq \alpha$, hence $\bigcap \Gamma < \alpha$ since $\alpha \in \Gamma$ but we are assuming that $\bigcap \Gamma \notin \Gamma$. But this means that $\forall \alpha \in \Gamma[\bigcap \Gamma \in \alpha]$. So $\bigcap \Gamma \in \bigcap \Gamma$, contradiction. $\square$

Ordinals are divided into three classes as follows. First there is 0, the empty set. An ordinal α is a *successor ordinal* if $\alpha = \beta +' 1$ for some β. Finally, α is a *limit ordinal* if it is nonzero and is not a successor ordinal. Thus 1, 2, etc. are successor ordinals.

To prove the existence of limit ordinals, we need the infinity axiom. Let x be as in the statement of the infinity axiom. Thus $0 \in x$, and $y \cup \{y\} \in x$ for all $y \in x$. We define

$$\omega = \bigcap \{z \subseteq x : 0 \in z \text{ and } y \cup \{y\} \in z \text{ for all } y \in z\}.$$

This definition does not depend on the choice of x. In fact, suppose that also $0 \in x'$, and $y \cup \{y\} \in x'$ for all $y \in x'$; we want to show that

$$\begin{aligned} &\bigcap \{z \subseteq x : 0 \in z \text{ and } y \cup \{y\} \in z \text{ for all } y \in z\} \\ &= \bigcap \{z \subseteq x' : 0 \in z \text{ and } y \cup \{y\} \in z \text{ for all } y \in z\}. \end{aligned}$$

Let $\mathscr{A} = \{z \subseteq x : 0 \in z \text{ and } y \cup \{y\} \in z \text{ for all } y \in z\}$ and $\mathscr{A}' = \{z \subseteq x' : 0 \in z \text{ and } y \cup \{y\} \in z \text{ for all } y \in z\}$. Suppose that $w \in \bigcap \mathscr{A}$, and suppose that $z \in \mathscr{A}'$. Clearly $z \cap x \in \mathscr{A}$, so $w \in z \cap x$, so $w \in z$. This shows that $w \in \bigcap \mathscr{A}'$. Hence $\bigcap \mathscr{A} \subseteq \bigcap \mathscr{A}'$. The other inclusion is proved in the same way.

The members of ω are *natural numbers.*

Theorem 7.14. *If $A \subseteq \omega$, $0 \in A$, and $y \cup \{y\} \in A$ for all $y \in A$, then $A = \omega$.*

Proof. With x as in the definition of ω, we clearly have $x \cap A \in \mathscr{A}$ where $\mathscr{A}$ is as above. Hence $\omega \subseteq x \cap A \subseteq A$, so $A = \omega$. $\square$

Proposition 7.15. $0 \in \omega$, *and for all $y \in \omega$, also $y +' 1 \in \omega$.*

Proof. With $\mathscr{A}$ as above, if $z \in \mathscr{A}$, then $0 \in z$. So $0 \in \bigcap \mathscr{A} = \omega$. Now suppose that $y \in \omega$ and $z \in \mathscr{A}$. Then $y \in z$, and it follows that $y +' 1 \in z$. Hence $y +' 1 \in \omega$. □

Theorem 7.16. *ω is the first limit ordinal.*

Proof. Let $A = \{y \in \omega : y \text{ is an ordinal}\}$. Then $0 \in A$ by Propositions 7.1 and 7.15. Suppose that $y \in A$. Then $y \in \omega$, so $y +' 1 \in \omega$ by Proposition 7.15. Also, y is an ordinal, so $y +' 1$ is an ordinal by Proposition 7.2. This shows that $y +' 1 \in A$. It follows that $A = \omega$, by Theorem 7.17. Hence every member of ω is an ordinal, and hence is transitive.

Next, let $B = \{y \in \omega : y \subseteq \omega\}$. Then $0 \in B$ by Proposition 7.15. Suppose that $y \in B$. Then $y \in \omega$, so $y +' 1 \in \omega$ by Proposition 7.15. Also, $y \subseteq \omega$. Since $y \in \omega$, it follows that $y \cup \{y\} \subseteq \omega$. So $y +' 1 \in B$. Hence $B = \omega$ by Theorem 7.17. This shows that ω is transitive, and hence is an ordinal.

Next, let $C = \{y \in \omega : y \text{ is not a limit ordinal}\}$. $0 \in \omega$ by Theorem 7.15, and by definition 0 is not a limit ordinal, so $0 \in C$. Suppose that $y \in C$. Then $y \in \omega$, so $y +' 1 \in \omega$. Also, by definition $y +' 1$ is not a limit ordinal. So $y +' 1 \in C$. It follows that $C = \omega$, and hence for every $\alpha < \omega$, α is not a limit ordinal.

Since $0 \in \omega$, $\omega \neq 0$. If $\omega = y +' 1$, then $y \in \omega$ and hence $\omega = y +' 1 \in \omega$, contradiction. Thus ω is a limit ordinal. □

Proposition 7.17. *The following conditions are equivalent:*

(i) α is a limit ordinal;

(ii) $\alpha \neq 0$, and for every $\beta < \alpha$ there is a γ such that $\beta < \gamma < \alpha$.

(iii) $\alpha = \bigcup \alpha \neq 0$.

Proof. (i)⇒(ii): suppose that α is a limit ordinal. So $\alpha \neq 0$, by definition. Suppose that $\beta < \alpha$. Then $\beta +' 1 \leq \alpha$ by Proposition 7.10. Hence $\beta +' 1 < \alpha$ since α is not a successor ordinal. Thus $\beta < \beta +' 1 < \alpha$.

(ii)⇒(iii): if $\beta \in \bigcup \alpha$, choose $\gamma \in \alpha$ such that $\beta \in \gamma$. Then $\beta \in \alpha$ since α is an ordinal. This shows that $\bigcup \alpha \subseteq \alpha$.

If $\beta \in \alpha$, choose γ with $\beta < \gamma < \alpha$. Thus $\beta \in \bigcup \alpha$. This proves that $\alpha = \bigcup \alpha$, and $\alpha \neq 0$ is given.

(iii)⇒(i): suppose that $\alpha = \beta +' 1$. Then $\beta \in \alpha = \bigcup \alpha$, so choose $\gamma \in \alpha$ such that $\beta \in \gamma$. Thus $\beta < \gamma \leq \beta$, so $\beta < \beta$, contradiction. □

Proposition 7.18. *If $\alpha = \beta +' 1$, then $\bigcup \alpha = \beta$.*

Proof. Assume that $\alpha = \beta +' 1$. Suppose that $\gamma \in \bigcup \alpha$. Choose $\delta \in \alpha$ such that $\gamma \in \delta$. Thus $\gamma < \delta < \alpha$, so $\delta \leq \beta$, hence $\gamma \in \beta$. This shows that $\bigcup \alpha \subseteq \beta$.

If $\gamma \in \beta$, then $\gamma \in \beta \in \alpha$, so $\gamma \in \bigcup \alpha$. So $\bigcup \alpha = \beta$. □

EXERCISES

These exercises give some equivalent definitions of ordinals. A *well-ordered set* is a pair $(A, <)$ such that A is a set, $<$ is a relation included in $A \times A$, $<$ is irreflexive on A ($a \not< a$ for all $a \in A$), $<$ is transitive ($a < b < c$ implies that $a < c$), $<$ is linear on A (for all

$a, b \in A$, either $a = b$, $a < b$, or $b < a$), and any nonempty subset X of A has a least element (an element $a \in X$ such that $a \leq b$ for all $b \in X$).

E7.1. Prove that if x is an ordinal, then x is transitive and $(x, \{(y, z) \in x \times x : y \in z\})$ is a well-ordered set.

E7.2. Assume that x is transitive and $(x, \{(y, z) \in x \times x : y \in z\})$ is a well-ordered set. Prove that for all $y, z \in x$, either $y = z$ or $y \in z$ or $z \in y$.

E7.3. Assume that x is transitve and for all $y, z \in x$, either $y = z$ or $y \in z$ or $z \in y$. Prove that for all y, if $y \subset x$ and y is transitive, then $y \in x$. Hint: apply the foundation axiom to $x \backslash y$.

E7.4. Assume that x is transitive and for all y, if $y \subset x$ and y is transitive, then $y \in x$. Show that x is an ordinal. Hint: let $y = \{z \in x : z \text{ is an ordinal}\}$, and get a contradiction from the assumption that $y \subset x$.

E7.5. Show that if x is an ordinal, then the following two conditions hold:

(i) For all $y \in x$, either $y \cup \{y\} = x$ or $y \cup \{y\} \in x$.

(ii) For all $y \subseteq x$, either $\bigcup y = x$ or $\bigcup y \in x$.

E7.6. Assume the two conditions of exercise E7.5. Show that x is an ordinal. Hint: Show that there is an ordinal α not in x. Taking such an ordinal α, show that there is a least $\beta \in \alpha \cup \{\alpha\}$ such that $\beta \notin x$. Work with such a β to show that x is an ordinal.

8. Recursion

In this chapter we prove a general recursion theorem which will be used many times in these notes. The theorem involves classes, so we begin with a few remarks about classes and sets.

Classes and sets

Although expressions like $\{x : x = x\}$ and $\{\alpha : \alpha \text{ is an ordinal}\}$ are natural, they cannot be put into the framework of our logic for set theory. These "collections" are "too big". It is intuitively indispensible to continue using such expressions. One should understand that when this is done, there is a rigorous way of reformulating what is said. These big collections are called *classes*; their rigorous counterparts are simply formulas of our set theoretic language. We can also talk about *class functions*, *class relations*, the *domain* of class functions, etc. Most of the notions that we have introduced so far have class counterparts. In particular, we have the important classes $\mathbf{V}$, the class of all sets, and $\mathbf{On}$, the class of all ordinals. They correspond to the formulas "$x = x$" and "α is an ordinal". We attempt to use bold face letters for classes; in some cases the classes in question are actually sets. A class which is not a set is called a *proper class.*

Well-founded class relations

If $\mathbf{A}$ is a class, a class relation $\mathbf{R}$ is *well-founded on* $\mathbf{A}$ iff $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$ and for every nonempty subset X of $\mathbf{A}$ there is an $x \in X$ such that for all $y \in X$ it is not the case that $(y, x) \in \mathbf{R}$. Such a set x is called $\mathbf{R}$-*minimal.*

This notion is important even if $\mathbf{A}$ and $\mathbf{R}$ are mere sets. Two important examples of well-founded class relations are as follows.

Proposition 8.1. *The class relation* $\{(x, y) : x \in y\}$ *is well-founded on* $\mathbf{V}$.

Proof. Let X be a nonempty subset of $\mathbf{V}$. This just means that X is a nonempty set. By the foundation axiom, choose $x \in X$ such that $x \cap X = \emptyset$. Then for all $y \in X$ it is not the case that $y \in x$. $\square$

Proposition 8.2. *The class relation* $\{(\alpha, \beta) : \alpha < \beta\}$ *is well-founded on* $\mathbf{On}$.

Proof. Let X be a nonempty subset of $\mathbf{On}$. Thus X is a nonempty set of ordinals. By Theorem 4.14 we have $\bigcap X \in X$ and for all $y \in X$ it is not the case that $y \in \bigcap X$. $\square$

On the other hand, the class relation $\mathbf{R} = \{(x, y) : y \in x\}$ is not well-founded on $\mathbf{V}$. In fact the set ω does not have an $\mathbf{R}$-minimal element, since if $m \in \omega$ then also $m +' 1 \in \omega$ and $(m +' 1, m) \in \mathbf{R}$.

Recall that our intuitive notion of class is made rigorous by using formulas instead. Thus we would talk about a formula $\varphi(x, y)$ being well-founded on another formula $\psi(x)$. In the case of $\in$, we are really looking at the formula $x \in y$ being well-founded on the formula $x = x$. So, rigorously we are associating with two formulas $\varphi(x, y)$ and $\psi(x)$ another formula "$\varphi(x, y)$ is well-founded on $\psi(x)$", namely the following formula:

$$\forall x \forall y[\varphi(x, y) \to \psi(x) \wedge \psi(y)] \wedge \forall X[\forall x \in X \psi(x) \wedge X \neq \emptyset \to \exists x \in X \forall y \in X \neg\varphi(y, x)].$$

Let $\mathbf{A}$ be a class and $\mathbf{R}$ a class relation with $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$. For any $x \in \mathbf{A}$ we define $\text{pred}_{\mathbf{AR}}(x) = \{y \in \mathbf{A} : (y, x) \in \mathbf{R}\}$. We say that $\mathbf{R}$ is *set-like* on $\mathbf{A}$ iff $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$ and $\text{pred}_{\mathbf{AR}}(x)$ is a set for all $x \in \mathbf{A}$.

For example, for $\mathbf{R} = \{(x, y) : x \in y\}$ we have $\text{pred}_{\mathbf{VR}}(x) = x$ for any set x, and $\mathbf{R}$ is set-like on $\mathbf{V}$. For $\mathbf{R} = \{(\alpha, \beta) : \alpha < \beta\}\}$ we have $\text{pred}_{\mathbf{OnR}}(\alpha) = \alpha$ for any ordinal α, and $\mathbf{R}$ is set-like on $\mathbf{On}$.

On the other hand, $\mathbf{R} = \{(\alpha, \beta) : \alpha > \beta\}$ is not set-like on $\mathbf{On}$, since for example $\text{pred}_{\mathbf{OnR}}(0) = \{\alpha : \alpha > 0\}$ and this is not a set, as otherwise $\mathbf{On} = \{0\} \cup \text{pred}_{\mathbf{OnR}}(0)$ would be a set.

Formally we are dealing with formulas $\varphi(x, y)$ and $\psi(x)$, such that $\forall x, y[\varphi(x, y) \rightarrow \psi(x) \wedge \psi(y)]$. Then $\text{pred}_{\varphi\psi}$ is the formula $\varphi(y, x)$, and "φ is set-like on ψ" is the formula

$$\forall x[\psi(x) \rightarrow \exists z \forall y[y \in z \leftrightarrow \varphi(y, x)]].$$

Now let $\mathbf{R}$ be a class relation. We define

$$\begin{aligned}\mathbf{R}^* = \{(a, b) : \exists n \in \omega \backslash 1 \exists f[&f \text{ is a function with domain } n +' 1 \text{ and} \\ &\forall i < n[(f(i), f(i +' 1)) \in \mathbf{R} \text{ and } f(0) = a \text{ and } f(n) = b]]\}.\end{aligned}$$

This is called the *transitive closure* of $\mathbf{R}$.

Formally, given a formula $\varphi(x, y)$, we define another formula φ^*:

$$\begin{aligned}&\exists n \in \omega \backslash 1 \exists f[f \text{ is a function with domain } n +' 1 \text{ and} \\ &\forall i < n[\varphi(f(i), f(i +' 1)) \text{ and } f(0) = x \text{ and } f(n) = y]]\}.\end{aligned}$$

The actual formula in our set-theoretical language is long, since we have to replace the definitions of ω, function, ordered pair, etc. by formulas involving $\in$ alone.

We actually do not need the fact that $\mathbf{R}^*$ is transitive, but we do need the following facts. If $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$ and $x \in \mathbf{A}$, let $\text{pred}'_{\mathbf{AR}}(x) = \{x\} \cup \text{pred}_{\mathbf{AR}}(x)$.

Theorem 8.3. *Let $\mathbf{R}$ be a class relation.*

(i) $\mathbf{R} \subseteq \mathbf{R}^*$.

(ii) If $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$, $x \in \mathbf{A}$, $(u, v) \in \mathbf{R}$, and $v \in \text{pred}'_{\mathbf{AR}^*}(x)$, then $u \in \text{pred}'_{\mathbf{AR}^*}(x)$.

Proof. (i): Suppose that $(a, b) \in \mathbf{R}$. Let f be the function with domain 2 such that $f(0) = a$ and $f(1) = b$. This function shows that $(a, b) \in \mathbf{R}^*$.

(ii): Assume the hypotheses. There are two cases.

Case 1. $v = x$. Then $(u, x) \in \mathbf{R}$, so by (i), $(u, x) \in \mathbf{R}^*$. Hence $u \in \text{pred}_{\mathbf{AR}}(x) \subseteq \text{pred}'_{\mathbf{AR}}(x)$.

Case 2. $v \in \text{pred}_{\mathbf{AR}^*}(x)$. Choose n and f correspondingly. Let

$$g = \{(0, u)\} \cup \{(i +' 1, f(i)) : i \leq n\}.$$

Then g is a function with domain $n +' 2$, $g(0) = u$, $g(n +' 1) = x$, and $\forall i < n +' 1[(g(i), g(i +' 1)) \in \mathbf{R}]$. Hence $u \in \text{pred}_{\mathbf{AR}^*}(x) \subseteq \text{pred}'_{\mathbf{AR}^*}(x)$. $\square$

Theorem 8.4. *If* $\mathbf{R}$ *is set-like on* $\mathbf{A}$*, then also* $\mathbf{R}^*$ *is set-like on* $\mathbf{A}$*.*

Proof. Let $x \in \mathbf{A}$; we want to show that $\mathrm{pred}_{\mathbf{AR}^*}(x)$ is a set. For each $n \in \omega\backslash 1$ let

$$\begin{aligned} D_n = \{y \in \mathbf{A} : &\text{there is a function } f \text{ with domain } n +' 1 \text{ such that} \\ &f(0) = y, f(n) = x \text{ and } \forall i < n[(f(i), f(i +' 1)) \in \mathbf{R}]\} \end{aligned}$$

We will prove by induction on n that each D_n is a set. First take $n = 1$. Now $D_1 = \{y \in \mathbf{A}$:there is a function f with domain 2 such that $f(0) = y$, $f(1) = x$, and $(y, x) \in \mathbf{R}\} = \mathrm{pred}_{\mathbf{AR}}(x)$, so D_1 is a set by hypothesis. Now assume that D_n is a set. Let $\mathbf{F}(y) = \mathrm{pred}_{\mathbf{AR}}(y)$ for each $y \in D_n$. This makes sense, since by hypothesis each class $\mathrm{pred}_{\mathbf{AR}}(y)$ is a set. So $\mathbf{F}$ is a function whose domain is the set D_n. By the replacement and comprehension axioms, its range is a set. That is, $\{\mathrm{pred}_{\mathbf{AR}}(y) : y \in D_n\}$ is a set. Now we claim

$$(*) \qquad D_{n+'1} = \bigcup\{\mathrm{pred}_{\mathbf{AR}}(y) : y \in D_n\}.$$

This claim shows that $D_{n+'1}$ is a set, completing the inductive proof.

To prove the claim, first suppose that $z \in D_{n+'1}$. Let f be a function with domain $n +' 2$ such that $f(0) = z$, $f(n +' 1) = x$, and $\forall i < n +' 1[(f(i), f(i +' 1) \in \mathbf{R}$. Define g with domain $n +' 1$ by setting $g(i) = f(i +' 1)$ for all $i < n +' 1$. Then $g(0) = f(1)$, $g(n) = f(n +' 1) = x$. and for all $i < n$, $(g(i), g(i +' 1)) = (f(i +' 1), f(i +' 2)) \in \mathbf{R}$. Hence $g(0) \in D_n$. Clearly $(z, g(0)) \in \mathbf{R}$, so $z \in \mathrm{pred}_{\mathbf{AR}}(g(0))$. Thus z is in the right side of $(*)$.

Second, suppose that z is in the right side of $(*)$. Say $z \in \mathrm{pred}_{\mathbf{AR}}(y)$ with $y \in D_n$. So $(z, y) \in \mathbf{R}$, and there is a function f with domain $n +' 1$ such that $f(0) = y$, $f(n) = x$, and $\forall i < n[(f(i), f(i +' 1)) \in \mathbf{R}]$. Define g with domain $n +' 2$ by setting $g(0) = z$ and $g(i +' 1) = f(i)$ for all $i < n +' 1$. Then $g(n +' 1) = f(n) = x$ and $\forall i < n +' 1[(g(i), g(i +' 1)) \in \mathbf{R}]$. Hence $z \in D_{n+'1}$, and the claim is proved.

Now for each $n \in \omega\backslash 1$ let $\mathbf{G}(n) = D_n$. Then $\mathbf{G}$ is a function whose domain is the set $\omega\backslash 1$, so by replacement and comprehension, it range is a set. Thus $\{D_n : n \in \omega\backslash 1\}$ is a set. Now we claim

$$\mathrm{pred}_{\mathbf{AR}^*}(x) = \bigcup\{D_n : n \in \omega\backslash 1\}.$$

This claim will finish the proof. We have

$$\begin{aligned} \bigcup\{D_n : n \in \omega\backslash 1\} = \{y \in \mathbf{A} : \exists n \in \omega\backslash 1 \exists f[&f \text{ is a function with domain } n +' 1 \\ &\text{such that } f(0) = y,\ f(n) = x, \text{ and } \forall i < n[(f(i), f(i +' 1)) \in \mathbf{R}]] \\ = \mathrm{pred}_{\mathbf{AR}^*}(x).& \end{aligned}$$

□

Theorem 8.5. *If* $\mathbf{R}$ *is well-founded and set-like on a class* $\mathbf{A}$*, then every nonempty subclass of* $\mathbf{A}$ *has an* $\mathbf{R}$*-minimal element.*

Proof. Suppose that $\mathbf{X}$ is a nonempty subclass of $\mathbf{A}$. Take any $x \in \mathbf{X}$. Now $\mathrm{pred}'_{\mathbf{AR}^*}(x) \cap \mathbf{X}$ is a nonempty subset of $\mathbf{A}$, by Theorem 8.4 and the comprehension axioms. Let y be an $\mathbf{R}$-minimal element of $\mathrm{pred}'_{\mathbf{AR}^*}(x) \cap \mathbf{X}$. In particular, $y \in \mathbf{X}$. Suppose that

$(z,y) \in \mathbf{R}$. Then $z \in \text{pred}_{\mathbf{AR}}(y)$. By Theorem 8.3(ii) it follows that $z \in \text{pred}'_{\mathbf{AR}^*}(x)$. Hence $z \notin \mathbf{X}$ by the choice of y. so y is the desired $\mathbf{R}$-minimal element of $\mathbf{X}$. $\square$

Theorem 8.6. *If $\mathbf{F}$ is a class function and a is a set contained in the domain of $\mathbf{F}$, then there is a (set) function f with domain a such that $f(x) = \mathbf{F}(x)$ for all $x \in a$.*

Proof. Let $\mathbf{G}(x) = (x, \mathbf{F}(x))$ for all $x \in \text{dmn}(\mathbf{F}$. By the replacement and comprehension axioms, the class $\{\mathbf{G}(x) : x \in a\}$ is a set. This class is $\{(x, \mathbf{F}(x)) : x \in a\}$. Thus it is the desired function f. $\square$

In terms of formulas, $\mathbf{F}$ corresponds to a formula $\varphi(x,y)$ such that for all x there is at most one y such that $\varphi(x,y)$. Then $\mathbf{G}$ corresponds to the formula $\psi(x,y) \stackrel{\text{def}}{=} \exists z[\varphi(x,z) \wedge y = (x,z)]$. Clearly for all x there is at most one y such that $\psi(x,y)$, and if $\psi(x,y)$ then $y = (x,z)$ where $\varphi(x,z)$ holds.

The function asserted to exist in Theorem 8.6 will be denoted by $\mathbf{F} \restriction a$.

Theorem 8.7. (The recursion theorem) *Suppose that $\mathbf{R}$ is a class relation which is well-founded and set like on a class $\mathbf{A}$, and $\mathbf{G}$ is a class function mapping $\mathbf{A} \times \mathbf{V}$ into $\mathbf{V}$. Then there is a class function $\mathbf{F}$ mapping $\mathbf{A}$ into $\mathbf{V}$ such that $\mathbf{F}(a) = \mathbf{G}(a, \mathbf{F} \restriction \text{pred}_{\mathbf{AR}}(a))$ for all $a \in \mathbf{A}$.*

Proof. We say that a function f is an *approximation to* $\mathbf{F}$ iff $\text{dmn}(f) \subseteq \mathbf{A}$ and for every $a \in \text{dmn}(f)$ we have $\text{pred}_{\mathbf{AR}}(a) \subseteq \text{dmn}(f)$ and $f(a) = \mathbf{G}(a, f \restriction \text{pred}_{\mathbf{AR}}(a))$.

(1) If f and f' are approximations to $\mathbf{F}$ and $a \in \text{dmn}(f) \cap \text{dmn}(f')$, then $f(a) = f'(a)$.

In fact, suppose that this is not true. Then the set $X = \{a \in \text{dmn}(f) \cap \text{dmn}(f') : f(a) \neq f'(a)\}$ is nonempty. Let a be an R-minimal element of X. Now if $b \in \text{pred}_{\mathbf{AR}}(a)$ then $b \in \text{dmn}(f) \cap \text{dmn}(f')$ and $(b,a) \in \mathbf{R}$, hence $b \notin X$; so $f(b) = f'(b)$. Thus $f \restriction \text{pred}_{\mathbf{AR}}(a) = f' \restriction \text{pred}_{\mathbf{AR}}(a)$. It follows that

$$f(a) = \mathbf{G}(a, f \restriction \text{pred}_{\mathbf{AR}}(a)) = \mathbf{G}(a, f' \restriction \text{pred}_{\mathbf{AR}}(a)) = f'(a),$$

contradiction. So (1) holds.

(2) If f is an approximation to $\mathbf{F}$, $x \in \text{dmn}(f)$, n is a positive integer, g is a function with domain $n +' 1$, $g(n) = x$, and $\forall i < n[(g(i), g(i +' 1)) \in \mathbf{R}]$, then $g(0) \in \text{dmn}(f)$.

To prove this, assuming that f is an approximation to $\mathbf{F}$ and $x \in \text{dmn}(f)$, we prove by induction on $n \geq 1$ that if g is a function with domain $n +' 1$, $g(n) = x$, and $\forall i < n[(g(i), g(i +' 1)) \in \mathbf{R}]$, then $g(0) \in \text{dmn}(f)$. For $n = 1$ we have $(g(0), x) \in \mathbf{R}$, so $g(0) \in \text{pred}_{\mathbf{AR}}(x)$ and hence $g(0) \in \text{dmn}(f)$. Assume that it is true for n, and now assume that g is a function with domain $n +' 2$, $g(n +' 1) = x$, and $\forall i < n +' 1[(g(i), g(i +' 1)) \in \mathbf{R}]$. Define $h(i) = g(i +' 1)$ for all $i < n$. Then $h(n) = g(n +' 1) = x$ and $\forall i < n[(h(i), h(i +' 1)) = (g(i +' 1), g(i +' 2)) \in \mathbf{R}]$. Hence $h(0) \in \text{dmn}(f)$ by the inductive hypothesis. Since $(g(0), h(0)) = (g(0), g(1))$ we have $(g(0), h(0)) \in \mathbf{R}$ and hence $g(0) \in \text{dmn}(f)$. This finishes the inductive proof of (2).

(3) If f is an approximation to $\mathbf{F}$ and $x \in \text{dmn}(f)$, then $\text{pred}'_{\mathbf{AR}^*}(x) \subseteq \text{dmn}(f)$.

This is clear from (2) and the definition of $\mathrm{pred}'_{\mathbf{AR}^*}(x)$.

(4) If f is an approximation to $\mathbf{F}$ and $x \in \mathrm{dmn}(f)$, then $f \restriction \mathrm{pred}'_{\mathbf{AR}^*}(x)$ is an approximation to $\mathbf{F}$.

In fact, if $a \in \mathrm{pred}'_{\mathbf{AR}^*}(x)$ then $\mathrm{pred}_{AR}(a) \subseteq \mathrm{pred}'_{\mathbf{AR}^*}(x)$ by Theorem 8.3(ii). Suppose that $a \in \mathrm{pred}'_{\mathbf{AR}^*}(x)$. Then

$$\begin{aligned}(f \restriction \mathrm{pred}'_{\mathbf{AR}^*}(x))(a) &= f(a) = \mathbf{G}(a, f \restriction \mathrm{pred}_{\mathbf{AR}}(a)) \\ &= \mathbf{G}(a, (f \restriction \mathrm{pred}'_{\mathbf{AR}^*}(x)) \restriction \mathrm{pred}_{\mathbf{AR}}(a)).\end{aligned}$$

This proves (4).

(5) For all $x \in \mathbf{A}$ there is an approximation f to $\mathbf{F}$ such that $x \in \mathrm{dmn}(f)$.

Suppose not. Let $\mathbf{X} = \{x \in \mathbf{A} :$ there does not exist an approximation f to $\mathbf{F}$ such that $x \in \mathrm{dmn}(f)\}$. So $\mathbf{X}$ is a nonempty subclass of $\mathbf{A}$. By Theorem 8.5, let x be an R-minimal element of $\mathbf{X}$. Now if $(y, x) \in \mathbf{R}$ then $y \notin \mathbf{X}$, and so there is an approximation f to $\mathbf{F}$ such that $y \in \mathrm{dmn}(f)$. Then by (4), also $f \restriction \mathrm{pred}'_{\mathbf{AR}^*}(y)$ is an approximation to $\mathbf{F}$. If also g is an approximation to $\mathbf{F}$ such that $y \in \mathrm{dmn}(g)$, then by (4) $g \restriction \mathrm{pred}'_{\mathbf{AR}^*}(y)$ is an approximation to $\mathbf{F}$. By (1), $f \restriction \mathrm{pred}'_{\mathbf{AR}^*}(y) = g \restriction \mathrm{pred}'_{\mathbf{AR}^*}(y)$. Thus there is a unique approximation to $\mathbf{F}$ whose domain is $\mathrm{pred}'_{\mathbf{AR}^*}(y)$. This is true for all $y \in \mathrm{pred}_{\mathbf{AR}}(x)$, so by replacement and comprehension there is a set

$$\begin{aligned}\mathscr{A} \stackrel{\text{def}}{=} \{f : f \text{ is an approximation to } \mathbf{F} \text{ with domain} \\ \mathrm{pred}'_{\mathbf{AR}^*}(y), \text{ for some } y \in \mathrm{pred}_{\mathbf{AR}}(x)\}.\end{aligned}$$

Let $g = \bigcup \mathscr{A}$. We claim that g is an approximation to $\mathbf{F}$. We prove this in several steps.

First, g is a function. For suppose that $(a, b), (a, c) \in g$. Choose $f, f' \in \mathscr{A}$ such that $(a, b) \in f$ and $(a, c) \in f'$. Since both f and f' are approximations to $\mathbf{F}$ and $a \in \mathrm{dmn}(f) \cap \mathrm{dmn}(f')$, it follows from (1) that $b = c$.

Second, the domain of g is $\bigcup\{\mathrm{pred}'_{\mathbf{AR}^*}(y) : y \in \mathrm{pred}_{\mathbf{AR}}(x)\}$. In fact, if $a \in \mathrm{dmn}(g)$ then there is an $f \in \mathscr{A}$ such that $a \in \mathrm{dmn}(f)$, and $\mathrm{dmn}(f) = \mathrm{pred}'_{\mathbf{AR}^*}(y)$ for some $y \in \mathrm{pred}_{\mathbf{AR}}(x)$; so a is in the indicated union. If $y \in \mathrm{pred}_{\mathbf{AR}}(x)$, then $\mathrm{pred}'_{\mathbf{AR}^*}(y)$ is the domain of some $f \in \mathscr{A}$, and hence $\mathrm{pred}'_{\mathbf{AR}^*}(y) \subseteq \mathrm{dmn}(g)$. So the domain of g is as indicated.

Next, if $a \in \mathrm{dmn}(g)$ then $\mathrm{pred}_{\mathbf{AR}}(a) \subseteq \mathrm{dmn}(g)$. For, suppose that $b \in \mathrm{pred}_{\mathbf{AR}}(a)$. Then $a \in \mathrm{dmn}(f)$ for some $f \in \mathscr{A}$, hence $a \in \mathrm{pred}'_{\mathbf{AR}^*}(y)$ for some $y \in \mathrm{pred}_{\mathbf{AR}}(x)$, so $b \in \mathrm{pred}'_{\mathbf{AR}^*}(y)$ by Theorem 8.3(ii), and it follows that $b \in \mathrm{dmn}(g)$. This proves that $\mathrm{pred}_{\mathbf{AR}}(a) \subseteq \mathrm{dmn}(g)$.

The final condition for g to be an approximation to $\mathbf{F}$ is shown as follows. Suppose that $a \in \mathrm{dmn}(g)$. Choose $y \in \mathrm{pred}_{\mathbf{AR}}(x)$ such that $a \in \mathrm{dmn}(f)$, where f is an approximation to $\mathbf{F}$ with domain $\mathrm{pred}'_{\mathbf{AR}}(y)$. Then

$$g(a) = f(a) = \mathbf{G}(a, f \restriction \mathrm{pred}'_{\mathbf{AR}}(y)) = \mathbf{G}(a, g \restriction \mathrm{pred}'_{\mathbf{AR}}(y)).$$

Now let $h = g \cup \{(x, \mathbf{G}(x, g \restriction \mathrm{pred}_{\mathbf{AR}}(x)))\}$. We claim that h is an approximation to $\mathbf{F}$. Since $x \in \mathrm{dmn}(h)$, this is a contradiction, proving (5).

To prove the claim, first note that h is a function, since $x \notin \mathrm{dmn}(g)$ by the choice of x, since g is an approximation. Clearly $\mathrm{dmn}(h) \subseteq \mathbf{A}$. Suppose that $a \in \mathrm{dmn}(h)$. if $a \in \mathrm{dmn}(g)$, then $\mathrm{pred}_{\mathbf{AR}}(a) \subseteq \mathrm{dmn}(g) \subseteq \mathrm{dmn}(h)$. If $a = x$ and $y \in \mathrm{pred}_{\mathbf{AR}}(x)$, then $y \in \mathrm{pred}'_{\mathbf{AR}^*} \subseteq \mathrm{dmn}(g) \subseteq \mathrm{dmn}(h)$. Hence $\mathrm{pred}_{\mathbf{AR}}(x) \subseteq \mathrm{dmn}(h)$. Finally, if $a \in \mathrm{dmn}(g)$, then

$$h(a) = g(a) = \mathbf{G}(a, g \restriction \mathrm{pred}_{\mathbf{AR}}(a)) = \mathbf{G}(a, h \restriction \mathrm{pred}_{\mathbf{AR}}(a)).$$

If $a = x$, then

$$h(a) = h(x) = \mathbf{G}(x, g \restriction \mathrm{pred}_{\mathbf{AR}}(x)) = \mathbf{G}(x, h \restriction \mathrm{pred}_{\mathbf{AR}}(x)).$$

So we have proved (5).

Now by (5), for all $a \in \mathbf{A}$ there is a b such that there is an f such that f is an approximation to $\mathbf{F}$ and $b \in \mathrm{dmn}(f)$. By (1), this b is uniquely determined by a. Hence there is a class function $\mathbf{F}'$ such that for all $a \in \mathbf{A}$, $\mathbf{F}'(a)$ is equal to such a b. Moreover, if f is as indicated and $b \in \mathrm{pred}_{\mathbf{AR}}(a)$, then $\mathbf{F}'(b) = f(b)$. Thus $\mathbf{F}' \restriction \mathrm{pred}_{\mathbf{AR}}(a) = f \restriction \mathrm{pred}_{\mathbf{AR}}(a)$. It follows that

$$\mathbf{F}'(a) = f(a) = \mathbf{G}(a, f \restriction \mathrm{pred}_{\mathbf{AR}}(a)) = \mathbf{G}(a, \mathbf{F}' \restriction \mathrm{pred}_{\mathbf{AR}}(a)). \qquad \square$$

Theorem 8.8. *Suppose that* $\mathbf{R}$ *is a class relation which is well-founded and set like on a class* $\mathbf{A}$*, and* $\mathbf{G}$ *is a class function mapping* $\mathbf{A} \times \mathbf{V}$ *into* $\mathbf{V}$*. Suppose that* $\mathbf{F}$ *and* $\mathbf{F}'$ *are class functions mapping* $\mathbf{A}$ *into* $\mathbf{V}$ *such that* $\mathbf{F}(a) = \mathbf{G}(a, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}(a))$ *and* $\mathbf{F}'(a) = \mathbf{G}(a, \mathbf{F}' \restriction \mathrm{pred}_{\mathbf{AR}}(a))$ *for all* $a \in \mathbf{A}$*. Then* $\mathbf{F} = \mathbf{F}'$*.*

Proof. Suppose not. Then $\mathbf{X} \stackrel{\text{def}}{=} \{a \in \mathbf{A} : \mathbf{F}(a) \neq \mathbf{F}'(a)\}$ is a nonempty subclass of $\mathbf{A}$. Hence by Theorem 8.5 let a be an $\mathbf{R}$-minimal element of $\mathbf{X}$. If $(b, a) \in \mathbf{R}$, then $b \notin \mathbf{X}$, and hence $\mathbf{F}(b) = \mathbf{F}'(b)$. Thus $\mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}(a) = \mathbf{F}' \restriction \mathrm{pred}_{\mathbf{AR}}(a)$. So

$$\mathbf{F}(a) = \mathbf{G}(a, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}(a)) = \mathbf{G}(a, \mathbf{F}' \restriction \mathrm{pred}_{\mathbf{AR}}(a)) = \mathbf{F}'(a),$$

contradiction. $\square$

We make some remarks about the rigorous formulation of Theorems 8.7 and 8.8. In Theorem 8.7 we are given formulas $\varphi(x, y)$, $\psi(x)$, and $\chi(x, y, z)$ corresponding to $\mathbf{R}$, $\mathbf{A}$, and $\mathbf{G}$. We assume that φ is well-founded and set-like on ψ. The assumption on χ is

$$\begin{aligned}&\forall x \forall y[\psi(x) \to \exists! z \chi(x, y, z)]\\ &\wedge \forall x \forall y \forall z[\chi(x, y, z) \to \psi(x)].\end{aligned}$$

The conclusion is that there is a formula $\theta(x, y)$ such that

$$\begin{aligned}(*) \quad &\forall x[\psi(x) \to \exists! y \theta(x, y)] \wedge \forall x \forall y[\theta(x, y) \to \psi(x)]\\ &\wedge \forall x \exists y \exists f[f \text{ is a function } \wedge \forall u \forall v[(u, v) \in f \leftrightarrow \varphi(u, x) \wedge \theta(u, v)] \wedge \chi(x, f, y)]]\end{aligned}$$

The proof defines the formula θ explicitly. Namely, let $\mu(f)$ be the following formula, the rigorous version of "f is an approximation to $\mathbf{F}$":

$$\begin{aligned}
&f \text{ is a function } \wedge \forall\, x[x \in \operatorname{dmn}(f) \to \varphi(x)] \wedge \\
&\forall x \forall y[x \in \operatorname{dmn}(f) \wedge \varphi(y,x) \to y \in \operatorname{dmn}(f)] \wedge \\
&\forall a \forall g[a \in \operatorname{dmn}(f) \wedge g \text{ is a function } \wedge \forall y[y \in \operatorname{dmn}(g) \leftrightarrow \varphi(y,x)] \wedge \\
&\forall y \in \operatorname{dmn}(g)[g(y) = f(y)] \to \chi(a, g, f(a))]
\end{aligned}$$

Then $\theta(x, y)$ is the formula $\exists f[\mu(f) \wedge x \in \operatorname{dmn}(f) \wedge f(x) = y]$.

The rigorous version of Theorem 8.8 is that if $\theta'(x, y)$ is another formula satisfying $(*)$ (with θ replaced by θ'), then $\forall x \forall y[\theta(x, y) \leftrightarrow \theta'(x, y)]$.

EXERCISES

E8.1. Give an example of $\mathbf{A}, \mathbf{R}$ such that $\mathbf{R}$ is not well-founded on $\mathbf{A}$ and is not set-like on $\mathbf{A}$.

E8.2. Give an example of $\mathbf{A}, \mathbf{R}$ such that $\mathbf{R}$ is not well-founded on $\mathbf{A}$ but is set-like on $\mathbf{A}$. Give one example with $\mathbf{R}$ and $\mathbf{A}$ are proper classes, and one example where they are sets.

E8.3. Give an example of $\mathbf{A}, \mathbf{R}$ such that $\mathbf{R}$ is well-founded on $\mathbf{A}$ but is not set-like on $\mathbf{A}$.

E8.4. Suppose that $\mathbf{R}$ is a class relation contained in $\mathbf{A} \times \mathbf{A}$, $x \in \mathbf{A}$, and $v \in \operatorname{pred}_{\mathbf{AR}^*}(x)$. Prove by induction on n that if $n \in \omega \backslash 1$, f is a function with domain $n +' 1$, $\forall i < n[(f(i), f(i +' 1)) \in \mathbf{R}]$ and $f(n) = v$, then $f(0) \in \operatorname{pred}_{\mathbf{AR}^*}(x)$.

E8.5. Suppose that $\mathbf{R}$ is a class relation contained in $\mathbf{A} \times \mathbf{A}$, $(u, v) \in \mathbf{R}^*$, and $(v, w) \in \mathbf{R}^*$. Show that $(u, w) \in \mathbf{R}^*$.

E8.6. Give an example of a proper class $\mathbf{X}$ which has a proper class of $\in$-minimal elements.

E8.7. Give an example of a proper class relation $\mathbf{R}$ contained in $\mathbf{A} \times \mathbf{A}$ for some proper class $\mathbf{A}$, and a class function $\mathbf{G}$ mapping $\mathbf{A} \times \mathbf{V}$ into $\mathbf{V}$ such that $\mathbf{R}$ is set-like on $\mathbf{A}$ but not well-founded on $\mathbf{A}$ and there is no class function $\mathbf{F}$ mapping $\mathbf{A}$ into $\mathbf{V}$ such that $\mathbf{F}(a) = \mathbf{G}(a, \mathbf{F}\langle \operatorname{pred}_{\mathbf{AR}}(a))$ for all $a \in \mathbf{A}$.

E8.8. Give an example of a proper class relation $\mathbf{R}$ contained in $\mathbf{A} \times \mathbf{A}$ for some proper class $\mathbf{A}$ and a class function $\mathbf{G}$ mapping $\mathbf{A} \times \mathbf{V}$ into $\mathbf{V}$ such that $\mathbf{R}$ is set-like on $\mathbf{A}$ but not well-founded on $\mathbf{A}$ but still there is a class function $\mathbf{F}$ mapping $\mathbf{A}$ into $\mathbf{V}$ such that $\mathbf{F}(a) = \mathbf{G}(a, \mathbf{F}\langle \operatorname{pred}_{\mathbf{AR}}(a))$ for all $a \in \mathbf{A}$.

9. Ordinals, II

Transfinite induction

The transfinite induction principles follow rather easily from the following generalization of Theorem 7.13.

Theorem 9.1. *Let* $\mathbf{A}$ *be an ordinal, or* $\mathbf{On}$*. Then every nonempty subclass of* $\mathbf{A}$ *has a least element.*

Proof. This follows from Theorem 8.5. □

There are two forms of the principle of transfinite induction, given in the following two theorems.

Theorem 9.2. *Let* $\mathbf{A}$ *be an ordinal or* $\mathbf{On}$*. Suppose that* $\mathbf{B} \subseteq \mathbf{A}$ *and the following condition holds:*

$$\forall \alpha \in \mathbf{A}[\alpha \subseteq \mathbf{B} \Rightarrow \alpha \in \mathbf{B}].$$

Then $\mathbf{B} = \mathbf{A}$.

Proof. Suppose not, and let α be the least element of $\mathbf{A}\backslash\mathbf{B}$. Thus $\alpha \subseteq \mathbf{B}$, so by hypothesis $\alpha \in \mathbf{B}$, contradiction. □

Corollary 9.3. *Suppose that* $\mathbf{B}$ *is a class of ordinals and the following condition holds:*

$$\forall \alpha[\alpha \subseteq \mathbf{B} \Rightarrow \alpha \in \mathbf{B}].$$

Then $\mathbf{B} = \mathbf{On}$. □

Corollary 9.4. *Suppose that* β *is an ordinal,* $X \subseteq \beta$*, and the following condition holds:*

$$\forall \alpha < \beta[\alpha \subseteq X \Rightarrow \alpha \in X].$$

Then $X = \beta$. □

Theorem 9.5. *Suppose that* $\mathbf{A}$ *is an ordinal or* $\mathbf{On}$*,* $\mathbf{B} \subseteq \mathbf{A}$*, and the following conditions hold:*

(i) If $0 \in \mathbf{A}$*, then* $0 \in \mathbf{B}$*.*
(ii) If $\alpha +' 1 \in \mathbf{A}$ *and* $\alpha \in \mathbf{B}$*, then* $\alpha +' 1 \in \mathbf{B}$*.*
(iii) If α *is a limit ordinal,* $\alpha \in \mathbf{A}$*, and* $\alpha \subseteq \mathbf{B}$*, then* $\alpha \in \mathbf{B}$*.*

Then $\mathbf{B} = \mathbf{A}$.

Proof. Suppose not, and let α be the least element of $\mathbf{A}\backslash\mathbf{B}$. Then $\alpha \neq 0$ by (i). If $\alpha = \beta +' 1$ for some β, then $\beta < \alpha$, so $\beta \in \mathbf{B}$, and then $\alpha \in \mathbf{B}$ by (ii), contradiction. Finally, suppose that α is a limit ordinal. Then $\alpha \subseteq \mathbf{B}$, and so $\alpha \in \mathbf{B}$ by (iii), conradiction. □

Corollary 9.6. *Suppose that* $\mathbf{B} \subseteq \mathbf{On}$ *and the following conditions hold:*

(i) $0 \in \mathbf{B}$.
(ii) If $\alpha \in \mathbf{B}$, *then* $\alpha +' 1 \in \mathbf{B}$.
(iii) If α *is a limit ordinal and* $\alpha \subseteq \mathbf{B}$, *then* $\alpha \in \mathbf{B}$.

Then $\mathbf{B} = \mathbf{On}$. □

Transfinite recursion

Theorem 9.7. *Suppose that* $\mathbf{G}$ *is a class function mapping* $\mathbf{On} \times \mathbf{V}$ *into* $\mathbf{V}$. *Then there is a unique class function* $\mathbf{F}$ *mapping* $\mathbf{On}$ *into* $\mathbf{V}$ *such that* $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F} \upharpoonright \alpha)$ *for every ordinal* α.

Proof. We apply Theorems 8.7 and 8.8 with $\mathbf{R} = \{(\alpha, \beta) : \alpha < \beta\}$. □

Well-order

A *partial order* is a pair $(P, <)$ such that P is a set, $<$ is a relation contained in $P \times P$, $<$ is irreflexive ($x \not< x$ for all $x \in P$), and $<$ is transitive (for all $x, y, z \in P$, $x < y < z$ implies that $x < z$). For $(P, <)$ a partial order, we define $p_1 \leq p_2$ iff $p_1 < p_2$ or $p_1 = p_2$. A *linear order* is a partial order $(P, <)$ such that for all $x, y \in P$, either $x < y$, $x = y$, or $y < x$. A *well-order* is a linear order $(P, <)$ such that for every nonempty $X \subseteq P$ there is an $x \in X$ such that $\forall y \in X[y \not< x]$. This element x is called the $<$-*least element* of X.

Proposition 9.8. *For any ordinal* α, $(\alpha, <)$ *is a well-order.* □

Proposition 9.9. *If* $(P, <)$ *is a well-order, then* $<$ *is well-founded.* □

If $(P, <)$ and $(Q, \prec)$ are partial orders, then a function $f : P \to Q$ is *strictly increasing* iff $\forall p_1, p_2 \in P[p_1 < p_2 \Rightarrow f(p_1) \prec f(p_2)]$.

Proposition 9.10. *If* $(A, <)$ *and* $(B, \prec)$ *are linearly orders and* $f : A \to B$ *is strictly increasing, then* $\forall a_0, a_1 \in A[a_0 < a_1 \Leftrightarrow f(a_0) \prec f(a_1)]$.

Proof. The direction $\Rightarrow$ is given by the definition. Now suppose that it is not true that $a_0 < a_1$. Then $a_1 \leq a_0$, so $f(a_1) \leq f(a_0)$. So $f(a_0) < f(a_1)$ is not true. □

Proposition 9.11. *If* $(A, <)$ *is a well-ordered set and* $f : A \to A$ *is strictly increasing, then* $x \leq f(x)$ *for all* $x \in A$.

Proof. Suppose not. Then then set $B \stackrel{\text{def}}{=} \{x \in A : f(x) < x\}$ is nonempty. Let b be the least element of B. Thus $f(b) < b$. Hence by the choice of b, we have $f(b) \leq f(f(b))$. Hence by Proposition 9.10, $b \leq f(b)$, contradiction. □

Let $(A, <)$ and $(B, \prec)$ be partial orders. An *isomorphism* from $(A, <)$ onto $(B, \prec)$ is a function f mapping A onto B such that $\forall a_1, a_2 \in A[a_1 < a_2$ iff $f(a_1) \prec f(a_2)]$.

Proposition 9.12. *If* $(A, <)$ *and* $(B, \prec)$ *are isomorphic well-orders, then there is a unique isomorphism* f *mapping* A *onto* B.

Proof. The existence of f follows from the definition. Suppose that both f and g are isomorphisms from A onto B. Then $f^{-1} \circ g$ is a strictly increasing function from A into A, so by Proposition 9.11 we get $x \le (f^{-1} \circ g)(x)$ for every $x \in A$; so $f(x) \le g(x)$ for every $x \in A$. Similarly, $g(x) \le f(x)$ for every $x \in A$, so $f = g$. □

Corollary 9.13. *If $\alpha \neq \beta$, then $(\alpha, <)$ and $(\beta, <)$ are not isomorphic.*

Proof. Suppose to the contrary that f is an isomorphism from $(\alpha, <)$ onto $(\beta, <)$, with $\beta < \alpha$. Then f is a strictly increasing function mapping α into α. Hence $\beta \le f(\beta) < \beta$ by Proposition 9.11, contradiction. □

The following theorem is fundamental. The proof is also of general interest; it can be followed in outline form in many other situations.

Theorem 9.14. *Every well-order is isomorphic to an ordinal.*

Proof. Let $(A, \prec)$ be a well-order. We may assume that $A \neq \emptyset$. We define a class function $\mathbf{G} : \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ as follows. For any ordinal α and set x,

$$\mathbf{G}(\alpha, x) = \begin{cases} \prec\text{-least element of } A \backslash \mathrm{rng}(x) & \text{if } x \text{ is a function and this set is nonempty,} \\ A & \text{otherwise.} \end{cases}$$

Now by Theorem 9.7 let $\mathbf{F} : \mathrm{On} \to \mathbf{V}$ be such that $\mathbf{F}(\beta) = \mathbf{G}(\beta, \mathbf{F} \restriction \beta)$ for each ordinal β.

(1) If $\beta < \gamma$ and $\mathbf{F}(\beta) = A$, then $\mathbf{F}(\gamma) = A$.

For, $A \backslash \mathrm{rng}(\mathbf{F} \restriction \gamma) \subseteq A \backslash \mathrm{rng}(\mathbf{F} \restriction \beta)$, so $A \backslash \mathrm{rng}(\mathbf{F} \restriction \beta)$ empty implies that $A \backslash \mathrm{rng}(\mathbf{F} \restriction \gamma)$ is empty, giving (1).

(2) if $\beta < \gamma$ and $\mathbf{F}(\gamma) \neq A$, then $\mathbf{F}(\beta) \neq A$ and $\mathbf{F}(\beta) \prec \mathbf{F}(\gamma)$.

The first assertion follows from (1). For the second assertion, note that $A \backslash \mathrm{rng}(\mathbf{F} \restriction \gamma) \subseteq A \backslash \mathrm{rng}(\mathbf{F} \restriction \beta)$, hence $\mathbf{F}(\gamma) \in A \backslash \mathrm{rng}(\mathbf{F} \restriction \beta)$, so $\mathbf{F}(\beta) \preceq \mathbf{F}(\gamma)$ by definition. Also $\mathbf{F}(\beta) \in \mathrm{rng}(\mathbf{F} \restriction \gamma)$, and $\mathbf{F}(\gamma) \notin \mathrm{rng}(\mathbf{F} \restriction \gamma)$, so $\mathbf{F}(\beta) \prec \mathbf{F}(\gamma)$, as desired in (2).

(3) There is an ordinal γ such that $\mathbf{F}(\gamma) = A$.

In fact, suppose not. Let $B = \{a \in A : \exists\alpha[\mathbf{F}(\alpha) = a]\}$. Then $\mathbf{F}^{-1}$ maps B onto $\mathbf{On}$, so by the replacement axiom, $\mathbf{On}$ is a set, contradiction.

Choose γ minimum such that $\mathbf{F}(\gamma) = A$. (Note that $\mathbf{F}(0) \neq A$, since A is nonempty and so has a least element.) By (2), $\mathbf{F} \restriction \gamma$ is strictly inceasing and maps onto A. Hence $\mathbf{F} \restriction \gamma$ is the desired isomorphism, using Proposition 9.10. □

Ordinal class functions

We say that $\mathbf{F}$ is an *ordinal class function* iff $\mathbf{F}$ is a class function whose domain is an ordinal, or the whole class $\mathbf{On}$, and whose range is contained in $\mathbf{On}$. We consider three properties of an ordinal class function $\mathbf{F}$ with domain $\mathbf{A}$:

- $\mathbf{F}$ is *strictly increasing* iff for any ordinals $\alpha, \beta \in \mathbf{A}$, if $\alpha < \beta$ then $\mathbf{F}(\alpha) < \mathbf{F}(\beta)$.

- $\mathbf{F}$ is *continuous* iff for every limit ordinal $\alpha \in \mathbf{A}$, $\mathbf{F}(\alpha) = \bigcup_{\beta<\alpha} \mathbf{F}(\beta)$.
- $\mathbf{F}$ is *normal* iff it is continuous and strictly increasing.

The following is a version of Proposition 9.11, with essentially the same proof.

Proposition 9.15. *If $\mathbf{F}$ is a strictly increasing ordinal class function with domain $\mathbf{A}$, then $\alpha \leq \mathbf{F}(\alpha)$ for every ordinal $\alpha \in \mathbf{A}$.*

Proof. Suppose not, and let α be the least member of $\mathbf{A}$ such that $\mathbf{F}(\alpha) < \alpha$. Then $\mathbf{F}(\mathbf{F}(\alpha)) < \mathbf{F}(\alpha)$, so that $\mathbf{F}(\alpha)$ is an ordinal β smaller than α such that $\mathbf{F}(\beta) < \beta$, contradiction. □

Proposition 9.16. *If $\mathbf{F}$ is a continuous ordinal class function with domain $\mathbf{A}$, and $\mathbf{F}(\alpha) < \mathbf{F}(\alpha +' 1)$ for every ordinal α such that $\alpha +' 1 \in \mathbf{A}$, then $\mathbf{F}$ is strictly increasing.*

Proof. Fix an ordinal $\gamma \in \mathbf{A}$, and suppose that there is an ordinal $\delta \in \mathbf{A}$ with $\gamma < \delta$ and $\mathbf{F}(\delta) \leq \mathbf{F}(\gamma)$; we want to get a contradiction. Take the least such δ.

Case 1. $\delta = \theta +' 1$ for some θ. Thus $\gamma \leq \theta$. If $\gamma = \theta$, then $\mathbf{F}(\gamma) < \mathbf{F}(\delta)$ by the hypothesis of the proposition, contradicting our supposition. Hence $\gamma < \theta$. Hence $\mathbf{F}(\gamma) < \mathbf{F}(\theta)$ by the minimality of δ, and $\mathbf{F}(\theta) < \mathbf{F}(\delta)$ by the assumption of the proposition, so $\mathbf{F}(\gamma) < \mathbf{F}(\delta)$, contradiction.

Case 2. δ is a limit ordinal. Then there is a $\theta < \delta$ with $\gamma < \theta$, and so by the minimality of δ we have

$$\mathbf{F}(\gamma) < \mathbf{F}(\theta) \leq \bigcup_{\varepsilon<\delta} \mathbf{F}(\varepsilon) = \mathbf{F}(\delta),$$

contradiction. □

Proposition 9.17. *Suppose that $\mathbf{F}$ is a normal ordinal class function with domain $\mathbf{A}$, and $\xi \in \mathbf{A}$ is a limit ordinal. Then $\mathbf{F}(\xi)$ is a limit ordinal too.*

Proof. Suppose that $\gamma < \mathbf{F}(\xi)$. Thus $\gamma \in \bigcup_{\eta<\xi} \mathbf{F}(\eta)$, so there is a $\eta < \xi$ such that $\gamma < \mathbf{F}(\eta)$. Now $\mathbf{F}(\eta) < \mathbf{F}(\xi)$. Hence $\mathbf{F}(\xi)$ is a limit ordinal. □

Proposition 9.18. *Suppose that $\mathbf{F}$ and $\mathbf{G}$ are normal ordinal class functions, with domains $\mathbf{A}, \mathbf{B}$ respectively, and the range of $\mathbf{F}$ is contained in $\mathbf{B}$. Then also $\mathbf{G} \circ \mathbf{F}$ is normal.*

Proof. Clearly $\mathbf{G} \circ \mathbf{F}$ is strictly increasing. Now suppose that $\xi \in \mathbf{A}$ is a limit ordinal. Then $\mathbf{F}(\xi)$ is a limit ordinal by Proposition 9.17.

Suppose that $\rho < \xi$. Then $\mathbf{F}(\rho) < \mathbf{F}(\xi)$, so $\mathbf{G}(\mathbf{F}(\rho)) \leq \bigcup_{\eta<\mathbf{F}(\xi)} \mathbf{G}(\eta) = \mathbf{G}(\mathbf{F}(\xi))$. Thus

$$(*) \qquad \bigcup_{\rho<\xi} \mathbf{G}(\mathbf{F}(\rho)) \leq \mathbf{G}(\mathbf{F}(\xi)).$$

Now if $\eta < \mathbf{F}(\xi)$, then by the continuity of $\mathbf{F}$, $\eta < \bigcup_{\rho<\xi} \mathbf{F}(\rho)$, and hence there is a $\rho < \xi$ such that $\eta < \mathbf{F}(\rho)$; so $\mathbf{G}(\eta) < \mathbf{G}(\mathbf{F}(\rho))$. So for any $\eta < \mathbf{F}(\xi)$ we have $\mathbf{G}(\eta) \leq \bigcup_{\rho<\xi} \mathbf{G}(\mathbf{F}(\rho))$. Hence

$$\mathbf{G}(\mathbf{F}(\xi)) = \bigcup_{\eta<\mathbf{F}(\xi)} \mathbf{G}(\eta) \leq \bigcup_{\rho<\xi} \mathbf{G}(\mathbf{F}(\rho));$$

together with $(*)$ this gives the continuity of $\mathbf{G} \circ F$. $\square$

Ordinal addition

We use the general recursion theorem to define ordinal addition:

Theorem 9.19. *There is a unique function $+$ mapping $\mathbf{On} \times \mathbf{On}$ into $\mathbf{On}$ such that the following conditions hold for any α:*

(i) $\alpha + 0 = \alpha$;

(ii) $\alpha + (\beta +' 1) = (\alpha + \beta) +' 1$;

(iii) $\alpha + \gamma = \bigcup_{\beta<\gamma}(\alpha + \beta)$ for γ a limit ordinal.

Proof. For the existence we use the main recursion theorem, Theorem 8.7. Let $\mathbf{A} = \mathbf{On} \times \mathbf{On}$, and let $\mathbf{R} = \{((\alpha, \beta), (\alpha, \gamma) : \beta < \gamma$. Then $\text{pred}_{\mathbf{AR}}(\alpha, \beta) = \{\alpha\} \times \beta$, a set. Thus $\mathbf{R}$ is set-like. Given a nonempty subset X of $\mathbf{A}$, choose $(\alpha, \gamma) \in X$, and then choose β minimum such that $(\alpha, \beta) \in X$. Clearly (α, β) is an $\mathbf{R}$-minimal element of X. Thus $\mathbf{R}$ is well-founded on $\mathbf{A}$.

Now we define $\mathbf{G} : \mathbf{A} \times \mathbf{V} \to \mathbf{V}$. For any α, β and any set x, let

$$\mathbf{G}((\alpha, \beta), x) = \begin{cases} \alpha & \text{if } \beta = 0, \\ x(\alpha, \gamma) +' 1 & \text{if } x \text{ is a function with domain } \{\alpha\} \times \beta \\ & \text{and } \beta = \gamma +' 1, \\ \bigcup_{\gamma<\beta} x(\alpha, \gamma) & \text{if } x \text{ is a function with domain } \{\alpha\} \times \beta \\ & \text{and } \beta \text{ is a limit ordinal,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Then by Theorem 8.7 let $\mathbf{F} : \mathbf{A} \to \mathbf{V}$ be such that $\mathbf{F}(y) = \mathbf{G}(y, \mathbf{F} \restriction \text{pred}_{\mathbf{AR}}(y))$ for any $y \in \mathbf{A}$. Then

$$\begin{aligned} \mathbf{F}(\alpha, 0) &= \mathbf{G}((\alpha, 0), \mathbf{F} \restriction \text{pred}_{\mathbf{AR}}((\alpha, 0))) = \alpha; \\ \mathbf{F}(\alpha, \beta +' 1) &= \mathbf{G}((\alpha, \beta +' 1), \mathbf{F} \restriction \text{pred}_{\mathbf{AR}}((\alpha, \beta +' 1))) \\ &= \mathbf{F}(\alpha, \beta) +' 1; \\ \mathbf{F}(\alpha, \beta) &= \mathbf{G}((\alpha, \beta), \mathbf{F} \restriction \text{pred}_{\mathbf{AR}}((\alpha, \beta))) \\ &= \bigcup_{\gamma<\beta} \mathbf{F}(\alpha, \gamma) \quad \text{if } \beta \text{ is a limit ordinal.} \end{aligned}$$

Thus writing $\alpha + \beta$ instead of $\mathbf{F}(\alpha, \beta)$ we see that $\mathbf{F}$ is as desired.

Now suppose that $+^o$ also satisfies the conditions of the theorem. We show that $\alpha + \beta = \alpha +^o \beta$ for all α, β, by fixing α and going by induction on β, using Corollary 9.9. We have $\alpha + 0 = \alpha = \alpha +^o \beta$. Assume that $\alpha + \beta = \alpha +^o \beta$. Then $\alpha + (\beta +' 1) = (\alpha + \beta) +' 1 = (\alpha +^o \beta) +' 1 = \alpha +^o (\beta +' 1)$. Assume that β is a limit ordinal and $\alpha + \gamma = \alpha +^o \gamma$ for every $\gamma < \beta$. Then $\alpha + \beta = \bigcup_{\gamma<\beta} \alpha + \gamma = \bigcup_{\gamma<\beta} \alpha +^o \gamma = \alpha +^o \beta$. $\square$

Proposition 9.20. $\alpha + 1 = \alpha +' 1$ *for any ordinal* α.

Proof. $\alpha + 1 = \alpha + (0 +' 1) = (\alpha + 0) +' 1 = \alpha +' 1$. $\square$

Now we can stop using the notation $\alpha +' 1$, using $\alpha + 1$ instead.

We state the simplest properties of ordinal addition in the following theorem., *Weakly increasing* means that $\alpha < \beta$ implies that $\mathbf{F}(\alpha) \leq \mathbf{F}(\beta)$.

Theorem 9.21. *(i) If $m, n \in \omega$, then $m + n \in \omega$.*

(ii) For any ordinal α, the class function $\mathbf{F}$ *which takes each ordinal β to $\alpha + \beta$ is a normal function.*

(iii) For any ordinal β, the class function $\mathbf{F}$ *which takes each ordinal α to $\alpha + \beta$ is weakly increasing.*

(iv) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

(v) $\beta \leq \alpha + \beta$.

(vi) $0 + \alpha = \alpha$.

(vii) $\alpha \leq \beta$ iff there is a δ such that $\alpha + \delta = \beta$.

(viii) $\alpha < \beta$ iff there is a $\delta > 0$ such that $\alpha + \delta = \beta$.

Proof. (i): with m fixed we use induction on n, thus appealing to Theorem 7.14. We have $m+0 = m \in \omega$. Assume that $n \in \omega$ and $m+n \in \omega$. then $m+(n+1) = (m+n)+1 \in \omega$, completing the induction.

(ii): by Proposition 9.19.

(iv): Fix α and β; we proceed by induction on γ. The case $\gamma = 0$ is obvious. Assume that $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$. Then

$$\begin{aligned}
\alpha + (\beta + (\gamma + 1)) &= \alpha + ((\beta + \gamma) + 1) \\
&= (\alpha + (\beta + \gamma)) + 1 \\
&= ((\alpha + \beta) + \gamma) + 1 \\
&= (\alpha + \beta) + (\gamma + 1).
\end{aligned}$$

Finally, suppose that γ is a limit ordinal and we know our result for all $\delta < \gamma$. Let $\mathbf{F}, \mathbf{G}, \mathbf{H}$ be the ordinal class functions such that, for any ordinal δ,

$$\begin{aligned}
\mathbf{F}(\delta) &= \alpha + \delta; \\
\mathbf{G}(\delta) &= (\alpha + \beta) + \delta; \\
\mathbf{H}(\delta) &= \beta + \delta.
\end{aligned}$$

Thus according to (ii), all three of these functions are normal. Hence, using Proposition 9.18,

$$\begin{aligned}
\alpha + (\beta + \gamma) &= \mathbf{F}(\mathbf{H}(\gamma)) \\
&= \bigcup_{\delta<\gamma} \mathbf{F}(\mathbf{H}(\delta)) \\
&= \bigcup_{\delta<\gamma} (\alpha + (\beta + \delta)) \\
&= \bigcup_{\delta<\gamma} ((\alpha + \beta) + \delta
\end{aligned}$$

$$\begin{aligned} &= \bigcup_{\delta<\gamma} \mathbf{G}(\delta) \\ &= \mathbf{G}(\gamma) \\ &= (\alpha+\beta)+\gamma. \end{aligned}$$

(v): by (ii) and Proposition 9.15.

(vi): induction on α. $0+0=0$. If $0+\alpha=\alpha$, then $0+(\alpha+1)=(0+\alpha)+1=\alpha+1$. If α is limit and $0+\beta=\beta$ for all $\beta<\alpha$, then $0+\alpha=\bigcup_{\beta<\alpha}(0+\beta)=\bigcup_{\beta<\alpha}\beta=\alpha$.

(vii): In the $\Rightarrow$ direction, assume that $\alpha\le\beta$. Now $\beta\le\alpha+\beta$ by (v). Let δ be minimum such that $\beta\le\alpha+\delta$. Suppose that $\beta<\alpha+\delta$. If $\delta=\varepsilon+1$ for some ε, then $\beta<(\alpha+\varepsilon)+1$ and hence $\beta\le\alpha+\varepsilon$ using Proposition 7.10. This contradicts the choice of δ. A similar contradiction is reached if δ is a limit ordinal. So $\beta=\alpha+\delta$.

For the $\Leftarrow$ direction, we prove that $\alpha\le\alpha+\delta$ for all δ by induction on δ. It is clear for $\delta=0$. Assume that $\alpha\le\alpha+\delta$. Now $\alpha+\delta<(\alpha+\delta)+1=\alpha+(\delta+1)$, so $\alpha\le\alpha+(\delta+1)$. Finally, suppose that δ is a limit ordinal and $\alpha\le\alpha+\gamma$ for all $\gamma<\delta$. Clearly then $\alpha\le\bigcup_{\gamma<\chi}(\alpha+\gamma)=\alpha+\delta$.

(viii): If $\alpha<\beta$, choose δ by (vii) so that $\alpha+\delta=\beta$. Since $\alpha\ne\beta$ we have $\delta>0$. For the other direction, if $\alpha+\delta=\beta$ with $\delta>0$, then $\alpha=\alpha+0<\alpha+\delta=\beta$, using (ii).

(iii): Suppose that $\gamma<\alpha$. By (viii), choose $\delta>0$ such that $\gamma+\delta=\alpha$. Then $\beta\le\delta+\beta$ by (v), and so by (ii) and (iv), $\gamma+\beta\le\gamma+(\delta+\beta)=(\gamma+\delta)+\beta=\alpha+\beta$. $\square$

Note that $+$ is not commutative. In fact, $1+\omega=\omega<\omega+1$. The ordinal class function $\mathbf{F}$, which for a fixed β takes each ordinal α to $\alpha+\beta$, is not continuous. For example, $\omega+1$ is not equal to $\bigcup_{m\in\omega}(m+1)$, as the latter is equal to ω.

Ordinal multiplication

Theorem 9.22. *There is a unique function $\cdot$ mapping* **On** $\times$ **On** *into* **On** *such that the following conditions hold:*

$$\begin{aligned} \alpha\cdot 0 &= 0; \\ \alpha\cdot(\beta+1) &= \alpha\cdot\beta+\alpha; \\ \alpha\cdot\beta &= \bigcup_{\gamma<\beta}(\alpha\cdot\gamma) \quad \textit{for } \beta \textit{ limit.} \end{aligned}$$

Proof. The proof is very similar to the proof of Theorem 9.19. We start with $\mathbf{A}$ and $\mathbf{R}$ as in that proof.

Now we define $\mathbf{G}:\mathbf{A}\times\mathbf{V}\to\mathbf{V}$. For any α,β and any set x, let

$$\mathbf{G}((\alpha,\beta),x)=\begin{cases} 0 & \text{if } \beta=0, \\ x(\alpha,\gamma)+\alpha & \text{if } x \text{ is a function with domain } \{\alpha\}\times\beta \\ & \text{and } \beta=\gamma+'1, \\ \bigcup_{\gamma<\beta}x(\alpha,\gamma) & \text{if } x \text{ is a function with domain } \{\alpha\}\times\beta \\ & \text{and } \beta \text{ is a limit ordinal,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Then by Theorem 8.7 let $\mathbf{F} : \mathbf{A} \to \mathbf{V}$ be such that $\mathbf{F}(y) = \mathbf{G}(y, \mathbf{F} \upharpoonright \mathrm{pred}_{\mathbf{AR}}(y))$ for any $y \in \mathbf{A}$. Then

$$\begin{aligned}
\mathbf{F}(\alpha, 0) &= \mathbf{G}((\alpha, 0), \mathbf{F} \upharpoonright \mathrm{pred}_{\mathbf{AR}}((\alpha, 0))) = 0;\\
\mathbf{F}(\alpha, \beta +' 1) &= \mathbf{G}((\alpha, \beta +' 1), \mathbf{F} \upharpoonright \mathrm{pred}_{\mathbf{AR}}((\alpha, \beta +' 1)))\\
&= \mathbf{F}(\alpha, \beta) + \alpha;\\
\mathbf{F}(\alpha, \beta) &= \mathbf{G}((\alpha, \beta), \mathbf{F} \upharpoonright \mathrm{pred}_{\mathbf{AR}}((\alpha, \beta)))\\
&= \bigcup_{\gamma<\beta} \mathbf{F}(\alpha, \gamma) \quad \text{if } \beta \text{ is a limit ordinal.}
\end{aligned}$$

Thus writing $\alpha \cdot \beta$ instead of $\mathbf{F}(\alpha, \beta)$ we see that $\mathbf{F}$ is as desired.

Now suppose that $\cdot^o$ also satisfies the conditions of the theorem. We show that $\alpha \cdot \beta = \alpha \cdot^o \beta$ for all α, β, by fixing α and going by induction on β, using Corollary 9.9. We have $\alpha \cdot 0 = 0 = \alpha \cdot^o \beta$. Assume that $\alpha \cdot \beta = \alpha \cdot^o \beta$. Then $\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha = (\alpha \cdot^o \beta) + 1 = \alpha \cdot^o (\beta + 1)$. Assume that β is a limit ordinal and $\alpha \cdot \gamma = \alpha \cdot^o \gamma$ for every $\gamma < \beta$. Then $\alpha \cdot \beta = \bigcup_{\gamma<\beta}(\alpha \cdot \gamma) = \bigcup_{\gamma<\beta}(\alpha \cdot^o \gamma) = \alpha \cdot^o \beta$. $\square$

Here are some basic properties of ordinal multiplication:

Theorem 9.23. *(i) If $m, n \in \omega$, then $m \cdot n \in \omega$.*

(ii) If $\alpha \neq 0$, then $\alpha \cdot \beta < \alpha \cdot (\beta + 1)$;

(iii) If $\alpha \neq 0$, then the class function assigning to each ordinal β the product $\alpha \cdot \beta$ is normal.

(iv) $0 \cdot \alpha = 0$;

(v) $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$;

(vi) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$;

(vii) If $\alpha \neq 0$, then $\beta \leq \alpha \cdot \beta$;

(viii) If $\alpha < \beta$ then $\alpha \cdot \gamma \leq \beta \cdot \gamma$;

(ix) $\alpha \cdot 1 = \alpha$.

(x) $\alpha \cdot 2 = \alpha + \alpha$.

(xi) If $\alpha, \beta \neq 0$ then $\alpha \cdot \beta \neq 0$.

Proof. (i): Induction on n, with m fixed. $m \cdot 0 = 0 \in \omega$. Assume that $m \cdot n \in \omega$. Then $m \cdot (n+1) = m \cdot n + m$; this is in ω by the inductive hypothesis and Theorem 9.21(i).

(ii): Using 9.21(ii), $\alpha \cdot \beta = \alpha \cdot \beta + 0 < \alpha \cdot \beta + \alpha = \alpha \cdot (\beta + 1)$.

(iii): this follows from (ii) and Proposition 9.16.

(iv): We prove this by induction on α. $0 \cdot 0 = 0$. Assuming that $0 \cdot \alpha = 0$, we have $0 \cdot (\alpha + 1) = 0 \cdot \alpha + 0 = 0 + 0 = 0$. Assuming that α is a limit ordinal and $0 \cdot \gamma = 0$ for all $\gamma < \alpha$, we have $0 \cdot \alpha = \bigcup_{\gamma<\alpha}(0 \cdot \gamma) = \bigcup_{\gamma<\alpha} 0 = 0$.

(v) Fix α and β. By (iv) we may assume that $\alpha \neq 0$; we then proceed by induction on γ. We define some ordinal class functions $\mathbf{F}, \mathbf{F}', \mathbf{G}$: for any γ, $\mathbf{F}(\gamma) = \beta + \gamma$; $\mathbf{F}'(\gamma) = \alpha \cdot \beta + \gamma$; $\mathbf{G}(\gamma) = \alpha \cdot \gamma$. These are normal functions by (iii) and Theorem 9.21(ii).

First of all,

$$\alpha \cdot (\beta + 0) = \alpha \cdot \beta = (\alpha \cdot \beta) + 0 = (\alpha \cdot \beta) + (\alpha \cdot 0),$$

so (v) holds for $\gamma = 0$. Now assume that (v) holds for γ. Then

$$\begin{aligned}
\alpha \cdot (\beta + (\gamma + 1)) &= \alpha \cdot ((\beta + \gamma) + 1) \\
&= \alpha \cdot (\beta + \gamma) + \alpha \\
&= (\alpha \cdot \beta) + (\alpha \cdot \gamma) + \alpha \\
&= (\alpha \cdot \beta) + (\alpha \cdot (\gamma + 1)),
\end{aligned}$$

as desired.

Finally, suppose that δ is a limit ordinal and we know (v) for all $\gamma < \delta$. Then

$$\begin{aligned}
\alpha \cdot (\beta + \delta) &= \mathbf{G}(\mathbf{F}(\delta)) \\
&= (\mathbf{G} \circ \mathbf{F})(\delta) \\
&= \bigcup_{\gamma < \delta} (\mathbf{G} \circ \mathbf{F})(\gamma) \\
&= \bigcup_{\gamma < \delta} (\alpha \cdot (\beta + \gamma)) \\
&= \bigcup_{\gamma < \delta} ((\alpha \cdot \beta) + (\alpha \cdot \gamma)) \\
&= \bigcup_{\gamma < \delta} \mathbf{F}'(\mathbf{G}(\gamma)) \\
&= \bigcup_{\gamma < \delta} (\mathbf{F}' \circ \mathbf{G})(\gamma) \\
&= (\mathbf{F}' \circ \mathbf{G})(\delta) \\
&= (\alpha \cdot \beta) + (\alpha \cdot \delta),
\end{aligned}$$

as desired. This completes the proof of (v).

(vi): For $\alpha = 0$, $0 \cdot (\beta \cdot \gamma) = 0$ by (iv), and by (iv) again, $(0 \cdot \beta) \cdot \gamma = 0 \cdot \gamma = 0$. For $\beta = 0$, $\alpha \cdot (0 \cdot \gamma) = \alpha \cdot 0 = 0$ using (iv), and $(\alpha \cdot 0) \cdot \gamma = 0 \cdot \gamma = 0$, using (iv) again.

So we assume that $\alpha, \beta \neq 0$. With fixed α, β we now proceed by induction on γ. Let $\mathbf{F}$ and $\mathbf{G}$ be the class functions defined by $\mathbf{F}(\delta) = \beta \cdot \delta$ and $\mathbf{G}(\delta) = \alpha \cdot \delta$ for all δ. These are normal functions by (iii). Then $\alpha \cdot (\beta \cdot 0) = \alpha \cdot 0 = 0 = (\alpha \cdot \beta) \cdot 0$. Assuming that $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$, we have

$$\begin{aligned}
\alpha \cdot (\beta \cdot (\gamma + 1)) &= \alpha \cdot (\beta \cdot \gamma + \beta) \\
&= \alpha \cdot (\beta \cdot \gamma) + \alpha \cdot \beta \\
&= (\alpha \cdot \beta) \cdot \gamma + \alpha \cdot \beta \\
&= (\alpha \cdot \beta) \cdot (\gamma + 1).
\end{aligned}$$

Finally, for δ limit, assuming that $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$ for all $\gamma < \delta$, we have

$$\alpha \cdot (\beta \cdot \delta) = \mathbf{G}(\mathbf{F}(\delta))$$

$$
\begin{aligned}
&= (\mathbf{G} \circ \mathbf{F})(\delta) \\
&= \bigcup_{\gamma<\delta} \mathbf{G}(\mathbf{F}(\gamma)) \\
&= \bigcup_{\gamma<\delta} \alpha \cdot (\beta \cdot \gamma) \\
&= \bigcup_{\gamma<\delta} (\alpha \cdot \beta) \cdot \gamma \\
&= (\alpha \cdot \beta) \cdot \delta.
\end{aligned}
$$

(vii): follows from (ii) and Proposition 9.15.

(viii): Fix $\alpha < \beta$. We prove that $\alpha \cdot \gamma \leq \beta \cdot \gamma$ by induction on γ. We have $\alpha \cdot 0 = 0 = \beta \cdot 0$, so $\alpha \cdot 0 \leq \beta \cdot 0$. Suppose that $\alpha \cdot \gamma \leq \beta \cdot \gamma$. Then

$$
\begin{aligned}
\alpha \cdot (\gamma + 1) &= \alpha \cdot \gamma + \alpha \\
&\leq \beta \cdot \gamma + \alpha \quad \text{induction hypothesis, Theorem 9.21(iii)} \\
&< \beta \cdot \gamma + \beta \quad \text{Theorem 9.21(ii)} \\
&= \beta \cdot (\gamma + 1).
\end{aligned}
$$

Finally, suppose that γ is a limit ordinal and $\alpha \cdot \delta \leq \beta \cdot \delta$ for every $\delta < \gamma$. Then

$$
\begin{aligned}
\alpha \cdot \gamma &= \bigcup_{\delta<\gamma} (\alpha \cdot \delta) \\
&\leq \bigcup_{\delta<\gamma} (\beta \cdot \delta) \quad \text{induction hypothesis, Proposition 7.8} \\
&= \beta \cdot \gamma.
\end{aligned}
$$

(ix): $\alpha \cdot 1 = \alpha \cdot (0 + 1) = \alpha \cdot 0 + \alpha = 0 + \alpha = \alpha$ using Proposition 9.21(vi).

(x): $\alpha \cdot 2 = \alpha \cdot (1 + 1) = \alpha \cdot 1 + \alpha = \alpha + \alpha$.

(xi): With $\alpha \neq 0$ fixed we go by induction on β, proving that $\beta \neq 0$ implies that $\alpha \cdot \beta \neq 0$. This is vacuously true for $\beta = 0$. Assume that the implication holds for β, and assume that $\beta + 1 \neq 0$. Then $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha > \alpha \cdot \beta + 0 = \alpha \cdot \beta$ using (iii); so $\alpha \cdot (\beta + 1) \neq 0$. Finally, suppose that β is a limit ordinal and the implication holds for all $\gamma < \beta$. Then $\alpha \cdot \beta = \bigcup_{\gamma<\beta} (\alpha \cdot \gamma) \geq \alpha \cdot 1 \neq 0$. □

The commutative law for multiplication fails in general. For example, $2 \cdot \omega = \omega$ while $\omega \cdot 2 = \omega + \omega > \omega$. Also the distributive law $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$ fails in general. For example, $(1 + 1) \cdot \omega = 2 \cdot \omega = \omega$, while $1 \cdot \omega + 1 \cdot \omega = \omega + \omega > \omega$. Here we use the fact that $1 \cdot \omega = \omega$. In fact, $1 \cdot \alpha = \alpha$ for any ordinal α, as is easily shown by induction on α.

Ordinal exponentiation

Theorem 9.24. *There is a unique function mapping* **On** $\times$ **On** *into* **On** *such that the following conditions hold, where we write the value of the function at an argument* (α, β)

as α^β*:*

$$\begin{aligned}
\alpha^0 &= 1;\\
\alpha^{\beta+1} &= \alpha^\beta \cdot \alpha;\\
\alpha^\beta &= \bigcup_{\gamma<\beta} (\alpha^\gamma) \quad \textit{for } \beta \textit{ limit.}
\end{aligned}$$

Proof. The proof is very similar to the proofs of Theorem 9.19 and 9.22. We start with $\mathbf{A}$ and $\mathbf{R}$ as in those proofs.

Now we define $\mathbf{G} : \mathbf{A} \times \mathbf{V} \to \mathbf{V}$. For any α, β and any set x, let

$$\mathbf{G}((\alpha,\beta),x) = \begin{cases} 1 & \text{if } \beta = 0,\\ x(\alpha,\gamma)\cdot\alpha & \text{if } x \text{ is a function with domain } \{\alpha\}\times\beta\\ & \text{and } \beta = \gamma +' 1,\\ \bigcup_{\gamma<\beta} x(\alpha,\gamma) & \text{if } x \text{ is a function with domain } \{\alpha\}\times\beta\\ & \text{and } \beta \text{ is a limit ordinal,}\\ \emptyset & \text{otherwise.} \end{cases}$$

Then by Theorem 8.7 let $\mathbf{F} : \mathbf{A} \to \mathbf{V}$ be such that $\mathbf{F}(y) = \mathbf{G}(y, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}(y))$ for any $y \in \mathbf{A}$. Then

$$\begin{aligned}
\mathbf{F}(\alpha,0) &= \mathbf{G}((\alpha,0), \mathbf{F}\restriction \mathrm{pred}_{\mathbf{AR}}((\alpha,0))) = 1;\\
\mathbf{F}(\alpha,\beta+'1) &= \mathbf{G}((\alpha,\beta+'1), \mathbf{F}\restriction \mathrm{pred}_{\mathbf{AR}}((\alpha,\beta+'1)))\\
&= \mathbf{F}(\alpha,\beta)\cdot\alpha;\\
\mathbf{F}(\alpha,\beta) &= \mathbf{G}((\alpha,\beta), \mathbf{F}\restriction \mathrm{pred}_{\mathbf{AR}}((\alpha,\beta)))\\
&= \bigcup_{\gamma<\beta} \mathbf{F}(\alpha,\gamma) \quad \text{if } \beta \text{ is a limit ordinal.}
\end{aligned}$$

Thus writing α^β instead of $\mathbf{F}(\alpha,\beta)$ we see that $\mathbf{F}$ is as desired.

Now suppose that $\mathbf{F}'$ also satisfies the conditions of the theorem. We show that $\alpha^\beta = \mathbf{F}'(\alpha,\beta)$ for all α,β, by fixing α and going by induction on β, using Corollary 9.9. We have $\alpha^0 = 1 = \mathbf{F}'(\alpha,\beta)$. Assume that $\alpha^\beta = \mathbf{F}'(\alpha,\beta)$. Then $\alpha^{\beta+1} = (\alpha^\beta)\cdot\alpha = \mathbf{F}'(\alpha,\beta)\cdot\alpha = \mathbf{F}'(\alpha,\beta+1)$. Assume that β is a limit ordinal and $\alpha^\gamma = \mathbf{F}'(\alpha,\gamma)$ for every $\gamma < \beta$. Then $\alpha^\beta = \bigcup_{\gamma<\beta}\alpha^\gamma = \bigcup_{\gamma<\beta}\mathbf{F}'(\alpha,\gamma) = \mathbf{F}'(\alpha,\beta)$. □

Now we give the simplest properties of exponentiation.

Theorem 9.25. *(i) If* $m, n \in \omega$*, then* $m^n \in \omega$*.*

(ii) $0^0 = 1$*;*

(iii) $0^{\beta+1} = 0$*;*

(iv) $0^\beta = 1$ *for* β *a limit ordinal;*

(v) $1^\beta = 1$*;*

(vi) If $\alpha \neq 0$*, then* $\alpha^\beta \neq 0$*;*

(vii) If $\alpha > 1$ *then* $\alpha^\beta < \alpha^{\beta+1}$*;*

(viii) If $\alpha > 1$, then the ordinal class function assigning to each ordinal β the value α^β is normal;

(ix) If $\alpha > 1$, then $\beta \leq \alpha^\beta$;

(x) If $0 < \alpha < \beta$, then $\alpha^\gamma \leq \beta^\gamma$;

(xi) For $\alpha \neq 0$, $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$;

(xii) For $\alpha \neq 0$, $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$.

Proof. (i): With m fixed we go by induction on n. $m^0 = 1 \in \omega$. Assume that $m^n \in \omega$. Then $m^{n+1} = m^n \cdot m \in \omega$ by the induction hypothesis and Theorem 9.23(i).

(ii): Obvious.

(iii): $0^{\beta+1} = 0^\beta \cdot 0 = 0$.

(iv): We prove by induction on β that

$$0^\beta = \begin{cases} 1 & \text{if } \beta = 0, \\ 0 & \text{if } \beta \text{ is a successor ordinal,} \\ 1 & \text{if } \beta \text{ is a limit ordinal.} \end{cases}$$

This is clearly true for $\beta = 0$, and if it is true for γ then it is true for $\gamma + 1$ by (iii). Now suppose that β is a limit ordinal and it is true for all $\gamma < \beta$. Thus 0^γ is 0 or 1 for each $\gamma < \beta$, and $0^0 = 1$ with $0 < \beta$, so $0^\beta = \bigcup_{\gamma<\beta} 0^\gamma = 1$.

(v): we prove this by induction on β. $1^0 = 1$. Assume that $1^\beta = 1$. Then $1^{\beta+1} = 1^\beta \cdot 1 = 1 \cdot 1 = 1$. Assume that β is a limit ordinal and $1^\gamma = 1$ for all $\gamma < \beta$. Then $1^\beta = \bigcup_{\gamma<\beta} 1^\gamma = 1$.

(vi) With $\alpha \neq 0$ fixed, we go by induction on β. $\alpha^0 = 1 \neq 0$. Assume that $\alpha^\beta \neq 0$. Then $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha \neq 0$ by the inductive hypothesis and Theorem 9.23(xi). Assume that β is a limit ordinal and $\alpha^\gamma \neq 0$ for all $\gamma < \beta$. Then $\alpha^\beta = \bigcup_{\gamma<\beta} \alpha^\gamma \neq 0$ by the inductive hypothesis, since $0 < \beta$ and $\alpha^0 \neq 0$.

(vii): We have $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha > \alpha^\beta \cdot 1 = \alpha^\beta$ using (vi) and Theorem 9.23(iii),(ix).

(viii): by (vii) and Theorem 9.16.

(ix): by (viii) and Theorem 9.15.

(x): With $0 < \alpha < \beta$, induction on γ. $\alpha^0 = 1 = \beta^0$. Assume that $\alpha^\gamma \leq \beta^\gamma$. Then $\alpha^{\gamma+1} = \alpha^\gamma \cdot \alpha \leq \beta^\gamma \cdot \alpha$ (by the inductive hypothesis and Theorem 9.23(viii)) $< \beta^\gamma \cdot \beta$ (by Theorem 9.23 (iii)) $= \beta^{\gamma+1}$. Now assume that $\alpha^\gamma \leq \alpha^\beta$ for all $\gamma < \delta$, where δ is a limit ordinal. Then $\alpha^\delta = \bigcup_{\gamma<\delta} \alpha^\gamma \leq \bigcup_{\gamma<\delta} \beta^\gamma = \beta^\delta$, using Proposition 7.12.

(xi): By (v) we may assume that $\alpha > 1$. Define $\mathbf{F}(\delta) = \beta+\delta$, $\mathbf{G}(\delta) = \alpha^\delta$, $\mathbf{H}(\delta) = \alpha^\beta \cdot \delta$. These are normal functions by Theorem 9.21(ii), Theorem 9.23(iii) and (vi), and (viii).

Now we go by induction on γ. $\alpha^{\beta+0} = \alpha^\beta = \alpha^\beta \cdot 1 = \alpha^\beta \cdot \alpha^0$. Assume that $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$. Then $\alpha^{\beta+\gamma+1} = \alpha^{\beta+\gamma} \cdot \alpha = \alpha^\beta \cdot \alpha^\gamma \cdot \alpha + a^\beta \cdot \alpha^{\gamma+1}$. Finally, suppose that δ is limit and $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$ for every $\gamma < \delta$. Then

$$\begin{aligned} \alpha^{\beta+\delta} &= \mathbf{G}(\mathbf{F}(\delta)) \\ &= \bigcup_{\gamma<\delta} \mathbf{G}(\mathbf{F}(\gamma)) \\ &= \bigcup_{\gamma<\delta} \alpha^{\beta+\gamma} \end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{\gamma<\delta}(\alpha^\beta \cdot \alpha^\gamma) \\
&= \bigcup_{\gamma<\delta} \mathbf{H}(\gamma) \\
&= \mathbf{H}(\delta) \\
&= \alpha^\beta \cdot \alpha^\delta.
\end{aligned}
$$

(xii): First note that it holds for $\beta = 0$, since $(\alpha^0)^\gamma = 1^\gamma = 1$ and $\alpha^{0\cdot\gamma} = \alpha^0 = 1$. Similarly, it holds for $\alpha = 1$. Now assume that $\alpha > 1$ and $\beta > 0$. Let $\mathbf{F}(\delta) = \alpha^\delta$ for any δ, and $G(\delta) = \beta \cdot \delta$ for any δ. Then $\mathbf{F}$ and $\mathbf{G}$ are normal functions. Now we prove the result by induction on γ. First, $(\alpha^\beta)^0 = 1 = \alpha^0 = \alpha^{\beta\cdot 0}$. Now assume that $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$. Then

$$(\alpha^\beta)^{\gamma+1} = (\alpha^\beta)^\gamma \cdot \alpha^\beta = \alpha^{\beta\cdot\gamma} \cdot \alpha^\beta = \alpha^{\beta\cdot\gamma+\beta} = \alpha^{\beta\cdot(\gamma+1)}.$$

Finally, suppose that δ is a limit ordinal and $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$ for all $\gamma < \delta$. Then

$$
\begin{aligned}
(\alpha^\beta)^\delta &= \bigcup_{\gamma<\delta}(\alpha^\beta)^\gamma \\
&= \bigcup_{\gamma<\delta} \alpha^{\beta\cdot\gamma} \\
&= \bigcup_{\gamma<\delta} \mathbf{F}(\mathbf{G}(\gamma)) \\
&= \mathbf{F}(\mathbf{G}(\delta)) \\
&= \alpha^{\beta\cdot\delta}
\end{aligned}
$$

□

Theorem 9.26. (division algorithm) *Suppose that α and β are ordinals, with $\beta \neq 0$. Then there are unique ordinals ξ, η such that $\alpha = \beta \cdot \xi + \eta$ with $\eta < \beta$.*

Proof. First we prove the existence. Note that $\alpha < \alpha + 1 \leq \beta \cdot (\alpha + 1)$. Thus there is an ordinal number ρ such that $\alpha < \beta \cdot \rho$; take the least such ρ. Obviously $\rho \neq 0$. If ρ is a limit ordinal, then because $\beta \cdot \rho = \bigcup_{\sigma<\rho}(\beta \cdot \sigma)$, it follows that there is a $\sigma < \rho$ such that $\alpha < \beta \cdot \sigma$, contradicting the minimality of ρ. Thus ρ is a successor ordinal $\xi + 1$. By the definition of ρ we have $\beta \cdot \xi \leq \alpha$. Hence there is an ordinal η such that $\beta \cdot \xi + \eta = \alpha$. We claim that $\eta < \beta$. Otherwise, $\alpha = \beta \cdot \xi + \eta \geq \beta \cdot \xi + \beta = \beta \cdot (\xi + 1) = \beta \cdot \rho$, contradicting the definition of ρ. This finishes the proof of existence.

For uniqueness, suppose that also $\alpha = \beta \cdot \xi' + \eta'$ with $\eta' < \beta$. Suppose that $\xi \neq \xi'$. By symmetry, say $\xi < \xi'$. Then

$$\alpha = \beta \cdot \xi + \eta < \beta \cdot \xi + \beta = \beta \cdot (\xi + 1) \leq \beta \cdot \xi' \leq \beta \cdot \xi' + \eta' = \alpha,$$

contradiction. Hence $\xi = \xi'$. Hence also $\eta = \eta'$. □

Theorem 9.27. (extended division algorithm)*Let α and β be ordinals, with $\alpha \neq 0$ and $1 < \beta$. Then there exist unique ordinals $\gamma, \delta, \varepsilon$ such that the following conditions hold:*

(i) $\alpha = \beta^\gamma \cdot \delta + \varepsilon$.
(ii) $\gamma \leq \alpha$.
(iii) $0 < \delta < \beta$,
(iv) $\varepsilon < \beta^\gamma$.

Proof. We have $\alpha < \alpha + 1 \leq \beta^{\alpha+1}$; so there is an ordinal φ such that $\alpha < \beta^\varphi$. We take the least such φ. Clearly φ is a successor ordinal $\gamma + 1$. So we have $\beta^\gamma \leq \alpha < \beta^{\gamma+1}$. Now $\beta^\gamma \neq 0$, since $\beta > 1$. Hence by the division algorithm there are ordinals δ, ε such that $\alpha = \beta^\gamma \cdot \delta + \varepsilon$, with $\varepsilon < \beta^\gamma$. Now $\delta < \beta$; for if $\beta \leq \delta$, then

$$\alpha = \beta^\gamma \cdot \delta + \varepsilon \geq \beta^\gamma \cdot \beta = \beta^{\gamma+1} > \alpha,$$

contradiction. We have $\delta \neq 0$, as otherwise $\alpha = \varepsilon < \beta^\gamma$, contradiction.. Also, $\gamma \leq \alpha$, since

$$\alpha = \beta^\gamma \cdot \delta + \varepsilon \geq \beta^\gamma \geq \gamma.$$

This proves the existence of $\gamma, \delta, \varepsilon$ as called for in the theorem.

Suppose that $\gamma', \delta', \varepsilon'$ also satisfy the indicated conditions; thus

(1) $\alpha = \beta^{\gamma'} \cdot \delta' + \varepsilon'$,

(2) $\gamma' \leq \alpha$,

(3) $0 < \delta' < \beta$,

(4) $\varepsilon' < \beta^{\gamma'}$.

Suppose that $\gamma \neq \gamma'$; by symmetry, say that $\gamma < \gamma'$. Then

$$\alpha = \beta^\gamma \cdot \delta + \varepsilon < \beta^\gamma \cdot \delta + \beta^\gamma = \beta^\gamma \cdot (\delta + 1) \leq \beta^\gamma \cdot \beta = \beta^{\gamma+1} \leq \beta^{\gamma'} \leq \alpha,$$

contradiction. Hence $\gamma = \gamma'$. Hence by the ordinary division algorithm we also have $\delta = \delta'$ and $\varepsilon = \varepsilon'$. $\square$

We can obtain an interesting normal form for ordinals by re-applying Theorem 9.27 to the "remainder" ε over and over again. That is the purpose of the following definitions and results. This generalizes the ordinary decimal and binary systems of notation, by taking $\beta = 10$ or $\beta = 2$ and restricting to natural numbers. For infinite ordinals it is useful to take $\beta = \omega$; this gives the *Cantor normal form*.

To abbreviate some long expressions, we let $N(\beta, m, \gamma, \delta)$ stand for the following statement:

β is an ordinal > 1, m is a positive integer, γ and δ are sequences of ordinals each of length m, and:

(1) $\gamma(0) > \gamma(1) > \cdots > \gamma(m-1)$;

(2) $0 < \delta(i) < \beta$ *for each* $i < m$.

If $N(\beta, m, \gamma, \delta)$, then we define

$$k(\beta, m, \gamma, \delta) = \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot \delta(1) + \cdots + \beta^{\gamma(m-1)} \cdot \delta(m-1).$$

Lemma 9.28. *Assume that $N(\beta, m, \gamma, \delta)$ and $N(\beta, n, \gamma', \delta')$. Then:*

(i) $k(\beta, m, \gamma, \delta) \geq \gamma(0)$.

(ii) $k(\beta, m, \gamma, \delta) < \beta^{\gamma(0)} \cdot (\delta(0) + 1) \leq \beta^{\gamma(0)+1}$.

(iii) If $\gamma(0) \neq \gamma'(0)$, then $k(\beta, m, \gamma, \delta) < k(\beta, n, \gamma', \delta')$ iff $\gamma(0) < \gamma'(0)$.

(iv) If $\gamma(0) = \gamma'(0)$ and $\delta(0) \neq \delta'(0)$, then $k(\beta, m, \gamma, \delta) < k(\beta, n, \gamma', \delta')$ iff $\delta(0) < \delta'(0)$.

(v) If $\gamma(j) = \gamma'(j)$ and $\delta(j) = \delta'(j)$ for all $j < i$, while $\gamma(i) \neq \gamma'(i)$, then $k(\beta, m, \gamma, \delta) < k(\beta, n, \gamma', \delta')$ iff $\gamma(i) < \gamma'(i)$.

(vi) If $\gamma(j) = \gamma'(j)$ and $\delta(j) = \delta'(j)$ for all $j < i$, while $\gamma(i) = \gamma'(i)$ and $\delta(i) \neq \delta'(i)$, then $k(\beta, m, \gamma, \delta) < k(\beta, n, \gamma', \delta')$ iff $\delta(i) < \delta'(i)$.

(vii) If $\gamma \leq \gamma'$, $\delta \leq \delta'$, and $m < n$, then $k(\beta, m, \gamma, \delta) < k(\beta, n, \gamma', \delta')$.

Proof. (i): $k(\beta, m, \gamma, \delta) \geq \beta^{\gamma(0)} \geq \gamma(0)$.

(ii): It is clear for $m = 1$. For $m > 1$,

$$\begin{aligned}
\beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot \delta(1) + \cdots + \beta^{\gamma(m-1)} \cdot \delta(m-1) &< \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)+1} \\
&\leq \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(0)} \\
&= \beta^{\gamma(0)} \cdot (\delta(0) + 1) \\
&\leq \beta^{\gamma(0)} \cdot \beta \\
&= \beta^{\gamma(0)+1}.
\end{aligned}$$

For (iii), assume the hypothesis, and suppose that $\gamma(0) < \gamma'(0)$. Then

$$\begin{aligned}
k(\beta, m, \gamma, \delta) < \beta^{\gamma(0)} \cdot (\delta(0) + 1) &\leq \beta^{\gamma(0)+1} \quad \text{by (ii)} \\
&\leq \beta^{\gamma'(0)} \\
&\leq k(\beta, n, \gamma', \delta').
\end{aligned}$$

By symmetry (iii) now follows.

For (iv), assume the hypothesis, and suppose that $\delta(0) < \delta'(0)$. Then

$$\begin{aligned}
k(\beta, m, \gamma, \delta) < \beta^{\gamma(0)} \cdot (\delta(0) + 1) &\leq \beta^{\gamma'(0)} \cdot (\delta(0) + 1) \quad \text{by (ii)} \\
&\leq \beta^{\gamma'(0)} \cdot \delta'(0) \\
&\leq k(\beta, n, \gamma', \delta')
\end{aligned}$$

By symmetry (iv) now follows.

(v) is clear from (iii), by deleting the first i summands of the sums.

(vi) is clear from (iv), by deleting the first i summands of the sums.

(vii) is clear. □

Theorem 9.29. (expansion theorem) *Let α and β be ordinals, with $\alpha \neq 0$ and $1 < \beta$. Then there exist a unique $m \in \omega$ and finite sequences $\langle \gamma(i) : i < m \rangle$ and $\langle \delta(i) : i < m \rangle$ of ordinals such that the following conditions hold:*

(i) $\alpha = \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot \delta(1) + \cdots + \beta^{\gamma(m-1)} \cdot \delta(m-1)$.
(ii) $\alpha \geq \gamma(0) > \gamma(1) > \cdots > \gamma(m-1)$.
(iii) $0 < \delta(i) < \beta$ *for each* $i < m$.

Proof. For the existence, with $\beta > 1$ fixed we proceed by induction on α. Assume that the theorem holds for every $\alpha' < \alpha$ such that $\alpha' \neq 0$, and suppose that $\alpha \neq 0$. By Theorem 9.27, let φ, ψ, θ be such that

(1) $\alpha = \beta^{\varphi} \cdot \psi + \theta$,

(2) $\varphi \leq \alpha$,

(3) $0 < \psi < \beta$,

(4) $\theta < \beta^{\varphi}$.

If $\theta = 0$, then we can take our sequences to be $\langle \gamma(0) \rangle$ and $\langle \delta(0) \rangle$, with $\gamma(0) = \varphi$ and $\delta(0) = \psi$. Now assume that $\theta > 0$. Then

$$\theta < \beta^{\varphi} \leq \beta^{\varphi} \cdot \psi + \theta = \alpha;$$

so $\theta < \alpha$. Hence by the inductive assumption we can write

$$\theta = \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot \delta(1) + \cdots + \beta^{\gamma(m-1)} \cdot \delta(m-1)$$

with

(5) $\theta \geq \gamma(0) > \gamma(1) > \cdots > \gamma(m-1)$.

(6) $0 < \delta(i) < \beta$ for each $i < m$.

Then our desired sequences for α are

$$\langle \varphi, \gamma(0), \gamma(1), \ldots, \gamma(m-1) \rangle \quad \text{and} \quad \langle \psi, \delta(0), \delta(1), \ldots, \delta(m-1) \rangle.$$

To prove this, we just need to show that $\varphi > \gamma(0)$. If $\varphi \leq \gamma(0)$, then

$$\beta^{\varphi} \leq \beta^{\gamma(0)} \leq \theta,$$

contradiction.

This finishes the existence part of the proof.

For the uniqueness, we use the notation introduced above, and proceed by induction on α. Suppose the uniqueness statement holds for all nonzero $\alpha' < \alpha$, and now we have $N(\beta, m, \gamma, \delta)$, $N(\beta, n, \gamma', \delta')$, and

$$\alpha = k(\beta, m, \gamma, \delta) = k(\beta, n, \gamma', \delta').$$

We suppose that the uniqueness fails. Say $m \leq n$. Then there is an $i < m$ such that $\gamma(i) \neq \gamma'(i)$ or $\delta(i) \neq \delta'(i)$; we take the least such i. Then we have a contradiction of Lemma 9.28. $\square$

Lemma 9.30. *(i) If $\omega \leq \alpha$, then $1 + \alpha = \alpha$.*

(ii) If $\delta \neq 0$, then $\omega \leq \omega^\delta$ and $1 + \omega^\delta = \omega^\delta$.

Proof. (i): By Theorem 9.21(vii) there is a β such that $\omega + \beta = \alpha$. Hence $1 + \alpha = 1 + \omega + \beta = \omega + \beta = \alpha$.

(ii) $\omega = \omega^1 \leq \omega^\delta$, and $1 + \omega^\delta = \omega^\delta$ by (i). □

Lemma 9.31. *If $\alpha < \omega^\beta$ then $\alpha + \omega^\beta = \omega^\beta$.*

Proof. First we prove

(1) If $\gamma < \beta$, then $\omega^\gamma + \omega^\beta = \omega^\beta$.

In fact, suppose that $\gamma < \beta$. Then there is a nonzero δ such that $\gamma + \delta = \beta$. Then

$$\omega^\gamma + \omega^\beta = \omega^\gamma + \omega^{\gamma+\delta} = \omega^\gamma + \omega^\gamma \cdot \omega^\delta = \omega^\gamma \cdot (1 + \omega^\delta) = \omega^\gamma \cdot \omega^\delta = \omega^\beta.$$

By an easy ordinary induction, we obtain from (1)

(2) If $\gamma < \beta$ and $m \in \omega$, then $\omega^\gamma \cdot m + \omega^\beta = \omega^\beta$.

Now we turn to the general case. If $\beta = 0$ or $\alpha < \omega$, the desired conclusion is clear. So suppose that $\omega \leq \alpha$ and $\beta > 0$. Then we can write $\alpha = \omega^\gamma \cdot m + \delta$ with $m \in \omega$ and $\delta < \omega^\gamma$. Then

$$\omega^\beta \leq \alpha + \omega^\beta = \omega^\gamma \cdot m + \delta + \omega^\beta \leq \omega^\gamma \cdot (m+1) + \omega^\beta = \omega^\beta$$ □

Theorem 9.32. *The following conditions are equivalent:*

(i) $\beta + \alpha = \alpha$ for all $\beta < \alpha$. (Absorption under addition)

(ii) For all $\beta, \gamma < \alpha$, also $\beta + \gamma < \alpha$.

(iii) $\alpha = 0$, or $\alpha = \omega^\beta$ for some β.

Proof. (i)⇒(ii): Assuming (i), if $\beta, \gamma < \alpha$, then $\beta + \gamma < \beta + \alpha = \alpha$.

(ii)⇒(iii): Assume (ii). If $\alpha = 0$ or $\alpha = 1$, condition (iii) holds, so suppose that $2 \leq \alpha$. Then clearly (ii) implies that $\alpha \geq \omega$. Choose β, m, γ such that $m \in \omega$, $\alpha = \omega^\beta \cdot m + \gamma$, and $\gamma < \omega^\beta$. If $\gamma \neq 0$, then $\omega^\beta \cdot m < \omega^\beta \cdot m + \gamma = \alpha$, and also $\gamma < \omega^\beta < \alpha$, so that (ii) is contradicted. So $\gamma = 0$. If $m > 1$, write $m = n + 1$ with $n \neq 0$. Then

$$\alpha = \omega^\beta \cdot m = \omega^\beta \cdot (n+1) = \omega^\beta \cdot n + \omega^\beta,$$

and $\omega^\beta \cdot n, \omega^\beta < \alpha$, again contradicting (ii). Hence $m = 1$, as desired in (iii).

Finally, (iii)⇒(i) by Lemma 9.31. □

Lemma 9.33. *If $\alpha \neq 0$ and m is a positive integer, then $m \cdot \omega^\alpha = \omega^\alpha$.*

Proof. Induction on α. It is clear for $\alpha = 1$. Assuming it true for α, we have $m \cdot \omega^{\alpha+1} = m \cdot \omega^\alpha \cdot \omega = \omega^\alpha \cdot \omega = \omega^{\alpha+1}$. Assuming it is true for every $\beta < \alpha$ with α a limit ordinal, we have $1 + \alpha = \alpha$, and so $m \cdot \omega^\alpha = m \cdot \omega \cdot \omega^\alpha = \omega \cdot \omega^\alpha = \omega^\alpha$. □

Theorem 9.34. *The following conditions are equivalent:*

(i) For all β, if $0 < \beta < \alpha$ then $\beta \cdot \alpha = \alpha$. (absorption under multiplication)
(ii) For all $\beta, \gamma < \alpha$, also $\beta \cdot \gamma < \alpha$.
(iii) $\alpha \in \{0, 1, 2\}$ or there is a β such that $\alpha = \omega^{(\omega^\beta)}$.

Proof. (i)⇒(ii): Assume (i), and suppose that $\beta, \gamma < \alpha$. If $\beta = 0$, then $\beta \cdot \gamma = 0 < \alpha$. If $\beta \neq 0$, then $\beta \cdot \gamma < \beta \cdot \alpha = \alpha$.

(ii)⇒(iii): Assume (ii), and suppose that $\alpha \notin \{0, 1, 2\}$. Clearly then $\omega \leq \alpha$. Now if $\beta, \gamma < \alpha$, then $\beta + \gamma < \alpha$. In fact, if $\beta \leq \gamma$, then $\beta + \gamma \leq \gamma + \gamma = \gamma \cdot 2 < \alpha$ by (ii); and if $\gamma < \beta$ then $\beta + \gamma < \beta + \beta = \beta \cdot 2 < \alpha$. Hence by Theorem 9.32 there is a γ such that $\alpha = \omega^\gamma$. Now if $\delta, \varepsilon < \gamma$, then $\omega^\delta, \omega^\varepsilon < \omega^\gamma = \alpha$, and hence $\omega^{\delta+\varepsilon} = \omega^\delta \cdot \omega^\varepsilon < \alpha = \omega^\gamma$, so that $\delta + \varepsilon < \gamma$. Hence by Theorem 9.32, $\gamma = \omega^\beta$ for some β.

(iii)⇒(i): Assume (iii). Clearly $0, 1, 2$ satisfy (i), so assume that $\alpha = \omega^{(\omega^\beta)}$. Take any $\gamma < \alpha$ with $\gamma \neq 0$. If $\gamma < \omega$, then $\gamma \cdot \alpha = \alpha$ by Lemma 9.33. So assume that $\omega \leq \gamma$. Write $\gamma = \omega^\delta \cdot m + \varepsilon$ with $m \in \omega$ and $\varepsilon < \omega^\delta$. Then $\delta < \beta$, and so

$$\begin{aligned}
\alpha = \omega^{(\omega^\beta)} \leq \gamma \cdot \omega^{(\omega^\beta)} &= (\omega^\delta \cdot m + \varepsilon) \cdot \omega^{(\omega^\beta)} \\
&\leq (\omega^\delta \cdot m + \omega^\delta) \cdot \omega^{(\omega^\beta)} \\
&= \omega^\delta \cdot (m+1) \cdot \omega^{(\omega^\beta)} \\
&\leq \omega^{\delta+1} \cdot \omega^{(\omega^\beta)} \\
&= \omega^{\delta+1+\omega^\beta} \\
&= \omega^{(\omega^\beta)} \\
&= \alpha
\end{aligned}$$

□

EXERCISES

E9.1. Let $(A, <)$ be a well order. Suppose that $B \subset A$ and $\forall b \in B \forall a \in A[a < b \rightarrow a \in B]$. Prove that there is an element $a \in A$ such that $B = \{b \in A : b < a\}$.

E9.2. Let $(A, <)$ be a well order. Suppose that $B \subset A$ and $\forall b \in B \forall a \in A[a < b \rightarrow a \in B]$. Prove that $(A, <)$ is not isomorphic to $(B, <)$.

E9.3. Suppose that f is a one-one function mapping an ordinal α onto a set A. Define a relation $\prec$ which is a subset of $A \times A$ such that $(A, <)$ is a well-order and f is an isomorphism of $(\alpha, <)$ onto $(A, \prec)$.

E9.4. Prove that $1 + m = m + 1$ for any $m \in \omega$.

E9.5. Prove that $m + n = n + m$ for any $m, n \in \omega$.

E9.6. Prove that $\omega \leq \alpha$ iff $1 + \alpha = \alpha$.

E9.7. For any ordinals α, β let

$$\alpha \oplus \beta = (\alpha \times \{0\}) \cup (\beta \times \{1\}).$$

We define a relation $\prec$ as follows. For any $x, y \in \alpha \oplus \beta$, $x \prec y$ iff one of the following three conditions holds:

(i) There are $\xi, \eta < \alpha$ such that $x = (\xi, 0)$, $y = (\eta, 0)$, and $\xi < \eta$.
(ii) There are $\xi, \eta < \beta$ such that $x = (\xi, 1)$, $y = (\eta, 1)$, and $\xi < \eta$.
(ii) There are $\xi < \alpha$ and $\eta < \beta$ such that $x = (\xi, 0)$ and $y = (\eta, 1)$.

Prove that $(\alpha \oplus \beta, \prec)$ is a well order which is isomorphic to $\alpha + \beta$.

E9.8. Given ordinals α, β, we define the following relation $\prec$ on $\alpha \times \beta$:

$$\begin{aligned}(\xi, \eta) \prec (\xi', \eta') \quad \text{iff} \quad & ((\xi, \eta) \text{ and } (\xi', \eta') \text{ are in } \alpha \times \beta \text{ and:} \\ & \eta < \eta', \text{ or } (\eta = \eta' \text{ and } \xi < \xi').\end{aligned}$$

We may say that this is the *anti-dictionary* or *anti-lexicographic* order.

Show that the set $\alpha \times \beta$ under the anti-lexicographic order is a well order which is isomorphic to $\alpha \cdot \beta$.

E9.9. Suppose that α and β are ordinals, with $\beta \neq 0$. We define

$$^{\alpha}\beta^{\mathrm{w}} = \{f \in {}^{\alpha}\beta : \{\xi < \alpha : f(\xi) \neq 0\} \text{ is finite}\}.$$

For $f, g \in {}^{\alpha}\beta^{\mathrm{w}}$ we write $f \prec g$ iff $f \neq g$ and $f(\xi) < g(\xi)$ for the **greatest** $\xi < \alpha$ for which $f(\xi) \neq g(\xi)$.

Prove that $({}^{\alpha}\beta^{\mathrm{w}}, \prec)$ is a well-order which is order-isomorphic to the ordinal exponent β^{α}. (A set X is *finite* iff there is a bijection from some natural number onto X.)

E9.10. Show that for every nonzero ordinal α there are only finitely many ordinals β such that $\alpha = \gamma \cdot \beta$ for some γ.

E9.11. Prove that $n^{(\omega^{\omega})} = \omega^{(\omega^{\omega})}$ for every natural number $n > 1$.

E9.12 Show that the following conditions are equivalent for any ordinals α, β:
(i) $\alpha + \beta = \beta + \alpha$.
(ii) There exist an ordinal γ and natural numbers k, l such that $\alpha = \gamma \cdot k$ and $\beta = \gamma \cdot l$.

E9.13. Suppose that $\alpha < \omega^{\gamma}$. Show that $\alpha + \beta + \omega^{\gamma} = \beta + \omega^{\gamma}$.

E9.14. Show that the following conditions are equivalent:
(i) α is a limit ordinal
(ii) $\alpha = \omega \cdot \beta$ for some $\beta \neq 0$.
(iii) For every $m \in \omega \backslash 1$ we have $m \cdot \alpha = \alpha$, and $\alpha \neq 0$.

E9.15. Show that $(\alpha + \beta) \cdot \gamma \leq \alpha \cdot \gamma + \beta \cdot \gamma$ for any ordinals α, β, γ.

10. The axiom of choice

We give a small number of equivalent forms of the axiom of choice; these forms should be sufficient for most mathematical purposes. The axiom of choice has been investigated a lot, and we give some references for this after proving the main theorem of this chapter.

The set of axioms of ZFC with the axiom of choice removed is denoted by ZF; so we work in ZF in this chapter.

The two main equivalents to the axiom of choice are as follows.

Zorn's Lemma. *If $(A, <)$ is a partial order such that $A \neq \emptyset$ and every subset of A simply ordered by $<$ has an upper bound, then A has a maximal element under $<$, i.e., an element a such that there is no element $b \in A$ such that $a < b$.*

Well-ordering principle. *For every set A there is a well-ordering of A, i.e., there is a relation $<$ such that $(A, <)$ is a well-order.*

In addition, the following principle, usually called the axiom of choice, is equivalent to the actual form that we have chosen:

Choice-function principle. *If A is a family of nonempty sets, then there is a function f with domain A such that $f(a) \in a$ for every $a \in A$. Such a function f is called a choice function for A.*

Lemma 10.1. *Suppose that $(A, <)$ is a partial order and $a \in A$. Then $A \not< a$.*

Proof. Suppose to the contrary that $A < a$. Then $(A, a) \in <\subseteq A \times A$, so $A \in A$, contradicting Theorem 7.5. $\square$

Theorem 10.2. *In ZF the following four statements are equivalent:*

(i) the axiom of choice;

(ii) the choice-function principle;

(iii) Zorn's lemma.

(iv) the well-ordering principle.

Proof. Axiom of choice $\Rightarrow$ choice-function principle. Assume the axiom of choice, and let A be a family of nonempty sets. Let

$$\mathscr{A} = \{X : \exists a \in A[X = \{(a, x) : x \in a\}]\}.$$

Since each member of A is nonempty, also each member of $\mathscr{A}$ is nonempty. Given $X, Y \in \mathscr{A}$ with $X \neq Y$, choose $a, b \in A$ such that $X = \{(a, x) : x \in a\}$ and $Y = \{(b, x) : x \in b\}$. Thus $a \neq b$ since $A \neq B$. The basic property of ordered pairs then implies that $A \cap B = \emptyset$.

So, by the axiom of choice, let $\mathscr{B}$ have exactly one element in common with each element of $\mathscr{A}$. Define $f = \{b \in \mathscr{B} : \text{there exist } a \in A \text{ and } x \text{ such that } b = (a, x)\}$. Clearly f is the desired choice function for A.

Choice-function principle $\Rightarrow$ Zorn's lemma. Assume the choice-function principle, and also assume the hypotheses of Zorn's lemma. Let f be a choice function for

$\mathscr{P}(A)\backslash\{\emptyset\}$. Define $\mathbf{G} : \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ by setting, for any ordinal α and any set x,

$$\mathbf{G}(\alpha, x) = \begin{cases} f(\{a \in A : x(\beta) < a \text{ for all } \beta < \alpha\}) & \text{if } x \text{ is a function with domain } \alpha \\ & \text{and this set is nonempty,} \\ A & \text{otherwise.} \end{cases}$$

Let $\mathbf{F}$ be obtained from $\mathbf{G}$ by the recursion theorem 6.7; thus for any ordinal α,

$$\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha) = \begin{cases} f(\{a \in A : \mathbf{F}(\beta) < a \text{ for all } \beta < \alpha\}) & \text{if this set is nonempty,} \\ A & \text{otherwise.} \end{cases}$$

(1) If $\alpha < \beta \in \mathbf{On}$ and $\mathbf{F}(\beta) \neq A$, then $\mathbf{F}(\alpha) \neq A$, and $\mathbf{F}(\alpha) < \mathbf{F}(\beta)$.

In fact, suppose that $\mathbf{F}(\alpha) = A$. Now by the definition of $\mathbf{F}(\beta)$, the set $\{a \in A : \mathbf{F}(\gamma) < a$ for all $\gamma < \beta\}$ is nonempty. Let a be a member of this set. Now $\alpha < \beta$, so $A = \mathbf{F}(\alpha) < a$, contradicting Lemma 10.1.

Since $\mathbf{F}(\beta) = f(\{a \in A : \mathbf{F}(\gamma) < a$ for all $\gamma < \beta\})$ and $\alpha < \beta$, it follows that $\mathbf{F}(\alpha) < \mathbf{F}(\beta)$.

(2) There is an ordinal α such that $\mathbf{F}(\alpha) = A$.

Otherwise, by (1), $\mathbf{F}$ is a one-one function from $\mathbf{On}$ into A. So by the comprehension axioms, rng$(\mathbf{F})$ is a set, and hence by the replacement axioms, $\mathbf{On} = \mathbf{F}^{-1}[\text{rng}\mathbf{F}]$ is a set, contradicting Theorem 7.6.

Let α be minimum such that $\mathbf{F}(\alpha) = A$. Now $\mathbf{F}[\alpha]$ is linearly ordered by (1), so by the hypothesis of Zorn's lemma, there is an $a \in A$ such that $\mathbf{F}(\beta) \leq a$ for all $\beta < \alpha$. Now the set $\{a \in A : f(\beta) < f(0)$ for all $\beta < 0\}$ is trivially nonempty, since A is nonempty, so $\mathbf{F}(0) \neq A$. Hence $\alpha > 0$. If α is a limit ordinal, then for any $\beta < \alpha$ we have $\mathbf{F}(\beta) < \mathbf{F}(\beta + 1) \leq a$, and hence $\mathbf{F}(\alpha) \neq A$, contradiction. Hence α is a successor ordinal $\beta + 1$, and so $\mathbf{F}(\beta)$ is a maximal element of A.

Zorn's lemma $\Rightarrow$ well-ordering principle. Assume Zorn's lemma, and let A be any set. We may assume that A is nonempty. Let

$$P = \{(B, <) : B \subseteq A \text{ and } (B <) \text{ is a well-ordering structure}\}.$$

We partially order P as follows: $(B, <) \prec (C, \ll)$ iff $B \subseteq C$, $\forall a, b \in B[a < b$ iff $a \ll b]$, and $\forall b \in B \forall c \in C\backslash B[b \ll c]$. Clearly this does partially order P. $P \neq \emptyset$, since $(\{a\}, \emptyset) \in P$ for any $a \in A$. Now suppose that Q is a nonempty subset of P simply ordered by $\prec$. Let

$$D = \bigcup_{(B,<) \in Q} B,$$

$$<_D = \bigcup_{(B,<) \in Q} < .$$

Clearly $(D, <_D)$ is a linear order with $D \subseteq A$. Suppose that X is a nonempty subset of D. Fix $z \in X$, and choose $(B, <) \in Q$ such that $z \in B$. Then $X \cap B$ is a nonempty subset of B; let x be its least element under $<$. Suppose that $y \in X$ and $y <_D x$. Choose $(C, \ll) \in Q$ such that $x, y \in C$ and $y \ll x$. Since Q is simply ordered by $\prec$, we have two cases.

Case 1. $(C, \ll) \preceq (B, <)$. Then $y \in C \subseteq B$ and $y \in X$. so $y < x$, contradicting the choice of x.

Case 2. $(B, <) \prec (C, \ll)$. If $y \in B$, then $y < x$, contradicting the choice of x. So $y \in C \backslash B$. But then $x \ll y$, contradiction.

Thus we have shown that x is the $<_D$-least element of X. So $(D, <_D)$ is the desired upper bound for Q.

Having verified the hypotheses of Zorn's lemma, we get a maximal element $(B, <)$ of P. Suppose that $B \neq A$. Choose any $a \in A \backslash B$, and let

$$\begin{aligned} C &= B \cup \{a\}, \\ <_C &= < \cup \{(b, a) : b \in B\}. \end{aligned}$$

Clearly this gives an element $(C, <_C)$ of P such that $(B, <) \prec (C, <_C)$, contradiction.

Well-ordering principle $\Rightarrow$ Axiom of choice. Assume the well-ordering principle, and let $\mathscr{A}$ be a family of pairwise disjoint nonempty sets. Let $C = \bigcup \mathscr{A}$, and let $\prec$ be a well-order of C. Define $B = \{c \in C : c$ is the $\prec$-least element of the $P \in A$ for which $c \in P\}$. Clearly B has exactly one element in common with each member of A. □

There are many statements which are equivalent to the axiom of choice on the basis of ZF. We list some striking ones. A fairly complete list is in

Rubin, H.; Rubin, J. **Equivalents of the axiom of choice.** North-Holland (1963), 134pp.

(About 100 forms are listed, with proofs of equivalence.)

1. For every relation R there is a function $f \subseteq R$ such that $\mathrm{dmn}(f) = \mathrm{dmn}(R)$.

2. For any sets A, B, either there is an injection of A into B or one of B into A.

3. For any transitive relation R there is a maximal $S \subseteq R$ which is a linear ordering.

4. Every product of compact spaces is compact.

5. Every formula having a model of size ω also has a model of any infinite size.

6. If A can be well-ordered, then so can $\mathscr{P}(A)$.

There are also statements which follow from the axiom of choice but do not imply it on the basis of ZF. A fairly complete list of such statement is in

Howard, P.; Rubin, J. **Consequences of the axiom of choice.** Amer. Math. Soc. (1998), 432pp.

(383 forms are listed)

Again we list some striking ones:

1. Every Boolean algebra has a maximal ideal.

2. Any product of compact Hausdorff spaces is compact.

3. The compactness theorem of first-order logic.

4. Every commutative ring has a prime ideal.

5. Every set can be linearly ordered.

6. Every linear ordering has a cofinal well-ordered subset.

7. The Hahn-Banach theorem.

8. Every field has an algebraic closure.

9. Every family of unordered pairs has a choice function.

10. Every linearly ordered set can be well-ordered.

EXERCISES

In the first four exercises, we assume elementary background and ask for the proofs of some standard mathematical facts that require the axiom of choice.

E10.1. Show that any vector space over a field has a basis (possibly infinite).

E10.2. A subset C of $\mathbb{R}$ is *closed* iff the following condition holds:

For every sequence $f \in {}^{\omega}C$, if f converges to a real number x, then $x \in C$.

Here to say that f converges to x means that

$$\forall \varepsilon > 0 \exists M \forall m \geq M[|f_m - x| < \varepsilon].$$

Prove that if $\langle C_m : m \in \omega \rangle$ is a sequence of nonempty closed subsets of $\mathbb{R}$, $\forall m \in \omega \forall x, y \in C_m[|x - y| < 1/(m+1)]$, and $C_m \supseteq C_n$ for $m < n$, then $\bigcap_{m \in \omega} C_m$ is nonempty. Hint: use the Cauchy convergence criterion.

E10.3. Prove that every nontrivial commutative ring with identity has a maximal ideal. Nontrivial means that $0 \neq 1$. Only very elementary definitions and facts are needed here; they can be found in most abstract algebra books. Hint: use Zorn's lemma.

E10.4. A function $g : \mathbb{R} \to \mathbb{R}$ is *continuous* at $a \in \mathbb{R}$ iff for every sequence $f \in {}^{\omega}\mathbb{R}$ which converges to a, the sequence $g \circ f$ converges to $g(a)$. (See Exercise E10.2.) Show that g is continuous at a iff the following condition holds:

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in \mathbb{R}[|x - a| < \delta \to |g(x) - g(a)| < \varepsilon].$$

Hint: for $\to$, argue by contradiction.

E10.5 Show by induction on m, without using the axiom of choice, that if $m \in \omega$ and $\langle A_i : i \in m \rangle$ is a system of nonempty sets, then there is a function f with domain m such that $f(i) \in A_i$ for all $i \in m$.

E10.6 Using AC, prove the following, which is called the *Principle of Dependent Choice* (which is also weaker than the axiom of choice, but cannot be proved in ZF). If A is a nonempty set, R is a relation, $R \subseteq A \times A$, and for every $a \in A$ there is a $b \in A$ such that aRb, then there is a function $f : \omega \to A$ such that $f(i)Rf(i+1)$ for all $i \in \omega$.

The remaining exercises outline proofs of some equivalents to the axiom of choice; so each exercise states something provable in ZF. We are interested in the following statements.

(1) If $<$ is a partial ordering and $\prec$ is a simple ordering which is a subset of $<$, then there is a maximal (under $\subseteq$) simple ordering $\ll$ such that $\prec$ is a subset of $\ll$, which in turn is a subset of $<$.

(2) For any two sets A and B, either there is a one-one function mapping A into B or there is a one-one function mapping B into A.

(3) For any two nonempty sets A and B, either there is a function mapping A onto B or there is a function mapping B onto A.

(4) A family $\mathscr{F}$ of subsets of a set A has *finite character* if for all $X \subseteq A$, $X \in \mathscr{F}$ iff every finite subset of X is in $\mathscr{F}$. Principle (4) says that every family of finite character has a maximal element under $\subseteq$.

(5) For any relation R there is a function $f \subseteq R$ such that $\operatorname{dmn} R = \operatorname{dmn} f$.

E10.7. Show that the axiom of choice implies (1). [Use Zorn's lemma]

E10.8. Prove that (1) implies (2). [Given sets A and B, define $f < g$ iff f and g are one-one functions which are subsets of $A \times B$, and $f \subset g$. Apply (1) to $<$ and the empty simple ordering.]

E10.9. Prove that (2) implies (3). [Easy]

E10.10. Show in ZF that for any set A there is an ordinal α such that there is no one-one function mapping α into A. Hint: consider all well-orderings contained in $A \times A$.

E10.11. Prove that (3) implies the axiom of choice. [Show that any set A can be well-ordered, as follows. Use exercise E10.10 to find an ordinal α which cannot be mapped one-one into $\mathscr{P}(A)$. Show that if $f : A \to \alpha$ maps onto α, then $\langle f^{-1}[\{\beta\}] : \beta < \alpha \rangle$ is a one-one function from α into $\mathscr{P}(A)$.

E10.12. Show that the axiom of choice implies (4). [Use Zorn's lemma.]

E10.13. Show that (4) implies (5). [Given a relation R, let $\mathscr{F}$ consist of all functions contained in R.]

E10.14. Show that (5) implies the axiom of choice. [Given a family $\langle A_i : i \in I \rangle$ of nonempty sets, let $R = \{(i, x) : i \in I \text{ and } x \in A_i\}$.]

11. The Banach-Tarski paradox

The Banach-Tarski paradox is that a unit ball in Euclidean 3-space can be decomposed into finitely many parts which can then be reassembled to form two unit balls in Euclidean 3-space (maybe some parts are not used in these reassemblings). Reassembling is done using distance-preserving transformations. This is one of the most striking consequences of the axiom of choice, and is good background for the study of measure theory (of course the parts of the decomposition are not measurable). We give a proof of the theorem here without going into any side issues. We follow Wagon, **The Banach-Tarski paradox**, where variations and connections to measure theory are explained in full. The proof involves very little set theory, only the axiom of choice. Some third semester calculus and some linear algebra and simple group theory are used. Altogether the proof should be accessible to a first-year graduate student who has seen some applications of the axiom of choice.

We start with some preliminaries on geometry and linear algebra. The "reassembling" mentioned in the Banach-Tarski paradox is entirely done by rotations and translations. Given a line in 3-space and an angle ξ, we imagine the rotation about the given line through the angle ξ. Mainly we will be interested in rotations about lines that go through the origin. We indicate how to obtain the matrix representations of such rotations. First suppose that φ is the rotation about the z-axis counterclockwise through the angle ξ. Then, using polar coordinates,

$$\begin{aligned}
\varphi\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \varphi\begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix} \\
&= \begin{pmatrix} r\cos(\theta+\xi) \\ r\sin(\theta+\xi) \\ z \end{pmatrix} \\
&= \begin{pmatrix} r\cos\theta\cos\xi - r\sin\theta\sin\xi \\ r\cos\theta\sin\xi + r\sin\theta\cos\xi \\ z \end{pmatrix} \\
&= \begin{pmatrix} x\cos\xi - y\sin\xi \\ x\sin\xi + y\cos\xi \\ z \end{pmatrix},
\end{aligned}$$

which gives the matrix representation of φ:

$$\begin{pmatrix} \cos\xi & -\sin\xi & 0 \\ \sin\xi & \cos\xi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, the matrix representations of rotations counterclockwise through the angle ξ about the x- and y-axes are, respectively,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\xi & -\sin\xi \\ 0 & \sin\xi & \cos\xi \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos\xi & 0 & \sin\xi \\ 0 & 1 & 0 \\ -\sin\xi & 0 & \cos\xi \end{pmatrix}.$$

Next, note that any rotation with respect to a line through the origin can be obtained as a composition of rotations about the three axes. This is easy to see using spherical coordinates. If l is a line through the origin and a point P different from the origin with spherical coordinates ρ, φ, θ, a rotation about l through an angle ξ can be obtained as follows: rotate about the z axis through the angle $-\theta$, then about the y-axis through the angle $-\varphi$ (thereby transforming l into the z-axis), then about the z-axis through the angle ξ, then back through φ about the y axis and through θ about the z-axis.

We want to connect this to linear algebra. Recall that a 3×3 matrix A is *orthogonal* provided that it is invertible and $A^T = A^{-1}$. Thus the matrices above are orthogonal. A matrix is orthogonal iff its columns form a basis for ${}^3\mathbb{R}$ consisting of mutually orthogonal unit vectors; this is easy to see. It is easy to check that a product of orthogonal matrices is orthogonal. Hence all of the rotations about lines through the origin are represented by orthogonal matrices.

Lemma 11.1. *If A is an orthogonal 3×3 real matrix and X and Y are 3×1 column vectors, then $(AX) \cdot (AY) = X \cdot Y$, where $\cdot$ is scalar multiplication.*

Proof. This is a simple computation:

$$(AX) \cdot (AY) = (AX)^T(AY) = X^T A^T AY = X^T A^{-1} AY = X^T Y = X \cdot Y. \qquad \square$$

It follows that any rotation about a line through the origin preserves distance, because $|P-Q| = \sqrt{(P-Q)\cdot(P-Q)}$ for any vectors P and Q. Such rotations have an additional property: their matrix representations have determinant 1. This is clear from the discussion above. It turns out that this additional property characterizes the rotations about lines through the origin (see M. Artin, **Algebra**), but we do not need to prove that. The following property of such matrices is very useful, however.

Lemma 11.2. *Suppose that A is an orthogonal 3×3 real matrix with determinant 1, A not the identity. Then there is a non-zero 3×1 matrix X such that for any 3×1 matrix Y,*

$$AY = Y \quad \text{iff} \quad \exists a \in \mathbb{R}[Y = aX].$$

Proof. Note that $A^T(A - I) = I - A^T = (I - A)^T$. Hence

$$-|A - I| = |I - A| = |(I - A)^T| = |A^T(A - I)| = |A^T|\,|A - I| = |A - I|.$$

It follows that $|A - I| = 0$. Hence the system of equations $(A - I)X = 0$ has a nontrivial solution, which gives the X we want. Namely, we then have $AX = X$, of course. Then $A(aX) = aAX = aX$. This proves $\Leftarrow$ in the equivalence of the lemma. It remains to do the converse. We may assume that X has length 1. Now we apply the Gram-Schmidt process to get a basis for ${}^3\mathbb{R}$ consisting of mutually orthogonal unit vectors, the first of which is X. We put them into a matrix B as column vectors, X the first column. Note that the first column of AB is X, since $AX = X$, and hence the first column of $B^{-1}AB$

is $(1\ 0\ 0)^T$. Since $B^{-1}AB$ is an orthogonal matrix, it follows because its columns are mutually orthogonal that it has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}.$$

Now suppose that $AY = Y$. Let $B^{-1}Y = (u\ e\ f)^T$. Then

(1) $e = f = 0$.

For, suppose that (1) fails. Now $B^{-1}ABB^{-1}Y = B^{-1}AY = B^{-1}Y$, while a direct computation using the above form of $B^{-1}AB$ yields $B^{-1}ABB^{-1}Y = (u\ \ ae + bf\ \ ce + df)^T$. So we get the two equations

$$\begin{aligned} ae + bf &= e \\ ce + df &= f \end{aligned} \quad \text{or} \quad \begin{aligned} (a-1)e + bf &= 0 \\ ce + (d-1)f &= 0 \end{aligned}$$

Since (1) fails, it follows that the determinant $\begin{vmatrix} a-1 & b \\ c & d-1 \end{vmatrix}$ is 0. Thus $ad-a-d+1-bc = 0$. Now $B^{-1}AB$ has determinant 1, and its determinant is $ad-bc$, so we infer that $a+d = 2$. But $a^2 + c^2 = 1$ and $b^2 + d^2 = 1$ since the columns of $B^{-1}AB$ are unit vectors, so $|a| \le 1$ and $|d| \le 1$. Hence $a = d = 1$ and $b = c = 0$. So $B^{-1}AB$ is the identity matrix, so A is also, contradiction. Hence (1) holds after all.

From (1) we get $Y = B(u\ 0\ 0)^T = uX$, as desired. □

One more remark on geometry: any rotation preserves distance. We already said this for rotations about lines through the origin. If l does not go through the origin, one can use a translation to transform it into a line through the origin, do the rotation then, and then translate back. Since translations clearly preserve distance, so arbitrary rotations preserve distance.

The first concrete step in the proof of the Banach, Tarski theorem is to describe a very special group of permutations of ${}^3\mathbb{R}$. Let φ be the counterclockwise rotation about the z-axis through the angle $\cos^{-1}(\frac{1}{3})$, and let ρ the similar rotation about the x-axis. The matrix representation of these rotations and their inverses is, by the above,

(1) $$\varphi^{\pm 1} = \begin{pmatrix} \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \rho^{\pm 1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} \\ 0 & \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}.$$

Let G_0 be the group of permutations of ${}^3\mathbb{R}$ generated by φ and ρ. By a *word* in φ and ρ we mean a finite sequence with elements in $\{\varphi, \varphi^{-1}, \rho, \rho^{-1}\}$. Given such a word $w = \langle \sigma_0, \ldots, \sigma_{m-1} \rangle$, we let $\overline{w}$ be the composition $\sigma_0 \circ \ldots \circ \sigma_{m-1}$. Further, we call w *reduced* if no two successive terms of w have any of the four forms $\langle \varphi, \varphi^{-1} \rangle$, $\langle \varphi^{-1}, \varphi \rangle$, $\langle \rho, \rho^{-1} \rangle$, or $\langle \rho^{-1}, \rho \rangle$.

Lemma 11.3. *If w is a reduced word of positive length, then $\overline{w}$ is not the identity.*

Proof. Suppose the contrary. Since $\rho \circ \overline{w} \circ \rho^{-1}$ is also the identity, we may assume that w ends with $\rho^{\pm 1}$ (on the right). [If w already ends with $\rho^{\pm 1}$, we do nothing. If it ends with $\varphi^{\pm 1}$, let $w' = \rho w \rho^{-1}$. Then w' is reduced, unless w has the form $\rho^{-1} w''$, in which case $w''\rho^{-1}$ is reduced, and still $\overline{w''\rho^{-1}} = \overline{w}$ =the identity.]

Since obviously $\rho^{\pm 1}$ is not the identity, w must have length at least 2. Now we claim

(1) For every terminal segment w' of w of nonzero even length the vector $\overline{w'}\,(1 \quad 0 \quad 0)^T$ has the form $(1/3^k)\,(a \quad b\sqrt{2} \quad c)^T$, with a divisible by 3 and b not divisible by 3.

We prove this by induction on the length of w'. First note that, by computation,

$$\rho\varphi = \frac{1}{9}\begin{pmatrix} 3 & -6\sqrt{2} & 0 \\ 2\sqrt{2} & 1 & -6\sqrt{2} \\ 8 & 2\sqrt{2} & 3 \end{pmatrix}; \quad \rho\varphi^{-1} = \frac{1}{9}\begin{pmatrix} 3 & 6\sqrt{2} & 0 \\ -2\sqrt{2} & 1 & -6\sqrt{2} \\ -8 & 2\sqrt{2} & 3 \end{pmatrix};$$

$$\rho^{-1}\varphi = \frac{1}{9}\begin{pmatrix} 3 & -6\sqrt{2} & 0 \\ 2\sqrt{2} & 1 & 6\sqrt{2} \\ -8 & -2\sqrt{2} & 3 \end{pmatrix}; \quad \rho^{-1}\varphi^{-1} = \frac{1}{9}\begin{pmatrix} 3 & 6\sqrt{2} & 0 \\ -2\sqrt{2} & 1 & 6\sqrt{2} \\ 8 & -2\sqrt{2} & 3 \end{pmatrix}.$$

Now we proceed by induction. For w' of length 2 we have

$$\rho\varphi\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 3 \\ 2\sqrt{2} \\ 8 \end{pmatrix}; \quad \rho\varphi^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 3 \\ -2\sqrt{2} \\ -8 \end{pmatrix};$$

$$\rho^{-1}\varphi\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 3 \\ 2\sqrt{2} \\ -8 \end{pmatrix}; \quad \rho^{-1}\varphi^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 3 \\ -2\sqrt{2} \\ 8 \end{pmatrix};$$

hence (1) holds in this case. The induction step:

$$\rho\varphi\left(\frac{1}{3^k}\begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix}\right) = \frac{1}{3^{k+2}}\begin{pmatrix} 3a - 12b \\ 2\sqrt{2}a + b\sqrt{2} - 6\sqrt{2}c \\ 8a + 4b + 3c \end{pmatrix};$$

$$\rho\varphi^{-1}\left(\frac{1}{3^k}\begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix}\right) = \frac{1}{3^{k+2}}\begin{pmatrix} 3a + 12b \\ -2\sqrt{2}a + b\sqrt{2} - 6\sqrt{2}c \\ -8a + 4b + 3c \end{pmatrix};$$

$$\rho^{-1}\varphi\left(\frac{1}{3^k}\begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix}\right) = \frac{1}{3^{k+2}}\begin{pmatrix} 3a - 12b \\ 2\sqrt{2}a + b\sqrt{2} + 6\sqrt{2}c \\ -8a - 4b + 3c \end{pmatrix};$$

$$\rho^{-1}\varphi^{-1}\frac{1}{3^{k+2}}\left(\frac{1}{3^k}\begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix}\right) = \begin{pmatrix} 3a + 12b \\ -2\sqrt{2}a + b\sqrt{2} + 6\sqrt{2}c \\ 8a - 4b + 3c \end{pmatrix}.$$

So, our assertion (1) is true. If w itself is of even length, then a contradiction has been reached, since b is not divisible by 3. If w is of odd length, then the following shows that the second entry of $\overline{w}\begin{pmatrix}1\\0\\0\end{pmatrix}^T$ is nonzero, still a contradiction:

$$\varphi\left(\frac{1}{3^k}\begin{pmatrix}a\\b\sqrt{2}\\c\end{pmatrix}\right)=\frac{1}{3^{k+1}}\begin{pmatrix}a-4b\\2\sqrt{2}a+b\sqrt{2}\\3c\end{pmatrix};$$

$$\varphi^{-1}\left(\frac{1}{3^k}\begin{pmatrix}a\\b\sqrt{2}\\c\end{pmatrix}\right)=\frac{1}{3^{k+1}}\begin{pmatrix}a+4b\\-2\sqrt{2}a+b\sqrt{2}\\3c\end{pmatrix}.$$

This finishes the proof of Lemma 11.3 □

This lemma really says that G_0 is (isomorphic to) the free group on two generators. But we do not need to go into that. We do need the following corollary, though.

Corollary 11.4. *For every $g \in G_0$ there is a unique reduced word w such that $g = \overline{w}$.*

Proof. Suppose that w and w' both work, and $w \neq w'$. Say $w = \langle \sigma_0, \ldots, \sigma_{m-1} \rangle$ and $w' = \langle \tau_0, \ldots, \tau_{n-1} \rangle$. If one is a proper segment of the other, say by symmetry w is a proper segment of w', then

$$\begin{aligned} g &= \overline{w} = \sigma_0 \circ \ldots \circ \sigma_{m-1} \\ &= \overline{w'} = \tau_0 \circ \ldots \circ \tau_{n-1}; \end{aligned}$$

since $\sigma_i = \tau_i$ for all $i < m$, we obtain $I = \tau_m \circ \ldots \circ \tau_{n-1}$, I the identity. But $\langle \tau_m, \ldots, \tau_{n-1} \rangle$ is reduced, contradicting 11.3.

Thus neither w nor w' is a proper initial segment of the other. Hence there is an $i < \min(m, n)$ such that $\sigma_i \neq \tau_i$, while $\sigma_j = \tau_j$ for all $j < i$ (maybe $i = 0$). But then we have by cancellation $\sigma_i \circ \ldots \circ \sigma_{m-1} = \tau_i \circ \ldots \circ \tau_{n-1}$, so $\tau_{n-1}^{-1} \circ \ldots \circ \tau_i^{-1} \sigma_i \circ \ldots \circ \sigma_{m-1} = I$. But since $\sigma_i \neq \tau_i$, the word $\langle \tau_{n-1}^{-1}, \ldots, \tau_i^{-1}, \sigma_i, \ldots, \sigma_{m-1} \rangle$ is reduced, again contradicting 11.3. □

If G is a group and X is a set, we say that G *acts on* X if there is a homomorphism from G into the group of all permutations of X. Usually this homomorphism will be denoted by $\check{}$, so that $\check{g}$ is the permutation of X corresponding to $g \in G$. (Most mathematicians don't even use $\check{}$, using the same symbol for elements of the group and for the image under the homomorphism.) An important example is: any group G acts on itself by left multiplication. Thus for any $g \in G$, $\check{g} : G \to G$ is defined by $\check{g}(h) = g \cdot h$, for all $h \in G$.

Let G act on a set X, and let $E \subseteq X$. Then we say that E is *G-paradoxical* if there are positive integers m, n and pairwise disjoint subsets $A_0, \ldots, A_{m-1}, B_0, \ldots, B_{n-1}$ of E, and elements $\langle g_i : i < m \rangle$ and $\langle h_i : i < n \rangle$ of G such that $E = \bigcup_{i<m} \check{g}_i[A_i]$ and $E = \bigcup_{j<n} \check{h}_j[B_j]$. Note that this comes close to the Banach-Tarski formulation, except that the sets X and E are unspecified.

Lemma 11.5. *G_0, acting on itself by left multiplication, is G_0-paradoxical.*

Proof. If σ is one of $\phi^{\pm 1}$, $\rho^{\pm 1}$, we denote by $W(\sigma)$ the set of all reduced words beginning on the left with σ, and $\overline{W}(\sigma) = \{\overline{w} : w \in W(\sigma)\}$. Thus, obviously,

$$G_0 = \{I\} \cup \overline{W}(\phi) \cup \overline{W}(\phi^{-1}) \cup \overline{W}(\rho) \cup \overline{W}(\rho^{-1}),$$

where I is the identity element of G_0. These five sets are pairwise disjoint by 11.4. Thus the lemma will be proved, with $m = n = 2$, by proving the following two statements:

(1) $G_0 = \overline{W}(\phi) \cup \check{\phi}[\overline{W}(\phi^{-1})]$.

To see this, suppose that $g \in G_0$ and $g \notin \overline{W}(\phi)$. Write $g = \overline{w}$, w a reduced word. Then w does not start with ϕ. Hence $\phi^{-1}w$ is still a reduced word, and $g = \phi \circ \phi^{-1} \circ \overline{w} \in \check{\phi}[\overline{W}(\phi^{-1})]$, as desired.

(2) $G_0 = \overline{W}(\rho) \cup \check{\rho}[\overline{W}(\rho^{-1})]$.

The proof is just like for (1). $\square$

We need two more definitions, given that G acts on a set X. For each $x \in X$, the *G-orbit* of x is $\{\check{g}(x) : g \in G\}$. The set of G-orbits forms a partition of X. We say that G *is without non-trivial fixed points* if for every non-identity $g \in G$ and every $x \in X$, $\check{g}(x) \neq x$.

The following lemma is the place in the proof of the Banach-Tarski paradox where the axiom of choice is used. Don't jump to the conclusion that the proof is almost over; our group G_0 above has non-trivial fixed points, and so does not satisfy the hypothesis of the lemma. Some trickery remains to be done even after this lemma. [For example, all points on the z-axis are fixed by φ.]

Lemma 11.6. *Suppose that G is G-paradoxical and acts on a set X without non-trivial fixed points. Then X is G-paradoxical.*

Proof. Let

$$A_0, \ldots, A_{m-1}, B_0, \ldots, B_{n-1}, g_0, \ldots, g_{m-1}, h_0, \ldots, h_{n-1}$$

be as in the defintion of paradoxical. By AC, let M be a subset of X having exactly one element in common with each G-orbit. Then we claim:

(1) $\langle \check{g}[M] : g \in G \rangle$ is a partition of X.

First of all, obviously each set $\check{g}[M]$ is nonempty. Next, their union is X, since for any $x \in X$ there is a $y \in M$ which is in the same G-orbit as x, and this yields a $g \in G$ such that $x = \check{g}(y)$ and hence $x \in \check{g}[M]$. Finally, if g and h are distinct elements of G, then $\check{g}[M]$ and $\check{h}[M]$ are disjoint. In fact, otherwise let y be a common element. Say $\check{g}(x) = y$, $x \in M$, and $\check{h}(z) = y$, $z \in M$. Then clearly x and z are in the same G-orbit, so $x = z$ since they are "both" in M. Then $(g^{-1} \cdot h)\check{}(z) = z$ and $g^{-1} \cdot h$ is not the identity, contradicting the no non-trivial fixed point assumption. So, (1) holds.

Now let $A_i^* = \bigcup_{g \in A_i} \check{g}[M]$ and $B_j^* = \bigcup_{g \in B_j} \check{g}[M]$, for all $i < m$ and $j < n$.

(2) $A_i^* \cap A_k^* = 0$ if $i < k < m$.

In fact, suppose that $x \in A_i^* \cap A_k^*$. Then we can choose $g \in A_i$ and $h \in A_k$ such that $x \in \check{g}[M] \cap \check{h}[M]$. But $g \neq h$ since $A_i \cap A_k = 0$, so this contradicts (1). Similarly the following two conditios hold:

(3) $B_i^* \cap B_k^* = 0$ if $i < k < n$.
(4) $A_i^* \cap B_j^* = 0$ if $i < m$ and $j < n$.

(5) $\bigcup_{i<m} \check{g}_i[A_i^*] = X$.

For, let $x \in X$. Say by (1) that $x \in \check{g}[M]$. Then by the choice of the A_i's there is an $i < m$ such that $g \in \check{g}_i[A_i]$. Say $h \in A_i$ and $g = \check{g}_i(h) = g_i \cdot h$. Since $x \in \check{g}[M]$, say $x = \check{g}(m)$ with $m \in M$. Then $x = (g_i \cdot h)\check{}(m) = \check{g}_i(\check{h}(m))$. Now $\check{h}(m) \in \check{h}[M] \subseteq A_i^*$, so $x \in \check{g}_i[A_i^*]$, as desired in (5).

(6) $\bigcup_{i<n} \check{h}_i[B_i^*] = X$.

This is proved similarly. $\square$

Let $S^2 = \{x \in {}^3\mathbb{R} : |x| = 1\}$ be the usual unit sphere. Now we can prove the first paradoxical result leading to the Banach-Tarski paradox:

Theorem 11.7. (Hausdorff) *There is a countable $D \subseteq S^2$ such that $S^2 \backslash D$ is G_0-paradoxical.*

Proof. Let D be the set of all fixed points of non-identity elements of G_0. By 11.2, D is countable. Now we claim that if $\sigma \in G_0$, then $\sigma[S^2\backslash D] = S^2\backslash D$. For, assume that $x \in S^2\backslash D$ and $\sigma(x) \in D$. Say $\tau \in G_0$, τ not the identity, and $\tau(\sigma(x)) = \sigma(x)$. Then $\sigma^{-1}\tau\sigma(x) = x$. Now $\sigma^{-1} \circ \tau \circ \sigma$ is not the identity, since τ isn't, so $x \in D$, contradiction. This proves that $\sigma[S^2\backslash D] \subseteq S^2\backslash D$. This holds for any $\sigma \in G_0$, in particular for σ^{-1}, and applying σ to that inclusion yields $S^2\backslash D \subseteq \sigma[S^2\backslash D]$, so the desired equality holds.

Thus G_0 acts on $S^2\backslash D$ without non-trivial fixed points. So by 11.5 and 11.6, $S^2\backslash D$ is G_0-paradoxical. $\square$

Let us see how far we have to go now. This theorem only looks at the sphere, not the ball. A countable subset is ignored. Since the sphere is uncountable, this makes the result close to what we want. But actually there is a countable subset of the sphere which is dense on it. [Take points whose spherical coordinates are rational.]

For the next step we need a new notion. Suppose that G is a group acting on a set X, and $A, B \subseteq X$. We say that *A and B are finitely G-equidecomposable* if A and B can be decomposed into the same number of parts, each part of A being carried into the corresponding part of B by an element of G. In symbols, there is a positive integer n such that there are partitions $A = \bigcup_{i<n} A_i$ and $B = \bigcup_{i<n} B_i$ and members $g_i \in G$ for $i < n$ such that $\check{g}_i[A_i] = B_i$ for all $i < n$. We then write $A \sim_G B$.

Lemma 11.8. *If G acts on a set X, then $\sim_G$ is an equivalence relation on $\mathscr{P}(X)$.*

Proof. Obviously $\sim_G$ is reflexive on $\mathscr{P}(X)$ and is symmetric. Now suppose that $A \sim_G B \sim_G C$. Then we get partitions $A = \bigcup_{i<m} A_i$ and $B = \bigcup_{i<m} B_i$ with elements $g_i \in G$ such that $\check{g}_i[A_i] = B_i$ for all $i < m$; and partitions $B = \bigcup_{j<n} B_j'$ and $C = \bigcup_{j<n} C_j$

with elements $h_i \in G$ such that $\check{h_j}[B'_j] = C_j$ for all $j < n$. Now for all $i < m$ and $j < n$ let $B_{ij} = B_i \cap B'_j$, $A_{ij} = \check{g_i}^{-1}[B_{ij}]$, and $C_{ij} = \check{h_j}[B_{ij}]$. Then $A = \bigcup_{i<m,j<n} A_{ij}$ is a partition of A, $C = \bigcup_{i<m,j<n} C_{ij}$ is a partition of C, and $(h_j \cdot g_i)\check{}[A_{ij}] = C_{ij}$. Some of the B_{ij} may be empty; eliminating the empty ones yields the desired nonemptiness of members of the partitions. □

Lemma 11.9. *Suppose that G acts on X, E and E' are finitely G-equidecomposable subsets of X, and E is G-paradoxical. Then also E' is G-paradoxical.*

Proof. Because E is G-paradoxical, we can find pairwise disjoint subsets

$$A_0, \ldots, A_{m-1}, B_0, \ldots, B_{n-1}$$

of E and corresponding elements $g_0, \ldots, g_{m-1}, h_0, \ldots, h_{n-1}$ of G such that

$$E = \bigcup_{i<m} \check{g_i}[A_i] = \bigcup_{j<n} \check{h_j}[B_i].$$

And because E and E' are finitely G=equidecomposable we can find partitions $E = \bigcup_{k<p} C_k$ and $E' = \bigcup_{k<p} D_k$ with elements $f_i \in G$ such that $\check{f_i}[C_k] = D_k$ for all $k < p$. Then the following sets are pairwise disjoint: $A_i \cap \check{g_i}^{-1}[C_k]$ for $i < m$ and $k < p$, and $B_j \cap \check{h_j}^{-1}[C_k]$ for $j < n$ and $k < p$. And

$$\begin{aligned} E' &= \bigcup_{k<p} D_k = \bigcup_{k<p} \check{f_k}[C_k] \\ &= \bigcup_{k<p} \check{f_k}[C_k \cap \bigcup_{i<m} \check{g_i}[A_i]] \\ &= \bigcup_{k<p,i<m} \check{f_k}[C_k \cap \check{g_i}[A_i]] \\ &= \bigcup_{k<p,i<m} (f_k \cdot g_i)\check{}[A_i \cap \check{g_i}^{-1}[C_k]], \end{aligned}$$

and similarly

$$E' = \bigcup_{k<p,j<n} (f_k \cdot h_j)\check{}[B_i \cap \check{h_j}^{-1}[C_k]].$$ □

Lemma 11.10. *Let D be a countable subset of S^2. Then there is a rotation σ with respect to a line through the origin such that if G_1 is the group of transformations of ${}^3\mathbb{R}$ generated by σ, then S^2 and $S^2\backslash D$ are G_1-equidecomposable.*

Proof. For each $d \in D$ let $f(d)$ be the line through the origin and d. Then f maps D into the set L of all lines through the origin, and the range of f is countable. But L itself is uncountable: for example, for each $\theta \in [0, \pi]$ one can take the line through the origin and $(\cos\theta, \sin\theta, 0)$. Hence there is a line $l \in L$ not in the range of f. This means that l does not pass through any point of D. Fix a direction in which to take rotations about l.

Note that if P and Q are distinct points of D, then there is at most one rotation about l which takes P to Q and is between 0 and 2π; this will be denoted by ψ_{PQ}, if it exists. Now let A consist of all $\theta \in (0, 2\pi)$ such that there is a positive integer n and a $P \in D$ such that if σ is the rotation about l through the angle $n\theta$, then $\sigma(P) \in D$. We claim that A is countable. For, if $P, Q \in D$, ψ_{PQ} is defined, $n \in \omega\backslash\{0\}$, $k \in \omega$, and $0 < \frac{1}{n}(\psi_{PQ} + 2\pi k) < 2\pi$, then $\frac{1}{n}(\psi_{PQ} + 2\pi k) \in A$; and every member of A can be obtained this way. [Given $\theta \in A$, we have $n\theta = \psi_{PQ} + 2\pi k$ for some $P, Q \in D$ and $n, k \in \omega$.] This really defines a function from $D \times D \times (\omega\backslash\{0\}) \times \omega$ onto A, so A is, indeed, countable. We choose $\theta \in (0, 2\pi)\backslash A$, and take the rotation σ about l through the angle θ. Let $\overline{D} = \bigcup_{n\in\omega} \sigma^n[D]$. The choice of σ says that $\sigma^n[D] \cap D = 0$ for every positive integer n. Hence if $n < m < \omega$ we have $\sigma^n[D] \cap \sigma^m[D] = 0$, since

$$\sigma^n[D] \cap \sigma^m[D] = \sigma^n[D \cap \sigma^{m-n}[D]] = \sigma^n[0] = 0.$$

Note that $\sigma[\overline{D}] = \overline{D}\backslash D$. Hence

$$S^2 = \overline{D} \cup (S^2\backslash\overline{D}) \sim_{G_1} \sigma[\overline{D}] \cup (S^2\backslash\overline{D}) = S^2\backslash D. \qquad \square$$

Now let G_2 be the group of permutations of ${}^3\mathbb{R}$ generated by $\{\varphi, \rho, \sigma\}$. We now have the first form of the Banach-Tarski paradox:

Theorem 11.11. (Banach, Tarski) *S^2 is G_2-paradoxical.* $\square$

One can loosely state this theorem as follows: one can decompose S^2 into a finite number of pieces, rotate some of these pieces finitely many times with respect to certain lines through the origin to reassemble S^2, and then similarly transform some of the remaining pieces to also reassemble S^2. The rotations are of three kinds: the very specific rotations φ and ρ defined at the beginning of this section, and the rotation σ in the preceding proof, for which we do not have an explicit description. One can apply the inverses of these rotations as well. After doing the second reassembling, one can apply a translation to make that copy of S^2 disjoint from the first copy.

Finally, let $B = \{x \in {}^3\mathbb{R} : |x| \le 1\}$ be the unit ball in 3-space. Let G_3 be the group generated by φ, ρ, σ, and the rotation τ about the line determined by $(0, 0, \frac{1}{2})$ and $(1, 0, \frac{1}{2})$, through the angle $\pi/\sqrt{2}$. Note that by the considerations at the beginning of this section, τ consists of the translation $(\,x \quad y \quad z\,)^T \mapsto (\,x \quad y \quad z - \frac{1}{2}\,)^T$, followed by the rotation through $\frac{\pi}{\sqrt{2}}$ about the x-axis, followed by the translation $(\,x \quad y \quad z\,)^T \mapsto (\,x \quad y \quad z + \frac{1}{2}\,)^T$.

Lemma 11.12. *For any positive integer k,*

$$\tau^k \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2}\sin\left(\frac{k\pi}{\sqrt{2}}\right) \\ -\frac{1}{2}\cos\left(\frac{k\pi}{\sqrt{2}}\right) + \frac{1}{2} \end{pmatrix}.$$

Hence $\tau^p\,(\,0 \quad 0 \quad 0\,)^T \ne (\,0 \quad 0 \quad 0\,)^T$ for every positive integer p.

Proof. The first equation is easily proved by induction on k. Then the second inequality follows since for any positive integer p, the argument $\frac{p\pi}{\sqrt{2}}$ is never equal to $m\pi$ for any integer m, since $\sqrt{2}$ is irrational. $\square$

Theorem 11.13. (Banach, Tarski) *B is G_3-paradoxical.*

Proof. By 11.11 there are pairwise disjoint subsets A_i and B_j of S^2 and members g_i, h_j of G_2 for $i < m$ and $j < n$ such that $S^2 = \bigcup_{i<m} g_i[A_i] = \bigcup_{j<n} h_j[B_j]$. For each $i < m$ and $j < n$ let $A'_i = \{\alpha x : x \in A_i,\ 0 < \alpha \leq 1\}$ and $B_j = \{\alpha x : x \in B_j,\ 0 < \alpha \leq 1\}$. Then

(1) The A'_i's and B'_j's are pairwise disjoint.

For example, suppose that $y \in A'_i \cap B'_j$. Then we can write $y = \alpha x$ with $x \in A_i$, $0 < \alpha \leq 1$, and also $y = \beta z$ with $z \in B_j$, and $0 < \beta \leq 1$. Hence $|y| = \alpha = \beta$. Hence $x = z$, contradiction.

(2) $B\backslash\{0\} = \bigcup_{i<m} g_i[A'_i] = \bigcup_{j<n} h_j[B'_j]$.

In fact, let $y \in B\backslash\{0\}$. Let $x = y/|y|$. Then $x \in S^2$, so there is an $i < m$ such that $x \in g_i[A_i]$. Say that $x = g_i(z)$ with $z \in A_i$. Then $|y|z \in A'_i$, and $g_i(|y|z) = |y|g_i(z) = |y|x = y$. So $y \in g_i[A'_i]$. This proves the first equality in (2), and the second equality is proved similarly.

So far, we have shown that $B\backslash\{0\}$ is G_2-paradoxical. Now we show that B and $B\backslash\{0\}$ are finitely G_3-equidecomposable, which will finish the proof. By lemma 11.12 we have

$$B = D \cup (B\backslash D) \sim_{G_3} \tau[D] \cup (B\backslash D) = B\backslash\{0\}.$$

This proves the desired equidecomposablity. $\square$

As in the case of S^2, a translation can be made if one wants one copy of B to be disjoint from the other.

12. Cardinals

This chapter is concerned with the basics of cardinal arithmetic.

Definition and basic properties

To abbreviate longer expressions, we say that sets A and B are *equipotent* iff there is a bijection between them. A *cardinal*, or *cardinal number*, is an ordinal α which is not equipotent with a smaller ordinal. We generally use Greek letters κ, λ, μ for cardinals. Obviously if κ and λ are distinct cardinals, then they are not equipotent.

Proposition 12.1. *For any set X there is an ordinal α equipotent with X.*

Proof. By the well-ordering principle, let $<$ be a well-ordering of X. Then X under $<$ is isomorphic to an ordinal. □

By this proposition, any set is equipotent with a cardinal—namely the least ordinal equipotent with it. This justifies the following definition. For any set X, the *cardinality*, or *size*, or *magnitude*, etc. of X is the unique cardinal $|X|$ equipotent with X. The basic property of this definition is given in the following theorem.

Theorem 12.2. *For any sets X and Y, the following conditions are equivalent:*
(i) $|X| = |Y|$.
(ii) X and Y are equipotent. □

The following proposition gives obvious facts about the particular way that we have defined the notion of cardinality.

Proposition 12.3. *(i) $|\alpha| \leq \alpha$.*
(ii) $|\alpha| = \alpha$ iff α is a cardinal. □

Lemma 12.4. *If $0 \neq m \in \omega$ then there is an $n \in \omega$ such that $m = n + 1$.*

Proof. Assume that $0 \neq m \in \omega$. By Theorem 7.16, m is a successor ordinal $\alpha + 1$. Since ω is transitive we have $\alpha \in \omega$. □

Proposition 12.5. *Every natural number is a cardinal.*

Proof. We prove by ordinary induction on n that for every natural number n and for every natural number m, if $m < n$ then there is no bijection from n to m. This is vacuously true for $n = 0$. Now assume it for n, but suppose that m is a natural number less than $n + 1$ and f is a bijection from $n + 1$ onto m. Since $n + 1 \neq 0$, obviously also $m \neq 0$. So $m = m' + 1$ for some natural number m', by Lemma 12.4. Let g be the bijection from m onto m which interchanges m' and $f(n)$ and leaves fixed all other elements of m. Then $g \circ f$ is a bijection from $n + 1$ onto m which takes n to m'. Hence $(g \circ f) \restriction n$ is a bijection from n onto m', and $m' < n$, contradicting the inductive hypothesis. □

Thus the natural numbers are the first cardinals, in the ordering of cardinals determined by the fact that they are special kinds of ordinals. A set is *finite* iff it is equipotent with some natural number; otherwise it is *infinite*. The following general lemma helps to prove that ω is the next cardinal.

Lemma 12.6. *If $(A, <)$ is a simple order, then every finite nonempty subset of A has a greatest element.*

Proof. We prove by induction on $m \geq 1$ that if $X \subseteq A$ and $|X| = m$ then X has a greatest element. For $m = 1$ this is obvious. Now assume the implication for m, and suppose that $X \subseteq A$ and $|X| = m + 1$. Let f be a bijection from $m + 1$ onto X, and let $X' = X \backslash \{f(m)\}$. So $|X'| = m$, and so X' has a largest element x. If $f(m) < x$, then x is the greatest element of X. If $x < f(m)$, then $f(m)$ is the greatest element of X. $\square$

Theorem 12.7. *ω is a cardinal.* $\square$

It is harder to find larger cardinals, but they exist; in fact the collection of cardinals is so big that, like the collection of ordinals, it does not exist as a set. We will see this a little bit later.

Note that a cardinal is infinite iff it is greater or equal ω. The following fact will be useful later.

Proposition 12.8. *Every infinite cardinal is a limit ordinal.*

Proof. Suppose not: κ is an infinite cardinal, and $\kappa = \alpha + 1$. We define $f : \alpha \to \kappa$ as follows: $f(0) = \alpha$, $f(m + 1) = m$ for all $m \in \omega$, and $f(\beta) = \beta$ for all $\beta \in \alpha \backslash \omega$. Clearly f is one-one and maps onto κ, contradiction. $\square$

Lemma 12.9. *If κ and λ are cardinals and $f : \kappa \to \lambda$ is one-one, then $\kappa \leq \lambda$.*

Proof. We define $\alpha \prec \beta$ iff $\alpha, \beta \in \kappa$ and $f(\alpha) < f(\beta)$. Clearly $\prec$ well-orders κ. Let g be an isomorphism from $(\kappa, \prec)$ onto an ordinal γ. Thus $\kappa \leq \gamma$ by the definition of cardinals. If $\alpha < \beta < \gamma$, then $g^{-1}(\alpha) \prec g^{-1}(\beta)$, hence by definition of $\prec$, $f(g^{-1}(\alpha)) < f(g^{-1}(\beta))$. Thus $f \circ g^{-1} : \gamma \to \lambda$ is strictly increasing. Hence by Proposition 6.15, $\alpha \leq (f \circ g^{-1})(\alpha)$ for all $\alpha < \gamma$, so $\lambda \not< \gamma$, hence $\gamma \leq \lambda$. We already know that $\kappa \leq \gamma$, so $\kappa \leq \lambda$. $\square$

The purpose of this lemma is to prove the following basic theorem.

Theorem 12.10. *If $A \subseteq B$, then $|A| \leq |B|$.*

Proof. Let $\kappa = |A|$, $\lambda = |B|$, and let f and g be one-one functions from κ onto A and of λ onto B, respectively. Then $g \circ f^{-1}$ is a one-one function from κ into λ, so $\kappa \leq \lambda$. $\square$

Corollary 12.11. *For any sets A and B the following conditions are equivalent:*
(i) $|A| \leq |B|$.
(ii) There is a one-one function mapping A into B.
(iii) $A = \emptyset$, or there is a function mapping B onto A.

Proof. Let f be a bijection from $|A|$ to A, and g a bijection from $|B|$ to B,

(i)$\Rightarrow$(ii): Assume that $|A| \leq |B|$. Then $|A| \subseteq |B|$ by Proposition 4.8, and $g \circ f^{-1}$ is a one-one mapping from A into B.

(ii)⇒(iii): Assume that $h : A \to B$ is one-one and $A \neq \emptyset$. Fix $a \in A$, and define $k : B \to A$ by setting, for any $b \in B$,

$$k(b) = \begin{cases} h^{-1}(b) & \text{if } b \in \mathrm{rng}(h), \\ a & \text{otherwise.} \end{cases}$$

Clearly k maps B onto A.

(iii)⇒(i): Obviously $A = \emptyset$ implies that $0 = |A| \leq |B|$. Now suppose that h maps B onto A. Then for any $\alpha < |A|$ there is a $b \in B$ such that $h(b) = f(\alpha)$, and hence there is a $\beta < |B|$ such that $h(g(\beta)) = f(\alpha)$. For each $\alpha < |A|$ let $k(\alpha) = \min\{\beta < |B| : h(g(\beta)) = f(\alpha)\}$. Then $h \circ g \circ k = f$, so k is one-one. By Lemma 12.9, $|A| \leq |B|$. □

Corollary 12.12. *If there is a one-one function from A into B and a one-one function from B into A, then there is a one-one function from A onto B.* □

This corollary is called the Cantor-Bernstein, or Schröder-Bernstein theorem. Our proof, if traced back, involves the axiom of choice. It can be proved without the axiom of choice, and this is sometimes desirable when describing a small portion of set theory to students. Some exercises outline such a proof.

Proposition 12.13. *If $m \in \omega$, A is a set with $|A| = m+1$, and $a \in A$, then $|A \backslash \{a\}| = m$.*

Proof. Let $f : A \to m+1$ be a bijection. Let g be a bijection from $m+1$ onto $m+1$ which interchanges m and $f(a)$, leaving other elements fixed. Then $g \circ f$ is a bijection of A onto $m+1$, and $(g \circ f)(a) = m$. Hence $(g \circ f)\langle(A \backslash \{a\})$ is a bijection from $A \backslash \{a\}$ onto m. □

Theorem 12.14. *Suppose that $m \in \omega$ and A and B are sets of size m. Let $f : A \to B$. Then f is one-one iff f is onto.*

Proof. We prove the statement

$$\forall m \in \omega \forall A, B, f[(|A| = |B| = m \text{ and } f : A \to B) \Rightarrow (f \text{ is one-one } \Leftrightarrow f \text{ is onto})]$$

by induction on m. It is obvious for $m = 0$. Suppose it is true for m, and $|A| = |B| = m+1$ and $f : A \to B$.

First suppose that f is one-one. Pick $a \in A$. Then by Proposition 12.13, $|A \backslash \{a\}| = |B \backslash \{f(a)\}| = m$. Now $f \restriction (A \backslash \{a\})$ maps into $B \backslash \{f(a)\}$, since if $x \in A \backslash \{a\}$ and $f(x) = f(a)$ then f being one-one is contradicted. Now $f \restriction (A \backslash \{a\})$ is one-one, so by the inductive hypothesis $f \restriction (A \backslash \{a\})$ is onto. Clearly then f is onto.

Second suppose that f is onto. Let $g : m+1 \to A$ be a bijection. Now for any $b \in B$ there is an $a \in A$ such that $f(a) = b$, hence there is an $i \in m+1$ such that $f(g(i)) = b$. Let $h(b)$ be the least such i. Then $(f \circ g \circ h)(b) = b$ for all $b \in B$. It follows that $h : B \to m+1$ is one-one. Hence by the first step above, h is onto. To show that f is one-one, suppose that $f(a_0) = f(a_1)$. Choose $i_0, i_1 \in m+1$ such that $g(i_0) = a_0$ and $g(i_1) = a_1$. Since h is onto, choose $b_0, b_1 \in B$ such that $h(b_0) = i_0$ and $h(b_1) = i_1$.

Then $b_0 = f(g(h(b_0))) = f(g(i_0)) = f(a_0) = f(a_1) = f(g(i_1)) = f(g(h(b_1))) = b_1$. Hence $i_0 = h(b_0) = h(b_1) = i_1$, and $a_0 = g(i_0) = g(i_1) = a_1$. □

Theorem 12.14 does not extend to infinite sets; see the exercises.

The following simple theorem is very important and basic for the theory of cardinals. It embodies in perhaps its simplest form the Cantor diagonal argument.

Theorem 12.15. *For any set A we have $|A| < |\mathscr{P}(A)|$.*

Proof. The function given by $a \mapsto \{a\}$ is a one-one function from A into $\mathscr{P}(A)$, and so $|A| \le |\mathscr{P}(A)|$. [Saying that $a \mapsto \{a\}$ is giving the value of the function at the argument a.] Suppose equality holds. Then there is a one-one function f mapping A onto $\mathscr{P}(A)$. Let $X = \{a \in A : a \notin f(a)\}$. Since f maps onto $\mathscr{P}(A)$, choose $a_0 \in A$ such that $f(a_0) = X$. Then $a_0 \in X$ iff $a_0 \notin X$, contradiction. □

By this theorem, for every ordinal α there is a larger cardinal, namely $|\mathscr{P}(\alpha)|$. Hence we can define α^+ to be the least cardinal $> \alpha$. Cardinals of the form κ^+ are called *successor cardinals*; other infinite cardinals are called *limit cardinals.* Is $\kappa^+ = |\mathscr{P}(\kappa)|$? The statement that this is true for every infinite cardinal κ is the famous *generalized continuum hypothesis* (GCH). The weaker statement that $\omega^+ = |\mathscr{P}(\omega)|$ is the *continuum hypothesis* (CH).

It can be shown that the generalized continuum hypothesis is consistent with our axioms. But also its negation is consistent; in fact, the negation of the weaker continuum hypothesis is consistent. All of this under the assumption that our axioms are consistent. (It is not possible to prove this consistency.)

Theorem 12.16. *If Γ is a set of cardinals, then $\bigcup \Gamma$ is also a cardinal.*

Proof. We know already that $\bigcup \Gamma$ is an ordinal. Suppose that $\kappa \stackrel{\text{def}}{=} |\bigcup \Gamma| < \bigcup \Gamma$. By definition of $\bigcup$, there is a $\lambda \in \Gamma$ such that $\kappa < \lambda$. (Membership is the same as $<$.) Now $\lambda \subseteq \bigcup \Gamma$. So $\lambda = |\lambda| \le |\bigcup \Gamma| = \kappa$, contradiction. □

We can now define the standard sequence of infinite cardinal numbers, by transfinite recursion.

Theorem 12.17. *There is a class ordinal function $\aleph$ with domain* **On** *such that the following conditions hold.*

(i) $\aleph_0 = \omega$.

(ii) $\aleph_{\beta+1} = \aleph_\beta^+$ *for any ordinal β.*

(iii) $\aleph_\beta = \bigcup_{\gamma<\beta} \aleph_\gamma$ *for every limit ordinal β.*

Proof. We define $\mathbf{G} : \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ as follows. For any ordinal α and any set x,

$$\mathbf{G}(\alpha, x) = \begin{cases} \omega & \text{if } \alpha = 0, \\ (x(\beta))^+ & \text{if } \alpha = \beta + 1 \text{ for some ordinal } \beta \text{ and} \\ & x \text{ is a function with domain } \alpha \text{ and } x(\beta) \text{ is an ordinal} \\ \bigcup_{\beta<\alpha} x(\beta) & \text{if } \alpha \text{ is a limit ordinal and } x \text{ is a function} \\ & \text{with domain } \alpha \text{ and ordinal values} \\ \emptyset & \text{otherwise.} \end{cases}$$

Now we apply Theorem 6.7 and get a function $\mathbf{F} : \mathbf{On} \to \mathbf{V}$ such that $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha)$ for every ordinal α. Then

$$\begin{aligned} \mathbf{F}(0) &= \mathbf{G}(0, \mathbf{F} \restriction 0) = \omega; \\ \mathbf{F}(\beta+1) &= \mathbf{G}(\beta+1, \mathbf{F} \restriction (\beta+1)) = (\mathbf{F}(\beta))^+ \\ \mathbf{F}(\alpha) &= \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha) = \bigcup_{\beta<\alpha} \mathbf{F}(\beta) \quad \text{for } \alpha \text{ limit} \end{aligned}$$ $\square$

For historical reasons, one sometimes writes ω_α in place of $\aleph_\alpha$. Now we get the following two results by Propositions 9.15 and 9.16.

Lemma 12.18. *If $\alpha < \beta$, then $\aleph_\alpha < \aleph_\beta$.* $\square$

Lemma 12.19. *$\alpha \leq \aleph_\alpha$ for every ordinal α.* $\square$

Theorem 12.20. *For every infinite cardinal κ there is an ordinal α such that $\kappa = \aleph_\alpha$.*

Proof. Let κ be any infinite cardinal. Then $\kappa \leq \aleph_\kappa < \aleph_{\kappa+1}$. Here $\kappa + 1$ refers to ordinal addition. This shows that there is an ordinal α such that $\kappa < \aleph_\alpha$; choose the least such α. Clearly $\alpha \neq 0$ and α is not a limit ordinal. Say $\alpha = \beta + 1$. Then $\aleph_\beta \leq \kappa < \aleph_{\beta+1}$, so $\kappa = \aleph_\beta$. $\square$

We can now say a little more about the continuum hypothesis. Not only is it consistent that it fails, but it is even consistent that $|\mathscr{P}(\omega)| = \aleph_2$, or $|\mathscr{P}(\omega)| = \aleph_{17}$, or $|\mathscr{P}(\omega)| = \aleph_{\omega+1}$; the possibilities have been spelled out in great detail. Some impossible situations are $|\mathscr{P}(\omega)| = \aleph_\omega$ and $|\mathscr{P}(\omega)| = \aleph_{\omega+\omega}$; we will establish this later in this chapter.

Addition of cardinals

Let κ and λ be cardinals. We define

$$\kappa + \lambda = |\{(\alpha, 0) : \alpha \in \kappa\} \cup \{(\beta, 1) : \beta \in \lambda\}|.$$

The idea is to take disjoint copies $\kappa \times \{0\}$ and $\lambda \times \{1\}$ of κ and λ and count the number of elements in their union.

Two immediate remarks should be made about this definition. First of all, this is not, in general, the same as the ordinal sum $\kappa + \lambda$. We depend on the context to distinguish the two notions of addition. For example, $\omega + 1 = \omega$ in the cardinal sense, but not in the ordinal sense. In fact, we know that $\omega < \omega + 1$ in the ordinal sense. To show that $\omega + 1 = \omega$ in the cardinal sense, it suffices to define a one-one function from ω onto the set

$$\{(m, 0) : m \in \omega\} \cup \{(0, 1)\}.$$

Let $f(0) = (0, 1)$ and $f(m + 1) = (m, 0)$ for any $m \in \omega$.

Secondly, the definition is consistent with our definition of addition for natural numbers (as a special case of ordinal addition), and thus it does coincide with ordinal addition when restricted to ω; this will be proved shortly.

Proposition 12.21. *If $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$.*

Proof. We have $|A| + |B| = |\{(\alpha, 0) : \alpha \in |A|\} \cup \{(\alpha, 1) : \alpha \in |B|\}|$. Now let $f : A \to |A|$ and $g : B \to |B|$ be bijections. Define h with domain $A \cup B$ by

$$h(a) = (f(a), 0) \quad \text{for all } a \in A,$$
$$h(b) = (g(b), 1) \quad \text{for all } b \in B.$$

Then it is clear that h is a bijection from $A \cup B$ onto $\{(\alpha, 0) : \alpha \in |A|\} \cup \{(\alpha, 1) : \alpha \in |B|\}$. Hence $|A \cup B| = |\{(\alpha, 0) : \alpha \in |A|\} \cup \{(\alpha, 1) : \alpha \in |B|\}| = |A| + |B|$. □

Proposition 12.22. *If m and n are natural numbers, then addition in the ordinal sense and in the cardinal number sense are the same.*

Proof. For this proof we denote ordinal addition by $+'$ and cardinal addition by $+$. With $m \in \omega$ fixed we prove that $m +' n = m + n$ by induction on n. The case $n = 0$ is clear. Now suppose that $m +' n = m + n$. Then $m +' (n +' 1) = (m +' n) +' 1 = (m +' n) \cup \{m +' n\}$. On the other hand,

$$\begin{aligned} m + (n +' 1) &= |\{(i, 0) : i \in m\} \cup \{(i, 1) : i \in n +' 1\}| \\ &= |\{(i, 0) : i \in m\} \cup \{(i, 1) : i \in n \cup \{n\}\}| \\ &= |\{(i, 0) : i \in m\} \cup \{(i, 1) : i \in n\} \cup \{(n, 1)\}| \\ &= |\{(i, 0) : i \in m\} \cup \{(i, 1) : i \in n\}| + 1 \\ &= (m +' n) +' 1 = m +' (n +' 1). \end{aligned}$$ □

Aside from simple facts about addition, there is the remarkable fact that $\kappa + \kappa = \kappa$ for every infinite cardinal κ. We shall prove this as a consequence of the similar result for multiplication.

The definition of cardinal addition can be extended to infinite sums, and very elementary properties of the binary sum are then special cases of more general results; so we proceed with the general definition. Let $\langle \kappa_i : i \in I \rangle$ be a system of cardinals (this just means that κ is a function with domain I whose values are always cardinals). Then we define

$$\sum_{i \in I} \kappa_i = \left| \bigcup_{i \in I} (\kappa_i \times \{i\}) \right|.$$

This is a generalization of summing two cardinals, as is immediate from the definitions:

Proposition 12.23. *If $\langle \kappa_i : i \in 2 \rangle$ is a system of cardinals (meaning that κ is a function with domain 2 such that both κ_0 and κ_1 are cardinals), then $\sum_{i \in 2} \kappa_i = \kappa_0 + \kappa_1$.* □

The following is easily proved by induction on $|I|$:

Proposition 12.24. *If $\langle m_i : i \in I \rangle$ is a system of natural numbers, with I finite, then $\sum_{i \in I} m_i$ is a natural number.* □

We mention some important but easy facts concerning the cardinalities of unions:

Proposition 12.25. *If $\langle A_i : i \in I\rangle$ is a system of pairwise disjoint sets, then $\left|\bigcup_{i\in I} A_i\right| = \sum_{i\in I} |A_i|$.* □

Proposition 12.26. *If $\langle A_i : i \in I\rangle$ is any system of sets, then $\left|\bigcup_{i\in I} A_i\right| \le \sum_{i\in I} |A_i|$.*

Proof. For each $i \in I$ let f_i be a bijection from $|A_i|$ onto A_i. (We use the axiom of choice here.) For any $i \in I$ and $\alpha \in |A_i|$ let $g((\alpha, i)) = f_i(\alpha)$. Then g maps $\bigcup_{i\in I}(|A_i| \times \{i\})$ onto $\bigcup_{i\in I} A_i$. Hence by Corollary 12.11,

$$\left|\bigcup_{i\in I} A_i\right| \le \left|\bigcup_{i\in I}(|A_i| \times \{i\})\right| = \sum_{i\in I} |A_i|.$$ □

Corollary 12.27. *If $\langle \kappa_i : i \in I\rangle$ is a system of cardinals, then $\bigcup_{i\in I} \kappa_i \le \sum_{i\in I} \kappa_i$.* □

Finally, we gather together some simple arithmetic of infinite sums:

Proposition 12.28. *(i) $\sum_{i\in I} 0 = 0$.*
(ii) $\sum_{i\in 0} \kappa_i = 0$.
(iii) $\sum_{i\in I} \kappa_i = \sum_{i\in I, \kappa_i\neq 0} \kappa_i$.
(iv) If $I \subseteq J$, then $\sum_{i\in I} \kappa_i \le \sum_{i\in J} \kappa_i$.
(v) If $\kappa_i \le \lambda_i$ for all $i \in I$, then $\sum_{i\in I} \kappa_i \le \sum_{i\in I} \lambda_i$.
(vi) $\sum_{i\in I} 1 = |I|$.
(vii) If κ is infinite, then $\kappa + 1 = \kappa$.

Proof. (i): $\sum_{i\in I} 0 = \left|\bigcup_{i\in I}(0 \times \{i\})\right| = |\emptyset| = 0$.
(ii): $\sum_{i\in 0} \kappa_i = \left|\bigcup_{i\in 0}(\kappa_i \times \{i\})\right| = |\emptyset| = 0$.
(iii): $\sum_{i\in I} \kappa_i = \left|\bigcup_{i\in I}(\kappa_i \times \{i\})\right| = \left|\bigcup_{i\in I,\kappa_i\neq 0}(\kappa_i \times \{i\}\right| = \sum_{i\in I,\kappa_i\neq 0} \kappa_i$.
(iv): Assume that $I \subseteq J$. Then $\bigcup_{i\in I}(\kappa_i \times \{i\}) \subseteq \bigcup_{i\in J}(\kappa_i \times \{i\})$ and so the desired conclusion follows by Theorem 12.10.
(v): Assume that $\kappa_i \le \lambda_i$ for all $i \in I$. Then $\bigcup_{i\in I}(\kappa_i \times \{i\}) \subseteq \bigcup_{i\in I}(\lambda_i \times \{i\})$, and Theorem 12.10 applies.
(vi): We have $\sum_{i\in I} 1 = \left|\bigcup_{i\in I}(1 \times \{i\})\right|$. Now the mapping $i \mapsto (0, i)$ is a bijection from I to $\bigcup_{i\in I}(1 \times \{i\})$, so the desired conclusion follows.
(vii) We define a function f mapping κ into $\{(\alpha, 0) : \alpha < \kappa\} \cup \{(0, 1)\}$ as follows. For any $\alpha < \kappa$,

$$f(\alpha) = \begin{cases} (0,1) & \text{if } \alpha = 0, \\ (\beta, 0) & \text{if } \alpha = \beta + 1 \in \omega, \\ (\alpha, 0) & \text{if } \omega \le \alpha < \kappa. \end{cases}$$

It is clear that f is a bijection, as desired. □

By definition,

$$\kappa \cdot \lambda = |\kappa \times \lambda|.$$

Again this is different from ordinal multiplication, and we depend on the context to distinguish between them. For example, in the ordinal sense $\omega \cdot 2 > \omega \cdot 1 = \omega$ but in the cardinal sense $\omega \cdot 2 = \omega$. One can see the latter by using the following function f from ω to $\omega \times 2$: $f(2m) = (m, 0)$ and $f(2m+1) = (m, 1)$ for any $m \in \omega$.

The following simple result can be used in verifying many simple facts concerning products.

Proposition 12.29. *If A is equipotent with C and B is equipotent with D, then $A \times B$ is equipotent with $C \times D$.*

Proof. Assume the hypothesis. Say $f : A \to C$ is a bijection, and $g : B \to D$ is a bijection. Define $h : A \times B \to C \times D$ by setting $h(a, b) = (f(a), g(b))$. Clearly h is a bijection from $A \times B$ onto $C \times D$. □

Proposition 12.30. *(i) $\kappa \cdot \lambda = \lambda \cdot \kappa$;*
(ii) $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$;
(iii) $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$;
(iv) $\kappa \cdot 0 = 0$;
(v) $\kappa \cdot 1 = \kappa$;
(vi) $\kappa \cdot 2 = \kappa + \kappa$;
(vii) $\sum_{i \in I} \kappa = \kappa \cdot |I|$;
(viii) If $\kappa \leq \mu$ and $\lambda \leq \nu$, then $\kappa \cdot \lambda \leq \mu \cdot \nu$.

Proof. (i): For any $\alpha \in \kappa$ and $\beta \in \lambda$ let $f(\alpha, \beta) = (\beta, \alpha)$. Clearly f is a bijection from $\kappa \times \lambda$ onto $\lambda \times \kappa$.

(ii): We have $\kappa \cdot (\lambda \cdot \mu) = |\kappa \times (\lambda \cdot \mu)| = |\kappa \times |\lambda \times \mu||$. Let f be a bijection from $\lambda \cdot \mu$ onto $\lambda \times \mu$. Define $g : \kappa \times |\lambda \times \mu| \to (\kappa \times \lambda) \times \mu$ by setting, for $\alpha \in \kappa$ and $\beta \in |\lambda \times \mu|$, $g(\alpha, \beta) = ((\alpha, 1^{\text{st}} f(\beta)), 2^{\text{nd}} f(\beta))$. Clearly g is a bijection.

We have $(\kappa \cdot \lambda) \cdot \mu = |(\kappa \cdot \lambda) \times \mu| = ||\kappa \times \lambda| \times \mu|$. Now let $h : \kappa \cdot \lambda \to \kappa \times \lambda$ be a bijection. Define $k : |\kappa \times \lambda| \times \mu \to (\kappa \times \lambda) \times \mu$ by setting, for $\alpha \in |\kappa \times \lambda|$ and $\beta \in \mu$, $k(\alpha, \beta) = (h(\alpha), \beta)$. Clearly h is a bijection.

Now $h^{-1} \circ g$ is a bijection from $\kappa \times |\lambda \times \mu|$ onto $|\kappa \times \lambda| \times \mu$, as desired.

(iii): We have

$$\begin{aligned}
\kappa \cdot (\lambda + \mu) &= |\kappa \times (\lambda + \mu)| \\
&= |\kappa \times |(\lambda \times \{0\}) \cup (\mu \times \{1\})|| \\
&= |\kappa \times (\lambda \times \{0\}) \cup (\mu \times \{1\}))| \quad \text{using Proposition 12.29;} \\
\kappa \cdot \lambda + \kappa \cdot \mu &= |(\kappa \cdot \lambda) \times \{0\}) \cup (\kappa \cdot \mu) \times \{1\})| \\
&= |(|\kappa \times \lambda| \times \{0\}) \cup (|\kappa \times \mu| \times \{1\}|.
\end{aligned}$$

Take bijections $g : |\kappa \times \lambda| \to \kappa \times \lambda$ and $h : |\kappa \times \mu| \to \kappa \times \mu$. Now we define

$$h : \kappa \times ((\lambda \times \{0\}) \cup (\mu \times \{1\})| \to (|\kappa \times \lambda| \times \{0\}) \cup (|\kappa \times \mu| \times \{1\}.$$

Let $\alpha \in \kappa$, $\beta \in \lambda$ and $\gamma \in \mu$. Then we define

$$\begin{aligned} h((\alpha, (\beta, 0))) &= (g^{-1}((\alpha, \beta)), 0); \\ h((\alpha, (\gamma, 1))) &= (g^{-1}(\alpha, \gamma)), 1) \end{aligned}$$

It suffices to show that h is a bijection. Clearly it is one-one. For ontoness, given $\alpha \in |\kappa \times \lambda|$ we have

$$h((1^{\text{st}}(g(\alpha), (2^{\text{nd}}(g(\alpha)), 0)) = (g^{-1}((1^{\text{st}}(g(\alpha)), 2^{\text{nd}}(g(\alpha)))), 0) = (\alpha, 0),$$

and similarly for $\alpha \in |\kappa \times \mu|$.

(iv): $\kappa \cdot 0 = |\kappa \times 0| = |0| = 0$.

(v): $\kappa \cdot 1 = |\kappa \times 1| = \kappa$, since $\alpha \mapsto (\alpha, 0)$ is a bijection from κ to $\kappa \times 1$.

(vi): $\kappa \cdot 2 = |\kappa \times 2|$ and $\kappa + \kappa = |(\kappa \times \{0\}) \cup (\kappa \times \{1\})| = |\kappa \times 2|$.

(vii): $\sum_{i \in I} \kappa = \left|\bigcup_{i \in I}(\kappa \cdot \{i\})\right|$ and $\kappa \cdot |I|| = |\kappa \times |I||$. Let f be a bijection from I to $|I|$. For any $\alpha \in \kappa$ and $i \in I$ let $g((\alpha, i)) = (\alpha, f(i))$. Clearly g is a bijection from $\bigcup_{i \in I}(\kappa \cdot \{i\})$ onto $\kappa \times |I|$.

(viii): Assume that $\kappa \le \mu$ and $\lambda \le \nu$. Then $\kappa \times \lambda \subseteq \mu \times \nu$, so the desired conclusion follows by Theorem 12.10. □

Proposition 12.31. *Multiplication of natural numbers means the same in the cardinal number sense as in ordinal sense.*

Proof. For this proof we use $\circ$ for ordinal multiplication and $\cdot$ for cardinal multiplication. We prove with fixed $m \in \omega$ that $m \circ n = m \cdot n$ for all $n \in \omega$. We have $m \circ 0 = 0$ and $m \cdot 0 = |m \times 0| = |0| = 0$. Assume that $m \circ n = m \cdot n$. Then $m \circ (n + 1) = m \circ n + m = m \cdot n + m = m \cdot n + m \cdot 1 = m \cdot (n + 1)$ using Proposition 12.30(iii). □

The basic theorem about multiplication of infinite cardinals is as follows.

Theorem 12.32. $\kappa \cdot \kappa = \kappa$ *for every infinite cardinal* κ.

Proof. Suppose not, and let κ be the least infinite cardinal such that $\kappa \cdot \kappa \neq \kappa$. Then $\kappa = \kappa \cdot 1 \le \kappa \cdot \kappa$, and so $\kappa < \kappa \cdot \kappa$. We now define a relation $\prec$ on $\kappa \times \kappa$. For all $\alpha, \beta, \gamma, \delta \in \kappa$,

$$\begin{aligned} (\alpha, \beta) \prec (\gamma, \delta) \text{ iff } & \max(\alpha, \beta) < \max(\gamma, \delta) \\ \text{or } & \max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha < \gamma \\ \text{or } & \max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha = \gamma \text{ and } \beta < \delta. \end{aligned}$$

Clearly this is a well-order. It follows that $(\kappa \times \kappa, \prec)$ is isomorphic to an ordinal α; let f be the isomorphism. We have $|\alpha| = |\kappa \times \kappa| = \kappa \cdot \kappa > \kappa$ by the remark at the beginning of this proof. So $\kappa < \alpha$. Therefore there exist $\beta, \gamma \in \kappa$ such that $f(\beta, \gamma) = \kappa$. Now

$$f[\{(\delta, \varepsilon) \in \kappa \times \kappa : (\delta, \varepsilon) \prec (\beta, \gamma)\}] = \kappa,$$

so, with $\varphi = \max(\beta, \gamma)+1$,

$$\begin{aligned}\kappa &= |\{(\delta, \varepsilon) \in \kappa \times \kappa : (\delta, \varepsilon) \prec (\beta, \gamma)\}| \\ &\leq |\varphi \times \varphi| = |\varphi| \cdot |\varphi|.\end{aligned}$$

But $\varphi < \kappa$, so either φ is finite, and $|\varphi| \cdot |\varphi|$ is then also finite, or else φ is infinite, and $|\varphi| \cdot |\varphi| = |\varphi|$ by the minimality of κ. In any case, $|\varphi| \cdot |\varphi| < \kappa$, contradiction. □

Corollary 12.33. *If κ and λ are nonzero cardinals and at least one of them is infinite, then $\kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda)$.*

Proof. Say wlog $\kappa \leq \lambda$. Then $\kappa + \lambda \leq \lambda + \lambda = \lambda \cdot 2 \leq \lambda \cdot \lambda = \lambda \leq \kappa + \lambda$. □

Corollary 12.34. *If $\langle A_i : i \in I\rangle$ is any system of sets, then*

$$\left|\bigcup_{i\in I} A_i\right| \leq |I| \cdot \bigcup_{i\in I} |A_i|.$$

Proof. For each $i \in I$ let $g_i : A_i \to |A_i|$ be a bijection (using the axiom of choice). Moreover, let c be a choice function for nonempty subsets of I. Now we define a function f mapping $\bigcup_{i\in I} A_i$ into $I \times \bigcup_{i\in I} |A_i|$. Take any $a \in \bigcup_{i\in I} A_i$, and let $j = c(\{i \in I : a \in A_i\})$. Then we set $f(a) = (j, g_j(a))$. Clearly f is one-one, and hence

$$\begin{aligned}\left|\bigcup_{i\in I} A_i\right| &\leq \left|I \times \bigcup_{i\in I} |A_i|\right| && \text{by Corollary 12.11} \\ &= \left||I| \times \bigcup_{i\in I} |A_i|\right| && \text{by Proposition 12.29} \\ &= |I| \cdot \bigcup_{i\in I} |A_i|.\end{aligned}$$

A set A is *countable* if $|A| \leq \omega$. So another corollary is

Corollary 12.35. *A countable union of countable sets is countable.* □

Proposition 12.36. *If $\langle \kappa_i : i \in I\rangle$ is a system of nonzero cardinals, and either I is infinite or some κ_i is infinite, then $\sum_{i\in I} \kappa_i = |I| \cdot \bigcup_{i\in I} \kappa_i$.*

Proof. We have

$$\begin{aligned}\sum_{i\in I} \kappa_i &\leq \sum_{i\in I} \bigcup_{j\in I} \kappa_j && \text{by Proposition 12.28(v)} \\ &= |I| \cdot \bigcup_{j\in I} \kappa_j && \text{by Proposition 12.30(vi)}\end{aligned}$$

This proves $\leq$ in the proposition.

Next, $\bigcup_{i\in I}\kappa_i \leq \sum_{i\in I}\kappa_i$ by Proposition 12.27, and $|I| = \sum_{i\in I} 1$ (by Proposition 12.28(vi)) $\leq \sum_{i\in I}\kappa_i$ (by Proposition 12.28(v)). Now the direction $\geq$ of the proposition follows from Corollary 12.33. □

By the above results, the binary operations of addition and multiplication of cardinals are trivial when applied to infinite cardinals; and the infinite sum is also easy to calculate. We now introduce infinite products which, as we shall see, are not so trivial. We need the following standard elementary notion: for $\langle A_i : i \in I\rangle$ a family of sets, we define

$$\prod_{i\in I} A_i = \{f : f \text{ is a function, } \mathrm{dmn}(f) = I, \text{ and } \forall i \in I[f(i) \in A_i]\}.$$

This is the *cartesian product* of the sets A_i. Now if $\langle \kappa_i : i \in I\rangle$ is a system of cardinals, we define

$$\prod_{i\in I}^{c} \kappa_i = \left|\prod_{i\in I}\kappa_i\right|.$$

Some elementary properties of this notion are summarized in the following proposition.

Proposition 12.37. *(i)* $\left|\prod_{i\in I} A_i\right| = \prod^c_{i\in I}|A_i|$.

(ii) If $\kappa_i = 0$ *for some* $i \in I$*, then* $\prod^c_{i\in I}\kappa_i = 0$.

(iii) $\prod^c_{i\in 0}\kappa_i = 1$.

(iv) $\prod^c_{i\in I}\kappa_i = \prod^c_{i\in I,\kappa_i\neq 1}\kappa_i$.

(v) $\prod^c_{i\in I} 1 = 1$.

(vi) If $\kappa_i \leq \lambda_i$ *for all* $i \in I$*, then* $\prod^c_{i\in I}\kappa_i \leq \prod^c_{i\in I}\lambda_i$.

(vii) $\prod^c_{i\in 2}\kappa_i = \kappa_0 \cdot \kappa_1$.

Proof. (i): For each $i \in I$, let f_i be a one-one function mapping A_i onto $|A_i$. (We are using the axiom of choice here.) Note that $\prod^c_{i\in I}|A_i| = \left|\prod_{i\in I}|A_i|\right|$. Thus we want to find a bijection from $\prod_{i\in I} A_i$ onto $\prod_{i\in I}|A_i|$. For each $x \in \prod_{i\in I} A_i$ and $j \in I$ let $(g(x))_j = f_j(x_j)$. Thus $g : \prod_{i\in I} A_i \to \prod_{i\in I}|A_i|$. Suppose that $g(x) = g(y)$. Then for any $j \in I$ we have $f_j(x_j) = ((g(x))_j = ((g(y))_j = f_j(y_j)$, and hence $x_j = y_j$; so $x = y$. Thus g is one-one. Given $y \in \prod_{i\in I}|A_i|$, define $x_j = f_j^{-1}(y_j)$ for any $j \in I$. Then $x \in \prod_{i\in I} A_i$ and $(g(x))_j = f_j(x_j) = f_j(f_j^{-1}(y_j)) = y_j$; so $g(x) = y$. This shows that g is onto.

(ii): If $\kappa_i = 0$ for some $i \in I$, then $\prod_{j\in I} A_j = \emptyset$, and hence $\prod^c_{j\in I} A_j = \left|\prod_{j\in I} A_j\right| = |\emptyset| = 0$.

(iii): We have $\prod^c_{i\in 0}\kappa_i = \left|\prod_{i\in 0}\kappa_i\right| = |\{\emptyset\}| = 1$.

(iv): We have $\prod^c_{i\in I}\kappa_i = \left|\prod_{i\in I}\kappa_i\right|$ and $\prod^c_{i\in I,\kappa_i\neq 1}\kappa_i = \left|\prod_{i\in I,\kappa_i\neq 1}\kappa_i\right|$, so we want a bijection from $\prod_{i\in I}\kappa_i$ onto $\prod_{i\in I,\kappa_i\neq 1}\kappa_i$. For each $x \in \prod_{i\in I}\kappa_i$ let $f(x) = x \restriction \{i \in I : \kappa_i \neq 1\}$. If $f(x) = f(y)$, then for any $i \in I$,

$$\begin{aligned} x(i) &= \begin{cases} (f(x))(i) & \text{if } \kappa_i \neq 1 \\ 0 & \text{if } \kappa_1 = 1 \end{cases} \\ &= \begin{cases} (f(y))(i) & \text{if } \kappa_i \neq 1 \\ 0 & \text{if } \kappa_1 = 1 \end{cases} \\ &= y(i). \end{aligned}$$

Thus f is one-one. Clearly it is onto.

(v): We have $\prod^c_{i\in I} 1 = \left|\prod_{i\in I} 1\right|$. Hence it suffices to show, using (iii), that if $I \neq \emptyset$ then $\prod_{i\in I} 1$ has only one element. This is clear.

(vi): Assume that $\kappa_i \leq \lambda_i$ for all $i \in I$. Then $\prod_{i\in I} \kappa_i \subseteq \prod_{i\in I} \lambda_i$, so (vi) follows from Theorem 12.10.

(vii): We have $\prod^c_{i\in 2} \kappa_i = \left|\prod_{i\in 2} \kappa_i\right|$, and $\kappa_0 \cdot \kappa_1 = |\kappa_0 \times \kappa_1|$. Hence it suffices to describe a bijection from $\prod_{i\in 2} \kappa_i$ onto $\kappa_0 \times \kappa_1$. For each $x \in \prod_{i\in 2} \kappa_i$ let $f(x) = (x(0), x(1))$. Clearly f is as desired. □

General commutative, associative, and distributive laws hold also:

Proposition 12.38. (Commutative law) *If $\langle \kappa_i : i \in I\rangle$ is a system of cardinals and $f : I \to I$ is one-one and onto, then*

$$\prod^c_{i\in I} \kappa_i = \prod^c_{i\in I} \kappa_{f(i)}.$$

Proof. For each $x \in \prod_{i\in I} \kappa_i$ define $g(x) \in \prod_{i\in I} \kappa_{f(i)}$ by setting $(g(x))_i = x_{f(i)}$. Clearly g is a bijection, and the proposition follows. □

Proposition 12.39. (Associative law) *If $\langle \kappa_{ij} : (i,j) \in I \times J\rangle$ is a system of cardinals, then*

$$\prod^c_{i\in I} \left(\prod^c_{j\in J} \kappa_{ij}\right) = \prod^c_{(i,j)\in I\times J} \kappa_{ij}.$$

Proof. Note that $\prod^c_{i\in I} \left(\prod^c_{j\in J} \kappa_{ij}\right) = \left|\prod_{i\in I} \left|\prod_{j\in J} \kappa_{ij}\right|\right|$. For each $i \in I$ let f_i be a bijection from $\left|\prod_{i\in J} \kappa_{ij}\right|$ onto $\prod_{i\in J} \kappa_{ij}$ (using the axiom of choice). Now we define g mapping $\prod_{i\in I} \left|\prod_{j\in J} \kappa_{ij}\right|$ to $\prod_{(i,j)\in I\times J} \kappa_{ij}$ by setting, for any $x \in \prod_{i\in I} \left|\prod_{j\in J} \kappa_{ij}\right|$ and any $(i,j) \in I \times J$, $(g(x))_{ij} = (f_i(x_i))_j$. To show that g is one-one, suppose that $g(x) = g(y)$. Take any $(i,j) \in I \times J$. Then $(f_i(x_i))_j = (g(x))_{ij} = (g(y))_{ij} = (f_i(y_i))_j$. Since j is arbitrary, $f_i(x_i) = f_i(y_i)$. Since f_i is one-one, $x_i = y_i$. Since i is arbitrary, $x = y$. Thus g is one-one. To show that g is onto, let $z \in \prod_{(i,j)\in I\times J} \kappa_{ij}$. Define $x \in \prod_{i\in I} \left|\prod_{j\in J} \kappa_{ij}\right|$ by setting $x_i = f_i^{-1}(\langle z_{ij} : j \in J\rangle)$. Then $(g(x))_{ij} = (f_i(x_i))_j = z_{ij}$; so $g(x) = z$. □

Proposition 12.40. (Distributive law) *If $\langle \lambda_i : i \in I\rangle$ is a system of cardinals, then*

$$\kappa \cdot \sum_{i\in I} \lambda_i = \sum_{i\in I} (\kappa \cdot \lambda_i).$$

Proof. We have

$$\kappa \cdot \sum_{i\in I} \lambda_i = \left|\kappa \times \left|\bigcup_{i\in I} (\lambda_i \times \{i\})\right|\right|;$$

$$\sum_{i\in I} (\kappa \cdot \lambda_i) = \left|\bigcup_{i\in I} ((\kappa \cdot \lambda_i) \times \{i\})\right|.$$

Let f be a bijection from $\left|\bigcup_{i\in I}(\lambda_i \times \{i\})\right|$ onto $\bigcup_{i\in I}(\lambda_i \times \{i\})$. For each $i \in I$ let g_i be a bijection from $\kappa \cdot \lambda_i$ onto $\kappa \times \lambda_i$ (using the axiom of choice). Now we define a function

$$h : \kappa \times \left|\bigcup_{i\in I}(\lambda_i \times \{i\})\right| \to \bigcup_{i\in I}((\kappa \cdot \lambda_i) \times \{i\}).$$

Let $(\alpha, \beta) \in \kappa \times \left|\bigcup_{i\in I}(\lambda_i \times \{i\})\right|$. Say $f(\beta) = (\gamma, i)$ with $i \in I$ and $\gamma \in \lambda_i$. Then we set $h((\alpha, \beta)) = (g_i^{-1}(\alpha, \gamma), i)$.

To show that h is one-one, suppose that $h((\alpha, \beta)) = h((\alpha', \beta'))$. Say $f(\beta) = (\gamma, i)$ and $f(\beta') = (\gamma', j)$. Then

$$(g_i^{-1}(\alpha, \gamma), i) = h((\alpha, \beta)) = h((\alpha', \beta')) = (g_j^{-1}(\alpha', \gamma'), j).$$

It follows that $i = j$ and $g_i^{-1}(\alpha, \gamma) = g_j^{-1}(\alpha', \gamma')$, hence $(\alpha, \gamma) = (\alpha', \gamma')$. So $\alpha = \alpha'$ and $\gamma = \gamma'$. Therefore $f(\beta) = f(\beta')$, so $\beta = \beta'$. We have shown that $(\alpha, \beta) = (\alpha', \beta')$. Hence h is one-one.

To show that h is onto, let $z \in \bigcup_{i\in I}((\kappa \cdot \lambda_i) \times \{i\})$; say $i \in I$ and $z = (\alpha, i)$ with $\alpha \in \kappa \cdot \lambda_i$. Let $g(\alpha) = (\beta, \gamma)$ with $\beta \in \kappa$ and $\gamma \in \lambda_i$. Then $(\gamma, i) \in \lambda_i \times \{i\}$. Let $\delta = f^{-1}(\gamma, i)$. Then we claim that $h((\beta, \delta)) = z$. For, we have $f(\delta) = (\gamma, i)$, and hence $h((\beta, \delta)) = (g_i^{-1}(\beta, \gamma), i) = (\alpha, i) = z$. □

Theorem 12.41. (König) *Suppose that $\langle \kappa_i : i \in I \rangle$ and $\langle \lambda_i : i \in I \rangle$ are systems of cardinals such that $\lambda_i < \kappa_i$ for all $i \in I$. Then*

$$\sum_{i\in I} \lambda_i < \prod_{i\in I}^{c} \kappa_i.$$

Proof. The proof is another instance of Cantor's diagonal argument. Suppose that this is not true; thus $\prod_{i\in I}^{c} \kappa_i \le \sum_{i\in I} \lambda_i$. It follows that there is a one-one function f mapping $\prod_{i\in I}^{c} \kappa_i$ into $\{(\alpha, i) : i \in I, \alpha < \lambda_i\}$. For each $i \in I$ let

$$K_i = \{(f^{-1}(\alpha, i))_i : \alpha < \lambda_i, (\alpha, i) \in \text{rng}(f)\}.$$

Clearly $K_i \subseteq \kappa_i$. Now $|K_i| \le \lambda_i < \kappa_i$, so we can choose $x_i \in \kappa_i \backslash K_i$ (using the axiom of choice). Say $f(x) = (\alpha, i)$. Then $x_i = (f^{-1}(\alpha, i))_i \in K_i$, contradiction. □

Exponentiation of cardinals

We define

$$\kappa^\lambda = |{}^\lambda\kappa|.$$

The following simple proposition will be useful.

Proposition 12.42. *If $|A| = |A'|$ and $|B| = |B'|$, then $|{}^A B| = |{}^{A'} B'|$.*

Proof. Let $f : A \to A'$ and $g : B \to B'$ be bijections. For any $x \in {}^A B$ let $F(x) = g \circ x \circ f^{-1}$. Thus $F(x) \in {}^{A'}B'$. If $F(x) = F(y)$, then $x = g^{-1} \circ F(x) \circ f = g^{-1} \circ F(y) \circ f = y$. So F is one-one. It is onto, since given $z \in {}^{A'}B'$ we have $g^{-1} \circ z \circ f \in {}^A B$, and $F(g^{-1} \circ z \circ f) = z$. $\square$

The elementary arithmetic of exponentiation is summarized in the following proposition:

Proposition 12.43. *(i)* $\kappa^0 = 1$.

(ii) If $\kappa \neq 0$, *then* $0^\kappa = 0$.

(iii) $\kappa^1 = \kappa$.

(iv) $1^\kappa = 1$.

(v) $\kappa^2 = \kappa \cdot \kappa$.

(vi) $\kappa^\lambda \cdot \kappa^\mu = \kappa^{\lambda+\mu}$.

(vii) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.

(viii) $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.

(ix) If $\kappa \leq \lambda \neq 0$ *and* $\mu \leq \nu$, *then* $\kappa^\mu \leq \lambda^\nu$.

(x) $\prod_{i \in I}^c \kappa = \kappa^{|I|}$.

(xi) $\kappa^{\sum_{i \in I} \lambda_i} = \prod_{i \in I}^c \kappa^{\lambda_i}$.

(xii) $\left(\prod_{i \in I}^c \kappa_i\right)^\lambda = \prod_{i \in I}^c \kappa_i^\lambda$.

Proof. (i): $\kappa^0 = |{}^0\kappa|$. Now ${}^0\kappa = \{\emptyset\}$, so $\kappa^0 = 1$.

(ii): if $\kappa \neq 0$, then $0^\kappa = |{}^\kappa 0|$ and ${}^\kappa 0 = \emptyset$, so $\kappa^0 = 0$.

(iii): $\kappa^1 = |{}^1\kappa|$, and ${}^1\kappa = \{\{(0, \alpha)\} : \alpha < \kappa\}$. The mapping $\alpha \mapsto \{(0, \alpha)\}$ is a bijection from κ onto $\{\{(0, \alpha)\} : \alpha < \kappa\}$.

(iv): $1^\kappa = |{}^\kappa 1|$, and ${}^\kappa 1$ has only one member, the function with domain κ and value always 0.

(v): $\kappa^2 = |{}^2\kappa|$ and $\kappa \cdot \kappa = |\kappa \times \kappa|$. For any $x \in {}^2\kappa$ let $f(x) = (x(0), x(1))$. Clearly f is a bijection from ${}^2\kappa$ onto $\kappa \times \kappa$.

(vi): $\kappa^\lambda \cdot \kappa^\mu = ||{}^\lambda\kappa| \times |{}^\mu\kappa|| = |{}^\lambda\kappa \times {}^\mu\kappa|$, using Proposition 12.29. Also, $\kappa^{\lambda+\mu} = |{}^{|(\lambda \times \{0\}) \cup (\mu \times \{1\})|}\kappa| = |{}^{(\lambda \times \{0\}) \cup (\mu \times \{1\})}\kappa|$, using Proposition 12.42. Hence it suffices to define a bijection from ${}^\lambda\kappa \times {}^\mu\kappa$ to ${}^{(\lambda \times \{0\}) \cup (\mu \times \{1\})}\kappa$. If $x \in {}^\lambda\kappa$ and $y \in {}^\mu\kappa$, define $(h(x, y))((\alpha, 0)) = x(\alpha)$ for any $\alpha \in \lambda$, and $(h(x, y))((\alpha, 1)) = y(\alpha)$ for any $\alpha \in \mu$. To show that h is one-one, suppose that $x, x' \in {}^\lambda\kappa$, $y, y' \in {}^\mu\kappa$, and $h(x, y) = h(x', y')$. To show that $x = x'$, take any $\alpha \in \lambda$. Then $x(\alpha) = (h(x, y))((\alpha, 0)) = (h(x', y'))((\alpha, 0)) = x'(\alpha)$. So $x = x'$. Similarly $y = y'$, so h is one-one. To show that h is onto, take any $z \in {}^{(\lambda \times \{0\}) \cup (\mu \times \{1\})}\kappa$. Define $x \in {}^\lambda\kappa$ by setting $x(\alpha) = z((\alpha, 0))$ for any $\alpha \in \lambda$, and define $y \in {}^\mu\kappa$ by setting $y(\alpha) = z((\alpha, 1))$ for any $\alpha \in \mu$. Then $h(x, y) = z$, since for any $\alpha \in \lambda$ we have $(h(x, y))((\alpha, 0)) = x(\alpha) = z((\alpha, 0))$ and for any $\alpha \in \mu$ we have $(h(x, y))((\alpha, 1)) = y(\alpha) = z((\alpha, 1))$.

(vii): $(\kappa \cdot \lambda)^\mu = |{}^\mu|\kappa \times \lambda|| = |{}^\mu(\kappa \times \lambda)|$ using Proposition 12.42. $\kappa^\mu \cdot \lambda^\mu = ||{}^\mu\kappa| \times |{}^\mu\lambda|| = |({}^\mu\kappa) \times ({}^\mu\lambda)|$ using Proposition 12.29. Hence it suffices to define a bijection from ${}^\mu(\kappa \times \lambda)$ onto $({}^\mu\kappa) \times ({}^\mu\lambda)$. For any $x \in \mu(\kappa \times \lambda)$, define $f(x) = (g(x), h(x))$, where $g(x)$ is the member of ${}^\mu\kappa$ such that $(g(x))(\alpha) = 1^{\text{st}}(x(\alpha))$ for any $\alpha \in \mu$, and $h(x)$ is the member of ${}^\mu\lambda$ such that $(h(x))(\alpha) = 2^{\text{nd}}(x(\alpha))$ for any $\alpha \in \mu$. To show that f is one-one, suppose that $f(x) = f(y)$. Then $g(x) = g(y)$, so for any $\alpha \in \mu$ we have $1^{\text{st}}(x(\alpha)) = (g(x))(\alpha) = (g(y))(\alpha) = 1^{\text{st}}(y(\alpha))$. Similarly, $2^{\text{nd}}(x(\alpha)) = 2^{\text{nd}}(y(\alpha))$ for any $\alpha \in \mu$. Hence $x(\alpha) = y(\alpha)$ for any $\alpha \in \mu$. Thus

$x = y$. So f is one-one. To show that f is onto, suppose that $(u, v) \in ({}^{\mu}\kappa) \times ({}^{\mu}\lambda)$. Define $x \in {}^{\mu}(\kappa \times \lambda)$ by setting $x(\alpha) = (u(\alpha), v(\alpha))$ for any $\alpha \in \mu$. Say $f(x) = (g(x), h(x))$. Then $(g(x))(\alpha) = 1^{\text{st}}(x(\alpha)) = u(\alpha)$ for any $\alpha \in \mu$; so $g(x) = u$. Similarly, $h(x) = v$. So $f(x) = (u, v)$, as desired.

(viii): $(\kappa^{\lambda})^{\mu} = |{}^{\mu}|{}^{\lambda}\kappa|| = |{}^{\mu}({}^{\lambda}\kappa)|$, using Proposition 12.42. $\kappa^{\lambda\cdot\mu} = |{}^{|\lambda\times\mu|}\kappa| = |{}^{\lambda\times\mu}\kappa|$ using Proposition 12.42. Hence it suffices to define a bijection from ${}^{\mu}({}^{\lambda}\kappa)$ onto ${}^{\lambda\times\mu}\kappa$. For any $x \in {}^{\mu}({}^{\lambda}\kappa)$ and any $\alpha \in \lambda$ and $\beta \in \mu$, let $(f(x))(\alpha, \beta) = (x(\alpha))(\beta)$. To show that f is one-one, suppose that $x, y \in {}^{\mu}({}^{\lambda}\kappa)$ and $f(x) = f(y)$. Take any $\alpha \in \lambda$ and $\beta \in \mu$. Then $(x(\alpha))(\beta) = (f(x))(\alpha, \beta) = (f(y))(\alpha, \beta) = (y(\alpha))(\beta)$. This being true for all $\beta \in \mu$, it follows that $x(\alpha) = y(\alpha)$. This is true for all $\alpha \in \lambda$, so $x = y$.

To see that f is onto, suppose that $z \in {}^{\lambda\times\mu}\kappa$. Define $x \in {}^{\lambda}({}^{\mu}\kappa)$ by setting $(x(\alpha))(\beta) = z(\alpha, \beta)$ for any $\alpha \in \lambda$ and $\beta \in \mu$. Then for any $\alpha \in \lambda$ and $\beta \in \mu$ we have $(f(x))(\alpha, \beta) = (x(\alpha))(\beta) = z(\alpha, \beta)$. So $f(x) = z$.

(ix): Assume that $\kappa \leq \lambda \neq 0$ and $\mu \leq \nu$. For every $x \in {}^{\mu}\kappa$ let $x+ \in {}^{\nu}\lambda$ be an extension of x. Then the mapping $x \mapsto x+$ is a one-one function from ${}^{\mu}\kappa$ into $\nu\lambda$. So (ix) follows.

(x): $\prod^{c}_{i\in I} \kappa = \left|\prod_{i\in i} \kappa\right|$ and $\kappa^{|I|} = |{}^{|I|}\kappa| = |{}^{I}\kappa|$ using Proposition 12.42. Note that actually $\prod_{i\in i} \kappa = {}^{I}\kappa$.

(xi): $\kappa^{\sum_{i\in I}\lambda_i} = \left|{}^{|\bigcup_{i\in I}(\lambda_i\times\{i\})|}\kappa\right| = \left|{}^{\bigcup_{i\in I}(\lambda_i\times\{i\})}\kappa\right|$, using Proposition 12.42. Also, $\prod^{c}_{i\in I} \kappa^{\lambda_i} = |\prod_{i\in I} |{}^{\lambda_i}\kappa|| = |\prod_{i\in I} {}^{\lambda_i}\kappa|$, using Proposition 12.37(i). Hence it suffices to define a bijection from ${}^{\bigcup_{i\in I}(\lambda_i\times\{i\})}\kappa$ onto $\prod_{i\in I} {}^{\lambda_i}\kappa$. Take any $x \in {}^{\bigcup_{i\in I}(\lambda_i\times\{i\})}\kappa$, $i \in I$, and $\alpha \in \lambda_i$. Define $(f(x))_i(\alpha) = x(\alpha, i)$. Then f is one-one. For, suppose that $f(x) = f(y)$. Take any $i \in I$ and $\alpha \in \lambda_i$. Then $x(\alpha, i) = (f(x))_i(\alpha) = (f(y))_i(\alpha) = y(\alpha, i)$. Hence $x = y$. To show that f is onto, let $z \in \prod_{i\in I} {}^{\lambda_i}\kappa$. Define $x \in {}^{\bigcup_{i\in I}(\lambda_i\times\{i\})}\kappa$ by setting, for any $i \in I$ and $\alpha \in \lambda_i$, $x(\alpha, i) = (z(i))(\alpha)$. Then for any $i \in I$ and $\alpha \in \lambda_i$, $(f(x))_i(\alpha) = x(\alpha, i) = (z(i))(\alpha)$. So $f(x) = z$.

(xii): $\left(\prod^{c}_{i\in I} \kappa_i\right)^{\lambda} = \left|{}^{\lambda}\left|\prod_{i\in I} \kappa_i\right|\right| = \left|{}^{\lambda}\prod_{i\in I} \kappa_i\right|$, using Proposition 12.42. Also, we have $\prod^{c}_{i\in I} \kappa_i^{\lambda} = \left|\prod_{i\in I} |{}^{\lambda}\kappa_i|\right| = \left|\prod_{i\in I} {}^{\lambda}\kappa_i\right|$, using Proposition 12.37(i). Hence it suffices to define a bijection from ${}^{\lambda}\prod_{i\in I} \kappa_i$ onto $\prod_{i\in I} {}^{\lambda}\kappa_i$. For any $x \in {}^{\lambda}\prod_{i\in I} \kappa_i$, $i \in I$, and $\alpha \in \lambda$, let $(f(x))_i(\alpha) = (x(\alpha))_i$. Then f is one-one. For, assume that $f(x) = f(y)$. Then for any $\alpha \in \lambda$ and $i \in I$ we have $(x(\alpha))_i = (f(x))_i(\alpha) = (f(y))_i(\alpha) = (y(\alpha))_i$. So $x = y$. Also, f is onto. For, suppose that $z \in \prod_{i\in I} {}^{\lambda}\kappa_i$. Define $x \in {}^{\lambda}\prod_{i\in I} \kappa_i$ by setting $(x(\alpha))_i = z_i(\alpha)$ for any $\alpha \in \lambda$ and $i \in I$. Then for any $i \in I$ and $\alpha \in \lambda$ we have $(f(x))_i(\alpha) = (x(\alpha))_i = z_i(\alpha)$. So $f(x) = z$. $\square$

Proposition 12.44. *If $m, n \in \omega$, then $m^n \in \omega$, and m^n has the same meaning in the ordinal or cardinal sense.*

Proof. For this proof, denote ordinal exponentiation by $\exp(m, n)$. With m fixed, we show that $m^n \in \omega$ and $m^n = \exp(m, n)$ by induction on n. We have $m^0 = 1$ by Proposition 12.43(i), and $\exp(m, 0) = 1$ also. Now assume that $m^n \in \omega$ and $m^n = \exp(m, n)$. Then $m^{n+1} = m^n \cdot m^1 = m^n \cdot m$ by Proposition 12.43(vi),(iii). We also have $\exp(m, n+1) = \exp(m, n) \cdot m$, so the inductive hypothesis gives the desired conclusion. $\square$

Proposition 12.45. $|\mathscr{P}(A)| = 2^{|A|}$.

For each $X \subseteq A$ define $\chi_X \in {}^A 2$ by setting

$$\chi_X(a) = \begin{cases} 1 & \text{if } a \in X, \\ 0 & \text{otherwise.} \end{cases}$$

[This is the *characteristic function* of X.] It is easy to check that χ is a bijection from $\mathscr{P}(A)$ onto ${}^A 2$. □

The calculation of exponentiation is not as simple as that for addition and multiplication. The following result gives one of the most useful facts about exponentiation, however.

Theorem 12.46. *If $2 \leq \kappa \leq \lambda \geq \omega$, then $\kappa^\lambda = 2^\lambda$.*

Proof. Note that each function $f : \lambda \to \lambda$ is a subset of $\lambda \times \lambda$. Hence ${}^\lambda \lambda \subseteq \mathscr{P}(\lambda \times \lambda)$, and so $\lambda^\lambda \leq |\mathscr{P}(\lambda \times \lambda)|$. Therefore,

$$2^\lambda \leq \kappa^\lambda \leq \lambda^\lambda \leq |\mathscr{P}(\lambda \times \lambda)| = 2^{\lambda \cdot \lambda} = 2^\lambda;$$

so all the entries in this string of inequalities are equal, and this gives $\kappa^\lambda = 2^\lambda$. □

Cofinality, and regular and singular cardinals

Further cardinal arithmetic depends on the notion of cofinality. For later purposes we define a rather general version of this notion. Let $(P, <)$ be a partial order. A subset X of P is *dominating* iff for every $p \in P$ there is an $x \in X$ such that $p \leq x$. The *cofinality* of P is the smallest cardinality of a dominating subset of P. We denote this cardinal by $\mathrm{cf}(P)$.

A subset X of P is *unbounded* iff there does not exist a $p \in P$ such that $x \leq p$ for all $x \in X$. If P is simply ordered without largest element, then these notions—dominating and unbounded—coincide. In fact, suppose that X is dominating but not unbounded. Since X is not unbounded, choose $p \in P$ such that $x \leq p$ for all $x \in X$. Since P does not have a largest element, choose $q \in P$ such that $p < q$. Then because X is dominating, choose $x \in X$ such that $q \leq x$. Then $q \leq x \leq p < q$, contradiction. Thus X dominating implies that X is unbounded. Now suppose that Y is unbounded but not dominating. Since Y is not dominating, there is a $p \in P$ such that $p \not\leq x$, for all $x \in Y$. Since P is a simple order, it follows that $x < p$ for all $x \in Y$. This contradicts Y being unbounded.

We apply these notions to infinite cardinals, which are simply ordered sets with no last element. Obviously any infinite cardinal κ is a dominating subset of itself; so $\mathrm{cf}(\kappa) \leq \kappa$. A cardinal κ is *regular* iff κ is infinite and $\mathrm{cf}(\kappa) = \kappa$. An infinite cardinal that is not regular is called *singular*.

Theorem 12.47. *For every infinite cardinal κ, the cardinal κ^+ is regular.*

Proof. Suppose that $\Gamma \subseteq \kappa^+$, Γ is unbounded in κ^+, and $|\Gamma| < \kappa^+$. Hence

$$\kappa^+ = \left| \bigcup_{\gamma \in \Gamma} \gamma \right| \leq \sum_{\gamma \in \Gamma} |\gamma| \leq \sum_{\gamma \in \Gamma} \kappa = \kappa \cdot \kappa = \kappa,$$

contradiction. The first equality here holds because Γ is unbounded in κ^+ and κ^+ is a limit ordinal. $\square$

This theorem almost tells the full story about when a cardinal is regular. Examples of singular cardinals are $\aleph_{\omega+\omega}$ and $\aleph_{\omega_1}$. But it is conceivable that there are regular cardinals not covered by Theorem 12.47. An uncountable regular limit cardinal is said to be *weakly inaccessible*. A cardinal κ is said to be *inaccessible* if it is regular, uncountable, and has the property that for any cardinal $\lambda < \kappa$, also $2^\lambda < \kappa$. Clearly every inaccessible cardinal is also weakly inaccessible. Under GCH, the two notions coincide. If it is consistent that there are weak inaccessibles, then it is consistent that 2^ω is weakly inaccessible; but of course it definitely is not inaccessible. It is consistent with ZFC that there are no uncountable weak inaccessibles at all. These consistency results will be proved later in these notes. It is reasonable to postulate the existence of inaccessibles, and they are useful in some situations. In fact, the subject of *large cardinals* is one of the most studied in contemporary set theory, with many spectacular results.

Theorem 12.48. *Suppose that $(A, <)$ is a simple order with no largest element. Then there is a strictly increasing function $f : \mathrm{cf}(A) \to A$ such that $\mathrm{rng}(f)$ is unbounded in A.*

Proof. Let X be a dominating subset of A of size $\mathrm{cf}(A)$, and let g be a bijection from $\mathrm{cf}(A)$ onto X. We define a function $f : \mathrm{cf}(A) \to X$ by recursion, as follows. If $f(\beta) \in X$ has been defined for all $\beta < \alpha$, where $\alpha < \mathrm{cf}(A)$, then $\{f(\beta) : \beta < \alpha\}$ has size less than $\mathrm{cf}(A)$, and hence it is not dominating. Hence there is an $a \in A$ such that $f(\beta) < a$ for all $\beta < \alpha$. We let $f(\alpha)$ be an element of X such that $a, g(\alpha) \leq f(\alpha)$.

Clearly f is strictly increasing. If $a \in A$, choose $\alpha < \mathrm{cf}(A)$ such that $a \leq g(\alpha)$. Then $a \leq f(\alpha)$. $\square$

Proposition 12.49. *Suppose that $(A, <)$ is a simple ordering with no largest element. Then $\mathrm{cf}(\mathrm{cf}(A)) = \mathrm{cf}(A)$.*

Proof. Clearly $\mathrm{cf}(\alpha) \leq \alpha$ for any ordinal α; in particular, $\mathrm{cf}(\mathrm{cf}(A)) \leq \mathrm{cf}(A)$. Now by Theorem 12.48, let $f : \mathrm{cf}(A) \to A$ be strictly increasing with $\mathrm{rng}(f)$ unbounded in A. Now $\mathrm{cf}(A)$ is an infinite cardinal, and hence it is a limit ordinal by Proposition 12.8. Hence Theorem 12.48 again applies, and we can let $g : \mathrm{cf}(\mathrm{cf}(A)) \to \mathrm{cf}(A)$ be strictly increasing with $\mathrm{rng}(g)$ unbounded in $\mathrm{cf}(A)$. Clearly $f \circ g : \mathrm{cf}(\mathrm{cf}(A)) \to A$ is strictly increasing. We claim that $\mathrm{rng}(f \circ g)$ is unbounded in A. For, given $a \in A$, choose $\alpha < \mathrm{cf}(A)$ such that $a \leq f(\alpha)$, and then choose $\beta < \mathrm{cf}(\mathrm{cf}(A))$ such that $\alpha \leq g(\beta)$. Then $a \leq f(\alpha) \leq f(g(\beta))$, proving the claim. It follows that $\mathrm{cf}(A) \leq \mathrm{cf}(\mathrm{cf}(A))$. $\square$

Proposition 12.50. *If κ is a regular cardinal, $\Gamma \subseteq \kappa$, and $|\Gamma| < \kappa$, then $\bigcup \Gamma < \kappa$.*

Proof. Since $\mathrm{cf}(\kappa) = \kappa$, from the definition of cf it follows that Γ is bounded in κ. Hence there is an $\alpha < \kappa$ such that $\gamma \leq \alpha$ for all $\gamma \in \Gamma$. So $\bigcup \Gamma \leq \alpha < \kappa$. $\square$

Proposition 12.51. *If A is a linearly ordered set with no greatest element, κ is a regular cardinal, and $f : \kappa \to A$ is strictly increasing with $\mathrm{rng}(f)$ unbounded in A, then $\kappa = \mathrm{cf}(A)$.*

Proof. By the definition of cf we have $\mathrm{cf}(A) \leq \kappa$. Suppose that $\mathrm{cf}(A) < \kappa$. By Theorem 12.48 let $g : \mathrm{cf}(A) \to A$ be strictly increasing with $\mathrm{rng}(g)$ unbounded in A. For each $\alpha < \mathrm{cf}(\alpha)$ choose $\beta_\alpha < \kappa$ such that $g(\alpha) \leq f(\beta_\alpha)$. Then $\{\beta_\alpha : \alpha < \mathrm{cf}(A)\} \subseteq \kappa$ and $|\{\beta_\alpha : \alpha < \mathrm{cf}(A)\}| < \kappa$, so by Proposition 12.50, $\bigcup_{\alpha<\mathrm{cf}(A)} \beta_\alpha < \kappa$. Let $\gamma < \kappa$ be such that $\beta_\alpha < \gamma$ for all $\alpha < \mathrm{cf}(A)$. Then $f(\gamma)$ is a bound for $\mathrm{rng}(g)$, contradiction. □

Proposition 12.52. *A cardinal κ is regular iff for every system $\langle \lambda_i : i \in I\rangle$ of cardinals less than κ, with $|I| < \kappa$, one also has $\sum_{i\in I} \lambda_i < \kappa$.*

Proof. $\Rightarrow$: Assume that κ is regular $\langle \lambda_i : i \in I\rangle$ is a system of cardinals less than κ, and $|I| < \kappa$. We have $\{\lambda_i : i \in I\} \subseteq \kappa$ and $|\{\lambda_i : i \in I\}| \leq |I|$, so by Proposition 12.50, $\bigcup_{i\in I} \lambda_i < \kappa$. Hence

$$\sum_{i\in I} \lambda_i \leq \sum_{i\in I} \bigcup_{i\in I} \lambda_i = |I| \cdot \bigcup_{i\in I} \lambda_i < \kappa.$$

$\Leftarrow$: Assume the indicated condition. Suppose that $\Gamma \subseteq \kappa$ and $|\Gamma| < \kappa$. Then $\langle |\alpha| : \alpha \in \Gamma\rangle$ is a system of cardinals less than κ, and $|\Gamma| < \kappa$. Hence $|\bigcup \Gamma| \leq \sum_{\lambda\in\Gamma} |\lambda| < \kappa$, so also $\bigcup \Gamma < \kappa$. Thus κ is regular. □

Proposition 12.53. *If κ is an infinite singular cardinal, then there is a strictly increasing sequence $\langle \lambda_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ of infinite successor cardinals such that $\kappa = \sum_{\alpha<\mathrm{cf}(\kappa)} \lambda_\alpha$.*

Proof. By Theorem 12.48, let $f : \mathrm{cf}(\kappa) \to \kappa$ be strictly increasing such that $\mathrm{rng}(f)$ is unbounded in κ. We define the desired sequence by recursion. Suppose that $\lambda_\beta < \kappa$ has been defined for all $\beta < \alpha$, with $\alpha < \mathrm{cf}(\kappa)$. Then $\bigcup_{\beta<\alpha} \lambda_\beta < \kappa$ by the definition of cofinality. So also

$$\left(\max\left(f(\alpha), \bigcup_{\beta<\alpha} \lambda_\beta\right)\right)^+ < \kappa,$$

and we define λ_α to be this cardinal.

Now $f(\delta) \leq \sum_{\alpha<\mathrm{cf}(\kappa)} \lambda_\alpha$ for each $\delta < \mathrm{cf}(\kappa)$, so

$$\kappa = \bigcup_{\delta\in\mathrm{cf}(\kappa)} f(\delta) \leq \sum_{\alpha<\mathrm{cf}(\kappa)} \lambda_\alpha \leq \sum_{\alpha<\mathrm{cf}(\kappa)} \kappa = \kappa \cdot \mathrm{cf}(\kappa) = \kappa.$$ □

The main theorem of cardinal arithmetic

Now we return to the general treatment of cardinal arithmetic.

Theorem 12.54. (König) *If κ is infinite and $\mathrm{cf}(\kappa) \leq \lambda$, then $\kappa^\lambda > \kappa$.*

Proof. If κ is regular, then $\kappa^\lambda \geq \kappa^\kappa = 2^\kappa > \kappa$. So, assume that κ is singular. Then by Theorem 12.53 there is a system $\langle \mu_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ of nonzero cardinals such that each μ_α is less than κ, and $\sum_{\alpha<\mathrm{cf}(\kappa)} \mu_\alpha = \kappa$. Hence, using Theorem 12.41,

$$\kappa = \sum_{\alpha<\mathrm{cf}(\kappa)} \mu_\alpha < \prod_{\alpha<\mathrm{cf}(\kappa)} \kappa = \kappa^{\mathrm{cf}(\kappa)} \leq \kappa^\lambda.$$ □

Corollary 12.55. *For λ infinite we have* $\mathrm{cf}(2^\lambda) > \lambda$.

Proof. Suppose that $\mathrm{cf}(2^\lambda) \le \lambda$. Then by Theorem 12.54, $(2^\lambda)^\lambda > 2^\lambda$. But $(2^\lambda)^\lambda = 2^{\lambda\cdot\lambda} = 2^\lambda$, contradiction. □

We can now verify a statement made earlier about possibilities for $|\mathscr{P}(\omega)|$. Since $|\mathscr{P}(\omega)| = 2^\omega$, the corollary says that $\mathrm{cf}(|\mathscr{P}(\omega)) > \omega$. So this implies that $|\mathscr{P}(\omega)|$ cannot be $\aleph_\omega$ or $\aleph_{\omega+\omega}$. Here $\omega+\omega$ is the ordinal sum of ω with ω. It rules out many other possibilities of this sort.

We now prove a lemma needed for the last major theorem of this subsection, which says how to compute exponents (in a way).

Lemma 12.56. *If κ is a limit cardinal and $\lambda \ge \mathrm{cf}(\kappa)$, then*

$$\kappa^\lambda = \left(\bigcup_{\substack{\mu<\kappa\\ \mu \text{ a cardinal}}} \mu^\lambda\right)^{\mathrm{cf}(\kappa)}.$$

Proof. By Theorem 12.48, let $\gamma : \mathrm{cf}(\kappa) \to \kappa$ be strictly increasing with $\mathrm{rng}(\gamma)$ unbounded in κ, and with $0 < \gamma_0$. We define $F : {}^\lambda\kappa \to \prod_{\alpha<\mathrm{cf}(\kappa)} {}^\lambda\gamma_\alpha$ as follows. If $f \in {}^\lambda\kappa$, $\alpha < \mathrm{cf}(\kappa)$, and $\beta < \lambda$, then

$$((F(f))_\alpha)_\beta = \begin{cases} f(\beta) & \text{if } f(\beta) < \gamma_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Now F is a one-one function. For, if $f, g \in {}^\lambda\kappa$ and $f \ne g$, say $\beta < \lambda$ and $f(\beta) \ne g(\beta)$. Choose $\alpha < \mathrm{cf}\kappa$ such that $f(\beta)$ and $g(\beta)$ are both less than γ_α. Then $((F(f))_\alpha)_\beta = f(\beta) \ne g(\beta) = ((F(g))_\alpha)_\beta$, from which it follows that $F(f) \ne F(g)$. Since F is one-one,

$$\begin{aligned}
\kappa^\lambda = |{}^\lambda\kappa| &\le \left|\prod_{\alpha<\mathrm{cf}(\kappa)} {}^\lambda\gamma_\alpha\right| \\
&\le \left|\prod_{\alpha<\mathrm{cf}(\kappa)} \left(\bigcup_{\substack{\mu<\kappa\\ \mu \text{ a cardinal}}} {}^\lambda\mu\right)\right| \\
&= \left(\bigcup_{\substack{\mu<\kappa\\ \mu \text{ a cardinal}}} \mu^\lambda\right)^{\mathrm{cf}(\kappa)} \\
&\le (\kappa^\lambda)^{\mathrm{cf}(\kappa)} = \kappa^{\lambda\cdot\mathrm{cf}(\kappa)} = \kappa^\lambda,
\end{aligned}$$

and the lemma follows. □

The following theorem is not needed for the main result, but it is a classical result about exponentiation.

Theorem 12.57. (Hausdorff) *If κ and λ are infinite cardinals, then $(\kappa^+)^\lambda = \kappa^\lambda \cdot \kappa^+$.*

Proof. If $\kappa^+ \leq \lambda$, then both sides are equal to 2^λ. Suppose that $\lambda < \kappa^+$. Then

$$\begin{aligned}(\kappa^+)^\lambda = |{}^\lambda(\kappa^+)| &= \left|\bigcup_{\alpha<\kappa^+} {}^\lambda\alpha\right| \\ &\leq \sum_{\alpha<\kappa^+} |\alpha|^\lambda \leq \kappa^\lambda \cdot \kappa^+ \leq (\kappa^+)^\lambda,\end{aligned}$$

as desired. □

Here is the promised theorem giving computation rules for exponentiation. It essentially reduces the computation of κ^λ to two special cases: 2^λ, and $\kappa^{\mathrm{cf}(\kappa)}$. Generalizations of the results mentioned about the continuum hypothesis give a pretty good picture of what can happen to 2^λ. The case of $\kappa^{\mathrm{cf}(\kappa}$ is more complicated, and there is still work being done on what the possibilities here are. Shelah used his PCF theory to prove that $\aleph_\omega^{\aleph_0} \leq 2^{\aleph_0} + \aleph_{\omega_4}$.

Theorem 12.58. (main theorem of cardinal arithmetic) *Let κ and λ be cardinals with $2 \leq \kappa$ and $\lambda \geq \omega$. Then*

(i) If $\kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$.

(ii) If κ is infinite and there is a $\mu < \kappa$ such that $\mu^\lambda \geq \kappa$, then $\kappa^\lambda = \mu^\lambda$.

(iii) Assume that κ is infinite and $\mu^\lambda < \kappa$ for all $\mu < \kappa$. Then $\lambda < \kappa$, and:

(a) if $\mathrm{cf}(\kappa) > \lambda$, then $\kappa^\lambda = \kappa$;

(b) if $\mathrm{cf}(\kappa) \leq \lambda$, then $\kappa^\lambda = \kappa^{\mathrm{cf}(\kappa)}$.

Proof. (i) has already been noted, in Theorem 12.46. Under the hypothesis of (ii),

$$\kappa^\lambda \leq (\mu^\lambda)^\lambda = \mu^\lambda \leq \kappa^\lambda,$$

as desired.

Now assume the hypothesis of (iii). In particular, $2^\lambda < \kappa$, so of course $\lambda < \kappa$. Next, assume the hypothesis of (iii)(a): $\mathrm{cf}(\kappa) > \lambda$. Then

$$\begin{aligned}\kappa^\lambda = |{}^\lambda\kappa| &= \left|\bigcup_{\alpha<\kappa} {}^\lambda\alpha\right| \quad (\text{since } \lambda < \mathrm{cf}(\kappa)) \\ &\leq \sum_{\alpha<\kappa} |\alpha|^\lambda \leq \kappa,\end{aligned}$$

giving the desired result.

Finally, assume the hypothesis of (iii)(b): $\mathrm{cf}(\kappa) \leq \lambda$. Since $\lambda < \kappa$, it follows that κ is singular, so in particular it is a limit cardinal. Then by Lemma 12.56,

$$\kappa^\lambda = \left(\bigcup_{\substack{\mu<\kappa \\ \mu \text{ a cardinal}}} \mu^\lambda\right)^{\mathrm{cf}(\kappa)} \leq \kappa^{\mathrm{cf}(\kappa)} \leq \kappa^\lambda. \qquad \square$$

In theory one can now compute κ^λ for infinite κ, λ as follows. If $\kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$. Suppose that $\kappa > \lambda$. Let κ' be minimum such that $(\kappa')^\lambda = \kappa^\lambda$. Then $\forall \mu < \kappa'[\mu^\lambda < \kappa']$. In fact, if $\mu < \kappa'$ and $\mu^\lambda \geq \kappa'$, then $(\kappa')^\lambda \leq (\mu^\lambda)^\lambda = \mu^{\lambda \cdot \lambda} = \mu^\lambda < \kappa^\lambda = (\kappa')^\lambda$, contradiction. Now $(\kappa')^\lambda$ is computed by 12.58(iii).

Under the generalized continuum hypothesis the computation of exponents is very simple:

Corollary 12.59. *Assume GCH, and suppose that κ and λ are cardinals with $2 \leq \kappa$ and λ infinite. Then:*

(i) If $\kappa \leq \lambda$, then $\kappa^\lambda = \lambda^+$.

(ii) If $\mathrm{cf}(\kappa) \leq \lambda < \kappa$, then $\kappa^\lambda = \kappa^+$.

(iii) If $\lambda < \mathrm{cf}(\kappa)$, then $\kappa^\lambda = \kappa$.

Proof. (i) is immediate from Theorem 12.58(i). For (ii), assume that $\mathrm{cf}(\kappa) \leq \lambda < \kappa$. Then κ is a limit cardinal, and so for each $\mu < \kappa$ we have $\mu^\lambda \leq (\max(\mu, \lambda))^+ < \kappa$; hence by Theorem 12.58(iii)(b) and Theorem 12.54 we have $\kappa^\lambda = \kappa^{\mathrm{cf}(\kappa)} > \kappa$; since $\kappa^{\mathrm{cf}(\kappa)} \leq \kappa^\kappa = \kappa^+$, it follows that $\kappa^\lambda = \kappa^+$. For (iii), assume that $\lambda < \mathrm{cf}(\kappa)$. If there is a $\mu < \kappa$ such that $\mu^\lambda \geq \kappa$, then by Theorem 12.58(ii), $\kappa^\lambda = \mu^\lambda \leq (\max(\lambda, \mu))^+ \leq \kappa$, as desired. If $\mu^\lambda < \kappa$ for all $\mu < \kappa$, then $\kappa^\lambda = \kappa$ by Theorem 12.58(iii)(a). □

EXERCISES

E12.1. Define sets A, B with $|A| = |B|$ such that there is a one-one function $f : A \to B$ which is not onto.

E12.2. Define sets A, B with $|A| = |B|$ such that there is an onto function $f : A \to B$ which is not one-one.

E12.3. Show that the restriction $\lambda \neq 0$ is necessary in Proposition 12.43(ix).

The next four exercises outline a proof of the Cantor-Schröder-Bernstein theorem without using the axiom of choice. This theorem says that if there is an injection from A into B and one from B into A, then there is a bijection from A to B. In the development in the text, using the axiom of choice, the hypothesis implies that $|A| \leq |B| \leq |A|$, and hence $|A| = |B|$. But it is of some interest that it can be proved in an elementary way, without using the axiom of choice or anything about ordinals and cardinals. Of course, the axiom of choice should not be used in these four exercises.

E12.4. Let $F : \mathscr{P}(A) \to \mathscr{P}(A)$, and assume that for all $X, Y \subseteq A$, if $X \subseteq Y$, then $F(X) \subseteq F(Y)$. Let $\mathscr{A} = \{X : X \subseteq A \text{ and } X \subseteq F(X)\}$, and set $X_0 = \bigcup_{X \in \mathscr{A}} X$. Then $X_0 \subseteq F(X_0)$.

E12.5. Under the assumptions of exercise E12.4 we actually have $X_0 = F(X_0)$.

E12.6. Suppose that $f : A \to B$ is one-one and $g : B \to A$ is also one-one. For every $X \subseteq A$ let $F(X) = A \backslash g[B \backslash f[X]]$. Show that for all $X, Y \subseteq A$, if $X \subseteq Y$ then $F(X) \subseteq F(Y)$.

E12.7. Prove the Cantor-Schröder-Bernstein theorem as follows. Assume that f and g are as in exercise E12.6, and choose F as in that exercise. Let X_0 be as in exercise E12.4.

Show that $A\backslash X_0 \subseteq \operatorname{rng}(g)$. Then define $h : A \to B$ by setting, for any $a \in A$,

$$h(a) = \begin{cases} f(a) & \text{if } a \in X_0, \\ g^{-1}(a) & \text{if } a \in A\backslash X_0. \end{cases}$$

Show that h is one-one and maps onto B.

E12.12. Show that if α and β are ordinals, then $|\alpha \dot{+} \beta| = |\alpha| + |\beta|$, where $\dot{+}$ is ordinal addition and $+$ is cardinal addition.

E12.9. Show that if α and β are ordinals, then $|\alpha \odot \beta| = |\alpha| \cdot |\beta|$, where $\odot$ is ordinal multiplication and $\cdot$ is cardinal multiplication.

E12.10. Show that if α and β are ordinals, $2 \leq \alpha$, and $\omega \leq \beta$, then $|{}^{\cdot}\alpha^{\beta}| = |\alpha| \cdot |\beta|$. Here the dot to the left of the first exponent indicates that ordinal exponentiation is involved. [**This is a good exercise to keep in mind. For example, ${}^{\cdot}2^{\omega}$ is a countable set, but 2^{ω} is not.**]

E12.11. Prove that if $|A| \leq |B|$ then $|\mathscr{P}(A)| \leq |\mathscr{P}(B)|$.

E12.12. Prove the following general distributive law:

$$\prod_{i \in I}^{c} \sum_{j \in J_i} \kappa_{ij} = \sum_{f \in P} \prod_{i \in I}^{c} \kappa_{i,f(i)},$$

where $P = \prod_{i \in I} J_i$.

E12.13. Show that for any cardinal κ we have $\kappa^{+} = \{\alpha : \alpha \text{ is an ordinal and } |\alpha| \leq \kappa\}$.

E12.14. For every infinite cardinal λ there is a cardinal $\kappa > \lambda$ such that $\kappa^{\lambda} = \kappa$.

E12.15. For every infinite cardinal λ there is a cardinal $\kappa > \lambda$ such that $\kappa^{\lambda} > \kappa$.

E12.16. Prove that for every $n \in \omega$, and every infinite cardinal κ, $\aleph_n^{\kappa} = 2^{\kappa} \cdot \aleph_n$.

E12.17. Prove that $\prod_{i \in I}^{c}(\kappa_i \cdot \lambda_i) = \prod_{i \in I}^{c} \kappa_i \cdot \prod_{i \in I}^{c} \lambda_i$.

E12.112. Prove that $\aleph_{\omega}^{\aleph_1} = 2^{\aleph_1} \cdot \aleph_{\omega}^{\aleph_0}$.

E12.19. Prove that $\aleph_{\omega}^{\aleph_0} = \prod_{n \in \omega}^{c} \aleph_n$.

E12.20. Prove that for any infinite cardinal κ, $(\kappa^{+})^{\kappa} = 2^{\kappa}$.

E12.21. Show that if κ is an infinite cardinal and C is the collection of all cardinals less than κ, then $|C| \leq \kappa$.

E12.22. Show that if κ is an infinite cardinal and C is the collection of all cardinals less than κ, then

$$2^{\kappa} = \left(\sum_{\nu \in C} 2^{\nu} \right)^{\operatorname{cf}(\kappa)}.$$

E12.23. Prove that for any limit ordinal τ, $\prod_{\xi < \tau}^{c} 2^{\aleph_{\xi}} = 2^{\aleph_{\tau}}$.

E12.24. Assume that κ is an infinite cardinal, and $2^\lambda < \kappa$ for every cardinal $\lambda < \kappa$. Show that $2^\kappa = \kappa^{\mathrm{cf}(\kappa)}$.

E12.25. Suppose that λ is a singular cardinal, $\mathrm{cf}(\lambda) = \omega$, and $2^\kappa < \lambda$ for every $\kappa < \lambda$. Prove that $2^\lambda = \lambda^\omega$.

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13. Boolean algebras and forcing orders

To introduce the apparatus of generic extensions and forcing in a clear fashion, it is necessary to go into a special set theoretic topic: Boolean algebras and their relation to certain orders.

A *Boolean algebra* (BA) is a structure $\langle A, +, \cdot, -, 0, 1\rangle$ with two binary operations $+$ and $\cdot$, a unary operation $-$, and two distinguished elements 0 and 1 such that the following axioms hold for all $x, y, z \in A$:

(A)	$x + (y + z) = (x + y) + z$;	(A′)	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
(C)	$x + y = y + x$;	(C′)	$x \cdot y = y \cdot x$;
(L)	$x + (x \cdot y) = x$;	(L′)	$x \cdot (x + y) = x$;
(D)	$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$;	(D′)	$x + (y \cdot z) = (x + y) \cdot (x + z)$;
(K)	$x + (-x) = 1$;	(K′)	$x \cdot (-x) = 0$.

The main example of a Boolean algebra is a field of sets: a set A of subsets of some set X, closed under union, intersection, and complementation with respect to X. The associated Boolean algebra is $\langle A, \cup, \cap, \backslash, 0, X\rangle$. Here $\backslash$ is treated as a one-place operation, producing $X\backslash a$ for any $a \in A$. This example is really all-encompassing—every BA is isomorphic to one of these. We will not prove this, or use it.

As is usual in algebra, we usually denote a whole algebra $\langle A, +, \cdot, -, 0, 1\rangle$ just by mentioning its *universe* A, everything else being implicit.

Some notations used in some treatments of Boolean algebras are: $\vee$ or $\cup$ for $+$; $\wedge$ or $\cap$ for $\cdot$; $'$ for $-$. These notations might be confusing if discussing logic, or elementary set theory. Our notation might be confusing if discussing ordinary algebra.

Now we give the elementary arithmetic of Boolean algebras. We recommend that the reader go through them, but then approach any arithmetic statement in the future from the point of view of seeing if it works in fields of sets; if so, it should be easy to derive from the axioms.

First we have the duality principle, which we shall not formulate carefully; our particular uses of it will be clear. Namely, notice that the axioms come in pairs, obtained from each other by interchanging $+$ and $\cdot$ and 0 and 1. This means that also if we prove some arithmetic statement, the *dual* statement, obtained by this interchanging process, is also valid.

Proposition 13.1. *$x + x = x$ and $x \cdot x = x$.*

Proof.

$$\begin{aligned} x + x &= x + x \cdot (x + x) \text{ by } (\mathrm{L}') \\ &= x \text{ by } (\mathrm{L}); \end{aligned}$$

the second statement follows by duality. □

Proposition 13.2. *$x + y = y$ iff $x \cdot y = x$.*

Proof. Assume that $x + y = y$. Then, by (L′),

$$x \cdot y = x \cdot (x + y) = x.$$

The converse follows by duality. $\square$

In any BA we define $x \leq y$ iff $x + y = y$. Note that the dual of $x \leq y$ is $y \leq x$, by 13.2 and commutativity. (The dual of a defined notion is obtained by dualizing the original notions.)

Proposition 13.3. *On any BA, $\leq$ is reflexive, transitive, and antisymmetric; that is, the following conditions hold:*

(i) $x \leq x$;

(ii) If $x \leq y$ and $y \leq z$, then $x \leq z$;

(iii) If $x \leq y$ and $y \leq x$, then $x = y$.

Proof. $x \leq x$ means $x + x = x$, which was proved in 13.1. Assume the hypothesis of (*ii*). Then

$$\begin{aligned} x + z &= x + (y + z) \\ &= (x + y) + z \\ &= y + z \\ &= z, \end{aligned}$$

as desired. Finally, under the hypotheses of (*iii*),

$$x = x + y = y + x = y. \qquad \square$$

Note that Proposition 13.3 says that $\leq$ is a partial order on the BA A. There are some notions concerning partial orders which we need. An element z is an *upper bound* for a set Y of elements of X if $y \leq z$ for all $y \in Y$; similarly for lower bounds. And z is a least upper bound for Y if it is an upper bound for Y and is $\leq$ any other upper bound for Y; simlarly for greatest lower bounds. By antisymmetry, in any partial order least upper bounds and greatest lower bounds are unique if they exist.

Proposition 13.4. *$x + y$ is the least upper bound of $\{x, y\}$, and $x \cdot y$ is the greatest lower bound of $\{x, y\}$.*

Proof. We have $x + (x + y) = (x + x) + y = x + y$, and similarly $y + (x + y) = y + (y + x) = (y + y) + x = y + x = x + y$; so $x + y$ is an upper bound for $\{x, y\}$. If z is any upper bound for $\{x, y\}$, then

$$(x + y) + z = (x + (y + z) = x + z = z,$$

as desired. The other part follows by duality(!). $\square$

Proposition 13.5. *(i) $x + 0 = x$ and $x \cdot 1 = x$;*

(ii) $x \cdot 0 = x$ and $x + 1 = 1$;

(iii) $0 \leq x \leq 1$.

Proof. By (K) and Proposition 13.4, 1 is the least upper bound of x and $-x$; in particular it is an upper bound, so $x \leq 1$. Everything else follows by duality, Proposition 13.2, and the definitions. $\square$

Proposition 13.6. *For any x and y, $y = -x$ iff $x \cdot y = 0$ and $x + y = 1$.*

Proof. $\Rightarrow$ holds by (K) and (K'). Now suppose that $x \cdot y = 0$ and $x + y = 1$. Then

$$y = y \cdot 1 = y \cdot (x + -x) = y \cdot x + y \cdot -x = 0 + y \cdot -x = y \cdot -x;$$
$$-x = -x \cdot 1 = -x \cdot (x + y) = -x \cdot x + -x \cdot y = 0 + -x \cdot y = -x \cdot y = y. \qquad \square$$

Proposition 13.7. *(i) $- - x = x$;*
(ii) if $-x = -y$ then $x = y$;
(iii) $-0 = 1$ and $-1 = 0$;
(iv) (DeMorgan's laws) $-(x + y) = -x \cdot -y$ and $-(x \cdot y) = -x + -y$.

Proof. If we apply Proposition 13.6 with x and y replaced respectively by $-x$ and x, we get $- - x = x$. Next, if $-x = -y$, then $x = - - x = - - y = y$. For (iii), by 13.5(iii), $0 \cdot 1 = 0$ and $0 + 1 = 1$, so by 13.6, $-0 = 1$. Then $-1 = 0$ by duality. For the first part of (iv),

$$\begin{aligned}(x + y) \cdot -x \cdot -y &= x \cdot -x \cdot -y + y \cdot -x \cdot -y \\ &= 0 + 0 = 0,\end{aligned}$$

and

$$\begin{aligned}(x + y) + -x \cdot -y &= x \cdot (y + -y) + y + -x \cdot -y \\ &= x \cdot y + x \cdot -y + y + -x \cdot -y \\ &= y + x \cdot -y + -x \cdot -y \\ &= y + -y = 1,\end{aligned}$$

so that $-(x + y) = -x \cdot -y$ by Proposition 13.6. Finally, the second part of (iv) follows by duality. $\square$

Proposition 13.8. *$x \leq y$ iff $-y \leq -x$.*

Proof. Assume that $x \leq y$. Then $x + y = y$, so $-x \cdot -y = -y$, i.e., $-y \leq -x$. For the converse, use the implication just proved, plus 13.7(i). $\square$

Proposition 13.9. *If $x \leq x'$ and $y \leq y'$, then $x + y \leq x' + y'$ and $x \cdot y \leq x' \cdot y'$.*

Proof. Assume the hypothesis. Then

$$(x + y) + (x' + y') = (x + x') + (y + y') = x' + y',$$

and so $x + y \leq x' + y'$; the second conclusion follows by duality. $\square$

Proposition 13.10. *$x \leq y$ iff $x \cdot -y = 0$.*

Proof. If $x \leq y$, then $x = x \cdot y$ and so $x \cdot -y = 0$. Conversely, if $x \cdot -y = 0$, then

$$x = x \cdot (y + -y) = x \cdot y + x \cdot -y = x \cdot y,$$

so that $x \leq y$. □

Elements $x, y \in A$ are *disjoint* if $x \cdot y = 0$. For any x, y we define

$$x \triangle y = x \cdot -y + y \cdot -x;$$

this is the *symmetric difference* of x and y.

Proposition 13.11. *(i) $x = y$ iff $x \triangle y = 0$;*
(ii) $x \cdot (y \triangle z) = (x \cdot y) \triangle (x \cdot z)$;
(iii) $x \triangle (y \triangle z) = (x \triangle y) \triangle z$.

Proof. For (i), $\Rightarrow$ is trivial. Now assume that $x \triangle y = 0$. Then $x \cdot -y = 0 = y \cdot -x$, so $x \leq y$ and $y \leq x$, so $x = y$.

For (ii), we have

$$\begin{aligned} x \cdot (y \triangle z) &= x \cdot y \cdot -z + x \cdot z \cdot -y \\ &= (x \cdot y) \cdot -(x \cdot z) + (x \cdot z) \cdot -(x \cdot y) \\ &= (x \cdot y) \triangle (x \cdot z), \end{aligned}$$

as desired.

Finally, for (iii),

$$\begin{aligned} x \triangle (y \triangle z) &= x \cdot -(y \cdot -z + -y \cdot z) + (y \cdot -z + -y \cdot -z) \cdot -x \\ &= x \cdot (-y + z) \cdot (y + -z) + -x \cdot y \cdot -z + -x \cdot -y \cdot z \\ &= x \cdot -y \cdot -z + x \cdot y \cdot z + -x \cdot y \cdot -z + -x \cdot -y \cdot z; \end{aligned}$$

if we apply the same argument to $z \triangle (y \triangle x)$ we get

$$z \triangle (y \triangle x) = z \cdot -y \cdot -x + z \cdot y \cdot x + -z \cdot y \cdot -x + -z \cdot -y \cdot x,$$

which is the same thing. So the obvious symmetry of $\triangle$ gives the desired result. □

One further useful result is that axiom (D′) is redundant:

Proposition 13.12. *(D′) is redundant.* (Assume all axioms except D'.)

Proof.

$$\begin{aligned} (x + y) \cdot (x + z) &= ((x + y) \cdot x) + ((x + y) \cdot z) \\ &= (x \cdot (x + y)) + (z \cdot (x + y)) \\ &= x + ((z \cdot x) + (z \cdot y)) \\ &= x + ((x \cdot z) + (y \cdot z)) \\ &= (x + (x \cdot z)) + (y \cdot z) \\ &= x + (y \cdot z). \end{aligned}$$

□

If M is a subset of a BA A, we denote by $\sum M$ its least upper bound (if it exists), and by $\prod M$ its greatest lower bound, if it exists. A is *complete* iff these always exist. Note that frequently people use $\bigvee M$ and $\bigwedge M$ instead of $\sum M$ and $\prod M$.

Proposition 13.13. *Assume that A is a complete BA.*

(i) $-\sum_{i\in I} a_i = \prod_{i\in I} -a_i$.

(ii) $-\prod_{i\in I} a_i = \sum_{i\in I} -a_i$.

Proof. For (i), let $a = \sum_{i\in I} a_i$; we show that $-a$ is the greatest lower bound of $\{-a_i : i \in I\}$. If $i \in I$, then $a_i \le a$, and hence $-a \le -a_i$; thus $-a$ is a lower bound for the indicated set. Now suppose that x is any lower bound for this set. Then for any $i \in I$ we have $x \le -a_i$, and so $a_i \le -x$. So $-x$ is an upper bound for $\{a_i : i \in I\}$, and so $a \le -x$. Hence $x \le -a$, as desired.

(ii) is proved similarly. $\square$

The following (possibly infinite) distributive law is frequently useful. One should be aware of the fact that more general infinite distributive laws do not hold, in general. Since this will not enter into our treatment, we do not go into a counterexample or further discussion of really general distributive laws.

Proposition 13.14. *If $\sum_{i\in I} a_i$ exists, then $\sum_{i\in I}(b \cdot a_i)$ exists and*

$$b \cdot \sum_{i\in I} a_i = \sum_{i\in I}(b \cdot a_i).$$

Proof. Let $s = \sum_{i\in I} a_i$; we shall show that $b \cdot s$ is the least upper bound of $\{b \cdot a_i : i \in I\}$. If $i \in I$, then $a_i \le s$ and so $b \cdot a_i \le b \cdot s$; so $b \cdot s$ is an upper bound for the indicated set. Now suppose that x is any upper bound for this set. Then for any $i \in I$ we have $b \cdot a_i \le x$, hence $b \cdot a_i \cdot -x = 0$ and so $a_i \le -(b \cdot -x) = -b + x$; so $-b + x$ is an upper bound for $\{a_i : i \in I\}$. It follows that $s \le -b + x$, and hence $s \cdot b \le x$, as desired. $\square$

Forcing orders

A forcing order is a triple $\mathbb{P} = (P, \le, 1)$ such that $\le$ is a reflexive and transitive relation on the nonempty set P, and $\forall p \in P(p \le 1)$. Note that we do not assume that $\le$ is antisymmetric. Partial orders are special cases of forcing orders in which this is assumed (but we do not assume the existence of 1 in partial orders). Note that we assume that every forcing order has a largest element. Many set-theorists use "partial order" instead of "forcing order".

Frequently we use just P for a forcing order; $\le$ and 1 are assumed.

We say that elements $p, q \in P$ are *compatible* iff there is an $r \le p, q$. We write $p \perp q$ to indicate that p and q are incompatible. A set A of elements of P is an *antichain* iff any two distinct members of A are incompatible. WARNING: sometimes "antichain" is used to mean pairwise incomparable, or in the case of Boolean algebras, pairwise disjoint. A subset Q of P is *dense* iff for every $p \in P$ there is a $q \in Q$ such that $q \le p$.

Now we are going to describe how to embed a forcing order into a complete BA. We take the regular open algebra of a certain topological space. We assume a very little bit of topology. To avoid assuming any knowledge of topology we now give a minimalist introduction to topology.

A *topology* on a set X is a collection $\mathscr{O}$ of subsets of X satisfying the following conditions:

(1) $X, \emptyset \in \mathscr{O}$.

(2) $\mathscr{O}$ is closed under arbitrary unions.

(3) $\mathscr{O}$ is closed under finite intersections.

The members of $\mathscr{O}$ are said to be *open*. The *interior* of a subset $Y \subseteq X$ is the union of all open sets contained in Y; we denote it by $\text{int}(Y)$.

Proposition 13.15. *(i)* $\text{int}(\emptyset) = \emptyset$.
(ii) $\text{int}(X) = X$.
(iii) $\text{int}(Y) \subseteq Y$.
(iv) $\text{int}(Y \cap Z) = \text{int}(Y) \cap \text{int}(Z)$.
(v) $\text{int}(\text{int}(Y)) = \text{int}(Y)$.
(vi) $\text{int}(Y) = \{x \in X : x \in U \subseteq Y \text{ for some open set } U\}$.

Proof. (i)–(iii), (v), and (vi) are obvious. For (iv), if U is an open set contained in $Y \cap Z$, then it is contained in Y; so $\text{int}(Y \cap Z) \subseteq \text{int}(Y)$. Similarly for Z, so $\subseteq$ holds. For $\supseteq$, note that the right side is an open set contained in $Y \cap Z$. (v) holds since $\text{int}(Y)$ is open. □

A subset C of X is *closed* iff $X \backslash C$ is open.

Proposition 13.16. *(i) $\emptyset$ and X are closed.*
(ii) The collection of all closed sets is closed under finite unions and intersections of any nonempty subcollection. □

For any $Y \subseteq X$, the *closure* of Y, denoted by $\text{cl}(Y)$, is the intersection of all closed sets containing Y.

Proposition 13.17. *(i)* $\text{cl}(Y) = X \backslash \text{int}(X \backslash Y)$.
(ii) $\text{int}(Y) = X \backslash \text{cl}(X \backslash Y)$.
(iii) $\text{cl}(\emptyset) = \emptyset$.
(iv) $\text{cl}(X) = X$.
(v) $Y \subseteq \text{cl}(Y)$.
(vi) $\text{cl}(Y \cup Z) = \text{cl}(Y) \cup \text{cl}(Z)$.
(vii) $\text{cl}(\text{cl}(Y)) = \text{cl}(Y)$.
(viii) $\text{cl}(Y) = \{x \in X : \text{for every open set } U, \text{ if } x \in U \text{ then } U \cap Y \neq \emptyset\}$.

Proof. (i): $\text{int}(X \backslash Y)$ is an open set contained in $X \backslash Y$, so Y is a subset of the closed set $X \backslash \text{int}(X \backslash Y)$. Hence $\text{cl}(Y) \subseteq X \backslash \text{int}(X \backslash Y)$. Also. $\text{cl}(Y)$ is a closed set containing

Y, so $X\backslash\text{cl}(Y)$ is an open set contained in $X\backslash Y$. Hence $X\backslash\text{cl}(Y) \subseteq \text{int}(X\backslash Y)$. Hence $X\backslash\text{int}(X\backslash Y \subseteq \text{cl}(Y)$. This proves (i).

(ii): Using (i),

$$X\backslash\text{cl}(X\backslash Y) = X\backslash(X\backslash\text{int}(X\backslash(X\backslash Y))) = \text{int}(Y).$$

(iii)–(v): clear.

(vi):

$$\begin{aligned}
\text{cl}(Y \cup Z) &= X\backslash\text{int}(X\backslash(Y \cup Z)) \quad \text{by (i)} \\
&= X\backslash\text{int}((X\backslash Y) \cap (X\backslash Z)) \\
&= X\backslash(\text{int}(X\backslash Y) \cap \text{int}(X\backslash Z)) \quad \text{by 13.15(iv)} \\
&= [X\backslash\text{int}(X\backslash Y)] \cup [X\backslash\text{int}(X\backslash Z)] \\
&= \text{cl}(Y) \cup \text{cl}(Z).
\end{aligned}$$

(vii):

$$\begin{aligned}
\text{cl}(\text{cl}(Y)) &= \text{cl}(X\backslash\text{int}(X\backslash Y)) \\
&= X\backslash\text{int}(X\backslash(X\backslash\text{int}(X\backslash Y))) \\
&= X\backslash\text{int}(\text{int}(X\backslash Y)) \\
&= X\backslash\text{int}(X\backslash Y) \\
&= \text{cl}(Y).
\end{aligned}$$

(vii): First suppose that $x \in \text{cl}(Y)$, and $x \in U$, U open. By (i) and Proposition 13.15(vi) we have $U \not\subseteq X\backslash Y$, i.e., $U \cap Y \neq \emptyset$, as desired. Second, suppose that $x \notin \text{cl}(Y)$. Then by (i) and 13.15(vi) there is an open U such that $x \in U \subseteq X\backslash Y$; so $U \cap Y = \emptyset$, as desired. □

Now we go beyond this minimum amount of topology and work with the notion of a regular open set, which is not a standard part of topology courses.

We say that Y is *regular open* iff $Y = \text{int}(\text{cl}(Y))$.

Proposition 13.18. *(i) If Y is open, then $Y \subseteq \text{int}(\text{cl}(Y))$.*

(ii) If U and V are regular open, then so is $U \cap V$.

(iii) $\text{int}(\text{cl}(Y))$ is regular open.

(iv) If U is open, then $\text{int}(\text{cl}(U))$ is the smallest regular open set containing U.

(v) If U is open then $U \cap \text{cl}(Y) \subseteq \text{cl}(U \cap Y)$.

(vi) If U is open, then $U \cap \text{int}(\text{cl}(Y)) \subseteq \text{int}(\text{cl}(U \cap Y))$.

(vii) If U and V are open and $U \cap V = \emptyset$, then $\text{int}(\text{cl}(U)) \cap V = \emptyset$.

(viii) If U and V are open and $U \cap V = \emptyset$, then $\text{int}(\text{cl}(U)) \cap \text{int}(\text{cl}(V)) = \emptyset$.

(ix) For any set M of regular open sets, $\text{int}(\text{cl}(\bigcup M)$ is the least regular open set containing each member of M.

Proof. (i): $Y \subseteq \text{cl}(Y)$, and hence $Y = \text{int}(Y) \subseteq \text{int}(\text{cl}(Y))$.

(ii): $U \cap V$ is open, and so $U \cap V \subseteq \operatorname{int}(\operatorname{cl}(U \cap V))$. For the other inclusion, $\operatorname{int}(\operatorname{cl}(U \cap V)) \subseteq \operatorname{int}(\operatorname{cl}(U)) = U$, and similarly for V, so the other inclusion holds.

(iii): $\operatorname{int}(\operatorname{cl}(X)) \subseteq \operatorname{cl}(X)$, so $\operatorname{cl}(\operatorname{int}(\operatorname{cl}(X))) \subseteq \operatorname{cl}(\operatorname{cl}(X)) = \operatorname{cl}(X)$; hence

$$\operatorname{int}(\operatorname{cl}(\operatorname{int}(\operatorname{cl}(X)))) \subseteq \operatorname{int}(\operatorname{cl}(X));$$

the other inclusion is clear.

(iv): By (iii), $\operatorname{int}(\operatorname{cl}(U))$ is a regular open set containing U. If V is any regular open set containing U, then $\operatorname{int}(\operatorname{cl}(U)) \subseteq \operatorname{int}(\operatorname{cl}(V)) = V$.

(v):

$$\begin{aligned}
U \cap (X \backslash (U \cap Y)) &\subseteq X \backslash Y, \quad \text{hence} \\
U \cap \operatorname{int}(X \backslash (U \cap Y)) &= \operatorname{int}(U) \cap \operatorname{int}(X \backslash (U \cap Y)) \\
&= \operatorname{int}(U \cap (X \backslash (U \cap Y))) \\
&\subseteq \operatorname{int}(X \backslash Y), \quad \text{hence} \\
X \backslash \operatorname{int}(X \backslash Y) &\subseteq X \backslash (U \cap \operatorname{int}(X \backslash (U \cap Y))) \\
&= (X \backslash U) \cup (X \backslash \operatorname{int}(X \backslash (U \cap Y))), \quad \text{hence} \\
U \cap (X \backslash \operatorname{int}(X \backslash Y)) &\subseteq (X \backslash \operatorname{int}(X \backslash (U \cap Y))),
\end{aligned}$$

and (v) follows.

(vi):

$$\begin{aligned}
U \cap \operatorname{int}(\operatorname{cl}(Y)) &= \operatorname{int}(U) \cap \operatorname{int}(\operatorname{cl}(Y)) \\
&= \operatorname{int}(U \cap \operatorname{cl}(Y)) \\
&\subseteq \operatorname{int}(\operatorname{cl}(U \cap Y)) \quad \text{by (v).}
\end{aligned}$$

(vii): $U \subseteq X \backslash V$, hence $\operatorname{cl}(U) \subseteq \operatorname{cl}(X \backslash V) = X \backslash V$, hence $\operatorname{cl}(U) \cap V = \emptyset$, and the conclusion of (vii) follows.

(viii): Apply (vii) twice.

(ix): If $U \in M$, then $U \subseteq \bigcup M \subseteq \operatorname{int}(\operatorname{cl}(\bigcup M)$. Suppose that V is regular open and $U \subseteq V$ for all $U \in M$. Then $\bigcup M \subseteq V$, and so $\operatorname{int}(\operatorname{cl}(\bigcup M)) \subseteq \operatorname{int}(\operatorname{cl}(V) = V$. □

We let $\mathrm{RO}(X)$ be the collection of all regular open sets in X. We define operations on $\mathrm{RO}(X)$ which will make it a Boolean algebra. For any $Y, Z \in \mathrm{RO}(X)$, let

$$\begin{aligned}
Y + Z &= \operatorname{int}(\operatorname{cl}(Y \cup Z)); \\
Y \cdot Z &= Y \cap Z; \\
-Y &= \operatorname{int}(X \backslash Y).
\end{aligned}$$

Theorem 13.19. *The structure*

$$\langle \mathrm{RO}(X), +, \cdot, -, \emptyset, X \rangle$$

is a complete BA. Moreover, the ordering $\leq$ coincides with $\subseteq$.

Proof. $\mathrm{RO}(X)$ is closed under $+$ by Proposition 13.18(ix), and is closed under $\cdot$ by Proposition 13.18(ii). Clearly it is closed under $-$, and $\emptyset, X \in \mathrm{RO}(X)$. Now we check the axioms. The following are completely obvious: (A′), (C′), (C). Now let unexplained variables range over $\mathrm{RO}(X)$. For (A), note by 13.18(i) that $U \subseteq U+V \subseteq (U+V)+W$; and similarly $V \subseteq (U+V)+W$ and $W \subseteq U+V \subseteq (U+V)+W$. If $U, V, W \subseteq Z$, then by 13.18(iv), $U+V \subseteq Z$ and hence $(U+V)+W \subseteq Z$. Thus $(U+V)+W$ is the least upper bound in $\mathrm{RO}(X)$ of U, V, W. This is true for all U, V, W. So $U+(V+W) = (V+W)+U$ is also the least upper bound of them; so (A) holds. For (L):

$$U + U \cdot V = \mathrm{int}(\mathrm{cl}(U \cup (U \cap V))) = \mathrm{int}(\mathrm{cl}(U)) = U.$$

(L′) holds by 13.18(i). For (D), first note that

$$\begin{aligned} Y \cdot (Z+W) &= Y \cap \mathrm{int}(\mathrm{cl}(Z \cup W)) \\ &\subseteq \mathrm{int}(\mathrm{cl}(Y \cap (Z \cup W))) \quad \text{by 13.18(vi)} \\ &= \mathrm{int}(\mathrm{cl}((Y \cap Z) \cup (Y \cap W))) \\ &= Y \cdot Z + Y \cdot W. \end{aligned}$$

On the other hand, $(Y \cap Z) \cup (Y \cap W) = Y \cap (Z \cup W) \subseteq Y, Z \cup W$, and hence easily

$$\begin{aligned} Y \cdot Z + Y \cdot W &= \mathrm{int}(\mathrm{cl}((Y \cap Z) \cup (Y \cap W))) \\ &\subseteq \mathrm{int}(\mathrm{cl}(Y) = Y \quad \text{and} \\ Y \cdot Z + Y \cdot W &= \mathrm{int}(\mathrm{cl}((Y \cap Z) \cup (Y \cap W))) \\ &\subseteq \mathrm{int}(\mathrm{cl}(Z \cup W) = Z + W; \end{aligned}$$

so the other inclusion follows, and (D) holds.

(K): For any regular open Y, from Proposition 13.17(ii) we get $-Y = \mathrm{int}(X \backslash Y) = X \backslash \mathrm{cl}(X \backslash (X \backslash Y)) = X \backslash \mathrm{cl}(Y)$. Hence

$$X = \mathrm{cl}(Y) \cup (X \backslash \mathrm{cl}(Y)) \subseteq \mathrm{cl}(Y) \cup \mathrm{cl}((X \backslash \mathrm{cl}(Y)) = \mathrm{cl}(Y \cup (X \backslash \mathrm{cl}(Y))),$$

and hence $X = Y + -Y$.

(K′): Clearly $\emptyset = Y \cap \mathrm{int}(X \backslash Y) = Y \cdot -Y$.

Thus we have now proved that $\langle \mathrm{RO}(X), +, \cdot, -, \emptyset, X \rangle$ is a BA. Since $\cdot$ is the same as $\cap$, $\leq$ is the same as $\subseteq$. Hence by Proposition 13.18(ix), $\langle \mathrm{RO}(X), +, \cdot, -, \emptyset, X \rangle$ is a complete BA. □

Now we return to our task of embedding a forcing order into a complete Boolean algebra. Let P be a given forcing order. For each $p \in P$ let $P \downarrow p = \{q : q \leq p\}$. Now we define

$$\mathscr{O}_P = \{X \subseteq P : (P \downarrow p) \subseteq X \text{ for every } p \in X\}.$$

We check that this gives a topology on P. Clearly $P, \emptyset \in \mathscr{O}$. To show that $\mathscr{O}$ is closed under arbitrary unions, suppose that $\mathscr{X} \subseteq \mathscr{O}$. Take any $p \in \bigcup \mathscr{X}$. Choose $X \in \mathscr{X}$ such that $p \in X$. Then $(P \downarrow p) \subseteq X \subseteq \bigcup \mathscr{X}$, as desired. If $X, Y \in \mathscr{O}_P$, suppose that $p \in X \cap Y$. Then $p \in X$, so $(P \downarrow p) \subseteq X$. Similarly $(P \downarrow p) \subseteq Y$, so $(P \downarrow p) \subseteq X \cap Y$. Thus $X \cap Y \in \mathscr{O}_P$, finishing the proof that $\mathscr{O}_P$ is a topology on P.

We denote the complete BA of regular open sets in this topology by $\mathrm{RO}(P)$.

Now for any $p \in P$ we define

$$e(p) = \mathrm{int}(\mathrm{cl}(P \downarrow p)).$$

Thus e maps P into $\mathrm{RO}(P)$.

This is our desired embedding. Actually it is not really an embedding in general, but it has several useful properties, and for many forcing orders it really is an embedding.

The useful properties mentioned are as follows. We say that a subset X of P is *dense below* p iff for every $r \leq p$ there is a $q \leq r$ such that $q \in X$.

Theorem 13.20. *Let P be a forcing order. Suppose that $p, q \in P$, F is a finite subset of P, $a, b \in \mathrm{RO}(P)$, and N is a subset of $\mathrm{RO}(P)$*

(i) $e[P]$ is dense in $\mathrm{RO}(P)$, i.e., for any nonzero $Y \in \mathrm{RO}(P)$ there is a $p \in P$ such that $e(p) \subseteq Y$.

(ii) If $p \leq q$ then $e(p) \subseteq e(q)$.

(iii) $p \perp q$ iff $e(p) \cap e(q) = \emptyset$.

(iv) If $e(p) \leq e(q)$, then p and q are compatible.

(v) The following conditions are equivalent:

(a) $e(p) \leq e(q)$.

(b) $\{r : r \leq p, q\}$ is dense below p.

(vi) The following conditions are equivalent, for F nonempty:

(a) $e(p) \leq \prod_{q \in F} e(q)$.

(b) $\{r : r \leq q$ for all $q \in F\}$ is dense below p.

(vii) The following conditions are equivalent:

(a) $e(p) \leq (\prod_{q \in F} e(q)) \cdot \sum N$.

(b) $\{r : r \leq q$ for all $q \in F$ and $e(r) \leq s$ for some $s \in N\}$ is dense below p.

(viii) $e(p) \leq -a$ iff there is no $q \leq p$ such that $e(q) \leq a$.

(ix) $e(p) \leq -a + b$ iff for all $q \leq p$, if $e(q) \leq a$ then $e(q) \leq b$.

Proof. (i): Assume the hypothesis. By the definition of the topology and since Y is nonempty and open, there is a $p \in P$ such that $P \downarrow p \subseteq Y$. Hence $e(p) = \mathrm{int}(\mathrm{cl}(P \downarrow p)) \subseteq \mathrm{int}(\mathrm{cl}(Y)) = Y$.

(ii): If $p \leq q$, then $P \downarrow p \subseteq P \downarrow q$, and so $e(p) = \mathrm{int}(\mathrm{cl}(P \downarrow p)) \subseteq \mathrm{int}(\mathrm{cl}(P \downarrow q) = e(q))$.

(iii): Assume that $p \perp q$. Then $(P \downarrow p) \cap (P \downarrow q) = \emptyset$, and hence by Proposition 13.18(viii), $e(p) \cap e(q) = \emptyset$.

Conversely, suppose that $e(p) \cap e(q) = \emptyset$. Then $(P \downarrow p) \cap (P \downarrow q) \subseteq e(p) \cap e(q) = \emptyset$, and so $p \perp q$.

(iv): If $e(p) \leq e(q)$, then $e(p) \cdot e(q) = e(p) \neq \emptyset$, so p and q are compatible by (iii).

(v): For (a)$\Rightarrow$(b), suppose that $e(p) \leq e(q)$ and $s \leq p$. Then $e(s) \leq e(p) \leq e(q)$, so s and q are compatible by (iv); say $r \leq s, q$. Then $r \leq s \leq p$, hence $r \leq p, q$, as desired.

For (b)⇒(a), suppose that $e(p) \not\leq e(q)$. Thus $e(p) \cdot -e(q) \neq 0$. Hence there is an s such that $e(s) \subseteq e(p) \cdot -e(q)$. Hence $e(s) \cdot e(q) = \emptyset$, so $s \perp q$ by (iii). Now $e(s) \subseteq e(p)$, so s and p are compatible by (iv); say $t \leq s, p$. For any $r \leq t$ we have $r \leq s$, and hence $r \perp q$. So (b) fails.

(vi): We proceed by induction on $|F|$. The case $|F| = 1$ is given by (v). Now assume the result for F, and suppose that $t \in P \backslash F$. First suppose that $e(p) \leq \prod_{q \in F} e(q) \cdot e(t)$. Suppose that $s \leq p$. Now $e(p) \leq \prod_{q \in F} e(q)$, so by the inductive hypothesis there is a $u \leq s$ such that $u \leq q$ for all $q \in F$. Thus $e(u) \leq e(s) \leq e(p) \leq e(t)$, so by (iv), u and t are compatible. Take any $v \leq u, t$. then $v \leq q$ for any $q \in F \cup \{t\}$, as desired.

Second, suppose that (b) holds for $F \cup \{t\}$. In particular, $\{r : r \leq q \text{ for all } q \in F\}$ is dense below p, and so $e(p) \leq \prod_{q \in F} e(q)$ by the inductive hypothesis. But also clearly $\{r : r \leq t\}$ is dense below p, so $e(p) \leq e(t)$ too, as desired.

(vii): First assume that $e(p) \leq (\prod_{q \in F} e(q)) \cdot \sum N$, and suppose that $u \leq p$. By (vi), there is a $v \leq u$ such that $v \leq q$ for each $q \in F$. Now $e(v) \leq e(u) \leq e(p) \leq \sum N$, so $0 \neq e(v) = e(v) \cdot \sum N = \sum_{s \in N}(e(v) \cdot e(s))$. Hence there is an $s \in N$ such that $e(v) \cdot e(s) \neq 0$. Hence by (iii), v and s are compatible; say $r \leq v, s$. Clearly r is in the set described in (b).

Second, suppose that (b) holds. Clearly then $\{r : r \leq q \text{ for all } q \in F\}$ is dense below p, and so $e(p) \leq \prod_{q \in F} e(q)$ by (vi). Now suppose that $e(p) \not\leq \sum N$. Then $e(p) \cdot -\sum N \neq 0$, so there is a q such that $e(q) \leq e(p) \cdot -\sum N$. By (iv), q and p are compatible; say $s \leq p, q$. Then by (b) choose $r \leq s$ and $t \in N$ such that $e(r) \leq t$. Thus $e(r) \leq e(s) \cdot t \leq e(p) \cdot t \leq (-\sum N) \cdot \sum N = 0$, contradiction.

(viii)⇒: Assume that $e(p) \leq -a$. Suppose that $q \leq p$ and $e(q) \leq a$. Then $e(q) \leq -a \cdot a = 0$, contradiction.

(viii)⇐: Assume that $e(p) \not\leq -a$. Then $e(p) \cdot a \neq 0$, so there is a q such that $e(q) \leq e(p) \cdot a$. By (vii) there is an $r \leq p, q$ with $e(r) \leq a$, as desired.

(ix)⇒: Assume that $e(p) \leq -a + b$, $q \leq p$, and $e(q) \leq a$. Then $e(q) \leq a \cdot (-a + b) \leq b$, as desired.

(ix)⇐: Assume the indicated condition, but suppose that $e(p) \not\leq -a + b$. Then $e(p) \cdot a \cdot -b \neq 0$, so there is a q such that $e(q) \leq e(p) \cdot a \cdot -b$. By (vii) with $F = \{p\}$ and $N = \{a \cdot -b\}$ we get q such that $q \leq p$ and $e(q) \leq a \cdot -b$. So $q \leq p$ and $e(q) \leq a$, so by our condition, $e(q) \leq b$. But also $e(q) \leq -b$, contradiction. □

We now expand on the remarks above concerning when e really is an embedding. Note that if P is a simple ordering, then the closure of $P \downarrow p$ is P itself, and hence P has only two regular open subsets, namely the empty set and P itself. If the ordering on P is trivial, meaning that no two elements are comparable, then every subset of P is regular open.

An important condition satisfied by many forcing orders is defined as follows. We say that P is *separative* iff it is a partial order (thus is an antisymmetric forcing order), and for any $p, q \in P$, if $p \not\leq q$ then there is an $r \leq p$ such that $r \perp q$.

Proposition 13.21. *Let P be a forcing order.*

(i) $\mathrm{cl}(P \downarrow p) = \{q : p \textit{ and } q \textit{ are compatible}\}$.

(ii) $e(p) = \{q : \textit{for all } r \leq q,\ r \textit{ and } p \textit{ are compatible}\}$.

(iii) The following conditions are equivalent:

(a) P is separative.

(b) e is one-one, and for all $p, q \in P$, $p \le q$ iff $e(p) \le e(q)$.

Proof. (i) and (ii) are clear. For (iii), (a)⇒(b), assume that P is separative. Take any $p, q \in P$. If $p \le q$, then $e(p) \le e(q)$ by 13.20(ii). Suppose that $p \not\le q$. Choose $r \le p$ such that $r \perp q$. Then $r \in e(p)$, while $r \notin e(q)$ by (ii). Thus $e(p) \not\le e(q)$.

Now suppose that $e(p) = e(q)$. Then $p \le q \le p$ by what was just shown, so $p = q$ since P is a partial order.

For (iii), (b)⇒(a), suppose that $p \le q \le p$. Then $e(p) \subseteq e(q) \subseteq e(p)$, so $e(p) = e(q)$, and hence $p = q$. So P is a partial order. Suppose that $p \not\le q$. Then $e(p) \not\subseteq e(q)$. Choose $s \in e(p) \backslash e(q)$. Since $s \notin e(q)$, by (ii) we can choose $t \le s$ such that $t \perp q$. Since $s \in e(p)$, it follows that t and p are compatible; choose $r \le t, p$. Clearly $r \perp q$. □

Now we prove a theorem which says that the regular open algebra of a forcing order is unique up to isomorphism.

Theorem 13.22. *Let P be a forcing order, A a complete BA, and j a function mapping P into $A \backslash \{0\}$ with the following properties:*

(i) $j[P]$ is dense in A, i.e., for any nonzero $a \in A$ there is a $p \in P$ such that $j(p) \subseteq a$.

(ii) For all $p, q \in P$, if $p \le q$ then $j(p) \le j(q)$.

(iii) For any $p, q \in P$, $p \perp q$ iff $j(p) \cdot j(q) = 0$.

Then there is a unique isomorphism f from $\mathrm{RO}(P)$ onto A such that $f \circ e = j$. That is, f is a bijection from $\mathrm{RO}(P)$ onto A, and for any $x, y \in \mathrm{RO}(P)$, $x \subseteq y$ iff $f(x) \le f(y)$; and $f \circ e = j$.

Note that since the Boolean operations are easily expressible in terms of $\le$ (as least upper bounds, etc.), the condition here implies that f preserves all of the Boolean operations too; this includes the infinite sums and products.

Proof. Before beginning the proof, we introduce some notation in order to make the situation more symmetric. Let $B_0 = \mathrm{RO}(P)$, $B_1 = A$, $k_0 = e$, and $k_1 = j$. Then for each $m < 2$ the following conditions hold:

(1) $k_m[P]$ is dense in B_m.

(2) For all $p, q \in P$, if $p \le q$ then $k_m(p) \le k_m(q)$.

(3) For all $p, q \in P$, $p \perp q$ iff $k_m(p) \cdot k_m(q) = 0$.

(4) For all $p, q \in P$, if $k_m(p) \le k_m(q)$, then p and q are compatible.

In fact, (1)–(3) follow from 13.20 and the assumptions of the theorem. Condition (4) for $m = 0$, so that $k_m = e$, follows from 13.20(iv). For $m = 1$, so that $k_m = j$, it follows easily from (iii).

Now we begin the proof. For each $m < 2$ we define, for any $x \in B_m$,

$$g_m(x) = \sum \{k_{1-m}(p) : p \in P,\ k_m(p) \le x\}.$$

The proof of the theorem now consists in checking the following, for each $m \in 2$:

(5) If $x, y \in B_m$ and $x \leq y$, then $g_m(x) \leq g_m(y)$.

(6) $g_{1-m} \circ g_m$ is the identity on B_m.

(7) $g_0 \circ k_0 = k_1$.

In fact, suppose that (5)–(7) have been proved. If $x, y \in \mathrm{RO}(P)$, then

$x \leq y$ implies that $g_0(x) \leq g_0(y)$ by (5);
$g_0(x) \leq g_0(y)$ implies that $x = g_1(g_0(x)) \leq g_1(g_0(y)) = y$ by (5) and (6).

Also, (6) holding for both $m = 0$ and $m = 1$ implies that g_0 is a bijection from $\mathrm{RO}(P)$ onto A. Moreover, by (7), $g_0 \circ e = g_0 \circ k_0 = k_1 = j$. So g_0 is the desired function f of the theorem.

Now (5) is obvious from the definition. To prove (6), assume that $m \in 2$. We first prove

(8) For any $p \in P$ and any $b \in B_m$, $k_m(p) \leq b$ iff $k_{1-m}(p) \leq g_m(b)$.

To prove (8), first suppose that $k_m(p) \leq b$. Then obviously $k_{1-m}(p) \leq g_m(b)$. Second, suppose that $k_{1-m}(p) \leq g_m(b)$ but $k_m(p) \not\leq b$. Thus $k_m(p) \cdot -b \neq 0$, so by the denseness of $k_m[P]$ in B_m, choose $q \in P$ such that $k_m(q) \leq k_m(p) \cdot -b$. Then p and q are compatible by (4), so let $r \in P$ be such that $r \leq p, q$. Hence

$$k_{1-m}(r) \leq k_{1-m}(p) \leq g_m(b) = \sum\{k_{1-m}(s) : s \in P,\ k_m(s) \leq b\}.$$

Hence $k_{1-m}(r) = \sum\{k_{1-m}(s) \cdot k_{1-m}(r) : s \in P,\ k_m(s) \leq b\}$, so there is an $s \in P$ such that $k_m(s) \leq b$ and $k_{1-m}(s) \cdot k_{1-m}(r) \neq 0$. Hence s and r are compatible; say $t \leq s, r$. Hence $k_m(t) \leq k_m(r) \leq k_m(q) \leq -b$, but also $k_m(t) \leq k_m(s) \leq b$, contradiction. This proves (8).

Now take any $b \in B_m$. Then

$$\begin{aligned} g_{1-m}(g_m(b)) &= \sum\{k_m(p) : p \in P,\ k_{1-m}(p) \leq g_m(b)\} \\ &= \sum\{k_m(p) : p \in P,\ k_m(p) \leq b\} \\ &= b. \end{aligned}$$

Thus (6) holds.

For (7), clearly $k_1(p) \leq g_0(k_0(p))$. Now suppose that $k_0(q) \leq k_0(p)$ but $k_1(q) \not\leq k_1(p)$. Then $k_1(q) \cdot -k_1(p) \neq 0$, so there is an r such that $k_1(r) \leq k_1(q) \cdot -k_1(p)$. Hence q and r are compatible, but $r \perp p$. Say $s \leq q, r$. Then $k_0(s) \leq k_0(q) \leq k_0(p)$, so s and p are compatible. Say $t \leq s, p$. Then $t \leq r, p$, contradiction. This proves (7).

This proves the existence of f. Now suppose that g is also an isomorphism from $\mathrm{RO}(P)$ onto A such that $g \circ e = j$, but suppose that $f \neq g$. Then there is an $X \in \mathrm{RO}(P)$ such that $f(X) \neq g(X)$. By symmetry, say that $f(X) \cdot -g(X) \neq 0$. By (ii), choose $p \in P$ such that $j(p) \leq f(X) \cdot -g(X)$. So $f(e(p)) = j(p) \leq f(X)$, so $e(p) \leq X$, and hence $j(p) = g(e(p)) \leq g(X)$. This contradicts $j(p) \leq -g(X)$. $\square$

E13.1. Let $(A, +, \cdot, -, 0, 1)$ be a Boolean algebra. Show that $(A, \triangle, \cdot, 0, 1)$ is a ring with identity in which every element is idempotent. This means that $x \cdot x = x$ for all x.

E13.2. Let $(A, +, \cdot, 0, 1)$ be a ring with identity in which every element is idempotent. Show that A is a commutative ring, and $(A, \oplus, \cdot, -, 0, 1)$ is a Boolean algebra, where for any $x, y \in A$, $x \oplus y = x + y + xy$ and for any $x \in A$, $-x = 1 + x$. Hint: expand $(x + y)^2$.

E13.3. Show that the processes described in exercises E13.1 and E13.2 are inverses of one another.

E13.4. A *filter* in a BA A is a subset F of A with the following properties:

(1) $1 \in F$.

(2) If $a \in F$ and $a \leq b$, then $b \in F$.

(3) If $a, b \in F$, then $a \cdot b \in F$.

An *ultrafilter* in A is a filter F such that $0 \notin F$, and for any $a \in A$, $a \in F$ or $-a \in F$.

Prove that a filter F is an ultrafilter iff F is maximal among the set of all filters G such that $0 \notin G$.

E13.5. (Continuing exercise E13.4) Prove that for any nonzero $a \in A$ there is an ultrafilter F such that $a \in F$.

E13.6. (Continuing exercise E13.4) Prove that any BA is isomorphic to a field of sets. (Stone's representation theorem) Hint: given a BA A, let X be the set of all ultrafilters on A and define $f(a) = \{F \in X : a \in F\}$.

E13.7 (Continuing exercise E13.4) Suppose that F is an ultrafilter on a BA A. Let 2 be the two-element BA. (This is, up to isomorphism, the BA of all subsets of 1.) For any $a \in A$ let

$$f(a) = \begin{cases} 1 & \text{if } a \in F, \\ 0 & \text{if } a \notin F. \end{cases}$$

Show that f is a homomorphism of A into 2. This means that for any $a, b \in A$, the following conditions hold:

$f(a + b) = f(a) + f(b)$;

$f(a \cdot b) = f(a) \cdot f(b)$;

$f(-a) = -f(a)$;

$f(0) = 0$;

$f(1) = 1$.

E13.8. (Lindenbaum-Tarski algebras; A knowledge of logic is assumed.) Suppose that $\mathscr{L}$ is a first-order language and T is a set of sentences of $\mathscr{L}$. Define $\varphi \equiv_T \psi$ iff φ and ψ are sentences of $\mathscr{L}$ and $T \models \varphi \leftrightarrow \psi$. Show that this is an equivalence relation on the set S of all sentences of $\mathscr{L}$. Let A be the collection of all equivalence classes under this equivalence relation. Show that there are operations $+$, $\cdot$, $-$ on A such that for any sentences φ, ψ,

$$\begin{aligned}[\varphi] + [\psi] &= [\varphi \vee \psi];\\ [\varphi] \cdot [\psi] &= [\varphi \wedge \psi];\\ -[\varphi] &= [\neg\varphi].\end{aligned}$$

Finally, show that $(A, +, \cdot, -, [\exists v_0(\neg(v_0 = v_0))], [\exists v_0(v_0 = v_0)])$ is a Boolean algebra.

E13.13. (A knowledge of logic is assumed.) Show that every Boolean algebra is isomorphic to one obtained as in exercise E13.8. Hint: Let A be a Boolean algebra. Let $\mathscr{L}$ be the first-order language which has a unary relation symbol R_a for each $a \in A$. Let T be the following set of sentences of $\mathscr{L}$:

$$\begin{aligned}&\forall x \forall y(x = y);\\ &\forall x[R_{-a}(x) \leftrightarrow \neg R_a(x)] \quad \text{for each } a \in A;\\ &\forall x[R_{a\cdot b}(x) \leftrightarrow R_a(x) \wedge R_b(x)] \quad \text{for all } a, b \in A;\\ &\forall x R_1(x).\end{aligned}$$

E13.10. Let A be the collection of all subsets X of $Y \stackrel{\text{def}}{=} \{r \in \mathbb{Q} : 0 \leq r\}$ such that there exist an $m \in \omega$ and $a, b \in {}^m(Y \cup \{\infty\})$ such that $a_0 < b_0 < a_1 < b_1 < \cdots < a_{m-1} < b_{m-1} \leq \infty$ and

$$X = [a_0, b_0) \cup [a_1, b_1) \cup \ldots \cup [a_{m-1}, b_{m-1}).$$

Note that $\emptyset \in A$ by taking $m = 0$, and $Y \in A$ since $Y = [0, \infty)$.

(i) Show that if X is as above, $c, d \in Y \cup \{\infty\}$ with $c < d$, $c \leq a_0$, then $X \cup [c, d) \in A$, and c is the first element of $X \cup [c, d)$.

(ii) Show that if X is as above and $c, d \in Y \cup \{\infty\}$ with $c < d$, then $X \cup [c, d) \in A$.

(iii) Show that $(A, \cup, \cap, \backslash, \emptyset, Y)$ is a Boolean algebra.

E13.11. (Continuing exercise E13.10.) For each $n \in \omega$ let $x_n = [n, n + 1)$, an interval in $\mathbb{Q}$. Show that $\sum_{n \in \omega} x_{2n}$ does not exist in A.

E13.12. Let A be the Boolean algebra of all subsets of some nonempty set X, under the natural set-theoretic operations. Show that if $\langle a_i : i \in I\rangle$ is a system of elements of A, then

$$\prod_{i \in I}(a_i + -a_i) = 1 = \sum_{\varepsilon \in {}^I 2} \prod_{i \in I} a_i^{\varepsilon(i)},$$

where for any y, $y^1 = y$ and $y^0 = -y$.

E13.13. Let M be the set of all finite functions $f \subseteq \omega \times 2$. For each $f \in M$ let

$$U_f = \{g \in {}^\omega 2 : f \subseteq g\}.$$

Let A consist of all finite unions of sets U_f.

(i) Show that A is a Boolean algebra under the set-theoretic operations.

(ii) For each $i \in \omega$, let $x_i = U_{\{(i,1)\}}$. Show that

$$^{\omega}2 = \prod_{i\in\omega}(x_i + -x_i)$$

while

$$\sum_{\varepsilon\in{}^{\omega}2}\prod_{i\in\omega} x_i^{\varepsilon(i)} = \emptyset,$$

where for any y, $y^1 = y$ and $y^0 = -y$.

This is an example of an infinite distributive law that holds in some BAs (by exercise E13.12), but does not hold in all BAs.

E13.14. Suppose that $(P, \leq, 1)$ is a forcing order. Define

$$p \equiv q \quad \text{iff} \quad p, q \in P,\ p \leq q, \text{ and } q \leq p.$$

Show that $\equiv$ is an equivalence relation, and if Q is the collection of all $\equiv$-classes, then there is a relation $\preceq$ on Q such that for all $p, q \in P$, $[p]_\equiv \preceq [q]_\equiv$ iff $p \leq q$. Finally, show that $(Q, \preceq)$ is a partial order, i.e., $\preceq$ is reflexive on Q, transitive, and antisymmetric ($q_1 \preceq q_2 \preceq q_1$ implies that $q_1 = q_2$); moreover, $q \leq [1]$ for all $q \in Q$.

E13.15. We say that $(P, <)$ is a *partial order in the second sense* iff $<$ is transitive and irreflexive. (Irreflexive means that for all $p \in P$, $p \not< p$.) Show that if $(P, <)$ is a partial order in the second sense and if we define $\preceq$ by $p \preceq q$ iff ($p, q \in P$ and $p < q$ or $p = q$), then $\mathscr{A}(P, <) \stackrel{\text{def}}{=} (P, \preceq)$ is a partial order. Furthermore, show that if $(P, \leq)$ is a partial order, and we define $p \prec q$ by $p \prec q$ iff ($p, q \in P$, $p \leq q$, and $p \neq q$), then $\mathscr{B}(P, \leq) \stackrel{\text{def}}{=} (P, \prec)$ is a partial order in the second sense.

Also prove that $\mathscr{A}$ and $\mathscr{B}$ are inverses of one another.

E13.16. Show that if $(P, \leq, 1)$ is a forcing order and we define $\prec$ by $p \prec q$ iff ($p, q \in P$, $p \leq q$ and $q \not\leq p$), then $(P, \prec)$ is a partial order in the second sense. Give an example where this partial order is not isomorphic to the one derived from $(P, \leq, 1)$ by the procedure of exercise E13.14.

E13.17. Prove that if $(P, \preceq, 1)$ is a forcing order such that the mapping e from P into $\mathrm{RO}(P)$ is one-one, then $(P, \preceq)$ is a partial order. Give an example of a forcing order such that e is not one-one. Give an example of an infinite forcing order Q such that e is not one-one, while for any $p, q \in Q$, $p \leq q$ iff $e(p) \subseteq e(q)$.

E13.18. (Continuing E13.14.) Let $\mathbb{P} = (P, \leq, 1)$ be a forcing order, and let $\mathbb{Q} = (Q, \preceq, [1])$ be as in exercise E13.14. Show that there is an isomorphism f of $\mathrm{RO}(\mathbb{P})$ onto $\mathrm{RO}(\mathbb{Q})$ such that $f \circ e_{\mathbb{P}} = e_{\mathbb{Q}} \circ \pi$, where $\pi : P \to Q$ is defined by $\pi(p) = [p]$ for all $p \in P$.

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14. Models of set theory

In this chapter we describe the basics concerning models of set theory, ending with a proof that if ZFC is consistent, then so is ZFC+"there are no uncountable inaccessibles".

A *set theory structure* is an ordered pair $\overline{A} = (A, R)$ such that A is a non-empty set and R is a binary relation contained in $A \times A$. The model-theoretic notions introduced in Chapter 2 can be applied here.

A notion similar to that of a model is *relativization.* Suppose that $\mathbf{M}$ is a class. We associate with each formula φ its *relativization to* $\mathbf{M}$, denoted by $\varphi^{\mathbf{M}}$. The definition goes by recursion on formulas:

$(x = y)^{\mathbf{M}}$ is $x = y$
$(x \in y)^{\mathbf{M}}$ is $x \in y$
$(\varphi \to \psi)^{\mathbf{M}}$ is $\varphi^{\mathbf{M}} \to \psi^{\mathbf{M}}$.
$(\neg\varphi)^{\mathbf{M}}$ is $\neg\varphi^{\mathbf{M}}$.
$(\forall x\varphi)^{\mathbf{M}}$ is $\forall x[x \in \mathbf{M} \to \varphi^{\mathbf{M}}]$.

The more rigorous version of this definition associates with each pair ψ, φ of formulas a third formula which is called the relativization of φ to ψ.

We say that φ *holds in* $\mathbf{M}$ or *is true in* $\mathbf{M}$, iff $\varphi^{\mathbf{M}}$ holds, i.e., iff ZFC $\vdash \varphi^{\mathbf{M}}$.

Theorem 14.1. *Let Γ be a set of sentences, φ a sentence, and $\mathbf{M}$ a class. Let $\Gamma^{\mathbf{M}} = \{\chi^{\mathbf{M}} : \chi \in \Gamma\}$. Suppose that $\Gamma \models \varphi$. Then*

$$\Gamma^{\mathbf{M}} \models \mathbf{M} \neq \emptyset \to \varphi^{\mathbf{M}}.$$

Proof. Assume the hypothesis of the theorem, let $\overline{A} = (A, E)$ be any set theory structure, assume that $\overline{A}$ is a model of $\Gamma^{\mathbf{M}}$, and suppose that $A \cap \mathbf{M} \neq \emptyset$. We want to show that $\overline{A}$ is a model of $\varphi^{\mathbf{M}}$. To do this, we define another structure $\overline{B} = (B, F)$ for our language. Let $B = A \cap \mathbf{M}$, and let $F = E \cap (B \times B)$. Now we claim:

(*) For any formula χ and any $c \in {}^{\omega}B$, $\overline{A} \models \chi^{\mathbf{M}}[c]$ iff $\overline{B} \models \chi[c]$.

We prove (*) by induction on χ:

$$\begin{aligned}
\overline{A} \models (v_i = v_j)^{\mathbf{M}}[c] \quad &\text{iff} \quad c_i = c_j \\
&\text{iff} \quad \overline{B} \models (v_i = v_j)[c]; \\
\overline{A} \models (v_i \in v_j)^{\mathbf{M}}[c] \quad &\text{iff} \quad c_i E c_j \\
&\text{iff} \quad c_i F c_j \\
&\text{iff} \quad \overline{B} \models (v_i \in v_j)[c]; \\
\overline{A} \models (\neg\chi)^{\mathbf{M}}[c] \quad &\text{iff} \quad \text{not}[\overline{A} \models \chi^{\mathbf{M}}[c]] \\
&\text{iff} \quad \text{not}[\overline{B} \models \chi[c]] \quad \text{(induction hypothesis)} \\
&\text{iff} \quad \overline{B} \models \neg\chi[c]; \\
\overline{A} \models (\chi \to \theta)^{\mathbf{M}}[c] \quad &\text{iff} \quad [\overline{A} \models \chi^{\mathbf{M}}[c] \text{ implies that } \overline{A} \models \theta^{\mathbf{M}}[c]] \\
&\text{iff} \quad [\overline{B} \models \chi[c] \text{ implies that } \overline{B} \models \theta[c] \\
&\qquad\qquad \text{(induction hypothesis)} \\
&\text{iff} \quad \overline{B} \models (\chi \to \theta)[c].
\end{aligned}$$

We do the quantifier step in each direction separately. First suppose that $\overline{A} \models (\forall v_i \chi)^{\mathbf{M}}[c]$. Thus $\overline{A} \models [\forall v_i[v_i \in \mathbf{M} \to \chi^{\mathbf{M}}][c]$. Take any $b \in B$. Then $b \in \mathbf{M}$, so $\overline{A} \models \chi^{\mathbf{M}}[c^i_b]$. By the inductive hypothesis, $\overline{B} \models \chi[c^i_b]$. This proves that $\overline{B} \models \forall v_i \chi[c]$.

Conversely, suppose that $\overline{B} \models \forall v_i \chi[c]$. Suppose that $a \in A$ and $\overline{A} \models (v_i \in \mathbf{M})[c^i_a]$. Then $a \in B$, so $\overline{B} \models \chi[c^i_a]$. By the inductive hypothesis, $\overline{A} \models \chi^{\mathbf{M}}[c^i_a]$. So we have shown that $\overline{A} \models \forall v_i[v_i \in \mathbf{M} \to \chi^M][c]$. That is, $\overline{A} \models (\forall v_i \chi)^{\mathbf{M}}[c]$.

This finishes the proof of (*).

Now $\overline{A}$ is a model of $\Gamma^{\mathbf{M}}$, so by (*), $\overline{B}$ is a model of Γ. Hence by assumption, $\overline{B}$ is a model of φ. So by (*) again, $\overline{A}$ is a model of $\varphi^{\mathbf{M}}$. $\square$

This theorem gives the basic idea of consistency proofs in set theory; we express this as follows. Remember by the completeness theorem that a set Γ of sentences is consistent iff it has a model.

Corollary 14.2. *Suppose that Γ and Δ are collections of sentences in our language of set theory. Suppose that $\mathbf{M}$ is a class, and $\Gamma \models [\mathbf{M} \neq \emptyset$ and $\varphi^{\mathbf{M}}]$ for each $\varphi \in \Delta$. Then Γ consistent implies that Δ is consistent.*

Proof. Suppose to the contrary that Δ does not have a model. Then trivially $\Delta \models \neg(x = x)$. By Theorem 14.1, $\Delta^{\mathbf{M}} \models \mathbf{M} \neq \emptyset \to \neg(x = x)$. Hence by hypothesis we get $\Gamma \models \neg(x = x)$, contradiction. $\square$

Later in this section we use this corollary with $\Gamma =$ ZFC and $\Delta =$ ZFC+"there are no inaccessibles"; the class $\mathbf{M}$ is more complicated to describe, and we defer that until we are actually ready to give the applications of the corollary.

The set-theoretical hierarchy

The hierarchy of sets is defined recursively as follows:

Theorem 14.3. *There is a class function $V : \mathbf{On} \to \mathbf{V}$ satisfying the following conditions:*

(i) $V_0 = \emptyset$.

(ii) $V_{\alpha+1} = \mathscr{P}(V_\alpha)$.

(iii) $V_\gamma = \bigcup_{\alpha<\gamma} V_\alpha$ for γ limit.

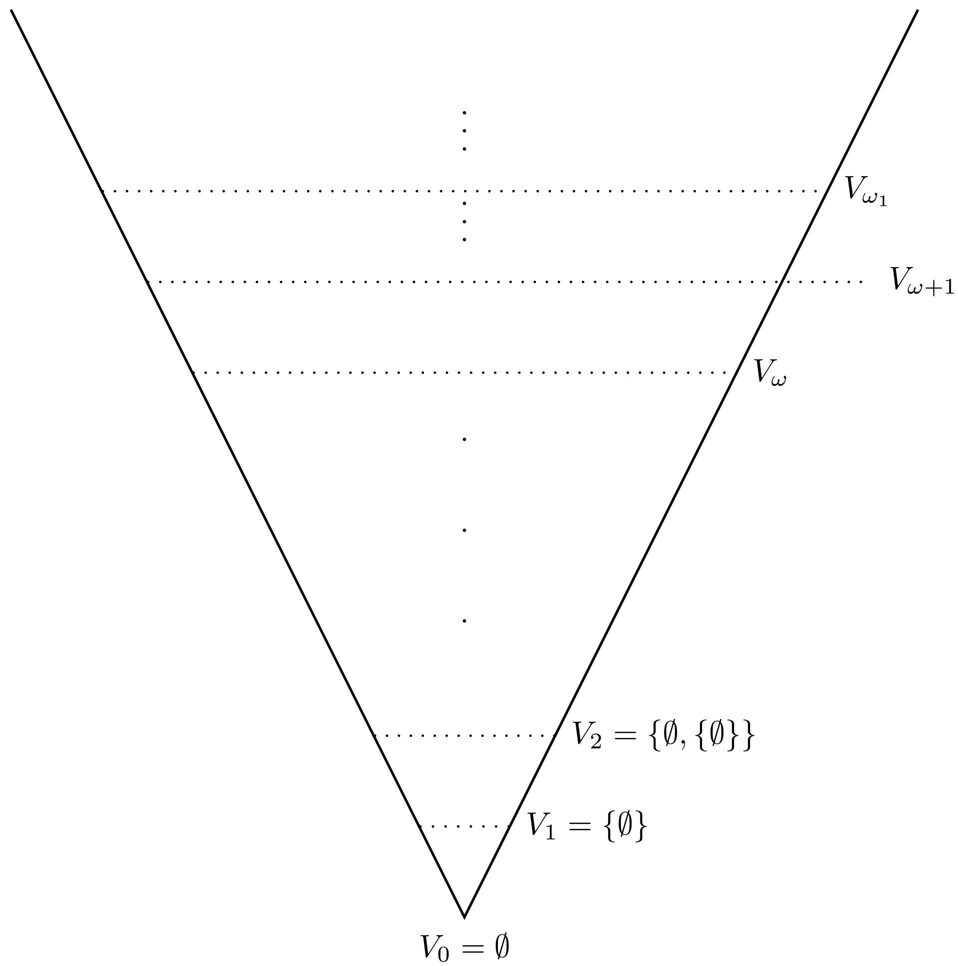
Proof. We apply Theorem 9.7. Define $\mathbf{G} : \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ as follows. For any ordinal α and set x, let

$$\mathbf{G}(\alpha, x) = \begin{cases} \emptyset & \text{if } x = \emptyset, \\ \mathscr{P}(x(\beta)) & \text{if } x \text{ is a function with domain } \alpha = \beta + 1, \\ \bigcup_{\beta<\alpha} x(\beta) & \text{if } x \text{ is a function with domain } \alpha, \text{ and } \alpha \text{ is a limit ordinal}, \\ \emptyset & \text{otherwise.} \end{cases}$$

So we apply Theorem 9.7 to obtain a class function $\mathbf{F} : \mathbf{On} \to \mathbf{V}$ such that for every ordinal α, $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha)$. Hence

$$\begin{aligned} \mathbf{F}(0) &= \mathbf{G}(0, \mathbf{F} \restriction 0) = \mathbf{G}(0, \emptyset) = \emptyset; \\ \mathbf{F}(\beta + 1) &= \mathbf{G}(\beta + 1, \mathbf{F} \restriction \beta + 1) = \mathscr{P}(\mathbf{F}(\beta)); \\ \mathbf{F}(\alpha) &= \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha) = \bigcup_{\beta<\alpha} \mathbf{F}(\beta) \quad \text{for } \alpha \text{ limit.} \end{aligned}$$

$\square$



Recall from chapter 7 the notion of a transitive set. We have used this notion only for defining ordinals so far. But the general notion will now play an important role in what follows.

Theorem 14.4. *For every ordinal α the following hold:*

(i) V_α is transitive.

(ii) $V_\beta \subseteq V_\alpha$ for all $\beta < \alpha$.

Proof. We prove these statements simultaneously by induction on α. They are clear for $\alpha = 0$. Assume that both statements hold for α; we prove them for $\alpha + 1$. First we prove

(1) $V_\alpha \subseteq V_{\alpha+1}$.

In fact, suppose that $x \in V_\alpha$. By (i) for α, the set V_α is transitive. Hence $x \subseteq V_\alpha$, so $x \in \mathscr{P}(V_\alpha) = V_{\alpha+1}$. So (1) holds.

Now (ii) follows. For, suppose that $\beta < \alpha + 1$. Then $\beta \leq \alpha$, so $V_\beta \subseteq V_\alpha$ by (ii) for α (or trivially if $\beta = \alpha$). Hence by (1), $V_\beta \subseteq V_{\alpha+1}$.

To prove (i) for $\alpha+1$, suppose that $x \in y \in V_{\alpha+1}$. Then $y \in \mathscr{P}(V_\alpha)$, so $y \subseteq V_\alpha$, hence $x \in V_\alpha$. By (1), $x \in V_{\alpha+1}$, as desired.

For the final inductive step, suppose that γ is a limit ordinal and (i) and (ii) hold for all $\alpha < \gamma$. To prove (i) for γ, suppose that $x \in y \in V_\gamma$. Then by definition of V_γ, there is an $\alpha < \gamma$ such that $y \in V_\alpha$. By (i) for α we get $x \in V_\alpha$. So $x \in V_\gamma$ by the definition of V_γ. Condition (ii) for γ is obvious. $\square$

A very important fact about this hierarchy is that every set is a member of some V_α. To prove this, we need the notion of transitive closure. We introduced and used this notion in Chapter 8, but we will prove the following independent of this.

Theorem 14.5. *For any set a there is a transitive set b with the following properties:*

(i) $a \subseteq b$.

(ii) For every transitive set c such that $a \subseteq c$ we have $b \subseteq c$.

Proof. We first make a definition by recursion. Define $\mathbf{G} : \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ by setting, for an $\alpha \in \mathbf{On}$ and any $x \in \mathbf{V}$

$$\mathbf{G}(\alpha, x) = \begin{cases} a & \text{if } x = \emptyset, \\ x(m) \cup \bigcup x(m) & \text{if } x \text{ is a function with domain } m+1 \text{ with } m \in \omega, . \\ 0 & \text{otherwise} \end{cases}$$

By Theorem 9.7 let $\mathbf{F} : \mathbf{On} \to \mathbf{V}$ be such that $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha)$ for any $\alpha \in \mathbf{On}$. Let $d = \mathbf{F} \restriction \omega$. Then $d_0 = \mathbf{F}(0) = \mathbf{G}(0, \mathbf{F} \restriction 0) = \mathbf{G}(0, \emptyset) = a$. For any $m \in \omega$ we have $d_{m+1} = \mathbf{F}(m+1) = \mathbf{G}(m+1, \mathbf{F} \restriction (m+1)) = \mathbf{F}(m) \cup \bigcup \mathbf{F}(m) = d_m \cup \bigcup d(m)$. Let $b = \bigcup_{m \in \omega} d_m$. Then $a = d_0 \subseteq b$. Suppose that $x \in y \in b$. Choose $m \in \omega$ such that $y \in d_m$. Then $x \in \bigcup d_m \subseteq d_{m+1} \subseteq b$. Thus b is transitive. Now suppose that c is a transitive set such that $a \subseteq c$. We show by induction that $d_m \subseteq c$ for every $m \in \omega$. First, $d_0 = a \subseteq c$, so this is true for $m = 0$. Now assume that it is true for m. Then $d_{m+1} = d_m \cup \bigcup d_m \subseteq c \cup \bigcup c = c$, completing the inductive proof.

Hence $b = \bigcup_{m \in \omega} d_m \subseteq c$. $\square$

The set shown to exist in Theorem 14.5 is called the *transitive closure* of a, and is denoted by $\mathrm{trcl}(a)$.

Theorem 14.6. *Every set is a member of some V_α.*

Proof. Suppose that this is not true, and let a be a set which is not a member of any V_α. Let $A = \{x \in \mathrm{trcl}(a \cup \{a\}) : x \text{ is not in any of the sets } V_\alpha\}$. Then $a \in A$, so A is nonempty. By the foundation axiom, choose $x \in A$ such that $x \cap A = 0$. Suppose that $y \in x$. Then $y \in \mathrm{trcl}(a \cup \{a\})$, so y is a member of some V_α. Let α_y be the least such α. Let $\gamma = \bigcup_{y \in x} \alpha_y$. Then by 13.1(ii), $x \subseteq V_\beta$. So $x \in V_{\beta+1}$, contradiction. $\square$

Thus by Theorem 14.6 we have $\mathbf{V} = \bigcup_{\alpha \in \mathbf{On}} V_\alpha$. An important technical consequence of Theorem 14.6 is the following definition, known as *Scott's trick*:

• Let R be a class equivalence relation on a class A. For each $a \in A$, let α be the smallest ordinal such that there is a $b \in V_\alpha$ with $(a, b) \in R$, and define

$$\mathrm{type}_R(a) = \{b \in V_\alpha : (a, b) \in R\}.$$

This is the "reduced" equivalence class of a. It could be that the collection of b such that $(a, b) \in R$ is a proper class, but $\mathrm{type}_R(a)$ is always a set.

On the basis of our hierarchy we can define the important notion of *rank* of sets:

- For any set x, the rank of x, denoted by $\mathrm{rank}(x)$, is the smallest ordinal α such that $x \in V_{\alpha+1}$.

We take $\alpha + 1$ here instead of α just for technical reasons. Some of the most important properties of ranks are given in the following theorem.

Theorem 14.7. *Let x be a set and α an ordinal. Then*
(i) $V_\alpha = \{y : \mathrm{rank}(y) < \alpha\}$.
(ii) For all $y \in x$ we have $\mathrm{rank}(y) < \mathrm{rank}(x)$.
(iii) $\mathrm{rank}(y) \le \mathrm{rank}(x)$ *for every* $y \subseteq x$.
(iv) $\mathrm{rank}(x) = \sup_{y \in x}(\mathrm{rank}(y) + 1)$.
(v) $\mathrm{rank}(\alpha) = \alpha$.
(vi) $V_\alpha \cap \mathbf{On} = \alpha$.

Proof. (i): Suppose that $y \in V_\alpha$. Then $\alpha \neq 0$. If α is a successor ordinal $\beta + 1$, then $\mathrm{rank}(y) \le \beta < \alpha$. If α is a limit ordinal, then $y \in V_\beta$ for some $\beta < \alpha$, hence $y \in V_{\beta+1}$ also, so $\mathrm{rank}(y) \le \beta < \alpha$. This proves $\subseteq$.

For $\supseteq$, suppose that $\beta \stackrel{\text{def}}{=} \mathrm{rank}(y) < \alpha$. Then $y \in V_{\beta+1} \subseteq V_\alpha$, as desired.

(ii): Suppose that $x \in y$. Let $\mathrm{rank}(y) = \alpha$. Thus $y \in V_{\alpha+1} = \mathscr{P}(V_\alpha)$, so $y \subseteq V_\alpha$ and hence $x \in V_\alpha$. Then by (i), $\mathrm{rank}(x) < \alpha$.

(iii): Let $\mathrm{rank}(x) = \alpha$. Then $x \in V_{\alpha+1}$, so $x \subseteq V_\alpha$. Let $y \subseteq x$. Then $y \subseteq V_\alpha$, and so $y \in V_{\alpha+1}$. Thus $\mathrm{rank}(y) \le \alpha$.

(iv): Let α be the indicated sup. Then $\ge$ holds by (ii). Now if $y \in x$, then $\mathrm{rank}(y) < \alpha$, and hence $y \in V_{\mathrm{rank}(y)+1} \subseteq V_\alpha$. This shows that $x \subseteq V_\alpha$, hence $x \in V_{\alpha+1}$, hence $\mathrm{rank}(x) \le \alpha$, finishing the proof of (iv).

(v): We prove this by transfinite induction. Suppose that it is true for all $\beta < \alpha$. Then by (iv),

$$\mathrm{rank}(\alpha) = \sup_{\beta<\alpha}(\mathrm{rank}(\beta) + 1) = \sup_{\beta<\alpha}(\beta + 1) = \alpha.$$

Finally, for (vi), using (i) and (v),

$$V_\alpha \cap \mathbf{On} = \{\beta \in \mathbf{On} : \beta \in V_\alpha\} = \{\beta \in \mathbf{On} : \mathrm{rank}(\beta) < \alpha\} = \{\beta \in \mathbf{On} : \beta < \alpha\} = \alpha. \quad \square$$

We now define a sequence of cardinals by recursion:

Theorem 14.8. *There is a function $\beth : \mathbf{On} \to \mathbf{V}$ such that the following conditions hold:*

$$\begin{aligned} \beth_0 &= \omega; \\ \beth_{\alpha+1} &= 2^{\beth_\alpha}; \\ \beth_\gamma &= \bigcup_{\alpha<\gamma} \beth_\alpha \quad \textit{for } \gamma \textit{ limit.} \end{aligned}$$

Proof. Define $\mathbf{G} : \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ by setting, for any ordinal α and any set x,

$$\mathbf{G}(\alpha, x) = \begin{cases} \omega & \text{if } x = \emptyset, \\ 2^{x(\beta)} & \text{if } x \text{ is a function with domain } \alpha = \beta + 1 \text{ and} \\ & \text{range a set of cardinals,} \\ \bigcup_{\beta<\alpha} x(\beta) & \text{if } x \text{ is a function with domain a limit ordinal } \alpha. \end{cases}$$

Then we obtain $\mathbf{F} : \mathbf{On} \to \mathbf{V}$ by Theorem 6.7: for any ordinal α, $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha)$. Hence

$$\begin{aligned} \mathbf{F}(0) &= \mathbf{G}(0, \mathbf{F} \restriction 0) = \mathbf{G}(0, \emptyset) = \omega; \\ \mathbf{F}(\beta + 1) &= \mathbf{G}(\beta + 1, \mathbf{F} \restriction (\beta + 1)) = 2^{\mathbf{F}(\beta)}; \\ \mathbf{F}(\alpha) &= \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha) = \bigcup_{\beta<\alpha} \mathbf{F}(\beta) \quad \text{for } \alpha \text{ limit.} \end{aligned}$$

□

Thus under GCH, $\aleph_\alpha = \beth_\alpha$ for every ordinal α; in fact, this is just a reformulation of GCH.

Theorem 14.9. *(i)* $n \le |V_n| \in \omega$ *for any* $n \in \omega$.
(ii) For any ordinal α, $|V_{\omega+\alpha}| = \beth_\alpha$.

Proof. (i) is clear by ordinary induction on n. We prove (ii) by the three-step transfinite induction (where γ is a limit ordinal below):

$$\begin{aligned} |V_\omega| &= \left| \bigcup_{n\in\omega} V_n \right| = \omega = \beth_0 \quad \text{by (i);} \\ |V_{\omega+\alpha+1}| &= |\mathscr{P}(V_{\omega+\alpha})| \\ &= 2^{|V_{\omega+\alpha}|} \\ &= 2^{\beth_\alpha} \quad \text{(inductive hypothesis)} \\ &= \beth_{\alpha+1}; \\ |V_{\omega+\gamma}| &= \left| \bigcup_{\beta<\gamma} V_{\omega+\beta} \right| \\ &\le \sum_{\beta<\gamma} |V_{\omega+\beta}| \\ &= \sum_{\beta<\gamma} \beth_\beta \quad \text{(inductive hypothesis)} \\ &\le \sum_{\beta<\gamma} \beth_\gamma \\ &= |\gamma| \cdot \beth_\gamma \\ &= \beth_\gamma \quad . \end{aligned}$$

To finish this last inductive step, note that for each $\beta < \gamma$ we have $\beth_\beta = |V_{\omega+\beta}| \leq |V_{\omega+\gamma}|$, and hence $\beth_\gamma \leq |V_{\omega+\gamma}|$. $\square$

Checking the axioms

Now we give some simple facts which will be useful in checking the axioms of ZFC in the transitive classes which we will define. See Chapter 5 for the original form of the axioms.

Theorem 14.10. *The extensionality axiom holds in any nonempty transitive class.*

Proof. Let $\mathbf{M}$ be any transitive class. The relativized version of the extensionality axiom is

$$\forall x \in \mathbf{M} \forall y \in \mathbf{M}[\forall z \in \mathbf{M}(z \in x \leftrightarrow z \in y) \rightarrow x = y].$$

To prove this, assume that $x, y \in \mathbf{M}$, and suppose that for all $z \in \mathbf{M}$, $z \in x$ iff $z \in y$. Take any $z \in x$. Because $\mathbf{M}$ is transitive, we get $z \in \mathbf{M}$. Hence $z \in y$. Thus $z \in x$ implies that $z \in y$. The converse is similar. So $x = y$. $\square$

The following theorem reduces checking the comprehension axioms to checking a closure property.

Theorem 14.11. *Suppose that* $\mathbf{M}$ *is a nonempty class, and for each formula* φ *with with free variables among* $x, z, w_1, \ldots, w_n$*,*

$$\forall z, w_1, \ldots, w_n \in \mathbf{M}[\{x \in z : \varphi^{\mathbf{M}}(x, z, w_1, \ldots, w_n)\} \in \mathbf{M}].$$

Then the comprehension axioms hold in $\mathbf{M}$*.*

Proof. The straightforward relativization of an instance of the comprehension axioms is

$$\forall z \in \mathbf{M} \forall w_1 \in \mathbf{M} \ldots \forall w_n \in \mathbf{M} \exists y \in \mathbf{M} \forall x \in \mathbf{M}(x \in y \leftrightarrow x \in z \wedge \varphi^{\mathbf{M}}).$$

So, we take $z, w_1, \ldots, w_n \in \mathbf{M}$. Let

$$y = \{x \in z : \varphi^{\mathbf{M}}(x, z, w_1, \ldots, w_n)\};$$

by hypothesis, we have $y \in \mathbf{M}$. Then for any $x \in \mathbf{M}$,

$$x \in y \quad \text{iff} \quad x \in z \text{ and } \varphi^{\mathbf{M}}(x, z, w_1, \ldots, w_n). \qquad \square$$

The following theorems are obvious from the forms of the pairing axiom and union axioms:

Theorem 14.12. *Suppose that* $\mathbf{M}$ *is a nonempty class and*

$$\forall x, y \in \mathbf{M} \exists z \in \mathbf{M}(x \in z \textit{ and } y \in z).$$

Then the pairing axiom holds in $\mathbf{M}$*.* $\square$

Theorem 14.13. *Suppose that* $\mathbf{M}$ *is a nonempty class and*

$$\forall x \in \mathbf{M} \exists z \in \mathbf{M} \left(\bigcup x \subseteq z \right).$$

Then the union axiom holds in $\mathbf{M}$. □

For the next result, recall that $z \subseteq x$ is an abbreviation for $\forall w(w \in z \rightarrow w \in x)$.

Theorem 14.14. *Suppose that* $\mathbf{M}$ *is a nonempty transitive class. Then the following are equivalent:*

(i) The power set axiom holds in $\mathbf{M}$.

(ii) For every $x \in \mathbf{M}$ *there is a* $y \in \mathbf{M}$ *such that* $\mathscr{P}(x) \cap \mathbf{M} \subseteq y$.

Proof. (i)⇒(ii): Assume (i). Thus

$$\forall x \in \mathbf{M} \exists y \in \mathbf{M} \forall z \in \mathbf{M} [\forall w \in \mathbf{M}(w \in z \rightarrow w \in x) \rightarrow z \in y]. \tag{1}$$

To prove (ii), take any $x \in \mathbf{M}$. Choose $y \in \mathbf{M}$ as in (1). Suppose that $z \in \mathscr{P}(x) \cap \mathbf{M}$. Clearly then $\forall w \in \mathbf{M}(w \in z \rightarrow w \in x)$, so by (1), $z \in y$, as desired in (ii).

(ii)⇒(i): Assume (ii). This time we want to prove (1). So, suppose that $x \in \mathbf{M}$. Choose $y \in \mathbf{M}$ as in (ii). Now suppose that $z \in \mathbf{M}$ and $\forall w \in \mathbf{M}(w \in z \rightarrow w \in x)$. Then the transitivity of $\mathbf{M}$ implies that $\forall w(w \in z \rightarrow w \in x)$, i.e., $z \subseteq x$. So by (ii), $z \in y$, as desired. □

We defer the discussion of the infinity axiom until we talk about absoluteness.

Theorem 14.15. *Suppose that* $\mathbf{M}$ *is a transitive class, and for every formula* φ *with free variables among* $x, y, A, w_1, \ldots, w_n$ *and for any* $A, w_1, \ldots, w_n \in \mathbf{M}$ *the following implication holds:*

$$\begin{aligned}&\forall x \in A \exists! y[y \in \mathbf{M} \wedge \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)] \quad \textit{implies that}\\ &\exists Y \in \mathbf{M}[\{y \in \mathbf{M} : \exists x \in A \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)\} \subseteq Y]].\end{aligned}$$

Then the replacement axioms hold in $\mathbf{M}$.

Proof. Assume the hypothesis of the theorem. We write out the relativized version of an instance of the replacement axiom in full, remembering to replace the quantifier $\exists!$ by its definition:

$$\begin{aligned}&\forall A \in \mathbf{M} \forall w_1 \in \mathbf{M} \ldots \forall w_n \in \mathbf{M}\\ &[\forall x \in \mathbf{M}[x \in A \rightarrow \exists y \in \mathbf{M}[\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \wedge \forall u \in \mathbf{M}\\ &\qquad [\varphi^{\mathbf{M}}(x, u, A, w_1, \ldots, w_n) \rightarrow y = u]]] \rightarrow\\ &\exists Y \in \mathbf{M} \forall x \in \mathbf{M}[x \in A \rightarrow \exists y \in \mathbf{M}[y \in Y \wedge \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)]]]].\end{aligned}$$

To prove this, assume that $A, w_1, \ldots, w_n \in \mathbf{M}$ and

$$\begin{aligned}&\forall x \in \mathbf{M}[x \in A \rightarrow \exists y \in \mathbf{M}[\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \wedge \forall u \in \mathbf{M}\\ &\qquad [\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \rightarrow y = u]]].\end{aligned}$$

Since $\mathbf{M}$ is transitive, we get

$$\forall x \in A \exists y \in \mathbf{M}[\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \wedge \forall u \in \mathbf{M}[\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \rightarrow y = u]],$$

so that

$$\forall x \in A \exists! y[y \in \mathbf{M} \wedge \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)]. \tag{1}$$

Hence by the hypothesis of the theorem we get $Y \in \mathbf{M}$ such that

$$\{y \in \mathbf{M} : \exists x \in A \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)\} \subseteq Y. \tag{2}$$

Suppose that $x \in \mathbf{M}$ and $x \in A$. By (1) we get $y \in \mathbf{M}$ such that $\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)$. Hence by (2) we get $y \in Y$, as desired. $\square$

Theorem 14.16. *If* $\mathbf{M}$ *is a transitive class, then the foundation axiom holds in* $\mathbf{M}$.

Proof. The foundation axiom, with the defined notion $\emptyset$ eliminated, is

$$\forall x[\exists y(y \in x) \rightarrow \exists y[y \in x \wedge \forall z \in y(z \notin x)]].$$

Hence the relativized version is

$$\forall x \in \mathbf{M}[\exists y \in \mathbf{M}(y \in x) \rightarrow \exists y \in \mathbf{M}[y \in x \wedge \forall z \in \mathbf{M}[z \in y \rightarrow z \notin x]]].$$

So, take any $x \in \mathbf{M}$, and suppose that there is a $y \in \mathbf{M}$ such that $y \in x$. Choose $y \in x$ so that $y \cap x = \emptyset$. Then $y \in \mathbf{M}$ by transitivity. If $z \in \mathbf{M}$ and $z \in y$, then $z \notin x$, as desired. $\square$

Absoluteness

To treat the infinity axiom and more complicated statements, we need to go into the important notion of absoluteness. Roughly speaking, a formula is absolute provided that its meaning does not change in going from one set to a bigger one, or vice versa. The exact definition is as follows.

• Suppose that $\mathbf{M} \subseteq \mathbf{N}$ are classes and $\varphi(x_1, \ldots, x_n)$ is a formula of our set-theoretical language. We say that φ is *absolute for* $\mathbf{M}, \mathbf{N}$ iff

$$\forall x_1, \ldots, x_n \in \mathbf{M}[\varphi^{\mathbf{M}}(x_1, \ldots, x_n) \text{ iff } \varphi^{\mathbf{N}}(x_1, \ldots, x_n)].$$

An important special case of this notion occurs when $\mathbf{N} = \mathbf{V}$. Then we just say that φ is *absolute for* $\mathbf{M}$.

More formally, we associate with three formulas $\mu(y, w_1, \ldots, w_m)$, $\nu(y, w_1, \ldots, w_m)$, $\varphi(x_1, \ldots, x_n)$ another formula "φ is absolute for μ, ν", namely the following formula:

$$\forall x_1, \ldots, x_n \left[\bigwedge_{1 \le i \le n} \mu(x_i) \rightarrow [\varphi^{\mu}(x_1, \ldots, x_n) \leftrightarrow \varphi^{\nu}(x_1, \ldots, x_n)] \right].$$

In full generality, very few formulas are absolute; for example, see Exercise E14.15. Usually we need to assume that the sets are transitive. Then there is an important set of formulas all of which are absolute; this class is defined as follows.

- The set of Δ_0-formulas is the smallest set Γ of formulas satisfying the following conditions:
  (a) Each atomic formula is in Γ.
  (b) If φ and ψ are in Γ, then so are $\neg\varphi$ and $\varphi \wedge \psi$.
  (c) If φ is in Γ, then so are $\exists x \in y\varphi$ and $\forall x \in y\varphi$.

Recall here that $\exists x \in y\varphi$ and $\forall x \in y\varphi$ are abbreviations for $\exists x(x \in y \wedge \varphi)$ and $\forall x(x \in y \to \varphi)$ respectively.

Theorem 14.17. *If* $\mathbf{M}$ *is transitive and* φ *is* Δ_0*, then* φ *is absolute for* $\mathbf{M}$*.*

Proof. We show that the collection of formulas absolute for $\mathbf{M}$ satisfies the conditions defining the set Δ_0. Absoluteness is clear for atomic formulas. It is also clear that if φ and ψ are absolute for $\mathbf{M}$, then so are $\neg\varphi$ and $\varphi \wedge \psi$. Now suppose that φ is absolute for $\mathbf{M}$; we show that $\exists x \in y\varphi$ is absolute for $\mathbf{M}$. Implicitly, φ can involve additional parameters $w_1, \ldots, w_n$. Assume that $y, w_1, \ldots, w_n \in \mathbf{M}$. First suppose that $\exists x \in y\varphi(x, y, w_1, \ldots, w_n)$. Choose $x \in y$ so that $\varphi(x, y, w_1, \ldots, w_n)$. Since $\mathbf{M}$ is transitive, $x \in \mathbf{M}$. Hence by the "inductive assumption", $\varphi^{\mathbf{M}}(x, y, w_1, \ldots, w_n)$ holds. This shows that $(\exists x \in y\varphi(x, y, w_1, \ldots, w_n))^{\mathbf{M}}$. Conversely suppose that $(\exists x \in y\varphi(x, y, w_1, \ldots, w_n))^{\mathbf{M}}$. Thus $\exists x \in \mathbf{M}[x \in y \wedge \varphi^{\mathbf{M}}(x, y, w_1, \ldots, w_n)$. By the inductive assumption, $\varphi(x, y, w_1, \ldots, w_n)$. So this shows that $\exists x \in y\varphi(x, y, w_1, \ldots, w_n)$. The case $\forall x \in y\varphi$ is treated similarly. $\square$

Ordinals and special kinds of ordinals are absolute since they could have been defined using Δ_0 formulas:

Theorem 14.18. *The following are absolute for any transitive class:*

(i) x *is an ordinal*	*(iii)* x *is a successor ordinal*	*(v)* x *is* ω
(ii) x *is a limit ordinal*	*(iv)* x *is a finite ordinal*	*(vi)* x *is* i *(each* $i < 10$*)*

Proof.

$$\begin{aligned}
x \text{ is an ordinal} &\leftrightarrow \forall y \in x \forall z \in y[z \in x] \wedge \forall y \in x \forall z \in y \forall w \in z[w \in y];\\
x \text{ is a limit ordinal} &\leftrightarrow \exists y \in x[y = y] \wedge x \text{ is an ordinal} \wedge \forall y \in x \exists z \in x(y \in z);\\
x \text{ is a successor ordinal} &\leftrightarrow x \text{ is an ordinal} \wedge x \neq \emptyset \wedge x \text{ is not a limit ordinal};\\
x \text{ is a finite ordinal} &\leftrightarrow \forall y[y \notin x] \vee (x \text{ is a successor ordinal}\\
&\qquad \wedge \forall y \in x(\forall z[z \notin y] \vee y \text{ is a successor ordinal}));\\
x = \omega &\leftrightarrow x \text{ is a limit ordinal} \wedge \forall y \in x(y \text{ is a finite ordinal});
\end{aligned}$$

finally, we do (vi) by induction on i. The case $i = 0$ is clear. Then

$$y = i + 1 \leftrightarrow \exists x \in y[x = i \wedge \forall z \in y[z \in x \vee z = x] \wedge \forall z \in x[z \in y] \wedge x \in y]. \qquad \square$$

The following theorem, while obvious, will be very useful in what follows.

Theorem 14.19. *Suppose that S is a set of sentences in our set-theoretic language, and $\mathbf{M}$ and $\mathbf{N}$ are classes which are models of S. Suppose that*

$$S \models \forall x_1, \ldots, x_n[\varphi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n)].$$

Then φ is absolute for $\mathbf{M}, \mathbf{N}$ iff ψ is. □

Of course we will usually apply this when S is a subset of ZFC.

Recall that all of the many definitions that we have made in our development of set theory are supposed to be eliminable in favor of our original language. To apply Theorem 14.19, we should note that the development of the very elementary set theory in Chapter 6 did not use the axiom of choice or the axiom of infinity. We let ZF be our axioms without the axiom of choice, and ZF − Inf the axioms ZF without the axiom of infinity.

The status of the functions that we have defined requires some explanation. Whenever we defined a function $\mathbf{F}$ of n arguments, we have implicitly assumed that there is an associated formula φ whose free variables are among the first $n+1$ variables, so that the following is derivable from the axioms assumed at the time of defining the function:

$$\forall v_0, \ldots, v_{n-1} \exists! v_n \varphi(v_0, \ldots, v_n).$$

Recall that "$\exists! v_n$" means "there is exactly one v_n". Now if we have a class model $\mathbf{M}$ in which this sentence holds, then we can define $\mathbf{F}^{\mathbf{M}}$ by setting, for any $x_0, \ldots, x_{n-1} \in \mathbf{M}$,

$$\mathbf{F}^{\mathbf{M}}(x_0, \ldots, x_{n-1}) = \text{ the unique } y \text{ such that } \varphi^{\mathbf{M}}(x_0, \ldots, x_{n-1}, y).$$

In case $\mathbf{M}$ satisfies the indicated sentence, we say that $\mathbf{F}$ *is defined in* $\mathbf{M}$. Given two class models $\mathbf{M} \subseteq \mathbf{N}$ in which $\mathbf{F}$ is defined, we say that $\mathbf{F}$ is *absolute for* $\mathbf{M}, \mathbf{N}$ provide that φ is. Note that for $\mathbf{F}$ to be absolute for $\mathbf{M}, \mathbf{N}$ it must be defined in both of them.

Proposition 14.20. *Suppose that $\mathbf{M} \subseteq \mathbf{N}$ are models in which $\mathbf{F}$ is defined. Then the following are equivalent:*

(i) $\mathbf{F}$ is absolute for $\mathbf{M}, \mathbf{N}$.

(ii) For all $x_0, \ldots, x_{n-1} \in \mathbf{M}$ we have $\mathbf{F}^{\mathbf{M}}(x_0, \ldots, x_{n-1}) = \mathbf{F}^{\mathbf{N}}(x_0, \ldots, x_{n-1})$.

Proof. Let φ be as above.

Assume (i), and suppose that $x_0, \ldots, x_{n-1} \in \mathbf{M}$. Let $y = \mathbf{F}^{\mathbf{M}}(x_0, \ldots, x_{n-1})$. Then $y \in \mathbf{M}$, and $\varphi^{\mathbf{M}}(x_0, \ldots, x_{n-1}, y)$, so by (i), $\varphi^{\mathbf{N}}(x_0, \ldots, x_{n-1}, y)$. Hence $\mathbf{F}^{\mathbf{N}}(x_0, \ldots, x_{n-1}) = y$.

Assume (ii), and suppose that $x_0, \ldots, x_{n-1}, y \in \mathbf{M}$. Then

$$\begin{aligned} \varphi^{\mathbf{M}}(x_0, \ldots, x_{n-1}, y) \quad &\text{iff} \quad \mathbf{F}^{\mathbf{M}}(x_0, \ldots, x_{n-1}) = y \quad (\text{definition of } \mathbf{F}) \\ &\text{iff} \quad \mathbf{F}^{\mathbf{N}}(x_0, \ldots, x_{n-1}) = y \quad (\text{by (ii)}) \\ &\text{iff} \quad \varphi^{\mathbf{N}}(x_0, \ldots, x_{n-1}, y) \quad (\text{definition of } \mathbf{F}). \end{aligned}$$ □

The following theorem gives many explicit absoluteness results, and will be used frequently along with some similar results below. Note that we do not need to be explicit about how the relations and functions were really defined in Chapter 6; in fact, we were not very explicit about that in the first place.

Theorem 14.21. *The following relations and functions were defined by formulas equivalent to Δ_0-formulas on the basis of* $\mathrm{ZF}-\mathrm{Inf}$, *and hence are absolute for all transitive class models of* $\mathrm{ZF}-\mathrm{Inf}$*:*

(i) $x \in y$
(ii) $x = y$
(iii) $x \subseteq y$
(iv) $\{x, y\}$
(v) $\{x\}$
(vi) (x, y)
(vii) $\emptyset$
(viii) $x \cup y$
(ix) $x \cap y$
(x) $x \backslash y$
(xi) $x \cup \{x\}$
(xii) x is transitive
(xiii) $\bigcup x$
(xiv) $\bigcap x$ *(with* $\bigcap \emptyset = \emptyset$*)*

Note here, for example, that in (iv) we really mean the 2-place function assigning to sets x, y the unordered pair $\{x, y\}$.

Proof. (i) and (ii) are already Δ_0 formulas. (iii):

$$x \subseteq y \leftrightarrow \forall z \in x(z \in y).$$

(iv):

$$z = \{x, y\} \leftrightarrow \forall w \in z(w = x \vee w = y) \wedge x \in z \wedge y \in z.$$

(v): Similarly. (vi):

$$z = (x, y) \leftrightarrow \forall w \in z[w = \{x, y\} \vee w = \{x\}] \wedge \exists w \in z[w = \{x, y\}] \wedge \exists w \in z[w = \{x\}].$$

(vii):

$$x = \emptyset \leftrightarrow \forall y \in x(y \neq y).$$

(viii):

$$z = x \cup y \leftrightarrow \forall w \in z(w \in x \vee w \in y) \wedge \forall w \in x(w \in z) \wedge \forall w \in y(w \in z).$$

(ix):

$$z = x \cap y \leftrightarrow \forall w \in z(w \in x \wedge w \in y) \wedge \forall w \in x(w \in y \rightarrow w \in z).$$

(x):

$$z = x \backslash y \leftrightarrow \forall w \in z(w \in x \wedge w \notin y) \wedge \forall w \in x(x \notin y \rightarrow w \in z).$$

(xi):

$$y = x \cup \{x\} \leftrightarrow \forall w \in y(w \in x \vee w = x) \wedge \forall w \in x(w \in y) \wedge x \in y.$$

(xii):

$$x \text{ is transitive} \leftrightarrow \forall y \in x(y \subseteq x).$$

(xiii):

$$y = \bigcup x \leftrightarrow \forall w \in y \exists z \in x(w \in z) \wedge \forall w \in x(w \subseteq y).$$

(xiv):

$$y = \bigcap x \leftrightarrow [x \neq \emptyset \wedge \forall w \in y \forall z \in x(w \in z)$$
$$\wedge \forall w \in x \forall t \in w[\forall z \in x(t \in z) \to t \in y] \vee [x = \emptyset \wedge y = \emptyset]. \qquad \square$$

A stronger form of Theorem 14.21. For each of the indicated relations and functions, we do not need $\mathbf{M}$ to be a model of all of ZF $-$ Inf. In fact, we need only finitely many of the axioms of ZF $-$ Inf: enough to prove the uniqueness condition for any functions involved, and enough to prove the equivalence of the formula with a Δ_0-formula, since Δ_0 formulas are absolute for any transitive class model. To be absolutely rigorous here, one would need an explicit definition for each relation and function symbol involved, and then an explicit proof of equivalence to a Δ_0 formula; given these, a finite set of axioms becomes clear. And since any of the relations and functions of Theorem 14.21 require only finitely many basic relations and functcions, this can always be done. For Theorem 14.21 it is easy enough to work this all out in detail. We will be interested, however, in using this fact for more complicated absoluteness results to come.

As an illustration, however, we do some details for the function $\{x, y\}$. The definition involved is naturally taken to be the following:

$$\forall x, y, z[z = \{x, y\} \leftrightarrow \forall w[w \in z \leftrightarrow w = x \vee x = y]].$$

The axioms involved are the pairing axiom and one instance of the comprehension axiom:

$$\forall x, y \exists w[x \in w \wedge y \in w];$$
$$\forall x, y, w \exists z \forall u(u \in z \leftrightarrow u \in w \wedge (u = x \vee u = y)).$$

$\{x, y\}$ is then absolute for any transitive class model of these three sentences, by the proof of (iv) in Theorem 14.21, for which they are sufficient.

For further absoluteness results we will not reduce to Δ_0 formulas. We need the following extensions of the absoluteness notion.

• Suppose that $\mathbf{M} \subseteq \mathbf{N}$ are classes, and $\varphi(w_1, \ldots, w_n)$ is a formula. Then we say that φ is *absolute upwards* for $\mathbf{M}, \mathbf{N}$ iff for all $w_1, \ldots, w_n \in \mathbf{M}$, if $\varphi^{\mathbf{M}}(w_1, \ldots, w_n)$, then $\varphi^{\mathbf{N}}(w_1, \ldots, w_n)$. It is *absolute downwards* for $\mathbf{M}, \mathbf{N}$ iff for all $w_1, \ldots, w_n \in \mathbf{M}$, if $\varphi^{\mathbf{N}}(w_1, \ldots, w_n)$, then $\varphi^{\mathbf{M}}(w_1, \ldots, w_n)$. Thus φ is absolute for $\mathbf{M}, \mathbf{N}$ iff it it is both absolute upwards for $\mathbf{M}, \mathbf{N}$ and absolute downwards for $\mathbf{M}, \mathbf{N}$.

Theorem 14.22. *Suppose that $\varphi(x_1, \ldots, x_n, w_1, \ldots, w_m)$ is absolute for $\mathbf{M}, \mathbf{N}$. Then*

(i) $\exists x_1, \ldots \exists x_n \varphi(x_1, \ldots, x_n, w_1, \ldots, w_m)$ is absolute upwards for $\mathbf{M}, \mathbf{N}$.

(ii) $\forall x_1, \ldots \forall x_n \varphi(x_1, \ldots, x_n, w_1, \ldots, w_m)$ is absolute downwards for $\mathbf{M}, \mathbf{N}$. $\square$

Theorem 14.23. *Absoluteness is preserved under composition. In detail: suppose that $\mathbf{M} \subseteq \mathbf{N}$ are classes, and the following are absolute for $\mathbf{M}, \mathbf{N}$:*

$\varphi(x_1, \ldots, x_n)$;

$\mathbf{F}$, *an* n*-ary function ;*
For each $i = 1, \ldots, n$, *an* m*-ary function* $\mathbf{G}_i$.

Then the following are absolute:

(i) $\varphi(\mathbf{G}_1(x_1, \ldots, x_m), \ldots, \mathbf{G}_n(x_1, \ldots, x_m))$.
(ii) *The* m*-ary function assigning to* $x_1, \ldots, x_m$ *the value*

$$\mathbf{F}(\mathbf{G}_1(x_1, \ldots, x_m), \ldots, \mathbf{G}_n(x_1, \ldots, x_m)).$$

Proof. We use Theorem 14.22:

$$\begin{aligned}
\varphi(\mathbf{G}_1(x_1, \ldots, x_m), \ldots, \mathbf{G}_n(x_1, \ldots, x_m)) \leftrightarrow & \exists z_1, \ldots \exists z_n \Bigg[\varphi(z_1, \ldots, z_n) \\
& \wedge \bigwedge_{i=1}^{n} (z_i = \mathbf{G}_i(x_1, \ldots, x_m)) \Bigg]; \\
\varphi(\mathbf{G}_1(x_1, \ldots, x_m), \ldots, \mathbf{G}_n(x_1, \ldots, x_m)) \leftrightarrow & \forall z_1, \ldots \forall z_n \Bigg[\bigwedge_{i=1}^{n} (z_i = \mathbf{G}_i(x_1, \ldots, x_m)) \\
& \rightarrow \varphi(z_1, \ldots, z_n) \Bigg]; \\
y = \mathbf{F}(\mathbf{G}_1(x_1, \ldots, x_m), \ldots, \mathbf{G}_n(x_1, \ldots, x_m)) \leftrightarrow & \exists z_1, \ldots \exists z_n \Bigg[(y = \mathbf{F}(z_1, \ldots, z_n)) \\
& \wedge \bigwedge_{i=1}^{n} (z_i = \mathbf{G}_i(x_1, \ldots, x_m)) \Bigg]; \\
y = \mathbf{F}(\mathbf{G}_1(x_1, \ldots, x_m), \ldots, \mathbf{G}_n(x_1, \ldots, x_m)) \leftrightarrow & \forall z_1, \ldots \forall z_n \Bigg[\bigwedge_{i=1}^{n} (z_i = \mathbf{G}_i(x_1, \ldots, x_m)) \\
& \rightarrow (y = \mathbf{F}(z_1, \ldots, z_n)) \Bigg]. \qquad \square
\end{aligned}$$

Theorem 14.24. *Suppose that* $\mathbf{M} \subseteq \mathbf{N}$ *are classes,* $\varphi(y, x_1, \ldots, x_m, w_1, \ldots, w_n)$ *is absolute for* $\mathbf{M}, \mathbf{N}$, *and* $\mathbf{F}$ *and* $\mathbf{G}$ *are* n*-ary functions absolute for* $\mathbf{M}, \mathbf{N}$. *Then the following are also absolute for* $\mathbf{M}, \mathbf{N}$*:*

(i) $z \in \mathbf{F}(x_1, \ldots, x_m)$.
(ii) $\mathbf{F}(x_1, \ldots, x_m) \in z$.
(iii) $\exists y \in \mathbf{F}(x_1, \ldots, x_m)\varphi(y, x_1, \ldots, x_m, w_1, \ldots, w_n)$.
(iv) $\forall y \in \mathbf{F}(x_1, \ldots, x_m)\varphi(y, x_1, \ldots, x_m, w_1, \ldots, w_n)$.
(v) $\mathbf{F}(x_1, \ldots, x_m) = \mathbf{G}(x_1, \ldots, x_m)$.
(vi) $\mathbf{F}(x_1, \ldots, x_m) \in \mathbf{G}(x_1, \ldots, x_m)$.

Proof.

$$\begin{aligned}
z \in \mathbf{F}(x_1, \ldots, x_m) &\leftrightarrow \exists w[z \in w \wedge w = \mathbf{F}(x_1, \ldots, x_m)]; \\
&\leftrightarrow \forall w[w = \mathbf{F}(x_1, \ldots, x_m) \rightarrow z \in w]; \\
\mathbf{F}(x_1, \ldots, x_m) \in z &\leftrightarrow \exists w \in z[w = \mathbf{F}(x_1, \ldots, x_m)];
\end{aligned}$$

$$\begin{aligned}
&\exists y \in \mathbf{F}(x_1,\ldots,x_m)\varphi(y,x_1,\ldots,x_m,w_1,\ldots,w_n)\\
&\leftrightarrow \exists w \exists y \in w[w = \mathbf{F}(x_1,\ldots,x_m) \wedge \varphi(y,x_1,\ldots,x_m,w_1,\ldots,w_n)];\\
&\leftrightarrow \forall w[w = \mathbf{F}(x_1,\ldots,x_m) \rightarrow \exists y \in w\varphi(y,x_1,\ldots,x_m,w_1,\ldots,w_n)];
\end{aligned}$$

(iv)–(vi) are proved similarly. □

We now give some more specific absoluteness results.

Theorem 14.25. *The following relations and functions are absolute for all transitive class models of* ZF − Inf*:*

(i) x is an ordered pair	*(iv)* $\mathrm{dmn}(R)$	*(vii)* $R(x)$
(ii) $A \times B$	*(v)* $\mathrm{rng}(R)$	*(viii) R is a one-one function*
(iii) R is a relation	*(vi) R is a function*	*(ix) x is an ordinal*

Note concerning (vii): This is supposed to have its natural meaning if R is a function and x is in its domain; otherwise, $R(x) = \emptyset$.

Proof.

$$\begin{aligned}
x \text{ is an ordered pair} &\leftrightarrow \left(\exists y \in \bigcup x\right)\left(\exists z \in \bigcup x\right)[x = (y,z)];\\
y = A \times B &\leftrightarrow (\forall a \in A)(\forall b \in B)[(a,b) \in y] \wedge\\
&\qquad (\forall z \in y)(\exists a \in A)(\exists b \in B)[z = (a,b)];\\
R \text{ is a relation} &\leftrightarrow \forall x \in R[x \text{ is an ordered pair}];\\
x = \mathrm{dmn}(R) &\leftrightarrow (\forall y \in x)\left(\exists z \in \bigcup\bigcup R\right)[(x,z) \in R] \wedge\\
&\qquad \left(\forall y \in \bigcup\bigcup R\right)\left(\forall z \in \bigcup\bigcup R\right)[(y,z) \in R \rightarrow y \in x];\\
x = \mathrm{rng}(R) &\leftrightarrow (\forall y \in x)\left(\exists z \in \bigcup\bigcup R\right)[(z,x) \in R] \wedge\\
&\qquad \left(\forall y \in \bigcup\bigcup R\right)\left(\forall z \in \bigcup\bigcup R\right)[(y,z) \in R \rightarrow z \in x];\\
R \text{ is a function} &\leftrightarrow R \text{ is a relation } \wedge \left(\forall x \in \bigcup\bigcup R\right)\left(\forall y \in \bigcup\bigcup R\right)\\
&\qquad \left(\forall z \in \bigcup\bigcup R\right)[(x,y) \in R \wedge (x,z) \in R \rightarrow y = z];\\
y = R(x) &\leftrightarrow [R \text{ is a function} \wedge (x,y) \in R] \vee\\
&\qquad [R \text{ is not a function} \wedge (\forall z \in y)(z \neq z)] \vee\\
&\qquad [x \notin \mathrm{dmn}(R) \wedge (\forall z \in y)(z \neq z)];\\
R \text{ is a one-one function} &\leftrightarrow R \text{ is a function} \wedge\\
&\qquad \forall x \in \mathrm{dmn}(R) \forall y \in \mathrm{dmn}(R)[R(x) = R(y) \rightarrow x = y];\\
x \text{ is an ordinal} &\leftrightarrow x \text{ is transitive} \wedge (\forall y \in x)(y \text{ is transitive}).
\end{aligned}$$

□

Theorem 14.26. *Suppose that* $\mathbf{M}$ *is a transitive class model of* $\mathrm{ZF} - \mathrm{Inf}$ *and* $\omega \in \mathbf{M}$. *Then the infinity axiom holds in* $\mathbf{M}$.

Proof. We have

$$\exists x \in \mathbf{M}[\emptyset \in x \wedge \forall y \in x(y \cup \{y\} \in x)],$$

so by the absoluteness of the notions here we get

$$[\exists x[\emptyset \in x \wedge \forall y \in x(y \cup \{y\} \in x)]]^{\mathbf{M}},$$

which means that the infinity axiom holds in $\mathbf{M}$. □

Theorem 14.27. *If* $\mathbf{M}$ *is a transitive class model of* ZF, *then* $\emptyset, \omega \in \mathbf{M}$ *and* $\mathbf{M}$ *is closed under the following set-theoretic operations:*

(i) $\cup$	*(iv)* $(a, b) \mapsto \{a, b\}$	*(vii)* $\bigcup$
(ii) $\cap$	*(v)* $(a, b) \mapsto (a, b)$	*(viii)* $\bigcap$
(iii) $(a, b) \mapsto a \backslash b$	*(vi)* $x \mapsto x \cup \{x\}$	

Moreover, $\omega + 1 \subseteq \mathbf{M}$ *and* $[\mathbf{M}]^{<\omega} \subseteq \mathbf{M}$.

Proof. $\mathbf{M}$ has elements x, y such that $(x = \emptyset)^{\mathbf{M}}$ and $(y = \omega)^{\mathbf{M}}$. So $x = \emptyset$ and $y = \omega$ by absoluteness. (See Theorem 14.18) (i)–(viii) are all very similar, so we only treat (i). Let $a, b \in \mathbf{M}$. Then because $\mathbf{M} \models \mathrm{ZF}$, there is a $c \in \mathbf{M}$ such that $(c = a \cup b)^{\mathbf{M}}$. By absoluteness, $c = a \cup b$.

Each $i \in \omega$ is in $\mathbf{M}$ by transitivity. Hence $\omega + 1 \subseteq \mathbf{M}$. Finally, we show by induction on n that if $x \subseteq \mathbf{M}$ and $|x| = n$ then $x \in \mathbf{M}$. This is clear for $n = 0$. Now suppose inductively that $x \subseteq \mathbf{M}$ and $|x| = n + 1$. Let $a \in x$ and set $y = x \backslash \{a\}$. So $|y| = n$. Hence $y \in \mathbf{M}$ by the inductive hypothesis. Hence $x = y \cup \{a\} \in \mathbf{M}$ by previous parts of this theorem. □

Our final abstract absoluteness result concerns recursive definitions.

Theorem 14.28. *Suppose that* $\mathbf{R}$ *is a class relation which is well-founded and set-like on* $\mathbf{A}$, *and* $\mathbf{G} : \mathbf{A} \times \mathbf{V} \to \mathbf{V}$. *Let* $\mathbf{F}$ *be given by Theorem 8.7: for all* $x \in \mathbf{A}$,

$$\mathbf{F}(x) = \mathbf{G}(x, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}(x)).$$

Let $\mathbf{M}$ *be a transitive class model of* $\mathrm{Z}\mathbf{F}$, *and assume the following additional conditions hold:*

(i) $\mathbf{G}$, $\mathbf{R}$, *and* $\mathbf{A}$ *are absolute for* $\mathbf{M}$.
(ii) $(\mathbf{R}$ *is set-like on* $\mathbf{A})^{\mathbf{M}}$.
(iii) $\forall x \in \mathbf{M} \cap \mathbf{A}[\mathrm{pred}_{\mathbf{AR}}(x) \subseteq \mathbf{M}]$.

Conclusion: $\mathbf{F}$ *is absolute for* $\mathbf{M}$.

Proof. First we claim

(1) ($\mathbf{R}$ is well-founded on $\mathbf{A})^{\mathbf{M}}$.

In fact, by absolutenss $\mathbf{R}^{\mathbf{M}} = \mathbf{R} \cap (\mathbf{M} \times \mathbf{M})$ and $\mathbf{A}^{\mathbf{M}} = \mathbf{A} \cap \mathbf{M}$, so it follows that in $\mathbf{M}$ every nonempty subset of $\mathbf{A}^{\mathbf{M}}$ has an $\mathbf{R}^{\mathbf{M}}$-minimal element. Hence we can apply the recursion theorem within $\mathbf{M}$ to define a function $\mathbf{H} : \mathbf{A}^{\mathbf{M}} \to \mathbf{M}$ such that for all $x \in \mathbf{A}^{\mathbf{M}}$,

$$\mathbf{H}(x) = \mathbf{G}^{\mathbf{M}}(x, \mathbf{H} \restriction \mathrm{pred}^{\mathbf{M}}_{\mathbf{A}^{\mathbf{M}}\mathbf{R}^{\mathbf{M}}}(x)).$$

We claim that $\mathbf{H} = \mathbf{F} \restriction \mathbf{A}^{\mathbf{M}}$, which will prove the theorem. In fact, suppose that x is $\mathbf{R}$-minimal such that $x \in \mathbf{A}^{\mathbf{M}}$ and $\mathbf{F}(x) \neq \mathbf{H}(x)$. Then using absoluteness again,

$$\mathbf{H}(x) = \mathbf{G}^{\mathbf{M}}(x, \mathbf{H} \restriction \mathrm{pred}^{\mathbf{M}}_{\mathbf{A}^{\mathbf{M}}\mathbf{R}^{\mathbf{M}}}(x)) = \mathbf{G}(x, \mathbf{H} \restriction \mathrm{pred}_{\mathbf{AR}}(x) = \mathbf{F}(x),$$

contradiction. □

Theorem 14.28 is needed for many deeper applications of absoluteness. We illustrate its use by the following result.

Theorem 14.29. *The following are absolute for transitive class models of* ZF*.*
(i) $\alpha + \beta$ *(ordinal addition)*
(ii) $\alpha \cdot \beta$ *(ordinal multiplication)*
(iii) α^β *(ordinal exponentiation)*
(iv) $\mathrm{rank}(x)$
(v) $\mathrm{trcl}(x)$

Proof. In each case it is mainly a matter of identifying $\mathbf{A}, \mathbf{R}, \mathbf{G}$ to which to apply Theorem 14.28; checking the conditions of that theorem are straightforward.

(i): $\mathbf{A} = \mathbf{On}$, $\mathbf{R} = \{(\alpha, \beta) : \alpha, \beta \in \mathbf{On}$, and $\alpha \in \beta\}$, and $\mathbf{G} : \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ is defined as follows:

$$\mathbf{G}(\alpha, f) = \begin{cases} \alpha & \text{if } f = \emptyset, \\ f(\beta) \cup \{f(\beta)\} & \text{if } f \text{ is a function with domain an ordinal } \beta + 1, \\ \bigcup_{\gamma \in \beta} f(\gamma) & \text{if } f \text{ is a function with domain a limit ordinal } \beta, \\ \emptyset & \text{otherwise.} \end{cases}$$

(ii) and (iii) are treated similarly. For (iv), take $\mathbf{R} = \{(x, y) : x \in y\}$, $\mathbf{A} = \mathbf{V}$, and define $\mathbf{G} : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ by setting, for all $x, f \in \mathbf{V}$,

$$\mathbf{G}(x, f) = \begin{cases} \bigcup_{y \in x}(f(y) \cup \{f(y)\}) & \text{if } f \text{ is a function with domain } x, \\ \emptyset & \text{otherwise.} \end{cases}$$

For (v), let $\mathbf{R} = \{(i, j) : i, j \in \omega \text{ and } i < j\}$, $\mathbf{A} = \omega$, and define $\mathbf{G} : \omega \times \mathbf{V} \to \mathbf{V}$ by setting, for all $m \in \omega$ and $f \in \mathbf{V}$,

$$\mathbf{G}(m, f) = \begin{cases} x & \text{if } m = 0, \\ f(\bigcup m) \cup \bigcup f(\bigcup m) & \text{if } m > 0 \text{ and } f \text{ is a function with domain } m, \\ \emptyset & \text{otherwise} \end{cases}$$

Then the function $\mathbf{F}$ obtained from Theorem 14.28 is absolute for transitive class models of ZF, and $\mathrm{trcl}(x) = \bigcup_{m \in \omega} \mathbf{F}(m)$. □

Theorem 14.30. *If* $\mathbf{M}$ *is a transitive model of ZF, then:*

(i) $\mathscr{P}^{\mathbf{M}}(x) = \mathscr{P}(x) \cap \mathbf{M}$ *for any* $x \in \mathbf{M}$*;*

(ii) $V_\alpha^{\mathbf{M}} = V_\alpha \cap \mathbf{M}$ *for any* $\alpha \in \mathbf{M}$.

Proof. (i): Assume that $x \in \mathbf{M}$. Then for any set y,

$$\begin{aligned} y \in \mathscr{P}^{\mathbf{M}}(x) \quad &\text{iff} \quad y \in \mathbf{M} \text{ and } (y \subseteq x)^{\mathbf{M}} \\ &\text{iff} \quad y \in \mathbf{M} \text{ and } y \subseteq x \\ &\text{iff} \quad y \in \mathscr{P}(x) \cap \mathbf{M}. \end{aligned}$$

(ii): Assume that $\alpha \in \mathbf{M}$. Then for any set x,

$$\begin{aligned} x \in V_\alpha^{\mathbf{M}} \quad &\text{iff} \quad x \in \mathbf{M} \text{ and } \operatorname{rank}^{\mathbf{M}}(x) < \alpha \\ &\text{iff} \quad x \in \mathbf{M} \text{ and } \operatorname{rank}(x) < \alpha \\ &\text{iff} \quad x \in \mathbf{M} \text{ and } \operatorname{rank}^{\mathbf{M}}(x) < \alpha \\ &\text{iff} \quad x \in V_\alpha \cap \mathbf{M} \end{aligned}$$

□

Proposition 14.31. *"R well-orders A" is absolute for models of ZF.*

Proof. Let $\mathbf{M}$ be a model of ZF. Suppose that $R, A \in \mathbf{M}$. Clearly

$$\begin{aligned} (R \text{ well-orders } A) \quad \text{iff} \quad &\exists x \exists f[x \text{ is an ordinal} \wedge f : x \to A \text{ is a bijection} \\ &\wedge \forall \beta, \gamma \in x[\beta < \gamma \text{ iff } (f(\beta), f(\gamma)) \in R]]. \end{aligned}$$

From this and elementary absoluteness results it is clear that $(R \text{ well-orders } A)^{\mathbf{M}}$ implies that (R well-orders A). Now suppose that (R well-orders A). Let x and f be such that x is an ordinal, $f : x \to A$ is a bijection, and $\forall \beta, \gamma \in x[\beta < \gamma \text{ iff } (f(\beta), f(\gamma)) \in R]$. Since $\mathbf{M}$ is a model of ZF, let $y, g \in \mathbf{M}$ be such that in $\mathbf{M}$ we have: y is an ordinal, $g : y \to A$ is a bijection, and $\forall \beta, \gamma \in y[\beta < \gamma \text{ iff } (g(\beta), g(\gamma)) \in R]$. By simple absoluteness results, this is really true. Then $x = y$ and $f = g$ by the uniqueness conditions in 9.12–9.14. □

Consistency of no inaccessibles

Theorem 14.32. *If* ZFC *is consistent, then so is* ZFC+ *"there do not exist uncountable inaccessible cardinals".*

Proof. For brevity we interpret "inaccessible" to mean "uncountable and inaccessible". Let

$$\mathbf{M} = \{x : \forall \alpha[\alpha \text{ inaccessible } \to x \in V_\alpha]\}$$

Thus $\mathbf{M}$ is a class, and $\mathbf{M} \subseteq V_\alpha$ for every inaccessible α (if there are such). We claim that $\mathbf{M}$ is a model of ZFC+"there do not exist uncountable inaccessible cardinals". To prove this, we consider two possibilities.

Case 1. $\mathbf{M} = \mathbf{V}$. Then of course $\mathbf{M}$ is a model of ZFC. Suppose that α is inaccessible. Then since $\mathbf{M} = \mathbf{V}$ we have $\mathbf{V} \subseteq V_\alpha$, which is not possible, since V_α is a set. Thus $\mathbf{M}$ is a model of ZFC + "there do not exist uncountable inaccessible cardinals".

Case 2. $\mathbf{M} \neq \mathbf{V}$. Let x be a set which is not in $\mathbf{M}$. Then there is an ordinal α such that α is inaccessible and $x \notin V_\alpha$. In particular, there is an inaccessible α, and we let κ be the least such.

(1) $\mathbf{M} = V_\kappa$.

In fact, if $x \in \mathbf{M}$, then $x \in V_\alpha$ for every inaccessible α, so in particular $x \in V_\kappa$. On the other hand, if $x \in V_\kappa$, then $x \in V_\alpha$ for every $\alpha \geq \kappa$, so $x \in V_\alpha$ for every inaccessible α, and so $x \in \mathbf{M}$. So (1) holds.

Now we show that V_κ is as desired. First, we need to check all the ZFC axioms. Here we use Theorems 14.10–14.16 and 14.26. Now V_κ is transitive, so by Theorem 14.10, the extensionality axiom holds in V_κ.

For the comprehension axioms, we are going to apply Theorem 14.11. Suppose that φ is a formula with free variables among $x, z, w_1, \ldots, w_n$, and we are given $z, w_1, \ldots, w_n \in V_\kappa$. Let $A = \{x \in z : \varphi^{V_\kappa}(x, z, w_1, \ldots, w_n)\}$. Then $A \subseteq z$. Say $z \in V_\beta$ with $\beta < \kappa$. Then $A \subseteq z \subseteq V_\beta$, so $A \in \mathscr{P}(V_\beta) = V_{\beta+1} \subseteq V_\kappa$. It follows from Theorem 14.11 that the comprehension axioms hold in V_κ.

Suppose that $x, y \in V_\kappa$. Say $x, y \in V_\beta$ with $\beta < \kappa$. Then $\{x, y\} \subseteq V_\beta$, so $\{x, y\} \in V_{\beta+1} \subseteq V_\kappa$. By Theorem 14.12, the pairing axiom holds in V_κ.

Suppose that $x \in V_\kappa$. Say $x \in V_\beta$ with $\beta < \kappa$. Then $\bigcup x \subseteq V_\beta$, so $\bigcup x \in V_{\beta+1} \subseteq V_\kappa$. By Theorem 14.13, the union axiom holds in V_κ.

Suppose that $x \in V_\kappa$. Say $x \in V_\beta$ with $\beta < \kappa$. Then $x \subseteq V_\beta$. Hence $y \subseteq V_\beta$ for each $y \subseteq x$, so $y \in \mathscr{P}(V_\beta) = V_{\beta+1}$ for each $y \in \mathscr{P}(x)$. This means that $\mathscr{P}(x) \subseteq V_{\beta+1}$, so $\mathscr{P}(x) \in V_{\beta+2}$. By Theorem 14.14, the power set axiom holds in V_κ.

To treat the replacement axioms, we need the following fact:

(1) $|V_\beta| < \kappa$ for every $\beta < \kappa$.

We prove (1) by induction on β. We have $|V_0| = |\omega| = \omega < \kappa$. If we have shown that $|V_\beta| < \kappa$, where $\beta < \kappa$, then $|V_{\beta+1}| = |\mathscr{P}(V_\beta)| = 2^{|V_\beta|} < \kappa$. If we know that $|V_\gamma| < \kappa$ for all $\gamma < \beta$, where $\beta < \kappa$, then by the regularity of κ we have $|V_\beta| = |\bigcup_{\gamma<\beta} V_\gamma| \leq \sum_{\gamma<\beta} |V_\gamma| < \kappa$.

Now for the replacement axioms, we will apply Theorem 14.15. So, suppose that φ is a formula with free variables among $x, y, A, w_1, \ldots, w_n$, any $A, w_1, \ldots, w_n \in V_\kappa$, and

$$\forall x \in A \exists! y[y \in V_\kappa \wedge \varphi^{V_\kappa}(x, y, A, w_1, \ldots, w_n)].$$

For each $x \in A$, let y_x be the unique set such that $y_x \in V_\kappa$ and $\varphi^{V_\kappa}(x, y_x, A, w_1, \ldots, w_n)$, and let $\alpha_x < \kappa$ be such that $y_x \in V_{\alpha_x}$. Choose $\beta < \kappa$ such that $A \in V_\beta$. Then $A \subseteq V_\beta$, and hence $|A| \leq |V_\beta| < \kappa$ by (1). It follows that $\gamma \stackrel{\text{def}}{=} \bigcup\{\alpha_x : x \in A\} < \kappa$. Let

$$Y = \{z \in V_\gamma : \exists x \in A \varphi^{V_\kappa}(x, z, A, w_1, \ldots, w_n)\}.$$

Thus $Y \subseteq V_\gamma$, so $Y \in V_{\gamma+1} \subseteq V_\kappa$. Suppose that $x \in A$ and $z \in V_\kappa$ is such that $\varphi^{V_\kappa}(x, z, A, w_1, \ldots, w_n)$. Then $z = y_x$ by the above, and so $z \in Y$, as desired.

By Theorem 14.16, the foundation axiom holds in V_κ.

We have now shown that V_κ is a model of ZF $-$ Inf.

For the infinity axiom, note that $\omega \in V_{\omega+1} \subseteq V_\kappa$. Hence the infinity axiom holds in V_κ by Theorem 14.26.

For the axiom of choice, suppose that $\mathscr{A} \in V_\kappa$ is a family of pairwise disjoint nonempty sets, and let $\mathscr{B} \subseteq \bigcup \mathscr{A}$ have exactly one element in common with each member of $\mathscr{A}$. Say $\mathscr{A} \in V_\alpha$ with $\alpha < \kappa$. Then $\mathscr{B} \subseteq \bigcup \mathscr{A} \subseteq V_\alpha$, so $\mathscr{B} \in V_{\alpha+1} \subseteq V_\kappa$, and the axiom of choice thus holds in V_κ.

So V_κ is a model of ZFC.

Finally, suppose that $x \in V_\kappa$ and $(x \text{ is an inaccessible cardinal})^{V_\kappa}$; we want to get a contradiction. In particular, $(x \text{ is an ordinal})^{V_\kappa}$, so by absoluteness, x is an ordinal. Absoluteness clearly implies that x is infinite. We claim that x is a cardinal. For, if $f : y \to x$ is a bijection with $y < x$, then clearly $f \in V_\kappa$, and hence by absoluteness $(f : y \to x \text{ is a bijection and } y < x)^{V_\alpha}$, contradiction. Similarly, x is regular; otherwise there is an injection $f : y \to x$ with $\operatorname{rng}(f)$ unbounded in x, so clearly $f \in V_\kappa$, and absolutenss again yields a contradiction. Thus x is a regular cardinal. Hence, since κ is the smallest inaccessible, there is a $y \in x$ such that there is a one-one function g from x into $\mathscr{P}(y)$. Again, $g \in V_\kappa$, and easy absoluteness results contradicts $(x \text{ is an inaccessible cardinal})^{V_\kappa}$. □

The Mostowski collapse

We describe an important procedure for obtaining structures $(A, \in)$ from structures (A, R) where R is not real membership.

Theorem 14.33. *Suppose that* $\mathbf{R}$ *is well-founded and set-like on* $\mathbf{A}$*. Then there is a class function* $\mathbf{F} : \mathbf{A} \to \mathbf{V}$ *such that for all* $a \in A$*,*

$$\mathbf{F}(a) = \{\mathbf{F}(b) : b \in \mathbf{A} \text{ and } (b, a) \in \mathbf{R}\}.$$

Proof. Define $\mathbf{G} : \mathbf{A} \times \mathbf{V} \to \mathbf{V}$ by setting, for any $a \in A$ and $x \in V$,

$$\mathbf{G}(a, x) = \begin{cases} \operatorname{rng}(x) & \text{if } x \text{ is a function with domain } \operatorname{pred}_{\mathbf{AR}}(a), \\ \emptyset & \text{otherwise.} \end{cases}$$

Then let $\mathbf{F}$ be obtained by the recursion theorem 8.7: $\mathbf{F} : \mathbf{A} \to \mathbf{V}$ and for any $a \in \mathbf{A}$, $\mathbf{F}(a) = \mathbf{G}(a, \mathbf{F} \restriction \operatorname{pred}_{\mathbf{AR}}(a))$. Thus for any $a \in \mathbf{A}$, $\mathbf{F}(a) = \operatorname{rng}(\mathbf{F} \restriction \operatorname{pred}_{\mathbf{AR}}(a)) = \{\mathbf{F}(b) : b \in \mathbf{A} \text{ and } (b, a) \in \mathbf{R}\}$. □

The *Mostowski collapse of* $\mathbf{A}, \mathbf{R}$ is defined as the range of this function $\mathbf{F}$.

Proposition 14.34. *Suppose that* $\mathbf{R}$ *is well-founded and set-like on* $\mathbf{A}$*,* $\mathbf{F}$ *is the Mostowski collapsing function for* $\mathbf{A}, \mathbf{R}$*, and* $\mathbf{M}$ *is the Mostowski collapse. Then*

(i) For all $x, y \in \mathbf{A}$*, if* $(x, y) \in \mathbf{R}$ *then* $\mathbf{G}(x) \in \mathbf{G}(y)$*.*

(ii) $\mathbf{M}$ *is transitive.*

Proof. (i) is obvious from the definition. If $a \in b \in \mathbf{M}$, choose $y \in \mathbf{A}$ such that $b = \mathbf{G}(y)$. Since $a \in b$, by the definition we have $a \in \operatorname{rng}(\mathbf{G}) = \mathbf{M}$. So (ii) holds. □

The Mostowski collapse is especially important for extensional relations, defined as follows.

• Let $\mathbf{R}$ be a class relation and $\mathbf{A}$ a class. We say that $\mathbf{R}$ *is extensional on* $\mathbf{A}$ iff the following generalization of the extensionality axiom holds:

$$\forall x, y \in \mathbf{A}[\forall z \in \mathbf{A}[(z, x) \in \mathbf{R} \text{ iff } (z, y) \in \mathbf{R}] \to x = y].$$

Proposition 14.35. *Suppose that* $\mathbf{R}$ *is well-founded and set-like on* $\mathbf{A}$*. Let* $\mathbf{F}$ *and* $\mathbf{M}$ *be the Mostowski collapsing function and Mostowski collapse, respectively. Then the following conditions are equivalent:*

(i) $\mathbf{R}$ *is extensional on* $\mathbf{A}$*.*

(ii) $\mathbf{F}$ *is one-one, and for all* $x, y \in \mathbf{A}$ *we have* $(x, y) \in \mathbf{R}$ *iff* $\mathbf{F}(x) \in \mathbf{F}(y)$*.*

Proof. (i)⇒(ii): Assume (i). Suppose that $\mathbf{F}$ is not one-one. Then the set

$$(*) \qquad \{x \in \mathbf{A} : \text{there is a } y \in \mathbf{A} \text{ such that } x \neq y \text{ and } \mathbf{F}(x) = \mathbf{F}(y)\}$$

is nonempty, and we take an $\mathbf{R}$-minimal element of this set. Also, let $y \in \mathbf{A}$ with $x \neq y$ and $\mathbf{F}(x) = \mathbf{F}(y)$. Since both x and y are in $\mathbf{A}$, and $x \neq y$, the extensionality condition gives two cases.

Case 1. There is a $z \in \mathbf{A}$ such that $(z, x) \in \mathbf{R}$ and $(z, y) \notin \mathbf{R}$. Since $(z, x) \in \mathbf{R}$, it follows that z is not in the set $(*)$. Now $\mathbf{F}(z) \in \mathbf{F}(x)$ by Proposition 14.34(i), so the fact that $\mathbf{F}(x) = \mathbf{F}(y)$ implies that $\mathbf{F}(z) \in \mathbf{F}(y)$. Hence by definition of $\mathbf{F}$ we can choose $w \in \mathbf{A}$ such that $(w, y) \in \mathbf{R}$ and $\mathbf{F}(z) = \mathbf{F}(w)$. Then from z not in $(*)$ we infer that $z = w$, hence $(z, y) \in \mathbf{R}$, contradiction.

Case 2. There is a $z \in \mathbf{A}$ such that $(z, y) \in \mathbf{R}$ and $(z, x) \notin \mathbf{R}$. Since $(z, y) \in \mathbf{R}$, by Proposition 14.34(i) we get $\mathbf{F}(z) \in \mathbf{F}(y) = \mathbf{F}(x)$, and so there is a $v \in \mathbf{A}$ such that $(v, x) \in \mathbf{R}$ and $\mathbf{F}(z) = \mathbf{F}(v)$. Now v is not in $(*)$ by the minimality of x, so $z = v$ and $(z, x) \in \mathbf{R}$, contradiction.

Therefore, $\mathbf{F}$ is one-one. Now the implication ⇒ in the second part of (ii) holds by Proposition 14.34(i). Suppose now that $\mathbf{F}(x) \in \mathbf{F}(y)$. Choose $w \in \mathbf{A}$ such that $(w, y) \in \mathbf{R}$ and $\mathbf{F}(x) = \mathbf{F}(w)$. Then $x = w$ since $\mathbf{F}$ is one-one, so $(x, y) \in \mathbf{R}$, as desired.

(ii)⇒(i): Assume (ii), and suppose that $x, y \in \mathbf{A}$, and $\forall z \in \mathbf{A}[(z, x) \in \mathbf{R}$ iff $(z, y) \in \mathbf{R}]$. Take any $u \in \mathbf{F}(x)$. By the definition of $\mathbf{F}$, choose $z \in \mathbf{A}$ such that $(z, x) \in \mathbf{R}$ and $u = \mathbf{F}(z)$. Then also $(z, y) \in \mathbf{R}$, so $u = \mathbf{F}(z) \in \mathbf{F}(y)$. This shows that $\mathbf{F}(x) \subseteq \mathbf{F}(y)$. Similarly $\mathbf{F}(y) \subseteq \mathbf{F}(x)$, so $\mathbf{F}(x) = \mathbf{F}(y)$. Since $\mathbf{F}$ is one-one, it follows that $x = y$. □

Theorem 14.36. *Suppose that* $\mathbf{R}$ *is a well-founded class relation that is set-like and extensional on a class* $\mathbf{A}$*. Then there are unique* $\mathbf{F}, \mathbf{M}$ *such that* $\mathbf{M}$ *is a transitive class and* $\mathbf{F}$ *is an isomorphism from* $(\mathbf{A}, \mathbf{R})$ *onto* $(\mathbf{M}, \in)$*.*

Note that we have formulated this in the usual fashion for isomorphism of structures, but of course we cannot form the ordered pairs $(\mathbf{A}, \mathbf{R})$ and $(\mathbf{M}, \in)$ if $\mathbf{A}, \mathbf{R}, \mathbf{M}$ are proper classes. So we understand the above as an abbreviation for a longer statement, that $\mathbf{F}$ is a bijection from $\mathbf{A}$ onto $\mathbf{M}$, etc.

Proof of 14.36: The existence of $\mathbf{F}$ and $\mathbf{M}$ is immediate from Propositions 14.33 and 14.35. Now suppose that $\mathbf{F}'$ and $\mathbf{M}'$ also work. Since $\mathbf{M}$ is the range of $\mathbf{F}$ and $\mathbf{M}'$ is the range of $\mathbf{F}'$, it suffices to show that $\mathbf{F} = \mathbf{F}'$. Suppose not. Let a be $\mathbf{R}$-minimal such that $\mathbf{F}(a) \neq \mathbf{F}'(a)$. Take any $x \in \mathbf{F}(a)$. Then since $\mathbf{M}$ is transitive and $\mathbf{F}(a) \in \mathbf{M}$, it follows that $x \in \mathbf{M}$. And since $\mathbf{F}$ maps onto $\mathbf{M}$, it then follows that there is a $b \in \mathbf{A}$ such that $\mathbf{F}(b) = x$. So $\mathbf{F}(b) \in \mathbf{F}(a)$ so, by the isomorphism property, $(b, a) \in \mathbf{R}$. Then the minimality of a yields That $\mathbf{F}'(b) = \mathbf{F}(b) \in \mathbf{F}(a)$. But also $(b, a) \in \mathbf{R}$ implies that $\mathbf{F}'(b) \in \mathbf{F}'(a)$ by the isomorphism property, so $x = \mathbf{F}(b) \in \mathbf{F}'(a)$. Thus we have proved that $\mathbf{F}(a) \subseteq \mathbf{F}'(a)$. By symmetry $\mathbf{F}'(a) \subseteq \mathbf{F}(a)$, so $\mathbf{F}(a) = \mathbf{F}'(a)$, contradiction. $\square$

Reflection theorems

We now want to consider to what extent sentences can reflect to proper subclasses of $\mathbf{V}$; this is a natural extension of our considerations for absoluteness.

Lemma 14.37. *Suppose that $\mathbf{M}$ and $\mathbf{N}$ are classes with $\mathbf{M} \subseteq \mathbf{N}$. Let $\varphi_0, \ldots, \varphi_n$ be a list of formulas such that if $i \leq n$ and ψ is a subformula of φ_i, then there is a $j \leq n$ such that φ_j is ψ. Then the following conditions are equivalent:*

(i) Each φ_i is absolute for $\mathbf{M}, \mathbf{N}$.

(ii) If $i \leq n$ and φ_i has the form $\forall x \varphi_j(x, y_1, \ldots, y_t)$ with $x, y_1, \ldots, y_t$ exactly all the free variables of φ_j, then

$$\forall y_1, \ldots, y_t \in \mathbf{M}[\forall x \in \mathbf{M}\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t) \to \forall x \in \mathbf{N}\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t)].$$

Proof. (i)$\Rightarrow$(ii): Assume (i) and the hypothesis of (ii). Suppose that $y_1, \ldots, y_t \in \mathbf{M}$ and $\forall x \in \mathbf{M}\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t)$. Thus by absoluteness $\forall x \in \mathbf{M}\varphi_j^{\mathbf{M}}(x, y_1, \ldots, y_t)$. Hence by absoluteness again, $\forall x \in \mathbf{N}\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t))$.

(ii)$\Rightarrow$(i): Assume (ii). We prove that φ_i is absolute for $\mathbf{M}, \mathbf{N}$ by induction on the length of φ_i. This is clear if φ_i is atomic, and it easily follows inductively if φ_i has the form $\neg\varphi_j$ or $\varphi_j \to \varphi_k$. Now suppose that φ_i is $\forall x \varphi_j(x, y_1, \ldots, y_t)$, and $y_1, \ldots, y_t \in \mathbf{M}$. then

$$\begin{aligned}
\varphi_i^{\mathbf{M}}(y_1, \ldots, y_t) &\leftrightarrow \forall x \in \mathbf{M}\varphi_j^{\mathbf{M}}(x, y_1, \ldots, y_t) \quad \text{(definition of relativization)}\\
&\leftrightarrow \forall x \in \mathbf{M}\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t) \quad \text{(induction hypothesis)}\\
&\leftrightarrow \forall x \in \mathbf{N}\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t) \quad \text{(by (ii)}\\
&\leftrightarrow \varphi_i^{\mathbf{N}}(y_1, \ldots, y_t) \quad \text{(definition of relativization)}
\end{aligned}$$

$\square$

Theorem 14.38. *Suppose that $Z(\alpha)$ is a set for every ordinal α, and the following conditions hold:*

(i) If $\alpha < \beta$, then $Z(\alpha) \subseteq Z(\beta)$.

(ii) If γ is a limit ordinal, then $Z(\gamma) = \bigcup_{\alpha<\gamma} Z(\alpha)$.

Let $\mathbf{Z} = \bigcup_{\alpha\in\mathbf{On}} Z(\alpha)$. Then for any formulas $\varphi_0, \ldots, \varphi_{n-1}$,

$$\forall\alpha\exists\beta > \alpha[\varphi_0, \ldots, \varphi_{n-1} \text{ are absolute for } Z(\beta), \mathbf{Z}].$$

Proof. Assume the hypothesis, and let an ordinal α be given. We are going to apply Lemma 14.37 with $\mathbf{N} = \mathbf{Z}$, and we need to find an appropriate $\beta > \alpha$ so that we can take $\mathbf{M} = Z(\beta)$ in 14.37.

We may assume that $\varphi_0, \ldots, \varphi_{n-1}$ is subformula-closed; i.e., if $i < n$, then every subformula of φ_i is in the list. Let A be the set of all $i < n$ such that φ_i begins with a universal quantifier. Suppose that $i \in A$ and φ_i is the formula $\forall x \varphi_j(x, y_1, \ldots, y_t)$, where $x, y_1, \ldots, y_t$ are exactly all the free variables of φ_j. We now define a class function $\mathbf{G}_i$ as follows. For any sets $y_1, \ldots, y_t$,

$$\mathbf{G}_i(y_1, \ldots, y_t) = \begin{cases} \text{the least } \eta \text{ such that } \exists x \in Z(\eta) \neg \varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t) & \text{if there is such,} \\ 0 & \text{otherwise.} \end{cases}$$

Then for each ordinal ξ we define

$$\mathbf{F}_i(\xi) = \sup\{\mathbf{G}_i(y_1, \ldots, y_t) : y_1, \ldots, y_t \in Z(\xi)\};$$

note that this supremum exists by the replacement axiom.

Now we define a sequence $\gamma_0, \ldots, \gamma_p, \ldots$ of ordinals by induction on $n \in \omega$. Let $\gamma_0 = \alpha + 1$. Having defined γ_p, let

$$\gamma_{p+1} = \max(\gamma_{p+1}, \sup\{\mathbf{F}_i(\xi) : i \in A, \xi \leq \gamma_p\} + 1).$$

Finally, let $\beta = \sup_{p \in \omega} \gamma_p$. Clearly $\alpha < \beta$ and β is a limit ordinal.

(1) If $i \in A$, $y_1, \ldots, y_t \in Z(\beta)$, and $\exists x \in \mathbf{Z} \neg \varphi_i^{\mathbf{Z}}(x, y_1, \ldots, y_t)$, then there is an $x \in Z(\beta)$ such that $\neg \varphi_i^{\mathbf{Z}}(x, y_1, \ldots, y_t)$.

In fact, choose p such that $y_1, \ldots, y_t \in Z(\gamma_p)$. Then $\mathbf{G}_i(y_1, \ldots, y_t) \leq \mathbf{F}_i(\gamma_p) < \gamma_{p+1}$. Hence an x as in (1) exists, with $x \in Z(\gamma_{p+1})$.

(1) clearly gives the desired conclusion. $\square$

Corollary 14.39. (The reflection theorem) *For any formulas* $\varphi_1, \ldots, \varphi_n$,

$$\text{ZF} \models \forall \alpha \exists \beta > \alpha [\varphi_1, \ldots, \varphi_n \text{ are absolute for } V_\beta]. \quad \square$$

Theorem 14.40. *Suppose that* $\mathbf{Z}$ *is a class and* $\varphi_1, \ldots, \varphi_n$ *are formulas. Then*

$$\forall X \subseteq \mathbf{Z} \exists A [X \subseteq A \subseteq \mathbf{Z} \text{ and } \varphi_1, \ldots, \varphi_n \text{ are absolute for } A, \mathbf{Z} \text{ and } |A| \leq \max(\omega, |X|)].$$

Proof. We may assume that $\varphi_1, \ldots, \varphi_n$ is subformula closed. For each ordinal α let $Z(\alpha) = \mathbf{Z} \cap V_\alpha$. Clearly there is an ordinal α such that $X \subseteq V_\alpha$, and hence $X \subseteq Z(\alpha)$. Now we apply Theorem 14.38 to obtain an ordinal $\beta > \alpha$ such that

(1) $$\varphi_1, \ldots, \varphi_n \text{ are absolute for } Z(\beta), \mathbf{Z}.$$

Let $\prec$ be a well-order of $Z(\beta)$. Let B be the set of all $i < n$ such that φ_i begins with a universal quantifier. Suppose that $i \in B$ and φ_i is the formula $\forall x \varphi_j(x, y_1, \ldots, y_t)$, where $x, y_1, \ldots, y_t$ are exactly all the free variables of φ_j. We now define a function H_i for each $i \in B$ as follows. For any sets $y_1, \ldots, y_t \in Z(\beta)$,

$$H_i(y_1, \ldots, y_t) = \begin{cases} \text{the } \prec\text{-least } x \in Z(\beta) \text{ such that } \neg\varphi_i^{Z(\beta)}(x, y_1, \ldots, y_t) & \text{if there is such,} \\ \text{the } \prec\text{-least element of } Z(\beta) & \text{otherwise.} \end{cases}$$

Let $A \subseteq Z(\beta)$ be closed under each function H_i, with $X \subseteq A$. We claim that A is as desired. To prove the absoluteness, it suffices by Lemma 14.37 to take any formula φ_i with $i \in A$, with notation as above, assume that $y_1, \ldots, y_t \in A$ and $\exists x \in \mathbf{Z} \neg\varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t)$, and find $x \in A$ such that $\neg\varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t)$. By (1) in the proof of Lemma 4.37, there is an $x \in Z(\beta)$ such that $\neg\varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t)$. Hence $H_i(y_1, \ldots, y_t)$ is an element of A such that $\neg\varphi_j^{\mathbf{Z}}(H_i(y_1, \ldots, y_t), y_1, \ldots, y_t)$, as desired.

It remains only to check the cardinality estimate. This is elementary. $\square$

Lemma 14.41. *Suppose that* $\mathbf{F}$ *is a bijection from* A *onto* $\mathbf{M}$*, and for any* $a, b \in A$ *we have* $a \in b$ *iff* $\mathbf{F}(a) \in \mathbf{F}(b)$*. Then for any formula* $\varphi(x_1, \ldots, x_n)$ *and any* $x_1, \ldots, x_n \in A$*,*

$$\varphi^A(x_1, \ldots, x_n) \leftrightarrow \varphi^{\mathbf{M}}(\mathbf{F}(x_1), \ldots, \mathbf{F}(x_n)).$$

Proof. An easy induction on φ. $\square$

Theorem 14.42. *Suppose that* $\mathbf{Z}$ *is a transitive class and* $\varphi_0, \ldots, \varphi_{m-1}$ *are sentences. Suppose that* X *is a transitive subset of* $\mathbf{Z}$*. Then there is a transitive set* M *such that* $X \subseteq M$*,* $|M| \le \max(\omega, |X|)$*, and for every* $i < m$*,* $\varphi_i^M \leftrightarrow \varphi_i^{\mathbf{Z}}$*.*

Proof. We may assume that the extensionality axiom is one of the φ_i's. Now we apply Theorem 14.40 to get a set A as indicated there. By Proposition 14.35, there is a transitive set M and a bijection G from A onto M such that for any $a, b \in A$, $a \in b$ iff $G(a) \in G(b)$. Hence all of the desired conditions are clear, except possibly $X \subseteq M$. We show that $G[X] = X$ by proving that $G(x) = x$ for all $x \in X$. In fact, suppose that $G(x) \neq x$ for some $x \in X$, and by the foundation axiom choose y such that $G(y) \neq y$ while $G(z) = z$ for all $z \in y$. Then if $z \in y$ we have $z, y \in X \subseteq A$, and hence $z = G(z) \in G(y)$. So $y \subseteq G(y)$. If $w \in G(y)$, then $w \in M = \text{rng}(G)$, so we can choose $z \in A$ such that $w = G(z)$. Then $G(z) \in G(y)$, so $z \in y$. Hence $w = G(z) = z$ and so $w \in y$. This gives $G(y) \subseteq y$, and finishes the proof. $\square$

Corollary 14.43. *Suppose that* S *is a set of sentences containing ZFC. Suppose also that* $\varphi_0, \ldots, \varphi_{n-1} \in S$*. Then*

$$S \models \exists M \left(M \text{ is transitive, } |M| = \omega, \text{ and } \bigwedge_{i<n} \varphi_i^M \right).$$

Proof. Take $\mathbf{Z} = \mathbf{V}$ and $X = \omega$ in Theorem 14.42. $\square$

The following corollary can be taken as a basis for working with countable transitive models of ZFC.

Theorem 14.44. *Suppose that S is a consistent set of sentences containing ZFC. Expand the basic set-theoretic language by adding an individual constant $\mathbf{M}$. Then the following set of sentences is consistent:*

$$S \cup \{\mathbf{M} \text{ is transitive}\} \cup \{|\mathbf{M}| = \omega\} \cup \{\varphi^{\mathbf{M}} : \varphi \in S\}.$$

Proof. Suppose that the indicated set is not consistent. Then there are $\varphi_0, \ldots, \varphi_{m-1}$ in S such that

$$S \models \mathbf{M} \text{ is transitive and } |\mathbf{M}| = \omega \to \neg \bigwedge_{i<n} \varphi_i^{\mathbf{M}};$$

it follows that

$$S \models \neg\exists\mathbf{M}\left(\mathbf{M} \text{ is transitive, } |\mathbf{M}| = \omega, \text{ and } \bigwedge_{i<n} \varphi_i^{\mathbf{M}}\right),$$

contradicting Corollary 14.43. □

EXERCISES

E14.1. Write out all the elements of V_α for $\alpha = 0, 1, 2, 3, 4$.

E14.2. Define by recursion

$$S(\alpha) = \bigcup_{\beta<\alpha} \mathscr{P}(S(\beta))$$

for every ordinal α. Prove that $V_\alpha = S(\alpha)$ for every ordinal α.

E14.3. Determine exactly the ranks of the following sets in terms of the ranks of the sets entering into their definitions. In some cases the rank is not completely determined by the ranks of the constituents; in such cases, describe all possibilities.

(i) $\{x\}$	(iv) $x \cup y$	(vii) $\bigcup x$	(x) R^{-1}
(ii) $\{x, y\}$	(v) $x \cap y$	(viii) $\mathrm{dmn}(R)$	
(iii) (x, y)	(vi) $x\backslash y$	(ix) $\mathscr{P}(x)$	

E14.4. Define $x\mathbf{R}y$ iff $(x, 1) \in y$. Show that $\mathbf{R}$ is well-founded and set-like on $\mathbf{V}$.

E14.5. (Continuing E14.4) By recursion let $\check{y} = \{(\check{x}, 1) : x \in y\}$ for any set y. Let $\mathbf{F}$ be the Mostowski collapsing function for $\mathbf{R}, \mathbf{V}$ in exercise E14.4. Prove that $\mathbf{F}(\check{y}) = y$ for every set y.

E14.6. Define $x\mathbf{R}y$ iff $x \in \mathrm{trcl}(y)$. Show that $\mathbf{R}$ is well-founded and set like on $\mathbf{V}$.

E14.7. (Continuing exercise E14.6) Let $\mathbf{F}$ be the Mostowski collapsing function for $\mathbf{R}, \mathbf{V}$. Show that $\mathbf{F}(x) = \mathrm{rank}(x)$ for every set x.

E14.8. Prove that if a is transitive, then $\{\mathrm{rank}(b) : b \in a\}$ is an ordinal.

E14.9. Show that for any set a we have $\mathrm{rank}(\mathrm{trcl}(a)) = \mathrm{rank}(a)$.

E14.10. For any infinite cardinal κ, let $H(\kappa)$ be the set of all x such that $|\mathrm{trcl}(x)| < \kappa$. Prove that $V_\omega = H(\omega)$. ($H(\omega)$ is the collection of all hereditarily finite sets.) Hint: $V_\omega \subseteq H(\omega)$ is easy. For the other direction, suppose that $x \in H(\omega)$, let $t = \mathrm{trcl}(x)$, and let $S = \{\mathrm{rank}(y) : y \in t\}$. Show that S is an ordinal.

E14.11. Which axioms of ZFC are true in **On**?

E14.12. Show that the power set operation is absolute for V_α for α limit.

E14.13. Let M be a countable transitive model of ZFC. Show that the power set operation is not absolute for M.

E14.14. Show that V_ω is a model of ZFC $-$ Inf.

E14.15. Show that the formula $\exists x(x \in y)$ is not absolute for all nonempty sets, but it is absolute for all nonempty transitive sets.

E14.16. Show that the formula $\exists z(x \in z)$ is not absolute for every nonempty transitive set.

E14.17. A formula is Σ_1 iff it has the form $\exists x\varphi$ with φ a Δ_0 formula; it is Π_1 iff it has the form $\forall x\varphi$ with φ a Δ_0 formula.

(i) Show that "X is countable" is equivalent on the basis of ZF to a Σ_1 formula.

(ii) Show that "α is a cardinal" is equivalent on the basis of ZF to a Π_1 formula.

E14.18. Prove that if κ is an infinite cardinal, then $H(\kappa) \subseteq V_\kappa$.

E14.19. Prove that for κ regular, $H(\kappa) = V_\kappa$ iff $\kappa = \omega$ or κ is inaccessible.

E14.20. Assume that κ is an infinite cardinal. Prove the following:

(a) $H(\kappa)$ is transitive.

(b) $H(\kappa) \cap \mathbf{On} = \kappa$.

(c) If $x \in H(\kappa)$, then $\bigcup x \in H(\kappa)$.

(d) If $x, y \in H(\kappa)$, then $\{x, y\} \in H(\kappa)$.

(e) If $y \subseteq x \in H(\kappa)$, then $y \in H(\kappa)$.

(f) If κ is regular and x is any set, then $x \in H(\kappa)$ iff $x \subseteq H(\kappa)$ and $|x| < \kappa$.

E14.21. Show that if κ is regular and uncountable, then $H(\kappa)$ is a model of all of the ZFC axioms except possibly the power set axiom.

15. Generic extensions and forcing

In this chapter we give the basic definitions and facts about generic extensions and forcing. Uses of these things will occupy much of remainder of these notes. We use "c.t.m." for "countable transitive model"; see Theorem 14.44.

Let $\mathbb{P} = (P, \leq, 1)$ be a forcing order. A *filter* on $\mathbb{P}$ is a subset G of P such that the following conditions hold:

(1) For all $p, q \in G$ there is an $r \in G$ such that $r \leq p$ and $r \leq q$.

(2) For all $p \in G$ and $q \in P$, if $p \leq q$ then $q \in G$.

Now let M be a c.t.m. of ZFC and let $\mathbb{P} = (P, \leq, 1) \in M$ be a forcing order. We say that G is $\mathbb{P}$-*generic over* M provided that the following conditions hold:

(3) G is a filter on $\mathbb{P}$.

(4) For every dense $D \subseteq P$ such that $D \in M$ we have $G \cap D \neq \emptyset$.

The definition of generic filter just given embodies a choice between two intuitive options. The option chosen corresponds to thinking of stronger conditions—those containing more information—as smaller in the forcing order. This may seen counter-intuitive, but it fits nicely with the embedding of forcing orders into Boolean algebras, as we will see. Many authors take the oppositite approach, considering stronger conditions as the greater ones. Of course this requires a corresponding change in the definition of generic filter (and denseness).

The following is the basic existence lemma for generic filters.

Lemma 15.1. *If M is a c.t.m. of* ZFC*, $\mathbb{P} = (P, \leq, 1) \in M$ is a forcing order, and $p \in P$, then there is a G which is $\mathbb{P}$-generic over M and $p \in G$.*

Proof. Let $\langle D_n : n \in \omega \rangle$ enumerate all of the dense subsets of P which are in M. We now define a sequence $\langle q_n : n \in \omega \rangle$ by recursion. Let $q_0 = p$. If $q_n \in P$ has been defined, choose $q_{n+1} \in D_n$ with $q_{n+1} \leq q_n$. Thus $p = q_0 \geq q_1 \geq \cdots$. Now we define

$$G = \{r \in P : q_n \leq r \text{ for some } n \in \omega\}.$$

We check that G is as desired. For (1), suppose that $r, s \in G$. Say $m, n \in \omega$ with $q_m \leq r$ and $q_n \leq s$. By symmetry, say $m \leq n$. Then $q_n \leq r, s$, and $q_n \in G$, as desired.

Condition (2) is clear. Hence (3) holds.

For (4), let $n \in \omega$. Then $q_{n+1} \in G \cap D_n$, as desired. □

It is important to realize that usually generic filters are not in the ground model M; this is expressed in the following lemma.

Lemma 15.2. *Suppose that M is a c.t.m. of ZFC and $\mathbb{P} = (P, \leq, 1) \in M$ is a forcing order. Assume the following:*

(1) For every $p \in P$ there are $q, r \in P$ such that $q \leq p$, $r \leq p$, and $q \perp r$.

Also suppose that G is $\mathbb{P}$-generic over M.

Then $G \notin M$.

Proof. Suppose to the contrary that $G \in M$. Then also $P \backslash G \in M$, since M is a model of ZFC and by absoluteness. We claim that $P \backslash G$ is dense. In fact, given $p \in P$, choose q, r as in (1). Then q, r cannot both be in G, by the definition of filter. So one at least is in $P \backslash G$, as desired. Since $P \backslash G$ is dense and in M, we contradict G being generic. $\square$

Most forcing orders used in forcing arguments satisfy the condition of Lemma 15.2; for more details on this lemma, see the exercises.

If $\mathbb{P}$ is a forcing order, a subset E of $\mathbb{P}$ is *predense* iff every $p \in \mathbb{P}$ is compatible with some member of E.

The following elementary proposition gives six equivalent ways to define generic filters.

Proposition 15.3. *Suppose that M is a c.t.m. of ZFC and $\mathbb{P}$ is a forcing order in M. Suppose that $G \subseteq P$ satisfies condition (2), i.e., if $p \in G$ and $p \leq q$, then $q \in G$. Then the following conditions are equivalent:*

(i) $G \cap D \neq \emptyset$ whenever $D \in M$ and D is dense in $\mathbb{P}$.

(ii) $G \cap A \neq \emptyset$ whenever $A \in M$ and A is a maximal antichain of $\mathbb{P}$.

(iii) $G \cap E \neq \emptyset$ whenever $E \in M$ and E is predense in $\mathbb{P}$.

Moreover, suppose that G satisfies (2) and one, hence all, of the conditions (i)–(iii). Then G is $\mathbb{P}$-generic over M iff the following condition holds:

(iv) For all $p, q \in G$, p and q are compatible.

Proof. (i)⇒(ii): Assume (i), and suppose that $A \in M$ is a maximal antichain of $\mathbb{P}$. Let $D = \{p \in P : p \leq q \text{ for some } q \in A\}$. We claim that D is dense. Suppose that r is arbitrary. Choose $q \in A$ such that r and q are compatible. Say $p \leq r, q$. Thus $p \in D$. So, indeed, D is dense. Clearly $D \in M$, since $A \in M$. By (i), choose $p \in D \cap G$. Say $p \leq q \in A$. Then $q \in G \cap A$, as desired.

(ii)⇒(iii): Assume (ii), and suppose that E is as in (iii). By Zorn's lemma, let A be a maximal member of

(1) $\{B \subseteq P : B \text{ is an antichain, and for every } p \in B \text{ there is a } q \in E \text{ such that } p \leq q\}$.

We claim that A is a maximal antichain. For, suppose that $p \perp q$ for all $q \in A$. Choose $s \in E$ such that p and s are compatible. Say $r \leq p, s$. Hence $r \perp q$ for all $q \in A$, so $r \notin A$. Thus $A \cup \{r\}$ is a member of (1), contradiction.

Clearly $A \in M$, since $E \in M$. So, since A is a maximal antichain, choose $p \in A \cap G$. Then choose $q \in E$ such that $p \leq q$. So $q \in E \cap G$, as desired.

(iii)⇒(i): Obvious.

Now we assume (2) in the definition, and (i)–(iii).

If G is $\mathbb{P}$-generic over M, clearly (iv) holds.

Now asume that (i)–(iv) hold, and suppose that $p, q \in G$; we want to find $r \in G$ such that $r \leq p, q$. Let

$$D = \{r : r \perp p \text{ or } r \perp q \text{ or } r \leq p, q\}.$$

We claim that D is dense in $\mathbb{P}$. For, let $s \in P$ be arbitrary. If $s \perp p$, then $s \leq s$ and $s \in D$, as desired. So suppose that s and p are compatible; say $t \leq s, p$. If $t \perp q$, then $t \leq s$ and $t \in D$, as desired. So suppose that t and q are compatible. Say $r \leq t, q$. Then $r \leq t \leq p$ and $r \leq t \leq s$, so $r \leq s$ and $r \leq p, q$, hence $r \in D$, as desired. This proves that D is dense.

Now by (i) choose $r \in D \cap G$. By (iv), r is compatible with p and r is compatible with q. So $r \leq p, q$, as desired. □

We are going to define the generic extension $M[G]$ by first defining *names* in M, and then producing the elements of $M[G]$ by using those names. The notion of a name is defined by recursion, using the following theorem.

Theorem 15.4. *Let P be any set. Then there is a function $\mathbf{F} : \mathbf{V} \to \mathbf{V}$ such that for any set τ,*

$$\mathbf{F}(\tau) = \begin{cases} 1 & \textit{if } \tau \textit{ is a relation and for all } \sigma, p \\ & \textit{if } (\sigma, p) \in \tau \textit{ then } p \in P \textit{ and } \mathbf{F}(\sigma) = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $\mathbf{R} = \{(\sigma, \tau) : \exists p \in P[(\sigma, p) \in \tau]\}$. Then $\mathbf{R}$ is well-founded on $\mathbf{V}$. In fact, let X be any nonempty set, and choose $\tau \in X$ of smallest rank. If $\sigma \mathbf{R} \tau$, then there is a $p \in P$ such that $(\sigma, p) \in \tau$, and then $\sigma \in \{\sigma\} \in \{\{\sigma\}, \{\sigma, p\}\} = (\sigma, p) \in \tau$, and hence $\text{rank}(\sigma) < \text{rank}(\tau)$. It follows that $\sigma \notin X$, as desired.

Also, $\mathbf{R}$ is set-like on $\mathbf{V}$. In fact, for any set τ we have

$$\text{pred}_{\mathbf{VR}}(\tau) = \{\sigma : \exists p \in P[(\sigma, p) \in \tau]\} = \left\{\sigma \in \bigcup\bigcup \tau : \exists p \in P[(\sigma, p) \in \tau]\right\}.$$

Now we define $\mathbf{G} : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ by setting

$$\mathbf{G}(\tau, f) = \begin{cases} 1 & \text{if } \tau \text{ is a relation, } f \text{ is a function with domain} \\ & \text{pred}_{\mathbf{VR}}(\tau), \text{ and } f(\sigma) = 1 \text{ for all } \sigma \in \text{pred}_{\mathbf{VR}}(\tau) \\ 0 & \text{otherwise.} \end{cases}$$

Now we obtain $\mathbf{F}$ by Theorem 5.7: for any set τ,

$$\begin{aligned} \mathbf{F}(\tau) &= \mathbf{G}(\tau, \mathbf{F} \restriction \text{pred}_{\mathbf{VR}}(\tau)) \\ &= \begin{cases} 1 & \text{if } \tau \text{ is a relation and } \mathbf{F}(\sigma) = 1 \\ & \text{for all } \sigma \in \text{pred}_{\mathbf{VR}}(\tau) \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} 1 & \text{if } \tau \text{ is a relation and for all } \sigma \text{ and } p \in P, \text{ if} \\ & (\sigma, p) \in \tau \text{ then } \mathbf{F}(\sigma) = 1, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

□

Now with $\mathbf{F}$ as in this theorem, a P name is a set τ such that $\mathbf{F}(\tau) = 1$.

Corollary 15.5. *Let P be any set. Then τ is a P-name iff τ is a relation and for all $(\sigma, p) \in \tau[\sigma$ is a P-name and $p \in P]$.*

Proof. $\Rightarrow$: suppose that τ is a P-name. Thus $\mathbf{F}(\tau) = 1$, so τ is a relation, and for all $(\sigma, p) \in \tau$, $\mathbf{F}(\sigma) = 1$ and $p \in P$. Hence for all $(\sigma, p) \in \tau[\sigma$ is a P-name and $p \in P]$.

Conversely, suppose that τ is a relation and for all $(\sigma, p) \in \tau[\sigma$ is a P-name and $p \in P]$. Then τ is a relation and for all $(\sigma, p) \in \tau[\mathbf{F}(\sigma) = 1$ and $p \in P]$. Hence $\mathbf{F}(\tau) = 1$, so τ is a P-name. $\square$

Note that "τ is a P-name" is absolute.

For any set P, we denote by $\mathbf{V}^P$ the (proper) class of all P-names. If M is a c.t.m. of ZFC, then we let $M^P = \mathbf{V}^P \cap M$. Note by absoluteness that

$$M^P = \{\tau \in M : (\tau \text{ is a } P\text{-name})^M\}.$$

If $G \subseteq P$, we define $\mathrm{val}(\tau, G)$ by recursion.

Theorem 15.6. *If $G \subseteq P$ then there is a function* val *such that for any set τ,*

$$\mathrm{val}(\tau, G) = \{\mathrm{val}(\sigma, G) : \exists p \in G[(\sigma, p) \in \tau]\}.$$

Proof. Let $\mathbf{R}$ be as in the proof of Theorem 15.4. Now we define $\mathbf{G} : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ by setting

$$\mathbf{G}(\tau, f) = \begin{cases} \{f(\sigma) : \exists p \in G[(\sigma, p) \in \tau]\} & \text{if } f \text{ is a function} \\ & \text{with domain } \mathrm{pred}_{\mathbf{VR}}(\tau), \\ 0 & \text{otherwise.} \end{cases}$$

Now we obtain $\mathbf{F}$ by Theorem 5.7; for any set τ,

$$\begin{aligned} \mathbf{F}(\tau) &= \mathbf{G}(\tau, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{VR}}(\tau)) \\ &= \{\mathbf{F}(\sigma) : \exists p \in G[(\sigma, p) \in \tau]\} \end{aligned}$$ $\square$

We also write τ_G in place of $\mathrm{val}(\tau, G)$. Notice that val is absolute for c.t.m. of ZFC.

Finally, if M is a c.t.m. of ZFC and $G \subseteq P \in M$, we define

$$M[G] = \{\tau_G : \tau \in M^P\}.$$

Note that $M^P \subseteq M$, and hence M^P is countable. Hence by the replacement axiom, $M[G]$ is also countable.

Now we can sketch the goals of chapters 15 and 16. We start with a c.t.m. M, and take $\kappa \in M$ such that κ is regular and greater than ω_1 (in the sense of M). Then we let $\mathbb{P}$ be the forcing order $(P, \supseteq, \emptyset)$, where P is the set of all finite functions contained in $\kappa \times 2$. Let G be any generic filter over $\mathbb{P}$. Then we show that $M[G]$ is a model of ZFC, in $M[G]$ we have $2^\omega = \kappa$, and cardinals in M and in $M[G]$ are the same. This shows the consistency of $\neg$CH.

On the other hand, starting with a c.t.m. M, we let $\mathbb{P}$ be the forcing order $(P, \supseteq, \emptyset)$ with P the set of all countable functions contained in $\omega_1 \times 2$. Then we show that $M[G]$

is a model of ZFC, in $M[G]$ we have $2^\omega = \omega_1$, and ω_1 is the same in M and $M[G]$. This proves the consistency of CH.

Lemma 15.7. *If M is a c.t.m. of ZFC, $\mathbb{P} \in M$ is a forcing order, and G is a filter on $\mathbb{P}$, then $M[G]$ is transitive.*

Proof. Suppose that $x \in y \in M[G]$. Then there is a $\tau \in M^P$ such that $y = \tau_G$. Since $x \in \tau_G$, there is a $\sigma \in M^P$ such that $x = \sigma_G$. So $x \in M[G]$. □

The following Lemma says that $M[G]$ is the smallest c.t.m. of ZFC which contains M as a subset and G as a member, once we show that it really is a model of ZFC. This lemma will be extremely useful in what follows.

Lemma 15.8. *Suppose that M is a c.t.m. of ZFC, $\mathbb{P} \in M$ is a forcing order, G is a filter on $\mathbb{P}$, N is a c.t.m. of ZFC, $M \subseteq N$, and $G \in N$. Then $M[G] \subseteq N$.*

Proof. Take any $x \in M[G]$. Say $x = \mathrm{val}(\sigma, G)$ with $\sigma \in M^{\mathbb{P}}$. Then $\sigma, G \in N$, so by absoluteness, $x = (\mathrm{val}(\sigma, G))^N \in N$. □

To show that M is a subset of $M[G]$, we need a function ˇ mapping M into the collection of all P-names. Again the definition is by recursion.

Theorem 15.9. *Suppose that $(P, \leq, 1)$ is a forcing order. Then there is a function $\mathbf{F} : \mathbf{V} \to \mathbf{V}$ such that for every set x, $\mathbf{F}(x) = \{(\mathbf{F}(y), 1) : y \in x\}$.*

Proof. Let $\mathbf{R} = \{(y, x) : y \in x\}$. Clearly $\mathbf{R}$ is well-founded and set-like on $\mathbf{V}$. Define $\mathbf{G} : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ by

$$\mathbf{G}(x, f) = \begin{cases} \{(f(y), 1) : y \in x\} & \text{if } f \text{ is a function with domain } x, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\mathbf{F}$ be obtained from $\mathbf{G}$ by Theorem 5.7. Then for any set x,

$$\mathbf{F}(x) = \mathbf{G}(x, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{VR}}(x)) = \{(\mathbf{F}(y), 1) : y \in x\}. \qquad \square$$

We denote $\mathbf{F}(x)$ by $\check{x}$. Thus for any set x,

$$\check{x} = \{(\check{y}, 1) : y \in x\}.$$

Note that this depends on $\mathbb{P}$; we could denote it by $\mathrm{check}(\mathbb{P}, x)$ to bring this out, if necessary. Again this function is absolute for transitive models of ZFC.

Lemma 15.10. *Suppose that M is a transitive model of ZFC, $\mathbb{P} \in M$ is a forcing order, and G is a non-empty filter on P. Then*

(i) For all $x \in M$, $\check{x} \in M^P$ and $\mathrm{val}(\check{x}, G) = x$.

(ii) $M \subseteq M[G]$.

Proof. Absoluteness implies that $\check{x} \in M^P$ for all $x \in M$. To prove $\mathrm{val}(\check{x}, G) = x$ for all x, suppose that this is not true, and by the foundation axiom take x such that $\mathrm{val}(\check{x}, G) = x$ while $\mathrm{val}(\check{y}, G) = y$ for all $y \in x$. (See Theorem 5.5.) Then

$$\begin{aligned}\mathrm{val}(\check{x}, G) &= \{\mathrm{val}(\sigma, G) : (\sigma, 1) \in \check{x}\}\\ &= \{\mathrm{val}(\check{y}, G) : y \in x\}\\ &= \{y : y \in x\}\\ &= x,\end{aligned}$$

contradiction.

Finally (ii) is immediate from (i). □

Next, for any partial order $\mathbb{P}$ we define a P-name Γ. It depends on $\mathbb{P}$ and could be defined as $\Gamma_{\mathbb{P}}$ to bring this out.

$$\Gamma = \{(\check{p}, p) : p \in P\}.$$

Lemma 15.11. *Suppose that M is a transitive model of ZFC, $\mathbb{P} \in M$ is a forcing order, and G is a non-empty filter on P. Then $\Gamma_G = G$. Hence $G \in M[G]$.*

Proof. $\Gamma_G = \{\check{p}_G : p \in G\} = \{p : p \in G\} = G$. □

Lemma 15.12. *Suppose that M is a transitive model of ZFC, $\mathbb{P} \in M$ is a forcing order, and G is a non-empty filter on P. Then $\mathrm{rank}(\tau_G) \leq \mathrm{rank}(\tau)$ for all $\tau \in M^P$.*

Proof. We prove this by induction on τ. Suppose that it is true for all $\sigma \in \mathrm{dmn}(\tau)$. If $x \in \tau_G$, then there is a $(\sigma, p) \in \tau$ such that $p \in G$ and $x = \sigma_G$. Hence by the inductive assumption, $\mathrm{rank}(x) \leq \mathrm{rank}(\sigma)$. Hence

$$\mathrm{rank}(\tau_G) = \sup_{x \in \tau_G} (\mathrm{rank}(x) + 1) \leq \mathrm{rank}(\tau). \qquad \square$$

Note by absoluteness of the rank function that $\mathrm{rank}(\tau)$ is the same within M or $M[G]$.

Lemma 15.13. *Suppose that M is a transitive model of ZFC, $\mathbb{P} \in M$ is a forcing order, and G is a non-empty filter on $\mathbb{P}$. Then M and $M[G]$ have the same ordinals.*

Proof. Since $M \subseteq M[G]$, every ordinal of M is an ordinal of $M[G]$. Now suppose that α is any ordinal of $M[G]$. Write $\alpha = \tau_G$, where $\tau \in M^P$. Now $\mathrm{rank}(\tau) = \mathrm{rank}^M(\tau) \in M$. So by Lemma 15.12, $\alpha = \mathrm{rank}(\alpha) = \mathrm{rank}(\tau_G) \leq \mathrm{rank}(\tau) \in M$, so $\alpha \in M$. □

The following lemma will be used often.

Lemma 15.14. *Suppose that M is a transitive model of ZFC, $\mathbb{P} \in M$ is a forcing order, $E \subseteq P$, $E \in M$, and G is a $\mathbb{P}$-generic filter over M. Then:*

(i) Either $G \cap E \neq \emptyset$, or there is a $q \in G$ such that $r \perp q$ for all $r \in E$.

(ii) If E is dense below p and $q \leq p$, then E is dense below q.

(iii) If $p \in G$ and E is dense below p, then $G \cap E \neq \emptyset$.

Proof. Let

$$D = \{p : p \leq r \text{ for some } r \in E\} \cup \{q : q \perp r \text{ for all } r \in E\}.$$

We claim that D is dense. For, suppose that $q \in P$. We may assume that $q \notin D$. So q is not in the second set defining D, and so there is an $r \in E$ which is compatible with q. Take p with $p \leq q, r$. then $p \in D$ and $p \leq q$, as desired.

Since D is dense, we can choose $s \in G \cap D$. Now to prove (i), suppose that $G \cap E = \emptyset$. Then s is not in the first set defining D, so it is in the second set, as desired.

(ii) is clear.

For (iii), suppose that $G \cap E = \emptyset$, and by (i) choose $q \in G$ such that $q \perp r$ for every $r \in E$. By the definition of filter, there is a $t \in G$ with $t \leq p, q$. Since E is dense below p, there is then a $u \in E$ with $u \leq t$. Thus $u \leq q$, so it is not the case that $u \perp q$, contradiction. □

Proposition 15.15. *Suppose that M is a transitive model of ZFC, $\mathbb{P} \in M$ is a forcing order, and G is a $\mathbb{P}$-generic filter over M. Suppose that $p \in P$ and p is compatible with each member of G. Then $p \in G$.*

Proof. The set $\{q \in P : q \leq p \text{ or } q \perp p\}$ is clearly dense in $\mathbb{P}$. □

Now we introduce the idea of forcing. Recall that the logical primitive notions are $\neg$, $\rightarrow$, $\forall$, and $=$.

With each formula $\varphi(v_0, \ldots, v_{m-1})$ of the language of set theory we define another formula

$$p \Vdash_{\mathbb{P},M} \varphi(\sigma_0, \ldots, \sigma_{m-1}),$$

which we read as "p forces $\varphi(\sigma_0, \ldots, \sigma_{m-1})$ with respect to $\mathbb{P}$ and M"; it is the statement

$\mathbb{P}$ is a forcing order, $\mathbb{P} \in M$, $\sigma_0, \ldots, \sigma_{m-1} \in M^P$, $p \in P$, and for every G which is $\mathbb{P}$-generic over M, if $p \in G$, then the relativization of φ to $M[G]$ holds for the elements $\sigma_{0G}, \ldots, \sigma_{(m-1)G}$.

Note that since G in general is not in the model M, this definition cannot be given in M. The main aim of the next part of this chapter is to show that the definition is equivalent to one which is definable in any countable transitive model of ZFC. We do this by defining a notion $\Vdash^*$ in M, and then proving the equivalence of $\Vdash^*$ with $\Vdash$. For the definition we first define a function to take care of atomic φ.

To understand the following theorem, see the definition and corollary following its proof.

Theorem 15.16. *Let $\mathbb{P}$ be a forcing order and e the embedding of $\mathbb{P}$ into $\mathrm{RO}(\mathbb{P})$. Then there is a class function F mapping $2 \times \mathbf{V}^P \times \mathbf{V}^P$ into $\mathrm{RO}(\mathbb{P})$ such that for any $\sigma, \tau \in \mathbf{V}^P$,*

$$F(0, \sigma, \tau) = \prod_{(\xi,p) \in \tau} [-e(p) + F(1, \xi, \sigma)] \cdot \prod_{(\eta,q) \in \sigma} [-e(q) + F(1, \eta, \tau)]$$

$$F(1, \sigma, \tau) = \sum_{(\xi,p) \in \tau} [e(p) \cdot F(0, \sigma, \xi)].$$

Proof. We are going to apply the recursion theorem 5.7. Let $\mathbf{A} = 2 \times \mathbf{V}^P \times \mathbf{V}^P$. Let

$$
\begin{array}{lll}
(\delta', \sigma', \tau')\mathbf{R}(\delta, \sigma, \tau) & \text{iff} & (\delta', \sigma', \tau'), (\delta, \sigma, \tau) \in \mathbf{A}, \text{ and} \\
 & & [\delta' = 1, \delta = 0, \tau' = \sigma \text{ and } \mathrm{rank}(\sigma') < \mathrm{rank}(\tau)] \text{ or} \\
 & & [\delta' = 1, \delta = 0, \tau' = \tau \text{ and } \mathrm{rank}(\sigma') < \mathrm{rank}(\sigma)] \text{ or} \\
 & & [\delta' = 0, \delta = 1, \sigma' = \sigma \text{ and } \mathrm{rank}(\tau') < \mathrm{rank}(\tau)]
\end{array}
$$

We claim that $\mathbf{R}$ is well-founded on $\mathbf{A}$. In fact, note that if $(0, \sigma'', \tau'')\mathbf{R}(1, \sigma', \tau')\mathbf{R}(0, \sigma, \tau)$, then

$$
\begin{aligned}
\sigma'' = \sigma', \ & \mathrm{rank}(\tau'') < \mathrm{rank}(\tau') \text{ and} \\
& [(\tau' = \sigma \text{ and } \mathrm{rank}(\sigma') < \mathrm{rank}(\tau)) \\
& \text{or } (\tau' = \tau \text{ and } \mathrm{rank}(\sigma') < \mathrm{rank}(\sigma))].
\end{aligned}
$$

Hence one of the following two conditions holds:

(1) $\tau' = \sigma$, $\mathrm{rank}(\sigma') < \mathrm{rank}(\tau)$, $\sigma'' = \sigma'$, and $\mathrm{rank}(\tau'') < \mathrm{rank}(\tau')$.

(2) $\tau' = \tau$, $\mathrm{rank}(\sigma') < \mathrm{rank}(\tau)$, $\sigma'' = \sigma'$, and $\mathrm{rank}(\tau'') < \mathrm{rank}(\tau')$.

In either case we clearly have $\max(\mathrm{rank}(\sigma''), \mathrm{rank}(\tau'')) < \max(\mathrm{rank}(\sigma), \mathrm{rank}(\tau)$. Hence there does not exist a sequence $\cdots a_2 \mathbf{R} a_1 \mathbf{R} a_0$. Hence $\mathbf{R}$ is well-founded on $\mathbf{A}$.

Next we claim that $\mathbf{R}$ is set-like on $\mathbf{A}$. For, let $(\delta, \sigma, \tau) \in \mathbf{A}$. Say $\sigma \in V_\alpha$ and $\tau \in V_\beta$. Then if $\delta = 0$ we have

$$
\begin{aligned}
\mathrm{pred}_{\mathbf{AR}}(0, \sigma, \tau) = \{ & (1, \sigma', \tau') \in 2 \times V^P \times V^P : [\tau' = \sigma \text{ and } \mathrm{rank}(\sigma') < \mathrm{rank}(\tau)] \text{ or} \\
& [\tau' = \tau \text{ and } \mathrm{rank}(\sigma') < \mathrm{rank}(\sigma)]\} \\
= \{ & (\delta', \sigma', \tau') \in 2 \times V_\beta \times \{\sigma\} : \delta' = 1 \text{ and } \sigma' \in V^P \\
& \text{and } \mathrm{rank}(\sigma') < \mathrm{rank}(\tau)\} \cup \\
& \{(\delta', \sigma', \tau') \in 2 \times V_\alpha \times \{\tau\} : \delta' = 1 \text{ and } \sigma' \in V^P \\
& \text{and } \mathrm{rank}(\sigma') < \mathrm{rank}(\sigma)\}.
\end{aligned}
$$

If $\delta = 1$, then

$$
\begin{aligned}
\mathrm{pred}_{\mathbf{AR}}(1, \sigma, \tau) &= \{(0, \sigma', \tau') : \sigma' = \sigma \text{ and } \mathrm{rank}(\tau') < \mathrm{rank}(\tau)\} \\
&= \{(\delta', \sigma', \tau') \in 2 \times \{\sigma\} \times V_\beta : \delta' = 0 \text{ and } \tau' \in V^P \\
&\quad \text{and } \mathrm{rank}(\tau') < \mathrm{rank}(\sigma)\}.
\end{aligned}
$$

This proves the claim.

Now we define $\mathbf{G} : 2 \times \mathbf{V}^P \times \mathbf{V}^P \to \mathbf{V}$. Let $\delta \in 2$, $\sigma \in \mathbf{V}^P$, $\tau \in \mathbf{V}^P$, and suppose that f is a function mapping $\mathrm{pred}_{\mathbf{AR}}(\delta, \sigma, \tau)$ into $\mathrm{RO}(\mathbb{P})$. Then we set

$$
\text{for } \delta = 0: \quad \mathbf{G}(0, \sigma, \tau, f) = \prod_{(\xi, p) \in \tau} [-e(p) + f(1, \xi, \sigma)] \cdot \prod_{(\eta, q) \in \sigma} [-e(q) + f(1, \eta, \tau)];
$$

$$
\text{for } \delta = 1: \quad \mathbf{G}(1, \sigma, \tau, f) = \sum_{(\xi, p) \in \tau} [e(p) \cdot f(0, \sigma, \xi)].
$$

Note that this makes sense, since $(\xi, p) \in \tau$ implies that $(1, \xi, \sigma)\mathbf{R}(0, \sigma, \tau)$, $(1, \eta, \tau)\mathbf{R}(0, \sigma, \tau)$ and $(0, \sigma, \xi)\mathbf{R}(1, \sigma, , \tau)$.

For any other $f \in \mathbf{V}$ let $\mathbf{G}(\delta, \sigma, \tau, f) = \emptyset$.

Now let $\mathbf{F}$ be obtained by Theorem 5.7: $\mathbf{F}(\delta, \sigma, \tau) = \mathbf{G}(\delta, \sigma, \tau, \mathbf{F} \restriction \text{pred}_{\mathbf{AR}}(\delta, \sigma, \tau))$. Then we have

$$\begin{aligned}
\mathbf{F}(0, \sigma, \tau) &= \mathbf{G}(0, \sigma, \tau, \mathbf{F} \restriction \text{pred}_{\mathbf{AR}}(0, \sigma, \tau)) \\
&= \prod_{(\xi,p)\in\tau} [-e(p) + \mathbf{F}(1, \xi, \sigma)] \cdot \prod_{(\eta,q)\in\sigma} [-e(q) + \mathbf{F}(1, \eta, \tau)]; \\
\mathbf{F}(1, \sigma, \tau) &= \mathbf{G}(1, \sigma, \tau, \mathbf{F} \restriction \text{pred}_{\mathbf{AR}}(1, \sigma, \tau)) \\
&= \sum_{(\xi,p)\in\tau} [e(p) \cdot \mathbf{F}(0, \sigma, \xi)]. \qquad \square
\end{aligned}$$

Now with $\mathbf{F}$ as in this theorem, we define $[\![\sigma = \tau]\!] = \mathbf{F}(0, \sigma, \tau)$ and $[\![\sigma \in \tau]\!] = \mathbf{F}(1, \sigma, \tau)$.

Corollary 15.17. *With $\mathbb{P}$ a forcing order and $\sigma, \tau \in \mathbf{V}^P$ we have*

$$\begin{aligned}
[\![\sigma = \tau]\!] &= \prod_{(\xi,p)\in\tau} [-e(p) + [\![\xi \in \sigma]\!]] \cdot \prod_{(\eta,q)\in\sigma} [-e(q) + [\![\eta \in \tau]\!]]; \\
[\![\sigma \in \tau]\!] &= \sum_{(\xi,p)\in\tau} [e(p) \cdot [\![\sigma = \xi]\!]]. \qquad \square
\end{aligned}$$

Thus we are defining $[\![\sigma = \tau]\!]$ to mean, in a sense, that every element of σ is an element of τ and every element of τ is an element of σ. And we define $[\![\sigma \in \tau]\!]$ to mean, in a sense, that there is some element of τ to which σ is equal. We now extend the definition of Boolean values to arbitrary formulas.

$$\begin{aligned}
[\![\neg\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!] &= -[\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!]; \\
[\![\varphi(\sigma_0, \ldots, \sigma_{m-1}) \to \psi(\sigma_0, \ldots, \sigma_{m-1})]\!] &= -[\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!] + [\![\psi(\sigma_0, \ldots, \sigma_{m-1})]\!]; \\
[\![\forall x\varphi(\sigma_0, \ldots, \sigma_{m-1}, x)]\!] &= \prod_{\tau\in\mathbf{V}^P} [\![\varphi(\sigma_0, \ldots, \sigma_{m-1}, \tau)]\!].
\end{aligned}$$

Note that the last big product has index set which is a proper class in general. But the values are all in the Boolean algebra $\text{RO}(\mathbb{P})$, so this makes sense. Namely, this part of the definition can be rewritten as follows:

$$[\![\forall x\varphi(\sigma_0, \ldots, \sigma_{m-1}, x)]\!] = \prod\{a \in \text{RO}(\mathbb{P}) : \exists\tau \in \mathbf{V}^P(a = [\![\varphi(\sigma_0, \ldots, \sigma_{m-1}, \tau)]\!])\}.$$

Lemma 15.18.

$$\begin{aligned}
[\![\varphi(\sigma_0, \ldots, \sigma_{m-1}) \vee \psi(\sigma_0, \ldots, \sigma_{m-1})]\!] &= [\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!] + [\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!]; \\
[\![\varphi(\sigma_0, \ldots, \sigma_{m-1}) \wedge \psi(\sigma_0, \ldots, \sigma_{m-1})]\!] &= [\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!] \cdot [\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!]; \\
[\![\exists x\varphi(\sigma_0, \ldots, \sigma_{m-1}, x)]\!] &= \sum_{\tau\in\mathbf{V}^P} [\![\varphi(\sigma_0, \ldots, \sigma_{m-1}, \tau)]\!].
\end{aligned}$$

Proof. Recall from Chapter 2 the definitions of $\vee$, $\wedge$, $\exists$. We omit the parameters $\sigma_0, \ldots, \sigma_{m-1}$.

$$\begin{aligned}
[\![\varphi \vee \psi]\!] &= [\![\neg\varphi \to \psi)]\!] \\
&= - - [\![\varphi]\!] + [\![\psi]\!]) \\
&= [\![\varphi]\!] + [\![\psi]\!]; \\
[\![\varphi \wedge \psi) &= [\![\neg(\varphi \to \neg\psi)]\!] \\
&= -(-[\![\varphi]\!] + -[\![\psi]\!]) \\
&= [\![\varphi]\!] \cdot [\![\psi]\!]; \\
[\![\exists v_i \varphi]\!] &= [\![\neg\forall v_i \neg\varphi]\!] \\
&= -\prod_{\tau \in \mathbf{V}^{\mathbb{P}}} -[\![\varphi(\tau)]\!] \\
&= \sum_{\tau \in \mathbf{V}^{\mathbb{P}}} [\![\varphi(\tau)]\!]. \qquad \square
\end{aligned}$$

Now we can give our alternate definition of forcing:

$$p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1}) \quad \text{iff} \quad e(p) \leq [\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!].$$

It is important that Boolean values and $\Vdash^*$ are definable in a c.t.m. M of ZFC. Note that the discussion of Boolean values and of $\Vdash^*$ has taken place in our usual framework of set theory. The complete BA RO($\mathbb{P}$) is in general uncountable. Given a c.t.m. M of ZFC, the definitions can take place within M, and while M may be a model of "RO($\mathbb{P}$) is uncountable", actually RO$(\mathbb{P})^M$ is countable. Thus even if $\sigma_0, \ldots, \sigma_{m-1}$ are members of M^P, the statements $p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1})$ and $(p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1}))^M$ are much different, since the products and sums involved in the definition of the former range over a possibly uncountable complete BA, while those in the latter range only over a countable BA (which is actually incomplete if it is infinite).

Now we prove the fundamental theorem connecting the notion $\Vdash^*$ in a c.t.m. M with the notion $\Vdash$, whose definition takes place outside M.

Theorem 15.19. (The Forcing Theorem) *Suppose that M is a c.t.m. of ZFC, $\mathbb{P} \in M$ is a forcing order, and G is $\mathbb{P}$-generic over M. Then the following conditions are equivalent:*

(i) There is a $p \in G$ such that $(p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1}))^M$.

(ii) $\varphi(\sigma_{0G}, \ldots, \sigma_{(m-1)G})$ holds in $M[G]$.

Proof. First we prove the equivalence for $\sigma = \tau$ and $\sigma \in \tau$ by induction on the well-founded relation $\mathbf{R}$ given in the proof of Theorem 15.16. For (i)$\Rightarrow$(ii), suppose that $p \in G$ and $(p \Vdash^* \sigma = \tau)^M$. We want to show that $\sigma_G = \tau_G$. Suppose that $a \in \sigma_G$. Then there is an $(\eta, q) \in \sigma$ such that $q \in G$ and $a = \eta_G$. Now $e(p) \leq -e(q) + [\![\eta \in \tau]\!]$, so $e(p) \cdot e(q) \leq [\![\eta \in \tau]\!]$. Since $p, q \in G$, choose $r \in G$ with $r \leq p, q$. Then $e(r) \leq [\![\eta \in \tau]\!]$. So $(r \Vdash^* \eta \in \tau)^M$, so by the inductive hypothesis $a = \eta_G \in \tau_G$. So we have shown that $\sigma_G \subseteq \tau_G$. Similarly, $\tau_G \subseteq \sigma_G$. So we have shown that (i)$\Rightarrow$(ii) for $\sigma = \tau$.

Now suppose that $p \in G$ and $(p \Vdash^* \sigma \in \tau)^M$. Thus $e(p) \leq \sum_{(\xi,s)\in\tau}[e(s) \cdot [\![\sigma = \xi]\!]$. Now we claim

(1) $\{q : \exists(\xi, s) \in \tau[q \leq s \text{ and } e(q) \leq [\![\sigma = \xi]\!]]\}$ is dense below p.

For, suppose that $r \leq p$. Then $e(r) \leq \sum_{(\xi,s)\in\tau}[e(s) \cdot [\![\sigma = \xi]\!]$, and hence

$$e(r) = e(r) \cdot \sum_{(\xi,s)\in\tau} [e(s) \cdot [\![\sigma = \xi]\!] = \sum_{(\xi,s)\in\tau} [e(r) \cdot e(s) \cdot [\![\sigma = \xi]\!].$$

It follows that there is a $(\xi, s) \in \tau$ such that $e(r) \cdot e(s) \cdot [\![\sigma = \xi]\!] \neq 0$. By Theorem 13.20(i) choose t so that $e(t) \leq e(r) \cdot e(s) \cdot [\![\sigma = \xi]\!]$. By Theorem 13.20(iii), t and r are compatible. Say $u \leq t, r$. Also, $e(u) \leq e(t) \leq e(s)$, so u and s are compatible. Say $v \leq u, s$. Then $e(v) \leq e(u) \leq e(t) \leq [\![\sigma = \xi]\!]$, so v is in the set of (1). So (1) holds.

Now by (1) and Theorem 15.14(iii), there exist a $q \in G$ with $q \leq p$ and $(\xi, s) \in \tau$ such that $e(q) \leq [\![\sigma = \xi]\!]$ and $q \leq s$. So $(q \Vdash^* \sigma = \xi)^M$, and by the inductive hypothesis we have $\sigma_G = \xi_G$. Now $q \leq s$ implies that $s \in G$, and so $(\xi, s) \in \tau$ yields $\xi_G \in \tau_G$ (by the definition of val). This proves (i)⇒(ii) for $\sigma \in \tau$.

Now for (ii)⇒(i), suppose that $\sigma_G = \tau_G$. Let

$$\begin{aligned} D =&\{r : (r \Vdash^* \sigma = \tau)^M \text{ or } \exists(\xi, p) \in \tau[r \leq p \text{ and } e(r) \leq -[\![\xi \in \sigma]\!]] \text{ or} \\ &\quad \exists(\eta, q) \in \sigma[r \leq q \text{ and } e(r) \leq -[\![\eta \in \tau]\!]]\}. \end{aligned}$$

We claim that D is dense. For, suppose that $s \in P$. Assume that $(s \not\Vdash^* \sigma = \tau)^M$. Thus $e(s) \not\leq [\![\sigma = \tau]\!]$, so

$$\begin{aligned} 0 &\neq e(s) \cdot -[\![\sigma = \tau]\!] \\ &= e(s) \cdot \Bigg(\sum_{(\xi,p)\in\tau} (e(p) \cdot -[\![\xi \in \sigma]\!]) + \sum_{(\eta,q)\in\sigma} (e(q) \cdot -[\![\eta \in \tau]\!]) \Bigg). \end{aligned}$$

It follows that one of the following conditions holds:

(2) There is a $(\xi, p) \in \tau$ such that $e(s) \cdot e(p) \cdot -[\![\xi \in \sigma]\!] \neq 0$.

(3) There is a $(\eta, q) \in \sigma$ such that $e(s) \cdot e(q) \cdot -[\![\xi \in \tau]\!] \neq 0$.

Suppose that (2) holds, with (ξ, p) as indicated. By Theorem 13.20(i) choose t such that $e(t) \leq e(s) \cdot e(p) \cdot -[\![\xi \in \sigma]\!]$. Since $e(t) \leq e(p)$, by Theorem 13.20(iv) we get $u \leq t, p$. Then $e(u) \leq e(t) \leq e(s)$, so again by Theorem 13.20(iv) we get v such that $v \leq u, s$. Then $v \leq u \leq p$ and $e(v) \leq e(u) \leq e(t) \leq -[\![\xi \in \sigma]\!]$. Thus $v \in D$, as desired.

By a similar argument, (3) gives an element of D below s. Hence D is dense.

Choose $r \in G \cap D$. We claim that $(r \Vdash^* \sigma = \tau)^M$. Otherwise one of the following conditions holds:

(4) $\exists(\xi, p) \in \tau[r \leq p \text{ and } e(r) \leq -[\![\xi \in \sigma]\!]]$.

(5) $\exists(\eta, q) \in \sigma[r \leq q \text{ and } e(r) \leq -[\![\eta \in \tau]\!]]$.

Suppose that (4) holds, with (ξ, p) as indicated. Now $e(r) \neq 0$, so $e(r) \not\leq [\![\xi \in \sigma]\!]$. Thus $(r \not\Vdash^* \xi \in \sigma)^M$. Hence by the inductive hypothesis, $\xi_G \notin \sigma_G$. But $r \leq p$, so $p \in G$, and hence $\xi_G \in \tau_G$. This contradicts our assumption that $\sigma_G = \tau_G$.

(5) leads to a contradiction similarly. Hence our claim holds, and we have proved (ii)⇒(i) for $\sigma = \tau$.

For (ii)⇒(i) for $\sigma \in \tau$, assume that $\sigma_G \in \tau_G$. Then there is a $(\xi, p) \in \tau$ such that $p \in G$ and $\sigma_G = \xi_G$. By the inductive hypothesis there is a $q \in G$ such that $(q \Vdash^* \sigma = \xi)^M$. Choose $r \in G$ with $r \leq p, q$. Then $e(r) \leq e(p) \cdot [\![\sigma = \xi]\!]$, and so $e(r) \leq [\![\sigma \in \tau]\!]$. Thus $(r \Vdash^* \sigma \in \tau)^M$.

Thus now the atomic cases are finished.

In the inductive steps we omit the parameters $\sigma_0, \ldots, \sigma_{m-1}$. Suppose that the equivalence holds for φ; we prove it for $\neg\varphi$. For (i)⇒(ii), suppose that $p \in G$ and $(p \Vdash^* \neg\varphi)^M$. We want to show that $\neg\varphi$ holds in $M[G]$. Suppose to the contrary that φ holds in $M[G]$. Then by the equivalence for φ, choose $q \in G$ such that $(q \Vdash^* \varphi)^M$. Choose $r \in G$ with $r \leq p, q$. Then $(r \Vdash^* \neg\varphi)^M$ and $(r \Vdash^* \varphi)^M$, contradiction.

For (ii)⇒(i), suppose that $\neg\varphi$ holds in $M[G]$. We claim that $D \stackrel{\text{def}}{=} \{p : (p \Vdash^* \varphi)^M$ or $(p \Vdash^* \neg\varphi)^M\}$ is dense. For, suppose that q is arbitrary. If $(q \Vdash^* \varphi)^M$, then $q \in D$. Suppose that $(q \not\Vdash^* \varphi)^M$. Then $e(q) \not\leq [\![\varphi]\!]$, so $e(q) \cdot -[\![\varphi]\!] \neq 0$. By Theorem 13.20(i) choose p so that $e(p) \leq e(q) \cdot -[\![\varphi]\!] \neq 0$. By Theorem 13.20(iv) choose $r \leq p, q$. Then $r \leq q$ and $e(r) \leq e(p) \leq -[\![\varphi]\!] = [\![\neg\varphi]\!]$. Hence $(r \Vdash \neg\varphi)^M$. This shows that $r \in D$. Thus D is dense. Choose $p \in D \cap G$. If $(p \Vdash^* \varphi)^M$, then $\varphi^{M[G]}$, contradiction. Hence $(p \Vdash^* \neg\varphi)^M$.

For $\rightarrow$, suppose that $p \in G$, $(p \Vdash^* \varphi \rightarrow \psi)^M$, and φ holds in $M[G]$. By the inductive hypothesis, choose $q \in G$ so that $(q \Vdash^* \varphi)^M$. Choose $r \in G$ with $r \leq p, q$. Then

$$e(r) \leq e(p) \cdot e(q) \leq [\![\varphi \rightarrow \psi]\!] \cdot [\![\varphi]\!] = (-[\![\varphi]\!] + [\![\psi]\!]) \cdot [\![\varphi]\!] \leq [\![\psi]\!].$$

Thus $(r \Vdash^* \psi)^M$, so by the inductive hypothesis, ψ holds in $M[G]$. So we have shown that $\varphi \rightarrow \psi$ holds in $M[G]$.

Conversely, suppose that $\varphi \rightarrow \psi$ holds in $M[G]$.

Case 1. φ holds in $M[G]$. Then also ψ holds in $M[G]$. By the inductive hypothesis we get $p \in G$ such that and $(p \Vdash^* \psi)^M$. Thus $e(p) \leq [\![\psi]\!]$, so $e(p) \leq -[\![\varphi]\!] + [\![\psi]\!] = [\![\varphi \rightarrow \psi]\!]$, hence $(p \Vdash^* \varphi \rightarrow \psi)^M$.

Case 2. φ does not hold in $M[G]$. By the case for $\neg$, there is a $p \in G$ such that $(p \Vdash^* \neg\varphi)^M$. Hence $e(p) \leq -[\![\varphi]\!] \leq -[\![\varphi]\!] + [\![\psi]\!] = [\![\varphi \rightarrow \psi]\!]$, hence $(p \Vdash^* \varphi \rightarrow \psi)^M$.

Finally, we deal with the formula $\forall x\varphi(x)$. For (i)⇒(ii), suppose that $\forall x\varphi(x)$ does not hold in $M[G]$. Then there is a name σ such that $\varphi(\sigma_G)$ does not hold. By the case for $\neg$ it follows that there is a $p \in G$ such that $(p \Vdash^* \neg\varphi(\sigma))^M$, so that $e(p) \leq -[\![\varphi(\sigma)]\!]$. Thus $e(p) \leq -\prod_{\tau \in V^{\mathbb{P}}}[\![\varphi(\tau)]\!]$ and so, since $e(p) \neq 0$, $e(p) \not\leq \prod_{\tau \in V^{\mathbb{P}}}[\![\varphi(\tau)]\!]$, so that it is not true that $(p \Vdash^* \forall x\varphi(x))^M$.

For (ii)⇒(i), suppose that $(p \Vdash^* \forall x\varphi(x))^M$, and suppose that σ is any name. Then $e(p) \leq [\![\varphi(\sigma)]\!]$, hence $(p \Vdash^* \varphi(\sigma))^M$, so $\varphi(\sigma_G)$ holds in $M[G]$, as desired. □

Corollary 15.20. *If M is a c.t.m. of ZFC, $\mathbb{P}$ is a forcing order in M, $p \in P$, and $\varphi(\tau_1, \ldots, \tau_m)$ is a formula, then*

$$p \Vdash \varphi(\tau_1, \ldots, \tau_m) \textit{ iff } (p \Vdash^* \varphi(\tau_1, \ldots, \tau_m))^M.$$

Proof. Again we omit the parameters $\tau_1,\ldots,\tau_m$. $\Rightarrow$: Assume that $p \Vdash \varphi$, but suppose that $(p \nVdash^* \varphi)^M$. Thus $e(p) \not\leq [\![\varphi]\!]$, so $e(p)\cdot -[\![\varphi]\!] \neq 0$. By Theorem 13.20(i) choose q such that $e(q) \leq e(p)\cdot -[\![\varphi]\!]$. By Theorem 13.20(iv) choose $r \leq p, q$. Then $e(r) \leq e(q) \leq -[\![\varphi]\!] = [\![\neg\varphi]\!]$. Hence $(r \Vdash \neg\varphi)^M$. Let G be $\mathbb{P}$-generic over M with $r \in G$. Then by Theorem 15.19, $\neg\varphi^{M[G]}$. But $r \leq p$, so by the definition of $\Vdash$, $\varphi^{M[G]}$, contradiction.

$\Leftarrow$: Assume that $(p \Vdash^* \varphi)^M$. Suppose that G is $\mathbb{P}$-generic over M and $p \in G$. Then by Theorem 15.19. $\varphi^{M[G]}$, as desired. $\square$

Corollary 15.21. *Let M be a c.t.m. of ZFC, $\mathbb{P} \in M$ a forcing order, and $G \subseteq M$ a $\mathbb{P}$-generic filter over M. Then*

$$\varphi(\tau_{1G},\ldots,\tau_{mG})^{M[G]} \text{ iff } \exists p \in G[p \Vdash \varphi(\tau_1,\ldots,\tau_m)].$$

Proof. $\Rightarrow$: Assume $\varphi(\tau_{1G},\ldots,\tau_{mG})^{M[G]}$. By Theorem 15.19, choose $p \in G$ such that $(p \Vdash^* \varphi(\tau_1,\ldots,\tau_m))^M$. By Corollary 15.20 we have $p \Vdash \varphi(\tau_1,\ldots,\tau_m)$.

$\Leftarrow$: by the definition of $\Vdash$. $\square$

Now if σ and τ are names, we define

$$\begin{aligned}
&\mathrm{up}(\sigma,\tau) = \{(\sigma,1),(\tau,1)\};\\
&\mathrm{op}(\sigma,\tau) = \mathrm{up}(\mathrm{up}(\sigma,\sigma),\mathrm{up}(\sigma,\tau)).
\end{aligned}$$

Lemma 15.22. *(i)* $(\mathrm{up}(\sigma,\tau))_G = \{\sigma_G,\tau_G\}$.

(ii) $(\mathrm{op}(\sigma,\tau))_G = (\sigma_G,\tau_G)$. $\square$

Theorem 15.23. *Let M be a c.t.m. of ZFC, $\mathbb{P} \in M$ a forcing order, $G \subseteq P$, and G $\mathbb{P}$-generic over M. Then $M[G]$ is a model of ZFC.*

Proof. We will apply theorems from Chapter 14. Recall from Lemma 15.7 that $M[G]$ is transitive. Hence extensionality and foundation hold in $M[G]$ by Theorems 14.10 and 14.16. For pairing, suppose that $x, y \in M[G]$. Say $x = \sigma_G$ and $y = \tau_G$. By Lemma 15.22 and Theorem 14.12, pairing holds. For union, suppose that $x \in M[G]$. Choose σ such that $x = \sigma_G$. Note that $\mathrm{dmn}(\sigma)$ is a set of $\mathbb{P}$-names, and hence so is $\tau \stackrel{\text{def}}{=} \bigcup \mathrm{dmn}(\sigma)$. We claim that $\bigcup x \subseteq \tau_G$; by Theorem 14.13 this will prove the union axiom. Let $y \in \bigcup x$. Say $y \in z \in x$. Then there exist $(\rho, r), (\xi, s)$ such that $y = \rho_G$, $r \in G$, $(\rho, r) \in \xi$, $z = \xi_G$, $s \in G$, and $(\xi, s) \in \sigma$. So $\xi \in \mathrm{dmn}(\sigma)$, and hence $(\rho, r) \in \bigcup \mathrm{dmn}(\sigma) = \tau$. It follows that $y = \rho_G \in \tau_G$, as desired.

To check comprehension, we apply Theorem 14.11. So, suppose $\varphi(x, z, w_1,\ldots,w_n)$ is a formula with the indicated free variables, and $\sigma, \tau_1,\ldots,\tau_n$ are $\mathbb{P}$ names. Let

$$y = \{x \in \sigma_G : \varphi^{M[G]}(x,\sigma_G,\tau_{1G},\ldots,\tau_{nG})\};$$

we want to show that $y \in M[G]$. Let

$$\rho = \{(\pi, p) \in \text{dmn}(\sigma) \times P : (p \Vdash^* (\pi \in \sigma \wedge \varphi(\pi, \sigma, \tau_1, \ldots, \tau_n))^M\}.$$

Thus $\rho \in M^P$. We claim that $\rho_G = y$, as desired. Suppose that $x \in \rho_G$. Then there is a $(\pi, p) \in \text{dmn}(\sigma) \times \mathbb{P}$ such that $p \in G$, $x = \pi_G$, and $(p \Vdash^* (\pi \in \sigma \wedge \varphi(\pi, \sigma, \tau_1, \ldots, \tau_n))^M$. By Corollary 15.20, $p \Vdash (\pi \in \sigma \wedge \varphi(\pi, \sigma, \tau_1, \ldots, \tau_n))$. Hence by definition of $\Vdash$, $\pi_G \in \sigma_G$ and $\varphi^{M[G]}(\pi_G, \sigma_G, \tau_{1G}, \ldots, \tau_{nG})$. Thus $x \in y$. Conversely, suppose that $x \in y$. Thus $x \in \sigma_G$ and $\varphi^{M[G]}(x, \sigma_G, \tau_{1G}, \ldots, \tau_{nG})$. Choose $(\pi, p) \in \sigma$ such that $x = \pi_G$ and $p \in G$. Thus $(\pi_G \in \sigma_G \wedge \varphi^{M[G]}(\pi_G, \sigma_G, \tau_{1G}, \ldots, \tau_{nG}))^{M[G]}$, so by Corollary 15.21 there is a $q \in G$ such that $q \Vdash \pi \in \sigma \wedge \varphi(\pi, \sigma, \tau_1, \ldots, \tau_n)$. Thus by Theorem 15.20 again we have $(\pi, q) \in \rho$, hence $x = \pi_G \in \rho_G$, as desired.

For the power set axiom, we will apply Theorem 14.14. Let σ be a P-name. It suffices to find another P-name ρ such that $\mathscr{P}(\sigma_G) \cap M[G] \subseteq \rho_G$. Let $\rho = S \times \{1\}$, where

$$S = \{\tau \in M^P : \text{dmn}(\tau) \subseteq \text{dmn}(\sigma)\}.$$

Suppose that $\mu \in M^P$ and $\mu_G \subseteq \sigma_G$; we want to show that $\mu_G \in \rho_G$. Let

$$\tau = \{(\pi, p) : \pi \in \text{dmn}(\sigma) \text{ and } p \Vdash \pi \in \mu\}.$$

Thus $\text{dmn}(\tau) \subseteq \text{dmn}(\sigma)$, so $\tau_G \in \rho_G$. It suffices now to show that $\tau_G = \mu_G$. First suppose that $x \in \mu_G$. Since $\mu_G \subseteq \sigma_G$, there is a $(\pi, q) \in \sigma$ such that $q \in G$ and $x = \pi_G$. Thus $\pi_G \in \sigma_G$, so by Theorem 15.21 there is a $p \in G$ such that $p \Vdash \pi \in \sigma$. Hence $(\pi, p) \in \tau$, and so $x = \pi_G \in \tau_G$. Second, suppose that $x \in \tau_G$. Choose $(\pi, p) \in \tau$ such that $p \in G$ and $x = \pi_G$. By definition of τ we have $\pi \in \text{dmn}(\sigma)$ and $p \Vdash \pi \in \mu$. By definition of $\Vdash$, $x = \pi_G \in \mu_G$. Hence we have shown that $\tau_G = \mu_G$, as desired.

For replacement, we apply Theorem 14.15. Let φ be a formula with free variables among $x, y, A, w_1, \ldots, w_n$, suppose that $\sigma, \tau_1, \ldots, \tau_n \in M^P$ and the following holds:

$$(\forall x \in \sigma_G \exists! y[\varphi(x, y, \sigma_G, \tau_{1G}, \ldots, \tau_{1G})])^{M[G]}. \tag{1}$$

We want to find $\rho \in M^P$ such that

$$(\forall y \exists x \in \sigma_G \varphi(x, y, \sigma_G, \tau_{1G}, \ldots, \tau_{nG}) \rightarrow y \in \rho_G)^{M[G]}.$$

In view of the uniqueness condition in (1) it suffices to find $\rho \in M^P$ such that

$$\forall x \in \sigma_G \exists y \in \rho_G(\varphi(x, y, \sigma_G, \tau_{1G}, \ldots, \tau_{nG})^{M[G]}. \tag{2}$$

In fact, if (2) holds, $y \in M[G]$, $x \in \sigma_G$, and $(\varphi(x, y, \sigma_G, \tau_{1G}, \ldots, \tau_{nG}))^{M[G]}$. by (2) choose $z \in \rho_G$ such that $(\varphi(x, z, \sigma_G, \tau_{1G}, \ldots, \tau_{nG})^{M[G]}$. Then by (1) we have $y = z$, and so $y \in \rho_G$.

Now we claim

(3) There is an $S \in M$ with $S \subseteq M^P$ such that

$$\begin{aligned}\forall \pi \in \text{dmn}(\sigma) \forall p \in P[\exists \mu \in M^P[(p \Vdash^* \varphi(\pi, \mu, \sigma, \tau_1, \ldots, \tau_n))^M]\\ \rightarrow \exists \mu \in S[(p \Vdash^* \varphi(\pi, \mu, \sigma, \tau_1, \ldots, \tau_n))^M]].\end{aligned}$$

To prove the claim, we make the following argument in M. For each $\pi \in \mathrm{dmn}(\sigma)$ and $p \in P$, if there is a $\mu \in M^P$ such that $(p \Vdash^* \varphi(\pi, \mu, \sigma, \tau_1, \ldots, \tau_n))M$, let $\alpha(\pi, p)$ be the least ordinal such that such a μ is in $V_{\alpha(\pi,p)}$, while $\alpha(\pi, p) = 0$ if there is no such ordinal. Let $\beta = \sup\{\alpha(\pi, p) : \pi \in \mathrm{dmn}(\alpha), p \in P\}$. Then

$$S = \{\mu \in V_\beta : \exists \pi \in \mathrm{dmn}(\sigma) \exists p \in P[(p \Vdash^* \varphi(\pi, \mu, \sigma, \tau_1, \ldots, \tau_n))^M]\}.$$

Clearly S is as desired in the claim.

Let $\rho = S \times \{1\}$. To show that ρ satisfies (2), let $x \in \sigma_G$. say $x = \pi_G$ with $(\pi, p) \in \rho$ and $p \in G$. Then by (1) there is a $\mu \in M^P$ such that $\varphi^{M[G]}(\pi_G, \mu_G, \sigma_G, \tau_{1G}, \ldots, \tau_{mG})$. By Corollary 15.21 choose $q \in G$ such that $q \Vdash \varphi(\pi, \mu, \sigma, \tau_1, \ldots, \tau_m)$. By Corollary 15.20, $(q \Vdash^* \varphi(\pi, \mu, \sigma, \tau_1, \ldots, \tau_m))^M$. Hence by (3) we may assume that $\mu \in S$. Hence $\mu_G \in \rho_G$, as desired.

For the infinity axiom, note that $\omega = \check{\omega}_G$ by Lemma 15.15. Hence the infinity axiom holds by Theorem 14.26.

Finally we consider the axiom of choice. We show that there is a choice function for any family A of nonempty sets, where $A \in M[G]$. By Theorem 14.27, $\bigcup A \in M[G]$. Say $\bigcup A = \sigma_G$. Let f be a bijection from some cardinal κ onto $\mathrm{dmn}(\sigma)$ (in M). Define $\tau = \{\mathrm{op}(\check{\alpha}, f(\alpha)) : \alpha < \kappa\} \times \{1\}$. Thus $\tau_G = \{(\mathrm{op}(\check{\alpha}, f(\alpha)))_G = \{(\alpha, (f(\alpha))_G) : \alpha < \kappa\}$. So τ_G is a function with domain κ. Each $x \in A$ is nonempty, and if $y \in x$ then $y \in \bigcup A$, and hence we can write $y = \tau_G$ with $(\tau, p) \in \sigma$ and $p \in G$. So there is an $\alpha < \kappa$ such that $f(\alpha) = \tau$; so $\tau_G(\alpha) = (f(\alpha))_G = \tau_g = y$. This shows that for each $x \in A$ there is an ordinal $\alpha < \kappa$ such that $\tau_G(\alpha) \in x$; we let α_x be the least such ordinal. Define $g(x) = \tau_G(\alpha_x)$ for all $x \in A$. Then $g(x) \in x$. $\square$

Parts of the following theorem will be used later without reference.

Theorem 15.24. *(i)* $[\![\sigma = \tau]\!] = [\![\tau = \sigma]\!]$.

(ii)

$$[\![\sigma = \tau]\!] = \left(\prod_{(\xi,p)\in\tau} (-e(p) + [\![\xi \in \sigma]\!]) \right) \cdot \left(\prod_{(\rho,q)\in\sigma} (-e(q) + [\![\rho \in \tau]\!]) \right).$$

(iii) $[\![\sigma = \sigma]\!] = 1$.

(iv) If $(\rho, r) \in \sigma$, *then* $e(r) \leq [\![\rho \in \sigma]\!]$, *and hence* $r \Vdash^* \rho \in \sigma$.

(v) $p \Vdash^* \sigma \in \tau$ *iff the set*

$$\{q : \exists(\pi, s) \in \tau(q \leq s \text{ and } q \Vdash^* \sigma = \pi)\}$$

is dense below p.

(vi) $p \Vdash^* \sigma = \tau$ *iff the following two conditions hold:*

(a) For all $(\pi, s) \in \sigma$, *the set*

$$\{q \leq p : \text{if } q \leq s, \text{ then there is a } (\rho, u) \in \tau \text{ such that } q \leq u \text{ and } q \Vdash^* \pi = \rho\}$$

is dense below p;

(b) For all $(\rho, u) \in \tau$, the set

$$\{q \le p : \text{if } q \le u, \text{ then there is a } (\pi, s) \in \sigma \text{ such that } q \le s \text{ and } q \Vdash^* \pi = \rho\}$$

is dense below p.

(vii)

$$p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1}) \wedge \psi(\sigma_0, \ldots, \sigma_{n-1}) \quad \text{iff} \\ p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1}) \text{ and } p \Vdash^* \psi(\sigma_0, \ldots, \sigma_{n-1}).$$

(viii) $p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1}) \vee \psi(\sigma_0, \ldots, \sigma_{n-1})$ iff for all $q \le p$ there is an $r \le q$ such that $r \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1})$ or $r \Vdash^* \psi(\sigma_0, \ldots, \sigma_{n-1})$.

(ix) $p \Vdash^* \neg\varphi(\sigma_0, \ldots, \sigma_{n-1})$ iff for all $q \le p$, $q \not\Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1})$.

(x) $\{p : p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1})$ or $p \Vdash^* \neg\varphi(\sigma_0, \ldots, \sigma_{n-1})\}$ is dense.

(xi) $p \Vdash^* \exists x\varphi(x, \sigma_0, \ldots, \sigma_{n-1})$ iff the set

$$\{r \le p : \text{there is a } \tau \in \mathbf{V}^P \text{ such that } r \Vdash^* \varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})\}$$

is dense below p.

(xii) $p \Vdash^* \forall x\varphi(x, \sigma_0, \ldots, \sigma_{m-1})$ iff for all $\tau \in \mathbf{V}^P$, $p \Vdash^* \varphi(\tau, \sigma_0, \ldots, \sigma_{m-1})$.

(xiiii) The following are equivalent:

(a) $p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1})$.

(b) For every $r \le p$, $r \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1})$.

(c) $\{r : r \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1})\}$ is dense below p.

(xiv) $p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1}) \to \psi(\sigma_0, \ldots, \sigma_{m-1})$ iff the set

$$\{q : q \Vdash^* \neg\varphi(\sigma_0, \ldots, \sigma_{m-1}) \text{ or } q \Vdash^* \psi(\sigma_0, \ldots, \sigma_{m-1})\}$$

is dense below p.

(xv) If $p \Vdash^* \neg\forall x\varphi(x, \sigma_0, \ldots, \sigma_{m-1})$, then the set

$$\{q : \text{there is a } \tau \in \mathbf{V}^P \text{ such that } q \Vdash \neg\varphi(\tau, \sigma_0, \ldots, \sigma_{m-1})\}$$

is dense below p.

(xvi) If $p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1})$ and $p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1}) \to \psi(\sigma_0, \ldots, \sigma_{m-1})$, then $p \Vdash^* \psi(\sigma_0, \ldots, \sigma_{m-1})$.

Proof.

(i): By induction:

$$[\![\sigma = \tau]\!] = \prod_{(\xi,p)\in\tau}\left[-e(p) + \sum_{(\rho,q)\in\sigma}(e(q)\cdot[\![\rho = \xi]\!])\right] \\ \cdot \prod_{(\rho,q)\in\sigma}\left[-e(q) + \sum_{(\xi,p)\in\tau}(e(p)\cdot[\![\rho = \xi]\!])\right]$$

$$
\begin{aligned}
&= \prod_{(\rho,q)\in\sigma} \left[-e(q) + \sum_{(\xi,p)\in\tau} (e(p) \cdot [\![\rho = \xi]\!]) \right] \\
&\cdot \prod_{(\xi,p)\in\tau} \left[-e(p) + \sum_{(\rho,q)\in\sigma} (e(q) \cdot [\![\rho = \xi]\!]) \right] \\
&= \prod_{(\rho,q)\in\sigma} \left[-e(q) + \sum_{(\xi,p)\in\tau} (e(p) \cdot [\![\xi = \rho]\!]) \right] \\
&\cdot \prod_{(\xi,p)\in\tau} \left[-e(p) + \sum_{(\rho,q)\in\sigma} (e(q) \cdot [\![\xi = \rho]\!]) \right] \\
&= [\![\tau = \sigma]\!].
\end{aligned}
$$

(ii):

$$
\begin{aligned}
[\![\sigma = \tau]\!] &= \prod_{(\xi,p)\in\tau} \left[-e(p) + \sum_{(\rho,q)\in\sigma} (e(q) \cdot [\![\rho = \xi]\!]) \right] \\
&\cdot \prod_{(\rho,q)\in\sigma} \left[-e(q) + \sum_{(\xi,p)\in\tau} (e(p) \cdot [\![\rho = \xi]\!]) \right] \\
&= \prod_{(\xi,p)\in\tau} \left[-e(p) + \sum_{(\rho,q)\in\sigma} (e(q) \cdot [\![\xi = \rho]\!]) \right] \\
&\cdot \prod_{(\rho,q)\in\sigma} \left[-e(q) + \sum_{(\xi,p)\in\tau} (e(p) \cdot [\![\rho = \xi]\!]) \right] \\
&= \left(\prod_{(\xi,p)\in\tau} (-e(p) + [\![\xi \in \sigma]\!]) \right) \cdot \left(\prod_{(\rho,q)\in\sigma} (-e(q) + [\![\rho \in \tau]\!]) \right).
\end{aligned}
$$

We prove (iii) and (iv) simultaneously by induction on the rank of σ; so suppose that they hold for all σ' of rank less than σ. Assume that $(\rho, r) \in \sigma$. Then by the definition of $[\![\rho \in \sigma]\!]$,

$$
[\![\rho \in \sigma]\!] = \sum_{(\mu,s)\in\sigma} (e(s) \cdot [\![\rho = \mu]\!]) \geq e(r) \cdot [\![\rho = \rho]\!] = e(r),
$$

as desired in (iv). Using this and (ii),

$$
[\![\sigma = \sigma]\!] = \prod_{(\rho,r)\in\sigma} (-e(r) + [\![\rho \in \sigma]\!]) = 1,
$$

as desired in (iii).

We now use Theorem 13.20(vii) in several of our arguments.

(v):

$$\begin{aligned}
p \Vdash^* \sigma \in \tau \quad &\text{iff} \quad e(p) \leq [\![\sigma \in \tau]\!] \\
&\text{iff} \quad e(p) \leq \sum_{(\pi,s)\in\tau} (e(s) \cdot [\![\sigma = \pi]\!]) \\
&\text{iff} \quad \{q : \exists(\pi, s) \in \tau[e(q) \leq e(s) \cdot [\![\sigma = \pi]\!]]\} \text{ is dense below } p.
\end{aligned}$$

We claim that the last statement here is equivalent to

$$(*) \qquad \{q : \exists(\pi, s) \in \tau[q \leq s \text{ and } e(s) \leq [\![\sigma = \pi]\!]]\} \text{ is dense below } p.$$

In fact clearly $(*)$ implies the above statement. Now suppose that

$$\{q : \exists(\pi, s) \in \tau[e(q) \leq e(s) \cdot [\![\sigma = \pi]\!]]\} \text{ is dense below } p.$$

Take any $r \leq p$, and choose $q \leq r$ and $(\pi, x) \in \tau$ such that $e(q) \leq e(s) \cdot [\![\sigma = \pi]\!]$. Then q and s are compatible; say $t \leq q, s$. Then $t \leq q \leq r$ and $e(t) \leq e(q) \leq e(s) \cdot [\![\sigma = \pi]\!]$. Thus $(*)$ holds.

Now $(*)$ is clearly equivalent to

$$\{q : \exists(\pi, s) \in \tau[q \leq s \text{ and } s \Vdash^* \sigma = \pi]\} \text{ is dense below } p.$$

(vi): Assume that $p \Vdash^* \sigma = \tau$.

For (a), suppose that $(\pi, s) \in \sigma$ and $r \leq p$. If $r \not\leq s$, then r itself is in the desired set; so suppose that $r \leq s$. Then

$$e(r) \leq e(s) \cdot e(p) \leq e(s) \cdot \left(-e(s) + \sum_{(\rho,u)\in\tau} (e(u) \cdot [\![\pi = \rho]\!]) \right) = e(s) \cdot \sum_{(\rho,u)\in\tau} (e(u) \cdot [\![\pi = \rho]\!]).$$

Hence there is a $(\rho, u) \in \tau$ such that $e(r) \cdot e(s) \cdot e(u) \cdot [\![\pi = \rho]\!] \neq 0$. Hence there exists a $v \leq r, s$ such that $e(v) \leq e(u) \cdot [\![\pi = \rho]\!]$. (See the argument for (v)). It follows that there is a $q \leq v, u$ with $e(q) \leq [\![\pi = \rho]\!]$. So $q \Vdash^* \pi = \rho$, and q is in the desired set.

(b) is treated similarly.

Now assume that (a) and (b) hold. We want to show that $p \Vdash^* \sigma = \tau$, i.e., that $e(p) \leq [\![\sigma = \tau]\!]$. To show that $e(p)$ is below the first big product in the definition of $[\![\sigma = \tau]\!]$, take any $(\xi, q) \in \tau$; we want to show that

$$e(p) \leq -e(q) + \sum_{(\rho,r)\in\sigma} (e(r) \cdot [\![\rho = \xi]\!]),$$

i.e., that

$$e(p) \cdot e(q) \leq \sum_{(\rho,r)\in\sigma} (e(r) \cdot [\![\rho = \xi]\!]).$$

Suppose that this is not the case. Then there is an s such that

$$e(s) \leq e(p) \cdot e(q) \cdot -\sum_{(\rho,r)\in\sigma}(e(r)\cdot[\![\rho=\xi]\!]) = e(p)\cdot e(q)\cdot\prod_{(\rho,r)\in\sigma}(-e(r)+-[\![\rho=\xi]\!]).$$

Hence there is a $u \leq s, p, q$. By (b) choose $v \leq u$ and $(\rho, r) \in \sigma$ such that $v \leq r$ and $v \Vdash^* \rho = \xi$. Then $e(v) \leq e(r) \cdot [\![\rho = \xi]\!]$, and also $e(v) \leq -e(r) + -[\![\rho = \xi]\!])$, contradiction.

Similarly, $e(p)$ is below the second big product in the definition of $[\![\sigma = \tau]\!]$.

(vii): Clear.

(viii): Since

$$p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1}) \vee \psi(\sigma_0, \ldots, \sigma_{n-1}) \quad \text{iff} \quad e(p) \leq [\![\varphi(\sigma_0, \ldots, \sigma_{n-1}) \vee \psi(\sigma_0, \ldots, \sigma_{n-1})]\!],$$

this is immediate from Theorem 16.20(vii).

(ix) $\Rightarrow$: if $p \Vdash^* \neg\varphi(\sigma_0, \ldots, \sigma_{n-1})$ and $q \leq p$, then

$$e(q) \leq e(p) \leq [\![\neg\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!] = -[\![\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!],$$

and hence $e(q) \not\leq [\![\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!]$, since $e(q) \neq 0$. Thus $q \not\Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1})$.

$\Leftarrow$: suppose that $p \not\Vdash^* \neg\varphi(\sigma_0, \ldots, \sigma_{n-1})$. Then

$$e(p) \not\leq [\![\neg\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!] = -[\![\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!],$$

and hence

$$e(p) \cdot [\![\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!] \neq 0,$$

so we can choose r such that

$$e(r) \leq e(p) \cdot [\![\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!],$$

hence there is a $q \leq p, r$, and so $q \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1})$.

(x): Let q be given. If $e(q) \cdot [\![\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!] \neq 0$, choose r such that $e(r) \leq e(q) \cdot [\![\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!]$, and then choose $p \leq q, r$. Thus $p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1})$, as desired. If $e(q) \cdot [\![\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!] = 0$, then $q \Vdash^* \neg\varphi(\sigma_0, \ldots, \sigma_{n-1})$.

(xi): Suppose that $p \Vdash^* \exists x\varphi(x, \sigma_0, \ldots, \sigma_{n-1})$, and suppose that $q \leq p$. Then $e(q) \leq \sum_{\tau\in M^P} [\![\varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})]\!]$, and so there is a $\tau \in M^P$ such that $e(q) \cdot [\![\varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})]\!] \neq 0$; hence we easily get $r \leq q$ such that $e(r) \leq [\![\varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})]\!]$. This implies that $r \Vdash^* \varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})$, as desired.

Conversely, suppose that the set

$$\{r \leq p : \text{there is a } \tau \in \mathbf{V}^P \text{ such that } r \Vdash^* \varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})\}$$

is dense below p, while $p \not\Vdash^* \exists x\varphi(x, \sigma_0, \ldots, \sigma_{n-1})$. Thus $e(p) \not\leq [\![\exists x\varphi(x, \sigma_0, \ldots, \sigma_{n-1})]\!]$, so

$$e(p) \cdot \prod_{\tau\in M^P} -[\![\varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})]\!] \neq 0.$$

Then we easily get $q \leq p$ such that

$$(4) \qquad e(q) \leq \prod_{\tau \in M^P} -[\![\varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})]\!].$$

By assumption, choose $r \leq q$ and $\tau \in M^P$ such that $r \Vdash^* \varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})$. Thus $e(r) \leq [\![\varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})]\!]$. This clearly contradicts (4).

(xii): Clear.

(xiii): Clearly (a)$\Rightarrow$(b)$\Rightarrow$(c). Now assume (c). Suppose that $p \not\Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1})$. Thus $e(p) \not\leq [\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!]$, so we easily get $q \leq p$ such that

$$(5) \qquad e(q) \leq -[\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!].$$

By (c), choose $r \leq q$ such that $r \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1})$. Clearly this contradicts (5).

(xiv): First suppose that $p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1}) \rightarrow \psi(\sigma_0, \ldots, \sigma_{m-1})$. Thus by the definition of $\rightarrow$ we have $p \Vdash^* \neg\varphi(\sigma_0, \ldots, \sigma_{m-1}) \vee \psi(\sigma_0, \ldots, \sigma_{m-1})$. Hence the desired conclusion follows by (viii). The converse follows by reversing these steps.

(xv): This is very similar to part of the proof of (xi), but we give it anyway. We have

$$\begin{aligned} i(p) &\leq [\![\neg\forall x\varphi(x, \sigma_0, \ldots, \sigma_{m-1})]\!] \\ &= \sum_{\tau \in M^P} -[\![\varphi(\tau, \sigma_0, \ldots, \sigma_{m-1})]\!]. \end{aligned}$$

Now suppose that $q \leq p$. Then $e(q)$ is $\leq$ the sum here, so we easily get $r \leq q$ and $\tau \in M^P$ such that $e(r) \leq [\![\neg\varphi(\tau, \sigma_0, \ldots, \sigma_{m-1})]\!]$. Hence $r \Vdash^* \neg\varphi(\tau, \sigma_0, \ldots, \sigma_{m-1})$, as desired.

(xvi): The hypotheses yield

$$\begin{aligned} &e(p) \leq [\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!] \quad \text{and} \\ &e(p) \leq -[\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!] + [\![\psi(\sigma_0, \ldots, \sigma_{m-1})]\!], \end{aligned}$$

so $e(p) \leq [\![\psi(\sigma_0, \ldots, \sigma_{m-1})]\!]$ and hence $p \Vdash^* \psi(\sigma_0, \ldots, \sigma_{m-1})]\!]$. $\square$

Exercises

E15.1. Let I and J be sets with I infinite and $|J| > 1$, and let $\mathbb{P} = (P, \leq, \emptyset)$, where P is the collection of all finite functions contained in $I \times J$ and $\leq$ is $\supseteq$ restricted to P. Show that $\mathbb{P}$ satisfies the condition of Lemma 15.2.

E15.2. Show that if the condition in the hypothesis of Lemma 15.2 fails, then there is a $\mathbb{P}$-generic filter G over M such that $G \in M$, and G intersects every dense subset of P (not only those in M). [Cf. Lemma 15.1.]

E15.3. Assume the hypothesis of Lemma 15.2. Show that there does not exist a $\mathbb{P}$-generic filter over M which intersects every dense subset of P (not only those which are in M). Hint: Take G generic, and show that $\{p \in P : p \notin G\}$ is dense. Thus in the definition of generic filter, the condition on dense sets being in M is necessary.

E15.4. Show that if $\mathbb{P}$ satisfies the condition of Lemma 15.2, then it has uncountably many dense subsets.

E15.5. Assume the hypothesis of Lemma 15.2. Show that there are 2^ω filters which are $\mathbb{P}$-generic over M.

E15.6. Let $\mathbb{P} = (\{1\}, \leq, 1)$. Prove that the collection of all $\mathbb{P}$-names is a proper class.

E15.7. Show that $p \Vdash \sigma = \tau$ iff the following two conditions hold.

(i) For every $(\xi, q) \in \sigma$ and every $r \leq p, q$ one has $r \Vdash \xi \in \tau$.

(ii) For every $(\xi, q) \in \tau$ and every $r \leq p, q$ one has $r \Vdash \xi \in \sigma$.

E15.8. Assume that $\mathbb{P} \in M$, $p, q \in P$, and $p \perp q$. Show that $\{\tau \in M^P : (p \Vdash \tau = \check{\emptyset})^M\}$ is a proper class in M.

E15.9. Recall that forcing order is *separative* iff it is antisymmetric ($p \leq q \leq p$ implies that $p = q$), and for all p, q, if $p \not\leq q$ then there is an $r \leq p$ such that $r \perp q$. Show that the forcing order of exercise E15.1 is separative.

E15.10. Assume that $\mathbb{P} \in M$ is separative and $p, q, r \in P$. Prove that the following two conditions are equivalent:

(i) $(p \Vdash \{(\{(\emptyset, q)\}, r)\} = \check{1})^M$.

(ii) $p \leq r$ and $p \perp q$.

E15.11. Suppose that $f : A \to M$ with $f \in M[G]$. Show that there is a $B \in M$ such that $f : A \to B$. Hint: let $f = \tau_G$ and $B = \{b : \exists p \in P[(p \Vdash \check{b} \in \operatorname{rng}(\tau))^M]\}$.

E15.12. Assume that $\mathbb{P} \in M$ and α is a cardinal of M. Then for any $\mathbb{P}$-generic G over M the following conditions are equivalent:

(1) For all $B \in M$, ${}^\alpha B \cap M = {}^\alpha B \cap M[G]$.

(2) ${}^\alpha M \cap M = {}^\alpha M \cap M[G]$.

E15.13. Suppose that $\mathbb{P} \in M$ is a forcing order satisfying the condition of Lemma 15.2. Assume that

$$M = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots \quad (n \in \omega),$$

where $M_{n+1} = M_n[G_n]$ for some G_n which is $\mathbb{P}$-generic over M_n, for each $n \in \omega$. Show that the power set axiom fails in $\bigcup_{n\in\omega} M_n$.

E15.14. Prove that the following conditions are equivalent:

$$[\![\varphi(\sigma_0, \ldots, \sigma_{m-1}) \leftrightarrow \psi(\sigma_0, \ldots, \sigma_{m-1})]\!] = 1$$
$$[\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!] = [\![\psi(\sigma_0, \ldots, \sigma_{m-1})]\!].$$

E15.15. Prove that $[\![\sigma = \tau]\!] \cdot [\![\tau = \rho]\!] \leq [\![\sigma = \rho]\!]$.

E15.16. Prove that if ZFC $\models \varphi$ then $[\![\varphi]\!] = 1$, for any sentence φ.

16. Independence of CH

The forcing orders used in this chapter are special cases of the following. For sets I, J and for λ an infinite cardinal,

$$\mathrm{Fn}(I, J, \lambda) = (\{f : f \text{ is a function contained in } I \times J \text{ and } |f| < \lambda\}, \supseteq, \emptyset).$$

We first show that $\neg$CH is consistent. The main part of the proof is given in the following theorem.

Theorem 16.1. (Cohen) *Let M be a c.t.m. of ZFC. Suppose that κ is any infinite cardinal of M. Let G be* $\mathrm{Fn}(\kappa, 2, \omega)$*-generic over M. Then $2^\omega \geq \kappa$ in $M[G]$.*

Proof. Let $g = \bigcup G$. Since any two members of G are compatible, g is a function.

(1) For each $\alpha \in \kappa$, the set $\{f \in \mathrm{Fn}(\kappa, 2, \omega) : \alpha \in \mathrm{dmn}(f)\}$ is dense in $\mathrm{Fn}(\kappa, 2, \omega)$ (and it is a member of M).

In fact, given $f \in \mathrm{Fn}(\kappa, 2, \omega)$, either f is already in the above set, or else $\alpha \notin \mathrm{dmn}(f)$ and then $f \cup \{(\alpha, 0)\}$ is an extension of f which is in that set. So (1) holds.

Since G intersects each set (1), it follows that g maps κ into 2. Let (in M) $h : \kappa \times \omega \to \kappa$ be a bijection. For each $\alpha < \kappa$ let $a_\alpha = \{m \in \omega : g(h(\alpha, m)) = 1\}$. We claim that $a_\alpha \neq a_\beta$ for distinct α, β; this will give our result. In fact, for distinct $\alpha, \beta < \kappa$, the set

$$\begin{aligned}&\{f \in \mathrm{Fn}(\kappa, 2, \omega) : \text{there is an } m \in \omega \text{ such that}\\ &\quad h(\alpha, m), h(\beta, m) \in \mathrm{dmn}(f) \text{ and } f(h(\alpha, m)) \neq f(h(\alpha, m))\}\end{aligned}$$

is dense in $\mathrm{Fn}(\kappa, 2, \omega)$ (and it is in M). In fact, let distinct α and β be given, and suppose that $f \in \mathrm{Fn}(\kappa, 2, \omega)$. Now $\{m : h(\alpha, m) \in f \text{ or } h(\beta, m) \in f\}$ is finite, so choose $m \in \omega$ not in this set. Thus $h(\alpha, m), h(\beta, m) \notin f$. Let $h = f \cup \{(h(\alpha, m), 0), (h(\beta, m), 1)\}$. Then h extends f and is in the above set, as desired.

It follows that G contains a member of this set. Hence $a_\alpha \neq a_\beta$. $\square$

By taking $\kappa > \omega_1$ in M, it would appear that we have shown the consistency of $\neg$CH. But there is a major detail that we have to take care of. Possibly ω_1 means something different in $M[G]$ than it does in M; maybe we have accidentally introduced a bijection from the ω_1 of M onto ω. Since M is countable, this is conceivable.

To illustrate this problem, let $\mathbb{P}$ be the forcing order consisting of all finite functions mapping a subset of ω into ω_1, ordered by $\supseteq$, with $\emptyset$ as "largest" element. Suppose that G is $\mathbb{P}$-generic over M. Now the following sets are dense:

$$\begin{aligned}A_m &\stackrel{\text{def}}{=} \{f \in P : m \in \mathrm{dmn}(f)\} \quad \text{for each } m \in \omega,\\ B_\alpha &\stackrel{\text{def}}{=} \{f \in P : \alpha \in \mathrm{rng}(f)\} \quad \text{for each } \alpha \in \omega_1^M.\end{aligned}$$

In fact, given any $g \in P$, if $m \in \omega\backslash\mathrm{dmn}(g)$, then $g \cup \{(m, 0)\}$ is a member of P which in A_m and contains g; and given any $g \in P$ and $\alpha < \omega_1^M$, choose $m \in \omega\backslash\mathrm{dmn}(g)$; then $g \cup \{(m, \alpha)\}$ is a member of P which in B_α and contains g. Now if G is $\mathbb{P}$-generic over M

and intersects all of the sets A_m and B_α, then clearly $\bigcup G$, which is a member of $M[G]$, is a function mapping ω onto ω_1^M. So ω_1^M gets "collapsed" to a countable ordinal in $M[G]$. Note that $\omega^M = \omega^{M[G]}$ by absoluteness.

Thus to finish the proof of consistency of $\neg$CH we need to study the preservation of cardinals in the passage from M to $M[G]$.

$\mathbb{P}$ *preserves cardinals* $\geq \kappa$ iff for every G which is $\mathbb{P}$-generic over M and every ordinal $\alpha \geq \kappa$ in M, α is a cardinal in M iff α is a cardinal in $M[G]$.

$\mathbb{P}$ *preserves cofinalities* $\geq \kappa$ iff for every G which is $\mathbb{P}$-generic over M and every limit ordinal α in M such that $(\mathrm{cf}(\alpha))^M \geq \kappa$, $(\mathrm{cf}(\alpha))^M = (\mathrm{cf}(\alpha))^{M[G]}$.

$\mathbb{P}$ *preserves regular cardinals* $\geq \kappa$ iff for every G which is $\mathbb{P}$-generic over M and every ordinal $\alpha \geq \kappa$ which is a regular cardinal of M, α is also a regular cardinal of $M[G]$.

If $\kappa = \omega$, we say simply that $\mathbb{P}$ preserves cardinals, cofinalities, or regular cardinals.

In these definitions, if we replace "$\geq$" by "$\leq$" we obtain new definitions which will be used below also.

The relationship between these notions that we want to give uses the following fact.

Lemma 16.2. *Suppose that α is a limit ordinal, κ and λ are regular cardinals, $f : \kappa \to \alpha$ is strictly increasing with $\mathrm{rng}(f)$ cofinal in α, and $g : \lambda \to \alpha$ is strictly increasing with $\mathrm{rng}(g)$ cofinal in α. Then $\kappa = \lambda$.*

Proof. Suppose not; say by symmetry $\kappa < \lambda$. For each $\xi < \kappa$ choose $\eta_\xi < \lambda$ such that $f(\xi) < g(\eta_\xi)$. Let $\rho = \sup_{\xi<\kappa} \eta_\xi$. Thus $\rho < \lambda$ by the regularity of λ. But then $f(\xi) < g(\rho) < \alpha$ for all $\xi < \kappa$, contradiction. □

Proposition 16.3. *Let M be a c.t.m. of ZFC, $\mathbb{P} \in M$ be a forcing order, and κ be a cardinal of M.*

(i) If $\mathbb{P}$ preserves regular cardinals $\geq \kappa$, then it preserves cofinalities $\geq \kappa$.

(ii) If $\mathbb{P}$ preserves cofinalities $\geq \kappa$, and κ is regular, then $\mathbb{P}$ preserves cardinals $\geq \kappa$.

Proof. (i): Let α be a limit ordinal of M with $(\mathrm{cf}(\alpha))^M \geq \kappa$. Then $(\mathrm{cf}(\alpha))^M$ is a regular cardinal of M which is $\geq \kappa$ and hence is also a regular cardinal of $M[G]$. Now we can apply Lemma 16.2 within $M[G]$ to $\kappa = (\mathrm{cf}(\alpha))^M$ and $\lambda = (\mathrm{cf}(\alpha))^{M[G]}$ to infer that $(\mathrm{cf}(\alpha))^M = (\mathrm{cf}(\alpha))^{M[G]}$.

(ii): Suppose that cardinals $\geq \kappa$ are not preserved, and let λ be the least cardinal of M which is $\geq \kappa$ but which is not a cardinal of $M[G]$. If λ is regular in M, then

$$\lambda = (\mathrm{cf}(\lambda))^M = (\mathrm{cf}(\lambda))^{M[G]},$$

and so λ is a regular cardinal in $M[G]$, contradiction. If λ is singular in M, then it is greater than κ since κ is regular in M, and so by the minimality of λ it is the supremum of cardinals in $M[G]$, and so it is a cardinal in $M[G]$, contradiction. □

We can replace "$\geq$" by "$\leq$" in this proposition and its proof; call this new statement *Proposition 16.3′*. The very last part of the proof of 16.3 can be simplified for $\leq$, and actually one does not need to assume that κ is regular in this case.

A forcing order $\mathbb{P}$ satisfies the *κ-chain condition*, abbreviated κ-c.c., iff every antichain in $\mathbb{P}$ has size less than κ.

The following theorem is very useful in forcing arguments.

Theorem 16.4. *Let M be a c.t.m. of ZFC, $\mathbb{P} \in M$ be a forcing order, κ be a cardinal of M, G be $\mathbb{P}$-generic over M, and suppose that $\mathbb{P}$ satisfies the κ-c.c. Suppose that $f \in M[G]$, $A, B \in M$, and $f : A \to B$. Then there is an $F : A \to \mathscr{P}(B)$ with $F \in M$ such that:*

(i) $f(a) \in F(a)$ for all $a \in A$.

(ii) $(|F(a)| < \kappa)^M$ for all $a \in A$.

Proof. Let $\tau \in M^P$ be such that $\tau_G = f$. Thus the statement "$\tau_G : A \to B$" holds in $M[G]$. Hence by Theorem 15.21 there is a $p \in G$ such that

$$p \Vdash \tau : \check{A} \to \check{B}.$$

Now for each $a \in A$ let

$$F(a) = \{b \in B : \text{there is a } q \le p \text{ such that } q \Vdash \mathrm{op}(\check{a}, \check{b}) \in \tau\}.$$

To prove (i), suppose that $a \in A$. Let $b = f(a)$. Thus $(a, b) \in f$, so by Theorem 15.21 there is an $r \in G$ such that $r \Vdash \mathrm{op}(\check{a}, \check{b}) \in \tau$. Let $q \in G$ with $q \le p, r$. Then q shows that $b \in F(a)$.

To prove (ii), again suppose that $a \in A$. By the axiom of choice in M, there is a function $Q : F(a) \to P$ such that for any $b \in F(a)$, $Q(b) \le p$ and $Q(b) \Vdash \mathrm{op}(\check{a}, \check{b}) \in \tau$.

(1) If $b, b' \in F(a)$ and $b \neq b'$, then $Q(b) \perp Q(b')$.

In fact, suppose that $r \le Q(b), Q(b')$. Then

$$r \Vdash \mathrm{op}(\check{a}, \check{b}) \in \tau \wedge \mathrm{op}(\check{a}, \check{b'}) \in \tau; \tag{2}$$

but also $r \le Q(b) \le p$, so $r \Vdash \tau : \check{A} \to \check{B}$, hence

$$r \Vdash \forall x, y, z[\mathrm{op}(x, y) \wedge \mathrm{op}(x, z) \to y = z]$$

and hence

$$r \Vdash \mathrm{op}(\check{a}, \check{b}) \in \tau \wedge \mathrm{op}(\check{a}, \check{b'}) \in \tau \to \check{b} = \check{b'}. \tag{3}$$

Now let H be $\mathbb{P}$-generic over M with $r \in H$. By the definition of forcing and (2) we have $(a, b) = (\mathrm{op}(\check{a}, \check{b}))_G \in \tau_G$ and $(a, b') = (\mathrm{op}(\check{a}, \check{b'}) \in \tau_G$. By (3) and the definition of forcing it follows that $b = b'$. Thus (1) holds.

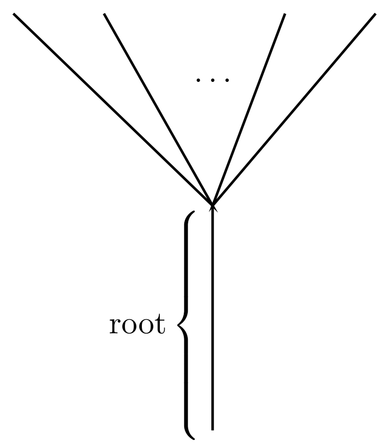
By (1), $\langle Q(b) : b \in F(a) \rangle$ is a one-one function onto an antichain of P. Hence $(|F(a)| < \kappa)^M$ by the κ-cc. $\square$

Proposition 16.5. *If M is a c.t.m. of ZFC, κ is a cardinal of M, and $\mathbb{P} \in M$ satisfies κ-cc in M, then $\mathbb{P}$ preserves regular cardinals $\ge \kappa$, and also preserves cofinalities $\ge \kappa$. If also κ is regular in M, then $\mathbb{P}$ preserves cardinals $\ge \kappa$.*

Proof. First we want to show that if $\lambda \geq \kappa$ is regular in M then also λ is regular in $M[G]$ (and hence is a cardinal of $M[G]$). Suppose that this is not the case. Hence in $M[G]$ there is an $\alpha < \lambda$ and a function $f : \alpha \to \lambda$ such that the range of f is cofinal in λ. Recall from Lemma 15.13 that M and $M[G]$ have the same ordinals. Thus $\alpha \in M$. By Theorem 16.4, let $F : \alpha \to \mathscr{P}(\lambda)$ be such that $f(\xi) \in F(\xi)$ and $(|F(\xi)| < \lambda)^M$ for all $\xi < \alpha$. Let $S = \bigcup_{\xi<\alpha} F(\xi)$. Then S is a subset of λ which is cofinal in λ and has size less than λ, contradiction.

The rest of the proposition follows from Proposition 16.3. □

To proceed we need an important theorem from infinite combinatorics. A collection $\mathscr{A}$ of sets forms a Δ-*system* iff there is a set r (called the *root* or *kernel* of the Δ-system) such that $A \cap B = r$ for any two distinct $A, B \in \mathscr{A}$. This is illustrated as follows:



The existence theorem for Δ-systems that is most often used is as follows.

Theorem 16.6. (Δ-system theorem) *If κ is an uncountable regular cardinal and $\mathscr{A}$ is a collection of finite sets with $|\mathscr{A}| \geq \kappa$, then there is a $\mathscr{B} \in [\mathscr{A}]^\kappa$ such that $\mathscr{B}$ is a Δ-system.*

Proof. First we prove the following special case of the theorem.

$(*)$ If $\mathscr{A}$ is a collection of finite sets each of size $m \in \omega$, with $|\mathscr{A}| = \kappa$, then there is a $\mathscr{B} \in [\mathscr{A}]^\kappa$ such that $\mathscr{B}$ is a Δ-system.

We prove this by induction on m. The hypothesis implies that $m > 0$. If $m = 1$, then each member of $\mathscr{A}$ is a singleton, and so $\mathscr{A}$ is a collection of pairwise disjoint sets; hence it is a Δ-system with root $\emptyset$. Now assume that $(*)$ holds for m, and suppose that $\mathscr{A}$ is a collection of finite sets each of size $m+1$, with $|\mathscr{A}| = \kappa$, and with $m > 0$. We consider two cases.

Case 1. There is an element x such that $\mathscr{C} \stackrel{\text{def}}{=} \{A \in \mathscr{A} : x \in A\}$ has size κ. Let $\mathscr{D} = \{A \backslash \{x\} : A \in \mathscr{C}\}$. Then $\mathscr{D}$ is a collection of finite sets each of size m, and $|\mathscr{D}| = \kappa$. Hence by the inductive assumption there is an $\mathscr{E} \in [\mathscr{D}]^\kappa$ which is a Δ-system, say with kernel r. Then $\{A \cup \{x\} : A \in E\} \in [\mathscr{A}]^\kappa$ and it is a Δ-system with kernel $r \cup \{x\}$.

Case 2. Case 1 does not hold. Let $\langle A_\alpha : \alpha < \kappa\rangle$ be a one-one enumeration of $\mathscr{A}$. Then from the assumption that Case 1 does not hold we get:

(∗∗) For every x, the set $\{\alpha < \kappa : x \in A_\alpha\}$ has size less than κ.

We now define a sequence $\langle \alpha(\beta) : \beta < \kappa\rangle$ of ordinals less than κ by recursion. Suppose that $\alpha(\beta)$ has been defined for all $\beta < \gamma$, where $\gamma < \kappa$. Then $\Gamma \stackrel{\text{def}}{=} \bigcup_{\beta<\gamma} A_{\alpha(\beta)}$ has size less than κ, and so by (∗∗), so does the set

$$\bigcup_{x\in\Gamma}\{\delta < \kappa : x \in A_\delta\}.$$

Thus we can choose $\alpha(\gamma) < \kappa$ such that for all $x \in \Gamma$ we have $x \notin A_{\alpha(\gamma)}$. This implies that $A_{\alpha(\gamma)} \cap A_{\alpha(\beta)} = \emptyset$ for all $\beta < \gamma$. Thus we have produced a pairwise disjoint system $\langle A_{\alpha(\beta)} : \beta < \kappa\rangle$, as desired. (The root is $\emptyset$ again.)

This finishes the inductive proof of (∗)

Now the theorem itself is proved as follows. Let $\mathscr{A}'$ be a subset of $\mathscr{A}$ of size κ. Then

$$\mathscr{A}' = \bigcup_{m\in\omega}\{A \in \mathscr{A}' : |A| = m\}.$$

Hence there is an $m \in \omega$ such that $\{A \in \mathscr{A}' : |A| = m\}$ has size κ. So (∗) applies to give the desired conclusion. □

By the *countable chain condition*, abbreviated ccc, we mean the ω_1-chain condition.

Lemma 16.7. *If κ is an infinite cardinal, then* $\mathrm{Fn}(\kappa, 2, \omega)$ *satisfies ccc.*

Proof. Suppose that $\mathscr{F} \subseteq \mathrm{Fn}(\kappa, 2, \omega)$ is uncountable. Since for each finite $F \subseteq \kappa$ there are only finitely many members of $\mathscr{F}$ with domain F, it is clear that $\{\mathrm{dmn}(f) : f \in \mathscr{F}\}$ is an uncountable collection of finite sets. By the Δ-system lemma, let $\mathscr{G}$ be an uncountable subset of this collection which forms a Δ-system, say with root R. Then

$$\mathscr{G} = \bigcup_{k\in {}^R2}\{f \in \mathscr{G} : f \restriction R = k\};$$

since ${}^R2$ is finite, there is a $k \in {}^R2$ such that

$$\mathscr{H} \stackrel{\text{def}}{=} \{f \in \mathscr{G} : f \restriction R = k\}$$

is uncountable. Clearly f and g are compatible for any $f, g \in \mathscr{H}$. □

Theorem 16.8. (Cohen) *Let M be a c.t.m. of ZFC. Suppose that κ is any cardinal of M. Let G be* $\mathrm{Fn}(\kappa, 2, \omega)$*-generic over M. Then $M[G]$ has the same cofinalities and cardinals as M, and $2^\omega \geq \kappa$ in $M[G]$.*

Proof. By Theorems 16.1, 16.5, and 16.7, also using the fact that ω is absolute. □

The method of proof of Theorem 16.8 is called *Cohen forcing.*

Theorem 16.9. (Cohen) *If* ZFC *is consistent, then so is* ZFC $+ \neg$CH.

Proof. Apply Theorem 16.8 with κ a cardinal of M greater than ω_1^M. $\square$

We now turn to the proof of consistency of CH. This depends on a new notion which is important in its own right.

Let λ be an infinite cardinal. A forcing order $\mathbb{P} = (P, \leq, 1)$ is λ-*closed* iff for all $\gamma < \lambda$ and every system $\langle p_\xi : \xi < \gamma \rangle$ of elements of P such that $p_\eta \leq p_\xi$ whenever $\xi < \eta < \gamma$, there is a $q \in P$ such that $q \leq p_\xi$ for all $\xi < \gamma$.

The importance of this notion for generic extensions comes about because of the following theorem, which is similar to Theorem 16.4.

Theorem 16.10. *Suppose that M is a c.t.m. of ZFC, $\mathbb{P} \in M$ is a forcing order, λ is a cardinal of M, $\mathbb{P}$ is λ-closed, $A, B \in M$, and $|A| < \lambda$. Suppose that G is $\mathbb{P}$-generic over M and $f \in M[G]$ with $f : A \to B$. Then $f \in M$.*

Proof. It suffices to prove this when A is an ordinal. For, suppose that this special case has been shown, and now suppose that A is arbitrary. In M, let j be a bijection from $\alpha \stackrel{\text{def}}{=} |A|^M$ onto A. Then $f \circ j : \alpha \to B$, so $f \circ j \in M$ by the special case. Hence $f \in M$.

So now we assume that $A = \alpha$, an ordinal less than λ. Let $K = ({}^\alpha B)^M$. Let $f = \tau_G$. We want to show that $f \in K$, for then $f \in M$. Suppose not. Now $\tau_G : \alpha \to B$ and $\tau_G \in K$. Hence by Theorem 15.21 there is a $p \in G$ such that

$$p \Vdash \tau : \check{\alpha} \to \check{B} \wedge \tau \notin \check{K}. \tag{1}$$

For a while we work entirely in M. We will define sequences $\langle p_\eta : \eta \leq \alpha \rangle$ of elements of P and $\langle z_\eta : \eta < \alpha \rangle$ of elements of B by recursion, so that the following conditions hold:

(2) $p_0 = p$.
(3) $p_\eta \leq p_\xi$ if $\xi < \eta$.
(4) $p_{\eta+1} \Vdash \tau(\check{\eta}) = \check{z}_\eta$.

Of course we start out by defining $p_0 = p$, so that (2) holds. Now suppose that p_η has been defined so that (2)–(4) hold; we define $p_{\eta+1}$. In fact, we claim that there exist a $p_{\eta+1} \leq p_\eta$ and a $z_\eta \in B$ such that $p_{\eta+1} \leq p_\eta$ and $p_{\eta+1} \Vdash \tau(\check{\eta}) = \check{z}_\eta$. To prove this claim, suppose that $p_\eta \in H$ where H is $\mathbb{P}$-generic over M. Then by (1), $\tau_H : \alpha \to B$ and $\tau_H \notin K$. Hence $\tau_H(\eta) \in B$; say $\tau_H(\eta) = z_\eta$. By Theorem 15.21, there is a $q \in H$ such that $q \Vdash \tau(\check{\eta}) = \check{z}_\eta$. Let $p_{\eta+1} \in H$ with $p_{\eta+1} \leq p_\eta, q$. This proves the claim. Thus (2)–(4) holds.

For η limit, p_η is given by the definition of λ-closed.

Note that the function z defined in this way is in K.

This finishes our argument within M. Now let H be $\mathbb{P}$-generic over M with $p_\alpha \in H$. Then $\tau_H(\eta) = z_\eta$ for each $\eta < \alpha$ by (4), so that $\tau_H = z \in K$. This contradicts (1), since $p_\alpha \leq p$. $\square$

Proposition 16.11. *Suppose that M is a c.t.m. of ZFC. $\mathbb{P} \in M$ is a forcing order, λ is a regular cardinal of M, and $\mathbb{P}$ is λ-closed. Then $\mathbb{P}$ preserves cofinalities and cardinals $\leq \lambda$.*

Proof. Otherwise, by Proposition 16.3′ there is a regular cardinal $\kappa \le \lambda$ of M which is not regular in $M[G]$. Thus there exist in $M[G]$ an ordinal $\alpha < \kappa$ and a function $f : \alpha \to \kappa$ such that $\text{rng}(f)$ is cofinal in κ. By Theorem 16.10, $f \in M$, contradiction. $\square$

Theorem 16.12. *Let M be a c.t.m. of ZFC, and let G be $\mathbb{F}(\omega_1, 2, \omega_1)$-generic over M. Then CH holds in $M[G]$, and $\omega_1^M = \omega_1^{M[G]}$.*

First we show that $\mathbb{F}(\omega_1, 2, \omega_1)$ is ω_1-closed. Let $\langle f_\xi : \xi < \alpha\rangle$ be a sequence of members of $\mathbb{F}(\omega_1, 2, \omega_1)$ such that $\alpha < \omega_1$ and $\forall \xi, \eta[\xi < \eta < \alpha \to f_\eta \supseteq f_\xi]$. Then clearly $\bigcup_{\xi<\alpha} f_\xi \in \mathbb{F}(\omega_1, 2, \omega_1)$ and $\bigcup_{\xi<\alpha} f_\xi \supseteq f_\eta$ for each $\eta < \xi$. Now it follows from Proposition 16.11 that $\omega_1^M = \omega_1^{M[G]}$.

By Theorem 16.10 we have $({}^\omega 2)^{M[G]} \subseteq M$. Let $F : \omega_1 \times \omega \to \omega_1$ be a bijection. For each $f \in {}^\omega 2$ let

$$D_f = \{g \in \text{Fn}(\omega_1, 2, \omega_1) : \exists \alpha < \omega_1 \forall n \in \omega[F(\alpha, n) \in \text{dmn}(g) \text{ and } g(F(\alpha, n)) = f(n)]\}\}.$$

Clearly D_f is dense in $\text{Fn}(\omega_1, 2, \omega_1)$. Define $h : \omega_1^M \to {}^\omega 2$ in $M[G]$ by: $(h(\alpha))(n) = (\bigcup G)(F(\alpha, n))$. Then h maps onto ${}^\omega 2$ by the denseness of the D_f's, as desired. $\square$

Theorem 16.13. (Gödel) *If ZFC is consistent, then so is ZFC + CH.* $\square$

Gödel also showed that if ZF is consistent, then so is ZFC + GCH. For this he introduced the notion of constructible sets.

In conclusion we give some elementary facts about forcing which will be used later.

Theorem 16.14. *$p \Vdash \check{a} \in \check{b}$ iff $a \in b$.* $\square$

Theorem 16.15. *$p \Vdash \exists x \varphi(x, \sigma_0, \ldots, \sigma_{m-1})$ iff the set*

$$\{r \le p : \textit{there is a } \tau \in M^P[r \Vdash \varphi(\tau, \sigma_0, \ldots, \sigma_{m-1})]\}$$

is dense below p.

Proof. $\Rightarrow$: Assume that $p \Vdash \exists x \varphi(x, \sigma_0, \ldots, \sigma_{m-1})$, and $q \le p$. Let G be $\mathbb{P}$-generic over M with $q \in G$. Then also $p \in G$, so $(\exists x \varphi(\sigma_0, \ldots, \sigma_{m-1}))^{M[G]}$. Hence there is a $\tau \in M^P$ such that $\varphi^{M[G]}(\tau_G, \sigma_{0G}, \ldots, \sigma_{(m-1)G})$ holds. Choose $s \in G$ such that $s \Vdash \varphi(\tau, \sigma_0, \ldots, \sigma_{m-1})$. Then choose $r \in G$ with $r \le q, s$. Thus r is in the indicated set, as desired.

$\Leftarrow$: Assume the indicated condition, and suppose that $p \in G$ with G $\mathbb{P}$-generic over M. Then there is an $r \in G$ with r in the indicated set. Hence $\varphi^{M[G]}(\tau_G, \sigma_{0G}, \ldots, \sigma_{(m-1)G})$, and so also $\exists x \varphi^{M[G]}(x, \sigma_{0G}, \ldots, \sigma_{(m-1)G})$, as desired. $\square$

Proposition 16.16. *If $p \Vdash \exists x \in \check{a} \varphi(x)$, then there exist a $q \le p$ and a $b \in a$ such that $q \Vdash \varphi(\check{b})$.*

Proof. We have $p \Vdash \exists x[x \in \check{a}$ and $\varphi(x)]$. Hence by Theorem 16.15, choose $q \le p$ and $b \in a$ such that $q \Vdash \check{b} \in \check{a}$ and $\varphi(\check{b})]$. So $q \Vdash \varphi(\check{b})$. $\square$

EXERCISES

E16.1. Show that $\mathrm{Fn}(\omega_1, 2, \omega_1)$ preserves cardinals $\ge \omega_2$.

E16.2. A system $\langle A_i : i \in I\rangle$ of sets is an *indexed* Δ*-system* iff there is a set r (again called the *root* such that $A_i \cap A_j = r$ for all distinct $i, j \in I$. Note that in an indexed system $\langle A_i : i \in I\rangle$ it is possible to have distinct $i, j \in I$ such that $A_i = A_j$; in fact, all of the A_i's could be equal, in which case the system is already an indexed Δ-system.

Prove that if κ is an uncountable regular cardinal and $\langle A_i : i \in I\rangle$ is a system of finite sets with $|I| \ge \kappa$, then there is a $J \in [I]^\kappa$ such that $\langle A_i : i \in J\rangle$ is an indexed Δ-system.

E16.3. Here we work only in ZFC (or in a fixed model of it). Suppose that $(X, <)$ is a linear order. Let P be the set of all pairs (p, n) such that $n \in \omega$ and $p \subseteq X \times n$ is a finite function. Define $(p, n) \le (q, m)$ iff $m \le n$, $\mathrm{dmn}(q) \subseteq \mathrm{dmn}(p)$, $\forall x \in \mathrm{dmn}(q)[p(x) \cap m = q(x)$, and

$$\forall x, y \in \mathrm{dmn}(q), \text{ if } x < y \text{ then } p(x)\backslash p(y) \subseteq m.$$

Show that $\mathbb{P}$ has ccc.

E16.4. Continuing exercise E16.3, suppose that we are working in a c.t.m. M of ZFC. Let G be $\mathbb{P}$-generic over M. For each $x \in X$ let

$$a_x = \bigcup\{p(x) : (p, n) \in G \text{ for some } n \in \omega, \text{ with } x \in \mathrm{dmn}(p)\}.$$

Thus $a_x \subseteq \omega$. Show that if $x < y$, then $a_x \backslash a_y$ is finite.

E16.5. Continuing exercises E16.3 and E16.4, show that if $x < y$, then $a_y \backslash a_x$ is infinite. Hint: for each $i < \omega$ let

$$E^i = \{(p, n) : x, y \in \mathrm{dmn}(p) \text{ and } |p(y)\backslash p(x)| \ge i\},$$

and show that E^i is dense.

E16.6. Define a set $\mathscr{A}$ of finite sets with $|\mathscr{A}| = \omega$ while there is no Δ-system $\mathscr{B} \in [\mathscr{A}]^\omega$.

E16.7. Let κ be singular. Define a set $\mathscr{A}$ of finite sets with $|\mathscr{A}| = \kappa$ while there is no Δ-system $\mathscr{B} \in [\mathscr{A}]^\kappa$.

17. Linear orders

In this chapter we prove some results about linear orders which form a useful background in much of set theory. Among these facts are: any two denumerable densely ordered sets are isomorphic, the existence of η_α sets, the existence of completions, a discussion of Suslin lines, and a proof of a very useful theorem of Hausdorff.

A linear order $(A,<)$ is *densely ordered* iff $|A|>1$, and for any $a<b$ in A there is a $c\in A$ such that $a<c<b$. A subset X of a linearly ordered set L is *dense in L* iff for any two elements $a<b$ in L there is an $x\in X$ such that $a<x<b$. Note that if X is dense in L and L has at least two elements, then L itself is dense.

Theorem 17.1. *If L is a dense linear order, then L is the disjoint union of two dense subsets.*

Proof. Let $\langle a_\alpha : \alpha<\kappa\rangle$ be a well-order of L, with $\kappa=|L|$. We put each a_α in A or B by recursion, as follows. Suppose that we have already done this for all $\beta<\alpha$. Let $C=\{a_\beta : \beta<\alpha \text{ and } a_\beta<a_\alpha\}$, and let $D=\{a_\beta : \beta<\alpha \text{ and } a_\beta>a_\alpha\}$. We take two possibilities.

Case 1. C has a largest element a_β, D has a smallest element a_γ, and $a_\beta,a_\gamma\in A$. Then we put a_α in B.

Case 2. Otherwise, we put a_α in A.

Now we want to see that this works. So, suppose that elements $a_\xi<a_\eta$ of L are given. Let $a_\beta<a_\gamma$ be the elements of L with smallest indices which are in the interval (a_ξ,a_η). If one of these is in A and the other in B, this gives elements of A and B in (a_ξ,a_η). So, suppose that they are both in A, or both in B. Let a_ν be the member of L with smallest index that is in (a_β,a_γ). Thus $a_\xi<a_\beta<a_\nu<a_\gamma<a_\eta$, so by the minimality of β and γ we have $\beta,\gamma<\nu$. Thus $\beta<\nu$ and $a_\beta<a_\nu$.

(1) a_β is the largest element of $\{a_\rho : \rho<\nu,\ a_\rho<a_\nu\}$.

In fact, a_β is in this set, as just observed. If $a_\beta<a_\rho$, $\rho<\nu$, and $a_\rho<a_\nu$, then also $a_\rho<a_\gamma$ since $a_\nu<a_\gamma$, so the definition of ν is contradicted. Hence (1) holds.

(2) a_γ is the smallest element of $\{a_\rho : \rho<\nu,\ a_\rho>a_\nu\}$.

In fact, $\gamma<\nu$ as observed just before (1), and $a_\gamma>a_\nu$ by the definition of a_ν. If $a_\rho<a_\gamma$, $\rho<\nu$, and $a_\rho>a_\nu$, then also $a_\rho>a_\beta$ since $a_\nu>a_\beta$, so the definition of ν is contradicted. Hence (2) holds.

So by construction, if $a_\beta,a_\gamma\in A$ then $a_\nu\in B$, while if $a_\beta,a_\gamma\in B$, then $a_\nu\in A$. So again we have found elements of both A and B which are in (a_ξ,a_η). □

The proof of the following result uses the important *back-and-forth* argument.

Theorem 17.2. *Any two denumerable densely ordered sets without first and last elements are order-isomorphic.*

Proof. Let $(A,<)$ and $(B,\prec)$ be denumerable densely ordered sets without first and last elements. Write $A=\{a_i : i\in\omega\}$ and $B=\{b_i : i\in\omega\}$. We now define by recursion sequences $\langle c_i : i \in \omega \rangle$ of elements of A and $\langle d_i : i \in \omega \rangle$ of elements of B. Let $c_0 = a_0$ and $d_0 = b_0$.

Now suppose that c_{2m} and d_{2m} have been defined so that the following condition hold:

(*) For all $i, j \leq 2m$, $c_i < c_j$ iff $d_i < d_j$.

(Note that then a similar equivalence holds for $=$ and for $>$.) We let $c_{2m+1} = a_{m+1}$. Now we consider several cases.

Case 1. $a_{m+1} = c_i$ for some $i \leq 2m$. Take the least such i, and let $d_{2m+1} = d_i$.

Case 2. $a_{m+1} < c_i$ for all $i \leq 2m$. Let d_{2m+1} be any element of B less than each d_i, $i \leq 2m$.

Case 3. $c_i < a_{m+1}$ for all $i \leq 2m$. Let d_{2m+1} be any element of B greater than each d_i, $i \leq 2m$.

Case 4. Case 1 fails, and there exist $i, j \leq 2m$ such that $c_i < a_{m+1} < c_j$. Let d_{2m+1} be any element b of B such that $d_i < b < d_j$ whenever $c_i < a_{m+1} < c_j$; such an element b exists by (*).

This finishes the definition of d_{2m+1}. d_{2m+2} and c_{2m+2} are defined similarly. Namely, we let $d_{2m+2} = b_{m+1}$ and then define c_{2m+2} similarly to the above, with a, b interchanged and c, d interchanged.

Note that each a_i appears in the sequence of c_i's, namely $c_0 = a_0$ and $c_{2i+1} = a_{i+1}$, and similarly each b_i appears in the sequence of d_i's. Hence it is clear that $\{(c_i, d_i) : i \in \omega\}$ is the desired order-isomorphism. □

Theorem 17.3. *If L is an infinite linear order, then there is a subset M of L which is order isomorphic to $(\omega, <)$, or to $(\omega, >)$.*

Proof. Suppose that L does not have a subset order isomorphic to $(\omega, >)$. We claim then that L is well-ordered, and therefore is isomorphic to an infinite ordinal and hence has a subset isomorphic to $(\omega, <)$. To prove this claim, suppose it is not true. So L has some nonempty subset P with no least element. We now define a sequence $\langle a_i : i \in \omega \rangle$ of elements of P by recursion. Let a_0 be any element of P. If $a_i \in P$ has been defined, then it is not the least element of P and so there is an $a_{i+1} \in P$ with $a_{i+1} < a_i$. This finishes the construction. Thus we have essentially produced a subset of L order isomorphic to $(\omega, >)$, contradiction. □

It would be natural to conjecture that Theorem 17.3 generalizes in the following way: for any infinite cardinal κ and any linear order L of size κ, there is a subset M of L order isomorphic to $(\kappa, <)$ or to $(\kappa, >)$. This is clearly false, as the real numbers under their usual order form a counterexample. (Given a set of real numbers order isomorphic to 2^ω, one could choose rationals between successive members of the set, and produce 2^ω rationals, contradiction.) We want to give an example that works for many cardinals. The construction we use is very important for later purposes too.

The following definitions apply to any infinite ordinal γ.

- If f and g are distinct elements of ${}^\gamma 2$, we define

$$\chi(f, g) = \min\{\alpha < \gamma : f(\alpha) \neq g(\alpha)\}.$$

• Let $f < g$ iff f and g are distinct elements of ${}^{\gamma}2$ and $f(\chi(f,g)) < g(\chi(f,g))$. (Thus $f(\chi(f,g)) = 0$ and $g(\chi(f,g)) = 1$.) Clearly $({}^{\gamma}2, <)$ is a linear order; this is called the *lexicographic order.*

We also need some general set-theoretic notation. If A is any set and κ any cardinal, then

$$\begin{aligned}[A]^{\kappa} &= \{X \subseteq A : |X| = \kappa\};\\ [A]^{<\kappa} &= \{X \subseteq A : |X| < \kappa\};\\ [A]^{\leq\kappa} &= \{X \subseteq A : |X| \leq \kappa\}.\end{aligned}$$

Theorem 17.4. *For any infinite cardinal κ, the linear order ${}^{\kappa}2$ does not contain a subset order isomorphic to κ^+ or to $(\kappa^+, >)$.*

Proof. The two assertions are proved in a very similar way, so we give details only for the first assertion. In fact, we assume that $\langle f_\alpha : \alpha < \kappa^+\rangle$ is a strictly increasing sequence of members of ${}^{\kappa}2$, and try to get a contradiction. The contradiction will follow rather easily from the following statement:

(1) If $\gamma \leq \kappa$, $\Gamma \in [\kappa^+]^{\kappa^+}$, and $f_\alpha \restriction \gamma < f_\beta \restriction \gamma$ for any $\alpha, \beta \in \Gamma$ such that $\alpha < \beta$, then there exist $\delta < \gamma$ and $\Delta \in [\Gamma]^{\kappa^+}$ such that $f_\alpha \restriction \delta < f_\beta \restriction \delta$ for any $\alpha, \beta \in \Delta$ such that $\alpha < \beta$.

To prove this, assume the hypothesis. For each $\alpha \in \Gamma$ let $f'_\alpha = f_\alpha \restriction \gamma$. Clearly Γ does not have a largest element. For each $\alpha \in \Gamma$ let α' be the least member of Γ which is greater than α. Then

$$\Gamma = \bigcup_{\xi<\gamma}\{\alpha \in \Gamma : \chi(f'_\alpha, f'_{\alpha'}) = \xi\}.$$

Since $|\Gamma| = \kappa^+$, it follows that there are $\delta < \gamma$ and $\Delta \in [\Gamma]^{\kappa^+}$ such that $\chi(f'_\alpha, f'_{\alpha'}) = \delta$ for all $\alpha \in \Delta$. We claim now that $f'_\alpha \restriction \delta < f'_\beta \restriction \delta$ for any two $\alpha, \beta \in \Delta$ such that $\alpha < \beta$, as desired in (1). For, take any such α, β. Suppose that $f'_\alpha \restriction \delta = f'_\beta \restriction \delta$. (Note that we must have $f'_\alpha \restriction \delta \leq f'_\beta \restriction \delta$.) Now from $\chi(f'_\alpha, f'_{\alpha'}) = \delta$ we get $f'_{\alpha'}(\delta) = 1$, and from $\chi(f'_\beta, f'_{\beta'}) = \delta$ we get $f'_\beta(\delta) = 0$. Now $f'_{\alpha'} \restriction \delta = f'_\alpha \restriction \delta = f'_\beta \restriction \delta$, so we get $f'_\beta < f'_{\alpha'} \leq f'_\beta$, contradiction. This proves (1).

Clearly from (1) we can construct an infinite decreasing sequence $\kappa > \gamma_1 > \gamma_2 > \cdots$ of ordinals, contradiction. $\square$

Now we give some more definitions, leading to a kind of generalization of Theorem 17.2.

• If $(L, <)$ is a linear order and $A, B \subseteq L$, we write $A < B$ iff $\forall x \in A \forall y \in B[x < y]$. If $A = \{a\}$ here, we write $a < B$; similarly for $A < b$.

• *Intervals* in linear orders are defined in the usual way. For example, $[a, b) = \{c : a \leq c < b\}$.

• An η_α*-set* is a linear order $(L, <)$ such that if $A, B \subseteq L$, $A < B$, and $|A|, |B| < \aleph_\alpha$, then there is a $c \in L$ such that $A < c < B$. Taking $A = \emptyset$ and $B = \{a\}$ for some $a \in L$, we see that η_α-sets do not have first elements; similarly they do not have last elements. Note that an η_0-set is just a densely ordered set without first or last elements.

• For any ordinal α, we define

$$H_\alpha = \{f \in {}^{\aleph_\alpha}2 : \text{there is a } \xi < \aleph_\alpha \text{ such that } f(\xi) = 1 \text{ and } f(\eta) = 0 \text{ for all } \eta \in (\xi, \aleph_\alpha)\}.$$

We take the order on H_α induced by that on ${}^{\aleph_\alpha}2$: $f < g$ in H_α iff $f < g$ as members of ${}^{\aleph_\alpha}2$.

Theorem 17.5. *Let α be an ordinal, and let* $\mathrm{cf}(\aleph_\alpha) = \aleph_\gamma$. *Then the following conditions hold:*

(i) H_α is an η_γ-set.
(ii) $\mathrm{cf}(H_\alpha, <) = \aleph_\gamma$.
(iii) $\mathrm{cf}(H_\alpha, >) = \aleph_\gamma$.
(iv) $|H_0| = \aleph_0$, *and for* $\alpha > 0$, $|H_\alpha| = \sum_{\beta<\alpha} 2^{\aleph_\beta}$.

Proof. For each $f \in H_\alpha$ let $\zeta_f < \aleph_\alpha$ be such that $f(\zeta_f) = 1$ and $f(\eta) = 0$ for all $\eta \in (\zeta_f, \aleph_\alpha)$.

For (i), suppose that $A, B \subseteq H_\alpha$ with $A < B$ and $|A|, |B| < \aleph_\gamma$. Obviously we may assume that one of A, B is nonempty. Then there are three possibilities:

Case 1. $A \neq \emptyset \neq B$. Let

$$\begin{aligned}
\xi &= \sup\{\zeta_f : f \in A\};\\
\rho &= \max(\xi + 1, \sup\{\zeta_f : f \in B\}).
\end{aligned}$$

Thus $\xi, \rho < \aleph_\alpha$ since $|A|, |B| < \aleph_\gamma = \mathrm{cf}(\aleph_\alpha)$. We now define $g \in {}^{\aleph_\alpha}2$ by setting, for each $\eta < \aleph_\alpha$,

$$g(\eta) = \begin{cases} 1 & \text{if } \eta \leq \xi \text{ and } \exists f \in A(f \restriction \eta = g \restriction \eta \text{ and } f(\eta) = 1);\\ 0 & \text{if } \eta \leq \xi \text{ and there is no such } f;\\ 0 & \text{if } \xi < \eta \leq \rho;\\ 1 & \text{if } \eta = \rho + 1;\\ 0 & \text{if } \rho + 1 < \eta < \aleph_\alpha. \end{cases}$$

Clearly $g \in H_\alpha$. We claim that $A < g < B$. Note that $g \notin A \cup B$ since $g(\rho + 1) = 1$ while $f(\rho + 1) = 0$ for any $f \in A \cup B$.

To prove the claim first suppose that $f \in A$. Assume that $g < f$; we will get a contradiction. Let $\eta = \chi(g, f)$. Then $g(\eta) = 0$ and $f(\eta) = 1$. It follows that $\eta \leq \xi$ and $g \restriction \eta = f \restriction \eta$, contradicting the definition of $g(\eta)$.

Second, suppose that $f \in B$. Assume that $f < g$; we will get a contradiction. Let $\eta = \chi(f, g)$. Thus $f(\eta) = 0$ and $g(\eta) = 1$. We claim that $\eta = \rho + 1$. For, otherwise since $g(\eta) = 1$ we must have $\eta \leq \xi$, and then there is an $h \in A$ such that $h \restriction \eta = g \restriction \eta$ and $h(\eta) = 1$. So $f \restriction \eta = g \restriction \eta = h \restriction \eta$, $f(\eta) = 0$, and $h(\eta) = 1$, so $f < h$. But $f \in B$ and $h \in A$, contradiction. This proves our claim that $\eta = \rho + 1$.

Now clearly $\zeta_f \leq \rho$. Since

$$g \restriction (\rho + 1) = g \restriction \eta = f \restriction \eta = f \restriction (\rho + 1),$$

it follows that $g(\zeta_f) = 1$. So from $\zeta_f \leq \rho$ we infer that $\zeta_f \leq \xi$. Thus since $g(\zeta_f) = 1$, it follows that there is a $k \in A$ such that $k \restriction \zeta_f = g \restriction \zeta_f$ and $k(\zeta_f) = 1$. But now we have

$k \restriction (\zeta_f + 1) = g \restriction (\zeta_f + 1) = f \restriction (\zeta_f + 1)$ and $f(\sigma) = 0$ for all $\sigma \in (\zeta_f, \aleph_\alpha)$. Hence $f \leq k$, which contradicts $f \in B$ and $k \in A$.

This finishes the proof of (i) in Case 1.

Case 2. $A = \emptyset \neq B$. Let

$$\rho = \sup\{\zeta_f : f \in B\}.$$

Define $g \in H_\alpha$ by setting, for each $\xi < \aleph_\alpha$,

$$g(\xi) = \begin{cases} 0 & \text{if } \xi \leq \rho, \\ 1 & \text{if } \xi = \rho + 1, \\ 0 & \text{if } \rho + 1 < \xi. \end{cases}$$

Clearly $g < B$, as desired.

Case 3. $A \neq \emptyset = B$. Let ξ be as in Case 1. Define $g \in H_\alpha$ by setting, for each $\eta < \aleph_\alpha$,

$$g(\eta) = \begin{cases} 1 & \text{if } \eta \leq \xi + 1, \\ 0 & \text{if } \xi + 1 < \eta. \end{cases}$$

Clearly $A < g$, as desired.

This finishes the proof of (i).

For (ii), let $\langle \delta_\xi : \xi < \aleph_\gamma \rangle$ be a strictly increasing sequence of ordinals with supremum $\aleph_\alpha$. For each $\xi < \aleph_\gamma$ define $f_\xi \in H_\alpha$ by setting, for each $\eta < \aleph_\alpha$,

$$f_\xi(\eta) = \begin{cases} 1 & \text{if } \eta \leq \delta_\xi, \\ 0 & \text{if } \delta_\xi < \eta. \end{cases}$$

Clearly $\langle f_\xi : \xi < \aleph_\gamma \rangle$ is a strictly increasing sequence of members of H_α and $\{f_\xi : \xi < \aleph_\gamma\}$ is cofinal in H_α. So (ii) holds.

For (iii), take $\langle \delta_\xi : \xi < \aleph_\gamma \rangle$ as in the proof of (ii). For each $\xi < \alpha_\gamma$ define $f_\xi \in H_\alpha$ by setting, for each $\eta < \aleph_\alpha$,

$$f_\xi(\eta) = \begin{cases} 0 & \text{if } \eta < \delta_\xi, \\ 1 & \text{if } \eta = \delta_\xi, \\ 0 & \text{if } \delta_\xi < \eta. \end{cases}$$

Clearly $\langle f_\xi : \xi < \aleph_\gamma$ is a strictly decreasing sequence of members of H_α and $\{f_\xi : \xi < \aleph_\gamma\}$ is cofinal in $(H_\alpha, >)$. So (iii) holds.

Finally, for (iv), for each $\delta < \aleph_\alpha$ let

$$L_\delta = \{f \in H_\alpha : f(\delta) = 1 \text{ and } f(\varepsilon) = 0 \text{ for all } \varepsilon \in (\delta, \aleph_\alpha)\}.$$

Clearly these sets are pairwise disjoint, and their union is H_α. For $\alpha = 0$,

$$|H_\alpha| = \sum_{\delta < \omega} |L_\delta| = \sum_{\delta < \omega} 2^{|\delta|} = \aleph_0.$$

For $\alpha > 0$,

$$\begin{aligned}
|H_\alpha| &= \sum_{\delta<\aleph_\alpha} |L_\delta| \\
&= \sum_{\delta<\omega} |L_\delta| + \sum_{\omega\le\delta<\aleph_\alpha} |L_\delta| \\
&= \aleph_0 + \sum_{\omega\le\delta<\aleph_\alpha} 2^{|\delta|} \\
&= \aleph_0 + \sum_{\beta<\alpha} \left(2^{\aleph_\beta} \cdot |\{\delta < \aleph_\alpha : |\delta| = \aleph_\beta\}|\right) \\
&= \aleph_0 + \sum_{\beta<\alpha} \left(2^{\aleph_\beta} \cdot \aleph_{\beta+1}\right) \\
&= \sum_{\beta<\aleph_\alpha} 2^{\aleph_\beta}.
\end{aligned}$$

□

Corollary 17.6. *If $\aleph_\alpha$ is regular, then*

(i) H_α is an η_α-set.
(ii) $\mathrm{cf}(H_\alpha, <) = \aleph_\alpha$.
(iii) $\mathrm{cf}(H_\alpha, >) = \aleph_\alpha$.
(iv) $|H_0| = \aleph_0$, and for $\alpha > 0$, $|H_\alpha| = \sum_{\beta<\alpha} 2^{\aleph_\beta}$. □

Corollary 17.7. *For each regular cardinal $\aleph_\alpha$ there is an η_α-set.* □

Corollary 17.8. *For each ordinal α there is an $\eta_{\alpha+1}$-set of size $2^{\aleph_\alpha}$.* □

Corollary 17.9. (GCH) *For each regular cardinal $\aleph_\alpha$ there is an η_α-set of size $\aleph_\alpha$.* □

One of the most useful facts about η_α-sets is their *universality*, expressed in the following theorem.

Theorem 17.10. *Suppose that $\aleph_\alpha$ is regular. If K is an η_α-set, then any linearly ordered set of size $\le \aleph_\alpha$ can be isomorphically embedded in K.*

Proof. Let L be a linearly ordered set of size at most $\aleph_\alpha$, and write $L = \{a_\xi : \xi < \aleph_\alpha\}$. We define a sequence $\langle f_\xi : \xi < \aleph_\alpha\rangle$ of functions by recursion. Suppose that f_η has been defined for all $\eta < \xi$ so that it is a strictly increasing function mapping a subset of L of size less than $\aleph_\alpha$ into K, and such that $f_\rho \subseteq f_\eta$ whenever $\rho < \eta < \xi$. Let $g = \bigcup_{\eta<\xi} f_\eta$. Then g is still a strictly increasing function mapping a subset of L of size less than $\aleph_\alpha$ into K. If $a_\xi \in \mathrm{dmn}(g)$, we set $f_\xi = g$. Suppose that $a_\xi \notin \mathrm{dmn}(g)$. Let

$$\begin{aligned}
A &= \{g(b) : b \in \mathrm{dmn}(g) \text{ and } b < a_\xi\};\\
B &= \{g(b) : b \in \mathrm{dmn}(g) \text{ and } a_\xi < b\}.
\end{aligned}$$

Then $A < B$, and $|A|, |B| < \aleph_\alpha$. So by the η_α-property, there is an element c of K such that $A < c < B$. We let $f_\xi = g \cup \{(a_\xi, c)\}$ for such an element c. (AC is used.)

This finishes the construction, and clearly $\bigcup_{\xi<\aleph_\alpha} f_\xi$ is as desired. $\square$

Given a linearly ordered set L, a subset X of L, and an element a of L, we call a an *upper bound* for X iff $x \le a$ for all $x \in X$. Thus every element of L is an upper bound of the empty set. We say that a is a *least upper bound* for X iff a is an upper bound for X and is $\le$ any upper bound for X. Clearly a least upper bound for X is unique if it exists. If a is the least upper bound of the empty set, then a is the smallest element of L. We use lub or sup to abbreviate least upper bound. Also "supremum" is synonymous with "least upper bound".

Similarly one defines *lower bound* and *greatest lower bound*. Any element is a lower bound of the empty set, and if a is the greatest lower bound of the empty set, then a is the largest element of L. We use glb or inf to abbreviate greatest lower bound. "infimum" is synonymous with "greatest lower bound".

A linear order L is *complete* iff every subset of L has a greatest lower bound and a least upper bound.

Proposition 17.11. *For any linear order L the following conditions are equivalent:*

(i) L is complete.

(ii) Every subset of L has a least upper bound.

(iii) Every subset of L has a greatest lower bound.

Proof. (i)⇒(ii): obvious. (ii)⇒(iii). Assume that every subset of L has a least upper bound, and let $X \subseteq L$; we want to show that X has a greatest lower bound. Let Y be the set of all lower bounds of X. Then let a be a least upper bound for Y. Take any $x \in X$. Then $\forall y \in Y[y \le x]$, so $a \le x$ since a is the lub of Y. This shows that a is a lower bound for X. Suppose that y is any lower bound for X. Then $y \in Y$, and hence $y \le a$ since a is an upper bound for Y.

(iii)⇒(i) is treated similarly. $\square$

Let $(L, <)$ be a linear order. We say that a linear order $(M, \prec)$ is a *completion* of L iff the following conditions hold:

(C1) $L \subseteq M$, and for any $a, b \in L$, $a < b$ iff $a \prec b$.

(C2) M is complete.

(C3) Every element of M is the lub of a set of elements of L.

(C4) If $a \in L$ is the lub in L of a subset X of L, then a is the lub of X in M.

Theorem 17.12. *Any linear order has a completion.*

Proof. Let $(L, <)$ be a linear order. We let M' be the collection of all $X \subseteq L$ such that the following conditions hold:

(1) For all $a, b \in L$, if $a < b \in X$ then $a \in X$.

(2) If X has a lub a in L, then $a \in X$.

We consider the structure $(M', \subset)$. It is clearly a partial order; we claim that it is a linear order. (Up to isomorphism it is the completion that we are after.) Suppose that $X, Y \in M'$ and $X \neq Y$; we want to show that $X \subset Y$ or $Y \subset X$. By symmetry take $a \in X \backslash Y$. Then we claim that $Y \subseteq X$ (hence $Y \subset X$). For, take any $b \in Y$. If $a < b$, then $a \in Y$ by (1), contradiction. Hence $b \leq a$, and so $b \in X$ by (1), as desired. This proves the claim.

Next we claim that $(M', \subset)$ is complete. For, suppose that $\mathscr{X} \subseteq M'$. Then $\bigcup \mathscr{X}$ satisfies (1). In fact, suppose that $c < d \in \bigcup \mathscr{X}$. Choose $X \in \mathscr{X}$ such that $d \in X$. Then $c \in X$ by (1) for X, and so $c \in \bigcup \mathscr{X}$. Now we consider two cases.

Case 1. $\bigcup \mathscr{X}$ does not have a lub in L. Then $\bigcup \mathscr{X} \in M'$, and it is clearly the lub of $\mathscr{X}$.

Case 2. $\bigcup \mathscr{X}$ has a lub in L; say a is its lub. Then

(3) $\bigcup \mathscr{X} \cup \{a\} = (-\infty, a]$.

In fact, $\subseteq$ is clear. Suppose that $b < a$. Then b is not an upper bound for $\bigcup \mathscr{X}$, so we can choose $c \in \bigcup \mathscr{X}$ such that $b < c$. Then $b \in \bigcup \mathscr{X}$ since $\bigcup \mathscr{X}$ satisfies (1). This proves (3).

Clearly $(-\infty, a] \in M'$. We claim that it is the lub of $\mathscr{X}$. Clearly it is an upper bound. Now suppose that Z is any upper bound. Then $\bigcup \mathscr{X} \subseteq Z$. If $a \notin Z$, then $\bigcup \mathscr{X} = Z$, contradicting (2) for Z. So $a \in Z$ and hence $(-\infty, a] \subseteq Z$.

Hence we have shown that $(M', \subset)$ is complete.

Now for each $a \in L$ let $f(a) = \{b \in L : b \leq a\}$. Clearly $f(a) \in M'$.

(4) For any $a, b \in L$ we have $a < b$ iff $f(a) \subset f(b)$.

For, suppose that $a, b \in L$. If $a < b$, clearly $f(a) \subseteq f(b)$, and even $f(a) \subset f(b)$ since $b \in f(b) \backslash f(a)$. The other implication in (4) follows easily from this implication by assuming that $b \leq a$.

(5) Every element of M' is a lub of elements of $f[L]$.

For, suppose that $X \in M'$, and let $\mathscr{X} = \{f(a) : a \in X\}$; we claim that X is the lub of $\mathscr{X}$. Clearly $f(a) \subseteq X$ for all $a \in X$, so X is an upper bound of $\mathscr{X}$. Suppose that $Y \in M'$ is any upper bound for $\mathscr{X}$. If $a \in X$, then $a \in f(a) \subseteq Y$, so $a \in Y$. Thus $X \subseteq Y$, as desired. So (5) holds.

(6) If $a \in L$ is the lub in L of $X \subseteq L$, then $f(a)$ is the lub in M' of $f[X]$.

For, assume that $a \in L$ is the lub in L of $X \subseteq L$. If $x \in X$, then $x \leq a$, so $f(x) \subseteq f(a)$. Thus $f(a)$ is an upper bound for $f[X]$ in M'. Now suppose that $Y \in M'$ and Y is an upper bound for $f[X]$. If $b \in L$ and $b < a$, then since a is the lub of X, there is a $d \in X$ such that $b < d \leq a$. So $f(d) \subseteq Y$, and hence $d \in Y$. Since $b < d$, we also have $b \in Y$. This shows that $f(a) \backslash \{a\} \subseteq Y$. If $a \in X$, then $f(a) \in f[X]$ and so $f(a) \subseteq Y$, as desired. Assume that $a \notin X$. Since a is the lub of X in L, there is no largest member of L which is less than a. Now suppose that $a \notin Y$. If $u \in Y$, then $u < a$, as otherwise $a \leq u$ and so $a \in Y$, contradiction. It follows that $Y = \{u \in L : u < a\}$. Clearly then a is the lub of Y. This contradicts (2). So (6) holds.

Thus M' is as desired, up to isomorphism.

Finally, we need to take care of the "up to isomorphism" business. Non-rigorously, we just identify a with $f(a)$ for each $a \in L$. This is the way things are done in similar contexts in mathematics. Rigorously we proceed as follows; and a similar method can be used in other contexts. Let A be a set disjoint from L such that $|A| = |M'\backslash f[L]|$. For example, we could take $A = \{(L, X) : X \in M'\backslash f[L]\}$; this set is clearly of the same size as $M'\backslash f[L]$, and it is disjoint from L by the foundation axiom. Let g be a bijection from A onto $M'\backslash f[L]$. Now let $N = L \cup A$, and define $h : N \to M'$ by setting, for any $x \in N$,

$$h(x) = \begin{cases} f(x) & \text{if } x \in L, \\ g(x) & \text{if } x \in A. \end{cases}$$

Thus h is a bijection from N to M', and it extends f. We now define $x \ll y$ iff $x, y \in N$ and $h(x) \subset h(y)$. We claim that $(N, \ll)$ really is a completion of L. (Not just up to isomorphism.) We check the conditions for this. Obviously $L \subseteq N$. Suppose that $a, b \in L$. Then $a < b$ iff $f(a) \subset f(b)$ iff $h(a) \subset h(b)$ iff $a \ll b$. Now h is obviously an order-isomorphism from $(N \subset)$ onto $(M' \subset)$, so N is complete. Now take any element a of N. Then by (5), $h(a)$ is the lub of a set $f[X]$ with $X \subseteq L$. By the isomorphism property, a is the lub of X. Finally, suppose that $a \in L$ is the lub of $X \subseteq L$. Then by (6), $f(a)$ is the lub of $f[X]$ in M', i.e., $h(a)$ is the lub of $h[X]$ in M'. By the isomorphism property, a is the lub of X in N. $\square$

Theorem 17.13. *If L is a linear order and M, N are completions of L, then there is an isomorphism f of M onto N such that $f \restriction L$ is the identity.*

Proof. It suffices to show that if N is a completion of L and M', f are as in the proof of Theorem 17.12, then there is an isomorphism g from N onto M' such that $g \restriction L = f$.

For any $x \in N$ let $g(x) = \{a \in L : a \leq_N x\}$. We claim that $g(x) \in M'$. Clearly condition (1) holds. Now suppose that $g(x)$ has a lub b in L. By (C4) for N, b is the lub of $g(x)$ in N. But obviously x is the lub of $g(x)$ in N, so $b = x \in g(x)$. So (2) holds for $g(x)$, and so $g(x) \in M'$.

If $x <_N y$, clearly $g(x) \leq g(y)$. By (C3) for N and y, there is an $a \in L$ such that $x <_N a \leq_N y$. So $a \in g(y)\backslash g(x)$. Hence $g(x) \subset g(y)$. Hence by Proposition 4.14, $x <_N y$ iff $g(x) \subset g(y)$, for any $x, y \in N$.

It remains only to show that g is a surjection. Let $X \in M'$. Set $x = \sup_N X$. If $a \in X$, then $a \leq_N x$ and so $a \in g(x)$. Thus $X \subseteq g(x)$. Now suppose that $a \in g(x)$. So $a \leq_N x$. If $a <_N x$, then there is a $y \in X$ such that $a <_N y \leq_N x$. It follows that $a \in X$. If $a = x$, then $a \in X$ by (2). So $g(x) \subseteq X$, showing that $g(x) = X$. $\square$

Corollary 17.14. *Suppose that L is a dense linear order, and M is a linear order. Then the following conditions are equivalent:*

(i) M is the completion of L.

(ii) (a) $L \subseteq M$

(b) M is complete.

(c) For any $a, b \in L$, $a <_L b$ iff $a <_M b$.

(d) For any $x, y \in M$, if $x <_M y$ then there is an $a \in L$ such that $x <_M a <_M y$.

Proof. (i)⇒(ii): Assume that M is the completion of L. then (a)–(c) are clear. Suppose that $x, y \in M$ and $x <_M y$. By (C3), choose $b \in L$ such that $x <_M b \leq_M y$. If

$x \in L$, then choose $a \in L$ such that $x <_L a <_L b$; so $x <_M a <_M y$, as desired. Assume that $x \notin L$. Then by (C4), b is not the lub in L of $\{u \in L : u <_M x\}$, so there is some $a \in L$ such that $a <_L b$ and a is an upper bound of $\{u \in L : u <_M x\}$. Since by (C3) x is the lub of $\{u \in L : u <_M x\}$, it follows that $x <_M a <_M b \leq_M y$, as desired.

(ii)⇒(i): Assume (ii). Then (C1) and (C2) are clear. For (C3), let $x \in M$, and let $X = \{a \in L : a < x\}$. Then x is an upper bound for X, and (ii)(d) clearly implies that it is the lub of X. For (C4), suppose that $a \in L$ is the lub in L of a set X of elements of L. Suppose that $x \in M$ is an upper bound for X and $x < a$. Then by (ii)(d) there is an element $b \in L$ such that $x < b < a$. Then there is an element $c \in X$ such that $b < c \leq a$. It follows that $c \leq x$, contradiction. $\square$

Note from this corollary that the completion of a dense linear order is also dense.

We now take up a special topic, Suslin lines.

• A subset U of a linear order L is *open* iff U is a union of open intervals (a, b) or $(-\infty, a)$ or (a, ∞). Here $(-\infty, a) = \{b \in L : b < a\}$ and $(a, \infty) = \{b \in L : a < b\}$. L itself is also counted as open. (If L has at least two elements, this follows from the other parts of this definition.) Note that if L has a largest element a, then $(a, \infty) = \emptyset$; similarly for smallest elements.

• An *antichain* in a linear order L is a collection of pairwise disjoint open sets.

• A linear order L has the *countable chain condition*, abbreviated ccc, iff every antichain in L is countable.

• A subset D of a linear order L is *topologically dense* in L iff $D \cap U \neq \emptyset$ for every nonempty open subset U of L. Then dense in the sense at the beginning of the chapter implies topologically dense. In fact, if D is dense in the original sense and U is a nonempty open set, take some non-empty open interval (a, b) contained in U. There is a $d \in D$ with $a < d < b$, so $D \cap U \neq \emptyset$. If $\emptyset \neq (a, \infty) \subseteq U$ for some a, choose $b \in (a, \infty)$, and then choose $d \in D$ such that $a < d < b$. Then again $D \cap U \neq \emptyset$. Similarly if $(-\infty, a) \subseteq U$ for some a.

Conversely, if L itself is dense, then topological denseness implies dense in the order sense; this is clear. On the other hand, take for example the ordered set ω; ω itself is topologically dense in ω, but ω is not dense in ω in the order sense.

• A linear order L is *separable* iff there is a countable subset C of L which is topologically dense in L. Note that if L is separable and (a, b) is a nonempty open interval of L, then (a, b), with the order induced by L ($x < y$ for $x, y \in (a, b)$ iff $x < y$ in L) is separable. In fact, if C is countable and topologically dense in L clearly $C \cap (a, b)$ is countable and topologically dense in (a, b). Similarly, $[a, b]$ is separable, taking $(C \cap [a, b]) \cup \{a, b\}$. This remark will be used shortly.

• A *Suslin line* is a linear ordered set $(S, <)$ satisfying the following conditions:
  (i) S has ccc.
  (ii) S is not separable.

• *Suslin's Hypothesis* (SH) is the statement that there do not exist Suslin lines.

Later in these notes we will prove that MA$+\neg$CH implies SH. Here MA is *Martin's axiom*, which we will define and discuss later. The consistency of MA $+$ $\neg$CH requires iterated forcing, and will be proven much later in these notes. Also later in these notes we will prove that $\diamondsuit$ implies $\neg$SH, and still later we will prove that $\diamondsuit$ is consistent with ZFC, namely it follows from $V = L$. Both $\diamondsuit$ and L are defined later.

For now we want to connect our notion of Suslin line with more familiar mathematics, and with the original conjecture of Suslin. The following is a theorem of elementary set theory.

Theorem 17.15. *For any linear order $(L, \prec)$ the following conditions are equivalent:*

(i) $(L, \prec)$ is isomorphic to $(\mathbb{R}, <)$.

(ii) The following conditions hold:

(a) L has no first or last elements.

(b) L is dense.

(c) Every nonempty subset of L which is bounded above has a least upper bound.

(d) L is separable.

Proof. (i)$\Rightarrow$(ii): standard facts about real numbers.

(ii)$\Rightarrow$(i): By (d), let C be a countable subset of L such that $(a, b) \cap C \neq \emptyset$ whenever $a < b$ in L. Clearly C is infinite, is dense, and has no first or last element. By Theorem 17.2, let f be an isomorphism from $(C, <)$ onto $(\mathbb{Q}, <)$. We now apply the procedure used at the end of the proof of 17.12. Let P be a set disjoint from $\mathbb{Q}$ such that $|L\backslash C| = |P|$, and let $R = \mathbb{Q} \cup P$. Let g be a bijection from $L\backslash C$ onto P, and define $h = f \cup g$. Define $x \prec y$ iff $x, y \in R$ and $h^{-1}(x) <_L h^{-1}(y)$. This makes R into a linearly ordered set with h an isomorphism from L onto R. Now we adjoin first and last elements a_R, b_R to R and similarly $a_{\mathbb{R}}, b_{\mathbb{R}}$ for $\mathbb{R}$; call the resulting linearly ordered sets R' and $\mathbb{R}'$. Then R' and $\mathbb{R}'$ are both completions of $\mathbb{Q}$ according to Corollary 17.14. Hence (i) holds by Theorem 17.13. $\square$

Originally, Suslin made the conjecture that separability in Theorem 17.15 can be replaced by the condition that every family of pairwise disjoint open intervals is countable. The following theorem shows that this conjecture and our statement of Suslin's hypothesis are equivalent.

Theorem 17.16. *The following conditions are equivalent:*

(i) There is a Suslin line.

(ii) There is a linearly ordered set $(L, <)$ satisfying the following conditions:

(a) L has no first or last elements.

(b) L is dense.

(c) Every nonempty subset of L which is bounded above has a least upper bound.

(d) No nonempty open subset of L is separable.

(e) L is ccc.

Proof. Obviously (ii) implies (i). Now suppose that (i) holds, and let S be a Suslin line. We obtain (ii) in two steps: first taking care of denseness, and then taking the completion to finish up.

We define a relation $\sim$ on S as follows: for any $a, b \in S$,

$$a \sim b \quad \text{iff} \quad \begin{aligned}[t] & a = b, \\ & \text{or } a < b \text{ and } [a, b] \text{ is separable,} \\ & \text{or } b < a \text{ and } [b, a] \text{ is separable.} \end{aligned}$$

Clearly $\sim$ is an equivalence relation on S. Let L be the collection of all equivalence classes under $\sim$.

(1) If $I \in L$, then I is convex, i.e., if $a < c < b$ with $a, b \in I$, then also $c \in I$.

For, $[a, b]$ is separable, so $[a, c]$ is separable too, and hence $a \sim c$; so $c \in I$.

(2) If $I \in L$, then I is separable.

For, this is clear if I has only one or two elements. Suppose that I has at least three elements. Then there exist $a, b \in I$ with $a < b$ and $(a, b) \neq \emptyset$. Let $\mathscr{M}$ be a maximal pairwise disjoint set of such intervals. Then $\mathscr{M}$ is countable. Say $\mathscr{M} = \{(x_n, y_n) : n \in \omega\}$. Since $x_n \sim y_n$, the interval $[x_n, y_n]$ is separable, so we can let D_n be a countable dense subset of it. We claim that the following countable set E is dense in I:

$$\begin{aligned} E = \bigcup_{n\in\omega} D_n & \cup \{e : e \text{ is the largest element of } I\} \\ & \cup \{a : a \text{ is the smallest element of } I\}. \end{aligned}$$

Thus e and a are added only if they exist. To show that E is dense in I, first suppose that $a, b \in I$, $a < b$, and $(a, b) \neq \emptyset$. Then by the maximality of $\mathscr{M}$, there is an $n \in \omega$ such that $(a, b) \cap (x_n, y_n) \neq \emptyset$. Choose $c \in (a, b) \cap (x_n, y_n)$. Then $\max(a, x_n) < c < \min(b, y_n)$, so there is a $d \in D_n \cap (\max(a, x_n), \min(b, y_n)) \subseteq (a, b)$, as desired. Second, suppose that $a \in I$ and $(a, \infty) \neq \emptyset$; here $(a, \infty) = \{x \in I : a < x\}$. We want to find $d \in E$ with $a < d$. If I has a largest element e, then e is as desired. Otherwise, there are $b, c \in I$ with $a < b < c$, and then an element of $(a, c) \cap E$, already shown to exist, is as desired. Similarly one deals with $-\infty$. Thus we have proved (2).

Now we define a relation $<$ on L by setting $I < J$ iff $I \neq J$ and $a < b$ for some $a \in I$ and $b \in J$. By (1) this is equivalent to saying that $I < J$ iff $I \neq J$ and $a < b$ for all $a \in I$ and $b \in J$. In fact, suppose that $a \in I$ and $b \in J$ and $a < b$, and also $c \in I$ and $d \in J$, while $d \leq c$. If $d \leq a$, then $d \leq a < b$ with $d, b \in J$ implies that $a \in J$, contradiction. Hence $a < d$. Since also $d \leq c$ this gives $d \in I$, contradiction.

Clearly $<$ makes L into a simply ordered set. Except for not being complete in the sense of (c), L is close to the linear order we want.

To see that L is dense, suppose that $I < J$ but $(I, J) = \emptyset$. Take any $a \in I$ and $b \in J$. Then $(a, b) \subseteq I \cup J$, and $I \cup J$ is separable by (2), so $a \sim b$, contradiction.

For (d), by a remark in the definition of separable it suffices to show that no open interval (I, J) is separable. Suppose to the contrary that (I, J) is separable. Let $\mathscr{A}$ be a countable dense subset of (I, J). Also, let $\mathscr{B} = \{K \in L : I < K < J \text{ and } |K| > 2\}$. Any two distinct members of $\mathscr{B}$ are disjoint, and hence by ccc $\mathscr{B}$ is countable. In fact, each

$K \in \mathscr{B}$ has the form (a, b), $[a, b)$, $(a, b]$, or $[a, b]$. since $|K| > 2$, in each case the open interval (a, b) is nonempty. So ccc applies.

Define $\mathscr{C} = \mathscr{A} \cup \mathscr{B} \cup \{I, J\}$. By (2), each member of $\mathscr{C}$ is separable, so for each $K \in \mathscr{C}$ we can let D_K be a countable dense subset of K. Let $E = \bigcup_{K \in \mathscr{C}} D_K$. So E is a countable set. Fix $a \in I$ and $b \in J$. We claim that $E \cap (a, b)$ is dense in (a, b). (Hence $a \sim b$ and so $I = J$, contradiction.) For, suppose that $a \leq c < d \leq b$ with $(c, d) \neq \emptyset$.

Case 1. $[c]_\sim = [d]_\sim = I$. Then $D_I \cap (c, d) \neq \emptyset$, so $E \cap (c, d) \neq \emptyset$, as desired.

Case 2. $[c]_\sim = [d]_\sim = J$. Similarly.

Case 3. $I < [c]_\sim = [d]_\sim < J$. Then $[c]_\sim \in \mathscr{B} \subseteq \mathscr{C}$, so the desired result follows again.

Case 4. $[c]_\sim < [d]_\sim$. Choose $K \in \mathscr{A}$ such that $[c]_\sim < K < [d]_\sim$. Hence $c < e < d$ for any $e \in D_K$, as desired.

Thus we have obtained a contradiction, which proves that (I, J) is not separable.

Next, we claim that L has ccc. In fact, suppose that $\mathscr{A}$ is an uncountable family of pairwise disjoint open intervals. Let $\mathscr{B}$ be the collection of all endpoints of members of $\mathscr{A}$, and for each $I \in \mathscr{B}$ choose $a_I \in I$. Then

$$\{(a_I, a_J) : (I, J) \in \mathscr{A}\}$$

is an uncountable collection of pairwise disjoint nonempty open intervals in S, contradiction. In fact, given $(I, J) \in \mathscr{A}$, choose K with $I < K < J$. then $a_K \in (a_I, a_J)$. So $(a_I, a_J) \neq \emptyset$. Suppose that $(I, J), (I', J')$ are distinct members of $\mathscr{A}$. Wlog $J \leq I'$. Then $a_J \leq a_{I'}$, and it follows that $(a_I, a_J) \cap (a_{I'}, a_{J'}) = \emptyset$.

This finishes the first part of the proof. We have verified that L satisfies (b), (d), and (e). Now let M be the completion of L, and let N be M without its first and last elements. We claim that N finally satisfies all of the conditions in (ii). Clearly N is dense, it has no first or last elements, and every nonempty subset of it bounded above has a least upper bound. Next, suppose that $a < b$ in N and C is a countable subset of (a, b) which is dense in (a, b). Choose $c, d \in L$ such that $a < c < d < b$. For any $u, v \in C$ with $c < u < v < d$ choose $e_{uv} \in L$ such that $u < e_{uv} < v$; such an element exists by Corollary 17.14. We claim that $\{e_{uv} : u, v \in C,\ u < v\}$ is dense in (c, d) in L, which is a contradiction. For, given x, y such that $c < x < y < d$ in L, by the definition of denseness we can find $u, v \in C$ such that $x < u < v < y$; and then $x < e_{uv} < y$, as desired.

It remains only to prove that N has ccc. Suppose that $\mathscr{A}$ is an uncountable collection of nonempty open intervals of N. By Corollary 17.14, for each $(a, b) \in \mathscr{A}$ we can find $c, d \in L$ such that $a < c < d < b$. So this gives an uncountable collection of nonempty open intervals in L, contradiction. $\square$

In the rest of this chapter we prove a very useful theorem on characters of points and gaps in linearly ordered sets due to Hausdorff.

For any cardinal κ, the order type which is the reverse of κ is denoted by κ^*. Reg is the class of all regular cardinals. We define *regular* so that every regular cardinal is infinite. If $\kappa < \lambda$ are cardinals, then $[\kappa, \lambda]_{\text{reg}}$ is the collection of all regular cardinals in the interval $[\kappa, \lambda]$; similarly for half-open and open intervals.

Let $R \subseteq \text{Reg} \times \text{Reg}$. We define

$$\chi_{\text{left}}(R) = \text{the least cardinal greater than each member of } \text{dmn}(R);$$
$$\chi_{\text{right}}(R) = \text{the least cardinal greater than each member of } \text{rng}(R).$$

Let L be a linear order, and let $x \in L$. If x is the first element of L, then its *left character* is 0. If x has an immediate predecessor, then its *left character* is 1. Finally, suppose that x is not the first element of L and does not have an immediate predecessor. Then the *left character* of x is the smallest cardinal κ such that there is a strictly increasing sequence of elements of L with supremum x. This cardinal κ is then regular. Similarly, if x is the last element of L, then its *right character* is 0. If x has an immediate successor, then its *right character* is 1. Finally, suppose that x is not the last element of L and it does not have an immediate successor. Then the *right character* of x is the smallest cardinal λ such that there is a strictly decreasing sequence of elements of L with infimum x. The *character* of x is the pair (κ, λ^*) where κ is the left character and λ is the right character. The *point-character set* of L is the collection of all characters of points of L; we denote it by $\text{Pchar}(L)$. Note that $\text{Pchar}(L) \neq \emptyset$.

A *gap* of L is an ordered pair (M, N) such that $M \neq \emptyset \neq N$, $L = M \cup N$, M has no largest element, N has no smallest element, and $\forall x \in M \forall y \in N (x < y)$. The definitions of left and right characters of a gap are similar to the above definitions for points; but they are always infinite regular cardinals. Again, the *character* of (M, N) is the pair (κ, λ^*) where κ is the left character and λ is the right character. The *gap-character set* of L is the collection of all characters of gaps of L; we denote it by $\text{Gchar}(L)$. We say that L is *Dedekind complete* iff every nonempty subset of L which is bounded above has a least upper bound. For L dense this is equivalent to saying that $\text{Gchar}(L) = \emptyset$.

The *full character set* of L is the pair $(\text{Pchar}(L), \text{Gchar}(L))$.

If L does not have a first element, then the *coinitiality* of L is the least cardinal κ such that there is a strictly decreasing sequence $\langle a_\alpha : \alpha < \kappa \rangle$ of elements of L such that $\forall x \in L \exists \alpha < \kappa [a_\alpha < x]$; we denote this cardinal by $\text{ci}(L)$. Similarly for the right end, if L does not have a greatest element then we define the *cofinality* of L, denoted by $\text{cf}(L)$.

L is *irreducible* iff it has no first or last elements, and the full character set of (x, y) is the same as the full character set of L for any two elements $x, y \in L$ with $x < y$.

Now a *complete character system* is a set $R \subseteq \text{Reg} \times \text{Reg}$ with the following properties:

(C1) $\text{dmn}(R) = [\omega, \chi_{\text{left}}(R))_{\text{reg}}$.

(C2) $\text{rng}(R) = [\omega, \chi_{\text{right}}(R))_{\text{reg}}$.

(C3) There is a κ such that $(\kappa, \kappa) \in R$.

Note these conditions do not mention orderings.

Proposition 17.17. *If L is an irreducible infinite Dedekind complete dense linear order, then* $\text{Pchar}(L)$ *is a complete character system. Moreover,* $\text{ci}(L) \leq \chi_{\text{right}}(\text{Pchar}(L))$ *and* $\text{cf}(L) \leq \chi_{\text{left}}(\text{Pchar}(L))$.

Proof. Let $R = \text{Pchar}(L)$. (C1): the inclusion $\subseteq$ is obvious. Now suppose that $\kappa \in [\omega, \chi_{\text{left}}(R))_{\text{reg}}$. Then there is a point x of L with character (μ, ν) such that $\kappa \leq \mu$.

Let $\langle a_\xi : \xi < \mu \rangle$ be a strictly increasing sequence of elements of L with supremum x. Let $y = \sup_{\xi<\kappa} a_\xi$. Clearly y has left character κ, as desired.

(C2): symmetric to (C1).

(C3): By a straightforward transfinite construction one gets (for some ordinal α) a strictly increasing sequence $\langle x_\xi : \xi < \alpha \rangle$ and a strictly decreasing sequence $\langle y_\xi : \xi < \alpha \rangle$ such that $x_\xi < y_\eta$ for all $\xi, \eta < \alpha$, and such that there is exactly one point z with $x_\xi < z < y_\eta$ for all $\xi, \eta < \alpha$. Then α is a limit ordinal, and z has character $(\mathrm{cf}(\alpha), \mathrm{cf}(\alpha))$, as desired.

Finally, suppose that $\mathrm{cf}(L) > \chi_{\mathrm{left}}(R)$. Then by the argument for (C1), L has a point with left character $\chi_{\mathrm{left}}(R)$, contradiction. A similar argument works for ci. $\square$

We shall use the sum construction for linear orders. If $\langle L_i : i \in I \rangle$ is a system of linear orders, and I itself is an ordered set, then by $\sum_{i\in I} L_i$ we mean the set

$$\{(i, a) : i \in I,\ a \in L_i\}$$

ordered lexicographically.

The following lemma is probably well-known.

Lemma 17.18. *If $\langle L_i : i \in I \rangle$ is a system of complete linear orders, and I is a complete linear order, then $\sum_{i\in I} L_i$ is also complete.*

Proof. Suppose that C is a nonempty subset of $\sum_{i\in I} L_i$. Let $i_0 = \sup\{i \in I : (i, a) \in C$ for some $a \in L_i\}$. We consider two cases.

Case 1. There is an $a \in L_{i_0}$ such that $(i_0, a) \in C$. Then we let $a_0 = \sup\{a \in L_{i_0} : (i_0, a) \in C\}$. Clearly (i_0, a_0) is the supremum of C.

Case 2. There is no $a \in L_{i_0}$ such that $(i_0, a) \in C$. Then the supremum of C is (i_0, a), where a is the first element of L_{i_0}. $\square$

Another construction we shall use is the infinite product. Suppose that I is a well-ordered set and $\langle L_i : i \in I \rangle$ is a system of linear orders. Then we make $\prod_{i\in I} L_i$ into a linear order by defining, for $f, g \in \prod_{i\in I} L_i$,

$$f < g \quad \text{iff} \quad f \neq g \text{ and } f(i) < g(i),$$

where $i = \mathrm{f.d.}(f, g)$, and $\mathrm{f.d.}(f, g)$ is the first $i \in I$ such that $f(i) \neq g(i)$.

Given such an infinite product, and given a strictly increasing sequence $x = \langle x_\alpha : \alpha < \lambda \rangle$ of members of it, with λ a limit ordinal, we call x of *argument type* if the following two conditions hold:

(A1) $\langle \mathrm{f.d.}(x_\alpha, x_{\alpha+1}) : \alpha < \lambda \rangle$ is strictly increasing

(A2) For each $\alpha < \lambda$, the sequence $\langle \mathrm{f.d.}(x_\alpha, x_\beta) : \alpha < \beta < \kappa \rangle$ is a constant sequence.

On the other hand, x is *of basis type* iff there is an $i \in I$ such that $\mathrm{f.d.}(x_\alpha, x_\beta) = i$ for all distinct $\alpha, \beta < \kappa$.

Lemma 17.19. *Let $\langle M_i : i \in I \rangle$ be a system of ordered sets, with I well-ordered. If $x < y < z$ in $\prod_{i\in I} M_i$, then $f.d.(x, z) = \min\{f.d.(x, y), f.d.(y, z)\}$.*

Proof. Let $i = \min\{\text{f.d.}(x,y), \text{f.d.}(y,z)\}$.

Case 1. $i = \text{f.d.}(x,y) = \text{f.d.}(y,z)$. Then $x \upharpoonright i = y \upharpoonright i = z \upharpoonright i$ and $x(i) < y(i) < z(i)$, so $\text{f.d.}(x,z) = i$.

Case 2. $i = \text{f.d.}(x,y) < \text{f.d.}(y,z)$. Then $x \upharpoonright i = y \upharpoonright i = z \upharpoonright i$ and $x(i) < y(i) = z(i)$, so $\text{f.d.}(x,z) = i$.

Case 3. $i = \text{f.d.}(y,z) < \text{f.d.}(x,y)$. Then $x \upharpoonright i = y \upharpoonright i = z \upharpoonright i$ and $x(i) = y(i) < z(i)$, so $\text{f.d.}(x,z) = i$. $\square$

The following is Satz XIV in Hausdorff [1908].

Theorem 17.20. *Let $\langle M_i : i \in I\rangle$ be a system of ordered sets, with I well-ordered. Suppose that κ is regular and $\langle x_\alpha : \alpha < \kappa\rangle$ is a strictly increasing sequence of elements of $\prod_{i\in I} M_i$. Then this sequence has a subsequence of length κ which is either of argument type or of basis type.*

Proof. First we claim

(1) For every $\alpha < \kappa$ there is a $\beta > \alpha$ and an $i \in I$ such that $\text{f.d.}(x_\alpha, x_\gamma) = i$ for all $\gamma \geq \beta$.

This is true because, by Lemma 17.19, if $\alpha < \beta < \gamma < \kappa$, then $\text{f.d.}(x_\alpha, x_\beta) \geq \text{f.d.}(x_\alpha, x_\gamma)$; hence

$$\text{f.d.}(x_\alpha, x_{\alpha+1}) \geq \text{f.d.}(x_\alpha, x_{\alpha+2}) \geq \cdots \geq \text{f.d.}(x_\alpha, x_{\alpha+\beta}) \geq \cdots$$

for all $\beta < \kappa$; so this sequence of elements of I has a minimum, and (1) holds.

Now for each $\alpha < \kappa$, let $\varphi(\alpha)$ be the least $\beta > \alpha$ so that an i as in (1) exists, and let $i(\alpha)$ be such an i. Thus

(2) For each $\alpha < \kappa$ we have $\alpha < \varphi(\alpha)$, and for all $\gamma \geq \varphi(\alpha)$ we have $\text{f.d.}(x_\alpha, x_\gamma) = i(\alpha)$.

Now we define a function $\alpha \in {}^\kappa\kappa$ by setting

$$\begin{aligned} \alpha(0) &= 0; \\ \alpha(\xi+1) &= \varphi(\alpha(\xi)); \\ \alpha(\eta) &= \sup_{\xi<\eta} \alpha(\xi) \quad \text{for } \eta \text{ limit} \end{aligned}$$

Then we clearly have

(3) α is a strictly increasing function, and $\text{f.d.}(x_{\alpha(\xi)}, x_{\alpha(\eta)}) = i(\alpha(\xi))$ for all $\xi, \eta < \kappa$ with $\xi < \eta$.

Moreover,

(4) If $\xi < \eta < \theta < \kappa$, then $i(\alpha(\xi)) = \text{f.d.}(x_{\alpha(\xi)}, x_{\alpha(\theta)}) \leq \text{f.d.}(x_{\alpha(\eta)}, x_{\alpha(\theta)}) = i(\alpha(\eta))$.

In fact, this is clear by Lemma 17.19.

Now we consider two cases.

Case 1. $\forall\xi<\kappa\exists\eta<\kappa[\xi<\eta$ and $i(\alpha(\xi))<i(\alpha(\eta))]$. Then there is a strictly increasing $\beta\in{}^\kappa\kappa$ such that for all $\xi,\eta\in\kappa$, if $\xi<\eta$ then $i(\alpha(\beta(\xi)))<i(\alpha(\beta(\eta)))$. Hence for $\xi<\eta<\kappa$ we have

$$\text{f.d.}(x_{\alpha(\beta(\xi))},x_{\alpha(\beta(\xi+1))})=i(\alpha(\beta(\xi)))<i(\alpha(\beta(\eta)))=\text{f.d.}(x_{\alpha(\beta(\eta))},x_{\alpha(\beta(\eta+1))});$$

moreover, if $\xi<\eta<\kappa$, then $\text{f.d.}(x_{\alpha(\beta(\xi))},x_{\alpha(\beta(\eta))})=i(\alpha(\beta(\xi)))$; so $\langle x_{\alpha(\beta(\xi))}:\xi<\kappa\rangle$ is of argument type.

Case 2. $\exists\xi<\kappa\forall\eta<\kappa[\xi<\eta$ implies that $i(\alpha(\xi))=i(\alpha(\eta))]$. Hence $\langle x_{\alpha(\xi+\eta)}:\eta<\kappa\rangle$ is of basis type. $\square$

A variant of the product construction will be useful. Let κ be an infinite regular cardinal. A *κ-system* is a pair (T,M) with the following properties:

(V1) For each $\alpha<\kappa$ and each $x\in T_\alpha$, $M_{x\alpha}$ is a linear order.

(V2) $T_0=\{\emptyset\}$.

(V3) For each $\alpha<\kappa$ we have

$$T_{\alpha+1}=\{f:\text{dmn}(f)=\alpha+1,\ (f\restriction\alpha)\in T_\alpha,\ f(\alpha)\in M_{(f\restriction\alpha)\alpha}\}$$

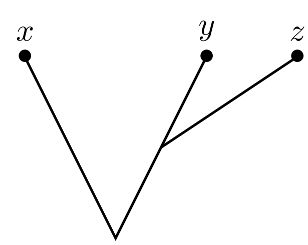
(V4) If $\beta\le\kappa$ is a limit ordinal $\le\kappa$, then $T_\beta=\{f:\text{dmn}(f)=\beta$ and $\forall\alpha<\beta[f\restriction\alpha\in T_\alpha]\}$.

We define a relation $<$ on T_κ by setting, for any $x,y\in T_\kappa$, $x<y$ iff $x\ne y$, and $x(\xi)<y(\xi)$, where $\xi=\text{f.d.}(x,y)$. Here the second $<$ relation is that of $M_{(x\restriction\xi)\xi}$.

Lemma 17.21. *Under the above assumptions, $<$ is a linear order on T_κ.*

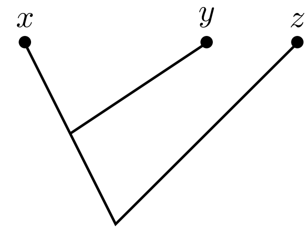
Proof. Clearly $<$ is irreflexive, and $\forall x,y\in T_\kappa[x<y$ or $x=y$ or $y<x]$. For transitivity, suppose that $x<y<z$. Let $\xi=$f.d.(x,y), $\eta=$f.d.(y,z), and $\theta=$f.d.(x,z).

Case 1. $\xi<\eta$. Then $\xi=\theta$ and $x(\theta)=x(\xi)<y(\xi)=z(\xi)$, so $x<z$.



Case 2. $\xi=\eta$. Then $\xi=\theta$ and $x(\theta)=x(\xi)<y(\xi)=y(\eta)<z(\eta)=z(\theta)$.

Case 3. $\eta<\xi$. Then $\theta=\eta$ and $x(\theta)=y(\theta)=y(\eta)<z(\eta)=z(\theta)$. $\square$



□

The idea is that this is a *variable product*: not all functions in a cartesian product are allowed. If $x \in T_\kappa$, then for each $\alpha < \kappa$ the value $x(\alpha)$ lies in an ordered set $M_{(x\restriction\alpha)\alpha}$ which depends on $x \restriction \alpha$. Thus the linear order has a tree-like property.

Theorem 17.22. *Assume the above notation. For each $\gamma < \kappa$ let $M'_\gamma = \{(x, y) : x \in T_\gamma, y \in M_{x\gamma}\}$. Let M'_γ have the lexicographic ordering. Then there is an isomorphism of T_κ into $\prod_{\gamma<\kappa} M'_\gamma$.*

Namely, for each $x \in T_\kappa$ define $f(x)$ by $(f(x))_\gamma = (x \restriction \gamma, x(\gamma))$ for any $\gamma < \kappa$. Then f is the indicated isomorphism. Moreover, for all $x, y \in T_\kappa$ we have f.d.$(x, y) =$ f.d.$(f(x), f(y))$.

Proof. Clearly f maps T_κ into $\prod_{\gamma<\kappa} M'_\gamma$. Suppose that $x, y \in T_\kappa$ and $x < y$. Choose α minimum such that $x(\alpha) \neq y(\alpha)$; so $x(\alpha) < y(\alpha)$. Hence $(x \restriction \alpha, x(\alpha)) < (x \restriction \alpha, y(\alpha)) = (y \restriction \alpha, y(\alpha))$. If $\beta < \alpha$, then $(x \restriction \beta, x(\beta)) = (y \restriction \beta, y(\beta))$. Hence $f(x) < f(y)$ and f.d.$(x, y) =$ f.d.$(f(x), f(y))$. On the other hand, suppose that $f(x) < f(y)$. Let $\alpha =$ f.d.$(f(x), f(y))$. If $\beta < \alpha$, then $(f(x))_\beta = (f(y))_\beta$, i.e., $(x \restriction \beta, x(\beta)) = (y \restriction \beta, y(\beta))$. Hence $x \restriction \alpha = y \restriction \alpha$. Since $(f(x))_\alpha < (f(y))_\alpha$, we have $(x \restriction \alpha, x(\alpha)) < (y \restriction \alpha, y(\alpha))$. Hence $x(\alpha) < y(\alpha)$. It follows that $x < y$. □

Theorem 17.23. *If (T, M) is a κ-system on a regular cardinal κ and each linear order $M_{x\alpha}$ is complete, then T_κ is complete.*

Proof. It suffices to take any regular cardinal ν, suppose that $x = \langle x_\theta : \theta < \nu\rangle$ is a strictly increasing sequence in T_κ, and show that it has a supremum. By Theorems 17.20 and 17.22 let $\langle f(x_{\theta(\xi)}) : \xi < \nu\rangle$ be a subsequence of $\langle f(x_\theta) : \theta < \nu\rangle$ which is of basis type or argument type.

Case 1. $\langle f(x_{\theta(\xi)}) : \xi < \nu\rangle$ is of basis type. Say $\gamma < \kappa$ and f.d.$(f(x_{\theta(\xi)}), f(x_{\theta(\eta)})) = \gamma$ for all distinct $\xi, \eta < \nu$. Thus by Theorem 17.22, f.d.$(x_{\theta(\xi)}, x_{\theta(\eta)}) = \gamma$ for all distinct $\xi, \eta < \nu$. Let $a = \sup\{x_{\theta(\xi)}(\gamma) : \xi < \nu\}$. Now define $y \in T_\kappa$ by setting

$$\begin{aligned}
y \restriction \gamma &= x_{\theta(0)} \restriction \gamma;\\
y(\gamma) &= (y \restriction \gamma) \cup \{(\gamma, a)\};\\
y(\delta + 1) &= (y \restriction \delta) \cup \{(\delta, b)\} \quad \text{with } b \text{ the least element of } M_{(y\restriction\delta)\delta)} \text{ for } \gamma \leq \delta;\\
y(\psi) &= \bigcup_{\delta<\psi} (y \restriction \delta) \quad \text{for } \psi \text{ limit } > \gamma.
\end{aligned}$$

Clearly y is an upper bound for $\langle x_\theta : \theta < \nu\rangle$. Now suppose that $z \in T_\kappa$ is any upper bound. If $\xi < \nu$, $\varphi < \gamma$, and $x_{\theta(\xi)}(\varphi) \neq z(\varphi)$, let $\rho =$ f.d.$(x_{\theta(\xi)}, z)$. Then $\rho < \gamma$ and

$x_{\theta(\xi)}(\rho) < z(\rho)$. Clearly then f.d.$(x_{\theta(\eta)}, z) = \rho$ for all $\eta < \nu$, and $y < z$. Thus we may assume that $x_{\theta(\xi)}(\varphi) = z(\varphi)$ for all $\xi < \nu$ and $\varphi < \gamma$. It follows that $a \le z(\gamma)$. If $a < z(\gamma)$, clearly $y < z$. Suppose that $a = z(\gamma)$. Then again clearly $y \le z$. So y is the least upper bound for $\langle x_\theta : \theta < \nu\rangle$.

Case 2. $\langle f(x_{\theta(\xi)}) : \xi < \nu\rangle$ is of argument type. By (A1), $\langle$f.d.$(x_{\theta(\xi)}, x_{\theta(\xi+1)} : \xi < \nu\rangle$ is strictly increasing. Thus $\nu \le \kappa$. Let $\beta = \sup\{$f.d.$(x_{\theta(\xi)}, x_{\theta(\xi+1)}) : \xi < \nu\}$. Thus $\beta \le \kappa$. Let $y_\xi = x_{\theta(\xi)} \restriction$ f.d.$(x_{\theta(\xi)}, x_{\theta(\xi+1)})$ for each $\xi < \nu$. Hence $y_\xi \in T_{\text{f.d.}(x_{\theta(\xi)}, x_{\theta(\xi+1)})}$. Now

(1) $y_\xi \subseteq y_\eta$ if $\xi < \eta < \nu$.

In fact, suppose that this is not true; say that $\xi < \eta < \nu$ but $y_\xi \not\subseteq y_\eta$. So there is an $\alpha <$ f.d.$(x_{\theta(\xi)}, x_{\theta(\xi+1)})$ such that $y_\xi(\alpha) \neq y_\eta(\alpha)$. Thus $x_{\theta(\xi)}(\alpha) \neq x_{\theta(\eta)}(\alpha)$; so f.d.$(x_{\theta(\xi)} x_{\theta(\eta)}) \le \alpha <$ f.d.$(x_{\theta(\xi)}, x_{\theta(\xi+1)})$, contradicting (A2).

From (1), clearly

(2) $y_\xi \subset y_\eta$ if $\xi < \eta < \nu$.

Now consider the function $z \stackrel{\text{def}}{=} \bigcup_{\xi<\nu} y_\xi$. We consider two cases.

Case 1. $\nu = \kappa$. Then $z \in T_\kappa$ and $\beta = \kappa$. We claim that z is the supremum of x in this case. If $\xi < \nu$, let $\alpha =$ f.d.$(x_{\theta(\xi)}, x_{\theta(\xi+1)})$. Then $z \restriction \alpha = y_\xi = x_{\theta(\xi)} \restriction \alpha$. Now $\alpha =$ f.d.$(x_{\theta(\xi)}, x_{\theta(\xi+1)}) <$ f.d.$(x_{\theta(\xi+1)}, x_{\theta(\xi+2)})$ by (A1) in the definition of argument type. So $x_{(\theta(\xi)}(\alpha) < x_{\theta(\xi+1)}(\alpha) = y_{\xi+1}(\alpha) \le z(\alpha)$. Thus $x_{\theta(\xi)} < z$. Now suppose that $w < z$. Let $\xi =$ f.d.(w, z). Since $\beta = \kappa$, choose $\eta < \nu$ such that $\xi <$ f.d.$(x_{\theta(\eta)}, x_{\theta(\eta+1)})$. Then $w \restriction \xi = z \restriction \xi = y_\eta \restriction \xi$, and $w(\xi) < z(\xi) = y_\eta(\xi)$. So $w < y_\eta$, as desired.

Case 2. $\nu < \kappa$. Then also $\beta < \kappa$. Also, $z \in T_\beta$. We define an extension $v \in T_\kappa$ of z by recursion. Let $w_0 = z$. If w_α has been defined as a member of $T_{\beta+\alpha}$, with $\beta + \alpha < \kappa$, let $a(\alpha)$ be the least member of $M_{w_\alpha \alpha}$, and set $w_{\alpha+1} = w_\alpha \cup \{(\alpha, a(\alpha))\}$. So $w_{\alpha+1} \in T_{\beta+\alpha+1}$. If γ is limit and w_α has been defined as a member of $T_{\beta+\alpha}$ for all $\alpha < \gamma$, and if $\beta + \gamma < \kappa$, let $w_\gamma = \bigcup_{\alpha<\gamma} w_\alpha$. Finally, let $v = \bigcup_{\alpha<\kappa} w_\alpha$. So $v \in T_\kappa$ and it is an extension of z. We claim that it is the l.u.b. of $\langle x_\alpha : \alpha < \nu\rangle$. First suppose that $\xi < \nu$. Then $x_{\theta(\xi)} \restriction$ f.d.$(x_{\theta(\xi)}, x_{\theta(\xi+1)}) = y_\xi = z \restriction$ f.d.$(x_{\theta(\xi)}, x_{\theta(\xi+1)})$, and

$$\begin{aligned} x_{\theta(\xi)}(\text{f.d.}(x_{\theta(\xi)}, x_{\theta(\xi+1)})) &< x_{\theta(\xi+1)}(\text{f.d.}(x_{\theta(\xi)}, x_{\theta(\xi+1)})) \\ &= y_{\xi+1}(\text{f.d.}(x_{\theta(\xi)}, x_{\theta(\xi+1)})) \\ &= z(\text{f.d.}(x_{\theta(\xi)}, x_{\theta(\xi+1)})). \end{aligned}$$

Thus $x_{\theta(\xi)} < v$. Now suppose that $t < v$. Then $\alpha \stackrel{\text{def}}{=}$ f.d.(t, v) is less than β by construction, so $t \restriction \alpha = z \restriction \alpha$ and $t(\alpha) < z(\alpha)$. By the definition of z this gives a $\xi < \nu$ such that $t \restriction \alpha = x_{\theta(\xi)} \restriction \alpha$ and $t(\alpha) < x_{\theta(\xi)}(\alpha)$. So $t < x_{\theta(\xi)}$, as desired. □

Our main theorem is as follows; it is Satz XVII of Hausdorff [1908].

Theorem 17.24. *Suppose that R is a complete character system, and κ, λ are regular cardinals with $\kappa \le \chi_{\text{right}}(R)$ and $\lambda \le \chi_{\text{left}}(R)$, and $\chi_{\text{left}}(R)$ and $\chi_{\text{right}}(R)$ are successor cardinals. Then there is an irreducible Dedekind complete dense order L such that* Pchar$(L) = R$, *with* ci$(L) = \lambda$ *and* cf$(L) = \kappa$.

Proof. We may assume that $\chi_{\mathrm{left}}(R) \geq \chi_{\mathrm{right}}(R)$; otherwise we replace R by $R^{-1} \stackrel{\mathrm{def}}{=} \{(\kappa,\lambda) : (\lambda,\kappa) \in R\}$, and replace the final order by its reverse. Let R be ordered lexicographically. Note by hypothesis that R has a largest element. We define some important orders which are components of the final order L. Let α and β be regular cardinals. Now we define

$$\varphi_{\alpha\beta} = \alpha + 1 + \beta^*;$$
$$\Phi = \sum_{(\alpha,\beta)\in R} \varphi_{\alpha\beta};$$
$$\mu(\alpha,\beta) = 1 + \alpha^* + \Phi + \beta + 1.$$

The symmetry of this definition will enable us to shorten several proofs below. Since we are using the standard notation for sums of order types, and some order types are repeated, it is good to have an exact notation for the indicated orders. m, f, l are new elements standing for "middle", "first", and "last" respectively. We suppose that with each ordinal ξ we associate a new element ξ', used in forming things like β^*. Thus more precisely,

$$\varphi_{\alpha\beta} = \alpha \cup \{m\} \cup \{\xi' : \xi < \beta\};$$

the ordering here is: α has its natural order; for $\xi, \eta < \beta$, we define $\xi' < \eta'$ iff $\eta < \xi$; $\xi < m$ for each $\xi < \alpha$; $m < \xi'$ for each $\xi < \beta$; and transitivity gives the rest.

$$\Phi = \{((\alpha,\beta),a) : (\alpha,\beta) \in R,\ a \in \varphi_{\alpha\beta}\} \quad \text{with lexicographic order;}$$

$$\mu(\alpha,\beta) = \{f\} \cup \{\xi' : \xi < \alpha\} \cup \Phi \cup \beta \cup \{l\};$$

the ordering should be obvious on the basis of the above remarks. We implicitly assume the distinctness of the various objects making up $\mu(\alpha,\beta)$.

(1) $\mu(\alpha,\beta)$ is a complete linear order, for any regular cardinals α,β.

This is clear on the basis of Lemma 17.18.

(2) If $\alpha \leq \chi_{\mathrm{right}}(R)$ is regular and $\beta \leq \chi_{\mathrm{left}}(R)$ is regular, then:

(a) the right character of the left end point of $\mu(\alpha,\beta)$ is α;

(b) the left character of the right end point of $\mu(\alpha,\beta)$ is β;

(c) if $a \in \mu(\alpha,\beta)$ is not an end point, and its character is (γ,δ^*), then $\gamma < \chi_{\mathrm{left}}(R)$ and $\delta < \chi_{\mathrm{right}}(R)$.

In fact, (a) and (b) are clear. Now suppose that $a \in \mu(\alpha,\beta)$ is not an end point. If a is in the α^* portion but not equal to $0'$, or it is in the β portion but is not the first element of β, the conclusion of (c) is clear. The character of $0'$ is $(1,1)$. The character of the first element of β is $(1,1)$ since R has a largest element. If a is within some $\varphi_{\alpha\beta}$ but is not the first or last element of $\varphi_{\alpha\beta}$, clearly the conclusion of (c) holds.

Now suppose that a is the first element of $\varphi_{\alpha\beta}$. If (α,β) is the first element of R, then the character of a is (1,1). If (α,β) has an immediate predecessor in R, then the character of a is (1,1). If (α,β) is not the first element of R but does not have an immediate predecessor, then the character of a is $(\gamma, 1)$, where γ is the cofinality of the set of predecessors of (α, β) in R. Clearly $\gamma < \chi_{\text{left}}(R)$.

Next suppose that a is the last element of $\varphi_{\alpha\beta}$. If (α, β) is the last element of R, then the character of a is $(1, 1)$. If (α, β) is not the last element of R then, since R is well-ordered, it has an immediate successor, and again the character of a is $(1, 1)$.

Let p be a new element, not appearing in any of the above orders. Let σ be the least regular cardinal such that $(\sigma, \sigma) \in R$; it exists by condition (C3) in the definition of a complete character system. For each regular $\alpha < \chi_{\text{left}}(R)$, let ξ_α be the least cardinal such that $(\alpha, \xi_\alpha) \in R$; it exists by (C1) in the definition of complete character set. Similarly, for each regular $\alpha \in \chi_{\text{right}}(R)$ let η_α be the least cardinal such that $(\eta_\alpha, \alpha) \in R$.

Now we define by recursion a σ-system (T, M). Let $T_0 = \{\emptyset\}$ and $M_{\emptyset 0} = \mu(\lambda, \kappa)$. Now except for $M_{\emptyset 0}$, $M_{x\alpha}$ will have the form $\{p\}$ or $\mu(\rho, \sigma)$ with $\rho < \chi_{\text{left}}(R)$ and $\sigma < \chi_{\text{right}}(R)$.

Suppose that $\gamma \leq \sigma$ is a limit ordinal. We let T_γ be the set of all x with domain γ such that $x \restriction \alpha \in T_\alpha$ for all $\alpha < \gamma$. Now suppose that $\gamma < \sigma$, still with γ a limit ordinal. Now if $x \in T_\gamma$ and $|M_{(x\restriction\alpha)\alpha}| > 1$ for all $\alpha < \gamma$, we set

$$M_{x\gamma} = \mu(\xi_{\text{cf}(\gamma)}, \eta_{\text{cf}(\gamma)}).$$

On the other hand, if $|M_{(x\restriction\alpha)\alpha}| = 1$ for some $\alpha < \gamma$, we set $M_{x\gamma} = \{p\}$.

Now suppose that $\gamma = \beta + 1$. Then we set

$$T_\gamma = \{x^\frown\langle b\rangle : x \in T_\beta \text{ and } b \in M_{x\beta}\}.$$

Now we define $M_{x\gamma}$ for each $x \in T_\gamma$.

(3) If $x(\beta) = p$, then $M_{x\gamma} = \{p\}$.

(4) If $x(\beta)$ is an endpoint of $M_{(x\restriction\beta)\beta}$ or has no immediate neighbors, then $M_{x\gamma} = \{p\}$.

(5) If $x(\beta)$ has a right neighbor but no left neighbor, and the left character of $x(\beta)$ in $M_{(x\restriction\beta)\beta}$ is α, then $M_{x\gamma} = \mu(\xi_\alpha, \sigma)$. Note that $\alpha < \chi_{\text{left}}(R)$.

(6) If $x(\beta)$ has a left neighbor but no right neighbor, and the right character of $x(\beta)$ in $M_{(x\restriction\beta)\beta}$ is α, then $M_{x\gamma} = \mu(\sigma, \eta_\alpha)$. Note that $\alpha < \chi_{\text{right}}(R)$.

(7) If $x(\beta)$ has both a left and a right neighbor, then $M_{x\gamma} = \mu(\sigma, \sigma)$.

This finishes the definition of (T, M). The linear order T_σ is close to the order we are after.

The following two facts are clear from the construction:

(8) If $x \in T_\sigma$, $\alpha < \beta < \sigma$, and $x(\alpha) = p$, then $x(\beta) = p$.

(9) If $x \in T_\sigma$ and $\alpha < \sigma$, then either $M_{(x\restriction\alpha)\alpha} = \{p\}$ or $M_{(x\restriction\alpha)\alpha} = \mu(\theta, \varphi)$ for some θ, φ; except for $M_{\emptyset 0}$ we have $\theta < \chi_{\text{left}}(R)$ and $\varphi < \chi_{\text{right}}(R)$.

From Theorem 17.23 we know that T_σ is complete. Now we find the characters of the elements of T_σ.

(10) The smallest element of T_σ has character $(0, \lambda^*)$.

To prove this, note that the smallest element of T_σ is $x \stackrel{\text{def}}{=} \langle f, p, p, \ldots\rangle$, where f is the first element of $M_\emptyset = \mu(\lambda, \kappa)$. For each $\alpha < \lambda$ let $y_\alpha = \langle(\alpha+1)', f, p, p, \ldots\rangle$. Clearly $x < y_\alpha$ for each $\alpha < \lambda$, and $y_\beta < y_\alpha$ if $\alpha < \beta < \lambda$. Now suppose that $x < w$. Hence $w(0) \neq f$. If $w(0)$ is not in the λ^* part, clearly $y_\alpha < w$ for every $\alpha < \lambda$. If $w(0) = \beta'$ for some $\beta < \lambda$, then $y_{\beta+1} < w$. This proves (10).

(11) The largest element of T_σ has character $(\kappa, 0)$.

In fact, the largest element of T_σ is $x \stackrel{\text{def}}{=} \langle l, p, p, \ldots\rangle$. For each $\alpha < \kappa$ let $y_\alpha = \langle \alpha, l, p, p, \ldots\rangle$. Clearly $y_\alpha < x$ for each $\alpha < \kappa$. Now suppose that $w < x$. Clearly then $w(0) < l$. If $w(0)$ is not in the κ part, clearly $w < y_\alpha$ for all $\alpha < \kappa$. If $w(0) = \alpha < \kappa$, then $w < y_{\alpha+1}$. This proves (11).

Let a be the first element of T_σ, and b the last element.

(12) If $a < x < b$ and $|M_{x\restriction\alpha}| > 1$ for every $\alpha < \sigma$, then x has character (σ, σ).

For, by symmetry it suffices to show that x has left character σ. For each $\alpha < \sigma$ let

$$y_\alpha = (x \restriction \alpha)^\frown\langle f, p, p, \ldots\rangle.$$

Clearly $y_\alpha < x$ for all $\alpha < \sigma$, and $\langle y_\alpha : \alpha < \sigma\rangle$ is strictly increasing. Now suppose that $z \in T_\sigma$ and $z < x$. Let $\alpha = \text{f.d.}(z, x)$. Then clearly $z < y_{\alpha+1}$, as desired. So (12) holds.

Now suppose that $a < x < b$ and $x(\alpha) = p$ for some $\alpha < \sigma$, and let α be minimum with this property. Then by construction, α is a successor ordinal $\gamma + 1$, and $x(\gamma)$ is an endpoint of $M_{(x\restriction\gamma)\gamma}$ or is an element of $M_{(x\restriction\gamma)\gamma}$ with no neighbor.

Case 1. $x(\gamma)$ is an element of $M_{(x\restriction\gamma)\gamma}$ with no neighbor. Then by definition, there is a $(\rho, \xi) \in R$ such that $x(\gamma) = ((\rho, \xi), m)$, i.e., $x(\gamma)$ is the middle element of $\varphi_{\rho\xi}$. We claim that x has character (ρ, ξ^*). To see this, for each $\alpha < \rho$ let

$$y_\alpha = (x \restriction \gamma)^\frown\langle((\rho, \xi), \alpha+1), f, p, p, \ldots\rangle.$$

Then $y_\alpha < x$, the sequence $\langle y_\alpha : \alpha < \rho\rangle$ is strictly increasing, and x is its supremum. So the left character of x is ρ, and similarly the right character of x is ξ.

Case 2. $x(\gamma)$ is the endpoint f of $M_{(x\restriction\gamma)\gamma}$. We now consider three subcases.

Subcase 2.1. $\gamma = 0$. This would imply that $x = a$, contradiction.

Subcase 2.2. γ is a limit ordinal. Then $\text{cf}(\gamma) < \sigma$. So by construction, $M_{(x\restriction\gamma)\gamma}$ is $\mu(\xi_{\text{cf}(\gamma)}, \eta_{\text{cf}(\gamma)})$. For each $\delta < \gamma$ let $y_\delta = (x \restriction \delta)^\frown\langle f, p, p, \ldots\rangle$. Clearly $\langle y_\delta : \delta < \gamma\rangle$ is strictly increasing with limit x. Hence x has character $(\text{cf}(\gamma), \xi_{\text{cf}(\gamma)}) \in R$.

Subcase 2.3. $\gamma = \beta + 1$ for some β. Then

$$x = (x \restriction \beta)^\frown\langle x(\beta), f, p, p, \ldots\rangle.$$

Clearly then one of (5)–(7) holds for $x(\beta)$.

Subsubcase 2.3.1. (5) holds for $x(\beta)$. So $x(\beta)$ has a right neighbor, but no left neighbor. Say the left character of $x(\beta)$ is α. Then by (5), $M_{(x\restriction\gamma)\gamma}$ is $\mu(\xi_\alpha, \sigma)$. We claim that x has character (α, ξ_α). To see this, let $\langle\alpha_\theta : \theta < \alpha\rangle$ be strictly increasing with supremum $x(\beta)$. Then for each $\theta < \alpha$ let y_θ be any element of N such that $x \upharpoonright \beta = y_\theta \upharpoonright \beta$ and $y_\theta(\beta) = \alpha_\theta$. So clearly $y_\theta < x$ and $\langle y_\theta : \theta < \alpha\rangle$ is strictly increasing. Now suppose that $z \in N$ and $z < x$. If f.d.$(z, x) < \beta$, then $z < y_0$. Suppose that f.d.$(z, x) = \beta$. Then $z(\beta) < x(\beta)$, so $z(\beta) < \alpha_\theta$ for some $\theta < \alpha$, and hence $z < y_\theta$. Clearly by the form of x, one of these possibilities for z must hold. Hence the left character of x is α. Clearly its right character is ξ_α.

Subsubcase 2.3.2. (6) holds for $x(\beta)$. So $x(\beta)$ has a left neighbor ε, but no right neighbor. Hence ε has a right neighbor, and hence (5) or (7) holds for $(x \upharpoonright \beta)^\frown\langle\varepsilon\rangle$ in place of x and β in place of γ. Hence $M_{(x\upharpoonright\beta)^\frown\langle\varepsilon\rangle,\beta+1}$ is $\mu(\xi_\alpha, \sigma)$ for some α, or $\mu(\sigma, \sigma)$. Now

(13) $y \stackrel{\text{def}}{=} (x \upharpoonright \beta)^\frown\langle\varepsilon, l, p, p, \ldots\rangle$ is the immediate predecessor of x.

In fact, clearly $y < x$. Suppose that $z < x$. Clearly f.d.$(z, x) \le \beta$. If f.d.$(z, x) < \beta$, obviously $z < y$. If f.d.$(z, x) = \beta$, then $z(\beta) \le \varepsilon$. If $z(\beta) < \varepsilon$, then $z < y$. If $z(\beta) = \varepsilon$, then $z(\gamma) \le l$, and so $z \le y$. So (13) holds.

Clearly the left character of y is σ. Now $M_{x\upharpoonright\gamma,\gamma}$ is $\mu(\sigma, \eta_\alpha)$ for some α, and so it is clear that the right character of x is also σ.

Subsubcase 2.3.3. (7) holds for $x(\beta)$. So $x(\beta)$ has a left neighbor ε and a right neighbor ρ. Then y as above is the immediate predecessor of x, and it has left character σ. Since $M_{(x\upharpoonright\gamma)\gamma}$ is $\mu(\sigma, \sigma)$, it is clear that the right character of x is σ.

Case 3. $x(\gamma)$ is the endpoint l of $M_{(x\upharpoonright\gamma)\gamma}$. We now consider three subcases.

Subcase 3.1. $\gamma = 0$. This would imply that $x = b$, contradiction.

Subcase 3.2. γ is a limit ordinal. Then $\mathrm{cf}(\gamma) < \sigma$. So by construction, $M_{(x\upharpoonright\gamma)\gamma}$ is $\mu(\xi_{\mathrm{cf}(\gamma)}, \eta_{\mathrm{cf}(\gamma)})$. Clearly the left character of x is $\eta_{\mathrm{cf}(\gamma)}$. Now for each $\delta < \gamma$ let $y_\delta = (x \upharpoonright \delta)^\frown\langle l, p, p, \ldots\rangle$. Thus $x < y_\delta$ and $\langle y_\delta : \delta < \gamma$ is strictly decreasing. Suppose that $x < z$. Then there is a $\delta < \gamma$ such that $x \upharpoonright \delta = z \upharpoonright \delta$ and $x(\delta) < z(\delta)$. Then $y_{\delta+1} < z$. This shows that the right character of x is $\mathrm{cf}(\gamma)$. So the character of x is $(\eta_{\mathrm{cf}(\gamma)}, \mathrm{cf}(\gamma))$.

Subcase 3.3. $\gamma = \beta + 1$ for some β. Then

$$x = (x \upharpoonright \beta)^\frown\langle x(\beta), l, p, p, \ldots\rangle.$$

Clearly then one of (5)–(7) holds for $x(\beta)$.

Subsubcase 3.3.1. (5) holds for $x(\beta)$. So $x(\beta)$ has a right neighbor, but no left neighbor. $M_{(x\upharpoonright\gamma)\gamma}$ is $\mu(\xi_\alpha, \sigma)$ for some α, so the left character of x is σ. Let ε be the right neighbor of $x(\beta)$, and set $y = (x \upharpoonright \beta)^\frown\langle\varepsilon, f, p, p, \ldots\rangle$. Then y is the right neighbor of x. y has a left neighbor, so one of (6),(7) holds, and hence the right character of y is σ.

Subsubcase 3.3.2. (6) holds for $x(\beta)$. So $x(\beta)$ has a left neighbor ε, but no right neighbor. Let the right character of $x(\beta)$ be α, and let $\langle\delta_\xi : \xi < \alpha\rangle$ be strictly decreasing with limit $x(\beta)$. For each $\xi < \alpha$ let

$$y_\xi = (x \upharpoonright \beta)^\frown\langle\delta_\xi, l, p, p, \ldots\rangle.$$

It is clear that $\langle y_\xi : \xi < \alpha\rangle$ is strictly decreasing with limit x. So the right character of x is α. Now $x_{x\upharpoonright\gamma,\gamma}$ is $\mu(\sigma, \eta_\alpha)$, so the left character of x is η_α. Thus x has character (η_α, α).

Subsubcase 3.3.3. (7) holds for $x(\beta)$. So $x \upharpoonright \beta$ has a right neighbor ρ. Moreover, $M_{x\upharpoonright\gamma,\gamma}$ is $\mu(\sigma, \sigma)$, so the left character of x is σ. Now let

$$y = (x \upharpoonright \beta)^\frown\langle\rho, f, p, p, \ldots\rangle.$$

Then y is the right neighbor of x. Since ρ, $M_{(x\upharpoonright\beta)^\frown\langle\rho\rangle,\gamma}$ is $\mu(\sigma,\eta_\alpha)$ for some α, or it is $\mu(\sigma,\sigma)$. Hence the right character of y is σ.

Summarizing our investigation of characters of elements of T_σ, we have:

(14) If $a < x < b$, then one of the following holds:

(a) x has no neighbors, and its character is in R.

(b) x has an immediate predecessor y, and the characters of x, y are $(1,\sigma)$ and $(\sigma, 1)$ respectively.

(c) x has an immediate successor y, and the characters of x, y are $(\sigma, 1)$ and $(1,\sigma)$ respectively.

Now we show that if $x < y$ in L_σ and y is not the immediate successor of x, then for every $(\xi,\eta) \in R$ there is a $z \in (x, y)$ with character (ξ,η). Let α be minimum such that $x(\alpha) \neq y(\alpha)$. Then $x(\alpha) \neq p$, as otherwise $M_{x\upharpoonright\alpha,\alpha} = \{p\}$ and so also $y(\alpha) = p$. Now we consider two cases.

Case I. $y(\alpha)$ is not the immediate successor of $x(\alpha)$. Choose z with $x(\alpha) < z < y(\alpha)$. Say $M_{(x\upharpoonright\alpha)^\frown\langle z\rangle} = \mu(\tau,\rho)$. In the Φ portion of $\mu(\tau,\rho)$ take the middle element $((\xi,\eta), m)$ of $\varphi(\xi,\eta)$. Let

$$w = (x \upharpoonright \alpha)^\frown\langle z\rangle^\frown\langle((\xi,\eta), m)\rangle, p, p, \ldots\rangle.$$

Then $x < w < y$ and w has character (ξ,η).

Case II. $y(\alpha)$ is the immediate successor of $x(\alpha)$. Then

$$\begin{aligned} x &\leq (x \upharpoonright \alpha)^\frown\langle x(\alpha)\rangle^\frown\langle l, p, p, \ldots\rangle \\ &< (x \upharpoonright \alpha)^\frown\langle y(\alpha)\rangle^\frown\langle f, p, p, \ldots\rangle \\ &\leq y. \end{aligned}$$

Since y is not the immediate successor of x, one of the $\leq$s here is really $<$.

Case IIa. $x < (x \upharpoonright \alpha)^\frown\langle x(\alpha)\rangle^\frown\langle l, p, p, \ldots\rangle$. Then $\alpha + 1$ is the first argument where these two sequences differ. Let $M_{(x\upharpoonright(\alpha+1)),\alpha+1} = \mu(\tau,\rho)$. Then $x(\alpha + 1)$ is not the immediate predecessor of l, and so the argument in Case I gives an element w with character (ξ,η) such that

$$x < w < (x \upharpoonright \alpha)^\frown\langle x(\alpha)\rangle^\frown\langle l, p, p, \ldots\rangle \leq y.$$

Case IIb. $(x \upharpoonright \alpha)^\frown\langle y(\alpha)\rangle^\frown\langle f, p, p, \ldots\rangle^y$. This is similar to Case IIa.

Now let L be obtained from T_σ by deleting the second element of any pair (x, y) of elements of T_σ such that y is the immediate successor of x. Clearly L is as desired in the theorem. □

Theorem 17.25. *Suppose that R is a complete character system and κ and λ are regular cardinals with $\kappa \leq \chi_{\text{right}}(R)$ and $\lambda \leq \chi_{\text{left}}(R)$, and $\chi_{\text{right}}(R)$ and $\chi_{\text{left}}(R)$ are successor cardinals. Also suppose that $R = R_0 \cup R_1$ with $R_0 \neq \emptyset$. Then there is an irreducible dense linear order M such that* $\text{Pchar}(M) = R_0$, $\text{Gchar}(M) = R_1$, $\text{ci}(M) = \lambda$, *and* $\text{cf}(M) = \kappa$.

Proof. Let L be given by Theorem 17.23. For each $(\alpha,\beta)\in R$ let $M_{\alpha\beta}=\{x\in L:$ the character of x is (α,β). Note that $M_{\alpha\beta}$ is dense in L. For each $(\alpha,\beta)\in R_0\cap R_1$ write $M_{\alpha\beta}=P_{\alpha\beta}\cup Q_{\alpha\beta}$ with $P_{\alpha\beta}\cap Q_{\alpha\beta}=\emptyset$ and both dense in $M_{\alpha\beta}$. Now we define

$$N=\bigcup_{(\alpha,\beta)\in R_0\cap R_1}P_{\alpha\beta}\cup\bigcup_{(\alpha,\beta)\in R_0\backslash R_1}M_{\alpha\beta}.$$

We claim that N is as desired. Take any $x<y$ in N. If $z\in(x,y)\cap N$, then $z\in M_{\alpha\beta}$ for some $(\alpha,\beta)\in R_0$, and so z has character $(a\alpha,\beta)$. If $(\alpha,\beta)\in R_0\cap R_1$, take any $z\in P_{\alpha\beta}$ such that $x<z<y$. Then $z\in N$ and it has character (α,β). Similarly, there is an element of $(x,y)\cap N$ whose character is a given member of $R_0\backslash R_1$. Thus $\mathrm{Pchar}(N)=R_0$. The elements of L that are omitted in N are all the members of

$$\bigcup_{(\alpha,\beta)\in R_0\cap R_1}Q_{\alpha\beta}\cup\bigcup_{(\alpha,\beta)\in R_1\backslash R_0}M_{\alpha\beta}.$$

It follows that $\mathrm{Gchar}(N)=R_1$. □

EXERCISES

E17.1. Show that Theorem 17.2 does not extend to ω_1. Hint: consider $\omega_1\times\mathbb{Q}$ and $\omega_1^*\times\mathbb{Q}$, both with the lexicographic order, where ω_1^* is ω_1 under the reverse order ($\alpha<^*\beta$ iff $\beta<\alpha$). Given two linear orders L and M, the *lexicographic order* on $L\times M$ is defined by: $(a,b)<(c,d)$ iff $a<c$, or $a=c$ and $b<d$.

E17.2. For any infinite cardinal κ, consider ${}^{\kappa}2$ under the lexicographic order, as for H_α. Show that it is a complete linear order.

E17.3. Suppose that κ and λ are cardinals, with $\omega\le\lambda\le\kappa$. Let μ be minimum such that $\kappa<\lambda^\mu$. Take the lexicographic order on ${}^{\mu}\lambda$, as for H_α. Show that this gives a dense linear order of size λ^μ with a dense subset of size κ.

E17.4. Show that $\mathscr{P}(\omega)$ under $\subseteq$ contains a chain of size 2^ω. Hint: remember that $|\omega|=|\mathbb{Q}|$.

E17.5. A subset S of a linear order L is *weakly dense* iff for all $a,b\in L$, if $a<b$ then there is an $s\in S$ such that $a\le s\le b$. Show that the following conditions are equivalent for any cardinals κ,λ such that $\omega\le\kappa\le\lambda$:

(i) There is a linear order of size λ with a weakly dense subset of size κ.

(ii) $\mathscr{P}(\kappa)$ has a chain of size λ.

E17.6. Suppose that L_i is a linear order with at least two elements, for each $i\in\omega$. Let $\prod_{i\in\omega}L_i$ have the lexicographic order. Show that it is not a well-order.

E17.7. Suppose that L is a ccc dense linear order. Show that L has a dense subset of size $\le\omega_1$. Hint: let $\prec$ be a well-order of L, and let

$$N=\{p\in L:\text{ there is an open set }U\text{ in }L\text{ such that }p\text{ is the }\prec\text{-first element of }U\},$$

and show that N is dense in L and has size at most ω_1.

E17.8. Let $\langle L_i : i \in I\rangle$ be a system of linear orders, with I itself an ordered set. Show that if each L_i is dense without first or last elements, then also $\sum_{i\in I} L_i$ is dense without first or last elements. Here $\sum_{i\in I} L_i = \bigcup_{i\in I}(\{i\} \times L_i)$, and $(i,a) < (j,b)$ iff $i < j$, or $i = j$ and $a < b$.

E17.9. Let κ be any infinite cardinal number. Let L_0 be a linear order similar to $\omega^*+\omega+1$; specifically, let it consist of a copy of $\mathbb{Z}$ followed by one element a greater than every integer, and let L_1 be a linear order similar to $\omega^* + \omega + 2$; say it consists of a copy of $\mathbb{Z}$ followed by two elements $a < b$ greater than every integer. For any $f \in {}^{\kappa}2$ let

$$M_f = \sum_{\alpha<\kappa} L_{f(\alpha)}.$$

Show that if $f, g \in {}^{\kappa}2$ then M_f and M_g are not isomorphic.

Conclude that there are exactly 2^κ linear orders of size κ up to isomorphism.

E17.10. Let κ be an uncountable cardinal. Let L_0 be a linear order similar to $\eta+1+\eta\cdot\omega_1^*$; specifically consisting of a copy of the rational numbers in the interval $(0,1]$ followed by $\mathbb{Q}\times\omega_1$, where $\mathbb{Q}\times\omega_1$ is ordered as follows: $(r,\alpha) < (s,\beta)$ iff $\alpha > \beta$, or $\alpha = \beta$ and $r < s$. Let L_1 be a linear order similar to $\eta\cdot\omega_1 + 1 + \eta\cdot\omega_1^*$; specifically, we take L_1 to be the set

$$\{(q,\alpha,0) : q \in \mathbb{Q}, \alpha < \omega_1\} \cup \{(0,0,1)\} \cup \{(q,\alpha,2) : q \in \mathbb{Q}, \alpha < \omega_1\},$$

with the following ordering:

$$\begin{aligned}
&(q,\alpha,0) < (r,\beta,0) \quad \text{iff} \quad \alpha < \beta, \text{ or } \alpha = \beta \text{ and } q < r;\\
&(q,\alpha,0) < (0,0,1) < (r,\beta,2) \text{ for all relevant } q,r,\alpha,\beta;\\
&(q,\alpha,2) < (r,\beta,2) \quad \text{iff} \quad \alpha > \beta, \text{ or } \alpha = \beta \text{ and } q < r.
\end{aligned}$$

For each $f \in {}^{\kappa}2$ let

$$M_f = \sum_{\alpha<\kappa} L_{f(\alpha)}.$$

Show that each M_f is a dense linear order without first or last elements, and if $f, g \in {}^{\kappa}2$ and $f \neq g$, then M_f and M_g are not isomorphic.

Conclude that for κ uncountable there are exactly 2^κ dense linear orders without first or last elements, of size κ, up to isomorphism.

18. Trees

In this chapter we study infinite trees. The main things we look at are König's tree theorem, Aronszajn trees, and Suslin trees.

A *tree* is a partially ordered set $(T, <)$ such that for each $t \in T$, the set $\{s \in T : s < t\}$ is well-ordered by the relation $<$. Thus every ordinal is a tree, but that is not so interesting in the present context. We introduce some standard terminology concerning trees.

• For each $t \in T$, the order type of $\{s \in T : s < t\}$ is called the *height* of t, and is denoted by $\mathrm{ht}(t, T)$ or simply $\mathrm{ht}(t)$ if T is understood.

• A *root* of a tree T is an element of T of height 0, i.e., it is an element of T with no elements of T below it. Frequently we will assume that there is only one root.

• For each ordinal α, the α-th *level* of T, denoted by $\mathrm{Lev}_\alpha(T)$, is the set of all elements of T of height α.

• The *height* of T itself is the least ordinal greater than the height of each element of T; it is denoted by $\mathrm{ht}(T)$.

• A *chain* in T is a subset of T linearly ordered by $<$.

• A *branch* of T is a maximal chain of T.

Note that chains and branches of T are actually well-ordered, and so we may talk about their *lengths*.

Some further terminology concerning trees will be introduced later. A typical tree is ${}^{<\omega}2$, which is by definition the set of all finite sequences of 0s and 1s, with $\subset$ as the partial order. More generally, one can consider ${}^{<\alpha}2$ for any ordinal α.

Theorem 18.1. (König) *Every tree of height ω in which every level is finite has an infinite branch.*

Proof. Let T be a tree of height ω in which every level is finite. We define a sequence $\langle t_m : m \in \omega\rangle$ of elements of T by recursion. Clearly $T = \bigcup_{r \text{ a root}}\{s \in T : r \leq s\}$, and the index set is finite, so we can choose a root t_0 such that $\{s \in T : t_0 \leq s\}$ is infinite. Suppose now that we have defined an element t_m of height m such that $\{s \in T : t_m \leq s\}$ is infinite. Let $S = \{u \in T : t_m < u \text{ and } u \text{ has height } \mathrm{ht}(t_m) + 1\}$. Clearly

$$\{s \in T : t_m \leq s\} = \{t_m\} \cup \bigcup_{u \in S}\{s \in T : u \leq s\}$$

and the index set of the big union is finite, so we can choose t_{m+1} of height $\mathrm{ht}(t_m) + 1$ such that $\{s \in T : t_{m+1} \leq s\}$ is infinite.

This finishes the construction. Clearly $\{t_m : m \in \omega\}$ is an infinite branch of T. □

In attempting to generalize König's theorem, one is naturally led to Aronszajn trees and Suslin trees. For the following definitions, let κ be any infinite cardinal.

• A tree $(T, <)$ is a *κ-tree* iff it has height κ and every level has size less than κ.

• A κ-*Aronszajn tree* is a κ-tree which has no chain of size κ.

• A subset X of a tree T is an *antichain* iff any two distinct members of X are incomparable. Note that each set $\mathrm{Lev}_\alpha(T)$ is an antichain. This notion is different from antichains as introduced in Chapters 13 and 17.

• A κ-*Suslin tree* is a tree of height κ which has no chains or antichains of size κ.

• An *Aronszajn tree* is an ω_1-Aronszajn tree, and a *Suslin tree* is an ω_1-Suslin tree.

It is natural to guess that Aronszajn trees and Suslin trees are the same thing, since the definition of κ-tree implies that all levels have size less than κ, and a guess is that this implies that all antichains are of size less than κ. This guess is not right though. Even our simplest example of a tree, ${}^{<\omega}2$, forms a counterexample. This tree has all levels finite, but it has infinite antichains, for example

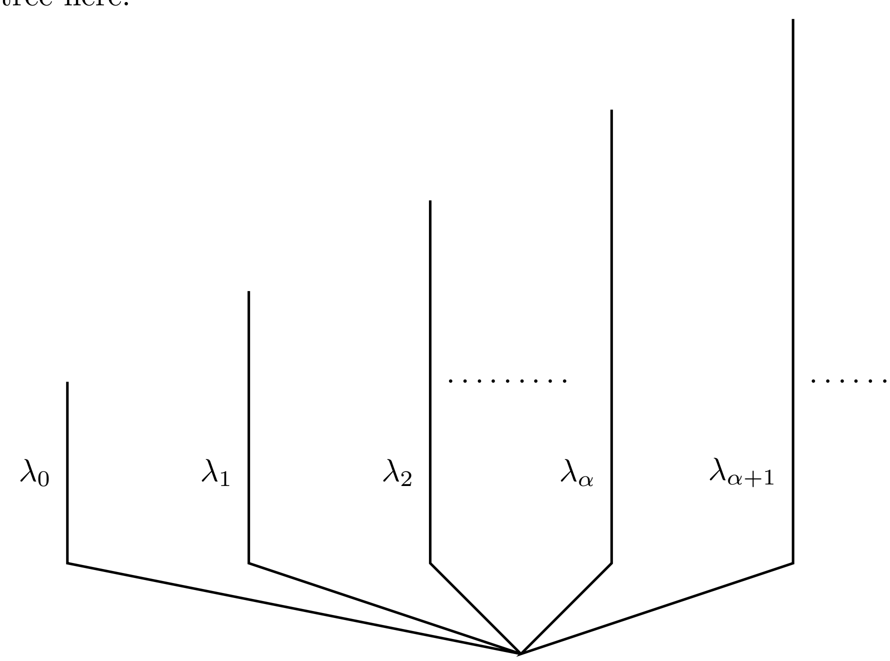
$$\{\langle 0\rangle, \langle 1,0\rangle, \langle 1,1,0\rangle, \langle 1,1,1,0\rangle, \ldots\}.$$

In the rest of this chapter we investigate these notions, and state some consistency results, some of which will be proved later. There is also one difficult natural open problem which we will formulate.

First we consider Aronszajn trees. Note that Theorem 18.1 can be rephrased as saying that there does not exist an ω-Aronszajn tree. As far as existence of Aronszajn trees is concerned, the following theorem takes care of the case of singular κ:

Theorem 18.2. *If κ is singular, then there is a κ-Aronszajn tree.*

Proof. Let $\langle \lambda_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing sequence of infinite cardinals with supremum κ. Consider the tree which has a single root, and above the root has disjoint chains which are copies of the λ_α's. Clearly this tree is a κ-Aronszajn tree. We picture this tree here:



Very rigorously, we could define T to be the set $\{0\} \cup \{(\alpha, \beta) : \alpha < \mathrm{cf}(\kappa)$ and $\beta < \lambda_\alpha\}$, with the ordering $0 < (\alpha, \beta)$ for all $\alpha < \mathrm{cf}(\kappa)$ and $\beta < \lambda_\alpha$, and $(\alpha, \beta) < (\alpha', \beta')$ iff $\alpha = \alpha'$ and $\beta < \beta'$. $\square$

Turning to regular κ, we first prove

Theorem 18.3. *There is an Aronszajn tree.*

Proof. We start with the tree

$$T = \{s \in {}^{<\omega_1}\omega : s \text{ is one-one}\}.$$

under $\subset$. This tree clearly does not have a chain of size ω_1. But all of its infinite levels are uncountable, so it is not an ω_1-Aronszajn tree. We will define a subset of it that is the desired tree. We define a system $\langle S_\alpha : \alpha < \omega_1\rangle$ of subsets of T by recursion; these will be the levels in the new tree.

Let $S_0 = \{\emptyset\}$. Now suppose that $\alpha > 0$ and S_β has been constructed for all $\beta < \alpha$ so that the following conditions hold for all $\beta < \alpha$:

(1_β) $S_\beta \subseteq {}^\beta\omega \cap T$.

(2_β) $\omega \backslash \mathrm{rng}(s)$ is infinite, for every $s \in S_\beta$.

(3_β) For all $\gamma < \beta$, if $s \in S_\gamma$, then there is a $t \in S_\beta$ such that $s \subset t$.

(4_β) $|S_\beta| \le \omega$.

(5_β) If $s \in S_\beta$, $t \in T$, and $\{\gamma < \beta : s(\gamma) \ne t(\gamma)\}$ is finite, then $t \in S_\beta$.

(6_β) If $s \in S_\beta$ and $\gamma < \beta$, then $s \restriction \gamma \in S_\gamma$.

(Vacuously these conditions hold for all $\beta < 0$.) If α is a successor ordinal $\varepsilon + 1$, we simply take

$$S_\alpha = \{s \cup \{(\varepsilon, n)\} : s \in S_\varepsilon \text{ and } n \notin \mathrm{rng}(s)\}.$$

Clearly (1_β)–(6_β) hold for all $\beta < \alpha + 1$.

Now suppose that α is a limit ordinal less than ω_1 and (1_β)–(6_β) hold for all $\beta < \alpha$. Since α is a countable limit ordinal, it follows that $\mathrm{cf}(\alpha) = \omega$. Let $\langle \delta_n : n \in \omega\rangle$ be a strictly increasing sequence of ordinals with supremum α. Now let $U = \bigcup_{\beta<\alpha} S_\beta$. Take any $s \in U$; we want to define an element $t_s \in {}^\alpha\omega \cap T$ which extends s. Let $\beta = \mathrm{dmn}(s)$.

Choose n minimum such that $\beta \le \delta_n$. Now we define a sequence $\langle u_i : i \in \omega\rangle$ of members of U; u_i will be a member of $S_{\delta_{n+i}}$. By (3_{δ_n}), let u_0 be a member of S_{δ_n} such that $s \subseteq u_0$. Having defined a member u_i of $S_{\delta_{n+i}}$, use $(3_{\delta_{n+i+1}})$ to get a member u_{i+1} of $S_{\delta_{n+i+1}}$ such that $u_i \subseteq u_{i+1}$. This finishes the construction. Let $v = \bigcup_{i\in\omega} u_i$. Thus $s \subseteq v \in {}^\alpha\omega \cap T$. Unfortunately, condition (2) may not hold for v, so this is not quite the element t_s that we are after. We define $t_s \in {}^\alpha\omega$ as follows. Let $\gamma < \alpha$. Then

$$t_s(\gamma) = \begin{cases} v(\delta_{2n+2i}) & \text{if } \gamma = \delta_{n+i} \text{ for some } i \in \omega, \\ v(\gamma) & \text{if } \gamma \notin \{\delta_{n+i} : i \in \omega\}. \end{cases}$$

Clearly $t_s \in {}^\alpha\omega \cap T$. Since $v(\delta_{2n+2i+1}) \notin \mathrm{rng}(t_s)$ for all $i \in \omega$, it follows that $\omega \backslash \mathrm{rng}(t_s)$ is infinite.

We now define

$$S_\alpha = \bigcup_{s\in U} \{w \in {}^\alpha\omega \cap T : \{\varepsilon < \alpha : w(\varepsilon) \neq t_s(\varepsilon)\} \text{ is finite}\}.$$

Now we want to check that (1_α)–(6_α) hold. Conditions (1_α) and (3_α) are very clear. For (2_α), suppose that $w \in S_\alpha$. Then $w \in {}^\alpha\omega \cap T$ and there is an $s \in U$ such that $\{\varepsilon < \alpha : w(\varepsilon) \neq t_s(\varepsilon)\}$ is finite. Since $\omega\backslash\mathrm{rng}(t_s)$ is infinite, clearly $\omega\backslash\mathrm{rng}(w)$ is infinite. For (4_α), note that U is countable by the assumption that (4_β) holds for every $\beta < \alpha$, while for each $s \in U$ the set

$$\{w \in {}^\alpha\omega \cap T : \{\varepsilon < \alpha : w(\varepsilon) \neq t_s(\varepsilon)\} \text{ is finite}\}$$

is also countable. So (4_α) holds. For (5_α), suppose that $w \in S_\alpha$, $x \in T$, and $\{\gamma < \alpha : w(\gamma) \neq x(\gamma)\}$ is finite. Choose $s \in U$ such that $\{\varepsilon < \alpha : w(\varepsilon) \neq t_s(\varepsilon)\}$ is finite. Then of course also $\{\varepsilon < \alpha : x(\varepsilon) \neq t_s(\varepsilon)\}$ is finite. So $x \in S_\alpha$, and (5_α) holds. Finally, for (6_α), suppose that $w \in S_\alpha$ and $\gamma < \alpha$; we want to show that $w \restriction \gamma \in S_\gamma$. Choose $s \in U$ such that $\{\varepsilon < \alpha : w(\varepsilon) \neq t_s(\varepsilon)\}$ is finite. Assume the notation introduced above when defining t_s (n, β, u, v). Choose $i \in \omega$ such that $\gamma \leq \delta_{n+i}$. Then

$$\begin{aligned}\{\varepsilon < \delta_{n+i} : w(\varepsilon) \neq u_i(\varepsilon)\} &= \{\varepsilon < \delta_{n+i} : w(\varepsilon) \neq v(\varepsilon)\}\\ &\subseteq \{\varepsilon < \delta_{n+i} : w(\varepsilon) \neq t_s(\delta)\} \cup \{\delta_{n+j} : j < i\},\end{aligned}$$

and the last union is clearly finite. It follows from $(5_{\delta_{n+1}})$ that $w \in S_\gamma$. So (6_α) holds.

This finishes the construction. Clearly $\bigcup_{\alpha<\omega_1} S_\alpha$ is the desired Aronszajn tree. □

We defer a discussion of possible generalizations of Theorem 18.3 until we discuss the closely related notion of a Suslin tree.

The proof of Theorem 18.2 gives

Theorem 18.4. *If κ is singular, then there is a κ-Suslin tree.* □

Note also that Theorem 18.1 implies that there are no ω-Suslin trees. There do not exist ZFC results about existence or non-existence of κ-Suslin trees for κ uncountable and regular. We limit ourselves at this point to some simple facts about Suslin trees.

Proposition 18.5. *If T is a κ-Suslin tree with κ uncountable and regular, then T is a κ-tree.* □

Proposition 18.6. *For any infinite cardinal κ, every κ-Suslin tree is a κ-Aronszajn tree.* □

This is a good place to notice that the construction of an ω_1-Aronszajn tree given in the proof of Theorem 18.3 does not give an ω_1-Suslin tree. In fact, assume the notation of that proof, and for each $n \in \omega$ let

$$A_n = \bigcup_{\alpha<\omega_1} \{s \in S_{\alpha+1} : s(\alpha) = n\}.$$

Clearly A_n is an antichain in $\bigcup_{\alpha<\omega_1} S_\alpha$, and $\bigcup_{n\in\omega} A_n = \bigcup_{\alpha<\omega_1} S_{\alpha+1}$. Hence $\left|\bigcup_{n\in\omega} A_n\right| = \omega_1$. It follows that some A_n is uncountable, so that $\bigcup_{\alpha<\omega_1} S_\alpha$ is not a Suslin tree.

We now introduce some notions that are useful in talking about κ-trees; these conditions were implicit in part of the proof of Theorem 18.3.

- A *well-pruned* κ-tree is a κ-tree T with exactly one root such that for all $\alpha < \beta < \mathrm{ht}(T)$ and for all $x \in \mathrm{Lev}_\alpha(T)$ there is a $y \in \mathrm{Lev}_\beta(T)$ such that $x < y$.

- A *normal subtree* of a tree $(T, <)$ is a tree $(S, \prec)$ satisfying the following conditions:
  (i) $S \subseteq T$.
  (ii) For any $s_1, s_2 \in S$, $s_1 \prec s_2$ iff $s_1 < s_2$.
  (iii) For any $s, t \in T$, if $s < t$ and $t \in S$, then $s \in S$.

Note that each level of a normal subtree is a subset of the corresponding level of T. Clearly a normal subtree of height κ of a κ-Aronszajn tree is a κ-Aronszajn tree; similarly for κ-Suslin trees.

- A tree T is *eventually branching* iff for all $t \in T$, the set $\{s \in T : t \leq s\}$ is not a chain.

Clearly a well-pruned κ-Aronszajn tree is eventually branching; similarly for κ-Suslin trees.

Theorem 18.7. *If κ is regular, then any κ-tree T has a normal subtree T' which is a well-pruned κ-tree. Moreover, if $x \in T$ and $|\{y \in T : x \leq y\}| = \kappa$ then we may assume that $x \in T'$.*

Proof. Let κ be regular, and let T be a κ-tree. We define

$$S = \{t \in T : |\{s \in T : t \leq s\}| = \kappa\}.$$

Clearly S is a normal subtree of T, although it may contain more than one root of T. Now we claim

(1) Some root of T is in S.

In fact, $\mathrm{Lev}_0(T)$ has size less than κ, and

$$T = \bigcup_{s\in \mathrm{Lev}_0(T)} \{t \in T : s \leq t\},$$

so there is some $s \in \mathrm{Lev}_0(T)$ such that $|\{t \in T : s \leq t\}| = \kappa$. This element s is in S, as desired in (1).

We now take an s as indicated. To satisfy the second condition in the Theorem, we can take s below the element x of that condition.

Now we let $S' = \{t \in S : s \leq t\}$. We claim that S' is as desired. Clearly it is a normal subtree of T, and it has exactly one root, namely s. To show that it has height κ and is well-pruned, it suffices now to prove

(2) If $u \in S'$, $\alpha < \beta < \kappa$, and $\mathrm{ht}(u, S') = \alpha$, then there is a $v \in S' \cap \mathrm{Lev}_\beta(T)$ such that $u < v$.

In fact,

$$\{t\in T : u\le t\} = \bigcup_{\alpha\le\gamma<\beta}\{t\in \mathrm{Lev}_\gamma(T) : u\le t\}\cup \bigcup_{\substack{v\in\mathrm{Lev}_\beta(T)\\ u<v}}\{t\in T : v\le t\},$$

and the first big union here is the union of fewer than κ sets, each of size less than κ. Hence there is a $v\in \mathrm{Lev}_\beta(T)$ such that $u<v$ and $|\{t\in T : v\le t\}|=\kappa$. So $v\in S'$ and $u<v$, as desired. □

Proposition 18.8. *Let κ be an uncountable regular cardinal. If T is an eventually branching κ-tree in which every antichain has size less than κ, then T is a Suslin tree.*

Proof. Suppose to the contrary that C is a chain of length κ. We may assume that C is maximal, so that it has elements of each level less than κ. For each $t\in T$ choose $f(t)\in T$ such that $t<f(t)\notin C$; this is possible by the eventually branching hypothesis. Now we define $\langle s_\alpha : \alpha<\kappa\rangle$ by recursion, choosing

$$s_\alpha\in\left\{t\in C : \sup_{\beta<\alpha}\mathrm{ht}(f(s_\beta),T)<\mathrm{ht}(t,T)\right\};$$

this is possible since κ is regular. Now $\langle f(s_\alpha) : \alpha<\kappa\rangle$ is an antichain. In fact, if $\beta<\alpha$ and $f(s_\beta)$ and $f(s_\alpha)$ are comparable, then by construction $\mathrm{ht}(f(s_\beta),T)<\mathrm{ht}(s_\alpha,T)<\mathrm{ht}(f(s_\alpha),T)$, and so $f(s_\beta)<f(s_\alpha)$. But then the tree property yields that $f(s_\beta)<s_\alpha$ and so $f(s_\beta)\in C$, contradiction.

Thus we have an antichain of size κ, contradiction. □

One of the main motivations for the notion of a Suslin tree comes from a correspondence between linear orders and trees. Under this correspondence, Suslin trees correspond to Suslin lines, and the existence of Suslin trees is equivalent to the existence of Suslin lines.

First we show how to go from a tree to a line, in a fairly general setting. Suppose that T is a well-pruned κ-tree, and let $\prec$ be a linear order of T. Here $\prec$ may have nothing to do with the order of the tree. Note that every branch of T has limit ordinal length. For each branch B of T, let $\mathrm{len}(B)$ be its length, and let $\langle b^B_\alpha : \alpha<\mathrm{len}(B)\rangle$ be an enumeration of B in increasing order. For distinct branches B_1, B_2, neither is included in the other, and so we can let $d(B_1,B_2)$ be the smallest ordinal $\alpha<\min(\mathrm{len}(B_1),\mathrm{len}(B_2))$ such that $b^{B_1}(\alpha)\ne b^{B_2}(\alpha)$. We define the *$\prec$-branch linear order of T*, denoted by $\mathscr{B}(T,\prec)$, to be the collection of all branches of T, where the order $<$ on $\mathscr{B}(T,\prec)$ is defined as follows: for any two distinct branches B_1, B_2,

$$B_1<B_2\quad\text{iff}\quad b^{B_1}(d(B_1,B_2))\prec b^{B_2}(d(B_1,B_2)).$$

This is a kind of lexicographic ordering of the branches. Clearly this is an irreflexive relation, and clearly any two branches are comparable. The following lemma gives that it is transitive.

Lemma 18.9. *Assume that $B_1 < B_2 < B_3$. Then exactly one of the following holds:*

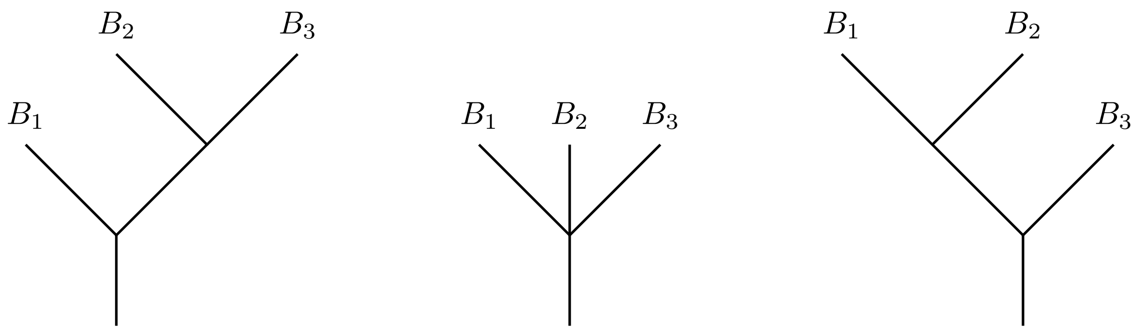
(i) $d(B_1, B_3) = d(B_1, B_2) < d(B_2, B_3)$.

(ii) $d(B_1, B_3) = d(B_1, B_2) = d(B_2, B_3)$.

(iii) $d(B_1, B_3) = d(B_2, B_3) < d(B_1, B_2)$.

In any case $B_1 < B_3$.

Clearly at most one of (i)–(iii) holds. These three conditions are illustrated as follows:



Case 1. $d(B_1, B_2) < d(B_2, B_3)$. Then, we claim, $d(B_1, B_3) = d(B_1, B_2)$. In fact, if $\alpha < d(B_1, B_2)$, then

$$b^{B_1}(\alpha) = b^{B_2}(\alpha) = b^{B_3}(\alpha),$$

while

$$b^{B_1}(d(B_1, B_2) \prec b^{B_2}(d(B_1, B_2)) = b^{B_3}(d(B_1, B_2)).$$

Hence the claim holds, and $B_1 < B_3$.

Case 2. $d(B_1, B_2) = d(B_2, B_3)$. Then, we claim, $d(B_1, B_3) = d(B_1, B_2)$. In fact, if $\alpha < d(B_1, B_2)$, then

$$b^{B_1}(\alpha) = b^{B_2}(\alpha) = b^{B_3}(\alpha),$$

while

$$b^{B_1}(d(B_1, B_2) \prec b^{B_2}(d(B_1, B_2)) \prec b^{B_3}(d(B_1, B_2)).$$

This proves the claim, and $B_1 < B_3$.

Case 3. $d(B_1, B_2) > d(B_2, B_3)$. Then, we claim, $d(B_1, B_3) = d(B_2, B_3)$. In fact, if $\alpha < d(B_2, B_3)$, then

$$b^{B_1}(\alpha) = b^{B_2}(\alpha) = b^{B_3}(\alpha),$$

while

$$b^{B_1}(d(B_2, B_3) = b^{B_2}(d(B_2, B_3)) \prec b^{B_3}(d(B_2, B_3)).$$

This proves the claim, and $B_1 < B_3$. □

Thus the construction gives a linear order.

Theorem 18.10. *If there is a Suslin tree then there is a Suslin line.*

Proof. By Theorem 18.7 we may assume that T is well-pruned. Take any linear order $\prec$ of T. To show that $\mathscr{B}(T, \prec)$ is ccc, suppose that $\mathscr{A}$ is an uncountable collection of nonempty pairwise disjoint open intervals in $\mathscr{B}(T, \prec)$. For each $(B, C) \in \mathscr{A}$ choose $E_{(B,C)} \in (B, C)$. Remembering that each branch has limit length, we can also select an ordinal $\alpha_{(B,C)}$ such that

$$d(B, E_{(B,C)}), d(E_{(B,C)}, C) < \alpha_{(B,C)} < \mathrm{len}(E_{(B,C)})$$

We claim that $\langle b^{E(B,C)}(\alpha_{(B,C)}) : (B, C) \in \mathscr{A} \rangle$ is a system of pairwise incomparable elements of T, which contradicts the definition of a Suslin tree. In fact, suppose that (B, C) and (B', C') are distinct elements of $\mathscr{A}$ and $b^{E(B,C)}(\alpha_{(B,C)}) \le b^{E(B',C')}(\alpha_{(B',C')})$. It follows that $\alpha_{(B,C)} \le \alpha_{(B',C')}$ and

(1) $b^{E(B,C)}(\beta) = b^{E(B',C')}(\beta)$ for all $\beta \le \alpha_{(B,C)}$.

Hence

(2) If $\beta < d(B, E_{(B,C)})$, then $\beta < \alpha_{(B,C)}$, and so $b^B(\beta) = b^{E_{(B,C)}}(\beta) = b^{E_{(B',C')}}(\beta)$.

Now recall that $d(B, E_{(B,C)}) < \alpha_{(B,C)}$. Hence

$$b^B(d(B, E_{(B,C)})) \prec b^{E_{(B,C)}}(d(B, E_{(B,C)})) = b^{E_{(B',C')}}(d(B, E_{(B,C)})),$$

and so $B < E_{(B',C')}$. Similarly, $E_{(B',C')} < C$, as follows:

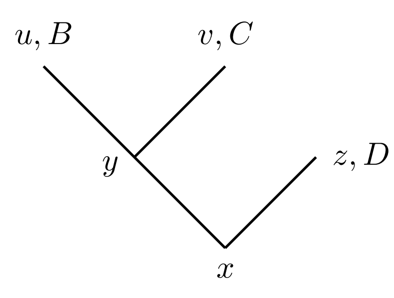
(3) If $\beta < d(C, E_{(B,C)})$, then $\beta < \alpha_{(B,C)}$, and so $b^C(\beta) = b^{E_{(B,C)}}(\beta) = b^{E_{(B',C')}}(\beta)$.

Now recall that $d(C, E_{(B,C)}) < \alpha_{(B,C)}$. Hence

$$b^C(d(C, E_{(B,C)})) \succ b^{E_{(B,C)}}(d(C, E_{(B,C)})) = b^{E_{(B',C')}}(d(C, E_{(B,C)})),$$

and so $C > E_{(B',C')}$. Hence $E_{(B',C')} \in (B, C)$. But also $E_{(B',C')} \in (B', C')$, contradiction.

To show that $\mathscr{B}(T, \prec)$ is not separable, it suffices to show that for each $\delta < \omega_1$ the set $\{B \in \mathscr{B}(T, \prec) : \mathrm{len}(B) < \delta\}$ is not dense in $\mathscr{B}(T, \prec)$. Take any $x \in T$ of height δ. Since $\{y : y > x\}$ has elements of every level greater than δ, it cannot be a chain, as this would give a chain of size ω_1. So there exist incomparable $y, z > x$. Similarly, there exist incomparable $u, v > y$. Let B, C, D be branches containing u, v, z respectively. By symmetry say $B < C$. Illustration:



(4) $\mathrm{ht}(y) < d(B, C)$

This holds since $y \in B \cap C$.

(5) $d(B,D) \le \text{ht}(y)$ and $d(C,D) \le \text{ht}(y)$; hence $d(B,D) < d(B,C)$ and $d(C,D) < d(B,C)$.

In fact, $y \in B \backslash D$, so $d(B,D) \le \text{ht}(y)$ follows. Similarly $d(C,D) \le \text{ht}(y)$. Now the rest follows by (4).

(6) $d(B,D) = d(C,D)$.

For, if $d(B,D) < d(C,D)$, then $b^C(d(B,D)) = b^D(d(B,D)) \ne b^B(d(B,D))$, contradicting $d(B,D) < d(B,C)$, part of (5). If $d(C,D) < d(B,D)$, then $b^B(d(C,D)) = b^D(d(C,D)) \ne b^C(d(C,D))$, contradicting $d(C,D) < d(B,C)$, part of (5).

By (6) we have $B, C < D$, or $D < B, C$. Since we are assuming that $B < C$, it follows that

(7) $B < C < D$ or $D < B < C$.

Case 1. $B < C < D$. Thus (B,D) is a nonempty open interval. Suppose that there is some branch E with $\text{len}(E) < \delta$ and $B < E < D$. Then $d(B,E), d(E,D) < \delta$. By Lemma 18.9 one of the following holds: $d(B,D) = d(B,E) < d(E,D)$; $d(B,D) = d(B,E) = d(E,D)$; $d(B,D) = d(E,D) < d(B,E)$. Hence $d(B,D) < \delta$. Since $x \in B \cap D$ and x has height δ, this is a contradiction.

Case 2. $D < B < C$. Thus (D,C) is a nonempty open interval. Suppose that there is some branch E with $\text{len}(E) < \delta$ and $D < E < C$. Then $d(D,E), d(E,C) < \delta$. By Lemma 18.9 one of the following holds: $d(D,C) = d(D,E) < d(E,C)$; $d(D,C) = d(D,E) = d(E,C)$; $d(D,C) = d(E,D) < d(D,E)$; hence $d(D,C) < \delta$. Since $x \in C \cap D$ and x is of height δ, this is a contradiction. $\square$

In the other direction, we prove:

Theorem 18.11. *If there is a Suslin line, then there is a Suslin tree.*

Proof. Assume that there is a Suslin line. Then by Theorem 17.16 we may assume that we have a linear order L satisfying the following conditions:

(1) L is dense, with no first or last elements.

(2) No nonempty open subset of L is separable.

(3) L is ccc.

(We do not need the other condition given in Theorem 17.16.) Let $\mathbb{I}$ be the collection of all open intervals (a,b) with $a < b$ in L. So by denseness, each such interval is nonempty. We are now going to define a sequence $\langle \mathbb{J}_\alpha : \alpha < \omega_1 \rangle$ of subsets of $\mathbb{I}$. Let $\mathbb{J}_0$ be a maximal disjoint subset of $\mathbb{I}$. Now suppose that $0 < \beta < \omega_1$ and we have defined $\mathbb{J}_\alpha$ for all $\alpha < \beta$ so that the following conditions hold:

(4_α) The elements of $\mathbb{J}_\alpha$ are pairwise disjoint.

(5_α) $\bigcup \mathbb{J}_\alpha$ is dense in L.

(6_α) If $\gamma < \alpha$, $I \in \mathbb{J}_\gamma$, and $J \in \mathbb{J}_\alpha$, then either $I \cap J = \emptyset$, or else $J \subseteq I$.

(7_α) If $\gamma < \alpha$ and $I \in \mathbb{J}_\gamma$, then there are at least two $J \in \mathbb{J}_\alpha$ such that $J \subseteq I$.

Note that (4_0)–(7_0) hold: (6_0) and (7_0) trivially hold, (4_0) holds by definition, and (5_0) holds by the maximality of $\mathbb{J}_0$.

First suppose that β is a successor ordinal $\delta+1$. For each $M \in \mathbb{J}_\delta$, choose disjoint members I_1, I_2 of $\mathbb{I}$ such that $I_1 \cup I_2 \subseteq M$, and let $\mathbb{K}_M$ be a maximal disjoint subset of

$$\{K \in \mathbb{I} : K \subseteq M\}$$

such that $I_1, I_2 \in \mathbb{K}_M$. The existence of I_1 and I_2 is clear by denseness. Then let $\mathbb{J}_\beta = \bigcup_{M \in \mathbb{J}_\delta} \mathbb{K}_M$. Clearly (4_β) holds.

For (5_β), suppose that $a, b \in L$ and $a < b$. By (5_δ), choose $c \in \bigcup \mathbb{J}_\delta$ such that $a < c < b$. Say $c \in (d,e) \in \mathbb{J}_\delta$. Thus $\max(a,d) < c < \min(b,e)$. We claim:

(8) There is a $K \in \mathbb{K}_{(d,e)}$ such that $(\max(a,d), \min(b,e)) \cap K \neq \emptyset$.

For, suppose that (8) fails. Choose u, v with $\max(a,d) < u < v < \min(b,e)$. Then $(u,v) \cap K = \emptyset$ for all $K \in \mathbb{K}_{(d,e)}$ and $(u,v) \subseteq (d,e)$. This contradicts the maximality of $\mathbb{K}_{(d,e)}$. So (8) holds.

Choose K as in (8). It follows that $\bigcup \mathbb{J}_\beta \cap (a,b) \neq \emptyset$, as desired for (5_β).

For (6_β), suppose that $\gamma \le \delta$, $I \in \mathbb{J}_\gamma$, and $J \in \mathbb{J}_\beta$. Choose $M \in \mathbb{J}_\delta$ such that $J \in \mathbb{K}_M$. Now we consider two cases.

Case 1. $\gamma = \delta$. Then by (4_δ), either $I \cap M = \emptyset$ or $I = M$. If $I \cap M = \emptyset$, then $I \cap J = \emptyset$ since $J \subseteq M$, as desired in (6_β). If $I = M$, then $J \subseteq I$ by definition.

Case 2. $\gamma < \delta$. In this case, by (6_δ) we have two further possibilities. If $I \cap M = \emptyset$, then also $I \cap J = \emptyset$, as desired in (6_β). Otherwise we have $M \subseteq I$ and clearly also $J \subseteq I$.

(7_β) is clear by construction.

Second, suppose that β is a limit ordinal. Let

$$\mathbb{K} = \{K \in \mathbb{I} : \text{for all } \alpha < \beta \text{ and all } I \in \mathbb{J}_\alpha [I \cap K = \emptyset \text{ or } K \subseteq I]\}.$$

Before defining $\mathbb{J}_\beta$, we need to know that $\mathbb{K}$ is nonempty. This follows from the following stronger statement.

(9) If $a, b \in L$ and $a < b$, then there is a $K \in \mathbb{K}$ such that $K \subseteq (a,b)$.

To prove this, suppose that $a, b \in L$ and $a < b$. Let E be the collection of all endpoints of the intervals in $\bigcup_{\gamma<\beta} \mathbb{J}_\gamma$. Since β is countable and each $\mathbb{J}_\gamma$ is countable by virtue of (4_γ) and the ccc for L, it follows that E is countable. Since (a,b) is not separable, there are $c, d \in L$ such that $a < c < d < b$ and $E \cap (c,d) = \emptyset$. For every $I \in \bigcup_{\gamma<\beta} \mathbb{J}_\gamma$, the interval (c,d) does not contain either of the endpoints of I, so it follows that $I \cap (c,d) = \emptyset$ or $(c,d) \subseteq I$. Hence $(c,d) \in \mathbb{K}$ and $(c,d) \subseteq (a,b)$, as desired in (9)

Now let $\mathbb{J}_\beta$ be a maximal pairwise disjoint subset of $\mathbb{K}$. So (4_β) holds, and also (6_β) is clear.

Now to prove (5_β), take any $a, b \in L$ with $a < b$, and choose $K \in \mathbb{K}$ such that $K \subseteq (a,b)$, as given in (9). By the maximality of $\mathbb{J}_\beta$, there is an $L \in \mathbb{J}_\beta$ such that $K \cap L \neq \emptyset$. Hence $(a,b) \cap \bigcup \mathbb{J}_\beta \neq \emptyset$, as desired in (5_β).

Finally, for (7_β), let $I \in \mathbb{J}_\gamma$ where $\gamma < \beta$. By $(7_{\gamma+1})$, there are two distinct $J, K \in \mathbb{J}_{\gamma+1}$ such that $J, K \subseteq I$, and then the construction gives $J' \subseteq J$ and $K' \subseteq K$ with $J', K' \in \mathbb{J}_\beta$, as desired.

This finishes the construction. Let $T = \bigcup_{\beta<\omega_1} \mathbb{J}_\beta$, with the ordering $\supset$. So this gives a partial order.

(10) If $I \in T$, then there is a unique $\alpha < \omega_1$ such that $I \in \mathbb{J}_\alpha$.

For, by definition there is some $\alpha < \omega_1$ such that $I \in \mathbb{J}_\alpha$. Suppose that also $I \in \mathbb{J}_\gamma$, with $\alpha \neq \gamma$. By symmetry, say that $\gamma < \alpha$. This contradicts (7_α).

We denote the α given by (10) by α_I.

(11) If $I \in T$ and $\gamma < \alpha_I$, then there is a unique $J \in \mathbb{J}_\gamma$ such that $I \subset J$.

In fact, by (5_γ) there is a $J \in \mathbb{J}_\gamma$ such that $I \cap J \neq \emptyset$. Then by (6_{α_I}), $I \subset J$. Hence by (4_{α_I}), also J is unique.

Let $I \in T$. For each $\gamma < \alpha_I$, let $f(\gamma)$ be the unique J given by (11). Then f is an order-isomorphism from α_I onto $\{J \in T : I \subset J\}$ under $\supset$. In fact, if $\gamma < \delta < \alpha_I$, then $I \subset f(\gamma) \cap f(\delta)$, and so (6_γ) and $6(_\delta)$ imply that $f(\delta) \subset f(\gamma)$. The function f maps onto, since if $I \subset J$ with $J \in \mathbb{J}_\beta$, then $\beta < \alpha$ by (6_{α_I}), and so $f(\beta) = J$.

It follows that T is a tree, and each $I \in T$ has level α_I. So T is a tree of height ω_1. If $\mathscr{A}$ is an antichain in T, then it is also an antichain in L, in the ordered set sense by (6_α), and so it is countable.

By Proposition 18.8, T is a Suslin tree. □

We mention without proof a result for higher cardinals. Assuming $V = L$, for each uncountable regular cardinal κ, there is a κ-Suslin tree iff κ is not weakly compact. (Weakly compact cardinals will be discussed later; they are inaccessible) It is a probably difficult open problem to show that it is consistent (relative to ZFC or even ZFC plus some large cardinals) that for each uncountable cardinal κ there is no κ^+-Aronszajn tree.

EXERCISES

E18.1. Let κ be an uncountable regular cardinal, and suppose that there is a κ-Aronszajn tree. Show that there is one which is a normal subtree of ${}^{<\kappa}2$. Hint: for each $\alpha < \kappa$ let g_α be an injection of $\mathrm{Lev}_\alpha(T)$ into ${}^{|\mathrm{Lev}_\alpha(T)|}2$ and glue these maps together.

E18.2. Do exercise E18.1 for κ-Suslin trees.

E18.3. Suppose that T and T' are κ-Aronszajn trees. Define an order $<$ on $T \times T'$ by $(s, s') < (t, t')$ iff $s < t$ and $s' < t'$. Show that $T \times T'$ is not a tree.

E18.4. Suppose that T and T' are κ-Aronszajn trees. Let

$$T \times' T' = \bigcup_{\alpha<\kappa} \mathrm{Lev}_\alpha(T) \times \mathrm{Lev}_\alpha(T');$$

$$(s, s') < (t, t') \text{ iff } (s, s'), (t, t') \in T \times' T',\ s < t, \text{ and } s' < t';$$

Show that $(T \times' T', <)$ is a κ-Aronszajn tree.

E18.5. Assume that κ is regular and uncountable. Suppose that T is a κ-Suslin tree. With the order on $T \times' T$ given in exercise E18.4, show that $T \times' T$ is not a κ-Suslin tree. Hint:

first show that for every $\alpha < \kappa$ there is an element s of T at level α such that there are incomparable $t, u > s$.

E18.6. A tree T is *everywhere branching* iff every $t \in T$ has at least two immediate successors. Show that every everywhere branching tree has at least 2^ω branches.

E18.7. Show that the hypothesis that all levels are finite is necessary in König's theorem.

E18.18. Show that if κ is singular with $\mathrm{cf}(\kappa) = \omega$, then there is no κ-Aronszajn tree with all levels finite.

E18.9. Prove that if κ is singular and there is a $\mathrm{cf}(\kappa)$-Aronszajn tree, then there is a κ-Aronszajn tree with all levels of power less than $\mathrm{cf}(\kappa)$.

E18.10. Show that for every infinite cardinal κ there is an eventually branching tree T of height κ such that for every subset S of T, if S is a tree under the order induced by T and every element of S has at least two immediate successors, then S has height ω.

E18.11. Show that if κ is an uncountable regular cardinal and T is a κ-Aronszajn tree, then T has a subset S such that under the order induced by T, S is a well-pruned κ-Aronszajn tree in which every element has at least two immediate successors.

Reference

Todorčević, S. *Trees and linearly ordered sets.* In **Handbook of set-theoretic topology.** North-Holland 1984, 235–293.

19. Clubs and stationary sets

Here we introduce the important notions of clubs and stationary sets. A basic result here is Fodor's theorem. We also give a combinatorial principle $\diamondsuit$, later proved consistent with ZFC, and use $\diamondsuit$ to construct a Suslin tree.

A subset Γ of an ordinal is *unbounded* iff for every $\beta < \alpha$ there is a $\gamma \in \Gamma$ such that $\beta \leq \gamma$. A subset C of α is *closed* in α provided that for every limit ordinal $\beta < \alpha$, if $C \cap \beta$ is unbounded in β then $\beta \in C$. Closed and unbounded subsets of α are called *clubs* of α.

The following simple fact about ordinals will be used below.

Lemma 19.1. *If α is an ordinal and $\Gamma \subseteq \alpha$, then o.t.$(\Gamma) \leq \alpha$.*

Proof. Let $\beta = \text{o.t.}(\Gamma)$, and let f be the isomorphism of β onto Γ. For all $\gamma < \beta$ we have $\gamma \leq f(\gamma) < \alpha$, so $\beta \subseteq \alpha$ and hence $\beta \leq \alpha$. $\square$

Note that $\emptyset$ is club in 0. If $\alpha = \beta + 1$, then $\{\beta\}$ is club in α. We are mainly interested in limit ordinals α. Then an equivalent way of looking at clubs is as follows.

Theorem 19.2. *Let α be a limit ordinal.*

(i) If C is club in α, then there exist an ordinal β and a normal function $f : \beta \to \alpha$ such that $\text{rng}(f) = C$.

(ii) If β is an ordinal and $f : \beta \to \alpha$ is a normal function such that $\text{rng}(f)$ is unbounded in α, then $\text{rng}(f)$ is club in α.

Proof. (i): Let β be the order type of C, and let $f : \beta \to C$ be the isomorphism of β onto C. Thus $f : \beta \to \alpha$, and f is strictly increasing. To show that f is continuous, suppose that $\gamma < \beta$ is a limit ordinal; we want to show that $f(\gamma) = \bigcup_{\delta<\gamma} f(\delta)$. Let $\varepsilon = \bigcup_{\delta<\gamma} f(\delta)$. Clearly ε is a limit ordinal. Now $C \cap \varepsilon$ is unbounded in ε. For, suppose that $\varphi < \varepsilon$. Then there is a $\delta < \gamma$ such that $\varphi < f(\delta)$. Since $\delta + 1 < \gamma$ and $f(\delta) < f(\delta + 1)$, we thus have $f(\delta) \in C \cap \varepsilon$. So, as claimed, $C \cap \varepsilon$ is unbounded in ε. Hence $\varepsilon \in C$. Since ε is the lub of $f[\gamma]$, it follows that $f(\gamma) = \varepsilon$, as desired. This proves (i).

(ii): Let $C = \text{rng}(f)$. We just need to show that C is closed in α. Suppose that $\gamma < \alpha$ is a limit ordinal, and $C \cap \gamma$ is unbounded in γ. We are going to show that $\psi \stackrel{\text{def}}{=} \bigcup f^{-1}[\gamma]$ is a limit ordinal less than β and $f(\psi) = \gamma$, thereby proving that $\gamma \in C$.

Choose $\delta \in C$ such that $\gamma < \delta$. Say $f(\varphi) = \delta$. Then $f^{-1}[\gamma] \subseteq \varphi$, since for every ordinal ε, if $\varepsilon \in f^{-1}[\gamma]$ then $f(\varepsilon) \in \gamma < \delta = f(\varphi)$ and so $\varepsilon < \varphi$. It follows that also $\bigcup f^{-1}[\gamma] \leq \varphi < \beta$.

Next, $\bigcup f^{-1}[\gamma]$ is a limit ordinal. For, if $\beta < \bigcup f^{-1}[\gamma]$, choose $\varepsilon \in f^{-1}[\gamma]$ such that $\beta \in \varepsilon$. Thus $f(\varepsilon) < \gamma$. Since γ is a limit ordinal and $C \cap \gamma$ is unbounded in γ, there is a θ such that $f(\varepsilon) < f(\theta) < \gamma$. Hence $\varepsilon < \theta \in f^{-1}[\gamma]$, so $\varepsilon \in \bigcup f^{-1}[\gamma]$. This shows that $\bigcup f^{-1}[\gamma]$ is a limit ordinal.

We have $f(\psi) = \bigcup_{\beta<\psi} f(\beta)$ by continuity. If $\beta < \psi$, choose $\varepsilon \in f^{-1}[\gamma]$ such that $\beta < \varepsilon$. then $f(\beta) < f(\varepsilon) \in \gamma$. This shows that $f(\psi) \leq \gamma$.

Finally, suppose that $\delta < \gamma$. Since $C \cap \gamma$ is unbounded in γ, choose θ such that $\delta < f(\theta) < \gamma$. Then $\theta \in f^{-1}[\gamma]$, so $\delta \in \bigcup f^{-1}[\gamma]$, i.e., $\delta < \psi$. Since ψ is a limit ordinal, say that $\delta < \varphi < \psi$. Then $\delta < \varphi \leq f(\varphi) \leq f(\psi)$. This shows that $\gamma \subseteq f(\psi)$, hence $f(\psi) = \gamma$. $\square$

Corollary 19.3. *If κ is a regular cardinal and $C \subseteq \kappa$, then the following conditions are equivalent:*

(i) C is club in κ.

(ii) There is a normal function $f : \kappa \to \kappa$ such that $\mathrm{rng}(f) = C$.

Proof. (i)⇒(ii): Suppose that C is club in κ. By Theorem 19.2(i) let β be an ordinal and $f : \beta \to \kappa$ a normal function with $\mathrm{rng}(f) = C$. Thus β is the order type of C, and so by Lemma 19.1, $\beta \leq \kappa$. The regularity of κ together with C being unbounded in κ imply that $\beta = \kappa$. Thus (ii) holds.

(ii)⇒(i): Suppose that $f : \kappa \to \kappa$ is a normal function such that $\mathrm{rng}(f) = C$. Then by Theorem 19.2(i), C is club in κ. □

Corollary 19.4. *If α is a limit ordinal, then there is club of α with order type $\mathrm{cf}(\alpha)$.*

Proof. By Theorem 8.48, let $f : \mathrm{cf}(\alpha) \to \alpha$ be a strictly increasing function with $\mathrm{rng}(f)$ unbounded in α. Define $g : \mathrm{cf}(\alpha) \to \alpha$ by recursion, as follows:

$$g(\xi) = \begin{cases} 0 & \text{if } \xi = 0, \\ \max(f(\eta), g(\eta) + 1) & \text{if } \xi = \eta + 1 \text{ for some } \eta, \\ \sup_{\eta<\xi} g(\eta) & \text{if } \xi \text{ is a limit ordinal.} \end{cases}$$

Clearly then g is a normal function from $\mathrm{cf}(\alpha)$ into α, with $\mathrm{rng}(g)$ unbounded in α. By Theorem 19.2(ii), the existence of the desired set C follows. □

If $\mathrm{cf}(\alpha) = \omega$, then Corollary 19.4 yields a strictly increasing function $f : \omega \to \alpha$ with $\mathrm{rng}(f)$ unbounded in α. Then $\mathrm{rng}(f)$ is club in α. The condition on limit ordinals in the definition of club is trivial in this case. Most of our results concern limit ordinals of uncountable cofinality.

If α is any limit ordinal and $\beta < \alpha$, then the interval $[\beta, \alpha)$ is a club of α. Another simple fact about clubs is that if C is club in a limit ordinal α of uncountable cofinality, then the set D of all limit ordinals which are in C is also club in α. (We need α of uncountable cofinality in order to have D unbounded.) Also, if C is club in α with $\mathrm{cf}(\alpha) > \omega$, then the set E of all limit points of members of C is also club in α. This set E is defined to be $\{\beta < \alpha : \beta \text{ is a limit ordinal and } C \cap \beta \text{ is unbounded in } \beta\}$; clearly $E \subseteq C$.

Now we give the first major fact about clubs.

Theorem 19.5. *If α is a limit ordinal with $\mathrm{cf}(\alpha) > \omega$, then the intersection of fewer than $\mathrm{cf}(\alpha)$ clubs of α is again a club.*

Proof. Suppose that $\beta < \mathrm{cf}(\alpha)$ and $\langle C_\xi : \xi < \beta \rangle$ is a system of clubs of α. Let $D = \bigcap_{\xi<\beta} C_\xi$. First we show that D is closed. To this end, suppose that $\gamma < \alpha$ is a limit ordinal, and $D \cap \gamma$ is unbounded in γ. Then for each $\xi < \beta$, the set C_ξ is unbounded in γ, and hence $\gamma \in C_\xi$ since C_ξ is closed in α. Therefore $\gamma \in D$.

To show that D is unbounded in α, take any $\gamma < \alpha$; we want to find $\delta > \gamma$ such that $\delta \in D$. We make a simple recursive construction of a sequence $\langle \varepsilon_n : n \in \omega \rangle$ of ordinals less than α. Let $\varepsilon_0 = \gamma$. Suppose that ε_n has been defined. Using the fact that each C_ξ is unbounded in α, for each $\xi < \beta$ choose $\theta_{n,\xi} \in C_\xi$ such that $\varepsilon_n < \theta_{n,\xi}$. Then let

$$\varepsilon_{n+1} = \sup_{\xi<\beta} \theta_{n,\xi};$$

we have $\varepsilon_{n+1} < \alpha$ since $\beta < \mathrm{cf}(\alpha)$. This finishes the recursive construction. Let $\delta = \sup_{n\in\omega} \varepsilon_n$. Then $\delta < \alpha$ since $\mathrm{cf}(\alpha) > \omega$. Clearly $C_\xi \cap \delta$ is unbounded in δ for each $\xi < \beta$, and hence $\delta \in C_\xi$. So $\delta \in D$, as desired. □

Again let α be any limit ordinal, and suppose that $\langle C_\xi : \xi < \alpha\rangle$ is a system of subsets of α. We define the *diagonal intersection* of this system:

$$\triangle_{\xi<\alpha} C_\xi = \{\beta \in \alpha : \forall \xi < \beta(\beta \in C_\xi)\}.$$

This construction is used often in discussion of clubs, in particular in the definition of some of the large cardinals.

Theorem 19.6. *Suppose that* $\mathrm{cf}(\alpha) > \omega$. *Assume that* $\langle C_\xi : \xi < \alpha\rangle$ *is a system of clubs of* α.

(i) If $\bigcap_{\xi<\beta} C_\xi$ *is unbounded in* α *for each* $\beta < \alpha$, *then* $\triangle_{\xi<\alpha} C_\xi$ *is club in* α.

(ii) If α *is regular, then* $\triangle_{\xi<\alpha} C_\xi$ *is club in* α.

Proof. Clearly (ii) follows from (i) (using Theorem 19.5 to verify the hypothesis of (i)), so it suffices to prove (i). Assume the hypothesis of (i).

For brevity set $D = \triangle_{\xi<\alpha} C_\xi$ First we show that D is closed in α. So, assume that β is a limit ordinal less than α, and $D \cap \beta$ is unbounded in β. To show that $\beta \in D$, take any $\xi < \beta$; we show that $\beta \in C_\xi$. Let $E = \{\gamma \in D \cap \beta : \xi < \gamma\}$. Then E is unbounded in β, and for each $\gamma \in E$ we have $\gamma \in C_\xi$, by the definition of D. So $\beta \in C_\xi$ since C_ξ is closed.

Second we show that D is unbounded in α. So, take any $\beta < \alpha$. We define a sequence $\langle \gamma_i : i < \omega\rangle$ of ordinals less than α by recursion. Let $\gamma_0 = \beta$. If γ_i has been defined, by the hypothesis of (i) let γ_{i+1} be a member of $\bigcap_{\xi<\gamma_i} C_\xi$ which is greater than γ_i. Finally, let $\delta = \sup_{i\in\omega} \gamma_i$. So $\delta < \alpha$ since $\mathrm{cf}(\alpha) > \omega$. We claim that $\delta \in D$. To see this, take any $\xi < \delta$. Choose $i \in \omega$ such that $\xi < \gamma_i$. Then $\gamma_j \in C_\xi$ for all $j \geq i$, and hence $C_\xi \cap \delta$ is unbounded in δ, so $\delta \in C_\xi$. This argument shows that $\delta \in D$. □

We give one more general fact about closed and unbounded sets; this one is frequently useful in showing that specific sets are closed and unbounded.

A *finitary partial operation* on a set A is a nonempty function whose domain is a subset of ${}^mA$ for some positive integer m and whose range is a subset of A. We say that a subset B of A is *closed* under such an operation iff for every $a \in ({}^mB) \cap \mathrm{dmn}(f)$ we have $f(a) \in B$.

Theorem 19.7. *Suppose that* κ *is an uncountable regular cardinal,* $X \in [\kappa]^{<\kappa}$, *and* $\mathscr{F}$ *is a collection of finitary partial operations on* κ, *with* $|\mathscr{F}| < \kappa$. *Then* $\{\alpha < \kappa : X \subseteq \alpha$ *and* α *is closed under each* $f \in \mathscr{F}\}$ *is club in* κ.

Proof. Denote the indicated set by C. To show that it is closed, suppose that α is a limit ordinal less than κ, and $C \cap \alpha$ is unbounded in α. To show that α is closed under any partial operation $f \in \mathscr{F}$, suppose that $\mathrm{dmn}(f) \subseteq {}^m\kappa$ and $a \in ({}^m\alpha) \cap \mathrm{dmn}(f)$. For each $i < m$ choose $\beta_i < \alpha$ such that $a_i \in \beta_i$. Since α is a limit ordinal, the ordinal $\gamma \stackrel{\text{def}}{=} \bigcup_{i<m} \beta_i$ is still less than α. Since $C \cap \alpha$ is unbounded in α, choose $\delta \in C \cap \alpha$ such that $\gamma < \delta$. Then $a \in {}^m\delta$ so, since $\delta \in C$, we have $f(a) \in \delta \subseteq \alpha$. Thus α is closed under f. Hence $\alpha \in C$; so C is closed in κ.

To show that C is unbounded in κ, take any $\alpha < \kappa$. We now define a sequence $\langle \beta_n : n \in \omega \rangle$ by recursion. Let $\beta_0 = \alpha$. Having defined $\beta_i < \kappa$, consider the set

$$\{f(a) : f \in \mathscr{F}, a \in \mathrm{dmn}(f), \text{ and each } a_j \text{ is in } \beta_i\}.$$

This set clearly has fewer than κ members. Hence we can take β_{i+1} to be some ordinal less than κ and greater than each member of this set. This finishes the construction.

Let $\gamma = \bigcup_{i\in\omega} \beta_i$. We claim that $\gamma \in C$, as desired. For, suppose that $f \in \mathscr{F}$, f has domain $\subseteq {}^n\kappa$, and $a \in ({}^n\gamma) \cap \mathrm{dmn}(f)$. Then for each $i < n$ choose $m_i \in \omega$ such that $a_i \in \beta_{m_i}$. Let p be the maximum of all the β_i's. Then $a \in ({}^n\beta_p) \cap \mathrm{dmn}(f)$, so by construction $f(a) \in \beta_{p+1} \subseteq \gamma$. □

Let α be a limit ordinal. A subset S of α is *stationary* iff S intersects every club of α. There are some obvious but useful facts about this notion. Assume that $\mathrm{cf}(\alpha) > \omega$. Then any club in α is stationary. An intersection of a stationary set with a club is again stationary. Any superset of a stationary set is again stationary. The union of fewer than $\mathrm{cf}(\alpha)$ nonstationary sets is again nonstationary. Every stationary set is unbounded in α. The following important fact is not quite so obvious:

Proposition 19.8. *If α is a limit ordinal and κ is a regular cardinal less than $\mathrm{cf}(\alpha)$, then the set*

$$S \stackrel{\text{def}}{=} \{\beta < \alpha : \mathrm{cf}(\beta) = \kappa\}$$

is stationary in α.

Proof. Let C be club in α. Let $f : \mathrm{cf}(\alpha) \to \alpha$ be strictly increasing, continuous, and with range cofinal in α. We define $g : \mathrm{cf}(\alpha) \to C$ by recursion. Let $g(0)$ be any member of C. For β a limit ordinal less than $\mathrm{cf}(\alpha)$, let $g(\beta) = \bigcup_{\gamma<\beta} g(\gamma)$. If $\beta < \mathrm{cf}(\alpha)$ and $g(\beta)$ has been defined, let $g(\beta+1)$ be a member of C greater than both $g(\beta)$ and $f(\beta)$. Clearly g is a strictly increasing continuous function mapping $\mathrm{cf}(\alpha)$ into C, and the range of g is cofinal in α. Thus $\mathrm{rng}(g)$ is club in α. Now $g(\kappa) \in C \cap S$, as desired. □

Let S be a set of ordinals. A function $f \in {}^S\mathbf{On}$ is *regressive* iff $f(\gamma) < \gamma$ for every $\gamma \in S\backslash\{0\}$. This is a natural notion, and leads to an important fact which is used in many of the deeper applications of stationary sets.

Theorem 19.9. (Fodor; also called the **pressing down lemma**) *Suppose that α is a limit ordinal of uncountable cofinality, S is a stationary subset of α, and $f : S \to \alpha$ is regressive. Then there is an $\beta < \alpha$ such that $f^{-1}[\beta]$ is stationary in α.*

In case α is regular, there is a $\gamma < \alpha$ such that $f^{-1}[\{\gamma\}]$ is stationary.

Proof. Assume the hypothesis of the first part of the theorem, but suppose that there is no β of the type indicated. So for every $\beta < \alpha$ we can choose a club C_β in α such that $C_\beta \cap f^{-1}[\beta] = \emptyset$. Let D be a club in α of order type $\mathrm{cf}(\alpha)$. Now for each $\beta < \alpha$ let $\tau(\beta)$ be the least member of D greater than β. For each $\beta < \alpha$ we define

$$E_\beta = \bigcap_{\gamma \in D \cap (\tau(\beta)+1)} C_\gamma.$$

We claim then that for every $\beta < \alpha$,

$$E_\beta \cap f^{-1}[\beta] = \emptyset. \tag{1}$$

In fact, $\beta < \tau(\beta) \in D \cap (\tau(\beta)+1)$, so $E_\beta \cap f^{-1}[\beta] \subseteq C_{\tau(\beta)} \cap f^{-1}[\tau(\beta)] = \emptyset$. So (1) holds.

Now by Theorem 19.5, each set E_β is club in α. Moreover, clearly $E_\beta \supseteq E_\delta$ if $\beta < \delta < \alpha$. Hence we can apply Theorem 19.6(i) to infer that $F \stackrel{\text{def}}{=} \triangle_{\beta<\alpha} E_\beta$ is club in α. Hence also the set G of all limit ordinals which are in F is club in α. Choose $\delta \in G \cap S$. Now $f(\delta) < \delta$; since δ is a limit ordinal, choose $\xi < \delta$ such that $f(\delta) < \xi$. But $\delta \in G \subseteq F$, so it follows by the definition of diagonal intersection that $\delta \in E_\xi$. From (1) we then see that $\delta \notin f^{-1}[\xi]$. This contradicts $f(\delta) < \xi$.

For the second part of the theorem, assume that α is regular. Note that, with β as in the first part, $f^{-1}[\beta] = \bigcup_{\gamma<\beta} f^{-1}[\{\gamma\}]$. Hence the second part follows from the fact mentioned above that a union of fewer than α nonstationary sets is nonstationary. □

To illustrate the use of Fodor's theorem we give the following result about Aronszajn trees which answers a natural question.

Theorem 19.10. *Suppose that κ is an uncountable regular cardinal, T is a κ-Aronszajn tree, and λ is an infinite cardinal less than κ. Further, suppose that $x \in T$ and $|\{y \in T : x < y\}| = \kappa$. Then there is an $\alpha > \mathrm{ht}(x)$ such that*

$$|\{y \in \mathrm{Lev}_\alpha(T) : x < y\}| \geq \lambda.$$

Proof. By Theorem 18.7 we may assume that T is well-pruned, and by taking $\{y \in T : x \leq y\}$ we may assume that x is the root of T. So now we want to find a level α such that $|\mathrm{Lev}_\alpha(T)| \geq \lambda$. We assume that this is not the case. So $|\mathrm{Lev}_\alpha(T)| < \lambda$ for all $\alpha < \kappa$.

Suppose that λ is singular. Then

$$\kappa = \bigcup_{\substack{\mu<\lambda \\ \mu \text{ a cardinal}}} \{\alpha < \kappa : |\mathrm{Lev}_\alpha(T)| < \mu^+\},$$

so there is a $\mu < \lambda$ such that $\Gamma \stackrel{\text{def}}{=} \{\alpha < \kappa : |\mathrm{Lev}_\alpha(T)| < \mu^+\}$ has power κ. Because T is well-pruned, we have $|\mathrm{Lev}_\alpha(T)| \leq |\mathrm{Lev}_\beta(T)$ whenever $\alpha < \beta$. It follows that $|\mathrm{Lev}_\alpha(T)| < \mu^+$ for all $\alpha < \kappa$, since Γ is clearly unbounded in κ. Thus we may assume that λ is regular.

For each $s \in T$ and each $\beta < \mathrm{ht}(s)$ let s_β be the unique element of height β less than s.

Let $\Delta = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \lambda\}$. So Δ is stationary in κ. Now we claim

(1) For every $\alpha \in \Delta$ and every $s \in \mathrm{Lev}_\alpha(T)$ there is a $\beta < \alpha$ such that the set $\{t \in T : s_\beta \leq t,\ \beta \leq \mathrm{ht}(t) < \alpha\}$ is a chain.

To prove this, suppose not. Thus we can choose $\alpha \in \Delta$ and $s \in \mathrm{Lev}_\alpha(T)$ such that

(2) For all $\beta < \alpha$ there is a $\gamma \in [\beta, \alpha)$ and a $t \in \mathrm{Lev}_\alpha(T)$ such that $s_\gamma < t \neq s$ and $s_{\gamma+1} \not\leq t$.

Now we use (2) to construct by recursion two sequences $\langle \gamma_\xi : \xi < \lambda\rangle$ and $\langle t_\xi : \xi < \lambda\rangle$. Suppose that these have been defined for all $\xi < \eta$, where $\eta < \lambda$, so that each $\gamma_\xi < \alpha$. Let $\delta = \bigcup_{\xi<\eta} \gamma_\xi$. So $\delta < \alpha$ since $\mathrm{cf}(\alpha) = \lambda$. By (2), choose $\gamma_\eta \in [\delta + 1, \alpha)$ and $t_\eta \in \mathrm{Lev}_\alpha(T)$ such that $s_{\gamma_\eta} < t_\eta \neq s$ and $s_{\gamma_\eta+1} \not\leq t_\eta$. Since $\mathrm{Lev}_\alpha(T)$ has size less than λ, there exist ξ, η with $\xi < \eta$ and $t_\xi = t_\eta$. Then $s_{\gamma_\xi+1} \leq s_{\gamma_\eta} < t_\eta = t_\xi$, contradiction. Hence (1) holds.

(3) For every $\alpha \in \Delta$ there is a $\beta < \alpha$ such that for each $s \in \mathrm{Lev}_\alpha(T)$ the set $\{t \in T : s_\beta \leq t,\ \beta \leq \mathrm{ht}(t) < \alpha\}$ is a chain.

To prove this, let $\alpha \in \Delta$. By (1), for each $s \in \mathrm{Lev}_\alpha(T)$ choose $\gamma_s < \alpha$ such tha the set $\{t \in T : s_{\gamma_s} \leq t,\ \gamma_s \leq \mathrm{ht}(t) < \alpha\}$ is a chain. Let $\beta = \sup_{\mathrm{ht}(s)=\alpha} \gamma_s$. Clearly β is as desired in (3).

Now for each $\alpha \in \Delta$ choose $f(\alpha)$ to be a β as in (3). So f is a regressive function defined on the stationary set Δ. Hence there is a $\beta < \alpha$ such that $f^{-1}[\{\beta\}]$ is stationary, and hence of size κ. So T does not branch beyond β, and hence has a branch of size κ because it is well-pruned, contradiction. $\square$

For the next result we need another important construction. Suppose that λ is an infinite cardinal, $f = \langle f_\rho : \rho < \lambda^+\rangle$ is a family of injections $f_\rho : \rho \to \lambda$, and S is a cofinal subset of λ^+. The (λ, f, S)-*Ulam matrix* is the function $A : \lambda \times \lambda^+ \to \mathscr{P}(\kappa)$ defined for any $\xi < \lambda$ and $\alpha < \lambda^+$ by

$$A^\xi_\alpha = \{\rho \in S\backslash(\alpha + 1) : f_\rho(\alpha) = \xi\}.$$

Theorem 19.11. (Ulam) *Let λ be an infinite cardinal, S is a stationary subset of λ^+, and I a collection of subsets of λ^+ having the following properties:*

(i) $\emptyset \in I$.

(ii) If $X \in [I]^{\leq\lambda}$, then $\bigcup X \in I$.

(iii) If $Y \subseteq X \in I$, then $Y \in I$.

(iv) If $\alpha < \lambda^+$, then $\{\alpha\} \in I$.

(v) $S \notin I$.

Then there is a system $\langle X_\alpha : \alpha < \lambda^+\rangle$ of subsets of S such that $X_\alpha \cap X_\beta = \emptyset$ for distinct $\alpha, \beta < \lambda^+$, and $X_\alpha \notin I$ for all $\alpha < \lambda^+$.

Proof. Let $f = \langle f_\rho : \rho < \lambda^+\rangle$ be a family of injections $f_\rho : \rho \to \lambda$, and let A be the (λ, f, S)-Ulam matrix. If $\xi < \lambda$, then for distinct $\alpha, \beta < \lambda^+$ we have $A^\xi_\alpha \cap A^\xi_\beta = \emptyset$, since the functions f_ρ are one-one. Moreover, for any $\alpha < \lambda^+$ we have

$$S\backslash \bigcup_{\xi<\lambda} A^\xi_\alpha \subseteq S \cap (\alpha + 1) \in I$$

by (ii)–(iv). By conditions (ii) and (v) it then follows that for each $\alpha < \lambda^+$ there is an $h(\alpha) < \lambda$ such that $A_\alpha^{h(\alpha)} \notin I$. Thus $h : \lambda^+ \to \lambda$, so there is a $\xi < \lambda$ such that $|h^{-1}[\{\xi\}]| = \lambda^+$. Hence $\{A_\alpha^\xi : \alpha < \lambda^+,\ h(\alpha) = \xi\}$ is as desired in the theorem. □

Theorem 19.12. *(i) If λ is an infinite cardinal and S is a stationary subset of λ^+, then we can partition S into λ^+-many stationary subsets.*

(ii) If κ is weakly inaccessible, then κ can be partitioned into κ many stationary subsets.

Proof. (i): Let I be the collection of all nonstationary subsets of λ^+. The conditions of Theorem 19.11 are all clear, and so by it we get a system $\langle X_\alpha : \alpha < \lambda^+\rangle$ of subsets of S such that $X_\alpha \cap X_\beta = \emptyset$ for distinct $\alpha, \beta < \lambda^+$, and $X_\alpha \notin I$ for all $\alpha < \lambda^+$. We can union $S \backslash \bigcup_{\alpha<\lambda^+} X_\alpha$ with X_0 to get the desired partition of S.

(ii) For each regular cardinal $\lambda < \kappa$, let $S_\lambda = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \lambda\}$. Thus S_λ is stationary by Proposition 19.8. By induction it is clear that if $\alpha < \kappa$, then $\aleph_{\alpha+1} < \kappa$. Hence there are κ regular cardinals less than κ. Thus we have κ many pairwise disjoint stationary subsets of κ, and these can be extended to a partition of κ as in the proof of (i). □

The first part of Theorem 19.12 can actually be extended to weak inaccessibles too, but the proof is longer.

Next we introduce an important combinatorial principle and show that it implies the existence of Suslin trees. $\diamondsuit$ is the following statement:

There exists a sequence $\langle A_\alpha : \alpha < \omega_1\rangle$ of sets with the following properties:

(i) $A_\alpha \subseteq \alpha$ for each $\alpha < \omega_1$.

(ii) For every subset A of ω_1, the set $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary in ω_1.

A sequence as in $\diamondsuit$ is called a $\diamondsuit$-sequence. Such a sequence in a sense captures all subsets of ω_1 in a sequence of length ω_1. Later in these notes we will show that $\diamondsuit$ follows from $V = L$.

Theorem 19.14. $\diamondsuit \Rightarrow$ CH.

Proof. Let $\langle A_\alpha : \alpha < \omega_1\rangle$ be a $\diamondsuit$-sequence. Then for every $A \subseteq \omega$ the set $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary in ω_1, and hence it has an infinite member; for such a member α we have $A = A_\alpha$. So we can let $f(A)$ be the least $\alpha < \omega_1$ such that $A = A_\alpha$, and we thus define an injection of $\mathscr{P}(\omega)$ into ω_1. □

Since $\diamondsuit$ is formulated in terms of subsets of ω_1, to construct a Suslin tree using $\diamondsuit$ it is natural to let the tree be ω_1 with some tree-order. The following lemma will be useful in doing the construction.

Lemma 19.15. *Suppose that $T = (\omega_1, \prec)$ is an ω_1-tree and A is a maximal antichain in T. Then*

$$\{\alpha < \omega_1 : (T \restriction \alpha) = \alpha \text{ and } A \cap \alpha \text{ is a maximal antichain in } T_\alpha\}$$

is club in ω_1.

Proof. Let C be the indicated set. For each $\alpha < \omega_1$ let $T \restriction \alpha = \{t \in T : \mathrm{ht}(t, T) < \alpha\}$. Suppose that $A \subseteq \omega_1$ is a maximal antichain in T. To see that C is closed in ω_1, let $\alpha < \omega_1$ be a limit ordinal, and suppose that $C \cap \alpha$ is unbounded in α. If $\beta \in (T \restriction \alpha)$, then there is a $\gamma < \alpha$ such that $\beta \in (T \restriction \gamma)$. Choose $\delta \in (C \cap \alpha)$ such that $\gamma < \delta$. Then $\beta \in (T \restriction \delta) = \delta$, so also $\beta \in \alpha$. This shows that $(T \restriction \alpha) \subseteq \alpha$. Conversely, suppose that $\beta \in \alpha$. Choose $\gamma \in C \cap \alpha$ such that $\beta < \gamma$. Then $\tau \in T \uparrow \gamma \subseteq T \uparrow \alpha$. Thus $(T \uparrow \alpha) = \alpha$.

To show that $A \cap \alpha$ is a maximal antichain in $T \restriction \alpha$, note first that at least it is an antichain. Now take any $\beta \in (T \restriction \alpha)$; we show that β is comparable under $\prec$ to some member of $A \cap \alpha$, which will show that $A \cap \alpha$ is a maximal antichain in $T \restriction \alpha$. Choose $\gamma < \alpha$ such that $\beta \in (T \restriction \gamma)$, and then choose $\delta \in (C \cap \alpha)$ such that $\gamma < \delta$. Thus $\beta \in (T \restriction \delta)$. Now $A \cap \delta$ is a maximal antichain in $T \restriction \delta$ since $\delta \in C$, so β is comparable with some $\varepsilon \in (A \cap \delta) \subseteq (A \cap \alpha)$, as desired.

To show that C is unbounded in κ we will apply Theorem 19.7 to the following three functions $f, g, h : \kappa \to \kappa$:

$$\begin{aligned}
f(\beta) &= \mathrm{ht}(\beta, T);\\
g(\beta) &= \sup(\mathrm{Lev}_\beta(T));\\
h(\beta) &= \text{ some member of } A \text{ comparable with } \beta \text{ under } \prec .
\end{aligned}$$

By Theorem 19.7, the set D of all $\alpha < \kappa$ which are closed under each of f, g, h is club in κ. We now show that $D \subseteq C$, which will prove that C is unbounded in κ. So, suppose that $\alpha \in D$. If $\beta \in (T \restriction \alpha)$, let $\gamma = \mathrm{ht}(\beta, T)$. Then $\gamma < \alpha$ and $\beta \in \mathrm{Lev}_\gamma(T)$, and so $\beta \leq g(\gamma) < \alpha$. Thus $(T \restriction \alpha) \subseteq \alpha$. Conversely, suppose that $\beta < \alpha$. Then $f(\beta) < \alpha$, i.e., $\mathrm{ht}(\beta, T) < \alpha$, so $\beta \in (T \restriction \alpha)$. Therefore $(T \restriction \alpha) = \alpha$. Now suppose that $\beta \in (T \restriction \alpha)$; we want to show that β is comparable with some member of $A \cap \alpha$, as this will prove that $A \cap \alpha$ is a maximal antichain in $T \restriction \alpha$. Since $\beta \in \alpha$ by what has already been shown, we have $h(\beta) < \alpha$, and so the element $h(\beta)$ is as desired. $\square$

Another crucial lemma for the construction is as follows.

Lemma 19.16. *Let* $T = (\omega_1, \prec)$ *be an eventually branching* ω_1*-tree and let* $\langle A_\alpha : \alpha < \omega_1 \rangle$ *be a* $\diamondsuit$*-sequence. Assume that for every limit* $\alpha < \omega_1$*, if* $T \restriction \alpha = \alpha$ *and* A_α *is a maximal antichain in* $T \restriction \alpha$*, then for every* $x \in \mathrm{Lev}_\alpha(T)$ *there is a* $y \in A_\alpha$ *such that* $y \prec x$*.*

Then T *is a Suslin tree.*

Proof. By Proposition 18.8 it suffices to show that every maximal antichain A of T is countable. By Lemma 19.15, the set

$$C \stackrel{\text{def}}{=} \{\alpha < \omega_1 : (T \restriction \alpha) = \alpha \text{ and } A \cap \alpha \text{ is a maximal antichain in } T_\alpha\}$$

is club in ω_1. Now by the definition of the $\diamondsuit$-sequence, the set $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary, so we can choose $\alpha \in C$ such that $A \cap \alpha = A_\alpha$. Now if $\beta \in T$ and $\mathrm{ht}(\beta, T) \geq \alpha$, then there is a $\gamma \in \mathrm{Lev}(\alpha, T)$ such that $\gamma \preceq \beta$, and the hypothesis of the lemma further yields a $\delta \in A_\alpha$ such that $\delta \prec \gamma$. Since $\delta \prec \beta$, it follows that $\beta \notin A$. So we have shown that for all $\beta \in T$, if $\text{ht}(\beta, T) \geq \alpha$ then $\beta \notin A$. Hence for any $\beta \in T$, if $\beta \in A$ then $\beta \in (T \upharpoonright \alpha) = \alpha$. So $A \subseteq \alpha$ and hence $A = A_\alpha$, so that A is countable. $\square$

Theorem 19.17. *$\diamondsuit$ implies that there is a Suslin tree.*

Proof. Assume $\diamondsuit$, and let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit$-sequence. We are going to construct a Suslin tree of the form $(\omega_1, \prec)$ in which for each $\alpha < \omega_1$ the α-th level is the set $\{\omega \cdot \alpha + m : m \in \omega\}$. We will do the construction by completely defining the tree up to heights $\alpha < \omega_1$ by recursion. Thus we define by recursion trees $(\omega \cdot \alpha, \prec_\alpha)$, so that really we are just defining the partial orders $\prec_\alpha$ by recursion.

We let $\prec_0 = \prec_1 = \emptyset$. Now suppose that $\beta > 1$ and $\prec_\alpha$ has been defined for all $\alpha < \beta$ so that the following conditions hold whenever $0 < \alpha < \beta$:

(1) $(\omega \cdot \alpha, \prec_\alpha)$ is a tree, denoted by T_α for brevity.

(2) If $\gamma < \alpha$ and $\xi, \eta \in T_\gamma$, then $\xi \prec_\gamma \eta$ iff $\xi \prec_\alpha \eta$.

(3) For each $\gamma < \alpha$, $\text{Lev}_\gamma(T_\alpha) = \{\omega \cdot \gamma + m : m \in \omega\}$.

(4) If $\gamma < \delta < \alpha$ and $m \in \omega$, then there is an $n \in \omega$ such that $\omega \cdot \gamma + m \prec_\alpha \omega \cdot \delta + n$.

(5) If $\delta < \alpha$, δ is a limit ordinal, $\omega \cdot \delta = \delta$, and A_δ is a maximal antichain in T_δ, then for every $x \in \text{Lev}_\delta(T_\alpha)$ there is a $y \in A_\delta$ such that $y \prec_\alpha x$.

Note that conditions (1)–(3) just say that the trees constructed have the special form indicated at the beginning, and are an increasing chain of trees. Condition (4) is to assure that the final tree is well-pruned. Condition (5) is connected to Lemma 19.16, which will be applied after the construction to verify that our tree is Suslin. Conditions (1)–(5) imply that if $x \in T_\alpha$, then it has the form $\omega \cdot \beta + m$ for some $\beta < \alpha$, and then $x \in \text{Lev}_\beta(T_\alpha)$ and for each $\gamma < \beta$ there is a unique element $\omega \cdot \gamma + n$ in T_α such that $\omega \cdot \gamma + n \prec_\alpha x$.

If β is a limit ordinal, let $\prec_\beta = \bigcup_{\alpha<\beta} \prec_\alpha$. Conditions (1)–(5) are then clear for any $\alpha \leq \beta$.

Next suppose that $\beta = \gamma + 2$ for some ordinal γ. Then we define

$$\begin{aligned}\prec_\beta = \prec_{\gamma+1} &\cup \{(\xi, \omega \cdot (\gamma+1) + 2m) : \xi \preceq_{\gamma+1} \omega \cdot \gamma + m,\ m \in \omega\} \\ &\cup \{(\xi, \omega \cdot (\gamma+1) + 2m+1) : \xi \preceq_{\gamma+1} \omega \cdot \gamma + m,\ m \in \omega\}.\end{aligned}$$

Clearly (1)–(5) hold for all $\alpha < \beta$.

The most important case is $\beta = \gamma + 1$ for some limit ordinal γ. To treat this case, we first associate with each $x \in T_\gamma$ a chain $B(x)$ in T_γ, and to do this we define by recursion a sequence $\langle y^x_n : n \in \omega \rangle$ of elements of T_γ. To define y^x_0 we consider two cases.

Case 1. $\omega \cdot \gamma = \gamma$ and A_γ is a maximal antichain in T_γ. Then x is comparable with some member z of A_γ, and we let y^x_0 be some element of T_γ such that $x, z \prec_\gamma y^x_0$.

Case 2. Otherwise, we just let $y^x_0 = x$.

Now let $\langle \xi_m : m \in \omega \rangle$ be a strictly increasing sequence of ordinals less than γ such that $\xi_0 = \text{ht}(y^x_0, T_\gamma)$ and $\sup_{m\in\omega} \xi_m = \gamma$. Now if y^x_i has been defined of height ξ_i, by (4) let y^x_{i+1} be an element of height ξ_{i+1} such that $y^x_i \prec_\gamma y^x_{i+1}$. Then we define

$$B(x) = \{z \in \omega \cdot \gamma : z \prec_\gamma y^x_i \text{ for some } i \in \omega\}.$$

Finally, let $\langle x(n) : n \in \omega \rangle$ be a one-one enumeration of $\omega \cdot \gamma$, and set

$$\prec_\beta = \prec_\gamma \cup \{(z, \omega \cdot \gamma + n) : n \in \omega,\ z \in B(x_n)\}.$$

Clearly (1)–(3) hold with γ in place of α. For (4), suppose that $\delta < \gamma$ and $m \in \omega$. Let $z = \omega \cdot \delta + m$. Thus $z \in \omega \cdot \gamma$, and hence there is an $n \in \omega$ such that $z = x(n)$. Hence $z \in B(x(n))$ and $z \prec_\beta \omega \cdot \gamma + n$, as desired.

For (5), suppose that $\omega \cdot \gamma = \gamma$, and A_γ is a maximal antichain in T_γ. Suppose that $w \in \mathrm{Lev}_\gamma(T_\beta)$. Choose n so that $w = \omega \cdot \gamma + n$. Then there is an $s \in A_\gamma$ such that $s < y_0^{x(n)}$. So $s \in B(x(n))$ and $s \prec_\beta \omega \cdot \gamma + n = w$, as desired.

Thus the construction is finished. Now we let $\prec = \bigcup_{\alpha<\omega_1} \prec_\alpha$. Clearly $T \stackrel{\text{def}}{=} (\omega_1, \prec)$ is an ω_1-tree. It is eventually branching by (4) and the $\beta = \gamma + 2$ step in the construction. The hypothesis of Lemma 19.16 holds by the step $\beta = \gamma + 1$, γ limit, in the construction. Therefore T is a Suslin tree by Lemma 19.16. □

We now introduce a generalization of clubs and stationary sets. Suppose that κ is an uncountable regular cardinal and A is a set such that $|A| \geq \kappa$. Then a subset X of $[A]^{<\kappa}$ is *closed* iff for every system $\langle a_\xi : \xi < \alpha \rangle$ of elements of X, with $\alpha < \kappa$ and with $a_\xi \subseteq a_\eta$ for all $\xi < \eta < \alpha$, also the union $\bigcup_{\xi<\alpha} a_\xi$ is in X. And we say that X is *unbounded* in $[A]^{<\kappa}$ iff for every $x \in [A]^{<\kappa}$ there is a $y \in X$ such that $x \subseteq y$. Club means closed and unbounded.

Theorem 19.18. *Suppose that κ is an uncountable regular cardinal, $|A| \geq \kappa$, and $a \in [A]^{<\kappa}$. Then $\{x \in [A]^{<\kappa} : a \subseteq x\}$ is club in $[A]^{<\kappa}$.*

Proof. Let C be the indicated set. Clearly C is closed. To show that it is unbounded, suppose that $y \in [A]^{<\kappa}$. Then $y \subseteq a \cup y \in C$, as desired. □

Theorem 19.19. *Suppose that κ is an regular cardinal $> \aleph_1$ and $|A| \geq \kappa$. Then $\{x \in [A]^{<\kappa} : |x| \geq \aleph_1\}$ is club in $[A]^{<\kappa}$.*

Proof. Let C be the indicated set. For closure, suppose that $\langle a_\xi : \xi < \alpha \rangle$ is a system of members of C, with $\alpha < \kappa$ and $a_\xi \subseteq a_\eta$ if $\xi < \eta < \alpha$. Since each a_ξ has size at least $\aleph_1$, so does $\bigcup_{\xi<\alpha} a_\xi$, and so $\bigcup_{\xi<\alpha} a_\xi \in C$. So C is closed. Given $x \in [A]^{<\kappa}$, let y be a subset of A of size $\aleph_1$. Then $x \subseteq x \cup y \in C$. So C is club in $[A]^{<\kappa}$. □

Theorem 19.20. *Suppose that κ is an uncountable regular cardinal and λ is a cardinal $> \kappa$. Then $\{x \in [\lambda]^{<\kappa} : x \cap \kappa \in \kappa\}$ is club in $[\lambda]^{<\kappa}$.*

Proof. Let C be the indicated set. To show that C is closed, suppose that $\langle a_\xi : \xi < \alpha \rangle$ is a system of members of C, with $\alpha < \kappa$ and $a_\xi \subseteq a_\eta$ if $\xi < \eta < \alpha$. Then $a_\xi \cap \kappa$ is an ordinal $\beta_\xi < \kappa$ for every $\xi < \alpha$. Since $\alpha < \kappa$ and κ is regular, it follows that

$$\left(\bigcup_{\xi<\alpha} a_\xi\right) \cap \kappa = \bigcup_{\xi<\alpha} (a_\xi \cap \kappa) = \bigcup_{\xi<\alpha} \beta_\xi$$

is an ordinal less than κ. So $\bigcup_{\xi<\alpha} a_\xi \in C$. Thus C is closed. To show that it is unbounded, let $y \in [\lambda]^{<\kappa}$. Let $x = (y \backslash \kappa) \cup (\bigcup(y \cap \kappa) + 1)$. Since κ is regular and $|y| < \kappa$, we have $|\bigcup(y \cap \kappa)| < \kappa$, and hence $|x| < \kappa$. Clearly $x \cap \kappa = \bigcup(y \cap \kappa) + 1 \in \kappa$. So $y \subseteq x \in C$, as desired. □

Theorem 19.21. *Suppose that κ is an uncountable regular cardinal and $|A| \geq \kappa$. Then the intersection of two clubs of $[A]^{<\kappa}$ is a club.*

Proof. Let C and D be club in $[A]^{<\kappa}$. Clearly $C \cap D$ is closed. To show that it is unbounded, take any $x \in [A]^{<\kappa}$. We define a sequence $\langle y_i : i \in \omega\rangle$ of members of $[A]^{<\kappa}$ by recursion. Let $y_0 = x$. Having defined y_{2i}, choose y_{2i+1} such that $y_{2i+1} \in C$ and $y_{2i} \subseteq y_{2i+1}$; and then choose y_{2i+2} such that $y_{2i+2} \in D$ and $y_{2i+1} \subseteq y_{2i+2}$. Then $x \subseteq \bigcup_{i\in\omega} y_i \in C \cap D$. □

Theorem 19.22. *Suppose that κ is an uncountable regular cardinal and $|A| \geq \kappa$. Then the intersection of fewer than κ clubs of $[A]^{<\kappa}$ is a club.*

Proof. Let $\langle C_\alpha : \alpha < \lambda\rangle$ be a system of clubs in κ, with $\lambda < \kappa$. We may assume that λ is an infinite cardinal. Clearly $\bigcap_{\alpha<\lambda} C_\alpha$ is closed in $[A]^{<\kappa}$. To show that it is unbounded, suppose that $x \in [A]^{<\kappa}$. We define a sequence $\langle y_\alpha : \alpha < \lambda \cdot \omega\rangle$ by recursion, where $\cdot$ is ordinal multiplication. Let $y_0 = x$. Suppose that y_α has been defined for all $\alpha < \beta$, with $\beta < \lambda \cdot \omega$, such that if $\alpha < \gamma < \beta$ then $y_\alpha \subseteq y_\gamma \in [A]^{<\kappa}$. If β is a successor ordinal $\lambda \cdot i + \gamma + 1$ with $i \in \omega$ and $\gamma < \lambda$, choose $y_\beta \in C_\gamma$ with $y_\gamma \subseteq y_\beta$. If β is a limit ordinal, let $y_\beta = \bigcup_{\alpha<\beta} y_\alpha$; so $y_\beta \in [A]^{<\kappa}$ by the regularity of κ. Finally, let $z = \bigcup_{\alpha<\lambda\cdot\omega} y_\alpha$. We claim that $x \subseteq z \in \bigcap_{\alpha<\lambda} C_\alpha$. Clearly $x \subseteq z$. Take any $\gamma < \lambda$. To show that $z \in C_\gamma$, it suffices to prove the following two things:

(1) $y_{\lambda\cdot i+\gamma+1} \in C_\gamma$ for all $i \in \omega$.

This is clear by construction.

(2) $z = \bigcup_{i\in\omega} y_{\lambda\cdot i+\gamma+1}$.

Since $\{\lambda \cdot i + \gamma + 1 : i \in \omega\}$ is cofinal in $\lambda \cdot \omega$, this is clear too. □

If κ is an uncountable regular cardinal, $|A| \geq \kappa$, and $\langle X_a : a \in A\rangle$ is a system of subsets of $[A]^{<\kappa}$, then the *diagonal intersection* of this system is the set

$$\triangle_{a\in A} X_a \stackrel{\text{def}}{=} \left\{ x \in [A]^{<\kappa} : x \in \bigcap_{a\in x} X_a \right\}.$$

Theorem 19.23. *Suppose that κ is an uncountable regular cardinal, $|A| \geq \kappa$, and $\langle X_a : a \in A\rangle$ is a system of clubs of $[A]^{<\kappa}$. Then $\triangle_{a\in A} X_a$ is club in $[A]^{<\kappa}$.*

Proof. For brevity let $D = \triangle_{a\in A} X_a$. To show that D is closed, suppose that $\langle x_\alpha : \alpha < \gamma\rangle$ is a system of members of D, with $\gamma < \kappa$, such that $x_\alpha \subseteq x_\beta$ if $\alpha < \beta < \gamma$. We want to show that $b \stackrel{\text{def}}{=} \bigcup_{\alpha<\gamma} x_\alpha$ is in D. To do this, by the definition of diagonal intersection we need to take any $a \in b$ and show that $b \in X_a$. Say $a \in x_\beta$ with $\beta < \gamma$. Then for any $\delta \in [\beta, \gamma)$ we have $a \in x_\delta$, and hence, since $x_\delta \in D$, by definition we get $x_\delta \in X_a$. Say $\beta + \tau = \gamma$. Then $\langle x_{\beta+\varepsilon} : \varepsilon < \tau \rangle$ is a system of elements of X_a, and $x_{\beta+\varepsilon} \subseteq x_{\beta+\xi}$ if $\varepsilon < \xi < \tau$. So because X_a is closed, we get

$$b = \bigcup_{\varepsilon < \tau} x_{\beta+\varepsilon} \in X_a.$$

So D is closed.

To show that D is unbounded, let $x \in [A]^{<\kappa}$ be given. We now define a sequence $\langle y_i : i \in \omega \rangle$ by recursion. Let $y_0 = x$. Having defined $y_i \in [A]^{<\kappa}$, by Theorem 19.22 the set $\bigcap_{a \in y_i} X_a$ is club in $[A]^{<\kappa}$. Hence we can choose y_{i+1} in this set such that $y_i \subseteq y_{i+1}$. This finishes the construction. Now let $z = \bigcup_{i \in \omega} y_i$. We claim that $x \subseteq z \in D$, as desired. For, clearly $x \subseteq z$. Now suppose that $a \in z$; we want to show that $z \in X_a$. Choose $i \in \omega$ so that $a \in y_i$. Then for any $j \geq i$ we have $a \in y_j$, and so by construction $y_{j+1} \in X_a$. Hence $z = \bigcup_{i \leq j} y_j \in X_a$, as desired. $\square$

Given an uncountable regular cardinal κ and a set A with $|A| \geq \kappa$, we say that a subset X of $[A]^{<\kappa}$ is *stationary* iff it intersects every club of $[A]^{<\kappa}$.

Theorem 19.24. *Suppose that κ is an uncountable regular cardinal, $|A| \geq \kappa$, S is a stationary subset of $[A]^{<\kappa}$, and f is a function with domain S such that $f(x) \in x$ for every nonempty $x \in S$. Then there exist a stationary subset T of S and an element $a \in A$ such that $f(x) = a$ for all $x \in T$.*

Proof. It suffices to show that there is an $a \in A$ such that $f^{-1}[\{a\}]$ is stationary. Suppose to the contrary that for each $a \in A$ there is a club C_a in $[A]^{<\kappa}$ such that $C_a \cap f^{-1}[\{a\}] = \emptyset$. By Theorem 19.23 choose $x \in S \cap \triangle_{a \in A} C_a$. Thus $x \in \bigcap_{a \in x} C_a$. In particular, $x \in C_{f(x)}$. So $x \in C_{f(x)} \cap f^{-1}[\{f(x)\}]$, contradiction. $\square$

Theorem 19.25. *Suppose that λ is regular, $\kappa^+ \leq \lambda$, and $S \subseteq [\lambda]^{<\kappa^+}$ is stationary. Then S is the disjoint union of λ stationary sets.*

Proof. For each nonempty $P \in [\lambda]^{<\kappa^+}$ write $P = \{\alpha^P_\xi : \xi < \kappa\}$.

(1) There is an $\eta < \kappa$ such that for all $\beta < \lambda$ the set $\{P \in S : \alpha^P_\eta \geq \beta\}$ is stationary.

Otherwise for every $\eta < \kappa$ there is a $\beta_\eta < \lambda$ such that $\{P \in S : \alpha^P_\eta \geq \beta_\eta\}$ is non-stationary. So there is a club C_η such that $C_\eta \cap \{P \in S : \alpha^P_\eta \geq \beta_\eta\} = \emptyset$. Let $\gamma = \sup_{\eta<\kappa} \beta_\eta$ and $D = \bigcap_{\eta<\kappa} C_\eta$. Note that D is club by Theorem 19.22. For all $P \in D \cap S$ and $\eta < \kappa$ we have $\alpha^P_\eta < \beta_\eta \leq \gamma$, so $P \subseteq \gamma$. Now by Theorem 19.18 the set $E \stackrel{\text{def}}{=} \{P \in [\lambda]^{<\kappa^+} : \gamma + 1 \subseteq P\}$ is club. So $E \cap D \cap S = \emptyset$, contradicting S stationary. So (1) holds.

Take $\eta < \kappa$ as in (1). For each $P \in S$ let $f(P) = \alpha^P_\eta$. Now for each $\beta < \lambda$ the set $T_\beta \stackrel{\text{def}}{=} \{P \in S : \alpha^P_\eta \geq \beta\}$ is stationary. For $P \in T_\beta$ we have $f(P) \in P$, so by Theorem 19.24 there is a stationary subset U_β of T_β and a $\delta_\beta < \lambda$ such that $f(P) = \delta_\beta$ for all $P \in U_\beta$. Let $V_\beta = \{P \in S : f(P) = \delta_\beta\}$. So $U_\beta \subseteq V_\beta$, hence U_β is stationary. We now define $\langle \varepsilon_\xi : \xi < \lambda \rangle$ by recursion. Suppose defined for all $\xi < \eta$. Let $\beta = \sup_{\xi<\eta}(\delta_{\varepsilon_\xi} + 1)$, and set $\varepsilon_\eta = \delta_\beta$. Clearly $U_{\varepsilon_\xi} \cap U_{\varepsilon\eta} = \emptyset$ for $\xi \neq \eta$. $\square$

Theorem 19.26. *Let κ be an uncountable regular cardinal. Thus $\kappa \subseteq [\kappa]^{<\kappa}$. Suppose that $C \subseteq [\kappa]^{<\kappa}$ is club. Then $C \cap \kappa$ is club in the usual sense.*

Proof. To show that $C \cap \kappa$ is closed, suppose that $\alpha < \kappa$ and $C \cap \kappa$ is unbounded in α in the usual sense. Let $\langle \beta_\xi : \xi < \mathrm{cf}(\alpha) \rangle$ be a system of elements of $C \cap \kappa$ with supremum α. Thus $\alpha = \bigcup_{\xi < \mathrm{cf}(\alpha)} \beta_\xi \in C$ since X is closed. This union is also in κ because κ is regular.

To show that $C \cap \kappa$ is unbounded in the usual sense, suppose that $\alpha < \kappa$. Since C is unbounded, choose $y_0 \in C$ such that $\alpha \subseteq y_0$. Now $y_0 \in [\kappa]^{<\kappa}$, so $\beta_0 \stackrel{\mathrm{def}}{=} \bigcup y_0 < \kappa$. Then choose $y_1 \in C$ such that $\beta_0 \subseteq y_1$. Continuing, we obtain $\alpha \subseteq y_0 \subseteq \beta_0 \subseteq y_1 \subseteq \beta_1 \subseteq \ldots$. The union of this sequence is in C since C is closed, and it is an ordinal $< \kappa$ since κ is regular, as desired. □

Theorem 19.27. *Let κ be an uncountable regular cardinal, and let $C \subseteq \kappa$ be club in the old sense. Then $\{X \in [\kappa]^{<\kappa} : \bigcup X \in C\}$ is club in the new sense.*

Proof. Let $C' = \{X \in [\kappa]^{<\kappa} : \bigcup X \in C\}$. Suppose that $\langle X_\xi : \xi < \alpha \rangle$ is an increasing sequence of members of C', with $\alpha < \kappa$. Then $\langle \bigcup X_\xi : \xi < \alpha \rangle$ is an increasing sequence of members of C, and so $\bigcup \bigcup_{\xi < \alpha} X_\xi \in C$. It follows that $\bigcup_{\xi < \alpha} X_\xi \in C'$.

Suppose that $X \in [\kappa]^{<\kappa}$. Then $\bigcup X$ is an ordinal less than κ, and so there is a limit ordinal $\alpha \in C$ such that $\bigcup X < \alpha$. Hence $X \subseteq \alpha = \bigcup \alpha$. So $\alpha \in C'$ is as desired. □

Theorem 19.28. *Let κ be an uncountable regular cardinal, and let $S \subseteq [\kappa]^{<\kappa}$ be stationary in the new sense. Then $\{\bigcup X : X \in S\}$ is stationary in the old sense.*

Proof. Let $S' = \{\bigcup X : X \in S\}$. Let C be a club in the old sense. With C' as in the proof of Theorem 19.27, choose $X \in S \cap C'$. Then $\bigcup X \in S' \cap C$, as desired. □

Theorem 19.29. *Let κ be an uncountable regular cardinal, and $S \subseteq \kappa$ be stationary in the old sense. Then S is stationary as a subset of $[\kappa]^{<\kappa}$.*

Proof. Let $X \subseteq [\kappa]^{<\kappa}$ be club. Then by Theorem 19.26, $X \cap \kappa$ is club in the old sense. Hence $S \cap X \cap \kappa \neq \emptyset$. □

EXERCISES

E19.1. Assume that κ is an uncountable regular cardinal and $\langle A_\alpha : \alpha < \kappa \rangle$ is a sequence of subsets of κ. Let $D = \triangle_{\alpha<\kappa} A_\alpha$. Prove the following:

(i) For all $\alpha < \kappa$, the set $D \backslash A_\alpha$ is nonstationary.

(ii) Suppose that $E \subseteq \kappa$ and for every $\alpha < \kappa$, the set $E \backslash A_\alpha$ is nonstationary. Show that $E \backslash D$ is nonstationary.

E19.2. Let $\kappa > \omega$ be regular. Show that there is a sequence $\langle S_\alpha : \alpha < \kappa \rangle$ of stationary subsets of κ such that $S_\beta \subseteq S_\alpha$ whenever $\alpha < \beta < \kappa$, and $\triangle_{\alpha<\kappa} S_\alpha = \{0\}$. Hint: use Theorem 19.12.

E19.3. Suppose that κ is uncountable and regular, and for each limit ordinal $\alpha < \kappa$ we are given a function $f_\alpha \in {}^\omega\alpha$. Suppose that S is a stationary subset of κ. Let $n \in \omega$. Show that there exist a $t \in {}^n\kappa$ and a stationary $S' \subseteq S$ such that for all $\alpha \in S'$, $f_\alpha \restriction n = t$.

E19.4. Suppose that $\mathrm{cf}(\kappa) > \omega$, $C \subseteq \kappa$ is club of order type $\mathrm{cf}(\kappa)$, and $\langle c_\beta : \beta < \mathrm{cf}(\kappa)\rangle$ is the strictly increasing enumeration of C. Let $X \subseteq \kappa$. Show that X is stationary in κ iff $\{\beta < \mathrm{cf}(\kappa) : c_\beta \in X\}$ is stationary in $\mathrm{cf}(\kappa)$.

E19.5. Suppose that κ is regular and uncountable, and $S \subseteq \kappa$ is stationary. Also, suppose that every $\alpha \in S$ is an uncountable regular cardinal. Show that

$$T \stackrel{\text{def}}{=} \{\alpha \in S : S \cap \alpha \text{ is non-stationary in } \alpha\}$$

is stationary in κ. Hint: given a club C in κ, let C' be the set of all limit points of C and let α be the least element of $C' \cap S$; show that $\alpha \in T \cap C$.

E19.6. Suppose that κ is uncountable and regular, and $\kappa \le |A|$. Suppose that C is a closed subset of $[A]^{<\kappa}$ and D is a directed subset of C with $|D| < \kappa$. (Directed means that if $x, y \in D$ then there is a $z \in D$ such that $x \cup y \subseteq z$.) Show that $\bigcup D \in C$. Hint: use induction on $|D|$.

E19.7. Let κ be uncountable and regular, and $\kappa \le |A|$. If $f : [A]^{<\omega} \to [A]^{<\kappa}$ let $C_f = \{x \in [A]^{<\kappa} : \forall s \in [x]^{<\omega}[f(s) \subseteq x]\}$. Show that C_f is club in $[A]^{<\kappa}$.

E19.8. (Continuing Exercise E19.7) Let κ be uncountable and regular, and $\kappa \le |A|$. Let D be club in $[A]^{<\kappa}$. Show that there is an $f : [A]^{<\omega} \to [A]^{<\kappa}$ such that $C_f \subseteq D$. Hint: show that there is an $f : [A]^{<\omega} \to C$ such that $\forall e \in [A]^{<\omega}[e \subseteq f(e)]$ and $\forall e_1, e_2 \in [A]^{<\omega}[e_1 \subseteq e_2 \to f(e_1) \subseteq f(e_2)]$.

E19.9. Let κ be uncountable and regular, $\kappa \le |A|$, and $A \subseteq B$. If $Y \in [A]^{<\kappa}$, let $Y^B = \{x \in [B]^{<\kappa} : x \cap A \in Y\}$. Show that if Y is club in $[A]^{<\kappa}$, then Y^B is club in $[B]^{<\kappa}$.

E19.10. Let κ be uncountable and regular, $\kappa \le |A|$, and $A \subseteq B$. If $Y \in [B]^{<\kappa}$, let $Y \restriction A = \{y \cap A : y \in Y\}$. Show that if Y is stationary in $[B]^{<\kappa}$ then $Y \restriction A$ is stationary in $[A]^{<\kappa}$.

E19.11. With κ, A, B as in exercise E19.9, suppose that $f : [B]^{<\omega} \to [B]^{<\kappa}$. For each $e \in [A]^{<\omega}$ define

$$\begin{aligned} x_0(e) &= e; \\ x_{n+1}(e) &= x_n(e) \cup \{f(s) : s \in [x_n(e)]^{<\omega}\}; \\ w(e) &= \bigcup_{n\in\omega} x_n(e). \end{aligned}$$

Also, for each $y \in [A]^{<\kappa}$ let $v(y) = \bigcup\{w(e) : e \in [y]^{<\omega}\}$.

Prove that $w(e) \in C_f$ for all $e \in [A]^{<\omega}$ and $v(y) \in C_f$ for all $y \in [A]^{<\kappa}$.

E19.12. With κ, A, B as in exercise E19.9, suppose that S is stationary in $[A]^{<\kappa}$. Show that S^B is stationary in $[B]^{<\kappa}$. Hint: use exercises E19.8 and E19.11.

Reference

Jech, T. *Stationary sets.* Chapter in **Handbook of set theory.**

20. Infinite combinatorics

In this chapter we survey the most useful theorems of infinite combinatorics; the best known of them is the infinite Ramsey theorem. We derive from it the finite Ramsey theorem.

Two sets A, B are *almost disjoint* iff $|A| = |B|$ while $|A \cap B| < |A|$. Of course we are mainly interested in this notion if A and B are infinite.

Theorem 20.1. *There is a family of 2^ω pairwise almost disjoint infinite sets of natural numbers.*

Proof. Let $X = \bigcup_{n\in\omega} {}^n2$. Then $|X| = \omega$, since X is clearly infinite, while

$$|X| \le \sum_{n\in\omega} 2^n \le \omega \cdot \omega = \omega.$$

Let f be a bijection from ω onto X. Then for each $g \in {}^\omega 2$ let $x_g = \{g \restriction n : n \in \omega\}$. So x_g is an infinite subset of X. If $g, h \in {}^\omega 2$ and $g \neq h$, choose n so that $g(n) \neq h(n)$. Then clearly $x_g \cap x_h \subseteq \{g \restriction i : i \le n\}$, and so this intersection is finite. Thus we have produced 2^ω pairwise almost disjoint infinite subsets of X. That carries over to ω. Namely, $\{f^{-1}[x_g] : g \in {}^\omega 2\}$ is a family of 2^ω pairwise almost disjoint infinite subsets of ω, as is easily checked. $\square$

Let X be an infinite set. A collection $\mathscr{A}$ of subsets of X is *independent* iff for any two finite disjoint subsets $\mathscr{B}, \mathscr{C}$ of $\mathscr{A}$ we have

$$\left(\bigcap_{Y\in\mathscr{B}} Y\right) \cap \left(\bigcap_{Z\in\mathscr{C}} (X\backslash Z)\right) \neq \emptyset.$$

Theorem 20.2. (Fichtenholz, Kantorovitch, Hausdorff) *For any infinite cardinal κ there is an independent family $\mathscr{A}$ of subsets of κ such that each member of $\mathscr{A}$ has size κ and $|\mathscr{A}| = 2^\kappa$; moreover, each of the above intersections has size κ.*

Proof. Let $\mathscr{F}$ be the family of all finite subsets of κ; thus $|\mathscr{F}| = \kappa$. Let Φ be the set of all finite subsets of $\mathscr{F}$; thus also $|\Phi| = \kappa$. It suffices now to work with $\mathscr{F} \times \Phi$ rather than κ itself.

For each $\Gamma \subseteq \kappa$ let

$$b_\Gamma = \{(\Delta, \varphi) \in \mathscr{F} \times \Phi : \Delta \cap \Gamma \in \varphi\}.$$

Note that each b_Γ has size κ; for example, $(\emptyset, \{\emptyset, \{\alpha\}\}) \in b_\Gamma$ for every $\alpha < \kappa$. So to finish the proof it suffices to take any two finite disjoint subsets H and K of $\mathscr{P}(\kappa)$ and show that

$$(*) \qquad \left(\bigcap_{A\in H} b_A\right) \cap \left(\bigcap_{B\in K} ((\mathscr{F} \times \Phi)\backslash b_B)\right)$$

has size κ. For distinct $A, B \in H \cup K$ pick $\alpha_{AB} \in A \triangle B$, and let $\Delta = \{\alpha_{AB} : A, B \in H \cup K,\ A \neq B\}$. Now it suffices to show that if $\beta \in \kappa \backslash \Delta$ and $\varphi = \{\Delta \cap A : A \in H\} \cup \{\{\beta\}\}$, then (Δ, φ) is a member of $(*)$. If $A \in H$, then $\Delta \cap A \in \varphi$, and so $(\Delta, \varphi) \in b_A$. Now suppose, to get a contradiction, that $B \in K$ and $(\Delta, \varphi) \in b_B$. Then $\Delta \cap B \in \varphi$. Since $\beta \notin \Delta$, it follows that there is an $A \in H$ such that $\Delta \cap B = \Delta \cap A$. Since $A \neq B$, we have $\alpha_{AB} \in A \triangle B$ and $\alpha_{AB} \in \Delta$, contradiction. □

We now give a generalization of the Δ-system theorem.

Theorem 20.3. *Suppose that κ and λ are cardinals, $\omega \leq \kappa < \lambda$, λ is regular, and for all $\alpha < \lambda$, $|[\alpha]^{<\kappa}| < \lambda$. Suppose that $\mathscr{A}$ is a collection of sets, with each $A \in \mathscr{A}$ of size less than κ, and with $|\mathscr{A}| \geq \lambda$. Then there is a $\mathscr{B} \in [\mathscr{A}]^\lambda$ which is a Δ-system.*

Proof.

(1) There is a regular cardinal μ such that $\kappa \leq \mu < \lambda$.

In fact, if κ is regular, we may take $\mu = \kappa$. If κ is singular, then $\kappa^+ \leq |[\kappa]^{<\kappa}| < \lambda$, so we may take $\mu = \kappa^+$.

We take μ as in (1). Let $S = \{\alpha < \lambda : \alpha \text{ is a limit ordinal and } \mathrm{cf}(\alpha) = \mu\}$. Then S is a stationary subset of λ.

Let $\mathscr{A}_0$ be a subset of $\mathscr{A}$ of size λ. Now $\left|\bigcup_{A \in \mathscr{A}_0} A\right| \leq \lambda$ since $\kappa < \lambda$. Let a be an injection of $\bigcup_{A \in \mathscr{A}_0} A$ into λ, and let A be a bijection of λ onto $\mathscr{A}_0$. Set $b_\alpha = a[A_\alpha]$ for each $\alpha < \lambda$. Now if $\alpha \in S$, then $|b_\alpha \cap \alpha| \leq |b_\alpha| = |A_\alpha| < \kappa \leq \mu = \mathrm{cf}(\alpha)$, so there is an ordinal $g(\alpha)$ such that $\sup(b_\alpha \cap \alpha) < g(\alpha) < \alpha$. Thus g is a regressive function on S. By Fodor's theorem, there exist a stationary $S' \subseteq S$ and a $\beta < \lambda$ such that $g[S'] = \{\beta\}$. For each $\alpha \in S'$ let $F(\alpha) = b_\alpha \cap \alpha$. Thus $F(\alpha) \in [\beta]^{<\kappa}$, and $|[\beta]^{<\kappa}| < \lambda$, so there exist an $S'' \in [S']^\lambda$ and a $B \in [\beta]^{<\kappa}$ such that $b_\alpha \cap \alpha = B$ for all $\alpha \in S''$.

Now we define $\langle \alpha_\xi : \xi < \lambda \rangle$ by recursion. For any $\xi < \lambda$, α_ξ is a member of S'' such that

(2) $\alpha_\eta < \alpha_\xi$ for all $\eta < \xi$, and
(3) $\delta < \alpha_\xi$ for all $\delta \in \bigcup_{\eta < \xi} b_{\alpha_\eta}$.

Since $\left|\bigcup_{\eta<\xi} b_{\alpha_\eta}\right| < \lambda$, this is possible by the regularity of λ.

Now let $\mathscr{A}_1 = A[\{\alpha_\xi : \xi < \lambda\}]$ and $r = a^{-1}[B]$. We claim that $C \cap D = r$ for distinct $C, D \in \mathscr{A}_1$. For, write $C = A_{\alpha_\xi}$ and $D = A_{\alpha_\eta}$. Without loss of generality, $\eta < \xi$. Suppose that $x \in r$. Thus $a(x) \in B \subseteq b_{\alpha_\xi}$, so by the definition of b_{α_ξ} we have $x \in A_{\alpha_\xi} = C$. Similarly $x \in D$. Conversely, suppose that $x \in C \cap D$. Thus $x \in A_{\alpha_\xi} \cap A_{\alpha_\eta}$, and hence $a(x) \in b_{\alpha_\xi} \cap b_{\alpha_\eta}$. By the definition of α_ξ, since $a(x) \in b_{\alpha_\eta}$ we have $a(x) < \alpha_\xi$. So $a(x) \in b_{\alpha_\xi} \cap \alpha_\xi = B$, and hence $x \in r$.

Clearly $|\mathscr{A}_1| = \lambda$. □

Another form of this theorem is as follows. An *indexed* Δ*-system* is a system $\langle A_i : i \in I \rangle$ of sets such that there is a set r (the *root*) such that $A_i \cap A_j = r$ for all distinct $i, j \in I$. Some, or even all, the A_i's can be equal.

Theorem 20.4. *Suppose that κ and λ are cardinals, $\omega \le \kappa < \lambda$, λ is regular, and for all $\alpha < \lambda$, $|[\alpha]^{<\kappa}| < \lambda$. Suppose that $\langle A_i : i \in I\rangle$ is a system of sets, with each A_i of size less than κ, and with $|I| \ge \lambda$. Then there is a $J \in [I]^\lambda$ such that $\langle A_i : i \in J\rangle$ is an indexed Δ-system.*

Proof. Define $i \equiv j$ iff $i, j \in I$ and $A_i = A_j$. If some equivalence class has λ or more elements, a subset J of that class of size λ is as desired. If every equivalence class has fewer than λ elements, then there are at least λ equivalence classes. Let $\mathscr{A}$ have exactly one element in common with λ equivalence classes. We apply Theorem 20.3 to get a subset $\mathscr{B}$ of $\mathscr{A}$ of size λ which is a Δ-system, say with kernel r. Say $\mathscr{B} = \{A_i : i \in J\}$ with $J \in [I]^\lambda$ and $A_i \ne A_j$ for $i \ne j$. Then $\langle A_i : i \in J\rangle$ is an indexed Δ-system with root r. $\square$

Now we give some important results of the *partition calculus*, which is infinitary Ramsey theory. The basic definition is as follows:

- Suppose that ρ is a nonzero cardinal number, $\langle \lambda_\alpha : \alpha < \rho\rangle$ is a sequence of cardinals, and σ, κ are cardinals. We also assume that $1 \le \sigma \le \lambda_\alpha \le \kappa$ for all $\alpha < \rho$. Then we write

$$\kappa \to (\langle \lambda_\alpha : \alpha < \rho\rangle)^\sigma$$

provided that the following holds:

For every $f : [\kappa]^\sigma \to \rho$ there exist $\alpha < \rho$ and $\Gamma \in [\kappa]^{\lambda_\alpha}$ such that $f[[\Gamma]^\sigma]] \subseteq \{\alpha\}$.

In this case we say that Γ is *homogeneous for f*.

The following colorful terminology is standard. We imagine that α is a color for each $\alpha < \rho$, and we color all of the σ-element subsets of κ. To say that Γ is homogeneous for f is to say that all of the σ-element subsets of Γ get the same color. Usually we will take σ and ρ to be a positive integers. If $\rho = 2$, we have only two colors, which are conventionally taken to be red (for 0) and blue (for 1). If $\sigma = 2$ we are dealing with ordinary graphs.

Note that if $\rho = 1$ then we are using only one color, and so the arrow relation obviously holds by taking $\Gamma = \kappa$. If κ is infinite and $\sigma = 1$ and ρ is a positive integer, then the relation holds no matter what σ is, since

$$\kappa = \bigcup_{i<\rho} \{\alpha < \kappa : f(\{\alpha\}) = i\},$$

and so there is some $i < \rho$ such that $|\{\alpha < \kappa : f(\{\alpha\}) = i\}| = \kappa \ge \lambda_i$, as desired.

For the first few theorems we assume that ρ is finite, and use the letter r instead of ρ.

The general infinite Ramsey theorem is as follows.

Theorem 20.5. (Ramsey) *If n and r are positive integers, then*

$$\omega \to (\underbrace{\omega, \ldots, \omega}_{r \text{ times}})^n.$$

Proof. We proceed by induction on n. The case $n = 1$ is trivial, as observed above. So assume that the theorem holds for $n \ge 1$, and now suppose that $f : [\omega]^{n+1} \to r$. For each $m \in \omega$ define $g_m : [\omega\backslash\{m\}]^n \to r$ by:

$$g_m(X) = f(X \cup \{m\}).$$

Then by the inductive hypothesis, for each $m \in \omega$ and each infinite $S \subseteq \omega$ there is an infinite $H_m^S \subseteq S\backslash\{m\}$ such that g_m is constant on $[H_m^S]^n$. We now construct by recursion two sequences $\langle S_i : i \in \omega\rangle$ and $\langle m_i : i \in \omega\rangle$. Each m_i will be in ω, and we will have $S_0 \supseteq S_1 \supseteq \cdots$. Let $S_0 = \omega$ and $m_0 = 0$. Suppose that S_i and m_i have been defined, with S_i an infinite subset of ω. We define

$$\begin{aligned} S_{i+1} &= H_{m_i}^{S_i} \quad \text{and} \\ m_{i+1} &= \text{the least element of } S_{i+1} \text{ greater than } m_i. \end{aligned}$$

Clearly $S_0 \supseteq S_1 \supseteq \cdots$ and $m_0 < m_1 < \cdots$. Moreover, $m_i \in S_i$ for all $i \in \omega$.

(1) For each $i \in \omega$, the function g_{m_i} is constant on $[\{m_j : j > i\}]^n$.

In fact, $\{m_j : j > i\} \subseteq S_{i+1}$ by the above, and so (1) is clear by the definition.

Let $p_i < r$ be the constant value of $g_{m_i} \restriction [\{m_j : j > i\}]^n$, for each $i \in \omega$. Hence

$$\omega = \bigcup_{j<r} \{i \in \omega : p_i = j\};$$

so there is a $j < r$ such that $K \stackrel{\text{def}}{=} \{i \in \omega : p_i = j\}$ is infinite. Let $L = \{m_i : i \in K\}$. We claim that $f[[L]^{n+1}] \subseteq \{j\}$, completing the inductive proof. For, take any $X \in [L]^{n+1}$; say $X = \{m_{i_0}, \ldots, m_{i_n}\}$ with $i_0 < \cdots < i_n$. Then

$$f(X) = g_{m_{i_0}}(\{m_{i_1}, \ldots, m_{i_n}\}) = p_{i_0} = j. \qquad \square$$

As a digression, we also prove the finite version of Ramsey's theorem:

Theorem 20.6. (Ramsey) *Suppose that $n, r, l_0, \ldots, l_{r-1}$ are positive integers, with $n \le l_i$ for each $i < r$. Then there is a $k \ge l_i$ for each $i < r$ and $k \ge n$ such that*

$$k \to (l_0, \ldots, l_{r-1})^n.$$

Proof. Assume the hypothesis, but suppose that the conclusion fails. Thus for every k such that $k \ge l_i$ for each $i < r$ with $k \ge n$ also, we have $k \not\to (l_0, \ldots, l_{r-1})^n$, which means that there is a function $f_k : [k]^n \to r$ such that for each $i < r$, there is no set $S \in [k]^{l_i}$ such that $f_k[[S]^n] \subseteq \{i\}$. We use these functions to define a certain $g : [\omega]^n \to r$ which will contradict the infinite version of Ramsey's theorem. Let $M = \{k \in \omega : k \ge l_i$ for each $i < r$ and $k \ge n\}$.

To define g, we define functions $h_i : [i]^n \to r$ by recursion. h_0 has to be the empty function. Now suppose that we have defined h_i so that $S_i \stackrel{\text{def}}{=} \{s \in M : f_s \restriction [i]^n = h_i\}$ is infinite. This is obviously true for $i = 0$. Then

$$S_i = \bigcup_{s:[i+1]^n \to r} \{k \in S_i : f_k \restriction [i+1]^n = s\},$$

and so there is a $h_{i+1} : [i+1]^n \to r$ such that $S_{i+1} \stackrel{\text{def}}{=} \{k \in S_i : f_k \restriction [i+1]^n = h_{i+1}\}$ is infinite, finishing the construction.

Clearly $h_i \subseteq h_{i+1}$ for all $i \in \omega$. Hence $g = \bigcup_{i\in\omega} h_i$ is a function mapping $[\omega]^n$ into r. By the infinite version of Ramsey's theorem choose $v < r$ and $Y \in [\omega]^\omega$ such that $g[[Y]^n] \subseteq \{v\}$. Take any $Z \in [Y]^{l_v}$. Choose i so that $Z \subseteq i$, and choose $k \in S_i$. Then for any $X \in [Z]^n$ we have

$$f_k(X) = h_i(X) = g(X) = v,$$

so Z is homogeneous for f_k, contradiction. □

According to the following theorem, the most obvious generalization of Ramsey's theorem does not hold.

Theorem 20.7. *For any infinite cardinal κ we have $2^\kappa \not\to (\kappa^+, \kappa^+)^2$.*

Proof. We consider ${}^\kappa 2$ under the lexicographic order; see the beginning of Chapter 13. Let $\langle f_\alpha : \alpha < 2^\kappa\rangle$ be a one-one enumeration of ${}^\kappa 2$. Define $F : 2^\kappa \to 2$ by setting, for any $\alpha < \beta < \kappa$,

$$F(\{\alpha,\beta\}) = \begin{cases} 0 & \text{if } f_\alpha < f_\beta, \\ 1 & \text{if } f_\beta < f_\alpha. \end{cases}$$

If $2^\kappa \to (\kappa^+, \kappa^+)^2$ holds, then there is a set $\Gamma \in [2^\kappa]^{\kappa^+}$ which is homogeneous for F. If $F(\{\alpha,\beta\}) = 0$ for all distinct $\alpha < \beta$ in Γ, then $\langle f_\alpha : \alpha \in \Gamma\rangle$ is a strictly increasing sequence of length o.t.(Γ), contradicting Theorem 17.4. A similar contradiction is reached if $F(\{\alpha,\beta\}) = 1$ for all distinct $\alpha < \beta$ in Γ. □

Corollary 20.8. *$\kappa^+ \not\to (\kappa^+, \kappa^+)^2$ for every infinite cardinal κ.*

Proof. Given $F : [\kappa^+]^2 \to 2$, extend F in any way to a function $G : [2^\kappa]^2 \to 2$. A homogeneous set for F yields a homogeneous set for G. So our corollary follows from Theorem 20.7. □

We can, however, generalize Ramsey's theorem in less obvious ways.

Theorem 20.9. (Dushnik-Miller) *If κ is an infinite regular cardinal, then $\kappa \to (\kappa, \omega)^2$.*

(This is also true for singular κ, but the proof is more complicated.)

Proof. Let $f : [\kappa]^2 \to 2$. Assume that

(1) For all $\Gamma \subseteq \kappa$, if $f[[\Gamma]^2] \subseteq \{0\}$, then $|\Gamma| < \kappa$.

Thus we want to find an infinite $\Delta \subseteq \kappa$ such that $f[[\Delta]^2] \subseteq \{1\}$. In order to do this, we will define by recursion subsets Θ_n, Ω_n of κ and elements α_n of κ, for all $n \in \omega$.

Let $\Omega_0 = \kappa$. Now suppose that Ω_n has been defined so that $|\Omega_n| = \kappa$; we define Ω_{n+1}, α_n, and Θ_n. Let Θ_n be a maximal subset of Ω_n such that $f[[\Theta_n]^2] \subseteq \{0\}$. Thus $|\Theta_n| < \kappa$ by (1). By this maximality,

$$\Omega_n \backslash \Theta_n = \bigcup_{\beta\in\Theta_n} \{\gamma \in \Omega_n\backslash\Theta_n : f(\{\beta,\gamma\}) = 1\}.$$

Hence since $|\Omega_n \backslash \Theta_n| = \kappa$, $|\Theta_n| < \kappa$, and κ is regular, there is an $\alpha_n \in \Theta_n$ such that $\Omega_{n+1} \stackrel{\text{def}}{=} \{\gamma \in \Omega_n \backslash \Theta_n : f(\{\alpha_n, \gamma\}) = 1\}$ has κ elements. This finishes the construction.

The following facts about this construction are clear:

(2) $\Omega_{n+1} \subseteq \Omega_n \backslash \Theta_n$;

(3) $\alpha_n \in \Theta_n \subseteq \Omega_n$;

(4) For all $\beta \in \Omega_{n+1}$ we have $f(\{\alpha_n, \beta\}) = 1$;

In addition:

(5) The α_n's are all distinct.

In fact, suppose that $i < n$. Then $\alpha_i \in \Theta_i$ and $\Theta_i \cap \Omega_{i+1} = \emptyset$. Since $\Omega_n \subseteq \Omega_{i+1}$, it follows that $\alpha_i \notin \Omega_n$ and so $\alpha_i \neq \alpha_n$. So (5) holds.

Now with $i < n$ we have $\alpha_n \in \Omega_n \subseteq \Omega_{i+1}$, and hence $f(\{\alpha_i, \alpha_n\}) = 1$. Thus $f[\{\alpha_n : n \in \omega\}]^2 \subseteq \{1\}$, as desired. $\square$

To formulate another generalization of Ramsey's theorem it is convenient to introduce a notation for a special form of the arrow notation. We write

$$\begin{aligned} &\kappa \to (\lambda)^\nu_\mu \quad \text{iff} \\ &\kappa \to (\langle \lambda : \alpha < \mu \rangle)^\nu \end{aligned}$$

In direct terms, then, $\kappa \to (\lambda)^\nu_\mu$ means that for every $f : [\kappa]^\nu \to \mu$ there is a $\Gamma \in [\kappa]^\lambda$ such that $|f[\Gamma]| = 1$.

The following cardinal notation is also needed for our next result: for any infinite cardinal κ we define

$$\begin{aligned} 2^\kappa_0 &= \kappa; \\ 2^\kappa_{n+1} &= 2^{(2^\kappa_n)} \quad \text{for all } n \in \omega. \end{aligned}$$

Theorem 20.10. (Erdös-Rado) *For every infinite cardinal κ and every positive integer n, $(2^\kappa_{n-1})^+ \to (\kappa^+)^n_\kappa$.*

Proof. Induction on n. For $n = 1$ we want to show that $\kappa^+ \to (\kappa^+)^1_\kappa$, and this is obvious. Now assume the statement for $n \geq 1$, and suppose that $f : [(2^\kappa_n)^+]^{n+1} \to \kappa$. For each $\alpha \in (2^\kappa_n)^+$ define $F_\alpha : [(2^\kappa_n)^+ \backslash \{\alpha\}]^n \to \kappa$ by setting $F_\alpha(x) = f(x \cup \{\alpha\})$.

(1) There is an $A \in [(2^\kappa_n)^+]^{2^\kappa_n}$ such that for all $C \in [A]^{2^\kappa_{n-1}}$ and all $u \in (2^\kappa_n)^+ \backslash C$ there is a $v \in A \backslash C$ such that $F_u \restriction [C]^n = F_v \restriction [C]^n$.

To prove this, we define a sequence $\langle A_\alpha : \alpha < (2^\kappa_{n-1})^+ \rangle$ of subsets of $(2^\kappa_n)^+$, each of size 2^κ_n. Let $A_0 = 2^\kappa_n$, and for α limit let $A_\alpha = \bigcup_{\beta < \alpha} A_\beta$. Now suppose that A_α has been defined, and $C \in [A_\alpha]^{2^\kappa_{n-1}}$. Define $u \equiv v$ iff $u, v \in (2^\kappa_n)^+ \backslash C$ and $F_u \restriction [C]^n = F_v \restriction [C]^n$. Now $|{}^{[C]^n}\kappa| = 2^\kappa_n$, so there are at most 2^κ_n equivalence classes. Let K_C have exactly one element in common with each equivalence class. Let $A_{\alpha+1} = A_\alpha \cup \{K_C : C \in [A_\alpha]^{2^\kappa_{n-1}}\}$.

Since $(2^\kappa_n)^{2^\kappa_{n-1}} = 2^\kappa_n$, we still have $|A_{\alpha+1}| = 2^\kappa_n$. This finishes the construction. Clearly $A \stackrel{\text{def}}{=} \bigcup_{\alpha\le(2^\kappa_{n-1})^+} A_\alpha$ is as desired in (1).

Take A as in (1), and fix $a \in (2^\kappa_n)^+\backslash A$. We now define a sequence $\langle x_\alpha : \alpha < (2^\kappa_{n-1})^+\rangle$ of elements of A. Given $C \stackrel{\text{def}}{=} \{x_\beta : \beta < \alpha\}$, by (1) let $x_\alpha \in A\backslash C$ be such that $F_{x_\alpha} \restriction [C]^n = F_a \restriction [C]^n$. This defines our sequence. Let $X = \{x_\alpha : \alpha < (2^\kappa_{n-1})^+\}$.

Now define $G : [X]^n \to \kappa$ by $G(x) = F_a(x)$. Suppose that $\alpha_0 < \cdots < \alpha_n < (2^\kappa_{n-1})^+$. Then

$$\begin{aligned} f(\{x_{\alpha_0},\ldots,x_{\alpha_n}\}) &= F_{x_{\alpha_n}}(\{x_{\alpha_0},\ldots,x_{\alpha_{n-1}}\}) \\ &= F_a(\{x_{\alpha_0},\ldots,x_{\alpha_{n-1}}\}) \\ &= G(\{x_{\alpha_0},\ldots,x_{\alpha_{n-1}}\}). \end{aligned}$$

Now by the inductive hypothesis there is an $H \in [X]^{\kappa^+}$ such that G is constant on $[H]^n$. By the above, f is constant on $[H]^{n+1}$. □

Corollary 20.11. $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ *for any infinite cardinal* κ. □

Theorem 20.12. *For any infinite cardinal* κ *we have* $2^\kappa \not\to (3)^2_\kappa$.

Proof. Define $F : [{}^\kappa 2]^2 \to \kappa$ by setting $F(\{f,g\}) = \chi(f,g)$ for any two distinct $f,g \in {}^\kappa 2$. If $\{f,g,h\}$ is homogeneous for F with f,g,h distinct, let $\alpha = \chi(f,g)$. Then $f(\alpha), g(\alpha), h(\alpha)$ are distinct members of 2, contradiction. □

Corollary 20.13. *For any infinite cardinal* κ *we have* $2^\kappa \not\to (\kappa^+)^2_\kappa$. □

Our final result in the partition calculus indicates that infinite exponents are in general hopeless.

Theorem 20.14. $\omega \not\to (\omega,\omega)^\omega$.

Proof. Let $<$ well-order $[\omega]^\omega$. We define for any $X \in [\omega]^\omega$

$$F(X) = \begin{cases} 0 & \text{if } Y < X \text{ for some } Y \in [X]^\omega, \\ 1 & \text{otherwise.} \end{cases}$$

Suppose that $H \in [\omega]^\omega$ is homogeneous for F. Let X be the $<$-least element of $[H]^\omega$. Thus $F(X) = 1$. So we must have $F(Y) = 1$ for all $Y \in [H]^\omega$. Write $H = \{m_i : i \in \omega\}$ without repetitions. For each $k \in \omega$ let

$$I_k = \{m_0, m_2, \ldots, m_{2k}\} \cup \{m_{2i+1} : i \in \omega\}.$$

Thus these are infinite subsets of H. Choose k_0 so that I_{k_0} is minimum among all of the I_k's. Then $I_{k_0} \subset I_{k_0+1}$ and $I_{k_0} < I_{k_0+1}$, so $F(I_{k_0+1}) = 0$, contradiction. □

We close this chapter with the following theorem of Comfort and Negrepontis.

Theorem 20.15. *Suppose that $\lambda = \lambda^{<\kappa}$. Then there is a system $\langle f_\alpha : \alpha < 2^\lambda \rangle$ of members of ${}^\lambda\lambda$ such that*

$$\forall M \in \left[2^\lambda\right]^{<\kappa} \forall g \in {}^M\lambda \exists \beta < \lambda \forall \alpha \in M[f_\alpha(\beta) = g(\alpha)].$$

Proof. Let

$$\mathscr{F} = \left\{(F, G, s) : F \in [\lambda]^{<\kappa},\ G \in [\mathscr{P}(F)]^{<\kappa},\ \text{and } s \in {}^G\lambda\right\}.$$

Now if $F \in [\lambda]^{<\kappa}$, say $|F| = \mu$, then

$$\left|[\mathscr{P}(F)]^{<\kappa}\right| \leq (2^\mu)^{<\kappa}| \leq (\lambda^\mu)^{<\kappa} \leq \lambda^{<\kappa} = \lambda,$$

and if $G \in [\mathscr{P}(F)]^{<\kappa}$ then $|{}^G\lambda| \leq \lambda^{<\kappa} = \lambda$. It follows that $|\mathscr{F}| = \lambda$. Let h be a bijection from λ onto $\mathscr{F}$, and let k be a bijection from 2^λ onto $\mathscr{P}(\lambda)$. Now for each $\alpha < 2^\lambda$ we define $f_\alpha \in {}^\lambda\lambda$ by setting, for each $\beta < \lambda$, with $h(\beta) = (F, G, s)$,

$$f_\alpha(\beta) = \begin{cases} s(k(\alpha) \cap F) & \text{if } k(\alpha) \cap F \in G, \\ 0 & \text{otherwise.} \end{cases}$$

Now to prove that $\langle f_\alpha : \alpha < 2^\lambda \rangle$ is as desired, suppose that $M \in [2^\lambda]^{<\kappa}$ and $g \in {}^M\lambda$. For distinct members α, β of M choose $\gamma(\alpha, \beta) \in k(\alpha) \triangle k(\beta)$. Then let

$$F = \{\gamma(\alpha, \beta) : \alpha, \beta \in M, \alpha \neq \beta\} \quad \text{and} \quad G = \{k(\alpha) \cap F : \alpha \in M\}.$$

Moreover, define $s : G \to \lambda$ by setting $s(k(\alpha) \cap F) = g(\alpha)$ for any $\alpha \in M$. This is possible since $k(\alpha) \cap F) \neq k(\beta) \cap F)$ for distinct $\alpha, \beta \in M$. Finally, let $\beta = h^{-1}(F, G, s)$. Then for any $\alpha \in M$ we have

$$f_\alpha(\beta) = s(k(\alpha) \cap F) = g(\alpha). \qquad \square$$

EXERCISES

E20.1. Suppose that $\kappa^\omega > \kappa$. Show that there is a family $\mathscr{A}$ of subsets of κ, each of size ω, with $|\mathscr{A}| = \kappa^+$ and the intersection of any two members of $\mathscr{A}$ is finite.

E20.2. Suppose that κ is any infinite cardinal, and λ is minimum such that $\kappa^\lambda > \kappa$. Show that there is a family $\mathscr{A}$ of subsets of κ, each of size λ, with the intersection of any two members of $\mathscr{A}$ being of size less than λ, and with $|\mathscr{A}| = \lambda^+$.

E20.3. Suppose that κ is uncountable and regular. Show that there is a family $\mathscr{A}$ of subsets of κ, each of size κ with the intersection of any two members of $\mathscr{A}$ of size less than κ, and with $|\mathscr{A}| = \kappa^+$. Hint: (1) show that there is a partition of κ into κ subsets, each of size κ; (2) Use Zorn's lemma to start from (1) and produce a maximal almost disjoint set; (3) Use a diagonal construction to show that the resulting family must have size $> \kappa$.

E20.4. Prove that if $\mathscr{F}$ is an uncountable family of finite functions each with range $\subseteq \omega$, then there are distinct $f, g \in \mathscr{F}$ such that $f \cup g$ is a function.

E20.5. (Double Δ-system theorem) Suppose that κ is a singular cardinal with $\mathrm{cf}(\kappa) > \omega$. Let $\langle \lambda_\alpha : \alpha < \mathrm{cf}(\kappa) \rangle$ be a strictly increasing sequence of successor cardinals with supremum κ, with $\mathrm{cf}(\kappa) < \lambda_0$, and such that for each $\alpha < \mathrm{cf}(\kappa)$ we have $(\sum_{\beta<\alpha} \lambda_\beta)^+ \leq \lambda_\alpha$. Suppose that $\langle A_\xi : \xi < \kappa \rangle$ is a system of finite sets. Then there exist a set $\Gamma \in [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$, a sequence $\langle \Xi_\alpha : \alpha \in \Gamma \rangle$ of subsets of κ, a sequence $\langle F_\alpha : \alpha \in \Gamma \rangle$ of finite sets, and a finite set G, such that the following conditions hold:

(i) $\langle \Xi_\alpha : \alpha \in \Gamma \rangle$ is a pairwise disjoint system, and $|\Xi_\alpha| = \lambda_\alpha$ for every $\alpha \in \Gamma$.

(ii) $\langle A_\xi : \xi \in \Xi_\alpha \rangle$ is a Δ-system with root F_α for every $\alpha \in \Gamma$.

(iii) $\langle F_\alpha : \alpha \in \Gamma \rangle$ is a Δ-system with root G.

(iv) If $\xi \in \Xi_\alpha$, $\eta \in \Xi_\beta$, and $\alpha \neq \beta$, then $A_\xi \cap A_\eta = G$.

E20.6. Suppose that $\mathscr{F}$ is a collection of countable functions, each with range $\subseteq 2^\omega$, and with $|\mathscr{F}| = (2^\omega)^+$. Show that there are distinct $f, g \in \mathscr{F}$ such that $f \cup g$ is a function.

E20.7. For any infinite cardinal κ, any linear order of size at least $(2^\kappa)^+$ has a subset of order type κ^+ or one similar to $(\kappa^+, >)$.

E20.8. For any infinite cardinal κ, any tree of size at least $(2^\kappa)^+$ has a branch or an antichain of size at least κ^+.

E20.9. Any uncountable tree either has an uncountable branch or an infinite antichain.

E20.10. Suppose that m is a positive integer. Show that any infinite set X of positive integers contains an infinite subset Y such that one of the following conditions holds:

(i) The members of Y are pairwise relatively prime.

(ii) There is a prime $p < m$ such that for any two $a, b \in Y$, p divides $a - b$.

(iii) If a, b are distinct members of Y, then a, b are not relatively prime, but the smallest prime divisor of $a - b$ is at least equal to m.

E20.11. Suppose that X is an infinite set, and $(X, <)$ and $(X, \prec)$ are two well-orderings of X. Show that there is an infinite subset Y of X such that for all $y, z \in Y$, $y < z$ iff $y \prec z$.

E20.12. Let S be an infinite set of points in the plane. Show that S has an infinite subset T such that all members of T are on the same line, or else no three distinct points of T are collinear.

E20.13. We consider the following variation of the arrow relation. For cardinals $\kappa, \lambda, \mu, \nu$, we define

$$\kappa \to [\lambda]^\mu_\nu$$

to mean that for every function $f : [\kappa]^\mu \to \nu$ there exist an $\alpha < \nu$ and a $\Gamma \in [\kappa]^\lambda$ such that $f[[\Gamma]^\mu] \subseteq \nu \backslash \{\alpha\}$. In coloring terminology, we color the μ-element subsets of κ with ν colors, and then there is a set which is anti-homogeneous for f, in the sense that there is a color α and a subset of size λ all of whose μ-element subsets *do not* get the color α.

Prove that for any infinite cardinal κ,

$$\kappa \not\to [\kappa]^\kappa_{2^\kappa}.$$

Hint: (1) Show that there is an enumeration $\langle X_\alpha : \alpha < 2^\kappa \rangle$ of $[\kappa]^\kappa$ in which every member of $[\kappa]^\kappa$ is repeated 2^κ times. (2) Show that $|[\kappa]^\kappa| = 2^\kappa$. (3) Show that there is a one-one

$\langle Y_\alpha : \alpha < 2^\kappa \rangle$ such that $Y_\alpha \in [X_\alpha]^\kappa$ for all $\alpha < 2^\kappa$. (4) Define $f : [\kappa]^\kappa \to 2^\kappa$ so that for all $\alpha < 2^\kappa$ one has

$$f(Y_\alpha) = \text{o.t.}(\{\beta < \alpha : X_\beta = X_\alpha\}).$$

Reference

Erdös, P.; Hajnal, A.; Máté, A.; Rado, R. **Combinatorial set theory: partition relations for cardinals.** Akad. Kiadó, Budapest 1984, 347pp.

21. Martin's axiom

Martin's axiom is not an axiom of ZFC, but it can be added to those axioms. It has many important consequences. Actually, the continuum hypothesis implies Martin's axiom, so it is of most interest when combined with the negation of the continuum hypothesis. The consistency of $\mathrm{MA}+\neg\mathrm{CH}$ involves iterated forcing, and is prove much later in these notes.

• For any infinite cardinal κ, the notation $\mathrm{MA}(\kappa)$ abbreviates the statement that for any ccc partial order $\mathbb{P}$ and any family $\mathscr{D}$ of dense sets in $\mathbb{P}$, with $|\mathscr{D}|\le\kappa$, there is a filter G on $\mathbb{P}$ such that $G\cap D\neq\emptyset$ for every $D\in\mathscr{D}$.

• *Martin's axiom*, abbreviated MA, is the statement that $\mathrm{MA}(\kappa)$ holds for every infinite $\kappa<2^\omega$.

Clearly if $\kappa<\lambda$ and $\mathrm{MA}(\lambda)$, then also $\mathrm{MA}(\kappa)$.

Theorem 21.1. $\mathrm{MA}(\omega)$ *holds.*

Proof. Let $\mathbb{P}$ be a ccc partial order and $\mathscr{D}$ a countable collection of dense sets in $\mathbb{P}$. If $\mathscr{D}$ is empty, we can fix any $p\in P$ and let $G=\{q\in\mathbb{P}:p\le q\}$. Then G is a filter on $\mathbb{P}$, which is all that is required in this case.

Now suppose that $\mathscr{D}$ is nonempty, and let $\langle D_n:n\in\omega\rangle$ enumerate all the members of $\mathscr{D}$; repetitions are needed if $\mathscr{D}$ is finite. We now define a sequence $\langle p_n:n\in\omega\rangle$ of elements of P by recursion. Let p_0 be any element of P. If p_n has been defined, by the denseness of D_n let p_{n+1} be such that $p_{n+1}\le p_n$ and $p_{n+1}\in D_n$. This finishes the construction. Let $G=\{q\in P:p_n\le q \text{ for some } n\in\omega\}$. Clearly G is as desired. $\square$

Note that ccc was not used in this proof.

Corollary 21.2. CH *implies* MA. $\square$

Theorem 21.3. $\mathrm{MA}(2^\omega)$ *does not hold.*

Proof. Suppose that it does hold. Let

$$\begin{aligned}
P&=\{f:f\text{ is a finite function with }\mathrm{dmn}(f)\subseteq\omega\text{ and }\mathrm{rng}(f)\subseteq 2\};\\
f\le g&\text{ iff }f,g\in P\text{ and }f\supseteq g;\\
\mathbb{P}&=(P,\le).
\end{aligned}$$

Then $\mathbb{P}$ has ccc, since P itself is countable. Now for each $n\in\omega$ let

$$D_n=\{f\in P:n\in\mathrm{dmn}(f)\}.$$

Each such set is dense in $\mathbb{P}$. For, if $g\in P$, either g is already in D_n, or $n\notin\mathrm{dmn}(g)$, and then $g\cup\{(n,0)\}$ is in D_n and it is $\le g$.

For each $h\in{}^\omega 2$ let

$$E_h=\{f\in P:\text{there is an }n\in\mathrm{dmn}(f)\text{ such that }f(n)\neq h(n)\}.$$

Again, each such set E_h is dense in $\mathbb{P}$. For, let $f \in P$. If $f \not\subseteq h$, then already $f \in E_h$, so suppose that $f \subseteq h$. Take any $n \in \omega \backslash \mathrm{dmn}(f)$, and let $g = f \cup \{(n, 1 - h(n))\}$. Then $g \in E_h$ and $g \leq f$, as desired.

So, by MA(2^ω) let G be a filter on $\mathbb{P}$ which intersects each of the sets D_n and E_h. Let $k = \bigcup G$.

(*) $k : \omega \to \omega$.

In fact, k is obviously a relation. Suppose that $(m, \varepsilon), (m, \delta) \in k$. Choose $f, g \in G$ such that $(m, \varepsilon) \in f$ and $(m, \delta) \in g$. Then choose $s \in G$ such that $s \leq f, g$. So $f, g \subseteq s$, and s is a function. It follows that $\varepsilon = \delta$. Thus k is a function.

If $n \in \omega$, choose $f \in G \cap D_n$. So $n \in \mathrm{dmn}(f)$, and so $n \in \mathrm{dmn}(k)$. So we have proved $(*)$.

Now take any $f \in G \cap E_k$. Choose $n \in \mathrm{dmn}(f)$ such that $f(n) \neq k(n)$. But $f \subseteq k$, contradiction. □

There is one more fact concerning the definition of MA which should be mentioned. Namely, for $\kappa > \omega$ the assumption of ccc is essential in the statement of MA(κ). (Recall our comment above that ccc is not needed in order to prove that MA(ω) holds.) To see this, define

$$
\begin{aligned}
P &= \{f : f \text{ is a finite function, } \mathrm{dmn}(f) \subseteq \omega, \text{ and } \mathrm{rng}(f) \subseteq \omega_1\};\\
f \leq g &\text{ iff } f, g \in P \text{ and } f \supseteq g;\\
\mathbb{P} &= (P, \leq).
\end{aligned}
$$

This example is similar to two of the partial orders above. Note that $\mathbb{P}$ does not have ccc, since for example $\{\{(0, \alpha)\} : \alpha < \omega_1\}$ is an uncountable antichain. Defining D_n as in the proof of Theorem 21.3, we clearly get dense subsets of $\mathbb{P}$. Also, for each $\alpha < \omega_1$ let

$$F_\alpha = \{f \in P : \alpha \in \mathrm{rng}(f)\}.$$

Then F_α is dense in $\mathbb{P}$. For, suppose that $g \in P$. If $\alpha \in \mathrm{rng}(g)$, then g itself is in F_α, so suppose that $\alpha \notin \mathrm{rng}(g)$. Choose $n \in \omega \backslash \mathrm{dmn}(g)$. Let $f = g \cup \{(n, \alpha)\}$. Then $f \in F_\alpha$ and $f \leq g$, as desired. Now if MA(ω_1) holds without the assumption of ccc, then we can apply it to our present partial order. Suppose that G is a filter on $\mathbb{P}$ which intersects each of these sets D_n and F_α. As in the proof of Theorem 21.3, $k \stackrel{\text{def}}{=} \bigcup G$ is a function mapping ω into ω_1. For any $\alpha < \omega_1$ choose $f \in G \cap F_\alpha$. Thus $\alpha \in \mathrm{rng}(f)$, and so $\alpha \in \mathrm{rng}(k)$. Thus k has range ω_1. This is impossible.

Now we proceed beyond the discussion of the definition of MA in order to give several typical applications of it. First we consider again almost disjoint sets of natural numbers. Our result here will be used to derive some important implications of MA for cardinal arithmetic. We proved in Theorem 20.1 that there is a family of size 2^ω of almost disjoint sets of natural numbers. Considering this further, we may ask what the size of maximal almost disjoint families can be; and we may consider the least such size. This is one of many min-max questions concerning the natural numbers which have been considered recently. There are many consistency results saying that numbers of this sort can be less than 2^ω; in particular, it is consistent that there is a maximal family of almost disjoint subsets of ω which has size less than 2^ω. MA, however, implies that this size, and most of the similarly defined min-max functions, is 2^ω.

Let $\mathscr{A} \subseteq \mathscr{P}(\omega)$. The *almost disjoint partial order for* $\mathscr{A}$ is defined as follows:

$$P_{\mathscr{A}} = \{(s, F) : s \in [\omega]^{<\omega} \text{ and } F \in [\mathscr{A}]^{<\omega}\};$$
$$(s', F') \leq (s, F) \text{ iff } s \subseteq s',\ F \subseteq F', \text{ and } x \cap s' \subseteq s \text{ for all } x \in F;$$
$$\mathbb{P}_{\mathscr{A}} = (P_{\mathscr{A}}, \leq).$$

We give some useful properties of this construction.

Lemma 21.4. *Let $\mathscr{A} \subseteq \mathscr{P}(\omega)$.*

(i) $\mathbb{P}_{\mathscr{A}}$ is a partial order.

(ii) Let $(s, F), (s', F') \in P_{\mathscr{A}}$. Then the following conditions are equivalent:

(a) (s, F) and (s', F') are compatible.

(b) $\forall x \in F(x \cap s' \subseteq s)$ and $\forall x \in F'(x \cap s \subseteq s')$.

(c) $(s \cup s', F \cup F') \leq (s, F), (s', F')$.

(iii) Suppose that $x \in \mathscr{A}$, and let $D_x = \{(s, F) \in P_{\mathscr{A}} : x \in F\}$. Then D_x is dense in $\mathbb{P}_{\mathscr{A}}$.

(iv) $\mathbb{P}_{\mathscr{A}}$ has ccc.

Proof. (i): Clearly $\leq$ is reflexive on $P_{\mathscr{A}}$ and it is antisymmetric, i.e. $(s, F) \leq (s', F') \leq (s, F)$ implies that $(s, F) = (s', F')$. Now suppose that $(s'', F'') \leq (s', F') \leq (s, F)$. Thus $s \subseteq s' \subseteq s''$, so $s \subseteq s''$. Similarly, $F \subseteq F''$. Now take any $x \in F$. Then $x \in F'$, so $x \cap s'' \subseteq s'$ because $(s'', F'') \leq (s', F')$. Hence $x \cap s'' \subseteq x \cap s'$. And $x \cap s' \subseteq s$ because $(s', F') \leq (s, F)$. So $x \cap s'' \subseteq s$, as desired.

(ii): For (a)⇒(b), assume (a). Choose $(s'', F'') \leq (s, F), (s', F')$. Now take any $x \in F$. Then $x \cap s' \subseteq x \cap s''$ since $s' \subseteq s''$, and $x \cap s'' \subseteq s$ since $(s'', F'') \leq (s, F)$; so $x \cap s' \subseteq s''$. The other part of (b) follows by symmetry.

(b)⇒(c): By symmetry it suffices to show that $(s \cup s', F \cup F') \leq (s, F)$, and for this we only need to check the last condition in the definition of $\leq$. So, suppose that $x \in F$. Then $x \cap (s \cup s') = (x \cap s) \cup (x \cap s') \subseteq s$ by (b).

(c)⇒(a): Obvious.

(iii): For any $(s, F) \in P_{\mathscr{A}}$, clearly $(s, F \cup \{x\}) \leq (s, F)$.

(iv) Suppose that $\langle (s_\xi, F_\xi) : \xi < \omega_1 \rangle$ is a pairwise incompatible system of elements of $P_{\mathscr{A}}$. Clearly then $s_\xi \neq s_\eta$ for distinct $\xi, \eta < \omega_1$, contradiction. □

Theorem 21.5. *Let κ be an infinite cardinal, and assume* MA(κ)*. Suppose that $\mathscr{A}, \mathscr{B} \subseteq \mathscr{P}(\omega)$, and $|\mathscr{A}|, |\mathscr{B}| \leq \kappa$. Also assume that*

(i) For all $y \in \mathscr{B}$ and all $F \in [\mathscr{A}]^{<\omega}$ we have $|y \backslash \bigcup F| = \omega$.

Then there is a $d \subseteq \omega$ such that $|d \cap x| < \omega$ for all $x \in \mathscr{A}$ and $|d \cap y| = \omega$ for all $y \in \mathscr{B}$.

Proof. For each $y \in \mathscr{B}$ and each $n \in \omega$ let

$$E^y_n = \{(s, F) \in \mathbb{P}_{\mathscr{A}} : s \cap y \not\subseteq n\}.$$

We claim that each such set is dense. For, suppose that $(s, F) \in \mathbb{P}_{\mathscr{A}}$. Then by assumption, $|y \backslash \bigcup F| = \omega$, so we can pick $m \in y \backslash \bigcup F$ such that $m > n$. Then $(s \cup \{m\}, F) \leq (s, F)$, since for each $z \in F$ we have $z \cap (s \cup \{m\}) \subseteq s$ because $m \notin z$. Also, $m \in y \backslash n$, so $(s \cup \{m\}) \in E^y_n$. This proves our claim.

There are clearly at most κ sets E^y_n; and also there are at most κ sets D_x with $x \in \mathscr{A}$, with D_x as in Lemma 21.4(iii). Hence by MA(κ) we can let G be a filter on $\mathbb{P}_{\mathscr{A}}$ intersecting all of these dense sets. Let $d = \bigcup_{(s,F) \in G} s$.

(1) For all $x \in \mathscr{A}$, the set $d \cap x$ is finite.

For, by the denseness of D_x, choose $(s, F) \in G \cap D_x$. Thus $x \in F$. We claim that $d \cap x \subseteq s$. To prove this, suppose that $n \in d \cap x$. Choose $(s', F') \in G$ such that $n \in s'$. Now (s, F) and (s', F') are compatible. By Lemma 21.4(ii), $\forall y \in F (y \cap s' \subseteq s)$; in particular, $x \cap s' \subseteq s$. Since $n \in x \cap s'$, we get $n \in s$. This proves our claim, and so (1) holds.

The proof will be finished by proving

(2) For all $y \in \mathscr{B}$, the set $d \cap y$ is infinite.

To prove (2), given $n \in \omega$ choose $(s, F) \in E^y_n \cap G$. Thus $s \cap y \nsubseteq n$, so we can choose $m \in s \cap y \backslash n$. Hence $m \in d \cap y \backslash n$, proving (2). $\square$

Corollary 21.6. *Let κ be an infinite cardinal and assume* MA(κ). *Suppose that $\mathscr{A} \subseteq \mathscr{P}(\omega)$ is an almost disjoint set of infinite subsets of ω of size κ. Then $\mathscr{A}$ is not maximal.*

Proof. If F is a finite subset of $\mathscr{A}$, then we can choose $a \in \mathscr{A} \backslash F$; then $a \cap \bigcup F = \bigcap_{b \in F} (a \cap b)$ is finite. Thus $\omega \backslash \bigcup F$ is infinite. Hence we can apply Theorem 21.5 to $\mathscr{A}$ and $\mathscr{B} \stackrel{\text{def}}{=} \{\omega\}$ to obtain the desired result. $\square$

Corollary 21.7. *Assuming MA, every maximal almost disjoint set of infinite sets of natural numbers has size 2^ω.* $\square$

Lemma 21.8. *Suppose that $\mathscr{B} \subseteq \mathscr{P}(\omega)$ is an almost disjoint family of infinite sets, and $|\mathscr{B}| = \kappa$, where $\omega \leq \kappa < 2^\omega$. Also suppose that $\mathscr{A} \subseteq \mathscr{B}$. Assume* MA($\kappa$).

Then there is a $d \subseteq \omega$ such that $|d \cap x| < \omega$ for all $x \in \mathscr{A}$ and $|d \cap x| = \omega$ for all $x \in \mathscr{B} \backslash \mathscr{A}$.

Proof. We apply 21.5 with $\mathscr{B} \backslash \mathscr{A}$ in place of $\mathscr{B}$. If $y \in \mathscr{B} \backslash \mathscr{A}$ and $F \in [\mathscr{A}]^{<\omega}$, then $y \cup F \subseteq \mathscr{B}$, and hence $y \cap z$ is finite for all $y \in F$. Hence also $y \cap \bigcup F$ is finite. Since y itself is infinite, it follows that $y \backslash \bigcup F$ is infinite.

Thus the hypotheses of 21.5 hold, and it then gives the desired result. $\square$

We now come to two of the most striking consequences of Martin's axiom.

Theorem 21.9. *If κ is an infinite cardinal and* MA(κ) *holds, then $2^\kappa = 2^\omega$.*

Proof. By Theorem 20.1 let $\mathscr{B}$ be an almost disjoint family of infinite subsets of ω such that $|\mathscr{B}| = \kappa$. For each $d \subseteq \omega$ let $F(d) = \{b \in \mathscr{B} : |b \cap d| < \omega\}$. We claim that F maps

$\mathscr{P}(\omega)$ onto $\mathscr{P}(\mathscr{B})$; from this it follows that $2^\kappa \le 2^\omega$, hence $2^\kappa = 2^\omega$. To prove the claim, suppose that $\mathscr{A} \subseteq \mathscr{B}$. A suitable d with $F(d) = \mathscr{A}$ is then given by Lemma 21.8. $\square$

Corollary 21.10. MA *implies that* 2^ω *is regular.*

Proof. Assume MA, and suppose that $\omega \le \kappa < 2^\omega$. Then $2^\kappa = 2^\omega$ by Theorem 21.9, and so $\mathrm{cf}(2^\omega) = \mathrm{cf}(2^\kappa) > \kappa$ by Corollary 12.25. $\square$

Another important application of Martin's axiom is to the existence of Suslin trees; in fact, Martin's axiom arose out of the proof of this theorem:

Theorem 21.11. $\mathrm{MA}(\omega_1)$ *implies that there are no Suslin trees.*

Proof. Suppose that $(T, \le)$ is a Suslin tree. By 18.7 and the remarks before it, we may assume that T is well-pruned. We are going to apply $\mathrm{MA}(\omega_1)$ to the partial order $(T, \ge)$, i.e., to T turned upside down. Because T has no uncountable antichains in the tree sense, $(T, \ge)$ has no uncountable antichains in the incompatibility sense. Now for each $\alpha < \omega_1$ let

$$D_\alpha = \{t \in T : \mathrm{ht}(t, T) > \alpha\}.$$

Then each D_α is dense in $(T, \ge)$. For, suppose that $s \in T$. By well-prunedness, choose $t \in T$ such that $s < t$ and $\mathrm{ht}(t, T) > \alpha$. Thus $t \in D_\alpha$ and $t > s$, as desired.

Now we let G be a filter on $(T, \ge)$ which intersects each D_α. Any two elements of G are compatible in $(T, \ge)$, so they are comparable in $(T, \le)$. Since $G \cap D_\alpha \ne \emptyset$ for all $\alpha < \omega_1$, G has a member of T of height greater than α, for each $\alpha < \omega_1$. Hence G is an uncountable chain, contradiction. $\square$

Our last application of Martin's axiom involves Lebesgue measure. In order not to assume too much about measures, we give some results of measure theory that will be used in our application but may have been omitted in your standard study of measure theory.

Lemma 21.12. *Suppose that* μ *is a measure and* E, F, G *are* μ*-measurable. Then*

$$\mu(E \triangle F) \le \mu(E \triangle G) + \mu(G \triangle F).$$

Proof.

$$\begin{aligned}
\mu(E \triangle F) &= \mu(E\backslash F) + \mu(F\backslash E)\\
&= \mu((E\backslash F) \cap G) + \mu((E\backslash F)\backslash G) + \mu(F\backslash E) \cap G) + \mu((F\backslash E)\backslash G)\\
&\le \mu(G\backslash F) + \mu(E\backslash G) + \mu(G\backslash E) + \mu(F\backslash G)\\
&= \mu(E \triangle G) + \mu(G \triangle F).
\end{aligned}$$

$\square$

Lemma 21.13. *If* E *is Lebesgue measurable with finite measure, then for any* $\varepsilon > 0$ *there is an open set* $U \supseteq E$ *such that* $\mu(E) \le \mu(U) \le \mu(E) + \varepsilon$*. Moreover, there is a system* $\langle K_j : j < \omega\rangle$ *of open intervals such that* $U = \bigcup_{j<\omega} K_j$ *and* $\mu(U) \le \sum_{j<\omega} \mu(K_j) \le \mu(E) + \varepsilon$*.*

Proof. By the basic definition of Lebesgue measure,

$$\mu(E) = \inf\left\{\sum_{j\in\omega}\mu(I_j) : \langle I_j : j\in\omega\rangle \text{ is a sequence of half-open intervals}\right.$$
$$\left.\text{such that } E\subseteq\bigcup_{j\in\omega} I_j\right\}.$$

Hence we can choose a sequence $\langle I_j : j\in\omega\rangle$ of half-open intervals such that $E\subseteq\bigcup_{j\in\omega} I_j$ and

$$\mu\left(\bigcup_{j\in\omega} I_j\right) \le \sum_{j\in\omega}\mu(I_j) \le \mu(E) + \frac{\varepsilon}{2}.$$

Write $I_j = [a_j, b_j)$ with $a_j < b_j$. Define

$$\begin{aligned}
K_j &= \left(a_j - \frac{\varepsilon}{2^{j+2}}, b_j\right); \quad \text{then}\\
E &\subseteq \bigcup_{j\in\omega} K_j \quad \text{and}\\
\mu\left(\bigcup_{j\in\omega} K_j\right) &\le \sum_{j\in\omega}\mu(K_j)\\
&= \sum_{j\in\omega}\left(\frac{\varepsilon}{2^{j+2}} + \mu(I_j)\right)\\
&= \sum_{j\in\omega}\frac{\varepsilon}{2^{j+2}} + \sum_{j\in\omega}\mu(I_j)\\
&\le \frac{\varepsilon}{2} + \mu(E) + \frac{\varepsilon}{2} = \mu(E) + \varepsilon.
\end{aligned}$$ □

The following is an elementary lemma concerning the topology of the reals.

Lemma 21.14. *Suppose that U is a bounded open set.*

(i) There is a collection $\mathscr{A}$ of pairwise disjoint open intervals such that $U = \bigcup\mathscr{A}$.

(ii) There exist a countable subset C of $\mathbb{R}$ and a collection $\mathscr{B}$ of pairwise disjoint open intervals with rational endpoints such that $U = C\cup\bigcup\mathscr{B}$ and $C\cap\bigcup\mathscr{B} = \emptyset$.

Proof. (i): For $x, y\in\mathbb{R}$, define $x\equiv y$ iff one of the following conditions holds: (1) $x = y$; (2) $x < y$ and $[x, y]\subseteq U$; (3) $y < x$ and $[y, x]\subseteq U$. Clearly $\equiv$ is an equivalence relation on $\mathbb{R}$. If $x < z < y$ and $x\equiv y$, then obviously $x\equiv z$. Thus each equivalence class is convex. If C is an equivalence class with more than one element, then it must be an open interval (a, b), since if for example the left endpoint a is in C then some real to the left of a must be in C, contradiction. It follows now that the collection $\mathscr{A}$ of all equivalence classes with more than one element is as desired in (i).

(ii): First note that the set $\mathscr{A}$ of (i) must be countable. Now take any $(a,b)\in\mathscr{A}$, $a<b$. Let $c_0<c_1<\cdots<c_m<\cdots$ be rational numbers in (a,b) which converge to b, and $c_0=d_0>d_1>\cdots>d_m>\cdots$ rational numbers which converge to a. Then let $L^{ab}_{2i}=(c_i,c_{i+1})$ and $L^{ab}_{2i+1}=(d_{i+1},d_i)$ for all $i\in\omega$. Let $D^{ab}=\{c_i:i<\omega\}\cup\{d_i:i<\omega\}$. Define $\mathscr{B}=\{L^{ab}_i:(a,b)\in\mathscr{A},i<\omega\}$ and $C=\bigcup_{(a,b)\in\mathscr{A}}D^{ab}$. Clearly this works for (ii). □

Lemma 21.15. *If E is Lebesgue measurable and $\varepsilon>0$, then there is an $m\in\omega$ and a sequence $\langle I_i:i<m\rangle$ of open intervals with rational endpoints such that $\mu\left(E\triangle\bigcup_{i<m}I_i\right)\le\varepsilon$.*

Proof. By Lemma 21.13 let $U\supseteq E$ be open such that $\mu(E)\le\mu(U)\le\mu(E)+\frac{\varepsilon}{2}$. Then choose C and $\mathscr{B}$ as in Lemma 21.14(ii). Let $W=\bigcup\mathscr{B}$. So $\mu(W)=\sum_{I\in\mathscr{B}}\mu(I)$. Then choose $m\in\omega$ and $\langle I_i:i<m\rangle$ elements of $\mathscr{B}$ such that $\sum_{I\in\mathscr{B}}\mu(I)-\sum_{i<m}\mu(I_i)\le\frac{\varepsilon}{2}$. Now $\mu(W)=\sum_{I\in\mathscr{B}}\mu(I)$ and $\mu(\bigcup_{i<m}I_i)=\sum_{i<m}\mu(I_i)$. Let $V=\bigcup_{i<m}I_i$. Thus $\mu(W)-\mu(V)\le\frac{\varepsilon}{2}$. Hence $V\subseteq W\subseteq U$, and

$$\begin{aligned}\mu(E\triangle V)&\le\mu(E\triangle U)+\mu(U\triangle W)+\mu(W\triangle V)\\&=\mu(U\backslash E)+\mu(C)+\mu(W\backslash V)\\&=\mu(U)-\mu(E)+\mu(W)-\mu(V)\\&\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$ □

Now we are ready for an application of Martin's axiom to Lebesgue measure.

Theorem 21.16. *Suppose that κ is an infinite cardinal and $\mathrm{MA}(\kappa)$ holds. If $\langle M_\alpha:\alpha<\kappa\rangle$ is a system of subsets of $\mathbb{R}$ each of Lebesgue measure 0, then also $\bigcup_{\alpha<\kappa}M_\alpha$ has Lebesgue measure 0.*

Proof. Let $\varepsilon>0$. We are going to find an open set U such that $\bigcup_{\alpha<\kappa}M_\alpha\subseteq U$ and $\mu(U)\le\varepsilon$; this will prove our result. Let

$$\mathbb{P}=\{p\subseteq\mathbb{R}:p\text{ is open and }\mu(p)<\varepsilon\}.$$

The ordering, as usual, is $\supseteq$.

(1) Elements $p,q\in\mathbb{P}$ are compatible iff $\mu(p\cup q)<\varepsilon$.

In fact, the direction $\Leftarrow$ is clear, while if p and q are compatible, then there is an $r\in\mathbb{P}$ with $r\supseteq p,q$, hence $p\cup q\subseteq r$ and $\mu(r)<\varepsilon$, hence $\mu(p\cup q)<\varepsilon$.

Next we check that $\mathbb{P}$ has ccc. Suppose that $\langle p_\alpha:\alpha<\omega_1\rangle$ is a system of pairwise incompatible elements of $\mathbb{P}$. Now

$$\omega_1=\bigcup_{n\in\omega}\left\{\alpha<\omega_1:\mu(p_\alpha)\le\varepsilon-\frac{1}{n+1}\right\},$$

so there exist an uncountable $\Gamma\subseteq\omega_1$ and a positive integer m such that $\mu(p_\alpha)\le\varepsilon-\frac{1}{m}$ for all $\alpha\in\Gamma$. Let $\mathscr{C}$ be the collection of all finite unions of open intervals with rational coefficients. Note that $\mathscr{C}$ is countable. By Lemma 21.15, for each $\alpha \in \Gamma$ let C_α be a member of $\mathscr{C}$ such that $\mu(p_\alpha \triangle C_\alpha) \leq \frac{1}{3m}$. Now take any two distinct members $\alpha, \beta \in \Gamma$. Then

$$\varepsilon \leq \mu(p_\alpha \cup p_\beta) = \mu(p_\alpha \cap p_\beta) + \mu(p_\alpha \triangle p_\beta) \leq \varepsilon - \frac{1}{m} + \mu(p_\alpha \triangle p_\beta),$$

and hence $\mu(p_\alpha \triangle p_\beta) \geq \frac{1}{m}$. Thus, using Lemma 21.12,

$$\frac{1}{m} \leq \mu(p_\alpha \triangle p_\beta) \leq \mu(p_\alpha \triangle C_\alpha) + \mu(C_\alpha \triangle C_\beta) + \mu(C_\beta \triangle p_\beta) \leq \frac{1}{3m} + \mu(C_\alpha \triangle C_\beta) + \frac{1}{3m};$$

Hence $\mu(C_\alpha \triangle C_\beta) \geq \frac{1}{3m}$. It follows that $C_\alpha \neq C_\beta$. But this means that $\langle C_\alpha : \alpha \in \Gamma \rangle$ is a one-one system of members of $\mathscr{C}$, contradiction. So $\mathbb{P}$ has ccc.

Now for each $\alpha < \kappa$ let

$$D_\alpha = \{p \in \mathbb{P} : M_\alpha \subseteq p\}.$$

To show that D_α is dense, take any $p \in \mathbb{P}$. Thus $\mu(p) < \varepsilon$. By Lemma 21.13, let U be an open set such that $M_\alpha \subseteq U$ and $\mu(U) < \varepsilon - \mu(p)$. Then $\mu(p \cup U) \leq \mu(p) + \mu(U) < \varepsilon$; so $p \cup U \in D_\alpha$ and $p \cup U \supset p$, as desired.

Now let G be a filter on $\mathbb{P}$ which intersects each D_α. Set $V = \bigcup G$. So V is an open set. For each $\alpha < \kappa$, choose $p_\alpha \in G \cap D_\alpha$. Then $M_\alpha \subseteq p_\alpha \subseteq V$. It remains only to show that $\mu(V) \leq \varepsilon$. Let $\mathscr{B}$ be the set of all open intervals with rational endpoints. We claim that $V = \bigcup(G \cap \mathscr{B})$. In fact, $\supseteq$ is clear, so suppose that $x \in V$. Then $x \in p$ for some $p \in G$, hence there is a $U \in \mathscr{B}$ such that $x \in U \subseteq p$, since p is open. Then $U \in G$ since G is a filter and the partial order is $\supseteq$. So we found a $U \in G \cap \mathscr{B}$ such that $x \in U$; hence $x \in \bigcup(G \cap \mathscr{B})$. This proves our claim. Now if F is a finite subset of G, then $\bigcup F \in G$ since G is a filter. In particular, $\bigcup F \in \mathbb{P}$, so its measure is less than ε. Now $G \cap \mathscr{B}$ is countable; let $\langle p_i : i \in \omega \rangle$ enumerate it. Define $q_i = p_i \backslash \bigcup_{j<i} p_j$ for all $i \in \omega$. Then by induction one sees that $\bigcup_{i<m} p_i = \bigcup_{i<m} q_i$, and hence $\bigcup(G \cap \mathscr{B}) = \bigcup_{i<\omega} q_i$. So

$$\begin{aligned}\mu(V) &= \mu\left(\bigcup(G \cap \mathscr{B})\right) = \mu\left(\bigcup_{i<\omega} q_i\right)\\ &= \sum_{i<\omega} \mu(q_i) = \lim_{m\to\infty} \sum_{i<m} \mu(q_i) = \lim_{m\to\infty} \mu\left(\bigcup_{i<m} q_i\right) = \lim_{m\to\infty} \mu\left(\bigcup_{i<m} p_i\right) \leq \varepsilon. \qquad \square\end{aligned}$$

EXERCISES

E21.1. Assume MA(κ). Suppose that X is a compact Hausdorff space, and any pairwise disjoint collection of open sets in X is countable. Suppose that U_α is dense open in X for each $\alpha < \kappa$. Show that $\bigcap_{\alpha<\kappa} U_\alpha \neq \emptyset$.

E21.2. A partial order P is said to have ω_1 as a *precaliber* iff for every system $\langle p_\alpha : \alpha < \omega_1 \rangle$ of elements of P there is an $X \in [\omega_1]^{\omega_1}$ such that for every finite subset F of X there is a $q \in P$ such that $q \leq p_\alpha$ for all $\alpha \in F$.

Show that MA(ω_1) implies that every ccc partial order P has ω_1 as a precaliber.

Hint: for each $\alpha < \omega_1$ let

$$W_\alpha = \{q \in P : \exists \beta > \alpha (q \text{ and } p_\alpha \text{ are compatible})\}.$$

Show that there is an $\alpha < \omega_1$ such that $W_\alpha = W_\beta$ for all $\beta > \alpha$, and apply $\mathrm{MA}(\omega_1)$ to W_α.

E21.3. Call a topological space X ccc iff every collection of pairwise disjoint open sets in X is countable. Show that $\prod_{i \in I} X_i$ is ccc iff $\forall F \in [I]^{<\omega}[\prod_{i \in F} X_i$ is ccc]. Hint: use the Δ-system theorem.

E21.4 Assuming $\mathrm{MA}(\omega_1)$, show that any product of ccc spaces is ccc.

E21.5. Assume $\mathrm{MA}(\omega_1)$. Suppose that P and Q are ccc partially ordered sets. Define $\leq$ on $P \times Q$ by setting $(a, b) \leq (c, d)$ iff $a \leq c$ and $b \leq d$. Show that $<$ is a ccc partial order on $P \times Q$. Hint: use exercise E21.2.

E21.6. We define $<^*$ on ${}^\omega\omega$ by setting $f <^* g$ iff $f, g \in {}^\omega\omega$ and $\exists n \forall m > n(f(m) < g(m)$. Suppose that $\mathrm{MA}(\kappa)$ holds and $\mathscr{F} \in [{}^\omega\omega]^\kappa$. Show that there is a $g \in {}^\omega\omega$ such that $f <^* g$ for all $f \in \mathscr{F}$. Hint: let P be the set of all pairs (p, F) such that p is a finite function mapping a subset of ω into ω and F is a finite subset of $\mathscr{F}$. Define $(p, F) \leq (q, G)$ iff $q \subseteq p$, $G \subseteq F$, and

$$\forall f \in G \forall n \in \operatorname{dmn}(p) \backslash \operatorname{dmn}(q)[p(n) > f(n)].$$

E21.7. Let $\mathscr{B} \subseteq [\omega]^\omega$ be almost disjoint of size κ, with $\omega \leq \kappa < 2^\omega$. Let $\mathscr{A} \subseteq \mathscr{B}$ with $\mathscr{A}$ countable. Assume $\mathrm{MA}(\kappa)$. Show that there is a $d \subseteq \omega$ such that $|d \cap x| < \omega$ for all $x \in \mathscr{A}$, and $|x \backslash d| < \omega$ for all $x \in \mathscr{B} \backslash \mathscr{A}$. Hint: Let $\langle a_i : i \in \omega \rangle$ enumerate $\mathscr{A}$. Let

$$\mathbb{P} = \{(s, F, m) : s \in [\omega]^{<\omega}, F \in [\mathscr{B} \backslash \mathscr{A}]^{<\omega}, \text{ and } m \in \omega\};$$
$$(s', F', m') \leq (s, F, m) \text{ iff } s \subseteq s', F \subseteq F', m \leq m', \text{ and}$$
$$\forall x \in F \left[\left(x \backslash \bigcup_{i \in m} a_i \right) \cap s' \subseteq s \right].$$

Show that $\mathbb{P}$ satisfies ccc. To apply $\mathrm{MA}(\kappa)$, one needs various dense sets. The most complicated is defined as follows. Let $\mathscr{D} = \{(s, F, m, i, n) : (s, F, m) \in \mathbb{P}, i < m$, and $n \in a_i \backslash s\}$. Clearly $|\mathscr{D}| = \kappa$. For each $(s, F, m, i, n) \in \mathscr{D}$ let

$$\begin{aligned} E_{(s,F,m,i,n)} = \{(s', F', m') \in \mathbb{P} : (s, F, m) \text{ and } (s', F', m') \text{ are incompatible} \\ \text{or } (s', F', m') \leq (s, F, m) \text{ and } n \in s'\}. \end{aligned}$$

E21.8. [The condition that $\mathscr{A}$ is countable is needed in exercise E21.7.] Show that there exist $\mathscr{A}, \mathscr{B}$ such that $\mathscr{B}$ is an almost disjoint family of infinite subsets of ω, $\mathscr{A} \subseteq \mathscr{B}$, $|\mathscr{A}| = |\mathscr{B} \backslash \mathscr{A}| = \omega_1$, and there does not exist a $d \subseteq \omega$ such that $|x \backslash d| < \omega$ for all $x \in \mathscr{A}$, and $|x \cap d| < \omega$ for all $x \in \mathscr{B} \backslash \mathscr{A}$. Hint: construct $\mathscr{A} = \{a_\alpha : \alpha < \omega_1\}$ and $\mathscr{B} \backslash \mathscr{A} = \{b_\alpha : \alpha < \omega_1\}$ by constructing a_α, b_α inductively, making sure that the elements are infinite and pairwise almost disjoint, and also $a_\alpha \cap b_\alpha = \emptyset$, while for $\alpha \neq \beta$ we have $a_\alpha \cap b_\beta \neq \emptyset$.

E21.9. Suppose that $\mathscr{A}$ is a family of infinite subsets of ω such that $\bigcap F$ is infinite for every finite subset F of $\mathscr{A}$. Suppose that $|\mathscr{A}| \le \kappa$. Assuming MA(κ), show that there is an infinite $X \subseteq \omega$ such that $X\backslash A$ is finite for every $A \in \mathscr{A}$. Hint: use Theorem 21.5.

E21.10. Show that MA(κ) is equivalent to MA(κ) restricted to ccc partial orders of cardinality $\le \kappa$. Hint: Assume the indicated special form of MA(κ), and assume given a ccc partially ordered set P and a family $\mathscr{D}$ of at most κ dense sets in P; we want to find a filter on P intersecting each member of $\mathscr{D}$. We introduce some operations on P. For each $D \in \mathscr{D}$ define $f_D : P \to P$ by setting, for each $p \in P$, $f_D(p)$ to be some element of D which is $\le p$. Also we define $g : P \times P \to P$ by setting, for all $p, q \in P$,

$$g(p,q) = \begin{cases} p & \text{if } p \text{ and } q \text{ are incompatible,} \\ r & \text{with } r \le p, q \text{ if there is such an } r. \end{cases}$$

Here, as in the definition of f_D, we are implicitly using the axiom of choice; for g, we choose any r of the indicated form.

We may assume that $\mathscr{D} \neq \emptyset$. Choose $D \in \mathscr{D}$, and choose $s \in D$. Now let Q be the intersection of all subsets of P which have s as a member and are closed under all of the operations f_D and g. We take the order on Q to be the order induced from P. Apply the special form to Q.

E21.11. Define $x \subset^* y$ iff $x, y \subseteq \omega$, $x\backslash y$ is finite, and $y\backslash x$ is infinite. Assume MA(κ), and suppose that $L, <)$ is a linear ordering of size at most κ. Show that there is a system $\langle a_x : x \in L\rangle$ of infinite subsets of ω such that for all $x, y \in L$, $x < y$ iff $a_x \subset^* a_y$. Hint: let P consist of all pairs (p, n) such that $n \in \omega$, p is a function whose domain is a finite subset of L, and $\forall x \in \text{dmn}(p)[p(x) \subseteq n]$. Define $(p, n) \le (q, m)$ iff $m \le n$, $\text{dmn}(q) \subseteq \text{dmn}(p)$, $\forall x \in \text{dmn}(q)[p(x) \cap m = q(x)]$, and $\forall x, y \in \text{dmn}(q)[x < y \to p(x)\backslash p(y) \subseteq m]$.

For the remaining exercises we use the following definitions.

$$\begin{aligned} a \subseteq^* b \quad &\text{iff} \quad a\backslash b \text{ is finite;} \\ a \subset^* b \quad &\text{iff} \quad a \subseteq^* b \text{ and } b\backslash a \text{ is infinite.} \end{aligned}$$

E21.12. If $\mathscr{A}, \mathscr{B}$ are nonempty countable subsets of $[\omega]^\omega$ and $a \subseteq^* b$ whenever $a \in \mathscr{A}$ and $b \in \mathscr{B}$, then there is a $c \in [\omega]^\omega$ such that $a \subseteq^* c \subseteq^* b$ whenever $a \in \mathscr{A}$ and $b \in \mathscr{B}$.

E21.13. Suppose that $\mathscr{A}$ is a nonempty countable family of members of $[\omega]^\omega$, and $\forall a, b \in \mathscr{A}[a \subseteq^* b$ or $b \subseteq^* a]$. Also suppose that $\forall a \in \mathscr{A}[a \subset^* d]$, where $d \in [\omega]^\omega$. Then there is a $c \in [\omega]^\omega$ such that $\forall a \in \mathscr{A}[a \subseteq^* c \subset^* d]$.

E21.14. If $a, b \in [\omega]^\omega$ and $a \subset^* b$, then there is a $c \in [\omega]^\omega$ such that $a \subset^* c \subset^* b$.

E21.15. Suppose that $\mathscr{A}$ and $\mathscr{B}$ are nonempty countable subsets of $[\omega]^\omega$, $\forall x, y \in \mathscr{A}[x \subseteq^* y$ or $y \subseteq^* x]$, $\forall x, y \in \mathscr{B}[x \subseteq^* y$ or $y \subseteq^* x]$, and $\forall x \in \mathscr{A} \forall y \in \mathscr{B}[a \subset^* b]$. Then there is a $c \in [\omega]^\omega$ such that $a \subset^* c \subset^* b$ for all $a \in \mathscr{A}$ and $b \in \mathscr{B}$.

Now we need some more terminology. Let $\mathscr{A} \subseteq [\omega]^\omega$, $b \in [\omega]^\omega$, and $\forall a \in \mathscr{A}[a \subset^* b]$. We say that b is *near* to $\mathscr{A}$ iff for all $m \in \omega$ the set $\{a \in \mathscr{A} : a\backslash b \subseteq m\}$ is finite.

E21.16. Suppose that $a_m \in [\omega]^\omega$ for all $m \in \omega$, $a_m \subset^* a_n$ whenever $m < n \in \omega$, $b \in [\omega]^\omega$, and $a_m \subset^* b$ for all $m \in \omega$. Then there is a $c \in [\omega]^\omega$ such that $\forall m \in \omega[a_m \subset^* c \subset^* b]$ and c is near to $\{a_n : n \in \omega\}$.

E21.17. Suppose that $\mathscr{A} \subseteq [\omega]^\omega$, $\forall x, y \in \mathscr{A}[x \subset^* y$ or $y \subset^* x]$, $b \in [\omega]^\omega$, $\forall x \in \mathscr{A}[x \subset^* b]$, and $\forall a \in \mathscr{A}[b$ is near to $\{d \in \mathscr{A} : d \subset^* a\}]$.

Then there is a $c \in [\omega]^\omega$ such that $\forall a \in \mathscr{A}[a \subset^* c \subset^* b]$ and c is near to $\mathscr{A}$.

E21.18. (The Hausdorff gap) There exist sequences $\langle a_\alpha : \alpha < \omega_1\rangle$ and $\langle b_\alpha : \alpha < \omega_1\rangle$ of members of $[\omega]^\omega$ such that $\forall \alpha, \beta < \omega_1[\alpha < \beta \to a_\alpha \subset^* a_\beta$ and $b_\beta \subset^* b_\alpha]$, $\forall \alpha, \beta < \omega_1[a_\alpha \subset^* b_\beta]$, and there does not exist a $c \subseteq \omega$ such that $\forall \alpha < \omega_1[a_\alpha \subset^* c$ and $c \subset^* b_\alpha]$.

Reference

Fremlin, D. **Consequences of Martin's axiom.** Cambridge Univ. Press, 325pp.

22. Large cardinals

The study, or use, of large cardinals is one of the most active areas of research in set theory currently. There are many provably different kinds of large cardinals whose descriptions are different from one another. We restrict ourselves in this chapter to three important kinds: Mahlo cardinals, weakly compact cardinals, and measurable cardinals. All of these large cardinals are uncountable regular limit cardinals (which are frequently called weakly inaccessible cardinals), and most of them are strongly inaccessible cardinals.

Mahlo cardinals

As we mentioned in the elementary part of these notes, one cannot prove in ZFC that uncountable weakly inaccessible cardinals exist (if ZFC itself is consistent). But now we assume that even the somewhat stronger inaccessible cardinals exist, and we want to explore, roughly speaking, how many such there can be. We begin with some easy propositions. A *strong limit cardinal* is an infinite cardinal κ such that $2^\lambda < \kappa$ for all $\lambda < \kappa$.

Proposition 22.1. *Assume that uncountable inaccessible cardinals exist, and suppose that κ is the least such. Then every uncountable strong limit cardinal less than κ is singular.* $\square$

The inaccessibles are a class of ordinals, hence form a well-ordered class, and they can be enumerated in a strictly increasing sequence $\langle \iota_\alpha : \alpha \in O\rangle$. Here O is an ordinal, or **On**, the class of all ordinals. The definition of Mahlo cardinal is motivated by the following simple proposition.

Proposition 22.2. *If $\kappa = \iota_\alpha$ with $\alpha < \kappa$, then the set $\{\lambda < \kappa : \lambda$ is regular$\}$ is a nonstationary subset of κ.*

Proof. Since κ is regular and $\alpha < \kappa$, we must have $\sup_{\beta<\alpha} \iota_\beta < \kappa$. Let $C = \{\gamma : \sup_{\beta<\alpha} \iota_\beta < \gamma < \kappa$ and γ is a strong limit cardinal$\}$. Then C is club in κ with empty intersection with the given set. $\square$

• κ is *Mahlo* iff κ is an uncountable inaccessible cardinal and $\{\lambda < \kappa : \lambda$ is regular$\}$ is stationary in κ.

• κ is *weakly Mahlo* iff κ is an uncountable weakly inaccessible cardinal and $\{\lambda < \kappa : \lambda$ is regular$\}$ is stationary in κ.

Since the function ι is strictly increasing, we have $\alpha \le \iota_\alpha$ for all α. Hence the following is a corollary of Proposition 22.2

Corollary 22.3. *If κ is a Mahlo cardinal, then $\kappa = \iota_\kappa$.* $\square$

Thus a Mahlo cardinal κ is not only inaccessible, but also has κ inaccessibles below it.

Proposition 22.4. *For any uncountable cardinal κ the following conditions are equivalent:*

(i) κ is Mahlo.

(ii) $\{\lambda < \kappa : \lambda$ *is inaccessible*$\}$ *is stationary in* κ.

Proof. (i)$\Rightarrow$(ii): Let $S = \{\lambda < \kappa : \lambda \text{ is regular}\}$, and $S' = \{\lambda < \kappa : \lambda \text{ is inaccessible}\}$. Assume that κ is Mahlo. In particular, κ is uncountable and inaccessible. Suppose that C is club in κ. The set $D = \{\lambda < \kappa : \lambda \text{ is strong limit}\}$ is clearly club in κ too. If $\lambda \in S \cap C \cap D$, then λ is inaccessible, as desired.

(ii)$\Rightarrow$(i): obvious. □

The following proposition answers a natural question one may ask after seeing Corollary 22.3.

Proposition 22.5. *Suppose that* κ *is minimum such that* $\iota_k = \kappa$. *Then* κ *is not Mahlo.*

Proof. Suppose to the contrary that κ is Mahlo, and let $S = \{\lambda < \kappa : \lambda$ is inaccessible$\}$ For each $\lambda \in S$, let $f(\lambda)$ be the $\alpha < \kappa$ such that $\lambda = \iota_\alpha$. Then $\alpha = f(\lambda) < \lambda$ by the minimality of κ. So f is regressive on the stationary set S, and hence there is an $\alpha < \kappa$ and a stationary subset S' of S such that $f(\lambda) = \alpha$ for all $\lambda \in S'$. But actually f is clearly a one-one function, contradiction. □

Mahlo cardinals are in a sense larger than "ordinary" inaccessibles. Namely, below every Mahlo cardinal κ there are κ inaccessibles. But now in principle one could enumerate all the Mahlo cardinals, and then apply the same idea used in going from regular cardinals to Mahlo cardinals in order to go from Mahlo cardinals to higher Mahlo cardinals. Thus we can make the definitions

• κ is *hyper-Mahlo* iff κ is inaccessible and the set $\{\lambda < \kappa : \lambda \text{ is Mahlo}\}$ is stationary in κ.

• κ is *hyper-hyper-Mahlo* iff κ is inaccessible and the set $\{\lambda < \kappa : \lambda \text{ is hyper-Mahlo}\}$ is stationary in κ.

Of course one can continue in this vein.

Weakly compact cardinals

• A cardinal κ is *weakly compact* iff $\kappa > \omega$ and $\kappa \to (\kappa, \kappa)^2$. There are several equivalent definitions of weak compactness. The one which justifies the name "compact" involves infinitary logic, and it will be discussed later. Right now we consider equivalent conditions involving trees and linear orderings.

• A cardinal κ has the *tree property* iff every κ-tree has a chain of size κ.

Equivalently, κ has the tree property iff there is no κ-Aronszajn tree.

• A cardinal κ has the *linear order property* iff every linear order $(L, <)$ of size κ has a subset with order type κ or κ^* under $<$.

Lemma 22.6. *For any regular cardinal* κ, *the linear order property implies the tree property.*

Proof. We are going to go from a tree to a linear order in a different way from the branch method of Chapter 18.

Assume the linear order property, and let $(T, <)$ be a κ-tree. For each $x \in T$ and each $\alpha \leq \mathrm{ht}(x, T)$ let x^α be the element of height α below x. Thus x^0 is the root which is below x, and $x^{\mathrm{ht}(x)} = x$. For each $x \in T$, let $T \upharpoonright x = \{y \in T : y < x\}$. If x, y are incomparable elements of T, then let $\chi(x, y)$ be the smallest ordinal $\alpha \leq \min(\mathrm{ht}(x), \mathrm{ht}(y))$ such that $x^\alpha \neq y^\alpha$. Let $<'$ be a well-order of T. Then we define, for any distinct $x, y \in T$,

$$x <'' y \quad \text{iff} \quad x < y, \text{ or } x \text{ and } y \text{ are incomparable and } x^{\chi(x,y)} <' y^{\chi(x,y)}.$$

We claim that this gives a linear order of T. To prove transitivity, suppose that $x <'' y <'' z$. Then there are several possibilities. These are illustrated in diagrams below.

Case 1. $x < y < z$. Then $x < z$, so $x <'' z$.

Case 2. $x < y$, while y and z are incomparable, with $y^{\chi(y,z)} <' z^{\chi(y,z)}$.

Subcase 2.1. $\mathrm{ht}(x) < \chi(y, z)$. Then $x = x^{\mathrm{ht}(x)} = y^{\mathrm{ht}(x)} = z^{\mathrm{ht}(x)}$ so that $x < z$, hence $x <'' z$.

Subcase 2.2. $\chi(y, z) \leq \mathrm{ht}(x)$. Then x and z are incomparable. In fact, if $z < x$ then $z < y$, contradicting the assumption that y and z are incomparable; if $x \leq z$, then $y^{\mathrm{ht}(x)} = x = x^{\mathrm{ht}(x)} = z^{\mathrm{ht}(x)}$, contradiction. Now if $\alpha < \chi(x, z)$ then $y^\alpha = x^\alpha = z^\alpha$; it follows that $\chi(x, z) \leq \chi(y, z)$. If $\alpha < \chi(y, z)$ then $\alpha \leq \mathrm{ht}(x)$, and hence $x^\alpha = y^\alpha = z^\alpha$; this shows that $\chi(y, z) \leq \chi(x, z)$. So $\chi(y, z) = \chi(x, z)$. Hence $x^{\chi(x,z)} = y^{\chi(x,z)} = y^{\chi(y,z)} <' z^{\chi(y,z)} = z^{\chi(x,z)}$, and hence $x <'' z$.

Case 3. x and y are incomparable, and $y < z$. Then x and z are incomparable. Now if $\alpha < \chi(x, y)$, then $x^\alpha = y^\alpha = z^\alpha$; this shows that $\chi(x, y) \leq \chi(x, z)$. Also, $x^{\chi(x,y)} <' y^{\chi(x,y)} = z^{\chi(x,y)}$, and this implies that $\chi(x, z) \leq \chi(x, y)$. So $\chi(x, y) = \chi(x, z)$. It follows that $x^{\chi(x,z)} = x^{\chi(x,y)} <' y^{\chi(x,y)} = z^{\chi(x,z)}$, and hence $x <'' z$.

Case 4. x and y are incomparable, and also y and z are incomparable. We consider subcases.

Subcase 4.1. $\chi(y, z) < \chi(x, y)$. Now if $\alpha < \chi(y, z)$, then $x^\alpha = y^\alpha = z^\alpha$; so $\chi(y, z) \leq \chi(x, z)$. Also, $x^{\chi(y,z)} = y^{\chi(y,z)} <' z^{\chi(y,z)}$, so that $\chi(x, z) \leq \chi(y, z)$. Hence $\chi(x, z) = \chi(y, z)$, and $x^{\chi(x,z)} = y^{\chi(y,z)} <' z^{\chi(y,z)}$, and hence $x <'' z$.

Subcase 4.2. $\chi(y, z) = \chi(x, y)$. Now $x^{\chi(x,y)} <' y^{\chi(x,y)} = y^{\chi(y,z)} <' z^{\chi(y,z)} = z^{\chi(x,y)}$. It follows that $\chi(x, z) \leq \chi(x, y)$. For any $\alpha < \chi(x, y)$ we have $x^\alpha = y^\alpha = z^\alpha$ since $\chi(y, z) = \chi(x, y)$. So $\chi(x, y) = \chi(x, z)$. Hence $x^{\chi(x,z)} = x^{\chi(x,y)} <' y^{\chi(x,y)} = y^{\chi(y,z)} <' z^{\chi(y,z)} = z^{\chi(x,z)}$, so $x <'' z$.

Subcase 4.3. $\chi(x, y) < \chi(y, z)$. Then $x^{\chi(x,y)} <' y^{\chi(x,y)} = z^{\chi(x,y)}$, and if $\alpha < \chi(x, y)$ then $x^\alpha = y^\alpha = z^\alpha$. It follows that $x <'' z$

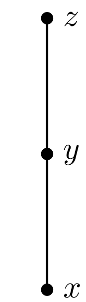
Clearly any two elements of T are comparable under $<''$, so we have a linear order. The following property is also needed.

(*) If $t < x, y$ and $x <'' a <'' y$, then $t < a$.

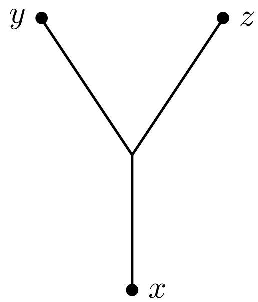
In fact, suppose not. If $a \leq t$, then $a < x$, hence $a <'' x$, contradiction. So a and t are incomparable. Then $\chi(a, t) \leq \mathrm{ht}(t)$, and hence $x <'' y <'' a$ or $a <'' x <'' y$, contradiction.

Now by the linear order property, $(T, <'')$ has a subset L of order type κ or κ^*. First suppose that L is of order type κ. Define

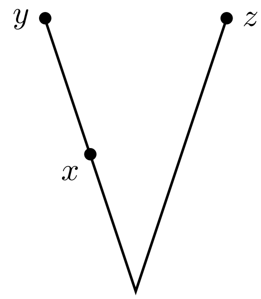
$$B = \{t \in T : \exists x \in L \forall a \in L[x \leq'' a \to t \leq a]\}.$$



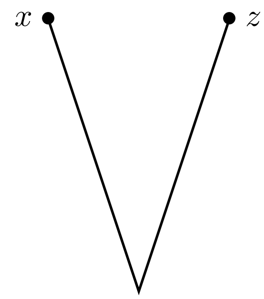
Case 1



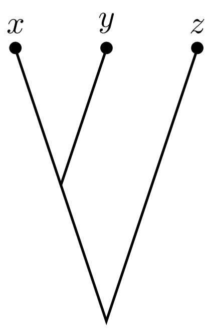
Subcase 2.1



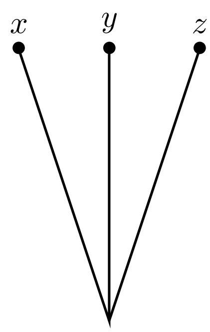
Subcase 2.2



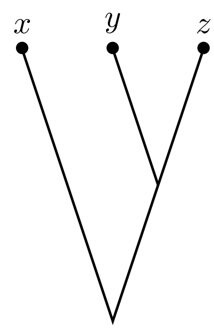
Case 3



Subcase 4.1



Subcase 4.2



Subcase 4.3

We claim that B is a chain in T of size κ. Suppose that $t_0, t_1 \in B$ with $t_0 \neq t_1$, and choose $x_0, x_1 \in L$ correspondingly. Say wlog $x_0 <'' x_1$. Now $t_0 \in B$ and $x_0 \leq'' x_1$, so $t_0 \leq x_1$. And $t_1 \in B$ and $x_1 \leq x_1$, so $t_1 \leq x_1$. So t_0 and t_1 are comparable.

Now let $\alpha < \kappa$; we show that B has an element of height α. For each t of height α let $V_t = \{x \in L : t \leq x\}$. Then

$$\{x \in L : \mathrm{ht}(x) \geq \alpha\} = \bigcup_{\mathrm{ht}(t)=\alpha} V_t;$$

since there are fewer than κ elements of height less than κ, this set has size κ, and so there is a t such that $\mathrm{ht}(t) = \alpha$ and $|V_t| = \kappa$. We claim that $t \in B$. To prove this, take any $x \in V_t$ such that $t < x$. Suppose that $a \in L$ and $x \leq'' a$. Choose $y \in V_t$ with $a <'' y$ and $t < y$. Then $t < x$, $t < y$, and $x \leq'' a <'' y$. If $x = a$, then $t \leq a$, as desired. If $x <'' a$, then $t < a$ by (*).

This finishes the case in which L has a subset of order type κ. The case of order type κ^* is similar, but we give it. So, suppose that L has order type κ^*. Define

$$B = \{t \in T : \exists x \in L \forall a \in L[a \leq'' x \rightarrow t \leq a]\}.$$

We claim that B is a chain in T of size κ. Suppose that $t_0, t_1 \in B$ with $t_0 \neq t_1$, and choose $x_0, x_1 \in L$ correspondingly. Say wlog $x_0 <'' x_1$. Now $t_0 \in B$ and $x_0 \leq x_0$, so $t_0 \leq x_0$. and $t_1 \in B$ and $x_0 \leq'' x_1$, so $t_1 \leq x_0$. So t_0 and t_1 are comparable.

Now let $\alpha < \kappa$; we show that B has an element of height α. For each t of height α let $V_t = \{x \in L : t \leq x\}$. Then

$$\{x \in L : \mathrm{ht}(x) \geq \alpha\} = \bigcup_{\mathrm{ht}(t)=\alpha} V_t;$$

since there are fewer than κ elements of height less than κ, this set has size κ, and so there is a t such that $\mathrm{ht}(t) = \alpha$ and $|V_t| = \kappa$. We claim that $t \in B$. To prove this, take any $x \in V_t$ such that $t < x$. Suppose that $a \in L$ and $a \leq'' x$. Choose $y \in V_t$ with $y <'' a$ and $t < y$. Then $t < x$, $t < y$, and $y <'' a \leq'' x$. If $a = x$, then $t < a$, as desired. If $a <'' x$, then $t < a$ by (*). □

Theorem 22.7. *For any uncountable cardinal κ the following conditions are equivalent:*

(i) κ is weakly compact.

(ii) κ is inaccessible, and it has the linear order property.

(iii) κ is inaccessible, and it has the tree property.

(iv) For any cardinal λ such that $1 < \lambda < \kappa$ we have $\kappa \to (\kappa)^2_\lambda$.

Proof. (i)⇒(ii): Assume that κ is weakly compact. First we need to show that κ is inaccessible.

To show that κ is regular, suppose to the contrary that $\kappa = \sum_{\alpha<\lambda} \mu_\alpha$, where $\lambda < \kappa$ and $\mu_\alpha < \kappa$ for each $\alpha < \lambda$. By the definition of infinite sum of cardinals, it follows that we can write $\kappa = \bigcup_{\alpha<\lambda} M_\alpha$, where $|M_\alpha| = \mu_\alpha$ for each $\alpha < \lambda$ and the M_α's are pairwise disjoint. Define $f : [\kappa]^2 \to 2$ by setting, for any distinct $\alpha, \beta < \kappa$,

$$f(\{\alpha, \beta\}) = \begin{cases} 0 & \text{if } \alpha, \beta \in M_\xi \text{ for some } \xi < \lambda, \\ 1 & \text{otherwise.} \end{cases}$$

Let H be homogeneous for f of size κ. First suppose that $f[[H]^2] = \{0\}$. Fix $\alpha_0 \in H$, and say $\alpha_0 \in M_\xi$. For any $\beta \in H$ we then have $\beta \in M_\xi$ also, by the homogeneity of H. So $H \subseteq M_\xi$, which is impossible since $|M_\xi| < \kappa$. Second, suppose that $f[[H]^2] = \{1\}$. Then any two distinct members of H lie in distinct M_ξ's. Hence if we define $g(\alpha)$ to be the $\xi < \lambda$ such that $\alpha \in M_\xi$ for each $\alpha \in H$, we get a one-one function from H into λ, which is impossible since $\lambda < \kappa$.

To show that κ is strong limit, suppose that $\lambda < \kappa$ but $\kappa \leq 2^\lambda$. Now by Theorem 20.7 we have $2^\lambda \not\to (\lambda^+, \lambda^+)^2$. So choose $f : [2^\lambda]^2 \to 2$ such that there does not exist an $X \in [2^\lambda]^{\lambda^+}$ with $f \restriction [X]^2$ constant. Define $g : [\kappa]^2 \to 2$ by setting $g(A) = f(A)$ for any $A \in [\kappa]^2$. Choose $Y \in [\kappa]^\kappa$ such that $g \restriction [Y]^2$ is constant. Take any $Z \in [Y]^{\lambda^+}$. Then $f \restriction [Z]^2$ is constant, contradiction.

So, κ is inaccessible. Now let $(L, <)$ be a linear order of size κ. Let $\prec$ be a well order of L. Now we define $f : [L]^2 \to 2$; suppose that $a, b \in L$ with $a \prec b$. Then

$$f(\{a, b\}) = \begin{cases} 0 & \text{if } a < b, \\ 1 & \text{if } b > a. \end{cases}$$

Let H be homogeneous for f and of size κ. If $f[[H]^2] = \{0\}$, then H is well-ordered by $<$. If $f[[H]^2] = \{1\}$, then H is well-ordered by $>$.

(ii)⇒(iii): By Lemma 22.6.

(iii)⇒(iv): Assume (iii). Suppose that $F : [\kappa]^2 \to \lambda$, where $1 < \lambda < \kappa$; we want to find a homogeneous set for F of size κ. We construct by recursion a sequence $\langle t_\alpha : \alpha < \kappa\rangle$ of members of ${}^{<\kappa}\kappa$; these will be the members of a tree T. Let $t_0 = \emptyset$. Now suppose that $0 < \alpha < \kappa$ and $t_\beta \in {}^{<\kappa}\kappa$ has been constructed for all $\beta < \alpha$. We now define t_α by recursion; its domain will also be determined by the recursive definition, and for this purpose it is convenient to actually define an auxiliary function $s : \kappa \to \kappa+1$ by recursion. If $s(\eta)$ has been defined for all $\eta < \xi$, we define

$$s(\xi) = \begin{cases} F(\{\beta,\alpha\}) & \text{where } \beta < \alpha \text{ is minimum such that } s \restriction \xi = t_\beta, \text{ if there is such a } \beta, \\ \\ \kappa & \text{if there is no such } \beta. \end{cases}$$

Now eventually the second condition here must hold, as otherwise $\langle s \restriction \xi : \xi < \kappa\rangle$ would be a one-one function from κ into $\{t_\beta : \beta < \alpha\}$, which is impossible. Take the least ξ such that $s(\xi) = \kappa$, and let $t_\alpha = s \restriction \xi$. This finishes the construction of the t_α's. Let $T = \{t_\alpha : \alpha < \kappa\}$, with the partial order $\subseteq$. Clearly this gives a tree.

By construction, if $\alpha < \kappa$ and $\xi < \mathrm{dmn}(t_\alpha)$, then $t_\alpha \restriction \xi \in T$. Thus the height of an element t_α is $\mathrm{dmn}(t_\alpha)$.

(2) The sequence $\langle t_\alpha : \alpha < \kappa\rangle$ is one-one.

In fact, suppose that $\beta < \alpha$ and $t_\alpha = t_\beta$. Say that $\mathrm{dmn}(t_\alpha) = \xi$. Then $t_\alpha = t_\alpha \restriction \xi = t_\beta$, and the construction of t_α gives something with domain greater than ξ, contradiction. Thus (2) holds, and hence $|T| = \kappa$.

(3) The set of all elements of T of level $\xi < \kappa$ has size less than κ.

In fact, let U be this set. Then

$$|U| \le \prod_{\eta<\xi} \lambda = \lambda^\xi < \kappa$$

since κ is inaccessible. So (3) holds, and hence, since $|T| = \kappa$, T has height κ and is a κ-tree.

(4) If $t_\beta \subset t_\alpha$, then $\beta < \alpha$ and $F(\{\beta,\alpha\}) = t_\alpha(\mathrm{dmn}(t_\beta))$.

This is clear from the definition.

Now by the tree property, there is a branch B of size κ. For each $\xi < \lambda$ let

$$H_\xi = \{\alpha < \kappa : t_\alpha \in B \text{ and } t_\alpha^\frown\langle\xi\rangle \in B\}.$$

We claim that each H_ξ is homogeneous for F. In fact, take any distinct $\alpha, \beta \in H_\xi$. Then $t_\alpha, t_\beta \in B$. Say $t_\beta \subset t_\alpha$. Then $\beta < \alpha$, and by construction $t_\alpha(\mathrm{dmn}(t_\beta)) = F(\{\alpha,\beta\})$. So $F(\{\alpha,\beta\}) = \xi$ by the definition of H_ξ, as desired. Now

$$\{\alpha < \kappa : t_\alpha \in B\} = \bigcup_{\xi<\lambda} \{\alpha < \kappa : t_\alpha \in H_\xi\},$$

so since $|B| = \kappa$ it follows that $|H_\xi| = \kappa$ for some $\xi < \lambda$, as desired.

(iv)$\Rightarrow$(i): obvious. □

Now we go into the connection of weakly compact cardinals with logic, thereby justifying the name "weakly compact". This is optional material.

Let κ and λ be infinite cardinals. The language $L_{\kappa\lambda}$ is an extension of ordinary first order logic as follows. The notion of a model is unchanged. In the logic, we have a sequence of λ distinct individual variables, and we allow quantification over any one-one sequence of fewer than λ variables. We also allow conjunctions and disjunctions of fewer than κ formulas. It should be clear what it means for an assignment of values to the variables to satisfy a formula in this extended language. We say that an infinite cardinal κ is *logically weakly compact* iff the following condition holds:

(*) For any language $L_{\kappa\kappa}$ with at most κ basic symbols, if Γ is a set of sentences of the language and if every subset of Γ of size less than κ has a model, then also Γ has a model.

Notice here the somewhat unnatural restriction that there are at most κ basic symbols. If we drop this restriction, we obtain the notion of a strongly compact cardinal. These cardinals are much larger than even the measurable cardinals discussed later. We will not go into the theory of such cardinals.

Theorem 22.8. *An infinite cardinal is logically weakly compact iff it is weakly compact.*

Proof. Suppose that κ is logically weakly compact.

(1) κ is regular.

Suppose not; say $X \subseteq \kappa$ is unbounded but $|X| < \kappa$. Take the language with individual constants c_α for $\alpha < \kappa$ and also one more individual constant d. Consider the following set Γ of sentences in this language:

$$\{d \neq c_\alpha : \alpha < \kappa\} \cup \left\{ \bigvee_{\beta\in X} \bigvee_{\alpha<\beta} (d = c_\alpha) \right\}.$$

If $\Delta \in [\Gamma]^{<\kappa}$, let A be the set of all $\alpha < \kappa$ such that $d = c_\alpha$ is in Δ. So $|A| < \kappa$. Take any $\alpha \in \kappa\backslash A$, and consider the structure $M = (\kappa, \gamma, \alpha)_{\gamma<\kappa}$. There is a $\beta \in X$ with $\alpha < \beta$, and this shows that M is a model of Δ.

Thus every subset of Γ of size less than κ has a model, so Γ has a model; but this is clearly impossible.

(2) κ is strong limit.

In fact, suppose not; let $\lambda < \kappa$ with $\kappa \leq 2^\lambda$. We consider the language with distinct individual constants c_α, d^i_α for all $\alpha < \kappa$ and $i < 2$. Let Γ be the following set of sentences in this language:

$$\left\{ \bigwedge_{\alpha<\lambda} [(c_\alpha = d^0_\alpha \vee c_\alpha = d^1_\alpha) \wedge d^0_\alpha \neq d^1_\alpha] \right\} \cup \left\{ \bigvee_{\alpha<\lambda} (c_\alpha \neq d^{f(\alpha)}_\alpha) : f \in {}^\lambda 2 \right\}.$$

Suppose that $\Delta \in [\Gamma]^{<\kappa}$. We may assume that Δ has the form

$$\left\{\bigwedge_{\alpha<\lambda}[(c_\alpha = d^0_\alpha \vee c_\alpha = d^1_\alpha) \wedge d^0_\alpha \neq d^1_\alpha]\right\} \cup \left\{\bigvee_{\alpha<\lambda}(c_\alpha \neq d^{f(\alpha)}_\alpha) : f \in M\right\},$$

where $M \in [{}^\lambda 2]^{<\kappa}$. Fix $g \in {}^\lambda 2 \backslash M$. Let $d^0_\alpha = \alpha$, $d^1_\alpha = \alpha+1$, and $c_\alpha = d^{g(\alpha)}_\alpha$, for all $\alpha < \lambda$. Clearly $(\kappa, c_\alpha, d^i_\alpha)_{\alpha<\lambda, i<2}$ is a model of Δ.

Thus every subset of Γ of size less than κ has a model, so Γ has a model, say $(M, u_\alpha, v^i_\alpha)_{\alpha<\lambda,i<2}$. By the first part of Γ there is a function $f \in {}^\lambda 2$ such that $u_\alpha = d^{f(\alpha)}_\alpha$ for every $\alpha < \lambda$. this contradicts the second part of Γ.

Hence we have shown that κ is inaccessible.

Finally, we prove that the tree property holds. Suppose that $(T, \leq)$ is a κ-tree. Let L be the language with a binary relation symbol $\prec$, unary relation symbols P_α for each $\alpha < \kappa$, individual constants c_t for each $t \in T$, and one more individual constant d. Let Γ be the following set of sentences:

all $L_{\kappa\kappa}$-sentences holding in the structure $M \stackrel{\text{def}}{=} (T, <, \mathrm{Lev}_\alpha(T), t)_{\alpha<\kappa, t\in T}$;

$\exists x[P_\alpha x \wedge x \prec d]$ for each $\alpha < \kappa$.

Clearly every subset of Γ of size less than κ has a model. Hence Γ has a model $N \stackrel{\text{def}}{=} (A, <', S'_\alpha, a_t, b)_{\alpha<\kappa,t\in T}$. For each $\alpha < \kappa$ choose $e_\alpha \in S'_\alpha$ with $e_\alpha <' b$. Now the following sentence holds in M and hence in N:

$$\forall x \left[P_\alpha x \leftrightarrow \bigvee_{s\in \mathrm{Lev}_\alpha(T)} (x = c_s)\right].$$

Hence for each $\alpha < \kappa$ we can choose $t(\alpha) \in T$ such that $e_a = a_{t(\alpha)}$. Now the sentence

$$\forall x, y, z[x < z \wedge y < z \rightarrow x \text{ and } y \text{ are comparable}]$$

holds in M, and hence in N. Now fix $\alpha < \beta < \kappa$. Now $e_\alpha, e_\beta <' b$, so it follows that e_α and e_β are comparable under $\leq'$. Hence $a_{t(\alpha)}$ and $a_{t(\beta)}$ are comparable under $\leq'$. It follows that $t(\alpha)$ and $t(\beta)$ are comparable under $\leq$. So $t(\alpha) < t(\beta)$. Thus we have a branch of size κ.

Now suppose that κ is weakly compact. Let L be an $L_{\kappa\kappa}$-language with at most κ symbols, and suppose that Γ is a set of sentences in L such that every subset Δ of Γ of size less than κ has a model M_Δ. We will construct a model of Γ by modifying Henkin's proof of the completeness theorem for first-order logic.

First we note that there are at most κ formulas of L. This is easily seen by the following recursive construction of all formulas:

$$F_0 = \text{ all atomic formulas;}$$

$$F_{\alpha+1} = F_\alpha \cup \{\neg\varphi : \varphi \in F_\alpha\} \cup \left\{\bigvee \Phi : \Phi \in [F_\alpha]^{<\kappa}\right\} \cup \{\exists\overline{x}\varphi : \varphi \in F_\alpha, \overline{x} \text{ of length } < \kappa\};$$

$$F_\alpha = \bigcup_{\beta<\alpha} F_\beta \text{ for } \alpha \text{ limit.}$$

By induction, $|F_\alpha| \leq \kappa$ for all $\alpha \leq \kappa$, and F_κ is the set of all formulas. (One uses that κ is inaccessible.)

Expand L to L' by adjoining a set C of new individual constants, with $|C| = \kappa$. Let Θ be the set of all subformulas of the sentences in Γ. Let $\langle \varphi_\alpha : \alpha < \kappa \rangle$ list all sentences of L' which are of the form $\exists \overline{x} \psi_\alpha(\overline{x})$ and are obtained from a member of Θ by replacing variables by members of C. Here $\overline{x}$ is a one-one sequence of variables of length less than κ; say that $\overline{x}$ has length β_α. Now we define a sequence $\langle d_\alpha : \alpha < \kappa \rangle$; each d_α will be a sequence of members of C of length less than κ. If d_β has been defined for all $\beta < \alpha$, then

$$\bigcup_{\beta<\alpha} \mathrm{rng}(d_\beta) \cup \{c \in C : c \text{ occurs in } \varphi_\beta \text{ for some } \beta < \alpha\}$$

has size less than κ. We then let d_α be a one-one sequence of members of C not in this set; d_α should have length β_α. Now for each $\alpha \leq \kappa$ let

$$\Omega_\alpha = \{\exists \overline{x} \psi_\beta(\overline{x}) \to \psi_\beta(\overline{d_\beta}) : \beta < \alpha\}.$$

Note that $\Omega_\alpha \subseteq \Omega_\gamma$ if $\alpha < \gamma \leq \kappa$. Now we define for each $\Delta \in [\Gamma]^{<\kappa}$ and each $\alpha \leq \kappa$ a model N^Δ_α of $\Delta \cup \Omega_\alpha$. Since $\Omega_0 = \emptyset$, we can let $N^\Delta_0 = M_\Delta$. Having defined N^Δ_α, since the range of d_α consists of new constants, we can choose denotations of those constants, expanding N^Δ_α to $N^\Delta_{\alpha+1}$, so that the sentence

$$\exists \overline{x} \psi_\alpha(\overline{x}) \to \psi_\alpha(\overline{d_\alpha})$$

holds in $N^\Delta_{\alpha+1}$. For $\alpha \leq \kappa$ limit we let $N^\Delta_\alpha = \bigcup_{\beta<\alpha} N^\Delta_\beta$.

It follows that N^Δ_κ is a model of $\Delta \cup \Omega_\kappa$. So each subset of $\Gamma \cup \Omega_\kappa$ of size less than κ has a model.

It suffices now to find a model of $\Gamma \cup \Omega_\kappa$ in the language L'. Let $\langle \psi_\alpha : \alpha < \kappa \rangle$ be an enumeration of all sentences obtained from members of Θ by replacing variables by members of C, each such sentence appearing κ times. Let T consist of all f satisfying the following conditions:

(3) f is a function with domain $\alpha < \kappa$.

(4) $\forall \beta < \alpha[(\psi_\beta \in \Gamma \cup \Omega_\kappa \to f(\beta) = \psi_\beta)$ and $\psi_\beta \notin \Gamma \cup \Omega_\kappa \to f(\beta) = \neg\psi_\beta)]$.

(5) $\mathrm{rng}(f)$ has a model.

Thus T forms a tree $\subseteq$.

(6) T has an element of height α, for each $\alpha < \kappa$.

In fact, $\Delta \overset{\text{def}}{=} \{\psi_\beta : \beta < \alpha, \psi_\beta \in \Gamma \cup \Omega_\kappa\} \cup \{\neg\psi_\beta : \beta < \alpha, \neg\psi_\beta \in \Gamma \cup \Omega_\kappa\}$ is a subset of $\Gamma \cup \Omega_\kappa$ of size less than κ, so it has a model P. For each $\beta < \alpha$ let

$$f(\beta) = \begin{cases} \psi_\beta & \text{if } P \models \psi_\beta, \\ \neg\psi_\gamma & \text{if } P \models \neg\psi_\beta. \end{cases}$$

Clearly f is an element of T with height α. So (6) holds.

Thus T is clearly a κ-tree, so by the tree property we can let B be a branch in T of size κ. Let $\Xi = \{f(\alpha) : \alpha < \kappa,\ f \in B,\ f \text{ has height } \alpha + 1\}$. Clearly $\Gamma \cup \Omega_\kappa \subseteq \Xi$ and for every $\alpha < \kappa$, $\psi_\alpha \in \Xi$ or $\neg\psi_\alpha \in \Xi$.

(7) If $\varphi, \varphi \to \chi \in \Xi$, then $\chi \in \Xi$.

In fact, say $\varphi = f(\alpha)$ and $\varphi \to \chi = f(\beta)$. Choose $\gamma > \alpha, \beta$ so that ψ_γ is χ. We may assume that $\mathrm{dmn}(f) \geq \gamma + 1$. Since $\mathrm{rng}(f)$ has a model, it follows that $f(\gamma) = \chi$. So (7) holds.

Let S be the set of all terms with no variables in them. We define $\sigma \equiv \tau$ iff $\sigma, \tau \in S$ and $(\sigma = \tau) \in \Xi$. Then $\equiv$ is an equivalence relation on S. In fact, let $\sigma \in S$. Say that $\sigma = \sigma$ is ψ_α. Since ψ_α holds in every model, it holds in any model of $\{f(\beta) : \beta \leq \alpha\}$, and hence $f(\alpha) = (\sigma = \sigma)$. So $(\sigma = \sigma) \in \Xi$ and so $\sigma \equiv \sigma$. Symmetry and transitivity follow by (7).

Let M be the collection of all equivalence classes. Using (7) it is easy to see that the function and relation symbols can be defined on M so that the following conditions hold:

(8) If F is an m-ary function symbol, then

$$F^M(\sigma_0/\equiv, \ldots, \sigma_{m-1}/\equiv) = F(\sigma_0, \ldots, \sigma_{m-1})/\equiv .$$

(9) If R is an m-ary relation symbol, then

$$\langle \sigma_0/\equiv, \ldots, \sigma_{m-1}/\equiv \rangle \in R^M \quad \text{iff} \quad R(\sigma_0, \ldots, \sigma_{m-1}) \in \Xi.$$

Now the final claim is as follows:

(10) If φ is a sentence of L', then $M \models \varphi$ iff $\varphi \in \Xi$.

Clearly this will finish the proof. We prove (10) by induction on φ. It is clear for atomic sentences by (8) and (9). If it holds for φ, it clearly holds for $\neg\varphi$. Now suppose that Q is a set of sentences of size less than κ, and (10) holds for each member of Q. Suppose that $M \models \bigwedge Q$. Then $M \models \varphi$ for each $\varphi \in Q$, and so $Q \subseteq \Xi$. Hence there is a $\Delta \in [\kappa]^{<\kappa}$ such that $Q = f[\Delta]$, with $f \in B$. Choose α greater than each member of Δ such that ψ_α is the formula $\bigwedge Q$. We may assume that $\alpha \in \mathrm{dmn}(f)$. Since $\mathrm{rng}(f)$ has a model, it follows that $f(\alpha) = \bigwedge Q$. Hence $\bigwedge Q \in \Xi$.

Conversely, suppose that $\bigwedge Q \in \Xi$. From (7) it easily follows that $\varphi \in \Xi$ for every $\varphi \in Q$, so by the inductive hypothesis $M \models \varphi$ for each $\varphi \in Q$, so $M \models \bigwedge Q$.

Finally, suppose that φ is $\exists\overline{x}\psi$, where (10) holds for shorter formulas. Suppose that $M \models \exists\overline{x}\psi$. Then there are members of S such that when they are substituted in ψ for $\overline{x}$, obtaining a sentence ψ', we have $M \models \psi'$. Hence by the inductive hypothesis, $\psi' \in \Xi$. (7) then yields $\exists\overline{x}\psi \in \Xi$.

Conversely, suppose that $\exists\overline{x}\psi \in \Xi$. Now there is a sequence $\overline{d}$ of members of C such that $\exists\overline{x}\psi \in \Xi \to \psi(\overline{d})$ is also in Ξ, and so by (7) we get $\psi(\overline{d}) \in \Xi$. By the inductive hypothesis, $M \models \psi(\overline{d})$, so $M \models \exists\overline{x}\psi \in \Xi$. $\square$

Next we want to show that every weakly compact cardinal is a Mahlo cardinal. To do this we need two lemmas.

Lemma 22.9. *Let A be a set of infinite cardinals such that for every regular cardinal κ, the set $A \cap \kappa$ is non-stationary in κ. Then there is a one-one regressive function with domain A.*

Proof. We proceed by induction on $\gamma \stackrel{\text{def}}{=} \bigcup A$. Note that γ is a cardinal; it is 0 if $A = \emptyset$. The cases $\gamma = 0$ and $\gamma = \omega$ are trivial, since then $A = \emptyset$ or $A = \{\omega\}$ respectively.

Next, suppose that γ is a successor cardinal κ^+. Then $A = A' \cup \{\kappa^+\}$ for some set A' of infinite cardinals less than κ^+. Then $\bigcup A' < \kappa^+$, so by the inductive hypothesis there is a one-one regressive function f on A'. We can extend f to A by setting $f(\kappa^+) = \kappa$, and so we get a one-one regressive function defined on A.

Suppose that γ is singular. Let $\langle \mu_\xi : \xi < \text{cf}(\gamma) \rangle$ be a strictly increasing continuous sequence of infinite cardinals with supremum γ, with $\text{cf}(\gamma) < \mu_0$. Note then that for every cardinal $\lambda < \gamma$, either $\lambda < \mu_0$ or else there is a unique $\xi < \text{cf}(\gamma)$ such that $\mu_\xi \le \lambda < \mu_{\xi+1}$. For every $\xi < \text{cf}(\gamma)$ we can apply the inductive hypothesis to $A \cap \mu_\xi$ to get a one-one regressive function g_ξ with domain $A \cap \mu_\xi$. We now define f with domain A. In case $\text{cf}(\gamma) = \omega$ we define, for each $\lambda \in A$,

$$f(\lambda) = \begin{cases} g_0(\lambda) + 2 & \text{if } \lambda < \mu_0, \\ \mu_\xi + g_{\xi+1}(\lambda) + 1 & \text{if } \mu_\xi < \lambda < \mu_{\xi+1}, \\ \mu_\xi & \text{if } \lambda = \mu_{\xi+1}, \\ 1 & \text{if } \lambda = \mu_0, \\ 0 & \text{if } \lambda = \gamma \in A. \end{cases}$$

Here the addition is ordinal addition. Clearly f is as desired in this case. If $\text{cf}(\gamma) > \omega$, let $\langle \nu_\xi : \xi < \text{cf}(\gamma) \rangle$ be a strictly increasing sequence of limit ordinals with supremum $\text{cf}(\gamma)$. Then we define, for each $\lambda \in A$,

$$f(\lambda) = \begin{cases} g_0(\lambda) + 1 & \text{if } \lambda < \mu_0, \\ \mu_\xi + g_{\xi+1}(\lambda) + 1 & \text{if } \mu_\xi < \lambda < \mu_{\xi+1}, \\ \nu_\xi & \text{if } \lambda = \mu_\xi, \\ 0 & \text{if } \lambda = \gamma \in A. \end{cases}$$

Clearly f works in this case too.

Finally, suppose that γ is a regular limit cardinal. By assumption, there is a club C in γ such that $C \cap \gamma \cap A = \emptyset$. We may assume that $C \cap \omega = \emptyset$. Let $\langle \mu_\xi : \xi < \gamma \rangle$ be the strictly increasing enumeration of C. Then we define, for each $\lambda \in A$,

$$f(\lambda) = \begin{cases} g_0(\lambda) + 1 & \text{if } \lambda < \mu_0, \\ \mu_\xi + g_{\xi+1}(\lambda) + 1 & \text{if } \mu_\xi < \lambda < \mu_{\xi+1}, \\ 0 & \text{if } \lambda = \gamma \in A. \end{cases}$$

Clearly f works in this case too. $\square$

Lemma 22.10. *Suppose that κ is weakly compact, and S is a stationary subset of κ. Then there is a regular $\lambda < \kappa$ such that $S \cap \lambda$ is stationary in λ.*

Proof. Suppose not. Thus for all regular $\lambda < \kappa$, the set $S \cap \lambda$ is non-stationary in λ. Let C be the collection of all infinite cardinals less than κ. Clearly C is club in κ, so

$S \cap C$ is stationary in κ. Clearly still $S \cap C \cap \lambda$ is non-stationary in λ for every regular $\lambda < \kappa$. So we may assume from the beginning that S is a set of infinite cardinals.

Let $\langle \lambda_\xi : \xi < \kappa \rangle$ be the strictly increasing enumeration of S. Let

$$T = \left\{ s : \exists \xi < \kappa \left[s \in \prod_{\eta < \xi} \lambda_\eta \text{ and } s \text{ is one-one} \right] \right\}.$$

For every $\xi < \kappa$ the set $S \cap \lambda_\xi$ is non-stationary in every regular cardinal, and hence by Lemma 22.9 there is a one-one regressive function s with domain $S \cap \lambda_\xi$. Now $S \cap \lambda_\xi = \{\lambda_\eta : \eta < \xi\}$. Hence $s \in T$.

Clearly T forms a tree of height κ under $\subseteq$. Now for any $\alpha < \kappa$,

$$\prod_{\beta < \alpha} \lambda_\beta \leq \left(\sup_{\beta < \alpha} \lambda_\beta \right)^{|\alpha|} < \kappa.$$

Hence by the tree property there is a branch B in T of size κ. Thus $\bigcup B$ is a one-one regressive function with domain S, contradicting Fodor's theorem. $\square$

Theorem 22.11. *Every weakly compact cardinal is Mahlo, hyper-Mahlo, hyper-hyper-Mahlo, etc.*

Proof. Let κ be weakly compact. Let $S = \{\lambda < \kappa : \lambda \text{ is regular}\}$. Suppose that C is club in κ. Then C is stationary in κ, so by Lemma 22.10 there is a regular $\lambda < \kappa$ such that $C \cap \lambda$ is stationary in λ; in particular, $C \cap \lambda$ is unbounded in λ, so $\lambda \in C$ since C is closed in κ. Thus we have shown that $S \cap C \neq \emptyset$. So κ is Mahlo.

Let $S' = \{\lambda < \kappa : \lambda \text{ is a Mahlo cardinal}\}$. Suppose that C is club in κ. Let $S'' = \{\lambda < \kappa : \lambda \text{ is regular}\}$. Since κ is Mahlo, S'' is stationary in κ. Then $C \cap S''$ is stationary in κ, so by Lemma 22.10 there is a regular $\lambda < \kappa$ such that $C \cap S'' \cap \lambda$ is stationary in λ. Hence λ is Mahlo, and also $C \cap \lambda$ is unbounded in λ, so $\lambda \in C$ since C is closed in κ. Thus we have shown that $S' \cap C \neq \emptyset$. So κ is hyper-Mahlo.

Let $S''' = \{\lambda < \kappa : \lambda \text{ is a hyper-Mahlo cardinal}\}$. Suppose that C is club in κ. Let $S^{iv} = \{\lambda < \kappa : \lambda \text{ is Mahlo}\}$. Since κ is hyper-Mahlo, S^{iv} is stationary in κ. Then $C \cap S^{iv}$ is stationary in κ, so by Lemma 22.10 there is a regular $\lambda < \kappa$ such that $C \cap S^{iv} \cap \lambda$ is stationary in λ. Hence λ is hyper-Mahlo, and also $C \cap \lambda$ is unbounded in λ, so $\lambda \in C$ since C is closed in κ. Thus we have shown that $S''' \cap C \neq \emptyset$. So κ is hyper-hyper-Mahlo.

Etc. $\square$

We now give another equivalent definition of weak compactness. For it we need several lemmas.

Lemma 22.12. *Suppose that* $\mathbf{R}$ *is a well-founded class relation on a class* $\mathbf{A}$*, and it is set-like and extensional. Also suppose that* $\mathbf{B} \subseteq \mathbf{A}$*,* $\mathbf{B}$ *is transitive,* $\forall a, b \in \mathbf{A}[a\mathbf{R}b \in \mathbf{B} \rightarrow a \in \mathbf{B}]$*, and* $\forall a, b \in \mathbf{B}[a\mathbf{R}b \leftrightarrow a \in b]$*. Let* $\mathbf{G}, \mathbf{M}$ *be the Mostowski collapse of* $(\mathbf{A}, \mathbf{R})$*. Then* $\mathbf{G} \restriction \mathbf{B}$ *is the identity.*

Proof. Suppose not, and let $\mathbf{X} = \{b \in \mathbf{B} : \mathbf{G}(b) \neq b\}$. Since we are assuming that $\mathbf{X}$ is a nonempty subclass of $\mathbf{A}$, choose $b \in \mathbf{X}$ such that $y \in \mathbf{A}$ and $y\mathbf{R}b$ imply that $y \notin \mathbf{X}$. Then

$$\begin{aligned}
\mathbf{G}(b) &= \{\mathbf{G}(y) : y \in \mathbf{A} \text{ and } y\mathbf{R}b\} \\
&= \{\mathbf{G}(y) : y \in \mathbf{B} \text{ and } y\mathbf{R}b\} \\
&= \{y : y \in \mathbf{B} \text{ and } y\mathbf{R}b\} \\
&= \{y : y \in \mathbf{B} \text{ and } y \in b\} \\
&= \{y : y \in b\} \\
&= b,
\end{aligned}$$

contradiction. □

Lemma 22.13. *Let κ be weakly compact. Then for every $U \subseteq V_\kappa$, the structure $(V_\kappa, \in, U)$ has a transitive elementary extension $(M, \in, U')$ such that $\kappa \in M$.*

(This means that $V_\kappa \subseteq M$ and a sentence holds in the structure $(V_\kappa, \in, U, x)_{x \in V_\kappa}$ iff it holds in $(M, \in, U', x)_{x \in V_\kappa}$.)

Proof. Let Γ be the set of all $L_{\kappa\kappa}$-sentences true in the structure $(V_\kappa, \in, U, x)_{x \in V_\kappa}$, together with the sentences

$$\begin{aligned}
&c \text{ is an ordinal}, \\
&\alpha < c \text{ (for all } \alpha < \kappa),
\end{aligned}$$

where c is a new individual constant. The language here clearly has κ many symbols. Every subset of Γ of size less than κ has a model; namely we can take $(V_\kappa, \in, U, x, \beta)_{x \in V_\kappa}$, choosing β greater than each α appearing in the sentences of Γ. Hence by weak compactness, Γ has a model $(M, E, W, k_x, y)_{x \in V_\kappa}$. This model is well-founded, since the sentence

$$\neg \exists v_0 v_1 \ldots \left[\bigwedge_{n \in \omega} (v_{n+1} \in v_n) \right]$$

holds in $(V_\kappa, \in, U, x)_{x \in V_\kappa}$, and hence in $(M, E, W, k_x, y)_{x \in V_\kappa}$.

Note that k is an injection of V_κ into M. Let F be a bijection from $M \backslash \mathrm{rng}(k)$ onto $\{(V_\kappa, u) : u \in M \backslash \mathrm{rng}(k)\}$. Then $G \stackrel{\text{def}}{=} k^{-1} \cup F^{-1}$ is one-one, mapping M onto some set N such that $V_\kappa \subseteq N$. We define, for $x, z \in N$, $xE'z$ iff $G^{-1}(x) E G^{-1}(z)$. Then G is an isomorphism from $(M, E, W, k_x, y)_{x \in V_\kappa}$ onto $\overline{N} \stackrel{\text{def}}{=} (N, E', G[W], x, G(y))_{x \in V_\kappa}$. Of course $\overline{N}$ is still well-founded. It is also extensional, since the extensionality axiom holds in $(V_\kappa, \in)$ and hence in (M, E) and (N, E'). Let H, P be the Mostowski collapse of (N, E'). Thus P is a transitive set, and

(1) H is an isomorphism from (N, E') onto $(P, \in)$.

(2) $\forall a, b \in N[aE'b \in V_\kappa \rightarrow a \in b]$.

In fact, suppose that $a, b \in N$ and $aE'b \in V_\kappa$. Let the individual constants used in the expansion of $(V_\kappa, \in, U)$ to $(V_\kappa, \in, U, x)_{a\in V_\kappa}$ be $\langle c_x : x \in V_\kappa\rangle$. Then

$$(V_\kappa, \in, U, x)_{a\in V_\kappa} \models \forall z \left[z \in k_b \to \bigvee_{w\in b} (z = k_w) \right],$$

and hence this sentence holds in $(N, E', G[W], x, G(y))_{x\in V_\kappa}$ as well, and so there is a $w \in b$ such that $a = w$, i.e., $a \in b$. So (2) holds.

(3) $\forall a, b \in V_\kappa[a \in b \to aE'b]$

In fact, suppose that $a, b \in V_\kappa$ and $a \in b$. Then the sentence $k_a \in k_b$ holds in $(V_\kappa, \in, U, x)_{x\in V_\kappa}$, so it also holds in $(N, E', G[W], x, G(y))_{x\in V_\kappa}$, so that $aE'b$.

We have now verified the hypotheses of Lemma 22.12. It follows that $H \restriction V_\kappa$ is the identity. In particular, $V_\kappa \subseteq P$. Now take any sentence σ in the language of $(V_\kappa, \in, U, x)_{x\in V_\kappa}$. Then

$$\begin{aligned}
(V_\kappa, \in, U, x)_{x\in V_\kappa} \models \sigma \quad &\text{iff} \quad (M, E, W, k_x)_{x\in V_\kappa} \models \sigma\\
&\text{iff} \quad (N, E', G[W], x)_{x\in V_\kappa} \models \sigma\\
&\text{iff} \quad (P, \in, H[G[W]], x)_{x\in V_\kappa} \models \sigma.
\end{aligned}$$

Thus $(P, \in, H[G[W]])$ is an elementary extension of $(V_\kappa, \in, U)$.

Now for $\alpha < \kappa$ we have

$$\begin{aligned}
&(M, E, W, k_x, y)_{x\in V_\kappa} \models [y \text{ is an ordinal and } k_\alpha Ey], \quad \text{hence}\\
&(N, E', G[W], x, G(y))_{x\in V_\kappa} \models [G(y) \text{ is an ordinal and } \alpha E' G(y)], \quad \text{hence}\\
&(P, \in, H[G[W]], x, H(G(y)))_{x\in V_\kappa} \models [H(G(y)) \text{ is an ordinal and } \alpha \in H(G(y))].
\end{aligned}$$

Thus $H(G(y))$ is an ordinal in P greater than each $\alpha < \kappa$, so since P is transitive, $\kappa \in P$. □

An infinite cardinal κ is *first-order describable* iff there is a $U \subseteq V_\kappa$ and a sentence σ in the language for $(V_\kappa, \in, U)$ such that $(V_\kappa, \in, U) \models \sigma$, while there is no $\alpha < \kappa$ such that $(V_\alpha, \in, U \cap V_\alpha) \models \sigma$.

Theorem 22.14. *If κ is infinite but not inaccessible, then it is first-order describable.*

Proof. ω is describable by the sentence that says that κ is the first limit ordinal; absoluteness is used. The subset U is not needed for this. Now suppose that κ is singular.

Let $\lambda = \mathrm{cf}(\kappa)$, and let f be a function whose domain is some ordinal $\gamma < \kappa$ with $\mathrm{rng}(f)$ cofinal in κ. Let $U = \{(\lambda, \beta, f(\beta)) : \beta < \lambda\}$. Let σ be the sentence expressing the following:

For every ordinal γ there is an ordinal δ with $\gamma < \delta$, U is nonempty, and there is an ordinal μ and a function g with domain μ such that U consists of all triples $(\mu, \beta, g(\beta))$ with $\beta < \mu$.

Clearly $(V_\kappa, \in, U) \models \sigma$. Suppose that $\alpha < \kappa$ and $(V_\alpha, \in, V_\alpha \cap U) \models \sigma$. Then α is a limit ordinal, and there is an ordinal $\gamma < \alpha$ and a function g with domain γ such that $V_\alpha \cap U$ consists of all triples $(\gamma, \beta, g(\beta))$ with $\beta < \gamma$. (Some absoluteness is used.) Now $V_\alpha \cap U$ is nonempty; choose $(\gamma, \beta, g(\beta))$ in it. Then $\gamma = \lambda$ since it is in U. It follows that $g = f$. Choose $\beta < \lambda$ such that $\alpha < f(\beta)$. Then $(\lambda, \beta, f(\beta)) \in U \cap V_\alpha$. Since $\alpha < f(\beta)$, it follows that α has rank less than α, contradiction.

Now suppose that $\lambda < \kappa \leq 2^\lambda$. A contradiction is reached similarly, as follows. Let f be a function whose domain is $\mathscr{P}(\lambda)$ with range κ. Let $U = \{(\lambda, B, f(B)) : B \subseteq \lambda\}$. Let σ be the sentence expressing the following:

For every ordinal γ there is an ordinal δ with $\gamma < \delta$, U is nonempty, and there is an ordinal μ and a function g with domain $\mathscr{P}(\mu)$ such that U consists of all triples $(\mu, B, g(B))$ with $B \subseteq \mu$.

Clearly $(V_\kappa, \in, U) \models \sigma$. Suppose that $\alpha < \kappa$ and $(V_\alpha, \in, V_\alpha \cap U) \models \sigma$. Then α is a limit ordinal, and there is an ordinal $\gamma < \alpha$ and a function g with domain $\mathscr{P}(\gamma)$ such that $V_\alpha \cap U$ consists of all triples $(\gamma, B, g(B))$ with $B \subseteq \gamma$. (Some absoluteness is used.) Clearly $\gamma = \lambda$; otherwise $U \cap V_\alpha$ would be empty. Note that $g = f$. Choose $B \subseteq \lambda$ such that $\alpha = f(B)$. Then $(\lambda, B, f(B)) \in U \cap V_\alpha$. Again this implies that α has rank less than α, contradiction. □

The new equivalent of weak compactness involves second-order logic. We augment first order logic by adding a new variable S ranging over subsets rather than elements. There is one new kind of atomic formula: Sv with v a first-order variable. This is interpreted as saying that v is a member of S.

Now an infinite cardinal κ is Π^1_1-*indescribable* iff for every $U \subseteq V_\kappa$ and every second-order sentence σ of the form $\forall S\varphi$, with no quantifiers on S within φ, if $(V_\kappa, \in, U) \models \sigma$, then there is an $\alpha < \kappa$ such that $(V_\alpha, \in, U \cap V_\alpha) \models \sigma$. Note that if κ is Π^1_1-indescribable then it is not first-order describable.

Theorem 22.15. *An infinite cardinal κ is weakly compact iff it is Π^1_1-indescribable.*

Proof. First suppose that κ is Π^1_1-indescribable. By Theorem 22.14 it is inaccessible. So it suffices to show that it has the tree property. By the proof of Theorem 22.7(iii)⇒(iv) it suffices to check the tree property for a tree $T \subseteq {}^{<\kappa}\kappa$. Note that ${}^{<\kappa}\kappa \subseteq V_\kappa$. Let σ be the following sentence in the second-order language of $(V_\kappa, \in, T)$:

$$\exists S[T \text{ is a tree under } \subset, \text{ and}$$
$$S \subseteq T \text{ and } S \text{ is a branch of } T \text{ of unbounded length}].$$

Thus for each $\alpha < \kappa$ the sentence σ holds in $(V_\alpha, \in, T \cap V_\alpha)$. Hence it holds in $(V_\kappa, \in, T)$, as desired.

Now suppose that κ is weakly compact. Let $U \subseteq V_\kappa$, and let σ be a Π^1_1-sentence holding in $(V_\kappa, \in, U)$. By Lemma 22.13, let $(M, \in, U')$ be a transitive elementary extension of $(V_\kappa, \in, U)$ such that $\kappa \in M$. Say that σ is $\forall S\varphi$, with φ having no quantifiers on S. Now

$$\forall X \subseteq V_\kappa[(V_\kappa, \in, U) \models \varphi(X)]. \tag{1}$$

Now since $\kappa \in M$ and $(M, \in)$ is a model of ZFC, V_κ^M exists, and by absoluteness it is equal to V_κ. Hence by (1) we get

$$(M, \in, U') \models \forall X \subseteq V_\kappa \varphi^{V_\kappa}(U' \cap V_\kappa).$$

Hence

$$(M, \in, U') \models \exists\alpha\forall X \subseteq V_\alpha \varphi^{V_\alpha}(U' \cap V_\alpha),$$

so by the elementary extension property we get

$$(V_\kappa, \in, U) \models \exists\alpha\forall X \subseteq V_\alpha \varphi^{V_\alpha}(U' \cap V_\alpha).$$

We choose such an α. Since $V_\kappa \cap \mathbf{On} = \kappa$, it follows that $\alpha < \kappa$. Hence $(V_\alpha, \in, U' \cap V_\alpha) \models \sigma$, as desired. □

Measurable cardinals

Our third kind of large cardinal is the class of measurable cardinals. Although, as the name suggests, this notion comes from measure theory, the definition and results we give are purely set-theoretical. Moreover, similarly to weakly compact cardinals, it is not obvious from the definition that we are dealing with large cardinals.

The definition is given in terms of the notion of an ultrafilter on a set.

• Let X be a nonempty set. A *filter* on X is a family $\mathscr{F}$ of subsets of X satisfying the following conditions:

(i) $X \in \mathscr{F}$.

(ii) If $Y, Z \in \mathscr{F}$, then $Y \cap Z \in \mathscr{F}$.

(iii) If $Y \in \mathscr{F}$ and $Y \subseteq Z \subseteq X$, then $Z \in \mathscr{F}$.

• A filter $\mathscr{F}$ on a set X is *proper* or *nontrivial* iff $\emptyset \notin \mathscr{F}$.

• An *ultrafilter* on a set X is a nontrivial filter $\mathscr{F}$ on X such that for every $Y \subseteq X$, either $Y \in \mathscr{F}$ or $X \backslash Y \in \mathscr{F}$.

• A family $\mathscr{A}$ of subsets of X has the *finite intersection property*, fip, iff for every finite subset $\mathscr{B}$ of $\mathscr{A}$ we have $\bigcap \mathscr{B} \neq \emptyset$.

• If $\mathscr{A}$ is a family of subsets of X, then the *filter generated by* $\mathscr{A}$ is the set

$$\{Y \subseteq X : \bigcap \mathscr{B} \subseteq Y \text{ for some finite } \mathscr{B} \subseteq \mathscr{A}\}.$$

[Clearly this is a filter on X, and it contains $\mathscr{A}$.]

Proposition 22.16. *If $x \in X$, then $\{Y \subseteq X : x \in Y\}$ is an ultrafilter on X.* □

An ultrafilter of the kind given in this proposition is called a *principal ultrafilter*. There are nonprincipal ultrafilters on any infinite set, as we will see shortly.

Proposition 22.17. *Let $\mathscr{F}$ be a proper filter on a set X. Then the following are equivalent:*

(i) $\mathscr{F}$ is an ultrafilter.

(ii) $\mathscr{F}$ is maximal in the partially ordered set of all proper filters (under $\subseteq$).

Proof. (i)$\Rightarrow$(ii): Assume (i), and suppose that $\mathscr{G}$ is a filter with $\mathscr{F} \subset \mathscr{G}$. Choose $Y \in \mathscr{G} \backslash \mathscr{F}$. Since $Y \notin \mathscr{F}$, we must have $X \backslash Y \in \mathscr{F} \subseteq \mathscr{G}$. So $Y, X \backslash Y \in \mathscr{G}$, hence $\emptyset = Y \cap (X \backslash Y) \in \mathscr{G}$, and so $\mathscr{G}$ is not proper.

(ii)$\Rightarrow$(i): Assume (ii), and suppose that $Y \subseteq X$, with $Y \notin \mathscr{F}$; we want to show that $X \backslash Y \in \mathscr{F}$. Let

$$\mathscr{G} = \{Z \subseteq X : Y \cap W \subseteq Z \text{ for some } W \in \mathscr{F}\}.$$

Clearly $\mathscr{G}$ is a filter on X, and $\mathscr{F} \subseteq \mathscr{G}$. Moreover, $Y \in \mathscr{G} \backslash \mathscr{F}$. It follows that $\mathscr{G}$ is not proper, and so $\emptyset \in \mathscr{G}$. Thus there is a $W \in \mathscr{F}$ such that $Y \cap W = \emptyset$. Hence $W \subseteq X \backslash Y$, and hence $X \backslash Y \in \mathscr{F}$, as desired. $\square$

Theorem 22.18. *For any infinite set X there is a nonprincipal ultrafilter on X. Moreover, if $\mathscr{A}$ is any collection of subsets of X with fip, then $\mathscr{A}$ can be extended to an ultrafilter.*

Proof. First we show that the first assertion follows from the second. Let $\mathscr{A}$ be the collection of all cofinite subsets of X—the subsets whose complements are finite. $\mathscr{A}$ has fip, since if $\mathscr{B}$ is a finite subset of $\mathscr{A}$, then $X \backslash \bigcap \mathscr{B} = \bigcup_{Y \in \mathscr{B}} (X \backslash B)$ is finite. By the second assertion, $\mathscr{A}$ can be extended to an ultrafilter F. Clearly F is nonprincipal.

To prove the second assertion, let $\mathscr{A}$ be a collection of subsets of X with fip, and let $\mathscr{C}$ be the collection of all proper filters on X which contain $\mathscr{A}$. Clearly the filter generated by $\mathscr{A}$ is proper, so $\mathscr{C} \neq \emptyset$. We consider $\mathscr{C}$ as a partially ordered set under inclusion. Any subset $\mathscr{D}$ of $\mathscr{C}$ which is a chain has an upper bound in $\mathscr{C}$, namely $\bigcup \mathscr{D}$, as is easily checked. So by Zorn's lemma $\mathscr{C}$ has a maximal member F. By Proposition 22.16, F is an ultrafilter. $\square$

• Let X be an infinite set, and let κ be an infinite cardinal. An ultrafilter F on X is κ-*complete* iff for any $\mathscr{A} \in [F]^{<\kappa}$ we have $\bigcap \mathscr{A} \in F$. We also say σ-complete synonomously with $\aleph_1$-complete.

This notion is clearly a generalization of one of the properties of ultrafilters. In fact, every ultrafilter is ω-complete, and every principal ultrafilter is κ-complete for every infinite cardinal κ.

Lemma 22.19. *Suppose that X is an infinite set, F is an ultrafilter on X, and κ is the least infinite cardinal such that there is an $\mathscr{A} \in [F]^{\kappa}$ such that $\bigcap \mathscr{A} \notin F$. Then there is a partition $\mathscr{P}$ of X such that $|\mathscr{P}| = \kappa$ and $X \backslash Y \in F$ for all $Y \in \mathscr{P}$.*

Proof. Let $\langle Y_\alpha : \alpha < \kappa \rangle$ enumerate $\mathscr{A}$. Let $Z_0 = X \backslash Y_0$, and for $\alpha > 0$ let $Z_\alpha = (\bigcap_{\beta < \alpha} Y_\beta) \backslash Y_\alpha$. Note that $Y_\alpha \subseteq X \backslash Z_\alpha$, and so $X \backslash Z_\alpha \in F$. Clearly $Z_\alpha \cap Z_\beta = \emptyset$ for $\alpha \neq \beta$. Let $W = \bigcap_{\alpha < \lambda} Y_\alpha$. Clearly $W \cap Z_\alpha = \emptyset$ for all $\alpha < \lambda$. Let

$$\mathscr{P} = (\{Z_\alpha : \alpha < \kappa\} \cup \{W\}) \backslash \{\emptyset\}.$$

So $\mathscr{P}$ is a partition of X and $X \backslash Z \in F$ for all $Z \in \mathscr{P}$. Clearly $|\mathscr{P}| \leq \kappa$. If $|\mathscr{P}| < \kappa$, then

$$\emptyset = \bigcap_{Z \in \mathscr{P}} (X \backslash Z) \in F,$$

contradiction. So $|\mathscr{P}| = \kappa$. $\square$

Theorem 22.20. *Suppose that κ is the least infinite cardinal such that there is a nonprincipal σ-complete ultrafilter F on κ. Then F is κ-complete.*

Proof. Assume the hypothesis, but suppose that F is not κ-complete. So there is a $\mathscr{A} \in [F]^{<\kappa}$ such that $\bigcap \mathscr{A} \notin F$. Hence by Lemma 22.19 there is a partition $\mathscr{P}$ of κ such that $|\mathscr{P}| < \kappa$ and $X \backslash P \in F$ for every $P \in \mathscr{P}$. Let $\langle P_\alpha : \alpha < \lambda \rangle$ be a one-one enumeration of $\mathscr{P}$, λ an infinite cardinal. We are now going to construct a nonprincipal σ-complete ultrafilter G on λ, which will contradict the minimality of κ.

Define $f : \kappa \to \lambda$ by letting $f(\beta)$ be the unique $\alpha < \lambda$ such that $\beta \in P_\alpha$. Then we define

$$G = \{D \subseteq \lambda : f^{-1}[D] \in F\}.$$

We check the desired conditions for G. $\emptyset \notin G$, since $f^{-1}[\emptyset] = \emptyset \notin F$. If $D \in G$ and $D \subseteq E$, then $f^{-1}[D] \in F$ and $f^{-1}[D] \subseteq f^{-1}[E]$, so $f^{-1}[E] \in F$ and hence $E \in G$. Similarly, G is closed under $\cap$. Given $D \subseteq \lambda$, either $f^{-1}[D] \in F$ or $f^{-1}[\lambda \backslash D] = \kappa \backslash f^{-1}[D] \in F$, hence $D \in G$ or $\lambda \backslash D \in G$. So G is an ultrafilter on λ. It is nonprincipal, since for any $\alpha < \lambda$ we have $f^{-1}[\{\alpha\}] = P_\alpha \notin F$ and hence $\{\alpha\} \notin G$. Finally, G is σ-complete, since if $\mathscr{D}$ is a countable subset of G, then

$$f^{-1}\left[\bigcap \mathscr{D}\right] = \bigcap_{P \in \mathscr{D}} f^{-1}[P] \in F,$$

and hence $\bigcap \mathscr{D} \in G$. $\square$

We say that an uncountable cardinal κ is *measurable* iff there is a κ-complete nonprincipal ultrafilter on κ.

Theorem 22.21. *Every measurable cardinal is weakly compact.*

Proof. Let κ be a measurable cardinal, and let U be a nonprincipal κ-complete ultrafilter on κ.

Since U is nonprincipal, $\kappa \backslash \{\alpha\} \in U$ for every $\alpha < \kappa$. Then κ-completeness implies that $\kappa \backslash F \in U$ for every $F \in [\kappa]^{<\kappa}$.

Now we show that κ is regular. For, suppose it is singular. Then we can write $\kappa = \bigcup_{\alpha<\lambda} \Gamma_\alpha$, where $\lambda < \kappa$ and each Γ_α has size less than κ. So by the previous paragraph, $\kappa \backslash \Gamma_\alpha \in U$ for every $\alpha < \kappa$, and hence

$$\emptyset = \bigcap_{\alpha<\lambda} (\kappa \backslash \Gamma_\alpha) \in U,$$

contradiction.

Next, κ is strong limit. For, suppose that $\lambda < \kappa$ and $2^\lambda \geq \kappa$. Let $S \in [{}^\lambda 2]^\kappa$. Let $\langle f_\alpha : \alpha < \kappa \rangle$ be a one-one enumeration of S. Now for each $\beta < \lambda$, one of the sets

$\{\alpha < \kappa : f_\alpha(\beta) = 0\}$ and $\{\alpha < \kappa : f_\alpha(\beta) = 1\}$ is in U, so we can let $\varepsilon(\beta) \in 2$ be such that $\{\alpha < \kappa : f_\alpha(\beta) = \varepsilon(\beta)\} \in U$. Then

$$\bigcap_{\beta<\lambda} \{\alpha < \kappa : f_\alpha(\beta) = \varepsilon(\beta)\} \in U;$$

this set clearly has only one element, contradiction.

Thus we now know that κ is inaccessible. Finally, we check the tree property. Let $(T, \prec)$ be a tree of height κ such that every level has size less than κ. Then $|T| = \kappa$, and we may assume that actually $T = \kappa$. Let $B = \{\alpha < \kappa : \{t \in T : \alpha \preceq t\} \in U\}$. Clearly any two elements of B are comparable under $\prec$. Now take any $\alpha < \kappa$; we claim that $\mathrm{Lev}_\alpha(T) \cap B \neq \emptyset$. In fact,

$$\kappa = \{t \in T : \mathrm{ht}(t, T) < \alpha\} \cup \bigcup_{t \in \mathrm{Lev}_\alpha(T)} \{s \in T : t \preceq s\}. \tag{1}$$

Now by regularity of κ we have $|\{t \in T : \mathrm{ht}(t, T) < \alpha\}| < \kappa$, and so the complement of this set is in U, and then (1) yields

$$\bigcup_{t \in \mathrm{Lev}_\alpha(T)} \{s \in T : t \preceq s\} \in U. \tag{2}$$

Now $|\mathrm{Lev}_\alpha(T)| < \kappa$, so from (2) our claim easily follows.

Thus B is a branch of size κ, as desired. □

A diagram of large cardinals

We define some more large cardinals, and then indicate relationships between them by a diagram.

All cardinals are assumed to be uncountable.

1. **regular limit cardinals**.

2. **inaccessible**.

3. **Mahlo**.

4. **weakly compact**.

5. **indescribable**. The ω-order language is an extension of first order logic in which one has variables of each type $n \in \omega$. For n positive, a variable of type n ranges over $\mathscr{P}^n(A)$ for a given structure A. In addition to first-order atomic formulas, one has formulas $P \in Q$ with P n-th order and Q $(n+1)$-order. Quantification is allowed over the higher order variables.

κ is *indescribable* iff for all $U \subseteq V_\kappa$ and every higher order sentence σ, if $(V_\kappa, \in, U) \models \sigma$ then there is an $\alpha < \kappa$ such that $(V_\alpha, \in, U \cap V_\alpha) \models \sigma$.

6. $\kappa \to (\omega)_2^{<\omega}$. Here in general

$$\kappa \to (\alpha)_m^{<\omega}$$

means that for every function $f : \bigcup_{n\in\omega}[\kappa]^n \to m$ there is a subset $H \subseteq \kappa$ of order type α such that for each $n \in \omega$, $f \restriction [H]^n$ is constant.

7. $0^\sharp$ exists. This means that there is a non-identity elementary embedding of L into L. Thus no actual cardinal is referred to. But $0^\sharp$ implies the existence of some large cardinals, and the existence of some large cardinals implies that $0^\sharp$ exists.

8. **Jónsson** κ is a Jónsson cardinal iff every model of size κ has a proper elementary substructure of size κ.

9. **Rowbottom** κ is a Rowbottom cardinal iff for every uncountable $\lambda < \kappa$, every model of type (κ, λ) has an elementary submodel of type (κ, ω).

10. **Ramsey** $\kappa \to (\kappa)_2^{<\omega}$.

11. **measurable**

12. **strong** κ is a strong cardinal iff for every set X there exists a nontrivial elementary embedding from V to $\mathbf{M}$ with κ the first ordinal moved and with $\kappa \in \mathbf{M}$.

13. **Woodin** κ is a Woodin cardinal iff

$$\forall A \subseteq V_\kappa \forall \lambda < \kappa \exists \mu \in (\lambda, \kappa) \forall \nu < \kappa \exists j [j \text{ is a nontrivial elementary embedding of } V$$
$$\text{into some set } \mathbf{M}, \text{ with } \mu \text{ the first ordinal moved, such that}$$
$$j(\mu) > \nu, V_\nu \subseteq \mathbf{M}, A \cap V_\nu = j(A) \cap V_\nu]$$

14. **superstrong** κ is superstrong iff there is a nontrivial elementary embedding $j : V \to \mathbf{M}$ with κ the first ordinal moved, such that $V_{j(\kappa)} \subseteq \mathbf{M}$.

15. **strongly compact** κ is strongly compact iff for any $L_{\kappa\kappa}$-language, if Γ is a set of sentences and every subset of Γ of size less than κ has a model, then Γ itself has a model.

16. **supercompact** κ is supercompact iff for every A with $|A| \geq \kappa$ there is normal measure on $P_\kappa(A)$.

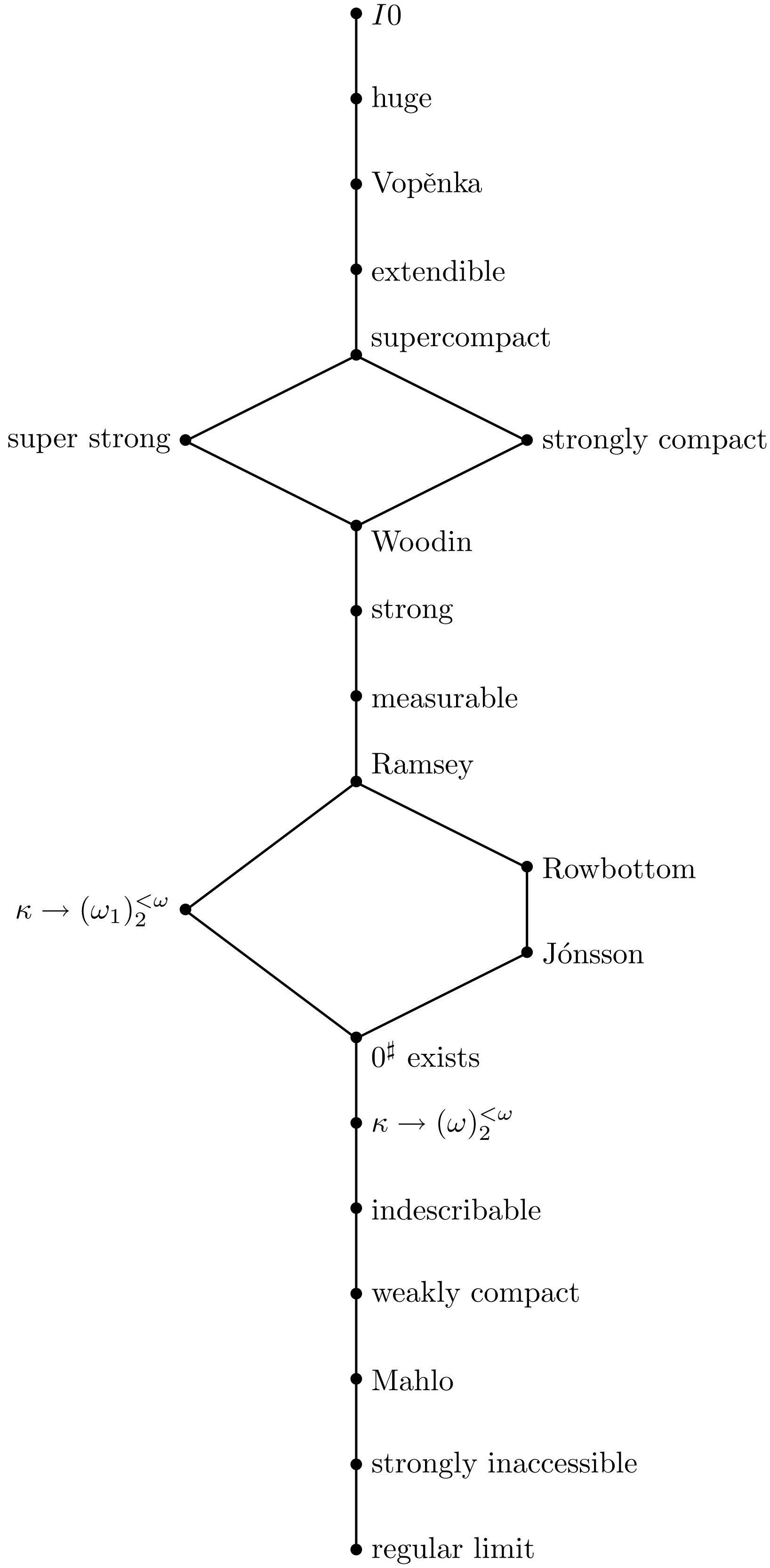
17. **extendible** For an ordinal η, we say that k is η-extendible iff there exist ζ and a nontrivial elementary embedding $j : V_{\kappa+\eta} \to V_\zeta$ with κ first ordinal moved, with $\eta < j(\kappa)$. κ is extendible iff it is η-extendible for every $\eta > 0$.

22. **Vopěnka's principle** If C is a proper class of models in a given first-order language, then there exist two distinct members $A, B \in C$ such that A can be elementarily embedded in B.

19. **huge** A cardinal κ is huge iff there is a nontrivial elementary embedding $j : V \to \mathbf{M}$ with κ the first ordinal moved, such that $\mathbf{M}^{j(\kappa)} \subseteq \mathbf{M}$.

20. $I0$. There is an ordinal δ and a proper elementary embedding j of $L(V_{\delta+1})$ into $L(V_{\delta+1})$ such that the first ordinal moved is less than δ.

In the diagram on the next page, a line indicates that (the consistency of the) existence of the cardinal above implies (the consistency of the) existence of the one below.



$I0$
huge
Vopěnka
extendible
supercompact
super strong
strongly compact
Woodin
strong
measurable
Ramsey
Rowbottom
$\kappa \to (\omega_1)_2^{<\omega}$
Jónsson
$0^\sharp$ exists
$\kappa \to (\omega)_2^{<\omega}$
indescribable
weakly compact
Mahlo
strongly inaccessible
regular limit

E22.1. Let κ be an uncountable regular cardinal. We define $S < T$ iff S and T are stationary subsets of κ and the following two conditions hold:

(1) $\{\alpha \in T : \text{cf}(\alpha) \leq \omega\}$ is nonstationary in κ.

(2) $\{\alpha \in T : S \cap \alpha$ is nonstationary in $\alpha)\}$ is nonstationary in κ.

Prove that if $\omega < \lambda < \mu < \kappa$, all these cardinals regular, then $E^\kappa_\lambda < E^\kappa_\mu$, where

$$E^\kappa_\lambda = \{\alpha < \kappa : \text{cf}(\alpha) = \lambda\},$$

and similarly for E^κ_μ.

E22.2. Continuing exercise E22.1: Assume that κ is uncountable and regular. Show that the relation $<$ is transitive.

E22.3. If κ is an uncountable regular cardinal and S is a stationary subset of κ, we define

$$\text{Tr}(S) = \{\alpha < \kappa : \text{cf}(\alpha) > \omega \text{ and } S \cap \alpha \text{ is stationary in } \alpha\}.$$

Suppose that A, B are stationary subsets of an uncountable regular cardinal κ and $A < B$. Show that $\text{Tr}(A)$ is stationary.

E22.4. (Real-valued measurable cardinals) We describe a special kind of measure. A measure on a set S is a function $\mu : \mathscr{P}(S) \to [0, \infty)$ satisfying the following conditions:

(1) $\mu(\emptyset) = 0$ and $\mu(S) = 1$.

(2) If $\mu(\{s\}) = 0$ for all $s \in S$,

(3) If $\langle X_i : i \in \omega\rangle$ is a system of pairwise disjoint subsets of S, then $\mu(\bigcup_{i\in\omega} X_i) = \sum_{i\in\omega} \mu(X_i)$. (The X_i's are not necessarily nonempty.)

Let κ be an infinite cardinal. Then μ is *κ-additive* iff for every system $\langle X_\alpha : \alpha < \gamma\rangle$ of nonempty pairwise disjoint sets, with $\gamma < \kappa$, we have

$$\mu\left(\bigcup_{\alpha<\gamma} X_\alpha\right) = \sum_{\alpha<\gamma} \mu(X_\alpha).$$

Here this sum (where the index set γ might be uncountable), is understood to be

$$\sup_{\substack{F\subseteq\gamma,\\ F \text{ finite}}} \sum_{\alpha\in F} \mu(X_\alpha).$$

We say that an uncountable cardinal κ is *real-valued measurable* iff there is a κ-additive measure on κ. Show that every measurable cardinal is real-valued measurable. Hint: let μ take on only the values 0 and 1.

E22.5. Suppose that μ is a measure on a set S. A subset A of S is a *μ-atom* iff $\mu(A) > 0$ and for every $X \subseteq A$, either $\mu(X) = 0$ or $\mu(X) = \mu(A)$. Show that if κ is a real-valued measurable cardinal, μ is a κ-additive measure on κ, and $A \subseteq \kappa$ is a μ-atom, then

$\{X \subseteq A : \mu(X) = \mu(A)\}$ is a κ-complete nonprincipal ultrafilter on A. Conclude that κ is a measurable cardinal if there exist such μ and A.

E22.6. Prove that if κ is real-valued measurable then either κ is measurable or $\kappa \leq 2^\omega$. Hint: if there do not exist any μ-atoms, construct a binary tree of height at most ω_1.

E22.7. Let κ be a regular uncountable cardinal. Show that the diagonal intersection of the system $\langle(\alpha+1, \kappa) : \alpha < \kappa\rangle$ is the set of all limit ordinals less than κ.

E22.8. Let F be a filter on a regular uncountable cardinal κ. We say that F is *normal* iff it is closed under diagonal intersections. Suppose that F is normal, and $(\alpha, \kappa) \in F$ for every $\alpha < \kappa$. Show that every club of κ is in F. Hint: use exercise E22.7.

E22.9. Let F be a proper filter on a regular uncountable cardinal κ. Show that the following conditions are equivalent.

(i) F is normal

(ii) For any $S_0 \subseteq \kappa$, if $\kappa \backslash S_0 \notin F$ and f is a regressive function defined on S_0, then there is an $S \subseteq S_0$ with $\kappa \backslash S \notin F$ and f is constant on S.

E22.10. A *probability measure* on a set S is a real-valued function μ with domain $\mathscr{P}(S)$ having the following properties:

(i) $\mu(\emptyset) = 0$ and $\mu(S) = 1$.

(ii) If $X \subseteq Y$, then $\mu(X) \leq \mu(Y)$.

(iii) $\mu(\{a\}) = 0$ for all $a \in S$.

(iv) If $\langle X_n : n \in \omega\rangle$ is a system of pairwise disjoint sets, then $\mu(\bigcup_{n\in\omega} X_n) = \sum_{n\in\omega} \mu(X_n)$. (Some of the sets X_n might be empty.)

Prove that there does not exist a probability measure on ω_1. Hint: consider an Ulam matrix.

E22.11. Show that if κ is a measurable cardinal, then there is a normal κ-complete nonprincipal ultrafilter on κ. Hint: Let D be a κ-complete nonprincipal ultrafilter on κ. Define $f \equiv g$ iff $f, g \in {}^\kappa\kappa$ and $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in D$. Show that $\equiv$ is an equivalence relation on ${}^\kappa\kappa$. Show that there is a relation $\prec$ on the collection of all $\equiv$-classes such that for all $f, g \in {}^\kappa\kappa$, $[f] \prec [g]$ iff $\{\alpha < \kappa : f(\alpha) < g(\alpha)\} \in D$. Here for any function $h \in {}^\kappa\kappa$ we use $[h]$ for the equivalence class of h under $\equiv$. Show that $\prec$ makes the collection of all equivalence classes into a well-order. Show that there is a $\prec$ smallest equivalence class x such that $\forall f \in x \forall\gamma < \kappa[\{\alpha < \kappa : \gamma < f(\alpha)\} \in D$. Let $E = \{X \subseteq \kappa : f^{-1}[X] \in D\}$. Show that E satisfies the requirements of the exercise.

Reference

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23. Constructible sets

This chapter is devoted to the exposition of Gödel's constructible sets. We will define a proper class $\mathbf{L}$, the class of all constructible sets. The development culminates in the proof of consistency of AC and GCH relative to the consistency of ZF. We also prove the relative consistency of $\diamondsuit$.

Sets are called constructible iff they are built up from the empty set using easily defined procedures. This essentially amounts to replacing the power set operation in the definition of the V_α's by an operation which produces only definable subsets. So first we have to indicate what we mean by "definable subsets".

We also introduce two variants on constructibility: starting from a set B, and allowing parameters from a set C. For the main notion of constructibility we take $B = C = \emptyset$.

We begin with some simple operations on sets, to express definability.

- $\mathrm{Rel}(A, C, n, i) = \{s \in {}^nA : s(i) \in C\}$ for $i < n < \omega$.
- $\mathrm{Proj}'(A, R, i, n) = \{s \in {}^nA : \forall t \in {}^nA[t \upharpoonright (n\backslash\{i\}) = s \upharpoonright (n\backslash\{i\}) \to t \in R]\}$ for $i < n < \omega$ and all $R \subseteq {}^nA$.
- $\mathrm{Diag}_\in(A, n, i, j) = \{s \in {}^nA : s(i) \in s(j)\}$ for $i, j < n < \omega$.
- $\mathrm{Diag}_=(A, n, i, j) = \{s \in {}^nA : s(i) = s(j)\}$ for $i, j < n < \omega$.

These basic functions are applied recursively in the following theorem.

Theorem 23.1. *There is a class function $\mathbf{F} : \omega \times \mathbf{V} \times \mathbf{V} \times \omega \to \mathbf{V}$ such that for any sets A, C and any $k, n \in \omega$ we have*

$$\begin{aligned}\mathbf{F}(0, A, C, n) &= \{\mathrm{Rel}(A, C, n, i) : i < n\} \cup \{\mathrm{Diag}_\in(A, n, i, j) : i, j < n\}\\ &\quad \cup \{\mathrm{Diag}_=(A, n, i, j) : i, j < n\};\\ \mathbf{F}(k+1, A, C, n) &= \mathbf{F}(k, A, C, n) \cup \{{}^nA\backslash R : R \in \mathbf{F}(k, A, C, n)\}\\ &\quad \cup \{({}^nA\backslash R) \cup S : R, S \in \mathbf{F}(k, A, C, n)\}\\ &\quad \cup \{\mathrm{Proj}'(A, R, i, n) : R \in \mathbf{F}(k, A, C, n), i < n\}.\end{aligned}$$

Proof. Let $\mathbf{A} = \omega \times \mathbf{V} \times \mathbf{V} \times \omega$, and define $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$ as follows:

$$(k, A, C, n)\mathbf{R}(k', A', C', n') \quad \text{iff} \quad k, k', n, n' \in \omega \text{ and } A = A' \text{ and } C = C' \text{ and } k < k'.$$

Then $\mathbf{R}$ is well-founded on $\mathbf{A}$. In fact, if X is a nonempty subset of $\mathbf{A}$, let $(k, A, C, n) \in X$ be a member of X with k minimum. Clearly (k, A, C, n) is $\mathbf{R}$-minimal. Also, $\mathbf{R}$ is set-like on $\mathbf{A}$. For, let $(k, A, C, n) \in \mathbf{A}$. Then

$$\mathrm{pred}_{\mathbf{AR}}(k, A, C, n) = \{(k', A', C', n') \in \omega \times \{A\} \times \{C\} \times \omega : k' < k\}.$$

Now define $\mathbf{G} : \mathbf{A} \times \mathbf{V} \to \mathbf{V}$ as follows. Suppose that $A, C \in V$, $n \in \omega$, and $f \in V$. Then we define

$$\begin{aligned}\mathbf{G}(0, A, C, n, f) &= \{\mathrm{Rel}(A, C, n, i) : i < n\} \cup \{\mathrm{Diag}_\in(A, n, i, j) : i, j < n\}\\ &\quad \cup \{\mathrm{Diag}_=(A, n, i, j) : i, j < n\},\end{aligned}$$

If $k \in \omega$, $k = k' + 1$, and f is a function with domain $\mathrm{pred}_{\mathrm{AR}}(k, A, C, n)$, then we define

$$\begin{aligned}\mathbf{G}(k, A, C, n, f) = f(k', A, C, n) &\cup \{{}^n A \backslash R : R \in f(k', A, C, n)\} \\ &\cup \{({}^n A \backslash R) \cup S : R, S \in f(k', A, C, n)\} \\ &\cup \{\mathrm{Proj}'(A, R, i, n) : R \in f(k', A, C, n), i < n\}.\end{aligned}$$

If f is not a function with domain $\mathrm{pred}_{\mathbf{AR}}(k, A, C, n)$, let $\mathbf{G}(k, A, C, n, f) = \emptyset$.

Now let $\mathbf{F} : \mathbf{A} \to \mathbf{V}$ be obtained by the recursion theorem 8.7: $\mathbf{F}(k, A, C, n) = \mathbf{G}(k, A, C, n, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}(k, A, C, n))$ for all $(k, A, C, n) \in \mathbf{A}$. Then for any sets A, C and any $k \in \omega$,

$$\begin{aligned}\mathbf{F}(0, A, C, n) &= \mathbf{G}(0, A, C, n, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}(0, A, C, n)) \\ &= \{\mathrm{Rel}(A, C, n, i) : i < n\} \cup \{\mathrm{Diag}_{\in}(A, n, i, j) : i, j < n\} \\ &\quad \cup \{\mathrm{Diag}_{=}(A, n, i, j) : i, j < n\}; \\ \mathbf{F}(k+1, A, C, n) &= \mathbf{G}(k+1, A, C, n, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}(k+1, A, C, n)) \\ &= \mathbf{F}(k, A, C, n) \cup \{{}^n A \backslash R : R \in \mathbf{F}(k, A, C, n)\} \\ &\quad \cup \{({}^n A \backslash R) \cup S : R, S \in \mathbf{F}(k, A, C, n)\} \\ &\quad \cup \{\mathrm{Proj}'(A, R, i, n) : R \in \mathbf{F}(k, A, C, n), i < n\}.\end{aligned}$$

□

We use $\mathbf{Df}'$ for the function in Theorem 23.1, and then we define $\mathbf{Df} : \mathbf{V} \times \mathbf{V} \times \omega \to \mathbf{V}$ by setting, for any sets A, C and any $n \in \omega$,

$$\mathbf{Df}(A, C, n) = \bigcup_{k \in \omega} \mathbf{Df}'(k, A, C, n).$$

Now let $\mathscr{L}'$ be the first order language for set theory augmented by a unary relation symbol $\mathbf{R}$. Given two sets A, C, a formula $\varphi(v_0, \ldots, v_{n-1})$ and elements $a_0, \ldots, a_{n-1}$ of A, we denote by $\varphi^{A,C}(a_0, \ldots, a_{n-1})$ the statement that φ holds with quantifiers relativized to A and a subformula $\mathbf{R}v_i$ interpreted as saying that $a_i \in C$. It is easy to give a recursive definition of $\varphi^{A,C}(a_0, \ldots, a_{n-1})$.

Lemma 23.2. *Let $n \in \omega$, $\varphi(v_0, \ldots, v_{n-1})$ be a formula of $\mathscr{L}'$ with variables among $v_0, \ldots, v_{n-1}$, and let A, C be any sets, $A \neq \emptyset$. Then*

$$\{s \in {}^n A : \varphi^{A,C}(s(0), \ldots, s(n-1))\} \in \mathbf{Df}(A, C, n).$$

Proof. Induction on φ:

If φ is $\mathbf{R}v_i$, then $i < n$ and

$$\begin{aligned}\{s \in {}^n A : \varphi^{A,C}(s(0), \ldots, s(n-1))\} &= \{s \in {}^n A : s(i) \in C\} \\ &= \mathrm{Rel}(A, C, n, i) \\ &\in \mathbf{Df}'(0, A, C, n) \\ &\subseteq \mathbf{Df}(A, C, n).\end{aligned}$$

If φ is $v_i \in v_j$, then

$$\begin{aligned}\{s \in {}^nA : \varphi^{A,C}(s(0),\ldots,s(n-1))\} &= \{x \in {}^nA : s(i) \in s(j)\}\\ &= \mathrm{Diag}_{\in}(A,n,i,j)\\ &\in \mathbf{Df}'(0,A,C,n)\\ &\subseteq \mathbf{Df}(A,C,n).\end{aligned}$$

If φ is $v_i = v_j$, then

$$\begin{aligned}\{s \in {}^nA : \varphi^{A,C}(s(0),\ldots,s(n-1))\} &= \{x \in {}^nA : s(i) = s(j)\}\\ &= \mathrm{Diag}_{=}(A,n,i,j)\\ &\in \mathbf{Df}'(0,A,C,n)\\ &\subseteq \mathbf{Df}(A,C,n).\end{aligned}$$

Suppose that φ is $\neg\psi$, where

$$\{s \in {}^nA : \psi^{A,C}(s(0),\ldots,s(n-1))\} \in \mathbf{Df}(A,C,n).$$

Say $\{s \in {}^nA : \psi^{A,C}(s(0),\ldots,s(n-1))\} \in \mathbf{Df}'(k,A,C,n)$. Then

$$\begin{aligned}\{s \in {}^nA : \varphi^{A,C}(s(0),\ldots,s(n-1))\} &= {}^nA\backslash\{s \in {}^nA : \psi^{A,C}(s(0),\ldots,s(n-1))\}\\ &\in \mathbf{Df}'(k+1,A,C,n)\\ &\subseteq \mathbf{Df}(A,C,n).\end{aligned}$$

Suppose that φ is $\psi \to \chi$, where

$$\begin{aligned}&\{s \in {}^nA : \psi^{A,C}(s(0),\ldots,s(n-1))\} \in \mathbf{Df}(A,C,n)\\ \text{and } &\{s \in {}^nA : \chi^{A,C}(s(0),\ldots,s(n-1))\} \in \mathbf{Df}(A,C,n).\end{aligned}$$

Say

$$\begin{aligned}&\{s \in {}^nA : \psi^{A,C}(s(0),\ldots,s(n-1))\} \in \mathbf{Df}'(k,A,C,n)\\ \text{and } &\{s \in {}^nA : \chi^{A,C}(s(0),\ldots,s(n-1))\} \in \mathbf{Df}'(l,A,C,n).\end{aligned}$$

We may assume that $k = l$. Then

$$\begin{aligned}&\{s \in {}^nA : \varphi^{A,C}(s(0),\ldots,s(n-1))\}\\ &= \{s \in {}^nA : (\neg\psi)^{A,C}(s(0),\ldots,s(n-1))\} \cup \{s \in {}^nA : \chi^{A,C}(s(0),\ldots,s(n-1))\}\\ &= ({}^nA\backslash\{s \in {}^nA : \psi^{A,C}(s(0),\ldots,s(n-1))\}) \cup \{s \in {}^nA : \chi^{A,C}(s(0),\ldots,s(n-1))\}\\ &\in \mathbf{Df}'(k+1,A,C,n)\\ &\subseteq \mathbf{Df}(A,C,n).\end{aligned}$$

Finally, suppose that φ is $\forall v_i\psi$. We assume inductively that

$$\{s \in {}^nA : \psi^{A,C}(s(0), \ldots, s(n-1))\} \in \mathbf{Df}(A, C, n);$$

say

$$\{s \in {}^nA : \psi^{A,C}(s(0), \ldots, s(n-1))\} \in \mathbf{Df}'(k, A, C, n).$$

Then

$$\begin{aligned}
&\{s \in {}^nA : \varphi^{A,C}(s(0), \ldots, s(n-1))\} \\
&\qquad = \{s \in {}^nA : \forall t \in {}^nA[t \restriction (n\backslash\{i\}) = s \restriction (n\backslash\{i\}) \rightarrow \psi^{A,C}(t(0), \ldots, t(n-1))] \\
&\qquad = \mathrm{Proj}'(A, \{s \in {}^{n+1}A : \psi^{A,C}(s(0), \ldots, s(n-1))\}, n) \\
&\qquad \in \mathbf{Df}'(k+1, A, C, n) \\
&\qquad \subseteq \mathbf{Df}(A, C, n) \qquad \square
\end{aligned}$$

Now we prove a sequence of lemmas leading up to the fact that **Df** is absolute for transitive models of ZF. To do this, we have to extend the definitions of our functions above so that they are defined for all sets, since absoluteness was developed only for such functions. We do this by just letting the values be 0 for arguments not in the domain of the original functions.

Lemma 23.3. *The function* Rel *is absolute for transitive models of* Zf*.*

Proof. $x = \mathrm{Rel}(A, C, n, i)$ iff one of the following conditions holds:

(1) $n \notin \omega$ and $x = 0$,

(2) $n \in \omega$ and $i \notin n$ and $x = 0$,

(3) $i < n \in \omega$ and the following condition holds:

$$\forall s \in x[s \in {}^nA \text{ and } s(i) \in C] \text{ and } \forall s \in {}^nA[s(i) \in C \rightarrow s \in x]. \qquad \square$$

Lemma 23.4. *The function* Proj′ *is absolute for transitive models of* ZF*.*

Proof. $x = \mathrm{Proj}'(A, R, n)$ iff one of the following conditions holds:

(1) $n \notin \omega$ and $x = 0$,

(2) $n \in \omega$ and not$(i < n)$ and $x = 0$.

(3) $n \in \omega$ and $i < n$ and $R \not\subseteq {}^{n+1}A$ and $x = 0$,

(3) $n \in \omega$ and $i < n$ and $R \subseteq {}^{n+1}A$ and the following conditions hold:

(4) $\forall s \in x[s \in {}^nA$ and $\forall t \in {}^nA[t \restriction (n\backslash\{i\}) = s \restriction (n\backslash\{i\}) \rightarrow t \in R]$

(5) $\forall s \in {}^nA[\forall t \in {}^nA[t \restriction (n\backslash\{i\}) = s \restriction (n\backslash\{i\}) \rightarrow t \in R] \rightarrow s \in x]$ $\square$

Lemma 23.5. *The function* $\mathrm{Diag}_\in$ *is absolute for transitive models of* ZF*.*

Proof. $x = \mathrm{Diag}_\in(A, n, i, j)$ iff not($n \in \omega$ and $i, j < n$) and $x = 0$, or $n \in \omega$, $i, j < n$, and the following condition holds:

$$\forall s \in x[s \in {}^nA \wedge s(i) \in s(j)] \wedge \forall s \in {}^nA[s(i) \in s(j) \rightarrow s \in x] \qquad \square$$

Lemma 23.6. *The function* $\mathrm{Diag}_=$ *is absolute for transitive models of* ZF.

Proof. Similar to that of Lemma 23.5. $\square$

Lemma 23.7. *For any sets A and C and any natural number n let*

$$\begin{aligned} T_1(A, C, n) = &\{\mathrm{Rel}(A, C, n, i) : i < n\} \cup \{\mathrm{Diag}_\in(A, n, i, j) : i, j < n\} \\ &\cup \{\mathrm{Diag}_=(A, n, i, j) : i, j < n\}. \end{aligned}$$

Then T_1 is absolute for transitive models of ZF.

Proof. $x = T_1(A, C, n)$ iff not($n \in \omega$) and $x = 0$, or $n \in \omega$ and the following condition holds:

$$\begin{aligned} &\forall y \in x[\exists i < n[y = \mathrm{Rel}(A, C, n, i)] \vee \\ &\qquad\qquad \exists i, j < n[y = \mathrm{Diag}_\in(A, n, i, j) \vee y = \mathrm{Diag}_=(A, n, i, j)]] \\ &\wedge \forall i < n[\mathrm{Rel}(A, C, n, i) \in x] \\ &\wedge \forall i, j < n[\mathrm{Diag}_\in(A, n, i, j) \in x] \\ &\wedge \forall i, j < n[\mathrm{Diag}_=(A, n, i, j) \in x] \end{aligned}$$

Lemma 23.8. *For any sets A, L and any natural number n let*

$$T_2(A, n, L) = \{{}^nA \backslash R : R \in L\}.$$

Then T_2 is absolute for transitive models of ZF.

Proof. $x = T_2(A, n, L)$ iff not($n \in \omega$) and $x = 0$, or $n \in \omega$ and the following condition holds:

$$\forall y \in x \exists R \in L[y = {}^nA \backslash R] \wedge \forall z[\exists R \in L(z = {}^nA \backslash R) \rightarrow z \in x].$$

Here we need a little argument. Let M be a transitive model of ZF. Suppose that $A, n, L \in M$, $n \in \omega$, z is a set, $R \in L$, and $z = {}^nA \backslash R$; we would like to show that $z \in M$. There is a $w \in M$ such that $w = ({}^nA \backslash R)^M$, since M is a model of ZF. By absoluteness, $z = w \in M$, as desired. $\square$

Lemma 23.9. *For all sets A, L and all $n \in \omega$ let $T_3(A, n, L) = \{({}^nA \backslash R) \cup S : R, S \in L\}$. Then T_3 is absolute for transitive models of ZF.*

Proof. $x = T_3(A, n, L)$ iff ($n \notin \omega$ and $x = 0$) or $n \in \omega$ and the following holds:

$$\forall y \in x \exists R, S \in L[y = ({}^nA \backslash R) \cup S] \text{ and } \forall R, S \in L[({}^nA \backslash R) \cup S \in x]. \qquad \square$$

Lemma 23.10. *For any sets A, L and any natural numbers i, n with $i < n$, let $T_4(A, i, n, L)$ be the set $\{\text{Proj}'(A, R, i, n) : R \in L\}$. Then T_4 is absolute for transitive models of* ZF.

Proof. $x = T_4(A, n, X)$ iff one of the following conditions holds:

(1) not$(n \in \omega)$ and $x = 0$.

(2) $n \in \omega$, not$(i < n)$, and $x = 0$.

(3) $i, n \in \omega$, $i < n$, and the following two conditions hold:
(a) $\forall R \in L[\text{Proj}'(A, R, i, n) \in x]$.
(b) $\forall y \in x \exists R \in L[y = \text{Proj}'(A, R, i, n)]$. □

Lemma 23.11. $\mathbf{Df}'$ *is absolute for transitive models of* ZF.

Proof. We are going to apply Theorem 14.28. Let $\mathbf{A}, \mathbf{R}, \mathbf{G}$ be as in the proof of Theorem 23.1. Clearly $\mathbf{A}$ and $\mathbf{R}$ are absolute for transitive models of ZF. Now suppose that A and C are sets, $k, n \in \omega$, and f, X are sets. Then $\mathbf{G}(k, A, C, n, f) = X$ iff one of the following conditions holds:

(1) f is a function with domain $\text{pred}_{\mathbf{AR}}(k, A, C, n)$, $k = 0$, and $X = T_1(A, C, n)$.

(2) f is a function with domain $\text{pred}_{\mathbf{AR}}(k, A, C, n)$, $k = k' + 1$ for some $k' \in \omega$, and

$$X = f(k, A, C, n) \cup T_2(A, n, f(k, A, C, n)) \cup T_3(f(k, A, C, n)) \cup T_4(A, n, f(k, A, C, n + 1).$$

(3) f is not a function with domain $\text{pred}_{\mathbf{AR}}(k, A, C, n)$ and $X = \emptyset$.

Thus $\mathbf{G}$ is absolute for transitive models of ZF.

Next, the statement ($\mathbf{R}$ is set-like on $\mathbf{A}$)^M, with M a model of ZF, can be written as follows:

For all sets $A, C \in M$ and all $k, n \in \omega$ there is a set $X \in M$ such that $X = \{(k', A, C, n') : k' < k\}$.

Clearly this holds in M. It is also clear that this set X is a subset of M.

This verifies the hypotheses of Theorem 14.28, and so $\mathbf{Df}'$ is absolute for transitive models of ZF. □

Lemma 23.12. Df *is absolute for all transitive models of* ZF.

Proof.

$$\begin{aligned} x = \mathbf{Df}(A, C, n) \quad \text{iff} \quad & \forall y \in x \exists k \in \omega[y \in \mathbf{Df}'(k, A, C, n)] \\ & \wedge \forall k \in \omega \forall y \in \mathbf{Df}'(k, A, C, n)[y \in x]. \end{aligned}$$ □

The following is the *definable power set operation*: For any sets A, C,

$$\mathscr{D}(A, C) = \{X \subseteq A : \exists n \in \omega \exists s \in {}^n A \exists R \in \mathbf{Df}(A, C, n + 1)[X = \{x \in A : s^\frown\langle x \rangle \in R\}]\}.$$

Here $s^\frown\langle x \rangle$ is the member t of ${}^{n+1}A$ such that $s \subseteq t$ and $t(n) = x$.

Lemma 23.13. *$\mathscr{D}(A,C)$ is absolute for all transitive models of* ZF*.*

Proof.

$$\begin{aligned} D=\mathscr{D}(A,C) \quad \text{iff} \quad & \forall X\in D\exists n\in\omega\exists s\in {}^nA\exists R\in \mathbf{Df}(A,C,n+1)\\ &\qquad [\forall x\in X(x\in A\wedge s^\frown\langle x\rangle\in R)\\ &\qquad\quad \wedge\forall x\in A(s^\frown\langle x\rangle\in R\to x\in X)]\\ &\quad \wedge\forall X[\exists n\in\omega\exists s\in {}^nA\exists R\in \mathbf{Df}(A,C,n+1)\\ &\qquad [\forall x\in X(x\in A\wedge s^\frown\langle x\rangle\in R)\\ &\qquad\quad \wedge\forall x\in A\wedge\forall s\in {}^nA(s^\frown\langle x\rangle\in R\to x\in X)]\to X\in D]\end{aligned}$$

Here there is an unbounded quantifier $\forall X$, and one must check that if the statement holds in a transitive class model of ZF, then it really holds. This is clear. □

Lemma 23.14. *Let $\varphi(v_0,\ldots,v_{n-1},x)$ be a formula of $\mathscr{L}'$ with the indicated free variables. Then*

$$\forall A\forall C\forall v_0,\ldots,v_{n-1}\in A[\{x\in A:\varphi^{A,C}(v_0,\ldots,v_{n-1},x)\}\in\mathscr{D}(A,C)].$$

Proof. Let $v_0,\ldots,v_{n-1}\in A$ and $R=\{s\in {}^{n+1}A:\varphi^A(s(0),\ldots,s(n))\}$. Then by Lemma 23.2, $R\in\mathbf{Df}(A,C,n+1)$. Clearly

$$\{x\in A:\varphi^{A,C}(v_0,\ldots,v_{n-1},x)\}=\{x\in A:v^\frown\langle x\rangle\in R\},$$

and hence $\{x\in A:\varphi^{A,C}(v_0,\ldots,v_{n-1},x)\}\in\mathscr{D}(A,C)$. □

Lemma 23.15. *For any sets A,C and any $n\in\omega$, $|\mathrm{Df}(A,C,n)|\le\omega$.* □

Lemma 23.16. *Let A and C be any sets. Then:*

(i) $\mathscr{D}(A,C)\subseteq\mathscr{P}(A)$.
(ii) If A is transitive, then $A\subseteq\mathscr{D}(A,C)$.
(iii) If $X\in[A]^{<\omega}$, then $X\in\mathscr{D}(A,C)$.
(iv) If A is infinite, then $|\mathscr{D}(A,C)|=|A|$.
(v) $C\cap A\in\mathscr{D}(A,C)$.

Proof. (i) is obvious. For (ii), let $\varphi(v,x)$ be the formula $x\in v$. Then for any $v\in A$ we have $v=\{x\in A:x\in v\}$ by the transitivity of A, and so $v\in\mathscr{D}(A,C)$ by Lemma 23.14.

For (iii), suppose that $X\in[A]^{<\omega}$. Then there exist an $n\in\omega$ and an $s:n\to A$ with $\mathrm{rng}(s)=X$. For each $i<n$ we have

$$\{s\in {}^{n+1}A:s(n)=s(i)\}=\mathrm{Diag}_=(A,n+1,i,n)\in\mathbf{Df}(A,C,n+1).$$

Hence

$$R\overset{\text{def}}{=}\{s\in {}^{n+1}A:s(n)\in\mathrm{rng}(s\restriction n)\}=\bigcup_{i<n}\{s\in {}^{n+1}A:s(n)=s(i)\}\in\mathbf{Df}(A,C,n+1).$$

Hence

$$X = \{x \in A : s^\frown\langle x\rangle \in R\} \in \mathscr{D}(A, C),$$

as desired.

For (iv), note that

$$\mathscr{D}(A, C) = \bigcup_{\substack{n\in\omega \\ s\in{}^nA}} \{\{x \in A : s^\frown\langle x\rangle \in R\} : R \in \mathbf{Df}(A, C, n)\}.$$

Hence

$$|\mathscr{D}(A, C)| \leq \sum_{n\in\omega}(|{}^nA| \cdot |\mathbf{Df}(A, C, n)|) \leq \omega \cdot |A| \cdot \omega = |A|.$$

On the other hand, $\{a\} \in \mathscr{D}(A, C)$ for each $a \in A$ by (iii), so $|A| \leq |\mathscr{D}(A, C)|$. So (iv) holds.

Finally, if $x \in C \cap A$, then $A \cap C = \{x \in A : x \in C\} \in \mathscr{D}(A, C)$ by Lemma 23.14. $\square$

Now we define the constructible hierarchy. Recall that for any set B, $\mathrm{trcl}(B)$ is the transitive closure of B, i.e., it is the smallest transitive set containing B. It can be defined recursively using the intuition $\mathrm{trcl}(B) = B \cup \bigcup B \cup \bigcup\bigcup B \cup \ldots$. See Theorem 14.5.

Theorem 23.17. *There is a class function* $\mathbf{F} : \mathbf{V} \times \mathbf{V} \times \mathbf{On} \to \mathbf{V}$ *such that for any sets* B, C *and any ordinal* α*,*

$$\begin{aligned}
\mathbf{F}(B, C, 0) &= \mathrm{trcl}(B), \\
\mathbf{F}(B, C, \alpha + 1) &= \mathscr{D}(\mathbf{F}(B, C, \alpha), C), \\
\mathbf{F}(B, C, \alpha) &= \bigcup_{\beta<\alpha} \mathbf{F}(B, C, \beta) \quad \textit{for } \alpha \textit{ limit.}
\end{aligned}$$

Proof. Let $\mathbf{A} = \mathbf{V} \times \mathbf{V} \times \mathbf{On}$ and $\mathbf{R} = \{((B, C, \alpha), (B, C, \beta)) : \alpha, \beta \in \mathbf{On} \text{ and } \alpha < \beta\}$. Clearly $\mathbf{R}$ is well-founded and set-like on $\mathbf{A}$. Define $\mathbf{G} : \mathbf{A} \times \mathbf{V} \to \mathbf{V}$ by setting, for any sets A, C, f and any ordinal α,

$$\mathbf{G}(B, C, \alpha, f) = \begin{cases}
\mathrm{trcl}(B) & \text{if } \alpha = 0, \\
\mathscr{D}(f(B, C, \alpha'), C) & \text{if } \alpha = \alpha' + 1 \text{ and } f \text{ is a function} \\
 & \text{with domain } \mathrm{pred}_{\mathbf{AR}}(B, C, \alpha) \\
\bigcup_{\beta<\alpha} f(B, C, \beta) & \text{if } \alpha \text{ is limit and } f \text{ is a function} \\
 & \text{with domain } \mathrm{pred}_{\mathbf{AR}}(B, C, \alpha) \\
\emptyset & \text{otherwise.}
\end{cases}$$

Let $\mathbf{F}$ be obtained by Theorem 5.7. Clearly $\mathbf{F}$ is as desired. $\square$

With $\mathbf{F}$ as in Theorem 23.17 we define

$$\begin{aligned}
L_\alpha^{BC} &= \mathbf{F}(B, C, \alpha); \\
L_\alpha(B) &= L_\alpha^{\{B\}\emptyset};
\end{aligned}$$

$$
\begin{aligned}
L_\alpha[C] &= L_\alpha^{\emptyset C};\\
\mathbf{L}^{BC} &= \bigcup_{\alpha\in\mathrm{On}} L_\alpha^{BC};\\
\mathbf{L} &= \mathbf{L}^{\emptyset\emptyset}\\
\mathbf{L}(B) &= \mathbf{L}^{B\emptyset};\\
\mathbf{L}[C] &= \mathbf{L}^{\emptyset C}.
\end{aligned}
$$

Corollary 23.18.

$$
\begin{aligned}
L_0^{BC} &= \mathrm{trcl}(B);\\
L_{\alpha+1}^{BC} &= \mathscr{D}(L_\alpha^{BC}, C);\\
L_\alpha^{BC} &= \bigcup_{\beta<\alpha} L_\beta^{BC} \quad \textit{for } \alpha \textit{ limit.}
\end{aligned}
$$

Corollary 23.19.

$$
\begin{aligned}
L_0 &= \emptyset;\\
L_{\alpha+1} &= \mathscr{D}(L_\alpha, \emptyset);\\
L_\alpha &= \bigcup_{\beta<\alpha} L_\beta \quad \textit{for } \alpha \textit{ limit.}
\end{aligned}
$$

Corollary 23.20.

$$
\begin{aligned}
L_0(B) &= \mathrm{trcl}(B);\\
L_{\alpha+1}(B) &= \mathscr{D}(L_\alpha(B), \emptyset);\\
L_\alpha(B) &= \bigcup_{\beta<\alpha} L_\beta(B) \quad \textit{for } \alpha \textit{ limit.}
\end{aligned}
$$

Corollary 23.21.

$$
\begin{aligned}
L_0[C] &= \emptyset;\\
L_{\alpha+1}[C] &= \mathscr{D}(L_\alpha[C], C);\\
L_\alpha[C] &= \bigcup_{\beta<\alpha} L_\beta[C] \quad \textit{for } \alpha \textit{ limit.}
\end{aligned}
$$

Lemma 23.22. *For any ordinal α,*

(i) L_α^{BC} is transitive.

(ii) $L_\beta^{BC} \subseteq L_\alpha^{BC}$ for all $\beta < \alpha$.

Proof. We prove both statements simultaneously by induction on α. Both statements are clear for $\alpha = 0$. Now assume them for α. By (i) for α and Lemma 23.16(ii), it follows that $V_\alpha^{BC} \subseteq \mathscr{D}(V_\alpha^{BC}, C) = V_{\alpha+1}^{BC}$, and this easily gives (ii) for $\alpha+1$. If $x \in y \in L_{\alpha+1}^{BC}$, then $y \in \mathscr{D}(L_\alpha^{BC}, C) \subseteq \mathscr{P}(L_\alpha^{BC})$, so $x \in L_\alpha^{BC} \subseteq L_{\alpha+1}^{BC}$. So $L_{\alpha+1}^{BC}$ is transitive.

If α is a limit ordinal and (i) and (ii) hold for all $\alpha' < \alpha$, clearly they hold for α too. $\square$

Now we have a notion of rank for constructible sets too. Let B, C be sets, with B transitive. For each $x \in \mathbf{L}^{BC}$, its $\mathbf{L}^{BC}$-*rank* is the least ordinal $\rho(x) = \alpha$ such that $x \in L_{\alpha+1}^{BC}$.

Theorem 23.23. *Suppose that B and C are sets, with B transitive. Let $x \in \mathbf{L}^{BC}$ and let α an ordinal. Then*

(i) $L_\alpha^{BC} = \{y \in \mathbf{L}^{BC} : \rho^{BC}(y) < \alpha\}$.

(ii) For all $y \in x$ we have $y \in \mathbf{L}^{BC}$, and $\rho^{BC}(y) < \rho^{BC}(x)$.

(iii) $\alpha \in \mathbf{L}^{\emptyset C}$, and $\rho^{\emptyset C}(\alpha) = \alpha$.

(iv) $L_\alpha^{\emptyset C} \cap \mathrm{On} = \alpha$.

(v) $L_\alpha^{BC} \in L_{\alpha+1}^{BC}$.

(vi) $L_\alpha \subseteq V_\alpha$ for all α.

(vii) $[L_\alpha]^{<\omega} \subseteq L_{\alpha+1}$ for every α.

(viii) $L_n = V_n$ for every $n \in \omega$.

(ix) $L_\omega = V_\omega$.

(x) $B \in L^{BC}$.

(xi) $C \cap L^{BC} \in L^{BC}$.

(xii) If $A \subseteq A'$, then $\mathscr{D}(A, C) \subseteq \mathscr{D}(A', C)$.

(xiii) $L_\alpha^{\emptyset C} \subseteq L_\alpha^{BC}$.

Proof. (i): Suppose that $y \in L_\alpha^{BC}$. Then $\alpha \neq 0$. If α is a successor ordinal $\beta+1$, then $\rho^{BC}(y) \leq \beta < \alpha$. If α is a limit ordinal, then $y \in L_\beta^{BC}$ for some $\beta < \alpha$, hence $y \in L_{\beta+1}^{BC}$ also, so $\rho^{BC}(y) \leq \beta < \alpha$. This proves $\subseteq$.

For $\supseteq$, suppose that $\beta \stackrel{\text{def}}{=} \rho^{BC}(y) < \alpha$. Then $y \in L_{\beta+1}^{BC} \subseteq L_\alpha^{BC}$, as desired.

(ii): Assume that $y \in x$. Let $\rho^{BC}(x) = \beta$. Then $x \in L_{\beta+1}^{BC} = \mathscr{D}(L_\beta^{BC}, C) \subseteq \mathscr{P}(L_\beta^{BC})$, and so $y \in L_\beta^{BC}$. Hence $\rho^{BC}(y) < \rho^{BC}(x)$.

(iv): We prove this by induction on α. It is obvious for $\alpha = 0$, and the inductive step when α is limit is clear. So, suppose that we know that $L_\beta^{\emptyset C} \cap \mathrm{On} = \beta$, and $\alpha = \beta + 1$. If $\gamma \in L_\alpha^{\emptyset C} \cap \mathrm{On}$, then $\gamma \in \mathscr{D}(L_\beta^{\emptyset C}, C) \subseteq \mathscr{P}(L_\beta^{\emptyset C})$, so $\gamma \subseteq L_\beta^{\emptyset C} \cap \mathrm{On} = \beta$; hence $\gamma \leq \beta$. This shows that $L_\alpha^{\emptyset C} \cap \mathrm{On} \subseteq \alpha$. If $\gamma < \beta$, then $\gamma \in L_\beta^{\emptyset C} \cap \mathrm{On} \subseteq L_\alpha^{\emptyset C} \cap \mathrm{On}$. Thus it remains only to show that $\beta \in L_\alpha^{\emptyset C}$. Now there is a natural Δ_0 formula $\varphi(x)$ which expresses that x is an ordinal:

$$\forall y \in x \forall z \in y (z \in x) \wedge \forall y \in x \forall z \in y \forall w \in z (w \in y);$$

this just says that x is transitive and every member of x is transitive. Since $L_\beta^{\emptyset C}$ is transitive, $\varphi(x)$ is absolute for it. Hence

$$\beta = L_\beta^{\emptyset C} \cap \mathrm{On} = \{x \in L_\beta^{\emptyset C} : \varphi^{L_\beta^{\emptyset C}}(x)\}.$$

Hence $\beta \in \mathscr{D}(L^{\emptyset C}_\beta, C) = L^{\emptyset C}_\alpha$, as desired.

(iii): By (iv) we have $L^{\emptyset C}_{\alpha+1} \cap \mathrm{On} = \alpha + 1$, and hence $\alpha \in \alpha + 1 \subseteq L^{\emptyset C}_{\alpha+1}$, so that $\alpha \in \mathbf{L}^{\emptyset C}$ and $\rho^{\emptyset C}(\alpha) \leq \alpha$. By (iv) again, we cannot have $\alpha \in L^{\emptyset C}_\alpha$, so $\rho^{\emptyset C}(\alpha) = \alpha$.

(v): $L^{BC}_\alpha = \{x \in L^{BC}_\alpha : (x = x)^{L^{BC}_\alpha}\} \in \mathscr{D}(L^{BC}_\alpha, C) = L^{BC}_{\alpha+1}$.

(vi): An easy induction on α.

(vii): Clearly $[L^{BC}_\alpha]^{<\omega} \subseteq \mathscr{D}(L^{BC}_\alpha, C) = L^{BC}_{\alpha+1}$.

(viii): By induction on n. It is clear for $n = 0$. Assume that $L_n = V_n$. Thus L_n is finite. Hence by (vii) and (vi),

$$V_{n+1} = \mathscr{P}(V_n) = \mathscr{P}(L_n) = [L_n]^{<\omega} \subseteq L_{n+1} \subseteq V_{n+1},$$

as desired.

(ix): Immediate from (viii).

(x): Obvious.

(xi): For each $c \in C \cap L^{BC}$ let $\alpha(c)$ be an ordinal such that $c \in L^{BC}_{\alpha(c)}$, and let $\beta = \bigcup_{c \in C} \alpha(c)$. Thus $C \cap L^{BC} \subseteq L^{BC}_\beta$. Then by Lemma 23.14,

$$C \cap L^{BC} = C \cap L^{BC}_\beta = \{x \in L^{BC} : x \in C\} \in \mathscr{D}(L^{BC}_\beta, C) = L^{BC}_{\beta+1} \subseteq L^{BC}.$$

(xii): obvious.

(xiii): Clear by induction on α, using (xii). □

Lemma 23.24. *If $\alpha \geq \omega$, then $|L_\alpha| = |\alpha|$.*

Proof. First note by Theorem 23.23(iv) that $\alpha \subseteq L_\alpha$. Hence $|\alpha| \leq |L_\alpha|$ for every ordinal α. So we just need to prove that $|L_\alpha| \leq |\alpha|$ for infinite α.

Now we prove the lemma by induction on α. We assume that for every infinite $\beta < \alpha$ we have $|L_\beta| = |\beta|$. Since L_n is finite for $n \in \omega$ by theorems 23.23(viii), it follows that $|L_\beta| \leq |\alpha|$ for every $\beta < \alpha$. If α is a limit ordinal, then

$$|L_\alpha| \leq \sum_{\beta<\alpha} |L_\beta| \leq |\alpha| \cdot |\alpha| = |\alpha|,$$

If $\alpha = \beta + 1$, then by 23.16(iv), $|L_\alpha| = |\mathscr{D}(L_\beta)| = |L_\beta| = |\beta| = |\alpha|$. □

Lemma 23.24 exhibits an important difference between the hierarchy of sets and the hierarchy of constructible sets. Although the two hierarchies agree up through stage ω, we have $|V_{\omega+1}| = 2^\omega$ and $|L_{\omega+1}| = \omega$, by Theorem 14.9 and Lemma 23.24. The hierarchy of sets continues to create many new sets at each stage, but the hierarchy of constructible sets builds new sets much more slowly. But since, as we will see, it is consistent that $\mathbf{V} = \mathbf{L}$ so that eventually the same sets could be created.

Theorem 23.25. *$\mathbf{L}^{BC}$ is a model of ZF.*

Proof. We take the axioms one by one, using the results of Chapter 14. Extensionality and foundation hold since $\mathbf{L}^{BC}$ is transitive (by Lemma 23.22(i)); see Theorems 14.10 and 14.16.

According to Theorem 14.11, to verify that the comprehension axioms hold in $\mathbf{L}^{BC}$ it suffices to take any formula φ with free variables among $x, z, w_1, \ldots, w_n$, assume that $z, w_1, \ldots, w_n \in \mathbf{L}^{BC}$, and prove that

$$(1) \qquad \{x \in z : \varphi^{\mathbf{L}^{BC}}(x, z, w_1, \ldots, w_n)\} \in \mathbf{L}^{BC}$$

Clearly there is an ordinal α such that $z, w_1, \ldots, w_n \in L^{BC}_\alpha$. By Theorem 14.38, choose an ordinal $\beta > \alpha$ such that the formula $x \in z \wedge \varphi$ is absolute for $L^{BC}_\beta, \mathbf{L}^{BC}$. Then

$$\begin{aligned}
&\{x \in z : \varphi^{\mathbf{L}^{BC}}(x, z, w_1, \ldots, w_n)\}\\
&\quad = \{x \in L^{BC}_\beta : (x \in z \wedge \varphi^{\mathbf{L}^{BC}}(x, z, w_1, \ldots, w_n))\}\\
&\quad = \{x \in L^{BC}_\beta : (x \in z \wedge \varphi^{L^{BC}_\beta}(x, z, w_1, \ldots, w_n))\} \quad \text{(absoluteness)}\\
&\quad \in \mathscr{D}(L^{BC}_\beta, C) \quad \text{(by Lemma 23.14)}\\
&\quad = L^{BC}_{\beta+1},
\end{aligned}$$

and (1) holds.

Pairing: See Theorem 14.12. Suppose that $x, y \in \mathbf{L}^{BC}$. Choose α so that $x, y \in L^{BC}_\alpha$. Then by Lemma 23.14,

$$\{x, y\} = \{z \in L^{BC}_\alpha : (z = x \vee z = y)^{L^{BC}_\alpha}\} \in \mathscr{D}(L^{BC}_\alpha, C) = L^{BC}_{\alpha+1},$$

as desired.

Union: See Theorem 14.13. Suppose that $x \in \mathbf{L}^{BC}$. Choose α so that $x \in L^{BC}_\alpha$. Then

$$\begin{aligned}
&\bigcup x = \{z : \exists u(z \in u \wedge u \in x)\}\\
&\quad = \{z \in L^{BC}_\alpha : (\exists u(z \in u \wedge u \in x))^{L^{BC}_\alpha}\} \quad \text{(since } L^{BC}_\alpha \text{ is transitive)}\\
&\quad \in \mathscr{D}(L^{BC}_\alpha, C) = L^{BC}_{\alpha+1},
\end{aligned}$$

as desired.

Power set: See Theorem 14.14. Suppose that $x \in \mathbf{L}^{BC}$. For each $z \in \mathbf{L}$ such that $z \subseteq x$ choose β_z such that $z \in L^{BC}_{\beta_z}$. Let $\gamma = \sup_{z \subseteq x} L^{BC}_{\beta_z}$. Then, using Theorem 23.23(v),

$$\mathscr{P}(x) \cap \mathbf{L}^{BC} = \{z \in \mathbf{L}^{BC} : z \subseteq x\} \subseteq L^{BC}_\gamma \in L^{BC}_{\gamma+1} \subseteq \mathbf{L}^{BC},$$

as desired.

Replacement: See Theorem 14.15. Suppose that φ is a formula with free variables among $x, y, A, w_1, \ldots, w_n$, we are given $A, w_1, \ldots, w_n \in \mathbf{L}$, and

$$(1) \qquad \forall x \in A \exists ! y[y \in \mathbf{L}^{BC} \wedge \varphi^{\mathbf{L}}(x, y, A, w_1, \ldots, w_n)].$$

For each $x \in A$ let z_x be such that $z_x \in \mathbf{L}^{BC}$ and $\varphi^{\mathbf{L}}(x, z_x, A, w_1, \ldots, w_n)$; we are using the replacement axiom here. Then for each $x \in A$ choose α_x so that $z_x \in L^{BC}_{\alpha_x}$. Let

$\beta = \sup_{x \in A} \alpha_x$. Suppose now that $y \in \mathbf{L}^{BC}$ and $\varphi^{\mathbf{L}^{BC}}(x, y, A, w_1, \ldots, w_n)$ for some $x \in A$. Then by (1), $y = z_x$, and hence $y \in L^{BC}_{\alpha_x} \subseteq L^{BC}_\gamma$. This proves that

$$\{y \in \mathbf{L}^{BC} : \exists x \in A \varphi^{\mathbf{L}^{BC}}(x, y, A, w_1, \ldots, w_n)\} \subseteq L^{BC}_\gamma.$$

Since $L^{BC}_\gamma \in L^{BC}_{\gamma+1} \subseteq \mathbf{L}^{BC}$, this is as desired.

Infinity: Since $\omega \in L^{\emptyset C}_{\omega+1} \subseteq L^{BC}_{\omega+1} \subseteq \mathbf{L}^{BC}$ by Theorem 23.23(iv),(xiii), the infinity axiom holds by Theorem 14.26. □

Theorem 23.26. *Suppose that* $\mathbf{M}$ *is a transitive proper class model of* ZF, $B \in \mathbf{M}$, *and* $C \cap \mathbf{M} \in \mathbf{M}$. *Then* $\mathbf{L}^{BC} = (\mathbf{L}^{BC})^{\mathbf{M}} \subseteq \mathbf{M}$.

Proof. Take any ordinal α. Then $\mathbf{M} \not\subseteq L^{BC}_\alpha$, since $\mathbf{M}$ is a proper class; so choose $x \in \mathbf{M} \backslash L^{BC}_\alpha$. Then $\rho^{BC}(x) \geq \alpha$. Now $\rho^{BC}(x) = (\rho^{BC})^{\mathbf{M}}(x)$ by absoluteness, so $\rho^{BC}(x) \in \mathbf{M}$, and hence $\alpha \in \mathbf{M}$. This proves that $\mathbf{On} \subseteq \mathbf{M}$.

It follows by absoluteness of L^{BC} that

$$(\mathbf{L}^{BC})^{\mathbf{M}} = \{x \in \mathbf{M} : (\exists \alpha (x \in L^{BC}_\alpha))^{\mathbf{M}}\} = \bigcup_{\alpha \in \mathrm{On}} (L^{BC}_\alpha)^{\mathbf{M}} = \bigcup_{\alpha \in \mathrm{On}} L^{BC}_\alpha = \mathbf{L}^{BC}.$$

Hence $\mathbf{L}^{BC} = (\mathbf{L}^{BC})^{\mathbf{M}} \subseteq \mathbf{M}$. □

By Lemma 23.13 we have:

Corollary 23.27. *The function* $\langle L_\alpha(B) : B \in V,\ \alpha \in \mathrm{On}\rangle$ *is absolute for transitive class models of ZF.* □

Corollary 23.28. $\mathbf{L}$ *is a model of* $\mathrm{ZF} + V = L$.

Proof. We want to prove that $\forall x \in \mathbf{L} \exists \alpha \in \mathbf{L}(x \in L_\alpha)^{\mathbf{L}})$. So, let $x \in \mathbf{L}$. Choose α such that $x \in L_\alpha$. Now $x \in L^{\mathbf{L}}_\alpha$ by Lemma 23.27. □

Now we turn to the proof that the axiom of choice holds in $\mathbf{L}$. In fact, we will define a well-order of $\mathbf{L}$.

We will deal frequently with some obvious lexicographic orders, which we uniformly denote by $<_{\mathrm{lex}}$, leaving to the reader exactly which lexicographic order is referred to.

Let A be any set, and n any natural number. For each $R \in \{\mathrm{Diag}_\in(A, n, i, j) : i, j < n\}$, let $(\mathrm{Ch}(0, A, n, R), \mathrm{Ch}(1, A, n, R))$ be the smallest pair (i, j), in the lexicographic order of $\omega \times \omega$ such that $i, j < n$ and $R = \mathrm{Diag}_\in(A, n, i, j)$. Note, for example, that $\mathrm{Diag}_\in(A, 2, 0, 0) = \mathrm{Diag}_\in(A, 2, 1, 1) = \emptyset$. Now we define

$$R <_{0An} S \quad \text{iff} \quad (\mathrm{Ch}(0, A, n, R), \mathrm{Ch}(1, A, n, R)) <_{\mathrm{lex}} \mathrm{Ch}(0, A, n, S), \mathrm{Ch}(1, A, n, S)).$$

Clearly this is a well-order of $\{\mathrm{Diag}_\in(A, n, i, j) : i, j < n\}$.

In a very analogous way we can define a well-order $<_{1An}$ of $\{\mathrm{Diag}_=(A, n, i, j) : i, j < n\}$.

Now we can define a well-order $<_{2An}$ of

$$\{\mathrm{Diag}_\in(A,n,i,j) : i,j<n\} \cup \{\mathrm{Diag}_=(A,n,i,j) : i,j<n\}$$

as follows. For any R,S in this union,

$$\begin{aligned} R <_{2An} S \quad \text{iff} \quad & R,S \in \{\mathrm{Diag}_\in(A,n,i,j) : i,j<n\} \text{ and } R <_{0An} S \\ & \text{or } R \in \{\mathrm{Diag}_\in(A,n,i,j) : i,j<n\}, S \notin \{\mathrm{Diag}_\in(A,n,i,j) : i,j<n\} \\ & \text{or } R,S \notin \{\mathrm{Diag}_\in(A,n,i,j) : i,j<n\} \text{ and } R <_{1An} S. \end{aligned}$$

For the next few constructions, suppose that X and A are sets, $n \in \omega$, and we are given a well-ordering $<$ of X. Then we well-order $\{{}^nA\backslash R : R \in X\}$ by setting

$$S \prec_{0,A,n,<,X} T \quad \text{iff} \quad \exists S',T' \in X[S'<T' \text{ and } S = {}^nA\backslash S', T = {}^nA\backslash T'].$$

We well-order $\{R \cap S : R,S \in X\}$ as follows. Suppose that $U,V \in \{R \cap S : R,S \in X\}$. Let (R,S) be lexicographically smallest in $X \times X$ (using $<$) such that $U = R \cap S$, and let (R',S') be lexicographically smallest in $X \times X$ (using $<$) such that $V = R' \cap S'$. Then $U <_{1,A,n,<,X} V$ iff $(R,S) <_{\mathrm{lex}} (R',S')$.

We well-order $\{\mathrm{Proj}'(A,R,n) : R \in X\}$ as follows. Suppose $U,V \in \{\mathrm{Proj}'(A,R,n) : R \in X\}$. Let R be $<$-minimum in X such that $U = \mathrm{Proj}'(A,R,n)$, and let S be $<$-minimum in X such that $V = \mathrm{Proj}'(A,S,n)$. Then $U <_{2,A,n,<,X} V$ iff $R<S$.

Next, for any set A and any $k,n \in \omega$ we define a well-order $<_{3kAn}$ of $\mathbf{Df}'(k,A,\emptyset,n)$ by induction on k. Let $<_{30An}$ be $<_{2An}$. Assume that $<_{3kAn}$ has been defined for all $n \in \omega$, and let $R,S \in \mathbf{Df}'(k+1,A,\emptyset,n)$. Then we define $R <_{3(k+1)An} S$ iff one of the following conditions holds:

(1) $R,S \in \mathbf{Df}'(k,A,\emptyset,n)$ and $R <_{3kAn} S$

(2) $R \in \mathbf{Df}'(k,A,\emptyset,n)$ and $S \notin \mathbf{Df}'(k,A,\emptyset,n)$.

(3) $R,S \notin \mathbf{Df}'(k,A,\emptyset,n)$, $R,S \in \{{}^nA\backslash T : T \in \mathbf{Df}'(k,A,\emptyset,n)\}$, and

$$R \prec_{0,A,n,<_{3kAn},\mathbf{Df}'(k,A,\emptyset,n)} S.$$

(4) $R,S \notin \mathbf{Df}'(k,A,\emptyset,n)$, $R \in \{{}^nA\backslash T : T \in \mathbf{Df}'(k,A,\emptyset,n)\}$, and $S \notin \{{}^nA\backslash T : T \in \mathbf{Df}'(k,A,\emptyset,n)\}$.

(5) $R,S \notin \mathbf{Df}'(k,A,\emptyset,n)$, $R,S \notin \{{}^nA\backslash T : T \in \mathbf{Df}'(k,A,\emptyset,n)\}$, $R,S \in \{T \cap U : T,U \in \mathbf{Df}'(k,A,\emptyset,n)\}$, and

$$R \prec_{1,A,n,<_{3kAn},\mathrm{Df}'(k,A,n)} S.$$

(6) $R,S \notin \mathbf{Df}'(k,A,\emptyset,n)$, $R,S \notin \{{}^nA\backslash T : T \in \mathbf{Df}'(k,A,\emptyset,n)\}$, $R \in \{T \cap U : T,U \in \mathbf{Df}'(k,A,\emptyset,n)\}$, and $S \notin \{T \cap U : T,U \in \mathbf{Df}'(k,A,\emptyset,n)\}$.

(7) $R,S \notin \mathbf{Df}'(k,A,\emptyset,n)$, $R,S \notin \{{}^nA\backslash T : T \in \mathbf{Df}'(k,A,\emptyset,n)\}$, $R,S \notin \{T \cap U : T,U \in \mathbf{Df}'(k,A,\emptyset,n)\}$, and

$$R \prec_{2,A,n,<_{3kA(n+1)},\mathrm{Df}'(k,A,n)} S.$$

Finally, for any set A and any natural number n, we well-order $\mathbf{Df}(A, \emptyset, n)$ as follows. Let $R, S \in \mathbf{Df}(A, \emptyset, n)$. Let k be minimum such that $R \in \mathbf{Df}'(k, A, \emptyset, n)$, and let l be minimum such that $S \in \mathbf{Df}'(l, A, \emptyset, n)$. Then we define

$$R <_{4An} S \quad \text{iff} \quad k < l, \text{ or } k = l \text{ and } R <_{3kAn} S.$$

We define a well-ordering $<_{5\alpha}$ of L_α by recursion. First of all, $<_{50} = \emptyset$. If α is a limit ordinal, then for any $x, y \in L_\alpha$ we define

$$x <_{5\alpha} y \quad \text{iff} \quad \rho(x) < \rho(y) \vee [\rho(x) = \rho(y) \text{ and } x <_{5\rho(x)} y].$$

Clearly this is a well-order of L_α.

Now suppose that a well-order $<_{5\alpha}$ of L_α has been defined. Then for each $n \in \omega$ we define the lexicographic order $<_{6n\alpha}$ on ${}^nL_\alpha$: for any $x, y \in {}^nL_\alpha$,

$$x <_{6n\alpha} y \quad \text{iff} \quad \exists k < n[x \restriction k = y \restriction k \text{ and } x(k) <_{5\alpha} y(k)].$$

Clearly this is a well-order of ${}^nL_\alpha$. Now for any $X \in L_{\alpha+1} = \mathscr{D}(L_\alpha)$, let $n(X)$ be the least natural number n such that

$$\exists s \in {}^nL_\alpha \exists R \in \mathrm{Df}(L_\alpha, n+1)[X = \{x \in L_\alpha : s^\frown\langle x\rangle \in R\}].$$

Then let $s(X)$ be the least member of ${}^{n(X)}L_\alpha$ (under the well-order $<_{6n(X)\alpha}$) such that

$$\exists R \in \mathrm{Df}(L_\alpha, n(X)+1)[X = \{x \in L_\alpha : s(X)^\frown\langle x\rangle \in R\}].$$

Then let $R(X)$ be the least member of $\mathrm{Df}(L_\alpha, n+1)$ (under the well-order $<_{4L_\alpha(n+1)}$) such that

$$X = \{x \in L_\alpha : s(X)^\frown\langle x\rangle \in R(X)\}].$$

Finally, for any $X, Y \in L_{\alpha+1}$ we define $X <_{5(\alpha+1)} Y$ iff one of the following conditions holds:

(i) $X, Y \in L_\alpha$ and $X <_{5\alpha} Y$.
(ii) $X \in L_\alpha$ and $Y \notin L_\alpha$.
(iii) $X, Y \notin L_\alpha$ and one of the following conditions holds:
  (a) $n(X) < n(Y)$.
  (b) $n(X) = n(Y)$ and $s(X) <_{6n(X)\alpha} s(Y)$.
  (c) $n(X) = n(Y)$ and $s(X) = s(Y)$ and $R(X) <_{4L_\alpha(n+1)} R(Y)$.

Clearly this gives a well-order of $L_{\alpha+1}$.

We denote the union of all the well-orders $<_{5\alpha}$ for $\alpha \in \mathbf{On}$ by $<_\mathbf{L}$. Under $\mathbf{V} = \mathbf{L}$ it is a well-ordering of the universe. Thus we have shown the following.

Theorem 23.29. *$<_\mathbf{L}$ is a well-order of $\mathbf{L}$. Hence $\mathbf{L}$ is a model of ZFC. It follows that if ZF is consistent, then so is ZFC.* □

Now we work up to proving GCH from $\mathbf{V} = \mathbf{L}$. For any set M, let $o(M) = M \cap \mathbf{On}$.

Lemma 23.30. *If M is a transitive set, then $o(M)$ is an ordinal, and is in fact the first ordinal not in M.*

Proof. Since $o(M)$ is a set of ordinals, for the first statement it suffices to show that $o(M)$ is transitive. Suppose that $\alpha \in \beta \in o(M)$. Then $\alpha \in M$ since M is transitive, as desired.

For the second statement, first of all, $o(M) \notin M$, as otherwise $o(M) \in o(M)$. Now suppose that $\alpha \notin M$. If $\alpha < o(M)$, then $\alpha \in M$, contradiction. $\square$

Theorem 23.31. *There is a sentence φ which is a finite conjunction of members of* ZF + $\mathbf{V} = \mathbf{L}$ *such that*

$$ZFC \vdash \forall M[M \text{ transitive } \wedge \varphi^M \to M = L_{o(M)}].$$

Proof. Let φ be a conjunction of $\mathbf{V} = \mathbf{L}$ together with enough of ZF to prove that $\langle L_\alpha : \alpha \in \mathbf{On}\rangle$ is absolute, and also enough to prove that there is no largest ordinal. Then for any transitive set M, if φ^M, then $o(M)$ is a limit ordinal, $(\forall x(x \in \mathbf{L}))^M$ and hence $M = \mathbf{L}^M$, and

$$M = \mathbf{L}^M = \{x \in M : (\exists\alpha(x \in L_\alpha))^M\} = \bigcup_{\alpha\in M} L_\alpha = L_{o(M)}. \qquad \square$$

Theorem 23.32. *If $\mathbf{V} = \mathbf{L}$, then for every infinite ordinal α we have $\mathscr{P}(L_\alpha) \subseteq L_{\alpha^+}$.*

Proof. Let φ be as in Theorem 23.31. Assume that $\mathbf{V} = \mathbf{L}$ and α is an infinite ordinal. Take any $A \in \mathscr{P}(L_\alpha)$. Let $X = L_\alpha \cup \{A\}$. Clearly X is transitive. By Lemma 23.24, $|X| = |\alpha|$. Now by Theorem 14.39 with $\mathbf{Z} = \mathbf{V}$, let M be a transitive set such that $X \subseteq M$, $|M| = |\alpha|$, and $\varphi^M \leftrightarrow \varphi^{\mathbf{V}}$. But $\varphi^{\mathbf{V}}$ actually holds, so φ^M holds. Hence $M = L_{o(M)}$ by Theorem 23.31. Now $o(M) = M \cap \mathbf{On}$, and $|M| = |\alpha|$, so $o(M) < \alpha^+$. Hence $A \in X \subseteq M = L_{o(M)} \subseteq L_{\alpha^+}$. $\square$

Theorem 23.33. $\mathbf{V} = \mathbf{L}$ *implies* AC + GCH.

Proof. Assume $\mathbf{V} = \mathbf{L}$. Then AC holds by 23.29. By Theorem 23.32 we have, for any infinite cardinal κ, $\mathscr{P}(\kappa) \subseteq \mathscr{P}(L_\kappa) \subseteq L_{\kappa^+}$. Since $|L_{\kappa^+}| = \kappa^+$ by Lemma 23.25, it follows that $2^\kappa = \kappa^+$. $\square$

Corollary 23.34. *If* ZF *is consistent, then so is* ZFC + GCH. $\square$

Theorem 23.35. $\mathbf{V} = \mathbf{L}$ *implies* $\diamondsuit$.

Proof. Assume $\mathbf{V} = \mathbf{L}$. By recursion, for each $\alpha < \omega_1$ we define (A_α, C_α) to be the $<_L$-first pair of subsets of α such that C_α is club in α and there is no $\xi \in C_\alpha$ such that $A_\alpha \cap \xi = A_\xi$, or $(A_\alpha, C_\alpha) = (0, 0)$ if this is not possible. Thus $f \stackrel{\text{def}}{=} \langle (A_\alpha, C_\alpha) : \alpha < \omega_1\rangle$ is defined by recursion, and is absolute for models of ZF (or certain finite fragments of it). We claim that $\langle A_\alpha : \alpha < \omega_1\rangle$ is a $\diamondsuit$-sequence.

To prove this, we suppose that it is not a $\diamondsuit$-sequence. Then there is a subset A of ω_1 such that $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is not stationary, and hence there is a club D in ω_1 such that $A \cap \alpha \neq A_\alpha$ for all $\alpha \in D$. We take the $<_L$-first such pair (A, D).

(1) $A, D \in L_{\omega_2}$.

This holds since $\omega_1 \in L_{\omega_1+1}$ by Theorem 23.23(iv), hence $\omega_1 \subseteq L_{\omega_1+1}$, and then (1) follows by Theorem 23.32.

Now we need the following elementary fact:

(2) If $x, y \in L_\alpha$, then $\{x, y\} \in L_{\alpha+1}$.

This is an application of Theorem 23.14

$$\{x, y\} = \{z \in L_\alpha : z = x \text{ or } z = y\} \in \mathscr{D}(L_\alpha, \emptyset) = L_{\alpha+1}.$$

(3) $f \in L_{\omega_2}$.

In fact, fix $\alpha < \omega_1$. Then $A_\alpha, C_\alpha \in L_{\omega_2}$ by the argument proving (1). Hence by (2), also $(\alpha, (A_\alpha, C_\alpha)) \in L_{\omega_2}$. Hence unfixing α, we see that there is a $\xi < \omega_2$ such that $f \subseteq L_\xi$; hence (3) follows by Theorem 23.32.

Now we apply Theorem 14.39 to $\mathbf{Z} = \mathbf{L}$ to obtain a transitive set P such that $L_{\omega_2} \subseteq P$ and certain formulas, relations, and functions in the rest of this proof are absolute for P.

Now by Theorem 14.41, let M be a set such that $\{\omega, \omega_1, f, (A, D)\} \subseteq M \subseteq P$, $M \preceq P$, and $|M| \leq \omega$.

(4) $\emptyset \in M$.

In fact, $\emptyset \in P$, so $\forall y(y \notin \emptyset)$ holds in P by absoluteness, hence $\exists x \forall y(y \notin x)$ holds in P, hence in M since $M \preceq P$, so choose $x \in M$ such that $\forall y(y \notin x)$ holds in M. Hence it holds in P since $M \preceq P$. Since P is transitive, it follows that $x = \emptyset$, as desired.

(5) If $\xi \in M \cap \mathbf{On}$, then $\xi + 1 \in M$.

The proof of (5) is similar to that of (4).

(6) $M \cap \omega_1$ is a countable limit ordinal.

To prove (6) it suffices to show that $M \cap \omega_1$ is an ordinal. In fact, then (5) implies that it is a limit ordinal, and hence since M is countable, it is countable. To show that $M \cap \omega_1$ is an ordinal it suffices to take any $\xi \in M \cap \omega_1$ and show that $\xi \subseteq M$, since this will show that $M \cap \omega_1$ is transitive; so as a transitive set of transitive sets, it is an ordinal. If $\xi < \omega$, clearly $\xi \subseteq M$ by (4) and (5). Suppose that $\omega \leq \xi$. Let g be a bijection from ω onto ξ. Then "g is a bijection from ω onto ξ" holds in P by absoluteness, so "$\exists g$(g is a bijection from ω onto ξ)" holds in P, and hence in M. Choose $h \in M$ such that "h is a bijection from ω onto ξ" holds in M; then it holds in P, and hence it is really true, by absoluteness. Now by similar arguments, $h(n) \in M$ for every $n \in \omega$, so $\xi \subseteq M$, as desired.

Let $\alpha = M \cap \omega_1$. Now M is extensional since P is. Let G, N be the Mostowski collapsing function and the Mostowski collapse, respectively.

(7) $G(\beta) = \beta$ for all $\beta < \alpha$.

We prove (7) by induction, using the fact from (6) that $\alpha \subseteq M$:

$$G(\beta) = \{G(\gamma) : \gamma \in M \text{ and } \gamma \in \beta\} = \{\gamma : \gamma \in \beta\} = \beta.$$

(8) $G(A) = A \cap \alpha$.

For,

$$\begin{aligned} G(A) &= \{G(\beta) : \beta \in M \text{ and } \beta \in A\} \\ &= \{G(\beta) : \beta \in M \cap \omega_1 \text{ and } \beta \in A\} \\ &= \{\beta : \beta \in \alpha \text{ and } \beta \in A\} \quad \text{using (7)} \\ &= A \cap \alpha. \end{aligned}$$

Similarly,

(9) $G(D) = D \cap \alpha$.

(10) $G(\omega_1) = \alpha$.

In fact,

$$G(\omega_1) = \{G(\beta) : \beta \in M \text{ and } \beta \in \omega_1\} = \{G(\beta) : \beta \in \alpha\} = \alpha$$

by (7).

Now by absoluteness,

$$\begin{aligned} P \models &(A, D) \text{ is } <_L\text{-first such that } A, D \subseteq \omega_1, \\ &D \text{ is club in } \omega_1, \text{ and } A \cap \beta \neq 1^{\text{st}}(f(\beta)) \text{ for all } \beta \in D. \end{aligned}$$

It follows that M is a model of this same formula, and hence applying the isomorphism G and using the above facts, we get

$$\begin{aligned} N \models &(A \cap \alpha, D \cap \alpha) \text{ is } <_L\text{-first such that } D \cap \alpha \text{ is a club in } \alpha \\ &\text{and } A \cap \alpha \cap \beta \neq 1^{\text{st}}(f(\beta)) \text{ for all } \beta \in D \cap \alpha. \end{aligned}$$

By absoluteness, since N is transitive this statment really holds. It follows that $A \cap \alpha = A_\alpha$. Moreover, $\alpha \in D$ since D is club in ω_1. This is a contradiction. $\square$

EXERCISES

E23.1. In the ordering $<_{\mathbf{L}}$ determine the first four sets and their order. Hint: use Corollary 23.5.

E23.2. Suppose that $\mathbf{M}$ is a nonempty transitive class satisfying the comprehension axioms, and also $\forall x \subseteq \mathbf{M} \exists y \in \mathbf{M}[x \subseteq y]$. Show that $\mathbf{M}$ is a model of ZF.

E23.3. Show that if $\mathbf{M}$ is a transitive proper class model of ZF, then $\forall x \subseteq \mathbf{M} \exists y \in \mathbf{M}[x \subseteq y]$.

E23.4. Show that for every ordinal $\alpha > \omega$, $|L_\alpha| = |V_\alpha|$ iff $\alpha = \beth_\alpha$.

E23.5. Assume $\mathbf{V} = \mathbf{L}$ and $\alpha > \omega$. Then $L_\alpha = V_\alpha$ iff $\alpha = \beth_\alpha$.

E23.6. Assume $\mathbf{V} = \mathbf{L}$ and prove that $L_\kappa = H(\kappa)$ for every infinite cardinal κ.

In the next exercises we develop the theory of ordinal definable sets. **OD** is the class of all sets a such that:

$$\exists\beta > \mathrm{rank}(a)\exists n \in \omega\exists s \in {}^n\beta\exists R \in \mathbf{Df}(V_\beta, \emptyset, n+1)$$
$$\forall x \in V_\beta[s^\frown\langle x\rangle \in R \leftrightarrow x = a].$$

E23.7. Show that if $\varphi(y_1, \ldots, y_n, x)$ is a formula with at most the indicated variables free, then

$$\forall\alpha_1, \ldots, \alpha_n\forall a[\forall x[\varphi(\alpha_1, \ldots, \alpha_n, x) \leftrightarrow x = a] \rightarrow a \in \mathbf{OD}].$$

Also show that $\emptyset \in \mathbf{OD}$.

E23.8. We define $s \lhd t$ iff $s, t \in {}^{<\omega}\mathbf{ON}$ and one of the following holds:

(i) $s = \emptyset$ and $t \neq \emptyset$;

(ii) $s, t \neq \emptyset$ and $\max(\mathrm{rng}(s)) < \max(\mathrm{rng}(t))$;

(iii) $s, t \neq \emptyset$ and $\max(\mathrm{rng}(s)) = \max(\mathrm{rng}(t))$ and $\mathrm{dmn}(s) < \mathrm{dmn}(t)$;

(iv) $s, t \neq \emptyset$ and $\max(\mathrm{rng}(s)) = \max(\mathrm{rng}(t))$ and $\mathrm{dmn}(s) = \mathrm{dmn}(t)$ and $\exists k \in \mathrm{dmn}(s)[s \upharpoonright k = t \upharpoonright k$ and $s(k) < t(k)]$.

Prove the following:

(v) $\lhd$ well-orders ${}^{<\omega}\mathbf{ON}$.

(vi) $\forall t \in {}^{<\omega}\mathbf{ON}[\{s : s \lhd t\}$ is a set].

(vii) For every infinite ordinal α we have $|{}^{<\omega}\alpha| = |\alpha|$.

(viii) For every uncountable cardinal κ, the set ${}^{<\omega}\kappa$ is well-ordered by $\lhd$ in order type κ and is an initial segment of ${}^{<\omega}\mathbf{ON}$.

(ix) ${}^{<\omega}\omega$ is well-ordered by $\lhd$ in order type ω^2.

We need to get a finer description of the definable sets. That is done by the following recursive definition of a function En of three variables m, A, n with $m, n \in \omega$ and A any set. The definition is by recursion on m, with A fixed. We assume that $\mathrm{En}(m', A, n')$ is defined for all $m' < m$ and for all $n' \in \omega$. Write $m = 2^i \cdot 3^j \cdot 5^k \cdot r$ with $r \in \omega$ not divisible by 2, 3, or 5.

$$\mathrm{En}(m, A, n) = \begin{cases} \mathrm{Diag}_\in(A, n, i, j) & \text{if } r = 1,\ k = 0, \text{ and } i, j < n, \\ \mathrm{Diag}_=(A, n, i, j) & \text{if } r = 1,\ k = 1, \text{ and } i, j < n, \\ {}^nA\backslash\mathrm{En}(i, A, n) & \text{if } r = 1,\ k = 2, \\ \mathrm{En}(i, A, n) \cap \mathrm{En}(j, A, n) & \text{if } r = 1,\ k = 3, \\ \mathrm{Proj}'(A, \mathrm{En}(i, A, n+1), n) & \text{if } r = 1,\ k = 4, \\ \emptyset & \text{otherwise.} \end{cases}$$

E23.9. Prove that for any $m \in \omega$ and any set A, $\mathbf{Df}(A, \emptyset, n) = \{\mathrm{En}(m, A, n) : m \in \omega\}$.

E23.10. Prove that if $\varphi(x_0, \ldots, x_{n-1})$ is a formula with free variables among $x_0, \ldots, x_{n-1}$, then there is an $m \in \omega$ such that for every set A,

$$\{s \in {}^nA : \varphi^A(s(0), \ldots, s(n-1))\} = \mathrm{En}(m, A, n).$$

E23.11. By exercise E23.8, for each uncountable cardinal κ there is an isomorphism f_κ from $(\kappa, \in)$ onto $({}^{<\omega}\kappa, \lhd)$. Then $f_\kappa \subseteq f_\lambda$ for $\kappa < \lambda$. It follows that there is a function Enon mapping **ON** onto ${}^{<\omega}$**ON** such that $\alpha < \beta$ iff $\mathrm{Enon}(\alpha) \lhd \mathrm{Enon}(\beta)$.

Now we define a class function Enod with domain **ON**, as follows. For any ordinal γ,

$$\mathrm{Enod}(\gamma) = \begin{cases} a & \text{if there exist } s, \beta, m, n \text{ such that } \mathrm{Enon}(\gamma) = s^\frown\langle \beta, n, m\rangle \\ & \text{with } m, n \in \omega,\ \beta \in \mathbf{ON},\ s \in {}^{<\omega}\beta,\ \mathrm{dmn}(s) = n, \text{ and} \\ & \forall x \in V_\beta[s^\frown x \in \mathrm{En}(m, V_\beta, n+1) \leftrightarrow x = a], \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $\mathbf{OD} = \{\mathrm{Enod}(\gamma) : \gamma \in \mathbf{ON}\}$.

E23.12. Now we define $\mathbf{HOD} = \{x \in \mathbf{OD} : \mathrm{trcl}(x) \subseteq \mathbf{OD}\}$.

Prove that $\mathbf{ON} \subseteq \mathbf{HOD}$ and **HOD** is transitive.

E23.13. Show that $(V_\alpha \cap \mathbf{HOD}) \in \mathbf{HOD}$ for every ordinal α.

E23.14. Prove without using the axiom of choice that **HOD** is a model of ZFC.

E23.15. Show that the axiom of choice holds in $L(B)$ iff $\mathrm{trcl}(\{A\})$ can be well-ordered in $L(B)$.

E23.16. Recall from elementary set theory the following definition of the standard well-ordering of On $\times$ On:

$$\begin{aligned} (\alpha, \beta) \prec (\gamma, \delta) \quad \text{iff} \quad & (\alpha \cup \beta < \gamma \cup \delta) \\ & \text{or } (\alpha \cup \beta = \gamma \cup \delta \text{ and } \alpha < \gamma) \\ & \text{or } (\alpha \cup \beta = \gamma \cup \delta \text{ and } \alpha = \gamma \text{ and } \beta < \delta). \end{aligned}$$

Prove that $\prec$ is absolute for transitive class models of ZF.

Now define $\Delta : \mathrm{On} \to \mathrm{On} \times \mathrm{On}$ by recursion as follows:

$$\Delta(0) = (0, 0);$$
$$\Delta(\alpha+1) = \begin{cases} (\beta, \gamma+1) & \text{if } \Delta(\alpha) = (\beta, \gamma) \text{ and } \gamma < \beta, \\ (0, \beta+1) & \text{if } \Delta(\alpha) = (\beta, \gamma) \text{ and } \gamma = \beta, \\ (\beta+1, \gamma) & \text{if } \Delta(\alpha) = (\beta, \gamma) \text{ and } \beta + 1 < \gamma, \\ (\gamma, 0) & \text{if } \Delta(\alpha) = (\beta, \gamma) \text{ and } \beta + 1 = \gamma; \end{cases}$$
$$\Delta(\alpha) = \prec\text{-least}(\beta, \gamma) \text{ such that } \forall \delta < \alpha[\Delta(\delta) \prec (\beta, \gamma)] \text{ if } \alpha \text{ is limit.}$$

Prove:

(1) If $\alpha < \beta$, then $\Delta(\alpha) \prec \Delta(\beta)$.
(2) Δ maps onto On $\times$ On.
(3) Δ is absolute for transitive class models of ZF.
(4) Δ^{-1} is absolute for transitive class models of ZF.

E23.17. Suppose that M is a transitive class model of ZFC, and every set of ordinals is in M. Show that $M = V$. Hint: take any set X. Let $\kappa = |\mathrm{trcl}(\{X\})|$, and let $f : \kappa \to \mathrm{trcl}(\{X\})$ be a bijection. Define $\alpha E \beta$ iff $\alpha, \beta < \kappa$ and $f(\alpha) \in f(\beta)$. Use exercise

E23.16 to show that $E \in M$. Take the Mostowski collapse of (κ, E) in M, and infer that $X \in M$.

E23.18. Show that if $X \subseteq \omega_1$ then CH holds in $L(X)$. Hint: show that if $A \in \mathscr{P}(\omega)$ in $L(X)$, then there are $\alpha, \beta < \omega_1$ such that $X \in L_\alpha(X \cap \beta)$.

E23.19. Show that if $X \subseteq \omega_1$ then GCH holds in $L(X)$.

24. Powers of regular cardinals

In this chapter we continue Chapter 16, and describe in more detail the possibilities for 2^κ when κ is regular, where the results are fairly complete. The case of singular κ is more involved, and there are still important open problems.

To obtain upper bounds on the size of powers the following concept will be used.

Suppose that $\mathbb{P}$ is a forcing order and $\sigma \in V^{\mathbb{P}}$. A *nice name* for a subset of σ is a member of $V^{\mathbb{P}}$ of the form

$$\bigcup_{\pi \in \mathrm{dmn}(\sigma)} (\{\pi\} \times A_\pi),$$

where each A_π is an antichain in $\mathbb{P}$.

Lemma 24.1. *If σ is a P-name and $(\pi, p) \in \sigma$, then $p \Vdash \pi \in \sigma$.*

Proof. Let G be generic with $p \in G$. Then by Theorem 15.6, $\pi_G \in \sigma_G$. $\square$

Proposition 24.2. *Let M be a c.t.m. of ZFC, $\mathbb{P} \in M$ a forcing order, and $\sigma \in M^P$.*

(i) For any $\mu \in M^P$ there is a nice name $\tau \in M^{\mathbb{P}}$ for a subset of σ such that

$$(*) \qquad 1 \Vdash \tau = \mu \cap \sigma.$$

(ii) If G is $\mathbb{P}$-generic over M and $a \subseteq \sigma_G$ in $M[G]$, then $a = \tau_G$ for some nice name τ for a subset of σ.

Proof. Assume the hypotheses of the proposition.

(i): Assume also that $\mu \in M^P$. For each $\pi \in \mathrm{dmn}(\sigma)$ let $A_\pi \subseteq P$ be such that

(1) $p \Vdash (\pi \in \mu \wedge \pi \in \sigma)$ for all $p \in A_\pi$.

(2) A_π is an antichain of $\mathbb{P}$.

(3) A_π is maximal with respect to (1) and (2).

Moreover, we do this definition inside M, so that $\langle A_\pi : \pi \in \mathrm{dmn}(\sigma)\rangle \in M$. Now let

$$\tau = \bigcup_{\pi \in \mathrm{dmn}(\sigma)} (\{\pi\} \times A_\pi).$$

To prove $(*)$, suppose that G is $\mathbb{P}$-generic over M; we want to show that $\tau_G = \mu_G \cap \sigma_G$.

First suppose that $a \in \mu_G \cap \sigma_G$. Choose $(\pi, p) \in \sigma$ such that $p \in G$ and $a = \pi_G$. By Lemma 24.1, $p \Vdash \pi \in \sigma$.

(4) $A_\pi \cap G \neq \emptyset$.

For, suppose that $A_\pi \cap G = \emptyset$. By Lemma 15.14(i), there is a $q \in G$ such that $q \perp r$ for all $r \in A_\pi$. Now since $\pi_G \in \mu_G$, by Corollary 15.21 there is a $q' \in G$ such that $q' \Vdash \pi \in \mu$. Let $r \in G$ with $r \leq q, q'$. Then $r \Vdash (\pi \in \mu \wedge \pi \in \sigma)$. It follows that $A_\pi \cup \{r\}$ is an antichain, contradicting (3). Thus (4) holds.

By (4), take $q \in A_\pi \cap G$. Then $(\pi, q) \in \tau$ and $q \in G$, so $a = \pi_G \in \tau_G$. Thus we have shown that $\mu_G \cap \sigma_G \subseteq \tau_G$.

Now suppose that $a \in \tau_G$. Choose $(\pi, p) \in \tau$ such that $p \in G$ and $a = \pi_G$. Thus $p \in A_\pi$, so by (1), $p \Vdash (\pi \in \mu \wedge \pi \in \sigma)$. By the definition of forcing, $a = \pi_G \in \mu_G \cap \sigma_G$. This shows that $\tau_G \subseteq \mu_G \cap \sigma_G$. Hence $\tau_G = \mu_G \cap \sigma_G$.

(ii): Assume the hypotheses of (ii). Write $a = \mu_G$. Taking τ as in (i), we have $a = \mu_G = \mu_G \cap \sigma_G = \tau_G$, as desired. □

Proposition 24.3. *Suppose that M is a c.t.m. of ZFC, and in M, $\mathbb{P}$ is a forcing order, $|P| = \kappa \geq \omega$, $\mathbb{P}$ has the λ-cc, and μ is an infinite cardinal. Suppose that G is $\mathbb{P}$-generic over M. Then there is a function in $M[G]$ mapping $((\kappa^{<\lambda})^\mu)^M$ onto a set containing $\mathscr{P}(\mu)^{M[G]}$.*

Proof. We do some calculations in M. Each antichain in $\mathbb{P}$ has size at most $\kappa^{<\lambda}$. Since $|\mathrm{dmn}(\check{\mu})|$ has size μ, we thus have at most $\nu \stackrel{\text{def}}{=} (\kappa^{<\lambda})^\mu$ nice names for subsets of $\check{\mu}$. Let $\langle \tau_\alpha : \alpha < \nu \rangle$ enumerate all of these names. Define

$$\pi = \{(\mathrm{op}(\check{\alpha}, \tau_\alpha), 1) : \alpha < \nu\}.$$

Now π_G is a function. For, if $x \in \pi_G$, then there is an $\alpha < \nu$ such that $x = (\alpha, (\tau_\alpha)_G)$, by Lemma 15.22. Thus π_G is a relation. Now suppose that $(x, y), (x, z) \in \pi_G$. Then there exist $\alpha, \beta < \nu$ such that $(x, y) = (\alpha, (\tau_\alpha)_G)$ and $(x, z) = (\beta, (\tau_\beta)_)$. Hence $\alpha = \beta$ and $y = z$. Clearly the domain of π_G is ν. By Proposition 24.2, $\mathscr{P}(\mu) \subseteq \mathrm{rng}(\pi_G)$ in $M[G]$, as desired. □

Now we can prove a more precise version of Theorem 16.8.

Theorem 24.4. (Solovay) *Let M be a c.t.m. of ZFC. Suppose that κ is a cardinal of M such that $\kappa^\omega = \kappa$. Let $\mathbb{P}$ be the partial order* $\mathrm{fin}(\kappa, 2)$ *ordered by $\supseteq$, and let G be $\mathbb{P}$-generic over M. Then $M[G]$ has the same cofinalities and cardinals as M, and $2^\omega = \kappa$ in $M[G]$.*

Moreover, if λ is any infinite cardinal in M, then $\kappa \leq (2^\lambda)^{M[G]} \leq (\kappa^\lambda)^M$.

Proof. By Theorem 16.8, $M[G]$ has the same cofinalities and cardinals as M and $\kappa \leq 2^\omega$.

Note that $|\mathrm{fin}(\kappa, 2)| = \kappa$ in M. Hence by Proposition 24.3, for any infinite cardinal λ of M we have

$$\kappa \leq (2^\omega)^{M[G]} \leq (2^\lambda)^{M[G]} \leq ((\kappa^{<\lambda})^\lambda)^M = (\kappa^\lambda)^M.$$

Applying this to $\lambda = \omega$ we get $2^\omega = \kappa$ in $M[G]$. □

By assuming that the ground model satisfies GCH, which is consistent by the theory of constructible sets, we can obtain a sharper result.

Corollary 24.5. *Suppose that M is a c.t.m. of $ZFC + GCH$. Suppose that κ is an uncountable regular cardinal of M. Let $\mathbb{P}$ be the partial order* $\mathrm{fin}(\kappa, 2)$ *ordered by $\supseteq$, and let G be $\mathbb{P}$-generic over M. Then $M[G]$ has the same cofinalities and cardinals as M, and $2^\omega = \kappa$ in $M[G]$.*

Moreover, for any infinite cardinal λ of M we have

$$(2^\lambda)^{M[G]} = \begin{cases} \kappa & \text{if } \lambda < \kappa, \\ \lambda^+ & \text{if } \kappa \leq \lambda. \end{cases}$$

Proof. By GCH we have $\kappa^\omega = \kappa$. Hence the hypothesis of Theorem 24.4 holds, and the conclusion follows using GCH in M. □

We give several more specific corollaries.

Corollary 24.6. *If ZFC is consistent, then so is each of the following:*
(i) $ZFC + 2^{\aleph_0} = \aleph_2$.
(ii) $ZFC + 2^{\aleph_0} = \aleph_{203}$.
(iii) $ZFC + 2^{\aleph_0} = \aleph_{\omega_1}$.
(iv) $ZFC + 2^{\aleph_0} = \aleph_{\omega_4}$. □

Corollary 24.7. *If ZFC+ "there is an uncountable regular limit cardinal" is consistent, so is ZFC+ "2^ω is a regular limit cardinal".* □

Corollary 24.8. *Suppose that M is a c.t.m. of ZFC. Then there is a generic extension $M[G]$ such that in it, $2^\omega = ((2^\omega)^+)^M$.*

Since clearly $((2^\omega)^+)^\omega = (2^\omega)^+$ in M, this is immediate from Theorem 24.4. □

Now we turn to powers of regular uncountable cardinals, where similar results hold. We need some elementary facts about cardinals. For cardinals κ, λ, we define

$$\kappa^{<\lambda} = \sup_{\alpha<\lambda} |{}^\alpha\kappa|.$$

Note here that the supremum is over all ordinals less than λ, not only cardinals.

Proposition 24.9. *Let κ and λ be cardinals with $\kappa \geq 2$ and λ infinite and regular. Then $(\kappa^{<\lambda})^{<\lambda} = \kappa^{<\lambda}$.*

Proof. Clearly $\geq$ holds. For $\leq$, by the fact that $\lambda \cdot \lambda = \lambda$ it suffices to find an injection from

$$\bigcup_{\alpha<\lambda} {}^\alpha\left(\bigcup_{\beta<\lambda} {}^\beta\kappa\right) \tag{1}$$

into

$$\bigcup_{\alpha,\beta<\lambda} {}^{\alpha\times\beta}(\kappa+1). \tag{2}$$

Let x be a member of (1), and choose $\alpha < \lambda$ accordingly. Then for each $\xi < \alpha$ there is a $\beta_{x,\xi} < \lambda$ such that $x(\xi) \in {}^{\beta_{x,\xi}}\kappa$. Let $\gamma_x = \sup_{\xi<\alpha} \beta_{x,\xi}$. Then $\gamma_x < \lambda$ by the regularity of λ. We now define $f(x)$ with domain $\alpha \times \gamma_x$ by setting, for any $\xi < \alpha$ and $\eta < \gamma_x$

$$(f(x))(\xi,\eta) = \begin{cases} (x(\xi))(\eta) & \text{if } \eta < \beta_{x,\xi}, \\ \kappa & \text{otherwise.} \end{cases}$$

Then f is one-one. In fact, suppose that $f(x) = f(y)$. Let the domain of $f(x)$ be $\alpha \times \gamma_x$ as above. Suppose that $\xi < \alpha$. If $\beta_{x,\xi} \neq \beta_{y,\xi}$, say $\beta_{x,\xi} < \beta_{y,\xi}$. Then $\gamma_x = \gamma_y \geq \beta_{y,\xi} > \beta_{x,\xi}$, and $(f(x))(\xi, \beta_{x,\xi}) = \kappa$ while $(f(y))(\xi, \beta_{x,\xi}) < \kappa$, contradiction. Hence $\beta_{x,\xi} = \beta_{y,\xi}$. Finally, take any $\eta < \beta_{x\xi}$. Then

$$(x(\xi))(\eta) = (f(x))(\xi,\eta) = (f(y))(\xi,\eta) = (y(\xi))(\eta);$$

it follows that $x = y$.

Now the direction $\leq$ follows.

Proposition 24.10. *For any cardinals κ, λ, $|[\kappa]^{<\lambda}| \leq \kappa^{<\lambda}$.*

Proof. For each cardinal $\mu < \lambda$ define $f : {}^{\mu}\kappa \to [\kappa]^{\leq\mu}\backslash\{\emptyset\}$ by setting $f(x) = \operatorname{rng}(x)$ for any $x \in {}^{\mu}\kappa$. Clearly f is an onto map. It follows that $|[\kappa]^{\leq\mu}| \leq |{}^{\mu}\kappa| \leq \kappa^{<\lambda}$. Hence

$$\begin{aligned} |[\kappa]^{<\lambda}| &= \left| \bigcup_{\substack{\mu<\lambda,\\ \mu \text{ a cardinal}}} [\kappa]^{\leq\mu} \right| \\ &\leq \sum_{\substack{\mu<\lambda,\\ \mu \text{ a cardinal}}} |[\kappa]^{\leq\mu}| \\ &\leq \sum_{\substack{\mu<\lambda,\\ \mu \text{ a cardinal}}} \kappa^{<\lambda} \\ &\leq \lambda \cdot \kappa^{<\lambda} \\ &= \kappa^{<\lambda}. \end{aligned}$$

□

Lemma 24.11. *If I, J are sets and λ is an infinite cardinal, then* $\operatorname{Fn}(I, J, \lambda)$ *has the* $(|J|^{<\lambda})^+$*-cc.*

Proof. Let $\theta = (|J|^{<\lambda})^+$, and suppose that $\{p_\xi : \xi < \theta\}$ is a collection of elements of $\operatorname{Fn}(I, J, \lambda)$; we want to show that there are distinct $\xi, \eta < \theta$ such that p_ξ and p_η are compatible. We want to apply the general indexed Δ-system theorem 20.4, with κ, λ, $\langle A_i : i \in I\rangle$ replaced by λ, θ, $\langle \operatorname{dmn}(p_\xi) : \xi < \theta\rangle$ respectively. Obviously θ is regular. If $\alpha < \theta$, then $|[\alpha]^{<\lambda}| \leq |\alpha|^{<\lambda}$ (by Proposition 24.10) $\leq (|J|^{<\lambda})^{<\lambda} = |J|^{<\lambda}$ (by Proposition 24.9) $< \theta$. Thus we can apply 20.4, and we get $J \in [\theta]^\theta$ such that $\langle \operatorname{dmn}(p_\xi) : \xi \in J\rangle$ is an indexed Δ-system, say with root r. Now $|{}^rJ| \leq |J|^{<\lambda} < \theta$, so there exist a $K \in [J]^\theta$ and an $f \in {}^rJ$ such that $p_\xi \restriction r = f$ for all $\xi \in K$. Clearly p_ξ and p_η are compatible for any two $\xi, \eta \in K$. □

Lemma 24.12. *If I, J are sets and λ is a regular cardinal, then* $\mathrm{Fn}(I, J, \lambda)$ *is λ-closed.*

Proof. Suppose that $\gamma < \lambda$ and $\langle p_\xi : \xi < \gamma\rangle$ is a system of elements of $\mathrm{Fn}(I, J, \lambda)$ such that $p_\eta \supseteq p_\xi$ whenever $\xi < \eta < \gamma$. Let $q = \bigcup_{\xi<\gamma} p_\xi$. Clearly $q \in \mathrm{Fn}(I, J, \lambda)$ and $q \supseteq p_\xi$ for each $\xi < \gamma$. □

We now need another little fact about cardinal arithmetic.

Lemma 24.13. *If λ is regular, then $\lambda^{<\lambda} = 2^{<\lambda}$.*

Proof. Note that if $\alpha < \lambda$, then by the regularity of λ,

$$|{}^\alpha\lambda| = \left|\bigcup_{\beta<\lambda} {}^\alpha\beta\right| \leq \sum_{\beta<\lambda} |\beta|^{|\alpha|} \leq \sum_{\beta<\lambda} |\max(\alpha, \beta)|^{|\max(\alpha,\beta)|} \leq \sum_{\beta<\lambda} 2^{|\max(\alpha,\beta)|} \leq 2^{<\lambda} \leq \lambda^{<\lambda};$$

hence the lemma follows. □

Lemma 24.14. *Suppose that M is a c.t.m. of ZFC, $I, J, \lambda \in M$, and in M, λ is a regular cardinal, $2^{<\lambda} = \lambda$ and $|J| \leq \lambda$. Then $Fn(I, J, \lambda)^M$ preserves cofinalities and cardinalities.*

Proof. By Lemma 24.12, the set $\mathrm{Fn}(I, J, \lambda)$ is λ-closed, and so by Proposition 12.11, $\mathrm{Fn}(I, J, \lambda)$ preserves cofinalities and cardinalities $\leq \lambda$. Now $|J|^{<\lambda} \leq \lambda^{<\lambda} = 2^{<\lambda} = \lambda$ by Lemma 24.13, Hence by Lemma 24.11, $\mathrm{Fn}(I, J, \lambda)$ has the λ^+-cc. By Proposition 16.5 $\mathrm{Fn}(I, J, \lambda)$ preserves cofinalities and cardinals $\geq \lambda^+$. □

Now we can give our main theorem concerning making 2^λ as large as we want, for any regular λ given in advance.

Theorem 24.15. *Suppose that M is a c.t.m. of ZFC and in M we have cardinals κ, λ such that $\lambda < \kappa$, λ is regular, $2^{<\lambda} = \lambda$, and $\kappa^\lambda = \kappa$. Let $P = \mathrm{Fn}(\kappa, 2, \lambda)$ ordered by $\supseteq$. Then P preserves cofinalities and cardinalities. Let G be $\mathbb{P}$-generic over M. Then*

(i) $(2^\lambda = \kappa)^{M[G]}$.

(ii) If μ and ν are cardinals of M and $\omega \leq \mu < \lambda$, then $(\nu^\mu)^M = (\nu^\mu)^{M[G]}$.

(iii) For any cardinal μ of M, if $\mu \geq \lambda$ then $(2^\mu)^{M[G]} = (\kappa^\mu)^M$.

Proof. Preservation of cofinalities and cardinalities follows from Lemma 24.14. Now we turn to (i). To show that $\kappa \leq (2^\lambda)^{M[G]}$, we proceed as in the proof of Theorem 16.1 Let $g = \bigcup G$. So g is a function mapping a subset of κ into 2.

(1) For each $\alpha \in \kappa$, the set $\{f \in \mathrm{Fn}(\kappa, 2, \lambda) : \alpha \in \mathrm{dmn}(f)\}$ is dense in $\mathbb{P}$ (and it is a member of M).

In fact, given $f \in \mathrm{Fn}(\kappa, 2, \lambda)$, either f is already in the above set, or else $\alpha \notin \mathrm{dmn}(f)$ and then $f \cup \{(\alpha, 0)\}$ is an extension of f which is in that set. So (1) holds.

Since G intersects each set (1), it follows that g maps κ into 2. Let (in M) $h : \kappa\times\lambda \to \kappa$ be a bijection. For each $\alpha < \kappa$ let $a_\alpha = \{\xi \in \lambda : g(h(\alpha, \xi)) = 1\}$. We claim that $a_\alpha \neq a_\beta$ for distinct α, β; this will give $\kappa \leq (2^\lambda)^{M[G]}$. The set

$$\begin{aligned}&\{f \in \mathrm{Fn}(\kappa, 2, \lambda) : \text{there is a } \xi \in \lambda \text{ such that}\\ &\quad h(\alpha, \xi), h(\beta, \xi) \in \mathrm{dmn}(f) \text{ and } f(h(\alpha, \xi)) \neq f(h(\alpha, \xi))\}\end{aligned}$$

is dense in $\mathbb{P}$ (and it is in M). In fact, let distinct α and β be given, and suppose that $f \in \mathrm{Fn}(\kappa, 2, \lambda)$. Now $\{\xi : h(\alpha, \xi) \in f$ or $h(\beta, \xi) \in f\}$ has size less than λ, so choose $\xi \in \lambda$ not in this set. Thus $h(\alpha, \xi), h(\beta, \xi) \notin f$. Let $h = f \cup \{(h(\alpha, \xi), 0), (h(\beta, \xi), 1)\}$. Then h extends f and is in the above set, as desired.

It follows that G contains a member of this set. Hence $a_\alpha \neq a_\beta$. Thus we have now shown that $\kappa \leq (2^\lambda)^{M[G]}$.

For the other inequality, note by Lemma 24.11 that $\mathbb{P}$ has the $(2^{<\lambda})^+$-cc, and by hypothesis $(2^{<\lambda})^+ = \lambda^+$. By the assumption that $\kappa^\lambda = \kappa$ we also have $|P| = \kappa$. Hence by Proposition 24.3 the other inequality follows. Thus we have finished the proof of (i).

For (ii), assume the hypothesis. If $f \in M[G]$ and $f : \mu \to \nu$, then $f \in M$ by Theorem 16.10. Hence (ii) follows.

Finally, for (iii), suppose that μ is a cardinal of M such that $\mu \geq \lambda$. By Proposition 24.3 with λ replaced by λ^+ we have $(2^\mu)^{M[G]} \leq (\kappa^\mu)^M$. Now $(\kappa^\mu)^M \leq (\kappa^\mu)^{M[G]} = ((2^\lambda)^\mu)^{M[G]} = (2^\mu)^{M[G]}$, so (iii) holds. $\square$

Corollary 24.16. *Suppose that M is a c.t.m. of ZFC+ GCH, and in M we have cardinals κ, λ, both regular, with $\lambda < \kappa$. Let $P = \mathrm{Fn}(\kappa, 2, \lambda)$ ordered by $\supseteq$. Then P preserves cofinalities and cardinalities. Let G be $\mathbb{P}$-generic over M. Then for any infinite cardinal μ,*

$$(2^\mu)^{M[G]} = \begin{cases} \mu^+ & \text{if } \mu < \lambda, \\ \kappa & \text{if } \lambda \leq \mu < \kappa, \\ \mu^+ & \text{if } \kappa \leq \mu. \end{cases}$$

Proof. Immediate from Theorem 24.15. $\square$

Theorem 24.15 gives quite a bit of control over what can happen to powers 2^κ for κ regular. We can apply this theorem to obtain a considerable generalization of it.

Theorem 24.17. *Suppose that $n \in \omega$ and M is a c.t.m. of ZFC. Also assume the following:*

(i) $\lambda_1 < \cdots < \lambda_n$ are regular cardinals in M.
(ii) $\kappa_1 \leq \cdots \leq \kappa_n$ are cardinals in M.
(iii) $(\mathrm{cf}(\kappa_i) > \lambda_i)^M$ for each $i = 1, \ldots, n$.
(iv) $(2^{<\lambda_i} = \lambda_i)^M$ for each $i = 1, \ldots, n$.
(v) $(\kappa_i^{\lambda_i})^M = \kappa_i$ for each $i = 1, \ldots, n$

Then there is a c.t.m. $N \supseteq M$ with the same cofinalities and cardinals such that:

(vi) $(2^{\lambda_i} = \kappa_i)^N$ for each $i = 1, \ldots, n$.
(vii) $(2^\mu)^N = (\kappa_n^\mu)^M$ for all $\mu > \lambda_n$.

Proof. The statement vacuously holds for $n = 0$. Suppose that it holds for $n-1$, and the hypothesis holds for n, where n is a positive integer. Let $\mathbb{P}_n = \mathrm{Fn}(\kappa_n, 2, \lambda_n)$. Then by Lemma 24.11, $\mathbb{P}_n$ has the $(2^{<\lambda_n})^+$-cc, i.e., by (iv) it has the λ_n^+-cc. By Lemma 24.12 it is λ_n-closed. So by Proposition 24.14, $\mathbb{P}_n$ preserves all cofinalities and cardinalities. Let G be $\mathbb{P}_n$-generic over M. By Theorem 24.15, $(2^{\lambda_n} = \kappa_n)^{M[G]}$, $(2^\mu)^{M[G]} = (\kappa_n^\mu)^M$ for all $\mu > \lambda_n$, and also conditions (i)-(v) hold for $M[G]$ for $i = 1, \ldots, n-1$. Hence by the inductive hypothesis, there is a c.t.m. N with $M[G] \subseteq N$ such that

(1) $(2^{\lambda_i} = \kappa_i)^N$ for each $i = 1, \ldots, n-1$.

(2) $(2^\mu)^N = (\kappa_{n-1}^\mu)^{M[G]}$ for all $\mu > \lambda_{n-1}$.

In particular,

$$\begin{aligned}(2^{\lambda_n})^N = (\kappa_{n-1}^{\lambda_n})^{M[G]} &\leq (\kappa_n^{\lambda_n})^{M[G]} = ((2^{\lambda_n})^{\lambda_n})^{M[G]} = (2^{\lambda_n})^{M[G]} \\ &= \kappa_n = (2^{\lambda_n})^{M[G]} \leq (2^{\lambda_n})^N.\end{aligned}$$

Thus $(2^{\lambda_n})^N = \kappa_n$. Furthermore, if $\mu > \lambda_n$ then

$$\begin{aligned}(2^\mu)^N = (\kappa_{n-1}^\mu)^{M[G]} &\leq (\kappa_n^\mu)^{M[G]} = ((2^{\lambda_n})^\mu)^{M[G]} = (2^\mu)^{M[G]} \\ &= (\kappa_n^\mu)^M \leq (\kappa_n^\mu)^N = ((2^{\lambda_n})^\mu)^N = (2^\mu)^N.\end{aligned}$$

It follows that $(2^\mu)^N = (\kappa_n^\mu)^M$. This completes the inductive proof. □

Corollary 24.18. *Suppose that $n \in \omega$ and M is a c.t.m. of ZFC + GCH. Also assume the following:*

(i) $\lambda_1 < \cdots < \lambda_n$ are regular cardinals in M.
(ii) $\kappa_1 \leq \cdots \leq \kappa_n$ are cardinals in M.
(iii) $(\mathrm{cf}(\kappa_i) > \lambda_i)^M$ for each $i = 1, \ldots, n$.

Then there is a c.t.m. $N \supseteq M$ with the same cofinalities and cardinals such that:

(iv) $(2^{\lambda_i} = \kappa_i)^N$ for each $i = 1, \ldots, n$.
(v) $(2^\mu)^N = (\kappa_n^\mu)^M$ for all $\mu > \lambda_n$. □

Corollary 24.19. *If ZFC is consistent, then so are each of the following:*

(i) ZFC + $2^{\aleph_0} = 2^{\aleph_1} = \aleph_3$.
(ii) ZFC $\cup \{2^{\aleph_n} = \aleph_{n+2} : n < 100\}$.
(iii) ZFC $\cup \{2^{\aleph_n} = \aleph_{\omega+1} : n < 300\}$.
(iv) ZFC $\cup \{2^{\aleph_n} = \aleph_{\omega+n} : n < 33\}$. □

Corollary 24.20. *If it is consistent with ZFC that there is an uncountable regular limit cardinal, then the following is consistent:*

ZFC $\cup \{2^{\aleph_n}$ is the first regular limit cardinal: $n < 1000\}$. □

Theorem 24.17 can itself be generalized; the following is the ultimate generalization, in some sense. We do not give the proof.

Theorem. (Easton) *Suppose that M is a c.t.m. of ZFC, and that in M E is a class function whose domain is the class of all regular cardinals, and whose range is contained in the class of cardinals of M. Also assume the following in M:*

(i) For any regular cardinal λ, $\mathrm{cf}(E(\lambda)) > \lambda$.
(ii) If $\lambda < \kappa$ are regular cardinals, then $E(\lambda) \leq E(\kappa)$.

Then there is a generic extension $M[G]$ of M preserving cofinalities and cardinals such that in $M[G]$, $2^\lambda = E(\lambda)$ for every regular λ.

Note that we have always been concerned with 2^λ for λ regular; 2^λ when λ is singular can be computed on the basis of what has been done for regular cardinals. It is difficult to directly do something about 2^λ when λ is singular, and there are even hard open problems remaining concerning this. PCF theory applies to these questions. (PCF = "possible cofinalities")

EXERCISES

E24.1. Show that $\mathrm{fin}(\omega, \omega_1)$ collapses ω_1 to ω, but preserves cardinals $\geq \omega_2$.

E24.2. Suppose that κ is an uncountable regular cardinal of M, and $\mathscr{P} \in M$ is a κ-cc forcing order. Assume that C is club in κ, with $C \in M[G]$. Show that there is a $C' \subseteq C$ such that $C' \in M$ and C' is club in κ. Hint: in $M[G]$ let $f : \kappa \to \kappa$ be such that $\forall \alpha < \kappa[\alpha < f(\alpha) \in C]$. Apply Theorem 12.4.

E24.3. Suppose that κ is an uncountable regular cardinal of M, and $\mathscr{P} \in M$ is a κ-cc forcing order. Assume that $S \in M$ is stationary in κ, in the sense of M. Show that it remains stationary in $M[G]$.

E24.4. Suppose that κ is an uncountable regular cardinal of M, and $\mathscr{P} \in M$ is a κ-closed forcing order. Assume that $S \in M$ is stationary in κ, in the sense of M. Show that it remains stationary in $M[G]$.

E24.5. Prove that if ZFC is consistent, then so is $\mathrm{ZFC} + \mathrm{GCH} + \neg(V = L)$.

In this chapter we prove that if ZF is consistent, then so is ZF + ¬AC. First, however, we go into the relationship of isomorphisms of forcing orders to forcing and generic sets; this is needed for the consistency proof, and is independently interesting and important.

Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing orders, and suppose that $f : P \to Q$. We define a function f_* with domain V^P by recursion by setting, for any $\tau \in V^P$,

$$f_*(\tau) = \{(f_*(\sigma), f(p)) : (\sigma, p) \in \tau\}.$$

Proposition 25.1. *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are forcing orders and f is a function mapping P into Q. Then*

(i) $f_*(\tau) \in V^{\mathbb{Q}}$ for any $\tau \in V^{\mathbb{P}}$.

(ii) f_* is absolute for c.t.m. of ZFC.

(iii) If M is a c.t.m. of ZFC, then f_* maps $M^{\mathbb{P}}$ into $M^{\mathbb{Q}}$.

Proof. (i) is easily proved by induction on τ. (ii) follows from absoluteness of recursive definitions. (iii) follows from (i), (ii). □

Again, let $\mathbb{P}$ and $\mathbb{Q}$ be forcing orders. An *isomorphism* from $\mathbb{P}$ to $\mathbb{Q}$ is a bijection f from P onto Q such that $f(1^{\mathbb{P}}) = 1^{\mathbb{Q}}$, and for any $p, r \in P$, $p \leq^{\mathbb{P}} r$ iff $f(p) \leq^{\mathbb{Q}} f(r)$.

As with other mathematical notions of isomorphisms, an isomorphism of forcing orders extends in a routine way to mappings of structures derived from the forcing orders. We give several results which carry out this routine analysis.

Lemma 25.2. *If $\mathbb{P}$ and $\mathbb{Q}$ are forcing orders and f is an isomorphism from $\mathbb{P}$ to $\mathbb{Q}$, then $f_*(\check{x}^{\mathbb{P}}) = \check{x}^{\mathbb{Q}}$ for every set x.*

Proof. The proof is by $\in$-induction:

$$\begin{aligned} f_*(\check{x}^{\mathbb{P}}) &= \{(f_*(\sigma), f(p)) : (\sigma, p) \in \check{x}^{\mathbb{P}}\} \\ &= \{(f_*(\check{y}^{\mathbb{P}}), f(1^{\mathbb{P}})) : y \in x\} \\ &= \{(\check{y}^{\mathbb{Q}}, 1^{\mathbb{Q}}) : y \in x\} \\ &= \check{x}^{\mathbb{Q}}. \end{aligned}$$ □

Lemma 25.3. *Suppose that $\mathbb{P}, \mathbb{Q}, \mathbb{R}$ are forcing orders, g is an isomorphism from $\mathbb{P}$ to $\mathbb{Q}$, and f is an isomorphism from $\mathbb{Q}$ to $\mathbb{R}$. Then $f \circ g$ is an isomorphism from $\mathbb{P}$ to $\mathbb{R}$, and $f_*(g_*(\tau)) = (f \circ g)_*(\tau)$ for every $\tau \in V^P$.*

Proof. Obviously $f \circ g$ is an isomorphism from $\mathbb{P}$ to $\mathbb{R}$. We prove the second statement by induction:

$$\begin{aligned} f_*(g_*(\tau)) &= \{(f_*(\sigma), f(p)) : (\sigma, p) \in g_*(\tau)\} \\ &= \{(f_*(g_*(\rho)), f(g(q))) : (\rho, q) \in \tau\} \\ &= \{((f \circ g)_*(\rho), (f \circ g)(q)) : (\rho, q) \in \tau\} \\ &= (f \circ g)_*(\tau). \end{aligned}$$ □

Corollary 25.4. *If f is an isomorphism from a forcing order $\mathbb{P}$ to a forcing order $\mathbb{Q}$, then f_* is a bijection from V^P to V^Q.* □

Recall from Chapter 13 the "embedding" of a forcing order $\mathbb{P}$ into RO($\mathbb{P}$); we denote that embedding by $e_{\mathbb{P}}$. If f is an isomorphism $\mathbb{P}$ with $\mathbb{Q}$, then it is easy to see that $e_{\mathbb{Q}} \circ f$ satisfies the following conditions:

(1) $e_{\mathbb{Q}}[f[P]]$ is dense in RO($\mathbb{Q}$).
(2) For all $p, q \in P$, if $p \leq q$ then $e_{\mathbb{Q}}(f(p)) \leq e_{\mathbb{Q}}(f(q))$.
(3) For any $p, q \in P$, $p \perp q$ iff $e_{\mathbb{Q}}(f(p)) \cdot e_{\mathbb{Q}}(f(q)) = 0$.

Hence from Theorem 13.22 it follows that there is a unique isomorphism f^* of RO($\mathbb{P}$) onto RO($\mathbb{Q}$) such that $f^* \circ e_{\mathbb{P}} = e_{\mathbb{Q}} \circ f$.

Lemma 25.5. *Let f be an isomorphism from a forcing order $\mathbb{P}$ onto a forcing order $\mathbb{Q}$. Then*

$$f^*([\![\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!]_{\mathrm{RO}(\mathbb{P})}) = [\![\varphi(f_*(\sigma_0), \ldots, f_*(\sigma_{n-1}))]\!]_{\mathrm{RO}(\mathbb{Q})}.$$

Proof. We omit the subscripts $_{\mathbb{P}}$ and $_{\mathbb{Q}}$.

First take atomic equality formulas, by well-founded induction:

$$\begin{aligned}
f^*([\![\sigma = \tau]\!]) = f^*\Bigg(& \prod_{(\xi,p)\in\tau} \left[-e(p) + \sum_{(\rho,q)\in\sigma} (e(q) \cdot [\![\rho = \xi]\!]) \right] \\
& \cdot \prod_{(\rho,q)\in\sigma} \left[-e(q) + \sum_{(\xi,p)\in\tau} (e(p) \cdot [\![\rho = \xi]\!]) \right] \Bigg) \\
&= \prod_{(\xi,p)\in\tau} \left[-f^*(e(p)) + \sum_{(\rho,q)\in\sigma} (f^*(e(q)) \cdot f^*([\![\rho = \xi]\!])) \right] \\
& \cdot \prod_{(\rho,q)\in\sigma} \left[-f^*(e(q)) + \sum_{(\xi,p)\in\tau} (f^*(e(p)) \cdot f^*([\![\rho = \xi]\!]) \right] \\
&= \prod_{(\xi,p)\in\tau} \left[-e(f(p)) + \sum_{(\rho,q)\in\sigma} (e(f(q)) \cdot [\![f_*(\rho) = f_*(\xi)]\!]) \right] \\
& \cdot \prod_{(\rho,q)\in\sigma} \left[-e(f(q)) + \sum_{(\xi,p)\in\tau} (e(f(p)) \cdot [\![f_*(\rho) = f_*(\xi)]\!]) \right] \\
&= \prod_{(\xi,p)\in f_*(\tau)} \left[-e(p) + \sum_{(\rho,q)\in f_*(\sigma)} (e(q) \cdot [\![\rho = \xi]\!]) \right] \\
& \cdot \prod_{(\rho,q)\in f_*(\sigma)} \left[-e(q) + \sum_{(\xi,p)\in f_*(\tau)} (e(p) \cdot [\![\rho = \xi]\!]) \right] \\
&= [\![f_*(\sigma) = f_*(\tau)]\!].
\end{aligned}$$

Now we prove the lemma itself by induction on φ, thus officially outside our usual mathematical language. The atomic equality case has already been treated. Here is the remaining argument, with obvious inductive assumptions:

$$\begin{aligned}
f^*([\![\sigma\in\tau]\!]) &= f^*\left(\sum_{(\rho,p)\in\tau}(e(p)\cdot[\![\sigma=\rho]\!])\right)\\
&= \sum_{(\rho,p)\in\tau}(f^*(e(p))\cdot f^*([\![\sigma=\rho]\!]))\\
&= \sum_{(\rho,p)\in\tau}(e(f(p))\cdot[\![f_*(\sigma)=f_*(\rho)]\!])\\
&= \sum_{(\varphi,q)\in f_*(\tau)}(e(q)\cdot[\![f_*(\sigma)=\varphi]\!])\\
&= [\![f_*(\sigma)=f_*(\tau)]\!];\\
f^*([\![\neg\varphi(\sigma_0,\ldots,\sigma_{n-1})]\!]) &= f^*(-[\![\varphi(\sigma_0,\ldots,\sigma_{n-1})]\!])\\
&= -f^*([\![\varphi(\sigma_0,\ldots,\sigma_{n-1})]\!])\\
&= -[\![\varphi(f_*(\sigma_0),\ldots,f_*(\sigma_{n-1}))]\!]\\
&= [\![\neg\varphi(f_*(\sigma_0),\ldots,f_*(\sigma_{n-1}))]\!];\\
f^*([\![\varphi(\sigma_0,\ldots,\sigma_{n-1})\vee\psi(\sigma_0,\ldots,\sigma_{n-1})]\!])&\\
&= f^*([\![\varphi(\sigma_0,\ldots,\sigma_{n-1})]\!]+[\![\psi(\sigma_0,\ldots,\sigma_{n-1})]\!])\\
&= f^*([\![\varphi(\sigma_0,\ldots,\sigma_{n-1})]\!])+f^*([\![\psi(\sigma_0,\ldots,\sigma_{n-1})]\!])\\
&= [\![\varphi(f_*(\sigma_0),\ldots,f_*(\sigma_{n-1}))]\!])+[\![\psi(f_*(\sigma_0),\ldots,f_*(\sigma_{n-1}))]\!])\\
&= [\![\varphi(f_*(\sigma_0),\ldots,f_*(\sigma_{n-1}))\vee\psi(f_*(\sigma_0),\ldots,f_*(\sigma_{n-1}))]\!]);\\
f^*([\![\exists x\varphi(\sigma_0,\ldots,\sigma_{n-1},x)]\!]) &= f^*\left(\sum_{\tau\in V^P}[\![\varphi(\sigma_0,\ldots,\sigma_{n-1},\tau)]\!]\right)\\
&= \sum_{\tau\in V^Q} f^*([\![\varphi(\sigma_0,\ldots,\sigma_{n-1},\tau)]\!])\\
&= \sum_{\tau\in V^Q}[\![\varphi(f_*(\sigma_0),\ldots,f_*(\sigma_{n-1}),f_*(\tau))]\!])\\
&= \sum_{\tau\in V^Q}[\![\varphi(f_*(\sigma_0),\ldots,f_*(\sigma_{n-1}),\tau)]\!])\\
&= [\![\exists x\varphi(f_*(\sigma_0),\ldots,f_*(\sigma_{n-1}),x)]\!]. \qquad \square
\end{aligned}$$

Lemma 25.6. *If $\mathbb{P}$ and $\mathbb{Q}$ are forcing orders and f is an isomorphism of $\mathbb{P}$ to $\mathbb{Q}$, then for any $p\in P$,*

$$p\Vdash_{\mathbb{P}}\varphi(\sigma_0,\ldots,\sigma_{n-1})\quad\text{iff}\quad f(p)\Vdash_{\mathbb{Q}}\varphi(f_*(\sigma_0),\ldots,f_*(\sigma_{n-1})).$$

Proof.

$$p\Vdash\varphi(\sigma_0,\ldots,\sigma_{n-1})\quad\text{iff}\quad e(p)\le[\![\varphi(\sigma_0,\ldots,\sigma_{n-1})]\!]$$

$$\begin{aligned}
&\text{iff} \quad f^*(e(p)) \leq f^*([\![\varphi(\sigma_0,\ldots,\sigma_{n-1})]\!])\\
&\text{iff} \quad e(f(p)) \leq [\![\varphi(f_*(\sigma_0),\ldots,f_*(\sigma_{n-1}))\\
&\text{iff} \quad f(p) \Vdash \varphi(f_*(\sigma_0),\ldots,f_*(\sigma_{n-1})). \qquad \square
\end{aligned}$$

Corollary 25.7. *If $\mathbb{P}$ and $\mathbb{Q}$ are forcing orders and f is an isomorphism of $\mathbb{P}$ to $\mathbb{Q}$, then for any $p \in P$,*

$$p \Vdash \varphi(\check{x}_0,\ldots,\check{x}_{n-1}) \quad \text{iff} \quad f(p) \Vdash \varphi(\check{x}_0,\ldots,\check{x}_{n-1}).$$

Proof. By Lemmas 25.2 and 25.6. $\square$

Lemma 25.8. *Suppose that M is a c.t.m. of ZFC, $\mathbb{P},\mathbb{Q} \in M$ are forcing orders, $f \in M$ is an isomorphism of $\mathbb{P}$ to $\mathbb{Q}$, and G is $\mathbb{P}$-generic over M. Then $f[G]$ is $\mathbb{Q}$-generic over M. Moreover, if $\sigma_G \in \tau_G$, then $(f_*(\sigma))_{f[G]} \in (f_*(\tau))_{f[G]}$, and if $\sigma_G = \tau_G$, then $(f_*(\sigma))_{f[G]} = (f_*(\tau))_{f[G]}$.*

Proof. We skip some details. If $D \subseteq Q$ is dense and $D \in M$, clearly $f^{-1}[D] \in M$ is dense in $\mathbb{P}$; choose $p \in f^{-1}[D] \cap G$. Then $f(p) \in D \cap f[G]$. So $f[G]$ is $\mathbb{P}$-generic over M.

Now suppose that $\sigma_G \in \tau_G$. Choose $p \in G$ such that $p \Vdash \sigma \in \tau$. Then $f(p) \in f[G]$ and $f(p) \Vdash f_*(\sigma) \in f_*(\tau)$. Hence $(f_*(\sigma))_{f[G]} \in (f_*(\tau))_{f[G]}$.

Similarly, $\sigma_G = \tau_G$ implies that $(f_*(\sigma))_{f[G]} = (f_*(\tau))_{f[G]}$. $\square$

Lemma 25.9. *Suppose that M is a c.t.m. of ZFC, $\mathbb{P},\mathbb{Q} \in M$ are forcing orders, $f \in M$ is an isomorphism of $\mathbb{P}$ to $\mathbb{Q}$, and G is $\mathbb{P}$-generic over M. Then $M[G] = M[f[G]]$, and there is an bijection $f^\star : M[G] \to M[G]$ such that $f^\star(\sigma_G) = (f_*(\sigma))_{f[G]}$ for every $\sigma \in M^{\mathbb{P}}$. Moreover, for $x, y \in M[G]$ we have $x \in y$ iff $f^\star(x) \in f^\star(y)$.*

Proof. Clearly $f[G] \in M[G]$. Hence by Lemma 15.8, $M[f[G]] \subseteq M[G]$. Applying this to f^{-1}, we get $M[G] = M[f^{-1}[f[G]]] \subseteq M[f[G]]$. So $M[G] = M[f[G]]$.

By Lemma 25.8 there is a function $f^\star : M[G] \to M[G]$ such that $f^\star(\sigma_G) = (f_*(\sigma))_{f[G]}$ for every $\sigma \in M^{\mathbb{P}}$. $f^\star$ is a bijection since

$$(f^{-1})^\star(f^\star(\sigma_G)) = (f^{-1})^\star((f_*(\sigma))_{f[G]}) = ((f^{-1})_*(f_*(\sigma))_{f^{-1}[f[G]]} = \sigma_G,$$

so that $f^{-1} \circ f$ is the identity on $M[G]$; and similarly $f \circ f^{-1}$ is the identity on $M[f[G]] = M[G]$. Finally, by Lemma 25.8, $\sigma_G \in \tau_G$ iff $f^\star(\sigma_G) \in f^\star(\tau_G)$. $\square$

We now turn to more special considerations needed for our proof of consistency of $\neg$AC.

$\mathbb{P}$ is *almost homogeneous* iff for all $p, q \in P$ there is an automorphism f of $\mathbb{P}$ such that $f(p)$ and q are compatible.

Lemma 25.10. *Let $\mathbb{P}$ be an almost homogeneous forcing order. Then*

(i) If there is a p such that $p \Vdash \varphi(\check{x}_0,\ldots,\check{x}_{n-1})$, then $1 \Vdash \varphi(\check{x}_0,\ldots,\check{x}_{n-1})$.

(ii) Either $1 \Vdash \varphi(\check{x}_0,\ldots,\check{x}_{n-1})$ or $1 \Vdash \neg\varphi(\check{x}_0,\ldots,\check{x}_{n-1})$.

Proof. (i): Assume that $p \Vdash \varphi(\check{x}_0,\ldots,\check{x}_{n-1})$, but suppose that $1 \not\Vdash \varphi(\check{x}_0,\ldots,\check{x}_{n-1})$. Thus $[\![\varphi(\check{x}_0,\ldots,\check{x}_{n-1})]\!] \neq 1$, so there is a q such that $e(q) \leq -[\![\varphi(\check{x}_0,\ldots,\check{x}_{n-1})]\!]$; so

$q \Vdash \neg\varphi(\check{x}_0, \ldots, \check{x}_{n-1})$. Let f be an automorphism such that $f(p)$ and q are compatible. By Lemma 25.7, $f(p) \Vdash \varphi(\check{x}_0, \ldots, \check{x}_{n-1})$. If $r \le f(p), q$ we then have $e(r) \le [\![\varphi(\check{x}_0, \ldots, \check{x}_{n-1})]\!] \cdot -[\![\varphi(\check{x}_0, \ldots, \check{x}_{n-1})]\!]$, contradiction.

(ii): Suppose that $1 \not\Vdash \varphi(\check{x}_0, \ldots, \check{x}_{n-1})$. By (i), $p \not\Vdash \varphi(\check{x}_0, \ldots, \check{x}_{n-1})$ for all p. Hence $1 \Vdash \neg\varphi(\check{x}_0, \ldots, \check{x}_{n-1})$. □

Lemma 25.11. *Suppose that I and J are sets, κ is an infinite cardinal, and $\pi = \langle \pi_i : i \in I \rangle$ is a system of permutations of J. For each $f \in \mathrm{Fn}(I, J, \kappa)$ define $\pi_\circ(f)$ to be the function given as follows:*

$$\begin{aligned} \mathrm{dmn}(\pi_\circ(f)) &= \mathrm{dmn}(f); \\ (\pi_\circ(f))(i) &= \pi_i(f(i)) \quad \textit{for each } i \in \mathrm{dmn}(f). \end{aligned}$$

Then $\pi_\circ$ is an automorphism of $\mathrm{Fn}(I, J, \kappa)$.

Proof. Clearly $\pi_\circ(f) \in \mathrm{Fn}(I, J, \kappa)$ for any $f \in \mathrm{Fn}(I, J, \kappa)$. Now $\pi_\circ$ is one-one: suppose that $\pi_\circ(f) = \pi_\circ(g)$. Then

$$\mathrm{dmn}(f) = \mathrm{dmn}(\pi_\circ(f)) = \mathrm{dmn}(\pi_\circ(g)) = \mathrm{dmn}(g),$$

and for any $i \in \mathrm{dmn}(f)$,

$$f(i) = \pi_i^{-1}(\pi_i(f(i))) = \pi_i^{-1}((\pi_\circ(f))(i) = \pi_i^{-1}((\pi_\circ(g))(i) = \pi_i^{-1}(\pi_i(g(i))) = g(i).$$

So $f = g$.

Next, $\pi_\circ$ maps onto $\mathrm{Fn}(I, J, \kappa)$. For, let $g \in \mathrm{Fn}(I, J, \kappa)$. Let $f(i) = \pi_i^{-1}(g(i))$ for all $i \in \mathrm{dmn}(g)$, with $\mathrm{dmn}(f) = \mathrm{dmn}(g)$. Clearly $\pi_\circ(f) = g$.

Now suppose that $f, g \in \mathrm{Fn}(I, J, \kappa)$ and $f \subseteq g$. Then for $i \in \mathrm{dmn}(f)$ we have $(\pi_\circ(f))(i) = \pi_i(f(i)) = \pi_i(g(i)) = (\pi_\circ(g))(i)$. So $\pi_\circ(f) \subseteq \pi_\circ(g)$. The other implication is proved similarly. □

Lemma 25.12. $\mathrm{Fn}(I, J, \kappa)$ *is almost homogeneous.*

Proof. Suppose that $f, g \in \mathrm{Fn}(I, J, \kappa)$. For each $i \in I$ let π_i be the following permutation of J: if $i \notin \mathrm{dmn}(f) \cap \mathrm{dmn}(g)$, then π_i is the identity on J. If $i \in \mathrm{dmn}(f) \cap \mathrm{dmn}(g)$, then π_i is the transposition $(f(i), g(i))$ (which may be the identity). Then we claim that $\pi_\circ(f)$ and g are compatible. For, suppose that $i \in \mathrm{dmn}(\pi_\circ(f)) \cap \mathrm{dmn}(g) = \mathrm{dmn}(f) \cap \mathrm{dmn}(g)$. Then $(\pi_\circ(f))(i) = \pi_i(f(i)) = g(i)$, as desired. □

We now need a basic result about product forcing; this result will be useful also when discussing iterated forcing later.

Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing orders. Their *product* $\mathbb{P} \times \mathbb{Q}$ is defined to be $(P \times Q, \le', 1')$, where

$$\begin{aligned} (p_1, q_1) \le (p_2, q_2) \quad &\text{iff} \quad p_1 \le p_2 \text{ and } q_1 \le q_2; \\ 1' &= (1^{\mathbb{P}}, 1^{\mathbb{Q}}). \end{aligned}$$

Theorem 25.13. *Suppose that M is a c.t.m. of ZFC, $\mathbb{P},\mathbb{Q}\in M$ are forcing orders, $G_1\subseteq P$, and $G_2\subseteq Q$. Then the following are equivalent:*

(i) $G_0\times G_1$ is $(\mathbb{P}\times\mathbb{Q})$-generic over M.

(ii) G_0 is $\mathbb{P}$-generic over M and G_1 is $\mathbb{Q}$-generic over $M[G_0]$.

(iii) G_1 is $\mathbb{Q}$-generic over M and G_0 is $\mathbb{P}$-generic over $M[G_1]$.

Moreover, if one of (i)–(iii) holds, then $M[G_0\times G_1]=M[G_0][G_1]=M[G_1][G_0]$.

Proof. (i)⇒(ii): Assume that $G_0\times G_1$ is $(\mathbb{P}\times\mathbb{Q})$-generic over M. Suppose that $p\in G_0$ and $p\le p'\in P$. Then $(p,1)\in G_0\times G_1$ and $(p,1)\le(p',1)$, so $(p',1)\in G_0\times G_1$; so $p'\in G_0$.

Suppose that $p,p'\in G_0$. Then $(p,1),(p',1)\in G_0\times G_1$, so they are compatible. Clearly this implies that p and p' are compatible.

Let $D\in M$ be a dense subset of P. Define $E=\{(p,q)\in P\times Q: p\in D\}$. Then $E\in M$ and it is clearly dense in $P\times Q$. Hence we can choose $(p,q)\in E\cap(G_0\times G_1)$. So $p\in D\cap G_0$, as desired.

Thus G_0 is $\mathbb{P}$-generic over M.

Now by arguments similar to the above, G_1 satisfies the conditions to be $\mathbb{Q}$-generic over $M[G_0]$ except possibly the denseness condition. So, suppose that $D\in M[G_0]$ is a dense subset of Q. Take $\tau\in M^P$ such that $\tau_{G_0}=D$. Then there is a $p\in G_0$ such that

$$p\Vdash \tau \text{ is dense in } \check{\mathbb{Q}}. \tag{1}$$

Define

$$E=\{(p',q): p'\le p \text{ and } p'\Vdash \check q\in\tau\}.$$

Thus E is a subset of $P\times Q$; we claim that it is dense below $(p,1)$. To prove this, take any $(p',q')\le(p,1)$. Now $p'\le p$, so by (1),

$$p'\Vdash \forall x\in\check Q\exists y\in\check Q(y\in\tau \text{ and } y\le x),$$

and hence

$$p'\Vdash \exists y\in\check Q(y\in\tau \text{ and } y\le\check{q}').$$

Hence by Proposition 16.16 there is a $p''\le p'$ and a $q''\in Q$ such that

$$p''\Vdash \check{q}''\in\tau \text{ and } \check{q}''\le\check{q}'.$$

Then by Lemma 25.11, $q''\le q'$. Hence $p''\le p'\le p$ and $p''\Vdash\check{q}''\in\tau$, so $(p'',q'')\in E$. Also $(p'',q'')\le(p',q')$. So this proves our claim that E is dense below $(p,1)$.

Since $p\in G_0$, we have $(p,1)\in G_0\times G_1$, so by the genericity of $G_0\times G_1$ we get $(G_0\times G_1)\cap E\neq 0$; say that $(r,s)\in(G_0\times G_1)\cap E$. Thus $r\in G_0$, $s\in G_1$, $r\le p$, and $r\Vdash\check s\in\tau$. Hence $s\in\tau_G=D$, so $D\cap G_1\neq\emptyset$. So we have proved (ii).

(ii)⇒(i): Assume (ii). First we check that $G_0\times G_1$ is a filter. Suppose that $(p,q)\in G_0\times G_1$ and $(p,q)\le(p',q')$. Then $p\le p'$, hence $p'\in G_0$; similarly, $q'\in G_1$. So $(p',q')\in G_0\times G_1$. Now suppose that $(p,q),(p',q')\in G_0\times G_1$. Thus $p,p'\in G_0$, so there is an $r\in G_0$ such that $r\le p,p'$. Similarly we get an $s\in G_1$ such that $s\le q,q'$. So $(r,s)\in G_0\times G_1$ and $(r,s)\le(p,q),(p',q')$. So $G_0\times G_1$ is a filter.

Now suppose that $D \subseteq P \times Q$ is dense and is in M. Let

$$D^* = \{q \in Q : \text{there is a } p \in G_0 \text{ such that } (p,q) \in D\}.$$

Thus $D^* \in M[G_0]$. Note that if $q \in D^* \cap G_1$, then with p as in the definition of D^* we get $(p,q) \in D \cap (G_0 \times G_1)$. Thus it suffices to show that D^* is dense in $\mathbb{Q}$. To this end, suppose that $q \in Q$. Let

$$E = \{p \in P : \text{there is a } q' \leq q \text{ such that } (p,q') \in D\}.$$

Clearly $E \in M$. Also, E is dense in P: if $p \in P$, choose $(p',q') \in D$ such that $(p',q') \leq (p,q)$; then $p' \in E$, as desired. Now since G_0 is $\mathbb{P}$-generic over M, choose $p \in G_0 \cap E$. Then by the definition of E, choose $q' \leq q$ such that $(p,q') \in D$. Thus $q' \in D^*$ and $q' \leq q$, as desired. This proves (i).

By symmetry, (i)$\leftrightarrow$(iii).

Now assume that one of (i)–(iii) holds, and hence all three hold. Now $M \subseteq M[G_0][G_1]$ and $G_0 \times G_1 \in M[G_0][G_1]$, so by Lemma 15.8, $M[G_0 \times G_1] \subseteq M[G_0][G_1]$. On the other hand, $M \subseteq M[G_0 \times G_1]$ and $G_0 \in M[G_0 \times G_1]$, so by Lemma 15.8, $M[G_0] \subseteq M[G_0 \times G_1]$. And $G_1 \in M[G_0 \times G_1]$, so by Lemma 15.8 yet again, $M[G_0][G_1] \subseteq M[G_0 \times G_1]$. This proves that $M[G_0 \times G_1] = M[G_0][G_1]$. By symmetry, $M[G_0 \times G_1] = M[G_1][G_0]$. $\square$

Lemma 25.14. *Suppose that M is a c.t.m. of ZFC, and $I, J \in M$ are uncountable (in the sense of M). Let $\mathbb{P}$ and $\mathbb{Q}$ be the partial orders* $\mathrm{Fin}(I,2,\omega)$ *and* $\mathrm{Fin}(J,2,\omega)$ *respectively. Then for any formula $\varphi(x)$ and any ordinal α,*

$$1^{\mathbb{P}} \Vdash_{\mathbb{P}} (\varphi(\check{\alpha}^{\mathbb{P}})^{\mathbf{L}(\mathscr{P}(\omega))} \quad \text{iff} \quad 1^{\mathbb{Q}} \Vdash_{\mathbb{Q}} (\varphi(\check{\alpha}^{\mathbb{Q}})^{\mathbf{L}(\mathscr{P}(\omega))}.$$

Proof. By symmetry, say $(|I| \leq |J|)^M$. Let $\mathbb{R}$ be the partial order $\mathrm{Fn}(I,J,\omega_1)$, and let H be $\mathbb{R}$-generic over M. Then by the usual argument, $\bigcup H$ is a function mapping I onto J in $M[H]$. Thus

(1) $(|I| = |J|)^{M[H]}$.

Next,

(2) If G is $\mathbb{P}$-generic over $M[H]$, then G is $\mathbb{P}$-generic over M and $M[G] \subseteq M[H][G] = M[G][H]$.

In fact, assume that G is $\mathbb{P}$-generic over $M[H]$. Obviously then G is $\mathbb{P}$-generic over M. By Lemma 15.8 we have $M[G] \subseteq M[H][G]$. By Theorem 25.13, $M[H][G] = M[G][H]$.

Recall that $\mathbb{P}$ preserves cardinalities. So $M[G]$ has the same cardinals as M.

Now by Lemma 24.12, we know that $\mathbb{R}$ is ω_1-closed in M.

(3) $\mathbb{R}$ is ω_1-closed in $M[G]$.

In fact, working in $M[G]$ suppose that $\beta < \omega_1$, $p = \langle p_\alpha : \alpha < \beta \rangle$ is a system of members of R, and $p_\gamma \leq p_\alpha$ whenever $\alpha < \gamma < \beta$. Let τ be a P-name with $\tau_G = p$, and let $q \in G$ be such that

$$\begin{aligned} q \Vdash &\tau \text{ is a function with domain } \check{\omega}_1 \text{ and range } \subseteq \check{R} \\ &\wedge \forall \alpha, \gamma < \check{\beta}[\alpha < \gamma \rightarrow \tau_\gamma \leq \tau_\alpha]. \end{aligned}$$

In particular we have

$$q \Vdash \forall\alpha < \check{\beta}\exists s \in \check{R}[\tau(\alpha) = s],$$

and so by Proposition 16.16, for each $\alpha < \beta$ we can choose $r_\alpha \in G$ and $s_\alpha \in R$ such that $r_\alpha \leq q$ and $r_\alpha \Vdash \tau(\check{\alpha}) = \check{s}_\alpha$. Then

(4) If $\alpha < \gamma < \beta$, then $s_\gamma \leq s_\alpha$.

In fact, choose $t \leq r_\alpha, r_\gamma$; this is possible since $r_\alpha, r_\gamma \in G$. Then

$$t \Vdash \tau(\check{\gamma}) \leq \tau(\check{\alpha}) \wedge \tau(\check{\alpha}) = \check{s}_\alpha \wedge \tau(\check{\gamma}) = \check{s}_\gamma,$$

hence $t \Vdash \check{s}_\gamma \leq \check{s}_\alpha$, hence it easily follows that $s_\gamma \leq s_\alpha$, so that (4) holds.

Thus in M we have a decreasing sequence $s = \langle s_\alpha : \alpha < \beta\rangle$, and so there is a $t \in R$ such that $t \leq s_\alpha$ for all $\alpha < \beta$. It follows that $r_\alpha \Vdash \check{t} \leq \tau(\check{\alpha})$ for all $\alpha < \beta$, and hence $t \leq p_\alpha$ for all $\alpha < \beta$, as desired for (3).

(5) $(\mathscr{P}(\omega))^{M[G]} = (\mathscr{P}(\omega))^{M[G][H]}$.

For, if $f \in {}^\omega 2$ and $f \in M[G][H]$, then by (3) and Theorem 16.10 we have $f \in M[G]$. So (5) holds.

$$(6) \qquad 1^{\mathbb{P}} \Vdash_{\mathbb{P},M} (\varphi(\check{\alpha}^{\mathbb{P}})^{\mathbf{L}(\mathscr{P}(\omega))} \quad \text{iff} \quad 1^{\mathbb{P}} \Vdash_{\mathbb{P},M[H]} (\varphi(\check{\alpha}^{\mathbb{P}})^{\mathbf{L}(\mathscr{P}(\omega))}.$$

In fact, first suppose that $1^{\mathbb{P}} \Vdash_{\mathbb{P},M} (\varphi(\check{\alpha}^{\mathbb{P}})^{\mathbf{L}(\mathscr{P}(\omega))}$; but suppose also that $1^{\mathbb{P}} \nVdash_{\mathbb{P},M[H]} (\varphi(\check{\alpha}^{\mathbb{P}})^{\mathbf{L}(\mathscr{P}(\omega))}$. Let G be $\mathbb{P}$-generic over $M[H]$ such that $\neg(\varphi(\check{\alpha}^{\mathbb{P}})^{\mathbf{L}(\mathscr{P}(\omega))})^{M[H][G]}$, and choose $p \in G$ such that $p \Vdash_{\mathbb{P},M[H]} \neg\varphi(\check{\alpha}^{\mathbb{P}})^{\mathbf{L}(\mathscr{P}(\omega))}$. Let G be $\mathbb{P}$-generic over $M[H]$ with $p \in G$. Hence $(\neg\varphi(\check{\alpha}^{\mathbb{P}})^{\mathbf{L}(\mathscr{P}(\omega))}$ holds in $M[H][G]$. By absoluteness and (5), $(\neg\varphi(\check{\alpha}^{\mathbb{P}})^{\mathbf{L}(\mathscr{P}(\omega))}$ holds in $M[G]$. But G is $\mathbb{P}$-generic over M, so this contradicts the supposition.

Second, suppose that $1^{\mathbb{P}} \Vdash_{\mathbb{P},M[H]} (\varphi(\check{\alpha}^{\mathbb{P}})^{\mathbf{L}(\mathscr{P}(\omega))}$. Take any G which is $\mathbb{P}$ generic over $M[H]$. Then $\varphi(\alpha)^{\mathbf{L}(\mathscr{P}(\omega))}$ holds in $M[H][G]$, and hence in $M[G]$ by absoluteness and (5). Thus G is $\mathbb{P}$-generic over M and $\varphi(\alpha)^{\mathbf{L}(\mathscr{P}(\omega))}$ holds in $M[G]$, so there is a $p \in G$ such that $p \Vdash_{\mathbb{P},M} (\varphi(\check{\alpha}^{\mathbb{P}})^{\mathbf{L}(\mathscr{P}(\omega))}$. Hence by Lemmas 25.12 and 25.14 applied to $\mathrm{Fin}(I,2) = \mathrm{Fn}(I,I,\omega)$ we get $1^{\mathbb{P}} \Vdash_{\mathbb{P},M} (\varphi(\check{\alpha}^{\mathbb{P}})^{\mathbf{L}(\mathscr{P}(\omega))}$. This proves (6).

By symmetry we have

$$(7) \qquad 1^{\mathbb{Q}} \Vdash_{\mathbb{Q},M} (\varphi(\check{\alpha}^{\mathbb{Q}})^{\mathbf{L}(\mathscr{P}(\omega))} \quad \text{iff} \quad 1^{\mathbb{Q}} \Vdash_{\mathbb{Q},M[H]} (\varphi(\check{\alpha}^{\mathbb{Q}})^{\mathbf{L}(\mathscr{P}(\omega))}.$$

Now in $M[H]$ we have $|I| = |J|$, as noted above. Hence in $M[H]$, the partial orders $\mathbb{P}$ and $\mathbb{Q}$ are isomomorphic. Hence the conclusion of the lemma follows from (6), (7), and Corollary 25.7. □

Theorem 25.17. *If ZF is consistent, then so is ZF* $+ \neg AC$.

Proof. Assume that ZF is consistent. By the theory of constructibility we know that also ZFC is consistent, so we take a c.t.m. M of ZFC. Let $\mathbb{P} = \mathrm{Fn}(\omega_1, 2, \omega)$, and let G be $\mathbb{P}$-generic over M, and let $N = \mathbf{L}(\mathscr{P}(\omega))^{M[G]}$. By Theorem 23.25, N is a model of ZF. We claim that AC fails in N, as desired.

For, suppose that AC holds in N, and in N let $\kappa = |\mathscr{P}(\omega)|$. Thus $(\check{\kappa} = |\mathscr{P}(\omega)|)^{\mathbf{L}(\mathscr{P}(\omega))}$ holds in $M[G]$, and so there is a $p \in G$ such that $p \Vdash (|\check{\kappa}| = |\mathscr{P}(\omega)|)^{\mathbf{L}(\mathscr{P}(\omega))}$. Hence $1 \Vdash_{\mathbb{P}} (|\check{\kappa}| = |\mathscr{P}(\omega)|)^{\mathbf{L}(\mathscr{P}(\omega))}$ by Lemmas 25.10 and 25.12.

Now let $\mathbb{Q}$ be the partial order $\mathrm{Fin}(|\kappa|^+, 2, \omega)$. By Lemma 25.14 and the preceding paragraph we have $1 \Vdash_{\mathbb{Q}} (|\check{\kappa}| = |\mathscr{P}(\omega)|)^{\mathbf{L}(\mathscr{P}(\omega))}$. Let H be $\mathbb{Q}$-generic over M. Then $(|\check{\kappa}| = |\mathscr{P}(\omega)|)^{\mathbf{L}(\mathscr{P}(\omega))}$ holds in $M[H]$. This means that there is a bijection from κ to $\mathscr{P}(\omega)$ in $M[H]$. But the argument used in Cohen forcing shows that ω has at least $|\kappa|^+$ subsets in $M[H]$, contradiction. □

EXERCISES

E25.1. Show that for any infinite cardinal κ, the partial order $\mathrm{Fn}(\kappa, 2, \omega)$ is isomorphic to $\mathrm{Fn}(\kappa \times \omega, 2, \omega)$.

E25.2. Prove that if $\mathbb{P}$ and $\mathbb{Q}$ are isomorphic forcing orders, then $\mathrm{RO}(\mathbb{P})$ and $\mathrm{RO}(\mathbb{Q})$ are isomorphic Boolean algebras.

E25.3. Give an example of non-isomorphic forcing orders $\mathbb{P}$ and $\mathbb{Q}$ such that $\mathrm{RO}(\mathbb{P})$ and $\mathrm{RO}(\mathbb{Q})$ are isomorphic Boolean algebras.

E25.4. For any system $\langle \mathbb{P}_i : i \in I \rangle$, we define the *weak product* $\prod^w_{i \in I} \mathbb{P}_i$ as follows: the underlying set is $\{f \in \prod_{i \in I} P_i : \{i \in I : f(i) \neq 1\}$ is finite$\}$, with $f \leq g$ iff $f(i) \leq_{\mathbb{P}_i} g(i)$ for all $i \in I$. Prove that for any infinite cardinal κ, the forcing order $\mathrm{fn}(\kappa, 2, \omega)$ is isomorphic to $\prod^w_{\alpha < \kappa} \mathbb{P}_\alpha$, where each $\mathbb{P}_\alpha$ is equal to $\mathrm{fin}(\omega, 2)$.

The remaining exercises are concerned with generalizations of the main theorem, Theorem 25.17, of this chapter. Proofs of the consistency of ¬CH date from the 1930's—long before forcing, and shortly before constructibility. The proofs were not relative to ZFC; one had to admit many "Urelemente"—elements with no members. But the basic idea of those proofs can be adapted to ZFC, and Theorem 25.17 is of this sort. We give some exercises which form an introduction to the rather extensive work that has been done in this area. Most of this work is to show that such-and-such a statement, while a consequence of ZFC, cannot be proved in ZF, but does not imply AC either.

E25.5. We expand the language of set theory by adding an individual constant $\emptyset$. An *Urelement* is an object a such that $a \neq \emptyset$ but a does not have any elements. (Plural is *Urelemente*.) A *set* is an object x which is either $\emptyset$ or has an element. Both of these are just definitions, formally like this:

$$Ur(a) \leftrightarrow a \neq \emptyset \wedge \forall x (x \notin a);$$
$$Set(x) \leftrightarrow x = \emptyset \vee \exists y (y \in x).$$

Now we let ZFU be the following set of axioms in this language:

All the axioms of ZF except extensionality and foundation.

$\forall x [\neg (x \in \emptyset)]$.

$\forall x, y[Set(x) \wedge Set(y) \wedge \forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y]$.
$\forall x[Set(x) \wedge x \neq \emptyset \rightarrow \exists y \in x \forall z(z \in x \rightarrow z \notin y)]$.

We also reformulate the axiom of choice for ZFU; it is the following statement:

$$\begin{aligned}\forall \mathscr{A}\{Set(\mathscr{A}) \wedge \forall x \in \mathscr{A}[Set(x) \wedge x \neq \emptyset]\\ \wedge \forall x \in \mathscr{A} \forall y \in \mathscr{A}[x \neq y \rightarrow \forall z[\neg(z \in x \wedge z \in y)]]\\ \rightarrow \exists \mathscr{B} \forall x \in \mathscr{A} \exists! y(y \in x \wedge y \in \mathscr{B})\}.\end{aligned}$$

We let $ZFCU$ be all of these axioms.

One can adapt most of elementary set theory to use these axioms; browsing through the first few chapters should convince one of this.

In this exercise, give a new definition of *ordinal*.

Also, show that if we add the axiom $\neg \exists a[Ur(a)]$ we get a theory equivalent to ZF.

E25.6. Let κ be an infinite cardinal. Let α be an infinite ordinal such that $\kappa \leq V_\alpha$ Since $|V_{\alpha+1}| = 2^{|V_\alpha|}$, we also have $\kappa \leq |V_{\alpha+1} \backslash V_\alpha|$. Let U be a subset of $V_{\alpha+1} \backslash V_\alpha$ of size κ. Let Z be any element of U, fixed for what follows. We define $\langle W_\beta : \beta \in \mathbf{On} \rangle$ by recursion:

$$\begin{aligned} W_0 &= U;\\ W_{\beta+1} &= W_\beta \cup (\mathscr{P}(W_\beta) \backslash \{\emptyset\});\\ W_\beta &= \bigcup_{\gamma<\beta} W_\gamma \quad \text{for } \beta \text{ limit};\\ W &= \bigcup_{\beta \in \text{Ord}} W_\beta. \end{aligned}$$

Prove the following:

(1) If $\gamma < \beta$, then $W_\gamma \subseteq W_\beta$.

(2) If $x \in y \in W_\beta \backslash U$, then $x \in W_\beta$. (Thus $y \in W \backslash U$ implies that $y \subseteq W$.)

(3) $W_\beta \cap V_\alpha = \emptyset$ for all β.

Now for each $x \in W$ we define its rank $\text{rank}_W(x)$ in this new hierarchy. Let β be minimum such that $x \in W_\beta$. If $\beta = 0$, let $\text{rank}_W(x) = -1$. Otherwise, β is a successor ordinal $\gamma + 1$ and we define $\text{rank}_W(x) = \gamma$.

(4) $W_\beta = \{x \in W : \text{rank}_W(x) < \beta\}$.

(5) If $x, y \in W$ and $x \in y$, then $\text{rank}_W(x) < \text{rank}_W(y)$.

(6) If $x \in W \backslash U$, then $\text{rank}_W(x) = \sup_{y \in x}(\text{rank}_W(y) + 1)$.

(7) If $x \in W$, then $\text{rank}(x) = \alpha + 1 + \text{rank}_W(x)$.

(8) If $a \in W \backslash U$, then $W \cap a \neq \emptyset$.

(9) For any $a \in W$ we have $Ur^W(a)$ iff $a \in U \backslash \{Z\}$.

(10) For any $a \in W$ we have $Set^W(a)$ iff $a \notin U\backslash\{Z\}$.

This is clear from (9).

E25.7. (Continuing E25.6) Show that (W, Z) is a model of ZFCU.

E25.8. (Continuing E25.7) Let f be a permutation of $U\backslash\{Z\}$. Show that f extends to an automorphism f^+ of the structure $(W, \in, Z)$ in a natural way, so that $f^+(a) = \{f^+(b) : b \in W, b \in a\}$ for every $a \in W$.

E25.9. (Continuing E25.8) An element a of W is *W-transitive* iff for all $b, c \in W$, if $b \in c \in a$ then $b \in a$. Note that each member of U, and even each set of members of U, are symmetric. Show that for any $a \in W\backslash U$ there is a smallest W-transitive set T such that either $T = a \in U$ or $a \notin U$ and $a \subseteq T$. We call this set (which is clearly unique) the *W-transitive closure* of a.

Also show that if f is a permutation of $U\backslash\{Z\}$ and $a \in W\backslash U$, then f^+ maps the W-transitive closure of a onto the W-transitive closure of $f^+(a)$.

E25.10. (Continuing E25.9) An element a of W is *symmetric* iff there is a finite subset F of $U\backslash\{Z\}$ such that $f^+(a) = a$ for every permutation f of $U\backslash\{Z\}$ such that $f(x) = x$ for all $x \in F$. Then we call *a hereditarily symmetric* iff every b in the W-transitive closure of $\{a\}$ is symmetric. Let H be the class of all hereditarily symmetric elements of W. Prove:

(i) Every element of U is hereditarily symmetric.

(ii) Prove that if a is symmetric and f is any permutation of $U\backslash\{Z\}$, then $f^+(a)$ is symmetric.

(iii) Prove that if f is any permutation of $U\backslash\{Z\}$, then $f^+ \restriction H$ is an automorphism of (H, Z).

(iv) Prove that if $\varphi(v_0, \ldots, v_{n-1})$ is any formula, $v_0, \ldots, v_{n-1} \in H$, φ^H holds, and f is any automorphism of (H, Z), then $\varphi^H(f^+(v_0), \ldots, f^+(v_{n-1}))$.

(v) Prove that (H, Z) is a model of ZFU. where H is the class of all hereditrily symmetric elements of W.

E25.11. (Continuing E25.10) (i) We make a metalanguage definition, associating with each natural number m a term $\overline{m}$ in a definitional extension of the language for ZFU: $\overline{0} = \emptyset$, and $\overline{m+1} = \overline{m} \cup \{\overline{m}\}$. Prove that ZFU $\models \overline{m} \in \omega$ for all $m \in \omega$.

(ii) Prove that if $m < n < \omega$, then ZFU $\models \overline{m} \in \overline{n} \wedge \overline{m} \neq \overline{n}$.

(iii) Let ZFUI be the theory ZFU together with each of the following sentences, for $m \in \omega$:

$$\exists v_0 \ldots v_m \left(\bigwedge_{i \leq m} Ur(v_i) \wedge \bigwedge_{0 \leq i < j \leq m} [\neg(v_i = v_j)] \right).$$

Prove that AC cannot be proved in ZFUI.

Hint: In fact, show that ZFUI cannot prove that there is a one-one function mapping ω into Ur. For this, take the above model (H, Z) with κ infinite, hence with U infinite. Assume that $f \in H$ is such that

$$(H, Z) \models f \text{ is a one-one function mapping } \omega \text{ into } Ur,$$

and get a contradiction.

References

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26. Embeddings, iterated forcing, and Martin's axiom

In this chapter we mainly develop iterated forcing. The idea of iterated forcing is to construct in succession $M[G_0]$, $M[G_0][G_1]$, etc., continuing transfinitely, but stopping at some stage $M[G_0][G_1]\ldots[G_\alpha]$. Here G_0 is P_0-generic over M, G_1 is P_1-generic over $M[G_0]$, etc., where P_0 is a forcing order in M, P_1 is a forcing order in $M[G_0]$, etc.

Note that after the famous "..." in such an iteration one cannot simply take the union of preceding models. This has already been observed in exercise E15.13. We give here a solution of that exercise, since this helps motivate the way that iterated forcing is defined. Take the simple case in which we are given a forcing order $\mathbb{P}$ in M such that for every $p\in\mathbb{P}$ there are incompatible $q,r\le p$, and form

$$M = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

where for each n, $M_{n+1} = M_n[G_n]$ for some G_n which is $\mathbb{P}$-generic over M_n. We claim that $\bigcup_{n\in\omega} M_n$ does not satisfy the power set axiom. For, assume that $R = \bigcup_{n\in\omega} M_n$ does satisfy the power set axiom. Then $R \models \exists y\forall z(z\subseteq P \to z\in y)$. Choose $y\in R$ so that $R\models \forall z(z\subseteq P\to z\in y)$. Say $y\in M_n$. Then $R\models G_n\subseteq P\to z\in y$. By absoluteness, $R\models G_n\subseteq P$. So $R\models G_n\in y$, hence $G_n\in y\in M_n$. This contradicts Lemma 11.2.

Thus care must be taken at limit steps in an iteration.

A remarkable fact about iteration is that the final stage can be defined as a simple generic extension of M with respect to a (complicated) forcing order. In fact, the official definition of iterated forcing will have this property built-in.

Usually a single step in an iteration is the most important, with the gluing together of all the single steps a technical matter. Such a single step amounts to seeing what happens in the situation $M[G][H]$, and we first deal with that in detail.

Since we will be dealing with at least two forcing orders at the same time, it is important to be rather precise with the notation. So we return to the official notation $\mathbb{P} = (P,\le,1)$ for forcing orders introduced in Chapter 13. Now we want to be even more precise, and write $\mathbb{P} = (P_{\mathbb{P}},\le_{\mathbb{P}},1_{\mathbb{P}})$ to indicate the dependence on the particular forcing order.

Let M be a c.t.m. of ZFC, and let $\mathbb{P}\in M$ be a forcing order. A $\mathbb{P}$-*name for a forcing order* is a $\mathbb{P}$-name $\pi = \mathrm{op}(\mathrm{op}(\pi^0,\pi^1),\pi^2)$ such that $\pi^2\in\mathrm{dmn}(\pi^0)$ and

$$1_{\mathbb{P}} \Vdash_{\mathbb{P}} \pi^2\in\pi^0 \text{ and } \pi^1 \text{ is a forcing order on } \pi^0 \text{ with largest element } \pi^2.$$

Sometimes we denote π^1,π^2 by $\le_\pi$, and 1_π respectively. Recall from before 15.22 the definition of op.

Thus if G is $\mathbb{P}$-generic over M, then π_G is a forcing order in $M[G]$. We now want to define a single forcing order in M which embodies both $\mathbb{P}$ and π_G, in a sense. So we define a forcing order $\mathbb{P}*\pi$ in M. The underlying set of this forcing order is

$$\{(p,\tau) : p\in P_{\mathbb{P}}, \tau\in\mathrm{dmn}(\pi^0), \text{ and } p\Vdash_{\mathbb{P}}\tau\in\pi^0\}.$$

The order in $\mathbb{P}*\pi$ is given by

$$(p,\tau)\le_{\mathbb{P}*\pi}(q,\sigma) \quad\text{iff}\quad p\le_{\mathbb{P}} q \text{ and } p\Vdash_{\mathbb{P}}\tau\le_\pi\sigma.$$

Finally, we let $1_{\mathbb{P}*\pi} = (1_{\mathbb{P}}, 1_\pi)$.

We have given the definition here replete with all necessary subscripts. But from now on we omit some subscripts when no confusion is likely. We illustrate this simplification by giving the proof of the following theorem first without subscripts and then with them.

Proposition 26.1. *Under the above notation, $\mathbb{P} * \pi$ is a forcing order in M.*

Proof. Suppose that $(p,\tau) \in \mathbb{P} * \pi$. Then $1 \Vdash \forall x \in \pi^0 (x \leq x)$ and $p \Vdash \tau \in \pi^0$, so $p \Vdash \tau \leq \tau$. Hence $(p,\tau) \leq (p,\tau)$.

Suppose that $(p,\tau) \leq (q,\sigma) \leq (r,\varphi)$. Then $p \leq q \leq r$, so $p \leq r$. Also, $p \Vdash \tau \leq \sigma$ and $q \Vdash \sigma \leq \varphi$, so, since $p \leq q$, $p \Vdash \tau \leq \varphi$. Thus $(p,\tau) \leq (r,\varphi)$.

If $(p,\tau) \in \mathbb{P} * \pi$, then $p \leq 1$ and $p \Vdash \tau \leq 1$, so $(p,\tau) \leq (1,1)$. $\square$

Proof with subscripts. Suppose that $(p,\tau) \in \mathbb{P} * \pi$. Then $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \forall x \in \pi^0 (x \leq_\pi x)$ and $p \Vdash_{\mathbb{P}} \tau \in \pi^0$, so $p \Vdash_{\mathbb{P}} \tau \leq_\pi \tau$. So $(p,\tau) \leq_{\mathbb{P}*\pi} (p,\tau)$.

Suppose that $(p,\tau) \leq_{\mathbb{P}*\pi} (q,\sigma) \leq_{\mathbb{P}*\pi} (r,\varphi)$. Then $p \leq_{\mathbb{P}} q \leq_{\mathbb{P}} r$, so $p \leq_{\mathbb{P}} r$. Also, $p \Vdash_{\mathbb{P}} \tau \leq_\pi \sigma$ and $q \Vdash_{\mathbb{P}} \sigma \leq_\pi \varphi$, so, since $p \leq_{\mathbb{P}} q$, $p \Vdash_{\mathbb{P}} \tau \leq_\pi \varphi$. Thus $(p,\tau) \leq_{\mathbb{P}*\pi} (r,\varphi)$.

If $(p,\tau) \in \mathbb{P} * \pi$, then $p \leq_{\mathbb{P}} 1$ and $p \Vdash_{\mathbb{P}} \tau \leq_\pi 1_\pi$, so $(p,\tau) \leq_{\mathbb{P}*\pi} (1_{\mathbb{P}}, 1_\pi)$. $\square$

We now need to digress into more of the general theory of forcing orders and forcing. If $\mathbb{P}$ and $\mathbb{Q}$ are forcing orders, a function $i : P_{\mathbb{P}} \to P_{\mathbb{Q}}$ is a *complete embedding* iff the following conditions hold:

(1) $i(1) = 1$.

(2) For all $p, p' \in P_{\mathbb{P}}$, if $p' \leq p$ then $i(p') \leq i(p)$.

(3) For all $p, p' \in P_{\mathbb{P}}$, $p \perp p'$ iff $i(p) \perp i(p')$.

(4) For any $q \in P_{\mathbb{Q}}$ there is a $p \in P_{\mathbb{P}}$, which is called a *reduction* of q to $\mathbb{P}$, such that for all $p' \in P_{\mathbb{P}}$, if $p' \leq p$ then $i(p')$ and q are compatible.

Proposition 26.2. *Suppose $\mathbb{P}$ and $\mathbb{Q}$ are forcing orders and $i : P_{\mathbb{P}} \to P_{\mathbb{Q}}$. Then i is a complete embedding iff (1)–(3) hold together with*
(5) For all $A \subseteq \mathbb{P}$, if A is a maximal antichain in $\mathbb{P}$, then $i[A]$ is a maximal antichain in $\mathbb{Q}$.

Proof. $\Rightarrow$: Assume that i is a complete embedding, and suppose that $A \subseteq \mathbb{P}$ is a maximal antichain in $\mathbb{P}$. Let $q \in Q$. Choose p by (4). Then there is a $p' \in A$ such that p and p' are compatible. Say $p'' \leq p, p'$. Then by (4), $i(p'')$ and q are compatible. By (2), $i(p'') \leq i(p')$. Thus $i(p')$ and q are compatible. This shows that $i[A]$ is a maximal antichain in $\mathbb{Q}$.

$\Leftarrow$: Assume (5), and suppose that $q \in \mathbb{Q}$. Let $A \subseteq \mathbb{P}$ be maximal such that the following hold:

(a) $p \perp p'$ for all distinct $p, p' \in A$.

(b) $i(p) \perp q$ for all $p \in A$.

Then by (b) and (3), $i[A]$ is an antichain in $\mathbb{Q}$, but it is not maximal. Hence by (5), A is not a maximal antichain. So there is a $p \in \mathbb{P}$ such that $p \perp p''$ for all $p'' \in A$. Suppose that

$p' \in \mathbb{P}$ and $p' \leq p$. Then also $p' \perp p''$ for all $p'' \in A$. By the maximality of A, it follows that $i(p')$ and q are compatible. $\square$

Theorem 26.3. *Suppose that M is a c.t.m. of ZFC, and in M i is a complete embedding of a forcing order $\mathbb{P}$ into a forcing order $\mathbb{Q}$. Suppose that H is $\mathbb{Q}$-generic over M. Then $i^{-1}[H]$ is $\mathbb{P}$-generic over M, and $M[i^{-1}[H]] \subseteq M[H]$.*

Proof. To show that $i^{-1}[H]$ is $\mathbb{P}$-generic over M we will apply Proposition 15.3 and show that $i^{-1}[H]$ is upward closed, any two members of it are compatible, and it intersects every dense set which is in M.

Suppose that $p \geq p' \in i^{-1}[H]$. Thus by (1), $i(p) \geq i(p') \in H$, so $i(p) \in H$ and hence $p \in i^{-1}[H]$.

Suppose that $p, q \in i^{-1}[H]$. Thus $i(p), i(q) \in H$, so $i(p)$ and $i(q)$ are compatible. Hence p and q are compatible by (2).

Suppose that $D \subseteq P_{\mathbb{P}}$ is dense and is in M. Let

$$E = \{q \in P_{\mathbb{Q}} : \text{there is a } u \in D \text{ such that } q \leq i(u)\}.$$

Clearly E is in M. We claim that it is dense in $\mathbb{Q}$. To prove this, suppose that $s \in P_{\mathbb{Q}}$. Let $t \in P_{\mathbb{P}}$ be a reduction of s to $\mathbb{P}$. Choose $u \in D$ such that $u \leq t$. Then by the definition of reduction it follows that $i(u)$ and s are compatible. Say $q \leq i(u), s$. Then $q \in E$ and $q \leq s$, as desired.

So, choose $q \in E \cap H$. Then there is a $u \in D$ such that $q \leq i(u)$. It follows that $i(u) \in H$, and hence $u \in i^{-1}[H] \cap D$, as desired.

So we have checked that $i^{-1}[H]$ is $\mathbb{P}$-generic over M.

Now $i \in M \subseteq M[H]$, so $i^{-1}[H] \in M[H]$ by absoluteness. It follows from Lemma 15.8 that $M[i^{-1}[H]] \subseteq M[H]$. $\square$

For the next theorem, recall the definition of i_* from Chapter 25.

Theorem 26.4. *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are forcing orders and i is a function mapping P into Q. Suppose that M is a c.t.m. of ZFC. Then*

(i) If $H \subseteq Q$ and $\tau \in M^{\mathbb{P}}$, then $\mathrm{val}(\tau, i^{-1}[H]) = \mathrm{val}(i_*(\tau), H)$.

(ii) Assume that i is a complete embedding of $\mathbb{P}$ into $\mathbb{Q}$. Suppose that H is $\mathbb{Q}$-generic over M and $\varphi(x_1, \ldots, x_n)$ is a formula which is absolute for c.t.m. of ZFC. Then for any $p \in P$,

$$p \Vdash_{\mathbb{P}} \varphi(\tau_1, \ldots, \tau_n) \quad \text{iff} \quad i(p) \Vdash_{\mathbb{Q}} \varphi(i_*(\tau_1), \ldots, i_*(\tau_n)).$$

Proof. (i): by induction on τ. First suppose that $x \in \mathrm{val}(\tau, i^{-1}[H])$. Choose $(\sigma, p) \in \tau$ such that $p \in i^{-1}[H]$ and $x = \mathrm{val}(\sigma, i^{-1}[H])$. Then $i(p) \in H$ and, by the inductive hypothesis, $x = \mathrm{val}(i_*(\sigma), H)$. Thus $(i_*(\sigma), i(p)) \in i_*(\tau)$. So $x \in \mathrm{val}(i_*(\tau), H)$.

Conversely, suppose that $x \in \mathrm{val}(i_*(\tau), H)$. Choose $(\sigma, p) \in \tau$ so that $i(p) \in H$ and $x = \mathrm{val}(i_*(\sigma), H)$. Then $p \in i^{-1}[H]$ and, by the inductive hypothesis, $x = \mathrm{val}(\sigma, i^{-1}[H])$. So $x \in \mathrm{val}(\tau, i^{-1}[H])$.

(ii): For $\Rightarrow$, assume that $p \Vdash_{\mathbb{P}} \varphi(\tau_1, \ldots, \tau_n)$. Let H be $\mathbb{Q}$-generic over $\mathbb{Q}$ with $i(p) \in H$. Then $p \in i^{-1}[H]$, and $i^{-1}[H]$ is $\mathbb{P}$-generic over M by 26.3. Hence by the external definition of forcing,

$$\varphi(\mathrm{val}(\tau_1, i^{-1}[H]), \ldots, \mathrm{val}(\tau_n, i^{-1}[H]))$$

holds in $M[i^{-1}[H]]$. Now by (i), $\mathrm{val}(\tau_j, i^{-1}[H]) = \mathrm{val}(i_*(\tau_j), H)$ for each $j = 1, \ldots, n$, and $M[i^{-1}[H]] \subseteq M[H]$ by 26.3, so by absoluteness we see that

$$\varphi(\mathrm{val}(i_*(\tau_1), H), \ldots, \mathrm{val}(i_*(\tau_n), H))$$

holds in $M[H]$. Hence by the definition of forcing, $i(p) \Vdash_{\mathbb{Q}} \varphi(i_*(\tau_1), \ldots, i_*(\tau_n))$.

For $\Leftarrow$, suppose that it is not the case that $p \Vdash_{\mathbb{P}} \varphi(\tau_1, \ldots, \tau_n)$. Then there is a $q \leq p$ such that $q \Vdash_{\mathbb{P}} \neg\varphi(\tau_1, \ldots, \tau_n)$. By the direction $\Rightarrow$, we then have $i(q) \Vdash_{\mathbb{Q}} \neg\varphi(i_*(\tau_1), \ldots, i_*(\tau_n))$. Since $i(q) \leq i(p)$, it follows that it is not the case that $i(p) \Vdash_{\mathbb{Q}} \neg\varphi(i_*(\tau_1), \ldots, i_*(\tau_n))$. □

We return to our discussion of two-stage iterations. With the notation introduced above, define $i(p) = (p, 1_\pi)$.

Proposition 26.5. *Under the above notation, i is a complete embedding of $\mathbb{P}$ into $\mathbb{P} * \pi$.*

Proof. (1) and (2) are easy to check. For (3), let $(p, \tau) \in \mathbb{P}*\pi$ be given. We claim that p is a reduction of (p, τ) to $\mathbb{P}$. For, suppose that $q \leq p$. Then $i(q) = (q, 1)$ is compatible with (p, τ), since $(q, \tau) \in \mathbb{P} * \pi$ and $(q, \tau) \leq (q, 1), (p, \tau)$. □

Proposition 26.6. *Again assume the above notation. Suppose that G is $\mathbb{P}$-generic over M, ρ is a $\mathbb{P}$-name, and $\rho_G \in \pi^0_G$. then there is a $(r, \sigma) \in \mathbb{P} * \pi$ such that $r \in G$ and $r \Vdash \rho = \sigma \wedge \sigma \in \pi^0$.*

Proof. Choose $q \in G$ such that $q \Vdash \rho \in \pi^0$. Then by Lemma 15.17 choose $r \leq q$ with $r \in G$ such that for some $(\sigma, s) \in \pi^0$ we have $r \leq s$ and $r \Vdash \rho = \sigma$. Since $r \leq q$, we have $r \Vdash \rho \in \pi^0$, so by the external definition of forcing, $r \Vdash \sigma \in \pi^0$. Clearly $(r, \sigma) \in \mathbb{P}*\pi$. □

Next, suppose that G is $\mathbb{P}$-generic over M and $H \subseteq \pi^0_G$. Then we define

$$G * H = \{(p, \tau) \in \mathbb{P} * \pi : p \in G \text{ and } \tau_G \in H\}.$$

Theorem 26.7. *Suppose that M is a c.t.m. of ZFC, $\mathbb{P}$ is a forcing order in M, and π is a $\mathbb{P}$-name for a forcing order in M.*

If G is $\mathbb{P}$-generic over M and H is π_G-generic over $M[G]$, then $G*H$ is $(\mathbb{P}*\pi)$-generic over M, and $M[G * H] = M[G][H]$.

Proof. Suppose that $(p, \tau) \in G * H$ and $(p, \tau) \leq (q, \sigma)$. Thus $p \in G$, $\tau_G \in H$, $p \leq q$, and $p \Vdash \tau \leq \sigma$. Hence $q \in G$. Also, $\tau_G \leq \sigma_G$, and so $\sigma_G \in H$. Hence $(q, \sigma) \in G * H$.

Suppose that $(p, \tau), (q, \sigma) \in G*H$. Then $p \in G$, $\tau_G \in H$, $q \in G$, and $\sigma_G \in H$. Choose $\rho_G \in H$ such that $\rho_G \leq \tau_G, \sigma_G$. By Proposition 26.6 choose $(r, \varphi) \in \mathbb{P} * \pi$ such that $r \in G$ and $r \Vdash \rho = \varphi$. Also choose $s, t \in G$ so that $s \Vdash \rho \leq \tau$ and $t \Vdash \rho \leq \sigma$. Finally, take $u \leq p, q, r, s, t$. Then $(u, \varphi) \in \mathbb{P} * \pi$ and $(u, \varphi) \leq (p, \tau), (q, \sigma)$.

Now suppose that $D \subseteq \mathbb{P} * \pi$ is dense in $\mathbb{P} * \pi$. Let

$$F = \{\sigma_G : (q, \sigma) \in D \text{ for some } q \in G \text{ and some } \sigma\}.$$

We claim that F is dense in π_G. To check this, let $x \in \pi_G$; say $x = \rho_G$ with $(\rho, q) \in \pi$ and $q \in G$. Thus $(q, \rho) \in \mathbb{P} * \pi$. Now we claim that

$$K \stackrel{\text{def}}{=} \{s \in \mathbb{P} : (s, \sigma) \leq (q, \rho) \text{ for some } \sigma \text{ such that } (s, \sigma) \in D\}$$

is dense below q. For, suppose that $r \leq q$. Then $r \Vdash \rho \in \pi$, so $(r, \rho) \in \mathbb{P} * \pi$. Choose $(s, \sigma) \in D$ such that $(s, \sigma) \leq (r, \rho)$. Thus $s \in K$ and $s \leq r$, as desired.

Now let $s \in G \cap K$; say $(s, \sigma) \leq (q, \rho)$ with $(s, \sigma) \in D$. Then $s \Vdash \sigma \leq \rho$, so $\sigma_G \leq \rho_G$. Since $s \in G$, this shows that F is dense.

Choose $\sigma_G \in F \cap H$. Say $(q, \sigma) \in D$ with $q \in G$. Then $(q, \sigma) \in D \cap (G * H)$, as desired.

For the final statement of the theorem, note that $G \in M[G * H]$, since $p \in G$ iff $(p, 1) \in G * H$. Hence $M[G] \subseteq M[G * H]$ by Lemma 15.8. Also, $H \in M[G * H]$, since

$$H = \{x : \text{there is a } (q, \tau) \in G * H \text{ such that } x = \tau_G\}.$$

Hence $M[G][H] \subseteq M[G*H]$. Conversely, clearly $G*H \in M[G][H]$, so $M[G*H] \subseteq M[G][H]$. □

Theorem 26.8. *Suppose that M is a c.t.m. of ZFC, $\mathbb{P}$ is a forcing order in M, and π is a $\mathbb{P}$-name for a forcing order in M. Let i be the complete embedding defined above.*

Suppose that K is $(\mathbb{P} * \pi)$-generic over M. Define $G = i^{-1}[K]$ and

$$H = \{\tau_G : \tau \in \operatorname{dmn}(\pi^0) \text{ and } (q, \tau) \in K \text{ for some } q\}.$$

Then G is $\mathbb{P}$-generic over M, H is π_G-generic over $M[G]$, $K = G * H$, and $M[K] = M[G][H]$.

Proof. By Theorem 26.3, G is $\mathbb{P}$-generic over M and $M[G] \subseteq M[K]$.

Now suppose that $x \in H$ and $x \leq y$; we want to show that $y \in H$. Write $x = \tau_G$ with $\tau \in \operatorname{dmn}(\pi^0)$ and $(q, \tau) \in K$ for some q. We are assuming that $y \in \pi^0_G$. So there is an $(\sigma, s) \in \pi^0$ such that $s \in G$ and $y = \sigma_G$. Since $\tau_G \leq \sigma_G$, choose $p \in G$ such that $p \Vdash \tau \leq \sigma$. Since $p \in G$, we have $(p, 1) \in K$. Also $(q, \tau) \in K$, so we can choose $(r, \rho) \in K$ such that $(r, \rho) \leq (p, 1), (q, \tau)$. Thus $r \leq p$, so $r \Vdash \tau \leq \sigma$. Also, from $(r, \rho) \leq (q, \tau)$ we see that $r \Vdash \rho \leq \tau$. So $r \Vdash \rho \leq \sigma$. Hence $(r, \rho) \leq (r, \sigma)$, and hence $(r, \sigma) \in K$. This shows that $y = \sigma_G \in H$.

Next suppose that $x, y \in H$; we want to find a common extension. Write $x = \tau_G$ with $\tau \in \operatorname{dmn}(\pi^0)$ and $(q, \tau) \in K$, and $y = \sigma_G$ with $\sigma \in \operatorname{dmn}(\pi^0)$ and $(s, \sigma) \in K$. Choose $(p, \rho) \leq (q, \tau), (s, \sigma)$ with $(p, \rho) \in K$. Then also $(p, 1) \in K$, so $p \in G$. Also, $p \Vdash \rho \leq \tau$, so $\rho_G \leq \tau_G = x$. Similarly, $\rho_G \leq \sigma_G$.

Next let $D \in M[G]$ be a dense subset of π_G; we want to show that $D \cap H \neq \emptyset$. Let δ be a $\mathbb{P}$-name such that $\delta_G = D$. Then there is a $p \in G$ such that

$$p \Vdash \text{“}\delta \text{ is dense in } \pi\text{”}.$$

Let

$$D' = \{(q, \tau) \in \mathbb{P} * \pi : q \Vdash \tau \in \delta\}.$$

We claim that D' is dense below $(p,1)$. For, suppose that $(r,\rho) \leq (p,1)$. Then $r \Vdash$ "δ is dense in π", and $r \Vdash \rho \in \pi^0$, so

$$r \Vdash \exists x \in \pi^0 (x \in \pi^0 \wedge x \in \delta \wedge x \leq \rho).$$

Hence by Proposition 6.16 there exist $s \leq r$ and $\tau \in \text{dmn}(\pi^0)$ such that

$$s \Vdash \tau \in \pi^0 \wedge \tau \in \delta \wedge \tau \leq \rho.$$

Thus $(s,\tau) \in \mathbb{P} * \pi$, $(s,\tau) \leq (r,\rho)$, and $s \Vdash \tau \in \delta$. Thus $(s,\tau) \in D'$ and $(s,\tau) \leq (r,\rho)$. This proves that D' is dense below $(p,1)$.

Now $p \in G$, so $(p,1) \in K$. Hence there is a $(q,\tau) \in D' \cap K$. Hence also $(q,1) \in K$, so $q \in G$. Since $q \Vdash \tau \in \delta$, it follows that $\tau_G \in \delta_G = D$. Clearly also $\tau_G \in H$. This finishes the proof that H is π_G-generic over $M[G]$.

Next we show that $K \subseteq G * H$. If $(p,\tau) \in K$, then also $(p,1) \in K$, and so $p \in G$. Clearly also $\tau_G \in H$, so $(p,\tau) \in G * H$.

Now we show that $G * H \subseteq K$. Let $(p,\tau) \in G * H$. Thus $p \in G$ and $\tau_G \in H$. Hence $(p,1) \in K$. By the definition of H, there exist $\sigma \in \text{dmn}(\pi^0)$ and q such that $(q,\sigma) \in K$ and $\tau_G = \sigma_G$. Choose $r \in G$ such that $r \Vdash \tau = \sigma$. So $(r,1) \in K$. Let $(s,\varphi) \in K$ be such that $(s,\varphi) \leq (p,1),(q,\sigma),(r,1)$. Since $(s,\varphi) \leq (q,\sigma)$, we have $s \Vdash \varphi \leq \sigma$. Also, $s \leq r$, so $s \Vdash \tau = \sigma$. Hence $s \Vdash \varphi \leq \tau$. Hence $(s,\varphi) \leq (p,\tau)$, and so $(p,\tau) \in K$, as desired.

The last part of the theorem follows from Theorem 26.7. $\square$

Lemma 26.9. *Suppose that M is a c.t.m. of ZFC, κ is a regular cardinal of M, $\mathbb{P}$ is a forcing order in M, and $\mathbb{P}$ satisifies κ-cc in M. Suppose that σ is a $\mathbb{P}$-name, and*

$$1 \Vdash \sigma \subseteq \check{\kappa} \wedge |\sigma| < \check{\kappa}.$$

Then there is a $\beta < \kappa$ such that $1 \Vdash \sigma \subseteq \check{\beta}$.

Proof. First we work in M. Let

$$E = \left\{\alpha < \kappa : \text{there is a } p \in P \text{ such that } p \Vdash \check{\alpha} = \bigcup \sigma \right\}.$$

For each $\alpha \in E$, pick $p_\alpha \in P$ such that $p_\alpha \Vdash \check{\alpha} = \bigcup \sigma$.

(1) $\{p_\alpha : \alpha \in E\}$ is an antichain in $\mathbb{P}$.

For, suppose that α and β are distinct elements of E. If $q \leq p_\alpha, p_\beta$, then $q \Vdash \check{\alpha} = \check{\beta}$, contradiction.

Thus by κ-cc, $|E| < \kappa$. Hence there is a $\beta < \kappa$ such that $E \subseteq \beta$.

This finishes our argument inside M. Now if G is $\mathbb{P}$-generic over M, then κ is regular in $M[G]$ by Proposition 16.5. Since $|\sigma_G| < \kappa$, it follows that $\bigcup \sigma_G < \kappa$. Let $\alpha = \bigcup \sigma_G$, and choose $p \in G$ such that $p \Vdash \check{\alpha} = \bigcup \sigma$. Thus $\alpha \in E$, and hence $\alpha \in \beta$. Thus $\bigcup \sigma_G < \beta$, and so $\sigma_G \subseteq \beta$. Since this is true for any generic G, it follows that $1 \Vdash \sigma \subseteq \check{\beta}$. $\square$

Theorem 26.10. *Suppose that M is a c.t.m. of ZFC, and in M, π is a $\mathbb{P}$-name for a forcing order, κ is a regular cardinal, $\mathbb{P}$ is κ-cc, and $1 \Vdash \pi$ is $\check{\kappa}$-cc.*

Then $\mathbb{P} * \pi$ is κ-cc.

Proof. Suppose not, and let $\langle (p_\xi, \tau_\xi) : \xi < \kappa \rangle$ be an antichain in $\mathbb{P} * \pi$ in M. Let $\sigma = \{(\check{\xi}, p_\xi) : \xi < \kappa\}$. Thus σ is a $\mathbb{P}$-name in M.

Now let G be $\mathbb{P}$-generic over M. Then

(1) $\sigma_G = \{\xi \in \kappa : p_\xi \in G\}$.

In fact, $x \in \sigma_G$ iff there is a $\xi < \kappa$ such that $p_\xi \in G$ and $x = \xi$, so (1) holds.

We claim now that if $\xi < \eta$ and both are in σ_G, then $(\tau_\xi)_G$ and $(\tau_\eta)_G$ are incompatible. For, the hypothesis yields $p_\xi, p_\eta \in G$. Suppose that $x \leq (\tau_\xi)_G, (\tau_\eta)_G$. Then $x \in \pi^0_G$, so there exists a $(\rho, q) \in \pi^0$ such that $q \in G$ and $x = \rho_G$. Clearly $q \Vdash \rho \in \pi$. Also, there are $s, t \in G$ such that $s \Vdash \rho \leq \tau_\xi$ and $t \Vdash \rho \leq \tau_\eta$. Let $u \in G$ be such that $u \leq p_\xi, p_\eta, q, s, t$. Then $(u, \rho) \in \mathbb{P} * \pi$ and $(u, \rho) \leq (p_\xi, \tau_\xi), (p_\eta, \tau_\eta)$, contradiction. This proves our claim.

However, $1 \Vdash \pi$ is $\check{\kappa}$-cc, so π_G is κ-cc. Hence by the preceding paragraph, $|\sigma_G| < \kappa$.

Thus our argument with an arbitrary generic G has shown that $1 \Vdash \sigma \subseteq \check{\kappa}$. Hence by Lemma 26.9 there is a $\beta < \kappa$ such that $1 \Vdash \sigma \subseteq \check{\beta}$. But clearly $p_\beta \Vdash \check{\beta} \in \sigma$, contradiction. $\square$

We are now ready for the definition of iterated forcing. Suppose that I is any set. An *ideal* of subsets of I is a collection $\mathscr{I}$ of subsets of I such that $\emptyset \in \mathscr{I}$, $\mathscr{I}$ is closed under $\cup$, and if $x \in \mathscr{I}$ and $y \subseteq x$ then $y \in \mathscr{I}$. Let M be a c.t.m. of ZFC, α an ordinal, and $\mathscr{I}$ an ideal of subsets of α containing all finite subsets of I. If $\langle \mathbb{P}_\xi : \xi < \alpha \rangle$ is a sequence of forcing orders and $p \in \prod_{\xi < \alpha} P_\xi$, then the *support* of p is the set

$$\mathrm{supp}(p) \stackrel{\mathrm{def}}{=} \{\xi < \alpha : p(\xi) \neq 1_\xi\}.$$

An *α-stage iterated forcing construction with supports in $\mathscr{I}$* is an ordered pair $(\mathbb{P}, \pi)$ in M with the following properties:

(K1) $\mathbb{P}$ is a sequence of length $\alpha + 1$ of forcing orders.

(K2) π is a sequence of length $\alpha + 1$; each π_ξ is a $\mathbb{P}_\xi$-name for a forcing order.

(K3) For each $\xi \leq \alpha$, P_ξ is a collection of sequences of length ξ.

(K4) If $\xi < \eta \leq \alpha$ and $p \in P_\eta$, then $p \restriction \xi \in P_\xi$.

(K5) If $\xi < \alpha$ and $p \in P_\alpha$, then $p(\xi) \in \mathrm{dmn}(\pi^0_\xi)$.

(K6) If $\xi \leq \alpha$, then $1_\xi = \langle 1_\eta : \eta < \xi \rangle$.

(K7) $\mathbb{P}_0 = (\{0\}, 0, 0)$.

(K8) For every $\xi < \alpha$ and every $(\xi + 1)$-termed sequence p,

$$p \in P_{\xi+1} \quad \text{iff} \quad p \restriction \xi \in P_\xi,\ p(\xi) \in \mathrm{dmn}(\pi^0_\xi),\ \text{and } p \restriction \xi \Vdash_{\mathbb{P}_\xi} p(\xi) \in \pi^0_\xi.$$

(K9) For all $\xi < \alpha$ and all $p, p' \in P_{\xi+1}$,

$$p \leq_{\mathbb{P}_{\xi+1}} p' \quad \text{iff} \quad p \restriction \xi \leq_{\mathbb{P}_\xi} p' \restriction \xi \text{ and } p \restriction \xi \Vdash_{\mathbb{P}_\xi} p(\xi) \leq_{\pi_\xi} p'(\xi).$$

(K10) If $\eta \leq \alpha$ is a limit ordinal and p is an η-termed sequence, then

$$p \in P_\eta \quad \text{iff} \quad p \restriction \xi \in P_\xi \text{ for all } \xi < \eta \text{ and } \mathrm{supp}(p) \in \mathscr{I}.$$

(K11) If $\eta \leq \alpha$ is a limit ordinal and $p, p' \in P_\eta$, then

$$p \leq_{\mathbb{P}_\eta} p' \quad \text{iff} \quad p \restriction \xi \leq_{\mathbb{P}_\xi} p \restriction \xi \text{ for every } \xi < \eta.$$

Given this situation, if $\xi \leq \eta \leq \alpha$ we define a function $i_{\xi\eta}$ with domain P_ξ as follows. For each $p \in P_\xi$, the sequence $i_{\xi\eta}(p)$ is such that $(i_{\xi\eta}(p)) \restriction \xi = p$ and $(i_{\xi\eta}(p))(\mu) = 1_{\pi_\mu}$ for all $\mu \in [\xi, \eta)$.

Now we give some elementary properties of iterated forcing constructions.

Theorem 26.11. *Let an iterated forcing construction be given, with notation as above.*

(i) For every $\beta \leq \alpha$ and every $p \in P_\beta$, the set supp(p) is in $\mathscr{I}$.
(ii) For each $\xi < \alpha$, the forcing order $\mathbb{P}_{\xi+1}$ is isomorphic to $\mathbb{P}_\xi * \pi_\xi$.
(iii) For $\xi \leq \eta \leq \alpha$, the function $i_{\xi\eta}$ maps P_ξ into P_η.
(iv) If $\xi \leq \eta \leq \zeta \leq \alpha$, then $i_{\xi\zeta} = i_{\eta\zeta} \circ i_{\xi\eta}$.
(v) If $\xi \leq \eta$, then $i_{\xi\eta}(1_{\mathbb{P}_\xi}) = 1_{\mathbb{P}_\eta}$.
(vi) If $\xi \leq \eta$, $p, p' \in P_\eta$, and $p \leq p'$, then $p \restriction \xi \leq p' \restriction \xi$.
(vii) If $\xi \leq \eta$, $p, p' \in P_\xi$, and $p \leq p'$, then $i_{\xi\eta}(p) \leq i_{\xi\eta}(p')$.
(viii) If $\xi \leq \eta$, $p, q \in P_\eta$, and $p \restriction \xi \perp q \restriction \xi$, then $p \perp q$.
(ix) If $\xi < \eta$, $p, q \in P_\eta$, and supp$(p) \cap$ supp$(q) \subseteq \xi$, then $p \restriction \xi \perp q \restriction \xi$ iff $p \perp q$.
(x) If $\xi \leq \eta$ and $p, q \in P_\xi$, then $p \perp q$ iff $i_{\xi\eta}(p) \perp i_{\xi\eta}(q)$.
(xi) If $\xi \leq \eta$, then $i_{\xi\eta}$ is a complete embedding of $\mathbb{P}_\xi$ into $\mathbb{P}_\eta$.

Proof. (i): An easy induction on β.

(ii): By (K2), π_ξ is a $\mathbb{P}_\xi$-name for a forcing order, so that $\mathbb{P}_\xi * \pi_\xi$ is defined. For each $(p, \tau) \in P_\xi * \pi_\xi$ let $f(p, \tau)$ be the sequence of length $\xi + 1$ such that $(f(p, \tau)) \restriction \xi = p$ and $(f(p, \tau))(\xi) = \tau$. Thus $f(p, \tau) \in P_{\xi+1}$ by the definition of $\mathbb{P}_\xi * \pi_\xi$ and (K8). Also it is clear that f is a bijection. The definitions also make clear that $(p, \tau) \leq (q, \sigma)$ iff $f(p, \tau) \leq f(q, \sigma)$. Finally, $f(1_{\mathbb{P}_\xi * \pi_\xi}) = 1_{\mathbb{P}_{\xi+1}}$ by (K6).

(iii): By induction on η, with ξ fixed.

(iv): Obvious.

(v): Obvious.

(vi): We prove this by induction on η, with ξ fixed. So, we assume that $\xi < \eta$ and (vi) holds for all $\eta' < \eta$. Suppose that $p, p' \in P_\eta$ and $p \leq p'$. If $\eta = \eta' + 1$, then $p \restriction \eta' \leq p' \restriction \eta'$ by (K9), and hence $p \restriction \xi \leq p' \restriction \xi$ by the inductive hypothesis. If η is a limit ordinal, then $p \restriction \xi \leq p' \restriction \xi$ by (K11).

(vii): We prove this by induction on η, with ξ fixed. So, we assume that $\xi < \eta$ and (vii) holds for all $\eta' < \eta$. Suppose that $p, p' \in P_\xi$ and $p \leq p'$. If $\eta = \eta' + 1$, then $i_{\xi\eta'}(p) \leq i_{\xi\eta'}(p')$ by the inductive hypothesis, and then $i_{\xi\eta}(p) \leq i_{\xi\eta}(p')$ by (K9), since

$$\begin{gathered} i_{\xi\eta}(p) \restriction \eta' = i_{\xi\eta'}(p),\ i_{\xi\eta}(p') \restriction \eta' = i_{\xi\eta'}(p'), \\ i_{\xi\eta'}(p) \Vdash 1 \leq_{\pi_{\xi'}} 1,\ (i_{\xi\eta}(p))(\eta') = 1, \text{ and } (i_{\xi\eta}(p'))(\eta') = 1. \end{gathered}$$

For η limit, the desired conclusion is immediate from (K11) and the inductive hypothesis.

(viii): immediate from (vi).

(ix): $\Rightarrow$ holds by (viii). For $\Leftarrow$, suppose that $r \in P_\xi$ and $r \leq p \restriction \xi, q \restriction \xi$; we want to show that p and q are compatible.

Define s with domain η by setting, for each $\rho < \eta$,

$$s(\rho) = \begin{cases} r(\rho) & \text{if } \rho < \xi, \\ p(\rho) & \text{if } \xi \leq \rho \in \text{supp}(p), \\ q(\rho) & \text{if } \xi \leq \rho \in \text{supp}(q)\backslash\text{supp}(p), \\ 1 & \text{otherwise.} \end{cases}$$

Now it suffices to prove the following statement:

(*) For all $\gamma \leq \eta$ we have $s \restriction \gamma \in P_\gamma$, $s \restriction \gamma \leq p \restriction \gamma$, and $s \restriction \gamma \leq q \restriction \gamma$.

We prove (*) by induction on γ. If $\gamma \leq \xi$, then $s \restriction \gamma = r \restriction \gamma \in P_\gamma$ by (K4), and $s \restriction \gamma \leq p \restriction \gamma, q \restriction \gamma$ since $r \leq p \restriction \xi, q \restriction \xi$, by (vi).

Now assume inductively that $\xi < \gamma \leq \eta$. First suppose that γ is a successor ordinal $\gamma' + 1$. Then $s \restriction \gamma' \in P_{\gamma'}$ by the inductive hypothesis. Now we consider several cases.

Case 1. $\gamma' \in \text{supp}(p)$. Then $s(\gamma') = p(\gamma') \in \text{dmn}(\pi^0_{\gamma'})$. Moreover, by the inductive hypothesis $s \restriction \gamma' \leq p \restriction \gamma'$, and $p \restriction \gamma' \Vdash p(\gamma') \in \pi^0_{\gamma'}$. It follows that $s \restriction \gamma' \Vdash s(\gamma') \in \pi^0_{\gamma'}$. Thus $s \restriction \gamma \in P_\gamma$ by (K8).

Case 2. $\gamma' \in \text{supp}(q)\backslash\text{supp}(p)$. This is treated similarly to Case 1.

Case 3. $\gamma' \notin \text{supp}(p) \cup \text{supp}(q)$. Then $s(\gamma') = 1$, and hence clearly $s \restriction \gamma \in P_\gamma$ by (K8).

So, we have shown that $s \restriction \gamma \in P_\gamma$ in any case.

To show that $s \restriction \gamma \leq p \restriction \gamma$, first note that $s \restriction \gamma' \leq p \restriction \gamma'$ by the inductive hypothesis. If $\gamma' \in \text{supp}(p)$, then $s(\gamma') = p(\gamma')$ and so obviously $p \restriction \gamma' \Vdash s(\gamma') \leq p(\gamma')$ and hence $s\langle\gamma \leq p \restriction \gamma$ by (K9). If $\gamma' \notin \text{supp}(p)$, then $p(\gamma') = 1$ and again the desired conclusion holds. Thus $s \restriction \gamma \leq p \restriction \gamma$.

For $s \restriction \gamma \leq q \restriction \gamma$, first note that $s \restriction \gamma' \leq q \restriction \gamma'$ by the inductive hypothesis. If $\gamma' \in \text{supp}(q)$, then $\gamma' \notin \text{supp}(p)$ by the hypothesis of (ix), since $\xi \leq \gamma'$. Hence the proof can continue as for p.

This finishes the successor case $\gamma = \gamma' + 1$. Now suppose that γ is a limit ordinal. By the inductive hypothesis, $s \restriction \rho \in P_\rho$ for each $\rho < \gamma$. Since clearly $\text{supp}(s) \subseteq \text{supp}(r) \cup \text{supp}(p) \cup \text{supp}(q)$, we have $\text{supp}(s) \in \mathscr{I}$. Hence $s \in P_\gamma$ by (K10). Finally, $s \restriction \gamma \leq p \restriction \gamma, q \restriction \gamma$ by the inductive hypothesis and (K11).

This finishes the proof of (ix).

(x): Immediate from (iii) and (ix).

(xi): Conditions (1) and (2) hold by (vii) and (ix). For (3), suppose that $q \in P_\eta$. Then $q \restriction \xi \in P_\xi$ by (vi); we claim that it is a reduction of q to $\mathbb{P}_\xi$. For, suppose that $q \restriction \xi \leq p$. Then $\text{supp}(q) \cap \text{supp}(i_{\xi\eta}(p)) \subseteq \xi$ and $q \restriction \xi$ and $(i_{\xi\eta}(p)) \restriction \xi = p$ are compatible since $q \restriction \xi \leq p$. So by (ix), q and $i_{\xi\eta}(p)$ are compatible, as desired. $\square$

Lemma 26.12. *Suppose that an iterated forcing construction is given, with notation as above. Also suppose that κ is an uncountable regular cardinal, and $\mathscr{I}$ is the collection of* *all finite subsets of α. Suppose that for each $\xi < \alpha$, $1 \Vdash_{\mathbb{P}_\xi} (\pi_\xi$ is $\check{\kappa} - cc)$. Then for each $\xi \leq \alpha$ the forcing order $\mathbb{P}_\xi$ is κ-cc in M.*

Proof. We proceed by induction on ξ. It is trivially true for $\xi = 0$, by (K7). The inductive step from ξ to $\xi + 1$ follows from Theorem 26.11(ii) and Theorem 26.10. Now suppose that ξ is limit and the assertion is true for all $\eta < \xi$. Suppose that $\langle p^\beta : \beta < \kappa\rangle$ is an antichain in $\mathbb{P}_\xi$. Let $M \in [\kappa]^\kappa$ be such that $\langle \mathrm{supp}(p_\xi) : \xi \in M\rangle$ is a Δ-system, say with root r. Choose $\eta < \xi$ such that $r \subseteq \eta$. Then by Theorem 26.11(ix), $\langle p_\nu \restriction \eta : \nu \in M\rangle$ is a system of incompatible elements of $\mathbb{P}_\eta$, contradiction. $\square$

Lemma 26.13. *Suppose that an iterated forcing construction is given, with notation as above.*

(i) Suppose that G is $\mathbb{P}_\alpha$-generic over M. For each $\xi \leq \alpha$ let $G_\xi = i_{\xi\alpha}^{-1}[G]$. Then

(a) For each $\xi \leq \alpha$, the set G_ξ is $\mathbb{P}_\xi$-generic over M.

(b) If $\xi \leq \eta \leq \alpha$, then $M[G_\xi] \subseteq M[G_\eta] \subseteq M[G]$.

(ii) Let $\xi < \alpha$. Define

$$\mathbb{Q}_\xi = (\pi_\xi)_{G_\xi};$$
$$H_\xi = \{\rho_{G_\xi} : \rho \in \mathrm{dmn}(\pi_\xi^0) \text{ and } \exists p(p^\frown\langle\rho\rangle \in G_{\xi+1})\}.$$

Then $H_\xi \in M[G_{\xi+1}]$ and H_ξ is $\mathbb{Q}_\xi$-generic over $M[G_\xi]$.

Proof. (i)(a) holds by Theorem 26.11(xi) and Theorem 26.3; and (i)(b) follows from these theorems too.

To prove (ii) we are going to apply Theorem 26.8 with $\mathbb{P}$ and π replaced by $\mathbb{P}_\xi$ and π_ξ; by (K2), π_ξ is a $\mathbb{P}_\xi$-name for a forcing order. Let j be the complete embedding of $\mathbb{P}_\xi$ into $\mathbb{P}_\xi * \pi_\xi$ given by $j(p) = (p, 1)$; this corresponds to i in Theorem 26.23. Now $G_{\xi+1}$ is $\mathbb{P}_{\xi+1}$-generic over M by (i). Let f be the isomorphism of $\mathbb{P}_\xi * \pi_\xi$ with $\mathbb{P}_{\xi+1}$ given in the proof of Theorem 26.11(ii). Clearly then $f^{-1}[G_{\xi+1}]$ is $\mathbb{P}_\xi * \pi_\xi$-generic over M, and we apply Theorem 26.8 with it in place of K. Note that $f \circ j = i_{\xi,\xi+1}$, and hence $j^{-1}[f^{-1}[G_{\xi+1}]] = G_\xi$; so G_ξ is the G in Theorem 26.23. Next,

$$\begin{aligned} H_\xi &= \{\rho_{G_\xi} : \rho \in \mathrm{dmn}(\pi_\xi^0) \text{ and } \exists p(p^\frown\langle\rho\rangle \in G_{\xi+1})\} \\ &= \{\rho_{G_\xi} : \rho \in \mathrm{dmn}(\pi_\xi^0) \text{ and } \exists p((p,\rho) \in f^{-1}[G_{\xi+1}])\}, \end{aligned}$$

so that Theorem 26.8 applies to yield that G_ξ is $\mathbb{P}_\xi$-generic over M (we already know this by (i)) and H_ξ is $\mathbb{Q}_\xi$-generic over $M[G_\xi]$. Clearly $H_\xi \in M[G_{\xi+1}]$. $\square$

Lemma 26.14. *Suppose that an iterated forcing construction is given, with notation as above, with α limit, and $\mathscr{I}$ the collection of all finite subsets of α. Suppose that G is $\mathbb{P}_\alpha$-generic over M, $S \in M$, $X \subseteq S$, $X \in M[G]$, and $(|S| < \mathrm{cf}(\alpha))^{M[G]}$.*

Then there is an $\eta < \alpha$ such that $X \in M[i_{\eta\alpha}^{-1}[G]]$.

Proof. Let σ be a $\mathbb{P}_\alpha$-name such that $X = \sigma_G$. Thus for any $s \in S$, $s \in X$ iff there is a $p \in G$ such that $p \Vdash_{\mathbb{P}_\alpha} \check{s} \in \sigma$. Now clearly $P_\alpha = \bigcup_{\xi<\alpha} i_{\xi\alpha}[P_\xi]$, and $G = \bigcup_{\xi<\alpha} i_{\xi\alpha}[i_{\xi\alpha}^{-1}[G]]$.

Hence for each $s \in X$ we can find $\xi(s) < \alpha$ such that there is a $p \in i^{-1}_{\xi(s)\alpha}[G]$ such that $i_{\xi(s)\alpha}(p) \Vdash_{\mathbb{P}_\alpha} \check{s} \in \sigma$. Let $\eta = \sup_{s\in X} \xi(s)$; so $\eta < \alpha$ by assumption.

Thus $X = \{s \in S : \exists p \in G_\eta(i_{\eta\alpha}(p) \Vdash_{\mathbb{P}_\alpha} \check{s} \in \sigma\}$. Hence $X \in M[G_\eta]$. □

This finishes our general exposition of iterated forcing. The main application of this method, which forms a starting point of further applications, is to the consistency of Martin's axiom with ¬CH. Before turning to this, however, there is another general fact about forcing which will be needed.

Lemma 26.16. *Suppose that M is a c.t.m. of ZFC and in M we have a forcing order $\mathbb{P}$, an antichain A of $\mathbb{P}$, and a system $\langle \sigma_q : q \in A\rangle$ of members of $M^{\mathbb{P}}$. Then there is a name $\pi \in M^{\mathbb{P}}$ such that $q \Vdash \pi = \sigma_q$ for every $q \in A$.*

Proof. We define

$$(\tau, r) \in \pi \quad \text{iff} \quad \begin{aligned}&(\tau, r) \in M^P \text{ and there is a } q \in A \text{ such that } r \le q\\ &\text{and } r \Vdash \tau \in \sigma_q \text{ and } \tau \in \mathrm{dmn}(\sigma_q).\end{aligned}$$

Fix $q \in A$ and fix a generic G for $\mathbb{P}$ over M such that $q \in G$; we want to show that $\pi_G = (\sigma_q)_G$.

First suppose that $x \in \pi_G$. Choose $(\tau, r) \in \pi$ such that $r \in G$ and $x = \tau_G$. By the definition of π, there is a $q' \in A$ such that $r \le q'$, $r \Vdash \tau \in \sigma_{q'}$, and $\tau \in \mathrm{dmn}(\sigma_{q'})$. Since $r \in G$, also $q' \in G$. But A is an antichain, $q, q' \in A$, and $q \in G$, so $q = q'$. So $r \Vdash \tau \in \sigma_q$, and since $r \in G$ it follows that $\tau_G \in (\sigma_q)_G$.

Second, suppose that $y \in (\sigma_q)_G$. Choose $(\tau, r) \in \sigma_q$ such that $r \in G$ and $y = \tau_G$. Since $\tau_G \in (\sigma_q)_G$, there is a $p \in G$ such that $p \Vdash \tau \in \sigma_q$. Also $q \in G$, so let $s \in G$ be such that $s \le p, q$. Then $(\tau, s) \in \pi$, and so $y = \tau_G \in \pi_G$. □

Theorem 26.17. (maximal principle) *Suppose that M is a c.t.m. of ZFC, $\mathbb{P} \in M$ is a forcing order, $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$, $p \in P$, and $p \Vdash \exists x\varphi(x, \tau_1, \ldots, \tau_n)$. Then there is a $\pi \in M^P$ such that $p \Vdash \varphi(\pi, \tau_1, \ldots, \tau_n)$.*

Proof. This argument takes place in M, unless otherwise indicated. By Zorn's lemma, let A be an antichain, maximal with respect to the property

(1) For all $q \in A$, $q \le p$ and $q \Vdash \varphi(\sigma, \pi_1, \ldots, \pi_n)$ for some $\sigma \in M^P$.

By the axiom of choice, for each $q \in A$ let $\sigma_q \in M^{\mathbb{P}}$ be such that $q \Vdash \varphi(\sigma_q, \pi_1, \ldots, \pi_n)$. By Lemma 26.17, let $\pi \in M^P$ be such that $q \Vdash \pi = \sigma_q$ for every $q \in A$. Since also $q \Vdash \varphi(\sigma_q, \tau_1, \ldots, \tau_n)$, an easy argument using the definition of forcing, thus external to M, shows that $q \Vdash \varphi(\pi, \tau_1, \ldots, \tau_n)$.

Now we show that $p \Vdash \varphi(\pi, \tau_1, \ldots, \tau_n)$. To this end we argue outside M. Suppose that G is $\mathbb{P}$-generic over M. We claim that $G \cap A \ne \emptyset$. In fact the set

$$\{r \le p : \text{there is a } \sigma \in M^{\mathbb{P}} \text{ such that } r \Vdash \varphi(\sigma, \tau_1, \ldots, \tau_n)\} \tag{2}$$

is dense below p, and hence there is an $r \in G$ which is also in (2). If $G \cap A = \emptyset$, then there is an element $q \in G$ incompatible with each member of A; in this case, choose $s \in G$ with

$s \le r, q$. Then s is in (2) and s is incompatible with each element of A, contradicting the maximality of A. So $G \cap A \neq \emptyset$.

Say $q \in G \cap A$. Choose $r \in G$ such that $r \le p, q$. Since $q \Vdash \varphi(\pi, \tau_1, \ldots, \tau_n)$, also $r \Vdash \varphi(\sigma, \tau_1, \ldots, \tau_n)$, and hence $\varphi(\tau_G, (\tau_1)_G, \ldots, (\tau_n)_G)$ holds in $M[G]$, as desired. □

Now we give a fact about Martin's axiom which is used below; it is an exercise in Chapter 21.

Lemma 26.18. *MA(κ) is equivalent to MA(κ) restricted to ccc forcing orders of cardinality $\le \kappa$.*

Proof. We assume the indicated special form of MA(κ), and assume given a ccc forcing order $\mathbb{P}$ and a family $\mathscr{D}$ of at most κ dense sets in $\mathbb{P}$; we want to find a filter on P intersecting each member of $\mathscr{D}$. We introduce some operations on P. For each $D \in \mathscr{D}$ define $f_D : P \to P$ by setting, for each $p \in P$, $f_D(p)$ to be some element of D which is $\le p$. Also we define $g : P \times P \to P$ by setting, for all $p, q \in P$,

$$g(p,q) = \begin{cases} p & \text{if } p \text{ and } q \text{ are incompatible,} \\ r & \text{with } r \le p, q \text{ if there is such an } r. \end{cases}$$

Here, as in the definition of f_D, we are implicitly using the axiom of choice; for g, we choose any r of the indicated form.

We may assume that $\mathscr{D} \neq \emptyset$. Choose $D \in \mathscr{D}$, and choose $s \in D$. Now let Q be the intersection of all subsets of P which have s as a member and are closed under all of the operations f_D and g. We take the order on Q to be the order induced from P.

(1) $|Q| \le \kappa$.

To prove this, we give an alternative definition of Q. Define

$$\begin{aligned} R_0 &= \{s\}; \\ R_{n+1} &= R_n \cup \{g(a,b) : a, b \in R_n\} \cup \{f_D(a) : D \in \mathscr{D} \text{ and } a \in R_n\}. \end{aligned}$$

Clearly $\bigcup_{n\in\omega} R_n = Q$. By induction, $|R_n| \le \kappa$ for all $n \in \omega$, and hence $|Q| \le \kappa$, as desired in (1).

We also need to check that Q is ccc. Suppose that X is a collection of pairwise incompatible elements of Q. Then these elements are also incompatible in P, since $x, y \in X$ with x, y compatible in P implies that $g(x,y) \le x, y$ and $g(x,y) \in Q$, so that x, y are compatible in Q. It follows that X is countable. So Q is ccc.

Next we claim that if $D \in \mathscr{D}$ then $D \cap Q$ is dense in Q. For, suppose $p \in Q$. Then $f_D(q) \in D \cap Q$. as desired.

Now we can apply our special case of MA(κ) to Q and $\{D \cap Q : D \in \mathscr{D}\}$; we obtain a filter G on Q such that $G \cap D \cap Q \neq \emptyset$ for all $D \in \mathscr{D}$. Let

$$G' = \{p \in P : q \le p \text{ for some } q \in G\}.$$

We claim that G' is the desired filter on P intersecting each $D \in \mathscr{D}$.

Clearly if $p \in G'$ and $p \leq r$, then $r \in G'$.

Suppose that $p_1, p_2 \in G'$. Choose $q_1, q_2 \in G$ such that $q_i \leq p_1$ for each $i = 1, 2$. Then there is an $r \in G$ such that $r \leq q_1, q_2$. Then $r \in G'$ and $r \leq p_1, p_2$. So G' is a filter on P.

Now for any $D \in \mathscr{D}$. Take $q \in G \cap D \cap Q$. Then $q \in G' \cap D$, as desired. □

Theorem 26.19. *Suppose that M is a c.t.m. of ZFC, and in M we have an uncountable regular cardinal κ such that $\sum_{\lambda<\kappa} 2^\lambda = \kappa$.*

Then there is a forcing order $\mathbb{P}$ in M such that $\mathbb{P}$ satisfies ccc, and for any $\mathbb{P}$-generic G over M, the extension $M[G]$ satisfies MA and $2^\omega = \kappa$.

Proof. The overall idea of the proof runs like this. We do an iterated forcing which has the effect of producing a chain

$$M = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq M_{\alpha+1} \subseteq \cdots \subseteq M_\kappa$$

of length $\kappa + 1$ of c.t.m.s of ZFC. We carry along in the construction a list of names of forcing orders. This list is of length κ. At the step from M_α to $M_{\alpha+1}$ we take care of one entry in this list, say $\mathbb{Q}$, by taking a $\mathbb{Q}$-generic filter G and setting $M_{\alpha+1} = M_\alpha[G]$, and we add to our list all names of forcing orders in M_α. By proper coding, we can do this so that at end we have taken care of all forcing orders in any model M_α. Then we show that any ccc forcing order in M_κ appeared already in an earlier stage and so a generic filter for it was added.

We begin by defining the coding which will be used.

Claim. There is a function f in M with the following properties:
(1) $f : \kappa \to \kappa \times \kappa$.
(2) For all $\xi, \beta, \gamma < \kappa$ there is an $\eta > \xi$ such that $f(\eta) = (\beta, \gamma)$.
(3) $1^{st}(f(\xi)) \leq \xi$ for all $\xi < \kappa$.

Proof of Claim. Let $g : \kappa \to \kappa \times \kappa \times \kappa$ be a bijection. For each $\xi < \kappa$ let $g(\xi) = (\alpha, \beta, \gamma)$, and set

$$f(\xi) = \begin{cases} (\beta, \gamma) & \text{if } \beta \leq \xi, \\ (0, 0) & \text{otherwise.} \end{cases}$$

So (1) and (3) obviously hold. For (2), suppose that $\xi, \beta, \gamma < \kappa$. Now $g^{-1}[\{(\alpha, \beta, \gamma) : \alpha < \kappa\}]$ has size κ, so there is an $\eta \in g^{-1}[\{(\alpha, \beta, \gamma) : \alpha < \kappa\}]$ such that $\xi, \beta < \eta$. Say $g(\eta) = (\alpha, \beta, \gamma)$. Then $\beta < \eta$, so $f(\eta) = (\beta, \gamma)$ and $\xi < \eta$, as desired.

Another preliminary is a cardinality bound. Note that if $\lambda < \kappa$, then $\kappa^\lambda = \kappa$, since, using regularity and $\sum_{\mu<\kappa} 2^\mu = \kappa$, we have

$$\kappa^\lambda = |{}^\lambda\kappa| = \left| \bigcup_{\mu<\kappa} {}^\lambda\mu \right| \leq \sum_{\rho<\kappa} 2^\rho = \kappa.$$

(4) If $\mathbb{Q}$ is a ccc forcing order in M of size less than κ, then there are at most κ pairs (λ, σ) such that $\lambda < \kappa$ and σ is a nice $\mathbb{Q}$-name for a subset of $(\lambda \times \lambda)\check{}$.

To prove (4), recall that a nice $\mathbb{Q}$-name for a subset of $(\lambda \times \lambda)\check{}$ is a set of the form

$$\bigcup\{\{\check{a}\} \times A_a : a \in \lambda \times \lambda\}$$

where for each $a \in \lambda \times \lambda$, A_a is an antichain in $\mathbb{Q}$. Now by ccc the number of antichains in $\mathbb{Q}$ is at most $|Q|^\omega \leq \kappa$. So for a fixed $\lambda < \kappa$ the number sets of the indicated form is at most $\kappa^\lambda = \kappa$. Hence (4) holds.

For brevity, we let $\mathrm{pord}(\lambda, W)$ abbreviate the statement that W is the order relation of a ccc forcing order on the set λ, with largest element 0.

Now we are going to define by recursion functions $\mathbb{P}$, π, λ, and σ with domain κ. Let $\mathscr{I}$ be the collection of all finite subsets of κ.

Let $\mathbb{P}_0$ be the trivial partial order $(\{0\}, 0, 0)$.

Now suppose that $\mathbb{P}_\alpha$ has been defined, so that it is a ccc forcing order in M. We now define π_α, λ^α, σ^α, and $\mathbb{P}_{\alpha+1}$. By (4), the set of all pairs (β, γ) such that $\beta < \kappa$ and γ is a nice $\mathbb{P}_\alpha$-name for a subset of $(\beta \times \beta)\check{}$ has size at most κ. We let $\{(\lambda^\alpha_\xi, \sigma^\alpha_\xi) : \xi < \kappa\}$ enumerate all of them. This defines λ^α and σ^α. Now let $f(\alpha) = (\beta, \gamma)$. So $\beta \leq \alpha$, and hence λ^β_γ and σ^β_γ are defined. We consider the complete embedding $i_{\beta\alpha}$ given in Theorem 26.11(xi). By Proposition 21.1, $i_{\beta\alpha *}(\sigma^\beta_\gamma)$ is a $\mathbb{P}_\alpha$-name.

(5) There is a $\mathbb{P}_\alpha$-name ρ such that

$$1_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \mathrm{pord}((\lambda^\beta_\gamma)\check{}, \rho) \text{ and } [\mathrm{pord}((\lambda^\beta_\gamma)\check{}, i_{\beta\alpha *}(\sigma^\beta_\gamma)) \to \rho = i_{\beta\alpha *}(\sigma^\beta_\gamma)].$$

In fact, clearly

$$1_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \exists W(\mathrm{pord}((\lambda^\beta_\gamma)\check{}, W) \text{ and } [\mathrm{pord}((\lambda^\beta_\gamma)\check{}, i_{\beta\alpha *}(\sigma^\beta_\gamma)) \to W = i_{\beta\alpha *}(\sigma^\beta_\gamma)],$$

so (5) follows by the maximal principle, Theorem 26.18.

We now let

$$\pi_\alpha = \mathrm{op}(\mathrm{op}(\lambda^\beta_\gamma)\check{}, \rho), 0),$$

which is a $\mathbb{P}_\alpha$-name for a forcing order. Finally, $\mathbb{P}_{\alpha+1}$ is determined by (K8) and (K9).

For limit $\alpha \leq \kappa$ we define $\mathbb{P}_\alpha$ by (K10) and (K11).

This finishes the construction.

By Lemma 26.12, each forcing order $\mathbb{P}_\alpha$ for $\alpha \leq \kappa$ satisfies ccc.

Now take any $\mathbb{P}_\kappa$-generic G over M; we want to show that $\mathrm{MA}(\mu)$ holds in $M[G]$ for every $\mu < \kappa$. (Later we show that $2^\omega = \kappa$ in $M[G]$.) Note that, by ccc, $\mathbb{P}_\kappa$ preserves cofinalities and cardinalities. Let $G_\xi = i^{-1}_{\xi\kappa}[G]$ for each $\xi < \kappa$.

Suppose that $\mathbb{Q}$ is a ccc forcing order in $M[G]$, $|Q| \leq \mu$, and $\mathscr{D}$ is a family of at most μ subsets of Q dense in $\mathbb{Q}$, with $\mathscr{D} \in M[G]$. By taking an isomorphic image, we may assume that Q is an ordinal φ less than κ, and it has maximal element 0.

(6) There are $\alpha, \beta, \gamma < \kappa$ such that $f(\alpha) = (\beta, \gamma)$, $Q = \lambda^\beta_\gamma$, $\leq_\mathbb{Q} = (\sigma^\beta_\gamma)_{G_\beta}$, and $\mathscr{D} \in M[G_\beta]$.

In fact, we have $\mathrm{pord}(\varphi, \leq_\mathbb{Q})$. Applying Lemma 26.14 with $\varphi \times \varphi$ in place of S and $\leq_\mathbb{Q}$ in place of X, we see that $\leq_\mathbb{Q}$ is a member of some $M[G_\beta]$ with $\beta < \kappa$. Let $\mathscr{D} = \{D_\alpha : \alpha < \mu\}$.

Then we can apply Lemma 26.14 to the set $\{(\alpha,\beta) : \alpha,\beta < \mu \text{ and } \beta \in D_\alpha\}$ to infer that $\mathscr{D} \in M[G_\eta]$ for some $\eta < \kappa$, and we may assume that $\eta = \beta$. By Proposition 4.9, there is a nice name $\mathbb{P}_\beta$-name ζ for a subset of $\varphi \times \varphi$ such that $\zeta_{G_\beta} = \leq_{\mathbb{Q}}$. By construction, we can then choose $\gamma < \kappa$ such that $(\varphi,\zeta) = (\lambda^\beta_\gamma, \sigma^\beta_\gamma)$. Next, choose α such that $f(\alpha) = (\beta,\gamma)$. Thus (6) holds.

Now we consider the construction of $\mathbb{P}_{\alpha+1}$. In this construction we chose a name ρ as in (5). Now we know that $\mathrm{pord}(\varphi, \leq_{\mathbb{Q}})$ (in $M[G]$). By absoluteness, this also holds in $M[G_\alpha]$. (It is still ccc, as otherwise it would fail to be ccc in $M[G]$.) We have $Q = \varphi = ((\lambda^\beta_\gamma)\check{\ })_{G_\beta}$ and $\leq_{\mathbb{Q}} = \zeta_{G_\beta} = (\sigma^\beta_\gamma)_{G_\beta}$. By Theorem 26.4(i), $(\sigma^\beta_\gamma)_{G_\beta} = (i_{\beta\alpha *}(\sigma^\beta_\gamma))_{G_\alpha}$. Take ρ as in (5). Then $\rho_{G_\alpha} = (i_{\beta\alpha *}(\sigma^\beta_\gamma))_{G_\alpha}$. Thus $\pi_\alpha = \mathrm{op}(\mathrm{op}(\lambda^\beta_\gamma\check{\ }, \rho), 0)$. Then $(\pi_\alpha)_{G_\alpha} = \mathbb{Q}$. Let

$$H = \{\psi_{G_\alpha} : \psi \in \mathrm{dmn}(\pi^0_\alpha) \text{ and } p^\frown\langle\psi\rangle \in G_{\alpha+1} \text{ for some } p\}.$$

Then H is $(\pi_\alpha)_{G_\alpha}$-generic over $M[G_\alpha]$ by Lemma 26.13. Since $\mathscr{D} \in M[G_\alpha]$, we get $H \cap D \neq \emptyset$ for all $D \in \mathscr{D}$, as desired.

Since $\mathrm{MA}(\mu)$ holds for every $\mu < \kappa$, it follows from Lemma 26.19 that in $M[G]$, $\kappa \leq 2^\omega$.

Now in M we have $\kappa^\omega = \kappa$, as observed early in this proof. Hence by Proposition 24.3 it follows that $2^\omega \leq \kappa$ in $M[G]$. Thus $2^\omega = \kappa$ in $M[G]$. $\square$

EXERCISES

E26.1. Let f be a complete embedding of $\mathbb{P}$ into $\mathbb{Q}$. Show that there is an isomorphism g of $\mathrm{RO}(\mathbb{P})$ into $\mathrm{RO}(\mathbb{Q})$ such that for any $X \subseteq \mathrm{RO}(\mathbb{P})$, $g(\sum^{\mathrm{RO}(\mathbb{P})} X) = \sum^{\mathrm{RO}(\mathbb{Q})}_{x\in X} g(x)$. Furthermore, show that the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "e_P"'] & Q \arrow[d, "e_Q"] \\ \mathrm{RO}(P) \arrow[r, "g"'] & \mathrm{RO}(Q) \end{tikzcd}$$

E26.2. Prove that a composition of complete embeddings is a complete embedding.

E26.3. Suppose that f is a complete embedding of $\mathbb{P}$ into $\mathbb{Q}$. Also suppose that $\forall p \in P \exists q, r \leq p(q \perp r)$. Show that $\forall p \in Q \exists q, r \leq p(q \perp r)$.

E26.4. Prove that every isomorphism is a complete embedding.

E26.5. Give an example of a complete embedding which is not an isomorphism.

E26.6. A *dense embedding* of $\mathbb{P}$ into $\mathbb{Q}$ is a function $f : P \to Q$ such that the following conditions hold:

(i) $\forall p, q \in P[p \leq q \to f(p) \leq f(q)]$.

(ii) $\forall p, q \in P[p \perp q \to f(p) \perp f(q)]$.

(iii) For any $q \in Q$ there is a $p \in P$ such that $f(p) \leq q$.

Show that every dense embedding is a complete embedding, and every isomorphism is a dense embedding.

E26.7. Prove that if f is a dense embedding of $\mathbb{P}$ into $\mathbb{Q}$, then $\mathrm{RO}(\mathbb{P})$ and $\mathrm{RO}(\mathbb{Q})$ are isomorphic.

E26.8. Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing orders in M. Recall the notion of product from Chapter 25. Let $\mathbb{Q}^* = \mathrm{op}(\mathrm{op}(\check{P}_{\mathbb{Q}}, \check{\leq}_{\mathbb{Q}}), \check{1}_{\mathbb{Q}})$. Note that $\check{}$ is with respect to $\mathbb{P}$ here; see the definition on page 38. Show that $\mathbb{Q}^*$ is a $\mathbb{P}$-name for a forcing order, and $\mathbb{P} \times \mathbb{Q}$ is isomorphic to $\mathbb{P} * \mathbb{Q}^*$.

E26.9. Give an example of a partial order $\mathbb{P}$ and a $\mathbb{P}$-name π for a partial order such that $\mathbb{P} * \pi$ is not a partial order. Hint: Let $\mathbb{P}$ be $\mathrm{fin}(\omega, 2)$. Let $p = \{(0,0)\}$ and $q = \{(0,1)\}$. Now define

$$\begin{aligned}
\pi^0 &= \{(\emptyset, p), (\{(\emptyset, q)\}, p), (\emptyset, q)\};\\
\pi^2 &= \emptyset;\\
\pi_1 &= \mathrm{up}(\mathrm{op}(\emptyset, \emptyset), \mathrm{op}(\emptyset, \emptyset)).
\end{aligned}$$

E26.10. Let κ be an infinite cardinal. For $f, g \in {}^{\kappa}\kappa$ we write $f <_{\kappa} g$ iff $|\{\alpha < \kappa : f(\alpha) \geq g(\alpha)\}| < \kappa$. We say that $\mathscr{F} \subseteq {}^{\kappa}\kappa$ is *almost unbounded* iff there is no $g \in {}^{\kappa}\kappa$ such that $f <_{\kappa} g$ for all $f \in \mathscr{F}$. Clearly ${}^{\kappa}\kappa$ itself is almost unbounded; it has size 2^{κ}

Show that if κ is a regular cardinal, then any almost unbounded subset of ${}^{\kappa}\kappa$ has size at least κ^+.

E26.11. Suppose that κ is an infinite cardinal and $\mathrm{MA}(\kappa)$ holds. Suppose that $\mathscr{F} \subseteq {}^{\omega}\omega$ and $|\mathscr{F}| = \kappa$. Then there is a $g \in {}^{\omega}\omega$ such that $f <_{\omega} g$ for all $f \in \mathscr{F}$. Hint: let P be the set of all pairs (p, F) such that p is a finite function contained in $\omega \times \omega$ and F is a finite subset of $\mathscr{F}$. Define $(p, F) \leq (q, G)$ iff $p \supseteq q$, $F \supseteq G$, and

$$\forall f \in G \forall n \in (\mathrm{dmn}(p) \backslash \mathrm{dmn}(q))[p(n) > f(n)].$$

E26.12. We begin exercises giving another application of iterated forcing.

(i) Show that there is a c.t.m. M of ZFC $+ 2^{\omega} = \omega_1 + 2^{\omega_1} = \omega_3$.

(ii) Show that if $\mathbb{Q}$ is a ccc forcing order of size $\leq \omega_1$ in the model M of (i), then there are at most ω_1 nice $\mathbb{Q}$-names for subsets of $(\omega \times \omega)\check{}$.

E26.13. (Continuing E26.12) Now we are going to define by recursion functions $\mathbb{P}$, π, and σ with domain ω_2.

Let $\mathbb{P}_0$ be the trivial partial order $(\{0\}, 0, 0)$.

Now suppose that $\mathbb{P}_\alpha$ has been defined, so that it is a ccc forcing order in M of size at most ω_1. We now define π_α, σ^α, and $\mathbb{P}_{\alpha+1}$. By E26.12(ii), the set of all nice $\mathbb{P}_\alpha$-names for subsets of $(\omega \times \omega)\check{}$ has size at most ω_1. We let $\{\tau^\alpha_\gamma : \gamma < \omega_1\}$ enumerate all of them. Prove:

(iii) For every $\gamma < \omega_1$ there is a $\mathbb{P}_\alpha$-name σ^α_γ such that

$$1_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \sigma^\alpha_\gamma : \check{\omega} \to \check{\omega} \text{ and } [\tau^\alpha_\gamma : \check{\omega} \to \check{\omega} \text{ implies that } \sigma^\alpha_\gamma = \tau^\alpha_\gamma].$$

E26.14. (Continuing E26.13) For each $H \in [\omega_1]^{<\omega}$ we define $\rho^\alpha_H = \{(\sigma^\alpha_\gamma, 1_{\mathbb{P}_\alpha}) : \gamma \in H\}$. So ρ^α_H is a $\mathbb{P}_\alpha$-name.

Next, define

$$\pi^0_\alpha = \{(\mathrm{op}(\check{p}, \rho^\alpha_H), 1) : p \in \mathrm{fin}(\omega, \omega) \text{ and } H \in [\omega_1]^{<\omega}\}.$$

Let G be $\mathbb{P}_\alpha$-generic over M. Prove:

(iv) $(\pi^0_\alpha)_G = \{(p, K) : p \in \mathrm{fin}(\omega, \omega) \text{ and } K \in [{}^\omega\omega]^{<\omega}\}$.

E26.15. (Continuing E26.14) Next, we define

$$\begin{aligned}\pi^1_\alpha = \{(\mathrm{op}(\mathrm{op}(\check{p}, \rho^\alpha_H), \mathrm{op}(\check{p}', \rho^\alpha_{H'})), q) &: p, p' \in \mathrm{fin}(\omega, \omega),\\ &H, H' \in [\omega_1]^{<\omega},\ p' \subseteq p,\ H' \subseteq H,\ q \in \mathbb{P}_\alpha, \text{ and for all } \gamma \in H'\\ &\text{and all } n \in \mathrm{dmn}(p) \backslash \mathrm{dmn}(p'), q \Vdash_{\mathbb{P}_\alpha} \sigma^\alpha_\gamma(\check{n}) < (p(n))\check{}\ \}.\end{aligned}$$

Again, suppose that G is $\mathbb{P}_\alpha$-generic over M. Prove:

(v) $$\begin{aligned}(\pi^1_\alpha)_G = \{((p, K), (p', K')) &: (p, K), (p', K') \in (\pi^0_\alpha)_G,\ p' \subseteq p,\ K' \subseteq K,\\ &\text{and for all } f \in K' \text{ and all } n \in \mathrm{dmn}(p) \backslash \mathrm{dmn}(p'),\ f(n) < p(n)\}.\end{aligned}$$

E26.16. (Continuing E26.15) Next, we let $\pi^2_\alpha = \{(\mathrm{op}(0, 0), 1_{\mathbb{P}_\alpha})\}$. Then for any generic G, $(\pi^2_\alpha)_G = (0, 0)$. Finally, let $\pi_\alpha = \mathrm{op}(\mathrm{op}(\pi^0_\alpha, \pi^1_\alpha), \pi^2_\alpha)$. This finishes the definition of π_α. Prove:

(vi) $1_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \pi_\alpha$ is $\check{\omega}_1 - cc$.

E26.17. (Continuing E26.16) Now $\mathbb{P}_{\alpha+1}$ is determined. The limit stages are clear. So the construction is finished, and $\mathbb{P}_\kappa$ is ccc.

Let G be $\mathbb{P}_\kappa$-generic over M. Prove

(vii) In $M[G]$, if $\mathscr{F} \subseteq {}^\omega\omega$ and $|\mathscr{F}| < \omega_2$, then there is a $g \in {}^\omega\omega$ such that $f <_\omega g$ for all $f \in \mathscr{F}$.

E26.18. (Continuing E26.17) Show that if ZFC is consistent, then there is a c.t.m. of ZFC with the following properties:

(i) $2^\omega = \omega_2$.
(ii) $2^{\omega_1} = \omega_3$.
(iii) Every almost unbounded set of functions from ω to ω has size 2^ω.
(iv) MA(ω_1) fails.

27. Various forcing orders

In this section we briefly survey various forcing orders which have been used. Many of them give rise to new real numbers, i.e., new subsets of ω. (It is customary to identify real numbers with subsets of ω, since these are simpler objects than Dedekind cuts; and a bijection in the ground model between $\mathbb{R}$ and $\mathscr{P}(\omega)$ transfers the newness to "real" real numbers.) For each kind of forcing we give a reference for further results concerning it. Of course our list of forcing orders is not complete, but we hope the treatment here can be a guide to further study.

Cohen forcing

The forcing used in Chapter 16 is, as indicated there, called *Cohen forcing.* If M is a c.t.m. of ZFC, $\mathbb{P}$ is $\mathrm{Fin}(\omega, 2)$, and G is $\mathbb{P}$-generic over M, then $\bigcup G$ is a *Cohen real.* More generally, if N is a c.t.m. of ZFC and $M \subseteq N$, then a Cohen real in N is a function $f : \omega \to 2$ in N such that there is a $\mathbb{P}$-generic filter G over M such that $M[G] \subseteq N$ and $f = \bigcup G$.

Theorem 27.1. *Suppose that M is a c.t.m. of ZFC, $I \in M$, $I = J_0 \cup J_1$ with $J_0 \cap J_1 = \emptyset$, and G is $\mathrm{Fin}(I, 2)$-generic over M.*

(i) Let $H_0 = G \cap \mathrm{Fin}(J_0, 2)$. Then H_0 is $\mathrm{Fin}(J_0, 2)$-generic over M.

(ii) Let $H_1 = G \cap \mathrm{Fin}(J_1, 2)$. Then H_1 is $\mathrm{Fin}(J_1, 2)$-generic over $M[H_0]$.

(iii) $M[G] = M[H_0][H_1]$.

Proof. We are going to use Theorem 25.13. Let $\mathbb{P}$ be the partial order $\mathrm{Fin}(J_0, 2)$ and $\mathbb{Q}$ the partial order $\mathrm{Fin}(J_1, 2)$. We claim that $\mathrm{Fin}(I, 2)$ is isomorphic to $\mathbb{P} \times \mathbb{Q}$. Define $f(p) = (p \restriction J_0, p \restriction J_1)$. Clearly this is an isomorphism. We claim that $f[G] = H_0 \times H_1$. For, suppose that $p \in G$. then $p \restriction J_0 \subseteq p$, so $p \restriction J_0 \in G$, and hence $p \restriction J_0 \in H_0$. Similarly, $p \restriction J_1 \in H_1$. So $f(p) \in H_0 \times H_1$. Conversely, if $(p, q) \in H_0 \times H_1$, then $p \in G$ and $q \in G$, so there is an $r \in G$ such that $p, q \subseteq r$. Now $p \cup q \subseteq r$, so $p \cup q \in G$. Clearly $f(p \cup q) = (p, q)$. So this proves that $f[G] = H_0 \times H_1$.

Now it follows from Lemma 25.9 that $H_0 \times H_1$ is $\mathbb{P} \times \mathbb{Q}$-generic over M, and $M[G] = M[H_0 \times H_1]$. Now we can apply Theorem 25.13 to get:

(1) H_0 is $\mathbb{P}$-generic over M.

(2) H_1 is $\mathbb{Q}$-generic over $M[H_0]$.

(3) $M[G] = M[H_0][H_1]$.

This proves our theorem. □

It follows that all of the subsets of ω given in the proof of Theorem 16.1 are Cohen reals:

Corollary 27.2. *Let M be a c.t.m. of ZFC and let κ be a cardinal of M such that $\kappa^\omega = \kappa$. Let $\mathbb{P} = \mathrm{Fin}(\kappa, 2)$ in M, and let G be $\mathbb{P}$-generic over M, and let $g = \bigcup G$. Let $h : \kappa \times \omega \to \kappa$ be a bijection in M. Then for each $\alpha < \kappa$, the set $\{m \in \omega : g(h(\alpha, m)) = 1\}$ is a Cohen real.*

Proof. Remember that subsets of ω and their characteristic functions are both considered as reals. Implicitly, one is a Cohen real iff the other is, by definition. So we will show that the function $l \stackrel{\text{def}}{=} \langle g(h(\alpha, m)) : m \in \omega\rangle$ is a Cohen real.

Fix $\alpha < \kappa$, and let $J = \{\beta < \kappa : h^{-1}(\beta)$ has the form (α, m) for some $m \in \omega\}$. Let $k(m) = h(\alpha, m)$ for all $m \in \omega$. Then k is a bijection from ω onto J. By 27.1, $G \cap \mathrm{Fin}(J, 2)$ is $\mathrm{Fin}(J, 2)$-generic over M. Define $k' : \mathrm{Fin}(J, 2) \to \mathrm{Fin}(\omega, 2)$ by setting $k'(p) = p \circ k$ for any $p \in \mathrm{Fin}(J, 2)$. So k' is an isomorphism from $\mathrm{Fin}(J, 2)$ onto $\mathrm{Fin}(\omega, 2)$. Clearly then $k'[G \cap \mathrm{Fin}(J, 2)]$ is $\mathrm{Fin}(\omega, 2)$-generic over M. So the proof is completed by checking that $\bigcup k'[G \cap \mathrm{Fin}(J, 2)] = l$. Take any $m \in \omega$. Then

$$
\begin{aligned}
(m, \varepsilon) \in \bigcup k'[G \cap \mathrm{Fin}(J, 2)] \quad &\text{iff} \quad \text{there is a } p \in k'[G \cap \mathrm{Fin}(J, 2)] \\
&\qquad\ \text{such that } (m, \varepsilon) \in p \\
&\text{iff} \quad \text{there is a } q \in G \cap \mathrm{Fin}(J, 2) \\
&\qquad\ \text{such that } (m, \varepsilon) \in k'(q) \\
&\text{iff} \quad \text{there is a } q \in G \cap \mathrm{Fin}(J, 2) \\
&\qquad\ \text{such that } (m, \varepsilon) \in q \circ k \\
&\text{iff} \quad g(k(m)) = \varepsilon \\
&\text{iff} \quad g(h(\alpha, m)) = \varepsilon \\
&\text{iff} \quad (m, \varepsilon) \in l
\end{aligned}
$$
□

Theorem 27.3. *Suppose that M is a c.t.m. of ZFC and G is $\mathrm{Fin}(\omega, 2)$-generic over M. Let $g = \bigcup G$ (so that g is a Cohen real). Then for any $f \in {}^{\omega}2$ which is in M, the set $\{m \in \omega : f(m) < g(m)\}$ is infinite.*

Proof. For each $n \in \omega$ let in M

$$D_n = \{h \in \mathrm{Fin}(\omega, 2) : \text{there is an } m > n \text{ such that } m \in \mathrm{dmn}(h) \text{ and } f(m) < h(m)\}.$$

Clearly D_n is dense. Hence the desired result follows. □

Thus if g is a Cohen real, then there is no f in the ground model such that $\{m \in \omega : g(m) \leq f(m)\}$ is finite. Put another way, if $A \subseteq \omega$ is a Cohen real, then there is no $B \subseteq \omega$ in the ground model such that $A \backslash B$ is finite.

Let $\langle P_i : i \in I\rangle$ be a system of forcing orders. We define the *product* of these orders to be the set

$$\prod_{i \in I}^{\mathrm{w}} P_i = \left\{ f \in \prod_{i \in I} P_i : \{j \in I : f(j) \neq 1\} \text{ is finite} \right\}$$

with the order

$$f \leq g \quad \text{iff} \quad \forall i \in I[f_i \leq g_i].$$

Theorem 27.4. *For any infinite cardinal κ, $\mathrm{Fin}(\kappa, 2)$ is isomorphic to $\prod_{\alpha<\kappa} \mathrm{Fin}(\omega, 2)$.*

Proof. Let $k : \kappa \to \kappa \times \omega$ be a bijection. For each $f \in \prod_{\alpha<\kappa} \mathrm{Fin}(\omega, 2)$ let

$$\mathrm{dmn}(F(f)) = \{\alpha < \kappa : 2^{\mathrm{nd}}(k(\alpha)) \in \mathrm{dmn}(f(1^{\mathrm{st}}(k(\alpha))) \text{ and}$$
$$(F(f))(\alpha) = (f(1^{\mathrm{st}}(k(\alpha))))(2^{\mathrm{nd}}(k(\alpha))).$$

Clearly F maps $\prod_{\alpha<\kappa} \mathrm{Fin}(\omega, 2)$ into $\mathrm{Fin}(\kappa, 2)$. To show that F is one-one, suppose that $f, g \in \prod_{\alpha<\kappa} \mathrm{Fin}(\omega, 2)$ and $f \neq g$; say $f(\alpha) \neq g(\alpha)$. Say $(n, \varepsilon) \in f(\alpha) \backslash g(\alpha)$. Let $\beta = k^{-1}(\alpha, n)$. Thus $\beta \in \mathrm{dmn}(F(f))$. We may assume that $\beta \in \mathrm{dmn}(F(g))$. It follows that $(F(f))(\beta) \neq (F(g))(\beta)$. So $F(f) \neq F(g)$.

To show that F maps onto, let $h \in \mathrm{Fin}(\kappa, 2)$. Define $f \in \prod_{\alpha<\kappa} \mathrm{Fin}(\omega, 2)$ by setting

$$\mathrm{dmn}(f(\alpha)) = \{n \in \omega : k^{-1}(\alpha, n) \in \mathrm{dmn}(h)\}$$
$$(f(\alpha))(n) = h(k^{-1}(\alpha, n)) \quad \text{if } k^{-1}(\alpha, n) \in \mathrm{dmn}(h).$$

Clearly $F(f) = h$.

Clearly $f \leq g$ iff $F(f) \subseteq F(g)$. □

Cohen reals are widely used in set theory.

Roitman, J. *Adding a random or a Cohen real.* Fund. Math. 103 (1979), 47–60.

Random forcing

The general idea of random forcing is to take a σ-algebra of measurable sets with respect to some measure, divide by the ideal of sets of measure zero, obtaining a complete Boolean algebra, and use it as the forcing algebra; the partially ordered set of nonzero elements is the forcing partial order.

We give fairly complete details for the case of the product measure on ${}^{\kappa}2$, for any infinite cardinal κ. To make our treatment self-contained we give a standard development of this measure, following

Fremlin, D. **Measure theory**, vol. 1.

Let κ be an infinite cardinal. For each $f \in \mathrm{Fn}(\kappa, 2, \omega)$ let $U_f = \{g \in {}^{\kappa}2 : f \subseteq g\}$. Hence $U_\emptyset = {}^{\kappa}2$. Note that the function taking f to U_f is one-one. For each $f \in \mathrm{Fn}(\kappa, 2, \omega)$ let $\theta_0(U_f) = 1/2^{|\mathrm{dmn}(f)|}$. Thus $\theta_0(U_\emptyset) = 1$. Let $\mathcal{C} = \{U_f : f \in \mathrm{Fn}(\kappa, 2, \omega)\}$. Note that ${}^{\kappa}2 \in \mathcal{C}$. For any $A \subseteq {}^{\kappa}2$ let

$$\theta(A) = \inf\left\{\sum_{n\in\omega} \theta_0(C_n) : C \in {}^{\omega}\mathcal{C} \text{ and } A \subseteq \bigcup_{n\in\omega} C_n\right\}.$$

An *outer measure* on a set X is a function $\mu : \mathscr{P}(X) \to [0, \infty]$ satisfying the following conditions:

(1) $\mu(\emptyset) = 0$.

(2) If $A \subseteq B \subseteq X$, then $\mu(A) \leq \mu(B)$.

(3) For every $A \in {}^{\omega}\mathscr{P}(X)$, $\mu(\bigcup_{n\in\omega} A_n) \leq \sum_{n\in\omega} \mu(A_n)$.

Proposition 27.4. *θ is an outer measure on ${}^{\kappa}2$.*

Proof. For (1), for any $m \in \omega$ let $f \in \mathrm{Fn}(\kappa, 2, \omega)$ have domain of size m. Then $\emptyset \subseteq U_f$ and $\theta_0(U_f) = \frac{1}{m}$. Hence $\theta(\emptyset) = 0$.

For (2), if $A \subseteq B \subseteq {}^{\kappa}2$, then

$$\left\{C \in {}^{\omega}\mathcal{C} : B \subseteq \bigcup_{n\in\omega} C_n\right\} \subseteq \left\{C \in {}^{\omega}\mathcal{C} : A \subseteq \bigcup_{n\in\omega} C_n\right\},$$

and hence $\mu(A) \leq \mu(B)$.

For (3), assume that $A \in {}^{\omega}\mathscr{P}({}^{\kappa}2)$. We may assume that $\sum_{n\in\omega} \theta(A_n) < \infty$. Let $\varepsilon > 0$; we show that $\theta(\bigcup_{n\in\omega} A_n) \leq \sum_{n\in\omega} \theta(A_n) + \varepsilon$, and the arbitrariness of ε then gives the desired result. For each $n \in \omega$ choose $C^n \in {}^{\omega}\mathcal{C}$ such that $A_n \subseteq \bigcup_{m\in\omega} C^n_m$ and $\sum_{m\in\omega} \theta_0(C^n_m) \leq \theta(A_n) + \frac{\varepsilon}{2^n}$. Then $\bigcup_{n\in\omega} A_n \subseteq \bigcup_{n\in\omega}\bigcup_{m\in\omega} C^n_m$ and

$$\theta\left(\bigcup_{n\in\omega} A_n\right) \leq \sum_{n\in\omega}\sum_{n\in\omega} \theta_0(C^n_m) \leq \sum_{n\in\omega} \theta(A_n) + \varepsilon,$$

as desired. $\square$

If A is a σ-algebra of subsets of X, then a *measure* on A is a function $\mu : A \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{i\in\omega} a_i) = \sum_{i\in\omega} \mu(a_i)$ if $a \in {}^{\omega}A$ and $a_i \cap a_j = \emptyset$ for all $i \neq j$. Note that $a_i = \emptyset$ is possible for some $i \in \omega$.

We give some important properties of measures:

Proposition 27.5. *Suppose that μ is a measure on a σ-algebra A of subsets of X. Then:*

(i) If $Y, Z \in A$ and $Y \subseteq Z$, then $\mu(Y) \leq \mu(Z)$.

(ii) If $Y \in {}^{\omega}A$, then $\mu(\bigcup_{n\in\omega} Y_n) \leq \sum_{n\in\omega} \mu(Y_n)$.

(iii) If $Y \in {}^{\omega}A$ and $Y_n \subseteq Y_{n+1}$ for all $n \in \omega$, then $\mu(\bigcup_{n\in\omega} Y_n) = \sup_{n\in\omega} \mu(Y_n)$.

Proof. (i): We have $\mu(Z) = \mu(Y) + \mu(Z\backslash Y) \geq \mu(Y)$.

(ii): Let $Z_n = Y_n\backslash \bigcup_{m<n} Y_m$. By induction, $\bigcup_{m\leq n} Z_m = \bigcup_{m\leq n} Y_m$, and hence $\bigcup_{m\in\omega} Z_m = \bigcup_{m\in\omega} Y_m$. Now

$$\mu\left(\bigcup_{m\in\omega} Y_m\right) = \mu\left(\bigcup_{m\in\omega} Z_m\right) = \sum_{m\in\omega} \mu(Z_m) \leq \sum_{m\in\omega} \mu(Y_m).$$

(iii): Again let $Z_n = Y_n\backslash \bigcup_{m<n} Y_m$. By induction, $Y_n = \bigcup_{m\leq n} Z_m$. Hence

$$\begin{aligned}\mu\left(\bigcup_{n\in\omega} Y_n\right) &= \mu\left(\bigcup_{n\in\omega} Z_n\right) \\ &= \sum_{n\in\omega} \mu(Z_n)\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \sum_{m\le n} \mu(Z_m) \\
&= \lim_{n\to\infty} \mu\left(\bigcup_{m\le n} Z_m\right) \\
&= \lim_{n\to\infty} \mu(Y_n) \\
&= \sup_{n\in\omega} \mu(Y_n). \qquad \square
\end{aligned}
$$

Proposition 27.6. *Let*

$$A = \{E \subseteq {}^{\kappa}2 : \forall X \subseteq {}^{\kappa}2[\theta(X) = \theta(X\cap E) + \theta(X\backslash E)]\}.$$

Then A is a σ-algebra of subsets of ${}^{\kappa}2$, and $\theta \restriction A$ is a measure on A.

Proof. $\emptyset \in A$ since for any $X \subseteq {}^{\kappa}2$ we have

$$\theta(X\cap\emptyset) + \theta(X\backslash\emptyset) = \theta(\emptyset) + \theta(X) = 0 + \theta(X) = \theta(X).$$

If $E \in A$, obviously also ${}^{\kappa}2\backslash E \in A$.

Next we show that if $E_1, E_2 \in A$ then $E_1 \cup E_2 \in A$. For any $X \subseteq {}^{\kappa}2$,

$$
\begin{aligned}
&\theta(X\cap(E_1\cup E_2)) + \theta(X\backslash(E_1\cup E_2)) \\
&\quad = \theta(X\cap(E_1\cup E_2)\cap E_1) + \theta(X\cap(E_1\cup E_2)\backslash E_1) + \theta(X\backslash(E_1\cup E_2)) \\
&\quad = \theta(X\cap E_1) + \theta((X\backslash E_1)\cap E_2) + \theta((X\backslash E_1)\backslash E_2) \\
&\quad = \theta(X\cap E_1) + \theta(X\backslash E_1) \\
&\quad = \theta(X).
\end{aligned}
$$

Now suppose that $E \in {}^{\omega}A$. Let $F = \bigcup_{i\in\omega} E_i$; we want to show that $F \in A$. For each $n \in \omega$ let $G_n = \bigcup_{i\le n} E_i$. So $G_n \in A$ by the binary case already considered. Let $H_0 = G_0$ and $H_{n+1} = G_{n+1}\backslash G_n$ for all $n \in \omega$. Hence $H_0 = E_0$ and $H_{n+1} = E_{n+1}\backslash G_n$ for all $n \in \omega$. Moreover, by induction $\bigcup_{i\le n} H_i = \bigcup_{i\le n} G_i$ for all $n \in \omega$, and hence $\bigcup_{i\in\omega} H_i = \bigcup_{i\in\omega} G_i = \bigcup_{i\in\omega} E_i = F$.

Now suppose that $n \ge 1$ and $X \subseteq {}^{\kappa}2$. Then

$$
\begin{aligned}
\theta(X\cap G_n) &= \theta(X\cap G_n\cap G_{n-1}) + \theta(X\cap G_n\backslash G_{n-1}) \\
&= \theta(X\cap G_{n-1}) + \theta(X\cap H_n).
\end{aligned}
$$

Hence by induction we get

$$\theta(X\cap G_n) = \sum_{m\le n} \theta(X\cap H_m) \quad \text{for all } n\in\omega. \tag{1}$$

Now $X = (X\cap F)\cup(X\backslash F)$, so by the outer measure property we have

$$\theta(X) \le \theta(X\cap F) + \theta(X\backslash F). \tag{2}$$

Now $X \cap F = \bigcup_{n\in\omega}(X \cap H_n)$, so by the outer measure property we have

$$\begin{aligned}\theta(X \cap F) &\le \sum_{n\in\omega} \theta(X \cap H_n)\\ &= \lim_{m\to\infty} \sum_{n\le m} \theta(X \cap H_n)\\ &= \lim_{m\to\infty} \theta(G_m) \quad \text{by (1)}\end{aligned}$$

Thus

(3) $$\theta(X \cap F) \le \lim_{m\to\infty} \theta(X \cap G_m)$$

Next, note that $G_n \subseteq \bigcup_{m\in\omega} G_m$ and hence $X\backslash \bigcup_{m\in\omega} G_m \subseteq X\backslash G_n$. Also $G_m \subseteq G_n$ for $m \le n$, and hence $X\backslash G_n \subseteq X\backslash G_m$. Thus by (2) in the definition of outer measure we have

$$\theta(X\backslash F) = \theta\left(X\backslash \bigcup_{m\in\omega} G_m\right) \le \inf_{m\in\omega} \theta(X\backslash G_m) = \lim_{m\to\omega} \theta(X\backslash G_m).$$

Together with (3) it then follows that

$$\theta(X \cap F) + \theta(X\backslash F) \le \lim_{m\to\omega} (\theta(X \cap G_m) + \theta(X\backslash G_m)) = \theta(X).$$

Together with (2) this implies that $F \in A$.

Thus A is a σ-algebra of subsets of ${}^\kappa 2$.

Now suppose that $E_n \cap E_m = \emptyset$ for $n \ne m$. Then

$$\theta(G_{n+1}) = \theta(G_{n+1} \cap E_{n+1}) + \theta(G_{n+1}\backslash E_{n+1}) = \theta(E_{n+1}) + \theta(G_n).$$

It follows by induction that $\theta(G_n) = \sum_{m\le n} \theta(E_m)$ for every $n \in \omega$. Hence

(4) $$\theta(F) \le \sum_{m\in\omega} \theta(E_m) \quad \text{since } \theta \text{ is an outer measure}$$

and

$$\theta(F) \ge \theta(\bigcup_{m\le n} E_m) = \theta(G_n) = \sum_{m\le n} \theta(E_m)$$

for each n, and hence $\theta(F) \ge \sum_{m\in\omega} E_m$. Now (4) gives $\theta(F) = \sum_{m\in\omega} \theta(E_m)$. □

Proposition 27.7. *If $\varepsilon \in 2$ and $\alpha < \kappa$, then $\{f \in {}^\kappa 2 : f(\alpha) = \varepsilon\} \in A$.*

Proof. Let $E = \{f \in {}^\kappa 2 : f(\alpha) = \varepsilon\}$, and let $X \subseteq {}^\kappa 2$; we want to show that $\theta(X) = \theta(X \cap E) + \theta(X\backslash E)$. $\le$ holds by the definition of outer measure. Now suppose that $\delta > 0$. Choose $C \in {}^\omega\mathcal{C}$ such that $X \subseteq \bigcup_{n\in\omega} C_n$ and $\sum_{n\in\omega} \theta_0(C_n) < \theta(X) + \delta$. For each $n \in \omega$ let $C_n = U_{f_n}$ with $f_n \in \mathrm{Fn}(\kappa, 2, \omega)$. For each $n \in \omega$, if $\alpha \notin \mathrm{dmn}(f_n)$, replace

C_n by U_g and U_h, where $g = f \cup \{(\alpha, 0)\}$ and $h = f \cup \{(\alpha, 1)\}$; let the new sequence be $C' \in {}^{\omega}\mathcal{C}$. Then $\sum_{n\in\omega} \theta(C_n) = \sum_{n\in\omega} \theta(C'_n)$ and $X \subseteq \bigcup_{n\in\omega} C'_n$. Then there is a partition M, N of ω such that $X \cap E \subseteq \bigcup_{n\in M} C'_n$ and $X \backslash E \subseteq \bigcup_{n\in N} C'_n$. Hence

$$\theta(X \cap E) + \theta(X \backslash E) \leq \sum_{n\in M} \theta(C'_n) + \sum_{n\in N} \theta(C'_n) = \sum_{n\in\omega} \theta(C'_n) < \theta(X) + \delta.$$

Since δ is arbitrary, it follows that $\theta(X) = \theta(X \cap E) + \theta(X \backslash E)$. $\square$

For $f : 2 \to \mathbb{R}$ we define $\int f = \frac{1}{2} f(0) + \frac{1}{2} f(1)$.

Proposition 27.8. *If $f_n : 2 \to [0, \infty)$ for each $n \in \omega$ and $\forall t < 2[\sum_{n\in\omega} f_n(t) < \infty]$, then $\sum_{n\in\omega} \int f_n < \infty$, and $\sum_{n\in\omega} \int f_n = \int \sum_{n\in\omega} f_n$.*

Proof.

$$\int \sum_{n\in\omega} f_n = \frac{1}{2} \sum_{n\in\omega} f_n(0) + \frac{1}{2} \sum_{n\in\omega} f_n(1) = \sum_{n\in\omega} \left(\frac{1}{2} f_n(0) + \frac{1}{2} f_n(1) \right) = \sum_{n\in\omega} \int f_n.$$

Proposition 27.9. $\theta({}^{\kappa}2) = 1$.

Proof. It is obvious that ${}^{\kappa}2 \in A$, and that $\theta({}^{\kappa}2) \leq \theta_0({}^{\kappa}2) = 1$. Suppose that $\theta({}^{\kappa}2) < 1$. Choose $C \in {}^{\omega}\mathcal{C}$ such that $2^{\kappa} = \bigcup_{n\in\omega} C_n$ and $\sum_{n\in\omega} \theta_0(C_n) < 1$. For each $n \in \omega$ let $C_n = U_{f_n}$, where $f_n \in \mathrm{Fn}(\kappa, 2, \omega)$.

(1) $\forall g \in \mathrm{Fn}(\kappa, 2, \omega) \exists n \in \omega[f_n \subseteq g$ or $g \subseteq f_n]$.

In fact, let $g \in \mathrm{Fn}(\kappa, 2, \omega)$. Let $h \in {}^{\kappa}2$ with $g \subseteq h$. Choose n such that $h \in C_n$. Then $f_n \subseteq h$. So $f_n \subseteq g$ or $g \subseteq f_n$.

(2) Let $M = \{n \in \omega : \forall m \neq n[f_m \not\subseteq f_n]\}$. Then ${}^{\kappa}2 \subseteq \bigcup_{n\in M} U_{f_n}$.

For, given $g \in {}^{\kappa}2$ choose $n \in \omega$ such that $g \in C_n$. Thus $f_n \subseteq g$. Let $m \in \omega$ with $f_m \subseteq f_n$ and $|\mathrm{dmn}(f_m)|$ minimum. Then $f_m \subseteq g$ and $m \in M$, as desired.

(3) $|M| \geq 2$.

In fact, obviously $M \neq \emptyset$. Suppose that $M = \{n\}$. Since $\sum_{n\in M} \theta_0(C_n) < 1$, we have $f_n \neq \emptyset$. Then ${}^{\kappa}2 \subseteq U_{f_n}$, contradiction.

(4) M is infinite.

In fact, suppose that M is finite, and let $m = \sup\{|\mathrm{dmn}(f_n)| : n \in M\}$. Let $g \in \mathrm{Fn}(\kappa, 2, \omega)$ be such that $|\mathrm{dmn}(g)| = m + 1$. Then by (1), $f_n \subseteq g$ for all $n \in M$. Because of (3), this contradicts (2).

Let $J = \bigcup_{n\in M} \mathrm{dmn}(f_n)$.

(5) J is infinite.

For, suppose that J is finite. Now $M = \bigcup_{G\subseteq J}\{n \in M : \mathrm{dmn}(f_n) = G\}$, so there is a $G \subseteq J$ such that $\{n \in M : \mathrm{dmn}(f_n) = G\}$ is infinite. But clearly $|\{n \in M : \mathrm{dmn}(f_n) = G\}| \leq 2^{|G|}$, contradiction.

Let $i : \omega \to J$ be a bijection. For $n, k \in \omega$ let f'_{nk} be the restriction of f_n to the domain $\{\alpha \in \text{dmn}(f_n) : \forall j < k[\alpha \neq i_j]\}$, and let

$$\alpha_{nk} = \frac{1}{2^{|\text{dmn}(f'_{nk})|}}.$$

Now for $n, k \in \omega$ and $t < 2$ we define

$$f_{nk}(t) = \begin{cases} \alpha_{n,k+1} & \text{if } i_k \notin \text{dmn}(f_n), \\ \alpha_{n,k+1} & \text{if } i_k \in \text{dmn}(f_n) \text{ and } f_n(i_k) = t, \\ 0 & \text{otherwise.} \end{cases}$$

(6) $\int f_{nk} = \alpha_{nk}$ for all $n, k \in \omega$.

In fact,

$$\begin{aligned} \int f_{nk} &= \frac{1}{2} f_{nk}(0) + \frac{1}{2} f_{nk}(1) \\ &= \begin{cases} \alpha_{n,k+1} & \text{if } i_k \notin \text{dmn}(f_n), \\ \frac{1}{2}\alpha_{n,k+1} & \text{if } i_k \in \text{dmn}(f_n) \end{cases} \\ &= \alpha_{nk}. \end{aligned}$$

Now we define by induction elements $t_k \in 2$ and subsets M_k of M. Let $M_0 = M$. Now suppose that M_k and t_i have been defined for all $i < k$, so that $\sum_{n \in M_k} \alpha_{nk} < 1$. Note that this holds for $k = 0$. Now

$$\begin{aligned} 1 > \sum_{n \in M_k} \alpha_{nk} &= \sum_{n \in M_k} \int f_{nk} \quad \text{by (6)} \\ &= \int \sum_{n \in M_k} f_{nk} \quad \text{by Proposition 1.} \end{aligned}$$

It follows that there is a $t_k < 2$ such that $\left(\sum_{n \in M_k} f_{nk}\right)(t_k) < 1$. Let

$$M_{k+1} = \{n \in M : \forall j < k+1[i_j \notin \text{dmn}(f_n), \text{ or } i_j \in \text{dmn}(f_n) \text{ and } f_n(i_j) = t_j]\}.$$

If $n \in M_{k+1}$, then $f_{nk}(t_k) = \alpha_{n,k+1}$. Hence

$$\sum_{n \in M_{k+1}} \alpha_{n,k+1} = \sum_{n \in M_{k+1}} f_{mk}(t_k) \leq \left(\sum_{n \in M_k} f_{nk}\right)(t_k) < 1.$$

Also, $M_{k+1} \neq \emptyset$. For, let $g \in {}^{\kappa}2$ such that $g(i_j) = t_j$ for all $j \leq k$. Say $g \in C_n$ with $n \in M$. Then $f_n \subseteq g$. Hence $i_j \notin \text{dmn}(f_n)$, or $i_j \in \text{dmn}(f_n)$ and $f_n(i_j) = t_j$. Thus $n \in M_{k+1}$.

This finishes the construction. Now let $g \in {}^{\kappa}2$ be such that $g(i_j) = t_j$ for all $j \in \omega$. Say $g \in C_n$ with $n \in M$. Then $f_n \subseteq g$. The domain of f_n is a finite subset of J. Choose

$k \in \omega$ so that $\mathrm{dmn}(f_n) \subseteq \{i_j : j < k\}$. Then $n \in M_k$. Hence $f'_{nk} = \emptyset$ and so $\alpha_{nk} = 1$. This contradicts $\sum_{m \in M_k} \alpha_{mk} < 1$. $\square$

Let ν be the tiny function with domain 2 which interchanges 0 and 1. For any $f \in {}^\kappa 2$ let $F(f) = \nu \circ f$.

Proposition 27.10.

(i) F is a permutation of ${}^\kappa 2$.

(ii) For any $f \in \mathrm{Fn}(\kappa, 2, \omega)$ we have $F[U_f] = U_{\nu \circ f}$.

(iii) For any $X \subseteq {}^\kappa 2$ we have $\theta(X) = \theta(F[X])$.

(iv) $\forall E \in A[F[E] \in A]$.

Proof. (i): Clearly F is one-one, and $F(F(f)) = f$ for any $f \in {}^\kappa 2$. So (i) holds.

(ii): For any $g \in {}^\kappa 2$,

$$\begin{aligned} g \in F[U_f] \quad &\text{iff} \quad \exists h \in U_f[g = F(h)] \\ &\text{iff} \quad \exists h \in {}^\kappa 2[f \subseteq h \text{ and } g = \nu \circ h] \\ &\text{iff} \quad \exists h \in {}^\kappa 2[\nu \circ f \subseteq \nu \circ h \text{ and } g = \nu \circ h] \\ &\text{iff} \quad \nu \circ f \subseteq g \\ &\text{iff} \quad g \in U_{\nu \circ f} \end{aligned}$$

(iii): Clearly $\theta_0(U_f) = \theta_0(F[U_f])$ for any $f \in \mathrm{Fn}(\kappa, 2, \omega)$. Also, $A \subseteq \bigcup_{n \in \omega C_n}$ iff $F[A] \subseteq \bigcup_{n \in \omega} F[C_n]$. So (iii) holds.

(iv): Suppose that $E \in A$. Let $X \subseteq {}^\kappa 2$. Then

$$\begin{aligned} \theta(X \cap F[E]) + \theta(X \backslash F[E]) &= \theta(F[F[X]] \cap F[E]) + \theta(F[F[X]] \backslash F[E]) \\ &= \theta(F[F[X] \cap E]) + \theta(F[F[X] \backslash E]) \\ &= \theta(F[X] \cap E) + \theta(F[X] \backslash E) \\ &= \theta(E) = \theta(F[E]). \end{aligned}$$

$\square$

Proposition 27.11. *If $\alpha < \kappa$ and $\varepsilon < 2$, then $\theta(U_{\{(\alpha,\varepsilon)\}}) = \frac{1}{2}$.*

Proof. By Proposition 27.10 we have $\theta(U_{\{(\alpha,\varepsilon)\}}) = \theta(U_{\{(\alpha,1-\varepsilon)\}})$, so the result follows from Proposition 27.9. $\square$

Proposition 27.12. *For each $f \in \mathrm{Fn}(\kappa, 2, \omega)$ we have $U_f \in A$ and $\theta(U_f) = \frac{1}{2^{|\mathrm{dmn}(f)|}}$.*

Proof. We have $U_f = \bigcap_{\alpha \in \mathrm{dmn}(f)} U_{\{(\alpha, f(\alpha))\}}$, so $U_f \in A$ by Proposition 27.7. We prove that $\theta(U_f) = \frac{1}{2^{|\mathrm{dmn}(f)|}}$ by induction on $|\mathrm{dmn}(f)|$. For $|\mathrm{dmn}(f)| = 1$, this holds by Proposition 27.11. Now assume that it holds for $|\mathrm{dmn}(f)| = m$. For any f with $|\mathrm{dmn}(f)| = m$ and $\alpha \notin \mathrm{dmn}(f)$ we have $2^{-|\mathrm{dmn}(f)|} = \theta(U_f) = \theta(U_{f \cup \{(\alpha,0)\}}) + \theta(U_{f \cup \{(\alpha,1)\}})$. Since $\theta(U_{f \cup \{(\alpha,\varepsilon)\}}) \le \theta_0(U_{f \cup \{(\alpha,\varepsilon)\}}) = 2^{-|\mathrm{dmn}(f)|-1}$ for each $\varepsilon \in 2$, it follows that $\theta(U_{f \cup \{(\alpha,\varepsilon)\}}) = 2^{-|\mathrm{dmn}(f)|-1}$ for each $\varepsilon \in 2$. $\square$

Proposition 27.13. *If F is a finite subset of ${}^{\kappa}2$, then $F \in A$ and $\theta(F) = 0$.*

Proof. This is obvious if $|F| \leq 1$, and then the general case follows. $\square$

This finishes our development of measure theory. Now we start to see how a forcing order is obtained.

For any BA A, an *ideal* of A is a nonempty subset of A such that if $a, b \in A$, $a \leq b$, and $b \in I$, then also $a \in I$; and if $a, b \in I$, then $a + b \in I$.

Proposition 27.14. *Let I be an ideal in a BA A. Define $\equiv_I = \{(a, b) : a, b \in A$ and $a \triangle b \in I\}$. Then $\equiv_I$ is an equivalence relation on A, and the collection of all equivalence classes can be made into a BA $(A/I, +, \cdot, -, [0]_I, [1]_I)$ such that the following conditions hold for all $a, b \in A$:*

(i) $[a]_I + [b]_I = [a + b]_I$.

(ii) $[a]_I \cdot [b]_I = [a \cdot b]_I$.

(iii) $-[a]_I = [-a]_I$.

Proof. Clearly $\equiv_I$ is reflexive on A and symmetric. Now suppose that $a \equiv_I b \equiv_I c$. Thus $a \triangle b \in I$ and $b \triangle c \in I$. Hence $a \cdot -c = a \cdot b \cdot -c + a \cdot -b \cdot -c \leq b \triangle c + a \triangle b \in I$. Hence $a \cdot -c \in I$. Similarly $c \cdot -a \in I$, so $a \triangle c \in I$; thus $a \equiv_I c$.

Suppose that $a \equiv_I a'$ and $b \equiv_I b'$. Then

$$(a + b) \cdot -(a' + b') = a \cdot -a' \cdot -b' + b \cdot -a' \cdot -b' \leq a \triangle a' + b \triangle b' \in I.$$

So $(a + b) \cdot -(a' + b') \in I$. Similarly $(a' + b') \cdot -(a + b) \in I$, so $(a + b) \triangle (a' + b') \in I$. Hence $(a + b) \equiv_I (a' + b')$. This shows that (i) is well-defined.

Similarly,

$$a \cdot b \cdot -(a' \cdot b') = a \cdot b \cdot -a' + a \cdot b \cdot -b' \leq a \triangle a' + b \triangle b' \in I,$$

so $a \cdot b \cdot -(a' \cdot b') \in I$. Similarly $a' \cdot b' \cdot -(a \cdot b) \in I$, so $(a \cdot b) \triangle (a' \cdot b') \in I$, so $[a \cdot b]_I = [a' \cdot b']_I$, and (ii) is well-defined.

Also, $(-a) \triangle (-a') = a \triangle a' \in I$, so $[-a]_I = [-a']_I$, and (iii) holds.

Now it is straightforward to check that $(A/I, +, \cdot, -, [0]_I, [1]_I)$ is a BA. $\square$

Now the *random forcing order on κ* is $((A/I) \backslash \{[0]_I\}, \leq, [1]_I)$, with A as in the above material on measure, and I is the ideal of members of A of measure 0. We denote it by ran_κ. For each $[a]_I$ in ran_κ we define $\theta([a]_I) = \theta(a)$. Clearly this definition is unambiguous.

Proposition 27.15. *ran_κ has ccc.*

Proof. Suppose to the contrary that $X \in [\mathrm{ran}_\kappa]^{\omega_1}$ is pairwise disjoint. Then $X = \bigcup_{n \in \omega} \{x \in X : \theta(x) \geq \frac{1}{n+1}\}$, so we can choose $X' \in [X]^{\omega_1}$ and n such that $\theta(x) \geq \frac{1}{n+1}$ for all $x \in X'$. Write $x = [a_x]_I$ for each $x \in X'$. Let $y : n + 2 \to X'$ be one-one. For each $i < n + 2$ let $b_i = a_{y_i} \prod_{j<i} -a_{y_j}$. Then $\langle b_i : i < n + 2 \rangle$ is a system of pairwise disjoint elements of A, and $\theta(b_i) = \theta(a_{j_i}) \geq \frac{1}{n+1}$ for all $i < n + 2$. Hence $\theta(\sum_{i<n+2} b_i) = \sum_{i<n+2} \theta(b_i) \geq \frac{n+2}{n+1}$, contradiction. $\square$

It follows that forcing with ran_κ preserves cofinalities and cardinals. If G is ran_κ-generic over a c.t.m. M, then for each $\alpha < \kappa$ one of the elements $[U_{\{(\alpha,0)\}}]_I$, $[U_{\{(\alpha,1)\}}]_I$ is in G since $\langle [U_{\{(\alpha,0)\}}]_I, [U_{\{(\alpha,1)\}}]_I \rangle$ is a maximal antichain. This gives a function $f : \kappa \to 2$. Its restriction to ω is a *random real.*

A BA A is *σ-complete* iff any countable subset of A has a sum.

Lemma 27.16. *If A is a σ-complete BA satisfying ccc, then A is complete.*

Proof. Let X be any subset of A; we want to show that it has a sum. By Zorn's lemma, let Y be a maximal set subject to the following conditions: Y consists of pairwise disjoint elements, and for any $y \in Y$ there is an $x \in X$ such that $y \leq x$. By ccc, Y is countable, and so $\sum Y$ exists. We claim that $\sum Y$ is the least upper bound of X.

Suppose that $x \in X$ and $x \not\leq \sum Y$. Then $x \cdot -\sum Y \neq 0$, and $Y \cup \{x \cdot -\sum Y\}$ properly contains Y and satisfies both of the conditions defining Y, contradiction. Hence $x \leq \sum Y$. So $\sum Y$ is an upper bound for X.

Suppose that z is any upper bound for X, but suppose that $\sum Y \not\leq z$. Thus $\sum Y \cdot -z \neq 0$, so by 2.2 there is a $y \in Y$ such that $y \cdot -z \neq 0$. Choose $x \in X$ such that $y \leq x$. Now $x \leq z$, so $y \cdot -z \leq z$, hence $y \cdot -z = 0$, contradiction. $\square$

Lemma 27.17. *A/I is complete.*

Proof. By Proposition 27.15 and Lemma 27.16 it suffices to show that it is σ-complete. So, suppose that X is a countable subset of A/I. We can write $X = \{[y]_I : y \in Y\}$ for some countable subset Y of A. We claim that $[\bigcup Y]_I$ is the least upper bound for X. For, if $x \in X$, choose $y \in Y$ such that $x = [y]_I$. Then $y \subseteq \bigcup Y$, so $x \leq [\bigcup Y]_I$. Now suppose that $[z]_I$ is any upper bound of X. Then $[y]_I \leq [z]_I$ for any $y \in Y$, so $y \backslash z \in I$, i.e., $\theta(y \backslash z) = 0$, for any $y \in Y$. Hence

$$\theta\left(\bigcup Y \backslash z\right) \leq \sum_{y \in Y} \theta(y \backslash z) = 0;$$

so $[\bigcup Y]_I \leq [z]_I$, as desired. $\square$

Theorem 27.18. *There is an isomorphism f of* $\mathrm{RO}(\mathrm{ran}_\kappa)$ *onto A/I such that $f(i(a)) = a$ for every $a \in \mathrm{ran}_\kappa$, where i is as in the definition of* RO.

Proof. Define $j : \mathrm{ran}_\kappa \to A/I$ by setting $j(a) = a$ for all $a \in \mathrm{ran}_\kappa$. Then the following conditions are clear:

(1) $j[\mathrm{ran}_\kappa]$ is dense in A/I. (In fact, $j[\mathrm{ran}_\kappa]$ consists of all nonzero elements of A/I.)

(2) If $a, b \in \mathrm{ran}_\kappa$ and $a \leq b$, then $j(a) \leq j(b)$.

(3) If $a, b \in \mathrm{ran}_\kappa$ and $a \perp b$, then $j(a) \cdot j(b) = 0$.

Hence our theorem follows from Theorem 13.22. $\square$

We now need the following general result about Boolean values.

Proposition 27.19. $[\![\exists x \in \check{A}\varphi(x)]\!] = \sum_{x\in A}[\![\varphi(\check{x})]\!]$.

Proof. Let $X = \{[\![\psi(\check{x})]\!] : x \in A\}$. First we show that $[\![\exists x \in \check{A}\varphi(x)]\!]$ is an upper bound for X. In fact, if $x \in A$ then $[\![\check{x} \in \check{A}]\!] = 1$ since $\check{x}_G = x \in A = \check{A}_G$ for any generic G, so that $1 \Vdash \check{x} \in \check{A}$. Hence

$$[\![\varphi(\check{x})]\!] = [\![\check{x} \in \check{A}]\!] \cdot [\![\varphi(\check{x})]\!] = [\![\check{x} \in \check{A} \wedge \varphi(\check{x})]\!] \leq [\![\exists x[x \in \check{A} \wedge \varphi(x)]]\!] = [\![\exists x \in \check{A}\varphi(x)]\!].$$

Now suppose that a is an upper bound for X, but $[\![\exists x \in \check{A}\varphi(x)]\!] \not\leq a$. Thus by definition,

$$\left(\sum_{\tau\in V^P}[\![\tau \in \check{A} \wedge \varphi(\tau)]\!]\right) \cdot -a \neq 0,$$

so there is a $\tau \in V^P$ such that $[\![\tau \in \check{A} \wedge \varphi(\tau)]\!] \cdot -a \neq 0$. Hence $\left(\sum_{b\in A}[\![\tau = \check{b} \wedge \varphi(\tau)]\!]\right) \cdot -a \neq 0$, so there is a $b \in A$ such that $[\![\tau = \check{b} \wedge \varphi(\tau)]\!] \cdot -a \neq 0$. But $[\![\tau = \check{b} \wedge \varphi(\tau)]\!] \leq [\![\varphi(\check{b})]\!]$, so this is a contradiction. □

Theorem 27.20. *Suppose that M is a c.t.m. of ZFC, and $A, I, \mathrm{ran}_\kappa$ are as above, all in M. Suppose that G is ran_κ-generic over M, and $f \in {}^\omega\omega$ in $M[G]$. Then there is an $h \in M \cap {}^\omega\omega$ such that $f(n) < h(n)$ for all $n \in \omega$.*

Proof. Let σ be a ran_κ-name such that $\sigma_G = f$, and let $p \in \mathrm{ran}_\kappa$ be such that $p \Vdash \sigma : \omega \to \omega$. We claim that

$$E \stackrel{\text{def}}{=} \{q \in \mathrm{ran}_\kappa : \text{there is an } h \in {}^\omega\omega \text{ such that } q \Vdash \forall n \in \omega(\sigma(n) < \check{h}(n))\}$$

is dense below p. Clearly this gives the conclusion of the theorem.

To prove this, take any $r \leq p$; we want to find $q \in E$ such that $q \leq r$. Let k be the isomorphism from $\mathrm{RO}(\mathrm{ran}_\kappa)$ to A/I given by Theorem 27.18. Now temporarily fix $n \in \omega$. Let $i : \mathrm{ran}_\kappa \to \mathrm{RO}(\mathrm{ran}_\kappa)$ be the mapping from Chapter 9. Then by Proposition 27.19,

$$i(r) \leq [\![\exists m \in \omega(\sigma(\check{n}) < m)]\!] = \sum_{m\in\omega}[\![\sigma(\check{n}) < \check{m}]\!],$$

Applying k, we get

$$r \leq \sum_{m\in\omega} k([\![\sigma(\check{n}) < \check{m}]\!]). \tag{1}$$

Let $r \cdot k([\![\sigma(\check{n}) < \check{m}]\!]) = [a_m]$ for each $m \in \omega$. Now clearly if $m < p$, then $r \Vdash \sigma(\check{n}) < \check{m} \to \sigma(\check{n}) < \check{p}$, so $[a_m] \leq [a_p]$. Let $b_m = \bigcup_{p\leq m} a_p$ for each $m \in \omega$. Then $[a_m] = [b_m]$ for each m.

(2) $\sum_{m\in\omega}[b_m] = \left[\bigcup_{m\in\omega} b_m\right]$.

In fact, $\left[\bigcup_{m\in\omega} b_m\right]$ is clearly an upper bound for $\{[b_m] : m \in \omega\}$. If $[c]$ is any upper bound, then $\mu(b_m \backslash c) = 0$ for each m, and hence $\theta\left(\bigcup_{m\in\omega} b_m \backslash c\right) = 0$, so that $\left[\bigcup_{m\in\omega} b_m\right] \leq [c]$. So (2) holds.

Note that $r = \left[\bigcup_{m\in\omega} b_m\right]$; so $\theta(r) = \theta\left(\bigcup_{m\in\omega} b_m\right)$. By Proposition 27.5(iii) we get $\theta(r) = \sup\{\theta(b_m) : m \in \omega\}$. So we can choose $m \in \omega$ such that $\theta(b_m) \geq \theta(r) - \frac{1}{2^{n+2}}\theta(r)$. Let $h(n)$ be the least such m. Thus

(3) $$\theta(r\backslash b_{h(n)}) = \theta(r) - \theta(b_{h(n)}) \leq \frac{1}{2^{n+2}}\theta(r).$$

Now

$$\begin{aligned}\theta\left(r\backslash \bigcap_{n\in\omega} b_{h(n)}\right) &= \theta\left(\bigcup_{n\in\omega}(r\backslash b_{h(n)})\right)\\ &\leq \sum_{n\in\omega}\frac{1}{2^{n+2}}\theta(r)\\ &= \frac{1}{2}\theta(r).\end{aligned}$$

It follows that

$$\theta\left(\bigcap_{n\in\omega} b_{h(n)}\right) > 0.$$

Let $q = \bigcap_{n\in\omega} b_{h(n)}$. So $[q] \in \mathrm{ran}_\kappa$. We claim that $[q] \leq r$ and $[q] \Vdash \forall n \in \omega(\sigma(n) < \check{h(n)})$. For, suppose that $[q] \in G$ with G ran_κ-generic over M, and suppose that $n \in \omega$. Then $[q] \leq [b_{h(n)} = [a_{h(n)} \leq r$. and also $[q] \leq k([\![\sigma(checkn) < \check{h(n)}]\!])$. Hence $i([q]) \leq [\![\sigma(checkn) < \check{h(n)}]\!]$, hence $[q] \Vdash \sigma(\check{n}) < \check{h(n)}$. Thus $[q] \in E$, as desired. $\square$

Corollary 27.21. *Suppose that M is a c.t.m. of ZFC, and $\mathbb{P}_r$ is considered in M. Suppose that G is $\mathbb{P}_r$-generic over M. Then no $f \in {}^\omega\omega$ in $M[G]$ is a Cohen real.*

Proof. By Theorem 27.3 and Theorem 27.20. $\square$

Thus we may say that adding a random real does not add a Cohen real.

Roitman, J. [79] *Adding a random or a Cohen real...* Fund. Math. 103 (1979), 47–60.

Sacks forcing

Let Seq be the set of all finite sequences of 0's and 1's. A *perfect tree* is a nonempty subset T of Seq with the following properies:

(1) If $t \in T$ and $m < \mathrm{dmn}(t)$, then $t \restriction m \in T$.

(2) For any $t \in T$ there is an $s \in T$ such that $t \subseteq s$ and $s^\frown\langle 0\rangle, s^\frown\langle 1\rangle \in T$.

Thus Seq itself is a perfect tree. *Sacks forcing* is the collection $\mathbb{Q}$ of all perfect trees, ordered by $\subseteq$ (not by $\supseteq$).

Note that an intersection of perfect trees does not have to be perfect. For example (with $\varepsilon_1, \varepsilon_2, \ldots$ any members of 2):

$$\begin{aligned}p &= \{\emptyset, \langle 0\rangle, \langle 0\varepsilon_1\rangle, \langle 0\varepsilon_1\varepsilon_2\rangle, \ldots\};\\ q &= \{\emptyset, \langle 1\rangle, \langle 1\varepsilon_1\rangle, \langle 1\varepsilon_1\varepsilon_2\rangle, \ldots\}.\end{aligned}$$

Also, one can have p, q perfect, $p \cap q$ not perfect, but $r \subseteq p \cap q$ for some perfect r:

$$\begin{aligned}
p = \{&\emptyset, \langle 1\rangle, \langle 1\varepsilon_1\rangle, \langle 1\varepsilon_1\varepsilon_2\rangle, \ldots \\
&\langle 0\rangle, \langle 01\rangle, \langle 01\varepsilon_2\rangle, \langle 01\varepsilon_2\varepsilon_3\rangle \ldots\};\\
q = \{&\emptyset, \langle 1\rangle, \langle 1\varepsilon_1\rangle, \langle 1, \varepsilon_1\varepsilon_2\rangle, \ldots \\
&\langle 0\rangle, \langle 00\rangle, \langle 00\varepsilon_2\rangle, \langle 00\varepsilon_2\varepsilon_3\rangle \ldots\};\\
r = \{&\emptyset, \langle 1\rangle, \langle 1\varepsilon_1\rangle, \langle 1, \varepsilon_1\varepsilon_2\rangle, \ldots\}.
\end{aligned}$$

Theorem 27.22. *Suppose that M is a c.t.m. of ZFC. Consider $\mathbb{Q}$ within M, and let G be $\mathbb{Q}$-generic over M. Then the set*

$$\{s \in Seq : s \in p \text{ for all } p \in G\}$$

is a function from ω into 2.

Proof. For each $n \in \omega$ let

$$D_n = \{p \in \mathbb{Q} : \text{there is an } s \in \text{Seq such that } \operatorname{dmn}(s) = n \text{ and } s \subseteq t \text{ or } t \subseteq s \text{ for all } t \in p\}.$$

Then D_n is dense: if $q \in \mathbb{Q}$, choose any $s \in q$ such that $\operatorname{dmn}(s) = n$, and let $p = \{t \in q : s \subseteq t \text{ or } t \subseteq s\}$. Clearly $p \in D_n$ and $p \subseteq q$.

Now for each $n \in \omega$ let $p^{(n)}$ be a member of $G \cap D_n$, and choose $s^{(n)}$ accordingly.

(1) If $m < n$, then $s^{(m)} \subseteq s^{(n)}$.

In fact, choose $r \in G$ such that $r \subseteq p^{(m)} \cap p^{(n)}$. Then $s^{(m)} \subseteq t$ and $s^{(n)} \subseteq t$ for all $t \in r$ with $\operatorname{dmn}(t) \geq n$, so $s^{(m)} \subseteq s^{(n)}$.

(2) $s^{(m)} \in q$ for all $q \in G$.

In fact, let $q \in G$, and choose $r \in G$ such that $r \subseteq q$ and $r \subseteq p^{(m)}$. Take $t \in r$ with $\operatorname{dmn}(t) = m$. then $t = s^{(m)}$ since $r \subseteq p^{(m)}$. Thus $s^{(m)} \in q$ since $r \subseteq q$.

(3) If $t \in q$ for all $q \in G$, then $t = s^{(m)}$ for some m.

For, let $\operatorname{dmn}(t) = m$. Since $t \in p^{(m)}$, we have $t = s^{(m)}$.

From (1)–(3) the conclusion of the theorem follows. □

The function described in Theorem 27.19 is called a *Sacks real.*

If $p \in \mathbb{Q}$, a member f of p is a *branching point* iff $f^\frown\langle 0\rangle, f^\frown\langle 1\rangle \in p$.

Sacks forcing does not satisfy ccc:

Proposition 27.23. *There is a family of 2^ω pairwise incompatible members of $\mathbb{Q}$.*

Proof. Let $\mathscr{A}$ be a family of 2^ω infinite pairwise almost disjoint subsets of ω. With each $A \in \mathscr{A}$ we define a sequence $\langle P_{A,n} : n \in \omega\rangle$ of subsets of Seq, by recursion:

$$P_{A,0} = \{\emptyset\};$$
$$P_{A,n+1} = \begin{cases} \{f^\frown\langle 0\rangle : f \in P_{A,n}\} & \text{if } n \notin A, \\ \{f^\frown\langle 0\rangle : f \in P_{A,n}\} \cup \{f^\frown\langle 1\rangle : f \in P_{A,n}\} & \text{if } n \in A. \end{cases}$$

Note that all members of $P_{A,n}$ have domain n. We set $p_A = \bigcup_{n\in\omega} P_{A,n}$. We claim that p_A is a perfect tree. Condition (1) is clear. For (2), suppose that $f \in p_A$; say $f \in P_{A,n}$. Let m be the least member of A greater than n. If g extends f by adjoining 0's from n to $m-1$, then $g^\frown\langle 0\rangle, g^\frown\langle 1\rangle \in p_A$, as desired in (2).

We claim that if $A, B \in \mathscr{A}$ and $A \neq B$, then p_A and p_B are incompatible. For, suppose that q is a perfect tree and $q \subseteq p_A, p_B$. Now $A \cap B$ is finite. Let m be an integer greater than each member of $A \cap B$. Let f be a branching point of q with $\mathrm{dmn}(f) \geq m$; it exists by (2) in the definition of perfect tree. Let $\mathrm{dmn}(f) = n$. Then $f \in P_{A,n}$ and $f^\frown\langle 0\rangle, f^\frown\langle 1\rangle \in P_{A,n+1}$, so $n \in A$ by construction. Similarly, $n \in B$, contradiction. □

Proposition 27.24. *$\mathbb{Q}$ is not ω_1-closed.*

Proof. For each $n \in \omega$ let

$$p_n = \{f \in \mathrm{Seq} : f(i) = 0 \text{ for all } i < n\}.$$

Clearly p_n is perfect, $p_n \subseteq p_m$ if $n > m$, and $\bigcap_{n\in\omega} P_n$ is $\{f\}$ with $f(i) = 0$ for all i, so that the descending sequence $\langle p_n : n \in \omega\rangle$ does not have any member of $\mathbb{Q}$ below it. □

By 27.23 and 27.24, the methods of chapters 16 and 24 cannot be used to show that forcing with $\mathbb{Q}$ preserves cardinals, even if we assume CH in the ground model. Nevertheless, we will show that it does preserve cardinals. To do this we will prove a modified version of ω_1-closure.

If p is a perfect tree, an *n-th branching point* of p is a branching point f of p such that there are exactly n branching points g such that $g \subseteq f$. Thus $n > 0$. For perfect trees p, q and n a positive integer, we write $p \leq_n q$ iff $p \subseteq q$ and every n-th branching point of q is a branching point of p. Also we write $p \leq_0 q$ iff $p \subseteq q$.

Lemma 27.25. *Suppose that $p \subseteq q$ are perfect trees, and $n \in \omega$. Then:*

(i) If $p \leq_n q$, then $p \leq_i q$ for every $i < n$.

(ii) If $p \leq_n q$ and f is an n-th branching point of q, then f is an n-th branching point of p.

(iii) For each positive integer n there is an $f \in p$ such that f is an n-th branching point of q.

(iv) The following conditions are equivalent:

(a) $p \leq_n q$.

(b) For every $f \in Seq$, if f is an n-th branching point of q, then $f^\frown\langle 0\rangle, f^\frown\langle 1\rangle \in p$.

(v) For each positive integer n there are exactly 2^{n-1} n-th branching points of a perfect tree p.

(vi) If p and q are perfect trees, then so is $p \cup q$.

(vii) If p and q are perfect trees, then $\{r : r$ is a perfect tree and $r \subseteq p$ or $r \subseteq q\}$ is dense below $p \cup q$.

Proof. (i): Assume that $p \leq_n q$, $i < n$, and f is an i-th branching point of q. Then since q is perfect there are n-th branching points g, h of q such that $f^\frown\langle 0\rangle \subseteq g$ and $f^\frown\langle 1\rangle \subseteq h$. So $g, h \in p$, hence $f \in p$. This shows that $p \leq_i q$.

(ii): Suppose that $p \le_n q$ and f is an n-th branching point of q. Let $r_0, \ldots, r_{n-1}$ be all of the branching points g of q such that $g \subseteq f$. Then by (i), $r_0, \ldots, r_{n-1}$ are all branching points of p. Hence f is an n-th branching point of p.

(iii): Let f be an n-th branching point of p. Then it is an m-th branching point of q for some $m \ge n$. Let r be an n-th branching point of q below f. Then $r \in p$, as desired. [But r might not be a branching point of p.]

(iv), (v), (vi): Immediate from the definitions.

(vii): Suppose that p, q, t are perfect trees and $t \subseteq p \cup q$; we want to find a perfect tree $r \subseteq t$ such that $r \subseteq p$ or $r \subseteq q$. If $t \subseteq p \cap q$, then $r = t$ works. Otherwise, there is some member f of t which is not in both p and q; say $f \in p \backslash q$. Then $r \stackrel{\text{def}}{=} \{g \in t : g \subseteq f$ or $f \subseteq g\}$ is a perfect tree with $r \subseteq t$ and $r \subseteq p$. □

Lemma 27.26. (Fusion lemma) *If $\langle p_n : n \in \omega\rangle$ is a sequence of perfect trees and $\cdots \le_n p_n \le_{n-1} \cdots \le_2 p_2 \le_1 p_1 \le_0 p_0$, then $q \stackrel{\text{def}}{=} \bigcap_{n\in\omega} p_n$ is a perfect tree, and $q \le_n p_n$ for all $n \in \omega$.*

Proof. Let n be a positive integer, and let s be an n-th branching point of p_n. If $n \le m$, then $p_m \le_n p_n$, so s is an n-th branching point of p_m; hence $s, s^\frown\langle 0\rangle, s^\frown\langle 1\rangle \in p_m$. It follows that $s, s^\frown\langle 0\rangle, s^\frown\langle 1\rangle \in q$, and s is a branching point of q. Thus we just need to show that q is a perfect tree.

Clearly if $t \in q$ and $n < \text{dmn}(t)$, then $t \restriction n \in q$. Now suppose that $s \in q$; we want to find a $t \in q$ with $s \le t$ and t is a branching point of q. Let $n = \text{dmn}(s)$. Now $s \in p_n$, and p_n has fewer than n elements less than s, so p_n has an n-th branching point $t \ge s$. By the first paragraph, $t \in q$. □

Let p be a perfect tree and $s \in p$. We define

$$p \restriction s = \{t \in p : t \subseteq s \text{ or } s \subseteq t\}.$$

Clearly $p \restriction s$ is still a perfect tree. Now for any positive integer n, let $t_0, \ldots, t_{2^n-1}$ be the collection of all immediate successors of n-th branching points of p. Suppose that for each $i < 2^n$ we have a perfect tree $q_i \le p \restriction t_i$. Then we define the *amalgamation of* $\{q_i : i < 2^n\}$ *into* p to be the set $\bigcup_{i<2^n} q_i$.

Lemma 27.27. *Under the above assumptions, the amalgamation r of $\{q_i : i < 2^n\}$ into p has the following properties:*

(i) r is a perfect tree.

(ii) $r \le_n p$.

Proof. (i): Suppose that $f \in r$, $g \in \text{Seq}$, and $g \subseteq f$. Say $f \in q_i$ with $i < 2^n$. Then $g \in q_i$, so $g \in r$. Now suppose that $f \in r$; we want to find a branching point of r above f. Say $f \in q_i$. Let g be a branching point of q_i with $f \subseteq g$. Clearly g is a branching point of r.

(ii): Suppose that f is an n-th branching point of p. Then there exist $i, j < 2^n$ such that $f^\frown\langle 0\rangle = t_i$ and $f^\frown\langle 1\rangle = t_j$. So $f^\frown\langle 0\rangle \in q_i \subseteq r$ and $f^\frown\langle 1\rangle = t_j \in q_j \subseteq r$, and so f is a branching point of r. □

Lemma 27.28. *Suppose that M is a c.t.m. of ZFC and we consider the Sacks partial order $\mathbb{Q}$ within M. Suppose that $B \in M$, $\tau \in M^{\mathbb{Q}}$, $p \in \mathbb{Q}$, and $p \Vdash \tau : \check{\omega} \to \check{B}$. Then there is a $q \leq p$ and a function $F : \omega \to [B]^{<\omega}$ in M such that $q \Vdash \tau(\check{n}) \in \check{F}_n$ for every $n \in \omega$.*

Proof. We work entirely within M, except as indicated. We construct two sequences $\langle q_n : n \in \omega\rangle$ and $\langle F_n : n \in \omega\rangle$ by recursion. Let $q_0 = p$. Suppose that q_n has been defined; we define F_n and q_{n+1}. Assume that $q_n \leq p$. Then $q_n \Vdash \tau : \check{\alpha} \to \check{B}$, so $q_n \Vdash \exists x \in \check{B}\tau(\check{n}) = x)$. Let $t_0, \ldots, t_{2^n-1}$ list all of the functions $f^\frown\langle 0\rangle$ and $f^\frown\langle 1\rangle$ such that f is an n-th branching point of q_n. Then for each $i < 2^n$ we have $q_n \restriction t_i \subseteq q_n$, and so $q_n \restriction t_i \Vdash \exists x \in \check{B}\tau(\check{n}) = x)$. Hence there exist an $r_i \subseteq q_n \restriction t_i$ and a $b_i \in B$ such that $r_i \Vdash \tau(\check{n}) = \check{b}_i$. Let q_{n+1} be the amalgamation of $\{r_i : i < 2^n\}$ into q_n, and let $F_n = \{b_i : i < 2^n\}$. Thus $q_{n+1} \leq_n q_n$ by 27.27. Moreover:

$$(1) \qquad q_{n+1} \Vdash \tau(\check{n}) \in \check{F}_n.$$

In fact, let G be $\mathbb{Q}$-generic over M with $q_{n+1} \in G$. By 27.22(vii), there is an i such that $r_i \in G$. Since $r_i \Vdash \tau(\check{n}) = \check{b}_i$, it follows that $\tau_G(n) \in F_n$, as desired in (1).

Now with (1) the construction is complete.

By the fusion lemma 27.26 we get $s \leq_n q_n$ for each n. Hence the conclusion of the lemma follows. □

Theorem 27.29. *If M is a c.t.m. of $ZFC + CH$ and $\mathbb{Q} \in M$ is the Sacks forcing partial order, and if G is $\mathbb{Q}$-generic over M, then cofinalities and cardinals are preserved in $M[G]$.*

Proof. Since $|\mathbb{Q}| \leq 2^\omega = \omega_1$ by CH, the poset $\mathbb{Q}$ satisfies the ω_2-chain condition, and so preserves cofinalities and cardinals $\geq \omega_2$. Hence it suffices to show that ω_1^M remains regular in $M[G]$. Suppose not: then there is a function $f : \omega \to \omega_1^M$ in $M[G]$ such that $\text{rng}(f)$ is cofinal in ω_1^M. Hence there is a name τ such that $f = \tau_G$, and hence there is a $p \in G$ such that $p \Vdash \tau : \check{\omega} \to \check{\omega}_1^M$. By Lemma 27.28, choose $q \leq p$ and $F : \omega \to [\omega_1^M]^{<\omega}$ in M such that $q \Vdash \tau(\check{n}) \in \check{F}_n$ for every $n \in \omega$. Take $\beta < \omega_1^M$ such that $\bigcup_{n\in\omega} F_n < \beta$. Now $q \Vdash \exists n \in \omega(\check{\beta} < \tau(\check{n})$, so there exist an $r \leq q$ and an $n \in \omega$ such that $r \Vdash \check{\beta} < \tau(\check{n})$. So we have:

(2) $r \Vdash \tau(\check{n}) \in \check{F}_n$;

(3) $\bigcup_{n\in\omega} F_n < \beta$;

(4) $r \Vdash \check{\beta} < \tau(\check{n})$.

These three conditions give the contradiction $r \Vdash \tau(\check{n}) < \tau(\check{n})$. □

Baumgartner, J.; Laver, R. [79] *Iterated perfect-set forcing.* Ann. Math. Logic 17 (1979), 271–288.

Hechler MAD forcing

A family $\mathscr{A}$ of infinite subsets of ω is *maximal almost disjoint (MAD)* iff any two members of $\mathscr{A}$ are almost disjoint, and $\mathscr{A}$ is maximal with this property. By Theorem 20.1, there is a MAD family of size 2^ω. (Apply Zorn's lemma.)

Theorem 27.30. *Every infinite MAD family of infinite subsets of ω is uncountable.*

Proof. Suppose that $\mathscr{A}$ is a denumerable pairwise almost disjoint family of infinite subsets of ω; we want to extend it. Write $\mathscr{A} = \{A_n : n \in \omega\}$, the A_n's distinct. We define $\langle a_n : n \in \omega\rangle$ by recursion. Suppose that a_m has been defined for all $m < n$. Now $\bigcup_{m<n}(A_m \cap A_n)$ is finite, so we can choose

$$a_n \in A_n \backslash \left(\{a_m : m < n\} \cup \bigcup_{m<n}(A_m \cap A_n)\right).$$

Note that then $a_n \notin A_m$ for any $m < n$. Let $B = \{a_n : n \in \omega\}$. Then B is infinite, and $B \cap A_n \subseteq \{a_m : m \leq n\}$. □

Also recall that Martin's axiom implies that every MAD family has size 2^ω; see Theorem 21.7. We now want to introduce a forcing which will make a MAD family of size ω_1, with $\neg$CH.

The members of our partial order $\mathbb{H}$ will be certain pairs (p, q); we define $(p, q) \in \mathbb{H}$ iff the following conditions hold:

(1) p is a function from a finite subset of ω_1 into ${}^n2$ for some $n \in \omega$. We write $n = n_p$.

(2) q is a function with domain contained in $[\mathrm{dmn}(p)]^2$ and range contained in n_p.

(3) If $\{\alpha, \beta\} \in \mathrm{dmn}(q)$ and $q(\{\alpha, \beta\}) = m$, then for every i with $m \leq i < n_p$ we have $(p(\alpha))(i) = 0$ or $(p(\beta))(i) = 0$.

Furthermore, for $(p_1, q_1), (p_2, q_2) \in \mathbb{H}$ we define $(p_1, q_1) \leq (p_2, q_2)$ iff the following conditions hold:

(4) $\mathrm{dmn}(p_1) \supseteq \mathrm{dmn}(p_2)$.

(5) $p_1(\alpha) \supseteq p_2(\alpha)$ for all $\alpha \in \mathrm{dmn}(p_2)$.

(6) $q_1 \supseteq q_2$.

Note that (5) implies that $n_{p_2} \leq n_{p_1}$.

The idea here is to produce almost disjoint sets a_α for $\alpha < \omega_1$; $p(\alpha)$ is the characteristic function of $a_\alpha \cap n_p$, and $a_\alpha \cap a_\beta \subseteq q(\{\alpha, \beta\})$.

Lemma 27.31. *Suppose that* $(p_3, q_3) \leq (p_1, q_1), (p_2, q_2)$. *Then* $(p_3, q_1 \cup q_2) \in \mathbb{H}$, *and* $(p_3, q_3) \leq (p_3, q_1 \cup q_2) \leq (p_1, q_1), (p_2, q_2)$.

Proof. Condition (1) clearly holds for $(p_3, q_1 \cup q_2)$, since it only involves p_3. Clearly $q_1 \cup q_2$ is a relation with domain $\subseteq [\mathrm{dmn}(p_1)]^2 \cup [\mathrm{dmn}(p_2)]^2 \subseteq [\mathrm{dmn}(p_3)]^2$. To show that it is a function, suppose that $\{\alpha, \beta\} \in \mathrm{dmn}(q_1) \cap \mathrm{dmn}(q_2)$. Then $q_1(\{\alpha, \beta\}) = q_3(\{\alpha, \beta\}) = q_2(\{\alpha, \beta\})$. So $q_1 \cup q_2$ is a function, and it clearly maps into $\max(n_{p_1}, n_{p_2}) \leq n_{p_3}$. Hence (2) holds for $(p_3, q_1 \cup q_2)$. Finally, suppose that $\{\alpha, \beta\} \in \mathrm{dmn}(q_1 \cup q_2)$. By symmetry, say $\{\alpha, \beta\} \in \mathrm{dmn}(q_1)$. Let $q_1(\{\alpha, \beta\}) = m$, and suppose that $m \leq i < n_{p_3}$. Then $q_3(\{\alpha, \beta\}) = q_1(\{\alpha, \beta\}) = m$, so $(p_3(\alpha))(i) = 0$ or $(p_3(\beta))(i) = 0$. So (3) holds. The final inequalities are clear. □

Lemma 27.32. $\mathbb{H}$ *satisfies ccc.*

Proof. Suppose that N is an uncountable subset of $\mathbb{H}$; we want to find two compatible members of N. Now $\langle \mathrm{dmn}(p) : (p,q) \in N\rangle$ is an uncountable system of finite sets, so there exist an uncountable $N' \subseteq N$ and a finite subset H of ω_1 such that $\langle \mathrm{dmn}(p) : (p,q) \in N'\rangle$ is a Δ-system with root H. Next,

$$\begin{aligned} N' = &\bigcup_{(f,g)\in J} \{(p,q) \in N' : p \restriction H = f \text{ and } q \restriction [H]^2 = g\}, \quad \text{where} \\ J = &\{(f,g) : f : H \to \omega,\ g \text{ is a function}, \\ &\mathrm{dmn}(g) \subseteq [H]^2, \ \text{ and } \mathrm{rng}(g) \subseteq \omega\}. \end{aligned}$$

Since J is countable, let $(f,g) \in J$ be such that $N'' \stackrel{\text{def}}{=} \{(p,q) \in N' : p \restriction H = f$ and $q \restriction [H]^2 = g\}$ is uncountable. Now we claim that any two members (p_1,q_1) and (p_2,q_2) of N'' are compatible. Since $p_1 \restriction H = p_2 \restriction H$ and $\mathrm{dmn}(p_1) \cap \mathrm{dmn}(p_2) = H$, the relation $p_1 \cup p_2$ is a function. Say $n_{p_1} \le n_{p_2}$. We now define a function p_3 with domain $\mathrm{dmn}(p_1) \cup \mathrm{dmn}(p_2)$. Let $\alpha \in \mathrm{dmn}(p_1) \cup \mathrm{dmn}(p_2)$. Then we define $p_3(\alpha) : n_{p_2} \to 2$ by setting, for any $i < n_{p_2}$,

$$(p_3(\alpha))(i) = \begin{cases} (p_2(\alpha))(i) & \text{if } \alpha \in \mathrm{dmn}(p_2), \\ (p_1(\alpha))(i) & \text{if } \alpha \in \mathrm{dmn}(p_1)\backslash\mathrm{dmn}(p_2) \text{ and } i < n_{p_1}, \\ 0 & \text{otherwise.} \end{cases}$$

To check that $(p_3, q_1 \cup q_2) \in \mathbb{H}$, first note that (1) is clear. To show that $q_1 \cup q_2$ is a function, suppose that $\{\alpha,\beta\} \in \mathrm{dmn}(q_1) \cap \mathrm{dmn}(q_2)$. Then $\mathrm{dmn}(q_1) \cap \mathrm{dmn}(q_2) \subseteq [\mathrm{dmn}(p_1)]^2 \cap [\mathrm{dmn}(p_2)]^2 = [H]^2$, and it follows that $q_1(\{\alpha,\beta\}) = q_2(\{\alpha,\beta\})$. Thus $q_1 \cup q_2$ is a function. Furthermore,

$$\begin{aligned} \mathrm{dmn}(q_1 \cup q_2) &= \mathrm{dmn}(q_1) \cup \mathrm{dmn}(q_2) \\ &\subseteq [\mathrm{dmn}(p_1)]^2 \cup [\mathrm{dmn}(p_2)]^2 \\ &\subseteq [\mathrm{dmn}(p_1) \cup \mathrm{dmn}(p_2)]^2 \\ &= [\mathrm{dmn}(p_3)]^2. \end{aligned}$$

The range of $q_1 \cup q_2$ is clearly contained in n_{p_2}. So we have checked (2). For (3), suppose that $\{\alpha,\beta\} \in \mathrm{dmn}(q_1 \cup q_2)$, $(q_1 \cup q_2)(\{\alpha,\beta\}) = m$, and $m \le i < n_{p_2}$. We consider some cases:

Case 1. $\{\alpha,\beta\} \in \mathrm{dmn}(q_2)$. Then $\alpha,\beta \in \mathrm{dmn}(p_2)$, so $p_3(\alpha) = p_2(\alpha)$ and $p_3(\beta) = p_2(\beta)$. Hence $(p_3(\alpha))(i) = 0$ or $(p_3(\beta))(i) = 0$, as desired.

Case 2. $\{\alpha,\beta\} \in \mathrm{dmn}(q_1)\backslash\mathrm{dmn}(q_2)$ and $i < n_{p_1}$. Thus $\alpha,\beta \in \mathrm{dmn}(p_1)$. If $\alpha \in \mathrm{dmn}(p_2)$, then $p_1(\alpha) = p_2(\alpha)$, and so $(p_3(\alpha))(i) = (p_1(\alpha))(i)$. If $\alpha \notin \mathrm{dmn}(p_2)$, still $(p_3(\alpha))(i) = (p_1(\alpha))(i)$. Similarly for β, so the desired conclusion follows.

Case 3. $\{\alpha,\beta\} \in \mathrm{dmn}(q_1)\backslash\mathrm{dmn}(q_2)$ and $n_{p_1} \le i$. Thus again $\alpha,\beta \in \mathrm{dmn}(p_1)$. If one of α,β is not in $\mathrm{dmn}(p_2)$, it follows that one of $(p_3(\alpha))(i)$ or $(p_3(\beta))(i)$ is 0, as desired. Suppose that both are in $\mathrm{dmn}(p_2)$. Then $\{\alpha,\beta\} \subseteq \mathrm{dmn}(p_1) \cap \mathrm{dmn}(p_2) = H$, and hence $\{\alpha,\beta\} \in \mathrm{dmn}(q_2)$, contradiction. $\square$

Theorem 27.33. *Let M be a c.t.m. of ZFC, and consider $\mathbb{H}$ in M. Let G be $\mathbb{H}$-generic over M. Then cofinalities and cardinals are preserved in $M[G]$, and in $M[G]$ there is a MAD family of size ω_1.*

Proof. Cofinalities and cardinals are preserved by 27.32. For each $\alpha < \omega_1$, let

$$x_\alpha = \bigcup\{p(\alpha) : (p,q) \in G \text{ for some } q, \text{ and } \alpha \in \mathrm{dmn}(p)\}.$$

We claim that x_α is a function. For, suppose that $(a,b),(a,c) \in x_\alpha$. By the definition, choose $(p_1,q_1),(p_2,q_2) \in G$ such that $\alpha \in \mathrm{dmn}(p_1)$, $\alpha \in \mathrm{dmn}(p_2)$, $(a,b) \in p_1(\alpha)$, and $(a,c) \in p_2(\alpha)$. Then choose $(p_3,q_3) \in G$ such that $(p_3,q_3) \le (p_1,q_1),(p_2,q_2)$. By (4) in the definition of $\mathbb{H}$ we have $\alpha \in \mathrm{dmn}(p_3)$, and by (5) we have $(a,b),(a,c) \in p_3(\alpha)$, so $a = c$.

Next we claim that in fact x_α has domain ω. (Its domain is clearly a subset of ω.) For, take any $m \in \omega$. It suffices to show that the set

$$D_{\alpha m} \stackrel{\text{def}}{=} \{(p,q) \in \mathbb{H} : \alpha \in \mathrm{dmn}(p) \text{ and } m \in \mathrm{dmn}(p(\alpha))\}$$

is dense. So, suppose that $(r,s) \in \mathbb{H}$. If $\alpha \in \mathrm{dmn}(r)$, let $t = r$. Suppose that $\alpha \notin \mathrm{dmn}(r)$. Extend r to t by adding the ordered pair $(\alpha, \langle 0 : i < n_r\rangle)$. Clearly $(t,s) \in \mathbb{H}$ and $(t,s) \le (r,s)$. If $m < n_t$, then $(t,s) \in D_{\alpha m}$, as desired. Suppose that $n_t \le m$. We now define

$$\begin{aligned} p = \{(\beta,g) : \beta \in \mathrm{dmn}(t),\ g \in {}^{m+1}2,\ t(\beta) \subseteq g, \text{ and}\\ g(i) = 0 \text{ for all } i \in [n_t,m]\}.\end{aligned}$$

Clearly $(p,s) \in \mathscr{H}$, in fact $(p,s) \in D_{\alpha m}$, and $(p,s) \le (t,s) \le (r,s)$, as desired.

So $D_{\alpha m}$ is dense, and hence each x_α is a function mapping ω into 2. We define $a_\alpha = \{m \in \omega : x_\alpha(m) = 1\}$. We claim that $\langle a_\alpha : \alpha < \omega_1$ is our desired MAD family.

Now we show that each a_α is infinite. For each $m \in \omega$ let

$$\begin{aligned} E_m = \{(p,q) \in \mathbb{H} : \alpha \in \mathrm{dmn}(p),\ m < n_p, \text{ and there is}\\ \text{an } i \in [m,n_p) \text{ such that } (p(\alpha))(i) = 1\}.\end{aligned}$$

Clearly in order to show that a_α is infinite it suffices to show that each set E_m is dense. So, suppose that $(r,s) \in \mathbb{H}$. First choose $(t,u) \le (r,s)$ with $(t,u) \in D_{\alpha 0}$. This is done just to make sure that α is in the domain of t. Let k be the maximum of $n_t + 1$ and $m + 1$. Define the function p as follows. $\mathrm{dmn}(p) = \mathrm{dmn}(t)$. For any $\gamma \in \mathrm{dmn}(t)$ and any $i < k$, let

$$(p(\gamma))(i) = \begin{cases} (t(\gamma))(i) & \text{if } i < n_t,\\ 0 & \text{if } n_t \le i \text{ and } \gamma \ne \alpha,\\ 1 & \text{if } n_t \le i \text{ and } \gamma = \alpha.\end{cases}$$

It is easy to check that $(p,u) \in \mathbb{H}$, in fact $(p,u) \in E_m$, and $(p,u) \le (r,s)$, as desired. So each a_α is infinite.

Next we show that distinct a_α, a_β are almost disjoint. Suppose that $\alpha, \beta < \omega_1$ with $\alpha \neq \beta$. Since $D_{\alpha 0}$ and $D_{\beta 0}$ are dense, there are $(p_1, q_1), (p_2, q_2) \in G$ with $\alpha \in \mathrm{dmn}(p_1)$ and $\beta \in \mathrm{dmn}(p_2)$. Choose $(p_3, q_3) \in G$ such that $(p_3, q_3) \leq (p_1, q_1), (p_2, q_2)$. Thus $\alpha, \beta \in \mathrm{dmn}(p_3)$. Next we claim:

$$F \stackrel{\text{def}}{=} \{(r, s) : \{\alpha, \beta\} \in \mathrm{dmn}(s)\}$$

is dense below (p_3, q_3). In fact, suppose that $(t, u) \leq (p_3, q_3)$. We may assume that $\{\alpha, \beta\} \notin \mathrm{dmn}(u)$. Let $\mathrm{dmn}(r) = \mathrm{dmn}(t)$, and for any $\gamma \in \mathrm{dmn}(r)$ let $r(\gamma)$ be the function with domain $n_t + 1$ such that $t(\gamma) \subseteq r(\gamma)$ and $(r(\gamma))(n_t) = 0$. Let $\mathrm{dmn}(s) = \mathrm{dmn}(u) \cup \{\{\alpha, \beta\}\}$, with $u \subseteq s$ and $s(\{\alpha, \beta\}) = n_t$. It is easily checked that $(r, s) \in \mathbb{H}$, in fact $(r, s) \in F$, and $(r, s) \leq (t, u)$. So, as claimed, F is dense below (p_3, q_3). Choose $(p_4, q_4) \in F \cap G$.

We claim that $a_\alpha \cap a_\beta \subseteq q_4(\{\alpha, \beta\})$. To prove this, assume that $m \in a_\alpha \cap a_\beta$, but suppose that $q_4(\{\alpha, \beta\}) \leq m$. Thus $x_\alpha(m) = 1 = x_\beta(m)$, so there are $(e, b), (c, d) \in G$ such that $\alpha \in \mathrm{dmn}(e)$, $m \in \mathrm{dmn}(e(\alpha))$, $(e(\alpha))(m) = 1$, and $\beta \in \mathrm{dmn}(c)$, $m \in \mathrm{dmn}(c(\beta))$, and $(c(\beta))(m) = 1$. Choose $(p_5, q_5) \in G$ with $(p_5, q_5) \leq (p_4, q_4), (a, b), (c, d)$. Then $\{\alpha, \beta\} \in \mathrm{dmn}(q_5)$, $q_5(\{\alpha, \beta\}) = q_4(\{\alpha, \beta\}) \leq m < n_{p_5}$, $(p_5(\alpha))(m) = (e(\alpha))(m) = 1$, and $(p_5(\beta))(m) = (c(\beta))(m) = 1$, contradiction. So we have shown that $\langle a_\alpha : \alpha < \omega_1 \rangle$ is an almost disjoint family.

To show that $\langle a_\alpha : \alpha < \omega_1 \rangle$ is MAD, suppose to the contrary that b is an infinite subset of ω such that $b \cap a_\alpha$ is finite for all $\alpha < \omega_1$. Let σ be a name such that $\sigma_G = b$. For each $\alpha < \omega_1$ let

$$\begin{aligned} \tau_\alpha = \{(\check{i}, (p, q)) : i \in \omega,\ (p, q) \in \mathbb{H},\ \alpha \in \mathrm{dmn}(p), \\ i \in \mathrm{dmn}(p(\alpha)),\ \text{and } (p(\alpha))(i) = 1\}. \end{aligned}$$

Cleary $\tau_{\alpha G} = a_\alpha$. For each $n \in \omega$ let $\mathscr{A}_n$ be maximal subject to the following conditions:

(1) $\mathscr{A}_n$ is a collection of pairwise incompatible members of $\mathbb{H}$.

(2) For each $(p, q) \in \mathscr{A}_n$, $(p, q) \Vdash \check{n} \in \sigma$ or $(p, q) \Vdash \check{n} \notin \sigma$.

Then

(3) $\mathscr{A}_n$ is maximal pairwise incompatible.

In fact, suppose that $(r, s) \perp (p, q)$ for all $(p, q) \in \mathscr{A}_n$. Now $(r, s) \Vdash \check{n} \in \sigma \vee \check{n} \notin \sigma$, so there is a $(t, u) \leq (r, s)$ such that $(t, u) \Vdash \check{n} \in \sigma$ or $(t, u) \Vdash \check{n} \notin \sigma$. Then $\mathscr{A}_n \cup \{(t, u)\}$ still satisfies (1) and (2), and $(t, u) \notin \mathscr{A}_n$, contradiction.

Now choose

$$\alpha \in \omega_1 \backslash \bigcup_{\substack{n \in \omega, \\ (p,q) \in \mathscr{A}_n}} \mathrm{dmn}(p).$$

Let $m \in \omega$ be such that $b \cap a_\alpha \subseteq m$. Choose $(p_1, q_1) \in G$ such that $(p_1, q_1) \Vdash \sigma \cap \tau_\alpha \subseteq \check{m}$. Using $D_{\alpha 0}$, we may assume that $\alpha \in \mathrm{dmn}(p_1)$. Choose $n \in b$ with $n > m$ and $n \geq n_{p_1}$. Then take $(p_2, q_2) \in G \cap \mathscr{A}_n$. Then $(p_2, q_2) \Vdash \check{n} \in \sigma$, since $n \in b = \sigma_G$. Choose $(p_3, q_3) \in G$ with $(p_3, q_3) \leq (p_1, q_1), (p_2, q_2)$. Then by 27.32 we have $(p_3, q_1 \cup q_2) \in \mathbb{H}$ and $(p_3, q_1 \cup q_2) \leq (p_1, q_1), (p_2, q_2)$.

Now choose $k > \max(n_{p_3}, n)$, and define p_4 as follows. The domain of p_4 is $\operatorname{dmn}(p_3)$. For each $\beta \in \operatorname{dmn}(p_3)$ we define $p_4(\beta) : k \to 2$ by setting, for each $i < k$,

$$(p_4(\beta))(i) = \begin{cases} (p_3(\beta))(i) & \text{if } i < n_{p_3}, \\ 0 & \text{if } n_{p_3} \le i \text{ and } \beta \ne \alpha, \\ 0 & \text{if } n_{p_3} \le i,\ \beta = \alpha, \text{ and } i \ne n, \\ 1 & \text{if } \beta = \alpha \text{ and } i = n. \end{cases}$$

We check that $(p_4, q_1 \cup q_2) \in \mathbb{H}$. Conditions (1) and (2) are clear. For (3), suppose that $\{\beta, \gamma\} \in \operatorname{dmn}(q_1 \cup q_2)$, and $(q_1 \cup q_2)(\{\beta, \gamma\}) \le i < n_{p_4}$. Remember that n_{p_4} is k. If $i < n_{p_3}$, then the desired conclusion follows since $(p_3, q_1 \cup q_2) \in \mathbb{H}$. If $n_{p_3} \le i$, then the desired conclusion follows since at least one of β, γ is different from α. Hence, indeed, $(p_4, q_1 \cup q_2) \in \mathbb{H}$.

Clearly $(p_4, q_1 \cup q_2) \le (p_2, q_2)$, and so $(p_4, q_1 \cup q_2) \Vdash \check{n} \in \sigma$. It is also clear that $(p_4, q_1 \cup q_2) \le (p_1, q_1)$, so $(p_4, q_1 \cup q_2) \Vdash \sigma \cap \tau_\alpha \subseteq \check{m}$. Since $m < n$, it follows that $(p_4, q_1 \cup q_2) \Vdash \check{n} \notin \tau_\alpha$. But $(\check{n}, (p_4, q_1 \cup q_2))$ is clearly a member of τ_α, and hence $(p_4, q_1 \cup q_2) \Vdash \check{n} \in \tau_\alpha$, contradiction. $\square$

Miller, A. [03] *A MAD Q-set.* Fund. Math. 178 (2003), 271–281.

Collapsing to ω_1

Theorem 27.34. *Let M be a c.t.m. of ZFC, and in M let λ be an infinite cardinal, and set $\kappa = \lambda^+$ in M. Let P be $\mathrm{Fn}(\omega, \lambda, \omega)$, and let G be P-generic over M. Then cardinals $\ge \kappa$ are preserved in going to $M[G]$, but $\omega_1^{M[G]} = \kappa$.*

Thus we may say that all cardinals μ such that $\omega < \mu < \kappa$ become countable ordinals in $M[G]$.

Proof. Let $g = \bigcup G$. Clearly g is a function with domain contained in ω and range contained in λ. We claim that actually its domain is ω and its range is λ. For, let $m \in \omega$ and $\alpha \in \lambda$. Let

$$D = \{f \in P : m \in \operatorname{dmn}(f) \text{ and } \alpha \in \operatorname{rng}(f)\}.$$

Clearly D is dense. Hence $m \in \operatorname{dmn}(g)$ and $\alpha \in \operatorname{rng}(g)$, as desired. It follows that in $M[G]$, $|\lambda| = \omega$, and so the same is true for every ordinal α such that $\omega \le \alpha \le \lambda$.

Now we can finish the proof by showing in M that P has the κ-cc. Let $X \subseteq P$ with $|X| = \kappa$. Then $\langle \operatorname{dmn}(f) : f \in X \rangle$ is a system of κ many finite sets, so by the Δ-system lemma 10.1 with κ, λ replaced by ω, κ, there is a $N \in [X]^\kappa$ such that $\langle \operatorname{dmn}(f) : f \in M \rangle$ is a Δ-system, say with root r. Since $|{}^r\lambda| \le \lambda < \kappa$, there are two members f, g of M such that $f \restriction r = g \restriction r$. So f and g are compatible, as desired. $\square$

We now want to do the same thing for regular limit cardinals κ. We introduce the *Lévy collapsing order*:

$$\begin{aligned} \mathrm{Lv}_\kappa = \{p : p &\text{ is a finite function, } \operatorname{dmn}(p) \subseteq \kappa \times \omega, \text{ and} \\ &\text{for all } (\alpha, n) \in \operatorname{dmn}(p),\ p(\alpha, n) \in \alpha\}. \end{aligned}$$

Again this set is ordered by $\supseteq$.

Lemma 27.35. *For κ regular uncountable,* Lv_κ *has the κ-cc.*

Proof. Very similar to part of the proof of 27.34. $\square$

Theorem 27.36. *Let M be a c.t.m. of ZFC, and suppose that in M κ is regular and uncountable. Let G be* Lv_κ*-generic over M. Then cardinals $\geq \kappa$ are preserved in $M[G]$, and $\omega_1^{M[G]} = \kappa$.*

Proof. Cardinals $\geq \kappa$ are preserved by 27.35. Suppose that $0 < \alpha < \kappa$; we will find a function mapping ω onto α in $M[G]$. Let $g = \bigcup G$. Clearly G is a function. Now for each $\alpha < \kappa$, and $m \in \omega$ let

$$D_{\alpha m} = \{p \in \mathrm{Lv}_\kappa : (\alpha, m) \in \mathrm{dmn}(p)\}.$$

Clearly $D_{\alpha m}$ is dense, so $(\alpha, m) \in \mathrm{dmn}(g)$. Thus $\mathrm{dmn}(g) = \kappa \times \omega$. Now suppose that $\alpha < \kappa$ and $\xi < \alpha$. Let

$$E_{\alpha\xi} = \{p \in \mathrm{Lv}_\kappa : \text{there is an } m \in \omega \text{ such that } (\alpha, m) \in \mathrm{dmn}(p) \text{ and } p(\alpha, m) = \xi\}.$$

We claim that $E_{\alpha\xi}$ is dense. For, suppose that $\alpha < \kappa$ and $\xi < \alpha$. Take any $q \in \mathrm{Lv}_\kappa$. Choose $m \in \omega$ such that $(\alpha, m) \notin \mathrm{dmn}(q)$, and let $p = q \cup \{((\alpha, m), \xi)\}$. Clearly $p \in E_{\alpha\xi}$, as desired.

It follows that $\langle g(\alpha, m) : m \in \omega\rangle$ maps ω onto α. $\square$

28. Proper forcing

The notion of proper forcing is defined in terms of clubs and stationary sets of the form $[\lambda]^{<\omega_1}$, where λ is an uncountable cardinal. We write $[\lambda]^{\leq\omega}$ in place of $[\lambda]^{<\omega_1}$.

Let A be uncountable. Let $\mathbf{K}_A$ be the collection of all algebras with universe A and countably many operations. We allow 0-ary operations, i.e., elements of the universe. For each $M \in \mathbf{K}_A$ let $Sm(M)$ be the set of all countable subuniverses of M. Clearly $Sm(M)$ is a club of $[A]^{\leq\omega}$.

Let $\mathcal{D}(A)$ be the collection of all subsets of $[A]^{\leq\omega}$ which include a club. Thus for A uncountable $\mathcal{D}(A)$ is a filter on $[A]^{\leq\omega}$ which is countably complete, by Theorem 19.22.

Theorem 28.1. *For A uncountable, for every club W of $[A]^{\leq\omega}$ there is an algebra $M \in \mathbf{K}_A$ such that $Sm(M) \subseteq W$.*

Proof. We define a function $s : {}^{<\omega}A \to W$. Let $s(\emptyset)$ be any member of W. Suppose that $s(a) \in W$ has been defined for any $a \in {}^mA$, where $m \in \omega$. Take any $a \in {}^{m+1}A$. Then we let $s(a)$ be any member of W containing the set $s(a \restriction m) \cup \{a_m\}$. This is possible since W is unbounded. Note that $\text{rng}(a) \subseteq s(a)$ for any $a \in {}^{<\omega}A$. Now for any positive m and any $a \in {}^mA$, let x_a be a function mapping ω onto $s(a)$. We now define for each positive integer m and each $i \in \omega$ an m-ary operation F^m_i on A by setting $F^m_i(a) = x_a(i)$. Let $M = (A, F^m_i)_{m,i\in\omega,m>0}$. We claim that $Sm(M) \subseteq W$.

To prove this, let $C \in Sm(M)$. Write $C = \{a_i : i \in \omega\}$. For each positive integer m let $t_m = s(a_0, \ldots, a_{m-1})$. Now by construction, $m < n$ implies that $t_m \subseteq t_n$. Moreover, t_m is the range of $x_{\langle a_0,\ldots,a_{m-1}\rangle}$, which is $\{F^m_i(a_0, \ldots, a_{m-1}) : i \in \omega\}$. Thus $t_m \subseteq C$. It follows that $C = \bigcup_{m>0} t_m \in W$. □

Theorem 28.2. *Suppose that P is a ccc forcing order in a c.t.m. M. Let G be P-generic over M. Let λ be an uncountable cardinal in the sense of M, and let C be club in $[\lambda]^{\leq\omega}$ in the sense of $M[G]$. Then C includes a club of $[\lambda]^{\leq\omega}$ in the sense of M.*

Proof. In $M[G]$ let $N = (\lambda, F^n_i)_{i,n\in\omega,0<n}$ be an algebra in $M[G]$ such that $Sm(N) \subseteq C$. Fix $n > 0$ and $a \stackrel{\text{def}}{=} \langle a_0, \ldots, a_{n-1}\rangle \in {}^n\lambda$. Define $f^{na} : \omega \to \lambda$ by setting $f^{na}(i) = F^n_i(a_0, \ldots, a_{n-1})$ for all $i \in \omega$. Thus $f^{na} : \omega \to \lambda$ and $f^{na} \in M[G]$. By Theorem 16.4 let $g^{na} : \omega \to \mathscr{P}(\lambda)$ be such that $\forall i \in \omega[f^{na}(i) \in g^{na}(i)]$ and $\forall i \in \omega[|g^{na}(i)| \leq \omega]$, with $g^{na} \in M$. In M, for each $i \in \omega$ let $h^{na}_i : \omega \to g^{na}(i)$ be a surjection. Now we define $H^n_{ik}(a_0, \ldots, a_{n-1}) = h^{na}_i(k)$ for all $i, k \in \omega$. Let $P = (\lambda, H^n_{ik})_{i,k,n<\omega,n>0}$. We claim that $Sm(P) \subseteq Sm(N)$. For, let $s \in Sm(P)$. Suppose that $n, i \in \omega$ with $n > 0$ and $a \stackrel{\text{def}}{=} \langle a_0, \ldots, a_{n-1}\rangle \in {}^ns$. Then $F^n_i(a_0, \ldots, a_{n-1}) = f^{na}(i) \in g^{na}(i)$, so there is a k such that $h^{na}_i(k) = f^{na}(i)$. Hence $F^n_i(a_0, \ldots, a_{n-1}) = f^{na}(i) = h^{na}_i(k) = H^n_{ik}(a_0, \ldots, a_{n-1}) \in s$. Thus $s \in Sm(N)$. Now since $Sm(N) \subseteq C$ we have $Sm(P) \subseteq C$. □

Corollary 28.3. *Suppose that P is a ccc forcing order in a c.t.m. M. Let G be P-generic over M. Let λ be an uncountable cardinal in the sense of M, and let S be a stationary subset of $[\lambda]^{\leq\omega}$ in M. Then S is also stationary in $M[G]$.* □

Definition. A forcing order P is *proper* iff for every uncountable cardinal λ, forcing with P preserves stationarity in $[\lambda]^{\leq\omega}$.

Corollary 28.4. *Every ccc forcing order is proper.* □

Lemma 28.5. *Assume that* $\mathbb{P}$ *is proper,* κ *is uncountable, and* $p \Vdash [\dot{A} \in [\kappa]^{\leq\omega}]$*. Then there exist a* $B \in [\kappa]^{\leq\omega}$ *and a* $q \leq p$ *such that* $q \Vdash [\dot{A} \subseteq \check{B}]$*.*

Proof. Obviously $([\kappa]^{\leq\omega})^M$ is club in $[\kappa]^{\leq\omega}$ and hence is also stationary in M. Let G be $\mathbb{P}$-generic over M with $p \in G$. Then $([\kappa]^{\leq\omega})^M$ is stationary in $M[G]$. Now obviously $\{X \in [\kappa]^{\leq\omega} : \dot{A}_G \subseteq X\}$ is club. Hence there is a $B \in ([\kappa]^{\leq\omega})^M$ such that $\dot{A}_G \subseteq B$. So there is a $q \leq p$ such that $q \Vdash [\dot{A} \subseteq \check{B}]$. □

Corollary 28.6. *Assume that* $\mathbb{P}$ *is proper, and* G *is* $\mathbb{P}$*-generic over* M*. Then* $\omega_1^M = \omega_1^{M[G]}$*.*

Proof. Suppose not. Then there is a $p \in G$ such that $p \Vdash \check{\omega}_1 \in [\omega_1]^{\leq\omega}$. Hence by Lemma 31.164 there exist a $B \in [\omega_1]^{\leq\omega}$ and a $q \leq p$ such that $q \Vdash [\check{\omega}_1 \subseteq \check{B}]$. Let $q \in H$ generic. Then $\omega_1 = \check{\omega}_{1G} \subseteq B$, contradiction. □

Proposition 28.7. *If* $\mathbb{P}$ *is proper and* $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} [\dot{\mathbb{Q}}$ *is proper*]*, then* $\mathbb{P} * \dot{\mathbb{Q}}$ *is proper.*

Proof. Assume the hypothesis, and suppose that κ is uncountable and $S \subseteq [\kappa]^{\leq\omega}$ is stationary. Let K be $(\mathbb{P} * \dot{\mathbb{Q}})$-generic over M. Form G and H as in Theorem 26.8. Then S is stationary in $M[G]$. Also, $\dot{\mathbb{Q}}_G$ is proper, so S is stationary in $M[G][H] = M[K]$. □

Definition. For $E \subseteq \mathbb{P}$, $p \perp E$ means $\forall q \in E[p \perp q]$. We say that E is *predense below* p, in symbols $p \leq \bigvee E$, iff $\forall q \leq p \exists r \in E[q$ and r are compatible].

Lemma 28.8. $q \perp E$ *iff* $q \Vdash [\check{E} \cap \Gamma = \emptyset]$*.*

Proof. $\Rightarrow$: Assume that $q \perp E$ and $q \in G$ generic. Suppose that $p \in E \cap G$. Then p and q are compatible, contradiction.

$\Leftarrow$: Suppose that $q \not\perp E$. Say $p \in E$ with p, q compatible. Say $r \leq p, q$. Let $r \in G$, generic. Then $p \in E \cap G$. Thus $E \cap G \neq \emptyset$. Hence $q \not\Vdash [\check{E} \cap \Gamma = \emptyset]$. □

Lemma 28.9. $p \leq \bigvee E$ *iff* $p \Vdash [\check{E} \cap \Gamma \neq \emptyset]$*.*

Proof. $\Rightarrow$: Assume that $p \leq \bigvee E$. Thus $\forall q \leq p[q \not\perp E]$, so by Lemma 28.8,

(1) $\forall q \leq p[q \not\Vdash [\check{E} \cap \Gamma = \emptyset]]$.

Now suppose that $p \not\Vdash [\check{E} \cap \Gamma \neq \emptyset]$. Then there is a generic G with $p \in G$ such that $E \cap G = \emptyset$. Hence there is a $q \in G$ such that $q \Vdash [\check{E} \cap \Gamma = \emptyset]$. Say $r \leq p, q$. Then $r \Vdash [\check{E} \cap \Gamma = \emptyset]$. This contradicts (1).

$\Leftarrow$: Assume that $p \Vdash [\check{E} \cap \Gamma \neq \emptyset]$. Suppose that $p \not\leq \bigvee E$. Then there is a $q \leq p$ such that $\forall r \in E[q \perp r]$; that is, such that $q \perp E$. By Lemma 28.8, $q \Vdash [\check{E} \cap \Gamma = \emptyset]$. But $q \leq p$ so $q \Vdash [\check{E} \cap \Gamma \neq \emptyset]$, contradiction. □

Definition. If $\mathbb{P} \in M$, then $p \in \mathbb{P}$ is $(M, \mathbb{P})$-*generic* iff for all dense $D \subseteq \mathbb{P}$ such that $D \in M$ we have $p \leq \bigvee(D \cap M)$.

For any $p \in \mathbb{P}$ we let $p{\downarrow}' = \{q : q \leq p\}$. A subset $X \subseteq \mathbb{P}$ is *open* iff $\forall p \in X[p{\downarrow}' \subseteq X]$.

For $D \subseteq \mathbb{P}$, $D{\downarrow} = \{p \in D : p{\downarrow} \subseteq D\}$

Proposition 28.10. *If θ is uncountable and regular, $\mathbb{P} \in M \preceq H(\theta)$ and $p \in \mathbb{P}$, then the following conditions are equivalent:*

(i) p is $(M, \mathbb{P})$-generic.

(ii) For all open dense $D \subseteq \mathbb{P}$ such that $D \in M$ we have $p \le (D \cap M)$.

(iii) For all predense $D \subseteq \mathbb{P}$ such that $D \in M$ we have $p \le (D \cap M)$.

Proof. First we claim:

(1) If $D \in M$, then $(D{\downarrow}') \in M$.

In fact, suppose that $D \in M$. Then

$$H(\theta) \models \exists X \forall x[x \in X \leftrightarrow \exists y \in D[x \le y]]; \quad \text{hence}$$
$$M \models \exists X \forall x[x \in X \leftrightarrow \exists y \in D[x \le y]];$$

taking $X \in M$ such that $M \models \forall x[x \in X \leftrightarrow \exists y \in D[x \le y]]$, we have $H(\theta) \models \forall x[x \in X \leftrightarrow \exists y \in D[x \le y]]$, so $X = (D{\downarrow}')$. So (1) holds.

Now obviously (i)⇒(ii). (ii)⇒(iii): Assume (ii), and suppose that $D \subseteq \mathbb{P}$ is predense and $D \in M$. Clearly $D{\downarrow}'$ is open dense, and by (1) $(D{\downarrow}') \in M$. Hence by (ii), $p \le \bigvee((D{\downarrow}') \cap M)$. Thus $\forall q \le p \exists r \in (D{\downarrow}') \cap M[q$ and r are compatible$]$. Suppose that $q \le p$, and choose $r \in (D{\downarrow}') \cap M$ so that q and r are compatible. Choose $s \in D$ such that $r \le s$. Then q and s are compatible. This shows that $p \le \bigvee(D \cap M)$, as desired in (iii).

(iii)⇒(i): similarly. □

Proposition 28.11. *If $D \subseteq \mathbb{P}$ is dense, then there is an $A \subseteq D$ such that A is a maximal antichain.*

Proof. Let $A \subseteq D$ be maximal such that it is an antichain. We claim that A is a maximal antichain in $\mathbb{P}$. For, suppose not; then there is a $b \in \mathbb{P}$ incompatible with each member of A. Choose $d \le b$ with $d \in D$. Then d is incompatible with each member of A; so $A \cup \{d\}$ is a subset of D which is an antichain, and $d \notin A$ since otherwise b would be incompatible with d. This contradiction proves the proposition. □

Proposition 28.12. *If θ is uncountable and regular, $\mathbb{P} \in M \preceq H(\theta)$ and $p \in \mathbb{P}$, then the following conditions are equivalent:*

(i) p is $(M, \mathbb{P})$-generic.

(ii) For every maximal antichain $D \subseteq \mathbb{P}$ such that $D \in M$ we have $p \le (D \cap M)$.

Proof. (i)⇒(ii): Assume (i), and suppose that $D \subseteq \mathbb{P}$ is a maximal antichain such that $D \in M$. Then $(D{\downarrow}') \in M$ by (1) in the proof of Proposition 28.10. We claim that $D{\downarrow}'$ is dense. For, let $q \in \mathbb{P}$. Choose $r \in D$ such that q and r are compatible; say $s \le q, r$. Thus $s \in (D{\downarrow}')$ and $s \le q$, as desired. It follows that $p \le ((D{\downarrow}') \cap M)$. As in the proof of 28.10 this shows that $p \le (D \cap M)$.

(ii)⇒(i): Assume (ii), and suppose that $D \subseteq M$ with D dense. Then

$$H(\theta) \models \exists X[X \text{ is a maximal antichain and } X \subseteq D], \quad \text{hence}$$
$$M \models \exists X[X \text{ is a maximal antichain and } X \subseteq D].$$

Take $X \in M$ such that $M \models [X$ is a maximal antichain and $X \subseteq D]$. Then $H(\theta) \models [X$ is a maximal antichain and $X \subseteq D]$; so X is a maximal antichain and $X \subseteq D$. Hence by (ii), $p \le \bigvee(X \cap M)$. To show that $p \le \bigvee(D \cap M)$, take any $q \le p$. Since $p \le \bigvee(X \cap M)$, choose $r \in X$ so that q and r are compatible. Then $r \in D$, as desired. $\square$

Definition. Let A be an uncountable set and $f : [A]^{<\omega} \to [A]^{\le\omega}$. A set $x \in [A]^{\le\omega}$ is a *closure point* of f iff $f(e) \subseteq x$ for every $e \in [x]^{<\omega}$. $\mathrm{Cl}(f)$ is the collection of all closure points of f.

If A is uncountable, $f : [A]^{<\omega} \to [A]^{\le\omega}$, and $x \in [A]^{\le\omega}$, we define $y_0 = x$ and $y_{i+1} = y_i \cup \bigcup\{f(e) : e \in [y_i]^{<\omega}\}$, and $\mathrm{Cl}_f(x) = \bigcup_{i\in\omega} y_i$.

Theorem 28.13. *In M suppose that θ is uncountable and regular and $\mathbb{P} \in H(\theta)$.*

Then in any generic extension $M[G]$ there is a club $C \subseteq [(H(\theta))^M]^{\le\omega}$ such that for all $N \in C$ and $p \in \mathbb{P}$ the following conditions hold:

(i) $N \preceq (H(\theta))^M$ and $\mathbb{P} \in N$.

(ii) For all $D \in N$, if D is a dense subset of $\mathbb{P}$, then $D \cap N \cap G \neq \emptyset$.

(iii) If $N \in M$ and $p \in G$, then there is a $q \le p$ that is $(N, \mathbb{P})$-generic.

Proof. Let C be the set of all $N \in [(H(\theta))^M]^{\le\omega}$ satisfying (i) and (ii). Clearly C is closed. To show that it is unbounded, let $K \in [(H(\theta))^M]^{\le\omega}$. Let c be a choice funtion for nonempty subsets of $(H(\theta))^M$. Let $\langle \exists x_n \varphi_n(x_n, \overline{y}_n) : n \in \omega\rangle$ list all formulas in the language of set theory that begin with an existential quantifier. We define $Y_0 = K$ and

$$\begin{aligned} Y_{m+1} = Y_m &\cup \{a \in (H(\theta))^M : n \in \omega, \exists \overline{b} \subseteq Y_m[(H(\theta))^M \models \varphi_n(a, \overline{b})]\} \\ &\cup \{c(D \cap G) : D \subseteq \mathbb{P},\ D \text{ dense},\ D \in Y_m\}. \end{aligned}$$

Finally, let $N = \bigcup_{m\in\omega} Y_m$. Clearly N satisfies (i) and (ii).

For (iii), suppose that $N \in M$ (and $N \in C$), and $p \in G$. Then there is a $q \in G$ with $q \le p$ such that $q \Vdash \forall D[D \text{ dense in } \check{\mathbb{P}} \to \exists x[x \in \check{N} \cap D \cap \Gamma]]$. Thus for all $D \in N$ such that $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$ we have $q \Vdash [(D \cap N)^v \cap \Gamma \neq \emptyset]$. Hence by Lemma 28.5 $q \le \bigvee(D \cap N)$ for all $D \in M$ such that $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$. $\square$

Lemma 28.14. *If A is an uncountable set, $C \subseteq [A]^{\le\omega}$ is club, and $D \subseteq C$ is countable and directed, then $\bigcup D \in C$.*

Proof. Let $D = \{d_i : i < \omega\}$. For each $i < \omega$ let $e_i \in D$ be such that $d_j \subseteq e_i$ for each $j < i$ and also $e_j \subseteq e_i$ for each $j < i$. Then $\bigcup D = \bigcup_{i<\omega} e_i \in C$. $\square$

Lemma 28.15. *If A is uncountable, $f : [A]^{<\omega} \to [A]^{\le\omega}$ and $\emptyset \neq x \in [A]^{\le\omega}$, then:*

(i) $\mathrm{Cl}_f(x) \in \mathrm{Cl}(f)$;

(ii) If $z \subseteq x$, then $\mathrm{Cl}_f(z) \subseteq \mathrm{Cl}_f(x)$;

(iii) $\mathrm{Cl}_f(x) = \bigcup\{\mathrm{Cl}_f(z) : z \in [x]^{<\omega}\}$;

(iv) If $y \subseteq x \in \mathrm{Cl}(f)$, then $\mathrm{Cl}_f(y) \subseteq x$;

(v) If $x \in \mathrm{Cl}(f)$, then $\mathrm{Cl}_f(x) = x$.

Proof. (i): suppose that $e \in [\mathrm{Cl}_f(x)]^{<\omega}$. With $\langle y_i : i \in \omega\rangle$ as in the definition of $\mathrm{Cl}_f(x)$, there is an $i \in \omega$ such that $e \in [y_i]^{<\omega}$. Hence $f(e) \subseteq y_{i+1} \subseteq \mathrm{Cl}_f(x)$. This proves (i).

(ii): if $\langle y_i^z : i \in \omega\rangle$ is as in the definition of $\mathrm{Cl}_f(z)$ and $\langle y_i^x : i \in \omega\rangle$ is as in the definition of $\mathrm{Cl}_f(x)$, then by induction $y_i^z \subseteq y_i^x$ for all $i \in \omega$, and (ii) follows.

(iii): $\supseteq$ holds by (ii). For $\subseteq$, with $\langle y_i : i \in \omega\rangle$ as in the definition of $\mathrm{Cl}_f(x)$, we prove that $y_i \subseteq$ rhs for all $i \in \omega$ by induction on i, where rhs is the right-hand side of (iii). Since $z \subseteq \mathrm{Cl}_f(z)$ for each $z \in [x]^{<\omega}$ we have $y_0 = x \subseteq$ rhs. Now suppose that $y_i \subseteq$ rhs. If $e \in [y_i]^{<\omega}$, then there is a $z \in [x]^{<\omega}$ such that $e \in \mathrm{Cl}_f(z)$, by (ii). Then $f(e) \subseteq \mathrm{Cl}_f(z) \subseteq$ rhs. Hence $y_{i+1} \subseteq$ rhs. This proves (iii).

(iv): clear.

(v): $x \subseteq \mathrm{Cl}_f(x) \subseteq x$ by (iv). □

Lemma 28.16. *If A is an uncountable set and $f : [A]^{<\omega} \to [A]^{\leq\omega}$, then $\mathrm{Cl}(f)$ is a club of $[A]^{\leq\omega}$.*

Proof. Clearly $\mathrm{Cl}(f)$ is closed. If $x \in [A]^{\leq\omega}$, then $x \subseteq \mathrm{Cl}_f(x) \in \mathrm{Cl}(f)$ by Lemma 28.14, so $\mathrm{Cl}(f)$ is unbounded. □

Lemma 28.17. *If A is an uncountable set and C is club in $[A]^{\leq\omega}$, then there is an $f : [A]^{<\omega} \to [A]^{\leq\omega}$ such that $\mathrm{Cl}(f) \subseteq C$.*

Proof. We define $f(e) \in C$ for all $e \in [A]^{<\omega}$ by recursion on $|e|$. For each $a \in A$ choose $f(\{a\}) \in C$ with $\{a\} \subseteq f(\{a\})$. Now suppose that $f(e)$ has been defined for all $e \in [A]^n$, and $e \in [A]^{n+1}$. Choose $f(e) \in C$ such that $e \cup \bigcup_{a\in e} f(e\backslash\{a\}) \subseteq f(e)$. Note that $e_1 \subseteq e_2$ implies that $f(e_1) \subseteq f(e_2)$.

We claim that $\mathrm{Cl}(f) \subseteq C$. For, suppose that $x \in \mathrm{Cl}(f)$. Then $\{f(e) : e \in [x]^{<\omega}\}$ is directed and $x = \bigcup\{f(e) : e \in [x]^{<\omega}\}$, so $x \in C$. □

Lemma 28.18. *Suppose that θ is uncountable and regular, and λ is uncountable and $\lambda \in H(\theta)$. Suppose that $C \subseteq [H(\theta)]^{\leq\omega}$ is club. Then there is a club C' in $[\lambda]^{\leq\omega}$ such that $C' \subseteq \{x \cap \lambda : x \in C\}$.*

Proof. Let $f : [H(\theta)]^{<\omega} \to [H(\theta)]^{\leq\omega}$ be such that $\mathrm{Cl}(f) \subseteq C$, by Lemma 28.16. Define $g : [\lambda]^{<\omega} \to [\lambda]^{\leq\omega}$ by setting $g(e) = \mathrm{Cl}_f(e) \cap \lambda$ for all $e \in [\lambda]^{<\omega}$. We claim that $\mathrm{Cl}(g) \subseteq \{x \cap \lambda : x \in \mathrm{Cl}(f)\}$. For, suppose that $y \in \mathrm{Cl}(g)$. Then

$$\begin{aligned}\mathrm{Cl}_f(y) \cap \lambda = \bigcup\{\mathrm{Cl}_f(z) \cap \lambda : z \in [y]^{<\omega}\} &= \bigcup\{g(z) : z \in [y]^{<\omega}\}\\ &\subseteq \bigcup\{\mathrm{Cl}_g(z) : z \in [y]^{<\omega}\} = y,\end{aligned}$$

so $y = \mathrm{Cl}_f(y) \cap \lambda$. □

Theorem 28.19. *Suppose that $\mathbb{P}$ is a forcing order. Then the following are equivalent:*

(i) $\mathbb{P}$ is proper,

(ii) For every uncountable regular cardinal θ with $\mathbb{P} \in H(\theta)$ there is a club $C \subseteq [H(\theta)]^{\leq\omega}$ such that for all $N \in C$ the following conditions hold:

(a) $\mathbb{P} \in N$.

(b) $N \preceq H(\theta)$

(c) for all $p \in \mathbb{P} \cap N$ there is a $q \leq p$ which is (N, P)-generic.

(iii) There is an infinite cardinal ρ such that for every uncountable regular cardinal $\theta \geq \rho$ with $\mathbb{P} \in H(\theta)$ there is a club $C \subseteq [H(\theta)]^{\leq\omega}$ such that for all $N \in C$ the following conditions hold:

(a) $\mathbb{P} \in N$.

(b) $N \preceq H(\theta)$

(c) for all $p \in \mathbb{P} \cap N$ there is a $q \leq p$ which is (N, P)-generic.

Proof. (i)⇒(ii): Assume that $\mathbb{P}$ is proper and θ is a regular uncountable cardinal such that $\mathbb{P} \in H(\theta)$. Let

$$S = \{N \subseteq [H(\theta)]^{\leq\omega} : \mathbb{P} \in N,\ N \preceq H(\theta),\ \exists p \in \mathbb{P} \cap N \forall q \leq p[q \text{ is not } (N, \mathbb{P})\text{-generic}]\}.$$

We claim that S is not stationary. For, suppose that it is. For each $N \in S$ choose $p \in \mathbb{P} \cap N$ as indicated. Then by Fodor's theorem, Theorem 19.24, there exist a $p \in \mathbb{P}$ and a stationary subset T of S such that for all $N \in T$, $p \in \mathbb{P} \cap N$ and $\forall q \leq p[q$ is not $(N, \mathbb{P})$-generic]. Suppose that G is $\mathbb{P}$-generic over M with $p \in G$. In $M[G]$ let C be a club given by Lemma 28.13. Since $\mathbb{P}$ is proper, T is stationary in $M[G]$, and so we can choose $N \in T \cap C$. Thus $N \in M$. By Lemma 28.13(iii), there is a $q \leq p$ which is $(N, \mathbb{P})$-generic. This contradicts the definition of T. So S is not stationary.

It follows that there is a club C in $[H(\theta)]^{\leq\omega}$ such that $C \cap S = \emptyset$. Let $C' = \{N \in [H(\theta)]^{\leq\omega} : \mathbb{P} \in N$ and $N \preceq H(\theta)\}$. Then C' is club. Now $C \cap C'$ is as desired in the theorem.

(ii)⇒(iii): obvious.

(iii)⇒(i): Assume (iii). Let λ be an uncountable cardinal, and suppose that $S \subseteq [\lambda]^{\leq\omega}$ is stationary, G is generic, and S is not stationary in $M[G]$. Say $C \subseteq [\lambda]^{\leq\omega}$ is club with $S \cap C = \emptyset$. Let $\langle f_i^n : i \in \omega, n \in \omega\rangle$ be a system of members of $M[G]$ with $f_i^n : {}^n\lambda \to \lambda$ and, with $M = (\lambda, \langle f_i^n : i \in \omega, n \in \omega\rangle)$ we have $\mathrm{Sm}(M) \subseteq C$. (See Theorem 28.2.) Then there is a name σ and a $p \in G$ such that $\forall i \in \omega \forall n \in \omega[p \Vdash [\sigma_i^n : {}^n\lambda \to \lambda$ and $\mathrm{Sm}(\mathrm{op}(\lambda, \sigma)) \cap S = \emptyset]$. Let $\theta \geq \rho$ be an uncountable cardinal such that $\lambda \in H(\theta)$. By (iii) let $C' \subseteq [H(\theta)]^{\leq\omega}$ satisfy the indicated conditions. By Lemma 28.13, let C'' be a club in $[\lambda]^{\leq\omega}$ such that $C'' \subseteq \{N \cap \lambda : N \in C'\}$. Since S is stationary, choose $N \in C' \cap C''$ such that $N \cap \lambda \in C'' \cap C' \cap S$. Now $H(\theta) \models \exists p[p \Vdash [\sigma_i^n : {}^n\lambda \to \lambda$ and $\mathrm{Sm}(\mathrm{op}(\lambda, \sigma)) \cap S = \emptyset]]$, so there is a $p \in N$ and $\sigma_i^n \in N$ such that $p \Vdash [\sigma_i^n : {}^n\lambda \to \lambda$ and $\mathrm{Sm}(\mathrm{op}(\lambda, \sigma)) \cap S = \emptyset]$. Choose $q \leq p$ such that q is $(N, \mathbb{P})$-generic. We claim that $q \Vdash N \cap \lambda \in \mathrm{Sm}(\mathrm{op}(\lambda, \sigma))$. Let $e \in {}^n(N \cap \lambda)$; we show that $q \Vdash \sigma_i^n(e) \in N \cap \lambda$. Let

$$A = \{r : \exists \alpha < \lambda[r \Vdash \sigma_i^n(e) = \alpha]\} \cup \{r : r \perp q\}.$$

Then A is dense and $A \in N$. From q being $(N, \mathbb{P})$-generic it follows that for all $r \leq q$, r is compatible with some $s \in A \cap N$. Hence for any $r \leq q$ there exist s, t such that $t \leq r, s$ and $s \in A \cap N$. Therefore there is an $\alpha < \lambda$ such that $s \Vdash \sigma_i^n(e) = \alpha$. Since α is definable from s, σ_i^n, e, it follows that $\alpha \in N$.

Thus we have shown that $q \Vdash N \cap \lambda \in \mathrm{Sm}(\mathrm{op}(\lambda, \sigma))$, $q \Vdash \mathrm{Sm}(\mathrm{op}(\lambda, \sigma)) \cap S = \emptyset]$, and $N \cap \lambda \in S$, contradiction. ∎

Theorem 28.20. *Every countably closed poset is proper.*

Proof. We apply Theorem 28.19. Let θ be an uncountable regular cardinal such that $\mathbb{P} \in H(\theta)$. Let $C = \{N : N \text{ countable}, \mathbb{P} \in N \text{ and } N \preceq H(\theta)\}$. Suppose that $p \in \mathbb{P} \cap N$. Let $\langle D_n : n \in \omega\rangle$ list all of the dense subsets of $\mathbb{P}$ which are in N. Define $\langle q_n : n \in \omega\rangle$ by recursion: $q_0 = p$. If q_n has been defined, let $q_{n+1} \leq q_n$ with $q_{n+1} \in D_n$. Let $r \leq q_n$ for all n. Clearly r is $(N, \mathbb{P})$-generic. $\square$

Definition. If M is a c.t.m. and $N \in M$, for G a generic filter over M we let $N[G] = \{\sigma_G : \sigma$ is a $\mathbb{P}$-name and $\sigma \in N$.

Theorem 28.21. *For any forcing poset $\mathbb{P}$ the following conditions are equivalent:*

(i) $\mathbb{P}$ is proper.

(ii) For every regular $\theta > 2^{|\mathrm{trcl}(\mathbb{P})|}$, every countable $N \preceq H(\theta)$ with $\mathbb{P} \in N$, and every $p \in \mathbb{P} \cap N$, there is a $q \leq p$ such that q is $(N, \mathbb{P})$-generic.

Proof. (i)$\Rightarrow$(ii): Assume that $\mathbb{P}$ is proper. Let $\theta > 2^{|\mathrm{trcl}(\mathbb{P})|}$, θ regular, let $N \preceq H(\theta)$ with $\mathbb{P} \in N$, N countable, and let $r \in \mathbb{P} \cap N$.

By Theorem 28.19 let $C \subseteq [H(|\mathrm{trcl}(\mathbb{P})|^+)]^{\leq\omega}$ be club such that for all $Q \in C$ the following conditions hold:

(1) $\mathbb{P} \in Q$.

(2) $Q \preceq H(|\mathrm{trcl}(\mathbb{P})|^+)$

(3) for all $p \in \mathbb{P} \cap Q$ there is a $q \leq p$ which is $(Q, \mathbb{P})$-generic.

Now by Lemma 28.17 there is an $f : [H(|\mathrm{trcl}(\mathbb{P})|^+]^{<\omega} \to [H(|\mathrm{trcl}(\mathbb{P})|^+)]^{\leq\omega}$ such that $\mathrm{Cl}(f) \subseteq C$. Now note that if $(M, N) \in f$ then $M \in [H(|\mathrm{trcl}(\mathbb{P})|^+)]^{<\omega}$ and $N \in [H(|\mathrm{trcl}(\mathbb{P})|^+)]^{\leq\omega}$. So $|M| < |\mathrm{trcl}(\mathbb{P})|^+$ and each member of M is in $H(|\mathrm{trcl}(\mathbb{P})|^+)$ and hence has size $< |\mathrm{trcl}(\mathbb{P})|^+$. So $M \in H(|\mathrm{trcl}(\mathbb{P})|^+)$. Similarly, $N \in H(|\mathrm{trcl}(\mathbb{P})|^+)$. Now $|f| = |H(|\mathrm{trcl}(\mathbb{P})|^+)]^{<\omega}| = |H(|\mathbb{P}|^+)| = 2^{<|\mathrm{trcl}(\mathbb{P})|^+} = 2^{|\mathrm{trcl}(\mathbb{P})|} < \theta$. So $f \in H(\theta)$.

Now for all $Q \in \mathrm{Cl}(f)$ the conditions (1)–(3) hold. We may assume that f is the least function in $H(\theta)$ with this property. Then clearly N is closed under f, and so $N \cap H(\theta)$ is closed under f. So $N \cap H(\theta) \in \mathrm{Cl}(f)$. Thus by (3) for $N \cap H(\theta)$, since $p \in H(\theta)$ because $p \in \mathbb{P} \in H(\theta)$, it follows that there is a $q \leq p$ which is $(N \cap H(\theta), \mathbb{P})$-generic. We claim that q is $(N, \mathbb{P})$-generic. For, suppose that $D \subseteq \mathbb{P}$ is dense and $D \in N$. Since $\mathbb{P} \in H(\theta)$, we have $D \in N \cap H(\theta)$. Hence $q \leq \bigvee(D \cap N \cap H(\theta))$. Hence for all $r \leq q$ there is an $s \in D \cap N \cap H(\theta)$ such that r and s are compatible. So $q \leq \bigvee(D \cap N)$, as desired.

(ii)$\Rightarrow$(i): Assume (ii), and let C be the set of all countable $N \preceq H(\theta)$ such that $\mathbb{P} \in N$. Then Theorem 31.188(iii) holds, and so $\mathbb{P}$ is proper. $\square$

Lemma 28.22. *If $N \preceq H(\lambda)$, then $N[G] \preceq H(\lambda)^{M[G]}$.*

Proof. We apply Tarski's criterion. Suppose that $H(\lambda)^{M[G]} \models \exists x\varphi(x, y_1, \ldots, y_n)$ with each y_i in $N[G]$. Say $y_i = \tau^i_G$ with $\tau_i \in N$. Thus $H(\lambda)^{M[G]} \models \exists x\varphi(x, \tau^1_G, \ldots, \tau^n_G)$. Choose $p \in G$ such that $p \Vdash (\exists x\varphi(\sigma, \tau^1, \ldots, \tau^n))^{H(\lambda)}$. Then since $N \preceq H(\lambda)$, it follows that $p \Vdash (\exists x\varphi(\sigma, \tau^1, \ldots, \tau^n))^N$. Hence $N[G] \models \exists x\varphi(x, y_1, \ldots, y_n)$. $\square$

Proposition 28.23. *Let $\mathbb{P}$ be a forcing order, σ a $\mathbb{P}$-name, $p \in \mathbb{P}$, and $p \Vdash [\sigma$ is an ordinal$]$. Then the set $\{q \in \mathbb{P} : \exists\alpha[q \Vdash [\sigma = \check{\alpha}]]\}$ is dense below p.*

Proof. Assume the hypotheses. Suppose that $r \le p$. Let G be generic with $r \in G$. Then σ_G is an ordinal α. Thus there is an $s \in G$ such that $s \Vdash [\sigma = \check{\alpha}]$. Choose $q \in G$ with $q \le r, s$. Then q is in the set of the proposition. $\square$

Proposition 28.24. *Suppose that $N \preceq H(\lambda)$, $\mathbb{P}$ is a forcing order in N, and $p \in \mathbb{P}$. Then the following are equivalent:*

(i) p is $(N, \mathbb{P})$-generic.

(ii) If $D \subseteq \mathbb{P}$, D is dense, and $D \in N$, then for every generic G, $p \in G$ implies that $D \cap N \cap G \ne \emptyset$.

Proof. (i)$\Rightarrow$(ii): Assume (i), $D \subseteq \mathbb{P}$, D is dense, $D \in N$, and $p \in G$ generic. By (i), $D \cap N$ is predense below p, so $D \cap N \cap G \ne \emptyset$.

(ii)$\Rightarrow$(i): Assume (ii) and suppose that $D \subseteq \mathbb{P}$ is dense, $D \in N$. Take any $q \le p$. Let G be generic with $q \in G$. By (ii), choose $r \in D \cap N \cap G$. Thus q is compatible with a member of $D \cap N$, as desired. $\square$

Proposition 28.25. *Suppose that $N \preceq H(\lambda)$, $\mathbb{P}$ is a forcing order in N, and $p \in \mathbb{P}$. Then the following are equivalent:*

(i) p is $(N, \mathbb{P})$-generic.

(ii) For every $\mathbb{P}$-name $\sigma \in N$ and every $q \le p$, if $q \Vdash \sigma$ is an ordinal, then for every generic G with $q \in G$, $\sigma_G \in N$.

Proof. (i)$\Rightarrow$(ii): Assume (i), $\sigma \in N$ is a $\mathbb{P}$-name, $q \le p$, $q \Vdash \sigma$ is an ordinal, and $q \in G$ generic. Then by Proposition 28.23, $D \stackrel{\text{def}}{=} \{r \in \mathbb{P} : \exists\alpha[r \Vdash \sigma = \check{\alpha}]\}$ is dense below q. Let f be a one-one function mapping D into **On** such that for each $r \in D$, $r \Vdash \sigma = f(r)^v$. Since $N \preceq H(\lambda)$, $f \in N$. Let $D' = D \cup \{r : r \perp q\}$. Then $D' \in N$ and D' is dense. So by (i) $D' \cap N$ is pre-dense below p, hence also pre-dense below q, so we can choose $r \in D \cap N \cap G$ with $r \le q$. Now $r \Vdash \sigma = f(r)^v$. Hence $\sigma_G = f(r) \in N$.

(ii)$\Rightarrow$(i): Assume (ii); we verify Proposition 28.24(ii). Suppose that $D \subseteq \mathbb{P}$, D is dense, and $D \in N$. Let $p \in G$ generic. Let f be a bijection from a cardinal κ onto D. Then $f, \kappa \in H(\lambda)$, so we get that f and κ are in N. Choose $a \in D \cap G$, and let $\alpha < \kappa$ be such that $f(\alpha) = a$. Let σ be a $\mathbb{P}$-name such that $\sigma_G = \alpha$. Choose $q \le p$ with $q \in G$ such that $q \Vdash \sigma$ is an ordinal. Then by (ii), $\sigma_G \in N$. Hence also $a \in N$, so $a \in D \cap N \cap G$. $\square$

Proposition 28.26. *Suppose that $\mathbb{P}$ is a forcing order, $N \preceq H(\lambda)$, and $p \in \mathbb{P}$. Then the following are equivalent:*

(i) p is $(N, \mathbb{P})$-generic.

(ii) If $p \in G$ generic and $\alpha \in N[G] \cap \mathrm{On}$, then $\alpha \in N$.

Proof. (i)$\Rightarrow$(ii): Assume (i), and suppose that $p \in G$ generic and $\alpha \in N[G] \cap \mathrm{On}$. Say $\alpha = \sigma_G$ with $\sigma \in N$. Choose $q \in G$ such that $q \Vdash \sigma$ is an ordinal, and choose $r \in G$ with $r \le p, q$. Then by Proposition 28.25(ii), $\sigma_G \in N$.

(ii)⇒(i): Assume (ii). We will check Proposition 28.25(ii). Suppose that σ is a $\mathbb{P}$-name, $\sigma \in N$, $q \le p$, $q \Vdash \sigma$ is an ordinal, and $q \in G$ generic. Then $\sigma_G \in N[G] \cap \mathrm{On}$, so by (ii), $\sigma_G \in N$. □

Definition. Now suppose that $\mathbb{P}$ is a forcing order and π is a $\mathbb{P}$-name for a forcing order. We now associate with each $(\mathbb{P} * \pi)$-name τ a $\mathbb{P}$-name τ^*, by recursion:

$$\tau^* = \{(\eta, p) : \exists \mu \exists \theta [\theta \in \mathrm{dmn}(\pi) \wedge \eta = \mathrm{op}(\mu^*, \theta) \wedge (\mu, (p, \theta)) \in \tau\}.$$

Proposition 28.27. *Suppose that $\mathbb{P}$ is a forcing order, π is a $\mathbb{P}$-name for a forcing order, τ is a $(\mathbb{P} * \pi)$-name, and G is $\mathbb{P}$-generic over M. Then τ^*_G is a π_G-name.*

Proof. By induction:

$$\begin{aligned}\tau^*_G &= \{\eta_G : \exists p \in G[(\eta, p) \in \tau^*]\} \\ &= \{\eta_G : \exists p \in G \exists \mu \exists \theta [\theta \in \mathrm{dmn}(\pi) \wedge \eta = \mathrm{op}(\mu^*, \theta) \wedge (\mu, (p, \theta)) \in \tau]\} \\ &= \{(\mu^*_G, \theta_G) : \theta \in \mathrm{dmn}(\pi) \wedge \exists p \in G[(\mu, (p, \theta)) \in \tau]\}.\end{aligned}$$

□

Proposition 28.28. *Suppose that $\mathbb{P}$ is a forcing order and π is a $\mathbb{P}$-name for a forcing order. Let $G * H$ be $(\mathbb{P} * \pi)$-generic. Then for any $(\mathbb{P} * \pi)$-name τ we have $\tau_{G*H} = (\tau^*_G)_H$.*

Proof. By induction:

$$\begin{aligned}\tau_{G*H} &= \{\sigma_{G*H} : \exists (p, \xi) \in G * H[(\sigma, (p, \xi)) \in \tau]\} \\ &= \{(\sigma^*_G)_H : \exists p \in G \exists \xi \in \mathrm{dmn}(\pi)[\xi_G \in H \wedge (\sigma, (p, \xi)) \in \tau]\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\tau^*_G)_H &= \{\rho_H : \exists q \in H[(\rho, q) \in \tau^*_G]\} \\ &= \{\mu^*_G : \exists p \in G \exists \theta \in \mathrm{dmn}(\pi)[\theta_G \in H \wedge (\mu, (p, \theta)) \in \tau]\}\end{aligned}$$

which is the same as above. □

Proposition 28.29. *Let P be a forcing order, π a P-name for a forcing order, $N \preceq H(\lambda)$, p is (N, P)-generic, $(p, \sigma) \in P * \pi$, and for all P-generic G, if $p \in G$ then σ_G is $(N[G], \pi_G)$-generic.*

Then (p, σ) is $(N, P * \pi)$-generic.

Proof. We will apply Proposition 28.26. Suppose that $G * H$ is generic, $(p, \sigma) \in G * H$, and $\alpha \in N[G * H] \cap \mathrm{On}$. Then there is a $\mathbb{P} * \pi$ name $\tau \in N$ such that $\alpha = \tau_{G*H}$. By Proposition 28.28 we have $\alpha = (\tau^*_G)_H$. Clearly $\tau^* \in N$, so $\tau^*_G \in N[G]$. Now $\sigma_G \in H$, $\alpha \in N[G][H]$ and σ_G is $(N[G], \pi_G)$-generic, it follows that $\alpha \in N[G] \cap \mathrm{On}$. Let ξ be a P-name, $\xi \in N$, such that $\xi_G = \alpha$. Since $p \in G$ and p is (N, P)-generic, it follows that $\alpha \in N$. □

Lemma 28.30. *Suppose that $\mathbb{P}$ is a forcing order, π is a $\mathbb{P}$-name for a forcing order, $i(p) = (p, 1)$ for all $p \in P$; so i is a complete embedding of $\mathbb{P}$ into $\mathbb{P} * \pi$. Suppose that σ is a P-name, and $G * H$ is generic. Then $(i_*(\sigma))_{G*H} = \sigma_G$.*

Proof. By induction on σ:

$$\begin{array}{lll} x \in (i_*(\sigma))_{G*H} & \text{iff} & \exists q \in G * H \exists \nu[(\nu, q) \in i_*(\sigma) \wedge x = \nu_{G*H}] \\ & \text{iff} & \exists q \in G * H \exists \nu \exists (\rho, r) \in \sigma[(\nu, q) = (i_*(\rho), i(r)) \wedge x = \nu_{G*H}] \\ & \text{iff} & \exists q \in G * H \exists (\rho, r) \in \sigma[x = (i_*(\rho))_{G*H} \wedge i(r) = q] \\ & \text{iff} & \exists (\rho, r) \in \sigma[x = \rho_G \wedge r \in G] \\ & \text{iff} & x \in \sigma_G. \end{array}$$

□

Theorem 28.31. *Suppose that $\mathbb{P}$ is a forcing poset and π is a full P-name such that $\Vdash$ [π is proper]. Let θ be a regular uncountable cardinal such that $\operatorname{trcl}(\mathbb{P}), \pi \in H(\theta)$, and suppose that $N \preceq H((2^\theta)^+)$, N countable, and $\mathbb{P} * \pi \in N$. Let i be the complete embedding of $\mathbb{P}$ into $\mathbb{P} * \pi$. Suppose that $p \in \mathbb{P}$ is $(N, \mathbb{P})$-generic, σ and η are $\mathbb{P}$-names, $\eta \in N$, and*

$$p \Vdash \sigma \in N^v \wedge \eta \in N^v \wedge \operatorname{op}(\sigma, \eta) \in (\mathbb{P} * \pi)^v \wedge \sigma \in \Gamma.$$

Then there is a $\xi \in \operatorname{dmn}(\pi)$ such that (p, ξ) is $(N, \mathbb{P} * \pi)$-generic and $(p, \xi) \Vdash i_*(\operatorname{op}(\sigma, \xi)) \in \Gamma$.

Proof. Let $p \in G$ generic. Then $\sigma_G \in N$, $\eta_G \in N$, $(\sigma_G, \eta_G) \in P * \pi$, and $\sigma_G \in G$. By Lemma 28.22, $N[G] \preceq (H((2^\theta)^+))^{M[G]}$. Now $\eta \in N$, so $\eta_G \in N[G]$. Also, $\pi \in H(\theta)$, so $\pi_G \in (H(\theta))^{M[G]}$. Note that π_G is a proper forcing order in $M[G]$. Now by Theorem 28.21 there is a $q \leq \eta_G$ such that q is $(N[G], \pi_G)$-generic. Thus

$$p \Vdash \exists \chi[\chi \in \pi \wedge \chi \leq \eta \wedge \chi \text{ is } (N[\Gamma], \pi) - \text{generic}.$$

By the maximal principle let τ be a $\mathbb{P}$-name such that

$$p \Vdash \tau \in \pi \wedge \tau \leq \eta \wedge \tau \text{ is } (N[\Gamma], \pi) - \text{generic}.$$

By the definition of full names, let $\xi \in \operatorname{dmn}(\pi)$ be such that $p \Vdash \xi = \tau$ and $(\xi, p) \in \pi$. Then $(p, \xi) \in \mathbb{P} * \pi$ and $p \Vdash \xi$ is $(N[\Gamma], \pi)$ – generic. Hence by Proposition 28.29, (p, ξ) is $(N, \mathbb{P} * \pi)$-generic. It remains to show that $(p, \xi) \Vdash i_*(\operatorname{op}(\sigma, \xi)) \in \Gamma$. Let $(p, \xi) \in G * H$, generic. Thus $p \in G$ and $\xi_G \in H$. Also, $\sigma_G \in G$ by assumption. Hence $(i_*(\operatorname{op}(\sigma, \xi)))_{G*H} = (\operatorname{op}(\sigma, \xi))_G = (\sigma_G, \xi_G) \in G * H$. □

Lemma 28.32. *Let $\alpha > 0$, and let $(\langle(\mathbb{P}_\xi : \xi \leq \alpha\rangle, \langle \pi_\xi : \xi < \alpha \rangle)$ be a countable support iteration, with each $\mathbb{P}_\xi$ for $\xi < \alpha$ proper, and each name π_ξ full. For $\xi \leq \eta \leq \alpha$, let $i_{\xi\eta}$ be the complete embedding of $\mathbb{P}_\xi$ into $\mathbb{P}_\eta$. For each $\xi \leq \alpha$ let Γ_ξ be the standard name for a generic filter over $\mathbb{P}_\xi$. Let λ be sufficiently large, and let $N \preceq H(\lambda)$ be countable with $\alpha, \mathbb{P}_\alpha, \langle \mathbb{P}_\xi : \xi \leq \alpha \rangle, \langle \pi_\xi : \xi < \alpha \rangle \in N$. Let $\gamma_0 \in \alpha \cap N$, and assume that $p_{\gamma_0} \in \mathbb{P}_{\gamma_0}$ is $(N, \mathbb{P}_{\gamma_0})$-generic and σ and τ are $\mathbb{P}_{\gamma_0}$-names such that*

$$p_{\gamma_0} \Vdash_{\mathbb{P}_{\gamma_0}} \sigma \in N^{\text{v}} \wedge \sigma \in \mathbb{P}_\alpha^{\text{v}} \wedge \tau \in \mathbb{P}_{\gamma_0}^v \wedge \tau \subseteq \sigma \wedge \tau \in \Gamma_{\gamma_0}.$$

Then there is a $(N,\mathbb{P}_\alpha)$-generic condition q such that $q \restriction \gamma_0 = p_{\gamma_0}$ and $q \Vdash_{\mathbb{P}_\alpha} (i_{\gamma_0\alpha})_*(\sigma) \in \Gamma_\alpha$.

Proof. Induction on α; so assume that the lemma holds for any positive ordinal less than α. First suppose that α is a successor ordinal $\beta+1$, and suppose that $\gamma_0 = \beta$. Let G be $\mathbb{P}_{\gamma_0}$-generic with $p_{\gamma_0} \in G$. Then $\sigma_G \in N$, $\sigma_G \in \mathbb{P}_\alpha$, $\tau_G \in \mathbb{P}_{\gamma_0}$, $\tau_G \subseteq \sigma_G$, and $\tau_G \in G$. Hence there is a ξ such that $\sigma_G = \tau_G{}^\frown\langle \xi_G \rangle$. Then

$$p_{\gamma_0} \Vdash \tau \in N^v \wedge \xi \in N^v \wedge \mathrm{op}(\tau,\xi) \in (\mathbb{P} * \pi_{\gamma_0})^v \wedge \tau \in \Gamma.$$

Then by Theorem 28.31 there is a $\xi \in \mathrm{dmn}(\pi_{\gamma_0})$ such that (p_{γ_0},ξ) is $(N,\mathbb{P}_{\gamma_0} * \pi_{\gamma_0})$-generic and $(p_{\gamma_0},\xi) \Vdash i_*(\mathrm{op}(\tau,\xi)) \in \Gamma$. Now for each $q \in \mathbb{P}_\alpha$ let $f(q) = (q \restriction \gamma_0, q(\gamma_0))$. Then f is an isomorphism from $\mathbb{P}_\alpha$ onto $\mathbb{P}_{\gamma_0} \times \pi_{\gamma_0}$. It follows that $p_{\gamma_0}{}^\frown\langle \xi \rangle$ is $(N,\mathbb{P}_\alpha)$-generic. We claim that $p_{\gamma_0}{}^\frown\langle \xi \rangle \Vdash ((i_{\gamma_0\alpha})_*)(\sigma) \in \Gamma$. For, suppose that $p_{\gamma_0}{}^\frown\langle \xi \rangle \in G$ generic. Then $(p_{\gamma_0},\xi) \in f[G]$, so $(i_*(\mathrm{op}(\tau,\xi)))_{f[G]} \in f[G]$. Now $f[G] = (G \restriction \gamma_0) * H$ for some H, so by Lemma 28.30, $(i_*(\mathrm{op}(\tau,\xi)))_{f[G]} = (\tau_{G\restriction\gamma_0}, \xi_{G\restriction\gamma_0}) = f(\sigma_{G\restriction\gamma_0})$. Since $(i_*(\mathrm{op}(\tau,\xi)))_{f[G]} \in f[G]$, it follows that $\sigma_{G\restriction\gamma_0} \in G$. Hence it remains only to show that $(((i_{\gamma_0\alpha})_*)(\sigma))_G = \sigma_{G\restriction\gamma_0}$:

$$\begin{aligned}
x \in (((i_{\gamma_0\alpha})_*)(\sigma))_G \quad & \text{iff} \quad \exists q \in G \exists \nu[(\nu,q) \in (i_{\gamma_0\alpha})_*(\sigma) \wedge x = \nu_G] \\
& \text{iff} \quad \exists q \in G \exists \nu \exists (\rho,r) \in \sigma[(\nu,q) = ((i_{\gamma_0\alpha})_*(\rho), i_{\gamma_0\alpha}(r)) \wedge x = \nu_G] \\
& \text{iff} \quad \exists q \in G \exists (\rho,r) \in \sigma[q = i_{\gamma_0\alpha}(r) \wedge x = ((i_{\gamma_0\alpha})_*(\rho))_G] \\
& \text{iff} \quad \exists q \in G \exists (\rho,r) \in \sigma[i_{\gamma_0,\alpha}(r) \in G \wedge x = \rho_{G\restriction\gamma_0}] \\
& \text{iff} \quad \exists (\rho,r) \in \sigma[r \in G \restriction \gamma_0 \wedge x = \rho_{G\restriction\gamma_0}] \\
& \text{iff} \quad x \in \sigma_{G\restriction\gamma_0}.
\end{aligned}$$

Now suppose that $\alpha = \beta+1$ and $\gamma_0 < \beta$. Now For any $\mathbb{P}_\alpha$-generic G let $\sigma_G = \rho_G{}^\frown\langle \xi_G \rangle$. Now

$$p_{\gamma_0} \Vdash \rho \in N^v \wedge \rho \in \mathbb{P}_\beta \wedge \tau \in \mathbb{P}^v_{\gamma_0} \wedge \tau \subseteq \rho \wedge \tau \in \Gamma_{\gamma_0},$$

so by the inductive hypothesis we get a $(N,\mathbb{P}_\beta)$-generic q such that $q \restriction \gamma_0 = p_{\gamma_0}$ and $q \Vdash (i_{\gamma_0\beta})_*(\rho) \in \Gamma_\beta$. Thus

$$\begin{aligned}
q \Vdash & (i_{\gamma_0\beta})_*(\sigma) \in N^v \wedge (i_{\gamma_0\beta})_*(\sigma) \in \mathbb{P}^v_\alpha \wedge (i_{\gamma_0\beta})_*(\rho) \in \mathbb{P}^v_\beta \wedge \\
& (i_{\gamma_0\beta})_*(\rho) \subseteq (i_{\gamma_0\beta})_*(\sigma) \wedge (i_{\gamma_0\beta})_*(\sigma) \in \Gamma_\beta.
\end{aligned}$$

Then by the first special case of this proof we get a $(N,\mathbb{P}_\alpha)$-generic r such that $r \restriction \beta = q$ and $r \Vdash (i_{\beta\alpha})_*((i_{\gamma_0\beta})_*(\sigma)) \in \Gamma_\alpha$. Now Proposition 28.29 finishes this part of the induction.

Now suppose that α is a limit ordinal. Now since $N \preceq H(\lambda)$, there is no largest ordinal in $\alpha \cap N$. Moreover, N is countable. Hence there is an increasing sequence $\langle \gamma_i : i \in \omega \rangle$ of ordinals in $\alpha \cap N$, cofinal in $\alpha \cap N$, starting with our given γ_0. Thus $\sup_{i\in\omega} \gamma_i = \alpha$. Let $\langle D_i : i \in \omega \rangle$ list all of the dense subsets of $\mathbb{P}_\alpha$ which are in N. Now we are going to define sequences $\langle q_i : i \in \omega \rangle$, $\langle \tau_i : i \in \omega \rangle$, $\langle \mu_i : i \in \omega \rangle$ so that the following conditions hold:

(1) $q_n \in \mathbb{P}_{\gamma_n}$ for each $n \in \omega$.

(2) $q_0 = p_{\gamma_0}$, q_n is $(N, \mathbb{P}_{\gamma_n})$-generic, and $q_{n+1} \restriction \gamma_n = q_n$.

(3) $\tau_0 = \sigma$, and for $n > 0$, τ_n is a $\mathbb{P}_{\gamma_n}$-name such that q_n forces (in $\mathbb{P}_{\gamma_n}$) each of the following:

(a) $\tau_n \in N^{\mathrm{v}} \wedge \tau_n \in \mathbb{P}^{\mathrm{v}}_\alpha$.

(b) μ_n is a $\mathbb{P}_{\gamma_n}$-name, $\mu_n \subseteq \tau_n$, and $\mu_n \in \Gamma_{\gamma_n}$.

(c) $\tau_n \le_\alpha (i_{\gamma_{n-1}\gamma_n})_*(\tau_{n-1})$.

(d) $\tau_n \in D^{\mathrm{v}}_{n-1}$.

We define $q_0 = p_{\gamma_0}$ and $\tau_0 = \sigma$. Now suppose that q_n and τ_n have been defined so that (1)–(3) hold. We claim

$$(4) \qquad q_n \Vdash_{\mathbb{P}_{\gamma_n}} \exists\chi[\chi \in \mathbb{P}^{\mathrm{v}}_\alpha \wedge \chi \in N^{\mathrm{v}} \wedge \exists\mu[\mu \in \Gamma_{\gamma_n} \wedge \mu \subseteq \chi] \wedge \chi \in D^{\mathrm{v}}_n \wedge \chi \le \tau_n]$$

To prove (4), let G be generic over $\mathbb{P}_{\gamma_n}$ with $q_n \in G$. Since (a)–(d) hold for n, we have $\tau_{nG} \in N$, $\tau_{nG} \in \mathbb{P}_\alpha$, $\mu_{nG} \subseteq \tau_{nG}$, $\mu_{nG} \in G$, $\tau_{nG} \le_\alpha ((i_{\gamma_{n-1}\gamma_n})_*(\tau_{n-1}))_G$, and $\tau_{nG} \in D_{n-1}$. Now let

$$D'_n = \{p \restriction \gamma_n : p \in D_n \wedge [p \le_\alpha \tau_{nG} \vee (p \restriction \gamma_n) \perp \mu_{nG})]\}.$$

Then $D'_n \in N$ since $N \preceq H(\lambda)$. We claim that D'_n is dense in $\mathbb{P}_{\gamma_n}$. To see this, let $r \in \mathbb{P}_{\gamma_n}$.

Case 1. $r \perp \mu_{nG}$. By the density of D_n, choose $s \in D_n$ such that $s \le_\alpha (i_{\gamma_n\alpha})_*(r)$. Then $s \restriction \gamma_n \le r$, and so $(s \restriction \gamma_n) \perp \mu_{nG}$. Hence $s \restriction \gamma_n \in D'_n$, as desired.

Case 2. r and μ_{nG} are compatible. Hence $(i_{\gamma_n\alpha})_*(r)$ and τ_{nG} are compatible; say $s \le (i_{\gamma_n\alpha})_*(r), \tau_{nG}$. By the density of D_n, let t be such that $t \in D_n$ and $t \le s$. Thus $t \le \tau_{nG}$, so $t \restriction \gamma_n \in D'_n$, and $t \restriction \gamma_n \le r$, as desired.

So D'_n is dense and $D'_n \in N$. Since q_n is $(N, \mathbb{P}_{\gamma_n})$-generic, it follows by definition that $D'_n \cap N$ is pre-dense below q_n. So we can choose $x \in G \cap D'_n \cap N$. Say $x = p \restriction \gamma_n$ with $p \in D_n$. Thus $H(\lambda) \models \exists p \in D_n[x = p \restriction \gamma_n]$, so since $N \preceq H(\lambda)$, we may assume that $p \in N$. Now $x, \mu_{nG} \in G$, so they are compatible. Hence $p \le_\alpha \tau_{nG}$. Thus with $\chi_G = p$ we have verified the conclusion of (4). So (4) holds.

By the maximal principle we get a $\mathbb{P}_{\gamma_n}$-name ρ such that

(5) $q_n \Vdash_{\mathbb{P}_{\gamma_n}} \rho \in \mathbb{P}^{\mathrm{v}}_\alpha \wedge \rho \in N^{\mathrm{v}} \wedge \exists\mu[\mu \in \Gamma_{\gamma_n} \wedge \mu \subseteq \rho] \wedge \rho \in D^{\mathrm{v}}_n \wedge \rho \le \tau_n$.

Now $q_n \Vdash_{\mathbb{P}_{\gamma_n}} \exists\xi[\xi \in \mathbb{P}^{\mathrm{v}}_{\gamma_{n+1}} \wedge \xi \in N^{\mathrm{v}} \wedge \xi \subseteq \rho]$. Hence by the maximal principle again, there is a $\mathbb{P}_{\gamma_n}$-name ξ such that $q_n \Vdash \xi \in \mathbb{P}^{\mathrm{v}}_{\gamma_{n+1}} \wedge \xi \in N^{\mathrm{v}} \wedge \xi \subseteq \rho$. Thus

$$q_n \Vdash_{\mathbb{P}_{\gamma_n}} \xi \in N^{\mathrm{v}} \wedge \xi \in \mathbb{P}^{\mathrm{v}}_{\gamma_{n+1}} \wedge \exists\mu[\mu \in \Gamma_{\gamma_n} \wedge \mu \subseteq \xi].$$

We now apply the inductive hypothesis to $\gamma_n, \gamma_{n+1}, q_n, \xi$ in place of $\gamma_0, \alpha, p_{\gamma_0}, \sigma$ to obtain a $(N, \mathbb{P}_{\gamma_{n+1}})$-generic condition q_{n+1} such that $q_{n+1} \restriction \gamma_n = q_n$ and $q_{n+1} \Vdash_{\mathbb{P}_{\gamma_{n+1}}} (i_{\gamma_n\gamma_{n+1}})_*(\xi) \in \Gamma_{\gamma_{n+1}}$. Let $\tau_{n+1} = (i_{\gamma_n\gamma_{n+1}})_*(\rho)$. Then we claim

$$(6) \qquad q_{n+1} \Vdash \tau_{n+1} \in N^{\mathrm{v}} \wedge \tau_{n+1} \in \mathbb{P}^{\mathrm{v}}_\alpha \wedge \exists\mu[\mu \subseteq \tau_{n+1} \wedge \mu \in \Gamma_{\gamma_{n+1}}] \wedge \tau_{n+1} \le i_{\gamma_n\gamma_{n+1}}(\tau_n) \wedge \tau_{n+1} \in D^{\mathrm{v}}_n$$

To prove (6) we apply Theorem 26.4. Let G be generic on $\mathbb{P}_{\gamma_{n+1}}$ with $q_{n+1} \in G$. Let $H = (i_{\gamma_n \gamma_{n+1}})^{-1}[G]$. Then H is $\mathbb{P}_{\gamma_n}$-generic by Theorem 26.3. Since $q_{n+1} \restriction \gamma_n = q_n$, we have $q_{n+1} \leq i_{\gamma_n \gamma_{n+1}}(q_n)$, hence $i_{\gamma_n \gamma_{n+1}}(q_n) \in G$ and $q_n \in H$. Hence by (5), $\rho_H \in N$, $\rho_H \in \mathbb{P}_\alpha$, $\rho_H \restriction \gamma_n \in H$, $\rho_H \leq_\alpha \tau_{nH}$, and $\rho_H \in D_n$. Now by Theorem 26.4 we have $\tau_{n+1,G} = ((i_{\gamma_n \gamma_{n+1}})_*(\rho))_G = \rho_H$. It follows that $\tau_{n+1,G} \in N$, $\tau_{n+1,G} \in \mathbb{P}_\alpha$, $\tau_{n+1,G} \in D_n$, and $\tau_{n+1,G} \leq_\alpha (i_{\gamma_n \gamma_{n+1}})_*(\tau_{nG})$. Finally,

$$\begin{aligned}
\tau_{n+1,G} \restriction \gamma_{n+1} &= ((i_{\gamma_n \gamma_{n+1}})_*(\rho))_G \restriction \gamma_{n+1} \\
&= \rho_H \restriction \gamma_{n+1} \\
&= \xi_H \\
&= ((i_{\gamma_n \gamma_{n+1}})_*(\xi))_G \\
&\in G \quad \text{since } q_{n+1} \Vdash_{\mathbb{P}_{\gamma_{n+1}}} (i_{\gamma_n \gamma_{n+1}})_*(\xi) \in \Gamma_{\gamma_{n+1}}.
\end{aligned}$$

This finishes the construction.

Let $r = \bigcup_{n \in \omega} q_n$. We claim that r is as desired in the Lemma. Clearly r has countable support. By (3), using Theorem 26.4, we have

(7) $i_{\gamma_n \alpha}(q_n) \Vdash (i_{\gamma_n \alpha})_*(\tau_n) \in N^{\text{v}} \wedge (i_{\gamma_n \alpha})_*(\tau_n) \in \mathbb{P}_\alpha^{\text{v}}$.

(8) $i_{\gamma_n \alpha}(q_n) \Vdash (i_{\gamma_n \alpha})_*(\tau_n) \restriction \gamma_n \in (i_{\gamma_n \alpha})_*(\Gamma_{\gamma_n})$.

(9) $i_{\gamma_{n+1} \alpha}(q_{n+1}) \Vdash (i_{\gamma_{n+1} \alpha})_*(\tau_{n+1}) \leq_\alpha (i_{\gamma_n \alpha})_*(\tau_n))$.

(10) $i_{\gamma_{n+1} \alpha}(q_{n+1}) \Vdash (i_{\gamma_{n+1} \alpha})_*(\tau_{n+1}) \in D_n^{\text{v}}$.

Since $r \leq i_{\gamma_n \alpha}(q_n)$ for all n, this gives

(11) $r \Vdash (i_{\gamma_n \alpha})_*(\tau_n) \in N^{\text{v}} \wedge (i_{\gamma_n \alpha})_*(\tau_n) \in \mathbb{P}_\alpha^{\text{v}}$.

(12) $r \Vdash (i_{\gamma_n \alpha})_*(\tau_n) \restriction \gamma_n \in (i_{\gamma_n \alpha})_*(\Gamma_{\gamma_n})$.

(13) $r \Vdash (i_{\gamma_{n+1} \alpha})_*(\tau_{n+1}) \leq_\alpha (i_{\gamma_n \alpha})_*(\tau_n))$.

(14) $r \Vdash (i_{\gamma_{n+1} \alpha})_*(\tau_{n+1}) \in D_n^{\text{v}}$.

Now to show that r is $(N, \mathbb{P}_\alpha)$-generic, we will apply Lemma 31.197. So suppose that G is generic on $\mathbb{P}_\alpha$, $r \in G$, D is dense in $\mathbb{P}_\alpha$, and $D \in N$; we want to show that $D \cap N \cap G \neq \emptyset$. Choose n so that $D = D_n$. Let $s = ((i_{\gamma_{n+1} \alpha})_*(\tau_{n+1}))_G$. Thus by (11) and (14), $s \in D \cap N$. By Theorem 26.4 and (12), $s = \tau_{n+1, G_{\eta_{n+1}}} \in G_{\eta_{n+1}} \subseteq G$. This finishes the proof that r is $(N, \mathbb{P}_\alpha)$-generic.

Clearly $r \restriction \gamma_0 = p_{\gamma_0}$, and $r \Vdash (i_{\gamma_0 \alpha})_*(\sigma) \in \Gamma_\alpha$ since $\sigma = \tau_0$. □

Theorem 28.33. *Let $\alpha > 0$, and let $(\langle (\mathbb{P}_\xi : \xi \leq \alpha \rangle, \langle \pi_\xi : \xi < \alpha \rangle)$ be a countable support iteration, with each $\mathbb{P}_\xi$ for $\xi < \alpha$ proper, and each name π_ξ full. Then $\mathbb{P}_\alpha$ is proper.*

Proof. Let N and λ be as in the statement of Lemma 28.32. We are going to apply Lemma 28.21 So, let $p \in \mathbb{P}_\alpha$ with $p \in N$. Let $\gamma_0 = 0$. Recall that $\mathbb{P}_0 = \{0\}$. Trivially, 0 is $(N, \mathbb{P}_0)$-generic. The hypothesis of Lemma 28.32 holds, with p^{v} in place of σ. Hence by Lemma 28.32 we get a $q \in \mathbb{P}_\alpha$ such that q is $(N, \mathbb{P}_\alpha)$-generic and $q \Vdash_\alpha (i_{0\alpha})_*(p^{\text{v}}) \in \Gamma_\alpha$.

Let G be $\mathbb{P}_\alpha$-generic with $q \in G$. Then $((i_{0\alpha})_*(p^{\mathrm{v}}))_G \in G$, i.e., $p \in G$. Choose $r \in G$ such that $r \leq p, q$. Then r is clearly $(N, \mathbb{P}_\alpha)$-generic. □

We now give an equivalent definition of properness which does not involve forcing at all. The new definition depends on a certain game of length ω. Let $\mathbb{P}$ be a forcing order; we describe a game $\Gamma(\mathbb{P})$ played between players I and II. First I chooses $p_0 \in P$ and a maximal antichain A_0 of $\mathbb{P}$. Then II chooses a countable subset B_0^0 of A_0. At the n-th pair of moves, I chooses a maximal antichain A_n and then II chooses countable sets $B_i^n \subseteq A_i$ for each $i \leq n$. Then we say that II wins iff there is a $q \leq p_0$ such that for every $i \in \omega$, the set

$$\bigcup_{i \leq n \in \omega} B_i^n$$

is predense below q.

We give a rigorous formulation of these ideas, not relying on informal notions of games. A *play* of the game $\Gamma(\mathbb{P})$ is an infinite sequence

$$\langle p_0, A_0, C_0, A_1, C_1, \ldots, A_n, C_n \ldots \rangle$$

satisfying the following conditions for each $n \in \omega$:

(1) $p_0 \in P$.

(2) A_n is a maximal antichain of $\mathbb{P}$.

(3) $C_n = \langle B_i^n : i \leq n \rangle$, where each B_i^n is a countable subset of A_i.

Given such a play, we say that II *wins* iff there is a $q \leq p_0$ such that for every $i \in \omega$, the set

$$\bigcup_{i \leq n \in \omega} B_i^n$$

is predense below q.

A *partial play of length m of* $\Gamma(\mathbb{P})$ is a sequence

$$\langle p_0, A_0, C_0, A_1, C_1, \ldots, A_{m-1}, C_{m-1}, A_m \rangle$$

satisfying the above conditions. Note that the partial play ends with one of the maximal antichains A_m. A *strategy* for II is a function S whose domain is the set of all partial plays of $\Gamma(\mathbb{P})$, such that if $\mathbb{P}$ is a partial play as above, then $S(\mathbb{P})$ is a set C_m satisfying the condition (3). A play is said to be *according to S* iff for every m, $C_m = S(\langle p_0, A_0, C_0, A_1, C_1, \ldots, A_{m-1}, C_{m-1}, A_m \rangle)$. The strategy S is *winning* iff II wins every play which is played according to S.

Proposition 28.34. *$\mathbb{P}$ is proper iff II has a winning strategy in $\Gamma(\mathbb{P})$.*

Proof. First suppose that $\mathbb{P}$ is proper, and suppose that $p_0 \in P$. Let λ be sufficiently large, and let $N \preceq H(\lambda)$ be such that $\mathbb{P}, p_0 \in N$. Now a strategy for II is as follows. After I chooses A_0, II chooses a countable set N_0 with $N \preceq N_0 \preceq H(\lambda)$ and with $A_0 \in N_0$; and II sets $B_0^0 = A_0 \cap N_0$. Suppose that I chooses A_n, and II has chosen $N_0 \preceq \cdots \preceq N_{n-1} \preceq H(\lambda)$. Then II chooses N_n so that $N_{n-1} \preceq N_n \preceq H(\lambda)$, and sets $B_i^n = A_i \cap N_n$ for all $i \leq n$. When the game is finished, let $N_\omega = \bigcup_{n\in\omega} N_n$. So $N_\omega \preceq H(\lambda)$. By Lemma 28.16 choose $q \leq p_0$ so that q is (N_ω, P)-generic. Since $N_\omega \preceq H(\lambda)$, we may assume that $q \in N_\omega$. Take any $i \in \omega$; we claim that $\bigcup_{i\leq n} B_i^n$ is predense below q. Say $q \in N_n$ with $i \leq n$. Again since $N_n \preceq H(\lambda)$, $A_n \in N_n$, and $B_i^n = A_i \cap N_n$, it follows that B_i^n is a maximal antichain in $P \cap N_n$. Let $D = \{r \in P \cap N_n : r \leq s \text{ for some } s \in B_i^n\}$. Then D is dense in $P \cap N_n$. Take any $r \leq q$. Then r is compatible with some $s \in D \cap N_n$. Hence r is compatible with some $t \in B_i^n$. This shows that II wins.

Conversely, suppose that II has a winning strategy σ. Let λ be sufficiently large, and let $N \preceq H(\lambda)$ be such that $P, p_0, \sigma \in N$, N countable. Then we take the game in which I lists all of the maximal antichains of P which are in N, and II plays using his strategy. All of the sets B_i^n which II plays are in N, since $\sigma \in N$. Since II wins, choose $q \leq p_0$ such that for each $i \in \omega$ the set $\bigcup_{i\leq n} B_i^n$ is predense below q. We claim that q is (P, N)-generic. For, let $D \subseteq P$, $D \in N$, be dense. Let C be maximal such that C is an antichain and $\forall p \in C \exists d \in D[p \leq d]$. Then C is a maximal antichain. For, suppose that $p \perp C$. Choose $d \in D$ such that $d \leq p$. Then $d \perp C$, so that $C \cup \{d\}$ still satisfies the conditions on C, contradiction. Say $C = A_i$. Say $q \in N_n$ with $i \leq n$. Now $\bigcup_{i\leq m} B_i^m$ is predense below q. Hence there is an $s \in \bigcup_{i\leq m} B_i^m$ such that s and q are compatible. Say $t \leq s, q$. Say $s \in B_i^m$ with $n \leq m$. Now $B_i = A_i \cap N$, so $s \in A_i = C$. Choose $d \in D$ such that $s \leq d$. Now t is compatible with $d \in D \cap N$ and $t \leq q$. This shows that q is (P, N)-generic. So P is proper. $\square$

The proper forcing axiom

The proper forcing axiom is similar to Martin's axiom. It is not really an axiom. It runs as follows:

(PFA) *If P is a proper poset and $\mathscr{D}$ is a collection of dense subsets of P with $|\mathscr{D}| \leq \omega_1$, then there is a filter G on P such that $G \cap D \neq \emptyset$ for all $D \in \mathscr{D}$.*

Corollary 28.35. *PFA implies $MA(\omega_1)$.* $\square$

First of all, PFA is relatively consistent. This requires large cardinals, however. Recall from Chapter 22 the definition of a supercompact cardinal.

Theorem. (Baumgartner) *If "ZFC + there is a supercompact cardinal" is consistent, then so is "ZFC + $2^\omega = \omega_2$ + PFA".*

Another important fact about PFA is as follows:

• *PFA implies $\neg(MA(\omega_2))$.*

Now we mention various important applications of PFA, giving the basic definitions, but no background. After this, to conclude this chapter we give one application in detail.

• An important algebra in set theory is $\mathscr{P}(\omega)/\text{fin}$. Here $\mathscr{P}(\omega)$ is the collection of all subsets of ω, and fin is the collection of finite subsets of ω. Clearly fin is an ideal in the

Boolean algebra $\mathscr{P}(\omega)$, so we are considering here the quotient algebra. Let A denote this quotient algebra.

Now let κ and λ be infinite cardinals. A (κ, λ^*)-*gap* is a pair (f, g) such that the following conditions hold:

(1) $f \in {}^\kappa A$ and $g \in {}^\lambda A$.

(2) If $\alpha < \beta < \kappa$, then $f_\alpha < f_\beta$.

(3) If $\alpha < \beta < \lambda$, then $g_\beta < g_\alpha$.

(4) If $\alpha < \kappa$ and $\beta < \lambda$, then $f_\alpha < g_\beta$.

We say that the gap is *unfilled* iff there is no $a \in A$ such that $f_\alpha < a < g_\beta$ for all $\alpha < \kappa$ and $\beta < \lambda$.

Hausdorff proved (in ZFC) that there is an unfilled (ω_1, ω_1^*)-gap. Under PFA, every unfilled gap has one of the forms (ω_1, ω_1^*), (ω, λ^*) with $\lambda \geq \omega_2$, or (κ, ω^*) with $\kappa \geq \omega_2$.

• PFA implies that there is a linear ordering of size 2^ω which cannot be embedded in $\mathscr{P}(\omega)/\text{fin}$.

• A set A of real numbers is $\aleph_1$-*dense* iff A intersects every open interval in exactly $\aleph_1$ points. Under PFA, any two $\aleph_1$-dense sets of real numbers are order-isomorphic.

• Under PFA, any uncountable Boolean algebra has an uncountable set of pairwise incomparable elements.

• Under PFA, any uncountable family of subsets of ω contains an uncountable simply ordered subset or an uncountable family of pairwise incomparable elements.

• PFA implies that every tree of height ω_2 with all levels of size less than ω_2 has a linearly ordered subset of size ω_2.

Now we begin to develop a detailed application of PFA. The application concerns closed unbounded subsets of ω_1, and the proof involves the notion of an indecomposable ordinal. An ordinal is called indecomposable iff it is 0 or has the form ω^β for some β; see Theorem 9.31. First we note the following:

Proposition 28.36. *If $\alpha < \omega_1$, then $\omega^\alpha < \omega_1$.*

Proof. By induction on α. □

Corollary 28.37. *The set of all indecomposable ordinals less than ω_1 is club in ω_1.* □

In particular, the union of a sequence of indecomposable ordinals is indecomposable.

Now we define a partial order which we discuss for the rest of this chapter. Let P_C be the set of all $f \in \text{Fn}(\omega_1, \omega_1, \omega)$ such that $f \subseteq g$ for some normal function g on ω_1. The order is $\supseteq$.

Lemma 28.38. *If $p \in P_C$, $\beta < \alpha < \omega_1$ for all $\beta \in \text{rng}(p)$, and α is indecomposable, then $p \cup \{(\alpha, \alpha)\} \in P_C$.*

Proof. We may assume that p is nonempty. Let f be a normal function on ω_1 such that $p \subseteq f$. Let β be the largest member of $\text{rng}(p)$. Then $(\beta \cap \text{rng}(f)) \cup (\omega_1 \backslash \beta)$ is club in ω_1; let g be a normal function with it as range. Choose $\gamma < \omega_1$ such that $f(\gamma) = \beta$. Then $g(\gamma + \delta) = \beta + \delta$ for every ordinal $\delta < \omega_1$, and so $g(\alpha) = g(\gamma + \alpha) = \beta + \alpha = \alpha$. So $p \cup \{(\alpha, \alpha)\} \subseteq g$, as desired.

Theorem 28.39. *P_C is proper.*

Proof. For each $p \in P_C$, let α_p be the least ordinal such that $p \subseteq \alpha_p \times \alpha_p$.
We claim

(1) If A is a maximal antichain in P_C and $\alpha < \omega_1$, then there is an ordinal $\beta < \omega_1$ such that for every $p \in P_C$ such that $p \subseteq \alpha \times \alpha$ there is a $q \in A$ such that q and p are compatible and $q \subseteq \beta \times \beta$.

To prove this, let A is a maximal antichain in P_C and $\alpha < \omega_1$. For each $p \in P_C$ such that $p \subseteq \alpha \times \alpha$ choose $q_p \in A$ such that q_p and p are compatible. Now $\{p \in P_C : p \subseteq \alpha \times \alpha\}$ is countable, so we can choose $\beta < \omega_1$ such that $\alpha_{q_p} \leq \beta$ for all $p \in P_C$ such that $p \subseteq \alpha \times \alpha$. Clearly β is as desired in (1).

We let $g(A, \alpha)$ be the smallest β satisfying (1). For each maximal antichain A, let $C(A) = \{\beta < \omega_1 : g(A, \alpha) < \beta \text{ for all } \alpha < \beta\}$. We claim that $C(A)$ is club in ω_1. To show that it is closed, suppose that γ is a limit ordinal less than ω_1 and $C(A) \cap \gamma$ is unbounded in γ. Take any $\alpha < \gamma$. Choose $\beta \in C(A) \cap \gamma$ such that $\alpha < \beta$. Then $g(A, \alpha) < \beta < \gamma$. This shows that $\gamma \in C(A)$, so $C(A)$ is closed. To show that it is unbounded, take any $\alpha < \omega_1$. Let $\beta_0 = \alpha$, and if β_n has been defined, let β_{n+1} be an ordinal such that $g(A, \alpha) < \beta_n$ for all $\alpha < \beta_n$. Clearly $\bigcup_{n \in \omega} \beta_n \in C(A)$, as desired.

Let Ind be the set of all indecomposable ordinals. So $C(A) \cap \text{Ind}$ is club for every maximal antichain A.

Now we define two functions S and f defined on all partial plays of the game $\Gamma(P_C)$. For a smallest play $\langle p_0, A_0 \rangle$, choose $f(\langle p_0, A_0 \rangle) \in \text{Ind} \cap C(A_0)$ such that $p_0 \subseteq f(\langle p_0, A_0 \rangle) \times f(\langle p_0, A_0 \rangle)$, and let

$$S(\langle p_0, A_0 \rangle) = \{p \in A_0 : p \subseteq f(\langle p_0, A_0 \rangle) \times f(\langle p_0, A_0 \rangle)\}$$

Now suppose that a partial play

$$\mathscr{P} \stackrel{\text{def}}{=} \langle p_0, A_0, C_0, A_1, C_1, \ldots, A_{m-1}, C_{m-1}, A_m \rangle$$

is given, with $m > 0$. Choose $f(\mathscr{P}) \in \text{Ind} \cap \bigcap_{i \leq m} C(A_i)$ with

$$f(\mathscr{P}) > f(\langle p_0, A_0, C_0, A_1, C_1, \ldots, A_{m-1} \rangle).$$

Then define for each $i \leq m$

$$B_i^m = \{p \in A_i : p \subseteq f(\langle p_0, A_0, C_0, A_1, C_1, \ldots, A_i \rangle) \times f(\langle p_0, A_0, C_0, A_1, C_1, \ldots, A_i \rangle)\}$$

and let $S(\mathscr{P}) = \langle B_i^m : i \leq m \rangle$.

This finishes the construction of these functions. Thus S is a strategy for II. We claim that it is a winning strategy. To prove this, let

$$\langle p_0, A_0, C_0, A_1, C_1, \ldots, A_n, C_n \ldots \rangle$$

be a play according to S, with associated function f. For each $m \in \omega$ let $\alpha_m = f(\langle p_0, A_0, C_0, A_1, C_1, \ldots, A_i \rangle)$. Let $\alpha = \bigcup_{m\in\omega} \alpha_m$. Thus α is an indecomposable ordinal less than ω_1. Since $p_0 \subseteq \alpha_0 \times \alpha_0 < \alpha$, it follows from 28.39 that $q \stackrel{\text{def}}{=} p_0 \cup \{(\alpha, \alpha)\} \in P_C$. We claim that q is what is needed to show that II has won. To prove this, suppose that $i \in \omega$. To show that $\bigcup_{i\le n\in\omega} B_i^n$ is predense below q, take any $r \le q$. Let $r' = r \cap (\alpha \times \alpha)$. Obviously $r' \in P_C$. Choose $n \in \omega$ with $n \ge i$ such that $r' \subseteq \alpha_n \times \alpha_n$. Since $\alpha_{n+1} \in C(A_i)$ and $\alpha_n < \alpha_{n+1}$, by definition of $C(A_i)$ we have $g(A_i, \alpha_n) < \alpha_{n+1}$, and it follows that there is an $s \in A_i$ such that s and r' are compatible and $s \subseteq g(A_i, \alpha_n) \times g(A_i, \alpha_n)$. We claim that s and r are compatible (as desired). For, choose normal functions h and k on ω_1 such that $s \cup r' \subseteq h$ and $r \subseteq k$. Now the set $(\mathrm{rng}(h) \cap \alpha_{n+1}) \cup (\alpha \backslash \alpha_{n+1}) \cup (\mathrm{rng}(k) \backslash \alpha)$ is club; let l be its strictly increasing enumeration. Choose $\gamma < \omega_1$ such that $h[\gamma] = \mathrm{rng}(h) \cap \alpha_{n+1}$. Now since $r \le q$ we have $\alpha \in \mathrm{dmn}(r)$ and $r(\alpha) = \alpha$; hence $k(\alpha) = \alpha$. It follows that

$$\begin{aligned} l(\delta) &= h(\delta) \quad \text{for all } \delta < \gamma, \\ l(\gamma + \varepsilon) &= \alpha_{n+1} + \varepsilon \quad \text{for all } \varepsilon < \alpha, \text{ and } l[[\gamma, \alpha)] = \alpha \backslash \alpha_{n+1}, \\ l(\alpha + \varepsilon) &= k(\alpha + \varepsilon) \quad \text{for all } \varepsilon < \omega_1. \end{aligned}$$

Now suppose that $\delta \in \mathrm{dmn}(s)$. Then $(\delta, s(\delta)) \in g(A_i, \alpha_n) \times g(A_i, \alpha_n) \subseteq \alpha_{n+1} \times \alpha_{n+1}$, and so $s(\delta) = h(\delta) = l(\delta)$. Next, suppose that $\delta \in \mathrm{dmn}(r)$ and $\delta < \alpha$. Since $r(\alpha) = \alpha$, it follows that $r(\delta) < \alpha$. So $(\delta, r(\delta)) \in \alpha \times \alpha$ and hence $r(\delta) = r'(\delta)$ and $(\delta, r(\delta)) \in \alpha_n \times \alpha_n$. So $r(\delta) = r'(\delta) = h(\delta) = l(\delta)$. Finally, suppose that $\delta \in \mathrm{dmn}(r)$ and $\alpha \le \delta$. Then $r(\delta) = k(\delta) = l(\delta)$. $\square$

Now we are ready for our one application of PFA.

Theorem 28.40. *Assume PFA, and suppose that $\langle A_\alpha : \alpha < \omega_1 \rangle$ is a collection of infinite subsets of ω_1. Then there is a club C in ω_1 such that for all $\alpha < \omega_1$, $A_\alpha \not\subseteq C$.*

Proof. We are going to apply PFA to the partial order P_C which we have been discussing. By 28.39, P_C is proper. We now define three families, each of size at most ω_1, of dense subsets of P_C. For each $\alpha < \omega_1$, let

$D_\alpha = \{p \in P_C :$ there is a $\gamma \in A_\alpha$ such that one of the following holds:
(1) $0 \in \mathrm{dmn}(p)$ and $\gamma < p(0)$;
(2) there is a β such that $\beta, \beta + 1 \in \mathrm{dmn}(p)$ and $p(\beta) < \gamma < p(\beta + 1)\}$.

To see that D_α is dense in P_C, let $q \in P_C$. Say $q \subseteq f$ with f a normal function on ω_1. Let $r = q \cup \{(0, f(0))\}$. Clearly $r \in P_C$ and $r \le q$. Let θ be the largest member of $\mathrm{rng}(r)$, and choose ρ so that $r(\rho) = \theta$.

If $\gamma < f(0)$ for some $\gamma \in A_\alpha$, then $r \in D_\alpha$, as desired. Suppose that $f(0) \leq \gamma$ for every $\gamma \in A_\alpha$.

Next, suppose that $A_\alpha \not\subseteq \text{rng}(f)$. Choose $\gamma \in A_\alpha \backslash \text{rng}(f)$, and let δ be minimum such that $\gamma < f(\delta)$. By the continuity of f, δ is a successor ordinal $\beta + 1$. Then $f(\beta) < \gamma < f(\beta+1)$, and so $r \cup \{(\beta, f(\beta)), (\beta+1, f(\beta+1))\}$ is the desired member of D_α extending q. So suppose that $A_\alpha \subseteq \text{rng}(f)$.

Suppose that $\theta < \gamma$ for some $\gamma \in A_\alpha$. Define $g : \omega_1 \to \omega_1$ by setting

$$g(\xi) = \begin{cases} f(\xi) & \text{if } \xi \leq \rho, \\ \gamma + \alpha & \text{if } \xi = \rho + \alpha \text{ with } \alpha \neq 0. \end{cases}$$

Clearly g is a normal function, and the function $s \stackrel{\text{def}}{=} r \cup \{(\rho+1, \gamma+1)\}$ is a subset of it. We have $s(\rho) = r(\rho) = \theta < \gamma < \gamma + 1 = s(\rho+1)$, so $s \in D_\alpha$, as desired. Hence we may assume that $A_\alpha \subseteq \theta + 1$.

Let $\text{dmn}(r) = \{\delta_i : i < m\}$ with $0 = \delta_0 < \delta_1 < \cdots < \delta_m$ and $r(\delta_m) = \theta$. The infinite set A_α is contained in the finite union $\bigcup_{i<m} [r(\delta_i), r(\delta_{i+1})]$, so we can choose $i < m$ such that $A_\alpha \cap [r(\delta_i), r(\delta_{i+1})]$ is infinite. Let $\langle \beta_j : j < \omega \rangle$ be such that $\langle f(\beta_j) : j < \omega \rangle$ enumerates the first ω elements of $A_\alpha \cap (r(\delta_i), r(\delta_{i+1}))$. For each $j < \omega$ let γ_j be such that $\beta_j + \gamma_j = \delta_{i+1}$. Then for $j < k < \omega$ we clearly have $\gamma_j \geq \gamma_k$. Hence we can choose $k < \omega$ such that $\gamma_k = \gamma_j$ for all $j \geq k$. Now we define $g : \omega_1 \to \omega_1$ by:

$$g(\xi) = \begin{cases} f(\xi) & \text{if } \xi \leq \beta_k, \\ f(\beta_{k+1} + \eta) & \text{if } \xi = \beta_k + \eta \text{ with } \eta \neq 0. \end{cases}$$

Clearly g is a normal function on ω_1. For any $j \leq i$ we have $r(\delta_j) = f(\delta_j) = g(\delta_j)$. Now suppose that $j \geq i+1$, and write $\delta_j = \delta_{i+1} + \sigma$. Then

$$r(\delta_j) = f(\delta_j) = f(\delta_{i+1} + \sigma) = f(\beta_{k+1} + \gamma_{k+1} + \sigma) = g(\beta_k + \gamma_k + \sigma) = g(\delta_{i+1} + \sigma) = g(\delta_j).$$

Thus $s \stackrel{\text{def}}{=} r \cup \{(\beta_k, g(\beta_k)), (\beta_k + 1, g(\beta_k + 1))\}$ is such that $s \subseteq g$, and

$$s(\beta_k) = g(\beta_k) = f(\beta_k) < f(\beta_{k+1}) < f(\beta_{k+1} + 1) = g(\beta_k + 1) = s(\beta_k + 1),$$

and $f(\beta_{k+1}) \in A_\alpha$. so $s \in D_\alpha$, as desired.

This finishes the proof that D_α is dense in P_C.

Next, for each ordinal $\alpha < \omega_1$ let

$$E_\alpha = \{p \in P_C : \alpha \in \text{dmn}(p)\}.$$

Clearly each E_α is dense in P_C.

Finally, for each limit ordinal $\alpha < \omega_1$ let

$$\begin{aligned} F_\alpha = \{p \in P_C : & \text{ one of the following holds:} \\ & (1)\ 0 \in \text{dmn}(p) \text{ and } \alpha < p(0), \\ & (2) \text{ there is a } \beta \in \text{dmn}(p) \text{ such that } p(\beta) = \alpha, \\ & (3) \text{ there is a } \beta < \omega_1 \text{ such that } \beta, \beta+1 \in \text{dmn}(p) \\ & \quad \text{and } p(\beta) < \alpha < p(\beta+1)\} \end{aligned}$$

To show that F_α is dense in P_C, suppose that $q \in P_C$. Let f be a normal function on ω_1 such that $q \subseteq f$. Let $r = q \cup \{(0, f(0)\}$. So $r \in P_C$ and $r \leq q$. If $\alpha < r(0)$, then $r \in F_\alpha$, as desired. So, assume that $r(0) \leq \alpha$. If $f(\beta) = \alpha$ for some β, then $r \cup \{(\beta, \alpha)\} \in F_\alpha$, as desired. So assume that $\alpha \notin \operatorname{rng}(f)$. Let δ be smallest such that $\alpha < f(\delta)$. Then δ is a successor ordinal $\beta + 1$ since f is continuous, and $f(\beta) < \alpha < f(\beta + 1)$. Hence $r \cup \{(\beta, f(\beta)), (\beta + 1, f(\beta + 1))\} \in F_\alpha$, as desired.

Now by PFA let G be a filter on P_C which intersects all of these dense sets. Let $f = \bigcup G$. We claim that $\operatorname{rng}(f)$ is the desired club.

First, f is a function. For, suppose that $(\alpha, \beta), (\alpha, \gamma) \in f$. Choose $p, q \in G$ such that $(\alpha, \beta) \in p$ and $(\alpha, \gamma) \in q$. Since G is a filter, $p \cup q \in F_C$, and so $\beta = \gamma$.

Since $G \cap E_\alpha \neq \emptyset$ for each $\alpha < \omega_1$, f has domain ω_1.

Next, f is strictly increasing. For, if $\alpha < \beta$, we can easily find $p \in G$ such that $\alpha, \beta \in \operatorname{dmn}(p)$, and hence $f(\alpha) = p(\alpha) < p(\beta) = f(\beta)$.

f is continuous: suppose that γ is a limit ordinal. Let $\alpha = \bigcup_{\beta<\gamma} f(\beta)$. Since f is strictly increasing, α is a limit ordinal too. Let $p \in G \cap F_\alpha$. Then by the definition of F_α there are three possibilities.

Case 1. $0 \in \operatorname{dmn}(p)$ and $\alpha < p(0)$. But $p(0) = f(0) \leq \alpha$, contradiction.

Case 2. There is a $\beta < \omega_1$ such that $\beta, \beta + 1 \in \operatorname{dmn}(p)$ and $p(\beta) < \alpha < p(\beta + 1)$. Choose $\delta < \gamma$ such that $f(\beta) = p(\beta) \leq f(\delta)$. Then clearly $\beta \leq \delta < \gamma$ so, since γ is a limit ordinal, also $\beta + 1 < \gamma$, and hence $p(\beta + 1) = f(\beta + 1) \leq \alpha$, contradiction.

Case 3. There is a $\beta \in \operatorname{dmn}(p)$ such that $p(\beta) = \alpha$. By Cases 1 and 2, this is the only possibility left. We claim that $\beta = \gamma$; this will prove continuity. In fact, if $\beta < \gamma$, then $\alpha = p(\beta) = f(\beta) < f(\beta + 1) \leq \alpha$, contradiction. Suppose that $\gamma < \beta$. If $\delta < \gamma$, then $f(\delta) < f(\gamma)$. So $\alpha \leq f(\gamma) < f(\beta) = \alpha$, contradiction. Thus $\beta = \gamma$.

So now we know that f is a normal function on ω_1, and hence $\operatorname{rng}(f)$ is club in ω_1. Now suppose that $\alpha < \omega_1$; we want to show that $A_\alpha \not\subseteq \operatorname{rng}(f)$. Choose $p \in D_\alpha \cap G$. Choose $\gamma \in A_\alpha$ in accordance with the definition of D_α. There are two possibilities. If $0 \in \operatorname{dmn}(p)$ and $\gamma < p(0)$, then $\gamma < f(0)$, and so $\gamma \notin \operatorname{rng}(f)$. If there is a $\beta < \omega_1$ such that $\beta, \beta + 1 \in \operatorname{dmn}(p)$ and $p(\beta) < \gamma < p(\beta + 1)$, then again $\gamma \notin \operatorname{rng}(p)$. $\square$

We give some more examples of iterated forcing. These are concerned with a certain partial order of functions. For any regular cardinal κ we define

$$f <_\kappa g \quad \text{iff} \quad f, g \in {}^\kappa\kappa \text{ and there is an } \alpha < \kappa \text{ such that } f(\beta) < g(\beta) \text{ for all } \beta \in [\alpha, \kappa).$$

This is clearly a partial order on ${}^\kappa\kappa$. We say that $\mathscr{F} \subseteq {}^\kappa\kappa$ is *almost unbounded* iff there is no $g \in {}^\kappa\kappa$ such that $f <_\kappa g$ for all $f \in \mathscr{F}$. Clearly ${}^\kappa\kappa$ itself is almost unbounded; it has size 2^κ.

Theorem 29.1. *Let κ be a regular cardinal. Then any almost unbounded subset of ${}^\kappa\kappa$ has size at least κ^+.*

Proof. Let $\mathscr{F} \subseteq {}^\kappa\kappa$ have size $\leq \kappa$; we want to find an almost bound for it. We may assume that $\mathscr{F} \neq \emptyset$. Write $\mathscr{F} = \{f_\alpha : \alpha < \kappa\}$, possibly with repetitions. (Since maybe $|\mathscr{F}| < \kappa$.) Define $g \in {}^\kappa\kappa$ by setting, for each $\alpha < \kappa$,

$$g(\alpha) = \left(\sup_{\beta \leq \alpha} f_\beta(\alpha)\right) + 1.$$

If $\beta < \kappa$, then $\{\alpha < \kappa : g(\alpha) \leq f_\beta(\alpha)\} \subseteq \beta$, and so $f_\beta <_\kappa g$. $\square$

Thus under GCH the size of almost unbounded sets has been determined. We are interested in what happens in the absence of GCH, more specifically, under $\neg$CH.

Theorem 29.2. *Suppose that κ is an infinite cardinal and* MA(κ) *holds. Suppose that $\mathscr{F} \subseteq {}^\omega\omega$ and $|\mathscr{F}| = \kappa$. Then there is a $g \in {}^\omega\omega$ such that $f <_\omega g$ for all $f \in \mathscr{F}$.*

Proof. Let $P = \{(p, F) : p \in \mathrm{Fn}(\omega, \omega, \omega) \text{ and } F \in [\mathscr{F}]^{<\omega}\}$. We partially order P by setting $(p, F) \leq (q, G)$ iff the following conditions hold:

(1) $p \supseteq q$.

(2) $F \supseteq G$.

(3) For all $f \in G$ and all $n \in (\mathrm{dmn}(p) \backslash \mathrm{dmn}(q))$, $p(n) > f(n)$.

To check that this really is a partial order, suppose that $(p, F) \leq (q, G) \leq (h, H)$. Obviously $p \supseteq h$ and $F \supseteq H$. Suppose that $f \in H$ and $n \in (\mathrm{dmn}(p) \backslash \mathrm{dmn}(h)$. If $n \in \mathrm{dmn}(q)$, then $p(n) = q(n) > f(n)$. If $n \notin \mathrm{dmn}(q)$, then $p(n) > f(n)$ since $f \in G$.

To show that $\mathscr{P}$ has ccc, suppose that $X \subseteq P$ is uncountable. Since $\mathrm{Fn}(\omega, \omega, \omega)$ is countable, there are $(p, F), (q, G) \in X$ with $p = q$. Then $(p, F \cup G) \in P$ and $(p, F \cup G) \leq (p, F), (p, G)$, as desired.

For each $h \in \mathscr{F}$ let $D_h = \{(p, F) \in P : h \in F\}$. Then D_h is dense. In fact, let $(q, G) \in P$ be given. Then $(q, G \cup \{h\}) \in P$ and $(q, G \cup \{h\}) \leq (q, G)$, as desired.

For each $n \in \omega$ let $E_n = \{(p, F) : n \in \mathrm{dmn}(f)\}$. Then E_n is dense. In fact, let $(q, G) \in P$ be given. We may assume that $n \notin \mathrm{dmn}(q)$. Choose $m > f(n)$ for each $f \in G$, and let $p = q \cup \{(n, m)\}$. Clearly $(p, G) \in E_n$ and $(p, G) \leq (q, G)$, as desired.

Now we apply MA(κ) to get a filter G on $\mathbb{P}$ intersecting all of these dense sets. Since G is a filter, the relation $g \stackrel{\text{def}}{=} \bigcup_{(p,F)\in G} p$ is a function. Since $G \cap E_n \neq \emptyset$ for each $n \in \omega$, g has domain ω. Let $f \in \mathscr{F}$. Choose $(p, F) \in G \cap D_f$. Let $m \in \omega$ be greater than each member of $\text{dmn}(p)$. We claim that $f(n) < g(n)$ for all $n \geq m$. For, suppose that $n \geq m$. Choose $(q, H) \in G$ such that $n \in \text{dmn}(q)$, and choose $(r, K) \in G$ such that $(r, K) \leq (p, F), (q, H)$. Then $f \in K$ since $F \subseteq K$. Also, $n \in \text{dmn}(r)$ since $q \subseteq r$. So $n \in \text{dmn}(r)\backslash\text{dmn}(p)$. Hence from $(r, K) \leq (p, F)$ we get $g(n) = r(n) > f(n)$. $\square$

As another illustration of iterated forcing, we now show that it is relatively consistent that every almost unbounded subset of ${}^\omega\omega$ has size 2^ω, while $\neg$ MA holds. This follows from the following theorem, using the fact that MA implies that $2^\kappa = 2^\omega$ for every infinite cardinal $\kappa < 2^\omega$.

Theorem 29.3. *There is a c.t.m. of ZFC with the following properties:*

(i) $2^\omega = \omega_2$.

(ii) $2^{\omega_1} = \omega_3$.

(iii) Every almost unbounded set of functions from ω *to* ω *has size* 2^ω.

Proof. Applying Theorem 24.15 to a model N of GCH, with $\lambda = \omega_1$ and $\kappa = \omega_3$, we get a c.t.m. M of ZFC such that in M, $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_3$. We are going to iterate within M, and iterate ω_2 times. At each successor step we will introduce a function almost greater than each member of ${}^\omega\omega$ at that stage. In the end, any subset of ${}^\omega\omega$ of size less than ω_2 appears at an earlier stage, and is almost bounded.

(1) If $\mathbb{Q}$ is a ccc forcing order in M of size $\leq \omega_1$, then there are at most ω_1 nice $\mathbb{Q}$-names for subsets of $(\omega \times \omega)\check{}$.

To prove (1), recall that a nice $\mathbb{Q}$-name for a subset of $(\omega \times \omega)\check{}$ is a set of the form

$$\bigcup\{\{\check{a}\} \times A_\alpha : a \in \omega \times \omega\}$$

where for each $a \in \omega \times \omega$, A_a is an antichain in $\mathbb{Q}$. Now by ccc the number of antichains in $\mathbb{Q}$ is at most $\sum_{\mu<\omega_1} |Q|^\mu \leq \omega_1$ by CH in M. So the number of sets of the indicated form is at most $\omega_1^\omega = \omega_1$. Hence (1) holds.

Now we are going to define by recursion functions $\mathbb{P}$, π, and σ with domain ω_2.

Let $\mathbb{P}_0$ be the trivial partial order $(\{0\}, 0, 0)$.

Now suppose that $\mathbb{P}_\alpha$ has been defined, so that it is a ccc forcing order in M of size at most ω_1. We now define π_α, σ^α, and $\mathbb{P}_{\alpha+1}$. By (1), the set of all nice $\mathbb{P}_\alpha$-names for subsets of $(\omega \times \omega)\check{}$ has size at most ω_1. We let $\{\tau^\alpha_\gamma : \gamma < \omega_1\}$ enumerate all of them.

(2) For every $\gamma < \omega_1$ there is a $\mathbb{P}_\alpha$-name σ^α_γ such that

$$1_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \sigma^\alpha_\gamma : \check{\omega} \to \check{\omega} \text{ and } [\tau^\alpha_\gamma : \check{\omega} \to \check{\omega} \text{ implies that } \sigma^\alpha_\gamma = \tau^\alpha_\gamma].$$

In fact, clearly

$$1_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \exists W[W : \check{\omega} \to \check{\omega} \text{ and } [\tau^\alpha_\gamma : \check{\omega} \to \check{\omega} \text{ implies that } W = \tau^\alpha_\gamma]],$$

and so (2) follows from the maximal principle.

This defines σ^α.

Now for each $H \in [\omega_1]^{<\omega}$ we define $\rho^\alpha_H = \{(\sigma^\alpha_\gamma, 1_{\mathbb{P}_\alpha}) : \gamma \in H\}$. So ρ^α_H is a $\mathbb{P}_\alpha$-name.

We now define

$$\pi^0_\alpha = \{(\mathrm{op}(\check{p}, \rho^\alpha_H), 1) : p \in \mathrm{Fn}(\omega, \omega, \omega) \text{ and } H \in [\omega_1]^{<\omega}\}.$$

Let G be $\mathbb{P}_\alpha$-generic over M. Then

$$(3) \qquad (\pi^0_\alpha)_G = \{(p, K) : p \in \mathrm{Fn}(\omega, \omega, \omega) \text{ and } K \in [{}^\omega\omega]^{<\omega}\}.$$

In fact, first suppose that $x \in (\pi^0_\alpha)_G$. Then there exist $p \in \mathrm{Fn}(\omega, \omega, \omega)$ and $H \in [\omega_1]^{<\omega}$ such that $x = (p, (\rho^\alpha_H)_G)$. Now $(\rho^\alpha_H)_G = \{(\sigma^\alpha_\gamma)_G : \gamma \in H\}$, and $(\sigma^\alpha_\gamma)_G \in {}^\omega\omega$ for each γ by (2). Thus x is in the right side of (3).

Second, suppose that $p \in \mathrm{Fn}(\omega, \omega, \omega)$ and $K \in [{}^\omega\omega]^{<\omega}$. For each $f \in K$ there is a $\gamma(f) < \omega_1$ such that $f = (\tau^\alpha_{\gamma(f)})_G$. Let $H = \{\gamma(f) : f \in K\}$. So H is a finite subset of ω_1, and hence is in N. By (1) we have $f = (\sigma^\alpha_{\gamma(f)})_G$ for each $f \in K$. Now $(\rho^\alpha_H)_G = K$, and so $(p, K) \in (\pi^0_\alpha)_G$, as desired. So (3) holds.

Next, we define

$$\begin{aligned}\pi^1_\alpha = \{(\mathrm{op}(\mathrm{op}(\check{p}, \rho^\alpha_H), \mathrm{op}(\check{p}', \rho^\alpha_{H'})), q) : {} & p, p' \in \mathrm{fin}(\omega, \omega),\\ & H, H' \in [\omega_1]^{<\omega},\ p' \subseteq p,\ H' \subseteq H,\ q \in \mathbb{P}_\alpha, \text{ and for all } \gamma \in H'\\ & \text{and all } n \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p'), q \Vdash_{\mathbb{P}_\alpha} \sigma^\alpha_\gamma(\check{n}) < (p(n))\check{}\ \}.\end{aligned}$$

Again, suppose that G is $\mathbb{P}_\alpha$-generic over M. Then

$$(4) \qquad \begin{aligned}(\pi^1_\alpha)_G = \{((p, K), (p', K')) : {} & (p, K), (p', K') \in (\pi^0_\alpha)_G,\ p' \subseteq p,\ K' \subseteq K,\\ & \text{and for all } f \in K' \text{ and all } n \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p'),\ f(n) < p(n)\}.\end{aligned}$$

To prove this, first suppose that $x \in (\pi^1_\alpha)_G$. Then there are $q \in G$, $p, p' \in \mathrm{Fn}(\omega, \omega, \omega)$ and $H, H' \in [\omega_1]^{<\omega}$ such that $x = ((p, (\rho^\alpha_H)_G), (p', (\rho^\alpha_{H'}))_G)$, $p' \subseteq p$, $H' \subseteq H$, and for all $\gamma \in H'$ and all $n \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p')$, $q \Vdash \sigma^\alpha_\gamma(\check{n}) < (p(n))\check{}$. Then with $K = (\rho^\alpha_H)_G$ and $K' = (\rho^\alpha_{H'}))_G$, the desired conditions clearly hold.

Second, suppose that p, p', K, K' exist as on the right side of (4). Then by the definition of π^0_α, there are $H, H' \in [\omega_1]^{<\omega}$ such that $K = (\rho^\alpha_H)_G$ and $K' = (\rho^\alpha_{H'})_G$. Then $K' = \{(\sigma^\alpha_\gamma)_G : \gamma \in H'\}$. Hence for every $\gamma \in H'$ and all $n \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p')$ we have $(\sigma^\alpha_\gamma)_G(n) < p(n)$. Since H' and $\mathrm{dmn}(p)\backslash\mathrm{dmn}(p')$ are finite, there is a $q \in G$ such that for every $\gamma \in H'$ and all $n \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p')$ we have $q \Vdash_{\mathbb{P}_\alpha} (\sigma^\alpha_\gamma)(\check{n}) < p(n)\check{}$. It follows now that $((p, K), (p', K')) \in (p^1_\alpha)_G$, as desired.

Next, we let $\pi^2_\alpha = \{(\mathrm{op}(0, 0), 1_{\mathbb{P}_\alpha})\}$. Then for any generic G, $(\pi^2_\alpha)_G = (0, 0)$. Finally, let $\pi_\alpha = \mathrm{op}(\mathrm{op}(\pi^0_\alpha, \pi^1_\alpha), \pi^2_\alpha)$. This finishes the definition of π_α.

By the argument in the proof of 29.2 we have

(5) $1_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \pi_\alpha$ is $\check{\omega}_1 - cc$.

Now $\mathbb{P}_{\alpha+1}$ is determined by (I7) and (I8).

At limit stages we take direct limits, so that ccc is maintained. So the construction is finished, and $\mathbb{P}_\kappa$ is ccc.

Let G be $\mathbb{P}_\kappa$-generic over M.

(6) In $N[G]$, if $\mathscr{F} \subseteq {}^\omega\omega$ and $|\mathscr{F}| < \omega_2$, then there is a $g \in {}^\omega\omega$ such that $f <_\omega g$ for all $f \in \mathscr{F}$.

For, let $\mathscr{F} = \{f_\xi : \xi < \omega_1\}$, possibly with repetitions. Let

$$\mathscr{F}' = \{(\xi, i, j) : \xi < \omega_1,\ i, j \in \omega,\ \text{and } f_\xi(i) = j\}.$$

Now there is an $\alpha < \omega_2$ such that $\mathscr{F}' \in N[i^{-1}_{\alpha\omega_2}[G]]$, and hence also $\mathscr{F} \in N[i^{-1}_{\alpha\omega_2}[G]]$. For brevity write $G_\xi = i^{-1}_{\xi\omega_2}[G]$ for every $\xi < \omega_2$. Let

$$H_\alpha = \{\eta_{G_\alpha} : \eta \in \mathrm{dmn}(\pi^0_\alpha) \text{ and } p^\frown\langle\eta\rangle \in G_{\alpha+1} \text{ for some } p\}.$$

Let $\mathbb{Q}_\alpha = (\pi_\alpha)_{G_\alpha}$. Thus by (3) and (4),

$$
\begin{aligned}
&(7) && Q_\alpha = \{(p, K) : p \in \mathrm{fin}(\omega, \omega) \text{ and } K \in [{}^\omega\omega]^{<\omega}\};\\
&(8) && \leq_{\mathbb{Q}_\alpha} = \{((p, K), (p', K')) : (p, K), (p', K') \in (\pi^0_\alpha)_G,\ p' \subseteq p,\ K' \subseteq K,\\
& && \qquad \text{and for all } f \in K' \text{ and all } n \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p'),\ f(n) < p(n)\}.
\end{aligned}
$$

Hence G_α is $\mathbb{P}$-generic over N, $H_\alpha \in N[G_{\alpha+1}]$, and H_α is $\mathbb{Q}_\alpha$-generic over $N[G_\alpha]$. Let $g = \bigcup_{(p,F)\in H_\alpha} p$. Clearly g is a function. For each $m \in \omega$, let

$$E_m = \{(p, K) :\in Q_\alpha : m \in \mathrm{dmn}(p)\}.$$

Then E_m is dense. (See the proof of 29.2.) It follows that $g \in {}^\omega\omega$.

Now take any $f \in {}^\omega\omega$ (in $N[G_\alpha]$). The set $D \stackrel{\text{def}}{=} \{(p, K) \in Q_\alpha : f \in K\}$ is dense, by the proof of 29.2. Hence we can choose $(p, K) \in D \cap H_\alpha$. We claim that $f(m) < g(m)$ for all m such that $m > n$ for each $n \in \mathrm{dmn}(p)$. For, suppose that such an m is given. Choose $(p', K') \in E_m \cap H_\alpha$, and then choose $(p'', K'') \in H_\alpha$ with $(p'', K'') \leq (p, K), (p', K')$. Now $m \in \mathrm{dmn}(p') \subseteq \mathrm{dmn}(p'')$, and $f \in K$. so from $(p'', K'') \leq (p, K)$ and $m \notin \mathrm{dmn}(p)$ we get $f(m) < p''(m) = g(m)$, as desired. This finishes the proof of (6).

By (6) we have $\omega_2 \leq 2^\omega$.

(9) $|P_\alpha| \leq \omega_1$ for all $\alpha < \omega_2$.

We prove this by induction on α. It is clear for $\alpha = 0$. Assume that $|P_\alpha| \leq \omega_1$. Clearly $|\pi^0_\alpha| = \omega_1$, so by (I7), $|P_{\alpha+1}| \leq \omega_1$. Suppose that α is limit, and $|P_\beta| \leq \omega_1$ for all $\beta < \alpha$. Since $\mathbb{P}_\alpha$ is the direct limit of previous $\mathbb{P}_\beta$s, clearly $|P_\alpha| \leq \omega_1$.

(10) $|P_{\omega_2}| \leq \omega_2$.

This is clear from (8), since P_{ω_2} is the direct limit of earlier $\mathbb{P}_\beta$s.

Now by Proposition 24.3, replacing κ, λ, μ there by $\omega_2, \omega_1, \omega$, we get $2^\omega \leq \omega_2$. So by the above, $2^\omega = \omega_2$ in $N[G]$. By Proposition 24.3, replacing κ, λ, μ there by $\omega_2, \omega_1, \omega_1$, we get $2^{\omega_1} \leq \omega_3$. Since $2^{\omega_1} = \omega_3$ in N, it follows that $2^{\omega_1} = \omega_3$ in $N[G]$. $\square$

We want to generalize 29.3 to higher cardinals. This requires some preparation.

Lemma 29.4. *Suppose that M is a c.t.m. of ZFC, and in M θ is a regular cardinal, $2^{<\theta} = \theta$, and $2^\theta = \theta^+$. We define a partial order $\mathbb{P}$ in M as follows:*

$$\begin{aligned}
&P = \{(p, F) : p \in \mathrm{Fn}(\theta, \theta, \theta), F \in \left[{}^\theta\theta\right]^{<\theta}\} \\
(p, F) \leq (q, G) \quad \textit{iff} \quad &q \subseteq p,\ G \subseteq F,\ \textit{and}\ \forall f \in G \forall \beta \in \mathrm{dmn}(p) \backslash \mathrm{dmn}(g)(p(\beta) > f(\beta)); \\
&1_{\mathbb{P}} = (0, 0).
\end{aligned}$$

Then the following conditions hold.

(i) $|P| \leq \theta^+$.

(ii) $\mathbb{P}$ is θ-closed.

(iii) $\mathbb{P}$ has the θ^+-cc.

(iv) $\mathbb{P}$ preserves cofinalities and cardinals.

(v) If G is $\mathbb{P}$-generic over M, then there is a function $g \in {}^\theta\theta$ in $M[G]$ such that $f <_\theta g$ for all $f \in ({}^\theta\theta)^M$.

Proof. Clearly (i) holds.

$\mathbb{P}$ satisfies the θ^+-c.c.: Suppose that $B \subseteq P$ with $|B| \geq \theta^+$. Then since $|\mathrm{Fn}(\theta, \theta, \theta)| = \theta$, wlog there is a q such that $p = q$ for all $(p, F) \in B$, and so θ^+-c.c. is clear.

$\mathbb{P}$ is θ-closed: Suppose that $\langle (p_\alpha, F_\alpha) : \alpha < \beta \rangle$ is decreasing, with $\beta < \theta$. Let $q = \bigcup_{\alpha<\beta} p_\alpha$ and $G = \bigcup_{\alpha<\beta} F_\alpha$. Suppose that $\alpha < \beta$; we claim that $(q, G) \leq (p_\alpha, F_\alpha)$. Suppose that $f \in F_\alpha$ and $\delta \in \mathrm{dmn}(q) \backslash \mathrm{dmn}(p_\alpha)$. Then there is a $\gamma < \beta$ such that $\delta \in \mathrm{dmn}(p_\gamma)$. We may assume that $\alpha < \gamma$. Hence $(p_\gamma, F_\gamma) \leq (p_\alpha, F_\alpha)$, so $q(\delta) = p_\gamma(\delta) > f(\delta)$, as desired.

Now it follows that (iv) holds.

Now suppose that G is $\mathbb{P}$-generic over M. Define

$$g = \bigcup_{(p,F) \in G} p.$$

Clearly g is a function with domain and range included in θ. To show that g has domain θ, take any $\alpha < \theta$. Let $D = \{(p, F) : \alpha \in \mathrm{dmn}(p)\}$. Then D is dense. In fact, suppose that $(q, H) \in \mathbb{P}$. Wlog $\alpha \notin \mathrm{dmn}(q)$. Let p be the extension of g by adding α to its domain and defining $p(\alpha)$ to be any ordinal less than θ which is greater than each $f(\alpha)$ for $f \in H$. Clearly $(p, H) \leq (q, H)$ and $(p, H) \in D$. So g has domain θ.

Finally, we claim that $f <^* g$ for all $f \in {}^\theta\theta \cap M$. In fact, clearly $E \stackrel{\text{def}}{=} \{(p, F) \in \mathbb{P} : f \in F\}$ is dense, and so we can choose $(p, F) \in E \cap G$. Take $\alpha < \theta$ such that $\sup(\mathrm{dmn}(p)) < \alpha$. Take any $\beta \in (\alpha, \theta)$. Choose (q, H) such that $\beta \in \mathrm{dmn}(q)$. Then choose $(r, K) \in G$ such that $(r, K) \leq (p, F), (q, H)$. Then $\beta \in \mathrm{dmn}(r) \backslash \mathrm{dmn}(p)$, and $f \in F$, so $g(\beta) = r(\beta) > f(\beta)$. This shows that $f <^* g$. $\square$

If π is a $\mathbb{P}$-name for a p.o., then we say that π is *full* for $\downarrow\theta$-sequences iff the following conditions (a)–(d) imply condition (e):

(a) $p \in \mathbb{P}$.

(b) $\alpha < \theta$.

(c) $\rho_\xi \in \operatorname{dmn}(\pi^0)$ for each $\xi < \alpha$.

(d) for all $\xi, \eta < \alpha$, if $\xi < \eta$, then $p \Vdash (\rho_\xi \in \pi^0) \wedge (\rho_\eta \in \pi^0) \wedge (\rho_\eta \leq \rho_\xi)$.

(e) There is a $\sigma \in \operatorname{dmn}(\pi^0)$ such that $p \Vdash \sigma \in \pi$ and $p \Vdash \sigma \leq \rho_\xi$ for each $\xi < \alpha$.

Lemma 29.5. *Let M be a c.t.m. of ZFC, and θ an infinite cardinal in M. Let $\mathscr{I}$ be the ideal in $\mathscr{P}(\theta)$ consisting of all sets of size less than θ. In M, let $(\mathbb{P}, \pi)$ be an α-stage iterated forcing construction with supports in $\mathscr{I}$ (Kunen's sense). Suppose that for each $\xi < \alpha$, the $\mathbb{P}_\xi$-name π_ξ is full for $\downarrow\theta$-sequences. Then $\mathbb{P}_\alpha$ is θ-closed.*

Proof. Let $\langle p^\nu : \nu < \sigma \rangle$ be a sequence of elements of $\mathbb{P}_\alpha$ such that $p^\nu \leq p^\mu$ if $\mu < \nu < \sigma$, and $\sigma < \theta$. We will define $p^\sigma = \langle p^\sigma_\xi : \xi < \alpha \rangle$ by recursion so that the following condition holds:

$$\text{For all } \xi < \alpha,\ p^\sigma \restriction \xi = \langle p^\sigma_\eta : \eta < \xi \rangle \in \mathbb{P}_\xi \text{ and } \forall \mu < \sigma (p^\sigma \restriction \xi \leq p^\mu \restriction \xi) \text{ and}$$
$$\operatorname{supp}(p^\sigma) = \bigcup_{\nu < \sigma} \operatorname{supp}(p^\nu).$$

The induction step to a limit ordinal ξ is clear, as is the case $\xi = 0$. Now we define p^σ_ξ, given $p^\sigma \restriction \xi$. By fullness we get $\rho^\sigma_\xi \in \operatorname{dmn}(\pi^0)$ such that

$$p^\sigma \restriction \xi \Vdash \rho^\sigma_\xi \in \pi \text{ and } p^\sigma \restriction \xi \Vdash \rho^\sigma_\xi \leq \rho^\sigma_\eta \text{ for each } \eta < \xi.$$

Clearly p^σ is as desired. $\square$

Here is our generalization of 29.3.

Theorem 29.6. *Let M be a c.t.m. of GCH, and let θ be an uncountable regular cardinal in M. Then there is a generic extension N of M preserving cofinalities and cardinals such that in N the following hold:*

(i) $2^\theta = \theta^{++}$.

(ii) $2^{(\theta^+)} = \theta^{+++}$.

(iii) Every subset of ${}^\theta\theta$ of size less than 2^θ is almost unbounded.

Proof. First we apply Corollary 24.16 with $\lambda = \theta^+$ and $\kappa = \theta^{+++}$ to get a generic extension M' of M preserving cofinalities and cardinals in which $2^{<\theta} = \theta$, $2^\theta = \theta^+$, and $2^{\theta^+} = \theta^{+++}$.

We are going to iterate within M', and iterate θ^{++} times. At each successor step we will introduce a function almost greater than each member of ${}^\theta\theta$ at that stage. In the end, any subset of ${}^\theta\theta$ of size less than θ^{++} appears at an earlier stage, and is almost bounded.

(1) If $\mathbb{Q}$ is a θ^+-cc forcing order in M' of size less $\leq \theta^+$, then there are at most θ^+ nice $\mathbb{Q}$-names for subsets of $(\theta \times \theta)\check{}$.

To prove (1), recall that a nice $\mathbb{Q}$-name for a subset of $(\theta \times \theta)\check{}$ is a set of the form

$$\bigcup\{\{\check{a}\} \times A_\alpha : a \in \theta \times \theta\}$$

where for each $a \in \theta \times \theta$, A_a is an antichain in $\mathbb{Q}$. Now by θ^+-cc, the number of antichains in $\mathbb{Q}$ is at most $\sum_{\mu<\theta^+} |Q|^\mu \leq \theta^+$ by $2^\theta = \theta^+$. So the number of sets of the indicated form is at most $(\theta^+)^\theta = \theta^+$. Hence (1) holds.

Now we are going to define by recursion functions $\mathbb{P}$, π, and σ with domain θ^{++}.

Let $\mathbb{P}_0$ be the trivial partial order $(\{0\}, 0, 0)$.

Now suppose that $\mathbb{P}_\alpha$ has been defined, so that it is a θ^+-cc forcing order in M' of size at most θ^+, it is θ-closed, and every element has support of size less than θ. Also we assume that π_ξ has been defined for every $\xi < \alpha$ so that π_ξ is a $\mathbb{P}_\xi$-name for a forcing order, and it is full for $\downarrow\theta$-sequences. We now define π_α, σ^α, and $\mathbb{P}_{\alpha+1}$. By (1), the set of all nice $\mathbb{P}_\alpha$-names for subsets of $(\theta \times \theta)\check{}$ has size at most θ^+. We let $\{\tau^\alpha_\gamma : \gamma < \theta^+\}$ enumerate all of them.

(2) For every $\gamma < \theta_1$ there is a $\mathbb{P}_\alpha$-name σ^α_γ such that

$$1_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \sigma^\alpha_\gamma : \check{\theta} \to \check{\theta} \text{ and } [\tau^\alpha_\gamma : \check{\theta} \to \check{\theta} \text{ implies that } \sigma^\alpha_\gamma = \tau^\alpha_\gamma].$$

In fact, clearly

$$1_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \exists W[W : \check{\theta} \to \check{\theta} \text{ and } [\tau^\alpha_\gamma : \check{\theta} \to \check{\theta} \text{ implies that } W = \tau^\alpha_\gamma]],$$

and so (2) follows from the maximal principle.

This defines σ^α.

Now for each $H \in [\theta^+]^{<\theta}$ we define $\rho^\alpha_H = \{(\sigma^\alpha_\gamma, 1_{\mathbb{P}_\alpha}) : \gamma \in H\}$. So ρ^α_H is a $\mathbb{P}_\alpha$-name.

We now define

$$\pi^0_\alpha = \{(\mathrm{op}(\check{p}, \rho^\alpha_H), 1) : p \in \mathrm{Fn}(\theta, \theta, \theta) \text{ and } H \in [\theta^+]^{<\theta}\}.$$

Let G be $\mathbb{P}_\alpha$-generic over M'. Then

(3) $$([\theta^+]^{<\theta})^{M'} = ([\theta^+]^{<\theta})^{M'[G]}.$$

In fact, $\subseteq$ is clear. Now suppose that $L \in ([\theta^+]^{<\theta})^{M'[G]}$. Then there exist an ordinal $\alpha < \theta$ and a bijection f from α onto L. Since $\mathbb{P}_\alpha$ is θ-closed, by 11.1 we have $f \in M'$, and hence $L \in M'$, as desired in (3). Similarly,

(4) $$(\mathrm{Fn}(\theta, \theta, \theta))^{M'} = (\mathrm{Fn}(\theta, \theta, \theta))^{M'[G]}.$$

(5) $$(\pi^0_\alpha)_G = \{(p, K) : p \in \mathrm{Fn}(\theta, \theta, \theta) \text{ and } K \in [{}^\theta\theta]^{<\theta}\}.$$

In fact, first suppose that $x \in (\pi^0_\alpha)_G$. Then there exist $p \in \mathrm{Fn}(\theta,\theta,\theta)$ and $H \in [\theta^+]^{<\theta}$ such that $x = (p, (\rho^\alpha_H)_G)$. Now $(\rho^\alpha_H)_G = \{(\sigma^\alpha_\gamma)_G : \gamma \in H\}$, and $(\sigma^\alpha_\gamma)_G \in {}^\theta\theta$ for each γ by (2). Thus x is in the right side of (3).

Second, suppose that $p \in \mathrm{Fn}(\theta,\theta,\theta)$ and $K \in [{}^\theta\theta]^{<\theta}$. For each $f \in K$ there is a $\gamma(f) < \theta^+$ such that $f = (\tau^\alpha_{\gamma(f)})_G$. Let $H = \{\gamma(f) : f \in K\}$. So H is a subset of θ^+ of size less than θ. By (2) we have $f = (\sigma^\alpha_{\gamma(f)})_G$ for each $f \in K$. Now $(\rho^\alpha_H)_G = K$, and so $(p, K) \in (\pi^0_\alpha)_G$, as desired. So (5) holds.

Next, we define

$$\begin{aligned}\pi^1_\alpha = \{(\mathrm{op}(\mathrm{op}(\check{p}, \rho^\alpha_H), \mathrm{op}(\check{p}', \rho^\alpha_{H'})), q) : {}& p, p' \in \mathrm{Fn}(\theta,\theta,\theta),\\ & H, H' \in [\theta^+]^{<\theta}, p' \subseteq p, H' \subseteq H, q \in \mathbb{P}_\alpha, \text{ and for all } \gamma \in H'\\ & \text{and all } \xi \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p'), q \Vdash_{\mathbb{P}_\alpha} \sigma^\alpha_\gamma(\check{\xi}) < (p(\xi))\check{\ }\}.\end{aligned}$$

Again, suppose that G is $\mathbb{P}_\alpha$-generic over M'. Then

$$\begin{aligned}(6)\qquad (\pi^1_\alpha)_G = \{((p,K),(p',K')) : {}& (p,K),(p',K') \in (\pi^0_\alpha)_G,\ p' \subseteq p,\ K' \subseteq K,\\ & \text{and for all } f \in K' \text{ and all } \xi \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p'),\ f(\xi) < p(\xi)\}.\end{aligned}$$

To prove this, first suppose that $x \in (\pi^1_\alpha)_G$. Then there are $q \in G$, $p, p' \in \mathrm{Fn}(\theta,\theta,\theta)$ and $H, H' \in [\theta^+]^{<\theta}$ such that $x = ((p, (\rho^\alpha_H)_G), (p', (\rho^\alpha_{H'}))_G)$, $p' \subseteq p$, $H' \subseteq H$, and for all $\gamma \in H'$ and all $n \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p')$, $q \Vdash \sigma^\alpha_\gamma(\check{\xi}) < (p(\xi))\check{\ }$. Then with $K = (\rho^\alpha_H)_G$ and $K' = (\rho^\alpha_{H'}))_G$, the desired conditions clearly hold.

Second, suppose that p, p', K, K' exist as on the right side of (4). Then by the definition of π^0_α, there are $H, H' \in [\theta^+]^{<\theta}$ such that $K = (\rho^\alpha_H)_G$ and $K' = (\rho^\alpha_{H'})_G$. Then $K' = \{(\sigma^\alpha_\gamma)_G : \gamma \in H'\}$. Hence for every $\gamma \in H'$ and all $\xi \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p')$ we have $(\sigma^\alpha_\gamma)_G(\xi) < p(\xi)$. Let $\langle(\xi_\nu, \psi_\nu) : \nu < \gamma\rangle$ enumerate all pairs (ξ, γ) such that $\xi \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p')$ and $\gamma \in H'$, with $\beta < \theta$, β limit. Now we define a system $\langle q_\nu : \nu \le \beta\rangle$ of members of $\mathbb{P}_\alpha$ by recursion. Let $q_0 = 1$. Suppose that q_ν has been defined so that $q_\nu \in G$. Now there is an $r \in G$ such that $r \Vdash \sigma^\alpha_{\gamma_\nu}(\hat{\xi}_\nu) < (p(\xi_\nu))\check{\ }$. Let $q_{\nu+1} \in G$ be such that $q_{\nu+1} \le r, q_\nu$. At limit stages $\le \beta$ we use that θ-closed property of $\mathbb{P}_\alpha$ to continue. Clearly q_β is as desired, showing that $((p,K),(p',K')) \in (p^1_\alpha)_G$.

Next, we let $\pi^2_\alpha = \{(\mathrm{op}(0,0), 1_{\mathbb{P}_\alpha})\}$. Then for any generic G, $(\pi^2_\alpha)_G = (0,0)$. Finally, let $\pi_\alpha = \mathrm{op}(\mathrm{op}(\pi^0_\alpha, \pi^1_\alpha), \pi^2_\alpha)$. This finishes the definition of π_α.

Using (5) and (6) it is clear that π_α is a $\mathbb{P}_\alpha$-name for a forcing order. To verify that it is full for $\downarrow\theta$-sequences, suppose that

$$\begin{aligned}&(7)\qquad p \in P_\alpha,\ \beta < \theta,\ \varphi_\xi \in \mathrm{dmn}(\pi^0_\alpha) \text{ for each } \xi < \beta,\\ &(8)\qquad \text{and if } \xi, \eta < \beta \text{ and } \xi < \eta, \text{ then } p \Vdash_{\mathbb{P}_\alpha} (\varphi_\xi \in \pi^0_\alpha) \wedge (\varphi_\eta \in \pi^0_\alpha) \wedge (\varphi_\eta \le_{\pi_\alpha} \varphi_\xi).\end{aligned}$$

We want to find $\psi \in \mathrm{dmn}(\pi^0_\alpha)$ such that

$$(9)\qquad p \Vdash \psi \in \pi^0_\alpha \text{ and } p \Vdash \psi \le_{\pi_\alpha} \varphi_\xi \text{ for each } \xi < \beta.$$

Since $\varphi_\xi \in \mathrm{dmn}(\pi^0_\alpha)$, there exist a $q_\xi \in \mathrm{Fn}(\theta,\theta,\theta)$ and an $H_\xi \in [\theta^+]^{<\theta}$ such that $\varphi_\xi = \mathrm{op}(\check{q}_\xi, \rho^\alpha_{H_\xi})$. Now if $\xi < \eta < \beta$, then $p \Vdash \varphi_\eta \le_{\pi_\alpha} \varphi_\xi$; hence $q_\xi \subseteq q_\eta$. Let $r = \bigcup_{\xi<\beta} \varphi_\xi$

and $K = \bigcup_{\xi<\beta} H_\xi$. Thus $r \in \mathrm{Fn}(\theta,\theta,\theta)$ and $K \in [\theta^+]^{<\theta}$. Let $\psi = \mathrm{op}(\check{r}, \rho^\alpha_K)$. Clearly $\psi \in \mathrm{dmn}(\pi^0_\alpha)$. Suppose that $\xi < \beta$. To show that $p \Vdash \psi \leq_{\pi_\alpha} q_\xi$, suppose that $p \in G$ with G $\mathbb{P}_\alpha$-generic over M'. Then $\psi_G = (r, (\rho^\alpha_K)_G)$, and clearly $(\rho^\alpha_K)_G) = K$. Suppose that $\gamma \in H_\xi$ and $\nu \in \mathrm{dmn}(r)\backslash\mathrm{dmn}(q_\xi)$. Say $\nu \in \mathrm{dmn}(q_\eta)$ with $\eta < \beta$. Clearly $\xi < \eta$. Since $(\varphi_\eta)_G \leq (\varphi_\xi)_G$ by (8), we have $r(\nu) = q_\eta(\nu) > (\sigma^\alpha_\gamma)_G(\nu)$. This proves that $\psi_G \leq (\varphi_\xi)_G$, and so (9) holds.

Now $\mathbb{P}_{\alpha+1}$ is defined by (I7) and (I8) in the definition of iteration. We now want to show that $\mathbb{P}_{\alpha+1}$ is θ^+-cc, and for this we will apply 15.10. we are assuming that $\mathbb{P}_\alpha$ is θ^+-cc, so it suffices to prove that $1 \Vdash_{\mathbb{P}_\alpha} \pi_\alpha - cc$. So, let G be $\mathbb{P}_\alpha$-generic over M'. As above, $2^{<\theta} = \theta$ in $M'[G]$. Now $|P_\alpha| \leq \theta^+$ by assumption. Hence $2^\theta = \theta^+$ in $M'[G]$ by 9.6 (with κ, λ, μ replaced by $\theta^+, \theta^+, \theta$ respectively). Hence $\pi_{\alpha G}$ is θ^+-cc by (5) and (6). So $\mathbb{P}_{\alpha+1}$ is θ^+-cc by 15.10.

$\mathbb{P}_{\alpha+1}$ is θ-closed by 29.5, since we have proved that π_α is full for $\downarrow\theta$-sequences. This finishes the recursion step from α to $\alpha+1$.

Now suppose that α is a limit ordinal $\leq \theta^{++}$. We let

$$\begin{aligned} P_\alpha = \{p : &p \text{ is a function with domain } \alpha,\ p_\xi \in P_\xi \text{ for all } \xi < \alpha \\ &\text{and } |\{\xi < \alpha : p_\xi \neq 1\}| < \theta\}. \end{aligned}$$

and for $p, q \in P_\alpha$, $p \leq q$ iff $p_\xi \leq q_\xi$ for all $\xi < \alpha$.

Now we show that $\mathbb{P}_\alpha$ has the θ^+-cc. Suppose that $\langle p^\gamma : \gamma < \theta^+\rangle$ is a system of members of P_α. Then we can apply the Δ-system theorem 10.1 to the system $\langle \mathrm{supp}(p^\gamma) : \gamma < \theta^+\rangle$, with κ, λ replaced by θ, θ^+ respectively. This gives us a set $L \in [\theta^+]^{\theta^+}$ and a set K such that for all distinct $\varphi, \gamma \in L$, $\mathrm{supp}(p^\gamma) \cap \mathrm{supp}(p^\delta) = K$. For $\gamma \in L$ and $\xi \in K$ we have $p^\gamma(\xi) \neq 1$, so we can write $p^\gamma(\xi) = \mathrm{op}(\check{q}^\gamma_\xi, \varphi_\xi)$ with $q^\gamma_\xi \in \mathrm{Fn}(\theta,\theta,\theta)$. Now for any $\gamma \in L$, the function $\langle a^\gamma_\xi : \xi \in K\rangle$ is a member of $\prod_{\xi\in K} \mathrm{Fn}(\theta,\theta,\theta)$, which has size at most θ. So there exist $L' \in [L]^{\theta+}$ and r such that $\langle q^\gamma_\xi : \xi \in K\rangle = \langle r_\xi : \xi \in K\rangle$ for all $\gamma \in L'$. Now it is clear that p^γ and p^δ are compatible for all $\gamma, \delta \in L'$, as desired.

By 29.5, $\mathbb{P}_\alpha$ is θ-closed. Clearly, for $\alpha < \theta^{++}$ $\mathbb{P}_\alpha$ has size at most θ^+.

This finishes the construction. For brevity let $\mathbb{R} = \mathbb{P}_{\theta^{++}}$.

Let G be $\mathbb{R}$-generic over M'.

(10) In $M'[G]$, if $\mathscr{F} \subseteq {}^\theta\theta$ and $|\mathscr{F}| < \theta^{++}$, then there is a $g \in {}^\theta\theta$ such that $f <_\theta g$ for all $f \in \mathscr{F}$.

For, let $\mathscr{F} = \{f_\xi : \xi < \theta^+\}$, possibly with repetitions. Let

$$\mathscr{F}' = \{(\xi, i, j) : \xi < \theta_1,\ i, j \in \theta, \text{ and } f_\xi(i) = j\}.$$

By 26.14 there is an $\alpha < \theta^{++}$ such that $\mathscr{F}' \in M'[i^{-1}_{\alpha\theta^{++}}[G]]$, and hence also $\mathscr{F} \in M'[i^{-1}_{\alpha\theta^{++}}[G]]$. For brevity write $G_\xi = i^{-1}_{\xi\theta^{++}}[G]$ for every $\xi < \theta^{++}$. Let

$$H_\alpha = \{\eta_{G_\alpha} : \eta \in \mathrm{dmn}(\pi^0_\alpha) \text{ and } p^\frown\langle\eta\rangle \in G_{\alpha+1} \text{ for some } p\}.$$

Let $\mathbb{Q}_\alpha = (\pi_\alpha)_{G_\alpha}$. Thus by (5) and (6),

$$
\begin{aligned}
&(11) && Q_\alpha = \{(p, K) : p \in \mathrm{fin}(\theta, \theta) \text{ and } K \in [{}^\theta\theta]^{<\theta}\};\\
&(12) && \leq_{\mathbb{Q}_\alpha} = \{((p, K), (p', K')) : (p, K), (p', K') \in (\pi_\alpha^0)_G,\ p' \subseteq p,\ K' \subseteq K,\\
&&& \qquad \text{and for all } f \in K' \text{ and all } \xi \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p'),\ f(\xi) < p(\xi)\}.
\end{aligned}
$$

Now (10) follows from 29.4.

Replacing κ, λ, μ in 24.3 by $\theta^{++}, \theta^{+}, \theta$ respectively, we get $2^\theta \leq \theta^{++}$ in $M'[G]$. Hence by (10), $2^\theta = \theta^{++}$ in $M'[G]$.

Replacing κ, λ, μ in 24.3 by $\theta^{++}, \theta^{+}, \theta^{+}$ respectively, we get $2^{\theta^+} \leq \theta^{+++}$ in $M'[G]$. Since $2^{\theta^+} = \theta^{+++}$ in M', it follows that $2^{\theta^+} = \theta^{+++}$ in $M'[G]$. $\square$

30. Cofinality of posets

We begin the study of possible cofinalities of partially ordered sets—the PCF theory. In this chapter we develop some combinatorial principles needed for the main results.

Ordinal-valued functions and their orderings

A *filter* on a set A is a collection F of subsets of A with the following properties:

(1) $A \in F$.
(2) If $X \in F$ and $X \subseteq Y \subseteq A$, then $Y \in F$.
(3) If $X, Y \in F$ then $X \cap Y \in F$.

A filter F is *proper* iff $F \neq \mathscr{P}(A)$.

Suppose that F is a filter on a set A and $R \subseteq \mathbf{On} \times \mathbf{On}$. Then for functions $f, g \in {}^A\mathbf{On}$ we define

$$f \, R_F \, g \quad \text{iff} \quad \{i \in A : f(i) \, R \, g(i)\} \in F.$$

The most important cases of this notion that we will deal with are $f <_F g$, $f \leq_F g$, and and $f =_F g$. Thus

$$\begin{aligned} f <_F g \quad &\text{iff} \quad \{i \in A : f(i) < g(i)\} \in F; \\ f \leq_F g \quad &\text{iff} \quad \{i \in A : f(i) \leq g(i)\} \in F; \\ f =_F g \quad &\text{iff} \quad \{i \in A : f(i) = g(i)\} \in F. \end{aligned}$$

Sometimes we use this notation for ideals rather than filters, using the duality between ideals and filters, which we now describe. An *ideal* on a set A is a collection I of subsets of A such that the following conditions hold:

(4) $\emptyset \in I$
(5) If $X \subseteq Y \in I$ then $X \in I$.
(6) If $X, Y \in I$ then $X \cup Y \in I$.

An ideal I is *proper* iff $I \neq \mathscr{P}(A)$.

If F is a filter on A, let $F' = \{X \subseteq A : A \backslash X \in F\}$. Then F' is an ideal on A. If I is an ideal on A, let $I^* = \{X \subseteq A : A \backslash X \in I\}$. Then I^* is a filter on A. If F is a filter on A, then $F'^* = F$. If I is an ideal on A, then $I^{*\prime} = I$.

Now if I is an ideal on A, then

$$\begin{aligned} f \, R_I \, g \quad &\text{iff} \quad \{i \in A : \neg(f(i) \, R_I \, g(i))\} \in I; \\ f <_I g \quad &\text{iff} \quad \{i \in A : f(i) \geq g(i)\} \in I; \\ f \leq_I g \quad &\text{iff} \quad \{i \in A : f(i) > g(i)\} \in I; \\ f =_I g \quad &\text{iff} \quad \{i \in A : f(i) \neq g(i)\} \in I. \end{aligned}$$

Some more notation: $R_I(f, g) = \{i \in I : f(i) \, R \, g(i)\}$. In particular, $<_I (f, g) = \{i \in I : f(i) < g(i)\}$ and $\leq_I (f, g) = \{i \in I : f(i) \leq g(i)\}$.

The following trivial proposition is nevertheless important in what follows.

Proposition 30.1. *Let F be a proper filter on A. Then*

(i) $<_F$ *is irreflexive and transitive.*
(ii) $\leq_F$ *is reflexive on* ${}^A\mathbf{On}$*, and it is transitive.*
(iii) $f \leq_F g <_F h$ *implies that* $f <_F h$*.*
(iv) $f <_F g \leq_F h$ *implies that* $f <_F h$*.*
(v) $f <_F g$ *or* $f =_F g$ *implies* $f \leq_F g$*.*
(vi) If $f =_F g$*, then* $g \leq_F f$*.*
(vii) If $f \leq_F g \leq_F f$*, then* $f =_F g$*.* □

Some care must be taken in working with these notions. The following examples illustrate this.

(1) An example with $f \leq_F g$ but neither $f <_F g$ nor $f =_F g$ nor $f = g$: Let $A = \omega$, $F = \{A\}$, and define $f, g \in {}^\omega\omega$ by setting $f(n) = n$ for all n and

$$g(n) = \begin{cases} n & \text{if } n \text{ is even,} \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$$

(2) An example where $f =_F g$ but neither $f <_F g$ nor $f = g$: Let $A = \omega$ and let F consist of all subsets of ω that contain all even natural numbers. Define f and g by

$$f(n) = \begin{cases} n & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd;} \end{cases} \quad g(n) = \begin{cases} n & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Products and reduced products

In the preceding section we were considering ordering-type relations on the proper classes ${}^A\mathbf{On}$. Now we restrict ourselves to sets. Suppose that $h \in {}^A\mathbf{On}$. We specialize the general notion by considering $\prod_{a\in A} h(a) \subseteq {}^A\mathbf{On}$. To eliminate trivialities, we usually assume that $h(a)$ is a limit ordinal for every $a \in A$; then we call h *non-trivial*.

Proposition 30.2. *If* F *is a proper filter on* A*,* $g, h \in {}^A\mathbf{On}$*,* h *is non-trivial, and* $g <_F h$*, then there is a* $k \in \prod_{a\in A} h(a)$ *such that* $g =_F k$*.*

Proof. For any $a \in A$ let

$$k(a) = \begin{cases} g(a) & \text{if } g(a) < h(a), \\ 0 & \text{otherwise.} \end{cases}$$

Thus $k \in \prod_{a\in A} h(a)$. Moreover,

$$\{a \in A : g(a) = k(a) \supseteq \{a \in A : g(a) < h(a)\} \in F,$$

so $g =_F k$. □

We will frequently consider the structure $(\prod_{a\in A} h(a), <_F, \leq_F)$ in what follows. For most considerations it is equivalent to consider the associated *reduced product*, which we define as follows. Note that $=_F$ is an equivalence relation on the set $\prod_{a\in A} h(a)$. We define the underlying set of the reduced product to be the collection of all equivalence classes under $=_F$; it is denoted by $\prod_{a\in A} h(a)/F$. Further, we define, for $x, y \in \prod_{a\in A} h(a)/F$,

$$x <_F y \quad \text{iff} \quad \exists f, g \in \prod A[x = [f],\ y = [g],\ \text{and } f <_F g];$$
$$x \leq_F y \quad \text{iff} \quad \exists f, g \in \prod A[x = [f],\ y = [g],\ \text{and } f \leq_F g].$$

Here $[h]$ denotes the equivalence class of $h \in \prod A$ under $=_F$.

Proposition 30.3. *Suppose that $h \in {}^A\mathbf{On}$ is nontrivial, and $f, g \in \prod_{a\in A} h(a)$. Then*
(i) $[f] <_F [g]$ iff $f <_F g$.
(ii) $[f] \leq_F [g]$ iff $f \leq_F g$.

Proof. (i): The direction $\Leftarrow$ is obvious. Now suppose that $[f] <_F [g]$. Then there are $f', g' \in \prod A$ such that $[f] = [f']$, $[g] = [g']$, and $f' <_F g'$. Hence

$$\begin{aligned}&\{\kappa \in A : f(\kappa) = f'(\kappa)\} \cap \{\kappa \in A : g(\kappa) = g'(\kappa)\} \cap \{\kappa \in A : f'(\kappa) < g'(\kappa)\}\\ &\quad \subseteq \{\kappa \in A : f(\kappa) < g(\kappa)\},\end{aligned}$$

and it follows that $\{\kappa \in A : f(\kappa) < g(\kappa)\} \in F$, and so $f <_F g$.
(ii): similarly. □

A filter F on A is an *ultrafilter* iff F is proper, and is maximal under all the proper filters on A. Equivalently, F is proper, and for any $X \subseteq A$, either $X \in F$ or $A\backslash X \in F$. The dual notion to an ultrafilter is a maximal ideal.

If F is an ultrafilter on A, then $\prod_{a\in A} h(a)/F$ is an *ultraproduct* of h.

Proposition 30.4. *If $h \in^A$ **On** is nontrivial and F is an ultrafilter on A, then $<_F$ is a linear order on $\prod_{a\in A} h(a)/F$, and $[f] \leq_F [g]$ iff $[f] <_F [g]$ or $[f] = [g]$.*

Proof. By Proposition 30.1(iii) and Proposition 30.3, $<_F$ is transitive. Also, from Proposition 30.3 it is clear that $<_F$ is irreflexive. Now suppose that $f, g \in \prod A$; we want to show that $[f]$ and $[g]$ are comparable. Assume that $[f] \neq [g]$. Thus $\{\kappa \in A : f(\kappa) = g(\kappa)\} \notin F$, so $\{\kappa \in A : f(\kappa) \neq g(\kappa)\} \in F$. Since

$$\{\kappa \in A : f(\kappa) \neq g(\kappa)\} = \{\kappa \in A : f(\kappa) < g(\kappa)\} \cup \{\kappa \in A : g(\kappa) < f(\kappa)\},$$

it follows that $[f] < [g]$ or $[g] < [f]$.

Thus $<_F$ is a linear order on $\prod A/F$.

Next,

$$\{\kappa \in A : f(\kappa) \leq g(\kappa)\} = \{\kappa \in A : f(\kappa) = g(\kappa)\} \cup \{\kappa \in A : f(\kappa) < g(\kappa)\},$$

so it follows by Proposition 30.3 that $[f] \leq_F [g]$ iff $[f] = [g]$ or $[f] <_F [g]$. □

Basic cofinality notions

In this section we allow partial orders P to be proper classes. We may speak of a partial ordering P if the relation $<_P$ is clear from the context. Recall the essential equivalence of the notion of a partial ordering with the "$\le$" version; see the easy exercise E13.15.

A *double ordering* is a system $(P, \le_P, <_P, =_P)$ such that the following conditions hold (cf. Proposition 30.1):

(i) $<_P$ is irreflexive and transitive.

(ii) $\le_P$ is reflexive on P, and it is transitive.

(iii) $f \le_P g <_P h$ implies that $f <_P h$.

(iv) $f <_P g \le_P h$ implies that $f <_P h$.

(v) $f <_P g$ or $f =_P g$ implies $f \le_P g$.

(vi) If $f =_P g$, then $g \le_P f$.

(vii) If $f \le_P g \le_P f$, then $f =_P g$.

Proposition 30.5. *For any set A any proper filter F on A, and any $P \subseteq {}^A\mathbf{On}$ the system $(P, \le_F, <_F, =_F)$ is a double ordering.* □

Proposition 30.6. *Let $h \in {}^A\mathbf{On}$, with h taking only limit ordinal values, and let F be a proper filter on A. Then $(\prod_{a\in A} h(a)/F, \le_F, <_F, =)$ is a double ordering.* □

We now give some general definitions, applying to any double ordering $(P, \le_P, <_P)$ unless otherwise indicated.

• A subclass $X \subseteq P$ is *cofinal* in P iff $\forall p \in P \exists q \in X(p \le_P q)$. By the condition (3) above, this is equivalent to saying that X is cofinal in P iff $\forall p \in P \exists q \in X(p <_P q)$.

• Since clearly P itself is cofinal in P, we can make the basic definition of the cofinality $\mathrm{cf}(P)$ of P, for a *set* P:

$$\mathrm{cf}(P) = \min\{|X| : X \text{ is cofinal in } P\}.$$

Note that $\mathrm{cf}(P)$ can be singular. For, let $A = \omega$, $h(a) = \omega_a$ for all $a \in \omega$, $I = \{\emptyset\}$, and $Y = \prod_{a\in A} h(a)$.. Suppose that X is cofinal in $\prod_{a\in A} h(a)$. Take any $a \in \omega$; we show that $\omega_a \le |X|$. We define a one-one sequence $\langle f_\alpha : \alpha < \omega_i\rangle$ of elements of X by recursion. Suppose that f_β has been defined for all $\beta < \alpha$. Let k be the member of $\prod_{a\in A} h(a)$ such that $k(b) = 0$ for all $b \neq a$, while $k(a) \in \omega_a \backslash \{f_\beta(a) : \beta < \alpha\}$. Choose $f_\alpha \in X$ such that $k <_I f_\alpha$.

• A sequence $\langle p_\xi : \xi < \lambda\rangle$ of elements of P is *$<_P$-increasing* iff $\forall \xi, \eta < \lambda(\xi < \eta \to p_\xi <_P p_\eta)$. Similarly for *$\le_P$-increasing.*

• Suppose that P is a double order and is a set. We say that P *has true cofinality* iff P has a linearly ordered subset which is cofinal.

Proposition 30.7. *Suppose that a set P is a double order, and $\langle p_\alpha : \alpha < \lambda\rangle$ is strictly increasing in the sense of P, is cofinal in P, and λ is regular. Then P has true cofinality, and its cofinality is λ.*

Proof. Obviously P has true cofinality. If X is a subset of P of size less than λ, for each $q \in X$ choose $\alpha_q < \lambda$ such that $q < p_{\alpha_q}$. Let $\beta = \sup_{q\in X} \alpha_q$. Then $\beta < \lambda$ since λ is regular. For any $q \in X$ we have $q < p_\beta$. This argument shows that $\mathrm{cf}(P) = \lambda$. □

Proposition 30.8. *Suppose that P is a double ordering, P a set, and P has true cofnality. Then:*

(i) $\mathrm{cf}(P)$ *is regular.*

(ii) $\mathrm{cf}(P)$ *is the least size of a linearly ordered subset which is cofinal in P.*

(iii) There is a $<_P$-increasing, cofinal sequence in P of length $\mathrm{cf}(P)$.

Proof. Let X be a linearly ordered subset of P which is cofinal in P, and let $\{y_\alpha : \alpha < \mathrm{cf}(P)\}$ be a subset of P which is cofinal in P; we do not assume that $\langle y_\alpha : \alpha < \mathrm{cf}(P)\rangle$ is $<_P$- or $\le_P$-increasing.

(iii): We define a sequence $\langle x_\alpha : \alpha < \mathrm{cf}(P)\rangle$ by recursion. Let x_0 be any element of X. If x_α has been defined, let $x_{\alpha+1} \in X$ be such that $x_\alpha, y_\alpha < x_{\alpha+1}$; it exists since X is cofinal, using condition (3). Now suppose that $\alpha < \mathrm{cf}(P)$ is limit and x_β has been defined for all $\beta < \alpha$. Then $\{x_\beta : \beta < \alpha\}$ is not cofinal in P, so there is a $z \in P$ such that $z \not\le x_\beta$ for all $\beta < \alpha$. Choose $x_\alpha \in X$ so that $z < x_\alpha$. Since X is linearly ordered, we must then have $x_\beta < x_\alpha$ for all $\beta < \alpha$. This finishes the construction. Since $y_\alpha < x_{\alpha+1}$ for all $\alpha < \mathrm{cf}(P)$, it follows that $\{x_\alpha : \xi < \mathrm{cf}(P)\}$ is cofinal in P. So (iii) holds.

(i): Suppose that $\mathrm{cf}(P)$ is singular, and let $\langle \beta_\xi : \xi < \mathrm{cf}(\mathrm{cf}(P))\rangle$ be a strictly increasing sequence cofinal in $\mathrm{cf}(P)$. With $\langle x_\alpha : \alpha < \mathrm{cf}(P)\rangle$ as in (iii), it is then clear that $\{x_{\beta_\xi} : \xi < \mathrm{cf}(\mathrm{cf}(P))\}$ is cofinal in P, contradiction (since $\mathrm{cf}(\mathrm{cf}(P)) < \mathrm{cf}(P)$ because $\mathrm{cf}(P)$ is singular).

(ii): By (iii), there is a linearly ordered subset of P of size $\mathrm{cf}(P)$ which is cofinal in P; by the definition of cofinality, there cannot be one of smaller size. □

For P with true cofinality, the cardinal $\mathrm{cf}(P)$ is called the *true cofinality* of P, and is denoted by $\mathrm{tcf}(P)$. We write $\mathrm{tcf}(P) = \lambda$ to mean that P has true cofinality, and it is equal to λ.

• P is λ-*directed* iff for any subset Q of P such that $|Q| < \lambda$ there is a $p \in P$ such that $q \le_P p$ for all $q \in Q$; equivalently, there is a $p \in P$ such that $q <_P p$ for all $q \in Q$.

Proposition 30.9. (Pouzet) *Assume that P is a double ordering which is a set. For any infinite cardinal λ, we have* $\mathrm{tcf}(P) = \lambda$ *iff the following two conditions hold:*

(i) P has a cofinal subset of size λ.

(ii) P is λ-directed.

Proof. $\Rightarrow$ is clear, remembering that λ is regular. Now assume that (i) and (ii) hold, and let X be a cofinal subset of P of size λ.

First we show that λ is regular. Suppose that it is singular. Write $X = \bigcup_{\alpha<\mathrm{cf}(\lambda)} Y_\alpha$ with $|Y_\alpha| < \lambda$ for each $\alpha < \mathrm{cf}(\lambda)$. Let p_α be an upper bound for Y_α for each $\alpha < \mathrm{cf}(\lambda)$, and let q be an upper bound for $\{p_\alpha : \alpha < \mathrm{cf}(\lambda)\}$. Choose $r > q$. Then choose $s \in X$ with $r \leq s$. Say $s \in Y_\alpha$. Then $s \leq p_\alpha \leq q < r \leq s$, contradiction.

So, λ is regular. Let $X = \{r_\alpha : \alpha < \lambda\}$. Now we define a sequence $\langle p_\alpha : \alpha < \lambda\rangle$ by recursion. Having defined p_β for all $\beta < \alpha$, by (ii) let p_α be such that $p_\beta < p_\alpha$ for all $\beta < \alpha$, and $r_\beta < p_\alpha$ for all $\beta < \alpha$. Clearly this sequence shows that $\mathrm{tcf}(P, <_P) = \lambda$. $\square$

Proposition 30.10. *Let P be a set. If G is a cofinal subset of P, then* $\mathrm{cf}(P) = \mathrm{cf}(G)$*. Moreover,* $\mathrm{tcf}(P) = \mathrm{tcf}(G)$*, in the sense that if one of them exists then so does the other, and they are equal.* (That is what we mean in the future too when we assert the equality of true cofinalities.)

Proof. Let H be a cofinal subset of P of size $\mathrm{cf}(P)$. For each $p \in H$ choose $q_p \in G$ such that $p \leq_P q_p$. Then $\{q_p : p \in H\}$ is cofinal in G. In fact, if $r \in G$, choose $p \in H$ such that $r \leq_P p$. Then $r \leq_P q_p$, as desired. This shows that $\mathrm{cf}(G) \leq \mathrm{cf}(P)$.

Now suppose that K is a cofinal subset of G. Then it is also cofinal in P. For, if $p \in P$ choose $q \in G$ such that $p \leq_P q$, and then choose $r \in K$ such that $q \leq_P r$. So $p \leq_P r$, as desired. This shows the other inequality.

For the true cofinality, we apply Theorem 30.9. So suppose that P has true cofinality λ. By Theorem 30.9 and the first part of this proof, G has a cofinal subset of size λ, since cofinality is the same as true cofinality when the latter exists. Now suppose that $X \subseteq G$ is of size $< \lambda$. Choose an upper bound p for it in P. Then choose $q \in G$ such that $p \leq_P q$. So q is an upper bound for X, as desired. Thus since Theorem 30.9(i) and 30.9(ii) hold for G, it follows from that theorem that $\mathrm{tcf}(G) = \lambda$.

The other implication, that the existence of $\mathrm{tcf}(G, <)$ implies that of $\mathrm{tcf}(P, <)$ and their equality, is even easier, since a sequence cofinal in G is also cofinal in P. $\square$

• A sequence $\langle p_\xi : \xi < \lambda\rangle$ of elements of P is *persistently cofinal* iff

$$\forall h \in P \exists \xi_0 < \lambda \forall \xi(\xi_0 \leq \xi < \lambda \Rightarrow h <_P p_\xi).$$

Proposition 30.11. *(i) If $\langle p_\xi : \xi < \lambda\rangle$ is $<_P$-increasing and cofinal in P, then it is persistently cofinal.*

(ii) If $\langle p_\xi : \xi < \lambda\rangle$ and $\langle p'_\xi : \xi < \lambda\rangle$ are two sequences of members of P, $\langle p_\xi : \xi < \lambda\rangle$ is persistently cofinal in P, and $p_\xi \leq_P p'_\xi$ for all $\xi < \lambda$, then also $\langle p'_\xi : \xi < \lambda\rangle$ is persistently cofinal in P. $\square$

• If $X \subseteq P$, then an *upper bound* for X is an element $p \in P$ such that $q \leq_P p$ for all $q \in X$.

• If $X \subseteq P$, then a *least upper bound* for X is an upper bound a for X such that $a \leq_P a'$ for every upper bound a' for X. So if a and b are least upper bounds for X, then $a \leq_P b \leq_P a$.

It is possible here to have $a \neq b$. For example, let $A = \omega$, $h(a) = \omega + \omega$ for all $a \in \omega$, $f_n(m) = m + n$ for all $m, n \in \omega$, $I = \{Y \subseteq \omega : \text{each member of } Y \text{ is odd}\}$. $X = \{f_n : n \in \omega\}$. We consider the double order $(\prod_{a\in\omega} h(a), \leq_I, <_I)$. Let

$$g(m) = \begin{cases} \omega & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd} \end{cases} \qquad h(m) = \begin{cases} \omega & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd} \end{cases}$$

Then g and h are least upper bounds for X, while $g \neq h$.

• If $X \subseteq P$, then a *minimal upper bound* for X is an upper bound a for X such that if b is an upper bound for X and $b \leq_P a$, then $a \leq_P b$.

Proposition 30.12. *If $X \subseteq P$ and a is a least upper bound for X, then a is a minimal upper bound for X.* □

Now we come to an ordering notion which is basic for pcf theory.

• If $X \subseteq P$ and for every $x \in X$ there is an $x' \in X$ such that $x <_P x'$, then an element $a \in P$ is an *exact upper bound* of X provided

(1) a is a least upper bound for X, and

(2) X is cofinal in $\{p \in P : p <_P a\}$.

Note that under the hypothesis here, $a \notin X$, and hence $x <_F a$ for all $x \in X$ by (1).

Here is an example of a set X with a least upper bound but no exact upper bound. Let $A = \omega$, $h(a) = \omega + \omega$ for all $a \in \omega$, and for $m, n \in \omega$,

$$f_n(m) = \begin{cases} n & \text{if } m \neq n, \\ 0 & \text{if } m = n, \end{cases}$$

$X = \{f_n : n \in \omega\}$, $I = \{\emptyset\}$. We consider the double order $(\prod_{a\in\omega} h(a), \leq_I, <_I)$. Then a least upper bound for X is the function a such that $a(m) = \omega$ for all $m \in \omega$, but X does not have an exact upper bound.

Ordinal-valued functions and exact upper bounds

In this section we give some simple facts about exact upper bounds in the case of most interest to us—the partial ordering of ordinal-valued functions.

First we note that the rough equivalence between products and reduced products continues to hold for the cofinality notions introduced above. We state this for the most important properties above:

Proposition 30.13. *Suppose that $h \in {}^A\mathbf{On}$, and h takes only limit ordinal values. Then*

(i) If $X \subseteq \prod_{a\in A} h(a)$, then X is cofinal in $(\prod_{a\in A} h(a), <_I, \leq_I)$ iff $\{[f] : f \in X\}$ is cofinal in $(\prod_{a\in A} h(a)/I, <_I, \leq_I)$.

(ii) $\mathrm{cf}(\prod_{a\in A} h(a), <_I, \leq_I) = \mathrm{cf}(\prod_{a\in A} h(a)/I, <_I, \leq_I)$.

(iii) $\mathrm{tcf}(\prod_{a\in A} h(a), <_I, \leq_I) = \mathrm{tcf}(\prod_{a\in A} h(a)/I, <_I, \leq_I)$.

(iv) If $X \subseteq \prod_{a\in A} h(a)$ and $f \in \prod_{a\in A} h(a)$, then f is an exact upper bound for X iff $[f]$ is an exact upper bound for $\{[g] : g \in X\}$.

Proof. (i) is immediate from Proposition 30.3. For (ii), if X is cofinal in the system $(\prod_{a\in A} h(a), <_I, \leq_I)$, then clearly $\{[f] : f \in X\}$ is cofinal in $(\prod_{a\in A} h(a)/I, <_I, \leq_I)$, by Proposition 30.3 again; so $\geq$ holds. Now suppose that $\{[f] : f \in Y\}$ is cofinal in $(\prod_{a\in A} h(a)/I, <_I, \leq_I)$. Given $g \in \prod_{a\in A} h(a)$, choose $f \in Y$ such that $[g] <_I [f]$. Then $g <_I f$. So Y is cofinal in $(\prod_{a\in A} h(a), <_I, \leq_I)$, and $\leq$ holds.

(iii) and (iv) are proved similarly. □

The following obvious proposition will be useful.

Proposition 30.14. *Suppose that $F \cup \{f, g\} \subseteq {}^A\mathbf{On}$, I is an ideal on A, and $f =_I g$. Suppose that f is an upper bound, least upper bound, minimal upper bound, or exact upper bound for F under $\le_I$. Then also g is an upper bound, least upper bound, minimal upper bound, or exact upper bound for F under $\le_I$, respectively.*

Here is our simplest existence theorem for exact upper bounds.

• If X is a collection of members of ${}^A\mathbf{On}$, then $\sup X \in {}^A\mathbf{On}$ is defined by

$$(\sup X)(a) = \sup\{f(a) : f \in X\}.$$

Proposition 30.15. *Suppose that $\lambda > |A|$ is a regular cardinal, and $f = \langle f_\xi : \xi < \lambda\rangle$ is an increasing sequence of members of ${}^A\mathbf{On}$ in the partial ordering $<$ of everywhere dominance. (That is, $f < g$ iff $f(a) < g(a)$ for all $a \in A$.) Then* $\sup f$ *is an exact upper bound for f, and* $\mathrm{cf}((\sup f)(a)) = \lambda$ *for every $a \in A$.*

Proof. For brevity let $h = \sup f$. Then clearly h is an upper bound for f. Now suppose that $f_\xi \le g \in {}^A\mathbf{On}$ for all $\xi < \lambda$. Then for any $a \in A$ we have $h(a) = \sup_{\xi<\lambda} f_\xi(a) \le g(a)$, so $h \le g$. Thus h is a least upper bound for f. Now suppose that $k \in {}^A\mathbf{On}$ and $k < h$. Then for every $a \in A$ we have $k(a) < h(a)$, and hence there is a $\xi_a < \lambda$ such that $k(a) < f_{\xi_a}(a)$. Let $\eta = \sup_{a\in A} \xi_a$. So $\eta < \lambda$ since λ is regular and greater than $|A|$. Clearly $k < f_\eta$, as desired. □

The next proposition gives equivalent definitions of least upper bounds for our special partial order.

Proposition 30.16. *Suppose that I is a proper ideal on A, $F \subseteq {}^A\mathbf{On}$, and $f \in {}^A\mathbf{On}$. Then the following conditions are equivalent.*

(i) f is a least upper bound of F under $\le_I$.

(ii) f is an upper bound of F under $\le_I$, and for any $f' \in {}^A\mathbf{On}$, if f' is an upper bound of F under $\le_I$ and $f' \le_I f$, then $f =_I f'$.

(iii) f is a minimal upper bound of F under $\le_I$.

Proof. (i)⇒(ii): Assume (i) and the hypotheses of (ii). Hence $f \le_I f'$, so $f =_I f'$ by Proposition 30.1(vii).

(ii)⇒(iii): Assume (ii), and suppose that $g \in {}^A\mathbf{On}$ is an upper bound for F and $g \le_I f$. Then $g =_I f$ by (ii), so $f \le_I g$.

(iii)⇒(i): Assume (iii). Let $g \in {}^A\mathbf{On}$ be any upper bound for F. Define $h(a) = \min(f(a), g(a))$ for all $a \in A$. Then h is an upper bound for F, since if $k \in F$, then $\{a \in A : k(a) > f(a)\} \in I$ and also $\{a \in A : k(a) > g(a)\} \in I$, and

$$\{a \in A : k(a) > \min(f(a), g(a))\} \subseteq \{a \in A : k(a) > f(a)\} \cup \{a \in A : k(a) > g(a)\} \in I,$$

so $k \leq_I h$. Also, clearly $h \leq_I f$. So by (iii), $f \leq_I h$, and hence $f \leq_I g$, as desired. $\square$

In the next proposition we see that in the definition of exact upper bound we can weaken the condition (1), under a mild restriction on the set in question.

Proposition 30.17. *Suppose that F is a nonempty set of functions in ${}^A\mathbf{On}$ and $\forall f \in F\exists f' \in F[f <_I f']$. Suppose that h is an upper bound of F, and $\forall g \in^A \mathbf{On}$, if $g <_I h$ then there is an $f \in F$ such that $g <_I f$. Then h is an exact upper bound for F.*

Proof. First note that $\{a \in A : h(a) = 0\} \in I$. In fact, choose $f \in F$. Then $f <_I h$, and so $\{a \in A : h(a) = 0\} \subseteq \{a \in A : f(a) \geq h(a)\} \in I$, as desired.

Now we show that h is a least upper bound for F. Let k be any upper bound. Let

$$l(a) = \begin{cases} k(a) & \text{if } k(a) < h(a), \\ 0 & \text{otherwise.} \end{cases}$$

Since $\{a \in A : l(a) \geq h(a)\} \subseteq \{a \in A : h(a) = 0\}$, it follows by the above that $\{a \in A : l(a) \geq h(a)\} \in I$, and so $l <_I h$. So by assumption, choose $f \in F$ such that $l <_I f$. Now $f \leq_I k$, so $l <_I k$ and hence

$$\{a \in A : k(a) < h(a)\} \subseteq \{a \in A : l(a) \geq k(a)\} \in I,$$

so $h \leq_I k$, as desired.

For the other property in the definition of exact upper bound, suppose that $g <_I h$. Then by assumption there is an $f \in F$ such that $g <_I f$, as desired. $\square$

Corollary 30.18. *If $h \in {}^A\mathbf{On}$ is non trivial and $F \subseteq \prod_{a\in A} h(a)$, then h is an exact upper bound of F with respect to an ideal I on A iff F is cofinal in $\prod_{a\in A} h(a)$.* $\square$

In the next proposition we use the standard notation I^+ for $A\backslash I$. The proposition shows that exact upper bounds restrict to smaller sets A.

Proposition 30.19. *Suppose that F is a nonempty subset of ${}^A\mathbf{On}$, I is a proper ideal on A, h is an exact upper bound for F with respect to I, and $\forall f \in F\exists f' \in F(f <_I f')$. Also, suppose that $A_0 \in I^+$. Then:*

(i) $J \stackrel{\text{def}}{=} I \cap \mathscr{P}(A_0)$ is a proper ideal on A_0.

(ii) For any $f, f' \in {}^A\mathbf{On}$, if $f <_I f'$ then $f \restriction A_0 <_J f' \restriction A_0$.

(iii) $h \restriction A_0$ is an exact upper bound for $\{f \restriction A_0 : f \in F\}$.

(i) is clear. Assume the hypotheses of (ii). Then

$$\{a \in A_0 : f'(a) \leq f(a)\} \subseteq \{a \in A : f'(a) \leq f(a)\} \in I,$$

and so $f \restriction A_0 <_J f' \restriction A_0$.

For (iii), by (ii) we see that $h \restriction A_0$ is an upper bound for $\{f \restriction A_0 : f \in F\}$. To see that it is an exact upper bound, we will apply Proposition 30.18. So, suppose that $k <_J h \restriction A_0$. Fix $f \in F$. Now define $g \in {}^A\mathbf{On}$ by setting

$$g(a) = \begin{cases} f(a) & \text{if } a \in A\backslash A_0, \\ k(a) & \text{if } a \in A_0. \end{cases}$$

Then

$$\{a \in A : g(a) \geq h(a)\} \subseteq \{a \in A : f(a) \geq h(a)\} \cup \{a \in A_0 : k(a) \geq h(a)\} \in I,$$

so $g <_I h$. Hence there is an $l \in F$ such that $g <_I l$. Hence

$$\{a \in A_0 : k(a) \geq l(a)\} \subseteq \{a \in A : g(a) \geq l(a)\} \in I,$$

so $k <_J l$, as desired. $\square$

Next, increasing the ideal maintains exact upper bounds:

Proposition 30.20. *Suppose that F is a nonempty subset of ${}^A\mathbf{On}$, I is a proper ideal on A, h is an exact upper bound for F with respect to I, and $\forall f \in F \exists f' \in F(f <_I f')$.*

Let J be a proper ideal on A such that $I \subseteq J$. Then h is an exact upper bound for F with respect to J.

Proof. We will apply Proposition 30.17. Note that h is clearly an upper bound for F with respect to J. Now suppose that $g <_J h$. Let $f \in F$. Define g' by

$$g'(a) = \begin{cases} g(a) & \text{if } g(a) < h(a), \\ f(a) & \text{otherwise.} \end{cases}$$

Then $\{a \in A : g'(a) \geq h(a)\} \subseteq \{a \in A : f(a) \geq h(a)\} \in I$, since $f <_I h$. So $g' <_I h$. Hence by the exactness of h there is a $k \in F$ such that $g' <_I k$. So

$$\begin{aligned} \{a : g(a) \geq k(a)\} \subseteq & \{a \in A : h(a) > g(a) \geq k(a)\} \cup \{a \in A : h(a) \leq g(a)\} \\ \subseteq & \{a \in A : g'(a) \geq k(a)\} \cup \{a \in A : h(a) \leq g(a)\}, \end{aligned}$$

and this union is in J since the first set is in I and the second one is in J. Hence $g <_J k$, as desired. $\square$

Again we turn from the general case of proper classes ${}^A\mathbf{On}$ to the sets $\prod_{a \in A} h(a)$, where $h \in^A \mathbf{On}$ has only limit ordinal values. We prove some results which show that under a weak hypothesis we can restrict attention to $\prod A$ for A a nonempty set of infinite regular cardinals instead of $\prod_{a \in A} h(a)$, as far as cofinality notions are concerned. Here $\prod A$ consists of all choice functions f with domain A; $f(a) \in a$ for all $a \in A$.

Proposition 30.21. *Suppose that $h \in {}^A\mathbf{On}$ and $h(a)$ is a limit ordinal for every $a \in A$. For each $a \in A$, let $S(a) \subseteq h(a)$ be cofinal in $h(a)$ with order type $\mathrm{cf}(h(a))$. Suppose that I is a proper ideal on A. Then*

(i) $\mathrm{cf}(\prod_{a \in A} h(a), <_I) = \mathrm{cf}(\prod_{a \in A} S(a), <_I)$ *and*

(ii) $\mathrm{tcf}(\prod_{a \in A} h(a), <_I) = \mathrm{tcf}(\prod_{a \in A} S(a), <_I)$.

Proof. For each $f \in \prod h$ define $g_f \in \prod_{a \in A} S(a)$ by setting

$$g_f(a) = \text{ least } \alpha \in S(a) \text{ such that } f(a) \leq \alpha.$$

We prove (i): suppose that $X \subseteq \prod h$ and X is cofinal in $(\prod h, <_I)$; we show that $\{g_f : f \in X\}$ is cofinal in $\mathrm{cf}(\prod_{a\in A} S(a), <_I)$, and this will prove $\geq$. So, let $k \in \prod_{a\in A} S(a)$. Thus $k \in \prod h$, so there is an $f \in X$ such that $k <_I f$. Since $f \leq g_f$, it follows that $k <_I g_f$, as desired. Conversely, suppose that $Y \subseteq \prod_{a\in A} S(a)$ and Y is cofinal in $(\prod_{a\in A} S(a), <_I)$; we show that also Y is cofinal in $\prod h$, and this will prove $\leq$ of the claim. Let $f \in \prod h$. Then $f \leq g_f$, and there is a $k \in Y$ such that $g_f <_I k$; so $f <_I k$, as desired.

This finishes the proof of (i).

For (ii), first suppose that $\mathrm{tcf}(\prod h, <_I)$ exists; call it λ. Thus λ is an infinite regular cardinal. Let $\langle f_i : i < \lambda\rangle$ be a $<_I$-increasing cofinal sequence in $\prod h$. We claim that $g_{f_i} \leq g_{f_j}$ if $i < j < \lambda$. In fact, if $a \in A$ and $f_i(a) < f_j(a)$, then $f_i(a) < f_j(a) \leq g_{f_j}(a) \in S(a)$, and so by the definition of g_{f_i} we get $g_{f_i}(a) \leq g_{f_j}(a)$. This implies that $g_{f_i} \leq_I g_{f_j}$. Now $\mathrm{cf}(\prod h, <_I) = \lambda$, so for any $B \in [\lambda]^{<\lambda}$ there is a $j < \lambda$ such that $g_{f_i} <_I f_j \leq g_{f_j}$. It follows that we can take a subsequence of $\langle g_{f_i} : i < \lambda\rangle$ which is strictly increasing modulo I; it is also clearly cofinal, and hence $\lambda = \mathrm{tcf}(\prod_{a\in A} S(a), <_I)$.

Conversely, suppose that $\mathrm{tcf}(\prod_{a\in A} S(a), <_I)$ exists; call it λ. Let $\langle f_i : i < \lambda\rangle$ be a $<_I$-increasing cofinal sequence in $\prod_{a\in A} S(a)$. Then it is also a sequence showing that $\mathrm{tcf}(\prod h, <_I)$ exists and equals $\mathrm{tcf}(\prod_{a\in A} S(a), <_I)$. □

Proposition 30.22. *Suppose that $\langle L_a : a \in A\rangle$ and $\langle M_a : a \in A\rangle$ are systems of linearly ordered sets such that each L_a and M_a has no last element. Suppose that L_a is isomorphic to M_a for all $a \in A$. Let I be any ideal on A. Then*

$$\left(\prod_{a\in A} L_a, <_I, \leq_I\right) \cong \left(\prod_{a\in A} M_a, <_I, \leq_I\right). \qquad \square$$

Putting the last two propositions together, we see that to determine cofinality and true cofinality of $(\prod h, <_I, \leq_I)$, where $h \in {}^A\mathbf{On}$ and $h(a)$ is a limit ordinal for all $a \in A$, it suffices to take the case in which each $h(a)$ is an infinite regular cardinal. (One passes from $h(a)$ to $S(a)$ and then to $\mathrm{cf}(h(a))$.) We can still make a further reduction, given in the following useful lemma.

Lemma 30.23. (Rudin-Keisler) *Suppose that c maps the set A into the class of regular cardinals, and $B = \{c(a) : a \in A\}$ is its range. For any ideal I over A, define its Rudin-Keisler projection J on B by*

$$X \in J \quad \textit{iff} \quad X \subseteq B \textit{ and } c^{-1}[X] \in I.$$

Then J is an ideal on B, and there is an isomorphism h of $\prod B/J$ into $\prod_{a\in A} c(a)/I$ such that for any $e \in \prod B$ we have $h(e/J) = \langle e(c(a)) : a \in A\rangle/I$.

If $|A| < \min(B)$, then the range of h is cofinal in $\prod_{a\in A} c(a)/I$, and we have

(i) $\mathrm{cf}(\prod B/J) = \mathrm{cf}(\prod_{a\in A} c(a)/I$ *and*

(ii) $\mathrm{tcf}(\prod B/J) = \mathrm{tcf}(\prod_{a\in A} c(a)/I)$.

Proof. Clearly J is an ideal. Next, for any $e \in \prod B$ let $\overline{e} = \langle e(c(a)) : a \in A\rangle$. Then for any $e_1, e_2 \in \prod B$ we have

$$\begin{aligned} e_1 =_J e_2 \quad &\text{iff} \quad \{b \in B : e_1(b) \neq e_2(b)\} \in J \\ &\text{iff} \quad c^{-1}[\{b \in B : e_1(b) \neq e_2(b)\}] \in I \\ &\text{iff} \quad \{a \in A : e_1(c(a)) \neq e_2(c(a))\} \in I \\ &\text{iff} \quad \overline{e_1} =_I \overline{e_2}. \end{aligned}$$

This shows that h exists as indicated and is one-one. Similarly, h preserves $<_I$ in each direction. So the first part of the lemma holds.

Now suppose that $|A| < \min(B)$. Let G be the range of h. By Proposition 30.11, (i) and (ii) follow from G being cofinal in $\prod_{a\in A} c(a)/I$. Let $g \in \prod_{a\in A} c(a)$. Define $e \in \prod B$ by setting, for any $b \in B$,

$$e(b) = \sup\{g(a) : a \in A \text{ and } c(a) = b\}.$$

The additional supposition implies that $e \in \prod B$. Now note that $\{a \in A : g(a) > e(c(a))\} = \emptyset \in I$, so that $g/I \leq h(e/J)$, as desired. $\square$

According to these last propositions, the calculation of true cofinalities for partial orders of the form $(\prod_{a\in A} h(a), <_I)$, with $h \in {}^A\mathbf{On}$ and $h(a)$ a limit ordinal for every $a \in A$, and with $|A| < \min(\mathrm{cf}(h(a)))$, reduces to the calculation of true cofinalities of partial orders of the form $(\prod B, <_J)$ with B a set of regular cardinals with $|B| < \min(B)$.

Lemma 30.24. *If $(P_i, <_i)$ is a partial order with true cofinality λ_i for each $i \in I$ and D is an ultrafilter on I, then* $\mathrm{tcf}(\prod_{i\in I} \lambda_i/D) = \mathrm{tcf}(\prod_{i\in I} P_i/D)$.

Proof. Note that $\prod_{i\in I} \lambda_i/D$ is a linear order, and so its true cofinality μ exists and equals its cofinality. So the lemma is asserting that the ultraproduct $\prod_{i\in I} P_i/D$ has μ as true cofinality.

Let $\langle g_\xi : \xi < \mu\rangle$ be a sequence of members of $\prod_{i\in I} \lambda_i$ such that $\langle g_\xi/D : \xi < \mu\rangle$ is strictly increasing and cofinal in $\prod_{i\in I} \lambda_i/D$. For each $i \in I$ let $\langle f_{\xi,i} : \xi < \lambda_i\rangle$ be strictly increasing and cofinal in $(P_i, <_i)$. For each $\xi < \mu$ define $h_\xi \in \prod_{i\in I} P_i$ by setting $h_\xi(i) = f_{g_\xi(i),i}$. We claim that $\langle h_\xi/D : \xi < \mu\rangle$ is strictly increasing and cofinal in $\prod_{i\in I} P_i/D$ (as desired).

To prove this, first suppose that $\xi < \eta < \mu$. Then

$$\{i \in I : h_\xi(i) < h_\eta(i)\} = \{i \in I : f_{g_\xi(i),i} <_i f_{g_\eta(i),i}\} = \{i \in I : g_\xi(i) < g_\eta(i)\} \in D;$$

so $h_\xi/D < h_\eta/D$.

Now suppose that $k \in \prod_{i\in I} P_i$; we want to find $\xi < \mu$ such that $k/D < h_\xi/D$. Define $l \in \prod_{i\in I} \lambda_i$ by letting $l(i)$ be the least $\xi < \mu$ such that $k(i) < f_{\xi,i}$. Choose $\xi < \mu$ such that $l/D < g_\xi/D$. Now if $l(i) < g_\xi(i)$, then $k(i) < f_{l(i),i} <_i f_{g_\xi(i),i} = h_\xi(i)$. So $k/D < h_\xi/D$. $\square$

We introduce several notions leading up to an existence theorem for exact upper bounds: projections, strongly increasing sequences, a partition property, and the bounding projection property.

We start with the important notion of **projections**. By a *projection framework* we mean a triple (A, I, S) consisting of a nonempty set A, an ideal I on A, and a sequence $\langle S_a : a \in A\rangle$ of nonempty sets of ordinals. Suppose that we are given such a framework. We define $\sup S$ in the natural way: it is a function with domain A, and $(\sup S)(a) = \sup(S_a)$ for every $a \in A$. Thus $\sup S \in {}^A\mathbf{On}$. Now suppose also that we have a function $f \in {}^A\mathbf{On}$. Then we define the *projection* of f onto $\prod_{a\in A} S_a$, denoted by $f^+ = \text{proj}(f, S)$, by setting, for any $a \in A$,

$$f^+(a) = \begin{cases} \min(S_a \backslash f(a)) & \text{if } f(a) < \sup(S_a), \\ \min(S_a) & \text{otherwise.} \end{cases}$$

Thus

$$f^+(a) = \begin{cases} f(a) & \text{if } f(a) \in S_a \text{ and } f(a) \text{ is not} \\ & \text{the largest element of } S_a, \\ \text{least } x \in S_a \text{ such that } f(a) < x & \text{if } f(a) \notin S_a \text{ and } f(a) < \sup(S_a), \\ \min(S_a) & \text{if } \sup(S_a) \le f(a). \end{cases}$$

Proposition 30.25. *Let a projection framework be given, with the notation above.*

(i) If $f \in {}^A\mathbf{On}$, then $f^+ \in \prod_{a\in A} S_a$.

(ii) If $f_1, f_2 \in {}^A\mathbf{On}$ and $f_1 =_I f_2$, then $f_1^+ =_I f_2^+$.

(iii) If $f \in {}^A\mathbf{On}$ and $f <_I \sup S$, then $f \le_I f^+$, and for every $g \in \prod_{a\in A} S_a$, if $f \le_I g$ then $f^+ \le_I g$.

Proof. (i) and (ii) are clear. For (iii), suppose that $f \in {}^A\mathbf{On}$ and $f <_I \sup S$. Then if $f(a) > f^+(a)$ we must have $f(a) \ge \sup(S_a)$. Hence $f \le_I f^+$. Now suppose that $g \in \prod_{a\in A} S_a$ and $f \le_I g$. If $f(a) \le g(a)$ and $f(a) < \sup(S_a)$, then $f^+(a) \le g(a)$. Hence

$$\{a \in A : g(a) < f^+(a)\} \subseteq \{a \in A : f(a) > g(a)\} \cup \{a \in A : f(a) \ge \sup(S_a)\} \in I,$$

so $f^+ \le_I g$. $\square$

Another important notion in discussing exact upper bounds is as follows. Let I be an ideal over A, L a set of ordinals, and $f = \langle f_\xi : \xi \in L\rangle$ a sequence of members of ${}^A\mathbf{On}$. Then we say that f is *strongly increasing under I* iff there is a system $\langle Z_\xi : \xi \in L\rangle$ of members of I such that

$$\forall \xi, \eta \in L[\xi < \eta \Rightarrow \forall a \in A\backslash(Z_\xi \cup Z_\eta)[f_\xi(a) < f_\eta(a)]].$$

Under the same assumptions we say that f is *very strongly increasing under I* iff there is a system $\langle Z_\xi : \xi \in L\rangle$ of members of I such that

$$\forall \xi, \eta \in L[\xi < \eta \Rightarrow \forall a \in A\backslash Z_\eta[f_\xi(a) < f_\eta(a)].$$

Proposition 30.26. *Under the above assumptions, f is very strongly increasing under I iff for every $\xi \in L$ we have*

$$(*) \qquad \sup\{f_\alpha + 1 : \alpha \in L \cap \xi\} \leq_I f_\xi.$$

Proof. $\Rightarrow$: suppose that f is very strongly increasing under I, with sets Z_ξ as indicated. Let $\xi \in L$. Suppose that $a \in A \backslash Z_\xi$. Then for any $\alpha \in L \cap \xi$ we have $f_\alpha(a) < f_\xi(a)$, and so $\sup\{f_\alpha(a) + 1 : \alpha \in L \cap \xi\} \leq f_\xi(a)$; it follows that $(*)$ holds.

$\Leftarrow$: suppose that $(*)$ holds for each $\xi \in L$. For each $\xi \in L$ let

$$Z_\xi = \{a \in A : \sup\{f_\alpha(a) + 1 : \alpha \in L \cap \xi\} > f_\xi(a)\};$$

it follows that $Z_\xi \in I$. Now suppose that $\alpha \in L$ and $\alpha < \xi$. Suppose that $a \in A \backslash Z_\xi$. Then $f_\alpha(a) < f_\alpha(a) + 1 \leq \sup\{f_\beta(a) + 1 : \beta \in L \cap \xi\} \leq f_\xi(a)$, as desired. □

Lemma 30.27. (The sandwich argument) *Suppose that $h = \langle h_\xi : \xi \in L\rangle$ is strongly increasing under I, L has no largest element, and ξ' is the successor in L of ξ for every $\xi \in L$. Also suppose that $f_\xi \in {}^A\mathbf{On}$ is such that*

$$h_\xi <_I f_\xi \leq_I h_{\xi'} \textit{ for every } \xi \in L.$$

Then $\langle f_\xi : \xi \in L\rangle$ is also strongly increasing under I.

Proof. Let $\langle Z_\xi : \xi \in L\rangle$ testify that h is strongly increasing under I. For every $\xi \in L$ let

$$W_\xi = \{a \in A : h_\xi(a) \geq f_\xi(a) \text{ or } f_\xi(a) > h_{\xi'}(a)\}.$$

Thus by hypothesis we have $W_\xi \in I$. Let $Z^\xi = W_\xi \cup Z_\xi \cup Z_{\xi'}$ for every $\xi \in L$; so $Z_\xi \in I$. Then if $\xi_1 < \xi_2$, both in L, and if $a \in A \backslash (Z^{\xi_1} \cup Z^{\xi_2})$, then

$$f_{\xi_1}(a) \leq h_{\xi_1'}(a) \leq h_{\xi_2}(a) < f_{\xi_2}(a);$$

these three inequalities hold because $a \in A \backslash W_{\xi_1}$, $a \in A \backslash (Z_{\xi_1'} \cup Z_{\xi_2})$, and $a \in A \backslash W_{\xi_2}$ respectively. □

Now we give a proposition connecting the notion of strongly increasing sequence with the existence of exact upper bounds.

Proposition 30.28. *Let I be a proper ideal over A, let $\lambda > |A|$ be a regular cardinal, and let $f = \langle f_\xi : \xi < \lambda\rangle$ be a $<_I$ increasing sequence of functions in ${}^A\mathbf{On}$. Then the following conditions are equivalent:*

(i) f has a strongly increasing subsequence of length λ under I.

(ii) f has an exact upper bound h such that $\{a \in A : \mathrm{cf}(h(a)) \neq \lambda\} \in I$.

(iii) f has an exact upper bound h such that $\mathrm{cf}(h(a)) = \lambda$ for all $a \in A$.

(iv) There is a sequence $g = \langle g_\xi : \xi < \lambda\rangle$ such that $g_\xi < g_\eta$ (everywhere) for $\xi < \eta$, and f is cofinally equivalent to g, in the sense that $\forall\xi < \lambda\exists\eta < \lambda(f_\xi <_I g_\eta)$ and $\forall\xi < \lambda\exists\eta < \lambda(g_\xi <_I f_\eta)$.

Proof. (i)⇒(ii): Let $\langle \eta(\xi) : \xi < \lambda \rangle$ be a strictly increasing sequence of ordinals less than λ, thus with supremum λ since λ is regular, and assume that $\langle f_{\eta(\xi)} : \xi < \lambda \rangle$ is strongly increasing under I. Hence for each $\xi < \lambda$ let $Z_\xi \in I$ be chosen correspondingly. We define for each $a \in A$

$$h(a) = \sup\{f_{\eta(\xi)}(a) : \xi < \lambda, a \notin Z_\xi\}.$$

To see that h is an exact upper bound for f, we are going to apply Proposition 30.17. If $f_{\eta(\xi)}(a) > h(a)$, then $a \in Z_\xi \in I$. Hence $f_{\eta(\xi)} \leq_I h$ for each $\xi < \lambda$. Then for any $\xi < \lambda$ we have $f_\xi \leq_I f_{\eta(\xi)} \leq_I h$, so h bounds every f_ξ. Now suppose that $d <_I h$. Let $M = \{a \in A : d(a) \geq h(a)\}$; so $M \in I$. For each $a \in A\backslash M$ we have $d(a) < h(a)$, and so there is a $\xi_a < \lambda$ such that $d(a) < f_{\eta(\xi_a)}(a)$ and $a \notin Z_{\xi_a}$. Since $|A| < \lambda$ and λ is regular, the ordinal $\rho \stackrel{\text{def}}{=} \sup_{a \in A\backslash M} \xi_a$ is less than λ. We claim that $d <_I f_{\eta(\rho)}$. In fact, suppose that $a \in A\backslash(M \cup Z_\rho)$. Then $a \in A\backslash(Z_{\xi_a} \cup Z_\rho)$, and so $d(a) < f_{\eta(\xi_a)}(a) \leq f_{\eta(\rho)}(a)$. Thus $d <_I f_{\eta(\rho)}$, as claimed. Now it follows easily from Proposition 30.17 that h is an exact upper bound for f.

For the final portion of (ii), it suffices to show

(1) There is a $W \in I$ such that $\text{cf}(h(a)) = \lambda$ for all $a \in A\backslash W$.

In fact, let

$$W = \{a \in A : \exists \xi_a < \lambda \forall \xi' \in [\xi_a, \lambda)[a \in Z_{\xi'}]\}.$$

Since $|A| < \lambda$, the ordinal $\rho \stackrel{\text{def}}{=} \sup_{a \in W} \xi_a$ is less than λ. Clearly $W \subseteq Z_\rho$, so $W \in I$. For $a \in A\backslash W$ we have $\forall \xi < \lambda \exists \xi' \in [\xi, \lambda)[a \notin Z_{\xi'}]$. This gives an increasing sequence $\langle \sigma_\nu : \nu < \lambda \rangle$ of ordinals less than λ such that $a \notin Z_{\sigma_\nu}$ for all $\nu < \lambda$. By the strong increasing property it follows that $f_{\eta(\sigma_0)}(a) < f_{\eta(\sigma_1)}(a) < \cdots$, and so $h(a)$ has cofinality λ. This proves (1), and with it, (ii).

(ii)⇒(iii): Let $W = \{a \in A : \text{cf}(h(a)) \neq \lambda\}$; so $W \in I$ by (ii). Since I is a proper ideal, choose $a_0 \in A\backslash W$, and define

$$h'(a) = \begin{cases} h(a) & \text{if } a \in A\backslash W, \\ h(a_0) & \text{if } a \in W. \end{cases}$$

Then $h =_I h'$, and it follows that h' satisfies the properties needed.

(iii)⇒(iv): For each $a \in A$, let $\langle \mu^a_\xi : \xi < \lambda \rangle$ be a strictly increasing sequence of ordinals with supremum $h(a)$. Define $g_\xi(a) = \mu^a_\xi$ for all $a \in A$ and $\xi < \lambda$. Clearly $g_\xi < g_\eta$ if $\xi < \eta$. Now suppose that $\xi < \lambda$. Then $f_\xi <_I h$. For each $a \in A$ such that $f_\xi(a) < h(a)$ choose $\rho_a < \lambda$ such that $f_\xi(a) < \mu^a_{\rho_a}$. Since $|A| < \lambda$, choose $\eta < \lambda$ such that $\rho_a < \eta$ for all $a \in A$. Then for any $a \in A$ such that $f_\xi(a) < h(a)$ we have $f_\xi(a) < \mu^a_\eta = g_\eta(a)$. Hence $f_\xi <_I g_\eta$, which is half of what is desired in (iv).

Now suppose that $\xi < \lambda$. Then $g_\xi < h$, so by the exactness of h, there is an $\eta < \lambda$ such that $g_\xi <_I f_\eta$, as desired.

(iv)⇒(i): Assume (iv). Define strictly increasing continuous sequences $\langle \eta(\xi) : \xi < \lambda \rangle$ and $\langle \rho(\xi) : \xi < \lambda \rangle$ of ordinals less than λ as follows. Let $\eta(0) = 0$, and choose $\rho(0)$ so that $g_0 <_I f_{\rho(0)}$. If $\eta(\xi)$ and $\rho(\xi)$ have been defined, choose $\eta(\xi + 1) > \eta(\xi)$ such that

$f_{\rho(\xi)} \leq_I g_{\eta(\xi+1)}$, and choose $\rho(\xi+1) > \rho(\xi)$ such that $g_{\eta(\xi+1)} <_I f_{\rho(\xi+1)}$. Thus for every $\xi < \lambda$ we have

$$g_{\eta(\xi)} <_I f_{\rho(\xi)} \leq_I g_{\eta(\xi+1)}.$$

since obviously $\langle g_{\eta(\xi)} : \xi < \lambda \rangle$ is strongly increasing under I, Lemma 30.27 gives (i). □

The notion of a strongly increasing sequence is clarified by giving an example of a sequence such that no subsequence is strongly increasing. This example depends on the following well-known lemma.

Lemma 30.29. *If κ is a regular cardinal and I is the ideal $[\kappa]^{<\kappa}$ on κ, then there is a sequence $f \stackrel{\text{def}}{=} \langle f_\xi : \xi < \kappa^+ \rangle$ of members of ${}^\kappa\kappa$ such that $f_\xi <_I f_\eta$ whenever $\xi < \eta < \kappa$.*

Proof. We construct the sequence by recursion. Let $f_0(\alpha) = 0$ for all $\alpha < \kappa$. If f_ξ has been defined, let $f_{\xi+1}(\alpha) = f_\xi(\alpha) + 1$ for all $\alpha < \kappa$. Now suppose that $\xi < \kappa$ is a limit ordinal, and f_η has been defined for every $\eta < \xi$. Let $\langle \eta(\beta) : \beta < \gamma \rangle$ be a strictly increasing sequence of ordinals with supremum ξ, where $\gamma = \mathrm{cf}(\xi)$. Thus $\gamma \leq \kappa$. Define

$$f_\xi(\alpha) = (\sup_{\beta \leq \alpha} f_{\eta(\beta)}(\alpha)) + 1.$$

The sequence constructed this way is as desired. For example, if ξ is a limit ordinal as above, then for each $\rho < \kappa$ we have $\{\alpha < \kappa : f_{\eta(\rho)}(\alpha) \geq f_\xi(\alpha)\} \subseteq \rho$, and so $f_{\eta(\rho)} <_I f_\xi$. □

Now let $A = \kappa$ and let I and f be as in the lemma. Suppose that f has a strongly increasing subsequence of length κ^+ under I. Then by proposition 30.28, f has an exact upper bound h such that $\mathrm{cf}(h(\alpha)) = \kappa^+$ for all $\alpha < \kappa$. Now the function k with domain κ taking the constant value κ is clearly an upper bound for f. Hence $h \leq_I k$. Hence there is an $\alpha < \kappa$ such that $h(\alpha) \leq k(\alpha) = \kappa$, contradiction.

A further fact along these lines is as follows.

Lemma 30.30. *Suppose that $I = [\omega]^{<\omega}$ and $f \stackrel{\text{def}}{=} \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of members of ${}^\omega\omega$ which has an exact upper bound h, where λ is an infinite cardinal. Then $\langle f_\xi : \xi < \lambda \rangle$ is a scale, i.e., for any $g \in {}^\omega\omega$ there is a $\xi < \lambda$ such that $g <_I f_\xi$.*

Proof. Let $k(m) = \omega$ for all $m < \omega$. Then k is an upper bound for f under $<_I$, and so $h \leq_I k$. Letting $h'(m) = \min(h(m), k(m))$ for all $m \in \omega$, we thus get $h =_I h'$. So by Proposition 30.14, h' is also an exact upper bound for f. Hence we may assume that $h(m) \leq \omega$ for every $m < \omega$. Now we claim

(1) $\exists n < \omega \forall p \geq n (0 < h(p))$.

In fact, the set $\{p \in \omega : f_0(p) \geq h(p)\}$ is in I, so there is an n such that $f_0(p) < h(p)$ for all $p \geq n$, as desired in (1).

Let n_0 be as in (1).

(2) $M \stackrel{\text{def}}{=} \{p \in \omega : h(p) \neq \omega\}$ is finite.

For, suppose that M is infinite. Define

$$l(p) = \begin{cases} h(p) - 1 & \text{if } 0 < h(p) < \omega, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $l <_I h$. For, $\{p : l(p) \geq h(p)\} \subseteq \{p : h(p) = 0\} \in I$. So our claim holds. Now by exactness, choose $\xi < \kappa$ such that $l <_I f_\xi$. Then we can choose $p \in M$ such that $l(p) < f_\xi(p) < h(p)$, contradiction.

Thus M is finite. Hence we may assume that $h(p) = \omega$ for all p, and the desired conclusion of the lemma follows. $\square$

Now there is a model M of ZFC in which there are no scales (see for example Blass [∞]), and yet it is easy to see that there is a sequence $f \stackrel{\text{def}}{=} \langle f_\xi : \xi < \omega_1 \rangle$ which is $<_I$-increasing. Hence by Lemma 30.30, this sequence does not have an exact upper bound.

Another fact which helps the intuition on exact upper bounds is as follows.

Lemma 30.31. *Let κ be a regular cardinal, and let $I = [\kappa]^{<\kappa}$. For each $\xi < \kappa$ let $f_\xi \in {}^\kappa\kappa$ be defined by $f_\xi(\alpha) = \xi$ for all $\alpha < \kappa$. Thus $f \stackrel{\text{def}}{=} \langle f_\xi : \xi < \kappa \rangle$ is increasing everywhere. Claim: f does not have a least upper bound under $<_I$. (Hence it does not have an exact upper bound.)*

Proof. Suppose that h is an upper bound for f under $<_I$. We find another upper bound k for f under $<_I$ such that h is not $\leq_I k$. First we claim

(1) $\forall \alpha < \kappa \exists \beta < \kappa \forall \gamma \geq \beta (\alpha \leq h(\gamma))$.

In fact, otherwise we get a $\xi < \kappa$ such that for all $\beta < \kappa$ there is a $\gamma > \beta$ such that $\xi > h(\gamma)$. But then $|\{\alpha < \kappa : f_\xi(\alpha) > h(\alpha)\}| = \kappa$, contradiction.

By (1) there is a strictly increasing sequence $\langle \beta_\alpha : \alpha < \kappa \rangle$ of ordinals less than κ such that for all $\alpha < \kappa$ and all $\gamma \geq \beta_\alpha$ we have $\alpha < h(\gamma)$. Now we define $k \in {}^\kappa\kappa$ by setting, for each $\gamma < \kappa$,

$$k(\gamma) = \begin{cases} \alpha & \text{if } \beta_{\alpha+1} \leq \gamma < \beta_{\alpha+2}, \\ h(\gamma) & \text{otherwise.} \end{cases}$$

To see that k is an upper bound for f under $<_I$, take any $\xi < \kappa$. If $\beta_{\xi+1} \leq \gamma$, then $h(\gamma) \geq \xi + 1$, and hence $k(\gamma) \geq \xi = f_\xi(\gamma)$, as desired. For each $\xi < \kappa$ we have $k(\beta_{\xi+1}) = \xi < h(\beta_{\xi+1})$, so h is not $\leq_I k$. $\square$

Now we define a partition property. Suppose that I is an ideal over a set A, λ is an uncountable regular cardinal $> |A|$, $f = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of members of ${}^A\mathbf{On}$, and κ is a regular cardinal such that $|A| < \kappa \leq \lambda$. The following property of these things is denoted by $(*)_\kappa$:

$(*)_\kappa$ For all unbounded $X \subseteq \lambda$ there is an $X_0 \subseteq X$ of order type κ such that $\langle f_\xi : \xi \in X_0 \rangle$ is strongly increasing under I.

Proposition 30.32. *Assume the above notation, with $\kappa < \lambda$. Then $(*)_\kappa$ holds iff the set*

$$\{\delta < \lambda : \mathrm{cf}(\delta) = \kappa \textit{ and } \langle f_\xi : \xi \in X_0 \rangle \textit{ is strongly increasing under } I \textit{ for some unbounded } X_0 \subseteq \delta\}$$

is stationary in λ.

Proof. Let S be the indicated set of ordinals δ.

$\Rightarrow$: Assume $(*)_\kappa$ and suppose that $C \subseteq \lambda$ is a club. Choose $C_0 \subseteq C$ of order type κ such that $\langle f_\xi : \xi \in C_0 \rangle$ is strongly increasing under I. Let $\delta = \sup(C_0)$. Clearly $\delta \in C \cap S$.

$\Leftarrow$: Assume that S is stationary in λ, and suppose that $X \subseteq \lambda$ is unbounded. Define

$$C = \{\alpha \in \lambda : \alpha \text{ is a limit ordinal and } X \cap \alpha \text{ is unbounded in } \alpha\}.$$

We check that C is club in λ. For closure, suppose that $\alpha < \lambda$ is a limit ordinal and $C \cap \alpha$ is unbounded in α; we want to show that $\alpha \in C$. So, we need to show that $X \cap \alpha$ is unbounded in α. To this end, take any $\beta < \alpha$; we want to find $\gamma \in X \cap \alpha$ such that $\beta < \gamma$. Since $C \cap \alpha$ is unbounded in α, choose $\delta \in C \cap \alpha$ such that $\beta < \delta$. By the definition of C we have that $X \cap \delta$ is unbounded in δ. So we can choose $\gamma \in X \cap \delta$ such that $\beta < \gamma$. Since $\gamma < \delta < \alpha$, γ is as desired. So, indeed, C is closed.

To show that C is unbounded in λ, take any $\beta < \lambda$; we want to find an $\alpha \in C$ such that $\beta < \alpha$. Since X is unbounded in λ, we can choose a sequence $\gamma_0 < \gamma_1 < \cdots$ of elements of X with $\beta < \gamma_0$. Now λ is uncountable and regular, so $\sup_{n\in\omega} \gamma_n < \lambda$, and it is the member of C we need.

Now choose $\delta \in C \cap S$. This gives us an unbounded set X_0 in δ such that $\langle f_\xi : \xi \in X_0 \rangle$ is strongly increasing under I. Now also $X \cap \delta$ is unbounded, since $\delta \in C$. Hence we can define by induction two increasing sequences $\langle \eta(\xi) : \xi < \kappa \rangle$ and $\langle \nu(\xi) : \xi < \kappa \rangle$ such that each $\eta(\xi)$ is in X_0, each $\nu(\xi)$ is in X, and $\eta(\xi) < \nu(\xi) \le \eta(\xi+1)$ for all $\xi < \kappa$. It follows by the sandwich argument, Lemma 30.28, that $X_1 \stackrel{\text{def}}{=} \{\nu(\xi) : \xi < \kappa\}$ is a subset of X as desired in $(*)_\kappa$. $\square$

Finally, we introduce the bounding projection property.

Suppose that $f = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of functions in ${}^A\mathbf{On}$, with λ a regular cardinal $> |A|$. Also suppose that κ is a regular cardinal and $|A| < \kappa \le \lambda$.

We say that f has the *bounding projection property for* κ iff whenever $\langle S(a) : a \in A \rangle$ is a system of nonempty sets of ordinals such that each $|S(a)| < \kappa$ and for each $\xi < \lambda$ we have $f_\xi <_I \sup(S(a))$, then for some $\xi < \lambda$, the function $\operatorname{proj}(f_\xi, \langle S(a) : a \in A \rangle)$ $<_I$-bounds f.

We need the following simple result.

Proposition 30.33. *Suppose that $f = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of functions in $\mathbf{On}^A$, with λ a regular cardinal $> |A|$. Also suppose that κ is a regular cardinal and $|A| < \kappa \le \lambda$. Assume that f has the bounding projection property for κ.*

Also suppose that $f' = \langle f'_\xi : \xi < \lambda \rangle$ is a sequence of functions in $\mathbf{On}^A$, and $f_\xi =_I f'_\xi$ for every $\xi < \lambda$.

Then f' has the bounding projection property for κ.

Proof. Clearly f' is $<_I$-increasing, so that the setup for the bounding projection property holds. Now suppose that $\langle S(a) : a \in A \rangle$ is a system of nonempty sets of ordinals such that each $|S(a)| < \kappa$ and for each $\xi < \lambda$ we have $f'_\xi <_I \sup(S)$. Then the same is true for f, so by the bounding projection property for f we can choose $\xi < \lambda$ such that the function $\operatorname{proj}(f_\xi, \langle S(a) : a \in A \rangle)$ $<_I$-bounds f. Now suppose that $\eta < \lambda$. Then

$f_\eta \leq_I \text{proj}(f_\xi, \langle S(a) : a \in A\rangle)$. Hence $f'_\eta \leq_I \text{proj}(f_\xi, \langle S(a) : a \in A\rangle)$, and $\text{proj}(f_\xi, \langle S(a) : a \in A\rangle) = \text{proj}(f'_\xi, \langle S(a) : a \in A\rangle)$, as desired. $\square$

The following proposition shows that we can weaken the bounded projection property somewhat, by replacing "$<_I$" by "$<$ (everywhere)".

Proposition 30.34. *Suppose that $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence of functions in $\mathbf{On}^A$, with λ a regular cardinal $> |A|$. Also suppose that κ is a regular cardinal and $|A| < \kappa \leq \lambda$. Then the following conditions are equivalent:*

(i) f has the bounding projection property for κ.

(ii) If $\langle S(a) : a \in A\rangle$ is a system of nonempty sets of ordinals such that each $|S(a)| < \kappa$ and for each $\xi < \lambda$ we have $f_\xi < \sup(S)$ (everywhere), then for some $\xi < \lambda$, the function proj$(f_\xi, \langle S(a) : a \in A\rangle)$ $<_I$-bounds f.

Proof. Obviously (i)⇒(ii). Now assume that (ii) holds, and suppose that $\langle S(a) : a \in A\rangle$ is a system of sets of ordinals such that each $|S(a)| < \kappa$ and for each $\xi < \lambda$ we have $f_\xi <_I \sup(S)$. Now for each $a \in A$ let

$$\gamma(a) = \begin{cases} \sup\{f_\xi(a) + 1 : \xi < \lambda \text{ and } f_\xi(a) \geq \sup(S(a))\} & \text{if this set is nonempty,} \\ \sup(S(a)) + 1 & \text{otherwise;} \end{cases}$$

$$S'(a) = S(a) \cup \{\gamma(a)\}.$$

Note that $f_\xi < \sup(S')$ everywhere. Hence by (ii), there is a $\xi < \lambda$ such that the function $\text{proj}(f_\xi, \langle S'(a) : a \in A\rangle)$ $<_I$-bounds f. Now let $\eta < \lambda$. If $f_\xi(a) < \sup(S(a))$ and $f_\eta(a) < (\text{proj}(f_\xi, \langle S'(a) : a \in A\rangle))(a)$, then

$$\begin{aligned}(\text{proj}(f_\xi, \langle S'(a) : a \in A\rangle))(a) &= \min(S'(a)\backslash f_\xi(a)) \\ &= \min(S(a)\backslash f_\xi(a)) \\ &= (\text{proj}(f_\xi, \langle S(a) : a \in A\rangle))(a).\end{aligned}$$

Hence $f_\eta <_I \text{proj}(f_\xi, \langle S(a) : a \in A\rangle)$, as desired. $\square$

Lemma 30.35. (Bounding projection lemma) *Suppose that I is an ideal over A, $\lambda > |A|$ is a regular cardinal, $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence satisfying $(*)_\kappa$ for a regular cardinal κ such that $|A| < \kappa \leq \lambda$. Then f has the bounding projection property for κ.*

Proof. Assume the hypothesis of the lemma and of the bounding projection property for κ. For every $\xi < \lambda$ let

$$f^+_\xi = \text{proj}(f_\xi, S).$$

Suppose that the conclusion of the bounding projection property fails. Then for every $\xi < \lambda$, the function f^+_ξ is not a bound for f, and so there is a $\xi' < \lambda$ such that $f_{\xi'} \not\leq_I f^+_\xi$. Since $f_\xi \leq_I f^+_\xi$, we must have $\xi < \xi'$. Clearly for any $\xi'' \geq \xi'$ we have $f_{\xi''} \not\leq_I f^+_\xi$. Thus for every $\xi'' \geq \xi'$ we have $<(f^+_\xi, f_{\xi''}) \in I^+$. Now we define a sequence $\langle \xi(\mu) : \mu < \lambda\rangle$ of elements of λ by recursion. Let $\xi(0) = 0$. Suppose that $\xi(\mu)$ has been defined. Choose $\xi(\mu+1) > \xi(\mu)$ so that $<(f^+_{\xi(\mu)}, f_{\xi''}) \in I^+$ for every $\xi'' \geq \xi(\mu+1)$. If ν is limit and $\xi(\mu)$ has been defined for all $\mu < \nu$, let $\xi(\nu) = \sup_{\mu<\nu} \xi(\mu)$. Then let X be the range of this sequence. Thus

$$\text{if } \xi, \xi' \in X \text{ and } \xi < \xi', \text{ then } <(f^+_\xi, f_{\xi'}) \in I^+.$$

Since $(*)_\kappa$ holds, there is a subset $X_0 \subseteq X$ of order type κ such that $\langle f_\xi : \xi \in X_0\rangle$ is strongly increasing under I. Let $\langle Z_\xi : \xi \in X_0\rangle$ be as in the definition of strongly increasing under I.

For every $\xi \in X_0$, let ξ' be the successor of ξ in X_0. Note that

$$<(f^+_\xi, f_{\xi'})\backslash(Z_\xi \cup Z_{\xi'} \cup \{a \in A : f_\xi(a) \geq \sup(S(a))\}) \in I^+,$$

and hence it is nonempty. So, choose

$$a_\xi \in <(f^+_\xi, f_{\xi'})\backslash(Z_\xi \cup Z_{\xi'} \cup \{a \in A : f_\xi(a) \geq \sup(S(a))\}).$$

Note that this implies that $f^+_\xi(a_\xi) \in S(a_\xi)$. Since $\kappa > |A|$, we can find a single $a \in A$ such that $a = a_\xi$ for all ξ in a subset X_1 of X_0 of size κ. Now for $\xi_1 < \xi_2$ with both in X_1, we have

$$f^+_{\xi_1}(a) < f_{\xi'_1}(a) \leq f_{\xi_2}(a) \leq f^+_{\xi_2}(a).$$

[The first inequality is a consequence of $a = a_{\xi_1} \in <(f^+_{\xi_1}, f_{\xi'_1})$, the second follows from $\xi'_1 \leq \xi_2$ and the fact that

$$a = a_{\xi_1} = a_{\xi_2} \in A\backslash(Z_{\xi'_1} \cup Z_{\xi_2}),$$

and the third is true by the definition of $f^+_{\xi_2}$.]

Thus $\langle f^+_\xi(a) : \xi \in X_1\rangle$ is a strictly increasing sequence of members of $S(a)$. This contradicts our assumption that $|S(a)| < \kappa$. $\square$

The next lemma reduces the problem of finding an exact upper bound to that of finding a least upper bound.

Lemma 30.36. *Suppose that I is a proper ideal over A, $\lambda \geq |A|^+$ is a regular cardinal, and $f = \langle f_\xi : \xi \in \lambda\rangle$ is a $<_I$-increasing sequence of functions in ${}^A\mathbf{On}$ satisfying the bounding projection property for $|A|^+$. Suppose that h is a least upper bound for f. Then h is an exact upper bound.*

Proof. Assume the hypotheses, and suppose that $g <_I h$; we want to find $\xi < \lambda$ such that $g <_I f_\xi$. By increasing h on a subset of A in the ideal, we may assume that $g < h$ everywhere. Define $S_a = \{g(a), h(a)\}$ for every $a \in A$. By the bounding projection property we get a $\xi < \lambda$ such that $f^+_\xi \stackrel{\text{def}}{=} \text{proj}(f_\xi, \langle S_a : a \in A\rangle)$ is an upper bound for f. We shall prove that $g <_I f_\xi$, as required.

Since h is a least upper bound, it follows that $h \leq_I f^+_\xi$. Thus $M \stackrel{\text{def}}{=} \{a \in A : h(a) > f^+_\xi(a)\} \in I$. Also, the set $N \stackrel{\text{def}}{=} \{a \in A : f_\xi(a) \geq \sup S_a\}$ is in I. Suppose that

$a \in A \backslash (M \cup N)$. Then $g(a) < h(a) \leq f_\xi^+(a) = \min(S_a \backslash f_\xi(a))$, and this implies that $g(a) < f_\xi(a)$. So $g <_I f_\xi$, as desired. □

Here is our first existence theorem for exact upper bounds.

Theorem 30.37. (Existence of exact upper bounds) *Suppose that I is a proper ideal over A, $\lambda > |A|^+$ is a regular cardinal, and $f = \langle f_\xi : \xi \in \lambda \rangle$ is a $<_I$-increasing sequence of functions in ${}^A\mathbf{On}$ that satisfies the bounding projection property for $|A|^+$. Then f has an exact upper bound.*

Proof. Assume the hypotheses. By Lemma 30.36 it suffices to show that f has a least upper bound, and to do this we will apply Proposition 30.16(ii). Suppose that f does not have a least upper bound. Since it obviously has an upper bound, this means, by Proposition 30.16(ii):

(1) For every upper bound $h \in {}^A\mathbf{On}$ for f there is another upper bound h' for f such that $h' \leq_I h$ and $\{a \in A : h'(a) < h(a)\} \in I^+$.

In fact, Proposition 30.16(ii) says that there is another upper bound h' for f such that $h' \leq_I h$ and it is not true that $h =_I h'$. Hence $\{a \in A : h(a) < h'(a)\} \in I$ and $\{a \in A : h(a) \neq h'(a)\} \in I^+$. So

$$\{a \in A : h(a) \neq h'(a)\} \backslash \{a \in A : h(a) < h'(a)\} \in I^+ \quad \text{and}$$
$$\{a \in A : h(a) \neq h'(a)\} \backslash \{a \in A : h(a) < h'(a)\} = \{a \in A : h'(a) < h(a)\},$$

so (1) follows.

Now we shall define by induction on $\alpha < |A|^+$ a sequence $S^\alpha = \langle S^\alpha(a) : a \in A \rangle$ of sets of ordinals satisfying the following conditions:

(2) $0 < |S^\alpha(a)| \leq |A|$ for each $a \in A$;

(3) $f_\xi(a) < \sup S^\alpha(a)$ for all $\xi \in \lambda$ and $a \in A$;

(4) If $\alpha < \beta$, then $S^\alpha(a) \subseteq S^\beta(a)$, and if δ is a limit ordinal, then $S^\delta(a) = \bigcup_{\alpha<\delta} S^\alpha(a)$.

We also define sequences $\langle h_\alpha : \alpha < |A|^+ \rangle$ and $\langle h'_\alpha : \alpha < |A|^+ \rangle$ of functions and $\langle \xi(\alpha) : \alpha < |A|^+ \rangle$ of ordinals.

The definition of S^α for α limit is fixed by (4), and the conditions (2)–(4) continue to hold. To define S^0, pick any function k that bounds f (everywhere) and define $S^0(a) = \{k(a)\}$ for all $a \in A$; so (2)–(4) hold.

Suppose that $S^\alpha = \langle S^\alpha(a) : a \in A \rangle$ has been defined, satisfying (2)–(4); we define $S^{\alpha+1}$. By the bounding projection property for $|A|^+$, there is a $\xi(\alpha) < \lambda$ such that $h_\alpha \overset{\text{def}}{=} \text{proj}(f_{\xi(\alpha)}, S^\alpha)$ is an upper bound for f under $<_I$. Then

(5) if $\xi(\alpha) \leq \xi' < \lambda$, then $h_\alpha =_I \text{proj}(f_{\xi'}, S^\alpha)$.

In fact, recall that $h_\alpha(a) = \min(S^\alpha(a) \backslash f_{\xi(\alpha)}(a))$ for every $a \in A$, using (3). Now suppose that $\xi(\alpha) < \xi' < \lambda$. Let $M = \{a \in A : f_{\xi(\alpha)}(a) \geq f_{\xi'}(a)\}$. So $M \in I$. For any $a \in A \backslash M$ we have $f_{\xi(\alpha)}(a) < f_{\xi'}(a)$, and hence

$$\min(S^\alpha(a) \backslash f_{\xi(\alpha)}(a)) \leq \min(S^\alpha(a) \backslash f_{\xi'}(a));$$

it follows that $h_\alpha \le_I \text{proj}(f_{\xi'}, S^\alpha)$. For the other direction, recall that h_α is an upper bound for f under $<_I$. So $f_{\xi'} \le_I h_\alpha$. If a is any element of A such that $f_{\xi'}(a) \le h_\alpha(a)$ then, since $h_\alpha(a) \in S^\alpha(a)$, we get $\min(S^\alpha(a)\backslash f_{\xi'}(a)) \le h_\alpha(a)$. Thus $\text{proj}(f_{\xi'}, S^\alpha) \le_I h_\alpha$.

This checks (5).

Now we apply (1) to get an upper bound h'_α for f such that $h'_\alpha \le_I h_\alpha$ and $<(h'_\alpha, h_\alpha) \in I^+$. We now define $S^{\alpha+1}(a) = S^\alpha(a) \cup \{h'_\alpha(a)\}$ for any $a \in A$.

(6) If $\xi(\alpha) \le \xi < \lambda$, then $\text{proj}(f_\xi, S^{\alpha+1}) =_I h'_\alpha$.

For, we have $f_\xi \le_I h'_\alpha$ and, by (5), $h_\alpha =_I \text{proj}(f_\xi, S^\alpha)$. If $a \in A$ is such that $f_\xi(a) \le h'_\alpha(a)$, $h'_\alpha(a) \le h_\alpha(a)$, and $h_\alpha(a) = \text{proj}(f_\xi, S^\alpha)(a)$, then $\min(S^\alpha(a)\backslash f_\xi(a)) = h_\alpha(a) \ge h'_\alpha(a) \ge f_\xi(a)$, and hence

$$\text{proj}(f_\xi, S^{\alpha+1})(a) = \min(S^{\alpha+1}(a)\backslash f_\xi(a)) = h'_\alpha(a).$$

It follows that $\text{proj}(f_\xi, S^{\alpha+1}) =_I h'_\alpha$, as desired in (6).

Now since $|A|^+ < \lambda$, let $\xi < \lambda$ be greater than each $\xi(\alpha)$ for $\alpha < |A|^+$. Define $H_\alpha = \text{proj}(f_\xi, S^\alpha)$ for each $\alpha < |A|^+$. Since $\xi > \xi(\alpha)$, we have $H_\alpha =_I h_\alpha$ by (5). Note that $H_{\alpha+1} = \text{proj}(f_\xi, S^{\alpha+1}) =_I h'_\alpha$; so $<(H_{\alpha+1}, H_\alpha) \in I^+$. Now clearly by the construction we have $S^{\alpha_1}(a) \subseteq S^{\alpha_2}(a)$ for all $a \in A$ when $\alpha_1 < \alpha_2 < |A|^+$. Hence we get

(7) if $\alpha_1 < \alpha_2 < |A|^+$, then $H_{\alpha_2} \le H_{\alpha_1}$, and $<(H_{\alpha_2}, H_{\alpha_1}) \in I^+$.

Now for every $\alpha < |A|^+$ pick $a_\alpha \in A$ such that $H_{\alpha+1}(a_\alpha) < H_\alpha(a_\alpha)$. We have $a_\alpha = a_\beta$ for all α, β in some subset of $|A|^+$ of size $|A|^+$, and this gives an infinite decreasing sequence of ordinals, contradiction. $\square$

Lemma 30.38. *Suppose that I is a proper ideal over A, $\lambda \ge |A|^+$ is a regular cardinal, $f = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of functions in ${}^A\mathbf{On}$, $|A|^+ \le \kappa \le \lambda$, f satisfies the bounding projection property for κ, and g is an exact upper bound for f. Then*

$$\{a \in A : g(a) \textit{ is non-limit, or } \text{cf}(g(a)) < \kappa\} \in I.$$

Proof. Let $P = \{a \in A : g(a)$ is non-limit, or $\text{cf}(g(a)) < \kappa\}$. If $a \in P$ and $g(a)$ is a limit ordinal, choose $S(a) \subseteq g(a)$ cofinal in $g(a)$ and of order type $< \kappa$. If $g(a) = 0$ let $S(a) = \{0\}$, and if $g(a) = \beta + 1$ for some β let $S(a) = \{\beta\}$. Finally, if $g(a)$ is limit but is not in P, let $S(a) = \{g(a)\}$.

Now for any $\xi < \lambda$ let

$$\begin{aligned} N_\xi &= \{a \in A : f_\xi(a) \ge f_{\xi+1}(a)\} \quad \text{and} \\ Q_\xi &= \{a \in A : f_{\xi+1}(a) \ge g(a)\}. \end{aligned}$$

Then clearly

($*$) If $a \in A\backslash(N_\xi \cup Q_\xi)$, then $f_\xi(a) < \sup(S(a))$.

It follows that $\{a \in A : f_\xi(a) \ge \sup S(a)\} \subseteq N_\xi \cup Q_\xi \in I$. Hence the hypothesis of the bounding projection property holds. Applying it, we get $\xi < \lambda$ such that $f^+_\xi \stackrel{\text{def}}{=}$

$\text{proj}(f_\xi, \langle S(a) : a \in A\rangle)$ $<_I$-bounds f. Since g is a least upper bound for f, we get $g \le_I f_\xi^+$, and hence $M \stackrel{\text{def}}{=} \{a \in A : f_\xi^+(a) < g(a)\} \in I$. By $(*)$, for any $a \in P\backslash(N_\xi \cup Q_\xi)$ we have $f_\xi^+(a) = \min(S(a)\backslash f_\xi(a)) < g(a)$. This shows that $P\backslash(N_\xi \cup Q_\xi) \subseteq M$, hence $P \subseteq N_\xi \cup Q_\xi \cup M \in I$, so $P \in I$, as desired. □

Now we give our main theorem on the existence of exact upper bounds.

Theorem 30.39. *Suppose that I is a proper ideal over A, $\lambda > |A|^+$ is a regular cardinal, $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence of functions in ${}^A\mathbf{On}$, and $|A|^+ \le \kappa$. Then the following are equivalent:*

(i) $(*)_\kappa$ holds for f.

(ii) f satisfies the bounding projection property for κ.

(iii) f has an exact upper bound g such that

$$\{a \in A : g(a) \textit{ is non-limit, or } \mathrm{cf}(g(a)) < \kappa\} \in I.$$

Proof. (i)⇒(ii): By the bounding projection lemma, Lemma 30.35.

(ii)⇒(iii): Since the bounding projection property for κ clearly implies the bounding projection property for $|A|^+$, this implication is true by Theorem 30.37 and Lemma 30.38.

(iii)⇒(i): Assume (iii). By modifying g on a set in the ideal we may assume that $g(a)$ is a limit ordinal and $\mathrm{cf}(g(a)) \ge \kappa$ for all $a \in A$. Choose a club $S(a) \subseteq g(a)$ of order type $\mathrm{cf}(g(a))$. Thus the order type of $S(a)$ is $\ge \kappa$. We prove that $(*)_\kappa$ holds. So, assume that $X \subseteq \lambda$ is unbounded; we want to find $X_0 \subseteq X$ of order type κ over which f is strongly increasing under I. To do this, we intend to define by induction on $\alpha < \kappa$ a function $h_\alpha \in \prod S$ and an index $\xi(\alpha) \in X$ such that

(1) $h_\alpha <_I f_{\xi(\alpha)} \le_I h_{\alpha+1}$.

(2) The sequence $\langle h_\alpha : \alpha < \kappa\rangle$ is $<$-increasing (increasing everywhere; and hence it certainly is strongly increasing under I).

(3) $\langle \xi(\alpha) : \alpha < \kappa\rangle$ is strictly increasing.

After we have done this, the sandwich argument (Lemma 30.27) shows that $\langle f_{\xi(\alpha)} : \alpha < \kappa\rangle$ is strongly increasing under I and of order type κ, giving the desired result.

The functions h_α are defined as follows.

$h_0 \in \prod S$ is arbitrary.

For a limit ordinal $\delta < \kappa$ let $h_\delta = \sup_{\alpha<\delta} h_\alpha$.

Having defined h_α, we define $h_{\alpha+1}$ as follows. Since g is an exact upper bound and $h_\alpha < g$, choose $\xi(\alpha)$ greater than all $\xi(\beta)$ for $\beta < \alpha$ such that $h_\alpha <_I f_{\xi(\alpha)}$. Also, since $f_\xi <_I g$ for all $\xi < \lambda$, the projections $f_\xi^+ = \text{proj}(f, S)$ are defined. We define

$$h_{\alpha+1}(a) = \begin{cases} \max(h_\alpha(a), f^+_{\xi(\alpha)}(a)) + 1 & \text{if } f_{\xi(\alpha)}(a) < g(a), \\ h_\alpha(a) + 1 & \text{if } f_{\xi(\alpha)}(a) \ge g(a). \end{cases}$$

Thus we have

$$h_\alpha <_I f_{\xi(\alpha)} \leq_I h_{\alpha+1}, \text{ for every } \alpha.$$

So conditions (1)–(3) hold. $\square$

Now we apply some infinite combinatorics to get information about $(*)_\kappa$.

Theorem 30.40. (Club guessing) *Suppose that κ is a regular cardinal, λ is a cardinal such that $\mathrm{cf}(\lambda) \geq \kappa^{++}$, and $S^\lambda_\kappa = \{\delta \in \lambda : \mathrm{cf}(\delta) = \kappa\}$. Then there is a sequence $\langle C_\delta : \delta \in S^\lambda_\kappa\rangle$ such that:*

(i) For every $\delta \in S^\lambda_\kappa$ the set $C_\delta \subseteq \delta$ is club, of order type κ.

(ii) For every club $D \subseteq \lambda$ there is a $\delta \in D \cap S^\lambda_\kappa$ such that $C_\delta \subseteq D$.

The sequence $\langle C_\delta : \delta \in S^\lambda_\kappa\rangle$ is called a *club guessing sequence* for S^λ_κ.

Proof. First we take the case of uncountable κ. Fix a sequence $C' = \langle C'_\delta : \delta \in S^\lambda_\kappa\rangle$ such that $C'_\delta \subseteq \delta$ is club in δ of order type κ, for every $\delta \in S^\lambda_\kappa$. For any club E of λ, let

$$C' \restriction E = \langle C'_\delta \cap E : \delta \in S^\lambda_\kappa \cap E'\rangle,$$

where $E' = \{\delta \in E : E \cap \delta \text{ is unbounded in } \delta\}$. Clearly E' is also club in λ. Also note that $C'_\delta \cap E$ is club in δ for each $\delta \in S^\lambda_\kappa \cap E'$. We claim:

(1) There is a club E of λ such that for every club D of λ there is a $\delta \in D \cap E' \cap S^\lambda_\kappa$ such that $C'_\delta \cap E \subseteq D$.

Note that if we prove (1), then the theorem follows by defining $C_\delta = C'_\delta \cap E$ for all $\delta \in E' \cap S^\lambda_\kappa$, and $C_\delta = C'_\delta$ for $\delta \in S^\kappa_\lambda \backslash E'$.

Assume that (1) is false. Hence for every club $E \subseteq \lambda$ there is a club $D_E \subseteq \lambda$ such that for every $\delta \in D_E \cap E' \cap S^\lambda_\kappa$ we have

$$C'_\delta \cap E \not\subseteq D_E.$$

We now define a sequence $\langle E^\alpha : \alpha < \kappa^+\rangle$ of clubs of λ decreasing under inclusion, by induction on α:

(2) $E^0 = \lambda$.

(3) If $\gamma < \kappa^+$ is a limit ordinal and E^α has been defined for all $\alpha < \gamma$, we set $E^\gamma = \bigcap_{\alpha<\gamma} E^\alpha$. Since $\gamma < \kappa^+ < \mathrm{cf}(\lambda)$, E^γ is club in λ.

(4) If E^α has been defined, let $E^{\alpha+1}$ be the set of all limit points of $E^\alpha \cap D_{E^\alpha}$, i.e., the set of all $\varepsilon < \lambda$ such that $E^\alpha \cap D_{E^\alpha} \cap \varepsilon$ is unbounded in ε.

This defines the sequence. Let $E = \bigcap_{\alpha<\kappa^+} E^\alpha$. Then E is club in λ. Take any $\delta \in S^\lambda_\kappa \cap E$. Since $|C'_\delta| = \kappa$ and the sequence $\langle E^\alpha : \alpha < \kappa^+\rangle$ is decreasing, there is an $\alpha < \kappa^+$ such that $C'_\delta \cap E = C'_\delta \cap E^\alpha$. So $C'_\delta \cap E^\alpha = C'_\delta \cap E^{\alpha+1}$. Hence $C'_\delta \cap E^\alpha \subseteq D_{E^\alpha}$, contradiction.

Thus the case κ uncountable has been finished.

Now we take the case $\kappa = \omega$. For $S = S^{\lambda}_{\aleph_0}$ fix $C = \langle C_\delta : \delta \in S\rangle$ so that C_δ is club in δ with order type ω. We denote the n-th element of C_δ by $C_\delta(n)$. For any club $E \subseteq \lambda$ and any $\delta \in S \cap E'$ we define

$$C^E_\delta = \{\max(E \cap (C_\delta(n) + 1)) : n \in \omega\},$$

where again E' is the set of limit points of members of E. This set is cofinal in δ. In fact, given $\alpha < \delta$, there is a $\beta \in E \cap \delta$ such that $\alpha < \beta$ since $\delta \in E'$, and there is an $n \in \omega$ such that $\beta < C_\delta(n)$. Then $\alpha < \max(E \cap (C_\delta(n) + 1))$, as desired. There may be repetitions in the description of C^E_δ, but $\max(E \cap (C_\delta(n) + 1)) \leq \max(E \cap (C_\delta(m) + 1))$ if $n < m$, so C^E_δ has order type ω. We claim

(5) There is a closed unbounded $E \subseteq \lambda$ such that for every club $D \subseteq \lambda$ there is a $\delta \in D \cap S \cap E'$ such that $C^E_\delta \subseteq D$. [This proves the club guessing property.]

Suppose that (5) fails. Thus for every closed unbounded $E \subseteq \lambda$ there exist a club $D_E \subseteq \lambda$ such that for every $\delta \in D_E \cap S \cap E'$ we have $C^E_\delta \not\subseteq D$. Then we construct a descending sequence E^α of clubs in λ as in the case $\kappa > \omega$, for $\alpha < \omega_1$. Thus for each $\alpha < \omega_1$ and each $\delta \in D_{E^\alpha} \cap S \cap (E^\alpha)'$ we have $C^{E^\alpha}_\delta \not\subseteq D_{E^\alpha}$. Let $E = \bigcap_{\alpha<\omega_1} E^\alpha$. Take any $\delta \in S \cap E$. For $n \in \omega$ and $\alpha < \beta$ we have

$$E^\alpha \cap (C_\delta(n) + 1) \supseteq E^\beta \cap (C_\delta(n) + 1),$$

and so $\max(E^\alpha \cap (C_\delta(n)+1)) \geq \max(E^\beta \cap (C_\delta(n)+1))$; it follows that there is an $\alpha_n < \omega_1$ such that $\max(E^\beta \cap (C_\delta(n) + 1)) = \max(E^{\alpha_n} \cap (C_\delta(n) + 1))$ for all $\beta > \alpha_n$. Choose γ greater than all α_n. Thus

(6) For all $\varepsilon > \gamma$ and all $n \in \omega$ we have $\max(E^\varepsilon \cap (C_\delta(n) + 1)) = \max(E^\gamma \cap (C_\delta(n) + 1))$.

But there is a $\rho \in C^{E^\gamma}_\delta \backslash D_{E^\gamma}$; say that $\rho = \max(E^\gamma \cap (C_\delta(n) + 1))$. Then $\rho = \max(E^{\gamma+1} \cap (C_\delta(n) + 1)) \in E^{\gamma+1} = (E^\gamma \cap D_{E^\gamma})' \in D_{E^\gamma}$, contradiction. □

Lemma 30.41. *Suppose that:*

(i) I is an ideal over A.

(ii) κ and λ are regular cardinals such that $|A| < \kappa$ and $\kappa^{++} < \lambda$.

(iii) $f = \langle f_\xi : \xi < \lambda\rangle$ is a sequence of length λ of functions in ${}^A\mathbf{On}$ that is $<_I$-increasing and satisfies the following condition:

For every $\delta < \lambda$ with $\mathrm{cf}(\delta) = \kappa^{++}$ there is a club $E_\delta \subseteq \delta$ such that for some $\delta' \geq \delta$ with $\delta' < \lambda$,

$$(\star) \qquad \sup\{f_\alpha : \alpha \in E_\delta\} \leq_I f_{\delta'}.$$

Under these assumptions, $(*)_\kappa$ holds for f.

Proof. Assume the hypotheses. Let $S = S^{\kappa^{++}}_\kappa$; so S is stationary in κ^{++}. By Theorem 30.40, let $\langle C_\delta : \delta \in S\rangle$ be a club guessing sequence for S; thus

(1) For every $\delta \in S$, the set $C_\delta \subseteq \delta$ is a club of order type κ.

(2) For every club $D \subseteq \kappa^{++}$ there is a $\delta \in D \cap S$ such that $C_\delta \subseteq D$.

Now let $U \subseteq \lambda$ be unbounded; we want to find $X_0 \subseteq U$ of order type κ such that $\langle f_\xi : \xi \in X_0 \rangle$ is strongly increasing under I. To do this we first define an increasing continuous sequence $\langle \xi(i) : i < \kappa^{++} \rangle \in {}^{\kappa^{++}}\lambda$ recursively.

Let $\xi(0) = 0$. For i limit, let $\xi(i) = \sup_{k<i} \xi(k)$.

Now suppose for some $i < \kappa^{++}$ that $\xi(k)$ has been defined for every $k \leq i$; we define $\xi(i+1)$. For each $\alpha \in S$ we define

$$h_\alpha = \sup\{f_\eta : \eta \in \xi[C_\alpha \cap (i+1)]\} \quad \text{and}$$
$$\sigma_\alpha = \begin{cases} \text{least } \sigma \in (\xi(i), \lambda) \text{ such that } h_\alpha \leq_I f_\sigma & \text{if there is such a } \sigma, \\ \xi(i) + 1 & \text{otherwise.} \end{cases}$$

Now we let $\xi(i+1)$ be the least member of U which is greater than $\sup\{\sigma_\alpha : \alpha \in S\}$. It follows that

(3) If $\alpha \in S$ and the first case in the definition of σ_α holds, then $h_\alpha <_I f_{\xi(i+1)}$.

Now the set $F \stackrel{\text{def}}{=} \{\xi(k) : k \in \kappa^{++}\}$ is closed, and has order type κ^{++}. Let $\delta = \sup F$. Then F is a club of δ, and $\mathrm{cf}(\delta) = \kappa^{++}$. Hence by the hypothesis (iii) of the lemma, there is a club $E_\delta \subseteq \delta$ and a $\delta' \in [\delta, \lambda)$ such that $(\star)$ in the lemma holds. Note that $F \cap E_\delta$ is club in δ.

Let $D = \xi^{-1}[F \cap E_\delta]$. Since ξ is strictly increasing and continuous, it follows that D is club in κ^{++}. Hence by (2) there is an $\alpha \in D \cap S$ such that $C_\alpha \subseteq D$. Hence

$$\overline{C}_\alpha \stackrel{\text{def}}{=} \xi[C_\alpha] \subseteq F \cap E_\delta$$

is club in $\xi(\alpha)$ of order type κ. Then by $(\star)$ we have

$$\sup\{f_\rho : \rho \in \overline{C}_\alpha\} \leq_I f_{\delta'}.$$

Now

(4) For every $\rho < \rho'$ both in $\overline{C}_\alpha$, we have $\sup\{f_\zeta : \zeta \in \overline{C}_\alpha \cap (\rho+1)\} <_I f_{\rho'}$.

To prove this, note that there is an $i < \kappa^{++}$ such that $\rho = \xi(i)$. Now follow the definition of $\xi(i+1)$. There C_α was considered (among all other closed unbounded sets in the guessing sequence), and h_α was formed at that stage. Now

$$h_\alpha = \sup\{f_\eta : \eta \in \xi[C_\alpha \cap (i+1)]\} \leq \sup\{f_\eta : \eta \in \xi[C_\alpha]\} = \sup\{f_\eta : \eta \in \overline{C}_\alpha\} \leq_I f_{\delta'},$$

so the first case in the definition of σ_α holds. Thus by (3), $h_\alpha <_I f_{\xi(i+1)}$. Clearly $\xi(i+1) \leq \rho'$, so (4) follows.

Now let $\langle \eta(\nu) : \nu < \kappa \rangle$ be the strictly increasing enumeration of $\overline{C}_\alpha$, and set

$$X_0 = \{\eta(\omega \cdot \rho + 2m + 1) : \rho < \kappa, 0 < m < \omega\},$$
$$X_1 = \{\eta(\omega \cdot \rho + 2m) : \rho < \kappa, 0 < m < \omega\},$$

and for each $\beta \in X_1$ let $f'_\beta = \sup\{f_\sigma + 1 : \sigma \in X_0 \cap \beta\}$. Then for $\beta < \beta'$, both in X_1, we have $f'_\beta < f'_{\beta'}$. Now suppose that $\zeta \in X_0$; say $\zeta = \eta(\omega \cdot \rho + 2m + 1)$ with $\rho < \kappa$ and $0 < m < \omega$. Then

$$\begin{aligned} f'_{\eta(\omega\cdot\rho+2m)} &= \sup\{f_\sigma + 1 : \sigma \in X_0 \cap \eta(\omega \cdot \rho + 2m)\} <_I f_\zeta \quad \text{by (4)} \\ &\leq \sup\{f_\sigma + 1 : \sigma \in X_0 \cap \eta(\omega \cdot \rho + 2m + 2)\} \\ &= f'_{\eta(\omega\cdot\rho+2m+2)}. \end{aligned}$$

Hence by Proposition 30.27, $\langle f_\zeta : \zeta \in X_0\rangle$ is very strongly increasing under I. □

Now we need a purely combinatorial proposition.

Proposition 30.42. *Suppose that κ and λ are regular cardinals, and $\kappa^{++} < \lambda$. Suppose that F is a function with domain contained in $[\lambda]^{<\kappa}$ and range contained in λ. Suppose that for every $\delta \in S^\lambda_{\kappa^{++}}$ there is a closed unbounded set $E_\delta \subseteq \delta$ such that $[E_\delta]^{<\kappa} \subseteq \mathrm{dmn}(F)$. Then the following set is stationary:*

$$\{\alpha \in S^\lambda_\kappa : \text{ there is a closed unbounded } D \subseteq \alpha \text{ such that for any } a, b \in D \\ \text{with } a < b,\ \{d \in D : d \leq a\} \in \mathrm{dmn}(F) \text{ and } F(\{d \in D : d \leq a\}) < b\}$$

Proof. We follow the proof of Theorem 30.41 closely. Call the indicated set T. Let U be a closed unbounded subset of λ. We want to find a member of $T \cap U$.

Let $S = S^{\kappa^{++}}_\kappa$; so S is stationary in κ^{++}. By Theorem 30.40, let $\langle C_\delta : \delta \in S\rangle$ be a club guessing sequence for S; thus

(1) For every $\delta \in S$, the set $C_\delta \subseteq \delta$ is a club of order type κ.

(2) For every club $D \subseteq \kappa^{++}$ there is a $\delta \in D \cap S$ such that $C_\delta \subseteq D$.

We define an increasing continuous sequence $\langle \xi(i) : i < \kappa^{++}\rangle \in {}^{\kappa^{++}}\lambda$ recursively.

Let $\xi(0)$ be the least member of U. For i limit, let $\xi(i) = \sup_{k<i} \xi(k)$.

Now suppose for some $i < \kappa^{++}$ that $\xi(k)$ has been defined for every $k \leq i$; we define $\xi(i + 1)$. For each $\alpha \in S$ we consider two possibilities. If $\xi[C_\alpha \cap (i + 1)] \in \mathrm{dmn}(F)$, we let σ_α be any ordinal greater than both $\xi(i)$ and $F(\xi[C_\alpha \cap (i + 1)])$. Otherwise, we let $\sigma_\alpha = \xi(i) + 1$. Since $|S| < \lambda$, we can let $\xi(i+1)$ be the least member of U greater than all σ_α for $\alpha \in S$. Hence

(3) If $\alpha \in S$ and the first case in the definition of σ_α holds, then $\xi[C_\alpha \cap (i + 1)] \in \mathrm{dmn}(F)$ and $F(\xi[C_\alpha \cap (i + 1)]) < \xi(i + 1)$.

Now the set $G = \mathrm{rng}(\xi)$ is closed and has order type κ^{++}. Let $\delta = \sup(G)$. Hence by the hypothesis of the proposition, there is a closed unbounded set $E_\delta \subseteq \delta$ such that $[E_\delta]^{<\kappa} \subseteq \mathrm{dmn}(F)$. Note that $G \cap E_\delta$ is also closed unbounded in δ.

Let $H = \xi^{-1}[G \cap E_\delta]$. Thus H is club in κ^{++}. Hence by (2) there is an $\alpha \in H \cap S$ such that $C_\alpha \subseteq H$. Hence $\overline{C}_\alpha \overset{\text{def}}{=} \xi[C_\alpha] \subseteq G \cap E_\delta$ is club in $\xi(\alpha)$ of order type κ. We claim that $\overline{C}_\alpha$ is as desired in the proposition. For, suppose that $a, b \in \overline{C}_\alpha$ and $a < b$. Write

$a = \xi(i)$. Then $\{d \in \overline{C}_\alpha : d \le a\} = \xi[C_\alpha \cap (i+1)] \subseteq E_\delta$, and so (3) gives the desired conclusion. $\square$

Next we give a condition under which $(*)_\kappa$ holds.

Lemma 30.43. *Suppose that I is a proper ideal over a set A of regular cardinals such that $|A| < \min(A)$. Assume that $\lambda > |A|$ is a regular cardinal such that $(\prod A, <_I)$ is λ-directed, and $\langle g_\xi : \xi < \lambda\rangle$ is a sequence of members of $\prod A$.*

Then there is a $<_I$-increasing sequence $f = \langle f_\xi : \xi < \lambda\rangle$ of length λ in $\prod A$ such that:

(i) $g_\xi < f_{\xi+1}$ for every $\xi < \lambda$.

(ii) $(*)_\kappa$ holds for f, for every regular cardinal κ such that $\kappa^{++} < \lambda$ and $\{a \in A : a \le \kappa^{++}\} \in I$.

Proof. Let f_0 be any member of $\prod A$. At successor stages, if f_ξ is defined, let $f_{\xi+1}$ be any function in $\prod A$ that $<$-extends f_ξ and g_ξ.

At limit stages δ, there are three cases. In the first case, $\mathrm{cf}(\delta) \le |A|$. Fix some $E_\delta \subseteq \delta$ club of order type $\mathrm{cf}(\delta)$, and define

$$f_\delta = \sup\{f_i : i \in E_\delta\}.$$

For any $a \in A$ we have $\mathrm{cf}(\delta) \le |A| < \min(A) \le a$, and so $f_\delta(a) < a$. Thus $f_\delta \in \prod A$.

In the second case, $\mathrm{cf}(\delta) = \kappa^{++}$, where κ is regular, $|A| < \kappa$, and $\{a \in A : a \le \kappa^{++}\} \in I$. Then we define f'_δ as in the first case. Then for any $a \in A$ with $a > \kappa^{++}$ we have $f'_\delta(a) < a$, and so $\{a \in A : a \le f'_\delta(a)\} \in I$, and we can modify f'_δ on this set which is in I to obtain our desired f_δ.

In the third case, neither of the first two cases holds. Then we let f_δ be any $\le_I$-upper bound of $\{f_\xi : \xi < \delta\}$; it exists by the λ-directedness assumption.

This completes the construction. Obviously (i) holds. For (ii), suppose that κ is a regular cardinal such that $\kappa^{++} < \lambda$ and $\{a \in A : a \le \kappa^{++}\} \in I$. If $|A| < \kappa$, the desired conclusion follows by Lemma 30.41. In case $\kappa \le |A|$, note that $\langle f_\xi : \xi < \kappa\rangle$ is $<$-increasing, and so is certainly strongly increasing under I. $\square$

Now we apply these results to the determination of true cofinality for some important concrete partial orders.

Notation. *For any set X of cardinals, let*

$$X^{(+)} = \{\alpha^+ : \alpha \in X\}.$$

Theorem 30.44. (Representation of μ^+ as a true cofinality, I) *Suppose that μ is a singular cardinal with uncountable cofinality. Then there is a club C in μ such that C has order type $\mathrm{cf}(\mu)$, every element of C is greater than $\mathrm{cf}(\mu)$, and*

$$\mu^+ = \mathrm{tcf}\left(\prod C^{(+)}, <_{J^{\mathrm{bd}}}\right),$$

where J^{bd} is the ideal of all bounded subsets of $C^{(+)}$.

Proof. Let C_0 be any closed unbounded set of limit cardinals less than μ such that $|C_0| = \mathrm{cf}(\mu)$ and all cardinals in C_0 are above $\mathrm{cf}(\mu)$. Then

(1) all members of C_0 which are limit points of C_0 are singular.

In fact, suppose on the contrary that $\kappa \in C_0$, κ is a limit point of C_0, and κ is regular. Thus $C_0 \cap \kappa$ is unbounded in κ, so $|C_0 \cap \kappa| = \kappa$. But $\mathrm{cf}(\mu) < \kappa$ and $|C_0| = \mathrm{cf}\mu$, contradiction. So (1) holds. Hence wlog every member of C_0 is singular.

Now we claim

(2) $(\prod C_0^{(+)}, <_{J^{\mathrm{bd}}})$ is μ-directed.

In fact, suppose that $F \subseteq \prod C_0^{(+)}$ and $|F| < \mu$. For $a \in C_0^{(+)}$ with $|F| < a$ let $h(a) = \sup_{f\in F} f(a)$; so $h(a) \in a$. For $a \in C_0^{(+)}$ with $a \le |F|$ let $h(a) = 0$. Clearly $f \le_{J^{\mathrm{bd}}} h$ for all $f \in F$. So (2) holds.

(3) $(\prod C_0^{(+)}, <_{J^{\mathrm{bd}}})$ is μ^+-directed.

In fact, by (2) it suffices to find a bound for a subset F of $\prod C_0^{(+)}$ such that $|F| = \mu$. Write $F = \bigcup_{\alpha<\mathrm{cf}(\mu)} G_\alpha$, with $|G_\alpha| < \mu$ for each $\alpha < \mathrm{cf}(\mu)$. By (2), each G_α has an upper bound k_α under $<_{J^{\mathrm{bd}}}$. Then $\{k_\alpha : \alpha < \mathrm{cf}(\mu)\}$ has an upper bound h under $<_{J^{\mathrm{bd}}}$. Clearly h is an upper bound for F.

Now we are going to apply Lemma 30.43 to J^{bd}, $C_0^{(+)}$, and μ^+ in place of I, A, and λ; and with anything for g. Clearly the hypotheses hold, so we get a $<_{J^{\mathrm{bd}}}$-increasing sequence $f = \langle f_\xi : \xi < \mu^+\rangle$ in $\prod C_0^{(+)}$ such that $(*)_\kappa$ holds for f and the bounding projection property holds for κ, for every regular cardinal $\kappa < \mu$. It also follows that the bounding projection property holds for $|A|^+$, and hence by 30.37, f has an exact upper bound h. Then by Lemma 30.38, for every regular $\kappa < \mu$ we have

$$(\star) \qquad \{a \in C_0^{(+)} : h(a) \text{ is non-limit, or } \mathrm{cf}(h(a)) < \kappa\} \in J^{\mathrm{bd}}.$$

Now the identity function k on $C_0^{(+)}$ is obviously is an upper bound for f, so $h \le_{J^{\mathrm{bd}}} k$. By modifying h on a set in J^{bd} we may assume that $h(a) \le a$ for all $a \in C_0^{(+)}$. Now we claim

$(\star\star)$ The set $C_1 \stackrel{\mathrm{def}}{=} \{\alpha \in C_0 : h(\alpha^+) = \alpha^+\}$ contains a club of μ.

Assume otherwise. Then for every club K, $K \cap (\mu \backslash C_1) \neq 0$. This means that $\mu\backslash C_1$ is stationary, and hence $S \stackrel{\mathrm{def}}{=} C_0\backslash C_1$ is stationary. For each $\alpha \in S$ we have $h(\alpha^+) < \alpha^+$. Hence $\mathrm{cf}(h(\alpha^+)) < \alpha$ since α is singular. Hence by Fodor's theorem $\langle \mathrm{cf}(h(\alpha^+)) : \alpha \in C_0\rangle$ is bounded by some $\kappa < \mu$ on a stationary subset of S. This contradicts $(\star)$.

Thus $(\star\star)$ holds, and so there is a club $C \subseteq C_0$ such that $h(\alpha^+) = \alpha^+$ for all $\alpha \in C$. Now $\langle f_\xi \restriction C^{(+)} : \xi < \mu^+\rangle$ is $<_{J^{\mathrm{bd}}}$-increasing. We claim that it is cofinal in $(\prod C^{(+)}, <_{J^{\mathrm{bd}}})$. For, suppose that $g \in \prod C^{(+)}$. Let g' be the extension of g to $\prod C_0^{(+)}$ such that $g'(a) = 0$ for any $a \in C_0\backslash C$. Then $g' <_{J^{\mathrm{bd}}} h$, and so there is a $\xi < \mu^+$ such that $g' <_{J^{\mathrm{bd}}} f_\xi$. So $g <_{J^{\mathrm{bd}}} f_\xi \restriction C^{(+)}$, as desired. This shows that $\mu^+ = \mathrm{tcf}(\prod C^{(+)}, <_{J^{\mathrm{bd}}})$. $\square$

Theorem 30.45. (Representation of μ^+ as a true cofinality, II) *If μ is a singular cardinal of countable cofinality, then there is an unbounded set $D \subseteq \mu$ of regular cardinals such that*

$$\mu^+ = \mathrm{tcf}\left(\prod D, <_{J^{\mathrm{bd}}}\right).$$

Proof. Let C_0 be a set of uncountable regular cardinals with supremum μ, of order type ω.

(1) $\prod C_0/J^{\mathrm{bd}}$ is μ-directed.

For, let $X \subseteq \prod C_0$ with $|X| < \mu$. For each $a \in C_0$ such that $|X| < a$, let $h(a) = \sup\{f(a) : f \in X\}$, and extend h to all of C_0 in any way. Clearly $h \in \prod C_0$ and it is an upper bound in the $<_{J^{\mathrm{bd}}}$ sense for X.

From (1) it is clear that $\prod C_0/J^{\mathrm{bd}}$ is also μ^+-directed. By Lemma 30.43 we then get a $<_{J^{\mathrm{bd}}}$-increasing sequence $\langle f_\xi : \xi < \mu^+\rangle$ which satisfies $(*)_\kappa$ for every regular $\kappa < \mu^+$. By Theorems 30.37 and 30.38 f has an exact upper bound h such that $\{a \in C_0 : h(a)$ is non-limit or $\mathrm{cf}(h(a)) < \kappa\} \in J^{\mathrm{bd}}$ for every regular $\kappa < \mu^+$. We may assume that $h(a) \le a$ for all $a \in C_0$, since the identity function is clearly an upper bound for f; and we may assume that each $h(a)$ is a limit ordinal of uncountable cofinality since $\{a \in C_0 : \mathrm{cf}(h(a)) < \omega_1\} \in J^{\mathrm{bd}}$.

(2) $\mathrm{tcf}\left(\prod_{a\in C_0} \mathrm{cf}(h(a)), <_{J^{\mathrm{bd}}}\right) = \mu^+$.

To prove this, for each $a \in C_0$ let D_a be club in $h(a)$ of order type $\mathrm{cf}(h(a))$, and let $\langle \eta_{a\xi} : \xi < \mathrm{cf}(h(a))\rangle$ be the strictly increasing enumeration of D_a. For each $\xi < \mu^+$ we define $f'_\xi \in \prod_{a\in C_0} \mathrm{cf}(h(a))$ as follows. Since $f_\xi <_{J^{\mathrm{bd}}} h$, the set $\{a \in C_0 : f_\xi(a) \ge h(a)\}$ is bounded, so choose $a_0 \in C_0$ such that for all $b \in C_0$ with $a_0 \le b$ we have $f_\xi(b) < h(b)$. For such a b we define $f'_\xi(b)$ to be the least ν such that $f_\xi(b) < \eta_{b\nu}$. Then we extend f'_α in any way to a member of $\prod_{a\in C_0} \mathrm{cf}(h(a)))$.

(3) $\xi < \sigma < \mu^+$ implies that $f'_\xi \le_{J^{\mathrm{bd}}} f'_\sigma$.

This is clear by the definitions.

Now for each $l \in \prod_{a\in C_0} \mathrm{cf}(h(a)))$ define $k_l \in \prod C_0$ by setting $k_l(a) = \eta_{al(a)}$ for all a. So $k_l < h$. Since h is an exact upper bound for f, choose $\xi < \mu^+$ such that $k_l <_{J^{\mathrm{bd}}} f_\xi$. Choose a such that $k_l(b) < f_\xi(b) < h(b)$ for all $b \ge a$. Then for all $b \ge a$, $\eta_{bl(b)} < \eta_{bf'_\xi(b)}$, and hence $l(b) < f'_\xi(b)$. This proves that $l <_{J^{\mathrm{bd}}} f'_\xi$. This proves the following statement.

(4) $\{f'_\xi : \xi < \mu^+\}$ is cofinal in $\left(\prod_{a\in C_0} \mathrm{cf}(h(a)), <_{J^{\mathrm{bd}}}\right)$.

Now (3) and (4) yield (2).

Now let $B = \{\mathrm{cf}(h(a)) : a \in C_0\}$. Define

$$X \in J \text{ iff } X \subseteq B \text{ and } h^{-1}[\mathrm{cf}^{-1}[X]] \in J^{\mathrm{bd}}.$$

By Lemma 30.24 we get $\mathrm{tcf}(\prod B/J) = \mu^+$. It suffices now to show that J is the ideal of bounded subsets of B. Suppose that $X \in J$, and choose $a \in C_0$ such that $h^{-1}[\mathrm{cf}^{-1}[X]] \subseteq \{b \in C_0 : b < a\}$. Thus $X \subseteq \{b \in A : \mathrm{cf}(h(b)) < a\} \in J^{\mathrm{bd}}$, so X is bounded. Conversely, if X is bounded, choose $a \in B$ such that $X \subseteq \{b \in B : b \le a\}$. Now

$$\begin{aligned} h^{-1}[\mathrm{cf}^{-1}[X]] &= \{b \in C_0 : \mathrm{cf}(h(b)) \in X\} \\ &\subseteq \{b \in C_0 : \mathrm{cf}(h(b)) \le a\}, \end{aligned}$$

and this is bounded by the choice of h. $\square$

EXERCISES

E30.1. Let κ and λ be regular cardinals, with $\kappa < \lambda$. We define two statements $CG(\kappa,\lambda)$ and $MAJ(\kappa,\lambda)$:

$$\begin{array}{rll}
CG(\kappa,\lambda) & \text{iff} & \text{there is a sequence } \langle C_\alpha : \alpha \in S^\lambda_\kappa\rangle \text{ such that:}\\
& & (i)\ \forall\alpha \in S^\lambda_\kappa[C_\alpha \subseteq \alpha \text{ is club, of order type } \kappa];\\
& & (ii)\ \forall D \subseteq \lambda[D \text{ club implies that } \exists\alpha \in S^\kappa_\lambda(C_\alpha \subseteq D)];\\
MAJ(\kappa,\lambda) & \text{iff} & \text{there is a sequence } \langle f_\alpha : \alpha \in S^\lambda_\kappa\rangle \text{ such that:}\\
& & (i)\ \forall\alpha \in S^\lambda_\kappa[f_\alpha : \kappa \to \lambda \text{ is strictly increasing;}\\
& & (ii)\ \forall g : \lambda^{<\kappa} \to \lambda\exists\alpha \in S^\lambda_\kappa\forall\beta < \kappa[g(\langle f_\alpha(\gamma) : \gamma < \beta\rangle) < f_\alpha(\beta)].
\end{array}$$

Thus Theorem 30.40 implies that $CG(\kappa,\kappa^{++})$ holds for any regular cardinal κ.

Prove that $CG(\kappa,\lambda)$ implies $MAJ(\kappa,\lambda)$.

E30.2. Recall the condition $\diamondsuit$:

There are sets $A_\alpha \subseteq \alpha$ for each $\alpha < \omega_1$ such that for every $A \subseteq \omega_1$ the set $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary.

We do not need any properties of $\diamondsuit$, but it is of interest that it follows from $V = L$ and implies CH.

Prove that $\diamondsuit$ implies $CG(\omega,\omega_1)$. (It is consistent that $CG(\omega,\omega_1)$ fails, but this is not part of this exercise.)

E30.3. Suppose that A is infinite, $h \in {}^A\mathbf{On}$ is a function having only limit ordinals as values, and F is a nonprincipal ultrafilter on A. [This means that each cofinite subset of A is in F; cofinite = complement of finite.] Prove that $\prod_{a\in A} h(a)/F$ is infinite.

E30.4. An ultrafilter F on a set A is *countably incomplete* iff there is a countably infinite partition P of A such that $A\backslash a \in F$ for every $a \in A$. Show that if F is countably incomplete, then $\prod_{a\in A} h(a)/F$ has size at least 2^ω. Hint: Let $\langle p_i : i \in \omega\rangle$ enumerate P, and for each $i \in \omega$ let $b_i = \bigcup_{i\le j<\omega} p_j$; thus $b_i \in F$. For each $a \in A$, let $c_a = \{i \in \omega : a \in b_i\}$; hence c_a is a finite set. Use these sets to solve the problem.

E30.5. (Continuing E30.3) Suppose that F is a countably incomplete ultrafilter on A. Show that $\prod_{a\in A} h(a)/F$ is not well-ordered.

E30.6. Let $F = \{\omega, \omega\backslash 1\}$, a filter on ω. Define a subset X of ${}^\omega\omega$ which has two distinct least upper bounds under $\le_F$.

E30.7. Give an example of a set A, a collection $F \subseteq {}^A\mathbf{On}$, and an ideal I on A, such that there is a subset X of F which has a unique least upper bound under $<_I$, but no exact upper bound. (See the example on page 121.)

E30.8. Suppose that P and Q are partially ordered sets. Show that the following two conditions are equivalent:

(i) There is a function $f : P \to Q$ such that $\forall q \in Q \exists p \in P \forall r \in P[p \le r$ implies that $q \le f(r)]$.

(ii) There is a function $g : Q \to P$ such that $\forall X \subseteq Q[X$ unbounded in Q implies that $g[X]$ is unbounded in $P]$.

E30.9. (Continuing E30.8) A partially ordered set P is *directed* iff $\forall p, q \in P \exists r \in P[p \le r$ and $q \le r]$. Suppose that P and Q are directed. If either of the conditions of E30.8 hold, we write $P \le Q$.

Assume that $P \le Q \le P$. Show that there exist $f' : P \to Q$ and $g' : Q \to P$ such that for any $p \in P$ and $q \in Q$ the following conditions hold:

(a) If $g'(q) \le p$, then $q \le f'(p)$.

(b) If $f'(p) \le q$, then $p \le g'(q)$.

E30.10. (Continuing E30.9) Suppose that P and Q are directed partially ordered sets. Show that the following conditions are equivalent:

(a) $P \le Q$ and $Q \le P$.

(b) There is a partially ordered set R such that both P and Q can be embedded in R as cofinal subsets. That is, there is an injection $f : P \to R$ such that $\forall p, p' \in P[p \le_P p'$ iff $f(p) \le_R f(p')$, with $f[P]$ cofinal in R, and similarly for Q: there is an injection $g : Q \to R$ such that $\forall q, q' \in Q[q \le_P q'$ iff $g(q) \le_R g(q')$, with $g[Q]$ cofinal in R.

Hint: (b)$\Rightarrow$(a) is easy. For (a)$\Rightarrow$(b), assume (a), and suppose that f', g' are as in E30.9. Also assume wlog that $P \cap Q = \emptyset$, let $R' = P \cup Q$, and let the order on R' extend both of the orders on P and Q, and in addition write, for $p \in P$ and $q \in Q$,

$$\begin{aligned} p \le_{R'} q \quad &\text{iff} \quad \exists p' \ge_P p[f'(p') \le_Q q]; \\ q \le_{R'} p \quad &\text{iff} \quad \exists q' \ge_Q q[g'(q') \le_P p]. \end{aligned}$$

Then $\le_{R'}$ is a quasiorder on R', and one can let R be the associated partial order.

E30.11. Suppose that κ, λ, μ are cardinals such that (1) $\kappa = 1$ or κ is infinite; (2) $\kappa \le \lambda$; (3) λ and μ are infinite. For $f, g \in {}^\lambda\mu$ we define $f \le_\kappa g$ iff $|\{\alpha < \lambda : f(\alpha) > g(\alpha)\}| < \kappa$. Let

$$\begin{aligned} \mathfrak{b}_{\kappa.\lambda.\mu} &= \min\{|B| : B \text{ is a } \le_\kappa\text{-unbounded subset of } {}^\lambda\mu\}; \\ \mathfrak{d}_{\kappa,\lambda,\mu} &= \min\{|B| : B \text{ is a } \le_\kappa\text{-cofinal subset of } {}^\lambda\mu\}. \end{aligned}$$

Prove that $\mathfrak{b}_{\kappa,\lambda,\mu}$ is regular, and $\mathfrak{b}_{\kappa,\lambda,\mu} \le \mathrm{cf}(\mathfrak{d}_{\kappa,\lambda,\mu})$.

E30.12. Suppose that $\lambda > |A|^+$ is a regular cardinal and $f = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence. Consider the following property of f and a regular cardinal κ such that $|A| < \kappa \le \lambda$:

Bad$_\kappa$: There exist:

(a) nonempty sets S_a of ordinals for $a \in A$, each of size less than κ, such that $f_\alpha <_I \langle \sup(S_a) : a \in A \rangle$ for all $\alpha < \lambda$, and

(b) an ultrafilter D over A extending the dual of I

such that for every $\alpha < \lambda$ there is a $\beta < \lambda$ such that $\mathrm{proj}(f_\alpha, S) <_D f_\beta$.

Show that the bounding projection property for κ implies $\neg\mathbf{Bad}_\kappa$.

E30.13. (Continuing E30.12) Suppose that $\lambda > |A|^+$ is a regular cardinal and $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence. Also suppose that κ is a regular cardinal and $|A| < \kappa \leq \lambda$. Let **Ugly** be the following statement:

There exists a function $g \in {}^A\,\mathbf{On}$ such that, defining $t_\alpha = \{a \in A : g(a) < f_\alpha(a)\}$, the sequence $\langle t_\alpha : \alpha < \lambda\rangle$ does not stabilize modulo I. That is, for every α there is a $\beta > \alpha$ in λ such that $t_\beta \backslash t_\alpha \in I^+$.

Show that if **Ugly** holds, then $\langle t_\alpha : \alpha < \lambda\rangle$ is $\subseteq_I$-increasing, i.e., if $\alpha < \beta < \lambda$ then $t_\alpha \backslash t_\beta \in I$.

Also show that if the bounding projection property holds for κ, then $\neg$**Ugly**.

E30.14. (Continuing E30.13) Suppose that $\lambda > |A|^+$ is a regular cardinal and $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence. Also suppose that κ is a regular cardinal, with $|A| < \kappa \leq \lambda$. Suppose that $\neg\mathbf{Bad}_\kappa$ and $\neg$**Ugly**. Show that the bounding projection property holds for κ.

Hint: Suppose that it does not hold. For brevity write f^+_α for $\mathrm{proj}(f_\alpha, S)$. For all $\xi, \alpha < \lambda$ let $t^\xi_\alpha = \{a \in A : f^+_\xi(a) < f_\alpha(a)\}$. Prove:

(1) For every $\xi < \lambda$ there is a $\beta_\xi > \xi$ such that $t^\xi_{\beta_\xi} \in I^+$ and for all $\gamma > \beta_\xi$ we have $t^\xi_\gamma \backslash t^\xi_{\beta_\xi} \in I$.

Now by (1) define strictly increasing sequences $\langle \xi(\nu) : \nu < \lambda\rangle$ and $\langle \beta(\nu) : \nu < \lambda\rangle$ such that for all $\nu < \lambda$, $t^{\xi(\nu)}_{\beta(\nu)} \in I^+$, $\xi(\nu) < \beta(\nu)$, $\beta(\nu) < \xi(\rho)$ if $\nu < \rho < \lambda$, and $t^{\xi(\nu)}_\gamma \backslash t^{\xi(\nu)}_{\beta(\nu)} \in I$ for all $\gamma > \beta(\nu)$. Prove:

(2) If $\nu < \rho < \lambda$, then

$$t^{\xi(\rho)}_{\beta(\rho)} \subseteq \left(t^{\xi(\nu)}_{\beta(\nu)} \cap t^{\xi(\rho)}_{\beta(\rho)}\right) \cup \left(t^{\xi(\nu)}_{\beta(\rho)} \backslash t^{\xi(\nu)}_{\beta(\nu)}\right) \cup \left\{a \in A : f^+_{\xi(\rho)}(a) < f^+_{\xi(\nu)}(a)\right\}.$$

Next, prove

(3) If $\nu_1 < \cdots < \nu_m < \lambda$, then $t^{\xi(\nu_1)}_{\beta(\nu_1)} \cap \ldots \cap t^{\xi(\nu_m)}_{\beta(\nu_m)} \in I^+$.

By (3), the set $I^* \cup \{t^{\xi(\nu)}_{\beta(\nu)} : \nu < \lambda\}$ has fip, and hence is contained in an ultrafilter D. This easily leads to a contradiction.

E30.15 (Continuing E30.14; The Trichotomy theorem) Suppose that $\lambda > |A|^+$ is a regular cardinal and $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence. Also let κ be a regular cardinal such that $|A| < \kappa \leq \lambda$. Let $\mathbf{Good}_\kappa$ be the statement that there exists an exact upper bound g for f such that $\mathrm{cf}(g(a)) \geq \kappa$ for every $a \in A$.

Prove that $\mathbf{Bad}_\kappa$, **Ugly**, or $\mathbf{Good}_\kappa$.

References

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31. Basic properties of PCF

For any set A of regular cardinals define

$$\mathrm{pcf}(A) = \left\{\mathrm{cf}\left(\prod A/D\right) : D \text{ is an ultrafilter on } A\right\}.$$

By definition, $\mathrm{pcf}(\emptyset) = \emptyset$. We begin with a very easy proposition which will be used a lot in what follows.

Proposition 31.1. *Let A and B be sets of regular cardinals.*

(i) $A \subseteq \mathrm{pcf}(A)$.

(ii) If $A \subseteq B$, *then* $\mathrm{pcf}(A) \subseteq \mathrm{pcf}(B)$.

(iii) $\mathrm{pcf}(A \cup B) = \mathrm{pcf}(A) \cup \mathrm{pcf}(B)$.

(iv) If $B \subseteq A$, *then* $\mathrm{pcf}(A)\backslash\mathrm{pcf}(B) \subseteq \mathrm{pcf}(A\backslash B)$.

(v) If A is finite, then $\mathrm{pcf}(A) = A$.

(vi) If $B \subseteq A$, B is finite, and A is infinite, then $\mathrm{pcf}(A) = \mathrm{pcf}(A\backslash B) \cup B$.

(vii) $\min(A) = \min(\mathrm{pcf}(A))$.

(viii) If A is infinite, then the first ω members of A are the same as the first ω members of $\mathrm{pcf}(A)$.

Proof. (i): For each $a \in A$, the principal ultrafilter with $\{a\}$ as a member shows that $a \in \mathrm{pcf}(A)$.

(ii): Any ultrafilter F on A can be extended to an ultrafilter G on B. The mapping $[f] \mapsto [f]$ is easily seen to be an isomorphism of $\prod A/F$ onto $\prod B/G$. Note here that $[f]$ is used in two senses, one for an element of $\prod A/F$, where each member of $[f]$ is in $\prod A$, and the other for an element of $\prod B/G$, with members in the larger set $\prod B$.

(iii): $\supseteq$ holds by (ii). Now suppose that D is an ultrafilter on $A \cup B$. Then $A \in D$ or $B \in D$, and this proves $\subseteq$.

(iv): Suppose that $B \subseteq A$ and $\lambda \in \mathrm{pcf}(A)\backslash\mathrm{pcf}(B)$. Let D be an ultrafilter on A such that $\lambda = \mathrm{cf}(\prod A/D)$. Then $B \notin D$, as otherwise $\lambda \in \mathrm{pcf}(B)$. So $A\backslash B \in D$, and so $\lambda \in \mathrm{pcf}(A\backslash B)$.

(v): If A is finite, then every ultrafilter on A is principal.

(vi): We have

$$\begin{aligned}\mathrm{pcf}(A) &= \mathrm{pcf}(A\backslash B) \cup \mathrm{pcf}(B) \quad \text{by (iii)}\\ &= \mathrm{pcf}(A\backslash B) \cup B \quad \text{by (v)}\end{aligned}$$

(vii): Let $a = \min(A)$. Thus $a \in \mathrm{pcf}(A)$ by (i). Suppose that $\lambda \in \mathrm{pcf}(A)$ with $\lambda < a$; we want to get a contradiction. Say $\langle [g_\xi] : \xi < \lambda\rangle$ is strictly increasing and cofinal in $\prod A/D$. Now define $h \in \prod A$ as follows: for any $b \in A$, $h(b) = \sup\{g_\xi(b) + 1 : \xi < \lambda\}$. Thus $[g_\xi] < [h]$ for all $\xi < \lambda$, contradiction.

(viii): Suppose that $\lambda \in \mathrm{pcf}(A)\backslash A$. Suppose that $\lambda \cap A$ is finite, and let $a = \min(A\backslash\lambda)$. So $\lambda \leq a$, and if $b \in A \cap a$ then $b < \lambda$. Thus $A \cap \lambda = A \cap a$. Hence $\lambda \in \mathrm{pcf}(A) = \mathrm{pcf}(A\backslash a) \cup (A \cap \lambda)$ by (vi), and so $a \leq \lambda$ by (vii). So $\lambda = a$, contradiction. Thus $\lambda \cap A$ is infinite, and this proves (viii). $\square$

The following result gives a connection with earlier material; of course there will be more connections shortly.

Proposition 31.2. *If A is a collection of regular cardinals, F is a proper filter on A, and $\lambda = \operatorname{tcf}(\prod A/F)$, then $\lambda \in \operatorname{pcf}(A)$.*

Proof. Let $\langle f_\xi : \xi < \lambda \rangle$ be a $<_F$-increasing cofinal sequence in $\prod A/F$. Let D be any ultrafilter containing F. Then clearly $\langle f_\xi : \xi < \lambda \rangle$ is a $<_D$-increasing cofinal sequence in $\prod A/D$. $\square$

Definitions. A set A is *progressive* iff A is an infinite set of regular cardinals and $|A| < \min(A)$.

If $\alpha < \beta$ are ordinals, then $(\alpha, \beta)_{\text{reg}}$ is the set of all regular cardinals κ such that $\alpha < \kappa < \beta$. Similarly for $[\alpha, \beta)_{\text{reg}}$, etc. All such sets are called *intervals of regular cardinals.*

Proposition 31.3. *Assume that A is a progressive set, then*

(i) Every infinite subset of A is progressive.

(ii) If α is an ordinal and $A \cap \alpha$ is unbounded in α, then α is a singular cardinal.

(iii) If A is an infinite interval of regular cardinals, then A does not have any weak inaccessible as a member, except possibly its first element. Moreover, there is a singular cardinal λ such that $A \cap \lambda$ is unbounded in λ and $A \backslash \lambda$ is finite.

Proof. (i): Obvious.

(ii): Obviously α is a cardinal. Now $A \cap \alpha$ is cofinal in α and $|A \cap \alpha| \le |A| < \min(A) < \alpha$. Hence α is singular.

(iii): If $\kappa \in A$, then by (ii), $A \cap \kappa$ cannot be unbounded in κ; hence κ is a successor cardinal, or is the first element of A. For the second assertion of (iii), let $\sup(A) = \aleph_{\alpha+n}$ with α a limit ordinal. Since A is an infinite interval of regular cardinals, it follows that $A \cap \aleph_\alpha$ is unbounded in $\aleph_\alpha$, and hence by (ii), $\aleph_\alpha$ is singular. Hence the desired conclusion follows. $\square$

Theorem 31.4. (Directed set theorem) *Suppose that A is a progressive set, and λ is a regular cardinal such that* $\sup(A) < \lambda$. *Suppose that I is a proper ideal over A containing all proper initial segments of A and such that $(\prod A, <_I)$ is λ-directed. Then there exist a set A' of regular cardinals and a proper ideal J over A' such that the following conditions hold:*

(i) $A' \subseteq [\min(A), \sup(A))$ and A' is cofinal in $\sup(A)$.

(ii) $|A'| \le |A|$.

(iii) J contains all bounded subsets of A'.

(iv) $\lambda = \operatorname{tcf}(\prod A', <_J)$.

Proof. First we note:

$(*)$ A does not have a largest element.

For, suppose that a is the largest element of A. Note that then $I = \mathscr{P}(A \backslash \{a\})$. For each $\xi < a$ define $f_\xi \in \prod A$ by setting

$$f_\xi(b) = \begin{cases} 0 & \text{if } b \neq a, \\ \xi & \text{if } b = a. \end{cases}$$

Since $a < \lambda$, choose $g \in \prod A$ such that $f_\xi <_I g$ for all $\xi \in a$. Thus $\{b \in A : f_\xi(b) \geq g(b)\} \in I$, so $f_\xi(a) < g(a)$ for all $\xi < a$. This is clearly impossible. So $(*)$ holds.

Now by Lemma 30.43 there is a $<_I$-increasing sequence $f = \langle f_\xi : \xi < \lambda \rangle$ in $\prod A$ which satisfies $(*)_\kappa$ for every $\kappa \in A$. Hence by 30.37–30.39, f has an exact upper bound $h \in {}^A\mathbf{On}$ such that

$$\{a \in A : h(a) \text{ is non-limit or } \mathrm{cf}(h(a)) < \kappa\} \in I \tag{1}$$

for every $\kappa \in A$. Now the identity function k on A is clearly an upper bound for f, so $h \leq_I k$; and by (1), $\{a \in A : h(a) \text{ is non-limit or } \mathrm{cf}(h(a)) < \min(A)\} \in I$. Hence by changing h on a set in the ideal we may assume that

$$\min(A) \leq \mathrm{cf}(h(a)) \leq a \quad \text{for all } a \in A. \tag{2}$$

Now f shows that $(\prod h, <_I)$ has true cofinality λ. Let $A' = \{\mathrm{cf}(h(a)) : a \in A\}$. By Lemma 30.23 there is a proper ideal J on A' such that $(\prod A', <_J)$ has true cofinality λ; namely,

$$X \in J \quad \text{iff} \quad X \subseteq A' \text{ and } h^{-1}[\mathrm{cf}^{-1}[X]] \in I.$$

Clearly (ii) and (iv) hold. By (2) we have $A' \subseteq [\min(A), \sup(A))$. Now to show that A' is cofinal in $\sup(A)$, suppose that $\kappa \in A$; we find $\mu \in A'$ such that $\kappa \leq \mu$. In fact, $\{a \in A : \mathrm{cf}(h(a)) < \kappa\} \in I$ by (1). Let $X = \{b \in A' : b < \kappa\}$. Then

$$h^{-1}[\mathrm{cf}^{-1}[X]] = \{a \in A : \mathrm{cf}(h(a)) < \kappa\} \in I,$$

and so $X \in J$. Taking any $\mu \in A' \backslash X$ we get $\kappa \leq \mu$. Thus (i) holds. Finally, for (iii), suppose that $\mu \in J$; we want to show that $Y \stackrel{\text{def}}{=} \{b \in A' : b < \mu\} \in J$. By (i), choose $\kappa \in A$ such that $\mu \leq \kappa$. Then $Y \subseteq \{b \in A' : b < \kappa\}$, and by the argument just given, the latter set is in J. So (iii) holds. $\square$

Corollary 31.5. *Suppose that A is progressive, is an interval of regular cardinals, and λ is a regular cardinal $> \sup(A)$. Assume that I is a proper ideal over A such that $(\prod A, <_I)$ is λ-directed. Then $\lambda \in \mathrm{pcf}(A)$.*

Proof. We may assume that I contains all proper initial segments of A. For, suppose that this is not true. Then there is a proper initial segment B of A such that $B \notin I$. With $a \in A \backslash B$ we then have $B \subseteq A \cap a$, and so $A \cap a \notin I$. Let a be the smallest element of A such that $A \cap a \notin I$. Then $J \stackrel{\text{def}}{=} I \cap \mathscr{P}(A \cap a)$ is a proper ideal that contains all proper initial segments of $A \cap a$. we claim that $(\prod(A \cap a), J)$ is λ-directed. For, suppose that $X \subseteq \prod(A \cap a)$ with $|X| < \lambda$. For each $g \in X$ let $g^+ \in \prod A$ be such that $g^+ \supseteq g$ and $g^+(b) = 0$ for all $b \in A \backslash a$. Choose $f \in \prod A$ such that $g^+ \leq_I f$ for all $g \in X$. So if $g \in X$ we have

$$\{b \in A \cap a : g(b) > f(b)\} = \{b \in A : g^+(b) > f(b)\} \in I \cap \mathscr{P}(A \cap a),$$

and so $g \leq_J (f \restriction (A \cap a)$ for all $g \in X$, as desired.

Now the corollary follows from the theorem. □

The ideal $J_{<\lambda}$

Let A be a set of regular cardinals. We define

$$J_{<\lambda}[A] = \{X \subseteq A : \mathrm{pcf}(X) \subseteq \lambda\}.$$

In words, $X \in J_{<\lambda}[A]$ iff X is a subset of A such that for any ultrafilter D over A, if $X \in D$, then $\mathrm{cf}(\prod A, <_D) < \lambda$. Thus X "forces" the cofinalities of ultraproducts to be below λ.

Clearly $J_{<\lambda}[A]$ is an ideal of A. If $\lambda < \min(A)$, then $J_{<\lambda}[A] = \{\emptyset\}$ by 31.1(vii). If $\lambda < \mu$, then $J_{<\lambda}[A] \subseteq J_{<\mu}[A]$. If $\lambda \notin \mathrm{pcf}(A)$, then $J_{<\lambda}[A] = J_{<\lambda^+}[A]$. If λ is greater than each member of $\mathrm{pcf}(A)$, then $J_{<\lambda}[A]$ is the improper ideal $\mathscr{P}(A)$. If $\lambda \in \mathrm{pcf}(A)$, then $A \notin J_{<\lambda}[A]$.

If A is clear from the context, we simply write $J_{<\lambda}$.

If I and J are ideals on a set A, then $I + J$ is the smallest ideal on A which contains $I \cup J$; it consists of all X such that $X \subseteq Y \cup Z$ for some $Y \in I$ and $Z \in J$.

Lemma 31.6. *If A is an infinite set of regular cardinals and B is a finite subset of A, then for any cardinal λ we have*

$$J_{<\lambda}[A] = J_{<\lambda}[A \backslash B] + \mathscr{P}(B \cap \lambda).$$

Proof. Let $X \in J_{<\lambda}[A]$. Thus $\mathrm{pcf}(X) \subseteq \lambda$. Using 31.1(vi) we have $\mathrm{pcf}(X) = \mathrm{pcf}(X \backslash B) \cup (X \cap B)$, so $X \backslash B \in J_{<\lambda}[A \backslash B]$ and $X \cap B \subseteq B \cap \lambda$, and it follows that $X \in J_{<\lambda}[A \backslash B] + \mathscr{P}(B \cap \lambda)$.

Now suppose that $X \in J_{<\lambda}[A \backslash B] + \mathscr{P}(B \cap \lambda)$. Then there is a $Y \in J_{<\lambda}[A \backslash B]$ such that $X \subseteq Y \cup (B \cap \lambda)$. Hence by 31.1(vi) again, $\mathrm{pcf}(X) \subseteq \mathrm{pcf}(Y) \cup (B \cap \lambda) \subseteq \lambda$, so $X \in J_{<\lambda}[A]$. □

Recall that for any ideal on a set Y, $I^* = \{a \subseteq Y : Y \backslash a \in I\}$ is the filter corresponding to I.

Proposition 31.7. *If A is a collection of regular cardinals and λ is a cardinal, then*

$$J^*_{<\lambda}[A] = \bigcap \left\{ D : D \textit{ is an ultrafilter and } \mathrm{cf}\left(\prod A/D\right) \geq \lambda \right\}.$$

The intersection is to be understood as being equal to $\mathscr{P}(A)$ if there is no ultrafilter D such that $\mathrm{cf}(\prod A/D) \geq \lambda$.

Proof. Note that for any $X \subseteq A$, $X \in J^*_{<\lambda}[A]$ iff $A \backslash X \in J_{<\lambda}[A]$ iff $\mathrm{pcf}(A \backslash X) \subseteq \lambda$. Now suppose that $X \in J^*_{<\lambda}[A]$ and D is an ultrafilter such that $\mathrm{cf}(\prod A/D) \geq \lambda$. If $X \notin D$, then $A \backslash X \in D$ and hence $\mathrm{pcf}(A \backslash X) \not\subseteq \lambda$, contradiction. Thus X is in the indicated intersection.

If X is in the indicated intersection, we want to show that $A\backslash X \subseteq \lambda$. To this end, suppose that D is an ultrafilter such that $A\backslash X \in D$, and to get a contradiction suppose that $\mathrm{cf}(\prod A/D) \geq \lambda$. Then $X \in D$ by assumption, contradiction.

Note that the argument gives the desired result in case there are no ultrafilters D as indicated in the intersection; in this case, $\mathrm{pcf}(A\backslash X) \subseteq \lambda$ for every $X \subseteq A$, and so $J^*_{<\lambda}[A] = \mathscr{P}(A)$. $\square$

Theorem 31.8. (λ-directedness) *Assume that A is progressive. Then for every cardinal λ, the partial order $(\prod A, <_{J_{<\lambda}[A]})$ is λ-directed.*

Proof. We may assume that there are infinitely many members of A less than λ. For, suppose not. Let $F \subseteq \prod A$ with $|F| < \lambda$. We define $g \in \prod A$ by setting, for any $a \in A$,

$$g(a) = \begin{cases} \sup\{f(a) : f \in F\} & \text{if } |F| < a, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $f \leq g \bmod J_{<\lambda}[A]$ for all $f \in F$. For, if $f(a) > g(a)$, then $\lambda > |F| \geq a$; thus $\{a : f(a) > g(a)\} \subseteq \lambda \cap A$. Now $\mathrm{pcf}(\lambda \cap A) = \lambda \cap A \subseteq \lambda$, so $\{a : f(a) > g(a)\} \in J_{<\lambda}[A]$.

So, we make the indicated assumption. By this assumption, the set $B \stackrel{\text{def}}{=} A \cap \{|A|^+, |A|^{++}, |A|^{+++}, |A|^{++++}\} \subseteq \lambda$. Suppose that we have shown that $(\prod(A\backslash B), J_{<\lambda}(A\backslash B))$ is λ-directed. Now let $Y \subseteq \prod A$ with $|Y| < \lambda$. Choose $g \in \prod(A\backslash B)$ such that $f \restriction (A\backslash B) <_{J_{<\lambda}[A\backslash B]} g$ for all $f \in Y$. Let $g^+ \in \prod A$ be an extension of g. Then

$$\begin{aligned}\{a : f(a) > g^+(a)\} &= \{a \in A\backslash B : f(a) > g(a)\} \cup \{a \in B : f(a) > g^+(a)\} \\ &\in J_{<\lambda}[A\backslash B] + \mathscr{P}(B \cap \lambda) \\ &= J_{<\lambda}[A] \quad \text{by Lemma 31.6.}\end{aligned}$$

Thus g^+ is an upper bound for Y mod $J_{<\lambda}[A]$.

Hence we may assume that $|A|^{+3} < \min(A)$.

Now we prove by induction on the cardinal λ_0 that if $\lambda_0 < \lambda$ and $F = \{f_i : i < \lambda_0\} \subseteq \prod A$ is a family of functions of size λ_0, then F has an upper bound in $(\prod A, <_{J_{<\lambda}})$. So, we assume that this is true for all cardinals less than λ_0. If $\lambda_0 < \min(A)$, then $\sup(F)$ is as desired. So, assume that $\min(A) \leq \lambda_0$.

First suppose that λ_0 is singular. Let $\langle \alpha_i : i < \mathrm{cf}(\lambda_0)\rangle$ be increasing and cofinal in λ_0, each α_i a cardinal. By the inductive hypothesis, let g_i be a bound for $\{f_\xi : \xi < \alpha_i\}$ for each $i < \mathrm{cf}\lambda_0$, and then let h be a bound for $\{g_i : i < \mathrm{cf}\lambda_0\}$. Clearly h is a bound for F.

So assume that λ_0 is regular. We are now going to define a $<_{J_{<\lambda}}$-increasing sequence $\langle f'_\xi : \xi < \lambda_0\rangle$ which satisfies $(*)_\kappa$, with $\kappa = |A|^+$, and such that $f_i \leq f'_i$ for all $i < \lambda_0$. To do this choose, for every $\delta \in S^{\lambda_0}_{\kappa^{++}}$ a club $E_\delta \subseteq \delta$ of order type κ^{++}. Now for such a δ we define

$$f'_\delta = \sup(\{f'_j : j \in E_\delta\} \cup \{f_\delta\}).$$

For ordinals $\delta < \lambda_0$ of cofinality $\neq \kappa^{++}$ we apply the inductive hypothesis to get f'_δ such that $f'_\xi <_{J_{<\lambda}} f'_\delta$ for every $\xi < \delta$ and also $f_\delta <_{J_{<\lambda}} f'_\delta$.

This finishes the construction. By Lemma 30.41, $(*)_{|A|^+}$ holds for f, and hence by Theorem 30.39, f has an exact upper bound $g \in {}^A\mathbf{On}$ with respect to $<_{J_{<\lambda}}$. The identity function on A is an upper bound for f, so we may assume that $g(a) \le a$ for all $a \in A$. Now we shall prove that $B \stackrel{\text{def}}{=} \{a \in A : g(a) = a\} \in J_{<\lambda}[A]$, so a further modification of g yields the desired upper bound for f.

To get a contradiction, suppose that $B \notin J_{<\lambda}[A]$. Hence $\mathrm{pcf}(B) \not\subseteq \lambda$, and so there is an ultrafilter D over A such that $B \in D$ and $\mathrm{cf}(\prod A/D) \ge \lambda$. Clearly $D \cap J_{<\lambda}[A] = \emptyset$, as otherwise $\mathrm{cf}(\prod A/D) < \lambda$. Now f has length $\lambda_0 < \lambda$, and so it is bounded in $\prod A/D$; say that $f_i <_D h \in \prod A$ for all $i < \lambda_0$. Thus $h(a) < a = g(a)$ for all $a \in B$. Now we define $h' \in \prod A$ by

$$h'(a) = \begin{cases} h(a) & \text{if } a \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Then $h' <_{J_{<\lambda}} g$, since

$$\{a \in A : h'(a) \ge g(a)\} = \{a \in A : g(a) = 0\} \subseteq \{a \in A : f_0(a) \ge g(a)\} \in J_{<\lambda}.$$

Hence by the exactness of g it follows that $h' <_{J_{<\lambda}} f_i$ for some $i < \lambda_0$. But $B \in D$ and hence $h =_D h'$. So $h <_D f_i$, contradiction. □

Corollary 31.9. *Suppose that A is progressive, D is an ultrafilter over A, and λ is a cardinal. Then:*

(i) $\mathrm{cf}(\prod A/D) < \lambda$ *iff* $J_{<\lambda}[A] \cap D \ne \emptyset$.

(ii) $\mathrm{cf}(\prod A/D) = \lambda$ *iff* $J_{<\lambda^+} \cap D \ne \emptyset = J_{<\lambda} \cap D$.

(iii) $\mathrm{cf}(\prod A/D) = \lambda$ *iff λ^+ is the first cardinal μ such that $J_{<\mu} \cap D \ne \emptyset$.*

Proof. (i): $\Rightarrow$: Assuming that $J_{<\lambda}[A] \cap D = \emptyset$, the fact from Theorem 31.8 that $<_{J_{<\lambda}}$ is λ-directed implies that also $\prod A/D$ is λ-directed, and hence $\mathrm{cf}(\prod A/D) \ge \lambda$.

$\Leftarrow$: Assume that $J_{<\lambda}[A] \cap D \ne \emptyset$. Choose $X \in J_{<\lambda} \cap D$. Then by definition, $\mathrm{pcf}(A) \subseteq \lambda$, and hence $\mathrm{cf}(\prod A/D) < \lambda$.

(ii): Immediate from (i).

(iii): Immediate from (ii). □

We now give two important theorems about pcf.

Theorem 31.10. *If A is progressive, then* $|\mathrm{pcf}(A)| \le 2^{|A|}$.

Proof. By Corollary 31.9, for each $\lambda \in \mathrm{pcf}(A)$ we can select an element $f(\lambda) \in J_{<\lambda^+} \backslash J_{<\lambda}$. Clearly f is a one-one function from $\mathrm{pcf}(A)$ into $\mathscr{P}(A)$. □

Notation. We write $J_{\le\lambda}$ in place of $J_{<\lambda^+}$.

Theorem 31.11. (The max pcf theorem) *If A is progressive, then* $\mathrm{pcf}(A)$ *has a largest element.*

Proof. Let

$$I = \bigcup_{\lambda \in \mathrm{pcf}(A)} J_{<\lambda}[A].$$

Now clearly each ideal $J_{<\lambda}$ is proper (since for example $\{\lambda\} \notin J_{<\lambda}$), so I is also proper. Extend the dual of I to an ultrafilter D, and let $\mu = \mathrm{cf}(\prod A/D)$. Then for each $\lambda \in \mathrm{pcf}(A)$ we have $J_{<\lambda} \cap D = \emptyset$ since $I \cap D = \emptyset$, and by Corollary 31.9 this means that $\mu \ge \lambda$. □

Corollary 31.12. *Suppose that A is progressive. If λ is a limit cardinal, then*

$$J_{<\lambda}[A] = \bigcup_{\theta<\lambda} J_{\leq\theta}[A].$$

Proof. The inclusion $\supseteq$ is clear. Now suppose that $X \in J_{<\lambda}[A]$. Thus $\mathrm{pcf}(X) \subseteq \lambda$. Let μ be the largest element of $\mathrm{pcf}(X)$. Then $\mu \in \lambda$, and $\mathrm{pcf}(X) \subseteq \mu^+$, so $X \in J_{<\mu^+}$, and the latter is a subset of the right side. $\square$

Theorem 31.13. (The interval theorem) *If A is a progressive interval of regular cardinals, then* $\mathrm{pcf}(A)$ *is an interval of regular cardinals.*

Proof. Let $\mu = \sup(A)$. By 31.3(iii) and 31.1(vi) we may assume that μ is singular. By Theorem 31.11 let $\lambda_0 = \max(\mathrm{pcf}(A))$. Thus we want to show that every regular cardinal λ in (μ, λ_0) is in $\mathrm{pcf}(A)$. By Theorem 31.8, the partial order $(\prod A, <_{J_{<\lambda}})$ is λ-directed. Clearly $J_{<\lambda}$ is a proper ideal, so $\lambda \in \mathrm{pcf}(A)$ by Corollary 31.5. $\square$

Definition. If κ is a cardinal $\leq |A|$, then we define

$$\mathrm{pcf}_\kappa(A) = \bigcup\{\mathrm{pcf}(X) : X \subseteq A \text{ and } |X| = \kappa\}.$$

Theorem 31.14. *If A is an interval of regular cardinals and $\kappa < \min(A)$, then* $\mathrm{pcf}_\kappa(A)$ *is an interval of regular cardinals.*

Note here that we do not assume that A is progressive.

Proof. Let $\lambda_0 = \sup \mathrm{pcf}_\kappa(A)$. Note that each subset X of A of cardinality κ is progressive, and so $\max(\mathrm{pcf}(X))$ exists by Theorem 31.11. Thus

$$\lambda_0 = \sup\{\max(\mathrm{pcf}(X)) : X \subseteq A \text{ and } |X| = \kappa\}.$$

To prove the theorem it suffices to take any regular cardinal λ such that $\min(A) < \lambda < \lambda_0$ and show that $\lambda \in \mathrm{pcf}_\kappa(A)$. In fact, this will show that $\mathrm{pcf}_\kappa(A)$ is an interval of regular cardinals, whether or not λ_0 is regular. Since $\lambda < \lambda_0$, there is an $X \subseteq A$ of size κ such that $\lambda \leq \max(\mathrm{pcf}(X))$. Hence $X \notin J_{<\lambda}[X]$. If there is a proper initial segment Y of X which is not in $J_{<\lambda}[X]$, we can choose the smallest $a \in X$ such that $X \cap a \notin J_{<\lambda}[X]$ and work with $X \cap a$ rather than X. So we may assume that every proper initial segment of X is in $J_{<\lambda}[X]$. If $\lambda \in A$, clearly $\lambda \in \mathrm{pcf}_\kappa(A)$. So we may assume that $\lambda \notin A$. If $\lambda < \sup(X)$, then $\lambda \in A$, contradiction. If $\lambda = \sup(X)$, then $\lambda = \sup(A)$ since $\lambda \notin A$, and this contradicts Proposition 31.3(ii). So $\sup(X) < \lambda$. Since $J_{<\lambda}[X]$ is λ-directed by Theorem 31.8, we can apply 31.4 to obtain $\lambda \in \mathrm{pcf}(X)$, and hence $\lambda \in \mathrm{pcf}_\kappa(A)$, as desired. $\square$

Another of the central results of pcf theory is as follows.

Theorem 31.15. (Closure theorem.) *Suppose that A is progressive, $B \subseteq \mathrm{pcf}(A)$, and B is progressive. Then* $\mathrm{pcf}(B) \subseteq \mathrm{pcf}(A)$. *In particular, if* $\mathrm{pcf}(A)$ *itself is progressive, then* $\mathrm{pcf}(\mathrm{pcf}(A)) = \mathrm{pcf}(A)$.

Proof. Suppose that $\mu \in \mathrm{pcf}(B)$, and let E be an ultrafilter on B such that $\mu = \mathrm{cf}(\prod B/E)$. For every $b \in B$ fix an ultrafilter D_b on A such that $b = \mathrm{cf}(\prod A/D_b)$. Define F by

$$X \in F \quad \text{iff} \quad X \subseteq A \text{ and } \{b \in B : X \in D_b\} \in E.$$

It is straightforward to check that F is an ultrafilter on A. The rest of the proof consists in showing that $\mu = \mathrm{cf}(\prod A/F)$.

By Proposition 30.22 we have

$$\mu = \mathrm{cf}\left(\prod_{b \in B}\left(\prod A/D_b\right)/E\right).$$

Hence it suffices by Proposition 30.10 to show that $\prod A/F$ is isomorphic to a cofinal subset of this iterated ultraproduct. To do this, we consider the Cartesian product $B \times A$ and define

$$H \in P \quad \text{iff} \quad H \subseteq B \times A \text{ and } \{b \in B : \{a \in A : (b,a) \in H\} \in D_b\} \in E.$$

Again it is straightforward to check that P is an ultrafilter over $B \times A$. Let $r(b,a) = a$ for any $(b,a) \in B \times A$. Then

$$(*) \qquad \left(\prod_{(b,a) \in B \times A} a\right)/P \cong \prod_{b \in B}\left(\prod A/D_b\right)/E.$$

To prove $(*)$, for any $f \in \prod_{\langle b,a\rangle \in B \times A} a$ we define $f' \in \prod_{b \in B}(\prod A/D_b)$ by setting

$$f'(b) = \langle f(b,a) : a \in A\rangle/D_b.$$

Then for any $f, g \in \prod_{\langle b,a\rangle \in B \times A} a$ we have

$$\begin{aligned} f =_P g \quad &\text{iff} \quad \{(b,a) : f(b,a) = g(b,a)\} \in P \\ &\text{iff} \quad \{b : \{a : f(b,a) = g(b,a)\} \in D_b\} \in E \\ &\text{iff} \quad \{b : f'(b) = g'(b)\} \in E \\ &\text{iff} \quad f' =_E g'. \end{aligned}$$

Hence we can define $k(f/P) = f'/E$, and we get a one-one function. To show that it is a surjection, suppose that $h \in \prod_{b \in B}(\prod A/D_b)$. For each $b \in B$ write $h(b) = h'_b/D_b$ with $h'_b \in \prod A$. Then define $f(b,a) = h'_b(a)$. Then

$$f'(b) = \langle f(b,a) : a \in A\rangle/D_b = \langle h'_b(a) : a \in A\rangle/D_b = h'_b/D_b = h(b),$$

as desired. Finally, k preserves order, since

$$\begin{aligned} f/P < g/P \quad &\text{iff} \quad \{(b,a) : f(b,a) < g(b,a)\} \in P \\ &\text{iff} \quad \{b : \{a : f(b,a) < g(b,a)\} \in D_b\} \in E \\ &\text{iff} \quad \{b : f'(b) < g'(b)\} \in E \\ &\text{iff} \quad k(f/P) < k(g/P). \end{aligned}$$

So $(*)$ holds.

Now we apply Lemma 30.23, with $r, B \times A, A, P$ in place of c, A, B, I respectively. Then F is the Rudin-Keisler projection on A, since for any $X \subseteq A$,

$$\begin{aligned} X \in F \quad &\text{iff} \quad \{b \in B : X \in D_b\} \in E \\ &\text{iff} \quad \{b \in B : \{a \in A : r(b,a) \in X\} \in D_b\} \in E \\ &\text{iff} \quad \{b \in B : \{a \in A : (b,a) \in r^{-1}[X]\} \in D_b\} \in E \\ &\text{iff} \quad r^{-1}[X] \in P. \end{aligned}$$

Thus by Lemma 30.23 we get an isomorphism h of $\prod A/F$ into $\prod_{(b,a)\in B\times A} a/P$ such that $h(e/F) = \langle e(r(b,a)) : (b,a) \in B \times A\rangle/P$ for any $e \in \prod A$. So now it suffices now to show that the range of h is cofinal in $\prod_{(b,a)\in B\times A} a/P$. Let $g \in \prod_{(b,a)\in B\times A} a$. For every $b \in B$ define $g_b \in \prod A$ by $g_b(a) = g(b,a)$. Let $\lambda = \min(B)$. Since B is progressive, we have $|B| < \lambda$. Hence by the λ-directness of $\prod A/J_{<\lambda}[A]$ (Theorem 31.8), there is a function $k \in \prod A$ such that $g_b <_{J_{<\lambda}} k$ for each $b \in B$. Now $\lambda \le b$ for all $b \in B$, so $J_{<\lambda} \cap D_b = \emptyset$, and so $g_b <_{D_b} k$. It follows that $g/P <_P h(k/D)$. In fact, let $H = \{(b,a) : g(b,a) < k(r(b,a))\}$. Then

$$\{b \in B : \{a \in A : (b,a) \in H\} \in D_b\} = \{b \in B : \{a \in A : g_b(a) < k(a)\} \in D_b\} = B \in E,$$

as desired. □

Generators for $J_{<\lambda}$

If I is an ideal on a set A and $B \subseteq A$, then $I + B$ is the ideal generated by $I \cup \{B\}$; that is, it is the intersection of all ideals J on A such that $I \cup \{B\} \subseteq J$.

Proposition 31.16. *Suppose that I is an ideal on A and $B, X \subseteq A$. Then the following conditions are equivalent:*

(i) $X \in I + B$.

(ii) There is a $Y \in I$ such that $X \subseteq Y \cup B$.

(iii) $X \backslash B \in I$.

Proof. Clearly (ii)⇒(i). The set

$$\{Z \subseteq A : \exists Y \in I[Z \subseteq Y \cup B]\}$$

is clearly an ideal containing $I \cup \{B\}$, so (i)⇒(ii). If Y is as in (ii), then $X\backslash B \subseteq Y$, and hence $X\backslash B \in I$; so (ii)⇒(iii). If $X\backslash B \in I$, then $X \subseteq (X\backslash B) \cup B$, so X satisfies the condition of (ii). So (iii)⇒(ii). □

The following easy lemma will be useful later.

Lemma 31.17. *Suppose that A is progressive and $B \subseteq A$.*

(i) $\mathscr{P}(B) \cap J_{<\lambda}[A] = J_{<\lambda}[B]$.

(ii) If $f, g \in \prod A$ and $f <_{J_{<\lambda}[A]} g$, then $(f \restriction B) <_{J_{<\lambda}[B]} (g \restriction B)$.

Proof. (i): Suppose that $X \in \mathscr{P}(B) \cap J_{<\lambda}[A]$ and $X \in D$, an ultrafilter on B. Extend D to an ultrafilter E on A. Then $\prod B/D \cong \prod A/E$, and $\mathrm{cf}(\prod A/E) < \lambda$. So $X \in J_{<\lambda}[B]$. The converse is proved similarly.

(ii): Assume that $f, g \in \prod A$ and $f <_{J_{<\lambda}[A]} g$. Then

$$\{a \in B : g(b) \leq f(b)\} \in \mathscr{P}(B) \cap J_{<\lambda}[A] = J_{<\lambda}[B]$$

by (i), as desired. □

Definitions. If there is a set X such that $J_{\leq\lambda}[A] = J_{<\lambda} + X$, then we say that λ is *normal.*

Let A be a set of regular cardinals, and λ a cardinal. A subset $B \subseteq A$ is a λ-*generator over* A iff $J_{\leq\lambda}[A] = J_{<\lambda}[A] + B$. We omit the qualifier "over A" if A is understood from the context.

Suppose that $\lambda \in \mathrm{pcf}(A)$. A *universal sequence for* λ is a sequence $f = \langle f_\xi : \xi < \lambda\rangle$ which is $<_{J_{<\lambda}[A]}$-increasing, and for every ultrafilter D over A such that $\mathrm{cf}(\prod A/D) = \lambda$, the sequence f is cofinal in $\prod A/D$.

Theorem 31.18. (Universal sequences) *Suppose that A is progressive. Then every $\lambda \in \mathrm{pcf}(A)$ has a universal sequence.*

Proof. First,

(1) We may assume that $|A|^+ < \min(A) < \lambda$.

In fact, suppose that we have proved the theorem under the assumption (1), and now take the general situation. Recall from Proposition 3.19(vii) that $\min(A) \leq \lambda$. If $\lambda = \min(A)$, define $f_\xi \in \prod A$, for $\xi < \lambda$, by $f_\xi(a) = \xi$ for all $a \in A$. Thus f is $<$-increasing, hence $<_{J_{<\lambda}[A]}$-increasing. Suppose that D is an ultrafilter on A such that $\mathrm{cf}(\prod A/D) = \lambda$. Then $\{\min(A)\} \in D$, as otherwise $A\backslash\{\min(A)\} \in D$ and hence $\mathrm{cf}(\prod A/D) > \lambda$ by Proposition 31.1(vii). Thus for any $g \in \prod A$, let $\xi = g(\min(A)) + 1$. Then $\{a \in A : g(a) < f_\xi(a)\} \supseteq \{\min(A)\} \in D$, so $[g] < [f_\xi]$. Hence $\langle [f_\xi] : \xi < \lambda\rangle$ is cofinal in $\prod A/D$.

Now suppose that $\min(A) < \lambda$. Let $a_0 = \min A$. Let $A' = A\backslash\{a_0\}$. If D is an ultrafilter such that $\lambda = \mathrm{cf}(\prod A/D)$, then $A' \in D$ since $a_0 < \lambda$, hence $\{a_0\} \notin D$. It follows that $\lambda \in \mathrm{pcf}(A')$. Clearly $|A'|^+ < \min A' \leq \lambda$. Hence by assumption we get a system $\langle f_\xi : \xi < \lambda\rangle$ of members of $\prod A'$ which is increasing in $<_{J_{<\lambda}[A']}$ such that for every ultrafilter D over A' such that $\lambda = \mathrm{cf}(\prod A'/D)$, f is cofinal in $\prod A'/D$. Extend each f_ξ to $g_\xi \in \prod A$ by setting $g_\xi(a_0) = 0$. If $\xi < \eta < \lambda$, then

$$\{a \in A : g_\xi(a) \geq g_\eta(a)\} \subseteq \{a \in A' : f_\xi(a) \geq f_\eta(a)\} \cup \{a_0\},$$

and $\{a \in A' : f_\xi(a) \geq f_\eta(a)\} \in J_{<\lambda}[A'] \subseteq J_{<\lambda}[A]$ and also $\{a_0\} \in J_{<\lambda}[A]$ since $a_0 < \lambda$, so $g_\xi <_{J_{<\lambda}} g_\eta$. Now let D be an ultrafilter over A such that $\lambda = \mathrm{cf}(\prod A/D)$. As above, $A' \in D$; let $D' = D \cap \mathscr{P}(A')$. Then $\lambda = \mathrm{cf}(\prod A'/D')$. To show that g is cofinal in $\prod A/D$, take any $h \in \prod A$. Choose $\xi < \lambda$ such that $(h \restriction A')/D' < f_\xi/D'$. Then

$$\{a \in A : h(a) \geq g_\xi(a)\} \supseteq \{a \in A' : h(a) \geq f_\xi(a)\},$$

so $h/D < g_\xi/D$, as desired.

Thus we can make the assumption as in (1). Suppose that there is no universal sequence for λ. Thus

(2) For every $<_{J_{<\lambda}}$-increasing sequence $f = \langle f_\xi : \xi < \lambda\rangle$ there is an ultrafilter D over A such that $\mathrm{cf}(\prod A/D) = \lambda$ but f is not cofinal in $\prod A/D$.

We are now going to construct a $<_{J_{<\lambda}}$-increasing sequence $f^\alpha = \langle f^\alpha_\xi : \xi < \lambda\rangle$ for each $\alpha < |A|^+$. We use the fact that $\prod A/J_{<\lambda}$ is λ-directed (Theorem 31.8).

Using this directedness, we start with any $<_{J_{<\lambda}}$-increasing sequence $f^0 = \langle f^0_\xi : \xi < \lambda\rangle$.

For δ limit $< |A|^+$ we are going to define f^δ_ξ by induction on ξ so that the following conditions hold:

(3) $f^\delta_i <_{J_{<\lambda}} f^\delta_\xi$ for $i < \xi$,

(4) $\sup\{f^\alpha_\xi : \alpha < \delta\} \le f^\delta_\xi$.

Suppose that f^δ_i has been defined for all $i < \xi$. By λ-directedness, choose g such that $f^\delta_i <_{J_{<\lambda}} g$ for all $i < \xi$. Now for any $a \in A$ we have $\sup\{f^\alpha_\xi(a) : \alpha < \delta\} < a$, since $\delta < |A|^+ < \min A \le a$. Hence we can define

$$f^\delta_\xi(a) = \max\{g(a), \sup\{f^\alpha_\xi(a) : \alpha < \delta\}\}.$$

Clearly the conditions (3), (4) hold.

Now suppose that f^α has been defined and is $<_{J_{<\lambda}}$-increasing; we define $f^{\alpha+1}$. By (2), choose an ultrafilter D_α over A such that

(5) $\mathrm{cf}(\prod A/D_\alpha) = \lambda$;

(6) The sequence f^α is bounded in $<_{D_\alpha}$.

By (6), choose $f^{\alpha+1}_0$ which bounds f^α in $<_{D_\alpha}$; in addition, $f^{\alpha+1}_0 \ge f^\alpha_0$. Let $\langle h_\xi/D_\alpha : \xi < \lambda\rangle$ be strictly increasing and cofinal in $\prod A/D_\alpha$. Now we define $f^{\alpha+1}_\xi$ by induction on ξ when $\xi > 0$. First, by λ-directness, choose k such that $f^{\alpha+1}_i <_{J_{<\lambda}} k$ for all $i < \xi$. Then for any $a \in A$ let

$$f^{\alpha+1}_\xi(a) = \max(k(a), h_\xi(a), f^\alpha_\xi(a)).$$

Then the following conditions hold:

(7) $f^{\alpha+1}$ is strictly increasing and cofinal in $\prod A/D_\alpha$;

(8) $f^{\alpha+1}_i \ge f^\alpha_i$ for every $i < \lambda$.

This finishes the construction. Clearly we then have

(9) If $i < \lambda$ and $\alpha_1 < \alpha_2 < |A|^+$, then $f^{\alpha_1}_i \le f^{\alpha_2}_i$.

(10) f^α is bounded in $\prod A/D_\alpha$ by $f^{\alpha+1}_0$.

(11) $f^{\alpha+1}$ is cofinal in $\prod A/D_\alpha$.

Now let $h = \sup_{\alpha<|A|^+} f_0^\alpha$. Then $h \in \prod A$, since $|A|^+ < \min(A)$. By (11), for each $\alpha < |A|^+$ choose $i_\alpha < \lambda$ such that $h <_{D_\alpha} f_{i_\alpha}^{\alpha+1}$. Since $\lambda > |A|^+$ is regular, we can choose $i < \lambda$ such that $i_\alpha < i$ for all $\alpha < |A|^+$. Now define

$$A^\alpha = \{a \in A : h(a) \leq f^\alpha(a)\}.$$

By (9) we have $A^\alpha \subseteq A^\beta$ for $\alpha < \beta < |A|^+$. We are going to get a contradiction by showing that $A^\alpha \subset A^{\alpha+1}$ for every $\alpha < |A|^+$.

In fact, this follows from the following two statements.

(12) $A^\alpha \notin D_\alpha$.

This holds because $f_i^\alpha <_{D_\alpha} f_i^{\alpha+1} \leq h$.

(13) $A^{\alpha+1} \in D_\alpha$.

This holds because $h <_{D_\alpha} f_i^{\alpha+1}$ by the choice of i and (7). $\square$

Proposition 31.19. *If A is a set of regular cardinals, λ is the largest member of* $\mathrm{pcf}(A)$, *and $\langle f_\xi : \xi < \lambda\rangle$ is universal for λ, then it is cofinal in $(\prod A, J_{<\lambda})$.*

Proof. Assume the hypotheses. Fix $g \in \prod A$; we want to find $\xi < \lambda$ such that $g <_{J_{<\lambda}} f_\xi$. Suppose that no such ξ exists. Then, we claim, the set

$$J^*_{<\lambda} \cup \{\{a \in A : g(a) \geq f_\xi(a)\} : \xi < \lambda\} \tag{1}$$

has fip. For, suppose that it does not have fip. Then there is a finite nonempty subset F of λ such that

$$\bigcup_{\xi\in F} \{a \in A : g(a) < f_\xi(a)\} : \xi < \lambda\} \in J^*_{<\lambda}. \tag{2}$$

Let η be the largest member of F. Note that the set

$$\{a \in A : f_\xi(a) < f_\rho(a) \text{ for all } \xi, \rho \in F \text{ such that } \xi < \rho\}$$

is also a member of $J^*_{<\lambda}$; intersecting this set with the set of (2), we get a member of $J^*_{<\lambda}$ which is a subset of $\{a \in A : g(a) < f_\eta(a)\}$, so that $g <_{J_{<\lambda}} f_\eta$, contradiction.

Thus the set (1) has fip. Let D be an ultrafilter containing it. Then $\mathrm{cf}(\prod A/D) = \lambda$, so by hypothesis there is a $\xi < \lambda$ such that $g <_D f_\xi$. Thus $\{a \in A : g(a) < f_\xi(a)\} \in D$. But also $\{a \in A : g(a) \geq f_\xi(a)\} \in D$, contradiction. $\square$

Theorem 31.20. *If A is progressive, then* $\mathrm{cf}(\prod A, <) = \max(\mathrm{pcf}(A))$. *In particular,* $\mathrm{cf}(\prod A, <)$ *is regular.*

Proof. First we prove $\geq$. Let $\lambda = \max(\mathrm{pcf}(A))$, and let D be an ultrafilter on A such that $\lambda = \mathrm{cf}(\prod A/D)$. Now for any $f, g \in \prod A$, if $f < g$ then $f <_D g$. Hence any cofinal set in $(\prod A, <)$ is also cofinal in $(\prod A, <_D)$, and so $\lambda = \mathrm{cf}(\prod A, <_D) \leq \mathrm{cf}(\prod A, <)$.

To prove $\leq$, we exhibit a cofinal subset of $(\prod A, <)$ of size λ. For every $\mu \in \text{pcf}(A)$ fix a universal sequence $f^\mu = \langle f_i^\mu : i < \mu \rangle$ for μ, by Theorem 31.18. Let F be the set of all functions of the form

$$\sup\{f_{i_1}^{\mu_1}, f_{i_2}^{\mu_2}, \ldots, f_{i_n}^{\mu_n}\},$$

where $\mu_1, \mu_2, \ldots, \mu_n$ is a finite sequence of members of $\text{pcf}(A)$, possibly with repetitions, and $i_k < \mu_k$ for each $k = 1, \ldots, n$. We claim that F is cofinal in $(\prod A, <)$; this will complete the proof.

To prove this claim, let $g \in \prod A$. Let

$$I = \{>(f, g) : f \in F\}.$$

(Recall that $>(f, g) = \{a \in A : f(a) > g(a)\}$.) Now I is closed under unions, since

$$>(f_1, g) \cup >(f_2, g) = >(\sup(f_1, f_2), g).$$

If $A \in I$, then $A = >(f, g)$ for some $f \in F$, as desired. So, suppose that $A \notin I$. Now $J \stackrel{\text{def}}{=} \{A \backslash X : X \in I\}$ has fip since I is closed under unions, and so this set can be extended to an ultrafilter D over A. Let $\mu = \text{cf}(\prod A/D)$. Then f^μ is cofinal in $(\prod A, <_D)$ since it is universal for μ. But $f_i^\mu \leq_I g$ for all $i < \mu$, since $f_i^\mu \in F$ and so $>(f_i^\mu, g) \in I$. This is a contradiction. $\square$

Note that Theorem 31.20 is not talking about true cofinality. In fact, clearly any increasing sequence of elements of $\prod A$ under $<$ must have order type at most $\min(A)$, and so true cofinality does not exist if A has more than one element.

Lemma 31.21. *Suppose that A is progressive, $\lambda \in \text{pcf}(A)$, and $f' = \langle f'_\xi : \xi < \lambda \rangle$ is a universal sequence for λ. Suppose that $f = \langle f_\xi : \xi < \lambda \rangle$ is $<_{J_{<\lambda}}$-increasing, and for every $\xi' < \lambda$ there is a $\xi < \lambda$ such that $f'_{\xi'} \leq_{J_{<\lambda}} f_\xi$. Then f is universal for λ.*

Proof. This is clear, since for any ultrafilter D over A such that $\text{cf}(\prod A/D) = \lambda$ we have $D \cap J_{<\lambda} = \emptyset$, and hence $f'_{\xi'} \leq_{J_{<\lambda}} f_\xi$ implies that $f'_{\xi'} \leq_D f_\xi$. $\square$

For the next result, note that if A is progressive, then $|A| < \min(A)$, and hence $|A|^+ \leq \min(A)$. So $A \cap |A|^+ = \emptyset \in J_{<\lambda}$ for any λ. So if μ is an ordinal and $A \cap \mu \notin J_{<\lambda}$, then $|A|^+ < \mu$.

Lemma 31.22. *Suppose that A is a progressive set of regular cardinals and $\lambda \in \text{pcf}(A)$.*

(i) Let μ be the least ordinal such that $A \cap \mu \notin J_{<\lambda}[A]$. Then there is a universal sequence for λ that satisfies $(*)_\kappa$ for every regular cardinal κ such that $\kappa < \mu$.

(ii) There is a universal sequence for λ that satisfies $(*)_{|A|^+}$.

Proof. First note that (ii) follows from (i) by the remark preceding this lemma. Now we prove (i). Note by the minimality of μ that either $\mu = \rho + 1$ for some $\rho \in A$, or μ is a limit cardinal and $A \cap \mu$ is unbounded in μ.

(1) $\mu \leq \lambda + 1$.

For, let D be an ultrafilter such that $\lambda = \mathrm{cf}(\prod A/D)$. Then $A \cap (\lambda+1) \in D$, as otherwise $\{a \in A : \lambda < a\} \in D$, and so $\mathrm{cf}(\prod A/D) > \lambda$ by 31.1(vii), contradiction. Thus $\lambda \in \mathrm{pcf}(A \cap (\lambda+1))$, and hence $\mathrm{pcf}((A \cap (\lambda+1)) \not\subseteq \lambda$, proving (1).

(2) $\mu \neq \lambda$.

For, $|A| < \min(A) \leq \lambda$, so $A \cap \lambda$ is bounded in λ because λ is regular. Hence $\mu \neq \lambda$ by an initial remark of this proof.

Now we can complete the proof for the case in which μ is $\rho + 1$ for some $\rho \in A$. In this case, actually $\rho = \lambda$. For, we have $A \cap \rho \in J_{<\lambda}[A]$ while $A \cap (\rho+1) \notin J_{<\lambda}[A]$. Let D be an ultrafilter on A such that $A \cap (\rho+1) \in D$ and $\mathrm{cf}(\prod A/D) \geq \lambda$. Then $A \cap \rho \notin D$, since $A \cap \rho \in J_{<\lambda}[A]$, so $\{\rho\} \in D$, and so $\rho \geq \lambda$. By (1) we then have $\rho = \lambda$.

Now define, for $\xi < \lambda$ and $a \in A$,

$$f_\xi(a) = \begin{cases} 0 & \text{if } a < \lambda, \\ \xi & \text{if } \lambda \leq a. \end{cases}$$

Thus $f_\xi \in \prod A$. The sequence $\langle f_\xi : \xi < \lambda \rangle$ is $<_{J_{<\lambda}[A]}$-increasing, since if $\xi < \eta < \lambda$ then $\{a \in A : f_\xi(a) \geq f_\eta(a)\} \subseteq A \cap \lambda \in J_{<\lambda}[A]$. It is also universal for λ. For, suppose that D is an ultrafilter on A such that $\mathrm{cf}(\prod A/D) = \lambda$. Suppose that $g \in \prod A$. Now $|A| < \min(A) \leq \lambda$, so $\xi \stackrel{\mathrm{def}}{=} (\sup_{a \in A} g(a)) + 1$ is less than λ. Now $\{a \in A : g(a) < f_\xi(a)\} = A \in D$, so $g <_D f_\xi$, as desired. Finally, $\langle f_\xi : \xi < \lambda \rangle$ satisfies $(*)_\lambda$, since it is itself strongly increasing under $J_{<\lambda}[A]$. In fact, if $\xi < \eta < \lambda$ and $a \in A \backslash \lambda$, then $f_\xi(a) = \xi < \eta = f_\eta(a)$, and $A \cap \lambda \in J_{<\lambda}[A]$.

Hence the case remains in which $\mu < \lambda$ and $A \cap \mu$ is unbounded in μ. Let $\langle f'_\xi : \xi < \lambda \rangle$ be any universal sequence for λ. We now apply Lemma 30.43 with I replaced by $J_{<\lambda}[A]$. (Recall that $(\prod A, I_{<\lambda}[A]$ is λ-directed by Theorem 31.8.) This gives us a $<_{J_{<\lambda}[A]}$-increasing sequence $f = \langle f_\xi : \xi < \lambda \rangle$ such that $f'_\xi < f_{\xi+1}$ for every $\xi < \lambda$, and $(*)_\kappa$ holds for f, for every regular cardinal κ such that $\kappa^{++} < \lambda$ and $\{a \in A : a \leq \kappa^{++}\} \in J_{<\lambda}[A]$. Clearly then f is universal for λ. If κ is a regular cardinal less than μ, then $\kappa^{++} < \mu < \lambda$, and $\{a \in A : a \leq \kappa^{++}\} \subseteq J_{<\lambda}[A]$ by the minimality of μ, so the conclusion of the lemma holds. $\square$

Lemma 31.23. *Suppose that A is a progressive set of regular cardinals, $B \subseteq A$, and λ is a regular cardinal. Then the following conditions are equivalent:*

(i) $J_{\leq\lambda}[A] = J_{<\lambda}[A] + B$.

(ii) $B \in J_{\leq\lambda}[A]$, and for every ultrafilter D on A, if $\mathrm{cf}(\prod A/D) = \lambda$, then $B \in D$.

Proof. (i)⇒(ii): Assume (i). Obviously, then, $B \in J_{\leq\lambda}[A]$. Now suppose that D is an ultrafilter on A and $\mathrm{cf}(\prod A/D) = \lambda$. By Corollary 31.9(ii) we have $J_{\leq\lambda}[A] \cap D \neq \emptyset = J_{<\lambda}[A] \cap D$. Choose $X \in J_{\leq\lambda}[A] \cap D$. Then by Proposition 31.16, $X \backslash B \in J_{<\lambda}[A]$, so since $J_{<\lambda}[A] \cap D = \emptyset$, we get $B \in D$.

(ii)⇒(i): $\supseteq$ is clear. Now suppose that $X \in J_{\leq\lambda}[A]$. If $X \subseteq B$, then obviously $X \in J_{<\lambda}[A] + B$. Suppose that $X \not\subseteq B$, and let D be any ultrafilter such that $X \backslash B \in D$. Then $\mathrm{cf}(\prod A/D) \leq \lambda$ since $\mathrm{pcf}(X) \subseteq \lambda^+$, and so $\mathrm{cf}(\prod A/D) < \lambda$ by the second assumption in (ii). This shows that $\mathrm{pcf}(X \backslash B) \subseteq \lambda$, so $X \backslash B \in J_{<\lambda}[A]$, and hence $X \in J_{<\lambda}[A] + B$ by Proposition 31.16. $\square$

Theorem 31.24. *If A is progressive, then every member of* $\mathrm{pcf}(A)$ *has a generator.*

Proof. First suppose that we have shown the theorem if $|A|^+ < \min(A)$. We show how it follows when $|A|^+ = \min(A)$. The least member of $\mathrm{pcf}(A)$ is $|A|^+$ by 31.1(vii). We have $J_{<|A|^+}[A] = \{\emptyset\}$ and $J_{\leq|A|^+}[A] = \{\emptyset, \{|A|^+\}\} = J_{<|A|^+}[A] + |A|^+$, so $|A|^+$ is a $|A|^+$-generator. Now suppose that $\lambda \in \mathrm{pcf}(A)$ with $\lambda > |A|^+$. Let $A' = A\backslash\{|A|^+\}$. By 31.1(vi) we also have $\lambda \in \mathrm{pcf}(A')$. By the supposed result there is a $b \subseteq A'$ such that $J_{\leq\lambda}[A'] = J_{<\lambda}[A'] + b$. Hence, applying Lemma 31.6 to λ^+ and $\{|A|^+\}$,

$$\begin{aligned} J_{\leq\lambda}[A] &= J_{\leq\lambda}[A'] + \{|A|^+\} \\ &= J_{<\lambda}[A'] + b + \{|A|^+\} \\ &= J_{<\lambda}[A] + b, \end{aligned}$$

as desired.

Thus we assume henceforth that $|A|^+ < \min(A)$. Suppose that $\lambda \in \mathrm{pcf}(A)$. First we take the case $\lambda = |A|^{++}$. Hence by Lemma 31.1(vii) we have $\lambda \in A$. Clearly

$$J_{\leq\lambda}[A] = \{\emptyset, \{\lambda\}\} = \{\emptyset\} + \{\lambda\} = J_{<\lambda}[A] + \{\lambda\},$$

so λ has a generator in this case. So henceforth we assume that $|A|^{++} < \lambda$.

By Lemma 31.22, there is a universal sequence $f = \langle f_\xi : \xi < \lambda\rangle$ for λ such that $(*)_{|A|^+}$ holds. Hence by Lemma 8.40, f has an exact upper bound h with respect to $<_{J_{<\lambda}}$. Since h is a least upper bound for f and the identity function on A is an upper bound for f, we may assume that $h(a) \leq a$ for all $a \in A$. We now define

$$B = \{a \in A : h(a) = a\}.$$

Thus we can finish the proof by showing that

$$(\star) \qquad J_{\leq\lambda}[A] = J_{<\lambda}[A] + B$$

First we show that $B \in J_{\leq\lambda}[A]$, i.e., that $\mathrm{pcf}(B) \subseteq \lambda^+$. Let D be any ultrafilter over A having B as an element; we want to show that $\mathrm{cf}(\prod A/D) \leq \lambda$. If $D \cap J_{<\lambda} \neq \emptyset$, then $\mathrm{cf}(\prod A/D) < \lambda$ by the definition of $J_{<\lambda}$. Suppose that $D \cap J_{<\lambda} = \emptyset$. Now since f is $<_{J_{<\lambda}}$-increasing and $D \cap J_{<\lambda} = \emptyset$, the sequence f is also $<_D$-increasing. It is also cofinal; for let $g \in \prod A$. Define

$$g'(a) = \begin{cases} g(a) & \text{if } a \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{a \in A : g'(a) \geq h(a)\} \subseteq \{a \in A : h(a) = 0\} \subseteq \{a \in A : f_0(a) \geq h(a)\} \in J_{<\lambda}$. So $g' <_{J_{<\lambda}} h$. Since h is an exact upper bound for f, there is hence a $\xi < \lambda$ such that $g' <_{J_{<\lambda}} f_\xi$. Hence $g' <_D f_\xi$, and clearly $g =_D g'$, so $g <_D f_\xi$. This proves that $\mathrm{cf}(\prod A/D) = \lambda$. So we have proved $\supseteq$ in $(\star)$.

For $\subseteq$, we argue by contradiction and suppose that there is an $X \in J_{\leq\lambda}$ such that $X \notin J_{<\lambda}[A] + B$. Hence (by Proposition 31.16), $X\backslash B \notin J_{<\lambda}$. Hence $J^*_{<\lambda} \cup \{X\backslash B\}$ has fip, so we extend it to an ultrafilter D. Since $D \cap J_{<\lambda} = \emptyset$, we have $\mathrm{cf}(\prod A/D) \geq \lambda$. But also $X \in D$ since $X \backslash B \in D$, and $X \in J_{\leq\lambda}$, so $\mathrm{cf}(\prod A/D) = \lambda$. By the universality of f it follows that f is cofinal in $\mathrm{cf}(\prod A/D)$. But $A \backslash B \in D$, so $\{a \in A : h(a) < a\} \in D$, and so there is a $\xi < \lambda$ such that $h <_D f_\xi$. This contradicts the fact that h is an upper bound of f under $<_{J_{<\lambda}}$. $\square$

Now we state some important properties of generators.

Lemma 31.25. *Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and $B \subseteq A$.*

(i) If B is a λ-generator, D is an ultrafilter on A, and $\mathrm{cf}(\prod A/D) = \lambda$, then $B \in D$.

(ii) If B is a λ-generator, then $\lambda \notin \mathrm{pcf}(A \backslash B)$.

(iii) If $B \in J_{\leq\lambda}$ and $\lambda \notin \mathrm{pcf}(A \backslash B)$, then B is a λ-generator.

(iv) If $\lambda = \max(\mathrm{pcf}(A))$, then A is a λ-generator on A.

(v) If B is a λ-generator, then the restrictions to B of any universal sequence for λ are cofinal in $(\prod B, <_{J_{<\lambda}[B]})$.

(vi) If B is a λ-generator, then $\mathrm{tcf}(\prod B, <_{J_{<\lambda}[B]}) = \lambda$.

(vii) If B is a λ-generator on A, then $\lambda = \max(\mathrm{pcf}(B))$.

(viii) If B is a λ-generator on A and D is an ultrafilter on A, then $\mathrm{cf}(\prod A/D) = \lambda$ iff $B \in D$ and $D \cap J_{<\lambda} = \emptyset$.

(ix) If B is a λ-generator on A and $B =_{J_{<\lambda}} C$, then C is a λ-generator on A. [Here $X =_I Y$ means that the symmetric difference of X and Y is in I, for any ideal I.]

(x) If B is a λ-generator, then so is $B \cap (\lambda + 1)$.

(xi) If B and C are λ-generators, then $B =_{J_{<\lambda}} C$.

(xii) If $\lambda = \max(\mathrm{pcf}(A))$ and B is a λ-generator, then $A \backslash B \in J_{<\lambda}$.

Proof. (i): By Corollary 31.9(ii), choose $C \in J_{\leq\lambda} \cap D$. Hence $C \subseteq X \cup B$ for some $X \in J_{<\lambda}$. By Corollary 31.9(ii) again, $J_{<\lambda} \cap D = \emptyset$, so $X \notin D$. Thus $C \backslash X \subseteq B$ and $C \backslash X \in D$, so $B \in D$.

(ii): Clear by (i).

(iii): Assume the hypothesis. We need to show that every member C of $J_{\leq\lambda}$ is a member of $J_{<\lambda} + B$. Now $\mathrm{pcf}(C) \subseteq \lambda^+$. Hence $\mathrm{pcf}(C \backslash B) \subseteq \lambda$, so $C \backslash B \in J_{<\lambda}$, and the desired conclusion follows from Proposition 31.16.

(iv): By (iii).

(v): Suppose not. Let $f = \langle f_\xi : \xi < \lambda \rangle$ be a universal sequence for λ such that there is an $h \in \prod B$ such that h is not bounded by any $f_\xi \restriction B$. Thus $\leq (f_\xi \restriction B, h) \notin J_{<\lambda}[B]$ for all $\xi < \lambda$. Now suppose that $\xi < \eta < \lambda$. Then

$$\begin{aligned} \leq (f_\eta \restriction B, h) \backslash (\leq (f_\xi \restriction B, h)) &= \{a \in B : f_\eta(a) \leq h(a) < f_\xi(a)\} \\ &\subseteq \{a \in A : f_\eta(a) < f_\xi(a)\} \in J_{<\lambda}[A]. \end{aligned}$$

Hence by Lemma 31.17(i) we have $\leq (f_\eta \restriction B, h) \backslash (\leq (f_\xi \restriction B, h)) \in J_{<\lambda}[B]$. It follows that if N is a finite subset of λ with largest element less than η, then

$$(*) \qquad (\leq (f_\eta \restriction B, h)) \backslash \bigcap_{\xi \in N} (\leq (f_\xi \restriction B, h)) \in J_{<\lambda}[B].$$

We claim now that

$$M \stackrel{\text{def}}{=} \{\leq (f_\xi \restriction B, h) : \xi < \lambda\} \cup (J_{<\lambda}[B])^*$$

has fip. Otherwise, there is a finite subset N of λ and a set $C \in J_{<\lambda}[B]$ such that

$$\left(\bigcap_{\xi\in N} \leq (f_\xi \restriction B, h)\right) \cap (B \backslash C) = \emptyset;$$

hence if ξ is the largest member of N we get $\leq (f_\xi \restriction B, h) \in J_{<\lambda}[B]$ by $(*)$, contradiction. So we extend the set M to an ultrafilter D on B, then to an ultrafilter E on A. Note that $B \in E$. Also, $E \cap J_{<\lambda}[A] = \emptyset$. In fact, if $X \in E \cap J_{<\lambda}[A]$, then $X \cap B \in J_{<\lambda}[A]$, so $X \cap B \in D \cap J_{<\lambda}[B]$ by Lemma 31.17(i). But $D \cap J_{<\lambda}[B] = \emptyset$ by construction. Now $B \in E \cap J_{\leq\lambda}[A]$, so $\mathrm{cf}(\prod A/E) = \lambda$, and h bounds all f_ξ in this ultraproduct, contradicting the universality of f.

(vi): By Lemma 31.17 and (v).

(vii): By (i) we have $\lambda \in \mathrm{pcf}(B)$. Now $B \in J_{\leq\lambda}[A]$, so $\mathrm{pcf}(B) \subseteq \lambda^+$. The desired conclusion follows.

(viii): For $\Rightarrow$, suppose that $\mathrm{cf}(\prod A/D) = \lambda$. Then $B \in D$ by (i), and obviously $D \cap J_{<\lambda} = \emptyset$. For $\Leftarrow$, assume that $B \in D$ and $D \cap J_{<\lambda} = \emptyset$. Now $B \in J_{\leq\lambda}$, so $\mathrm{cf}(\prod A/D) = \lambda$ by Corollary 31.9(ii).

(ix): We have $B \in J_{\leq\lambda}$ and $C = (C\backslash B) \cup (C \cap B)$, so $C \in J_{\leq\lambda}$. Suppose that $\lambda \in \mathrm{pcf}(A\backslash C)$. Let D be an ultrafilter on A such that $\mathrm{cf}(\prod A/D) = \lambda$ and $A\backslash C \in D$. Now $B \in D$ by (i), so $B\backslash C \in D$. This contradicts $B\backslash C \in J_{<\lambda}$. So $\lambda \notin \mathrm{pcf}(A\backslash C)$. Hence C is a λ-generator, by (iii).

(x): Let $B' = B \cap (\lambda + 1)$. Clearly $B' \in J_{\leq\lambda}$. Suppose that $\lambda \in \mathrm{pcf}(A\backslash B')$. Say $\lambda = \mathrm{cf}(\prod A/D)$ with $A\backslash B' \in D$. Also $A \cap (\lambda + 1) \in D$, since $A\backslash(\lambda + 1) \in D$ would imply that $\mathrm{cf}(\prod A/D) > \lambda$ by Proposition 31.1(vii). Since clearly

$$(A\backslash B') \cap (A \cap (\lambda + 1)) \subseteq A\backslash B,$$

this yields $A\backslash B \in D$, contradicting (ii). Therefore, $\lambda \notin \mathrm{pcf}(A\backslash B')$. So B' is a λ-generator, by (iii).

(xi): This is clear from Proposition 31.16.

(xii): Clear by (iv) and (xi). □

Lemma 31.26. *Suppose that A is a progressive set, F is a proper filter over A, and λ is a cardinal. Then the following are equivalent.*

(i) $\mathrm{tcf}(\prod A/F) = \lambda$.

(ii) $\lambda \in \mathrm{pcf}(A)$, *$F$ has a λ-generator on A as an element, and $J^*_{<\lambda} \subseteq F$.*

(iii) $\mathrm{cf}(\prod A/D) = \lambda$ *for every ultrafilter D extending F.*

Proof. (i)$\Rightarrow$(iii): obvious.

(iii)$\Rightarrow$(ii): Obviously $\lambda \in \mathrm{pcf}(A)$. Let B be a λ-generator on A. Suppose that $B \notin F$. Then there is an ultrafilter D on A such that $A\backslash B \in D$ and D extends F. Then $\mathrm{cf}(\prod A/D) = \lambda$ by (iii), and this contradicts Lemma 31.25(i).

(ii)$\Rightarrow$(i): Let $B \in F$ be a λ-generator. By Lemma 31.25(vi) we have $\mathrm{tcf}(\prod B/J_{<\lambda}) = \lambda$, and hence $\mathrm{tcf}(\prod A/F) = \lambda$ since $B \in F$ and $J^*_{<\lambda} \subseteq F$. □

Proposition 31.27. *Suppose that A is a progressive set of regular cardinals, and λ is any cardinal. Then the following conditions are equivalent:*

(i) $\lambda = \max(\mathrm{pcf}(A))$.

(ii) $\lambda = \mathrm{tcf}(\prod A/J_{<\lambda}[A])$.

(iii) $\lambda = \mathrm{cf}(\prod A/J_{<\lambda}[A])$.

Proof. (i)⇒(ii): By Lemma 31.25(iv),(vi).

(ii)⇒(iii): Obvious.

(iii)⇒(ii): Assume (iii). Let $\mu = \max(\mathrm{pcf}(A))$. By (i)⇒(iii) we have $\lambda = \mu$. □

Lemma 31.28. *Suppose that A is progressive, $A_0 \subseteq A$, and $\lambda \in \mathrm{pcf}(A_0)$. Suppose that B is a λ-generator on A. Then $B \cap A_0$ is a λ-generator on A_0.*

Proof. Since $B \in J_{\leq\lambda}[A]$, we have $\mathrm{pcf}(B) \subseteq \lambda^+$ and hence $\mathrm{pcf}(B \cap A_0) \subseteq \lambda^+$ and so $B \cap A_0 \in J_{\leq\lambda}[A_0]$. If $\lambda \in \mathrm{pcf}(A_0 \backslash B)$, then also $\lambda \in \mathrm{pcf}(A \backslash B)$, and this contradicts Lemma 31.25(ii). Hence $\lambda \notin \mathrm{pcf}(A_0 \backslash B)$, and hence $B \cap A_0$ is a λ-generator for A_0 by Lemma 31.25(iii). □

Definition. If A is progressive, a *generating sequence for A* is a sequence $\langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ such that B_λ is a λ-generator on A for each $\lambda \in \mathrm{pcf}(A)$.

Theorem 31.29. *Suppose that A is progressive, $\langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a generating sequence for A, and $X \subseteq A$. Then there is a finite subset N of $\mathrm{pcf}(X)$ such that $X \subseteq \bigcup_{\mu\in N} B_\mu$.*

Proof. We show that for all $X \subseteq A$, if $\lambda = \max(\mathrm{pcf}(X))$, then there is a finite subset N as indicated, using induction on λ. So, suppose that this is true for every cardinal $\mu < \lambda$, and now suppose that $X \subseteq A$ and $\max(\mathrm{pcf}(X)) = \lambda$. Then $\lambda \notin \mathrm{pcf}(X \backslash B_\lambda)$ by Lemma 31.25(ii), and so $\mathrm{pcf}(X \backslash B_\lambda) \subseteq \lambda$. Hence $\max(\mathrm{pcf}(X \backslash B_\lambda)) < \lambda$. Hence by the inductive hypothesis there is a finite subset N of $\mathrm{pcf}(X \backslash B_\lambda)$ such that $X \backslash B_\lambda \subseteq \bigcup_{\mu\in N} B_\mu$. Hence

$$X \subseteq B_\lambda \cup \bigcup_{\mu\in N} B_\mu,$$

and $\{\lambda\} \cup N \subseteq \mathrm{pcf}(X)$. □

Corollary 31.30. *Suppose that A is progressive, $\langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a generating sequence for A, and $X \subseteq A$. Suppose that λ is any infinite cardinal. Then $X \in J_{<\lambda}[A]$ iff $X \subseteq \bigcup_{\mu\in N} B_\mu$ for some finite subset N of $\lambda \cap \mathrm{pcf}(A)$.*

Proof. ⇒: Assume that $X \in J_{<\lambda}[A]$. Thus $\mathrm{pcf}(X) \subseteq \lambda$, and Theorem 31.29 gives the desired conclusion.

⇐: Assume that a set N is given as indicated. Suppose that $\rho \in \mathrm{pcf}(X)$. Say $\rho = \mathrm{cf}(\prod A/D)$ with $X \in D$. Then $B_\mu \in D$ for some $\mu \in N$. By the definition of generator, $B_\mu \in J_{\leq\mu}[A]$, and hence $\rho \leq \mu < \lambda$. Thus we have shown that $\mathrm{pcf}(X) \subseteq \lambda$, so $X \in J_{<\lambda}[A]$. □

Lemma 31.31. *Suppose that A is progressive and $\langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a generating sequence for A. Suppose that D is an ultrafilter on A. Then there is a $\lambda \in \mathrm{pcf}(A)$ such that $B_\lambda \in D$, and if λ is minimum with this property, then $\lambda = \mathrm{cf}(\prod A/D)$.*

Proof. Let $\mu = \mathrm{cf}(\prod A/D)$. Then $\mu \in \mathrm{pcf}(A)$ and $B_\mu \in D$ by Lemma 31.25(i). Suppose that $B_\lambda \in D$ with $\lambda < \mu$. Now $B_\lambda \in J_{\leq\lambda} \subseteq J_{<\mu}$, contradicting Lemma 31.25(viii), applied to μ. □

Lemma 31.32. *If A is progressive and also $\mathrm{pcf}(A)$ is progressive, and if $\lambda \in \mathrm{pcf}(A)$ and B is a λ-generator for A, then $\mathrm{pcf}(B)$ is a λ-generator for $\mathrm{pcf}(A)$.*

Proof. Note by Theorem 31.15 that $\mathrm{pcf}(\mathrm{pcf}(B)) = \mathrm{pcf}(B)$ and $\mathrm{pcf}(\mathrm{pcf}(A\backslash B)) = \mathrm{pcf}(A\backslash B)$. Since $B \in J_{\leq\lambda}[A]$, we have $\mathrm{pcf}(B) \subseteq \lambda^+$, and hence $\mathrm{pcf}(\mathrm{pcf}(B)) \subseteq \lambda^+$ and so $\mathrm{pcf}(B) \in J_{\leq\lambda}[\mathrm{pcf}(A)]$. Now suppose that $\lambda \in \mathrm{pcf}(\mathrm{pcf}(A)\backslash\mathrm{pcf}(B))$. Then by Lemma 31.1(iv) we have $\lambda \in \mathrm{pcf}(\mathrm{pcf}(A\backslash B)) = \mathrm{pcf}(A\backslash B)$, contradicting Lemma 31.25(ii). So $\lambda \notin \mathrm{pcf}(\mathrm{pcf}(A)\backslash\mathrm{pcf}(B))$. It now follows by Lemma 31.25(iii) that $\mathrm{pcf}(B)$ is a λ-generator for $\mathrm{pcf}(A)$. □

The following result is relevant to Theorem 30.44. Let μ be a singular cardinal, C a club of μ, and suppose that $X \in J_{<\mu}[C^{(+)}]$. Now $\mathrm{pcf}(X)$ has a maximal element, and so there is an $\alpha < \mu$ such that $X \subseteq \mathrm{pcf}(X) \subseteq \alpha$. Thus $J_{<\mu}[C^{(+)}] \subseteq J^{\mathrm{bd}}$.

Lemma 31.33. *If μ is a singular cardinal of uncountable cofinality, then there is a club $C \subseteq \mu$ such that $\mathrm{tcf}(\prod C^{(+)}/J_{<\mu}[C^{(+)}]) = \mu^+$.*

Proof. Let C_0 be a club in μ such that such that $\mu^+ = \mathrm{tcf}(\prod C_0^{(+)}/J^{bd})$, by Theorem 30.44. Let $C_1 \subseteq C_0$ be such that the order type of C_1 is $\mathrm{cf}(\mu)$, C_1 is cofinal in μ, and $\forall\kappa \in C_1[\mathrm{cf}(\mu) < \kappa]$. Hence $C_1^{(+)}$ is progressive. Now $\mu^+ \in \mathrm{pcf}(C_1^{(+)})$ by Lemma 31.26. Let B be a μ^+-generator for $C_1^{(+)}$. Define $C = \{\delta \in C_1 : \delta^+ \in B\}$. Now $C_1\backslash C$ is bounded. Otherwise, let $X = C_1^{(+)}\backslash B = (C_1\backslash C)^{(+)}$. So X is unbounded, and hence clearly $\mu^+ = \mathrm{tcf}(\prod X/J^{\mathrm{bd}})$. Hence $\mu^+ \in \mathrm{pcf}(X)$. This contradicts Lemma 31.25(ii).

So, choose $\varepsilon < \mu$ such that $C_1\backslash C \subseteq \varepsilon$. Hence $C_1\backslash\varepsilon \subseteq C\backslash\varepsilon \subseteq C_1\backslash\varepsilon$, so $C_1\backslash\varepsilon = C\backslash\varepsilon$. Clearly $\mu^+ = \mathrm{tcf}(\prod(C_1\backslash\varepsilon)^{(+)}/J^{\mathrm{bd}})$, so $\mu^+ \in \mathrm{pcf}((C_1\backslash\varepsilon)^{(+)})$. We claim that $\mathrm{tcf}(\prod(C_1\backslash\varepsilon)^{(+)}/J_{<\mu^+}[(C_1\backslash\varepsilon)^{(+)}]) = \mu^+$. To show this, we apply Lemma 31.26. Suppose that D is any ultrafilter on $(C_1\backslash\varepsilon)^{(+)}$ such that $J_{<\mu^+}[(C_1\backslash\varepsilon)^{(+)}] \cap D = \emptyset$. Now by Lemma 31.28, $B \cap (C_1\backslash\varepsilon)^{(+)}$ is a μ^+-generator for $(C_1\backslash\varepsilon)^{(+)}$. Note that $C^+ \subseteq B$. Now $B \cap (C_1\backslash\varepsilon)^{(+)} = B \cap (C\backslash\varepsilon)^{(+)} = (C\backslash\varepsilon)^{(+)}$. It follows by Lemma 31.25(viii) that $\mathrm{cf}(\prod(C_1\backslash\varepsilon)^{(+)}/D) = \mu^+$. This proves that $\mathrm{tcf}(\prod(C_0\backslash\varepsilon)^{(+)}/J_{<\mu^+}[(C_1\backslash\varepsilon)^{(+)}]) = \mu^+$. Now we claim that $J_{<\mu^+}[(C_1\backslash\varepsilon)^{(+)}] = J_{<\mu}[(C_1\backslash\varepsilon)^{(+)}]$. For, suppose that $X \in J_{<\mu^+}[(C_1\backslash\varepsilon)^{(+)}]$. So $\mathrm{pcf}(X) \subseteq \mu^+$. Since X is progressive (because $C_1\backslash\varepsilon)^{(+)}$ is), we have $\max(\mathrm{pcf}(X)) < \mu$, hence $\mathrm{pcf}(X) \subseteq \mu$. □

By essentially the same proof as for Lemma 31.33 we get

Lemma 31.34. *If μ is a singular cardinal of countable cofinality, then there is an unbounded subset C of μ consisting of regular cardinals such that $\mathrm{tcf}(\prod C/J_{<\mu}[C]) = \mu^+$.*

Proof. Let C_0 be an unbounded collection of regular cardinals in μ such that $\mu^+ = \mathrm{tcf}(\prod C_0/J^{bd})$, by Theorem 30.45. Let $C_1 \subseteq C_0$ be such that the order type of C_1 is $\mathrm{cf}(\mu)$, C_1 is cofinal in μ, and $\forall\kappa \in C_1[\omega < \kappa]$. Hence C_1 is progressive. Now $\mu^+ \in \mathrm{pcf}(C_1)$

by Lemma 31.26. Let B be a μ^+-generator for C_1. Define $C = B \cap C_1$. Now $C_1 \backslash C$ is bounded. Otherwise, let $X = C_1 \backslash B = C_1 \backslash C$. So X is unbounded, and hence clearly $\mu^+ = \mathrm{tcf}(\prod X/J^{\mathrm{bd}})$. Hence $\mu^+ \in \mathrm{pcf}(X)$. This contradicts Lemma 31.25(ii).

So, choose $\varepsilon < \mu$ such that $C_1 \backslash C \subseteq \varepsilon$. Hence $C_1 \backslash \varepsilon \subseteq C \backslash \varepsilon \subseteq C_1 \backslash \varepsilon$, so $C_1 \backslash \varepsilon = C \backslash \varepsilon$. Clearly $\mu^+ = \mathrm{tcf}(\prod (C_1 \backslash \varepsilon)/J^{\mathrm{bd}})$, so $\mu^+ \in \mathrm{pcf}(C_1 \backslash \varepsilon)$. We claim that $\mathrm{tcf}(\prod (C_1 \backslash \varepsilon)/J_{<\mu^+}[C_1 \backslash \varepsilon] = \mu^+$. To show this, we apply Lemma 31.26. Suppose that D is any ultrafilter on $C_1 \backslash \varepsilon$ such that $J_{<\mu^+}[C_1 \backslash \varepsilon] \cap D = \emptyset$. Now by Lemma 31.28, $B \cap (C_1 \backslash \varepsilon)$ is a μ^+-generator for $C_1 \backslash \varepsilon$. Note that $C \subseteq B$. Now $B \cap (C_1 \backslash \varepsilon) = B \cap (C \backslash \varepsilon) = (C \backslash \varepsilon)$. It follows by Lemma 31.25(viii) that $\mathrm{cf}(\prod (C_1 \backslash \varepsilon)/D) = \mu^+$. This proves that $\mathrm{tcf}(\prod (C_0 \backslash \varepsilon)/J_{<\mu^+}[C_1 \backslash \varepsilon]) = \mu^+$. Now we claim that $J_{<\mu^+}[C_1 \backslash \varepsilon] = J_{<\mu}[C_1 \backslash \varepsilon]$. For, suppose that $X \in J_{<\mu^+}[C_1 \backslash \varepsilon]$. So $\mathrm{pcf}(X) \subseteq \mu^+$. Since X is progressive (because $C_1 \backslash \varepsilon$ is), we have $\max(\mathrm{pcf}(X)) < \mu$, hence $\mathrm{pcf}(X) \subseteq \mu$. $\square$

Proposition 31.35. *Suppose that F is a proper filter over a progressive set A of regular cardinals. Define*

$$\mathrm{pcf}_F(A) = \left\{\mathrm{cf}\left(\prod A/D\right) : D \textit{ is an ultrafilter extending } F\right\}.$$

Then:

(i) $\max(\mathrm{pcf}_F(A))$ *exists.*

(ii) $\mathrm{cf}(\prod A/F) = \max(\mathrm{pcf}_F(A))$*.*

(iii) If $B \subseteq \mathrm{pcf}_F(A)$ *is progressive, then* $\mathrm{pcf}(B) \subseteq \mathrm{pcf}_F(A)$*.*

(iv) If A is a progressive interval of regular cardinals with no largest element, and

$$F = \{X \subseteq A : A \backslash X \textit{ is bounded}\}$$

is the filter of co-bounded subsets of A, then $\mathrm{pcf}_F(A)$ *is an interval of regular cardinals.*

Proof. (i): Clearly $\mathrm{pcf}_F(A) \subseteq \mathrm{pcf}(A)$, and so if $\lambda = \max(\mathrm{pcf}(A))$, then $A \in F \cap J_{<\lambda^+}[A]$. Hence we can choose μ minimum such that $F \cap J_{<\mu}[A] \neq \emptyset$. By Corollary 31.12, μ is not a limit cardinal; write $\mu = \lambda^+$. Then $F \cap J_{<\lambda} = \emptyset$, and so $F \cup J^*_{<\lambda}$ has fip; let D be an ultrafilter containing this set. Then $D \cap J_{\leq\lambda} \supseteq F \cap J_{\leq\lambda} \neq \emptyset$, while $D \cap J_{<\lambda} = \emptyset$. Hence $\mathrm{cf}(\prod A/D) = \lambda$ by Corollary 31.9. On the other hand, since $F \cap J_{\leq\lambda}[A] \neq \emptyset$, any ultrafilter E containing F must be such that $\mathrm{cf}(\prod A/E) \leq \lambda$.

(ii): Cf. the proof of Theorem 31.20. Let $\lambda = \max(\mathrm{pcf}_F(A))$, and let D be an ultrafilter extending F such that $\lambda = \mathrm{cf}(\prod A/D)$. Let $\langle f_\alpha : \alpha < \lambda\rangle$ be strictly increasing and cofinal mod D. Now if $g < h$ mod F, then also $g < h$ mod D. So a cofinal subset of $\prod A$ mod F is also a cofinal subset mod D, so $\lambda \leq \mathrm{cf}(\prod A/F)$. Hence it suffices to exhibit a cofinal subset of $\prod A$ mod F of size λ. For every $\mu \in \mathrm{pcf}_F(A)$ fix a universal sequence $f^\mu = \langle f^\mu_i : i < \mu\rangle$ for μ, by Theorem 31.18. Let G be the set of all functions of the form

$$\sup\{f^{\mu_1}_{i_1}, f^{\mu_2}_{i_2}, \ldots, f^{\mu_n}_{i_n}\},$$

where $\mu_1, \mu_2, \ldots, \mu_n$ is a finite sequence of members of $\mathrm{pcf}_F(A)$, possibly with repetitions, and $i_k < \mu_k$ for each $k = 1, \ldots, n$. We claim that G is cofinal in $(\prod A, <_F)$; this will complete the proof of (ii).

To prove this claim, let $g \in \prod A$. Suppose that $g \not< f \bmod F$ for all $f \in G$. Then, we claim, the set

$$(*) \qquad F \cup \{\{a \in A : f(a) \leq g(a)\} : f \in G\}$$

has fip. For, suppose not. Then there is a finite subset G' of G such that $\bigcup_{g \in G'} \{a \in A : g(a) < f(a)\} \in F$. Let $h = \sup_{f \in G'} f$. Then $g < h \bmod F$ and $h \in G$, contradiction. Thus $(*)$ has fip, and we let D be an ultrafilter containing it. Let $\mu = \mathrm{cf}(\prod A/D)$. Then $\mu \in \mathrm{pcf}_F(A)$, and $f \leq g \bmod D$ for all $f \in G$. Since the members of a universal sequence for μ are in G, this is a contradiction. This completes the proof of (ii).

For (iii), we look at the proof of Theorem 31.15. Let F' be the ultrafilter named F at the beginning of that proof. Since $B \subseteq \mathrm{pcf}_F(A)$, each $b \in B$ is in $\mathrm{pcf}_F(A)$, and hence the ultrafilters D_b can be taken to extend F. Hence $F \subseteq F'$, and so $\mu \in \mathrm{pcf}_F(A)$, as desired in (iii).

Finally, we prove (iv). Let $\lambda_0 = \min(\mathrm{pcf}_F(A))$ and $\lambda_1 = \max(\mathrm{pcf}_F(A))$, and suppose that μ is a regular cardinal such that $\lambda_0 < \mu < \lambda_1$. Let D be an ultrafilter such that $F \subseteq D$ and $\mathrm{cf}(\prod A/D) = \lambda_1$. Then by Corollary 31.9(ii), $D \cap J_{<\lambda_1} = \emptyset$, so $J^*_{\lambda_1} \subseteq D$. Thus $F \cup J^*_{<\mu} \subseteq F \cup J^*_{<\lambda_1} \subseteq D$, so $F \cup J^+_{<\mu}$ generates a proper filter G. Since $(\prod A, <_{J_{<\mu}})$ is μ-directed by Theorem 31.8, so is $(\prod A, <_G)$. Note that if $a \in A$, then $\{b \in A : a < b\} \in F$. It follows that $\sup(A) \leq \lambda_0 < \mu$. Hence we can apply Theorem 31.4 and get a subset A' of A (since A is an interval of regular cardinals) and a proper ideal K over A' such that A' is cofinal in A, K contains all proper initial segments of A', and $\mathrm{tcf}(\prod A, <_K) = \mu$. Let $\langle f_\alpha : \alpha < \mu \rangle$ be strictly increasing and cofinal mod K. Extend K^* to a filter L on A, and extend each function f_α to a function f^+_α on A. Then clearly $\langle f^+_\alpha : \alpha < \mu \rangle$ is strictly increasing and cofinal mod L, and L contains F. This shows that $\mu \in \mathrm{pcf}_F(A)$. □

EXERCISES

E31.1. A set A of regular cardinals is *almost progressive* iff A is infinite, and $A \cap |A|$ is finite. Prove the following:

(i) Every progressive set is almost progressive.

(ii) If A is an infinite set of regular cardinals and $|A| < \aleph_\omega$, then A is almost progressive.

(iii) Every infinite subset of an almost progressive set is almost progressive.

(iv) If A is almost progressive, then $A \backslash |A|^+$ is progressive, $A \cap |A|^+$ is finite, and $A = (A \backslash |A|^+) \cup (A \cap |A|^+)$.

(v) If α is an ordinal, A is almost progressive, and $A \cap \alpha$ is unbounded in α, then α is a singular cardinal.

(vi) If A is an almost progressive interval of regular cardinals, then A does not have any weak inaccessible as a member, except possibly its first element. If in addition A is infinite, then there is a singular cardinal λ such that $A \cap \lambda$ is unbounded in λ and $A \backslash \lambda$ is finite.

E31.2. Show that Theorem 31.4 and Corollary 31.5 also hold if A is almost progressive.

E31.3. Suppose that A is progressive and $\kappa \leq |A|$ is regular. Show that $\sup(\mathrm{pcf}_\kappa(A)) \leq \sup(A)^\kappa$.

E31.4. Suppose that A is progressive, and ν is a cardinal such that $A \cap \nu$ is unbounded in ν. Show that $\nu^+ \in \mathrm{pcf}(A)$.

E31.5. Assume GCH and suppose that A is progressive. Show that

$$\mathrm{pcf}(A) = A \cup \{\nu^+ : \nu \text{ a cardinal, } A \cap \nu \text{ unbounded in } \nu\}.$$

E31.6. Suppose that α is a limit ordinal and A is an infinite set such that $|A| < \mathrm{cf}(\alpha)$. Determine all regular cardinals λ such that $l = \mathrm{cf}({}^A\alpha/D)$ for some ultrafilter D on A.

E31.7. Suppose that κ is a regular cardinal and D is an ultrafilter on κ such that $\kappa \backslash \alpha \in D$ for every $\alpha < \kappa$. Show that $\mathrm{cf}({}^\kappa\kappa/D) > \kappa$.

E31.8. Let A be a progressive set of regular cardinals and λ an infinite cardinal. Suppose that J is an ideal on A. Show that the following conditions are equivalent:

(i) $J = I_{<\lambda}[A]$.

(ii) J is the intersection of all ideals K on A which satisfy the following condition: For each $X \subseteq A$ with $X \notin K$ there is an ultrafilter D on X such that $D \cap K = \emptyset$ and $\mathrm{cf}(\prod X/D) \geq \lambda$.

E31.9. Suppose that A is progressive and J is a proper ideal on A.

(i) Show that if $X \in \mathscr{P}(A) \backslash J$, then $J + (A \backslash X)$ is proper.

(ii) Show that there is an $X \in \mathscr{P}(A) \backslash J$ such that $\mathrm{tcf}(\prod A, <_{J+(A\backslash X)})$ exists.

E31.10. Suppose that A is progressive and κ is an infinite cardinal with $\kappa \leq |A|$. Then $|\mathrm{pcf}_\kappa(A)| \leq |A|^\kappa$.

32. Main cofinality theorems

The sets H_Ψ

We will shortly give several proofs involving the important general idea of making elementary chains inside the sets H_Ψ. Recall that H_Ψ, for an infinite cardinal Ψ, is the collection of all sets hereditarily of size less than Ψ, i.e., with transitive closure of size less than Ψ. We consider H_Ψ as a structure with $\in$ together with a well-ordering $<^*$ of it, possibly with other relations or functions, and consider elementary substructures of such structures.

Recall that A is an *elementary substructure* of B iff A is a subset of B, and for every formula $\varphi(x_0, \ldots, x_{m-1})$ and all $a_0, \ldots, a_{m-1} \in A$, $A \models \varphi(a_0, \ldots, a_{m-1})$ iff $B \models \varphi(a_0, \ldots, a_{m-1})$.

The basic downward Löwenheim-Skolem theorem will be used a lot. This theorem depends on the following lemma.

Lemma 32.1. (Tarski) *Suppose that A and B are first-order structures in the same language, with A a substructure of B. Then the following conditions are equivalent:*

(i) A is an elementary substructure of B.

(ii) For every formula of the form $\exists y\varphi(x_0, \ldots, x_{m-1}, y)$ and all $a_0, \ldots, a_{m-1} \in A$, if $B \models \exists y\varphi(a_0, \ldots, a_{m-1}, y)$ then there is a $b \in A$ such that $B \models \varphi(a_0, \ldots, a_{m-1}, b)$.

Proof. (i)⇒(ii): Assume (i) and the hypotheses of (ii). Then by (i) we see that $A \models \exists y\varphi(a_0, \ldots, a_{m-1}, y)$, so we can choose $b \in A$ such that $A \models \varphi(a_0, \ldots, a_{m-1}, b)$. Hence $B \models \varphi(a_0, \ldots, a_{m-1}, b)$, as desired.

(ii)⇒(i): Assume (ii). We show that for any formula $\varphi(x_0, \ldots, x_{m-1})$ and any elements $a_0, \ldots, a_{m-1} \in A$, $A \models \varphi(a_0, \ldots, a_{m-1})$ iff $B \models \varphi(a_0, \ldots, a_{m-1})$, by induction on φ. It is true for φ atomic by our assumption that A is a substructure of B. The induction steps involving $\neg$ and $\vee$ are clear. Now suppose that $A \models \exists y\varphi(a_0, \ldots, a_{m-1}, y)$, with $a_0, \ldots, a_{m-1} \in A$. Choose $b \in A$ such that $A \models \varphi(a_0, \ldots, a_{m-1}, b)$. By the inductive assumption, $B \models \varphi(a_0, \ldots, a_{m-1}, b)$. Hence $B \models \exists y\varphi(a_0, \ldots, a_{m-1}, y)$, as desired.

Conversely, suppose that $B \models \exists y\varphi(a_0, \ldots, a_{m-1}, y)$. By (ii), choose $b \in A$ such that $B \models \varphi(a_0, \ldots, a_{m-1}, b)$. By the inductive assumption, $A \models \varphi(a_0, \ldots, a_{m-1}, b)$. Hence $A \models \exists y\varphi(a_0, \ldots, a_{m-1}, y)$, as desired. □

Theorem 32.2. *Suppose that A is an L-structure, X is a subset of A, κ is an infinite cardinal, and κ is $\geq$ both $|X|$ and the number of formulas of $\mathscr{L}$, while $\kappa \leq |A|$. Then A has an elementary substructure B such that $X \subseteq B$ and $|B| = \kappa$.*

Proof. Let a well-order $\prec$ of A be given. We define $\langle C_n : n \in \omega\rangle$ by recursion. Let C_0 be a subset of A of size κ with $X \subseteq C_0$. Now suppose that C_n has been defined. Let M_n be the collection of all pairs of the form $(\exists y\varphi(x_0, \ldots, x_{m-1}, y), a)$ such that a is a sequence of elements of C_n of length m. For each such pair we define $f(\exists y\varphi(x_0, \ldots, x_{m-1}, y), a)$ to be the $\prec$-least element b of A such that $A \models \varphi(a_0, \ldots, a_{m-1}, b)$, if there is such an element, and otherwise let it be the least element of C_n. Then we define

$$C_{n+1} = C_n \cup \{f(\exists y\varphi(x_0, \ldots, x_{m-1}, y), a) : (\exists y\varphi(x_0, \ldots, x_{m-1}, y), a) \in M_n\}.$$

Finally, let $B = \bigcup_{n\in\omega} C_n$.

By induction it is clear that $|C_n| = \kappa$ for all $n \in \omega$, and so also $|B| = \kappa$.

Now to show that B is an elementary substructure of A we apply Lemma 32.1. First we show that B is a substructure of A; this amounts to showing that B is closed under each fundamental operation F^A. Say F is m-ary, and $b_0, \ldots, b_{m-1} \in B$. Then there is an n such that $b_0, \ldots, b_{m-1} \in C_n$. Now $(\exists y[Fx_0 \ldots x_{m-1} = y], \langle b_0, \ldots, b_{m-1}\rangle) \in M_n$. Let $c = F^A(b_0, \ldots, b_{m-1})$; so $f((\exists y[Fx_0 \ldots x_{m-1} = y], \langle b_0, \ldots, b_{m-1}\rangle) = c \in C_{n+1} \subseteq B$.

Now suppose that we are given a formula of the form $\exists y\varphi(x_0, \ldots, x_{m-1}, y)$ and elements $a_0, \ldots, a_{m-1}$ of B, and $A \models \exists y\varphi(a_0, \ldots, a_{m-1}, y)$. Clearly there is an $n \in \omega$ such that $a_0, \ldots, a_{m-1} \in C_n$. Then $(\exists y\varphi(x_0, \ldots, x_{m-1}, y), a) \in M_n$, and $f(\exists y\varphi(x_0, \ldots, x_{m-1}, y), a)$ is an element b of $C_{n+1} \subseteq B$ such that $A \models \varphi(a_0, \ldots, a_{m-1}, b)$. This is as desired in Lemma 32.1. $\square$

Given an elementary substructure A of a set H_Ψ, we will frequently use an argument of the following kind. A set theoretic formula holds in the real world, and involves only sets in A. By absoluteness, it holds in H_Ψ, and hence it holds in A. Thus we can transfer a statement to A even though A may not be transitive; and the procedure can be reversed.

To carry this out, we need some facts about transitive closures first of all.

Lemma 32.3. *(i) If $X \subseteq A$, then $\operatorname{tr\,cl}(X) \subseteq \operatorname{tr\,cl}(A)$.*

(ii) $\operatorname{tr\,cl}(\mathscr{P}(A)) = \mathscr{P}(A) \cup \operatorname{tr\,cl}(A)$.

(iii) If $\operatorname{tr\,cl}(A)$ is infinite, then $|\operatorname{tr\,cl}(\mathscr{P}(A))| \le 2^{|\operatorname{tr\,cl}(A)|}$.

(iv) $\operatorname{tr\,cl}(A \cup B) = \operatorname{tr\,cl}(A) \cup \operatorname{tr\,cl}(B)$.

(v) $\operatorname{tr\,cl}(A\times B) = (A\times B) \cup \{\{a\} : a \in A\} \cup \{\{a, b\} : a \in A, b \in B\} \cup \operatorname{tr\,cl}(A) \cup \operatorname{tr\,cl}(B)$.

(vi) If $\operatorname{tr\,cl}(A)$ or $\operatorname{tr\,cl}(B)$ is infinite, then $|\operatorname{tr\,cl}(A \times B)| \le \max(\operatorname{tr\,cl}(A), \operatorname{tr\,cl}(B)$.

(vii) $\operatorname{tr\,cl}({}^A B) \subseteq ({}^A B) \cup \operatorname{tr\,cl}(A \times B)$.

(viii) If $\operatorname{tr\,cl}(A)$ or $\operatorname{tr\,cl}(B)$ is infinite, then $|\operatorname{tr\,cl}({}^A B)| \le 2^{\max(|\operatorname{tr\,cl}(A)|,|\operatorname{tr\,cl}(A)|)}$.

(ix) If $\operatorname{tr\,cl}(A)$ is infinite, then $|\operatorname{tr\,cl}(\prod A)| \le 2^{|\operatorname{tr\,cl}(A)|}$.

(x) If $\operatorname{tr\,cl}(A)$ or $\operatorname{tr\,cl}(B)$ is infinite, then $|\operatorname{tr\,cl}({}^A(\prod B))| \le 2^{2^{\max(|\operatorname{tr\,cl}(A)|,|\operatorname{tr\,cl}(B)|)}}$.

(xi) If A is an infinite set of regular cardinals, then $|\operatorname{tr\,cl}(\operatorname{pcf}(A))| \le 2^{|\operatorname{tr\,cl}(A)|}$.

Proof. (i)–(viii) are clear. For (ix), note that $\prod A \subseteq {}^A\bigcup A$, so (ix) follow from (viii). For (x),

$$\begin{aligned}|\operatorname{tr\,cl}\left({}^A\left(\prod B\right)\right)| &\le 2^{\max\left(|\operatorname{tr\,cl}(A),|\operatorname{tr\,cl}\left(\prod B\right)|\right)} \quad \text{by (viii)}\\ &\le 2^{\max(|\operatorname{tr\,cl}(A),2^{|\operatorname{tr\,cl}(B)|})}\\ &\le 2^{2^{\max(|\operatorname{tr\,cl}(A)|,|\operatorname{tr\,cl}(B)|)}}.\end{aligned}$$

Finally, for (xi), note that $\operatorname{tr\,cl}(\operatorname{pcf}(A)) = \operatorname{pcf}(A) \cup \bigcup \operatorname{pcf}(A)$. Now $|\operatorname{pcf}(A)| \le 2^{|A|} \le 2^{|\operatorname{tr\,cl}(A)|}$ by Theorem 31.10. $\square$

We also need the fact that some rather complicated formulas and functions are absolute for sets H_Ψ. Note that H_Ψ is transitive. Many of the indicated formulas are not absolute for H_Ψ in general, but only under the assumptions given that Ψ is much larger than the sets in question.

Lemma 32.4. *Suppose that Ψ is an uncountable regular cardinal. Then the following formulas (as detailed in the proof) are absolute for H_Ψ.*

(i) $B = \mathscr{P}(A)$.

(ii) "D is an ultrafilter on A".

(iii) κ *is a cardinal.*

(iv) κ *is a regular cardinal.*

(v) "κ and λ are cardinals, and $\lambda = \kappa^+$".

(vi) $\kappa = |A|$.

(vii) $B = \prod A$.

(viii) $A = {}^B C$.

(ix) "A is infinite", if Ψ is uncountable.

(x) "A is an infinite set of regular cardinals and D is an ultrafilter on A and λ is a regular cardinal and $f \in {}^\lambda \prod A$ and f is strictly increasing and cofinal modulo D", provided that $2^{|\mathrm{tr\,cl}(A)|} < \Psi$.

(xi) "A is an infinite set of regular cardinals, and $B = \mathrm{pcf}(A)$", if $2^{|\mathrm{tr\,cl}(A)|} < \Psi$.

(xii) "A is an infinite set of regular cardinals and $f = \langle J_{<\lambda}[A] : \lambda \in \mathrm{pcf}(A)\rangle$", provided that $2^{|\mathrm{tr\,cl}(A)|} < \Psi$.

(xiii) "A is an infinite set of regular cardinals and $B = \langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ and $\forall\lambda \in \mathrm{pcf}(A)(B_\lambda$ is a λ-generator)", if $2^{2^{|\mathrm{tr\,cl}(A)|}} < \Psi$.

Proof. Absoluteness follows by easy arguments upon producing suitable formulas, as follows.

(i): Suppose that $A, B \in H_\Psi$. We may take the formula $B = \mathscr{P}(A)$ to be

$$\forall x \in B[\forall y \in x(y \in A)] \wedge \forall x[\forall y \in x(y \in A) \to x \in B].$$

The first part is obviously absolute for H_Ψ. If the second part holds in V it clearly holds in H_Ψ. Now suppose that the second part holds in H_Ψ. Suppose that $x \subseteq A$. Hence $x \in H_\Psi$ and it follows that $x \in B$.

(ii): Assume that $A, D \in H_\Psi$. We can take the statement "D is an ultrafilter on A" to be the following statement:

$$\begin{aligned}&\forall X \in D(X \subseteq A) \wedge A \in D \wedge \forall X, Y \in D(X \cap Y \in D) \wedge \emptyset \notin D\\ &\wedge \forall Y \forall X \in D[X \subseteq Y \wedge Y \subseteq A \to Y \in D] \wedge \forall Y[Y \subseteq A \to Y \in D \vee (A \backslash Y) \in D].\end{aligned}$$

Again this is absolute because $Y \subseteq A$ implies that $Y \in H_\Psi$.

(iii): Suppose that $\kappa \in H_\Psi$. Then

$$\begin{aligned}\kappa \text{ is a cardinal} \quad \text{iff} \quad &\kappa \text{ is an ordinal and } \forall f[f \text{ is a function and}\\ &\mathrm{dmn}(f) = \kappa \text{ and } \mathrm{rng}(f) \in \kappa \to f \text{ is not one-to-one}].\end{aligned}$$

Note here that if f is a function with $\mathrm{dmn}(f) = \kappa$ and $\mathrm{rng}(f) \subseteq \kappa$, then $f \subseteq \kappa \times \kappa$, and hence $f \in H_\Psi$.

(iv): Assume that $\kappa \in H_\Psi$. Then

$$\begin{aligned}\kappa \text{ is a regular cardinal} \quad \text{iff} \quad & \kappa \text{ is a cardinal, } 1 < \kappa, \text{ and } \forall f[f \text{ is a function} \\ & \text{and } \mathrm{dmn}(f) \in \kappa \text{ and } \mathrm{rng}(f) \subseteq \kappa \text{ and} \\ & \forall \alpha, \beta \in \mathrm{dmn}(f)(\alpha < \beta \rightarrow f(\alpha) < f(\beta)) \\ & \rightarrow \exists \gamma < \kappa \forall \alpha \in \mathrm{dmn}(f)(f(\alpha) \in \gamma)].\end{aligned}$$

(v): Assume that $\kappa, \lambda \in H_\Psi$. Then ($\kappa$ and λ are cardinals and $\lambda = \kappa^+$) iff

$$\begin{aligned}& \kappa \text{ is a cardinal and } \lambda \text{ is a cardinal and } \kappa < \lambda \\ & \text{and } \forall \alpha < \lambda[\kappa < \alpha \rightarrow \exists f[f \text{ is a function and } \mathrm{dmn}(f) = \kappa \\ & \text{and } \mathrm{rng}(f) = \alpha \text{ and } f \text{ is one-one and } \mathrm{rng}(f) = \alpha]].\end{aligned}$$

(vi): Suppose that $\kappa, A \in H_\Psi$. Then

$$\begin{aligned}\kappa = |A| \quad \text{iff} \quad & \kappa \text{ is a cardinal and } \exists f[f \text{ is a function} \\ & \text{and } \mathrm{dmn}(f) = \kappa \text{ and } \mathrm{rng}(f) = A \text{ and } f \text{ is one-to-one}]\end{aligned}$$

(vii): Assume that $A, B \in H_\Psi$. Then

$$\begin{aligned}B = \prod A \quad \text{iff} \quad & \forall f \in B[f \text{ is a function and } \mathrm{dmn}(f) = A \text{ and} \\ & \forall x \in A[f(x) \in x]] \text{ and } \forall f[f \text{ is a function and} \\ & \mathrm{dmn}(f) = A \text{ and } \forall x \in A[f(x) \in x] \rightarrow f \in B].\end{aligned}$$

Note that if f is a function with domain A and $f(x) \in x$ for all $x \in A$, then $f \subseteq A \times \bigcup A$, and hence $f \in H_\Psi$.

(viii): Suppose that $A, B, C \in H_\Psi$. Then

$$\begin{aligned}A = {}^B C \quad \text{iff} \quad & \forall f \in A[f \text{ is a function and } \mathrm{dmn}(f) = B \\ & \text{and } \mathrm{rng}(f) \subseteq C] \text{ and } \forall f[f \text{ is a function} \\ & \text{and } \mathrm{dmn}(f) = B \text{ and } \mathrm{rng}(f) \subseteq C \rightarrow f \in A].\end{aligned}$$

(ix): "A is infinite" iff $\exists f(f$ is a one-one function, $\mathrm{dmn}(f) = \omega$, and $\mathrm{rng}(f) \subseteq A)$.

(x): Suppose that $A, D, \lambda, f \in H_\Psi$, and $2^{|\mathrm{tr\,cl}(A)|} < \Psi$. Then $\prod A \in H_\Psi$ by Lemma 32.3(ix). Now

$$\begin{aligned}& A \text{ is an infinite set of regular cardinals and } D \text{ is an ultrafilter on } A \\ & \text{and } \lambda \text{ is a regular cardinal and } f \in {}^\lambda \prod A \text{ and } f \text{ is strictly} \\ & \text{increasing and cofinal modulo } D\end{aligned}$$

iff

A is infinite and $\forall x \in A[x$ is a regular cardinal$]$ and D is an ultrafilter on A and λ is a regular cardinal and $\exists B\Big[B = \prod A$ and f is a function

and $\mathrm{dmn}(f) = \lambda$ and $\mathrm{rng}(f) \subseteq B$ and

$\forall \xi, \eta < \lambda \forall X \subseteq A[\forall a \in A[a \in X \Leftrightarrow f_\xi(a) < f_\eta(a)] \to X \in D]$

and $\forall g \in B \exists \xi < \lambda \forall X \subseteq A[\forall a \in A[a \in X \Leftrightarrow g(a) < f_\xi(a)] \to X \in D]\Big]$.

(xi): Assume that $2^{|\mathrm{tr\,cl}(A)|)} < \Psi$, and $A, B \in H_\Psi$. Let $\varphi(A, D, \lambda, f)$ be the statement of (x). Note:

(1) If $\varphi(A, D, \lambda, f)$, then $D, \lambda, f \in H_\Psi$, and $\max(\lambda, |\mathrm{tr\,cl}(A)|) \le 2^{|\mathrm{tr\,cl}(A)|}$.

In fact, $D \subseteq \mathscr{P}(A)$, so $\mathrm{tr\,cl}(D) \subseteq \mathrm{tr\,cl}(\mathscr{P}(A)) = \mathscr{P}(A) \cup \mathrm{tr\,cl}(A)$, and so $|\mathrm{tr\,cl}(D)| < \Psi$ by Lemma 32.3(iii); so $D \in H_\Psi$. Now f is a one-one function from λ into $\prod A$, so $\lambda \le |\prod A| < \Psi$, and hence $\lambda \in H_\Psi$ and $\max(\lambda, |\mathrm{tr\,cl}(A)|) \le 2^{|\mathrm{tr\,cl}(A)|}$. Finally, $f \subseteq \lambda \times \prod A$, so it follows that $f \in H_\Psi$.

Thus (1) holds. Hence the following equivalence shows the absoluteness of the statement in (xi):

A is an infinite set of regular cardinals and $B = \mathrm{pcf}(A)$

iff

A is infinite, and $\forall \mu \in A(\mu$ is a regular cardinal$) \wedge \forall \lambda \in B \exists D \exists f \varphi(A, D, \lambda, f)$
$\wedge \forall D \forall \lambda \forall f[\varphi(A, D, \lambda, f) \to \lambda \in B]$.

(xii): Assume that $2^{|\mathrm{tr\,cl}(A)|)} < \Psi$. By Lemma 32.3(xi) we have $\mathrm{pcf}(A) \in H_\Psi$. Hence

A is an infinite set of regular cardinals $\wedge f = \langle J_{<\lambda}[A] : \lambda \in \mathrm{pcf}(A) \rangle$

iff

A is infinite and $\forall \kappa \in A(\kappa$ is a regular cardinal and
f is a function and $\exists B[B = \mathrm{pcf}(A) \wedge B = \mathrm{dmn}(f)]$
$\forall \lambda \in \mathrm{dmn}(f) \forall X \subseteq A[A \in f(\lambda)$ iff $\exists C[C = \mathrm{pcf}(X) \wedge C \subseteq \lambda]]$

(xiii): Assume that $2^{2^{|\mathrm{tr\,cl}(A)|}} < \Psi$, and $A, B \in H_\Psi$. Note as above that $\mathrm{pcf}(A) \in H_\Psi$. Note that for any cardinal λ we have $J_{<\lambda}[A] \subseteq \mathscr{P}(A)$ and, with f as in (xi),

$f \subseteq \mathrm{pcf}(A) \times \mathscr{P}(\mathscr{P}(A))$; so $f \in H_\Psi$. Let $\varphi(f, A)$ be the formula of (xii). Thus

A is a set of regular cardinals and $B = \langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$
and $\forall \lambda \in \mathrm{pcf}(A)(B_\lambda$ is a λ-generator)

iff

B is a function and $\exists C[C = \mathrm{pcf}(A) \wedge C = \mathrm{dmn}(B)] \wedge \exists f[\varphi(f, A)\wedge$
$\forall \lambda \in \mathrm{dmn}(B)\forall \mu \in \mathrm{dmn}(B)[\lambda$ is a cardinal and μ is a cardinal and
$\mu = \lambda^+ \rightarrow B_\lambda \in f(\mu) \wedge \forall X \subseteq A[X \in f(\mu)$ iff $X\backslash B_\lambda \in f(\lambda)]]]$ $\square$

Now we turn to the consideration of elementary substructures of H_Ψ. The following lemma gives basic facts used below.

Lemma 32.5. *Suppose that Ψ is an uncountable cardinal, and N is an elementary substructure of H_Ψ (under $\in$ and a well-order of H_Ψ).*

(i) For every ordinal α, $\alpha \in N$ iff $\alpha + 1 \in N$.

(ii) $\omega \subseteq N$.

(iii) If $a \in N$, then $\{a\} \in N$.

(iv) If $a, b \in N$, then $\{a, b\}, (a, b) \in N$.

(v) If $A, B \in N$, then $A \times B \in N$.

(vi) If $A \in N$ then $\bigcup A \in N$.

(vii) If $f \in N$ is a function, then $\mathrm{dmn}(f), \mathrm{rng}(f) \in N$.

(viii) If $f \in N$ is a function and $a \in N \cap \mathrm{dmn}(f)$, then $f(a) \in N$.

(ix) If $X, Y \in N$, $X \subseteq N$, and $|Y| \le |X|$, then $Y \subseteq N$.

(x) If $X \in N$ and $X \neq \emptyset$, then $X \cap N \neq \emptyset$.

(xi) $\mathscr{P}(A) \in N$ if $A \in N$ and $2^{|\mathrm{tr\,cl}(A)|} < \Psi$.

(xii) If ρ is an infinite ordinal, $|\rho|^+ < \Psi$, and $\rho \in N$, then $|\rho| \in N$ and $|\rho|^+ \in N$.

(xiii) If $A \in N$, then $\prod A \in N$ if $2^{|\mathrm{tr\,cl}(A)|} < \Psi$.

(xiv) If $A \in N$, A is a set of regular cardinals, and $A \subseteq H_\Psi$, then $\mathrm{pcf}(A) \in N$ if $2^{|\mathrm{tr\,cl}(A)|} < \Psi$.

(xv) If $A \in N$, A is a set of regular cardinals, then $\langle J_{<\lambda}[A] : \lambda \in \mathrm{pcf}(A)\rangle \in N$ if $2^{2^{|\mathrm{tr\,cl}(A)|}} < \Psi$.

(xvi) If $A \in N$ and A is a set of regular cardinals, then there is a function $\langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle \in N$, where for each $\lambda \in \mathrm{pcf}(A)$, the set B_λ is a λ-generator, if $2^{2^{|\mathrm{tr\,cl}(A)|}} < \Psi$.

Proof. (i): Let α be an ordinal, and suppose that $\alpha \in N$. Then $\alpha \in H_\Psi$, and hence $\alpha \cup \{\alpha\} \in H_\Psi$. By absoluteness, $H_\Psi \models \exists x(x = \alpha \cup \{\alpha\})$, so $N \models \exists x(x = \alpha \cup \{\alpha\})$. Choose $b \in N$ such that $N \models b = \alpha \cup \{\alpha\}$. Then $H_\Psi \models b = \alpha \cup \{\alpha\}$, so by absoluteness, $b = \alpha \cup \{\alpha\}$. This proves that $\alpha \cup \{\alpha\} \in N$.

The method used in proving (i) can be used in the other parts; so it suffices in most other cases just to indicate a formula which can be used.

(ii): An easy induction, using the formulas $\exists x \forall y \in x(y \neq y)$ and $\exists x[a \subseteq x \wedge a \in x \wedge \forall y \in x[y \in a \vee y = a]]$.

(iii): Use the formula $\exists x[\forall y \in x(y = a) \wedge a \in x]$.

(iv): Similar to (ii).

(v): Use the formula

$$\exists C[\forall a \in A \forall b \in B[(a, b) \in C] \wedge \forall x \in C \exists a \in A \exists b \in B[x = (a, b)]].$$

(vi): Use the formula $\exists B[\forall x \in A[x \subseteq B] \wedge \forall y \in B \exists x \in A(y \in x)]$.

(vii): Use the formula $\exists A[\forall x \forall y[(x, y) \in f \rightarrow x \in A] \wedge \forall x \in A \exists y[(x, y) \in f]]$. Note that this formula is absolute for H_Ψ for example $(x, y) \in f \in H_\Psi$ implies that $x, y \in H_\Psi$.

(viii): Use the formula $\exists x[(a, x) \in f]$.

(ix): Let f be a function mapping X onto Y (assuming, as we may, that $Y \neq \emptyset$). Then $f \in H_\Psi$, so by the above method, we get another function $g \in N$ which maps X onto Y. Now (viii) gives the conclusion of (ix).

(x): Use the formula $\exists x \in X[x = x]$.

(xi): $\mathscr{P}(A) \in H_\Psi$ by Lemma 32.3(iii). Hence we can use the formula

$$\exists B[\forall x \in B(x \subseteq A) \wedge \forall x[x \subseteq A \rightarrow x \in B]].$$

(xii): Assume that ρ is an infinite ordinal and $\rho \in N$. Then

$$H_\Psi \models \exists \alpha \leq \rho[(\exists f : \rho \rightarrow \alpha, \text{ a bijection}) \wedge \forall \beta \leq \rho[(\exists g : \rho \rightarrow \beta, \text{ a bijection}) \rightarrow \alpha \leq \beta]].$$

Hence by the standard argument, there are $\alpha, f \in N$ such that

$$H_\Psi \models f : \rho \rightarrow \alpha \text{ is a bijection } \wedge \forall \beta \leq \rho[(\exists g : \rho \rightarrow \beta, \text{ a bijection}) \rightarrow \alpha \leq \beta].$$

Clearly then $\alpha = |\rho|$.

For $|\rho|^+$, use the formula

$$\begin{aligned}\exists \beta \exists \Gamma \Big[& \forall \gamma \in \Gamma \exists f[f \text{ is a bijection from } \rho \text{ onto } \gamma] \\ & \wedge \forall \gamma \forall f[f \text{ is a bijection from } \rho \text{ onto } \gamma \rightarrow \gamma \in \Gamma] \\ & \wedge \beta = \bigcup \Gamma \Big].\end{aligned}$$

(xiii): Note that $\prod A \in H_\Psi$ by Lemma 32.3(ix). Then use the formula

$$\begin{aligned}\exists B \Big[& \forall f \in B(f \text{ is a function } \wedge \operatorname{dmn}(f) = A \wedge \forall a \in A(f(a) \in a)) \\ & \wedge \forall f[f \text{ is a function } \wedge \operatorname{dmn}(f) = A \wedge \forall a \in A(f(a) \in a) \rightarrow f \in B] \Big].\end{aligned}$$

(xiv): $\operatorname{pcf}(A) \in H_\Psi$ by Lemma 32.3(xi), so by Lemma 32.4(xi) we can use the formula $\exists B[B = \operatorname{pcf}(A)]$.

(xv): We have $\mathrm{pcf}(A) \in H_\Psi$ and $\mathscr{P}(\mathscr{P}(H_\Psi))$ by Lemma 32.3(iii),(xi). It follows that $\langle J_{<\lambda}[A] : \lambda \in \mathrm{pcf}(A)\rangle \in H_\Psi$. Hence by Lemma 32.4(xii) we can use the formula $\exists f[f = \langle J_{<\lambda}[A] : \lambda \in \mathrm{pcf}(A)\rangle]$.

(xvi): By Lemma 32.3(iii),(xi) and Lemma 32.4(xiii) we can use the formula

$$\exists B[B : \mathrm{pcf}(A) \to \mathscr{P}(A) \wedge \forall\lambda \in \mathrm{pcf}(A)[B_\lambda \text{ is a } \lambda \text{ generator for } A]]. \qquad \square$$

Definition. Let κ be a regular cardinal. An elementary substructure N of H_Ψ is κ-*presentable* iff there is an increasing and continuous chain $\langle N_\alpha : \alpha < \kappa\rangle$ of elementary substructures of H_Ψ such that:

(1) $|N| = \kappa$ and $\kappa + 1 \subseteq N$.

(2) $N = \bigcup_{\alpha<\kappa} N_\alpha$.

(3) For every $\alpha < \kappa$, the function $\langle N_\beta : \beta \leq \alpha\rangle$ is a member of $N_{\alpha+1}$.

It is obvious how to construct a κ-presentable substructure of H_Ψ.

Lemma 32.6. *If N is a κ-presentable substructure of H_Ψ, with notation as above, and if $\alpha < \kappa$, then $\alpha + \omega \subseteq N_\alpha \in N_{\alpha+1}$.*

Proof. First we show that $\alpha \subseteq N_\alpha$ for all $\alpha < \kappa$, by induction. It is trivial for $\alpha = 0$, and the successor step is immediate from the induction hypothesis and Lemma 32.5(vii). The limit step is clear.

Now it follows that $\alpha + \omega \subseteq N_\alpha$ by an inductive argument using Lemma 32.5(i). Finally, $N_\alpha \in N_{\alpha+1}$ by (3) and Lemma 32.5(viii). $\square$

For any set M, we let $\overline{M}$ be the set of all ordinals α such that $\alpha \in M$ or $M \cap \alpha$ is unbounded in α.

Lemma 32.7. *If N is a κ-presentable substructure of H_Ψ, with notation as above, then*

(i) If $\alpha < \kappa$, then $\overline{N_\alpha} \subseteq N$.

(ii) If $\kappa < \alpha \in \overline{N}\backslash N$, then α is a limit ordinal and $\mathrm{cf}(\alpha) = \kappa$, *and in fact there is a closed unbounded subset E of α such that $E \subseteq N$ and E has order type κ.*

Proof. First we consider (i). Suppose that $\gamma \in \overline{N_\alpha}$. We may assume that $\gamma \notin N_\alpha$.

Case 1. $\gamma = \sup(N_\alpha \cap \mathbf{On})$. Then

$$H_\Psi \models \exists\gamma'[\forall\delta(\delta \in N_\alpha \to \delta \leq \gamma') \wedge \forall\varepsilon[\forall\delta(\delta \in N_\alpha \to \delta \leq \varepsilon) \to \gamma' \leq \varepsilon]];$$

in fact, our given γ is the unique γ' for which this holds. Hence this statement holds in N, as desired.

Case 2. $\exists\theta \in N_\alpha(\gamma < \theta)$. We may assume that θ is minimum with this property. Now for any $\beta \in N_\alpha$ we can let $\rho(\beta)$ be the supremum of all ordinals in N_α which are less than β. So $\rho(\theta) = \gamma$. By absoluteness we get

$$\begin{aligned} H_\Psi \models &\forall\beta \in N_\alpha\exists\rho[\forall\varepsilon \in N_\alpha(\varepsilon < \beta \to \varepsilon < \rho) \\ &\wedge \forall\chi[\forall\varepsilon \in N_\alpha(\varepsilon < \beta \to \varepsilon < \chi) \to \rho \leq \chi]]; \end{aligned}$$

Hence N models this formula too; applying it to θ in place of β, we get $\rho \in N$ such that

$$N \models \forall \varepsilon \in N_\alpha(\varepsilon < \theta \to \varepsilon < \rho) \\ \wedge \forall \chi[\forall \varepsilon \in N_\alpha(\varepsilon < \theta \to \varepsilon < \chi) \to \rho \leq \chi].$$

Thus $\gamma = \rho \in N$, as desired. This proves (i).

For (ii), suppose that $\kappa < \alpha \in \overline{N} \backslash N$. Let $E = \{\sup(\alpha \cap N_\xi) : \xi < \kappa\}$. Note that if $\xi < \kappa$, then by (i), $\sup(\alpha \cap N_\xi) \in N$. So $E \subseteq N$. It is clearly closed in α. It is unbounded, since for any $\beta \in \alpha \cap N$ there is a $\xi < \kappa$ such that $\beta \in N_\xi$, and so $\beta \leq \sup(\alpha \cap N_\xi) \in N$. □

For any set N we define the *characteristic function* of N; it is defined for each regular cardinal μ as follows:

$$\mathrm{Ch}_N(\mu) = \sup(N \cap \mu).$$

Proposition 32.8. *Let κ be a regular cardinal, let N be a κ-presentable substructure of H_Ψ, and let μ be a regular cardinal.*

(i) If $\mu \leq \kappa$, then $\mathrm{Ch}_N(\mu) = \mu \in N$.

(ii) If $\kappa < \mu$, then $\mathrm{Ch}_N(\mu) \notin N$, $\mathrm{Ch}_N(\mu) < \mu$, and $\mathrm{Ch}_N(\mu)$ has cofinality κ.

(iii) For every $\alpha \in \overline{N} \cap \mu$ we have $\alpha \leq \mathrm{Ch}_N(\mu)$.

Proof. (i): True since $\kappa + 1 \subseteq N$.

(ii): Since $|N| = \kappa < \mu$ and μ is regular, we must have $\mathrm{Ch}_N(\mu) \notin N$ and $\mathrm{Ch}_N(\mu) < \mu$. Then $\mathrm{Ch}_N(\mu)$ has cofinality κ by Lemma 32.7.

(iii): clear. □

Theorem 32.9. *Suppose that M and N are elementary substructures of H_Ψ and $\kappa < \mu$ are cardinals, with $\mu < \Psi$.*

(i) If $M \cap \kappa \subseteq N \cap \kappa$ and $\sup(M \cap \nu^+) = \sup(M \cap N \cap \nu^+)$ for every successor cardinal $\nu^+ \leq \mu$ such that $\nu^+ \in M$, then $M \cap \mu \subseteq N \cap \mu$.

(ii) If M and N are both κ-presentable and if $\sup(M \cap \nu^+) = \sup(N \cap \nu^+)$ for every successor cardinal $\nu^+ \leq \mu$ such that $\nu^+ \in M$, then $M \cap \mu = N \cap \mu$.

Proof. (i): Assume the hypothesis. We prove by induction on cardinals δ in the interval $[\kappa, \mu]$ that $M \cap \delta \subseteq N \cap \delta$. This is given for $\delta = \kappa$. If, inductively, δ is a limit cardinal, then the desired conclusion is clear. So assume now that δ is a cardinal, $\kappa \leq \delta < \mu$, and $M \cap \delta \subseteq N \cap \delta$. If $\delta^+ \notin M$, then by Lemma 32.5(xii), $[\delta, \delta^+] \cap M = \emptyset$, so the desired conclusion is immediate from the inductive hypothesis. So, assume that $\delta^+ \in M$. Then the hypothesis of (i) implies that there are ordinals in $[\delta, \delta^+]$ which are in $M \cap N$, and hence by Lemma 32.5(xii) again, $\delta^+ \in N$. Now to show that $M \cap [\delta, \delta^+] \subseteq N \cap [\delta, \delta^+]$, take any ordinal $\gamma \in M \cap [\delta, \delta^+]$. We may assume that $\gamma < \delta^+$. Since $\sup(M \cap \delta^+) = \sup(M \cap N \cap \delta^+)$ by assumption, we can choose $\beta \in M \cap N \cap \delta^+$ such that $\gamma < \beta$. Let f be the $<^*$-smallest bijection from β to δ. So $f \in M \cap N$. Since $\gamma \in M$, we also have $f(\gamma) \in M$ by Lemma 32.5(viii). Now $f(\gamma) < \delta$, so by the inductive assumption that $M \cap \delta \subseteq N \cap \delta$, we have $f(\gamma) \in N$. Since $f \in N$, so is f^{-1}, and $f^{-1}(f(\gamma)) = \gamma \in N$, as desired. This finishes the proof of (i).

(ii): Assume the hypothesis. Now we want to check the hypothesis of (i). By the definition of κ-presentable we have $\kappa = M \cap \kappa = N \cap \kappa$. Now suppose that ν is a cardinal and $\nu^+ \leq \mu$ with $\nu^+ \in M$. We may assume that $\kappa < \nu^+$. Let $\gamma = \mathrm{Ch}_M(\nu^+)$; this is the same as $\mathrm{Ch}_N(\nu^+)$ by the hypothesis of (ii). By Lemma 32.8 we have $\gamma \notin M \cup N$; hence by Lemma 32.7 there are clubs P, Q in γ such that $P \subseteq M$ and $Q \subseteq N$. Hence $\sup(M \cap \nu^+) = \sup(M \cap \nu^+) = \sup(M \cap N \cap \nu^+)$. This verifies the hypothesis of (i) for the pair M, N and also for the pair N, M. So our conclusion follows. $\square$

Minimally obedient sequences

Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and B is a λ-generator for A. A sequence $\langle f_\xi : \xi < \lambda \rangle$ of members of $\prod A$ is called *persistently cofinal* for λ, B provided that $\langle (f_\xi \restriction B) : \xi < \lambda \rangle$ is persistently cofinal in $(\prod B, <_{J_{<\lambda}[B]})$. Recall that this means that for all $h \in \prod B$ there is a $\xi_0 < \lambda$ such that for all ξ, if $\xi_0 \leq \xi < \lambda$, then $h <_{J_{<\lambda}[B]} (f_\xi \restriction B)$.

Lemma 32.10. *Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and B and C are λ-generators for A. A sequence $\langle f_\xi : \xi < \lambda \rangle$ of members of $\prod A$ is persistently cofinal for λ, B iff it is persistently cofinal for λ, C.*

Proof. Suppose that $\langle f_\xi : \xi < \lambda \rangle$ is persistently cofinal for λ, B, and suppose that $h \in \prod C$. Let $k \in \prod B$ be any function such that $h \restriction (B \cap C) = k \restriction (B \cap C)$. Choose $\xi_0 < \lambda$ such that for all $\xi \in [\xi_0, \lambda)$ we have $k <_{J_{<\lambda}[B]} (f_\xi \restriction B)$. Then for any $\xi \in [\xi_0, \lambda)$ we have

$$\begin{aligned}\{a \in C : h(a) \geq f_\xi(a)\} &= \{a \in B \cap C : h(a) \geq f_\xi(a)\} \cup \{a \in C \backslash B : h(a) \geq f_\xi(a)\} \\ &\subseteq \{a \in B : k(a) \geq f_\xi(a)\} \cup (C \backslash B);\end{aligned}$$

Now $(C \backslash B) \in J_{<\lambda}[A]$ by Lemma 31.25(xi), so $h <_{J_{<\lambda}[C]} (f_\xi \restriction C)$. By symmetry the lemma follows. $\square$

Because of this lemma we say that f is persistently cofinal for λ iff it is persistently cofinal for λ, B for some λ-generator B.

Lemma 32.11. *Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and $f \stackrel{\text{def}}{=} \langle f_\xi : \xi < \lambda \rangle$ is universal for λ. Then f is persistently cofinal for λ.*

Proof. Let B be a λ-generator. Then by Lemma 31.25(vii), λ is the largest member of $\mathrm{pcf}(B)$. By Lemma 31.17, $\langle (f_\xi \restriction B) : \xi < \lambda \rangle$ is strictly increasing under $<_{J_{<\lambda}[B]}$, and by Lemma 31.25(v) it is cofinal in $(\prod B, <_{J_{<\lambda}[B]})$. By Proposition 30.11, it is thus persistently cofinal in $(\prod B, <_{J_{<\lambda}[B]})$. $\square$

Lemma 32.12. *Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and $A \in N$, where N is a κ-presentable elementary substructure of H_Ψ, with $|A| < \kappa < \min(A)$ and $2^{|\mathrm{tr\,cl}(A)|} < \Psi$. Suppose that $f = \langle f_\xi : \xi < \lambda \rangle$ is a sequence of functions in $\prod A$.*

Then for every $\xi < \lambda$ there is an $\alpha < \kappa$ such that for any $a \in A$,

$$f_\xi(a) < \mathrm{Ch}_N(a) \quad \text{iff} \quad f_\xi(a) < \mathrm{Ch}_{N_\alpha}(a).$$

Proof.

$$\begin{aligned}
\mathrm{Ch}_N(a) &= \sup(N \cap a)\\
&= \bigcup(N \cap a)\\
&= \bigcup\left(a \cap \bigcup_{\alpha<\kappa} N_\alpha\right)\\
&= \bigcup_{\alpha<\kappa}\bigcup(N_\alpha \cap a)\\
&= \bigcup_{\alpha<\kappa} \mathrm{Ch}_{N_\alpha}(a).
\end{aligned}$$

Hence for every $a \in A$ for which $f_\xi(a) < \mathrm{Ch}_N(a)$, there is an $\alpha_a < \kappa$ such that $f_\xi(a) < Ch_{N_{\alpha_a}}(a)$. Hence the existence of α as indicated follows. $\square$

Lemma 32.13. *Suppose that A is progressive, κ is regular, $\lambda \in \mathrm{pcf}(A)$, and $A, \lambda \in N$, where N is a κ-presentable elementary substructure of H_Ψ, with $|A| < \kappa < \min(A)$ and Ψ is big. Suppose that $f = \langle f_\xi : \xi < \lambda\rangle \in N$ is a sequence of functions in $\prod A$ which is persistently cofinal in λ. Then for every $\xi \geq \mathrm{Ch}_N(\lambda)$ the set*

$$\{a \in A : \mathrm{Ch}_N(a) \leq f_\xi(a)\}$$

is a λ-generator for A.

Proof. Assume the hypothesis, including $\xi \geq \mathrm{Ch}_N(\lambda)$. Let α be as in Lemma 32.12. We are going to apply Lemma 31.25(ix). Since $A, f, \lambda \in N$, we may assume that $A, f, \lambda \in N_0$, by renumbering the elementary chain if necessary. Now $\kappa \subseteq N$, and $|A| < \kappa$, so we easily see that there is a bijection $f \in N$ mapping an ordinal $\alpha < \kappa$ onto A; hence $A \subseteq N$ by Lemma 32.5(viii), and so $A \subseteq N_\beta$ for some $\beta < \kappa$. We may assume that $A \subseteq N_0$. By Lemma 32.5(xvi),(viii), there is a λ-generator B which is in N_0.

Now the sequence f is persistently cofinal in $\prod B/J_{<\lambda}$, and hence

$$\begin{aligned}
H_\Psi &\models \forall h \in \prod B \exists \eta < \lambda \forall \rho \geq \eta[h \restriction B <_{J_{<\lambda}} f_\rho \restriction B]; \quad \text{hence}\\
N &\models \forall h \in \prod B \exists \eta < \lambda \forall \rho \geq \eta[h \restriction B <_{J_{<\lambda}} f_\rho \restriction B];
\end{aligned}$$

Hence for every $h \in N$, if $h \in \prod B$ then there is an $\eta < \lambda$ with $\eta \in N$ such that $N \models \forall \rho \geq \eta[h \restriction B <_{J_{<\lambda}} f_\varphi \restriction B]$; going up, we see that really for every $h \in N \cap \prod A$ there is an $\eta_h \in N \cap \lambda$ such that for all ρ with $\rho \geq \eta_h$ we have $h \restriction B <_{J_{<\lambda}} f_\rho \restriction B$. Since ξ, as given in the statement of the Lemma, is $\geq$ each member of $N \cap \lambda$, hence $\geq \eta_h$ for each $h \in N \cap \prod A$, we see that

$$h \restriction B <_{J_{<\lambda}} f_\xi \restriction B \text{ for every } h \in N \cap \prod A. \tag{1}$$

Now we can apply (1) to $h = \mathrm{Ch}_{N_\alpha}$, since this function is clearly in N. So $\mathrm{Ch}_{N\alpha} \restriction B <_{J_{<\lambda}[B]} f_\xi \restriction B$. Hence by the choice of α (see Lemma 32.12)

$$\mathrm{Ch}_N \restriction B \leq_{J_{<\lambda}[B]} f_\xi \restriction B. \tag{2}$$

Note that (2) says that $B\backslash\{a \in A : \mathrm{Ch}_N(a) \leq f_\xi(a)\} \in J_{<\lambda}[A]$.

Now $\lambda \notin \mathrm{pcf}(A\backslash B)$ by Lemma 31.25(ii), and hence $J_{<\lambda}[A\backslash B] = J_{\leq\lambda}[A\backslash B]$. So by Theorem 31.8 we see that $\prod(A\backslash B)/J_{<\lambda}[A\backslash B]$ is λ^+-directed, so $\langle f_\xi \restriction (A\backslash B) : \xi < \lambda\rangle$ has an upper bound $h \in \prod(A\backslash B)$. We may assume that $h \in N$, by the usual argument. Hence

$$f_\xi \restriction (A\backslash B) <_{J_{<\lambda}[A\backslash B]} h < \mathrm{Ch}_N \restriction (A\backslash B);$$

hence $\{a \in A\backslash B : \mathrm{Ch}_N(a) \leq f_\xi(a)\} \in J_{<\lambda}[A]$, and together with (2) and using Lemma 31.25(ix) this finishes the proof. $\square$

Now suppose that A is progressive, δ is a limit ordinal, $f = \langle f_\xi : \xi < \delta\rangle$ is a sequence of members of $\prod A$, $|A|^+ \leq \mathrm{cf}(\delta) < \min(A)$, and E is a club of δ of order type $\mathrm{cf}(\delta)$. Then we define

$$h_E = \sup\{f_\xi : \xi \in E\}.$$

We call h_E the *supremum along E of f*. Thus $h_E \in \prod A$, since $\mathrm{cf}(\delta) < \min(A)$. Note that if $E_1 \subseteq E_2$ then $h_{E_1} \leq h_{E_2}$.

Lemma 32.14. *Let A, δ, f be as above. Then there is a unique function g in $\prod A$ such that the following two conditions hold.*

(i) There is a club C of δ of order type $\mathrm{cf}(\delta)$ *such that $g = h_C$.*

(ii) If E is any club of C of order type $\mathrm{cf}(\delta)$*, then $g \leq h_E$.*

Proof. Clearly such a function g is unique if it exists.

Now suppose that there is no such function g. Then for every club C of δ of order type $\mathrm{cf}(\delta)$ there is a club D of order type $\mathrm{cf}(\delta)$ such that $h_C \not\leq h_D$, hence $h_C \not\leq h_{C\cap D}$. Hence there is a decreasing sequence $\langle E_\alpha : \alpha < |A|^+\rangle$ of clubs of δ such that for every $\alpha < |A|^+$ we have $h_{E_\alpha} \not\leq h_{E_{\alpha+1}}$. Now note that

$$|A|^+ = \bigcup_{a\in A}\{\alpha < |A|^+ : h_{E_\alpha}(a) > h_{E_{\alpha+1}}(a)\}.$$

Hence there is an $a \in A$ such that $M \overset{\mathrm{def}}{=} \{\alpha < |A|^+ : h_{E_\alpha}(a) > h_{E_{\alpha+1}}(a)\}$ has size $|A|^+$. Now $h_{E_\alpha}(a) \geq h_{E_\beta}(a)$ whenever $\alpha < \beta < |A|^+$, so this gives an infinite decreasing sequence of ordinals, contradiction. $\square$

The function g of this lemma is called the *minimal club-obedient bound* of f.

Corollary 32.15. *Suppose that A is progressive, δ is a limit ordinal, $f = \langle f_\xi : \xi < \delta\rangle$ is a sequence of members of $\prod A$, $|A|^+ \leq \mathrm{cf}(\delta) < \min(A)$, J is an ideal on A, and f is $<_J$-increasing. Let g be the minimal club-obedient bound of f. Then g is a $\leq_J$-bound for f.* $\square$

Now suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and κ is a regular cardinal such that $|A| < \kappa < \min(A)$. We say that $f = \langle f_\alpha : \alpha < \lambda\rangle$ is *κ-minimally obedient for λ* iff f is a universal sequence for λ and for every $\delta < \lambda$ of cofinality κ, f_δ is the minimal club-obedient bound of f.

A sequence f is *minimally obedient for* λ iff $|A|^+ < \min(A)$ and f is minimally obedient for every regular κ such that $|A| < \kappa < \min(A)$.

Lemma 32.16. *Suppose that* $|A|^+ < \min(A)$ *and* $\lambda \in \mathrm{pcf}(A)$. *Then there is a minimally obedient sequence for* λ.

Proof. By Theorem 31.18, let $\langle f^0_\xi : \xi < \lambda\rangle$ be a universal sequence for λ. Now by induction we define functions f_ξ for $\xi < \lambda$. Let $f_0 = f^0_0$, and choose $f_{\xi+1}$ so that $\max(f_\xi, f^0_\xi) < f_{\xi+1}$.

For limit $\delta < \lambda$ such that $|A| < \mathrm{cf}(\delta) < \min(A)$, let f_δ be the minimally club-obedient bound of $\langle f_\xi : \xi < \delta\rangle$.

For other limit $\delta < \lambda$, use the λ-directedness (Theorem 31.8) to get f_δ as a $<_{J_{<\lambda}}$-bound of $\langle f_\xi : \xi < \delta\rangle$.

Thus we have assured the minimally obedient property, and it is clear that $\langle f_\xi : \xi < \lambda\rangle$ is universal. $\square$

Lemma 32.17. *Suppose that A is progressive, and κ is a regular cardinal such that $|A| < \kappa < \min(A)$. Also assume the following:*

(i) $\lambda \in \mathrm{pcf}(A)$.

(ii) $f = \langle f_\xi : \xi < \lambda\rangle$ *is a κ-minimally obedient sequence for* λ.

(iii) N is a κ-presentable elementary substructure of H_Ψ, with Ψ large, such that $\lambda, f, A \in N$.

Then the following conditions hold:

(iv) For every $\gamma \in \overline{N} \cap \lambda \backslash N$ *we have:*

(a) $\mathrm{cf}(\gamma) = \kappa$.

(b) There is a club C of γ of order type κ such that $f_\gamma = \sup\{f_\xi : \xi \in C\}$ *and* $C \subseteq N$.

(c) $f_\gamma(a) \in \overline{N} \cap a$ *for every* $a \in A$.

(v) If $\gamma = \mathrm{Ch}_N(\lambda)$, *then:*

(a) $\gamma \in \overline{N} \cap \lambda \backslash N$; *hence we let C be as in (iv)(b), with* $f_\gamma = \sup\{f_\xi : \xi \in C\}$.

(b) $f_\xi \in N$ *for each* $\xi \in C$.

(c) $f_\gamma \le (\mathrm{Ch}_N \restriction A)$.

(vi) $\gamma = \mathrm{Ch}_N(\lambda)$ *and C is as in (iv)(b), with* $f_\gamma = \sup\{f_\xi : \xi \in C\}$, *and B is a λ generator, then for every* $h \in N \cap \prod A$ *there is a* $\xi \in C$ *such that* $(h \restriction B) <_{J_{<\lambda}} (f_\xi \restriction B)$.

Proof. Assume (i)–(iii). Note that $A \subseteq N$, by Lemma 32.5(ix).

For (iv), suppose also that $\gamma \in \overline{N} \cap \lambda \backslash N$. Then by Lemma 32.7 we have $\mathrm{cf}(\gamma) = \kappa$, and there is a club E in γ of order type κ such that $E \subseteq N$. By (ii), we have $f_\gamma = f_C$ for some club C of γ of order type κ. By the minimally obedient property we have $f_C = f_{C\cap E}$, and thus we may assume that $C \subseteq E$. For any $\xi \in C$ and $a \in A$ we have $f_\xi(a) \in N$ by Lemma 32.5(viii). So (iv) holds.

For (v), suppose that $\gamma = \mathrm{Ch}_N(\lambda)$. Then $\gamma \in \overline{N} \cap \lambda \backslash N$ because $|N| = \kappa < \min(A) \le \lambda$. For each $\xi \in C$ we have $f_\xi \in N$ by Lemma 32.5(viii). For (c), if $a \in A$, then $f_\gamma(a) = \sup_{\xi\in C} f_\xi(a) \le \mathrm{Ch}_N(a)$, since $f_\xi(a) \in N \cap a$ for all $\xi \in C$.

Next, assume the hypotheses of (vi). By Lemma 32.11, f is persistently cofinal in λ, so by Lemma 32.13, B' is a λ-generator. By Lemma 31.25(v) there is a $\xi \in C$ such that $h \restriction B' <_{J_{<\lambda}} f_\xi \restriction B'$. Now $B =_{J_{<\lambda}[A]} B'$ by Lemma 31.25(xi), so

$$\{a \in B : h(a) \geq f_\xi(b)\} \subseteq (B \backslash B') \cup \{a \in B' : h(a) \geq f_\xi(b)\} \in J_{<\lambda}[A]. \qquad \square$$

We now define some abbreviations.

$H_1(A, \kappa, N, \Psi)$ abbreviates

A is a progressive set of regular cardinals, κ is a regular cardinal such that $|A| < \kappa <$ $\min(A)$, and N is a κ-presentable elementary substructure of H_Ψ, with Ψ big and $A \in N$.

$H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ abbreviates

$H_1(A, \kappa, N, \Psi)$, $\lambda \in \mathrm{pcf}(A)$, $f = \langle f_\xi : \xi < \lambda\rangle$ is a sequence of members of $\prod A$, $f \in N$, and $\gamma = \mathrm{Ch}_N(\lambda)$.

$P_1(A, \kappa, N, \Psi, \lambda, f, \gamma)$ abbreviates

$H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ and $\{a \in A : \mathrm{Ch}_N(a) \leq f_\gamma(a)\}$ is a λ-generator.

$P_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ abbreviates

$H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ and the following hold:
(i) $f_\gamma \leq (\mathrm{Ch}_N \restriction A)$.
(ii) For every $h \in N \cap \prod A$ there is a $d \in N \cap \prod A$ such that for any λ-generator B,

$$(h \restriction B) <_{J_{<\lambda}} (d \restriction B) \quad \text{and} \quad d \leq f_\gamma.$$

Thus $H_1(A, \kappa, N, \Psi)$ is part of the hypothesis of Lemma 32.17, and $H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ is a part of the hypotheses of Lemma 32.17(v).

Lemma 32.18. *If $H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ holds and f is persistently cofinal for λ, then $P_1(A, \kappa, N, \Psi, \lambda, f, \gamma)$ holds.*

Proof. This follows immediately from Lemma 32.13. $\square$

Lemma 32.19. *If $H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ holds and f is κ-minimally obedient for λ, then both $P_1(A, \kappa, N, \Psi, \lambda, f, \gamma)$ and $P_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ hold.*

Proof. Since f is κ-minimally obedient for λ, it is a universal sequence for λ, by definition. Hence by Lemma 32.11 f is persistently cofinal for λ, and so property P_1 follows from Lemma 32.18.

For P_2, note that $\lambda, A \in N$ since $f \in N$, by Lemma 32.5(vii),(ix). Hence the hypotheses of Lemma 32.17(v) hold. So (i) in P_2 holds by Lemma 32.17(v)(c). For condition (ii), suppose that $h \in N \cap \prod A$. Take B and C as in Lemma 32.17(vi), and choose $\xi \in C$ such that $h \restriction B <_{J_{<\lambda}} f_\xi \restriction B$. Let $d = f_\xi$. Clearly this proves condition (ii). $\square$

The following obvious extension of Lemma 32.19 will be useful below.

Lemma 32.20. *Assume $H_1(A,\kappa,N,\Psi)$, and also assume that $\gamma = \mathrm{Ch}_N(\lambda)$ and*

(i) $f \stackrel{\text{def}}{=} \langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a sequence of sequences $\langle f^\lambda_\xi : \xi < \lambda\rangle$ each of which is a κ-minimally obedient for λ.

Then for each $\lambda \in N \cap \mathrm{pcf}(A)$, $P_1(A,\kappa,N,\Psi,\lambda,f^\lambda,\gamma)$ and $P_2(A,\kappa,N,\Psi,\lambda,f^\lambda,\gamma)$ hold. □

Lemma 32.21. *Suppose that $P_1(A,\kappa,N,\Psi,\lambda,f,\gamma)$ and $P_2(A,\kappa,N,\Psi,\lambda,f,\gamma)$ hold. Then*

(i) $\{a \in A : \mathrm{Ch}_N(a) = f_\gamma(a)\}$ is a λ-generator.

(ii) If $\lambda = \max(\mathrm{pcf}(A))$, then

$$< (f_\gamma, \mathrm{Ch}_N \restriction A) = \{a \in A : f_\gamma(a) < \mathrm{Ch}_N(a)\} \in J_{<\lambda}[A].$$

Proof. By (i) of $P_2(A,\kappa,N,\Psi,\lambda,f,\gamma)$ we have $f_\gamma \leq (\mathrm{Ch}_N \restriction A)$, so (i) holds by $P_1(A,\kappa,N,\Psi,\lambda,f,\gamma)$. (ii) follows from $P_1(A,\kappa,N,\Psi,\lambda,f,\gamma)$ and Lemma 31.25(xii). □

Lemma 32.22. *Assume that $P_1(A,\kappa,N,\Psi,\lambda,f,\gamma)$ and $P_2(A,\kappa,N,\Psi,\lambda,f,\gamma)$ hold. Let*

$$b = \{a \in A : \mathrm{Ch}_N(a) = f_\gamma(a)\}.$$

Then

(i) b is a λ-generator.

(ii) There is a set $b' \subseteq b$ such that:

(a) $b' \in N$;

(b) $b \backslash b' \in J_{<\lambda}[A]$;

(c) b' is a λ-generator.

Proof. (i) holds by Lemma 32.21(i). For (ii), by Lemma 32.12 choose $\alpha < \kappa$ such that, for every $a \in A$,

$$f_\gamma(a) < \mathrm{Ch}_N(a) \quad \text{iff} \quad f_\gamma(a) < \mathrm{Ch}_{N_\alpha}(a). \tag{1}$$

Now by (i) of $P_2(A,\kappa,N,\Psi,\lambda,f,\gamma)$ we have $f_\gamma \leq (\mathrm{Ch}_N \restriction A)$. Hence by (1) we see that for every $a \in A$,

$$a \in b \quad \text{iff} \quad \mathrm{Ch}_{N_\alpha}(a) \leq f_\gamma(a). \tag{2}$$

Now by (ii) of $P_2(A,\kappa,N,\Psi,\lambda,f,\gamma)$ applied to $h = \mathrm{Ch}_{N_\alpha} \restriction A$, there is a $d \in N \cap \prod A$ such that the following conditions hold:

(3) $(\mathrm{Ch}_{N_\alpha} \restriction b) <_{J_{<\lambda}} (d \restriction b)$.

(4) $d \leq f_\gamma$.

Now we define

$$b' = \{a \in A : \mathrm{Ch}_{N_\alpha}(a) \leq d(a)\}.$$

Clearly $b' \in N$. Also, by (3),

$$b \backslash b' = \{a \in b : d(a) < \mathrm{Ch}_{N_\alpha}(a)\} \in J_{<\lambda},$$

and so (ii)(b) holds. Thus $b \subseteq_{J_{<\lambda}} b'$. If $a \in b'$, then $\mathrm{Ch}_{N_\alpha}(a) \leq d(a) \leq f_\gamma(a)$ by (4), so $a \in b$ by (2). Thus $b' \subseteq b$. Now (ii)(c) holds by Lemma 31.25(ix). $\square$

Lemma 32.23. *Assume $H_1(A, \kappa, N, \Psi)$ and $A \in N$. Suppose that $\langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle \in N$ is an array of sequences $\langle f^\lambda_\xi : \xi < \lambda\rangle$ with each $f^\lambda_\xi \in \prod A$. Also assume that for every $\lambda \in N \cap \mathrm{pcf}(A)$, both $P_1(A, \kappa, N, \Psi, \lambda, f^\lambda, \gamma(\lambda))$ and $P_2(A, \kappa, N, \Psi, \lambda, f^\lambda, \gamma(\lambda))$ hold.*

Then there exist cardinals $\lambda_0 > \lambda_1 > \cdots > \lambda_n$ in $\mathrm{pcf}(A) \cap N$ such that

$$(\mathrm{Ch}_N \restriction A) = \sup\{f^{\lambda_0}_{\gamma(\lambda_0)}, \ldots, f^{\lambda_n}_{\gamma(\lambda_n)}\}.$$

Proof. We will define by induction a descending sequence of cardinals $\lambda_i \in \mathrm{pcf}(A) \cap N$ and sets $A_i \in \mathscr{P}(A) \cap N$ (strictly decreasing under inclusion as i grows) such that if $A_i \neq \emptyset$ then $\lambda_i = \max(\mathrm{pcf}(A_i))$ and

$$(1) \qquad (\mathrm{Ch}_N \restriction (A \backslash A_{i+1})) = \sup\{(f^{\lambda_0}_{\gamma(\lambda_0)} \restriction (A \backslash A_{i+1})), \ldots, (f^{\lambda_i}_{\gamma(\lambda_i)} \restriction (A \backslash A_{i+1}))\}.$$

Since the cardinals are decreasing, there is a first i such that $A_{i+1} = \emptyset$, and then the lemma is proved. To start, $A_0 = A$ and $\lambda_0 = \max(\mathrm{pcf}(A))$. Clearly $\lambda_0 \in N$. Now suppose that λ_i and A_i are defined, with $A_i \neq 0$. By Lemma 32.22(i) and Lemma 31.25(x), the set

$$\{a \in A \cap (\lambda_i + 1) : \mathrm{Ch}_N(a) = f^{\lambda_i}_{\gamma(\lambda_i)}(a)\}$$

is a λ_i-generator. Hence by Lemma 32.22(ii) we get another λ_i-generator b'_{λ_i} such that

(2) $b'_{\lambda_i} \in N$.

(3) $b'_{\lambda_i} \subseteq \{a \in A \cap (\lambda_i + 1) : \mathrm{Ch}_N(a) = f^{\lambda_i}_{\gamma(\lambda_i)}(a)\}$.

Note that $b'_{\lambda_i} \neq \emptyset$. Let $A_{i+1} = A_i \backslash b'_{\lambda_i}$. Thus $A_{i+1} \in N$. Furthermore,

(4) $A \backslash A_{i+1} = (A \backslash A_i) \cup b'_{\lambda_1}$.

Now by Lemma 9.25(ii) and $\lambda_i = \max(\mathrm{pcf}(A_i))$ we have $\lambda_i \notin \mathrm{pcf}(A_{i+1})$. If $A_{i+1} \neq \emptyset$, we let $\lambda_{i+1} = \max(\mathrm{pcf}(A_{i+1}))$. Now by (i) of $P_2(A, \kappa, N, \Psi, \lambda, f^{\lambda_j}, \gamma(\lambda_j))$ we have

(5) $f^{\lambda_j}_{\gamma(\lambda_j)} \leq (\mathrm{Ch}_N \restriction A)$ for all $j \leq i$.

Now suppose that $a \in A \backslash A_{i+1}$. If $a \in A_i$, then by (4), $a \in b'_{\lambda_1}$, and so by (3), $\mathrm{Ch}_N(a) = f^{\lambda_i}_{\gamma(\lambda_1)}(a)$, and (1) holds for a. If $a \notin A_i$, then $A \neq A_i$, so $i \neq 0$. Hence by the inductive hypothesis for (1),

$$\mathrm{Ch}_N(a) = \sup\{f^{\lambda_0}_{\gamma(\lambda_0)}(a), \ldots, f^{\lambda_{i-1}}_{\gamma(\lambda_{i-1})}(a)\},$$

and (1) for a follows by (5). $\square$

The cofinality of $([\mu]^\kappa, \subseteq)$

First we give some simple properties of the sets $[\mu]^\kappa$, not involving pcf theory.

Proposition 32.24. *If $\kappa \le \mu$ are infinite cardinals, then*

$$(*) \qquad |[\mu]^\kappa| = \mathrm{cf}([\mu]^\kappa, \subseteq) \cdot 2^\kappa.$$

Proof. Let $\lambda = \mathrm{cf}([\mu]^\kappa, \subseteq)$, and let $\langle Y_i : i < \lambda \rangle$ be an enumeration of a cofinal subset of $\mathrm{cf}([\mu]^\kappa, \subseteq)$. For each $i < \lambda$ let f_i be a bijection from Y_i to κ. Now the inequality $\ge$ in $(*)$ is clear. For the other direction, we define an injection g of $[\mu]^\kappa$ into $\lambda \times \mathscr{P}(\kappa)$, as follows. Given $E \in [\mu]^\kappa$, let $i < \lambda$ be minimum such that $E \subseteq Y_i$, and define $g(E) = (i, f_i[E])$. Clearly g is one-one. $\square$

Proposition 32.25. *(i) If $\kappa_1 < \kappa_2 \le \mu$, then*

$$\mathrm{cf}([\mu]^{\kappa_1}, \subseteq) \le \mathrm{cf}([\mu]^{\kappa_2}, \subseteq) \cdot \mathrm{cf}([\kappa_2]^{\kappa_1}, \subseteq).$$

(ii) $\mathrm{cf}([\kappa^+]^\kappa, \subseteq) = \kappa^+$.
(iii) If $\kappa^+ \le \mu$, then $\mathrm{cf}([\mu]^\kappa, \subseteq) \le \mathrm{cf}([\mu]^{\kappa^+}, \subseteq) \cdot \kappa^+$.
(iv) If $\kappa \le \mu_1 < \mu_2$, then $\mathrm{cf}([\mu_1]^\kappa, \subseteq) \le \mathrm{cf}([\mu_2]^\kappa, \subseteq)$.
(v) If $\kappa \le \mu$, then $\mathrm{cf}([\mu^+]^\kappa, \subseteq) \le \mathrm{cf}([\mu]^\kappa, \subseteq) \cdot \mu^+$.
(vi) $\mathrm{cf}([\aleph_0]^{\aleph_0}, \subseteq) = 1$, *while for $m \in \omega \backslash 1$,* $\mathrm{cf}([\aleph_m]^{\aleph_0}) = \aleph_m$.
(vii) $\mathrm{cf}([\mu]^{\le\kappa}, \subseteq) = \mathrm{cf}([\mu]^\kappa, \subseteq)$.

Proof. (i): Let $M \subseteq [\mu]^{\kappa_2}$ be cofinal in $([\mu]^{\kappa_2}, \subseteq)$ of size $\mathrm{cf}([\mu]^{\kappa_2}, \subseteq)$, and let $N \subseteq ([\kappa_2]^{\kappa_1}, \subseteq)$ be cofinal in $([\kappa_2]^{\kappa_1}, \subseteq)$ of size $\mathrm{cf}([\kappa_2]^{\kappa_1}, \subseteq)$. For each $X \in M$ let $f_X : \kappa_2 \to X$ be a bijection. It suffices now to show that $\{f_X[Y] : X \in M, Y \in N\}$ is cofinal in $([\mu]^{\kappa_1}, \subseteq)$. Suppose that $W \in [\mu]^{\kappa_1}$. Choose $X \in M$ such that $W \subseteq X$. Then $f_X^{-1}[W] \in [\kappa_2]^{\kappa_1}$, so there is a $Y \in N$ such that $f_X^{-1}[W] \subseteq Y$. Then $W \subseteq f_X[Y]$, as desired.

(ii): The set $\{\gamma < \kappa^+ : |\gamma \backslash \kappa| = \kappa\}$ is clearly cofinal in $([\kappa^+]^\kappa$. If M is a nonempty subset of $[\kappa^+]^\kappa$ of size less than κ^+, then $|\bigcup M| = \kappa$, and $(\bigcup M) + 1$ is a member of $[\kappa^+]^\kappa$ not covered by any member of M. So (ii) holds.

(iii): Immediate from (i) and (ii).

(iv): Let $M \subseteq [\mu_2]^\kappa$ be cofinal of size $\mathrm{cf}([\mu_2]^\kappa, \subseteq)$. Let $N = \{X \cap \mu_1 : X \in M\} \backslash [\mu_1]^{<\kappa}$. It suffices to show that N is cofinal in $\mathrm{cf}([\mu_1]^\kappa, \subseteq)$. Suppose that $X \in [\mu_1]^\kappa$. Then also $X \in [\mu_2]^\kappa$, so we can choose $Y \in M$ such that $X \subseteq Y$. Clearly $X \subseteq Y \cap \mu_1 \in N$, as desired.

(v): For each $\gamma \in [\mu, \mu^+)$ let f_γ be a bijection from γ to μ. Let $E \subseteq [\mu]^\kappa$ be cofinal in $([\mu]^\kappa, \subseteq)$ and of size $\mathrm{cf}([\mu]^\kappa, \subseteq)$. It suffices to show that $\{f_\gamma^{-1}[X] : \gamma \in [\mu, \mu^+), X \in E\}$ is cofinal in $([\mu^+]^\kappa, \subseteq)$. So, take any $Y \in [\mu^+]^\kappa$. Choose $\gamma \in [\mu, \mu^+)$ such that $Y \subseteq \gamma$. Then $f_\gamma[Y] \in [\mu]^\kappa$, so we can choose $X \in E$ such that $f[Y] \subseteq X$. Then $Y \subseteq f_\gamma^{-1}[X]$, as desired.

(vi): Clearly $\mathrm{cf}([\aleph_0]^{\aleph_0}, \subseteq) = 1$. By induction it is clear from (v) that $\mathrm{cf}([\aleph_m]^{\aleph_0}) \le \aleph_m$. For $m > 0$ equality must hold, since if $X \subseteq [\aleph_m]^{\aleph_0}$ and $|X| < \aleph_m$, then $\bigcup X < \aleph_m$, and no denumerable subset of $\aleph_m \backslash \bigcup X$ is contained in a member of X.

(vii): Clear. $\square$

The following elementary lemmas will also be needed.

Lemma 32.26. *If $\alpha < \beta$ are limit ordinals, then*

$$|[\alpha, \beta]| = |\{\gamma : \alpha < \gamma < \beta,\ \gamma \text{ a successor ordinal}\}|.$$

Proof. For every $\delta \in [\alpha, \beta)$ let $f(\delta) = \delta + 1$. Then f is a one-one function from $[\alpha, \beta)$ onto $\{\gamma : \alpha < \gamma < \beta,\ \gamma \text{ a successor ordinal}\}$. □

Lemma 32.27. *If $\alpha < \theta \le \beta$ with θ limit, then*

$$|[\alpha, \beta]| = |\{\gamma : \alpha \le \gamma \le \beta,\ \gamma \text{ a successor ordinal}\}|.$$

Proof. Write $\beta = \delta + m$ with δ limit and $m \in \omega$. Then

$$[\alpha, \beta] = [\alpha, \alpha + \omega) \cup [\alpha + \omega, \delta] \cup (\delta, \beta],$$

and the desired conclusion follows easily from Lemma 32.26. □

Theorem 32.28. *Suppose that μ is singular and $\kappa < \mu$ is an uncountable regular cardinal such that $A \stackrel{\text{def}}{=} (\kappa, \mu)_{\text{reg}}$ has size $< \kappa$. Then*

$$\text{cf}([\mu]^\kappa, \subseteq) = \max(\text{pcf}(A)).$$

Proof. Note by the progressiveness of A that every limit cardinal in the interval (κ, μ) is singular, and hence every member of A is a successor cardinal.

First we prove $\ge$. Suppose to the contrary that $\text{cf}([\mu]^\kappa, \subseteq) < \max(\text{pcf}(A))$. For brevity write $\max(\text{pcf}(A)) = \lambda$. let $\{X_i : i \in I\} \subseteq [\mu]^\kappa$ be cofinal and of cardinality less than λ. Pick a universal sequence $\langle f_\xi : \xi < \lambda\rangle$ for λ by Theorem 31.18. For every $\xi < \lambda$, $\text{rng}(f_\xi)$ is a subset of μ of size $\le |A| \le \kappa$, and hence $\text{rng}(f_\xi)$ is covered by some X_i. Thus $\lambda = \bigcup_{i \in I}\{\xi < \lambda : \text{rng}(f_\xi) \subseteq X_i\}$, so by $|I| < \lambda$ and the regularity of λ we get an $i \in I$ such that $|\{\xi < \lambda : \text{rng}(f_\xi) \subseteq X_i\}| = \lambda$. Now define for any $a \in A$,

$$h(a) = \sup(a \cap X_i).$$

Since $\kappa < a$ for each $a \in A$, we have $h \in \prod A$. Now the sequence $\langle f_\xi : \xi < \lambda\rangle$ is cofinal in $\prod A$ under $<_{J_{<\lambda}}$ by Lemma 31.25(v),(iv). So there is a $\xi < \lambda$ such that $h <_{J_{<\lambda}} f_\xi$. Thus there is an $a \in A$ such that $h(a) < f_\xi(a) \in X_i$, contradicting the definition of h.

Second we prove $\le$, by exhibiting a cofinal subset of $[\mu]^\kappa$ of size at most $\max(\text{pcf}(A))$. Take N and Ψ so that $H_1(A, \kappa, N, \Psi)$. Let $\mathscr{M}$ be the set of all κ-presented elementary substructures M of H_Ψ such that $A \subseteq M$, and let

$$F = \{M \cap \mu : M \in \mathscr{M}\} \backslash [\mu]^{<\kappa}.$$

Since $|M| = \kappa$, we have $|M \cap \mu| \leq \kappa$, and so $\forall M \in F(|M \cap \mu| = \kappa)$.

(1) F is cofinal in $[\mu]^\kappa$.

In fact, for any $X \in [\mu]^\kappa$ we can find $M \in \mathscr{M}$ such that $X \subseteq M$, and (1) follows.

By (1) it suffices to prove that $|F| \leq \max(\mathrm{pcf}(A))$.

Claim. If $M, N \in \mathscr{M}$ are such that $\mathrm{Ch}_M \restriction A = \mathrm{Ch}_N \restriction A$, then $M \cap \mu = N \cap \mu$.

For, if ν^+ is a successor cardinal $\leq \mu$, then $\sup(M \cap \nu^+) = \mathrm{Ch}_M(\nu^+) = \mathrm{Ch}_N(\nu^+) = \sup(N \cap \nu^+)$. So the claim holds by Theorem 32.9.

Now for each $M \in \mathscr{M}$, let $g(M)$ be the sequence $\langle(\lambda_0, \gamma_0), \ldots, (\lambda_n, \gamma_n)\rangle$ given by Lemma 32.23. Clearly the range of g has size $\leq \max(\mathrm{pcf}(A))$. Now for each $X \in F$, choose $M_X \in \mathscr{M}$ such that $X = M_X \cap \mu$. Then for $X, Y \in F$ and $X \neq Y$ we have $M_X \cap \mu \neq M_Y \cap \mu$, hence by the claim $\mathrm{Ch}_{M_X} \restriction A \neq \mathrm{Ch}_{M_Y} \restriction A$, and hence by Lemma 32.23, $g(M_X) \neq g(M_Y)$. This proves that $|F| \leq \max(\mathrm{pcf}(A))$. □

Corollary 32.29. *Let $A = \{\aleph_m : 0 < m < \omega\}$. Then for any $m \in \omega$ we have $\mathrm{cf}([\aleph_\omega]^{\aleph_m}) = \max(\mathrm{pcf}(A))$.* □

Elevations and transitive generators

We start with some simple general notions about cardinals. If B is a set of cardinals, then a *walk* in B is a sequence $\lambda_0 > \lambda_1 > \cdots > \lambda_n$ of members of B. Such a walk is necessarily finite. Given cardinals $\lambda_0 > \lambda$ in B, a *walk from λ_0 to λ* is a walk as above with $\lambda_n = \lambda$. We denote by $F_{\lambda_0,\lambda}(B)$ the set of all walks from λ_0 to λ.

Now suppose that A is progressive and $\lambda_0 \in \mathrm{pcf}(A)$. A *special walk from λ_0 to λ_n in* $\mathrm{pcf}(A)$ is a walk $\lambda_0 > \cdots > \lambda_n$ in $\mathrm{pcf}(A)$ such that $\lambda_i \in A$ for all $i > 0$. We denote by $F'_{\lambda_0,\lambda}(A)$ the collection of all special walks from λ_0 to λ in $\mathrm{pcf}(A)$.

Next, suppose in addition that $f \overset{\mathrm{def}}{=} \langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a sequence of sequences, where each f^λ is a sequence $\langle f^\lambda_\xi : \xi < \lambda\rangle$ of members of $\prod A$. If $\lambda_0 > \cdots > \lambda_n$ is a special walk in $\mathrm{pcf}(A)$, and $\gamma_0 \in \lambda_0$, then we define an associated sequence of ordinals by setting

$$\gamma_{i+1} = f^{\lambda_i}_{\gamma_i}(\lambda_{i+1})$$

for all $i < n$. Note that $\gamma_i < \lambda_i$ for all $i = 0, \ldots, n$. Then we define

$$\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) = \gamma_n.$$

Now we define the *elevation* of the sequence f, denoted by $f^e \overset{\mathrm{def}}{=} \langle f^{\lambda,e} : \lambda \in \mathrm{pcf}(A)\rangle$, by setting, for any $\lambda_0 \in \mathrm{pcf}(A)$, any $\gamma_0 \in \lambda_0$, and any $\lambda \in A$,

$$f^{\lambda_0,e}_{\gamma_0}(\lambda) = \begin{cases} f^{\lambda_0}_{\gamma_0}(\lambda) & \text{if } \lambda_0 \leq \lambda, \\ \max(\{\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) : (\lambda_0, \ldots, \lambda_n) \in F'_{\lambda_0,\lambda}\}) & \text{if } \lambda < \lambda_0, \\ & \quad \text{and this maximum exists,} \\ f^{\lambda_0}_{\gamma_0}(\lambda) & \text{if } \lambda < \lambda_0\text{, otherwise.} \end{cases}$$

Note here that the superscript ^{e} is only notational, standing for "elevated".

Lemma 32.30. *Assume the above notation. Then $f^{\lambda_0}_{\gamma_0} \le f^{\lambda_0,e}_{\gamma_0}$ for all $\lambda_0 \in \mathrm{pcf}(A)$ and all $\gamma_0 \in \lambda_0$.*

Proof. Take any $\gamma_0 \in \lambda_0$ and any $\lambda \in A$. If $\lambda_0 \le \lambda$, then $f^{\lambda_0,e}_{\gamma_0}(\lambda) = f^{\lambda_0}_{\gamma_0}(\lambda)$. Suppose that $\lambda < \lambda_0$. If the above maximum does not exist, then again $f^{\lambda_0,e}_{\gamma_0}(\lambda) = f^{\lambda_0}_{\gamma_0}(\lambda)$. Suppose the maximum exists. Now $(\lambda_0, \lambda) \in F'_{\lambda_0,\lambda}(A)$, so

$$f^{\lambda_0}_{\gamma_0}(\lambda) = \mathrm{El}_{\lambda_0,\lambda}(\gamma_0) \le \max(\{\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) : (\lambda_0, \ldots, \lambda_n) \in F'_{\lambda_0,\lambda}\}) = f^{\lambda_0,e}_{\gamma_0}(\lambda). \qquad \square$$

Lemma 32.31. *Suppose that A is progressive, κ is a regular cardinal such that $|A| < \kappa < \min(A)$, and $f \stackrel{\text{def}}{=} \langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a sequence of sequences f^λ such that f^λ is κ-minimally obedient for λ. Assume also $H_1(A, \kappa, N, \Psi)$ and $f \in N$.*

Then also $f^e \in N$.

Proof. The proof is a more complicated instance of our standard procedure for going from V to H_Ψ to N and then back. We sketch the details.

Assume the hypotheses. In particular, $A \in N$. Hence also $\mathrm{pcf}(A) \in N$. Also, $|A| < \kappa$, so $A \subseteq N$. Now clearly $F' \in N$. Also, $\mathrm{El} \in N$. (Note that El depends upon A.) Then by absoluteness,

$$H_\Psi \models \exists g \ \ g \text{ is a function, } \mathrm{dmn}(g) = \mathrm{pcf}(A) \wedge \forall \lambda_0 \in \mathrm{pcf}(A) \forall \gamma_0 \in \lambda_0 \forall \lambda \in A$$

$$g(\lambda) = \begin{cases} f^{\lambda_0}_{\gamma_0}(\lambda) & \text{if } \lambda_0 \le \lambda, \\ \max(\{\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) : (\lambda_0, \ldots, \lambda_n) \in F'_{\lambda_0,\lambda}\}) & \text{if } \lambda < \lambda_0, \\ & \quad \text{and this maximum exists,} \\ f^{\lambda_0}_{\gamma_0}(\lambda) & \text{if } \lambda < \lambda_0, \text{ otherwise.} \end{cases}$$

Now the usual procedure can be applied. $\square$

Lemma 32.32. *Suppose that A is progressive, κ is a regular cardinal such that $|A| < \kappa < \min(A)$, and $f \stackrel{\text{def}}{=} \langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a sequence of sequences f^λ such that f^λ is κ-minimally obedient for λ. Assume $H_1(A, \kappa, N, \Psi)$ and $f \in N$.*

Suppose that $\lambda_0 \in \mathrm{pcf}(A) \cap N$, and let $\gamma_0 = \mathrm{Ch}_N(\lambda_0)$.

(i) If $\lambda_0 > \cdots > \lambda_n$ is a special walk in $\mathrm{pcf}(A)$, and $\gamma_1, \ldots, \gamma_n$ are formed as above, then $\gamma_i \in \overline{N}$ for all $i = 0, \ldots, n$.

(ii) For every $\lambda \in A \cap \lambda_0$ we have $f^{\lambda_0,e}_{\gamma_0}(\lambda) \in \overline{N}$.

Proof. (i): By Lemma 32.17(iv)(c), $f^{\lambda_0}_{\gamma_0}(\lambda) \in \overline{N}$, and (i) follows by induction using Lemma 32.17(iv)(c).

(ii): immediate from (i). $\square$

Lemma 32.33. *Assume the hypotheses of Lemma 32.32. Then* *(i) For any special walk $\lambda_0 > \cdots > \lambda_n = \lambda$ in $F'_{\lambda_0,\lambda}$, we have*

$$El_{\lambda_0,\ldots,\lambda_n}(\gamma_0) \leq \mathrm{Ch}_N(\lambda).$$

(ii) $f^{\lambda_0,e}_{\gamma_0} \leq \mathrm{Ch}_N \restriction A$ for every $\gamma_0 < \lambda_0$.
(iii) If there is a special walk $\lambda_0 > \cdots > \lambda_n = \lambda$ in $F'_{\lambda_0,\lambda}$ such that

$$El_{\lambda_0,\ldots,\lambda_n}(\gamma_0) = \mathrm{Ch}_N(\lambda),$$

then

$$\mathrm{Ch}_N(\lambda) = f^{\lambda_0,e}_{\gamma_0}(\lambda).$$

(iv) Suppose that $\mathrm{Ch}_N(\lambda) = f^{\lambda_0,e}_{\gamma_0}(\lambda) = \gamma$. If there is an $a \in A \cap \lambda$ such that $f^{\lambda,e}_{\gamma}(a) = \mathrm{Ch}_N(a)$, then also $f^{\lambda_0,e}_{\gamma_0}(a) = \mathrm{Ch}_N(a)$.

Proof. (i) is immediate from Lemma 32.32(i) and Lemma 32.8(iii). (ii) and (iii) follow from (i). For (iv), by Lemma 32.32(i) and (i) there are special walks $\lambda_0 > \cdots > \lambda_n = \lambda$ and $\lambda = \lambda'_0 > \cdots > \lambda'_m = a$ such that

$$\begin{aligned} &f^{\lambda_0,e}_{\gamma_0}(\lambda) = \mathrm{Ch}_N(\lambda) = \mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) \quad \text{and} \\ &f^{\lambda,e}_{\gamma}(a) = \mathrm{Ch}_N(a) = \mathrm{El}_{\lambda'_0,\ldots,\lambda'_m}(a). \end{aligned}$$

It follows that

$$\mathrm{El}_{\lambda_0,\ldots,\lambda_n,\lambda'_1,\ldots,a}(\gamma_0) = \mathrm{Ch}_N(a),$$

and (iii) then gives $f^{\lambda_0,e}_{\gamma_0}(a) = \mathrm{Ch}_N(a)$. $\square$

Definition. Suppose that A is progressive and $A \subseteq P \subseteq \mathrm{pcf}(A)$. A system $\langle b_\lambda : \lambda \in P\rangle$ of subsets of A is *transitive* iff for all $\lambda \in P$ and all $\mu \in b_\lambda$ we have $b_\mu \subseteq b_\lambda$.

Theorem 32.34. *Suppose that $H_1(A, \kappa, N, \Psi)$, and $f = \langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a system of functions, and each f^λ is κ-minimally obedient for λ. Let f^e be the derived elevated array. For every $\lambda_0 \in \mathrm{pcf}(A) \cap N$ put $\gamma_0 = \mathrm{Ch}_N(\lambda_0)$ and define*

$$b_{\lambda_0} = \{a \in A : \mathrm{Ch}_N(a) = f^{\lambda_0,e}_{\gamma_0}(a)\}.$$

Then the following hold for each $\lambda_0 \in \mathrm{pcf}(A) \cap N$:
(i) b_{λ_0} is a λ_0-generator.
(ii) There is a $b'_{\lambda_0} \subseteq b_{\lambda_0}$ such that
(a) $b_{\lambda_0} \backslash b'_{\lambda_0} \in J_{<\lambda_0}[A]$.
(b) $b'_{\lambda_0} \in N$ (each one individually, not the sequence).
(c) b'_{λ_0} is a λ_0-generator.
(iii) The system $\langle b_\lambda : \lambda \in \mathrm{pcf}(A) \cap N\rangle$ is transitive.

Proof. Note that $H_2(A, \kappa, N, \Psi, \lambda_0, f^{\lambda_0,e}, \gamma_0)$ holds by Lemma 32.31. By definition, minimally obedient implies universal, so f^{λ_0} is persistently cofinal by Lemma 32.11. Hence by Lemma 32.24, $f^{\lambda_0,e}$ is persistently cofinal, and so $P_1(A, \kappa, N, \Psi, \lambda_0, f^{\lambda_0,e}, \gamma_0)$ holds by

Lemma 32.18. Also, by Lemma 32.19 $P_2(A,\kappa,N,\Psi,\lambda_0,f^{\lambda_0},\gamma_0)$ holds, so the condition $P_2(A,\kappa,N,\Psi,\lambda_0,f^{\lambda_0,e},\gamma_0)$ holds by Lemmas 32.30 and 32.33(ii). Now (i) and (ii) hold by Lemma 32.22.

Now suppose that $\lambda_0 \in \mathrm{pcf}(A)\cap N$ and $\lambda \in b_{\lambda_0}$. Thus

$$\mathrm{Ch}_N(\lambda) = f^{\lambda_0,e}_{\gamma_0}(\lambda),$$

where $\gamma_0 = \mathrm{Ch}_N(\lambda_0)$. Write $\gamma = \mathrm{Ch}_N(\lambda)$. We want to show that $b_\lambda \subseteq b_{\lambda_0}$. Take any $a \in b_\lambda$. So $\mathrm{Ch}_N(a) = f^{\lambda,e}_\gamma(a)$. By Lemma 32.33(iv) we get $f^{\lambda_0,e}_{\gamma_0}(a) = \mathrm{Ch}_N(a)$, so $a \in b_{\lambda_0}$, as desired. □

Localization

Theorem 32.35. *Suppose that A is a progressive set. Then there is no subset $B \subseteq \mathrm{pcf}(A)$ such that $|B| = |A|^+$ and, for every $b \in B$, $b > \max(\mathrm{pcf}(B\cap b))$.*

Proof. Assume the contrary. We may assume that $|A|^+ < \min(A)$. In fact, if we know the result under this assumption, and now $|A|^+ = \min(A)$, suppose that $B \subseteq \mathrm{pcf}(A)$ with $|B| = |A|^+$ and $\forall b \in B[b > \max(\mathrm{pcf}(B\cap b))]$. Let $A' = A\backslash\{|A|^+\}$. Then let $B' = B\backslash\{|A|^+\}$. Hence we have $B' \subseteq \mathrm{pcf}(A')$. Clearly $|B'| = |A'|^+$ and $\forall b \in B'[b > \max(\mathrm{pcf}(B'\cap b))]$, contradiction.

Also, clearly we may assume that B has order type $|A|^+$.

Let $E = A \cup B$. Then $|E| < \min(E)$. Let $\kappa = |E|$. By Lemma 32.16, we get an array $\langle f^\lambda : \lambda \in \mathrm{pcf}(E)\rangle$, with each f^λ κ-minimally obedient for λ. Choose N and Ψ so that $H_1(A,\kappa,N,\Psi)$, with N containing $A, B, E, \langle f^\lambda : \lambda \in \mathrm{pcf}(E)\rangle$. Now let $\langle b_\lambda : \lambda \in \mathrm{pcf}(E)\cap N\rangle$ be the set of transitive generators as guaranteed by Theorem 32.34. Let $b'_\lambda \in N$ be such that $b'_\lambda \subseteq b_\lambda$ and $b_\lambda\backslash b'_\lambda \in J_{<\lambda}$.

Now let F be the function with domain $\{a \in A : \exists\beta \in B(a \in b_\beta)\}$ such that for each such a, $F(a)$ is the least $\beta \in B$ such that $a \in b_\beta$. Define $B_0 = \{\gamma \in B : \exists a \in \mathrm{dmn}(F)(\gamma \le F(a)\}$. Thus B_0 is an initial segment of B of size at most $|A|$. Clearly $B_0 \in N$. We let $\beta_0 = \min(B\backslash B_0)$; so $B_0 = B \cap \beta_0$.

Now we claim

(1) There exists a finite descending sequence $\lambda_0 > \cdots > \lambda_n$ of cardinals in $N \cap \mathrm{pcf}(B_0)$ such that $B_0 \subseteq b_{\lambda_0} \cup \ldots \cup b_{\lambda_n}$.

We prove more: we find a finite descending sequence $\lambda_0 > \cdots > \lambda_n$ of cardinals in $N\cap\mathrm{pcf}(B_0)$ such that $B_0 \subseteq b'_{\lambda_0} \cup \ldots \cup b'_{\lambda_n}$. Let $\lambda_0 = \max(\mathrm{pcf}(B_0))$. Since $B_0 \in N$, we clearly have $\lambda_0 \in N$ and hence $b'_{\lambda_0} \in N$. So $B_1 \stackrel{\mathrm{def}}{=} B_0\backslash b'_{\lambda_0} \in N$. Now suppose that $B_k \subseteq B_0$ has been defined so that $B_k \in N$. If $B_k = \emptyset$, the construction stops. Suppose that $B_k \ne \emptyset$. Let $\lambda_k = \max(\mathrm{pcf}(B_k))$. Clearly $\lambda_k \in N$, so $b'_{\lambda_k} \in N$ and $B_{\kappa+1} \stackrel{\mathrm{def}}{=} B_k\backslash b'_{\lambda_k} \in N$. Since $B_{\kappa+1} = B_k\backslash b'_{\lambda_k}$ and b'_{λ_k} is a λ_k-generator, from Lemma 9.25(xii) it follows that

$\lambda_0 > \lambda_1 > \cdots$. So the construction eventually stops; say that $B_{n+1} = \emptyset$. So $B_n \subseteq b'_{\lambda_n}$. So

$$\begin{aligned}
B_0 &\subseteq b'_{\lambda_0} \cup (B_0 \backslash b'_{\lambda_0}) \\
&= b'_{\lambda_0} \cup B_1 \\
&\subseteq b'_{\lambda_0} \cup b'_{\lambda_1} \cup B_2 \\
&\ldots\ldots\ldots \\
&\subseteq b'_{\lambda_0} \cup b'_{\lambda_1} \cup \ldots \cup B_n \\
&\subseteq b'_{\lambda_0} \cup b'_{\lambda_1} \cup \ldots \cup b'_{\lambda_n}.
\end{aligned}$$

This proves (1).

Note that $\beta_0 > \max(\mathrm{pcf}(B \cap \beta_0) = \max(\mathrm{pcf}(B_0)) \geq \lambda_0, \ldots, \lambda_n$ by the initial assumption of the proof. Next, we claim

(2) $b_{\beta_0} \subseteq b_{\lambda_0} \cup \ldots \cup b_{\lambda_n}$.

To prove this, first note that $b_{\beta_0} \subseteq A \cup B_0$. For, $b_{\beta_0} \subseteq E$ by definition, and $E = A \cup B$; $b_{\beta_0} \cap B = B_0$, so indeed $b_{\beta_0} \subseteq A \cup B_0$. Also, $B_0 \subseteq b_{\lambda_0} \cup \ldots \cup b_{\lambda_n}$. So it suffices to prove that $b_{\beta_0} \cap A \subseteq b_{\lambda_0} \cup \ldots \cup b_{\lambda_n}$.

Consider any cardinal $a \in b_{\beta_0} \cap A$. Since $\beta_0 \in B$, we have $a \in \mathrm{dmn}(F)$, and since $\beta_0 \notin B_0$ we have $F(a) < \beta_0$. Let $\beta = F(a)$. So $a \in b_\beta$, and $\beta < \beta_0$, so by the minimality of β_0, $\beta \in B_0$. Since $B_0 \subseteq b_{\lambda_0} \cup \ldots \cup b_{\lambda_n}$, it follows that $\beta \in b_{\lambda_i}$ for some $i = 0, \ldots, n$. But transitivity implies that $b_\beta \subseteq b_{\lambda_i}$, and hence $a \in b_{\lambda_i}$, as desired. So (2) holds.

By (2) we have

$$\mathrm{pcf}(b_{\beta_0}) \subseteq \mathrm{pcf}(b_{\lambda_0}) \cup \ldots \cup \mathrm{pcf}(b_{\lambda_n}),$$

and hence by Lemma 31.25(vii) we get $\beta_0 = \max(\mathrm{pcf}(b_{\beta_0})) \leq \max\{\lambda_i : i = 0, \ldots, n\} < \beta_0$, contradiction. □

Theorem 32.36. (Localization) *Suppose that A is a progressive set of regular cardinals. Suppose that $B \subseteq \mathrm{pcf}(A)$ is also progressive. Then for every $\lambda \in \mathrm{pcf}(B)$ there is a $B_0 \subseteq B$ such that $|B_0| \leq |A|$ and $\lambda \in \mathrm{pcf}(B_0)$.*

Proof. We prove by induction on λ that if A and B satisfy the hypotheses of the theorem, then the conclusion holds. Let C be a λ-generator over B. Thus $C \subseteq B$ and $\lambda = \max(\mathrm{pcf}(C))$ by Lemma 31.25(vii). Now $C \subseteq \mathrm{pcf}(A)$ and C is progressive. It suffices to find $B_0 \subseteq C$ with $|B_0| \leq |A|$ and $\lambda \in \mathrm{pcf}(B_0)$.

Let $C_0 = C$ and $\lambda_0 = \lambda$. Suppose that $C_0 \supseteq \cdots \supseteq C_i$ and $\lambda_0 > \cdots > \lambda_i$ have been constructed so that $\lambda = \max(\mathrm{pcf}(C_i))$ and C_i is a λ-generator over B. If there is no maximal element of $\lambda \cap \mathrm{pcf}(C_i)$ we stop the construction. Otherwise, let λ_{i+1} be that maximum element, let D_{i+1} be a λ_{i+1}-generator over B, and let $C_{i+1} = C_i \backslash D_{i+1}$. Now $D_{i+1} \in J_{\leq\lambda_{i+1}}[B] \subseteq J_{<\lambda}[B]$, so C_{i+1} is still a λ-generator of B by Lemma 9.25(ix), and $\lambda = \max(\mathrm{pcf}(C_{i+1}))$ by Lemma 31.25(vii). Note that $\lambda_{i+1} \notin \mathrm{pcf}(C_{i+1})$, by Lemma 31.25(ii).

This construction must eventually stop, when $\lambda \cap C_i$ does not have a maximal element; we fix the index i.

(1) There is an $E \subseteq \lambda \cap \mathrm{pcf}(C_i)$ such that $|E| \le |A|$ and $\lambda \in \mathrm{pcf}(E)$.

In fact, suppose that no such E exists. We now construct a strictly increasing sequence $\langle \gamma_j : j < |A|^+ \rangle$ of elements of $\mathrm{pcf}(C_i)$ such that $\gamma_k > \max(\mathrm{pcf}(\{\gamma_j : j < k\})$ for all $k < |A|^+$. (This contradicts Theorem 32.35.) Suppose that $\{\gamma_j : j < k\} = E$ has been defined. Now $\lambda \notin \mathrm{pcf}(E)$ by the supposition after (1), and $\lambda < \max(\mathrm{pcf}(E))$ is impossible since $\mathrm{pcf}(E) \subseteq \mathrm{pcf}(C_i)$ and $\lambda = \max(\mathrm{pcf}(C_i))$. So $\lambda > \max(\mathrm{pcf}(E))$. Hence, because $\lambda \cap C_i$ does not have a maximal element, we can choose $\gamma_k \in \lambda \cap C_i$ such that $\gamma_k > \max(\mathrm{pcf}(E))$, as desired. Hence (1) holds.

We take E as in (1). Apply the inductive hypothesis to each $\gamma \in E$ and to A, E in place of A, B; we get a set $G_\gamma \subseteq E$ such that $|G_\gamma| \le |A|$ and $\gamma \in \mathrm{pcf}(G_\gamma)$. Let $H = \bigcup_{\gamma \in E} G_\gamma$. Note that $|H| \le |A|$. Thus $E \subseteq \mathrm{pcf}(H)$. Since $\mathrm{pcf}(E) \subseteq \mathrm{pcf}(H)$ by Theorem 9.15, we have $\lambda \in \mathrm{pcf}(H)$, completing the inductive proof.

The size of pcf(A)

Theorem 32.37. *If A is a progressive interval of regular cardinals, then* $|\mathrm{pcf}(A)| < |A|^{+4}$.

Proof. Assume that A is a progressive interval of regular cardinals but $|\mathrm{pcf}(A)| \ge |A|^{+4}$. Let $\rho = |A|$. We will define a set B of size ρ^+ consisting of cardinals in $\mathrm{pcf}(A)$ such that each cardinal in B is greater than $\max(\mathrm{pcf}(B \cap b))$. This will contradict Theorem 32.35.

Let $S = S^{\rho^{+3}}_{\rho^+}$; so S is a stationary subset of ρ^{+3}. By Theorem 30.40 let $\langle C_k : k \in S \rangle$ be a club guessing sequence. Thus

(1) C_k is a club in k of order type ρ^+, for each $k \in S$.

(2) If D is a club in ρ^{+3}, then there is a $k \in D \cap S$ such that $C_k \subseteq D$.

Let σ be the ordinal such that $\aleph_\sigma = \sup(A)$. Now $\mathrm{pcf}(A)$ is an interval of regular cardinals by Theorem 31.13. So $\mathrm{pcf}(A)$ contains all regular cardinals in the set $\{\aleph_{\sigma+\alpha} : \alpha < \rho^{+4}\}$.

Now we are going to define a strictly increasing continuous sequence $\langle \alpha_i : i < \rho^{+3} \rangle$ of ordinals less than ρ^{+4}.

1. Let $\alpha_0 = \rho^{+3}$.
2. For i limit let $\alpha_i = \bigcup_{j<i} \alpha_j$.
3. Now suppose that α_j has been defined for all $j \le i$; we define α_{i+1}. For each $k \in S$ let $e_k = \{\aleph_{\sigma+\alpha_j} : j \in C_k \cap (i+1)\}$. Thus $e_k^{(+)}$ is a subset of $\mathrm{pcf}(A)$. If $\max(\mathrm{pcf}(e_k^{(+)})) < \aleph_{\sigma+\rho^{+4}}$, let β_k be an ordinal such that $\max(\mathrm{pcf}(e_k^{(+)})) < \aleph_{\sigma+\beta_k}$ and $\beta_k < \rho^{+4}$; otherwise let $\beta_k = 0$. Let α_{i+1} be greater than α_i and all β_k for $k \in S$, with $\alpha_{i+1} < \rho^{+4}$. This is possible because $|S| = \rho^{+3}$. Thus

(3) For every $k \in S$, if $\max(\mathrm{pcf}(e_k^{(+)})) < \aleph_{\sigma+\rho^{+4}}$, then $\max(\mathrm{pcf}(e_k^{(+)})) < \aleph_{\sigma+\alpha_{i+1}}$.

This finishes the definition of the sequence $\langle \alpha_i : i < \rho^{+3} \rangle$. Let $D = \{\alpha_i : i < \rho^{+3}\}$, and let $\delta = \sup(D)$. Then D is club in δ. Let $\mu = \aleph_{\sigma+\delta}$. Thus μ has cofinality ρ^{+3}, and it is singular since $\delta > \alpha_0 = \rho^{+3}$. Now we apply Corollary 9.35: there is a club C_0 in μ such that $\mu^+ = \max(\mathrm{pcf}(C_0^{(+)}))$. We may assume that $C_0 \subseteq [\aleph_\sigma, \mu)$. so we can write $C_0 = \{\aleph_{\sigma+i} : i \in D_0\}$ for some club D_0 in δ. Let $D_1 = D_0 \cap D$. So D_1 is a club of δ. Let

$E = \{i \in \rho^{+3} : \alpha_i \in D_1\}$. It is clear that E is a club in ρ^{+3}. So by (2) choose $k \in E \cap S$ such that $C_k \subseteq E$. Let $C'_k = \{\beta \in C_k :$ there is a largest $\gamma \in C_k$ such that $\gamma < \beta\}$. Set $B = \{\aleph^+_{\sigma+\alpha_i} : i \in C'_k\}$. We claim that B is as desired. Clearly $|B| = \rho^+$.

Take any $j \in C'_k$. We want to show that

$$(*) \qquad \aleph^+_{\sigma+\alpha_j} > \max(\mathrm{pcf}(B \cap \aleph^+_{\sigma+\alpha_j})).$$

Let $i \in C_k$ be largest such that $i < j$. So $i+1 \leq j$. We consider the definition given above of α_{i+1}. We defined $e_k = \{\aleph_{\sigma+\alpha_l} : l \in C_k \cap (i+1)\}$. Now

(4) $B \cap \aleph^+_{\sigma+\alpha_j} \subseteq e_k^{(+)}$.

For, if $b \in B \cap \aleph^+_{\sigma+\alpha_j}$, we can write $b = \aleph^+_{\sigma+\alpha_l}$ with $l \in C'_k$ and $l < j$. Hence $l \leq i$ and so $b = \aleph^+_{\sigma+\alpha_l} \in e_k^{(+)}$. So (4) holds.

Now if $l \in C_k \cap (i+1)$, then $l \in E$, and so $\alpha_l \in D_1 \subseteq D_0$. Hence $\aleph_{\sigma+\alpha_l} \in C_0$. This shows that $e_k^{(+)} \subseteq C_0^{(+)}$. So $\max(\mathrm{pcf}(e_k^{(+)})) \leq \max(\mathrm{pcf}(C_0^{(+)})) = \mu^+ < \aleph_{\sigma+\rho^{+4}}$. Hence by (3) we get $\max(\mathrm{pcf}(e_k^{(+)})) < \aleph_{\sigma+\alpha_{i+1}}$. So

$$\begin{aligned}\max(\mathrm{pcf}(B \cap \aleph^+_{\sigma+\alpha_j})) &\leq \max(\mathrm{pcf}(e_k^{(+)})) \quad \text{by (4)}\\ &< \aleph^+_{\sigma+\alpha_{i+1}}\\ &\leq \aleph^+_{\sigma+\alpha_j},\end{aligned}$$

which proves $(*)$. $\square$

Theorem 32.38. *If $\aleph_\delta$ is a singular cardinal such that $\delta < \aleph_\delta$, then*

$$\mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq) < \aleph_{|\delta|^{+4}}.$$

Proof. Let $\kappa = |\delta|^+$ and $A = (\kappa, \aleph_\delta)_{\mathrm{reg}}$. By Lemma 32.25(iii) and Lemma 32.28,

$$\begin{aligned}\mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq) &\leq \max(|\delta|^+, \mathrm{cf}([\aleph_\delta]^{|\delta|^+}, \subseteq))\\ &\leq \max(|\delta|^+, \max(\mathrm{pcf}(A))).\end{aligned}$$

Hence it suffices to show that $\max(\mathrm{pcf}(A)) < \aleph_{|\delta|^{+4}}$.

By Theorem 32.37, $|\mathrm{pcf}(A)| < |A|^{+4}$. Write $\max(\mathrm{pcf}(A)) = \aleph_\alpha$ and $\kappa = \aleph_\beta$. We want to show that $\alpha < |\delta|^{+4}$. Now $\mathrm{pcf}(A) = (\kappa, \max(\mathrm{pcf}(A))]_{\mathrm{reg}} = (\aleph_\beta, \aleph_\alpha]_{\mathrm{reg}}$. By Lemma 32.27, $|(\beta, \alpha)| = |\mathrm{pcf}(A)| < |A|^{+4} \leq |\delta|^{+4}$. Also, $\beta \leq \aleph_\beta = \kappa = |\delta|^+ < |\delta|^{+4}$. So $|\alpha| < |\delta|^{+4}$, and hence $\alpha < |\delta|^{+4}$. $\square$

Theorem 32.39. *If δ is a limit ordinal, then*

$$\aleph_\delta^{\mathrm{cf}(\delta)} < \max\left(\left(|\delta|^{\mathrm{cf}(\delta)}\right)^+, \aleph_{|\delta|^{+4}}\right).$$

Proof. If $\delta = \aleph_\delta$, then $|\delta| = \aleph_\delta$ and the conclusion is obvious. So assume that $\delta < \aleph_\delta$. Now

(1) $\aleph_\delta^{\mathrm{cf}(\delta)} \leq |\delta|^{\mathrm{cf}(\delta)} \cdot \mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq)$.

In fact, let $B \subseteq [\aleph_\delta]^{|\delta|}$ be cofinal and of size $\mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq)$. Now $\mathrm{cf}(\delta) \leq |\delta|$, so

$$[\aleph_\delta]^{\mathrm{cf}(\delta)} = \bigcup_{Y \in B} [Y]^{\mathrm{cf}(\delta)},$$

and (1) follows. Hence the theorem follows by Theorem 32.38. $\square$

Corollary 32.40. $\aleph_\omega^{\aleph_0} < \max\left((2^{\aleph_0})^+, \aleph_{\omega_4}\right)$. $\square$

We define $f \le g$ iff $f, g \in {}^{\omega}\omega$ and $f(m) \le g(m)$ for all $m \in \omega$.

We define $f \le^* g$ iff $f, g \in {}^{\omega}\omega$ and $\exists m \forall n \ge m[f(n) \le g(n)]$.

A family $\mathscr{D} \subseteq {}^{\omega}\omega$ is *almost dominating* iff $\forall f \in {}^{\omega}\omega \exists g \in \mathscr{D}[f \le^* g]$. Let $\mathfrak{d}$ be the smallest size of a almost dominating family; this is the *dominating* number.

A family $\mathscr{B} \subseteq {}^{\omega}\omega$ is *almost unbounded* iff there is no $f \in {}^{\omega}\omega$ such that $\forall g \in \mathscr{B}[g \le^* f]$. Let $\mathfrak{b}$ be the smallest size of an almost unbounded family.

Theorem 33.1. $\aleph_1 \le \mathrm{cf}(\mathfrak{b}) = \mathfrak{b} \le \mathrm{cf}(\mathfrak{d}) \le \mathfrak{d} \le 2^{\omega}$.

Proof. If $\{f_n : n \in \omega\} \subseteq {}^{\omega}\omega$, define $g \in {}^{\omega}\omega$ by setting $g(n) = \sup\{f_m(n) : m \le n\}$ for all $n \in \omega$. Thus if $m \le n$, then $f_m(n) \le g(n)$, so $f_m \le^* g$. Hence g is a $\le^*$ bound for $\{f_n : n \in \omega\} \subseteq {}^{\omega}\omega$. This argument shows that $\aleph_1 \le \mathfrak{b}$. Suppose that $\mathrm{cf}(\mathfrak{b}) < \mathfrak{b}$. Let X be almost unbounded with $|X| = \mathfrak{b}$. Then we can write $X = \bigcup_{\alpha<\mathrm{cf}(\mathfrak{b})} Y_\alpha$ with $|Y_\alpha| < \mathfrak{b}$ for all $\alpha < \mathrm{cf}(\mathfrak{b})$. Choose a bound g^α for Y_α for each $\alpha < \mathrm{cf}(\mathfrak{b})$, and then by the above argument choose a bound h for $\{g^\alpha : \alpha < \mathrm{cf}(\mathfrak{b})\}$. Then h is a bound for X, contradiction. Thus $\mathrm{cf}(\mathfrak{b}) = \mathfrak{b}$.

To prove that $\mathfrak{b} \le \mathrm{cf}(\mathfrak{d})$, let D be a almost dominating family of size $\mathfrak{d}$, and write $D = \bigcup_{\alpha<\mathrm{cf}(\mathfrak{d})} E_\alpha$, with each E_α of size less than $\mathfrak{d}$. Since then E_α is not almost dominating, there is an $f^\alpha \in {}^{\omega}\omega$ such that for all $g \in E_\alpha$ we have $f^\alpha \not\le^* g$. Suppose that $\mathrm{cf}(\mathfrak{d}) < \mathfrak{b}$, and accordingly let $h \in {}^{\omega}\omega$ be such that $f^\alpha \le^* h$ for all $\alpha < \mathrm{cf}(\mathfrak{d})$. Choose $k \in D$ such that $h \le^* k$. Say $k \in E_\alpha$. But $f^\alpha \le^* h \le^* k$, contradiction.

Finally, obviously ${}^{\omega}\omega$ is almost dominating, so $\mathfrak{d} \le 2^{\omega}$. □

An *interval partition* is a partition P of ω whose members are finite intervals. For such a partition, we introduce the following notation:

$$P = \{[i_n^P, i_{n+1}^P) : n \in \omega\},$$

where $0 = i_0^P < i_1^P < \cdots$. Given two interval partitions P, Q, we say that P *almost dominates* Q iff $\exists m \forall n \ge m \exists k[[i_k^Q, i_{k+1}^Q) \subseteq [i_n^P, i_{n+1}^P)]$.

Now with each interval partition P we associate a function $\mathrm{func}_P \in {}^{\omega}\omega$ as follows. For each $x \in \omega$, choose n such that $x \in [i_n^P, i_{n+1}^P)$, and let $\mathrm{func}_P(x) = i_{n+2}^P - 1$. Conversely, with each $g \in {}^{\omega}\omega$ we associate an interval partition $\mathrm{part}_g = Q$ as follows. We define $\langle i_k^Q : k \in \omega\rangle$ by recursion. Let $i_0^Q = 0$. If i_k^Q has been defined, let i_{k+1}^Q be minimum such that $i_{k+1}^Q > i_k^Q$ and for any $x \le i_k^Q$ we have $g(x) < i_{k+1}^Q$.

Theorem 33.2. *(i) If P is an interval partition and $g \in {}^{\omega}\omega$, and if* $\mathrm{func}_P \le^* g$, *then* part_g *almost dominates P.*

(ii) If P is an interval partition and $g \in {}^{\omega}\omega$, and if P almost dominates part_g, *then* $g \le^* \mathrm{func}_P$.

(iii) $\mathfrak{d}$ is the smallest size of a family $\mathscr{P}$ of interval partitions such that every interval partition is almost dominated by some member of $\mathscr{P}$.

(iv) $\mathfrak{b}$ is the smallest size of a family $\mathscr{P}$ of interval partitions such that there is no interval partition which almost dominates each member of $\mathscr{P}$.

Proof. For both (i) and (ii) let, for brevity, $Q = \mathrm{part}_g$.

(i) Choose p such that $\mathrm{func}_P(n) \le g(n)$ for all $n \ge p$. Take any $n \ge p$. Choose k such that $i^Q_n \in [i^P_k, i^P_{k+1})$. Take any $x \in [i^P_{k+1}, i^P_{k+2})$. Then $p \le n \le i^Q_n$, so

$$i^Q_n < i^P_{k+1} \le x \le i^P_{k+2} - 1 = f(i^Q_n) \le g(i^Q_n) < i^Q_{n+1}.$$

Thus $[i^P_{k+1}, i^P_{k+2}) \subseteq [i^Q_n, i^Q_{n+1})$, as desired.

(ii) By definition, choose m so that for all $n \ge m$ there is a k such that $[i^Q_k, i^Q_{k+1}) \subseteq [i^P_n, i^P_{n+1})$. Take any $x \ge i^P_m$; we claim that $g(x) \le \mathrm{func}_P(x)$. For take n such that $x \in [i^P_n, i^P_{n+1})$. Then $n+1 \ge m$, so we can choose $k \in \omega$ such that $[i^Q_k, i^Q_{k+1}) \subseteq [i^P_{n+1}, i^P_{n+2}))$ Now $x < i^P_{n+1} \le i^Q_k$, so $g(x) \le i^Q_{k+1} - 1 \le i^P_{n+2} - 1 = \mathrm{func}_P(x)$, as desired.

(iii) and (iv) follow immediately from (i) and (ii). $\square$

Given $X, Y \in [\omega]^\omega$, we say that X *splits* Y iff $Y \cap X$ and $Y \backslash X$ are infinite. A *splitting family* is a subset $S \subseteq [\omega]^\omega$ such that every $Y \in [\omega]^\omega$ is split by some member of S. The *splitting number* $\mathfrak{s}$ is the smallest cardinality of a splitting family.

If P is an interval partition, let $\varphi(P) = \bigcup_{n\in\omega}[i^P_{2n}, i^P_{2n+1})$. If $X \in [\omega]^\omega$ define an interval partition $\psi(X) = Q$ as follows:

$$\begin{aligned} i^Q_0 &= 0; \\ i^Q_{n+1} &= \text{ least } j > i^Q_n \text{ such that } [i^Q_n, j) \cap X \ne \emptyset. \end{aligned}$$

Lemma 33.3. *If P almost dominates $\psi(X)$, then $\varphi(P)$ splits X.*

Proof. Let $\Psi(X) = Q$, as above. Choose m such that $\forall n \ge m \exists k[[i^Q_k, i^Q_{k+1}) \subseteq [i^P_n, i^P_{n+1})]$; hence $\forall n[X \cap [i^P_n, i^P_{n+1}) \ne \emptyset]$. So the lemma follows. $\square$

Theorem 33.4. $\mathfrak{s} \le \mathfrak{d}$.

Proof. Let D be an almost dominating family of interval partitions, with $|D| = \mathfrak{d}$. Then $\{\varphi(P) : P \in D\}$ is a splitting family by Lemma 33.3. $\square$

A family $X \subseteq [\omega]^\omega$ is *unsplittable* iff there is no $a \in [\omega]^\omega$ such that $\forall x \in X[x \cap a$ is infinite and $x \backslash a$ is infinite].

Proposition 33.5. *$[\omega]^\omega$ is unsplittable.*

Proof. Suppose, on the contrary that a splits $[\omega]^\omega$. Applying this to a itself gives a contradiction. $\square$

We define the *reaping number* $\mathfrak{r}$ to be the smallest cardinality of an unsplittable family.

Proposition 33.6. $\mathfrak{b} \le \mathfrak{r}$.

Proof. Let $R \subseteq [\omega]^\omega$ be unsplittable with $|R| = \mathfrak{r}$. We claim that no interval partition P almost dominates each member of $\{\psi(X) : X \in R\}$. In fact, otherwise by Lemma 33.3, $\varphi(P)$ splits R, contradiction. So the proposition follows by Theorem 33.2(iv). $\square$

If $\mathscr{F}$ is a family of sets, a *pseudo-intersection* of $\mathscr{F}$ is an infinite set A such that $A \subseteq^* B$ for all $B \in \mathscr{F}$.

A *tower* is a sequence $\langle T_\xi : \xi < \alpha \rangle$ with the following properties:

(1) α is an ordinal, and each T_ξ is an infinite subset of ω.

(2) If $\xi < \eta < \alpha$, then $T_\eta \subseteq^* T_\xi$.

(3) $\{T_\xi : \xi < \alpha\}$ does not have a pseudo-intersection.

The *tower number* $\mathfrak{t}$ is the smallest ordinal α which is the length of a tower.

A set $\mathscr{D} \subseteq [\omega]^\omega$ is *open* iff $\forall X, Y \in [\omega]^\omega[X \subseteq^* Y \in \mathscr{D} \to X \in \mathscr{D}]$. $\mathscr{D}$ is *dense* iff $\forall Y \in [\omega]^\omega \exists X \in \mathscr{D}[X \subseteq Y]$. Obviously $[\omega]^\omega$ itself is dense open. We say that $\mathscr{D}$ is *weakly dense* iff $\forall Y \in [\omega]^\omega \exists X \in \mathscr{D}[X \subseteq^* Y]$.

Proposition 33.7. *If $\mathscr{D}$ is weakly dense, then there is a $\mathscr{D}'$ such that $\mathscr{D} \subseteq \mathscr{D}' \subseteq [\omega]^\omega$, $|\mathscr{D}| = |\mathscr{D}'|$, and $\mathscr{D}'$ is dense.*

Proof. Let $\mathscr{D}' = \{X \in [\omega]^\omega :$ there is a finite $F \subseteq \omega$ such that $X \cup F \in \mathscr{D}\}$. $\square$

Proposition 33.8. *For every $X \in [\omega]^\omega$ there is a dense open family $\mathscr{D}$ such that $X \notin \mathscr{D}$.*

Proof. Let $X = Y \cup Z$ with $Y, Z \in [\omega]^\omega$ and $Y \cap Z = \emptyset$. Define

$$\mathscr{D} = \{W \in [\omega]^\omega : W \subseteq^* Y \text{ or } W \subseteq^* Z \text{ or } W \cap X \text{ is finite}\}.$$

Clearly $\mathscr{D}$ is as desired. $\square$

We now consider the Boolean algebra $\mathscr{P}(\omega)/\mathrm{fin}$. A *partition* of this algebra is a system of pairwise disjoint nonzero elements with sum 1. A partition $\langle a_i : i \in I \rangle$ is a *refinement* of a partition $\langle b_j : j \in J \rangle$ provided that $\forall j \in J \exists i \in I[a_j \le b_i]$. We say that $\mathscr{P}(\omega)/\mathrm{fin}$ is (κ, ∞)-*distributive* iff every family of at most κ partitions has a common refinement. $\mathfrak{h}$ is the least κ such that $\mathscr{P}(\omega)/\mathrm{fin}$ is not (κ, ∞)-distributive, or 0 if there is no such κ.

Proposition 33.9. $\mathfrak{h} = \min\{|\mathscr{A}| :$ *$\mathscr{A}$ is a family of open weakly dense sets and* $\bigcap \mathscr{A} = \emptyset$.

Proof. Let $\mathfrak{h}' = \min\{|\mathscr{A}| : \mathscr{A}$ is a family of open weakly dense sets and $\bigcap \mathscr{A} = \emptyset\}$.

$\mathfrak{h}' \le \mathfrak{h}$: Suppose that $\mathscr{P}(\omega)/\mathrm{fin}$ is not (κ, ∞)-distributive. So there is a family $\mathscr{P}$ of partitions of $\mathscr{P}(\omega)/\mathrm{fin}$ such that $|\mathscr{P}| \le \kappa$ and $\mathscr{P}$ does not have a common refinement. Let Q be maximal subject to the following conditions: Q is a family of pairwise disjoint nonzero members of $\mathscr{P}(\omega)/\mathrm{fin}$ and $\forall a \in Q \forall P \in \mathscr{P} \exists b \in P[a \le b]$. Then Q is not a partition, as otherwise it would refine $\mathscr{P}$. Let $X \in [\omega]^\omega$ be such that $a \cdot [X] = 0$ for all $a \in Q$. Let $f : \omega \to X$ be a bijection. For each $P \in \mathscr{P}$ let

$$\mathscr{D}_P = \{Y \in [\omega]^\omega : \exists Z \in [\omega]^\omega([Z] \in P \text{ and } [f[Y]] \le [Z])\}.$$

We claim that $\mathscr{D}_P$ is dense open. It is clearly open. Now suppose that $W \in [\omega]^\omega$. Since $[f[W]] \ne 0$, there is a $Z \in [\omega]^\omega$ such that $[Z] \in P$ and $[f[W]] \cdot [Z] \ne 0$. Let $Y = W \cap f^{-1}[Z]$. Then $Y \in [\omega]^\omega$ and $[f[Y]] \le [Z]$. So $Y \in \mathscr{D}_P$. So the claim is established.

Now suppose that $\bigcap_{P\in\mathscr{P}}\mathscr{D}_P \neq \emptyset$; we will get a contradiction, and this will prove $\mathfrak{h} \leq \mathfrak{h}''$. Take $Y \in \bigcap_{P\in\mathscr{P}}\mathscr{D}_P$. For any $P \in \mathscr{P}$ we have $Y \in \mathscr{D}_P$, and so we can choose $Z \in [\omega]^\omega$ such that $[Z] \in P$ and $[f[Y]] \leq [Z]$. Then $[f[Y]] \leq [X]$ and $Q \cup \{[f[Y]]\}$ satisfies the conditions defining Q, contradiction.

$\mathfrak{h} \leq \mathfrak{h}'$: Suppose that $\mathscr{A}$ is a family of open weakly dense sets with empty intersection. Let $\kappa = |\mathscr{A}|$. We show that $\mathscr{P}(\omega)/\mathrm{fin}$ is not (κ, ∞)-distributive, and this will prove $\mathfrak{h} \leq \mathfrak{h}'$. If $\mathscr{D} \in \mathscr{A}$, let $P_{\mathscr{D}}$ be a maximal set of nonzero pairwise disjoint elements of $\mathscr{P}(\omega)/\mathrm{fin}$ such that $\forall a \in P \exists X \in \mathscr{D}(a \leq [X])$. Clearly $P_{\mathscr{D}}$ is a partition. Suppose that Q is a common refinement of $\{\mathscr{D}_P : P \in \mathscr{A}\}$; we will get a contradiction, which will finish the proof. Take any $[X] \in Q$. If $\mathscr{D} \in \mathscr{A}$, then there is a $[Y] \in P_{\mathscr{D}}$ such that $[X] \leq [Y]$. By the definition of $P_{\mathscr{D}}$, there is a $Z \in \mathscr{D}$ such that $[Y] \leq [Z]$. Then $X \in \mathscr{D}$ since $\mathscr{D}$ is open. So $X \in \bigcap\mathscr{A}$, contradiction. □

Proposition 33.10. $\mathfrak{t} \leq \mathfrak{h}$.

Proof. Suppose that $\mathscr{A}$ is a family of dense open sets with $|\mathscr{A}| < \mathfrak{t}$; we want to find a member of $\bigcap\mathscr{A}$. Write $\mathscr{A} = \{\mathscr{D}_\alpha : \alpha < \kappa\}$ with $\kappa < \mathfrak{t}$. We now define a sequence $\langle T_\alpha : \alpha \leq \kappa\}$ by recursion. Let $T_0 = \omega$. If $T_\alpha \in [\omega]^\omega$ has been chosen, let $T_{\alpha+1} \in \mathscr{D}_\alpha$ be a subset of T_α; this is possible because $\mathscr{D}_\alpha$ is dense. For $\alpha \leq \kappa$ limit, let T_α be a pseudo-intersection of $\{T_\beta : \beta < \alpha\}$; this is possible because $\alpha < \mathfrak{t}$. This finishes the construction.

By openness we have $T_\kappa \in \mathscr{A}$. □

Let $n \in \omega\backslash 1$, $k \in \omega\backslash 2$, $f : [\omega]^n \to k$, and $H \subseteq \omega$. Then H is *homogeneous for f* iff $f \restriction [H]^n$ is constant; it is *almost homogeneous for f* iff there is a finite set $F \subseteq H$ such that $H\backslash F$ is homogeneous for f.

Proposition 33.11. *If $n \in \omega\backslash 1$ and $k \in \omega\backslash 2$, then there is no $H \in [\omega]^\omega$ such that H is almost homogeneous for all $f \in {}^{[\omega]^n}k$.*

Proof. Suppose that such an H exists. Let $H = \{m_0, m_1, \ldots\}$ with $m_0 < m_1 < \cdots$. Define $f : [\omega]^n \to k$ by setting, for each $Y \in [\omega]^n$,

$$f(Y) = \begin{cases} 0 & \text{if } Y \notin [H]^n, \\ 0 & \text{if } Y \in [H]^n \text{ and } \min(Y) \text{ has the form } m_{2i} \text{ for some } i, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly H is not almost homogeneous for f, contradiction. □

We now define, for every positive integer n,

$$\mathrm{par}_n = \min\{|F| : F \subseteq {}^{[\omega]^n}2 : \neg\exists X \in [\omega]^\omega \forall f \in F[X \text{ is almost homogeneous for } f\}.$$

Proposition 33.12. $\mathfrak{s} = \mathrm{par}_1$.

Proof. First suppose that F satisfies the condition in the definiton of par_1, with $|F| = \mathrm{par}_1$. For each $f \in F$ let $P_f = \{m \in \omega : f(m) = 1\}$, and let $M = \{P_f : f \in F\}$. We claim that M is a splitting family; this will prove $\mathfrak{s} \leq \mathrm{par}_1$. So, suppose that $Y \in [\omega]^\omega$. Choose $f \in F$ such that Y is not almost homogeneous for f. Then $Y \cap P_f$ is infinite, as otherwise, since f has the constant value 0 on $Y \backslash P_f$, $Y \backslash P_f$ would be homogeneous for f. Similarly $Y \backslash P_f$ is infinite.

Second, suppose that S is a splitting family. Let F be the collection of all characteristic functions of members of S. So if we show that F satisfies the conditions in the definition of par_n, this will prove that $\mathfrak{s} \geq \mathrm{par}_1$. Suppose that $Y \in [\omega]^\omega$, and choose $M \in S$ which splits Y. Let f be the characteristic function of M. If N is any finite subset of Y, then $(Y \backslash N) \cap S$ and $(Y \backslash N) \backslash S$ are both infinite, and so f is not constant on $Y \backslash N$. $\square$

Proposition 33.13. *Suppose that $2 \leq k \in \omega$ and n is a positive integer. Then*

$$\mathrm{par}_n = \min\{|F| : F \subseteq {}^{[\omega]^n}k : \neg\exists X \in [\omega]^\omega \forall f \in F[X \text{ is almost homogeneous for } f]\}$$

Proof. If F is as in the definition of par_n, clearly F works as in the right side. So $\geq$ holds. Now suppose that F is as in the right side. For each $f \in F$ and $i < k$ define $g_{fi} : [\omega]^n \to 2$ by setting, for any $x \in [\omega]^n$,

$$g_{fi}(x) = \begin{cases} 0 & \text{if } f(x) = i, \\ 1 & \text{otherwise.} \end{cases}$$

Thus $G \stackrel{\text{def}}{=} \{g_{fi} : f \in F, i < k\}$ has the same size as X, so it suffices, in order to prove $\leq$, to show that G satisfies the condition in the definition of par_n. So suppose that $X \in [\omega]^\omega$ and X is almost homogeneous for each g_{fi}. We claim that X is almost homogeneous for each $f \in F$ (contradiction). For, take any $f \in F$. For each $i < k$ let M_i be a finite subset of X such that g_{fi} is constant on $[X \backslash M_i]^n$. We claim that f is constant on $\left[X \backslash \bigcup_{i<k} M_i\right]$, as desired. For, take any two $x, y \in X \backslash \bigcup_{i<k} M_i$. Say $f(x) = i$. Then since $x, y \in X \backslash M_i$, we get $g_{fi}(y) = g_{fi}(x) = 0$, and hence $f(y) = i$, as desired. $\square$

Example 33.14. *If n is a positive integer and $k \in \omega \backslash 2$, then there is a countable $F \subseteq {}^{[\omega]^n}k$ such that there is no $M \in [\omega]^\omega$ such that M is homogeneous for each $f \in F$.*

Proof. Let $[\omega]^n = \{a_\alpha : \alpha < \omega\}$, with $a_\alpha \neq a_\beta$ if $\alpha \neq \beta$. For each $\alpha < \omega$ we define $g_\alpha : [\omega]^n \to k$ by setting, for each $x \in [\omega]^n$,

$$g_\alpha(x) = \begin{cases} 1 & \text{if } x = a_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Let $F = \{g_\alpha : \alpha < \omega\}$. Suppose that $M \in [\omega]^\omega$. Choose $\alpha < \omega$ so that $a_\alpha \in [M]^n$, and choose $x \in [M]^n$ with $x \neq a_\alpha$. Then $g_\alpha(a_\alpha) = 1$ and $g_\alpha(x) = 0$, so M is not homogeneous for g_α. $\square$

Lemma 33.15. *If $m \leq n$, then* $\mathrm{par}_n \leq \mathrm{par}_m$. $\square$

Corollary 33.16. $\mathrm{par}_n \leq \mathfrak{s}$ *for every positive integer n.* $\square$

Theorem 33.17. $\omega < \mathfrak{s}$.

Proof. Suppose that $\{Y_i : i < \omega\}$ is a splitting family. It is clear how to construct by recursion an $\varepsilon \in {}^\omega 2$ such that $\bigcap_{j<i} Y_j^{\varepsilon(j)}$ is infinite for every $i < \omega$. Now construct $\langle m_i : i < \omega$ by letting $m_i \in \bigcap_{j<i} Y_j^{\varepsilon(j)} \backslash \{m_j : j < i\}$ for every $i < \omega$. Clearly $Z \stackrel{\text{def}}{=} \{m_i : i < \omega\}$ is not split by any Y_i. $\square$

Theorem 33.18. *For every integer $n \geq 2$, $\text{par}_n = \min(\mathfrak{b}, \mathfrak{s})$.*

Proof. By Corollary 33.16, $\text{par}_n \leq \mathfrak{s}$. Next we show that $\text{par}_n \leq \mathfrak{b}$. By Lemma 33.15 it suffices to take the case $n = 2$. Let B be an almost unbounded subset of ${}^\omega\omega$ with $|B| = \mathfrak{b}$. We may assume that each member of B is strictly increasing. For each $g \in B$ define $f_g : {}^{[\omega]^2} \to 2$ by setting for any $\{x, y\} \in [\omega]^2$ with $x < y$,

$$f_g(\{x,y\}) = \begin{cases} 1 & \text{if } g(x) < y, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that there is no set $H \in [\omega]^\omega$ which is almost homogeneous for all f_g's; this will prove $\text{par}_n \leq \mathfrak{b}$. Suppose that there is such an H.

(1) If $K \subseteq \omega$ and $f_g[[K]^2] \subseteq \{0\}$, then K is finite.

For, assume that $K \neq \emptyset$, and let x be its first element. If $z \in K\backslash\{x\}$, then $f_g(\{x, z\}) = 0$, and hence $z \leq g(x)$. So (1) holds.

Now we define $h, k : \omega \to \omega$ as follows. For any $x \in \omega$, $h(x)$ and $k(x)$ are the first and second elements of H which are greater than x. Now take any $g \in B$; we will show that $g <^* k$ (contradiction). Let F be a finite subset of H such that $f_g \restriction [H\backslash F]$ is constant. By (1), this constant value is 1. Thus if $x > F$, we have $h(x), k(x) \in H\backslash F$ and $h(x) < k(x)$, so $f_g(\{h(x), k(x)\}) = 1$, and hence $g(h(x)) < k(x)$. So $g(x) < g(h(x)) < k(x)$. Thus $g <^* k$, as desired.

So we have shown $\leq$ in the theorem.

For $\geq$, we prove the following statement by induction on n:

(2) If n is a positive integer, $\langle f_\xi : \xi < \kappa\rangle$ is a system of members of ${}^{[\omega]^n}2$, and $\kappa < \min(\mathfrak{s}, \mathfrak{b})$, then there is a set homogeneous for all of the f_ξ's.

This holds for $n = 1$ by Proposition 33.12. Now suppose that $n > 2$ and we know the result for $n - 1$. Suppose that $\langle f_\xi : \xi < \kappa\rangle$ is a sequence of members of ${}^{[\omega]^n}2$ with $\kappa < \min(\mathfrak{b}, \mathfrak{s})$. We want to find a set almost homogeneous for all of them. Let $c : \omega \to [\omega]^{n-1}$ be a bijection. For each $\xi < \kappa$ and $p \in \omega$ let

$$f_{\xi,p}(m) = \begin{cases} f_\xi(c(p) \cup \{m\}) & \text{if } m \notin c(p), \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\{f_{\xi,p} : \xi < \kappa, p \in \omega\}$ is a family of less than $\mathfrak{s}$ functions mapping ω into 2 (using Theorem 33.17). Hence by Proposition 33.12 there is an infinite set A almost homogeneous for all of them. So for each $\xi < \kappa$ and $p \in \omega$ we can choose $g_\xi(p) \in \omega$ and $j_\xi(p) \in 2$ such that $f_{\xi,p}(x) = j_\xi(p)$ for all $x \in A$ such that $x \geq g_\xi(p)$. Write $A = \{m_i : i < \omega\}$, m strictly increasing. For each $a \in [\omega]^{n-1}$ let $c_\xi(a) = j_\xi(c^{-1}(\{m_i : i \in a\}))$. By the inductive hypothesis, let M be an infinite set almost homogeneous for each j_ξ. Choose b_ξ and k_ξ such that c_ξ takes on the constant value k_ξ on $[M\backslash b_\xi]^{n-1}$. Let $B = \{m_i : i \in M\}$.

(3) If $a \in [B\backslash m_{b_\xi}]^{n-1}$ then $j_\xi(c^{-1}(a)) = k_\xi$.

In fact, write $a = \{m_i : i \in s\}$. Then $s \subseteq M$, and $m_i \geq m_{b_\xi}$ and hence $i \geq b_\xi$, for each $i \in s$. So $s \in [M\backslash b_\xi]^{n-1}$, so $k_\xi = c_\xi(s) = j_\xi(c^{-1}(\{m_i : i \in s\})) = j_\xi(c^{-1}(a))$. So (3) holds.

Since $\kappa < \mathfrak{b}$, choose h such that $g_\xi \leq^* h$ for all $\xi < \kappa$. Choose a_ξ so that $g_\xi(p) \leq h(p)$ for all $p \geq a_\xi$.

Now we define $x_0 < x_1 < \cdots$ in B by recursion. Suppose that x_s has been defined for all $s < t$. Choose $x_t \in B$ so that $x_s < x_t$ for all $s < t$, and also $h(p) < x_t$ for all p such that $c(p) \in [\{x_0, \ldots, x_{t-1}\}]^{n-1}$. Let $H = \{x_i : i < \omega\}$. We claim that H is almost homogeneous for each f_ξ. Let $\xi < \kappa$. Choose t such that $t > c(p)$ for each $p < a_\xi$, and also $t \geq m_{b_\xi}$. Suppose that $a \in [H\backslash t]^n$. Let m be the largest element of a, and let $p = c^{-1}(a\backslash\{m\})$. Then $c(p)$ consists of members of H which are $\geq t$, so $a_\xi \leq p$. Thus $g_\xi(p) \leq h(p) < m$. Also note that $a\backslash\{m\} \in [B\backslash m_{b_\xi}]^{n-1}$. So

$$f_\xi(a) = f_{\xi,p}(m) = j_\xi(p) = j_\xi(c^{-1}(a\backslash\{m\})) = k_\xi. \qquad \square$$

Proposition 33.19. $\mathfrak{h} \leq \mathfrak{b}, \mathfrak{s}$.

Proof. By Proposition 33.18 it suffices to show that $\mathfrak{h} \leq \mathrm{par}_2$. So, suppose that $F \subseteq {}^{[\omega]^2}2$ and $|F| < \mathfrak{h}$; we want to find $X \in [\omega]^\omega$ which is almost homogeneous for all $f \in F$. For each $f \in F$ let $\mathscr{D}_f = \{X \in [\omega]^\omega : X \text{ is almost homogeneous for } f\}$. We claim that each $\mathscr{D}_f$ is dense open. For openness, suppose that $X \subseteq^* Y \in \mathscr{D}_f$. Choose G, H finite such that $f \restriction [Y\backslash G]^2$ is constant and $X\backslash Y = H$. Then $f \restriction [X\backslash(G \cup H)]^2 \subseteq f \restriction [Y\backslash G]^2$ is constant; so $X \in \mathscr{G}_f$. For denseness, take any $Y \in [\omega]^\omega$. Then $f \restriction [Y]^2 : [Y]^2 \to 2$, so by Ramsey's theorem there is an infinite $X \subseteq Y$ such that $f \restriction [X]^2$ is constant; so $X \in \mathscr{D}_f$, as desired.

Take $H \in \bigcap_{f\in F} \mathscr{D}_f$. Clearly H is almost homogeneous for all $f \in F$, as desired. $\square$

A family $\mathscr{F} \subseteq [\omega]^\omega$ has the *strong finite intersection property*, SFIP, iff the intersection of any finite subset of $\mathscr{F}$ is infinite. We define

$$\mathfrak{p} = \min\{|\mathscr{F}| : \mathscr{F} \subseteq [\omega]^\omega,\ \mathscr{F} \text{ has SFIP, but has no pseudo-intersection}\}.$$

Proposition 33.20. $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{t}$.

Proof. Obviously $\mathfrak{p} \leq \mathfrak{t}$. Now suppose that $\mathscr{F} = \{A_m : m \in \omega\}$ has SFIP; we show that it has a pseudo-intersection, thus proving the first inequality in the proposition. Let $B_m = A_0 \cap \ldots \cap A_m$ for all $m \in \omega$. Each B_m is infinite. By the argument for the proof of Theorem 33.17, $\{B_m : m \in \omega\}$ has a pseudo-intersection. Hence so does $\mathscr{F}$. $\square$

Sets $a, b \in [\omega]^\omega$ are *almost disjoint* iff $A \cap B$ is finite. A set $A \subseteq [\omega]^\omega$ is *maximal almost disjoint*, MAD, iff any two distinct members of A are almost disjoint, and A is maximal under inclusion with this property. $\mathfrak{a}$ is the least size of a MAD family.

Proposition 33.21. $\mathfrak{b} \leq \mathfrak{a}$.

Proof. Suppose that $\mathscr{A}$ is an infinite MAD family. Let $\langle C_n : n \in \omega \rangle$ be a one-one enumeration of some of the members of $\mathscr{A}$. We define

$$D_0 = C_0 \cup \left(\omega \backslash \bigcup_{n \in \omega} C_n \right);$$
$$D_{n+1} = C_{n+1} \backslash \bigcup_{m \leq n} C_m.$$

Clearly $\langle D_n : n \in \omega \rangle$ is a partition of ω into infinite subsets. For each $n \in \omega$, let $f_n : D_n \to \omega$ be a bijection. Let $\mathscr{A}' = \mathscr{A} \backslash \{C_n : n \in \omega\}$. For each $A \in \mathscr{A}$ define $g_A : \omega \to \omega$ by letting $g_A(n)$ be the least natural number such that $\forall m \in A \cap D_n(f_n(m) < g_A(n))$, for any $n \in \omega$. We claim that $\{g_A : A \in \mathscr{A}'\}$ is unbounded, as desired.

To see this, suppose that $g_A \leq^* h$ for all $A \in \mathscr{A}'$. Define $X = \{f_n^{-1}(h(n)) : n \in \omega\}$. Thus $|D_n \cap X| = 1$ for all $n \in \omega$. Hence X is infinite. Now take any $A \in \mathscr{A}'$; we show that $X \cap A$ is finite. Choose $p \in \omega$ so that $g_A(n) \leq h(n)$ for all $n \geq p$. Then, we claim,

$$(*) \qquad A \cap X \subseteq \bigcup_{n<p} (A \cap D_n).$$

(Hence $A \cap X$ is finite, as desired.) To prove $(*)$, suppose that $m \in A \cap X$. Choose $n \in \omega$ so that $m = f_n^{-1}(h(n))$. Then $m \in A \cap D_n$, so $f_n(m) < g_A(n)$. But $f_n(m) = h(n)$, so it follows that $n < p$. $\square$

A set $X \subseteq [\omega]^\omega$ is *independent* iff for any finite disjoint $F, G \subseteq X$ we have $\bigcap F \cap \bigcap_{a \in G}(\omega \backslash a) \neq \emptyset$. We let $\mathfrak{i}$ be the least size of a maximal independent set.

Proposition 33.22. $\mathfrak{r} \leq \mathfrak{i}$.

Proof. Let $\mathscr{I} \subseteq [\omega]^\omega$ be maximal independent, with size $\mathfrak{i}$. Let R be the set of all monomials over $\mathscr{I}$. By maximality, R satisfies the conditions defining $\mathfrak{r}$. $\square$

Lemma 33.23. *Suppose that $\langle C_n : n \in \omega \rangle$ is a sequence of infinite subsets of ω such that $C_n \subseteq^* C_m$ if $m < n$. Suppose that $\mathscr{A}$ is a family of size less than $\mathfrak{d}$ of infinite subsets of ω, each of which has infinite intersection with each C_n. Then $\{C_n : n \in \omega\}$ has a pseudo-intersection B that has infinite intersection with each member of $\mathscr{A}$.*

Proof. Let $C'_n = \bigcap_{m \leq n} C_m$ for all $n \in \omega$. If $A \in \mathscr{A}$, then

$$A \cap C'_n = (A \cap C_n) \backslash \bigcup_{m<n} (C_n \backslash C_m),$$

so $A \cap C'_n$ is still infinite. So it suffices to work with the C'_n's rather than the C_n's.

For each $h \in {}^\omega\omega$ let $B_h = \bigcup_{n \in \omega}(C'_n \cap h(n))$. Then $B_h \backslash C'_n \subseteq \bigcup_{m<n} h(m)$, so that $B_h \subseteq^* C'_n$. Hence it suffices to find $h \in {}^\omega\omega$ so that B_h has infinite intersection with each member of $\mathscr{A}$.

For each $A \in \mathscr{A}$ and each $n \in \omega$, let $f_A(n)$ be the n-th element of the infinite set $A \cap C'_n$ (starting the numbering at 0). Since $|\mathscr{A}| < \mathfrak{d}$, the set $\{f_A : A \in \mathscr{A}\}$ is not almost dominating, and so we can choose $h \in {}^\omega\omega$ such that $h \not\leq^* f_A$ for all $A \in \mathscr{A}$. Thus for each $A \in \mathscr{A}$, the set $\{n \in \omega : h(n) > f_A(n)\}$ is infinite, so that $h(n) \cap A \cap C'_n$ has at least n elements for infinitely many n, and so $B_h \cap A$ is infinite, as desired. □

Proposition 33.24. $\mathfrak{d} \leq \mathfrak{i}$.

Proof. Suppose that $\mathscr{I} \subseteq [\omega]^\omega$ is independent and $|\mathscr{I}| < \mathfrak{d}$; we show that it is not maximal.

Let $\langle D_n : n \in \omega\rangle$ be a one-one enumeration of some of the elements of $\mathscr{I}$, and let $\mathscr{I}' = \mathscr{I}\backslash\{D_n : n \in \omega\}$. For each $\varepsilon \in {}^\omega 2$ and each $n \in \omega$ define

$$C_n^\varepsilon = \bigcap_{k<n} D_k^{\varepsilon(k)}.$$

Let

$$\mathscr{A} = \left\{\bigcap_{X\in F} X \cap \bigcap_{X\in G}(\omega\backslash X) : F, G \text{ are finite disjoint subsets of } \mathscr{I}\right\}.$$

We apply Lemma 33.23 to $\langle C_n^\varepsilon : n \in \omega\rangle$ and $\mathscr{A}$ to get a pseudo-intersection B^ε of $\{C_n^\varepsilon : n \in \omega\}$ which has infinite intersection with each element of $\mathscr{A}$. Thus

(1) $B^\varepsilon \subseteq^* \bigcap_{k<n} D_k^{\varepsilon(k)}$ for all $n \in \omega$.

(2) B^ε has infinite intersection with each element of $\mathscr{A}$.

(3) $B^\varepsilon \cap B^\delta$ is finite for distinct $\varepsilon, \delta \in {}^\omega 2$.

This is clear from (1).

(4) There are countable disjoint $Q, Q' \subseteq {}^\omega 2$ such that for every $p \in {}^{<\omega}2$ there are $f \in Q$ and $g \in Q'$ such that $p \subseteq f$ and $p \subseteq g$.

In fact, enumerate ${}^{<\omega}2$ as $\langle p_n : n \in \omega\}$. Now we define functions $f_n, g_n \in {}^\omega 2$ by induction as follows: they are distinct elements of the set

$$\{h \in {}^\omega 2 : p_n \subseteq h\}\backslash\{f_m, g_m : m < n\}.$$

Then we let $Q = \{f_n : n \in \omega\}$ and $Q' = \{g_n : n \in \omega\}$. Clearly (4) holds.

(5) There exists $\langle E^\varepsilon : \varepsilon \in Q \cup Q'\rangle$ such that the E^ε's are pairwise disjoint, $E^\varepsilon \subseteq B^\varepsilon$, and $B^\varepsilon\backslash E^\varepsilon$ is finite.

To prove this, enumerate $Q \cup Q'$ as $\langle \varepsilon_n : n \in \omega\rangle$ without repetitions, and let $E^{\varepsilon_n} = B^{\varepsilon_n}\backslash\bigcup_{m<n} B^{\varepsilon_m}$ for all m; clearly (5) then holds.

Now we define

$$Z = \bigcup_{\varepsilon\in Q} E^\varepsilon, \quad \text{and} \quad Z' = \bigcup_{\varepsilon\in Q'} E^\varepsilon.$$

(6) Z has infinite intersection with each set $\left(\bigcap_{X\in F} X\right) \cap \left(\bigcap_{X\in G}(\omega\backslash X)\right)$ with F, G finite disjoint subsets of $\mathscr{I}$.

In fact, take such F, G. Let $F' = F \cap \mathscr{I}'$ and $G' = G \cap \mathscr{I}'$. Choose $n \in \omega$ such that for all $k \in \omega$, if $D_k \in F \cup G$ then $k < n$. Define $p \in {}^n2$ by setting, for each $k < n$,

$$p(k) = \begin{cases} 1 & \text{if } D_k \in F, \\ 0 & \text{otherwise.} \end{cases}$$

Choose $\varepsilon \in Q$ such that $p \subseteq \varepsilon$. Then

$$\begin{aligned}
\left(\bigcap_{X\in F} X\right) \cap \left(\bigcap_{X\in G}(\omega\backslash X)\right) &= \left(\bigcap_{X\in F'} X\right) \cap \left(\bigcap_{X\in G'}(\omega\backslash X)\right) \cap \left(\bigcap_{D_k\in F\cup G} D_k^{\varepsilon(k)}\right) \\
&\supseteq \left(\bigcap_{X\in F'} X\right) \cap \left(\bigcap_{X\in G'}(\omega\backslash X)\right) \cap \left(\bigcap_{k<n} D_k^{\varepsilon(k)}\right) \\
&\supseteq^* \left(\bigcap_{X\in F'} X\right) \cap \left(\bigcap_{X\in G'}(\omega\backslash X)\right) \cap B^{\varepsilon}. \\
&\supseteq^* \left(\bigcap_{X\in F'} X\right) \cap \left(\bigcap_{X\in G'}(\omega\backslash X)\right) \cap E^{\varepsilon}.
\end{aligned}$$

The last intersection is infinite, and is a subset of Z since $\varepsilon \in Q$, as desired; (6) holds.

Similarly,

(7) Z' has infinite intersection with each set $\bigcap_{X\in F} X \cap \bigcap_{X\in G}(\omega\backslash X)$, with F, G finite disjoint subsets of $\mathscr{I}$.

Since $\omega\backslash Z \supseteq Z'$, this finishes the proof. $\square$

$\mathfrak{u}$ is the least size of a set which filter-generates a nonprincipal ultrafilter on $\mathscr{P}(\omega)/\text{fin}$.

Proposition 33.25. $\mathfrak{r} \leq \mathfrak{u}$.

Proof. Let X filter-generate a nonprincipal ultrafilter U. We may assume that X is closed under $\cap$. For any $a \subseteq \omega$, either $a \in U$ or $(\omega\backslash a) \in U$; so there is a $b \in X$ such that $b \subseteq a$ or $b \subseteq (\omega\backslash a)$. $\square$

A set $X \subseteq [\omega]^\omega$ is *ideal independent* iff $\forall F \in [X]^{<\omega}\forall a \in X\backslash F[a\backslash \bigcup F$ is infinite]. Let $\mathfrak{s}_{\mathrm{mm}} = \min\{|X| : X$ is maximal ideal independent$\}$.

Proposition 33.26. $\mathfrak{r} \leq \mathfrak{s}_{\mathrm{mm}}$.

Proof. Denote by $[x]$ the equivalence class of $x \subseteq \omega$ in the algebra $A \overset{\text{def}}{=} \mathscr{P}(\omega)/\text{fin}$. Suppose that $X \subseteq A$ is maximal ideal independent. Let

$$Y = X \cup \{-\sum F : F \in [X]^{<\omega}\} \cup \{b \cdot -\sum F : b \notin F,\ F \cup \{b\} \in [X]^{<\omega}\}.$$

Clearly the members of Y are nonzero. We claim that Y is weakly dense in A. For, suppose that $a \in A\backslash X$. Then $X \cup \{a\}$ is no longer ideal independent, so we have two cases.

Case 1. $a \leq \sum F$ for some $F \in [X]^{<\omega}$. Then $-\sum F \leq -a$, as desired.

Case 2. There exist a finite subset F of X and a $b \in X\backslash F$ such that $b \leq \sum F + a$. Then $b \cdot -\sum F \leq a$, as desired. □

Theorem 33.27. $\mathfrak{d} \leq \mathfrak{s}_{\mathrm{mm}}$.

Proof. Suppose, in order to get a contradiction, that $\mathfrak{s}_{\mathrm{mm}} < \mathfrak{d}$. Let $X \subseteq [\omega]^\omega$ be such that $\{[x] : x \in X\}$ is maximal ideal independent and $\omega \leq |X| < \mathfrak{d}$, with $[x] \neq [y]$ for $x, y \in X$ and $x \neq y$, where $[x]$ is the equivalence class of x mod finite. Let $\langle A_i : i \in \omega\rangle$ be a sequence of distinct elements of X. Define $A'_i = A_i \cup \{i\}$. Let

$$X' = (X\backslash\{A_i : i \in \omega\}) \cup \{A'_i : i \in \omega\}$$

Define $C_i = A'_i \backslash \bigcup_{j<i} A'_j$ for each $i \in \omega$. By ideal independence, each C_i is infinite.

(1) If $F \in [X']^{<\omega}$, $B \in X'\backslash(F \cup \{A'_i : i \in \omega\})$, and $n \in \omega$, then there is a $j \geq n$ such that $C_j \cap B\backslash \bigcup F \neq \emptyset$.

In fact, otherwise we have $\left(\bigcup_{j\geq n} C_j\right) \cap B\backslash \bigcup F) = \emptyset$, hence $B\backslash \left(\bigcup_{j<n} C_j \cup \bigcup F\right) = \emptyset$, hence $B\backslash \left(\bigcup_{j<n} A_j \cup \bigcup F\right) = \emptyset$, contradicting ideal independence.

By (1) we can make the following definition. For $F \in [X']^{<\omega}$, $B \in X'\backslash(F \cup \{A'_i : i \in \omega\})$ and $n \in \omega$, let

$$\varphi_{FB}(n) = \min\{k \in \omega : \exists j \geq n[C_j \cap B \cap k\backslash \bigcup F \neq \emptyset]\}.$$

The number of pairs (F, B) as above is less than $\mathfrak{d}$, so the set of all such functions φ_{FB} is not dominating. Hence there is a function $h_0 \in {}^\omega\omega$ not dominated by any of them. We may assume that h_0 is strictly increasing. For each $n \in \omega$ let $D_n = C_n\backslash h_0(n)$.

(2) If $F \in [X']^{<\omega}$ and $n \in \omega$, then there is a $j \geq n$ such that $D_j\backslash \bigcup F \neq \emptyset$.

In fact, otherwise we have $\left(\bigcup_{j\geq n} D_j\right) \backslash \bigcup F = \emptyset$, i.e., $\left(\bigcup_{j\geq n}(C_j\backslash h_0(j))\right) \backslash \bigcup F = \emptyset$. Hence $\left(\bigcup_{j\geq n}(C_j\backslash h_0(j))\right) \subseteq \bigcup F$. Choose $j \geq n$ so that $A'_j \notin F$. Then $C_j\backslash h_0(j) \subseteq \bigcup F$, i.e., $(A'_j\backslash \bigcup_{k<j} A'_k)\backslash h_0(j) \subseteq \bigcup F$. Hence $A'_j \subseteq^* \bigcup_{k<j} A'_k \cup \bigcup F$, contradicting ideal independence.

By (2) we can make the following definition. For $F \in [X']^{<\omega}$ and $n \in \omega$ let

$$\varphi'_F(n) = \min\{k : \exists j \geq n[D_j \cap k\backslash \bigcup F \neq \emptyset]\}.$$

Again, there are fewer than $\mathfrak{d}$ of these functions φ'_F, so there is a function $k \in {}^\omega\omega$ not dominated by any of them. For any $n \in \omega$ let

$$h_1(n) = \min\{h_1(n-1) + 1, k(n) + 1, \min(C_n\backslash h_0(n)) + 1),$$

with $h_1(n-1)+1$ omitted if $n=0$.

Now let

$$Y = \bigcup_{n\in\omega}[D_n \cap h_1(n)].$$

We claim that $[Y] \notin \{[a] : a \in X'\}$ and $\{[Y]\} \cup \{[a] : a \in X'\}$ is ideal independent. (Contradiction.)

(3) $\forall F \in [X']^{<\omega}[Y \not\subseteq^* \bigcup F]$; in particular, $[Y] \notin \{[a] : a \in X'\}$.

In fact, let $n \in \omega$; we will find $j \geq n$ such that $D_j \cap h_1(j)\backslash \bigcup F \neq \emptyset$. We have $k \not\leq^* \varphi'_F$, so choose $m \geq n$ so that $\varphi'_F(m) < k(m)$. By the definition of $\varphi'_F(m)$, there is a $j \geq m$ such that $D_j \cap \varphi'_F(m)\backslash \bigcup F \neq \emptyset$. We have $\varphi'_F(m) < k(m) < h_1(m) < h_1(j)$, so $D_j \cap h_1(j)\backslash \bigcup F \neq \emptyset$. This proves (3).

(4) For all $F \in [X']^{<\omega}$ and all $n \in \omega$ with $A'_n \notin F$ we have $A'_n \not\subseteq^* Y \cup \bigcup F$.

In fact, assume otherwise. Now for $m > n$ we have $A_n \cap C_m = \emptyset$, and hence $A_n \cap D_m = \emptyset$. Hence $A_n \subseteq^* \bigcup_{m\leq n}[D_n \cap h_1(n)] \cup \bigcup F$. Since $\bigcup_{m\leq n}[D_n \cap h_1(n)]$ is finite, it follows that $A_n \subseteq^* \bigcup F$, contradiction.

(5) If $F \in [X']^{<\omega}$ and $B \in X'\backslash(F \cup \{A'_n : n \in \omega\})$, then $B \not\subseteq^* Y \cup \bigcup F$.

In fact, let $n \in \omega$. We will find $j \geq n$ such that $C_j \cap B \cap h_0(j)\backslash \bigcup F \neq \emptyset$. Since $C_j \cap B \cap h_0(j)\backslash \bigcup F \subseteq B\backslash(Y \cup \bigcup F)$, this suffices to prove (5). Since h_0 is not dominated by φ_{FB}, choose $m \geq n$ such that $\varphi_{FB}(m) < h_0(m)$. Then by definition of φ_{FB} there is a $j \geq m$ such that $C_j \cap B \cap \varphi_{FB}(m)\backslash \bigcup F \neq \emptyset$. So $C_j \cap B \cap h_0(m)\backslash \bigcup F \neq \emptyset$, hence $C_j \cap B \cap h_0(j)\backslash \bigcup F \neq \emptyset$.

A *free sequence* is a sequence $\langle a_\xi : \xi < \alpha\rangle$ of members of $[\omega]^\omega$ such that for any finite $F, G \subseteq \alpha$ such that $\forall \xi \in F \forall \eta \in G[\xi < \eta]$ we have $\bigcap_{\xi\in F} a_\xi \cap \bigcap_{\eta\in G}(\omega\backslash a_\eta) \neq \emptyset$. We let $\mathfrak{f}$ be the least $|\alpha|$ such that there is a maximal free sequence of infinite length α.

Theorem 33.28. $\mathfrak{r} \leq \mathfrak{f}$.

Proof. Suppose that $\langle a_\xi : \xi < \alpha\rangle$ is a maximal free sequence in $\mathfrak{P}(\omega)/\mathrm{fin}$. We claim that

$$\left\{\prod_{\xi\in F} a_\xi : F \in [\alpha]^{<\omega}\right\} \cup \left\{\prod_{\xi\in F} a_\xi \cdot \prod_{\xi\in G} -a_\xi : F, G \in [\alpha]^{<\omega}, F < G\right\}$$

is weakly dense. To see this, let $b \in \mathfrak{P}(\omega)/\mathrm{fin}.$ of A. If $\prod_{\xi\in F} a_\xi \cdot \prod_{\xi\in G} -a_\xi \cdot -b = 0$ for some finite $F < G$, this is as desired. If there is a finite $F \subseteq \alpha$ such that $\prod_{\xi\in F} a_\xi \cdot b = 0$, this is also as desired. □

A forcing order P is *centered* iff for every finite $F \subseteq P$ there is a $p \in P$ such that $\forall q \in F[p \leq q]$. P is *σ-centered* iff it is a countable union of centered subsets. Clearly a σ-centered forcing order is ccc. We define

$$\begin{aligned}\mathfrak{m}_\sigma = \min\{\kappa :\ & \text{there is a } \sigma\text{-centered forcing order } P \text{ and a system}\\ & \mathscr{D} \text{ of dense subsets of } P \text{ with } |P| \leq \kappa \text{ such that there}\\ & \text{does not exist a filter on } P \text{ which intersects each member of } \mathscr{D}\}\end{aligned}$$

Lemma 33.29. *Let κ be an infinite cardinal. Suppose that for every σ-centered forcing order P of size $\leq \kappa$ and for every family $\mathscr{A}$ of dense subsets of P with $|\mathscr{A}| \leq \kappa$ there is a filter on P which intersects each member of $\mathscr{A}$.*

Then for every σ-centered forcing order P and for every family $\mathscr{A}$ of dense subsets of P with $|\mathscr{A}| \leq \kappa$ there is a filter on P which intersects each member of $\mathscr{A}$.

Proof. We prove the contrapositive. Thus suppose that P is a σ-centered forcing order, $\mathscr{A}$ is a family of dense subsets of P with $|\mathscr{A}| \leq \kappa$, and there is no filter on P which intersects each member of $\mathscr{A}$. By the downward Löwenheim, Skolem, Tarski theorem, let $\mathbb{Q} \stackrel{\text{def}}{=} (Q, \leq, 1, D')_{D \in \mathscr{A}}$ be an elementary substructure of $\mathbb{P} \stackrel{\text{def}}{=} (P, \leq, 1, D)_{D \in \mathscr{A}}$ with $|Q| \leq \kappa$. Write $P = \bigcup \mathscr{B}$, where $\mathscr{B}$ is countable and each of its members is centered. Then $Q = \bigcup_{A \in \mathscr{B}} (A \cap Q)$. Moreover, if $A \in \mathscr{B}$ and $q_0, \ldots, q_{m-1} \in A \cap Q$, then the formula $\exists x[x \leq q_0 \wedge \ldots \wedge z \leq q_{m-1}]]$ holds in $\mathbb{P}$, and hence in $\mathbb{Q}$. Hence Q is σ-centered. Now take any $D \in \mathscr{A}$. Then $\forall x \exists y \in D[y \leq x]$ holds in $\mathbb{P}$, and hence in $\mathbb{Q}$. It follows that $D \cap Q$ is dense in Q.

Suppose that G is a filter on Q such that $G \cap D \cap Q \neq \emptyset$ for all $D \in \mathscr{A}$. Let $G' = \{p \in P : \exists q \in G[q \leq p]\}$. Then G' is a filter on P, and $G' \cap D \neq \emptyset$ for all $D \in \mathscr{A}$, contradiction. Thus there does not exist such a G. So we have shown that the hypothesis of the Lemma is false. □

A subset X of a forcing order P is *open* iff $\forall p \in X \forall q \leq p[q \in X]$.

A subset X of a forcing order P is *linked* iff any two members of X are compatible.

Lemma 33.30. *Let κ be an infinite cardinal. Suppose that P is a forcing order, and for every family $\mathscr{A}$ of dense open subsets of P with $|\mathscr{A}| \leq \kappa$ there is a linked $G \subseteq P$ such that $G \cap D \neq \emptyset$ for all $D \in \mathscr{A}$.*

Then for every family $\mathscr{A}$ of dense subsets of P with $|\mathscr{A}| \leq \kappa$ there is a filter $G \subseteq P$ such that $G \cap D \neq \emptyset$ for all $D \in \mathscr{A}$.

Proof. First note:

(1) If $D \subseteq P$ is dense, and $E \subseteq D$ is maximal among subsets of D consisting of pairwise incompatible elements, then E is maximal incompatible in P.

In fact, suppose that $p \notin E$, and p is incompatible with each element of E. Choose $q \in D$ with $q \leq p$. Then q is incompatible with each element of E and $q \in D$, contradiction.

Now assume the hypothesis of the lemma, and suppose that $\mathscr{A}$ is a family of dense subsets of P with $|\mathscr{A}| \leq \kappa$. For each $X \subseteq P$ let $X{\downarrow} = \{p \in P : \exists q \in X[p \leq q]\}$. For each $D \in \mathscr{A}$ the set $D{\downarrow}$ is dense open. For each $D \in \mathscr{A}$ let $D' \subseteq (D{\downarrow})$ be maximal among the subsets of $D{\downarrow}$ consisting of pairwise incompatible elements. By (1), each D' is maximal incompatible in P. Clearly $D'{\downarrow}$ is dense open. Let $F_0 = \{D'{\downarrow} : D \in \mathscr{A}\}$. Now suppose that F_n has been defined and consists of dense open sets. For each pair $(B, C) \in F_n \times F_n$, the intersection $B \cap C$ is dense open; let E_{BC} be maximal among subsets of $B \cap C$ consisting of pairwise incompatible elements. Then E_{BC} is maximal incompatible in P. Let $F_{n+1} = F_n \cup \{E_{BC}{\downarrow} : (B, C) \in F_n \times F_n\}$. Let $H = \bigcup_{n \in \omega} F_n$. Note that if $B, C \in H$, then there is a dense open set $D \in H$ such that $D \subseteq B \cap C$. The members of H are dense open. Each $D \in H$ has the form $M_D\downarrow$ for some maximal incompatible set M_D. By the hypothesis of the Lemma let $L \subseteq P$ be linked with $L \cap D \neq \emptyset$ for all $D \in H$. Choose $p_D \in L \cap D$ for all $D \in H$. Then choose $q_D \in M_D$ such that $p_D \leq q_D$. Then $\{p_D : D \in H\}$ is linked, and so also $\{q_D : D \in H\}$ is linked.

(2) $\forall D, E \in H \exists X \in H[q_X \leq q_D, q_E]$.

For, choose $X \in H$ such that $X \subseteq D \cap E$. Suppose that $q_X \not\leq q_D$. Now $q_X \in M_X \subseteq X \subseteq D \cap E$, so $q_X \in D = (M_D \downarrow)$. Hence choose $t \in M_D$ such that $q_X \leq t$. Now $q_D, t \in M_D$, $q_X \leq t$, and $q_X \not\leq q_D$. So $q_X \neq t$. Hence q_X and t are incompatible. But q_X and Q_D are compatible, and $q_D \leq t$, contradiction. Hence $q_X \leq q_D$. Similarly $q_X \leq q_E$.

Now let $G = \{s \in P : \exists D \in H[q_D \leq s]\}$. Then by (2), G is a filter. If $D \in \mathscr{A}$, then $D'\downarrow \in F_0 \subseteq H$. Hence $q_{D'\downarrow} \in G$. Now $q_{D'\downarrow} \in M_{D'\downarrow} = D'\downarrow \subseteq D\downarrow$, so there is an $s \in D$ such that $q_{D'\downarrow} \leq s$. Thus $s \in G \cap D$. $\square$

Theorem 33.31. $\mathfrak{m}_\sigma = \mathfrak{p}$.

Proof. First we prove that $\mathfrak{m}_\sigma \leq \mathfrak{p}$; in fact, fix any $\kappa < \mathfrak{m}_\sigma$; we show that $\kappa < \mathfrak{p}$. Let $\mathscr{A} \subseteq [\omega]^\omega$ have SFIP with $|\mathscr{A}| = \kappa$; we want to find a pseudo-intersection of $\mathscr{A}$. Let

$$P = \{(s, W) : s \in [\omega]^{<\omega},\ W \in [\mathscr{A}]^{<\omega}\}$$
$$(s, W) \leq (s', W') \text{ iff } s \supseteq s',\ W \supseteq W',\ \forall x \in W'[(s \backslash s') \subseteq x].$$

For each $p \in P$ let $p = (s_p, W_p)$. Note that if $t \in [\omega]^{<\omega}$, then $\{p \in P : s_p = t\}$ is centered; so P is σ-centered.

For each $n \in \omega$ let $D_n = \{p \in P : |s_p| \geq n\}$. Then each D_n is dense. For, suppose that $p \in P$. Then $\bigcap W_p$ is infinite by the SFIP, so choose $t \in [\bigcap W_p]^n$. Then $(s_p \cup t, W_p) \leq p$ and $(s_p \cup t, W_p) \in D_n$.

For each $a \in \mathscr{A}$ let $E_a = \{p \in P : a \in W_p\}$. Then E_a is dense. For suppose that $p \in P$. Then $(s_p, W_p \cup \{a\}) \leq p$ and $(s_p, W_p \cup \{a\}) \in E_a$.

Now let G be a filter on P intersecting each of these dense sets. Let $b = \bigcup_{p \in G} s_p$. Then b is infinite since $D_n \cap G \neq \emptyset$ for all $n \in \omega$. We claim that b is a pseudo-intersection of $\mathscr{A}$. For, suppose that $a \in \mathscr{A}$; we want to show that $b \backslash a$ is finite. Fix $p \in E_a \cap G$. We claim that $b \backslash a \subseteq s_p$ (as desired). It suffices to show that $b \backslash s_p \subseteq a$. Take any $x \in b \backslash s_p$. Say $x \in s_q$ with $q \in G$. Choose $r \in G$ such that $r \leq p, q$. Since $r \leq q$, we have $x \in s_r$. Since $r \leq p$ and $a \in W_p$ we have $s_r \backslash s_p \subseteq a$. Hence $x \in a$.

This proves that $\mathfrak{m}_\sigma \leq \mathfrak{p}$.

For the other direction, fix $\kappa < \mathfrak{p}$. To show that $\kappa < \mathfrak{m}_\sigma$ it suffices by Lemmas 13.27 and 13.28 to take a σ-centered forcing order P of size $\leq \kappa$, let $\langle D_\alpha : \alpha < \kappa \rangle$ be a system of dense open subsets of P, and find a linked $G \subseteq P$ which intersects each D_α.

If there is a $p \in P$ such that $\forall q, r \leq p$ q and r are compatible, let $G = \{q \in P : p \leq q\}$; then G is as desired. So we assume that $\forall p \in P \exists q, r \leq p[$ q and r are incompatible$]$.

Let $P = \bigcup_{l \in \omega} C_l$, where each C_l is centered. We may assume that $1 \in C_l$ for each $l \in \omega$. For $\alpha < \kappa$ and $p \in \mathbb{P}$, let

(1) $$B_\alpha(p) = \{l \in \omega : D_\alpha \cap C_l \cap p\downarrow \neq \emptyset\}.$$

Now we claim that for each $m \in \omega$, the set $\{B_\alpha(p) : p \in C_m \text{ and } \alpha < \kappa\}$ has SFIP. For, suppose that $\langle p_i : i \in n\rangle$ is a system of elements of C_m and $\langle \alpha_i : i < n\rangle$ is a system of members of κ; we want to show that $\bigcap_{i<n} B_{\alpha_i}(p_i)$ is infinite. Let $E = \bigcap_{i<n} D_{\alpha_i}$. So E is dense open. Since C_m is centered, choose q so that $q \le p_i$ for each i. Then

$$I \stackrel{\text{def}}{=} \{l \in \omega : E \cap C_l \cap q\downarrow \neq \emptyset\} \subseteq \bigcap_{i<n} B_{\alpha_i}(p). \tag{2}$$

In fact, if $l \in I$ and $i < n$, then $E \cap C_l \cap q\downarrow \subseteq D_{\alpha_i} \cap C_i \cap p_i\downarrow$, and so $D_{\alpha_i} \cap C_l \cap p_i\downarrow \neq \emptyset$, as desired.

Clearly there is a system $\langle a_j : j \in \omega\rangle$ of pairwise incompatible elements $\le q$. Choose $b_j \le a_j$ with $b_j \in E$ for every $j \in \omega$. Say $b_j \in C_{l_j}$ for all $j \in \omega$. Since the b_j's are pairwise incompatible, the sequence $\langle l_j : j \in \omega\rangle$ is one-one. Clearly each l_j is in I. So our claim follows from (2).

Now by the claim, since $\kappa < \mathfrak{p}$, for each $m \in \omega$ let $Z_m \in [\omega]^\omega$ be such that $Z_m \subseteq^* B_\alpha(p)$ for all $p \in C_m$ and all $\alpha < \kappa$.

For each $\tau \in {}^{<\omega}\omega$ let $\Lambda(\tau) = \emptyset$ if $\tau = \emptyset$, and otherwise let $\Lambda(\tau) = \tau(\text{dmn}(\tau) - 1)$.

Let $T = \{\tau \in {}^{<\omega}\omega : \forall n < \text{dmn}(\tau)[\tau(n) \in Z_{\Lambda(\tau\restriction n)}]\}$. Thus $\emptyset \in T$. If $\tau \in T$ and $m \in Z_{\Lambda(\tau)}$, then $\tau^\frown\langle m\rangle \in T$.

For each $\alpha < \kappa$ fix $\Delta_\alpha : T \to \mathbb{P}$ with the following properties:

(3) $\Delta_\alpha(\emptyset) = \mathbb{1}$.

(4) If $\tau^\frown\langle l\rangle \in T$, then
- (a) If $l \in B_\alpha(\Delta_\alpha(\tau))$, then $\Delta_\alpha(\tau^\frown\langle l\rangle) \in D_\alpha \cap C_l \cap \Delta_\alpha(\tau)\downarrow$.
- (b) If $l \notin B_\alpha(\Delta_\alpha(\tau))$, then $\Delta_\alpha(\tau^\frown\langle l\rangle) = \mathbb{1}$.

Note that

$$\forall \tau \in T[\Delta_\alpha(\tau) \in C_{\Lambda(\tau)}], \tag{5}$$

In fact, if $\tau \neq \emptyset$ and $\tau(\text{dmn}(\tau) - 1) \in B_\alpha(\Delta_\alpha(\tau \restriction (\text{dmn}(\tau) - 1)))$, then this follows from (4)(a). Otherwise it follows since $\mathbb{1} \in C_{\Lambda(\tau)}$.

$$\forall\alpha\forall\tau \in T[Z_{\Lambda(\tau)} \subseteq^* B_\alpha(\Delta_\alpha(\tau))]. \tag{6}$$

In fact, suppose that $\alpha < \kappa$ and $\tau \in T$. Since $\Delta_\alpha(\tau) \in C_{\Lambda(\tau)}$ by (5), it follows that $Z_{\Lambda(\tau)} \subseteq^* B_\alpha(\Delta_\alpha(\tau))$, proving (6).

$$\forall\alpha\forall\tau \in T\forall l \in \omega[\tau^\frown\langle l\rangle \in T \to l \in Z_{\Lambda(\tau)}]. \tag{7}$$

This follows from the definition of T.

By (6), for each $\alpha < \kappa$ there is a function $\Phi_\alpha : T \to \omega$ such that:

$$\forall \tau \in T\forall l \in Z_{\Lambda(\tau)}[l \ge \Phi_\alpha(\tau) \to l \in B_\alpha(\Delta_\alpha(\tau))]. \tag{8}$$

Now $|T| = \omega$; let $h : \omega \to T$ be a bijection. Also, $\kappa < \mathfrak{p} \leq \mathfrak{b}$, so there is a $\Gamma : \omega \to \omega$ such that $\Phi_\alpha \circ h \leq^* \Gamma$ for all $\alpha < \kappa$.

$$(9) \qquad \forall \tau \in T \forall m \in \omega \exists l \geq m [\tau^\frown \langle l \rangle \in T].$$

In fact, $Z_{\Lambda(\tau)}$ is infinite, so (9) follows.

By (9), there is a function $g : \omega \to \omega$ such that $g \restriction n \in T$ for all $n \in \omega$ and $g(n)$ is the least $l \geq \Gamma(h^{-1}(g \restriction n))$ such that $(g \restriction n)^\frown \langle l \rangle \in T$.

Since $\Phi_\alpha \circ h \leq^* \Gamma$, there is a $k : \kappa \to \omega$ such that $\Phi_\alpha(h(n)) \leq \Gamma(n)$ for all $n \geq k_\alpha$. Hence if $h^{-1}(g \restriction n) \geq k_\alpha$ then $g(n) \geq \Gamma(h^{-1}(g \restriction n)) \geq \Phi_a(g \restriction n)$; hence by (8), $g(n) \in B_\alpha(\Delta_a(g \restriction n))$.

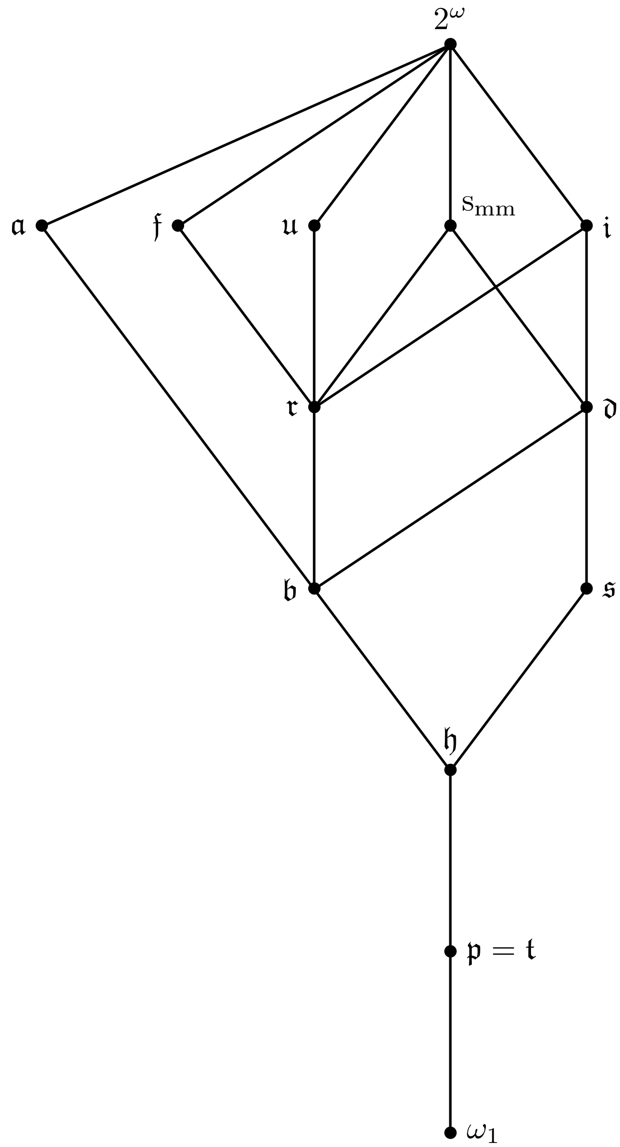
Now $(g \restriction n)^\frown \langle g(n) \rangle = g \restriction (n+1) \in T$ and $g(n) \in B_\alpha(\Delta_\alpha(g \restriction n))$, so by (4)(a) we get

$$(10 \qquad \Delta_\alpha(g \restriction (n+1)) \in D_\alpha \cap C_{g(n)} \cap \Delta_\alpha(g \restriction n)\downarrow,$$

and in particular

$$(11) \qquad \Delta_\alpha(g \restriction (n+1)) \leq \Delta_\alpha(g \restriction n).$$

Let $p_\alpha = \Delta_\alpha(g \restriction (k_\alpha + 1))$. So $p_\alpha \in D_\alpha \cap C_{g(k_\alpha)} \cap [\Delta_\alpha(g \restriction k_\alpha))$. Let $L = \{L_\alpha : \alpha < \kappa\}$. Hence $L \cap D_\alpha \neq \emptyset$ for all α. To show that L is linked, suppose that $\alpha, \beta < \kappa$. Say $k_\alpha \leq k_\beta$. If $k_\alpha = k_\beta$, then $p_\alpha, p_\beta \in C_{g(k_\alpha)}$, hence $p_\alpha \not\perp p_\beta$. Suppose that $k_\alpha < k_\beta$. By (11), $\Delta_\alpha(g \restriction (k_\beta + 1)) \leq \Delta_\alpha(g \restriction (k_\alpha + 1)) = p_\alpha$. Now $\Delta_\alpha(g \restriction (k_\beta + 1)), \Delta_\beta(g \restriction k_{\beta+1})) \in C_{g(k_\beta)}$ by (10), so $p_\alpha \not\perp p_\beta$. $\square$



For $\mathfrak{p} = \mathfrak{t}$ see the next chapter.

For the proof we need a form of Łoś's theorem on ultraproducts. In fact, our most complicated use of it is in the case of two-sorted structures. The language is as follows. There are two sorts of variables: $v_0, v_1, \ldots$ and $w_0, w_1, \ldots$. There is a four-place relation symbol Q. Atomic formulas have the form $v_i = v_j$, $w_i = w_j$, or $Qv_iw_jw_kw_l$. We have connectives $\neg$, $\rightarrow$, $\forall v_i$, and $\forall w_i$. A structure for this language is a triple (A, B, C) such that A and B are nonempty sets and $C \subseteq A \times B \times B \times B$. Given $a \in {}^\omega A$ and $b \in {}^\omega B$ and any formula φ, we define $(A, B, C) \models \varphi[a, b]$ as follows:

$$\begin{aligned}
&(A,B,C) \models (v_i = v_j)[a,b] \quad \text{iff} \quad a_i = a_j;\\
&(A,B,C) \models (w_i = w_j)[a,b] \quad \text{iff} \quad b_i = b_j;\\
&(A,B,C) \models (\neg\varphi)[a,b] \quad \text{iff} \quad \text{not}((A,B,C) \models \varphi[a,b])\\
&(A,B,C) \models (\varphi \rightarrow \psi)[a,b] \quad \text{iff} \quad \text{not}((A,B,C) \models \varphi[a,b]) \text{ or } ((A,B,C) \models \varphi)[a,b])\\
&(A,B,C) \models \forall v_i\varphi[a,b] \quad \text{iff} \quad \text{for all } u \in A((A,B,C) \models \varphi[a^i_u, b])\\
&(A,B,C) \models \forall v_i\varphi[a,b] \quad \text{iff} \quad \text{for all } u \in B((A,B,C) \models \varphi[a, b^i_u])
\end{aligned}$$

Suppose that $\langle (A_i, B_i, C_i) : i \in I\rangle$ is a system of structures, and F is an ultrafilter on I. For each $i \in I$ let $M_i = (A_i, B_i, C_i)$. Further, let $A' = \prod_{i\in I} A_i$, $B' = \prod_{i\in I} B_i$, and $C' = \{(a, b, c, d) : a \in A',\ b, c, d \in B'\}$. We define

$$\begin{aligned}
a \equiv_0 b \quad &\text{iff} \quad a, b \in A' \text{ and } \{i \in I : a_i = b_i\} \in F;\\
c \equiv_1 d \quad &\text{iff} \quad c, d \in B' \text{ and } \{i \in I : c_i = d_i\} \in F.
\end{aligned}$$

It is easy to check that $\equiv_0$ is an equivalence relation on A' and $\equiv_1$ is an equivalence relation on B'. We let A'' be the set of all $\equiv_0$-classes, and B'' the set of all $\equiv_1$-classes. We also define

$$C'' = \{([a], [b], [c], [d]) : \{i \in I : (a_i, b_i, c_i, d_i) \in C_i\} \in FWY\}.$$

Proposition 34.1. *For any $(a, b, c, d) \in A' \times B' \times B' \times B'$ the following are equivalent:*

(i) $([a], [b], [c], [d]) \in C''$.

(ii) $\{i \in I : (a_i, b_i, c_i, d_i) \in C_i\} \in F$.

Proof. (ii)$\Rightarrow$(i) holds by definition. Now assume (i). Then there are $a' \in A'$ and $b', c', d' \in B'$ such that $[a] = [a']$, $[b] = [b']$, $[c] = [c']$, $[d] - [d']$ and $\{i \in I : (a'_i, b'_i, c'_i, d'_i) \in C_i\} \in F$. Then $\{i \in I : a_i = a'_i\} \in F$, $\{i \in I : b_i = b'_i\} \in F$, $\{i \in I : c_i = c'_i\} \in F$, and $\{i \in I : d_i = d'_i\} \in F$. Now

$$\begin{aligned}
&\{i \in I : a_i = a'_i\} \cap \{i \in I : b_i = b'_i\} \cap \{i \in I : c_i = c'_i\} \cap \{i \in I : d_i = d'_i\} \cap\\
&\{i \in I : (a'_i, b'_i, c'_i, d'_i) \in C_i\} \subseteq \{i \in I : (a_i, b_i, c_i, d_i) \in C_i\};
\end{aligned}$$

it follows that $\{i \in I : (a_i, b_i, c_i, d_i) \in C_i\} \in F$. $\square$

The *ultraproduct* of $\langle M_i : i \in I\rangle$ is the structure (A'', B'', C''); it is denoted by $\prod_{i\in I} M_i/F$.

Theorem 34.2. (Łoś) *Let $\langle M_i : i \in I\rangle$ be a system of structures as above, and let F be an ultrafilter on I. Suppose that $a \in {}^{\omega}A'$ and $b \in {}^{\omega}B'$. Let $\pi : A' \to A''$ be the natural map; we use π also for the natural map from B' to B''. Then the following conditions are equivalent:*

(i) $\prod_{i\in I} M_i/F \models \varphi[\pi\circ a, \pi\circ b]$.

(ii) $\{i \in I : M_i \models \varphi[\mathrm{pr}_i \circ a, \mathrm{pr}_i \circ b]\} \in F$.

Proof. For brevity let $N = \prod_{i\in I} M_i/F$. We prove the theorem by induction on φ:

$$\begin{aligned}
N \models (v_k = v_j)[\pi\circ a, \pi\circ b] \quad &\text{iff} \quad (\pi\circ a)(k) = (\pi\circ a)(j)\\
&\text{iff} \quad [a_k] = [a_j]\\
&\text{iff} \quad \{i \in I : a_k(i) = a_j(i)\} \in F\\
&\text{iff} \quad \{i \in I : (\mathrm{pr}_i\circ a)(k) = (\mathrm{pr}_i\circ a)(j)\} \in F\\
&\text{iff} \quad \{i \in I : M_i \models (v_k = v_j)[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \in F;
\end{aligned}$$

similarly for $w_k = w_j$

$$\begin{aligned}
N \models \neg\varphi[\pi\circ a, \pi\circ b] \quad &\text{iff} \quad \mathrm{not}(N \models \varphi[\pi\circ a, \pi\circ b])\\
&\text{iff} \quad \mathrm{not}(\{i \in I : M_i \models \varphi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \in F)\\
&\text{iff} \quad (I\backslash\{i \in I : M_i \models \varphi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\}) \in F\\
&\text{iff} \quad \{i \in I : M_i \models \neg\varphi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \in F
\end{aligned}$$

$N \models (\varphi \to \psi)[\pi\circ a, \pi\circ b]$ iff

not($N \models \varphi[\pi\circ a, \pi\circ b]$) or $N \models \psi[\pi\circ a, \pi\circ b]$

iff

$\{i \in I : M_i \models \neg\varphi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \in F$

or $\{i \in I : M_i \models \psi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \in F$

iff

$\{i \in I : M_i \models \neg\varphi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \cup \{i \in I : M_i \models \psi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \in F$

iff $\{i \in I : M_i \models (\neg\varphi \vee \psi)[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \in F$

iff $\{i \in I : M_i \models (\varphi \to \psi)[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \in F$.

Now suppose that not($N \models (\forall v_k\varphi)[\pi\circ a, \pi\circ b]$). Then there is a $u \in M'$ such that not($N \models \varphi[\pi\circ a^k_u, \pi\circ b]$), so by the inductive hypothesis we get $\{i \in I : M_i \models \varphi[\mathrm{pr}_i\circ a^k_u, \mathrm{pr}_i\circ b]\} \notin F$. Since

$$\{i \in I : M_i \models \forall v_k\varphi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \subseteq \{i \in I : M_i \models \varphi[\mathrm{pr}_i\circ a^k_u, \mathrm{pr}_i\circ b]\},$$

it follows that $\{i \in I : M_i \models \forall v_k\varphi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \notin F$.

On the other hand, suppose that $\{i \in I : M_i \models \forall v_k\varphi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \notin F$. Then $P \stackrel{\mathrm{def}}{=} \{i \in I : M_i \models \exists v_k\neg\varphi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \in F$. For each $i \in P$ choose $u_i \in A_i$ such that $M_i \models \neg\varphi[(\mathrm{pr}_i\circ a)^k_{u_i}, \mathrm{pr}_i\circ b]$. For $i \in I\backslash P$ let $u_i \in A_i$ be arbitrary. Then for each $i \in P$ we have $M_i \models \neg\varphi[\mathrm{pr}_i\circ a^k_u, \mathrm{pr}_i\circ b]$, so $\{i \in I : M_i \models \neg\varphi[\mathrm{pr}_i\circ a^k_u, \mathrm{pr}_i\circ b]\} \in F$, hence $\{i \in I :$

$M_i \models \varphi[\mathrm{pr}_i \circ a_u^k, \mathrm{pr}_i \circ b]\} \notin F$, hence by the inductive hypothesis not$(N \models \varphi[\pi \circ a_u^k, \pi \circ b])$, so not$(N \models (\forall v_k \varphi)[\pi \circ a, \pi \circ b])$.

The case $\forall w_k \varphi$ is similar. $\square$

Let μ be an infinite cardinal. We define

$$L(\mu, F) = \left\{ \prod_{\alpha<\mu} L_\alpha/F : \text{each } L_\alpha \text{ is a finite linear order, and} \right.$$
$$\left. F \text{ is an ultrafilter on } \mu \text{ and } \forall n \in \omega[\{\alpha < \mu : |L_\alpha| > n\} \in F] \right\}$$

$$P(\mu, F) = \left\{ \prod_{\alpha<\mu} P_\alpha/F : \text{each } P_\alpha \text{ is a finite tree with a unique root, and} \right.$$
$$\left. F \text{ is an ultrafilter on } \mu \text{ and } \forall n \in \omega[\{\alpha < \mu : |P_\alpha| > n\} \in F] \right\}$$

Proposition 34.3. *(i) $L(\mu, F) \subseteq P(\mu, F)$.*

(ii) If $A \in P(\mu, F)$, then A has a maximal element, a unique minimum element, every non maximal element has at least one immediate successor, and every non-minimum element has a unique immediate predecessor.

(iii) If $A \in L(\mu, F)$, then A has a maximum element, and every non maximum element has a unique immediate successor.

Proof. Immediate, by Łoś's Theorem. $\square$

If $A \in L(\mu, F)$, then 1 is the greatest element of A, and 0 the least element. x is *near* 1 iff there is an $n \in \omega$ such that $x + n = 1$.

Proposition 34.4. *In any $A \in L(\mu, F)$, 0 is not near 1.*

Proof. Suppose that 0 is near 1; say $0+n = 1$. Then A has only $n+1$ elements. $\square$

For infinite regular cardinals κ, λ, and a linear order $(X, <)$, a (κ, λ)-*gap in* $(X, <)$ is a pair (a, b) with $a \in {}^\kappa X$ and $b \in {}^\lambda X$ such that:

(1) $\forall \alpha, \beta < \kappa \forall \gamma, \delta < \lambda[\alpha < \beta$ and $\gamma < \delta$ imply that $a_\alpha < \alpha_\beta < b_\delta < b_\gamma]$.

(2) There is no $x \in X$ such that $\forall \alpha < \kappa \forall \beta < \lambda[a_\alpha < x < b_\beta]$.

We define

$$\begin{aligned}
C(\mu, F) &= \{(\kappa, \lambda) : \text{there is a } (\kappa, \lambda)\text{-gap in some } (X, <) \in L(\mu, F)\}; \\
\mathfrak{p}(\mu, F) &= \min\{\kappa : \exists(\kappa_1, \kappa_2) \in C(\mu, F)[\max(\kappa_1, \kappa_2) = \kappa]\}; \\
\mathfrak{t}(\mu, F) &= \min\{\kappa \geq \omega : \kappa \text{ is regular and there is a strictly increasing} \\
&\qquad \text{unbounded } x \in {}^\kappa A \text{ for some } (A, \leq) \in P(\mu, F)\}; \\
D(\mu, F) &= \{(\kappa, \lambda) \in C(\mu, F) : \max(\kappa, \lambda) < \mathfrak{t}(\mu, F)\}
\end{aligned}$$

A subset M of $\prod_{\alpha<\mu} A_\alpha/F$ is *internal* iff there is a system $\langle B_\alpha : \alpha < \mu\rangle$ such that $\forall\alpha < \mu[B_\alpha \subseteq A_\alpha]$ and $M = \{[x] : \{\alpha < \mu : x_\alpha \in B_\alpha\} \in F\}$.

A function $f : {}^m(\prod_{\alpha<\mu} A_\alpha/F) \to (\prod_{\alpha<\mu} A_\alpha/F$ is *internal* iff there is a system $\langle f_\alpha : \alpha < \mu\rangle$ such that each $f_\alpha : {}^m A_\alpha \to A_\alpha$ and for all $x_0, \ldots, x_{m-1}, y \in \prod_{\alpha<\mu} A_\alpha$ we have

$$f([x_0], \ldots, [x_{m-1}]) = [y] \quad \text{iff} \quad \{\alpha < \mu : f_\alpha(x_{0\alpha}, \ldots, x_{m-1,\alpha}) = y_\alpha\} \in F.$$

Proposition 34.5. *The collection of internal subsets of $\prod_{\alpha<\mu} A_\alpha/F$ is a field of subsets of $\prod_{\alpha<\mu} A_\alpha/F$ containing all singletons.*

Proof. Let X and Y be internal subsets of $\prod_{\alpha<\mu} A_\alpha/F$, say given by $\langle B_\alpha : \alpha < \mu\rangle$ and $\langle C_\alpha : \alpha < \mu\rangle$. For each $\alpha < \mu$ let $D_\alpha = B_\alpha \cup C_\alpha$. Let $Z = \{[y] : \{\alpha < \mu : y_\alpha \in D_\alpha\} \in F\}$. We claim that $X \cup Y = Z$. First suppose that $a \in X$. Then there is an $x \in \prod_{\alpha<\mu} A_\alpha$ such that $a = [x]$ and $\{\alpha < \mu : x_\alpha \in B_\alpha\} \in F$. So also $\{\alpha < \mu : x_\alpha \in D_\alpha\} \in F$; so $a \in Z$. Similarly, $a \in Y$ implies that $a \in Z$.

Now suppose that $a \in Z$. Say $a = [y]$ with $\{\alpha < \mu : y_\alpha \in D_\alpha\} \in F$. Then

$$\{\alpha < \mu : y_\alpha \in B_\alpha\} \cup \{\alpha < \mu : y_\alpha \in C_\alpha\} \in F,$$

so we have two cases.

Case I. $W \stackrel{\text{def}}{=} \{\alpha < \mu : y_\alpha \in B_\alpha\} \in F$. Then $a \in X$

Case II. $\{\alpha < \mu : y_\alpha \in C_\alpha\} \in F$. Similarly, $a \in Y$.

Thus $X \cup Y = Z$.

Now let X be an internal subset of $\prod_{\alpha<\mu} A_\alpha/F$, say given by $\langle B_\alpha : \alpha < \mu\rangle$. Let $C_\alpha = A_\alpha \backslash B_\alpha$ for all $\alpha < \mu$, and let $Y = \{[y] : \{\alpha < \mu : y_\alpha \in C_\alpha\} \in F\}$. Suppose that $a \in X \cap Y$. Say $a = [f]$ with $\{\alpha < \mu : f_\alpha \in B_\alpha\} \in F$ and $x = [g]$ with $\{\alpha < \mu : g_\alpha \in C_\alpha\} \in F$. Then $[f] = [g]$, so $\{\alpha < \mu : f_\alpha = g_\alpha\} \in F$. Then

$$\emptyset = \{\alpha < \mu : f_\alpha \in B_\alpha\} \cap \{\alpha < \mu : g_\alpha \in C_\alpha\} \cap \{\alpha < \mu : f_\alpha = g_\alpha\} \in F,$$

contradiction.

Now $\prod_{\alpha<\mu} A_\alpha/F = X \cup Y$ by the argument for unions. So $(\prod_{i\in I} A_\alpha/F)\backslash X = Y$.

So we have a field of sets.

Given $[f] \in \prod_{\alpha<\mu} A_\alpha/F$, $\{[f]\}$ is internal, given by $\langle\{f_\alpha\} : \alpha < \mu\rangle$. $\square$

If $\psi(v_0)$ is a formula with one free variable v_0 and X is a structure, then $\psi(X) = \{x \in X : X \models \psi[x]\}$.

Proposition 34.6. *Let $X = \prod_{\alpha<\mu} A_\alpha/F$, and let $\psi(v_0)$ be a formula with one free variable v_0. Then $\psi(X)$ is an internal subset of X.*

Proof. We claim that $\langle\psi(A_\alpha) : \alpha < \mu\rangle$ shows that $\psi(X)$ is internal. In fact,

$$\begin{aligned}
[x] \in \psi(X) \quad &\text{iff} \quad X \models \psi[[x]] \\
&\text{iff} \quad \{\alpha < \mu : A_\alpha \models [x_\alpha]\} \in F \\
&\text{iff} \quad \{\alpha < \mu : x_\alpha \in \psi(A_\alpha)\} \in F.
\end{aligned}$$

$\square$

Proposition 34.7. *If κ is uncountable regular, $\kappa < \mathfrak{t}(\mu, F)$, and $\kappa \leq p(\mu, F)$, then $(\kappa, \kappa) \notin C(\mu, F)$.*

Proof. Suppose not. Let $\langle L_\alpha : \alpha < \mu\rangle$ be a system of finite linear orders, and $X = \prod_{\alpha<\mu} L_\alpha/F$. Suppose that X has a (κ, κ)-gap $a, b \in {}^\kappa X$. For each $\alpha < \mu$ let P_α be the set of all functions p such that:

(1) $\mathrm{dmn}(p)$ is an initial segment of L_α, and $\mathrm{rng}(p) \subseteq L_\alpha \times L_\alpha$.

(2) $\forall d, d' \in \mathrm{dmn}(p)[d <_\alpha d' \to \pi_0(p(d)) <_\alpha \pi_0(p(d')) <_\alpha \pi_1(p(d')) <_\alpha \pi_1(p(d))]$.

The partial order on P_α is inclusion. Clearly this gives a tree with the unique root $\emptyset$. Let $(Q, \leq_Q) = \prod_{\alpha<\mu} P_\alpha/F$.

For each $\alpha < \mu$ let $G_\alpha = \{(p, a, b, c) : p \in P_\alpha,\ a, b, c \in L_\alpha,\ a \in \mathrm{dmn}(p),\ p(a) = (b, c)\}$. Let $H = \prod_{\alpha<\mu} G_\alpha/F$. For details below it is convenient to apply Łoś's Theorem to the two-sorted structure $\overline{A} = (Q, X, \subseteq_Q, \leq_X, H)$. Now suppose that $q \in Q$, $x, y, z \in X$, and $(q, x, y, z) \in H$. Say $q = [q^*]$, $x = [x^*]$, $y = [y^*]$, and $z = [z^*]$. Then

$$\begin{aligned}
&\{\alpha < \mu : (q^*_\alpha, x^*_\alpha, y^*_\alpha, z^*_\alpha) \in G_\alpha\} \in F, \text{i.e.,}\\
&\{\alpha < \mu : q^*_\alpha \in P_\alpha, x^*_\alpha, y^*_\alpha, z^*_\alpha \in L_\alpha, x^*_\alpha \in \mathrm{dmn}(q^*_\alpha), q^*_\alpha(x^*_\alpha) = (y^*_\alpha, z^*_\alpha)\} \in F\\
&\{\alpha < \mu : q^*_\alpha \in P_\alpha, x^*_\alpha, y^*_\alpha, z^*_\alpha \in L_\alpha, x^*_\alpha \in \mathrm{dmn}(q^*_\alpha), q^*_\alpha(x^*_\alpha) = (y^*_\alpha, z^*_\alpha)\}\\
&\quad \subseteq \{\alpha < \mu : q^*_\alpha \in P_\alpha, x^*_\alpha \in L_\alpha, x^*_\alpha \in \mathrm{dmn}(q^*_\alpha), \exists \text{ unique } u, v \in P_\alpha[q^*_\alpha(x^*_\alpha) = (u, v)\}.
\end{aligned}$$

Hence using $\overline{A}$, if $q \in Q$, $x \in X$, and there are $y, z \in X$ such that $(q, x, y, z) \in H$, then there are unique $u, v \in X$ such that $(q, x, u, v) \in H$. Hence we can make the following definition. For any $q \in Q$ let $\mathrm{dmn}(f_q) = \{x \in X : \exists y, z \in X[(q, x, y, z) \in H]\}$, and set $f_q(x) = (y, z)$ with $(q, x, y, z) \in H$.

(3) If $d, d' \in \mathrm{dmn}(f_q)$ and $d < d'$, then $\pi_0(f_q(d)) < \pi_0(f_q(d')) < \pi_1(f_q(d')) < \pi_1(f_q(d))$.

In fact, let $f_q(d) = (r, s)$ and $f_q(d') = (t, u)$. Thus $(q, d, r, s), (q, d', t, u) \in H$. Write $x = [x^*]$ for any x in X or Q. Hence the following sets are in F:

$$\begin{aligned}
&\{\alpha < \mu : q^*_\alpha \in P_\alpha, d^*_\alpha, r^*_\alpha, s^*_\alpha \in L_\alpha, d^*_\alpha \in \mathrm{dmn}(q^*_\alpha), q^*_\alpha(d^*_\alpha) = (r^*_\alpha, s^*_\alpha)\};\\
&\{\alpha < \mu : q^*_\alpha \in P_\alpha, d'^*_\alpha, t^*_\alpha, u^*_\alpha \in L_\alpha, d'^*_\alpha \in \mathrm{dmn}(q^*_\alpha), q^*_\alpha(d'^*_\alpha) = (t^*_\alpha, u^*_\alpha)\};\\
&\{\alpha < \mu : d^*_\alpha < d'^*_\alpha\}.
\end{aligned}$$

Take any α in the intersection of these three sets. Then by (2) we have $r^*_\alpha < t^*_\alpha < u^*_\alpha < s^*_\alpha$. Hence (3) follows.

(4) $\forall q \in Q[\mathrm{dmn}(f_q)$ has a maximal element].

For, let $q \in Q$. Say $q = [q']$. Then for all $\alpha < \mu$, $q'_\alpha \in P_\alpha$ and so $\mathrm{dmn}(q'_\alpha)$ has a maximal element d_α. Thus

$$\exists b, c[(q'_\alpha, d_\alpha, b, c) \in G_\alpha \wedge \forall e, b', c'[(q'_\alpha, e, b', c') \in G_\alpha \to e \leq d_\alpha].$$

It follows that

$$\exists b, c[(q, [d_\alpha], b, c) \in H] \wedge \forall e, b', c'[(q, e, b', c') \in H \to d_\alpha \leq e].$$

Now (4) follows.

(5) $\forall q, r \in Q[q < r \to f_q \subseteq f_r]$.

In fact, suppose that $q, r \in Q$ and $q < r$. Write $q = [q']$ and $r = [r']$. Then $M \stackrel{\text{def}}{=} \{\alpha < \mu : q'_\alpha \subseteq r'_\alpha\} \in F$. For any $\alpha \in M$ we have

$$\forall a, b, c[(q'_\alpha, a, b, c) \in G_\alpha \to (r'_\alpha, a, b, c) \in G_\alpha];$$

hence

$$\forall a, b, c[(q, a, b, c) \in H \to (r, a, b, c) \in H];$$

(5) follows.

Now we define $c_\alpha \in Q$ for $\alpha < \kappa$. Let $a_\alpha = [a'_\alpha]$ and $b_\alpha = [b'_\alpha]$ for all $\alpha < \kappa$. For each $\alpha < \mu$ let $p_\alpha = \{(\min L_\alpha, (a'_{0\alpha}, b'_{0\alpha}))\}$, and set $c_0 = [\langle p_\alpha : \alpha < \mu\rangle]$. Note that for each $\alpha < \mu$ we have $(p_\alpha, \min L_\alpha, a'_{0\alpha}, b'_{0\alpha}) \in G_\alpha$, so $(c_0, 0, a_0, b_0) \in H$. Hence $f_{c_0} = \{(0, (a_0, b_0))\}$. By Proposition 34.4, 0 is not near to 1. Now suppose that $c_\alpha = [c'_\alpha]$ has been defined so that $\mathrm{dmn}(f_{c_\alpha})$ has a maximum element $d_\alpha = [d'_\alpha]$ which is not near to 1. Let

$$\begin{aligned} M \stackrel{\text{def}}{=} \{\beta < \mu : {} & d'_{\alpha\beta} \text{ is the maximum element of } \mathrm{dmn}(c'_{\alpha\beta}) \\ & \text{and } d'_{\alpha\beta} \text{ is not the maximum element of } L_\alpha\} \in F. \end{aligned}$$

For $\beta \in M$ let $c'_{\alpha+1,\beta} = c'_{\alpha\beta} \cup \{(d'_{\alpha\beta}+1, (a'_{\alpha+1,\beta}, b'_{\alpha+1,\beta}))\}$, with $c'_{\alpha+1,\beta}$ arbitrary otherwise. Let $c_{\alpha+1} = [\langle c'_{\alpha+1,\beta} : \beta < \mu\rangle]$. Then $f_{c_{\alpha+1}} = f_{c_\alpha} \cup \{(d_\alpha + 1, (a_{\alpha+1}, b_{\alpha+1}))\}$. Note that the maximum element $d_\alpha + 1$ of $\mathrm{dmn}(f_{c_{\alpha+1}})$ is not near to 1.

Now suppose that α is a limit ordinal less than κ and c_β has been defined for all $\beta < \alpha$, with $c_\beta < c_\gamma$ for $\beta < \gamma < \alpha$; also, for all $\beta < \alpha$ the domain of the function f_{c_β} has a maximum element d_β which is not near to 1. Since $\alpha < \kappa < \mathfrak{t}(\mu, F)$, there is an $e \in Q$ such that $c_\beta < e$ for all $\beta < \alpha$. Say $e = [e']$. For each $\gamma < \mu$ let g'_γ be the maximal member of $\mathrm{dmn}(e'_\gamma)$, and set $g = [g']$. Let

$$N = \{\gamma < \mu : g'_\gamma \text{ is the maximum member of } \mathrm{dmn}(e'_\gamma) \text{ and } a'_{0\gamma} < a'_{\alpha\gamma} < b'_{\alpha\gamma} < b'_{0\gamma}\}.$$

Thus $N \in F$. Now

$$N \subseteq \{\gamma < \mu : \exists s \le g'_\gamma \exists u, v[(e'_\gamma, s, u, v) \in G_\gamma \wedge u < a'_{\alpha\gamma} < b'_{\alpha\gamma} < v]\}.$$

The set on the right is thus in F. For each $\gamma \in N$ let w'_γ be the maximum s as indicated. and let w_γ be arbitrary for γ not in this set. Then $w \stackrel{\text{def}}{=} [\langle w'_\gamma : \gamma < \mu\rangle]$ is maximum such that $w \le g$ and $(e, w, u, v) \in H$ for some u, v with $u < a_\alpha < b_\alpha < v$. Now $\{\gamma < \mu : w'_\gamma \in \mathrm{dmn}(e'_\gamma)\} \in F$. For any γ in this set, let $q'_\gamma = e'_\gamma \restriction w'_\gamma$. Thus $q'_\gamma \in P_\gamma$. Then

$$\forall s, t, u[G_\gamma(q'_\gamma, s, t, u) \to G_\gamma(e'_\gamma, s, t, u)] \wedge \forall s[\exists t, u[G_\gamma(q'_\gamma, s, t, u)] \leftrightarrow s < w'_\gamma].$$

Then by Łoś's theorem we have

$$\forall s, t, h[H([q'], s, t, u) \to H(e, s, t, u)] \wedge \forall s[\exists t, u[H([q'], s, t, u) \leftrightarrow s < w].$$

Thus $f_{[q']} = f_e \upharpoonright w$.

Case 1. w is not near to 1. For any $\gamma < \mu$ let $c'_{\alpha\gamma} = q'_\gamma \cup \{(w'_\gamma, (a'_{\alpha\gamma}, b'_{\alpha\gamma}))\}$. Let $c_\alpha = [c'_\alpha]$. Now if $\beta < \alpha$, then $c_\beta < e$ and $d_\beta < w$, so $c_\beta < c_\alpha$. This completes the recursive definition. Since $\kappa < t(\mu, F)$, there is a $u \in Q$ such that $c_\alpha < u$ for all $\alpha < \kappa$. Let v be the largest element of the domain of f_u. Then for any $\alpha < \kappa$ we have

$$a_\alpha = \pi_0(f_u(\alpha)) < \pi_0(f_u(v)) < p_1(f_u(v)) < \pi_1(f_u(\alpha)) = b_\alpha,$$

contradicting a, b being a gap.

Case 2. w is near to 1. Let $\langle \beta_\xi : \xi < \mathrm{cf}(\alpha) \rangle$ be strictly increasing with supremum α. Now $\langle d_{\beta_\xi} : \xi < \mathrm{cf}(\alpha) \rangle$, $\langle w - n : n \in \omega \rangle$ is not a $(\mathrm{cf}(\alpha), \omega)$-gap since κ is uncountable and $\leq p(\mu, F)$, so there is a $\tilde{d}_\alpha \in X$ such that $\forall \xi < \mathrm{cf}(\alpha) \forall n \in \omega [d_{\beta_\xi} < \tilde{d}_\alpha < w - n \rangle$. Now we get a contradiction as in Case 1. Namely, for each $\gamma < \mu$ let $q''_\gamma = e'_\gamma \upharpoonright \tilde{d}_{\alpha\gamma}$ and $c'_{\alpha\gamma} = q''_\gamma \cup \{(\tilde{d}_{\alpha\gamma}, (a'_{\alpha\gamma}, b'_{\alpha\gamma}))\}$ and $c_\alpha = [c']$, then proceed as before to get a contradiction. $\square$

If L is a linear order and X is any nonempty set, then $X^{<L}$ is the set of all functions f such that the domain of f is an initial segment of L and the range of f is contained in X. Under $\subseteq$, $X^{<L}$ is a tree with a unique root, the empty set.

Theorem 34.8. *Every finite tree with a unique root can be isomorphically embedded in $X^{<L}$ for some finite linear order L and some finite set X.*

Proof. Let T be a finite tree with a unique root r. Let κ be the height of T, and let $L = \kappa$ under its natural order. Let $X = Y \backslash \{r\}$. For each $t \in T$ let $f(t) = \langle a_0, \ldots, a_m \rangle$ where $a_0, \ldots, a_m$ is a list in strictly increasing order of all elements different from r which are $\leq t$. In particular, $f(r) = \emptyset$. Clearly f is the desired isomorphic embedding. $\square$

Proposition 34.9. *If $\kappa = t(\mu, F)$, then $(\kappa, \kappa) \in C(\mu, F)$.*

Proof. Let $P = \prod_{\alpha<\mu} P_\alpha / F$, each P_α a finite tree with a unique root, with $c \in {}^\kappa P$ strictly increasing and unbounded. By Theorem 34.8 we may assume that $P_\alpha \subseteq X_\alpha^{<M_\alpha}$ where M_α is a finite linear order and X_α is a finite set. If $p, q \in P_\alpha$ are incomparable, then there exist s_{pq}, m_{pq}, n_{pq} such that $s_{pq}{}^\frown\langle m_{pq} \rangle \in X_\alpha^{<M_\alpha}$, $s_{pq}{}^\frown\langle n_{pq} \rangle \in X_\alpha^{<M_\alpha}$, $m_{pq} \neq n_{pq}$, $s_{pq}{}^\frown\langle m_{pq} \rangle \subseteq p$, and $s_{pq}{}^\frown\langle n_{pq} \rangle \subseteq q$. For each $\alpha < \mu$ let $<_\alpha$ be the linear order on X_α.

Now we define a relation $\prec_\alpha$ on $Q_\alpha \overset{\text{def}}{=} P_\alpha \times \{0, 1\}$:

(1) $(p, 0) \prec_\alpha (p, 1)$.

(2) If $p \subset q$, then $(p, 0) \prec_\alpha (q, 0), (q, 1)$ and $(q, 0), (q, 1) \prec_\alpha (p, 1)$.

(3) If p and q are incomparable and $m_{pq} <_\alpha n_{pq}$, then $(p, h) \prec_\alpha (q, j)$ for all $h, j \in \{0, 1\}$.

Now we claim:

(4) $\prec_\alpha$ is a linear order on Q_α.

In fact, clearly $\prec_\alpha$ is irreflexive, and any two distinct elements of Q are comparable. Now suppose that $(p, l) \prec_\alpha (q, j) \prec_\alpha (r, k)$. The following cases can be distinguished, and either the case is impossible, or $(p, l) <_\alpha (r, k)$.

Case 1. $p = q = r$. Then $l = 0$ and $j = 1$.Clearly not possible.

Case 2. $p = q \subset r$. Then $l = 0$ and $j = 1$, and again the situation is impossible.

Case 3. $p = q \supset r$. Then $l = 0$, $j = 1$, $k = 1$.

Case 4. $p = q$, q and r are incomparable. and $m_{qr} <_\alpha n_{qr}$.

Case 5. $p \subset q = r$. Then $l = 0$, $j = 0$, and $k = 1$.

Case 6. $p \subset q \subset r$. Then $l = j = 0$.

Case 7. $p \subset q \supset r$. So $l = 0$ and $k = 1$. Now p and r are comparable.

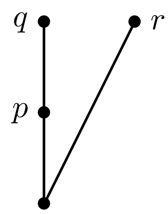
Subcase 7.1. $p \subset r$.

Subcase 7.2. $p = r$.

Subcase 7.3. $p \supset r$.

Case 8. $p \subset q$, q, r incomparable, $m_{qr} <_\alpha n_{qr}$. So $l = 0$.

Subcase 8.1. p, r incomparable. Note that $s_{qr} \subset p$, and hence $s_{pr} = s_{qr}$ and so $m_{pr} = m_{qr} <_\alpha n_{qr} = n_{pr}$.



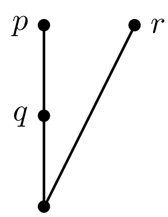
Subcase 8.2. p, r comparable. Then $p \subset r$.

Case 9. $p \supset q = r$. Then $j = 1$, contradiction.

Case 10. $p \supset q \subset r$. Then $k = 1$, contradiction.

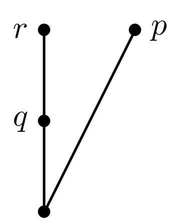
Case 11. $p \supset q \supset r$. Then $k = 1$ and $l = 1$.

Case 12. $p \supset q$, q, r incomparable. Then p, r incomparable. $m_{pr} = m_{qr}$ and $m_{qr} <_\alpha n_{qr} = n_{pr}$.



Case 13. p, q incomparable, $q = r$.

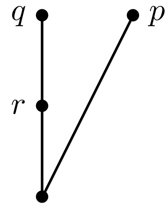
Case 14. p, q incomparable, $q \subset r$. Then p, r incomparable, $m_{pr} = m_{pq} <_\alpha n_{pq} = n_{pr}$.



Case 15. p, q incomparable, $q \supset r$. Then $k = 1$.

Subcase 15.1. p and r are comparable. Then $p \supset r$.

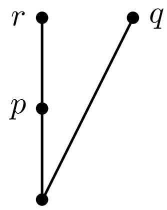
Subcase 15.2. p and r are incomparable. Then $m_{pr} = m_{pq} < n_{pq} = n_{pr}$.



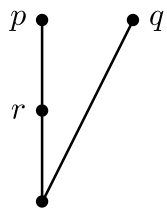
Case 16. p, q incomparable, q, r incomparable. So $m_{pq} <_\alpha n_{pq}$ and $m_{qr} <_\alpha n_{qr}$.

Subcase 16.1. $p = r$. Not possible.

Subcase 16.2. $p \subset r$. Then $m_{pq} <_\alpha n_{pq} = m_{qr} <_\alpha n_{qr} = m_{pq}$, contradiction.

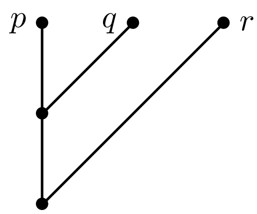


Subcase 16.3. $p \supset r$. Then $m_{pq} <_\alpha n_{pq} = m_{qr} <_\alpha n_{qr} = m_{pq}$, contradiction.

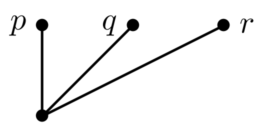


Subcase 16.4. p, r incomparable.

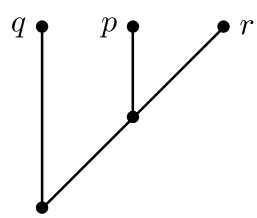
Subsubcase 16.4.1. $s_{pr} \subset s_{pq}$. Then $m_{pr} = m_{qr} < n_{qr} = n_{pr}$.



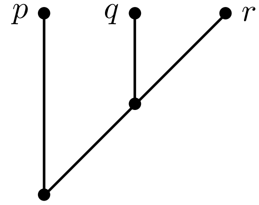
Subsubcase 16.4.2. $s_{pr} = s_{pq}$. Then $m_{pr} = m_{pq} < n_{pq} = m_{qr} < n_{qr} = n_{pr}$.



Subsubcase 16.4.3. $s_{qr} \subset s_{pr}$. Then $n_{qr} = m_{pq} <_\alpha n_{pq} = m_{qr}$, contradiction.



Subsubcase 16.4.4. $s_{pr} \subset s_{qr}$. Then $m_{pr} = m_{pq} < n_{pq} = n_{pr}$.



This finishes the proof of (4).

Now let $Q = \prod_{\alpha<\mu} Q_\alpha/F$. For each $\alpha < \kappa$ let $c_\alpha = [c'_\alpha]$. Note that $c'_\alpha \in \prod_{\beta<\mu} P_\beta$. and so $c'_{\alpha\beta} \in P_\beta$ for all $\beta < \mu$. Now define

$$c'^0_\alpha = \langle (c'_{\alpha\beta}, 0) : \beta < \mu \rangle \in \prod_{\beta<\mu} Q_\beta,$$
$$c'^1_\alpha = \langle (c'_{\alpha\beta}, 1) : \beta < \mu \rangle \in \prod_{\beta<\mu} Q_\beta.$$

Now we claim

(5) $(\langle [c'^0_\alpha] : \alpha < \kappa \rangle, \langle [c'^1_\alpha] : \alpha < \kappa \rangle)$ is a (κ, κ)-gap in Q.

In fact, take $\beta < \alpha < \kappa$. Then $c_\beta < c_\alpha$, so $[c'_\beta] < [c'_\alpha]$, hence $\{\gamma < \mu : c'_{\beta\gamma} \subset c'_{\alpha\gamma}\} \in F$. Now

$$\{\gamma < \mu : c'_{\beta\gamma} \subset c'_{\alpha\gamma}\} \subseteq \{\gamma < \mu : c'^0_{\beta\gamma} \prec_\gamma c'^0_{\alpha\gamma} \prec_\gamma c'^1_{\alpha\gamma} \prec_\gamma c'^1_{\beta\gamma}\}.$$

It follows that for $\beta < \alpha < \kappa$ we have $[c'^0_\beta] < [c'^0_\alpha] < [c'^1_\alpha] < [c'^1_\beta]$.]

Now suppose that $q \in Q$ and $[c'^0_\alpha] < q < [c'^1_\alpha]$ for all $\alpha < \kappa$. Write $q = [q']$ with $q' \in \prod_{\beta<\mu} Q_\beta$. Say $q'_\beta = (p_\beta, \varepsilon_\beta)$ for all $\beta < \mu$. Now $[\langle p_\beta : \beta < \mu \rangle]$ is not a bound for $\langle c_\alpha : \alpha < \kappa \rangle$, so there is an $\alpha < \kappa$ such that $c_\alpha \not< [\langle p_\beta : \beta < \mu \rangle]$. Hence the following sets are in F:

$$R \stackrel{\text{def}}{=} \{\beta < \mu : c'_{\alpha\beta} \not\subseteq p_\beta\};$$
$$N \stackrel{\text{def}}{=} \{\beta < \mu : (c'_{\alpha\beta}, 0) \prec_\beta (p_\beta, \varepsilon_\beta)\};$$
$$S \stackrel{\text{def}}{=} \{\beta < \mu : (p_\beta, \varepsilon_\beta) \prec_\beta (c'_{\alpha\beta}, 1)\}.$$

Now $\{\beta < \mu : \varepsilon_\beta = 0\} \cup \{\beta < \mu : \varepsilon_\beta = 1\} = \mu \in F$, so we have two cases.

Case 1. $M \stackrel{\text{def}}{=} \{\beta < \mu : \varepsilon_\beta = 0\} \in F$. Now suppose that $\beta \in M \cap N \cap R \cap S$. Then $(c'_{\alpha\beta}, 0) \prec_\beta (p_\beta, 0)$ and $c'_{\alpha\beta} \not\subseteq p_\beta$. $c'_{\alpha\beta} \supset p_\beta$ is also ruled out. So (3) in the definition of $\prec_\beta$ holds, and hence $(c'_{\alpha\beta}, h) \prec_\beta (p_\beta, k)$ for all $h, k \in \{0, 1\}$, contradicting $(p_\beta, 0) \prec_\beta (c'_{\alpha\beta}, 1)$.

Case 2. $M \stackrel{\text{def}}{=} \{\beta < \mu : \varepsilon_\beta = 1\} \in F$. Now suppose that $\beta \in M \cap N \cap R \cap S$. Then $(p_\beta, 1) \prec_\beta (c'_{\alpha\beta}, 1)$, so $p_\beta \not\subset c'_{\alpha\beta}$, and hence (3) holds again. So $(p_\beta, h) \prec_\beta (c'_{\alpha\beta}, k)$ for all $h, k \in \{0, 1\}$, contradicting $(c'_{\alpha\beta}, 0) \prec_\beta (p_\beta, 1)$. ☐

Corollary 34.10. $p(\mu, F) \leq t(\mu, F)$. ☐

Proposition 34.11. *If L is a linear order, a, b is a κ, λ-gap in L, and a, c is a κ, μ-gap in L, then $\lambda = \mu$.*

Proof. Say $\lambda < \mu$. For each $\xi < \lambda$ there is an $\eta < \mu$ such that $c_\eta < b_\xi$, as otherwise b_ξ would be below each c_η and hence would fill the gap a, c. So for each $\xi < \lambda$ choose $\eta_\xi < \mu$ such that $c_{\eta_\xi} < b_\xi$. Let $\theta = \sup_{\xi<\lambda} \eta_\xi + 1$. Then c_θ is below each b_ξ, again a contradiction. $\square$

Proposition 34.12. *If L_α is a finite linear order for each $\alpha < \mu$, then we have $(\prod_{\alpha<\mu} L_\alpha/F, \leq) \cong (\prod_{\alpha<\mu} L_\alpha/F, \geq)$.*

Proof. For each $\alpha < \mu$ let h_α be an isomorphism of $(L_\alpha, \leq)$ onto $(L_\alpha, \geq)$. Define $k : \prod_{\alpha<\mu} L_\alpha/F \to \prod_{\alpha<\mu} L_\alpha/F$ by setting $h([x]) = [y]$, where $y_\alpha = h_\alpha(x_\alpha)$ for all $\alpha < \mu$. Clearly k is a well-defined bijection. Now take any $x, y \in \prod_{\alpha<\mu} L_\alpha$.

$$\begin{aligned}
[x] \leq [y] \quad &\text{iff} \quad \{\alpha < \mu : x_\alpha \leq y_\alpha\} \in F \\
&\text{iff} \quad \{\alpha < \mu : h_\alpha(x_\alpha) \geq h_\alpha(y_\alpha)\} \in F \\
&\text{iff} \quad k([x]) \geq k([y]).
\end{aligned}$$

$\square$

Theorem 34.13. *Suppose that κ is regular and $\kappa \leq p(\mu, F)$. Let $\langle L_\alpha : \alpha < \mu\rangle$ be a system of finite linear orders, and let $(L, \leq) = \prod_{\alpha<\mu} L_\alpha/F$. Then there is a regular θ such that $(L, \leq)$ has a (κ, θ)-gap; and there is a regular θ' such that $(L, \leq)$ has a (θ', κ)-gap.*

Proof. By Proposition 34.12, L has an infinite increasing sequence $\langle c_n : n \in \omega\rangle$. Thus each c_n is not near to 1. Now suppose that $\alpha < \kappa$ and c_α has been defined and is not near to 1. Let $c_{\alpha+1} = c_\alpha + 1$. Suppose that $\alpha < \kappa$ is limit and c_β has been defined for all $\beta < \alpha$, each c_β not near to 1. Now $(\langle c_\beta : \beta < \alpha\rangle, \langle 1 - n : n \in \omega\rangle)$ is not a gap, since $|\alpha|, \omega < \kappa \leq p(\mu, F)$. Hence there is a c_α not near to 1 such that $c_\beta < c_\alpha$ for all $\beta < \alpha$. So we have constructed $c \in {}^\kappa L$ strictly increasing with no element near to 1. Let $A = \{d \in L : \forall \alpha < \kappa[c_\alpha < d]\}$. Note that A is nonempty, since e.g. $1 \in A$. If A has a first element d, then there is an $\alpha < \kappa$ such that $d - 1 \leq c_\alpha < c_{\alpha+1} < d$, contradiction. So A does not have a first element. Let $\langle e_\xi : \xi < \theta\rangle$ be a strictly decreasing coinitial sequence of elements of A. This gives θ as required in the theorem; for the second conclusion, apply Proposition 34.12. $\square$

Theorem 34.14.. *Suppose that κ is regular, $\kappa < t(\mu, F)$, and $\kappa \leq p(\mu, F)$. Then there is a unique regular θ such that $(\kappa, \theta) \in C(\mu, F)$.*

Proof. Existence was proved in Theorem 34.13. Now suppose that $\langle M_\alpha : \alpha < \mu\rangle$ and $\langle N_\alpha : \alpha < \mu\rangle$ are systems of finite linear orders, $M = \prod_{\alpha<\mu} M_\alpha/F$, $N = \prod_{\alpha<\mu} N_\alpha/F$, $a^0 \in {}^\kappa M$, $b^0 \in {}^{\theta_0} M$, (a^0, b^0) is a (κ, θ_0)-gap, $a^1 \in {}^\kappa N$, $b^1 \in {}^{\theta_1} N$, and (a^1, b^1) is a (κ, θ_1)-gap. We may assume that $M_\alpha \cap N_\alpha = \emptyset$ for all $\alpha < \mu$, and we define an order on $M_\alpha \cup N_\alpha$ by putting each member of M_α before each member of N_α. For each $\alpha < \mu$ let P_α be the set of all functions p such that

(1) $\text{dmn}(p)$ is an initial segment of $M_\alpha \cup N_\alpha$, and $\text{rng}(p) \subseteq M_\alpha \times N_\alpha$.

(2) $\forall d, d' \in \text{dmn}(p)[d < d' \to \pi_0(p(d)) < \pi_0(p(d'))$ and $\pi_1(p(d)) < \pi_1(p(d'))]$.

Let $P = \prod_{\alpha<\mu} P_\alpha/F$. For each $\alpha < \mu$ let $G_\alpha = \{(p, a, b, c) : p \in P_\alpha, a \in \mathrm{dmn}(p), p(a) = (b, c)\}$. Let $H = \prod_{\alpha<\mu} G_\alpha/F$. For $p \in P$ let $\mathrm{dmn}(f_p) = \{x : \exists y, z[(p, x, y, z) \in H\}$, and set $f_p(x) = (y, z)$. Note that if $d, d' \in \mathrm{dmn}(f_p)$ and $d < d'$, then $\pi_0(f_p(d)) < \pi_0(f_p(d'))$ and $\pi_1(f_p(d)) < \pi_1(f_p(d'))$.

We now construct $\langle c_\alpha : \alpha < \kappa\rangle \in {}^\kappa P$ stictly increasing such that for all $\alpha < \kappa$, if d_α is the maximum member of $\mathrm{dmn}(f_{c_\alpha})$ then d_α is not near to 1. Say $a^\varepsilon_\xi = [a^{\varepsilon\prime}_\xi]$ for $\varepsilon \in \{0, 1\}$. For each $\alpha < \mu$ let $p_\alpha = \{(0, (a^{0\prime}_0, a^{1\prime}_0))\}$. Let $c_0 = [\langle p_\alpha : \alpha < \mu\rangle]$. Note that 0 is not near to 1, since F is nonprincipal. We have $f_{c_0} = \{(0, (a^0_0, a^1_0))\}$.

If c_α has been defined so that the domain d_α of f_{c_α} is not near to 1 and d_α is the maximum member of $\mathrm{dmn}(f_{c_\alpha})$. Let $c_\alpha = [c'_\alpha]$, $d_\alpha = [d'_\alpha]$. Then

$$M \stackrel{\text{def}}{=} \{\beta < \mu : d'_{\alpha\beta} \text{ is the maximum element of } \mathrm{dmn}(c'_\alpha)$$
$$\text{but is not the greatest element of } M\alpha \cup N_\alpha\} \in F.$$

Then for $\beta \in M$ let $c'_{\alpha+1,\beta} = c'_{\alpha\beta} \cup \{(d'_{\alpha\beta} + 1, (a^{0\prime}_{\alpha+1}, a^{1\prime}_{\alpha+1}))\}$. $c'_{\alpha+1,\beta}$ is arbitrary otherwise. Then

$$f_{c_{\alpha+1}} = f_{c_\alpha} \cup \{(d_\alpha + 1, (a^0_{\alpha+1}, a^1_{\alpha+1}))\}.$$

Now suppose that α is limit. Since $\alpha < \kappa < t(\mu, F)$, there is an $e \in P$ such that $\forall \beta < \alpha[c_\beta < e]$. Now there is a maximum member d of $\mathrm{dmn}(f_e)$. Let $d = [d']$ and $e = [e']$. Let

$$N = \{\gamma < \mu : d'_\gamma \text{ is the maximum member of } \mathrm{dmn}(e'_\gamma) \text{ and } a^{0\prime}_0 < a^{0\prime}_\gamma \text{ and } a^{1\prime}_0 < a^{1\prime}_\gamma\}.$$

Thus $N \in F$. Now

$$N \subseteq \{\gamma < \mu : \exists s \le d'_\gamma \exists u, v[(e'_\gamma, s, u, v) \in G_\gamma \wedge u < a^{0\prime}_{\alpha\gamma} \text{ and } v < a^{1\prime}_{\alpha\gamma}\}.$$

The set on the right is thus in F, and it is finite. For each $\gamma \in N$ let w'_γ be the maximum s as indicated. and let w_γ be arbitrary for γ not in this set. Then $w \stackrel{\text{def}}{=} [\langle w'_\gamma : \gamma < \mu\rangle]$ is maximum such that $w \le d$ and $(e, w, u, v) \in H$ for some u, v with $u < a^0_\alpha$ and $v < a^1_\alpha$. Thus w is the maximum member of P which is $\le \mathrm{dmn}(f_e)$ such that $\pi_0(f_e(w)) < a^0_\alpha$ and $\pi_1(f_e(w)) < a^1_\alpha$. Now $\{\gamma < \mu : w'_\gamma \in \mathrm{dmn}(e'_\gamma)\} \in F$. For any γ in this set, let $q_\gamma = e'_\gamma \restriction w'_\gamma$. Then

$$\forall s, t, u[G_\gamma(q_\gamma, s, t, u) \to G_\gamma(e'_\gamma, s, t, u)] \wedge \forall s[\exists t, u[G_\gamma(q_\gamma, s, t, u)] \leftrightarrow s < w'_\gamma].$$

Then by Łoś's theorem we have

$$\forall s, t, h[H([q], s, t, u) \to H(e, s, t, u)] \wedge \forall s[\exists t, u[H([q], s, t, u) \leftrightarrow s < w]].$$

Thus $f_{[q]} = f_e \restriction w$.

Case 1. w is not near to 1. For any $\gamma < \mu$ let $c'_{\alpha\gamma} = q_\gamma \cup \{(w'_\gamma, (a'_{\alpha\gamma}, b'_{\alpha\gamma}))\}$. Then $[c'_\alpha] = [q] \cup \{(w, (a_\alpha, b_\alpha))\}$. Let $d_\alpha = w$. Let $c_\alpha = [c'_\alpha]$.

Case 2. w is near to 1. Let $\langle \beta_\xi : \xi < \mathrm{cf}(\alpha)\rangle$ be strictly increasing with supremum α. Now $\langle d_{\beta_\xi} : \xi < \mathrm{cf}(\alpha)\rangle$, $\langle w - n : n \in \omega\rangle$ is not a $(\mathrm{cf}(\alpha), \omega)$-gap since κ is uncountable and $\leq p(\mu, F)$, so there is a $d_\alpha \in X$ such that $\forall \xi < \mathrm{cf}(\alpha) \forall n \in \omega [d_\alpha < d_\alpha < w - n]\rangle$. As in Case 1 we get c' such that $[c'_\alpha] = [q] \cup \{(d_\alpha, (a_\alpha, b_\alpha))\}$. Let $c_\alpha = [c'_\alpha]$.

Since $\kappa < t(\mu, F)$, there is an $e \in P$ such that $\forall \alpha < \kappa [c_\alpha < e]$. Let $s = \max(\mathrm{dmn}(f_e))$. We construct $d^0 \in {}^{\theta_0}\prod_{\alpha<\mu}(M_\alpha \cup N_\alpha)/F$ so that (c, d^0) is a (κ, θ_0)-gap. A similar construction will give a (κ, θ_1)-gap, so $\theta_0 = \theta_1$ by Proposition 34.11. Let $d^0_0 = s$. Then $\forall \gamma < \kappa [d_\gamma < s]$. Suppose that $\xi < \theta_0$ and d^0_ξ has been constructed so that $\forall \gamma < \kappa [d_\gamma < d^0_\xi]$.

(3) There is an $x \in \mathrm{dmn}(f_e)$ such that $\pi_0(f_e(x)) \leq b^0_{\xi+1}$ and $x < d^0_\xi$.

In fact, take any $\eta < \kappa$. Then $\pi_0(f_e(d_\eta)) = a^0_\eta \leq b^0_{\xi+1}$ and $d_\eta < d^0_\xi$. So (3) holds. Let

$$d^0_{\xi+1} = \max\{x \in \mathrm{dmn}(f_e) : \pi_0(f_e(x)) \leq b^0_{\xi+1} \text{ and } x < d^0_\xi\}.$$

Note that by the argument for (3), $d_\eta < d^0_{\xi+1}$ for all $\eta < \kappa$.

Now suppose that ξ is limit and d^0_η has been defined for every $\eta < \xi$ so that $d_\theta < d^0_\eta$ for all $\theta < \kappa$ and $\eta < \xi$. We claim that there is a x such that the following conditions hold:

(4) $x \in \mathrm{dmn}(f_e)$.

(5) $\pi_0(f_e(x)) \leq b^0_\xi$.

(6) $d_\gamma < x$ for all $\gamma < \kappa$.

(7) $x < d^0_\eta$ for all $\eta < \xi$.

Suppose there is no such x. Now we claim

(8) $\forall \eta < \xi \exists \gamma < \theta_0 [b^0_\gamma < \pi_0(f_e(d^0_\eta))]$.

For, suppose that $\eta < \xi$. Then for any $\gamma < \kappa$, $d_\gamma < d^0_\eta$, and hence $\pi_0(f_e(d^0_\eta)) > \pi_0(f_e(d_\gamma)) = a^0_\gamma$. Now since a^0, b^0 is a gap, it follows that there is a $\delta < \theta_0$ such that $b^0_\delta < \pi_0(f_e(d^0_\eta)$, as desired in (8).

Now for each $\eta < \xi$ let

$$g(\eta) = \min\{\gamma < \theta_0 : \pi_0(f_e(d^0_\eta)) > b^0_\gamma\}.$$

We claim that $\mathrm{rng}(g)$ is cofinal in θ_0 (contradicting θ_0 regular). For, suppose that $\xi < \gamma < \theta_0$. Let

$$y = \max\{u \in \mathrm{dmn}(f_e) : \pi_0(f_e(u)) \leq b^0_\gamma\}.$$

Clearly y satisfies (4) and (6). Now $\pi_0(f_e(y)) \leq b^0_\gamma < b^0_\xi$, so (5) holds. Hence by assumption, (7) does not hold. Hence there is an $\eta < \xi$ such that $d^0_\eta \leq y$. Now $\pi_0(f_e(d^0_\eta)) \leq \pi_0(f_e(y)) \leq b^0_\gamma$. It follows that $g(\eta) > \gamma$, proving the claim.

So there is a x satisfying (4)–(7); we let d^0_ξ be such a x.

This finishes the construction of d^0. We claim that (d, d^0) is a (κ, θ_0)-gap. Suppose that x fills the gap. Take any $\xi < \kappa$ and $\eta < \theta_0$. Then

$$a^0_\xi = \pi_0(f_e(d_\xi)) < \pi_0(f_e(x)) < \pi_0(f_e(d^0_\eta)) \leq b^0_\eta.$$

Thus $\pi_0(f_\chi(x))$ fills the gap (a^0, b^0), contradiction.

So (d, d^0) is a (κ, θ_0)-gap. Similarly we get a (k, θ_1)-gap. By Proposition 34.13, $\theta_0 = \theta_1$. □

Corollary 34.15. *If κ is a regular cardinal, $\kappa < t(\mu, F)$, $\kappa \leq p(\mu, F)$, and there is an $L \in L(\mu, F)$ such that there is no (κ, θ)-gap in L, then $(\kappa, \theta) \notin C(\mu, F)$.*

Proof. Assume the hypothesis, but suppose that $(\kappa, \theta) \in C(\mu, F)$. Let $(X, <) \in L(\mu, F)$ be such that it has a (κ, θ)-gap. By Theorem 34.13, there is a (κ, θ')-gap in L. Thus $(\kappa, \theta), (\kappa, \theta') \in C(\mu, F)$, so by Theorem 34.14, $\theta = \theta'$. Hence L has a (κ, θ)-gap, contradiction. □

Theorem 34.16. *Suppose that $\langle (X_\alpha, \leq_\alpha) : \alpha < \mu \rangle$ is a system of finite linear orders, $X = \prod_{\alpha<\mu} X_i/F$, U is an infinite subset of X, Z is a family of internal subsets of X, $|U|, |Z| < t(\mu, F), p(\mu, F)$, and $U \subseteq z$ for all $z \in Z$.*

Then there is an internal Y such that $U \subseteq Y \subseteq \bigcap Z$.

Proof. Let $z \in {}^\kappa Z$ enumerate Z. For each $\alpha < \mu$ let Q_α be the set of all functions f satisfying the following conditions:

(1) $\text{dmn}(f)$ is an initial segment of X_α.

(2) $\text{rng}(f) \subseteq \mathscr{P}(X_\alpha)$.

(3) $\forall x, y \in X_\alpha[x \leq y \in \text{dmn}(f) \to f(y) \subseteq f(x)]$.

Let $Q = \prod_{\alpha<\mu}(Q_\alpha, \subseteq)/F$. For $\alpha < \mu$ let $G_\alpha = \{(f, a, b) : f \in Q_\alpha,\ a \in \text{dmn}(f),\ b \in f(a)\}$. Let $H = \prod_{\alpha<\mu} G_\alpha/F$. For $c \in Q$ let $\text{dmn}(f_c) = \{a : \exists b[(c, a, b) \in H]\}$ and for any $a \in \text{dmn}(f_c)$ let $f_c(a) = \{b : (c, a, b) \in H\}$. Now we construct $q \in {}^\kappa Q$ by recursion so that the following conditions hold:

(4) $\text{dmn}(f_{q_\alpha})$ has a maximal element d_α.

(5) d_α is not near to 1.

(6) $U \subseteq f_{q_\alpha}(d)$ for all $d \leq d_\alpha$.

(7) $f_{q_\alpha}(d_\alpha) \subseteq z_\alpha$.

(8) $f_{q_\alpha}(d_\alpha)$ is an internal subset of X.

(9) If $\beta < \alpha$, then $q_\beta \leq q_\alpha$. (Hence $f_{q_\beta} \subseteq f_{q_\alpha}$.)

Since Z is a family of internal subsets of X, for each $\alpha < \mu$ let $z'_\alpha \subseteq X_\alpha$ with $z_\alpha = \{[x] : \{\alpha < \mu : x_\alpha \in z'_\alpha\} \in F\}$. Now let $q'_{0\beta} = \{(0, z'_{0\beta})\}$ for all β, μ. Let $q_0 = [q'_0]$. Then

$$\begin{aligned}
f_{q_0}(0) &= \{b : (q_0, 0, b) \in H\} \\
&= \{[b'] : \{\beta < \mu : (q'_{0\beta}, 0, b'_\beta) \in G_\beta\} \in F\} \\
&= \{[b'] : \{\beta < \mu : b'_\beta \in z'_{0\beta}\} \in F\} \\
&= z_0.
\end{aligned}$$

Hence (4)–(9) hold for $\alpha = 0$.

Now assume that q_α has been defined satisfying (4)–(8). For each $\beta < \mu$ let

$$q'_{\alpha+1,\beta} = q'_{\alpha\beta} \cup \{(d'_{\alpha\beta} + 1, z'_{\alpha+1,\beta} \cap q'_{\alpha\beta}(d'_{\alpha\beta}))\}.$$

Let $q_{\alpha+1} = [q'_{\alpha+1}]$. Then $f_{q_{\alpha+1}} = f_{q_\alpha} \cup \{(d_\alpha + 1, z_{\alpha+1} \cap f_{q_\alpha}(d_\alpha))\}$. Then (4)–(9) hold for $\alpha + 1$.

Now suppose that α is a limit ordinal $\leq \kappa$. Then $\langle f_{q_\beta} : \beta < \alpha\rangle$ is strictly increasing by (9), and $\kappa < t(\mu, F)$, so there is an $r \in Q$ such that $q_\beta \leq r$ for all $\beta < \alpha$. Now note that if $\beta < \alpha$ and $u \in U$, then by (6), $u \in f_{q_\beta}(d_\beta) \subseteq f_r(d_\beta)$. For any $u \in U$ let

$$e_u = \max\{d \in \mathrm{dmn}(f_r) : u \in f_r(d)\}.$$

Thus $d_\beta < e_u$ for all $\beta < \alpha$. Since $|U|, |\alpha| < p(\mu, F)$, there is a d_α such that $d_\beta \leq d_\alpha \leq e_u$ for all $\beta < \alpha$ and $u \in U$. Let

$$q'_{\alpha\beta} = \begin{cases} r'_\beta \restriction d'_{\alpha\beta} \cup \{(d'_{\alpha\beta}, z'_{\alpha\beta} \cap r'_\beta(d'_{\alpha\beta}))\} & \text{if } \alpha < \kappa, \\ r'_\beta \restriction d'_{\alpha\beta} \cup \{(d'_{\alpha\beta}, r'_\beta(d'_{\alpha\beta}))\} & \text{if } \alpha = \kappa \end{cases}$$

Then let $q_\alpha = [q'_\alpha]$. Then

$$f_{c_\alpha} = \begin{cases} f_r \cup \{(d_\alpha, z_\alpha \cap r(d_\alpha))\} & \text{if } \alpha < \kappa, \\ f_r \cup \{(d_\alpha, r(d_\alpha))\} & \text{if } \alpha = \kappa \end{cases}$$

This finishes the construction. Clearly $f_{q_\kappa}(d_\kappa)$ is as desired. $\square$

Theorem 34.17. *Suppose that $\langle X_\alpha : \alpha < \mu\rangle$ is a system of finite linear orders. Let $X = \prod_{\alpha<\mu} X_\alpha/F$. Suppose that $d \in {}^\kappa X$ is strictly decreasing, with $\kappa < t(\mu, F)$, and let $D = \mathrm{rng}(d)$. Suppose that $G : D \to X$. Then there is an internal $H : X \to X$ such that $G \subseteq H$.*

Proof. Write $d_\alpha = [d'_\alpha]$ for all $\alpha < \mu$, and let $G' : D \to \prod_{\alpha<\mu} X_\alpha$ be such that $G(d_\alpha) = [G'(d_\alpha)]$ for all $\alpha < \kappa$.

For each $\alpha < \mu$ let P_α be the set of all functions f such that for some $d \in X_\alpha$, $\mathrm{dmn}(f) = \{x \in X_\alpha : d \leq x\}$, and $\mathrm{rng}(f) \subseteq P_\alpha$. Let $G_\alpha = \{(f, x, y) : f \in P_\alpha,\ x, y \in X_\alpha,\ f(x) = y\}$. We take the two-sorted structure $\overline{A}_\alpha = (X_\alpha, \leq, P_\alpha, \subseteq, G_\alpha)$ and let $\overline{B} = \prod_{\alpha<\mu} \overline{A}_\alpha/F$. Write $\overline{B} = (X, P, H)$. Then for any $\alpha < \mu$,

$$\begin{aligned} \overline{A}_\alpha \models & \forall f \in P_\alpha[\forall x, y, z \in X_\alpha[G(f, x, y) \wedge G(f, x, z) \to y = z] \\ & \wedge \exists! d \in X_\alpha \forall x \in X_\alpha[\exists y \in X_\alpha[G(f, x, y)] \leftrightarrow d \leq x]. \end{aligned}$$

It follows by Łoś's theorem that for every $c \in P$ there exist $d \in X$ and $f_c \in P$ such that f_c is a function with domain $\{x \in X : d \leq x\}$ and range contained in X, with $f_{[c']}([x']) = [y']$ iff $\{\alpha < \mu : c'_\alpha(x'_\alpha) = y'_\alpha\} \in F$.

We now define by recursion $c \in {}^\kappa P$ so that the following conditions hold:

(1) $\mathrm{dmn}(f_{c_\alpha}) = \{x \in X : d_\alpha \le x\}$.

(2) If $\alpha < \beta$, then $c_\alpha < c_\beta$.

(3) $f_{c_\alpha}(d_\alpha) = G(d_\alpha)$.

Now define $c'_{0\alpha}(x) = (G'(d_0))_\alpha$ for all $\alpha < \kappa$ and $x \ge d'_{0\alpha}$, and $c_0 = [c'_0]$. Clearly (1)–(3) hold.

Suppose that $c_\alpha = [c'_\alpha]$ has been defined. For any $\alpha < \mu$ and $d'_{\alpha+1,\beta} \le x$ define

$$c'_{\alpha+1,\beta}(x) = \begin{cases} c'_{\alpha\beta}(x) & \text{if } d'_{\alpha\beta} \le x, \\ G'(d_{\alpha+1})_\beta & \text{if } d'_{\alpha+1,\beta} \le x < d'_{\alpha\beta}. \end{cases}$$

Let $c_\alpha = [c'_\alpha]$. Clearly (1)–(3) hold.

Now suppose that α is limit and $c_\beta = [c'_\beta]$ has been defined for all $\beta < \alpha$. Since $\kappa < t(\mu, F)$, there is an $r = [r'] \in P$ such that $c_\beta < r$ for all $\beta < \alpha$. Define

$$c'_{\alpha\beta}(x) = \begin{cases} r'_\beta(x) & \text{if } x \ge d'_{\gamma\beta} \text{ for some } \gamma < \alpha, \\ G'(d_\alpha)_\beta & \text{if } d'_{\alpha\beta} \le x < d'_{\gamma\beta} \text{ for all } \gamma < \alpha. \end{cases}$$

Then let $c_\alpha = [c'_\alpha]$. Clearly (1)–(3) hold.

This completes the construction. Since $\kappa < t(\mu, F)$, choose $c = [c'] \in P$ such that $c_\alpha < c$ for all $\alpha < \kappa$. Now define

$$H'(x)_\beta = \begin{cases} c'_{\gamma\beta}(x) & \text{if } d'_{\gamma\beta} \le x \text{ for some } \gamma < \kappa, \\ x & \text{othewise.} \end{cases}$$

Then $H = [H']$ is as desired. □

Theorem 34.18. *Suppose that $\langle X_\alpha : \alpha < \mu\rangle$ is a system of finite linear orders. Let $X = \prod_{\alpha<\mu} X_\alpha/F$. Suppose that $d \in {}^\kappa X$ is strictly decreasing, with $\kappa < t(\mu, F)$, and let $D = \mathrm{rng}(d)$. Suppose that $G : {}^2D \to X$. Then there is an internal $H : {}^2X \to X$ such that $G \subseteq H$.*

Proof. Write $d_\alpha = [d'_\alpha]$ for all $\alpha < \mu$, and let $G' : D \times D \to \prod_{\alpha<\mu} X_\alpha$ be such that $G(d_\alpha, d_\beta) = [G'(d_\alpha, d_\beta)]$ for all $\alpha, \beta < \kappa$.

For each $\alpha < \mu$ let P_α be the set of all functions f such that for some $d \in X_\alpha$, $\mathrm{dmn}(f) = \{(x, y) \in X_\alpha \times X_\alpha : x, y \ge d\}$ and $\mathrm{rng}(f) \subseteq X_\alpha$. Let $G_\alpha = \{(f, x, y, z) : f \in P_\alpha,\ x, y, z \in X_\alpha,\ f(x, y) = z\}$. We take the two-sorted structure $\overline{A}_\alpha = (X_\alpha, \le, P_\alpha, \subseteq, G_\alpha)$ and let $\overline{B} = \prod_{\alpha<\mu} \overline{A}_\alpha/F$. Write $\overline{B} = (X, P, H)$. Then for any $\alpha < \mu$,

$$\begin{aligned}\overline{A}_\alpha \models &\forall f \in P_\alpha[\forall x, y, z, w \in X_\alpha[G(f, x, y, z) \wedge G(f, x, y, w) \to z = w] \\ &\wedge \exists! d \in X_\alpha \forall x, y \in X_\alpha[\exists z \in X_\alpha[G(f, x, y, z)] \leftrightarrow d \le x, y].\end{aligned}$$

It follows by Łoś's theorem that for every $c \in P$ there exist $d \in X$ and $f_c \in P$ such that f_c is a function with domain $\{(x, y) \in X \times X : d \le x, y\}$ and range contained in X, with $f_{[c']}([x'], [y']) = [z']$ iff $\{\alpha < \mu : c'_\alpha(x'_\alpha, y'_\alpha) = z'_\alpha\} \in F$.

Now we define by recursion $c \in {}^\kappa P$ so that the following conditions hold:

(1) $\mathrm{dmn}(f_{c_\alpha}) = \{(x, y) \in P \times P : d_\alpha \le x, y\}$.

(2) If $\alpha < \beta$, then $c_\alpha < c_\beta$.

(3) If $\alpha \le \beta$, then $f_{c_\beta}(d_\alpha, d_\beta) = G(d_\alpha, d_\beta)$.

Let $c'_{0\alpha}(x, y) = (G'(d_0, d_0))_\alpha$ for all $x, y \ge d'_{0\alpha}$ and all $\alpha < \kappa$; and let $c_0 = [c'_0]$. Clearly (1)–(3) hold. Now suppose that c_α has been defined. By Theorem 34.17 let $g_\alpha : X \to X$ be internal such that $g_\alpha(d_\eta) = G(d_{\alpha+1}, d_\eta)$ for all $\eta < \kappa$, and let $h_\alpha : X \to X$ be internal such that $h_\alpha(x) = G(d_\eta, d_{\alpha+1})$ for all $\eta < \kappa$. Say $g_\alpha = [g'_\alpha]$ with $g'_{\alpha\beta} : X_\beta \to X_\beta$ and $h_\alpha = [h'_\alpha]$ with $h'_{\alpha\beta} : X_\beta \to X_\beta$. Then define

$$c'_{\alpha+1,\beta}(x, y) = \begin{cases} c'_{\alpha\beta}(x, y) & \text{if } x, y \ge d'_{\alpha\beta}, \\ g'_{\alpha\beta}(y) & \text{if } x = d'_{\alpha+1,\beta} \text{ and } y \ge d'_{\alpha+1,\beta}, \\ h'_{\alpha\beta}(x) & \text{if } x > d'_{\alpha+1,\beta} \text{ and } y = d'_{\alpha+1,\beta} \end{cases}$$

Then let $c_{\alpha+1} = [c'_{\alpha+1}]$. Clearly (1)–(3) hold.

Now suppose that α is limit and $c_\beta = [c'_\beta]$ has been defined for all $\beta < \alpha$. Since $\kappa < t(\mu, F)$, there is an $r = [r'] \in P$ such that $c_\beta < r$ for all $\beta < \alpha$. By Theorem 34.17 let $g_\alpha : X \to X$ be internal such that $g_\alpha(d_\eta) = G(d_\alpha, d_\eta)$ for all $\eta < \kappa$, and let $h_\alpha : X \to X$ be internal such that $h_\alpha(x) = G(d_\eta, d_\alpha)$ for all $\eta < \kappa$. Say $g_\alpha = [g'_\alpha]$ with $g'_{\alpha\beta} : X_\beta \to X_\beta$ and $h_\alpha = [h'_\alpha]$ with $h'_{\alpha\beta} : X_\beta \to X_\beta$. Then define

$$c'_{\alpha\beta}(x, y) = \begin{cases} r'_\beta(x, y) & \text{if there is a } \gamma < \alpha \text{ such that } d_{\gamma\beta} \le x, y, \\ g'_{\alpha\beta}(y) & \text{if } x = d'_{\alpha\beta} \text{ and } y \ge d'_{\alpha\beta}, \\ h'_{\alpha\beta}(x) & \text{if } x > d'_{\alpha\beta} \text{ and } y = d'_{\alpha\beta} \end{cases}$$

Let $c_\alpha = [c'_\alpha]$. Clearly (1)–(3) hold.

This completes the construction. Since $\kappa < t(\mu, F)$, choose $c \in P$ such that $c_\alpha < c$ for all $\alpha < \kappa$. Now define

$$H'(x, y)_\beta = \begin{cases} c'_{\gamma\beta}(x, y) & \text{if } d_{\gamma\beta} \le x, y \text{ for some } \gamma < \kappa, \\ x & \text{otherwise.} \end{cases}$$

Clearly $H = [H']$ is as desired. □

Theorem 34.19. *If κ is a regular cardinal, then there is a function $F : [\kappa]^2 \to \kappa$ such that for every cofinal $A \subseteq \kappa^+$ we have $|F[[A]^2]| = \kappa$.*

Proof. For each $\alpha < \kappa^+$ let $g_\alpha : \alpha \to \kappa$ be an injection. Define $F(\{\alpha, \beta\}) = g_\alpha(\beta)$ where $\beta < \alpha$. Suppose that $A \subseteq \kappa^+$ is cofinal. Let $\langle \alpha_\xi : \xi < \kappa^+ \rangle$ be the strictly increasing enumeration of A. Then for any $\xi < \kappa$ we have $F(\{\alpha_\xi, \alpha_\kappa\}) = g_{\alpha_\kappa}(\alpha_\xi)$, and so

$$\kappa = |\mathrm{rng}(g_\kappa)| = |\{F[\{\alpha_\xi, \alpha_\kappa\}] : \xi < \kappa\}| \le |F[[A]^2]|$$ □

Theorem 34.20. $\mathrm{p}(\mu, F) = \mathrm{t}(\mu, F)$.

Proof. By Corollary 34.10, $\mathrm{p}(\mu, F) \leq \mathrm{t}(\mu, F)$, so we need to show that $\mathrm{t}(\mu, F) \leq \mathrm{p}(\mu, F)$. Let $\langle X_\alpha : \alpha < \mu \rangle$ be a system of finite linear orders such that $X \stackrel{\text{def}}{=} \prod_{\alpha<\mu} X_\alpha / F$ has a (κ, θ)-gap with $\theta \leq \kappa = \mathrm{p}(\mu, F)$. If $\theta = \kappa$, then $\mathrm{t}(\mu, F) = \mathrm{p}(\mu, F)$ by Proposition 34.7. Now suppose that $\theta < \kappa = \mathrm{p}(\mu, F) < \mathrm{t}(\mu, F)$; we want to get a contradiction. Let (x^1, x^0) be a (κ, θ)-gap in X. If $x \in X_\alpha$ let $X_\alpha \restriction x = \{x' \in X_\alpha : x' \leq x\}$. Define $P_\alpha = \{f : f$ is a function and $\exists D \subseteq X_\alpha[\mathrm{dmn}(f) = {}^2D]$ and $\mathrm{rng}(f) \subseteq X_\alpha\}$. Set

$$R_\alpha = \{(f, D, x, y, z) : f \in P_\alpha,\ \mathrm{dmn}(f) = {}^2D,\ x, y \in \mathrm{dmn}(f),\ f(x, y) = z\}.$$

Let $S = \prod_{\alpha<\mu} R_\alpha / F$, $E_p = \pi_1[S]$, $P = \pi_0[S] = \prod_{\alpha<\mu} P_\alpha / F$. For $p \in P$ let $\mathrm{dmn}(f_p) = \{(x, y) : \exists z, d[(p, d, x, y, z) \in S]\}$, and let $f_p(x, y) = z$. Thus

(1) $\forall p \in P[\mathrm{dmn}(p) = {}^2E_p \wedge \forall x, y \in E_p[f_p(x, y) \in P]]$.

(2) Assume that $\xi < \kappa$ and:

(a) $u \in {}^\xi X$ is strictly decreasing.

(b) $H : {}^2\xi \to \{x \in X : x > w\}$ is a function.

(c) $\overline{p} \in P$ is such that $f_{\overline{p}}(u_\alpha, u_\beta) = H(\alpha, \beta)$ for all $\alpha, \beta \in \xi$ such that $(u_\alpha, u_\beta) \in \mathrm{dmn}(f_{\overline{p}})$.

Then there is a $p \in P$ such that:

(d) $\forall \alpha, \beta \in \xi[(u_\alpha, u_\beta) \in \mathrm{dmn}(f_p)$ and $f_p(u_\alpha, u_\beta) = H(\alpha, \beta)]$.

(e) $f_p(x, y) > w$ for all x, y such that $(x, y) \in \mathrm{dmn}(f_p)$.

(f) If $(x, y) \in \mathrm{dmn}(f_p) \cap \mathrm{dmn}(f_{\overline{p}})$, then $f_p(x, y) = f_{\overline{p}}(x, y)$.

To prove (2), first we apply Theorem 34.18 to get an internal $\rho_1 : {}^2X \to X$ such that for all $\alpha, \beta < \xi$, $\rho_1(u_\alpha, u_\beta) = H(\alpha, \beta)$. Now for any $x, y \in X$ let

$$\rho_2(x, y) = \begin{cases} f_{\overline{p}}(x, y) & \text{if } (x, y) \in \mathrm{dmn}(f_{\overline{p}}), \\ \rho_1(x, y) & \text{otherwise.} \end{cases}$$

Note that ρ_2 is internal. For each $\alpha < \xi$ let $Z_\alpha = \{x \in X : \rho_2(x, u_\alpha) > w\}$. Let $U = \{u_\alpha : \alpha < \xi\}$. Then $U \subseteq Z_\alpha$ for each $\alpha < \xi$ by (2)(c) and the definition of ρ_2. Now by Theorem 34.16 there is an internal Y such that $U \subseteq Y \subseteq \bigcap_{\alpha<\xi} Z_\alpha$. Define

$$Y^* = Y \backslash \{y \in Y : \exists y' \in Y[\rho_2(y', y) \leq w]\}.$$

(3) $U \subseteq Y^*$.

For, let $\alpha < \xi$. Then $u_\alpha \in Y$. Suppose that $y' \in Y$. Then $y' \in Z_\alpha$, so $\rho_2(y', u_\alpha) > w$. It follows that $u_\alpha \in Y^*$. So (3) holds.

Clearly $\rho_2(x, y) > w$ for all $x, y \in Y^*$. Let $\rho = \rho_2 \restriction Y^*$. Thus (2)(e) holds. If $\alpha, \beta \in \xi$, then $u_\alpha, u_\beta \in Y^*$ by (3); so (2)(d) holds by (2)(c) and the definition of ρ_2. (2)(f) is clear.

Thus (2) holds.

Let Q_α be the set of all functions ψ such that

(4) $\mathrm{dmn}(\psi) = X_\alpha \restriction x$ for some $x \in X_\alpha$.

(5) $\mathrm{rng}(\psi) \subseteq X_\alpha \times P_\alpha$.

(6) $\forall z \leq x \forall (a,b) \in \mathrm{dmn}(\pi_2(\psi(z))[(\pi_2(\psi(z)))(a,b) \geq \pi_1(\psi(z))]$.

(7) $\forall z, z'[z \leq z' \leq x \to \pi_1(\psi(z)) \leq \pi_1(\psi(z'))$.

(8) $$\begin{aligned}\forall z, z', a, b[z \leq z' \leq x \wedge \forall w[z \leq w \leq z' \to (a,b) \in \mathrm{dmn}(\pi_2(\psi(w)))] \to \\ \forall w[z \leq w \leq z' \to (\pi_2(\psi(z)))(a,b) = (\pi_2(\psi(w)))(a,b)]].\end{aligned}$$

For each $\alpha < \mu$ let

$$\begin{aligned} T_\alpha &= \{(\psi, x, y, a, p, u, v, w) : \psi \in Q_\alpha,\ \mathrm{dmn}(\psi) = X_\alpha \restriction x, \\ &\qquad y \in X_\alpha,\ y \leq x,\ \varphi(y) = (a,p),\ (u,v) \in p, p(u,v) = w\}; \\ U &= \prod_{\alpha<\mu} T_\alpha/F; \\ Q &= \pi_0(U) = \prod_{\alpha<\mu} Q_\alpha/F. \end{aligned}$$

Now for any $\psi \in Q$ we define a function g_ψ. The domain of g_ψ is

(9) $$\{y : \exists x, a, p, u, v, w[(\psi, x, y, a, p, u, v, w) \in U]\},$$

and $g_\psi(y) = (a,p)$. We then have:

(10) $\mathrm{dmn}(g_\psi) = X \restriction x$, with x as in (9).

(11) $\mathrm{rng}(g_\psi) \subseteq X \times P$.

(12) $\forall z \leq x \forall (a,b) \in \mathrm{dmn}(f_{\pi_2(g_\psi(z))})[f_{\pi_2(g_\psi(z))}(a,b) \geq \pi_1(g_\psi(z))$.

(13) $\forall z, z'[z \leq z' \leq x \to \pi_1(g_\psi(z)) \leq \pi_1(g_\psi(z'))]$.

(14) $$\begin{aligned}\forall z, z', a, b[z \leq z' \leq x \wedge \forall w[z \leq w \leq z' \to (a,b) \in \mathrm{dmn}(f_{\pi_2(g_\psi(w))})] \to \\ \forall w[z \leq w \leq z' \to f_{\pi_2(g_\psi(z))}(a,b) = f_{\pi_2(g_\psi(w))}(a,b)]]\end{aligned}$$

For any $c \in Q$ let d_c be the maximum element of $\mathrm{dmn}(g_c)$. For any $c \in Q$ and $z \in \mathrm{dmn}(g_c)$ let $D_c(z)$ be such that $\mathrm{dmn}(f_{\pi_2(g_c(z))}) = {}^2D_c(z)$. Let $D_c = D_c(d_c)$. Further, let $g_c(z) = (c^1(z), c^2(z))$ and $g_c(d_c) = (c^1, c^2)$.

By Theorem 34.19, let $G_0 : [\theta^+]^2 \to \theta$ be such that for every cofinal $A \subseteq \theta^+$ we have $|G_0[[A]^2]| = \theta$. For $\alpha, \beta \in \kappa$ with $\alpha \neq \beta$ define

$$G(\{\alpha,\beta\}) = \begin{cases} G_0(\{\alpha,\beta\}) & \text{if } \alpha,\beta < \theta^+, \\ 0 & \text{otherwise.} \end{cases}$$

Now we construct $c \in {}^\kappa Q$ and $y \in {}^\kappa X$ by recursion so that the following conditions hold:

(16) d_{c_α} is not near to 1, and y_α is not near to 0.

(17) If $\alpha < \beta$, then $c_\alpha < c_\beta$ and $y_\beta < y_\alpha$.

(18) For all $\alpha, \beta, \gamma < \kappa$, if $y_\beta, y_\gamma \in D_{c_\alpha}$, then $f_{c_\alpha^2}(y_\beta, y_\gamma) = x^0_{G(\beta,\gamma)}$.

(19) $c^1_\alpha = x^1_\alpha + 1$.

(20) $\{y_\beta : \beta \le \alpha\} \subseteq D_{c_\alpha}$.

(21) $y_\alpha = \min(D_{c_\alpha})$.

(22) $\forall \alpha, \beta \forall z[\beta < \alpha < \kappa \wedge d_{c_\beta} \le z \le d_{c_\alpha} \rightarrow y_\beta \in D_{c_\alpha}(z)]$.

Let $y_0 \in X$ be such that y_0 is not near to 0, and let $c_0 = \{(0, (x^1_0 + 1, \{((y_0, y_0), x^0_{G(0,0)})\}))\}$. Clearly $c_0 \in Q$ and (16)–(22) hold.

Now suppose that c_β has been defined for all $\beta \le \alpha$ satisfying (16)–(22). Let $y_{\alpha+1} = y_\alpha - 1$. We apply (2) with ξ replaced by $\alpha + 2$, u replaced by $\langle y_\beta : \beta \le \alpha + 1\rangle$, H given by $H(\beta, \gamma) = x^0_{G(\beta,\gamma)}$ for all $\beta, \gamma \le \alpha + 1$, w replaced by $x^1_{\alpha+1}$, with $\overline{\rho}$ replaced by c^2_α. Note that if $\beta, \gamma \in \alpha + 2$ and $(y_\beta, y_\gamma) \in \mathrm{dmn}(c^2_\alpha)$, then $\beta, \gamma \neq \alpha + 1$ by (21), and so $f_{c^2_\alpha}(y_\beta, y_\gamma) = x^0_{G(\beta,\gamma)} = H(\beta, \gamma)$. So (2) gives a function ρ satisfying

(23) $\forall \beta, \gamma \in \alpha + 2[(y_\beta, y_\gamma) \in \mathrm{dmn}(f_\rho)$ and $f_\rho(y_\beta, y_\gamma) = H(\beta, \gamma)]$.

(24) $f_\rho(x, y) > x^1_{\alpha+1}$ for all x, y such that $(x, y) \in \mathrm{dmn}(f_\rho)$.

(25) If $(x, y) \in \mathrm{dmn}(f_\rho) \cap \mathrm{dmn}(f_{c^2_\alpha})$, then $f_\rho(x, y) = f_{c^2_\alpha}(x, y)$.

Let $\rho' = \rho \restriction {}^2\{w : y_{\alpha+1} \le w\}$, and $c_{\alpha+1} = c_\alpha \cup \{(d_{c_\alpha} + 1, (x^1_{\alpha+1} + 1, \rho'))\}$. We check (12) and (14) for $c_{\alpha+1}$. For (12), suppose that $z \le d_\alpha + 1$; we may assume that $z = d_\alpha + 1$. Suppose that $(a, b) \in \mathrm{dmn}(f_{\pi_2(g_{c_{\alpha+1}})}) = \mathrm{dmn}(f_{\rho'})$. Then $f_{\rho'}(a, b) \ge x^1_{\alpha+1} + 1$, as desired. (14) is clear by (25).

Clearly (16)–(22) hold.

Now suppose that α is limit. Then $\mathrm{cf}(\alpha) < t(\mu, F)$, so there is a c such that $c_\beta < c$ for all $\beta < \alpha$. For $\beta < \alpha$ we have $y_\beta \in D_{c_\beta} = D_{c_\beta}(d_{c_\beta}) = D_c(d_{c_\beta})$ by (21), and so $y_\beta \in D_c(c_\beta)$. Hence the following is well-defined.

$$e_\beta = \max\{z \in \mathrm{dmn}(g_c) : \forall z'[d_{c_\beta} \le z' \le z \rightarrow y_\beta \in D_c(z')]\}.$$

(26) $\forall \xi, \beta < \alpha[d_{c_\xi} < e_\beta]$.

For, suppose that $\xi, \beta < \alpha$ and suppose that $d_{c_\beta} \le z' \le d_{c_{\xi+1}}$. Hence $\beta < \xi$. Now by (22), $y_\beta \in D_{c_{\xi+1}}(z') = D_c(z')$, and so $d_{c_\xi} < d_{c_{\xi+1}} \le e_\beta$, as desired, proving (26).

Now $|\alpha| < \kappa = p(\mu, F)$, so there is a d_α such that $\forall \beta < a[d_{c_\beta} \le d_\alpha \le e_\beta]$. Clearly d_α is not near to 1. Let $c' = c \restriction d_\alpha$. Now $\mathrm{cf}(\alpha) < \kappa < t(\mu, F)$, so there is a y_α such that $\forall \eta < \alpha[y_\alpha < y_\eta]$, and y_α is not near to 0. Now we apply (2), with ξ replaced by $\alpha + 1$, u replaced by $\langle y_\xi : \xi \le \alpha\rangle$, H given by $H(\beta, \gamma) = x^0_{G(\beta,\gamma)}$ for all $\beta, \gamma \le \alpha + 1$, w replaced by x^1_α, with $\overline{\rho} = c'^2 \restriction {}^2\{x \in D_{c'} : x \le y_\alpha\}$. Note that if $\beta, \gamma \in \alpha + 1$ and $(\beta, \gamma) \in \mathrm{dmn}(\overline{\rho})$, then, with $\delta = \max(\beta, \gamma)$, $f_{\overline{\rho}}(y_\beta, y_\gamma) = f_{c'^2}(y_\beta, y_\gamma) = f_{c_\delta}(y_\beta, y_\gamma) = x^0_{G(\beta,\gamma)} = H(\beta, \gamma)$. So we obtain ρ such that

(27) $\forall \beta, \gamma \in \alpha + 1[(y_\beta, y_\gamma) \in \mathrm{dmn}(f_\rho)$ and $f_\rho(y_\beta, y_\gamma) = H(\beta, \gamma)]$.

(28) $f_\rho(x, y) > x^1_\alpha$ for all x, y such that $(x, y) \in \mathrm{dmn}(f_\rho)$.

(29) If $(x,y) \in \mathrm{dmn}(f_\rho) \cap \mathrm{dmn}(f_{\overline{\rho}})$, then $f_\rho(x,y) = f_{\overline{\rho}}(x,y)$.

Let $\rho' = \rho \restriction {}^2\{s : y_\alpha \le s\}$ and $c_\alpha = c' \cup \{(d_\alpha, (x^1_\alpha + 1, \rho'))\}$. Clearly $c_\alpha \in Q$ and (16)-(22) hold.

This finishes the construction of $\langle c_\alpha : \alpha < \kappa\rangle$.

Suppose that $\langle c_\alpha : \alpha < \kappa\rangle$ is bounded; say $c_\alpha < c$ for all $\alpha < \kappa$. For each $\eta < \theta^+$ we have by (21) $y_\eta \in D_{c_\eta} = D_{c_\eta}(d_{c_\eta}) = D_c(d_{c_\eta})$, so we can let z_η be the maximum element of

$$H_\eta \stackrel{\text{def}}{=} \{z \in \mathrm{dmn}(g_c) : \forall z'[d_{c_\eta} \le z' \le z \to y_\eta \in D_c(z')]\}.$$

(30) $d_{c_\alpha} \le z_\eta$ for all $\alpha < \kappa$.

For, suppose that $\alpha < \kappa$. Wlog $\eta < \alpha$. Suppose that $d_{c_\eta} \le z' \le d_{c_\alpha}$. By (22), $y_\eta \in D_{c_\alpha}(z') = D_c(z')$, so by the definition of H_η we have $d_{c_\alpha} \le z_\eta$, and (30) holds.

By (13) and (30), for each $\eta < \theta^+$, $c^1(z_\eta) \ge c^1(d_{c_\alpha}) = c^1_\alpha > x^1_\alpha$ for all $\alpha < \kappa$. So there is a $K(\eta) < \theta$ such that $x^0_{K(\eta)} < c^1_{z_\eta}$. Let $A \subseteq \theta^+$ and $\gamma \in \theta$ be such that $\forall \eta \in A[K(\eta) = \gamma]$. Choose $\zeta, \eta \in A$ such that $G(\eta,\zeta) > \gamma$. Let $z^* = \min(z_\eta, z_\zeta)$. So $d_{c_\eta}, d_{c_\zeta} \le z^* \le z_\eta, z_\zeta$, so $[y_\eta, y_\zeta\} \subseteq D_c(z^*)$. Hence with $\mu = \max(\eta, \zeta+1)$,

$$\begin{aligned}
&(g_c(z^*))(y_\eta, y_\zeta) = (g_c(d_{c_\mu}))(y_\eta, y_\zeta) = c^2_\mu(y_\eta, y_\zeta) = x^0_{G(\eta,\zeta)} < x^0_\gamma;\\
&(g_c(z^*))(y_\eta, y_\zeta) \ge c^1(z^*) > x^0_{K(\eta)} = x^0_\gamma,
\end{aligned}$$

contradiction. □

Theorem 34.21.

$$\begin{aligned}
\mathfrak{p} = \min\{\kappa : \exists A \in {}^\kappa([\omega]^\omega)[\forall \xi, \eta < \kappa \exists \rho < \kappa[A_\rho \subseteq A_\xi \cap A_\eta]\\
\wedge \neg\exists C \in [\omega]^\omega \forall \xi < \kappa[|C\backslash A_\xi| < \omega]]\}.
\end{aligned}$$

Proof. Clearly $\le$ holds. Now suppose that $\mathscr{A} \subseteq [\omega]^\omega$, $|\mathscr{A}| = \mathfrak{p}$, $\forall F \in [\mathscr{A}]^{<\omega}[\bigcap F$ is infinite], and there is no $C \in [\omega]^\omega$ such that $C\backslash A$ is finite for all $A \in \mathscr{A}$. Let $\mathscr{B} = \{\bigcap F : F \in \mathscr{A}^{<\omega}\}$, and let $A \in {}^{\mathfrak{p}}\mathscr{B}$ be a bijection. Clearly A satisfies the conditions of the theorem. □

Theorem 34.22. *Suppose that $\mathfrak{p} < \mathfrak{t}$ and $A \in {}^{\mathfrak{p}}([\omega]^\omega)$ is as in Theorem 34.21 with κ replaced by $\mathfrak{p}$. Then there exist a regular uncountable $\kappa < \mathfrak{p}$ and a $B \in {}^\kappa([\omega]^\omega)$ such that:*

(i) $\forall \xi < \mathfrak{p} \forall \alpha < \kappa[B_\alpha \cap A_\xi$ is infinite].

(ii) $\forall \alpha, \beta < \kappa[\beta \le \alpha \to B_\alpha \backslash B_\beta$ is finite].

(iii) $\neg\exists C \in [\omega]^\omega[\forall \alpha < \kappa[|C\backslash B_\alpha| < \omega]$ and $\forall \xi < \mathfrak{p}[C \cap A_\xi$ is infinite]].

Proof. We define ζ and $B' \in {}^\zeta([\omega]^\omega)$ by recursion so that

$$(1) \qquad \forall \xi < \zeta \forall \alpha < \mathfrak{p}[B'_\xi \cap A_\alpha \text{ is infinite}].$$

$B'_0 = \omega$, and $B'_{\xi+1} = B'_\xi \cap A_\xi$. Clearly (1) holds for $\xi + 1$. For ξ limit we consider two cases.

Case 1. There is an infinite $C \subseteq \omega$ such that $\forall \eta < \xi[C \backslash B'_\eta$ is finite] and $\forall \gamma < \mathfrak{p}[C \cap A_\gamma$ is infinite]. Then we let B'_ξ be such a C. Clearly (1) holds for ξ.

Case 2. Otherwise let $\zeta = \xi$ and stop.

(2) $\zeta \leq \mathfrak{p}$.

For, suppose otherwise. Then there is a $C \in [\omega]^\omega$ such that $\forall \eta < \mathfrak{p}[C \backslash B'_\eta$ is finite]. Now for any $\xi < \mathfrak{p}$ we have $B'_{\xi+1} \subseteq A_\xi$, so $C \backslash A_\xi \subseteq C \backslash B'_{\xi+1}$; so $C \backslash A_\xi$ is finite. This contradicts the hypothesis on A.

(3) $\zeta = \mathfrak{p}$.

Suppose that $F \in [\zeta]^{<\omega}$. Note that if $\xi < \eta < \zeta$ then $B'_\eta \backslash B'_\xi$ is finite. Choose $\xi < \zeta$ such that $F < \xi$. Then $B'_\xi \backslash \bigcap_{\eta \in F} B'_\eta = \bigcup_{\eta \in F}(B'_\xi \backslash B_\eta)$ is finite. Since B'_ξ is infinite, it follows that $\bigcap_{\eta \in F} B'_\eta$ is infinite. Suppose that $D \in [\omega]^\omega$ and $D \backslash B'_\xi$ is finite for all $\xi < \mathfrak{p}$. Then by the definition of ζ (Case 1 failing), there is a $\gamma < \mathfrak{p}$ such that $D \cap A_\gamma$ is finite. Then $D \cap B'_{\gamma+1} = D \cap B'_\gamma \cap A_\gamma$ is finite. Hence D is finite, contradiction. Together with (2) this proves (3).

Let $\kappa = \mathrm{cf}(\zeta)$, and let $\langle \zeta_\alpha : \alpha < \kappa \rangle$ be strictly increasing with supremum ζ. Define $B_\alpha = B'_{\zeta_\alpha}$ for all $\alpha < \kappa$.

(4) $\kappa > \omega$.

For, suppose that $\kappa = \omega$. For each $n \in \omega$ let $\overline{B}_n = \bigcap_{i \leq n} B_i \backslash n$. Thus $\langle \overline{B}_n : n \in \omega \rangle$ is decreasing, and $\bigcap_{n \in \omega} \overline{B}_n = \emptyset$. Moreover, $B_n \triangle \overline{B}_n$ is finite for all $n \in \omega$.

Now we define $f_\xi \in {}^\omega\omega$ for $\xi < \mathfrak{p}$ by setting $f_\xi(n) = \min(A_\xi \cap \overline{B}_n)$ for all $n \in \omega$. Since $\mathfrak{p} < \mathrm{t} \leq \mathfrak{b}$, there is an $f \in {}^\omega\omega$ such that $f_\xi \leq^* f$ for all $\xi < \mathfrak{p}$. Let $C = \bigcup_{n \in \omega}[(f(n)+1) \cap \overline{B}_n]$.

(5) $\forall \xi < \mathfrak{p}[A_\xi \cap C$ is infinite].

For, take any $\xi < \mathfrak{p}$. Choose N such that $\forall n \geq N[f_\xi(n) \leq f(n)]$. Then $f_\xi(n) \in [(f(n) + 1) \cap A_\xi \cap \overline{B}_n] \subseteq C$. Since $\bigcap_{n \in \omega} \overline{B}_n = \emptyset$, it follows that $A_\xi \cap C$ is infinite.

(6) $\forall n \in \omega[C \backslash \overline{B}_n$ is finite].

For, suppose that $m, n \in \omega$ and $m \in C \backslash \overline{B}_n$. Then there is a $p \in \omega$ such that $m \in [(f(p) + 1) \cap \overline{B}_p]$. Since $\overline{B}_q \subseteq \overline{B}_n$ for all $q \geq n$, we have $p < n$. So $m \in \bigcup_{p<n}(f(p) + 1)$. This proves (6).

(7) $\forall \xi < \zeta[C \backslash B'_\xi$ is finite].

In fact, take any $\xi < \zeta$. Choose $n \in \omega$ such that $\xi < \zeta_n$. Then in $\mathscr{P}(\omega)/\mathrm{fin}$ we have

$$[C] \leq [\overline{B}_n] = [B_n] = [B'_{\zeta_n}] \leq [B'_\xi],$$

which proves (7).

Now (5) and (7) contradict the choice of ζ. Hence (4) holds. □

Theorem 34.23. *Suppose that $\zeta < t$ is an ordinal, $A \in {}^\zeta([\omega]^\omega)$, $B \in {}^\omega([\omega]^\omega)$, and:*

(i) $B_n \backslash B_m$ is finite if $m < n$.

(ii) $\forall \xi < \zeta \forall n \in \omega[A_\xi \cap B_n$ is infinite].

(iii) $\forall \xi, \eta < \zeta[\eta \leq \xi \to \exists n \in \omega[B_n \cap A_\xi \backslash A_\eta$ *is finite*$]]$.

Then there is a $C \in [\omega]^\omega$ *such that* $\forall \xi < \zeta \forall n \in \omega[C \backslash A_\xi$ *and* $C \backslash B_n$ *are finite*$]$.

Proof. Replacing B_n by $\bigcap_{m \leq n} B_m \backslash n$, we may assume that $\forall n[B_{n+1} \subseteq B_n]$ and $\bigcap_{n \in \omega} B_n = \emptyset$. For all $\xi < \zeta$ let $f_\xi \in {}^\omega\omega$ be strictly increasing such that $\forall n \in \omega[f_\xi(n) \in B_n \cap A_\xi]$. Since $\zeta < \mathfrak{t} \leq \mathfrak{b}$, there is an $f \in {}^\omega\omega$ such that $f_\xi <^* f$ for all $\xi < \zeta$. Define $B^* = \bigcup_{n \in \omega}(B_n \cap f(n))$.

(1) $\forall \xi < \zeta[A_\xi \cap B^*$ is infinite$]$.

In fact, let $\xi < \zeta$. Choose N so that $\forall n \geq N[f_\xi(n) < f(n)]$. Then for any $n \geq N$ we have $f_\xi(n) \in A_\xi \cap f(n)$. So (1) holds.

(2) $\forall \xi, \eta < \zeta[\eta \leq \xi \to B^* \cap (A_\xi \backslash A_\eta)$ is finite$]$.

In fact, for any $n \in \omega$ we have $B^* \backslash B_n \subseteq \bigcup_{i<n}(B_i \cap f(i))$, so $B^* \backslash B_n$ is finite. Now suppose that $\xi, \eta < \zeta$ and $\eta \leq \xi$. By (iii), choose $n \in \omega$ such that $B_n \cap (A_\xi \backslash A_\eta)$ is finite. Then, in $\mathscr{P}(\omega)/\text{fin}$ we have $[B^*] \leq [B_n] \leq [\omega \backslash (A_\xi \backslash A_\eta)]$, and (2) follows.

Now by (2), $\langle B^* \cap A_\xi : \xi < \zeta \rangle$ is decreasing mod finite. Since $\zeta < t$, choose $C \in [\omega]^\omega$ such that $[C] \leq [B^* \cap A_\xi]$ for all $\xi < \zeta$. Clearly C is as desired. (See the argument following (2).) $\square$

If σ and τ are finite subsets of ω, we write $\sigma \lhd \tau$ if σ is a proper initial segment of τ. Now we define:

$$\Sigma = \{S \in [[\omega]^{<\omega} \backslash \{\emptyset\}]^\omega : \forall \sigma, \tau \in S[\sigma \neq \tau \to \min(\sigma) \neq \min(\tau)]\}$$
$$S \prec_\Sigma S' \quad \text{iff} \quad S, S' \in \Sigma \text{ and } \exists F \in [S']^{<\omega} \forall \tau \in S' \backslash F \exists \sigma \in S[\sigma \lhd \tau].$$

Note that the $\sigma \in S$ asserted to exist here is unique, since $\sigma \lhd \tau$ implies that the first element of σ is the same as the first element of τ, and distinct members of S have different first members.

Proposition 34.24. *If* $S \in \Sigma$ *and* $m \in \omega$, *then* $\{\sigma \in S : \min(\sigma) < m\}$ *and* $\{\sigma \in S : \max(\sigma) < m\}$ *are finite.*

Proof. For, obviously $\{\sigma \in S : \min(\sigma) < m\}$ is finite. Since $\{\sigma \in S : \max(\sigma) < m\} \subseteq \{\sigma \in S : \min(\sigma) < m\}$, also $\{\sigma \in S : \max(\sigma) < m\}$ is finite. $\square$

Theorem 34.25. $\prec_\Sigma$ *is transitive.*

Proof. Suppose that $S \prec S' \prec S''$. Choose n_0, n_1 such that

$$\forall \tau \in S'[\min(\tau) > n_0 \to \exists \sigma \in S[\sigma \lhd \tau]];$$
$$\forall \nu \in S''[\min(\nu) > n_1 \to \exists \tau \in S'[\tau \lhd \nu]].$$

Let

$$m = \max\{n_1, \sup\{\max(\tau) : \tau \in S',\ \min(\tau) \leq n_0\}.$$

Now suppose that $\nu \in S''$ and $\min(\nu) > m$. Since $\min(\nu) > n_1$, it follows that there is a $\tau \in S'$ such that $\tau \triangleleft \nu$. If $\min(\tau) \le n_0$, then $\min(\nu) > m \ge \max(\tau)$, contradiction. So $\min(\tau) > n_0$ and so there is a $\sigma \in S$ such that $\sigma \triangleleft \tau$, hence $\sigma \triangleleft \nu$. $\square$

Theorem 34.26. *Suppose that $\zeta < t$ is an ordinal, $S \in {}^{\zeta}\Sigma$, and $\forall \eta, \xi < \zeta[\eta < \xi \to S_\eta \prec S_\xi]$. Then there is a $T \in \Sigma$ such that $S_\xi \prec T$ for all $\xi < \zeta$.*

Proof. For each $n \in \omega$ let $B_n = \{\sigma \in [\omega]^{<\omega} \backslash \{\emptyset\} : \min(\sigma) \ge n\}$. For each $\xi < \zeta$ let

$$A_\xi = \{\tau \in [\omega]^{<\omega} \backslash \{\emptyset\} : \exists \sigma \in S_\xi[\sigma \triangleleft \tau]\}$$

Suppose that $\eta < \xi < \zeta$. Since $S_\eta \prec S_\xi$, choose n_0 so that $\forall \tau \in S_\xi[\min(\tau) \ge n_0 \to \exists \sigma \in S_\eta[\sigma \triangleleft \tau]]$. Thus $S_\xi \cap B_{n_0} \subseteq A_\eta$. Choose n such that $\max(\sigma) < n$ whenever $\sigma \in S_\xi \backslash B_{n_0}$.

(1) $A_\xi \cap B_n \subseteq A_\eta$.

In fact, suppose that $\tau \in A_\xi \cap B_n$. So $\min(\tau) \ge n$. Choose $\sigma \in S_\xi$ such that $\sigma \triangleleft \tau$. Then $\min(\sigma) \ge n$, so by the choice of n, $\sigma \in B_{n_0}$. So $\min(\sigma) \ge n_0$. Hence $\sigma \in S_\xi \cap B_{n_0} \subseteq A_\eta$, so there is a $\rho \in S_\eta$ such that $\rho \triangleleft \sigma$. Thus $\rho \triangleleft \tau$. This shows that $\tau \in A_\eta$, proving (1).

Now obviously each A_ξ is infinite, so also $A_\xi \cap B_n$ is infinite. Clearly $B_m \subseteq B_n$ if $n < m$. So the hypotheses of Theorem 34.23 hold. So let C be infinite such that $\forall \xi < \zeta \forall n \in \omega[C \backslash B_n$ and $C \backslash A_\xi$ are finite$]$.

Now suppose that $\xi < \zeta$; we show that $S_\xi \prec C$. Choose $n \in \omega$ such that $\forall \sigma \in C[\min(\sigma) \ge n \to \sigma \in A_\xi]$. Now take any $\tau \in C$ such that $\min(\tau) \ge n$. Then $\tau \in A_\xi$, so there is a $\sigma \in S_\xi$ such that $\sigma \triangleleft \tau$. $\square$

Theorem 34.27. *Suppose that $\mathfrak{p} < \mathfrak{t}$, $A \in {}^{\mathfrak{p}}([\omega]^\omega)$ as in Theorem 34.21, $\kappa < \mathfrak{p}$ and $B \in {}^{\kappa}([\omega]^\omega)$ as in Theorem 34.22.*

Then there is a function S with domain $\mathfrak{p} + 1$ such that:

(i) $\forall \xi \le \mathfrak{p}[S_\xi \in \Sigma]$.

(ii) $\forall \xi < \mathfrak{p} \forall \alpha < \kappa[S_\xi \backslash [B_\alpha]^{<\omega}$ is finite$]$.

(iii) $\forall \eta, \xi \le \mathfrak{p}[\eta < \xi \to S_\eta \prec S_\xi]$.

(iv) $\forall \xi < \mathfrak{p} \forall \sigma \in S_{\xi+1}[\max(\sigma) \in A_\xi]$.

Proof. We define S_ξ for $\xi \le \mathfrak{p}$ by recursion.

$\xi = 0$: Since $\kappa < \mathfrak{t}$ and $\forall \alpha, \beta < \kappa[\alpha < \beta \to B_\beta \backslash B_\alpha$ is finite$]$, there is an infinite $C \subseteq \omega$ such that $C \backslash B_\alpha$ is finite for all $\alpha < \kappa$. Let $S_0 = \{\{n\} : n \in C\}$. Clearly (i) holds for $\xi = 0$. Suppose that $\alpha < \kappa$. Then $C \backslash B_\alpha$ is finite, so there is an N such that $\forall n \ge N[n \in C \to n \in B_\alpha]$. Hence $\forall x \in S_0 \backslash \{\{n\} : n < N\}[x \in [B_\alpha]^{<\omega}]$. So $S_0 \backslash [B_\alpha]^{<\omega}$ is finite.

ξ to $\xi + 1$: If $\alpha < \beta < \kappa$ then $B_\beta \cap A_\xi \backslash (B_\alpha \cap A_\xi)$ is finite; and by (i) of Theorem 34.22, $B_\alpha \cap A_\xi$ is infinite for all $\alpha < \kappa$. Hence since $\kappa < t$, there is an infinite $C \subseteq A_\xi$ such that $C \backslash B_\alpha$ is finite for all $\alpha < \kappa$. For each $\sigma \in S_\xi$ there is a $\sigma' \in [\omega]^{<\omega} \backslash \{\emptyset\}$ such that $\sigma \triangleleft \sigma' \subseteq \sigma \cup C$; namely one can take any $n \in C$ with $n > \sigma$ (possible since σ is finite and C is infinite), and let $\sigma' = \sigma \cup \{n\}$. Let $S_{\xi+1} = \{\sigma' : \sigma \in S_\xi\}$. Then (i) is clear for $\xi + 1$. Concerning (ii), suppose that $\alpha < \kappa$. Choose N so that $\forall n \ge N[n \in C \to n \in B_\alpha]$, and choose M so that $\forall \sigma \in S_\xi[\min(\sigma) \ge M \to \sigma \in [B_\alpha]^{<\omega}]$. Suppose that $\tau \in S_{\xi+1}$ and

$\min(\tau) \geq M, N$. Say $\tau = \sigma'$ with $\sigma \in S_\xi$. Since $\min(\sigma) \geq M$ it follows that $\sigma \in [B_\alpha]^{<\omega}$. Now $\sigma' \backslash \sigma \subseteq C$ and $\min(\sigma' \backslash \sigma) \geq N$, so $\sigma' \backslash \sigma \subseteq B_\alpha$. Hence $\tau \in [B_\alpha]^{<\omega}$. This proves (ii). Clearly $S_\xi \prec S_{\xi+1}$. Finally, (iv) holds since for $\sigma \in S_{\xi+1}$ we have $\max(\sigma) \in C \subseteq A_\xi$.

ξ limit, $\xi < \mathfrak{p}$: Let $\zeta = \max(\xi, \kappa)$. Further, let

$$\begin{aligned}
&\mathbb{I} = \{I \in [[\omega]^{<\omega} \backslash \{\emptyset\}]^{<\omega} : \forall \sigma, \tau \in I[\sigma \neq \tau \to \min(\sigma) \neq \min(\tau)]\};\\
&\mathbb{P} = \mathbb{I} \times [\zeta]^{<\omega};\\
(I, J) \leq (I', J') \quad \text{iff} \quad &(I, J), (I', J') \in \mathbb{P},\ I \subseteq I',\ J \subseteq J', \text{ and } \forall \sigma \in I' \backslash I :\\
&(1)\ \forall \eta \in J \cap \xi \exists \tau \in S_\eta[\tau \triangleleft \sigma];\\
&(2)\ \forall \alpha \in J \cap \kappa[\sigma \subseteq B_\alpha].
\end{aligned}$$

Clearly $\leq$ is reflexive on $\mathbb{P}$. Suppose that $(I, J) \leq (I', J') \leq (I'', J'')$ and $\sigma \in I'' \backslash I$.

Case 1. $\sigma \notin I'$. Then (1) and (2) hold for J', hence also for J.

Case 2. $\sigma \in I'$. Clearly (1) and (2) hold for J.

It follows that $\leq$ is transitive. Clearly $\leq$ is antisymmetric. So $\leq$ is a partial order on $\mathbb{P}$. Also, $\mathbb{P}$ is σ-centered upwards. For, $\mathbb{P} = \bigcup_{I_0 \in \mathbb{I}} \{(I, J) \in \mathbb{P} : I = I_0\}$, and for any $I_0 \in \mathbb{I}$ the set $\{(I, J) \in \mathbb{P} : I = I_0\}$ is centered: if $(I_0, J_0), \ldots, (I_0, J_m) \in \mathbb{P}$, then for each $k \leq m$, $(I_0, J_k) \leq (I_0, J_0 \cup \ldots \cup J_m)$.

(1) $\forall \eta < \zeta[Q'_\eta \stackrel{\text{def}}{=} \{(I, J) \in \mathbb{P} : \eta \in J\}$ is cofinal in $\mathbb{P}]$.

For, given $\eta < \zeta$ and $(I, J) \in \mathbb{P}$ we have $(I, J) \leq (I, J \cup \{\eta\})$.

(2) $\forall k \in \omega[Q_k \stackrel{\text{def}}{=} \{(I, J) \in \mathbb{P} : \exists \sigma \in I[\sigma \not\subseteq k]\}$ is cofinal in $\mathbb{P}]$.

To prove (2), let $k \in \omega$ and $(I, J) \in \mathbb{P}$. By (1) we may assume that $0 \in J$. Let $\eta^* = \max(J \cap \xi)$ and $B^* = \bigcap_{\alpha \in J \cap \kappa} B_\alpha$.

(3) $\bigcup S_{\eta^*} \backslash B^*$ is finite.

For, let α be the maximum element of $J \cap \kappa$. Then by Theorem 34.22(ii), $B_\alpha \backslash B^*$ is finite. Choose $m \in \omega$ such that $\forall p \geq m[p \in B_\alpha \to p \in B^*]$. Also by (ii), there is an $n \in \omega$ such that $\forall \sigma \in S_{\eta^*}[\max(\sigma) \geq n \to \sigma \in [B_\alpha]^{<\omega}]$. Take any $p \geq m, n$ and suppose that $l \in \bigcup S_{\eta^*}$ with $l \geq p$. Say $l \in \sigma \in S_{\eta^*}$. Then $\max(\sigma) \geq l \geq p \geq n$, so $\sigma \in [B_\alpha]^{<\omega}$. Hence $l \in B_\alpha$ and $l \geq m$, so $l \in B^*$. This proves (3).

By (3) there is a $k' \geq k$ such that $\bigcup S_{\eta^*} \backslash B^* \subseteq k'$.

(4) $\exists \sigma \in S_{\eta^*}[k' \leq \min(\sigma)$ and $\forall \eta \in J \cap \eta^* \exists \tau \in S_\eta[\tau \triangleleft \sigma]]$.

In fact, $S_\eta \prec S_{\eta^*}$ for all $\eta \in J \cap \eta^*$, so for every $\eta \in J \cap \eta^*$ there is an $s_\eta \in \omega$ such that $\forall \tau \in S_{\eta^*}[\min(\tau) \geq s_\eta \to \exists \sigma \in S_\eta[\sigma \triangleleft \tau]]$. Let $t = \max\{s_\eta : \eta \in J \cap \eta^*\}$. Let $\sigma \in S_{\eta^*}$ be such that $\min(\sigma) \geq k', t$. Suppose that $\eta \in J \cap \eta^*$. Then there is a $\tau \in S_\eta$ such that $\tau \triangleleft \sigma$. So (4) holds.

Take σ as in (4). Since $\sigma \in S_{\eta^*}$ and $k' \leq \min(\sigma)$, it follows that $\sigma \subseteq B^*$. Since B^* is infinite, choose $m \in B^*$ such that $m > \max(\sigma)$. Let $\sigma' = \sigma \cup \{m\}$. Now $m > \max(\sigma) \geq \min(\sigma) \geq k' \geq k$, so $(I, J) \leq (I \cup \{\sigma'\}, J) \in Q_k$. This proves (2).

The following two statements are clear:

(5) If $\eta < \zeta$, $(I,J) \in Q'_\eta$, and $(I,J) \le (I',J')$, then $(I',J') \in Q'_\eta$.

(6) If $k \in \omega$, $(I,J) \in Q_k$, and $(I,J) \le (I',J')$, then $(I',J') \in Q_k$.

Now by Theorem 33.31 applied to $(\mathbb{P}, \ge)$, there is an $R \subseteq \mathbb{P}$ such that R is upwards directed and R intersects every Q'_η and every Q_k. Let $T = \bigcup\{I : (I,J) \in R\}$.

(7) T is infinite.

For, suppose that $\sigma_i \in T$ for each $i < m$. Say $\sigma_i \in I_i$ with $(I_i, J_i) \in R$, for each $i < m$. Let k be larger than each member of $\bigcup_{i<m} \sigma_i$, and choose $(I',J') \in R \cap Q_k$. Choose $\tau \in I'$ such that $\tau \nsubseteq k$. Then $\tau \neq \sigma_i$ for all $i < m$. So (7) holds.

(8) $\forall \eta < \xi[S_\eta \prec T]$.

For, suppose that $\eta < \xi$. Choose $(I,J) \in R \cap Q'_\eta$. Thus $\eta \in J$. Let $k \in \omega$ be greater than $\max(\sigma)$ for all $\sigma \in I$. Suppose that $\tau \in T$ and $\min(\tau) \ge k$. Then $\tau \notin I$. Say $\tau \in I'$ with $(I',J') \in R$. Choose $(I'',J'') \in R$ such that $(I,J),(I',J') \le (I'',J'')$. Then $\tau \in I''$ since $I' \subseteq I''$. Since $(I,J) \le (I'',J'')$ it follows that there is a $\sigma \in S_\eta$ such that $\sigma \triangleleft \tau$. This proves (8).

(9) $\forall \alpha < \kappa[T \backslash [B_\alpha]^{<\omega}$ is finite$]$.

For, let $\alpha < \kappa$. Choose $(I,J) \in R \cap Q'_\alpha$. Let k be greater than $\max(\sigma)$ for all $\sigma \in I$. Suppose that $\sigma \in T$ and $\min(\sigma) \ge k$. Say $\sigma \in I'$ with $(I',J') \in R$. So $\sigma \notin I$. Choose $(I'',J'') \in R$ such that $(I,J),(I',J') \le (I'',J'')$. Hence $\alpha \in J''$. It follows that $\sigma \subseteq B_\alpha$, i.e., $\sigma \in [B_\alpha]^{<\omega}$. So (9) holds.

Let $S_\xi = T$. Clearly (i)–(iii) hold.

$\xi = \mathfrak{p}$. Since $\mathfrak{p} < \mathfrak{t}$, we can apply Theorem 34.26 to get $S_\mathfrak{p}$ such that $S_\xi \prec S_\mathfrak{p}$ for all $\xi < \mathfrak{p}$. This completes the construction. $\square$

A *strict partial order* is a pair $(P,<)$ such that $<$ is irreflexive and transitive. Generalizing the notion for linear orderings, for infinite regular cardinals κ, λ, and a strict partial order $(X,<)$, a (κ,λ)-*gap in* $(X,<)$ is a pair (a,b) with $a \in {}^\kappa X$ and $b \in {}^\lambda X$ such that:

(1) $\forall \alpha, \beta < \kappa \forall \gamma, \delta < \lambda[\alpha < \beta$ and $\gamma < \delta$ imply that $a_\alpha < \alpha_\beta < b_\delta < b_\gamma]$.

(2) There is no $x \in X$ such that $\forall \alpha < \kappa \forall \beta < \lambda[a_\alpha < x < b_\beta]$.

A gap $(a,b) \in ({}^\kappa X) \times ({}^\lambda X)$ is *linear* provided the following conditions hold:

$$\forall x \in X[\forall \xi < \kappa[a_\xi < x] \to \exists \eta < \lambda[b_\eta < x]];$$
$$\forall x \in X[\forall \xi < \lambda[x < b_\xi] \to \exists \eta < \kappa[x < a_\eta]].$$

The gap is then called a *linear* (κ,λ)-*gap*.

Note that $\mathscr{P}(\omega)/\text{fin}$ under $<$ is a strict partial order. So is ${}^\omega\omega$ under the relation $<^*$, where $f <^* g$ iff $\exists k \forall n \ge k[f(n) \le g(n)]$ and $\{n : f(n) < g(n)\}$ is infinite.

Theorem 34.28. *If* $\mathfrak{p} < \mathfrak{t}$, *then there is a regular uncountable* $\kappa < \mathfrak{p}$ *such that there is a linear* $(\mathfrak{p},\kappa)$-*gap in* $(\mathscr{P}(\omega)/\text{fin},<)$ *and also one in* $({}^\omega\omega,<^*)$.

Proof. Let $A \in {}^{\mathfrak{p}}([\omega]^{\omega})$ be as in Theorem 34.21, κ and $B \in {}^{\kappa}([\omega]^{\omega})$ as in Theorem 34.22, $S \in {}^{\mathfrak{p}+1}\Sigma$ as in Theorem 34.27. Recall that κ is regular and uncountable. Let $A' = \{\min(\sigma) : \sigma \in S_{\mathfrak{p}}\}$, and for each $n \in A'$ let $\sigma_n \in S_{\mathfrak{p}}$ be such that $n = \min(\sigma_n)$. Clearly A' is infinite. For any $\alpha < \kappa$ and $n \in A'$ let

$$f_\alpha(n) = \begin{cases} \min(\sigma_n \backslash B_\alpha) & \text{if } \sigma_n \backslash B_\alpha \neq \emptyset, \\ 1 + \max(\sigma_n) & \text{otherwise.} \end{cases}$$

For each $\xi < \mathfrak{p}$ and each $n \in A'$ let

$$g_\xi(n) = \begin{cases} \max(\sigma) \text{ such that } \sigma \in S_{\xi+1} \text{ and } \sigma \triangleleft \sigma_n\} & \text{if there is such a } \sigma \\ 1 + \max(\sigma_n) & \text{otherwise.} \end{cases}$$

If $\alpha < \kappa$ and $n \in A'$, then $(\sigma_n \backslash B_\alpha) \cap f_\alpha(n) = \emptyset$, so

(1) For all $\alpha < \kappa$ and $n \in A'$ we have $\sigma_n \cap f_\alpha(n) \subseteq B_\alpha$.

(2) $\forall \xi < \mathfrak{p} \exists k \in \omega \forall n \in A' \backslash k [g_\xi(n) \in A_\xi]$.

In fact, $S_{\xi+1} \prec S_{\mathfrak{p}}$ by Theorem 34.27(iii). Hence there is a $k \in \omega$ such that for all $\tau \in S_{\mathfrak{p}}$, if $\min(\tau) \geq k$ then $\exists \mu \in S_{\xi+1}[\mu \triangleleft \tau]$. Suppose that $n \geq k$ and $n \in A'$. Then $\min(\sigma_\nu) = n \geq k$, so there is a $\tau \in S_{\xi+1}$ such that $\tau \triangleleft \sigma_n$. Hence the first clause in the definition of $g_\xi(n)$ applies, and we then get $g_\xi(n) \in A_\xi$ by Theorem 34.27(iv).

(3) $\forall n \in A'[n \leq g_\xi(n)]$.

In fact, let $n \in A'$. If the first clause in the definition of $g_\xi(n)$ holds, then $n = \min(\sigma_n) \leq \max(\sigma) = g_\xi(n)$, where $\sigma \triangleleft \sigma_n$ as in the definition. If the second clause holds, then $n = \min(\sigma_n) \leq 1 + \max(\sigma_n) = g_\xi(n)$. This proves (3).

(4) If $\alpha \leq \beta < \kappa$, then there is an $n_0 \in \omega$ such that $\forall n \in A' \backslash n_0 [\sigma_n \cap (B_\beta \backslash B_\alpha) = \emptyset]$.

For, assume that $\alpha \leq \beta < \kappa$. Then $B_\beta \backslash B_\alpha$ is finite by Theorem 34.22(ii). Let $n_0 > (B_\beta \backslash B_\alpha)$. Suppose that $n \in A' \backslash n_0$. Then $\min(\sigma_n) = n \geq n_0$, so $\sigma_n \cap (B_\beta \backslash B_\alpha) = \emptyset$.

(5) If $\alpha \leq \beta < \kappa$, and with n_0 as in (4), we have $\forall n \in A' \backslash n_0 [f_\beta(n) \leq f_\alpha(n)]$.

In fact, by (4) we have $\sigma_n \backslash B_\alpha \subseteq \sigma_n \backslash B_\beta$.

Case 1. $\sigma_n \backslash B_\alpha \neq \emptyset$. Then $f_\beta(n) \leq f_\alpha(n)$ by definition.

Case 2. $\sigma_n \backslash B_\alpha = \emptyset \neq \sigma_n \backslash B_\beta$. Then $f_\beta(n) = \min(\sigma_n \backslash B_\beta) \leq 1 + \max(\sigma_n) = f_\alpha(n)$.

Case 3. $\sigma_n \backslash B_\beta = \emptyset$. Clearly $f_\beta(n) = f_\alpha(n)$.

So (5) holds.

(6) $\forall \alpha < \kappa \exists \beta \in (\alpha, \kappa)[f_\beta <^* f_\alpha]$.

Suppose not; so there is an $\alpha < \kappa$ such that for all $\beta \in (\alpha, \kappa)[f_\beta \not<^* f_\alpha]$. For each $\beta < \kappa$ let $C_\beta = \{n \in A' : f_\beta(n) \geq f_\alpha(n)\}$; thus C_β is infinite since $f_\beta \not<^* f_\alpha$ if $\alpha < \beta$ and $f_\alpha \leq^* f_\beta$ if $\beta \leq \alpha$.

(7) $\forall \beta, \gamma < \kappa [\beta < \gamma \rightarrow C_\gamma \subseteq^* C_\beta]$.

In fact, suppose that $\beta < \gamma < \kappa$. By (5) choose $k \in \omega$ so that $\forall n \in A'\backslash k[f_\gamma(n) \le f_\beta(n)]$. Hence if $n \ge k$ and $n \in C_\gamma$ then $f_\alpha(n) \le f_\gamma(n) \le f_\beta(n)$. So (7) holds.

Since $\kappa < \mathfrak{p} < \mathfrak{t}$, let $D \in [\omega]^\omega$ be such that $\forall \beta \in \kappa[D \subseteq^* C_\beta]$. In particular, $D \subseteq^* C_0$, so $D\backslash C_0$ is finite. Since $C_0 \subseteq A'$, we may assume that $D \subseteq A'$. Now let

$$E = \bigcup_{n \in D} (\sigma_n \cap [n, f_\alpha(n))).$$

(8) $\forall \beta < \kappa \exists k \in \omega \forall n \in A'\backslash k \exists \tau_n \in S_\beta[\tau_n \lhd \sigma_n \text{ and } \tau_n \subseteq B_\alpha]]$.

This is clear since $S_\beta \prec S_\mathfrak{p}$ and $S_\beta\backslash[B_\alpha]^{<\omega}$ is finite, by Theorem 34.27(ii).

(9) E is infinite.

For, take any $\beta < \kappa$ and let k be as in (8). Take any $n \in D\backslash k$. Now $\max(\tau_n) < \min(\sigma_n\backslash B_\alpha)$, so $\tau_n \subseteq f_\alpha(n)$. Hence $\sigma_n \cap [n, f_\alpha(n)) \ne \emptyset$. Thus (9) holds.

(10) $\forall \beta \in (\alpha, \kappa)[D \subseteq^* \{n \in A' : f_\beta(n) = f_\alpha(n)\}]$.

In fact, suppose that $\alpha < \beta < \kappa$. Then $D \subseteq^* C_\beta$ and $f_\beta \le^* f_\alpha$, so (10) is clear.

(11) $\forall \beta \in \kappa[E \subseteq^* B_\beta]$.

For, suppose that $\beta < \kappa$. By Theorem 34.22(ii) we may assume that $\alpha < \beta$. By (10) chooose $k \in \omega$ such that $\forall n \ge k[n \in D \to f_\beta(n) = f_\alpha(n)]$. Suppose that m is greater than $\max(\sigma_p)$ for all $p < k$, and $m \in E$. Say $p \in D$ and $m \in \sigma_p \cap [p, f_\alpha(p))$. Then $p \ge k$ and so $f_\alpha(p) = f_\beta(p)$. Since $f_\beta(p) \subseteq B_\beta$ we get $m \in B_\beta$. This proves (11).

(12) $\forall \xi < \mathfrak{p}[E \cap A_\xi$ is infinite].

For, suppose that $\xi < \mathfrak{p}$. By (8) choose $k \in \omega$ such that $\forall n \in A'\backslash k \exists \tau_n \in S_{\xi+1}[\tau_n \lhd \sigma_n$ and $\tau_n \subseteq B_\alpha]$. Take any $n \in D\backslash k$. Then $\tau_n \in S_{\xi+1}$, so $\max(\tau_n) \in A_\xi$ by Theorem 34.27(iv). Also $\max(\tau_n) \in \sigma_n$. Since $\tau_n \subseteq f_\alpha(n)$, because $\tau_n \subseteq B_\alpha$, we have $\max(\tau_n) \in E$. So (12) holds.

Now (9), (11), (12) contradict Theorem 34.22(iii). Hence (6) holds.

(13) If $\xi < \eta < \mathfrak{p}$, then $\exists k \in \omega \forall n \in A'\backslash k[g_\xi(n) < g_\eta(n)]$.

For, assume that $\xi < \eta < \mathfrak{p}$. Then $S_{\xi+1} \prec S_{\eta+1} \prec S_\mathfrak{p}$, so

$$\begin{aligned}&\exists k \in \omega \forall \sigma[\min(\sigma) \ge k \to \exists \tau \in S_{\eta+1}[\tau \lhd \sigma]];\\ &\exists l \in \omega \forall \tau[\min(\tau) \ge l \to \exists \rho \in S_{\xi+1}[\rho \lhd \tau]].\end{aligned}$$

Let $s = \max(k, l)$. Suppose that $n \ge s$ and $n \in A'$. Then $\min(\sigma_n) = n \ge k$, so there is a $\tau \in S_{\eta+1}$ such that $\tau \lhd \sigma_n$. Also, $\min(\tau) \ge \min(\sigma_n) = n \ge l$, so there is a $\rho \in S_{\xi+1}$ such that $\rho \lhd \tau$. Now $g_\xi(n) = \max(\rho) < \max(\tau) = g_\eta(n)$. This proves (13).

(14) $\forall \xi < \mathfrak{p} \forall \alpha < \kappa[g_\xi \le^* f_\alpha]$.

For, suppose that $\xi < \mathfrak{p}$ and $\alpha < \kappa$. By (8), $\exists N \forall n \in A'\backslash N \exists \tau \in S_{\xi+1}[\tau \lhd \sigma_n$ and $\tau \subseteq B_\alpha]$. Thus for any $n \in A'\backslash N$,

$$\begin{aligned} g_\xi(n) &= \max(\tau) \le \min(\sigma_n\backslash B_\alpha) = f_\alpha(n)\\ \text{or } g_\xi(n) &= \max(\tau) \le 1 + \max(\sigma_n) = f_\alpha(n),\end{aligned}$$

proving (14).

(15) Suppose that $f \in {}^{A'}\omega$ and $\forall\alpha < \kappa[f \leq^* f_\alpha]$. Then $\exists\xi < \mathfrak{p}[f \leq^* g_\xi]$.

For, suppose that $f \in {}^{A'}\omega$ and $\forall\alpha < \kappa[f \leq^* f_\alpha]$. For any $n \in A'$ let

$$f'(n) = \begin{cases} f(n) & \text{if } f(n) \leq 1 + \max(\sigma_n), \\ 1 + \max(\sigma_n) & \text{otherwise.} \end{cases}$$

(16) $\forall\alpha < \kappa[f' \leq^* f_\alpha]$.

For, choose $k \in \omega$ such that $\forall n \in A'\backslash k[f(n) \leq f_\alpha(n)]$. Now $\forall n \in A'\backslash k[f'(n) \leq 1 + \max(\sigma_n)]$, hence $\forall n \in A'\backslash k[f'(n) \leq f_\alpha(n)]$. Thus $f' \leq^* f_\alpha$.

(17) If $\xi < \mathfrak{p}$ and $f' \leq^* g_\xi$, then $f \leq^* g_\xi$.

In fact, suppose that $\xi < \mathfrak{p}$ and $f' \leq^* g_\xi$. Choose k so that $\forall n \in A'\backslash k[f'(n) \leq g_\xi(n)]$. Since always $g_\xi(n) \leq 1 + \max(\sigma_n)$, it follows that $\forall n \in A'\backslash k[f(n) \leq g_\xi(n)]$. Thus $f \leq^* g_\xi$.

By (16) and (17) we may assume that $f(n) \leq 1 + \max(\sigma_n)$ for all $n \in A'$. Now let $D = \bigcup_{n\in A'}(\sigma_n \cap f(n))$.

(18) $\forall\alpha < \kappa[D\backslash B_\alpha$ is finite].

In fact, fix $\alpha < \kappa$. Then $D\backslash B_\alpha = \bigcup_{n\in A'}((\sigma_n \cap f(n))\backslash B_\alpha)$. Choose k so that $\forall n \in A'\backslash k[f(n) \leq f_\alpha(n)]$. Then if $n \in A'\backslash k$ and $\sigma_n\backslash B_\alpha \neq \emptyset$, then $f(n) \leq \min(\sigma_n\backslash B_\alpha)$, and so $(\sigma_n\backslash B_\alpha) \cap f(n) = \emptyset$. It follows that

$$D\backslash B_\alpha = \bigcup\{\sigma_n \cap f(n) : n \in A' \text{ and } n < k\},$$

and so $D\backslash B_\alpha$ is finite.

It now follows by Theorem 34.22(iii) that there is a $\xi < \mathfrak{p}$ such that $D \cap A_\xi$ is finite. By (2) let k be such that $\forall n \in A'\backslash k[g_\xi(n) \in A_\xi]$. Let $l \in \omega$ be such that $\forall n \in A'\backslash l\exists\tau \in S_{\xi+1}[\tau \triangleleft \sigma_n]]$. Hence if $n \in A'$, $n \geq k, l$, and $g_\xi(n) < f(n)$, then $g_\xi(n) = \max(\tau) \in \sigma_n \cap f(n) \cap A_\xi$, i.e., $g_\xi(n) \in D \cap A_\xi$. Since $D \cap A_\xi$ is finite and $g_\xi(n) \geq n$ for all n by (3), it follows that $f \leq^* g_\xi$, proving (15).

(19) Suppose that $f \in {}^{A'}\omega$ and $\forall\xi < \mathfrak{p}[g_\xi \leq^* f]$. Then $\exists\alpha < \kappa[f_\alpha \leq^* f]$.

In fact, suppose that $f \in {}^{A'}\omega$ and $\forall\xi < \mathfrak{p}[g_\xi \leq^* f]$, but $\forall\alpha < \kappa[f_\alpha \not\leq^* f]$.

(20) There is an infinite $C \subseteq A'$ such that $\forall\alpha < \kappa[C\backslash\{n \in A' : f_\alpha(n) > f(n)\}$ is finite].

For, let $a_\alpha = \{n \in A' : f_\alpha(n) > f(n)\}$ for each $\alpha < \kappa$. Suppose that F is a finite nonempty subset of κ, and let β be the largest member of F. By (5), $\{n \in A' : f_\beta(n) > f_\alpha(n)\}$ is finite for all $\alpha \in F$, hence also $\bigcup_{\alpha\in F}\{n \in A' : f_\beta(n) > f_\alpha(n)\}$ is finite. Hence the first set in the following sequence is infinite:

$$\begin{aligned} &\{n \in A' : f_\beta(n) > f(n)\} \cap \bigcap_{\alpha\in F}\{n \in A' : f_\beta(n) \leq f_\alpha(n)\} \\ &\subseteq \{n \in A' : \forall\alpha \in F[f_\alpha(n) > f(n)]\} \\ &= \bigcap_{\alpha\in F} a_\alpha. \end{aligned}$$

Now since $\kappa < \mathfrak{p}$, the existence of C as in (20) follows.

Let $D = \bigcup_{n\in C}(\sigma_n \cap f(n))$.

(21) $\forall\alpha < \kappa[D\backslash B_\alpha$ is finite].

In fact, fix $\alpha < \kappa$. Then $D\backslash B_\alpha = \bigcup_{n\in C}((\sigma_n \cap f(n))\backslash B_\alpha)$. By the definition of C, choose k so that $\forall n \geq k[n \in C \rightarrow f(n) < f_\alpha(n)]$. Then if $n \geq k$, $n \in C$, and $\sigma_n\backslash B_\alpha \neq \emptyset$, then $f(n) \leq \min(\sigma_n\backslash B_\alpha)$, and so $(\sigma_n\backslash B_\alpha) \cap f(n) = \emptyset$. It follows that

$$D\backslash B_\alpha = \bigcup\{\sigma_n \cap f(n) : n \in C \text{ and } n < k\},$$

and so $D\backslash B_\alpha$ is finite.

Now by Theorem 34.22(iii) there is a $\xi < \mathfrak{p}$ such that $D\cap A_\xi$ is finite. By (2), let $k \in \omega$ be such that $\forall n \in A'\backslash k[g_\xi(n) \in A_\xi]$. Let $l \in \omega$ be such that $\forall n \in A'\backslash l\exists\tau \in S_{\xi+1}[\tau \triangleleft \sigma_n]$. Since $g_\xi <^* g_{\xi+1} \leq^* f$, choose s so that $\forall n \geq s[g_\xi(n) < f(n)]$. Hence if $n \in C$, $n \geq s, k, l$, $g_\xi(n) = \max(\tau) \in \sigma_n \cap f(n) \cap A_\xi$, i.e., $g_\xi(n) \in D \cap A_\xi$. Since C is infinite and $g_\xi(n) \geq n$ for all $n \in C$, this contradicts $D \cap A_\xi$ being finite. So (19) holds.

Now by (6) and using $\kappa < \mathfrak{t}$, there is a strictly increasing $\alpha \in {}^\kappa\kappa$ such that $\forall\gamma, \delta < \kappa[\gamma < \delta \rightarrow f_{\alpha_\delta} <^* f_{\alpha_\gamma}]$.

Now let $M = A' \times \omega$ and define

$$U_\xi = \{(n, i) : n \in A',\ i \leq g_\xi(n)\} \quad \text{for all } \xi < \mathfrak{p}$$
$$V_\gamma = \{(n, i) : n \in A',\ i \leq f_{\alpha_\gamma}(n)\} \quad \text{for all } \gamma < \kappa.$$

(22) $(\langle[U_\xi] : \xi < \mathfrak{p}\rangle, \langle[V_\gamma] : \gamma < \kappa\rangle$ is a linear $(\mathfrak{p}, \kappa)$-gap in $(\mathscr{P}(M)/\text{fin}, <)$.

To prove (22), first note:

(23) $\forall\beta, \gamma < \kappa[\beta < \gamma \rightarrow [V_\gamma] < [V_\beta]]$.

In fact, suppose that $\beta < \gamma < \kappa$. Then $f_{\alpha_\gamma} <^* f_{\alpha_\beta}$. Hence there is a $k \in \omega$ such that $\forall n \in A'\backslash k[f_{\alpha_\gamma}(n) \leq f_{\alpha_\beta}(n)]$. Hence $V_\gamma\backslash V_\beta = \{(n, i) : n \in A' \cap k, f_{\alpha_\beta}(n) < i \leq f_{\alpha_\gamma}(n)\}$ is finite. So $[V_\gamma] \leq [V_\beta]$.

Now because $f_{\alpha_\gamma} <^* f_{\alpha_\beta}$, the set $\{n \in A' : f_{\alpha_\gamma}(n) < f_{\alpha_\beta}(n)\}$ is infinite. Hence $V_\beta\backslash V_\gamma = \{(n, i) : n \in A' \cap k, f_{\alpha_\gamma}(n) < i \leq f_{\alpha_\beta}(n)\}$ is infinite. So $[V_\gamma] < [V_\beta]$.

(24) $\forall\xi, \eta < \mathfrak{p}[\xi < \eta \rightarrow [U_\xi] < [U_\eta]]$.

For, assume $\xi < \eta < \mathfrak{p}$. By (13) $\exists k \in \omega\forall n \in A'\backslash k[g_\xi(n) < g_\eta(n)]$. Hence $U_\xi\backslash U_\eta = \{(n, i) : n < k, g_\eta(n) < i \leq g_\xi(n)\}$ is finite, so $[U_\xi] \leq [U_\eta]$.

Now because $g_\xi <^* g_\eta$, the set $\{n \in A' : g_\xi(n) < g_\eta(n)\}$ is infinite. Hence $U_\eta\backslash U_\xi = \{(n, i) : n \in A' \cap k, g_\xi(n) < i \leq g_\eta(n)\}$ is infinite. So $[U_\xi] < [U_\eta]$.

(25) $\forall\xi < \mathfrak{p}\forall\gamma < \kappa[[U_\xi] \leq [V_\gamma]]$.

In fact, let $\xi < \mathfrak{p}$ and $\gamma < \kappa$. By (14) choose k such that $\forall n \in A'\backslash k[g_\xi(n) \leq f_{\alpha_\gamma}(n)]$. Hence $U_\xi\backslash V_\gamma = \{(n, i) : n < k, f_{\alpha_\gamma}(n) < i \leq g_\xi(n)\}$ is finite, and so $[U_\xi] \leq [V_\gamma]$.

Now suppose that $W \subseteq M$ and $[W] \leq [V_\gamma]$ for all $\gamma < \kappa$.

(26) For all $n \in A'$, the set $\{i : (n,i) \in W\}$ is finite.

For, let $F \in [M]^{<\omega}$ be such that $\forall (m,i) \in M[(m,i) \notin F$ and $(m,i) \in W \to (m,i) \in V_0]$. Then

$$\{i : (n,i) \in W\} \subseteq \{i : \exists m[(m,i) \in F]\} \cup \{i : i \le f_{\alpha_0}(n)\}.$$

So (26) holds.

Now for each $n \in A'$ let $f(n) = \sup\{i : (n,i) \in W\}$, with $\sup \emptyset = 0$.

(27) $f \le^* f_{\alpha_\gamma}$ for all $\gamma < \kappa$.

In fact, with $\gamma < \kappa$ let $F \in [M]^{<\omega}$ be such that $\forall (n,i) \in M[(n,i) \notin F$ and $(n,i) \in W \to (n,i) \in V_\gamma]$. Let k be greater than all m such that $(m,i) \in F$ for some i. Suppose that $n \in A'$ and $n \ge k$. If $(n,i) \in W$, then $(n,i) \notin F$, hence $(n,i) \in V_\gamma$; it follows that $i \le f_{\alpha_\gamma}(n)$. Hence $f(n) \le f_{\alpha_\gamma}(n)$. This proves (27).

Now α is strictly increasing, so $\gamma \le \alpha_\gamma$ for all $\gamma < \kappa$. Hence by (27) we have $f \le^* f_\gamma$ for all $\gamma < \kappa$. Hence by (15) there is a $\xi < \mathfrak{p}$ such that $f \le^* g_\xi$. Say $k \in \omega$ and $\forall n \ge k[f(n) \le g_\xi(n)]$. Let $F = \{(m,i) : m < k$ and $(m,i) \in W\}$. Suppose that $(n,i) \in M\backslash F$ and $(n,i) \in W$. Then $n \ge k$, so $f(n) \le g_\xi(n)$. Also $i \le f(n)$, so $(n,i) \in U_\xi$. Thus we have shown:

(28) $[W] \le [U_\xi]$.

Now suppose that $X \subseteq M$ and $[U_\xi] \le [X]$ for all $\xi < \mathfrak{p}$. For each $n \in \omega$ let

$$f(n) = \begin{cases} \min\{i \in \omega : (n,i) \notin X\} & \text{if this set is nonempty,} \\ 2 + \max(\sigma_n) & \text{otherwise.} \end{cases}$$

Suppose that $\xi < \mathfrak{p}$. Then $U_\xi \backslash X$ is finite. So for $F = U_\xi \backslash X$ we have $\forall (n,i) \in M\backslash F[(n,i) \in U_\xi \to (n,i) \in X]$. Let k be greater than each $n \in A'$ such that $(n,i) \in F$ for some i. Suppose that $n \ge k$. Then $(n,i) \in U_\xi \backslash F$ for all $i \le g_\xi(n)$, so $(n,i) \in X$ for all $i \le g_\xi(n)$. Hence $f(n) > g_\xi(n)$. Thus $g_\xi \le^* f$.

This is true for all $\xi < \mathfrak{p}$. By (19) there is a $\gamma < \kappa$ such that $f_\gamma <^* f$. Hence $f_{\alpha_\gamma} <^* f$. So there is a $k \in \omega$ such that $\forall n \ge k[f_{\alpha_\gamma}(n) < f(n)]$. Let $F = \{(m,i) : m < k, i \le f_{\alpha_\gamma}(m)\}$. Suppose that $(n,i) \in M\backslash F$ and $(m,i) \in V_\gamma$. Then $i \le f_{\alpha_\gamma}(n)$. It follows that $n \ge k$. Hence $f_{\alpha_\gamma}(n) < f(n)$. Hence $(n,i) \in X$. Thus $[V_\gamma] \le [X]$.

This finishes the proof of (22), which gives the first conclusion of the theorem.

Now let $h : \omega \to A'$ be the strictly increasing enumeration of A'.

(29) If $u, v \in {}^{A'}\omega$, then $u \le^* v$ iff $(u \circ h) \le^* (v \circ h)$.

For, suppose that $u, v \in {}^{A'}\omega$. First suppose that $u \le^* v$. Choose k so that $\forall n \in A'\backslash k[u(n) \le v(n)]$. Then for any $n \ge k$ we have $h(n) \ge n \ge k$, so $u(h(n)) \le v(h(n))$. Thus $(u \circ h) \le^* (v \circ h)$. Second suppose that $(u \circ h) \le^* (v \circ h)$. Let k be such that $\forall n \ge k[u(h(n) \le v(h(n))]$. Suppose that $m \in A'\backslash h(k)$. Say $m = h(n)$. Then $n \ge k$, so $u(m) = u(h(n)) \le v(h(n)) = vm)$. Hence $u \le^* v$. This proves (29).

(30) If $u, v \in {}^{A'}\omega$, then $u <^* v$ iff $(u \circ h) <^* (v \circ h)$.

This is true since $\{n \in A : u(n) < v(n)\}$ is infinite iff $\{n \in \omega : u(h(n)) < v(h(n))\}$ is infinite.

Now the second conclusion of the theorem follows from (5), (6). (13), (14), (15), (19), (29), and (30). $\square$

Proposition 34.29. *The forcing order* $([\omega]^\omega, \leq^*, \omega)$ *is* $\mathfrak{t}$*-closed.* $\square$

For brevity let $\mathbb{P} = ([\omega]^\omega, \leq^*, \omega)$.

Proposition 34.30. *Suppose that G is M-generic over $\mathbb{P}$. Then G is an ultrafilter on ω.*

Proof. Let $A \subseteq \omega$ in $M[G]$; we want to show that $A \in G$ or $(\omega \backslash A) \in G$. By Theorem 16.10, $A \in M$. Let $D = \{a \in P : a \leq^* A \text{ or } a \leq^* (\omega \backslash A)\}$. Then D is dense in P, since if $B \in [\omega]^\omega$ then $B \cap A$ or $B \backslash A$ is infinite. Take $a \in G \cap D$. Then $A \in G$ or $(\omega \backslash A) \in G$. $\square$

Proposition 34.31. *If $A, B \in [\omega]^\omega$ in M and G is M generic over $\mathbb{P}$, then $A \leq^* B$ iff $M[G] \models \check{A} \leq^* \check{B}$.*

Proof. We take $A \leq^* B$ to mean that there exist an $m \in \omega$ and a bijection f from m onto $A \backslash B$. If such f, g exist in M, then $m, f \in M[G]$ and so $A \leq^* B$ in $M[G]$. If they exist in $M[G]$, then $f, g \in M$ by Theorem 16.10. $\square$

Proposition 34.32. *If $A, B \in [\omega]^\omega$, and G is M generic over $\mathbb{P}$, then $A <^* B$ in M iff $M[G] \models \check{A} <^* \check{B}$.*

Proof. We take $A <^* B$ to mean that $A \leq^* B$ and there is an injection g from ω into $B \backslash A$. Hence the result follows by the above argument. $\square$

Proposition 34.33. *If G is M generic over $\mathbb{P}$, then $M[G] \models \mathfrak{t} \leq \mathfrak{t}^M$.*

Proof. Suppose in M that $\langle A_\alpha : \alpha < \mathfrak{t} \rangle$ is strictly decreasing under $\leq^*$, each $A_\alpha \in [\omega]^\omega$, such that there is no $B \in [\omega]^\omega$ such that $\forall \alpha < \mathfrak{t}[B \leq^* A_\alpha]$. Then in $M[G]$ the sequence $\langle A_\alpha : \alpha < \mathfrak{t} \rangle$ is strictly decreasing under $\leq^*$ by Proposition 34.32. Suppose in $M[G]$ that $B \in [\omega]^\omega$ such that $\forall \alpha < \mathfrak{t}[B \leq^* A_\alpha]$. Then by Theorem 16.10 $B \in M$, contradiction. $\square$

Proposition 34.34. *If G is M generic over $\mathbb{P}$, then $M[G] \models \mathfrak{t} = \mathfrak{t}^M$.*

Proof. Suppose that $\kappa < \mathfrak{t}$ and $A \in {}^\kappa([\omega]^\omega) \in M[G]$ is strictly decreasing under $\leq^*$. Let $B : \kappa \times \omega \to 2$ be defined by

$$B(\xi, n) = \begin{cases} 1 & \text{if } n \in A_\xi, \\ 0 & \text{if } n \notin A_\xi. \end{cases}$$

Then $B \in M$ by Theorem 16.10. Hence $A \in M$. Let $C \in [\omega]^\omega$ be such that $C \leq^* A_\xi$ for all $\xi < \kappa$. Then this is true in $M[G]$ also by Proposition 34.31. It follows that $\kappa < \mathfrak{t}$ in the sense of $M[G]$. Since κ is arbitrary, $M[G] \models \mathfrak{t}^M \leq \mathfrak{t}$. The other inequailty holds by Proposition 34.33. $\square$

Proposition 34.35. *If G is M generic over $\mathbb{P}$, then $M[G] \models \mathfrak{p} = \mathfrak{p}^M$.*

Proof. Let A be as in Theorem 34.21 (in M) with $\kappa = \mathfrak{p}$. Suppose that $B \in [\omega]^\omega$ in $M[G]$ and $B \leq^* A_\xi$ for all $\xi < \mathfrak{p}$. Then $B \in M$ by Theorem 16.10, contradiction. Hence $M[G] \models \mathfrak{p} \leq \mathfrak{p}^M$.

Now suppose that A is as in Theorem 34.21 (in $M[G]$) for any $\kappa < \mathfrak{p}$. Then $A \in M$ by the argument in the proof of Proposition 34.33. Hence there is a $C \in [\omega]^\omega$ in M such that $\forall \xi < \kappa[C \leq^* A_\xi]$. Since $C \in M[G]$, it follows that $\kappa < \mathfrak{p}$ in the sense of $M[G]$. Hence $\mathfrak{p}^M \leq \mathfrak{p}$. □

Theorem 34.36. $M[G] \models \mathfrak{t} \leq \mathfrak{t}(\omega, G)$.

Proof. Working in $M[G]$, suppose that $\kappa < \mathfrak{t}$, $\langle P_n : n \in \omega \rangle$ is a sequence of finite trees each with a single root, and $A \in {}^\kappa(\prod_{n\in\omega} P_n)$ is such that $\langle [A_\xi] : \xi < \kappa \rangle$ is strictly increasing and unbounded. Wlog $\forall n \in \omega[P_n \subseteq \omega]$ and $\forall m, n \in \omega[m \neq n \to P_n \cap P_m = \emptyset]$. Let $B : \kappa \times \omega \to \omega$ be defined by $B(\xi, n) = A_\xi(n)$. Then $B \in M$ by Theorem 16.10. Hence $A \in M$. This is a contradiction. □

Proposition 34.37. *If there is a (κ_1, κ_2)-linear gap in $({}^\omega\omega, <')$, then there is a (κ_2, κ_1)-linear gap in $({}^\omega\omega, <')$.*

Suppose that $f \in {}^{\kappa_1}({}^\omega\omega)$, $g \in {}^{\kappa_2}({}^\omega\omega)$ and (f, g) is a (κ_1, κ_2)-linear gap in $({}^\omega\omega, <')$. For $\alpha < \kappa_1$, $\beta < \kappa_2$, and $n \in \omega$, let

$$g'_\alpha(n) = \begin{cases} g_0(n) - f_\alpha(n) & \text{if } f_\alpha(n) \leq g_\alpha(n), \\ 0 & \text{otherwise;} \end{cases}$$

$$f'_\beta(n) = \begin{cases} g_0(n) - g_\beta(n) & \text{if } g_\beta(n) \leq g_0(n), \\ 0 & \text{otherwise;} \end{cases}$$

Suppose that $\alpha < \beta < \kappa_2$. Choose $k \in \omega$ such that $\forall n \geq k[g_\beta(n) \leq g_\alpha(n)]$. Suppose that $n \geq k$.

Case 1. $g_\alpha(n) \leq g_0(n)$. Then $f'_\alpha(n) \leq f'_\beta(n)$.

Case 2. $g_\alpha(n) > g_0(n)$. Then $f'_\alpha(n) = 0$.

Choose $l \in \omega$ so that $\forall n \geq l[g_\alpha(n) \leq g_0(n)]$. Then for $n \geq k, l$ and $g_\beta(n) < g_\alpha(n)$ we have $f_\alpha(n) < f_\beta(n)$. It follows that $f'_\alpha <' f'_\beta$.

Now suppose that $\alpha < \beta < \kappa_1$. Choose $k \in \omega$ such that $\forall n \geq k[f_\alpha(n) \leq f_\beta(n)]$. Suppose that $n \geq k$.

Case 1. $f_\beta(n) \leq g_0(n)$. Then $g'_\beta(n) \leq g'_\alpha(n)$.

Case 2. $f_\beta(n) > g_0(n)$. Then $g'_\beta(n) = 0$.

Choose $l \in \omega$ so that $\forall n \geq l[f_\beta(n) \leq g_0(n)]$. Then for $n \geq k, l$ and $f_\alpha(n) < f_\beta(n)$ we have $g_\beta(n) < g_\alpha(n)$. It follows that $g'_\beta < g'_\alpha$.

Next suppose that $\alpha < \kappa_1$ and $\beta < \kappa_2$. Choose $k \in \omega$ such that $\forall n \geq k[f_\alpha(n) \leq g_\beta(n)]$ and choose $l \in \omega$ such that $\forall n \geq l[g_\beta(n) \leq g_0(n)]$. Then for all $n \geq k, l$ we have $f'_\beta(n) \leq g'_\alpha(n)$.

Suppose that $h \in {}^\omega\omega$ and $h <' g'_\xi$ for all $\xi < \kappa_1$. Fix $\xi < \kappa_1$. Choose $k \in \omega$ so that $\forall n \geq k[h(n) \leq g'_\xi(n)]$ and choose $l \in \omega$ so that $\forall n \geq l[f_\xi(n) \leq g_0(n)]$. Then for all $n \geq k, l$

we have $h(n) \le g_0(n) - f_\xi(n)$, and hence $f_\xi(n) \le g_0(n) - h(n)$. It follows that there is an $\eta < \kappa_2$ such that $\exists r \forall n \ge r[g_\eta(n) \le g_0(n) - h(n)]$. Hence $h(n) \le g_0(n) - g_\eta(n) = f'_\eta(n)$. Thus $h <' f'_\eta$.

Suppose that $h \in {}^\omega\omega$ and $f'_\xi <' h$ for all $\xi < \kappa_2$. Fix $\xi < \kappa_2$. Choose $k \in \omega$ so that $\forall n \ge k[f'_\xi(n) \le h(n)]$ and choose $l \in \omega$ so that $\forall n \ge l[g_\xi(n) \le g_0(n)]$. Then for all $n \ge k, l$ we have $g_0(n) - g_\xi(n) \le h(n)$. Let $s(n) = \min(g_0(n), h(n))$ for all $n \in \omega$. Then for all $n \ge k, l$ we have $g_0(n) - g_\xi(n) \le s(n)$, so $g_0(n) - s(n) \le g_\xi(n)$. It follows that there is an $\eta < \kappa_1$ such that for some r, $\forall n \ge r[g_0(n) - s(n) \le f_\eta(n)$. Take t so that $\forall n \ge t[f_\eta(n) \le g_0(n)]$. Then $\forall n \ge r, t[g'_\eta(n) = g_0(n) - f_\eta(n) \le s(n) \le h(n)$. So $g'_\eta \le h$. □

Theorem 34.38. *Suppose that $\mathfrak{p} < \mathfrak{t}$ and G is M-generic over $\mathbb{P}$. Then $M[G] \models \mathfrak{p}(\omega, G) \le \mathfrak{p}$.*

Proof. By Theorem 34.28 let κ be an uncountable regular cardinal less than $\mathfrak{p}$ and (f, g) be a linear $(\mathfrak{p}, \kappa)$-gap in $({}^\omega\omega, <^*)$, in $M[G]$. Note that f_ξ and g_α are in M, by Theorem 16.10. Let $k \in \omega$ be such that $\forall n \ge k[g_1(n) < g_0(n)]$. Define

$$g'_0(n) = \begin{cases} 1 & \text{if } n < k, \\ g_0(n) & \text{if } n \ge k. \end{cases}$$

Thus $\forall n \in \omega[g'_0(n) > 0]$. Now for each $\alpha \in \kappa\backslash\{0\}$ let $l_\alpha \in \omega$ be such that $\forall n \ge l_\alpha[g_\alpha(n) < g_0(n)]$. Define

$$g'_\alpha(n) = \begin{cases} g_\alpha(n) & \text{if } n \ge l_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\forall \alpha < \kappa \forall n \in \omega[g'_\alpha(n) < g_0(n)]$.

For each $\xi < \mathfrak{p}$ let $m_\xi \in \omega$ be such that $\forall n \ge m_\xi[f_\xi(n) < g_0(n)]$. Define

$$f'_\xi(n) = \begin{cases} f_\xi(n) & \text{if } n \ge m_\xi, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\forall \xi < \kappa \forall n \in \omega[f'_\xi(n) < g_0(n)]$.

Thus we may assume that $\forall n \in \omega[g_0(n) > 0]$, $\forall \alpha < \kappa \forall n \in \omega[g_\alpha(n) < g_0(n)]$, and $\forall \xi < \mathfrak{p} \forall n \in \omega[f_\xi(n) < g_0(n)]$.

Now for any $n \in \omega$ let $X_n = (g_0(n), \le)$ and $X = \prod_{n \in \omega} X_n/G$. Let $h_\alpha(n) = g_{1+\alpha}(n)$ for all $\alpha < \kappa$. It suffices now to show that $(\langle [f_\xi] : \xi < \mathfrak{p}\rangle, \langle [h_\eta] : \eta < \kappa\rangle)$ is a gap in X.

If $\xi < \eta < \mathfrak{p}$, then $\{n \in \omega : f_\xi(n) \ge f_\eta(n)\}$ is finite, and hence its complement is in G; so $[f_\xi] < [f_\eta]$. Similarly, $\alpha < \beta < \kappa$ implies that $[h_\beta] < [h_\alpha]$. Also, by the same argument $[f_\xi] < [h_\alpha]$ for all $\xi < \mathfrak{p}$ and $\alpha < \kappa$.

Suppose that $k \in {}^\omega\omega$ and $\forall \eta < \kappa[[k] \le [h_\eta]]$. Thus $\forall \eta < \kappa[\{n \in \omega : k(n) \le h_\eta(n)\} \in G]$. Let $p \in G$ such that

$$p \Vdash \forall \eta < \kappa[\{n \in \omega : \check{k}(n) \le \check{h}_\eta(n)\} \in \Gamma].$$

For each $\eta < \kappa$ let $A_\eta = \{n \in \omega : k(n) \le h_\eta(n)\}$.

(1) $\forall\eta < \kappa[p \subseteq^* A_\eta]$.

For, fix $\eta < \kappa$ and suppose that $p \not\subseteq^* A_\eta$. Then $p\backslash A_\eta$ is infinite. Let H be M-generic over $\mathbb{P}$ with $p\backslash A_\eta \in H$. Now $p \in H$, so $[k]_H \leq [h_\eta]_H$. Also $\omega\backslash A_\eta = \{n \in \omega : h_\eta(n) < k(n)\}$ and $(\omega\backslash A_\eta) \in H$, so $[h_\eta]_H < [k]_H$, contradiction. This proves (1).

Now for any $n \in \omega$ let

$$\tilde{k}(n) = \begin{cases} k(n) & \text{if } n \in p, \\ 0 & \text{otherwise.} \end{cases}$$

(2) $\forall\eta < \kappa[\tilde{k} \leq^* h_\eta]$.

For, take any $\eta < \kappa$. By (1) let $k \in \omega$ be such that $\forall n \geq k[n \in p \to n \in A_\eta]$. Thus $\forall n \geq k[n \in p \to k(n) \leq h_\eta(n)]$. So $\forall n \geq k[\tilde{k}(n) \leq h_\eta(n)]$. So (2) holds.

It follows that there is a $\gamma < \mathfrak{p}$ such that $\tilde{k} <^* f_\gamma$. Since $p \in G$, we get $[k] = [\tilde{k}] < [f_\gamma]$. This proves that $(\langle [f_\xi] : \xi < \mathfrak{p}\rangle, \langle [h_\eta] : \eta < \kappa\rangle)$ is a gap in X. □

Theorem 34.39. p=t

Proof. Suppose that $\mathfrak{p} < \mathfrak{t}$. Let G be M-generic over $\mathbb{P}$. Then

$$M[G] \models \mathfrak{p}(\omega, G) \leq \mathfrak{p} < \mathfrak{t} \leq \mathfrak{t}(\omega, G) = \mathfrak{p}(\omega, G),$$

contradiction. □

35. Consistency results concerning $\mathscr{P}(\omega)/\mathrm{fin}$

We give relative consistency theorems which show that consistently each of the functions described in the diagram at the end of chapter 33 can be less than 2^ω. For the first consistency result, concerning $\mathfrak{a}$, we need to go into the theory of products of forcing orders.

If $\mathbb{P}_0$ and $\mathbb{P}_1$ are forcing orders, their *product* is the cartesian product $\mathbb{P}_0 \times \mathbb{P}_1$ with the order relation

$$(p_0, p_1) \le (q_0, q_1) \quad \text{iff} \quad p_0 \le q_0 \text{ and } p_1 \le q_1.$$

We define $i_0 : \mathbb{P}_0 \to \mathbb{P}_0 \times \mathbb{P}_1$ and $i_0 : \mathbb{P}_0 \to \mathbb{P}_0 \times \mathbb{P}_1$ by $i_0(p) = (p, 1)$ and $i_1(p) = (1, p)$.

Proposition 35.1. *i_0 and i_1 are complete embeddings.*

Proof. See the definition of complete embedding just before Proposition 26.2. Only (4) needs thought. For i_0, given $(p_0, p_1) \in \mathbb{P}_0 \times \mathbb{P}_1$ we take p_0 to be the reduction. Suppose that $p' \in \mathbb{P}_0$ and $p' \le p_0$. Then $i(p') = (p', 1)$ is compatible with (p_0, p_1); namely, (p', p_1) is below both of them. Similarly for i_1. □

Proposition 35.2. *Suppose that G is $(\mathbb{P}_0 \times \mathbb{P}_1)$-generic over M. Then $i_0^{-1}[G]$ is $\mathbb{P}_0$-generic over M, and $G = (i_0^{-1}[G] \times i_1^{-1}[G])$.*

Proof. The first assertion follows from Theorem 26.3. For the second assertion, $\subseteq$ is obvious. Now suppose that $(p_0, p_1) \in (i_0^{-1}[G] \times i_1^{-1}[G])$. Then $(p_0, 1) \in G$ and $(1, p_1) \in G$. Choose $(q_0, q_1) \in G$ below both of these. Then $(q_0, q_1) \le (p_0, p_1)$, so $(p_0, p_1) \in G$. □

Theorem 35.3. *Suppose that $G_0 \subseteq \mathbb{P}_0 \in M$ and $G_1 \subseteq \mathbb{P}_1 \in M$. Then the following conditions are equivalent:*

(i) $G_0 \times G_1$ is $(\mathbb{P}_0 \times \mathbb{P}_1)$-generic over M.

(ii) G_0 is $\mathbb{P}_0$-generic over M and G_1 is $\mathbb{P}_1$-generic over $M[G_0]$.

(iii) G_1 is $\mathbb{P}_1$-generic over M and G_0 is $\mathbb{P}_0$-generic over $M[G_1]$.

Proof. By symmetry it suffices to show that (i) and (ii) are equivalent. First suppose that $G_0 \times G_1$ is $(\mathbb{P}_0 \times \mathbb{P}_1)$-generic over M. Clearly $i_0^{-1}[G_0 \times G_1] = G_0$, so G_0 is $\mathbb{P}_0$-generic over M by Proposition 35.2. To show that G_1 is $\mathbb{P}_1$-generic over $M[G_0]$, take any dense $D \subseteq \mathbb{P}_1$, in $M[G_0]$. Let τ be a $\mathbb{P}_0$-name such that $D = \tau_{G_0}$. Choose $p_0 \in G_0$ such that

$$p_0 \Vdash (\tau \text{ is dense in } \mathbb{P}_1).$$

Let

$$D' = \{(q_0, q_1) \in (\mathbb{P}_0 \times \mathbb{P}_1) : q_0 \le p_0 \text{ and } q_0 \Vdash (\check{q}_1 \in \tau)\}.$$

(1) D' is dense below $(p_0, 1)$.

For, suppose that $(r_0, r_1) \le (p_0, 1)$. Since $r_0 \le p_0$ we have

$$r_0 \Vdash \exists x \in \check{\mathbb{P}}_1 [x \in \tau \text{ and } x \le \check{r}_1].$$

Hence by Proposition 16.16 there exist $q_0 \leq r_0$ and $q_1 \in \mathbb{P}_1$ such that

$$q_0 \Vdash (\check{q}_1 \in \tau \text{ and } \check{q}_1 \leq \check{r}_1).$$

By Theorem 16.14 we then get $q_1 \in r_1$. Hence $(q_0, q_1) \leq (r_0, r_1)$ and $(q_0, q_1) \in D'$. So (1) holds.

By (1), choose $(q_0, q_1) \in (G_0 \times G_1) \cap D'$. Then $q_0 \Vdash \check{q}_1 \in \tau$, and $q_0 \in G_0$, so $q_1 \in \tau_{G_0} = D$. Also $q_1 \in G_1$. This proves (ii).

Conversely, assume (ii).

(2) $G_0 \times G_1$ is a filter on $\mathbb{P}_0 \times \mathbb{P}_1$.

For, clearly $G_0 \times G_1$ is closed upwards. Now suppose that $(p_0, p_1), (q_0, q_1) \in (G_0 \times G_1)$. Choose $s_0 \in G_0$ with $s_0 \leq p_0, q_0$, and choose $s_1 \in G_1$ so that $s_1 \leq p_1, q_1$. Then $(s_0, s_1) \in (G_0 \times G_1)$ and $(s_0, s_1) \leq (p_0, p_1), (q_0, q_1)$. so (2) holds.

To show that $G_0 \times G_1$ is generic, suppose that $D \in M$, $D \subseteq (\mathbb{P}_0 \times \mathbb{P}_1)$, D dense. Let

$$D^* = \{p_1 \in \mathbb{P}_1 : \exists p_0 \in G_0[(p_0, p_1) \in D]\}.$$

(3) D^* is dense in $\mathbb{P}_1$.

For, take $r_1 \in \mathbb{P}_1$. Let

$$D_0 = \{p_0 \in \mathbb{P}_0 : \exists p_1 \leq r_1[(p_0, p_1) \in D]\}.$$

Then D_0 is dense in $\mathbb{P}_0$, for if $s \in \mathbb{P}_0$ then there is a $(p_0, p_1) \in D$ with $(p_0, p_1) \leq (s, r_1)$, and then $p_0 \leq s$ and $p_0 \in D_0$. It follows that there is a $p_0 \in D_0 \cap G_0$. Take $p_1 \leq r_1$ such that $(p_0, p_1) \in D$. Then $p_1 \in D^*$ and $p_1 \leq r_1$. This proves (3).

Choose $r_1 \in D^* \cap G_1$; then take $p_0 \in G_0$ such that $(p_0, p_1) \in D$. So $(p_0, p_1) \in D \cap (G_0 \times G_1)$. $\square$

Theorem 35.4. *Suppose that $G_0 \subseteq \mathbb{P}_0 \in M$ and $G_1 \subseteq \mathbb{P}_1 \in M$. Also suppose that $G_0 \times G_1$ is $(\mathbb{P}_0 \times \mathbb{P}_1)$-generic over M. (See Theorem 35.3.)*

Then $M[G_0 \times G_1] = M[G_0][G_1] = M[G_1][G_0]$.

Proof. We have $M \subseteq M[G_0][G_1]$ and $(G_0 \times G_1) \in M[G_0][G_1]$. Hence $M[G_0 \times G_1] \subseteq M[G_0][G_1]$ by Lemma 15.8. Also, $M \subseteq M[G_0 \times G_1]$ and $G_0 \in M[G_0 \times G_1]$, so $M[G_0] \subseteq M[G_0 \times G_1]$. Next, $G_1 \in M[G_0 \times G_1]$, so by Lemma 15.8, $M[G_0][G_1] \subseteq M[G_0 \times G_1]$. This proves that $M[G_0][G_1] = M[G_0 \times G_1]$. Similarly, $M[G_1][G_0] = M[G_0 \times G_1]$. $\square$

Theorem 35.5. *Suppose that $I = I_0 \cup I_1$ with $I_0, I_1 \in M$. Let G be $\mathrm{Fn}(I, 2, \omega)$-generic over M. Let $G_0 = G \cap \mathrm{Fn}(I_0, 2, \omega)$ and $G_1 = G \cap \mathrm{Fn}(I_1, 2, \omega)$. Then:*

(i) G_0 is $\mathrm{Fn}(I_0, 2, \omega)$-generic over M.

(ii) G_1 is $\mathrm{Fn}(I_1, 2, \omega)$-generic over $M[G_0]$.

(iii) $M[G] = M[G_0][G_1]$.

Proof. Define $f : \mathrm{Fn}(I_0, 2, \omega) \times \mathrm{Fn}(I_1, 2, \omega)$ by setting $f(p, q) = p \cup q$ for any $p \in \mathrm{Fn}(I_0, 2, \omega)$ and $q \in \mathrm{Fn}(I_1, 2, \omega)$. Clearly f is an isomorphism. Note that $f^{-1}(r) =$

$(r \cap \mathrm{Fn}(I_0, 2, \omega), r \cap \mathrm{Fn}(I_1, 2, \omega))$. By Lemma 25.9, $M[G] = M[f^{-1}[G]] = M[\mathrm{Fn}(I_0, 2, \omega) \times \mathrm{Fn}(I_1, 2, \omega)]$. Now (i) and (ii) hold by Theorem 35.3(ii). (iii) holds by Theorem 35.4. $\square$

Lemma 35.6. *Let M be a c.t.m. and let $I, S \in M$. Let G be $\mathrm{Fn}(I, 2, \omega)$-generic over M. Suppose that $X \in M[G]$ and $X \subseteq S$. Then $X \in M[G \cap \mathrm{Fn}(I_0, 2, \omega)]$ for some $I_0 \subseteq I$ such that $I_0 \in M$ and $|I_0| \le |S|)^M$.*

Proof. If S is finite, then $X \in M$; so assume that S is infinite. By Proposition 24.2 let τ be a nice name for a subset of $\check{S}$ such that $X = \tau_G$. Say $\tau = \bigcup_{s \in S}(\check{s} \times A_s)$, where each A_s is an antichain in $\mathrm{Fn}(I, 2, \omega)$. Let

$$I_0 = \bigcup\{\mathrm{dmn}(p) : \exists s \in S[p \in A_s]\}.$$

Let $G_0 = G \cap \mathrm{Fn}(I_0, 2, \omega)$. Thus $X \in M[G \cap \mathrm{Fn}(I_0, 2, \omega)]$. Now by ccc in M (see Lemma 16.7), each A_s is countable. Hence $|I_0| \le |S|)^M$. $\square$

Theorem 35.7. *Suppose that M satisfies CH. and $I \in M$. Let G be $\mathrm{Fn}(I, 2, \omega)$-generic over M. Then in $M[G]$ there is a mad family of size ω_1.*

Proof. For a while we work with $\mathbb{P} \stackrel{\text{def}}{=} \mathrm{Fn}(\omega, 2, \omega)$. Note that if A is an antichain in $\mathbb{P}$ then $|A| \le |\mathbb{P}| = \omega$. Hence there are at most ω_1 pairs (p, τ) such that $p \in \mathbb{P}$ and τ is a nice name for a subset of $\check{\omega}$. Let $\langle (p_\xi, \tau_\xi) : \xi < \omega_1 \rangle$ list all such pairs.

Now we define $A \in {}^{\omega_1}([\omega]^\omega)$ by recursion. Let $\langle A_n : n \in \omega \rangle$ be a system of infinite pairwise disjoint subsets of ω. Now suppose that $\xi \in [\omega, \omega_1)$ and A_η has been defined for all $\eta < \xi$. Then:

(1) There is a $B \in [\omega]^\omega$ such that the following conditions hold:
  (i) $\forall \eta < \xi[|A_\eta \cap B| < \omega]$.
  (ii) If
    (I) $p_\xi \Vdash (|\tau_\xi| = \omega)$ and $\forall \eta < \xi[p_\xi \Vdash |\tau_\xi \cap \check{A}_\eta| < \omega]$, then
    (II) $\forall n \in \omega \forall q \le p_\xi \exists r \le q \exists m \ge n[m \in B$ and $r \Vdash \check{m} \in \tau_\xi]$.

To prove (1), first note that if (1)(ii)(I) fails to hold, then we can use the proof described after Proposition 33.20 to construct B satisfying (i). So we may assume that (1)(ii)(I) holds. Now let $\langle C_i : i \in \omega \rangle$ enumerate $\{A_\eta : \eta < \xi\}$ without repetitions, and let $\langle (n_i, q_i) : i \in \omega \rangle$ enumerate $\omega \times \{q : q \le p_\xi\}$. Clearly for all $i \in \omega$ we have $p_\xi \Vdash (|\tau_\xi \backslash (\check{C}_0 \cup \ldots \cup \check{C}_i)| = \check{\omega})$; hence each q_i also forces this. So

$$q_i \Vdash \exists m \ge \check{n}_i[m \in \tau_\xi \text{ and } m \notin (\check{C}_0 \cup \ldots \cup \check{C}_i)].$$

By Theorems 16.14 and 16.15 there exist $r_i \le q_i$ and $m_i \ge n_i$ such that $m_i \notin (C_0 \cup \ldots \cup C_i)$ and $r_i \Vdash (\check{m}_i \in \tau_\xi)$. Let $B = \{m_i : i \in \omega\}$. Clearly (i) and (ii)(II) hold. Let $A_\xi = B$. This finishes the construction.

Let $\mathscr{A} = \{A_\xi : \xi < \omega_1\}$.

(2) If G is $\mathbb{P}$-generic over M, then $\mathscr{A}$ is mad in $M[G]$.

In fact, otherwise there is a $\xi < \omega_1$ such that $p_\xi \Vdash |\tau_\xi| = \omega$ and $p_\xi \Vdash \forall X \in \mathscr{A}[|\tau_\xi \cap X| < \omega]$. So (1)(i)(I) holds, and also $p_\xi \Vdash |\tau_\xi \cap \check{A}_\xi| < \omega$. So there exist a $q \leq p$ and an $n \in \omega$ such that $q \Vdash [\tau_\xi \cap A_\xi \subseteq \check{n}]$. This contradicts (1)(ii)(II). Thus (2) holds.

Now suppose that G os $\mathrm{Fn}(I, 2, \omega)$-generic over M, $X \in M[G]$, $|X| = \omega$, and $\forall Y \in \mathscr{A}[|X \cap Y| < \omega]$. By Lemma 35.6, $X \in M[G \cap \mathrm{Fn}(I_0, 2, \omega)]$for some $I_0 \subseteq I$ with $|I_0| = \omega$. Now $\mathrm{Fn}(I_0, 2, \omega) \cong \mathrm{Fn}(\omega, 2, \omega)$, so by Lemma 25.9 $M[G \cap \mathrm{Fn}(I_0, 2, \omega)] = M[H]$ for some H which is $\mathbb{P}$-generic over M. This contradicts (2). $\square$

Corollary 35.8. *It is relatively consistent that $\mathfrak{a} < 2^\omega$.* $\square$

Theorem 35.9. *It is relatively consistent that $\mathfrak{u} < 2^\omega$.*

Proof. Let M be a c.t.m. such that $2^\omega > \omega_1$ in M. Let U a nonprincipal ultrafilter on ω in M. Define

$$\begin{aligned}
&\mathbb{P} = \{(F, H) : F \in [U]^{<\omega},\ H \in [\omega]^{<\omega}\};\\
&(F, H) \leq (F', H') \quad \text{iff} \quad F \supseteq F',\ H \supseteq H',\ \forall x \in F \forall m \in H \backslash H'[m \in x].
\end{aligned}$$

Clearly $\mathbb{P}$ is ccc, by considering second coordinates.

For each $x \in U$ let

$$D_x = \{(F, H) : x \in F\}.$$

Clearly D_x is dense. For each $m \in \omega$ let

$$E_m = \{(F, H) : \exists n \geq m[n \in H]\}.$$

This is dense too: given $(F, H) \in \mathbb{P}$, choose $n \in \bigcap F \backslash m$; then $(F, H \cup \{m\}) \leq (F, H)$.

Now let G be generic over M for $\mathbb{P}$. Define

$$a = \bigcup_{(F,H) \in G} H.$$

By the density of the E_m's, a is infinite. Now suppose that $x \in U$. Choose $(F, H) \in G$ such that $x \in F$. We claim that $a \backslash x \subseteq H$. For, suppose that $m \in a \backslash H$. Say $m \in H'$ with $(F', H') \in G$. Choose $(F'', H'') \in G$ such that $(F'', H'') \leq (F, H), (F', H')$. Then $m \in H' \subseteq H''$, $m \notin H$, and $x \in F$, so $m \in x$, as desired.

Now we do an iterated forcing, using the above construction at successor steps, obtaining:

(1) an increasing sequence $\langle M_\alpha : \alpha \leq \omega_1 \rangle$ of c.t.m., with $M_0 = M$;

(2) a sequence $\langle a_\alpha : \alpha < \omega_1 \rangle$ with each a_α an infinite subset of ω in M_α;

(3) an increasing sequence $\langle U_\alpha : \alpha < \omega_1 \rangle$ of ultrafilters on ω, each $U_\alpha \in M_\alpha$;

(4) for each $\alpha < \omega_1$ we have $\forall x \in U_\alpha[a_\alpha \leq x]$;

(5) $\{a_\beta : \beta < \alpha\} \subseteq U_\alpha$ for all $\alpha < \omega_1$.

Then we let U_{ω_1} be the filter generated by $\{a_\alpha : \alpha < \omega_1\}$ in M_{ω_1}. It is an ultrafilter, since each subset of ω in M_{ω_1} is in some M_α with $\alpha < \omega_1$, by Lemma 26.14. $\square$

Lemma 35.10. *Let M be a c.t.m. of ZFC, and suppose that I is an ideal in $\mathscr{P}(\omega)^M$ containing all singletons. Define*

$$P = \{(b, y) : b \in I,\ y \in [\omega]^{<\omega}\};$$
$$(b, y) \le (b', y') \quad \textit{iff} \quad b \supseteq b',\ y \supseteq y',\ y \cap b' \subseteq y'.$$

Then P is ccc. Let G be P-generic over M, and define $d = \bigcup_{(b,y)\in G} y$. Then the following conditions hold:

(i) If $c \subseteq \omega$ and $c \notin I$, then $c \cap d$ is infinite.

(ii) If $c \subseteq \omega$ and $c \notin I$ then $c \backslash d$ is infinite.

(iii) If $b \in I$, then $b \cap d$ is finite.

Proof. Assume the hypotheses. Clearly P is ccc. For (i) and (ii), suppose that $c \subseteq \omega$ and $c \notin I$. For each $n \in \omega$ let

$$E_n = \{(b, y) : \exists m > n[m \in c \cap y\}.$$

To show that E_n is dense, let $(b, y) \in P$. Then $c\backslash b$ is infinite, as otherwise $c \subseteq b \cup (c\backslash b) \in I$. Choose $m > n$ with $m \in c\backslash b$. Then $(b, y \cup \{m\}) \in E_n$ and $(b, y \cup \{m\}) \le (b, y)$, showing that E_n is dense.

The denseness of each set E_n clearly implies (i).

Next, define for any $n \in \omega$

$$H_n = \{(b, y) \in P : \exists m > n[m \in b \cap c\backslash y]\}.$$

To show that H_n is dense, let $(b, y) \in P$ be given. Since every finite subset of ω is in I, the set c is infinite. Choose $m \in c\backslash y$ with $m > n$. Then $(b \cup \{m\}, y) \in H_n$ and $(b \cup \{m\}, y) \le (b, y)$. This shows that H_n is dense.

Now given $n \in \omega$, choose $(b, y) \in H_n \cap G$, and then choose $m > n$ such that $m \in b \cap c\backslash y$. We claim that $m \notin d$. For, suppose that $m \in d$; say $m \in y'$ with $(b', y') \in G$. Choose $(b'', y'') \in G$ such that $(b'', y'') \le (b, y), (b', y')$. Thus $y'' \cap b \subseteq y$ and $y'' \cap b' \subseteq y'$. Now $m \in y'$, so $m \in y''$; also $m \in b$, so $m \in y$, contradiction. This finishes the proof of (ii).

For (iii), suppose that $b \in I$. Now the set $\{(c, y) \in P : b \subseteq c\}$ is clearly dense, so choose $(c, y) \in G$ such that $b \subseteq c$. We claim that $b \cap d \subseteq y$. In fact, suppose that $m \in b \cap d$. Say $(e, z) \in G$ with $m \in z$. Choose $(u, v) \in G$ such that $(u, v) \le (c, y), (e, z)$. So $v \cap c \subseteq y$ and $v \cap e \subseteq z$. Now $m \in z \subseteq v$, and $m \in b \subseteq c$, so $m \in v \cap c \subseteq y$, as desired; (iii) holds. $\square$

Lemma 35.11. *We work within a c.t.m. M. Suppose that κ is an infinite cardinal, and $\langle a_\xi : \xi < \kappa\rangle$ is a system of infinite subsets of ω which is an independent system. Let $\mathscr{A} = \{a_\xi : \xi < \kappa\}$. Thus $\langle [a_\xi] : \xi < \kappa\rangle$ is a system of independent elements of $\mathscr{P}(\omega)/\mathrm{fin}$. Let A be the completion of Fr(κ), and let $\langle x_\xi : \xi < \kappa\rangle$ be the free generators of Fr(κ). Then by Sikorski's extension theorem, there is a homomorphism f from $\mathscr{P}(\omega)/\mathrm{fin}$ into*

A such that $f([a_\xi]) = x_\xi$ for every $\xi < \kappa$. Let $h(b) = f([b])$ for any $b \subseteq \omega$. So h is a homomorphism from $\mathscr{P}(\omega)$ into A such that $h(a_\xi) = x_\xi$ for every $\xi < \kappa$. Also, $h(M) = 0$ for every finite $M \subseteq \omega$.

Apply Lemma 35.10 to the ideal $\ker(h)$*, obtaining P, G, d as indicated there. Then:*

(i) If R is a finite subset of κ and $\varepsilon \in {}^R2$, then $\bigcap_{\alpha\in R} a_\alpha^{\varepsilon(\alpha)} \cap d$ is infinite.

(ii) If R is a finite subset of κ and $\varepsilon \in {}^R2$, then $\bigcap_{\alpha\in R} a_\alpha^{\varepsilon(\alpha)} \backslash d$ is infinite.

(iii) If $b \in ker(h)$, then $b \cap d$ is finite.

(iv) If $x \in \mathscr{P}(\omega) \cap M\backslash\mathscr{A}$, then $\mathscr{A} \cup \{x, d\}$ is not independent.

Proof.

(i) Let R and ε be as in (i). Then $\bigcap_{\alpha\in R} a_\alpha^{\varepsilon(\alpha)} \notin I$ by assumption, so the desired conclusion follows from (i) of Lemma 35.10.

(ii) is proved similarly, and (iii) follows from (iii) of Lemma 35.10.

Finally, for (iv), we show that if $x \in \mathscr{P}(\omega) \cap M\backslash\mathscr{A}$, then $\mathscr{A} \cup \{x, d\}$ is not an independent family.

Case 1. $h(x) = 0$. Then $x \cap d$ is finite by (iii).

Case 2. $h(x) \neq 0$. Then there is a finite subset R of κ and a $\varepsilon \in {}^R2$ such that $\bigcap_{\alpha\in R} x_\alpha^{\varepsilon(\alpha)} \leq h(x)$. It follows that $\bigcap_{\alpha\in R} a_\alpha^{\varepsilon(\alpha)} \backslash x$ is in the kernel of h, and so $\bigcap_{\alpha\in R} a_\alpha^{\varepsilon(\alpha)} \backslash x \cap d$ is finite. □

Theorem 35.12. *It is relatively consistent to have* $\mathfrak{i} < 2^\omega$.

Proof. We start with a c.t.m. M such that $2^\omega > \omega_1$ in M and with an independent family $\langle a_n : n \in \omega\rangle$ in $\mathscr{P}(\omega)$ in M. Then we do an iteration of length ω_1, applying Lemma 35.11 at successor steps, building an independent sequence $\langle a_\alpha : \alpha < \omega_1\rangle$. The final model is as desired, using Lemma 26.14. □

Lemma 35.13. *Let M be a c.t.m. of ZFC. Suppose that κ is an infinite cardinal and $\langle a_i : i < \kappa\rangle$ is a system of infinite subsets of ω such that $\langle [a_i] : i < \kappa\rangle$ is ideal independent, where $[x]$ denotes the equivalence class of x modulo the ideal fin of $\mathscr{P}(\omega)$. Then there is a generic extension $M[G]$ of M using a ccc partial order such that in $M[G]$ there is a $d \subseteq \omega$ with the following two properties:*

(i) $\langle [a_i] : i < \kappa\rangle^\frown\langle[\omega\backslash d]\rangle$ is ideal independent.

(ii) If $x \in (\mathscr{P}(\omega) \cap M)\backslash(\{a_i : i < \kappa\} \cup \{\omega\backslash d\})$, then $\langle [a_i] : i < \kappa\rangle^\frown\langle[\omega\backslash d], [x]\rangle$ is not ideal independent.

Proof. Let I be the ideal on $\mathscr{P}(\omega)$ generated by

$$\{\{m\} : m \in \omega\} \cup \{a_i \cap a_j : i, j < \kappa, i \neq j\},$$

and let f be the natural homomorphism from $\mathscr{P}(\omega)$ onto $\mathscr{P}(\omega)/I$. Note that $f(a_i) \neq 0$ for all $i < \kappa$, by ideal independence. Let B be the subalgebra of $\mathscr{P}(\omega)/I$ generated by $\{f(a_i) : i < \kappa\}$. Thus B is an atomic BA, with $\{f(a_i) : i < \kappa\}$ its set of atoms. Thus f is a homomorphism from $\mathscr{P}(\omega)$ onto B.

Now we apply Lemma 35.10 to the ideal $\ker(f)$, obtaining P, G, d as indicated there.

(1) If R is a finite subset of κ and $i \in \kappa\backslash R$, then $a_i \cap \bigcap_{j\in R}(\omega\backslash a_j) \cap d$ is infinite.

In fact, $a_i \cap \bigcap_{j\in R}(\omega\backslash a_j)$ is clearly not in the kernel of f, so (1) follows from (i) of Lemma 35.10.

(2) If R is a finite subset of κ, then $\omega\backslash \left(d \cup \bigcup_{i\in R} a_i\right)$ is infinite.

In fact, $\omega\backslash \bigcup_{i\in R} a_i$ is clearly not in the kernel of f, so (2) follows from (ii) of Lemma 35.10.

Now we can show that $\langle [a_i] : i < \kappa\rangle^\frown\langle[\omega\backslash d]\rangle$ is ideal independent. Suppose not. Then there are two possibilities.

Case 1. There exist a finite $R \subseteq \kappa$ and an $i \in \kappa\backslash R$ such that $[a_i] \le [\omega\backslash d] + \sum_{j\in R}[a_j]$. This contradicts (1).

Case 2. There is a finite $R \subseteq \kappa$ such that $[\omega\backslash d] \le \sum_{i\in R}[a_i]$. This contradicts (2).

This proves ideal independence.

It remains only to prove (ii). So, assume that $x \in (\mathscr{P}(\omega)\cap M)\backslash(\{a_i : i < \kappa\} \cup \{\omega\backslash d\})$.

Case 1. $x \in \ker(f)$. Then $[x] \le [\omega\backslash d]$ by (iii) of Lemma 1, as desired.

Case 2. $x \notin \ker(f)$. Choose $i < \kappa$ such that $f(a_i) \le f(x)$. Thus $a_i\backslash x \in \ker(f)$, Hence by (iii) of Lemma 1, $(a_i\backslash x) \cap d$ is finite. So $[a_i] \le [x] + [\omega\backslash d]$, as desired. □

Theorem 35.14. *It is relatively consistent that* $\mathrm{s_{mm}} < 2^\omega$.

Lemma 35.15. *Let M be a c.t.m. of ZFC. Suppose that α is an infinite ordinal, and $\langle a_\xi : \xi < \alpha\rangle$ is a system of infinite subsets of ω such that $\langle [a_\xi] : \xi < \alpha\rangle$ is a free sequence in* $\mathscr{P}(\omega)/\mathrm{fin}$*, where $[a_\xi]$ denotes the equivalence class of a_ξ modulo the ideal fin.*

Then there is a generic extension $M[G]$ of M using a ccc partial order such that in $M[G]$ there exist infinite $d, e \subseteq \omega$ with the following properties:

(i) $\langle [a_\xi] : \xi < \alpha\rangle^\frown\langle[\omega\backslash d], [e]\rangle$ is a free sequence.

(ii) If $x \in (\mathscr{P}(\omega) \cap M)\backslash(\{a_\xi : \xi < \alpha\} \cup \{\omega\backslash d, e\})$, then $\langle [a_\xi] : \xi < \alpha\rangle^\frown\langle[\omega\backslash d], [e], [x]\rangle$ is not a free sequence.

Proof. For each $\xi \le \alpha$, the set $\{[a_\eta] : \eta < \xi\} \cup \{-[a_\eta] : \xi \le \eta < \alpha\}$ has the fip, by the free sequence property, and we let F_ξ be an ultrafilter on $\mathscr{P}(\omega)/\mathrm{fin}$ containing this set. Let $I = \{x : -[x] \in F_\xi$ for all $\xi \le \alpha\}$. Clearly I is an ideal on $\mathscr{P}(\omega)$ and $\{m\} \in I$ for all $m \in \omega$.

(1) If $\xi < \eta < \alpha$, then $[a_\eta]_I < [a_\xi]_I$.

In fact, suppose that $\xi < \eta < \alpha$. If $\nu \le \alpha$ and $[a_\eta]\cdot -[a_\xi] \in F_\nu$, then $\eta < \nu$, hence $\xi < \nu$ and so $[a_\xi] \in F_\nu$, contradiction. Hence $-([a_\eta] \cdot -[a_\xi]) \in F_\nu$ for all $\nu \le \alpha$, and so $[a_\eta]_I \le [a_\xi]_I$. Now suppose that $[a_\eta]_I = [a_\xi]$. Then $a_\xi \cdot -a_\eta \in I$, so in particular $-[a_\xi] + [a_\eta] \in F_{\xi+1}$. Since also $[a_\xi] \in F_{\xi+1}$, it follows that $[a_\eta] \in F_{\xi+1}$. But $\xi < \eta$, contradiction. So (1) holds.

(2) $[a_0]_I \ne 1$.

This holds since $-[a_0] \in F_0$, and hence $(\omega\backslash a_0) \notin I$.

(3) If $\alpha = \beta + 1$, then $[a_\beta]_I \ne 0$.

This is true since $[a_\beta] \in F_\alpha$, and hence $[a_\beta] \notin I$.

Now let J be an ideal in $\mathscr{P}(\omega)$ which is maximal subject to the following conditions:

(4) $I \subseteq J$.

(5) If $\xi < \eta < \alpha$, then $a_\xi \backslash a_\eta \notin J$.

(6) $\omega \backslash a_0 \notin J$.

(7) If $\alpha = \beta + 1$, then $a_\beta \notin J$.

Clearly then we have:

(8) For any $x \subseteq \omega$ one of the following conditions holds.
 (a) $x \in J$.
 (b) There exist $\xi < \eta < \alpha$ such that $a_\xi \cdot -a_\eta \cdot -x \in J$.
 (c) $-a_0 \cdot -x \in J$.
 (d) $\alpha = \beta + 1$ and $a_\beta \cdot -x \in J$.

Also we have

(9) If $F, K \in [\alpha]^{<\omega}$ and $F < K$, then $\bigcap_{\xi \in F} a_\xi \cap \bigcap_{\eta \in K} -a_\eta \notin J$.

Now we apply Lemma 1 to the ideal J to obtain a generic extension $M[G]$ such that, with $d = \bigcup_{(b,y) \in G} y$, the following conditions hold:

(10) If $c \subseteq \omega$ and $c \notin J$, then $c \cap d$ is infinite.

(11) If $c \subseteq \omega$ and $c \notin J$ then $c \backslash d$ is infinite.

(12) If $b \in J$, then $b \cap d$ is finite.

Hence by (9) we get

(13) If $F, K \in [\alpha]^{<\omega}$ and $F < K$, then $\bigcap_{\xi \in F} a_\xi \cap \bigcap_{\eta \in K} -a_\eta \cap d$ is infinite.

(14) If $F, K \in [\alpha]^{<\omega}$ and $F < K$, then $\bigcap_{\xi \in F} a_\xi \cap \bigcap_{\eta \in K} -a_\eta \backslash d$ is infinite.

Now let K be the ideal in $\mathscr{P}(\omega)^{M[G]}$ generated by J.

(15) If F is a finite subset of α, then $\bigcap_{\xi \in F} a_\xi \cap (\omega \backslash d) \notin K$.

In fact, otherwise we get a $c \in J$ such that $\bigcap_{\xi \in F} a_\xi \cap (\omega \backslash d) \subseteq c$, and so $\left(\bigcap_{\xi \in F} a_\xi \backslash c\right) \cap (\omega \backslash d) = \emptyset$. But clearly $\left(\bigcap_{\xi \in F} A_\xi \backslash c\right) \notin J$, so this contradicts (11). Similarly,

(16) If $F, L \in [\alpha]^{<\omega}$ and $F < L$, then $\bigcap_{\xi \in F} a_\xi \cap \bigcap_{\eta \in L} -a_\eta \cap d \notin K$.

Now we apply Lemma 1 with I replaced by K to obtain a generic extension $M[G][H]$ and an infinite subset e of ω such that $\langle [a_\xi : \xi < \alpha\rangle^\frown \langle [\omega \backslash d], [e]\rangle$ is a free sequence and the following condition holds:

(16) If $b \in K$, then $b \cap e$ is finite.

(17) If $x \in (\mathscr{P}(\omega) \cap M) \backslash (\{a_\xi : \xi < \alpha\} \cup \{\omega \backslash d, e\}$, then $\langle [a_\xi] : \xi < \alpha\rangle^\frown \langle [\omega \backslash d], [e], [x]\rangle$ is not a free sequence.

To prove this, we consider cases.

Case 1. $x \in K$. Then $x \cap e$ is finite by (16), as desired.

Case 2. $x \notin K$. Then $x \notin J$, and so by (8) we have three subcases.

Subcase 2.1. There exist $\xi < \eta < \alpha$ such that $a_\xi \cdot -a_\eta \cdot -x \in J$. Then by (12), $a_\xi \cdot -a_\eta \cdot -x \cdot d$ is finite, as desired.

Subcase 2.2. $-a_0 \cdot -x \in J$. Then by (12), $-a_0 \cdot -x \cdot d$ is finite, as desired.

Subcase 2.3. $\alpha = \beta + 1$ and $a_\beta \cdot -x \in J$. Then by (12), $a_\beta \cdot -x \cdot d$ is finite, as desired. □

Theorem 35.16. *It is relatively consistent that* $\mathfrak{f} < 2^\omega$. □

In this appendix we define and develop the main properties of the integers. The development is based upon Chapter 6, in which properties of natural numbers were given. At the end of that chapter a sketch of the construction of integers was given, and we now give full details.

Let $A = \omega \times \omega$. We define a relation $\sim$ on A by setting, for any $m, n, p, q \in \omega$,

$$(m, n) \sim (p, q) \quad \text{iff} \quad m + q = n + p.$$

This definition is motivated by thinking of (m, n) as representing, in some sense, $m - n$.

Lemma 36.1. *$\sim$ is an equivalence relation on A.*

Proof. For reflexivity, given $m, n \in \omega$ we want to show that $(m, n) \sim (m, n)$. By definition, this means that we want to show that $m+n = n+m$. This is given by 6.14(iv).

For symmetry, assume that $(m, n) \sim (p, q)$; we want to show that $(p, q) \sim (m, n)$. The assumption means, by definition, that $m + q = n + p$. Hence $p + n = q + m$ by 6.14(iv) again. Hence $(p, q) \sim (m, n)$. [In the definition, replace m, n, p, q by p, q, m, n respectively.]

For transitivity, assume that $(m, n) \sim (p, q) \sim (r, s)$. Thus $m + q = n + p$ and $p + s = q + r$. Hence $m + q + s = n + p + s = n + q + r$, so using 6.15(iii) we get $m + s = n + r$, so that $(m, n) \sim (r, s)$. □

We now let $\mathbb{Z}'$ be the collection of all equivalence classes under $\sim$. Elements of $\mathbb{Z}'$ are denoted by $[(m, n)]$ with $m, n \in \omega$.

For the purposes of this appendix, we treat binary operations on $\mathbb{Z}'$ as functions mapping $\mathbb{Z}' \times \mathbb{Z}'$ into $\mathbb{Z}'$.

Proposition 36.2. *There is a binary operation $+$ on $\mathbb{Z}'$ such that for any $m, n, p, q \in \omega$, $[(m, n)] + [(p, q)] = [(m + p, n + q)]$.*

Proof. Let

$$\begin{aligned} R = \{(x, y) : &\text{ there exist } m, n, p, q \in \omega \text{ such that} \\ &x = ([(m, n)], [(p, q)]) \text{ and } y = [(m + p, n + q)]\}. \end{aligned}$$

We claim that R is a function. For, assume that $(x, y), (x, z) \in R$. Then we can choose $m, n, p, q, m', n', p', q' \in \omega$ such that the following conditions hold:

$$x = ([(m, n)], [(p, q)]); \tag{1}$$

$$y = [(m + p, n + q)]; \tag{2}$$

$$x = ([(m', n')], [(p', q')]); \tag{3}$$

$$z = [(m' + p', n' + q')]. \tag{4}$$

From (1) and (3) we get $[(m, n)] = [(m', n')]$ and $[(p, q)] = [(p', q')]$, hence $(m, n) \sim (m', n')$ and $(p, q) \sim (p', q')$, hence $m + n' = n + m'$ and $p + q' = q + p'$. Hence

$$m + p + n' + q' = m + n' + p + q' = n + m' + q + p' = n + q + m' + p',$$

from which it follows that $(m+p,n+q) \sim (m'+p',n'+q')$, hence $[(m+p,n+q)] = [(m'+p',n'+q')]$, hence $y = z$ by (2) and (4). This shows that R is a function.

Knowing that R is a function, the definition of R then says that for any $m,n,p,q \in \omega$, $([(m,n)],[(p,q)])$ is in the domain of R, and $R(([(m,n)],[(p,q)])) = [(m+p,n+q)]$. This is as desired in the proposition. $\square$

Proposition 36.3. *The operation $+$ on $\mathbb{Z}'$ is associative and commutative. That is, if $x,y,z \in \mathbb{Z}'$, then $x+(y+z) = (x+y)+z$ and $x+y = y+x$.*

Proof. For any $a,b,c,d,e,f \in \omega$ we have

$$\begin{aligned}
[(a,b)] + ([(c,d)] + [(e,f)]) &= [(a,b)] + [(c+e,d+f)] \\
&= [(a+c+e,b+d+f)] \\
&= [(a+c,b+d)] + [(e,f)] \\
&= ([(a,b)] + [(c,d)]) + [(e,f)]; \\
[(a,b)] + [(c,d)] &= [(a+c,b+d)] \\
&= [(c+a,d+b)] \\
&= [(c,d)] + [(a,b)].
\end{aligned}$$

$\square$

Now we define $0' = [(0,0)]$.

Proposition 36.4. *For any $a,b \in \omega$, $[(a,b)] + 0' = [(a,b)]$.* $\square$

Proposition 36.5. *For any $x \in \mathbb{Z}'$ there is a $y \in \mathbb{Z}'$ such that $x+y = 0'$.*

Proof. Let $x \in \mathbb{Z}'$; hence there are $a,b \in \omega$ such that $x = [(a,b)]$. Let $y = [(b,a)]$. Then $x+y = [(a,b)] + [(b,a)] = [(a+b,b+a)] = [(0,0)] = 0'$. $\square$

There are little group-theoretic facts that say that $0'$ and y above are unique:

Proposition 36.6. *If z is an element of $\mathbb{Z}'$ such that $x+z = x$ for all $x \in \mathbb{Z}'$, then $z = 0'$.*

Proof. $z = 0' + z$ (by 36.4) $= 0'$ (by assumption). $\square$

Proposition 36.7. *If $x,y,z \in \mathbb{Z}'$ and $x+y = 0' = x+z$, then $y = z$.*

Proof. $y = 0' + y = x+z+y = z+x+y = z+0' = z$. $\square$

These are all of the properties of $+$ that we need.

Proposition 36.8. *There is a binary operation $\cdot$ on $\mathbb{Z}'$ such that for all $m,n,p,q \in \omega$, $[(m,n)] \cdot [(p,q)] = [(m \cdot p + n \cdot q, m \cdot q + n \cdot p)]$.*

Proof. Let

$$\begin{aligned}
R = \{(x,y) :\ & \text{there exist } m,n,p,q \in \omega \text{ such that} \\
& x = ([(m,n)],[(p,q)]) \text{ and } y = [(m \cdot p + n \cdot q, m \cdot q + n \cdot p)]\}.
\end{aligned}$$

We claim that R is a function. For, assume that $(x,y),(x,z) \in R$. Then we can choose $m,n,p,q,m',n',p',q' \in \omega$ such that the following conditions hold:

$$x = ([(m,n)],[(p,q)]); \tag{1}$$

$$y = [(m\cdot p + n\cdot q, m\cdot q + n\cdot p)]; \tag{2}$$

$$x = ([(m',n')],[(p',q')]); \tag{3}$$

$$z = [(m'\cdot p' + n'\cdot q', m'\cdot q' + n'\cdot p')]. \tag{4}$$

From (1) and (3) we get $[(m,n)] = [(m',n')]$ and $[(p,q)] = [(p',q')]$, hence $(m,n) \sim (m',n')$ and $(p,q) \sim (p',q')$, hence $m+n' = n+m'$ and $p+q' = q+p'$. Hence

$$m\cdot p + m\cdot q' + n\cdot q + n\cdot p' = m\cdot q + m\cdot p' + n\cdot p + n\cdot q'. \tag{1}$$

Also,

$$m\cdot p' + n'\cdot p' + n\cdot q' + m'\cdot q' = n\cdot p' + m'\cdot p' + m\cdot q' + n'\cdot q'. \tag{2}$$

Hence

$$\begin{aligned} m\cdot p + n\cdot q + m'\cdot q' + n'\cdot p' + m\cdot p' + n\cdot q' \\ &= m\cdot p + n\cdot q + m\cdot p' + n'\cdot p' + n\cdot q' + m'\cdot q' \\ &= m\cdot p + n\cdot q + n\cdot p' + m'\cdot p' + m\cdot q' + n'\cdot q' \quad \text{by (2)} \\ &= m\cdot p + m\cdot q' + n\cdot q + n\cdot p' + m'\cdot p' + n'\cdot q' \\ &= m\cdot q + m\cdot p' + n\cdot p + n\cdot q' + m'\cdot p' + n'\cdot q' \quad \text{by (1)} \end{aligned}$$

Considering the first side of the top equation and the last part, we can cancel $m\cdot p'$ and $n\cdot q'$ by 6.15(iii), and we get

$$m\cdot p + n\cdot q + m'\cdot q' + n'\cdot p' = m\cdot q + n\cdot p + m'\cdot p' + n'\cdot q',$$

which easily yields $y = z$.

Thus R is a function, and this clearly proves the proposition. $\square$

Proposition 36.9. *Let $x,y,z \in \mathbb{Z}'$. Then*

(i) $x\cdot y = y\cdot x$.

(ii) $x\cdot(y\cdot z) = (x\cdot y)\cdot z$.

(iii) $x\cdot(y+z) = x\cdot y + x\cdot z$.

Proof. Write $x = [(m,n)]$, $y = [(p,q)$, and $z = [(r,s)]$. Then

$$\begin{aligned} x\cdot y &= [(m,n)]\cdot[(p,q)] \\ &= [(m\cdot p + n\cdot q, m\cdot q + n\cdot p)] \\ &= [(p\cdot m + q\cdot n, p\cdot n + q\cdot m)] \\ &= [(p,q)]\cdot[(m,n)] \end{aligned}$$

$$
\begin{aligned}
&= y \cdot x;\\
x \cdot (y \cdot z) &= [(m,n)] \cdot ([(p,q)] \cdot [(r,s)])\\
&= [(m,n)] \cdot [(p \cdot r + q \cdot s, p \cdot s + q \cdot r)]\\
&= [(m \cdot p \cdot r + m \cdot q \cdot s + n \cdot p \cdot s + n \cdot q \cdot r,\\
&\quad m \cdot p \cdot s + m \cdot q \cdot r + n \cdot p \cdot r + n \cdot q \cdot s)];\\
(x \cdot y) \cdot z &= ([(m,n)] \cdot [(p,q)]) \cdot [(r,s)]\\
&= [(m \cdot p + n \cdot q, m \cdot q + n \cdot p)] \cdot [(r,s)]\\
&= [(m \cdot p \cdot r + n \cdot q \cdot r + m \cdot q \cdot s + n \cdot p \cdot s,\\
&\quad m \cdot p \cdot s + n \cdot q \cdot s + m \cdot q \cdot r + n \cdot p \cdot r)]\\
&= x \cdot (y \cdot z) \quad \text{by the above;}\\
x \cdot (y + z) &= [(m,n)] \cdot ([(p,q)] + [(r,s)])\\
&= [(m,n)] \cdot [(p + r, q + s)]\\
&= [(m \cdot p + m \cdot r + n \cdot q + n \cdot s, m \cdot q + m \cdot s + n \cdot p + n \cdot r)];\\
x \cdot y + x \cdot z &= [(m,n)] \cdot [(p,q)] + [(m,n)] \cdot [(r,s)]\\
&= [(m \cdot p + n \cdot q, m \cdot q + n \cdot p)] + [(m \cdot r + n \cdot s, m \cdot s + n \cdot r)]\\
&= [(m \cdot p + n \cdot q + m \cdot r + n \cdot s, m \cdot q + n \cdot p + m \cdot s + n \cdot r)]\\
&= x \cdot (y + z) \quad \text{by the above.}
\end{aligned}
$$
□

Now we define $1' = [(1,0)]$.

Proposition 36.10. *$1' \cdot x = x$ and $0' \cdot x = 0'$ for all $x \in \mathbb{Z}'$.*

Proof. Take any $x \in \mathbb{Z}$; say that $x = [(m,n)]$. Then

$$1' \cdot x = [(1,0)] \cdot [(m,n)] = [(1 \cdot m + 0 \cdot m, 1 \cdot n + 0 \cdot m)] = [(m,n)] = x,$$

as desired.

For the second statement, note that $x \cdot 0' + x = x \cdot 0' + x \cdot 1' = x \cdot (0' + 1') = x \cdot 1' = x$, so $x \cdot 0' = 0'$. □

Proposition 36.11. *If $x, y \in \mathbb{Z}'$ and $x \cdot y = 0'$, then $x = 0'$ or $y = 0'$.*

Proof. Write $x = [(m,n)]$ and $y = [(p,q)]$. Now $x \cdot y = [(m,n)] \cdot [(p,q)] = [m \cdot p + n \cdot q, m \cdot q + n \cdot p)]$ and also $x \cdot y = 0' = [0,0)]$, so $(m \cdot p + n \cdot q, m \cdot q + n \cdot p) \sim (0,0)$, so

$$m \cdot p + n \cdot q = m \cdot q + n \cdot p. \tag{1}$$

Suppose that $x \neq 0'$; we will show that $y = 0'$, which will prove the proposition. Thus $[(m,n)] = x \neq 0' = [(0,0)]$, so $m \neq n$. Hence $m < n$ or $n < m$.

Case 1. $m < n$. Then there is a nonzero natural number s such that $m + s = n$. Substituting this into (1) we get

$$
\begin{aligned}
m \cdot p + n \cdot q &= m \cdot p + (m + s) \cdot q\\
&= m \cdot p + m \cdot q + s \cdot q \quad \text{and}\\
m \cdot q + n \cdot p &= m \cdot q + (m + s) \cdot p\\
&= m \cdot q + m \cdot p + s \cdot p,
\end{aligned}
$$

and hence

$$m \cdot p + m \cdot q + s \cdot q = m \cdot q + m \cdot p + s \cdot p.$$

Then by 6.15(iii) we get $s \cdot q = s \cdot p$, and 6.20(viii) yields $q = p$. Hence $(p, q) \sim (0, 0)$, so $y = [(p, q)] = [(0, 0)] = 0'$.

Case 2. $n < m$. This is very similar to case 1. There is a nonzero natural number s such that $n + s = m$. Substituting this into (1) we get

$$\begin{aligned} m \cdot p + n \cdot q &= (n + s) \cdot p + n \cdot q \\ &= n \cdot p + s \cdot p + n \cdot q; \\ m \cdot q + n \cdot p &= (n + s) \cdot q + n \cdot p \\ &= n \cdot q + s \cdot q + n \cdot p, \end{aligned}$$

and hence

$$n \cdot p + s \cdot p + n \cdot q = n \cdot q + s \cdot q + n \cdot p.$$

Then by 6.15(iii) we get $s \cdot p = s \cdot q$, and 6.20(viii) yields $p = q$. Hence $(p, q) \sim (0, 0)$, so $y = [(p, q)] = [(0, 0)] = 0'$. □

This is all of the arithmetic properties of $\mathbb{Z}'$ that is needed. Now we introduce the order. First we only define the collection of positive elements:

$$P = \{[(m, n)] : m, n \in \omega \text{ and } m > n\}.$$

Note that this really means

$$P = \{x : \text{there exist } m, n \in \omega \text{ such that } x = [(m, n)] \text{ and } m > n\}.$$

Proposition 36.12. *For any $m, n \in \omega$, $[(m, n)] \in P$ iff $m > n$.*

Proof. $\Leftarrow$: true by definition. $\Rightarrow$: Suppose that $[(m, n)] \in P$. Choose $p, q \in \omega$ such that $p > q$ and $[(m, n)] = [(p, q)]$. Thus $(m, n) \sim (p, q)$, so $m + q = n + p$. If $m \leq n$, then by 6.17,

$$m + q \leq n + q < n + p = m + q,$$

contradiction. Hence $m < n$. □

Proposition 36.13. *For any $a, b \in \mathbb{Z}'$ we have:*

(i) If $a \neq 0'$, then $a \in P$ or $-a \in P$, but not both.

(ii) If $a, b \in P$, then $a + b \in P$.

(iii) If $a, b \in P$, then $a \cdot b \in P$.

Proof. Let $a = [(m, n)]$ and $b = [(p, q)]$. For (i), since $0' = [(0, 0)]$ we see that if $a \neq [(0, 0)]$ then $(m, n) \not\sim (0, 0)$ and so $m \neq n$. If $m < n$, then $-a = [(n, m)] \in P$. If $m > n$, then $a \in P$. If $a, -a \in P$, then by 36.12, $m < n$ and $n < m$, contradiction.

(ii): Assume that $a, b \in P$. Then by 36.12, $m > n$ and $p > q$. Clearly then $m+p > n+q$ by 6.17, so $a + b = [(m + p, n + q)] \in P$.

(iii): Assume that $a, b \in P$. Then by 36.12, $m > n$ and $p > q$. Write $n + s = m$ and $q + t = p$, with $s, t \neq 0$. Hence $s \cdot t \neq 0$. Now

$$(*) \qquad a \cdot b = [(m, n)] \cdot [(p, q)] = [(m \cdot p + n \cdot q, m \cdot q + n \cdot p)].$$

Now

$$\begin{aligned} m \cdot q + n \cdot p + s \cdot t &= m \cdot q + n \cdot (q + t) + s \cdot t \\ &= m \cdot q + n \cdot q + n \cdot t + s \cdot t \\ &= m \cdot q + n \cdot q + (n + s) \cdot t \\ &= m \cdot q + n \cdot q + m \cdot t \\ &= m \cdot (q + t) + n \cdot q \\ &= m \cdot p + n \cdot q, \end{aligned}$$

and so $m \cdot q + n \cdot p < m \cdot p + n \cdot q$, so that $a \cdot b \in P$ by $(*)$ and 36.12. □

Now we can define the order: $a < b$ iff $b - a \in P$. The main properties of $<$ are given in the following proposition.

Proposition 36.14. *Let $x, y, z \in \mathbb{Z}'$. Then*
(i) $x \not< x$.
(ii) If $x < y < z$, then $x < z$.
(iii) $x < y$, $x = y$, or $y < x$.
(iv) $x < y$ iff $x + z < y + z$.
(v) If $0' < x$ and $0' < y$, then $0' < x \cdot y$.
(vi) If $0' < z$, then $x < y$ implies that $x \cdot z < y \cdot z$.

Proof. (i): $x - x = 0'$, so (i) follows from 36.13(i).

(ii): Assume that $x < y < z$. So $y - x \in P$ and $z - y \in P$. Hence $z - x = (z - y) + y - x) \in P$ by 6.13(ii), so $x < z$.

(iii): We have $x = y$ or $x - y \in P$ or $y - x \in P$, so (iii) follows.

(iv): $x < y$ iff $y - x \in P$ iff $(y + z) - (x + z) \in P$ iff $x + z < y + z$.

(v): This is immediate from 36.13(iii).

(vi): Assume that $0' < z$ and $x < y$. So $z, y - x \in P$, so by 36.13(iii), $y \cdot z - x \cdot z = z \cdot (y - x) \in P$, and so $x \cdot z < y \cdot z$. □

This finishes our treatment of $\mathbb{Z}'$. Now we need to relate it to ω, and define our final version $\mathbb{Z}$ of the integers.

For any $m \in \omega$ let $f(m) = [(m, 0)]$.

Proposition 36.15. *f is a one-one function mapping ω into $\mathbb{Z}$. Moreover, for any $m, n \in \omega$ we have*
(i) $f(m + n) = f(m) + f(n)$.
(ii) $f(m \cdot n) = f(m) \cdot f(n)$.
(iii) $m < n$ iff $f(m) < f(n)$.

Proof. Suppose that $f(m) = f(n)$. Thus $[(m, 0)] = [(n, 0)]$, so $(m, 0) \sim (n, 0)$, hence $m + 0 = 0 + n$, hence $m = n$. So f is one-one. Next,

$$f(m + n) = [(m + n, 0)] = [(m, 0)] + [(n, 0)] = f(m) + f(n);$$
$$f(m \cdot n) = [(m \cdot n, 0)] = [(m \cdot n + 0 \cdot 0, m \cdot 0 + 0 \cdot n)] = [(m, 0)] \cdot [(n, 0)] = f(m) \cdot f(n)$$
$$\begin{aligned} f(m) < f(n) \quad &\text{iff} \quad [(m, 0)] < [(n, 0)] \\ &\text{iff} \quad m + 0 < 0 + n \\ &\text{iff} \quad m < n. \end{aligned}$$

□

We have now identified a part of $\mathbb{Z}'$ which acts like the natural numbers. We now want to apply the replacement process to officially define $\mathbb{Z}$.

Proposition 36.16. $\omega \cap \mathbb{Z}' = \emptyset$.

Proof. Suppose that $m \in \omega \cap \mathbb{Z}'$. Choose $n, p \in \omega$ such that $m = [(n, p)]$. But $[(n, p)]$ is an infinite set, since it contains all of the pairs $(n, p), (n + 1, p + 1), (n + 2, p + 2), \ldots$, contradiction. □

Now we define $\mathbb{Z} = (\mathbb{Z}' \backslash \mathrm{rng}(f)) \cup \omega$. There is a one-one function $g : \mathbb{Z} \to \mathbb{Z}'$, defined by $g([(m, n)]) = [(m, n)]$ if $[(m, n)] \in \mathbb{Z}' \backslash \mathrm{rng}(f)$, and $g(m) = f(m)$ for $m \in \omega$. Clearly g is a bijection. Now the operations $+'$ and $\cdot'$ are defined on $\mathbb{Z}$ as follows. For any $a, b \in \mathbb{Z}$,

$$a +' b = g^{-1}(g(a) + g(b));$$
$$a \cdot' b = g^{-1}(g(a) \cdot g(b)).$$

moreover, we define $a <' b$ iff $g(a) < g(b)$. With these definitions, g becomes an isomorphism of $\mathbb{Z}$ onto $\mathbb{Z}'$. Namely, if $a, b \in \mathbb{Z}$, then

$$g(a +' b) = g(g^{-1}(g(a) + g(b))) = g(a) + g(b);$$
$$g(a \cdot' b) = g(g^{-1}(g(a) \cdot g(b))) = g(a) \cdot g(b);$$
$$a <' b \quad \text{iff} \quad g(a) < g(b).$$

Moreover, the operations $+'$ and $\cdot'$ on ω coincide with the ones defined in Chapter 6, since if $m, n \in \omega$, then

$$m +' n = g^{-1}(g(m) + g(n)) = g^{-1}(f(m) + f(n)) = g^{-1}(f(m + n)) = m + n;$$
$$m \cdot' n = g^{-1}(g(m) \cdot g(n)) = g^{-1}(f(m) \cdot f(n)) = g^{-1}(f(m \cdot n)) = m \cdot n;$$
$$\begin{aligned} m <' n \quad &\text{iff} \quad g(m) < g(n) \\ &\text{iff} \quad f(m) < f(n) \\ &\text{iff} \quad m < n. \end{aligned}$$

All of the properties above, like the associative, commutative, and distributive laws, hold for $\mathbb{Z}$ since g is an isomorphism. Of course we use $+, \cdot, <$ now rather than $+', \cdot', <'$.

Here we define the rational numbers and give their fundamental properties. For brevity we denote multiplication of integers by justaposition, as is usually done.

Let $A = \mathbb{Z} \times (\mathbb{Z} \backslash \{0\})$. We define a relation $\sim$ on A as follows:

$$(a, b) \sim (c, d) \quad \text{iff} \quad ad = bc$$

This definition and succeeding ones are well-motivated if you think of (a, b) as being $\frac{a}{b}$ intuitively.

Lemma 37.1. *$\sim$ is an equivalence relation on A.*

Proof. Reflexivity: If $(a, b) \in A$, then $ab = ba$, so $(a, b) \sim (a, b)$.

Symmetry: Assume that $(a, b) \sim (c, d)$. Thus $ad = bc$, so $cb = da$, and hence $(c, d) \sim (a, b)$.

Transitivity: Assume that $(a, b) \sim (c, d) \sim (e, f)$. Thus $ad = bc$ and $cf = de$. Hence $adf = bcf = bde$, so $0 = adf - bde = d(af - be)$. Since $d \neq 0$, it follows that $af - be = 0$, and hence $af = be$. This shows that $(a, b) = (e, f)$. □

We let $\mathbb{Q}'$ be the set of all equivalence classes under $\sim$.

Proposition 37.2. *There is a binary operation $+$ on $\mathbb{Q}'$ such that for any $(a, b), (c, d) \in A$, $[(a, b)] + [(c, d)] = [(ad + bc, bd)]$.*

Proof. First note that if $(a, b), (c, d) \in A$, then $bd \neq 0$, so that at least the pair $(ad + bc, bd)$ is in A. Now let

$$\begin{aligned} R = \{(x, y) : &\text{there exist } (a, b), (c, d) \in A \text{ such that} \\ &x = ([(a, b)], [(c, d)]) \text{ and } y = [(ad + bc, bd)]\}. \end{aligned}$$

We claim that R is a function. For, suppose that $(x, y), (x, z) \in R$. Then we can choose $(a, b), (c, d), (a', b'), (c', d') \in A$ such that $x = ([(a, b)], [(c, d)])$, $y = [(ad + bc, bd)]$, $x = ([(a', b')], [(c', d')])$, and $y = [(a'd' + b'c', b'd')]$. so $([(a, b)], [(c, d)]) = ([(a', b')], [(c', d')])$, hence $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$, hence $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, hence

$$ab' = ba' \tag{1}$$

$$cd' = dc' \tag{2}$$

Hence

$$\begin{aligned} (ad + bc)b'd' &= adb'd' + bcb'd' \\ &= ab'dd' + cd'bb' \\ &= ba'dd' + dc'bb' \quad \text{by (1), (2)} \\ &= a'd'bd + b'c'bd \\ &= (a'd' + b'c')bd, \end{aligned}$$

and hence $(ad+bc, bd) \sim (a'd'+b'c', b'd')$. Thus $y = [(ad+bc, bd)] = [(a'd'+b'c', b'd')] = y'$. This proves that R is a function. The proposition is now clear. $\square$

Proposition 37.3. *If $x, y, z \in \mathbb{Q}'$, then*
(i) $x + (y + z) = (x + y) + z$.
(ii) $x + y = y + x$.

Proof. Let $x = [(a, b)]$, $y = [(c, d)]$, and $z = [(e, f)]$. Then

$$\begin{aligned}
x + (y + z) &= [(a, b)] + ([(c, d)] + [(e, f)]) \\
&= [(a, b)] + [(cf + de, df)] \\
&= [(adf + b(cf + de), bdf)]; \\
(x + y) + z &= ([(a, b)] + [(c, d)]) + [(e, f)] \\
&= [(ad + bc, bd)] + [(e, f)] \\
&= [((ad + bc)f + bde, bdf)] \\
&= [(adf + bcf + bde, bdf)] \\
&= x + (y + z); \\
x + y &= [(a, b)] + [(c, d)] \\
&= [(ad + bc, bd)] \\
&= [(cb + da, db)] \\
&= [(c, d)] + [(a, b)] \\
&= y + x.
\end{aligned}$$

$\square$

Now we define $0' = [(0, 1)]$.

Proposition 37.4. *$x + 0' = x$ for any $x \in \mathbb{Q}$. Moreover, for any $x \in \mathbb{Q}'$ there is a $y \in \mathbb{Q}'$ such that $x + y = 0'$.*

Proof. Let $x = [(a, b)]$. Then

$$\begin{aligned}
x + 0' &= [(a, b)] + [(0, 1)] \\
&= [(a \cdot 1 + b \cdot 0, b \cdot 1)] \\
&= [(a, b)] \\
&= x.
\end{aligned}$$

Next, let $y = [(-a, b)]$. Then

$$x + y = [(a, b)] + [(-a, b)] = [(ab + b(-a), bb)] = [(0, bb)] = [(0, 1)].$$

Here the last equality holds because $0 \cdot 1 = 0 = bb \cdot 0$. $\square$

The following two facts are proved as in appendix B, proof of B6 and B7.

Proposition 37.5. *If r is an element of $\mathbb{Q}'$ such that $x + r = x$ for all $x \in \mathbb{Q}'$, then $r = 0'$.*

Proposition 37.6. *If $x, y, z \in \mathbb{Q}'$ and $x + y = 0' = x + z$, then $y = z$.*

These are all of the properties of $+$ that we need.

Proposition 37.7. *There is a binary operation $\cdot$ on $\mathbb{Q}'$ such that for all $(a, b), (c, d) \in A$, $[(a, b)] \cdot [(c, d)] = [(ac, bd)]$.*

Proof. First note that if $(a, b), (c, d) \in A$, then $bd \neq 0$, so that $(ac, bd) \in A$. Now let

$$\begin{aligned} R = \{(x, y) :\ & \text{there exist } (a, b), (c, d) \in A \text{ such that} \\ & x = ([(a, b)], y = [(c, d)]), \text{ and } z = [(ac, bd)]\}. \end{aligned}$$

We claim that R is a function. For, suppose that $(x, y), (x, z) \in R$. Then we can choose $(a, b), (c, d), (a', b'), (c', d') \in A$ such that $x = ([(a, b)], [(c, d)])$, $y = [(ac, bd)]$, $x = ([(a', b')], [(c', d')])$, and $z = [(a'c', b'd')]$. So $([(a, b)], [(c, d)]) = ([(a', b')], [(c', d')])$, and hence $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$, hence $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, hence $ab' = ba'$ and $cd' = dc'$. Hence

$$\begin{aligned} & acb'd' = ab'cd' = ba'dc' = bda'c', \\ & \text{hence } (ac, bd) \sim (a'c', b'd'), \\ & \text{hence } y = [(ac, bd)] = [(a'c', b'd')] = z. \end{aligned}$$

So R is a function, and the conclusion is clear. $\square$

Proposition 37.8. *For any $x, y, z \in \mathbb{Q}'$ we have*
(i) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
(ii) $x \cdot y = y \cdot x$.
(iii) $x \cdot (y + z) = x \cdot y + x \cdot z$.

Proof. Write $x = [(a, b)]$, $y = [(c, d)$, and $z = [(e, f)]$. Then

$$\begin{aligned} x \cdot (y \cdot z) &= [(a, b)] \cdot ([(c, d)] \cdot [(e, f)]) \\ &= [(a, b)] \cdot [(ce, df)] \\ &= [(ace, bdf)] \\ &= [(ac, bd)] \cdot [(e, f)] \\ &= ([(a, b)] \cdot [(c, d)]) \cdot [(e, f)] \\ &= (x \cdot y) \cdot z; \\ x \cdot y &= [(a, b)] \cdot [(c, d)] \\ &= [(ac, bd)] \\ &= [(ca, db)] \\ &= [(c, d)] \cdot [(a, b)] \\ &= y \cdot x; \\ x \cdot (y + z) &= [(a, b)] \cdot ([(c, d)] + [(e, f)]) \end{aligned}$$

$$
\begin{aligned}
&= [(a,b)] \cdot [(cf+de, df)] \\
&= [(a(cf+de), bdf)] \\
&= [(acf+ade, bdf)]; \\
x \cdot y + x \cdot z &= [(a,b)] \cdot [(c,d)] + [(a,b)] \cdot [(e,f)] \\
&= [(ac, bd)] + [(ae, bf)] \\
&= [(acbf + bdae, bdbf)].
\end{aligned}
$$

Thus for the distributive law (iii) we just need to show that $[(acf+ade, bdf)] = [(acbf + bdae, bdbf)]$, or equivalently that $(acf+ade, bdf) \sim (acbf+bdae, bdbf)$, or equivalently that $(acf+ade)bdbf = bdf(acbf+bdae)$. This last statement is proved as follows:

$$(acf+ade)bdbf = abbcdff + abbddef \text{ and } bdf(acbf+bdae) = abbcdff + abbddef. \quad \square$$

Next, we define $1' = [(1,1)]$.

Proposition 37.9. *Let $x \in \mathbb{Q}'$.*

(i) $x \cdot 1' = x$.

(ii) If $x \neq 0'$ then there is a unique $y \in \mathbb{Q}'$ such that $x \cdot y = 1'$.

Proof. Write $x = [(a,b)]$. Then $x \cdot 1' = [(a,b)] \cdot [(1,1)] = [(a,b)] = x$. For (ii), assume that $x \neq 0'$. Thus $[(a,b)] \neq [(0,1)]$, so $a \cdot 1 \neq b \cdot 0$, i.e., $a \neq 0$. Let $y = [(b,a)]$. Then $x \cdot y = [(a,b)] \cdot [(b,a)] = [(ab, ba)]$, and this is equal to $[(1,1)] = 1'$ since $ab1 = ba1$. Suppose that also $x \cdot z = 1'$. Write $z = [(c,d)]$. then $[(1,1)] = x \cdot z = [(a,b)] \cdot [(c,d)] = [ac, bd)$, and so $ac = bd$, and hence $y = [(b,a)] = [(c,d)] = z$. $\square$

We turn to the order of the rationals. In general outline, we follow the procedure used for the integers.

First we define the set P of positive rationals:

$$P = \{[(a,b)] \in \mathbb{Q}' : ab > 0\}.$$

As for the similar definition for integers, this definition says that if $ab > 0$ then $[(a,b)] \in P$, but does not say anything about the converse, so we prove this converse:

Proposition 37.10. $[(a,b)] \in P$ *iff* $ab > 0$.

Proof. As mentioned, $\Leftarrow$ holds by definition. Now assume that $[(a,b)] \in P$. This means that there is a $[(c,d)] \in \mathbb{Q}'$ such that $[(a,b)] = [(c,d)]$ and $cd > 0$. So $(a,b) \sim (c,d)$, and hence $ad = bc$. Hence $adbd = bcbd$. Now we need the following little general fact:

(1) If $x \in \mathbb{Z}$ and $x \neq 0$, then $xx > 0$.

In fact, we have $x > 0$ or $-x > 0$ by B13(i) and the definition of $<$ for integers, so by B14(v), $xx > 0$ or $xx = (-x)(-x) > 0$, as desired in (1).

Now by (1) and B14(v) we have $abdd = bcbd > 0$. In particular, $ab \neq 0$. If $ab < 0$, then $abdd < 0dd = 0$, contradiction. So $ab > 0$. $\square$

Proposition 37.11. *Suppose that $r, s \in \mathbb{Q}'$.*

(i) If $r \neq 0'$, then $r \in P$ or $-r \in P$, but not both.

(ii) If $r, s \in P$, then $r + s \in P$.

(iii) If $r, s \in P$, then $r \cdot s \in P$.

Proof. Let $r = [(a, b)]$ and $s = [(c, d)]$.

(i): Assume that $r \neq 0'$. Then $ab \neq 0$, since $ab = 0$ would imply that $a = 0$ (since $b \neq 0$), and so $(a, b) = (0, b) \sim (0, 0)$ and hence $r = [(a, b)] = [(0, 0)] = 0'$, contradiction. If $ab > 0$, then $r \in P$, and if $-(ab) > 0$, then $(-a)b > 0$, so $-r = [(-a, b)] \in P$. Thus $r \in P$ or $-r \in P$. Suppose that $r \in P$ and $-r \in P$. Thus $ab > 0$ and $(-a)b > 0$, contradiction.

(ii): Suppose that $r, s \in P$. Then $ab > 0$ and $cd > 0$. Now $r + s = [(ad + bc, bd)]$, and $(ad + bc)bd = abdd + bbcd$. By (1) in the proof of 37.10, $dd > 0$ and $bb > 0$. Hence by properties of integers, $abdd + bbcd > 0$.

(iii): Suppose that $r, s \in P$. Then $ab > 0$ and $cd > 0$. Now $rs = [(ac, bd)]$ and $acbd = abcd > 0$. $\square$

Now we can define the order: $a < b$ iff $b - a \in P$. The main properties of $<$ are given in the following proposition.

Proposition 37.12. *Let $x, y, z \in \mathbb{Q}'$. Then*

(i) $x \not< x$.

(ii) If $x < y < z$, then $x < z$.

(iii) $x < y$, $x = y$, or $y < x$.

(iv) $x < y$ iff $x + z < y + z$.

(v) If $0' < x$ and $0' < y$, then $0' < x \cdot y$.

(vi) If $0' < z$, then $x < y$ implies that $x \cdot z < y \cdot z$.

Proof. (i): $x - x = 0'$, so (i) follows from 37.11(i).

(ii): Assume that $x < y < z$. So $y - x \in P$ and $z - y \in P$. Hence $z - x = (z - y) + y - x) \in P$ by 37.11(ii), so $x < z$.

(iii): We have $x = y$ or $x - y \in P$ or $y - x \in P$, so (iii) follows.

(iv): $x < y$ iff $y - x \in P$ iff $(y + z) - (x + z) \in P$ iff $x + z < y + z$.

(v): This is immediate from 37.11(iii).

(vi): Assume that $0' < z$ and $x < y$. So $z, y - x \in P$, so by 37.11(iii), $y \cdot z - x \cdot z = z \cdot (y - x) \in P$, and so $x \cdot z < y \cdot z$. $\square$

This finishes the main construction of the rational numbers. There are still two things to do, though: identify the integers among the rationals, and make a replacement so that the integers are a subset of the rationals.

For every integer a we define $f(a) = [(a, 1)]$.

Proposition 37.13. *f is an isomorphism of $\mathbb{Z}$ into $\mathbb{Q}'$. That is, f is an injection, and for any $a, b \in \mathbb{Z}$ we have $f(a + b) = f(a) + f(b)$ and $f(a \cdot b) = f(a) \cdot f(b)$.*

Proof. Suppose that $f(a) = f(b)$. Thus $[(a, 1)] = [(b, 1)]$, hence $(a, 1) \sim (b, 1)$, hence $a = a1 = 1b = b$. So f is an injection.

Now suppose that $a, b \in \mathbb{Z}$. Then

$$\begin{aligned}
f(a) + f(b) &= [(a,1)] + [(b,1)] = [(a1 + 1b, 1)] = [(a+b,1)] = f(a+b);\\
f(a) \cdot f(b) &= [(a,1)] \cdot [(b,1)] = [(ab,1)] = f(ab).
\end{aligned}$$

□

Proposition 37.14. $\mathbb{Z} \cap \mathbb{Q}' = \emptyset$.

Proof. To show that $\omega \cap \mathbb{Q}' = \emptyset$ it suffices to show that each element of $\mathbb{Q}'$ is infinite. If $[(a,b)] \in \mathbb{Q}'$, then $(ca, cb) \in [(a,b)]$ for every $c \in \mathbb{Z}$, and $cb \neq db$ for $c \neq d$, and so $(ca, cb) \neq (da, db)$ for $c \neq d$; hence $[(a,b)]$ is infinite.

Now suppose that $x \in \mathbb{Z} \cap \mathbb{Q}'$ with $x \notin \omega$. Temporarily denote the equivalence relation used to define $\mathbb{Z}'$ by $\equiv$. Then there exist $m, n \in \omega$ such that $x = [(m,n)]_\equiv$, and there exists $(a,b) \in A$ such that $x = [(a,b)]_\sim$. Then $(a,b) \sim (2a, 2b)$, so also $[(2a,2b)]_\sim = [(a,b)]_\sim = x = [(m,n)]_\equiv$. Hence $(a,b), (2a,2b) \in [(m,n)]_\equiv$, and it follows that $(a,b) \equiv (2a,2b)$. So $a + 2b = b + 2a$, and hence $a = b$. Then $(0,0) \equiv (a,b)$, so $(0,0) \in [(a,b)]_\equiv = [(a,b)]_\sim$, and we infer that $(0,0) \in A$, contradiction. □

We can now proceed very much like for $\mathbb{Z}$ and $\mathbb{Z}'$. We define $\mathbb{Q} = (\mathbb{Q}' \backslash \operatorname{rng}(f)) \cup \mathbb{Z}$. There is a one-one function $g : \mathbb{Q} \to \mathbb{Q}'$, defined by $g([(a,b)]) = [(a,b)]$ if $[(a,b)] \in \mathbb{Q}' \backslash \operatorname{rng}(f)$, and $g(a) = f(a)$ for $a \in \mathbb{Z}$. Clearly g is a bijection. Now the operations $+'$ and $\cdot'$ are defined on $\mathbb{Q}$ as follows. For any $a, b \in \mathbb{Q}$,

$$\begin{aligned}
a +' b &= g^{-1}(g(a) + g(b));\\
a \cdot' b &= g^{-1}(g(a) \cdot g(b)).
\end{aligned}$$

moreover, we define $a <' b$ iff $g(a) < g(b)$. With these definitions, g becomes an isomorphism of $\mathbb{Q}$ onto $\mathbb{Q}'$. Namely, if $a, b \in \mathbb{Q}$, then

$$\begin{aligned}
g(a +' b) &= g(g^{-1}(g(a) + g(b))) = g(a) + g(b);\\
g(a \cdot' b) &= g(g^{-1}(g(a) \cdot g(b))) = g(a) \cdot g(b);\\
a <' b \quad &\text{iff} \quad g(a) < g(b).
\end{aligned}$$

Moreover, the operations $+'$ and $\cdot'$ on $\mathbb{Z}$ coincide with the ones defined previously, since if $a, b \in \mathbb{Z}$, then

$$\begin{aligned}
a +' b &= g^{-1}(g(a) + g(b)) = g^{-1}(f(a) + f(b)) = g^{-1}(f(a+b)) = a + b;\\
a \cdot' b &= g^{-1}(g(a) \cdot g(b)) = g^{-1}(f(a) \cdot f(b)) = g^{-1}(f(a \cdot b)) = a \cdot b;\\
a <' b \quad &\text{iff} \quad g(a) < g(b)\\
&\text{iff} \quad f(a) < f(b)\\
&\text{iff} \quad a < b.
\end{aligned}$$

All of the properties above, like the associative, commutative, and distributive laws, hold for $\mathbb{Z}$ since g is an isomorphism. Of course we use $+, \cdot, <$ now rather than $+', \cdot', <'$.

A subset A of $\mathbb{Q}$ is a *Dedekind cut* provided the following conditions hold:

(1) $\mathbb{Q} \neq A \neq \emptyset$;
(2) For all $r, s \in \mathbb{Q}$, if $r < s$ and $s \in A$, then $r \in A$.
(3) A has no largest element.

Let $\mathbb{R}'$ be the set of all Dedekind cuts.

If A and B are Dedekind cuts, then we define

$$A + B = \{x : \text{there are } a \in A \text{ and } b \in B \text{ such that } x = a + b\}.$$

Proposition 38.1. *If A and B are Dedekind cuts, then so is $A + B$.*

Proof. Since A and B are both nonempty, clearly $A + B$ is nonempty. Now take $r \in \mathbb{Q}\backslash A$ and $s \in \mathbb{Q}\backslash B$. So $t < r$ for all $t \in A$, and $u < s$ for all $u \in B$. Then $a + b < r + s$ for all $a \in A$ and $b \in B$, so that $x < r + s$ for all $x \in A + B$. In particular, $r + s \notin A + B$, by the irreflexivity of $<$. So we have shown that (1) holds for $A + B$.

Now suppose that $r < s \in A + B$. Write $s = a + b$ with $a \in A$ and $b \in B$. Then $r < s = a + b$, so $r - a < b$, and hence $r - a \in B$ by (2) for B. Hence $r = a + (r - a)$ shows that $r \in A + B$. So (2) holds for $A + B$.

Suppose that $x \in A + B$. Write $x = a + b$ with $a \in A$ and $b \in B$. Since a is not the greatest element of A, by (3) choose $a' \in A$ such that $a < a'$. Then $x = a + b < a' + b \in A + B$, proving (3) for $A + B$. □

Proposition 38.2. *Let A, B, C be Dedekind cuts. Then*
(i) $A + B = B + A$.
(ii) $A + (B + C) = (A + B) + C$.

Proof. (i): obvious. (ii): Suppose that $x \in A + (B + C)$. Then there are $a \in A$ and $y \in (B + C)$ such that $x = a + y$; and there are $b \in B$ and $c \in C$ such that $y = b + c$. So $x = a + b + c$. Now $a + b \in (A + B)$, so $x \in ((A + B) + C)$. This shows that $A + (B + C) \subseteq (A + B) + C$. Since this is generally true for all Dedekind cuts A, B, C, we also have $(A + B) + C = C + (B + A) \subseteq (C + B) + A = A + (B + C)$. □

Now we define, following Chapter 6,

$$Z = \{r \in \mathbb{Q} : r < 0\}.$$

Clearly Z is a Dedekind cut.

Proposition 38.3. $A + Z = A$ *for every Dedekind cut A.*

Proof. Let $a \in A$. Since A does not have a largest element, choose $b \in A$ such that $a < b$. Then $a - b < 0$, hence $a - b \in Z$, and so $a = b + (a - b)$ shows that $a \in A + Z$.

Conversely, suppose that $x \in A + Z$. Then there exist $a \in A$ and $b \in Z$ such that $x = a + b$. Since $b < 0$, we have $x < a$, and so $x \in A$, as desired. □

It is easy to check that Z is the only element of $\mathbb{R}'$ such that $A+Z=A$ for all A.

Next, for any Dedekind cut A we define

$$-A=\{r\in\mathbb{Q}: \text{there is an } s\in\mathbb{Q} \text{ such that } r<s \text{ and } -s\notin A\}.$$

Proposition 38.4. *$A+-A=Z$ for any Dedekind cut A.*

Proof. First we show that $-A$ is itself a Dedekind cut. Since $A\neq\mathbb{Q}$, choose $r\in\mathbb{Q}\backslash A$. Then also $r+1\notin A$. so $-(r+1)<-r$ and $-(-r)=r\notin A$. It follows that $-(r+1)\in -A$. Hence $-A\neq\emptyset$. Next, choose $r\in A$. Then $-r\notin -A$, as otherwise there is an s such that $-r<s$ and $-s\notin A$; but $-s<r$, contradiction. So $-A\neq\mathbb{Q}$. Finally, suppose that $r\in -A$; we want to find a larger element in A. Choose s such that $r<s$ and $-s\notin A$. Take $t\in\mathbb{Q}$ such that $r<t<s$; for example, take $t=(r+s)/2$. Clearly then $t\in -A$, as desired. This checks that $-A$ is a Dedekind cut.

Now suppose that $x\in A+-A$. Then there are $a\in A$ and $b\in -A$ such that $x=a+b$. Choose $c\in\mathbb{Q}$ such that $b<c$ and $-c\notin A$. Suppose that $0\leq x$. Then $x=a+b<a+c$, and so $-c<a+-x\leq a$, and hence $-c\in A$, contradiction. Hence $x<0$, so that $x\in Z$.

Second suppose that $r\in Z$. Fix $b\notin A$.

(1) There is a positive integer p such that $b+\frac{pr}{2}\in A$.

In fact, to prove (1), also fix $a\in A$. Then $a<b$, as otherwise we would have $b\in A$. Hence there are positive integers s,t such that $b-a=\frac{s}{t}$. Since $\frac{r}{2}<0$, there are also positive integers u,v such that $\frac{r}{2}=-\frac{u}{v}$. Then $b-a=\frac{s}{t}\leq s\leq su=sv(-\frac{r}{2})$. Hence $b+sv\frac{r}{2}\leq a$, and so $b+sv\frac{r}{2}\in A$, proving (1).

Let p be the smallest positive integer such that $b+p\frac{r}{2}\in A$. Recall that $b\notin A$, so that even if $p=1$ we can assert that $b+(p-1)\frac{r}{2}\notin A$. Now

$$r=b+pr+(-b+(-p+1)\frac{r}{2}+\frac{r}{2}),$$

and $(-b+(-p+1)\frac{r}{2}+\frac{r}{2})<(-b+(-p+1)\frac{r}{2}$, and $-(-b+(-p+1)\frac{r}{2})=b+(p-1)\frac{r}{2}\notin A$. This shows that $r\in A+-A$. □

The element $-A$ is unique: if $A+B=Z$, then $B=-A$. In particular, $-Z=Z$.

Next, we call a Dedekind cut A *positive* iff if has at least one positive member.

Proposition 38.5. *For any Dedekind cut A, exactly one of the following holds:*
(i) A is positive;
(ii) $A=Z$;
(iii) $-A$ is positive.

Proof. Suppose that A is not positive, and $A\neq Z$. Since A is not positive, all its members are negative or zero; since it has no largest element, $0\notin A$. Thus $A\subseteq Z$. Since $A\neq Z$, we actually have $A\subset Z$. Choose $r\in Z\backslash A$. Now $r+r<0+r=r<0$, and so $r<\frac{r}{2}<0$. Hence $0<-\frac{r}{2}<-r$. So $-\frac{r}{2}\in -A$, since $-(-r)=r\notin A$. This shows that $-A$ is positive.

So we have shown that one of (i)–(iii) holds.

Obviously (i) and (ii) do not simultaneously hold. Suppose that both A and $-A$ are positive. Hence there is a positive element $r \in A$, and a positive element $s \in -A$. By the definition of $-A$, choose t such that $s < t$ and $-t \notin A$. Then $-t < -s < 0 < r$, so $-t \in A$, contradiction. Thus (i) and (iii) do not simultaneously hold. Finally, suppose that $-Z$ is positive. Let r be a positive element of $-Z$. Then by definition there is an s such that $r < s$ and $-s \notin Z$. So $0 \leq -s < -r$, contradicting r being positive. $\square$

On the basis of Proposition 38.5, the following definition makes sense. For any Dedekind cut A,

$$|A| = \begin{cases} A & \text{if } A = Z \text{ or } A \text{ is positive,} \\ -A & \text{if } -A \text{ is positive.} \end{cases}$$

Now we repeat the definition of product from Chapter 6. Let A and B be Dedekind cuts.

$$(a) \qquad \begin{aligned} A \cdot B = \{r \in \mathbb{Q} : & \text{ there are } s \in A \text{ and } t \in B \text{ such that } 0 < s \\ & \text{and } 0 < t \text{ and } r < s \cdot t\} \text{ if } A \text{ and } B \text{ are positive,} \end{aligned}$$

$$(b) \qquad A \cdot B = Z \text{ if } A = Z \text{ or } B = Z,$$

$$(c) \qquad A \cdot B = -(|A| \cdot |B|) \text{ if } A \neq Z \neq B \text{ and exactly one of } A, B \text{ is positive}$$

$$(d) \qquad A \cdot B = (-A) \cdot (-B) \text{ if } -A \text{ and } -B \text{ are both positive.}$$

Proposition 38.6. *Let A, B, C be Dedekind cuts.*
(i) $A \cdot B = B \cdot A$.
(ii) $(-A) \cdot B = -(A \cdot B) = A \cdot (-B)$.
(iii) $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.
(iv) $A \cdot (B + C) = A \cdot B + A \cdot C$.

Proof. (i): this is clear if both A and B are positive, or if one of them is Z. If both are different from Z and exactly one of them is positive, then $|A|$ and $|B|$ are both positive, and

$$A \cdot B = -(|A| \cdot |B|) = -(|B| \cdot |A|) = B \cdot A.$$

If $-A$ and $-B$ are both positive, then

$$A \cdot B = (-A) \cdot (-B) = (-B) \cdot (-A) = B \cdot A.$$

Thus (i) holds.

(ii): First we prove that $(-A) \cdot B = -(A \cdot B)$. This is true by (b) if one of A, B is Z, since $-Z = Z$. If A and B are positive, then

$$(-A) \cdot B = -(A \cdot B) \quad \text{by (c).}$$

If $-A$ and B are positive, then

$$\begin{aligned} -(A \cdot B) &= -(-((-A) \cdot B)) \quad \text{by (c)} \\ &= (-A) \cdot B. \end{aligned}$$

If A and $-B$ are positive, then

$$\begin{aligned} (-A) \cdot B = B \cdot (-A) & \quad \text{by (i)} \\ = -(B \cdot A) & \quad \text{by the previous case} \\ = -(A \cdot B) & \quad \text{by (i).} \end{aligned}$$

Finally, if $-A$ and $-B$ are positive, then

$$\begin{aligned} (-A) \cdot B &= -((-A) \cdot (-B)) \quad \text{by (c)} \\ &= -(A \cdot B) \quad \text{by (d).} \end{aligned}$$

Thus $(-A) \cdot B = -(A \cdot B)$ in general. The other part of (ii) follows from (i).

(iii):

(1) If A, B, C are all positive, then $A \cdot (B \cdot C) \subseteq (A \cdot B) \cdot C$.

For, assume that A, B, C are all positive. Clearly then $A \cdot B$ and $B \cdot C$ are positive. Now let $x \in A \cdot (B \cdot C)$. Then there exist s, t such that $x < s \cdot t$, $0 < s \in A$, and $0 < t \in B \cdot C$. Since $t \in B \cdot C$, there exist u, v such that $t < u \cdot v$, $0 < u \in B$, and $0 < v \in C$. Choose $s' \in A$ such that $s < s'$. Then $s \cdot u < s' \cdot u$, $0 < s' \in A$, and $0 < u \in B$, so $s \cdot u \in A \cdot B$. Then $x < s \cdot u \cdot v$, $0 < s \cdot u \in A \cdot B$, and $0 < v \in C$, so $x \in (A \cdot B) \cdot C$. This proves (1).

(2) If one of A, B, C is equal to Z, then $A \cdot (B \cdot C) = Z = (A \cdot B) \cdot C$.

This is clear.

(3) If A, B, C are all positive, then $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.

In fact,

$$\begin{aligned} A \cdot (B \cdot C) &\subseteq (A \cdot B) \cdot C \quad \text{by (1)} \\ &= C \cdot (B \cdot A) \quad \text{by (i)} \\ &\subseteq (C \cdot B) \cdot A \quad \text{by (1)} \\ &= A \cdot (B \cdot C) \quad \text{by (i).} \end{aligned}$$

So (3) holds.

Now we can use (ii) to finish (iii):

$$\begin{aligned} A, B, -C \text{ positive: } A \cdot (B \cdot C) &= A \cdot -(B \cdot -C) \\ &= -(A \cdot (B \cdot -C) \\ &= -((A \cdot B) \cdot -C) \end{aligned}$$

$$
\begin{aligned}
&= (A \cdot B) \cdot C; \\
A, -B, C \text{ positive: } A \cdot (B \cdot C) &= A \cdot -(-B \cdot C) \\
&= -(A \cdot (-B \cdot C)) \\
&= -((A \cdot -B) \cdot C) \\
&= (A \cdot B) \cdot C; \\
A, -B, -C \text{ positive: } A \cdot (B \cdot C) &= A \cdot ((-B) \cdot (-C)) \\
&= (A \cdot -B) \cdot -C \\
&= (A \cdot B) \cdot C; \\
C \text{ positive: } (A \cdot B) \cdot C &= C \cdot (B \cdot A) \\
&= (C \cdot B) \cdot A \\
&= A \cdot (B \cdot C); \\
-A, B, -C \text{ positive: } A \cdot (B \cdot C) &= A \cdot -(B \cdot -C) \\
&= -((-A) \cdot -(B \cdot -C)) \\
&= (-A) \cdot (B \cdot -C) \\
&= ((-A) \cdot B) \cdot -C \\
&= (A \cdot B) \cdot C; \\
-A, -B, -C \text{ positive: } A \cdot (B \cdot C) &= A \cdot ((-B) \cdot (-C)) \\
&= -((-A) \cdot ((-B) \cdot (-C))) \\
&= -(((-A) \cdot (-B)) \cdot -C) \\
&= (A \cdot B) \cdot C.
\end{aligned}
$$

(iv): Clearly

(4) If one of A, B, C is Z, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

(5) If A, B, C are positive, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

For, first suppose that $x \in A \cdot (B + C)$. Then we can choose s, t so that $0 < s \in A$, $0 < t \in B + C$, and $x < s \cdot t$. Since $t \in B + C$, there are $b \in B$ and $c \in C$ such that $t = b + c$. Now choose $b' \in B$ with $b \leq b'$ and $0 < b'$, and choose $c' \in C$ such that $c \leq c'$ and $0 < c'$. Now $x = s \cdot b' + (x - s \cdot b')$, and clearly $s \cdot b' \in A \cdot B$, while

$$x - s \cdot b' < s \cdot (b' + c') - s \cdot b' = s \cdot c',$$

and clearly $s \cdot c' \in A \cdot C$. This proves $\subseteq$ in (5).

Now suppose that $y \in A \cdot B + A \cdot C$. Then we can write $y = u + v$ with $u \in A \cdot B$ and $v \in A \cdot C$. Say $u < s \cdot t$ with $0 < s \in A$ and $0 < t \in B$, and $v < a \cdot c$ with $0 < a \in A$ and $0 < c \in C$. Let s' be the maximum of s and a. Then $y < s' \cdot (t + c)$, $0 < s' \in A$, and $t + c \in B + C$. So $y \in A \cdot (B + C)$. This proves $\supseteq$ in (5).

(6) If $A, B, -C$ are positive, and also $B + C$ is positive, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

For,

$$\begin{aligned} A \cdot B &= A \cdot (B + C + -C) \\ &= A \cdot (B + C) + A \cdot (-C) \quad \text{by (5)} \\ &= A \cdot (B + C) + -(A \cdot C), \quad \text{by (ii)} \end{aligned}$$

and (6) follows.

(7) If $A, B, -C$ are positive, and $B + C$ is negative, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

For,

$$\begin{aligned} -(A \cdot (B + C)) &= A \cdot (-(B + C)) \quad \text{by (ii)} \\ &= A \cdot (-B + -C) \\ &= A \cdot (-B) + A \cdot (-C) \quad \text{by (6)} \\ &= -(A \cdot B) + -(A \cdot C), \quad \text{by (ii)} \end{aligned}$$

and (7) follows.

(8) If $A, B, -C$ are positive, and $B + C = Z$, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

For, under these hypotheses, $C = -B$, and so

$$A \cdot (B + C) = A \cdot Z = Z = A \cdot B + -(A \cdot B) = A \cdot B + A \cdot (-B) = A \cdot B + A \cdot C.$$

(9) If $A, -B, C$ are positive, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

This follows from (6)–(8) since $+$ is commutative.

(10) If $A, -B, -C$ are positive, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

For,

$$\begin{aligned} A \cdot (B + C) &= -(A \cdot (-B + -C)) \quad \text{by (ii)} \\ &= -(A \cdot (-B) + A \cdot (-C)) \quad \text{by (5)} \\ &= -(-(A \cdot B) + -(A \cdot C)) \quad \text{by (ii)} \\ &= A \cdot B + A \cdot C. \end{aligned}$$

(11) If A is positive, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

This is true by (6)–(10).

(12) If $-A$ is positive, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

In fact, $(-A) \cdot (B + C) = (-A) \cdot B + (-A) \cdot C$ by (11), and (12) follows, using (ii). $\square$

Now we define

$$I = \{r \in \mathbb{Q} : r < 1\}.$$

Clearly I is a Dedekind cut.

Proposition 38.7. *$A \cdot I = A$ for any Dedekind cut A.*

Proof. This is clear if $A = Z$. Now suppose that A is positive. Suppose that $r \in A \cdot I$. Then there are $s, t \in \mathbb{Q}$ such that $0 < s \in A$, $0 < t \in I$, and $r < s \cdot t$. Clearly then $r < s$, so $r \in A$ by the definition of Dedekind cut.

Conversely, suppose that $r \in A$. Choose $r', r'' \in A$ such that $r < r' < r''$ and $0 < r'$. Let $s = \frac{r'}{r''}$. Then $0 < s < 1$, so $s \in I$. Since $r < r' = r'' \cdot s$, it follows that $r \in A \cdot I$. Thus we have shown that $A \cdot I = A$ for A positive.

If $-A$ is positive, then $A \cdot I = -((-A) \cdot I) = -(-A) = A$, using D6(ii). $\square$

Proposition 38.8. *If A is a Dedekind cut and $A \neq Z$, then there is a Dedekind cut B such that $A \cdot B = I$.*

Proof. First suppose that A is positive. Let

$$B = \{r \in \mathbb{Q} : r < 0, \text{ or } 0 \leq r \text{ and } r \cdot s < 1 \text{ for every } s \in A \text{ for which } 0 < s\}.$$

Then $B \neq \emptyset$, since clearly $0 \in B$. Clearly if $r' < r \in B$, then also $r' \in B$. If $0 < s \in A$, then $\frac{1}{s} \notin B$. So B is a Dedekind cut.

We claim that $A \cdot B = I$. Suppose that $r \in A \cdot B$. Choose s, t so that $0 < s \in A$, $0 < t \in B$, and $r < s \cdot t$. Then by the definition of B, $s \cdot t < 1$, so $r < 1$. Hence $r \in I$.

Conversely, suppose that $r \in I$, so that $r < 1$. Choose r', r'', r''' so that $0, r < r' < r'' < r''' < 1$. Let $C = \{s \in \mathbb{Q} : s < r'''\}$. Clearly C is a Dedekind cut.

(1) $(A \cdot C) \subset A$.

In fact, clearly $(A \cdot C) \subseteq A$. Suppose that $A \cdot C = A$. Now

$$A = A \cdot I = (A \cdot C) + (A \cdot (I - C)) = A + (A \cdot (I - C)),$$

so $A \cdot (I - C) = Z$. Choose s, t so that $r''' < s < t < 1$. Then $-s < -r'''$ and $r''' \notin C$, so $-s \in -C$. Hence $0 < t - s \in (I - C)$. So $I - C$ is positive. Since A is also positive, it follows that $A \cdot (I - C)$ is positive, contradiction. Hence (1) holds.

By (1), choose $s \in A \backslash (A \cdot C)$. We may assume that $0 < s$. Thus

(2) For all a, c, if $0 < a \in A$ and $0 < c \in C$, then $a \cdot c \leq s$.

Now let $v = \frac{r'}{s}$. Thus $s \cdot v = r' > r$. Hence we will get $r \in A \cdot B$ as soon as we show that $v \in B$. Suppose that $0 < a \in A$. Now $0 < r'' \in C$, so by (2) we have $a \cdot r'' \leq s$. Hence

$$a \cdot v = a \cdot \frac{r'}{s} < a \cdot \frac{r''}{s} \leq 1,$$

so that $a \cdot v < 1$, as desired.

Thus we have finished the proof in the case that A is positive. If $-A$ is positive, then choose B so that $(-A) \cdot B = I$. Then $(A \cdot (-B)) = (-A) \cdot B = I$, using 38.7(ii). $\square$

This finishes the purely arithmetic part of the construction of the real numbers. Now we discuss ordering. We define $A < B$ iff $B - A$ is positive. Elementary properties of $<$ are given in the following proposition.

Proposition 38.9. *Let $A, B, C \in \mathbb{R}'$. Then*

(i) $A \not< A$.

(ii) If $A < B < C$, then $A < C$.

(iii) $A < B$, $A = B$, or $B < A$.

(iv) $A < B$ iff $A + C < B + C$.

(v) $Z < I$.

(vi) If $Z < A$ and $Z < B$, then $Z < A \cdot B$.

(vii) If $Z < C$, then $A < B$ implies that $A \cdot C < B \cdot C$.

(viii) $A < B$ iff $A \subset B$.

Proof. (i): $A - A = Z$, so $A \not< A$ by 38.5.

(ii) Suppose that $A < B < C$. Thus $B - A$ and $C - B$ are positive. Hence clearly also $C - A = C - B + B - A$ is positive.

(iii): Given A, B, by 38.5 we have $A - B$ positive, $A - B = Z$, or $-(A - B) = B - A$ is positive. By definition this gives $A < B$, $A = B$, or $B < A$.

(iv): First suppose that $A < B$. Thus $B - A$ is positive. Since $B + C - (A + C) = B - A$, it follows that $A + C < B + C$.

Second, suppose that $A + C < B + C$. Thus $B - A = B + C - (A + C)$ is positive, so $A < B$.

(v): Obviously I is positive.

(vi): Assume that $Z < A$ and $Z < B$. Thus A and B are positive. Clearly then $A \cdot B$ is positive. So $Z < A \cdot B$.

(vii): Assume that $Z < C$ and $A < B$. Then C and $B - A$ are positive, so also $C \cdot (B - A) = C \cdot B - (A \cdot C)$ is positive, and so $A \cdot C < B \cdot C$.

(viii): Suppose that $A < B$. Thus $B - A$ is positive. Choose x so that $0 < x \in B - A$. Then we can write $x = b + a$ with $b \in B$ and $a \in -A$. By the definition of $-A$, choose $s \in \mathbb{Q}$ so that $a < s$ and $-s \notin A$. Then $-s < -a$, so also $-a \notin A$. Also $b + a > 0$, so $b > -a$, and it follows that $b \notin A$. Now if $y \in A$, then $y < b$, as otherwise $b \le y$ would imply that $b \in A$. But then $y \in B$. So $A \subseteq B$, and since $b \in B \backslash A$, even $A \subset B$.

Conversely, suppose that $A \subset B$. Choose $b \in B \backslash A$. Choose c, d such that $b < c < d \in B$. Now $-c < -b$ and $b \notin A$, so $-c \in -A$. Thus $d - c$ is a positive element of $B - A$, hence $B - A$ is positive and $A < B$. □

The following theorem expresses the essential new property of the reals as opposed to the rationals.

Theorem 38.10. *Every nonempty subset of $\mathbb{R}'$ which is bounded above has a least upper bound. That is, if $\emptyset \neq \mathscr{X} \subseteq \mathbb{R}'$, and there is a Dedekind cut D such that $A \le D$ for all $A \in \mathbb{R}'$, then there is a Dedekind cut B such that the following two conditions hold:*

(i) $A \le B$ for every $A \in \mathscr{X}$.

(ii) For any Dedekind cut C, if $A \le C$ for every $A \in \mathscr{X}$, then $B \le C$.

Proof. Let $B = \bigcup_{A \in \mathscr{X}} A$. Since $\mathscr{X}$ is nonempty, and each Dedekind cut is nonempty, it follows that B is nonempty. To show that B does not consist of all rationals, we use the assumption that $\mathscr{X}$ has an upper bound. Let D be an upper bound for $\mathscr{X}$. Thus $A \le D$ for all $A \in \mathscr{X}$. By 38.9(viii), $A \subseteq D$ for all $A \in \mathscr{X}$, and hence $B \subseteq D$. Since $D \neq \mathbb{Q}$, also

$B \neq \mathbb{Q}$. If $x < y \in B$, then $y \in A$ for some $A \in \mathscr{X}$, hence $x \in A$, hence $x \in B$. Thus B is a Dedekind cut.

For any $A \in \mathscr{X}$ we have $A \subseteq B$, and hence $A \leq B$ by D9(viii).

Now suppose that $A \subseteq C$ for all $A \in \mathscr{X}$, where C is a Dedekind cut. Then $B \subseteq C$, hence $B \leq C$ by 38.9(viii). □

Next we want to embed the rationals into $\mathbb{R}'$. For every rational r we define $f(r) = \{q \in \mathbb{Q} : q < r\}$. Clearly $f(r)$ is a Dedekind cut.

Proposition 38.11. *(i) f is one-one.*

(ii) $f(r+s) = f(r) + f(s)$ for any $r, s \in \mathbb{Q}$.

(iii) $f(r \cdot s) = f(r) \cdot f(s)$ for any $r, s \in \mathbb{Q}$.

Proof. (i): Suppose that $r, s \in \mathbb{Q}$; say $r < s$. Then $r \in f(s) \backslash f(r)$, so $f(r) \neq f(s)$.

(ii): First suppose that $x \in f(r+s)$. Thus $x < r+s$, so $x - s < r$. Let r' be a rational number such that $x - s < r' < r$. Then $x = r' + (x - r')$, and $x - r' < s$, so $x \in f(r) + f(s)$.

Conversely, suppose that $x \in f(r) + f(s)$. Choose $a \in f(r)$ and $b \in f(s)$ so that $x = a + b$. Then $a < r$ and $b < s$, so $x < r + s$, and so $x \in f(r+s)$.

(iii): Note that $f(0) = Z$; hence (iii) is clear if $r = 0$ or $s = 0$. Suppose that $r, s > 0$. Suppose that $x \in f(r \cdot s)$. So $x < r \cdot s$. Hence $\frac{x}{s} < r$. Choose $r' \in \mathbb{Q}$ such that $\frac{x}{s} < r' < r$ and $0 < r'$. Hence $\frac{x}{r'} < s$. Choose $s' \in \mathbb{Q}$ such that $\frac{x}{r'} < s' < s$ and $0 < s'$. Then $x < r' \cdot s'$, $0 < r' \in f(r)$, and $0 < s' \in f(s)$, so $x \in f(r) \cdot f(s)$.

Conversely, suppose that $x \in f(r) \cdot f(s)$. Then there are $r' \in f(r)$ and $s' \in f(s)$ such that $0 < r'$, $0 < s'$, and $x < r' \cdot s'$. Hence $x < r \cdot s$, so $x \in f(r \cdot s)$, as desired. This finishes the case in which $r, s > 0$.

To continue we need the following little fact:

(1) $-f(r) = \{q \in \mathbb{Q} : q < -r\}$ for any rational number r.

In fact, suppose that $q \in -f(r)$. Then there is a rational t such that $q < t$ and $-t \notin f(r)$. thus $-t \not< r$, so $r \leq -t$. Hence $t \leq -r$, so $q < -r$. Conversely, suppose that $q < -r$. Now $r \notin f(r)$, so $q \in -f(r)$. Thus (1) holds.

Now suppose that $r < 0 < s$. Then, using (1),

$$f(r) \cdot f(s) = -((-f(r)) \cdot f(s)) = -(f(-r) \cdot f(s)) = -f((-r) \cdot s) = f(r \cdot s).$$

Similarly if $s < 0 < r$. If $r, s < 0$, then

$$(f(r) \cdot f(s) = (-f(r)) \cdot (-f(s)) = f(-r) \cdot f(-s) = f((-r) \cdot (-s)) = f(r \cdot s).$$ □

Proposition 38.12. $\mathbb{Q} \cap \mathbb{R}' = \emptyset$.

Proof. First, $\omega \cap \mathbb{R}' = \emptyset$, since the members of ω are all finite, while the members of $\mathbb{R}'$ are all infinite.

Now suppose that $a \in \mathbb{Z} \cap \mathbb{R}'$. Then $a \notin \omega$ by the preceding paragraph, so $a = [(m, n)]$ for some $m, n \in \omega$. But also $a \in \mathbb{R}'$, so a is a set of rationals. In particular, (m, n) is a rational. Now (m, n) has either one or two elements; the only rationals with only one or two elements are 1 and 2. Since $\emptyset \in 1$ and $\emptyset \in 2$, we get $\emptyset \in (m, n)$, contradiction.

A similar argument shows that $a \in \mathbb{Q} \cap \mathbb{R}'$ leads to a contradiction. $\square$

We can now proceed very much like in previous appendices. We define $\mathbb{R} = (\mathbb{R}' \backslash \mathrm{rng}(f)) \cup \mathbb{Q}$. There is a one-one function $g : \mathbb{R} \to \mathbb{R}'$, defined by $g(A) = A$ if $A \in \mathbb{R}' \backslash \mathrm{rng}(f)$, and $g(A) = f(A)$ for $A \in \mathbb{Q}$. Clearly g is a bijection. Now the operations $+'$ and $\cdot'$ are defined on $\mathbb{R}$ as follows. For any $a, b \in \mathbb{R}$,

$$\begin{aligned} a +' b &= g^{-1}(g(a) + g(b)); \\ a \cdot' b &= g^{-1}(g(a) \cdot g(b)). \end{aligned}$$

moreover, we define $a <' b$ iff $g(a) < g(b)$. With these definitions, g becomes an isomorphism of $\mathbb{R}$ onto $\mathbb{R}'$. Namely, if $a, b \in \mathbb{R}$, then

$$\begin{aligned} g(a +' b) &= g(g^{-1}(g(a) + g(b))) = g(a) + g(b); \\ g(a \cdot' b) &= g(g^{-1}(g(a) \cdot g(b))) = g(a) \cdot g(b); \\ a <' b \quad &\text{iff} \quad g(a) < g(b). \end{aligned}$$

Moreover, the operations $+'$ and $\cdot'$ on $\mathbb{Q}$ coincide with the ones defined in appendix C, since if $a, b \in \mathbb{Q}$, then

$$\begin{aligned} a +' b &= g^{-1}(g(a) + g(b)) = g^{-1}(f(a) + f(b)) = g^{-1}(f(a + b)) = a + b; \\ a \cdot' b &= g^{-1}(g(a) \cdot g(b)) = g^{-1}(f(a) \cdot f(b)) = g^{-1}(f(a \cdot b)) = a \cdot b; \end{aligned}$$

$$\begin{aligned} a <' b \quad &\text{iff} \quad g(a) < g(b) \\ &\text{iff} \quad f(a) < f(b) \\ &\text{iff} \quad a < b. \end{aligned}$$

All of the properties above, like the associative, commutative, and distributive laws, hold for $\mathbb{R}$ since g is an isomorphism. Of course we use $+, \cdot, <$ now rather than $+', \cdot', <'$.