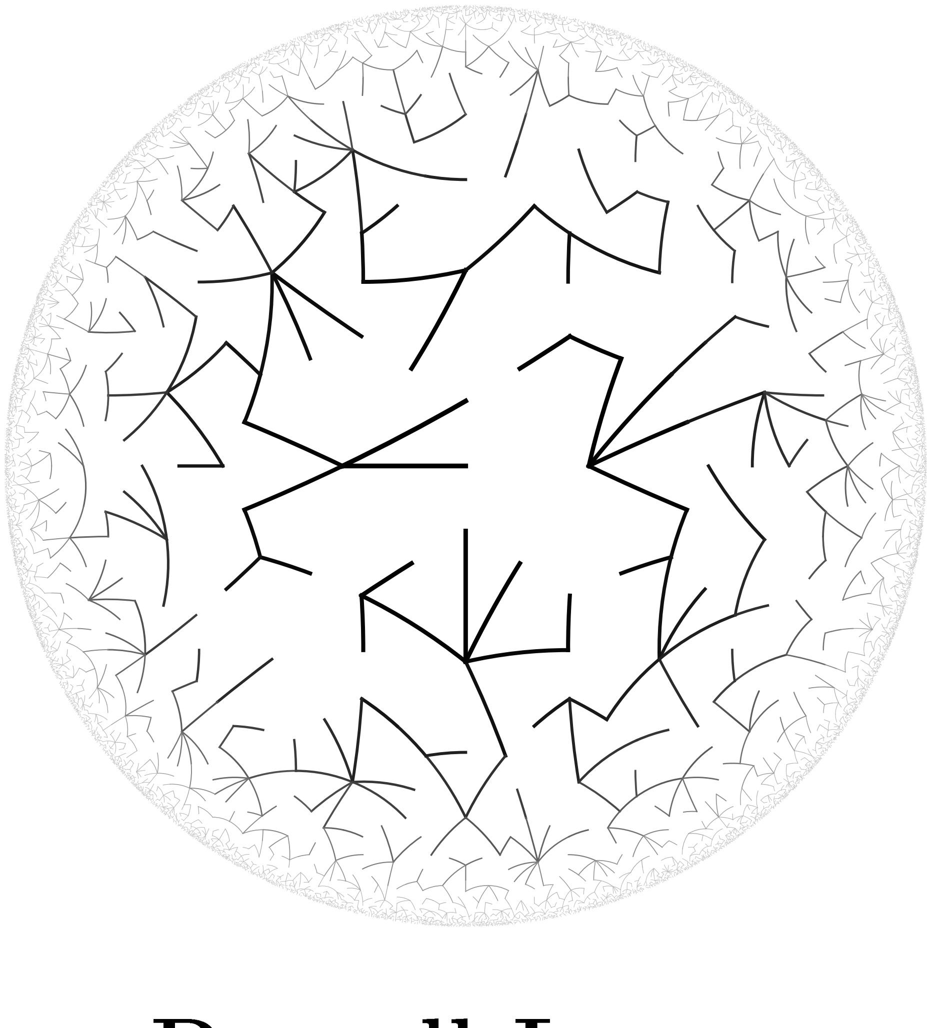
Probability on Trees and Networks



Russell Lyons

with Yuval Peres

A love and respect of trees has been characteristic of mankind since the beginning of human evolution. Instinctively, we understood the importance of trees to our lives before we were able to ascribe reasons for our dependence on them.

— *America's Garden Book*, James and Louise Bush-Brown, rev. ed. by The New York Botanical Garden, Charles Scribner's Sons, New York, 1980, p. 142.

The cover shows a sample from the wired uniform spanning forest on the edges of the $(2,3,7)$*-triangle tessellation of the hyperbolic plane.*

Table of Contents

Preface

This book began as lecture notes for an advanced graduate course called "Probability on Trees" that Lyons gave in Spring 1993. We are grateful to Rabi Bhattacharya for having suggested that he teach such a course. We have attempted to preserve the informal flavor of lectures. Many exercises are also included, so that real courses can be given based on the book. Indeed, previous versions have already been used for courses or seminars in several countries. The most current version of the book can be found on the web. A few of the authors' results and proofs appear here for the first time. At this point, almost all of the actual writing was done by Lyons. We hope to have a more balanced co-authorship eventually.

This book is concerned with certain aspects of discrete probability on infinite graphs that are currently in vigorous development. We feel that there are three main classes of graphs on which discrete probability is most interesting, namely, trees, Cayley graphs of groups (or more generally, transitive, or even quasi-transitive, graphs), and planar graphs. Thus, this book develops the general theory of certain probabilistic processes and then specializes to these particular classes. In addition, there are several reasons for a special study of trees. Since in most cases, analysis is easier on trees, analysis can be carried further. Then one can often either apply the results from trees to other situations or can transfer to other situations the techniques developed by working on trees. Trees also occur naturally in many situations, either combinatorially or as descriptions of compact sets in Euclidean space $\mathbb{R}^d$. (More classical probability, of course, has tended to focus on the special and important case of the Euclidean lattices $\mathbb{Z}^d$.)

It is well known that there are many connections among probability, trees, and groups. We survey only some of them, concentrating on recent developments of the past twenty years. We discuss some of those results that are most interesting, most beautiful, or easiest to prove without much background. Of course, we are biased by our own research interests and knowledge. We include the best proofs available of recent as well as classic results. Much more is known about almost every subject we present. The only prerequisite is knowledge of conditional expectation with respect to a σ-algebra, and even that is rarely used. For part of Chapter 13 and all of Chapter 16, basic knowledge of ergodic theory is also required.

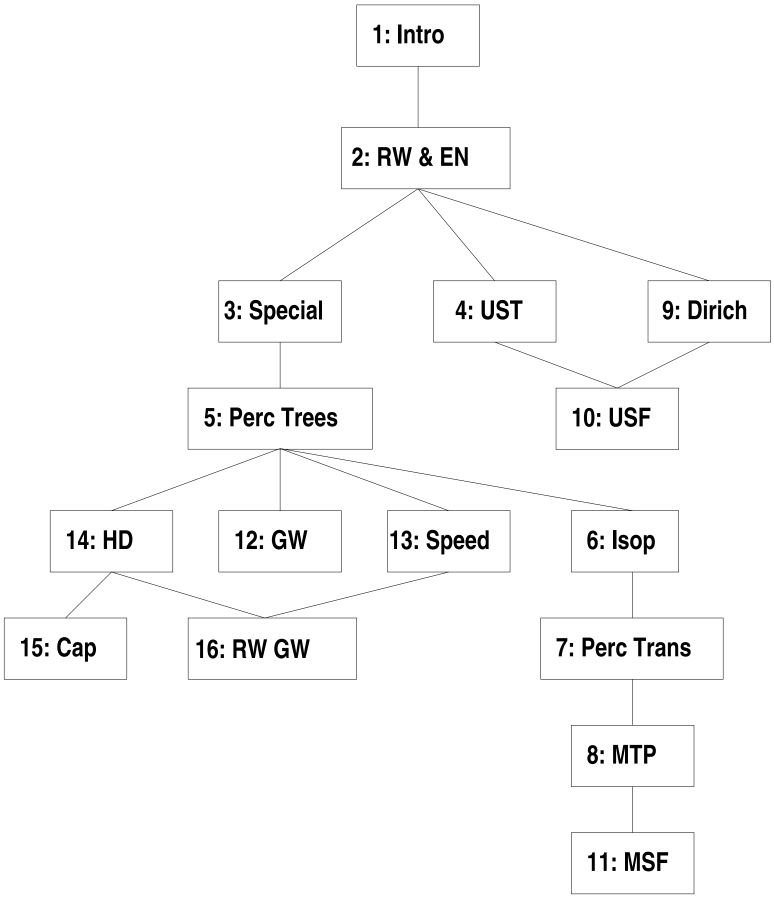
Most exercises that appear in the text, as opposed to those at the end of the chapters, are ones that will be particularly helpful to do when they are reached. They either facilitate one's understanding or will be used later in the text. These in-text exercises are also collected at the end of each chapter for easy reference, just before additional exercises are presented.

Some notation we use is $\langle \cdots \rangle$ for a sequence (or, sometimes, more general function), $\restriction$

for the restriction of a function or measure to a set, $\mathbf{E}[X\,;\,A]$ for the expectation of X on the event A, and $|\bullet|$ for the cardinality of a set. Some definitions are repeated in different chapters to enable more selective reading.

A question labelled as **Question** $m.n$ is one to which the answer is unknown, where m and n are numbers. Unattributed results are usually not due to us.

Major chapter dependencies are indicated in this figure:



Lyons is grateful to the Institute for Advanced Studies and the Institute of Mathematics, both at the Hebrew University of Jerusalem, for support during some of the writing. We are grateful to Brian Barker, Jochen Geiger, Janko Gravner, Svante Janson, Tri Minh Lai, Steve Morrow, Peter Mörters, Jason Schweinsberg, Jeff Steif, Ádám Timár, and especially Jacob Magnusson for noting several corrections to the manuscript.

Russell Lyons
Indiana University
rdlyons@indiana.edu
http://mypage.iu.edu/~rdlyons/

Yuval Peres
Microsoft Corporation
peres@microsoft.com
http://research.microsoft.com/en-us/um/people/peres/

Chapter 1

Some Highlights

This chapter gives some of the highlights to be encountered in this book. Some of the topics in the book do not appear at all here since they are not as suitable to a quick overview. Also, we concentrate in this overview on trees since it is easiest to use them to illustrate most of the themes.

Notation and terminology for graphs is the following. A ***graph*** is a pair $G = (\mathsf{V}, \mathsf{E})$, where V is a set of ***vertices*** and E is a symmetric subset of $\mathsf{V} \times \mathsf{V}$, called the ***edge*** set. The word "symmetric" means that $(x, y) \in \mathsf{E}$ iff $(y, x) \in \mathsf{E}$; here, x and y are called the ***endpoints*** of (x, y). The symmetry assumption is usually phrased by saying that the graph is ***undirected*** or that its edges are ***unoriented***. Without this assumption, the graph is called ***directed***. If we need to distinguish the two, we write an unoriented edge as $[x, y]$, while an oriented edge is written as $\langle x, y \rangle$. An unoriented edge can be thought of as the pair of oriented edges with the same endpoints. If $(x, y) \in \mathsf{E}$, then we call x and y ***adjacent*** or ***neighbors***, and we write $x \sim y$. The ***degree*** of a vertex is the number of its neighbors. If this is finite for each vertex, we call the graph ***locally finite***. If the degree of every vertex is the same number d, then the graph is called ***regular*** or d***-regular***. If x is an endpoint of an edge e, we also say that x and e are ***incident***, while if two edges share an endpoint, then we call those edges ***adjacent***. If we have more than one graph under consideration, we distinguish the vertex and edge sets by writing $\mathsf{V}(G)$ and $\mathsf{E}(G)$. A ***path***** in a graph is a sequence of vertices where each successive pair of vertices is an edge in the graph. A finite path with at least one edge and whose first and last vertices are the same is called a ***cycle***. A graph is ***connected*** if there is a path from any of its vertices to any other. A graph with no cycles is called a ***forest***; a connected forest is a ***tree***.

If there are numbers (weights) $c(e)$ assigned to the edges e of a graph, the resulting object is called a ***network***. Sometimes we work with more general objects than graphs, called multigraphs. A ***multigraph*** is a pair of sets, V and E, together with a pair of maps

* In graph theory, a "path" is necessarily self-avoiding. What we call a "path" is called in graph theory a "walk". However, to avoid confusion with random walks, we do not adopt that terminology. When a path does not pass through any vertex (resp., edge) more than once, we will call it ***vertex simple*** (resp., ***edge simple***). We'll just say ***simple*** also to mean vertex simple, which implies edge simple.

from $\mathsf{E} \to \mathsf{V}$ denoted $e \mapsto e^-$ and $e \mapsto e^+$. The images of e are called the ***endpoints*** of e, the former being its ***tail*** and the latter its ***head***. If $e^- = e^+ = x$, then e is a ***loop*** at x. Edges with the same pair of endpoints are called ***parallel*** or ***multiple***. If the multigraph is undirected, then for every edge $e \in \mathsf{E}$, there is an edge $-e \in \mathsf{E}$ such that $(-e)^- = e^+$ and $(-e)^+ = e^-$. Given a network $G = (\mathsf{V}, \mathsf{E})$ with weights $c(\bullet)$ and a subset of its vertices K, the ***induced subnetwork*** $G{\restriction}K$ is the subnetwork with vertex set K, edge set $(K \times K) \cap \mathsf{E}$, and weights $c{\restriction}(K \times K) \cap \mathsf{E}$.

Items such as theorems are numbered in this book as $C.n$, where C is the chapter number and n is the item number in that chapter; C is omitted when the item appears in the same chapter as the reference to it. In particular, in this chapter, items to be encountered in later chapters are numbered with their chapter numbers.

§1.1. Branching Number.

A tree is called ***locally finite*** if the degree of each vertex is finite (but not necessarily uniformly bounded). Our trees will usually be ***rooted***, meaning that some vertex is designated as the root, denoted o. We imagine the tree as growing (upwards) away from its root. Each vertex then has branches leading to its children, which are its neighbors that are further from the root. For the purposes of this chapter, we do not allow the possibility of leaves, i.e., vertices without children.

How do we assign an average branching number to an arbitrary infinite locally finite tree? If the tree is a binary tree, as in Figure 1.1, then clearly the answer will be "2". But in the general case, since the tree is infinite, no straight average is available. We must take some kind of limit or use some other procedure.

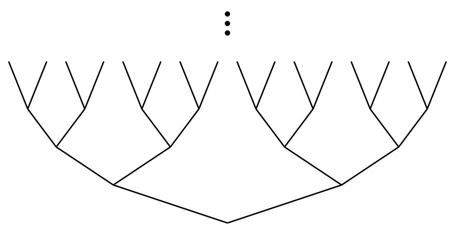


Figure 1.1. The binary tree.

One simple idea is as follows. Let T_n be the set of vertices at distance n from the root, o. Define the ***lower (exponential) growth rate*** of the tree to be

$$\underline{\mathrm{gr}}\, T := \liminf_{n\to\infty} |T_n|^{1/n} .$$

This certainly will give the number "2" to the binary tree. One can also define the ***upper (exponential) growth rate***

$$\overline{\operatorname{gr}}\, T := \limsup_{n\to\infty} |T_n|^{1/n}$$

and the ***(exponential) growth rate***

$$\operatorname{gr} T := \lim_{n\to\infty} |T_n|^{1/n}$$

when the limit exists. However, notice that these notions of growth barely account for the structure of the tree: only $|T_n|$ matters, not how the vertices at different levels are connected to each other. Of course, if T is ***spherically symmetric***, meaning that for each n, every vertex at distance n from the root has the same number of children (which may depend on n), then there is really no more information in the tree than that contained in the sequence $\langle |T_n| \rangle$. For more general trees, however, we will use a different approach.

Consider the tree as a network of pipes and imagine water entering the network at the root. However much water enters a pipe leaves at the other end and splits up among the outgoing pipes (edges). Consider the following sort of restriction: Given $\lambda \ge 1$, suppose that the amount of water that can flow through an edge at distance n from o is only λ^{-n}. If λ is too big, then perhaps no water can flow. In fact, consider the binary tree. A moment's thought shows that water can still flow throughout the tree provided that $\lambda \le 2$, but that as soon as $\lambda > 2$, then no water at all can flow. Obviously, this critical value of 2 for λ is the same as the branching number of the binary tree. So let us make a general definition: the ***branching number*** of a tree T is the supremum of those λ that admit a positive amount of water to flow through T; denote this critical value of λ by $\operatorname{br} T$. As we will see, this definition is the exponential of what Furstenberg (1970) called the "dimension" of a tree, which is the Hausdorff dimension of its boundary.

It is not hard to check that $\operatorname{br} T$ is related to $\underline{\operatorname{gr}}\, T$ by

$$\operatorname{br} T \le \underline{\operatorname{gr}}\, T\,. \tag{1.1}$$

Often, as in the case of the binary tree, equality holds here. However, there are many examples of strict inequality.

▷ **Exercise 1.1.**
Prove (1.1).

▷ **Exercise 1.2.**
Show that $\operatorname{br} T = \underline{\operatorname{gr}}\, T$ when T is spherically symmetric.

Example 1.1. If T is a tree such that vertices at even distances from o have 2 children while the rest have 3 children, then $\operatorname{br} T = \operatorname{gr} T = \sqrt{6}$. It is easy to see that $\operatorname{gr} T = \sqrt{6}$, whence by (1.1), it remains to show that $\operatorname{br} T \geq \sqrt{6}$, i.e., given $\lambda < \sqrt{6}$, to show that a positive amount of water can flow to infinity with the constraints given. Indeed, we can use the water flow with amount $6^{-n/2}$ flowing on those edges at distance n from the root when n is even and with amount $6^{-(n-1)/2}/3$ flowing on those edges at distance n from the root when n is odd.

Example 1.2. (The 1–3 Tree) Let T be a tree embedded in the upper half plane with o at the origin. List T_n in clockwise order as $\langle x_1^n, \ldots, x_{2^n}^n \rangle$. Let x_k^n have 1 child if $k \leq 2^{n-1}$ and 3 children otherwise; see Figure 1.2. Now, a ***ray*** is an infinite path from the root that doesn't backtrack. If x is a vertex of T that does not have the form $x_{2^n}^n$, then there are only finitely many rays that pass through x. This means that water cannot flow to infinity through such a vertex x when $\lambda > 1$. That leaves only the possibility of water flowing along the single ray consisting of the vertices $x_{2^n}^n$, which is impossible too. Hence $\operatorname{br} T = 1$, yet $\operatorname{gr} T = 2$.

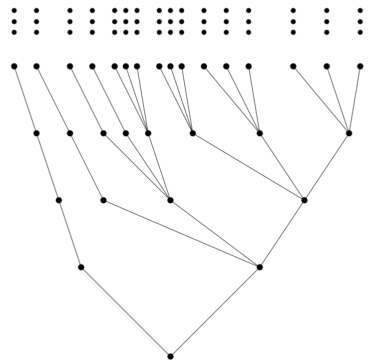


Figure 1.2. A tree with branching number 1 and growth rate 2.

Example 1.3. If $T^{(1)}$ and $T^{(2)}$ are trees, form a new tree $T^{(1)} \vee T^{(2)}$ from disjoint copies of $T^{(1)}$ and $T^{(2)}$ by joining their roots to a new point taken as the root of $T^{(1)} \vee T^{(2)}$ (Figure 1.3). Then

$$\operatorname{br}(T^{(1)} \vee T^{(2)}) = \operatorname{br} T^{(1)} \vee \operatorname{br} T^{(2)}$$

since water can flow in the join $T^{(1)} \vee T^{(2)}$ iff water can flow in one of the trees.

We will denote by $|x|$ the distance of a vertex x to the root.

Example 1.4. We will put two trees together such that $\operatorname{br}(T^{(1)} \vee T^{(2)}) = 1$ but $\underline{\operatorname{gr}}(T^{(1)} \vee T^{(2)}) > 1$. Let $n_k \uparrow \infty$. Let $T^{(1)}$ (resp., $T^{(2)}$) be a tree such that x has 1 child (resp.,

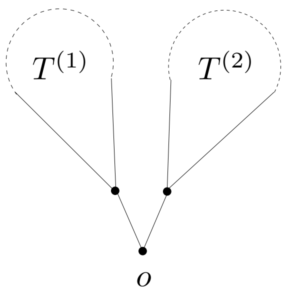


Figure 1.3. Joining two trees.

2 children) for $n_{2k} \le |x| \le n_{2k+1}$ and 2 (resp., 1) otherwise. If n_k increases sufficiently rapidly, then $\operatorname{br} T^{(1)} = \operatorname{br} T^{(2)} = 1$, so $\operatorname{br}(T^{(1)} \vee T^{(2)}) = 1$. But if n_k increases sufficiently rapidly, then $\underline{\operatorname{gr}}(T^{(1)} \vee T^{(2)}) = \sqrt{2}$.

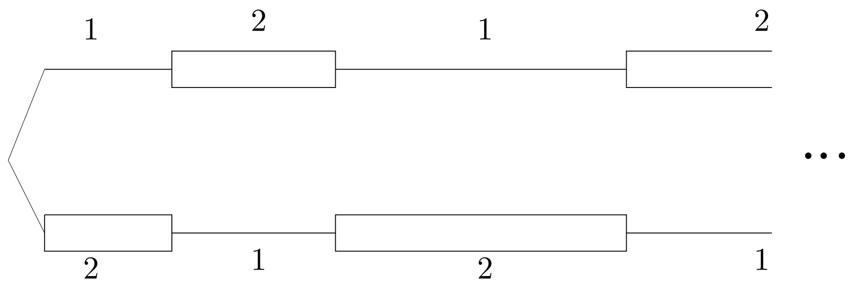


Figure 1.4. A schematic representation of a tree with branching number 1 and growth rate $\sqrt{2}$.

▷ **Exercise 1.3.**

Verify that if $\langle n_k \rangle$ increases sufficiently rapidly, then $\underline{\operatorname{gr}}(T^{(1)} \vee T^{(2)}) = \sqrt{2}$. Furthermore, show that the set of possible values of $\underline{\operatorname{gr}}(T^{(1)} \vee T^{(2)})$ over *all* sequences $\langle n_k \rangle$ is $[\sqrt{2}, 2]$.

While $\underline{\operatorname{gr}}\, T$ is easy to compute, $\operatorname{br} T$ may not be. Nevertheless, it is the branching number which is much more important. Fortunately, Furstenberg (1967) gave a useful condition sufficient for equality in (1.1): Given a vertex x in T, let T^x denote the subtree of T formed by the descendants of x. This tree is rooted at x.

Theorem 3.8. *If for all vertices $x \in T$, there is an isomorphism of T^x as a rooted tree to a subtree of T rooted at o, then $\operatorname{br} T = \operatorname{gr} T$.*

We call trees satisfying the hypothesis of this theorem ***subperiodic***; actually, we will later broaden slightly this definition. As we will see, subperiodic trees arise naturally, which accounts for the importance of Furstenberg's theorem.

§1.2. Electric Current.

We can ask another flow question on trees, namely: If λ^{-n} is the conductance of edges at distance n from the root of T and a battery is connected between the root and "infinity", will current flow? Of course, what we mean is that we establish a unit potential between the root and level N of T, let $N \to \infty$, and see whether the limiting current is positive. If so, the tree is said to have positive effective conductance and finite effective resistance. (All electrical terms will be carefully explained in Chapter 2.)

Example 1.5. Consider the binary tree. By symmetry, all the vertices at a given distance from o have the same potential, so they may be identified ("soldered" together) without changing any voltages or currents. This gives a new graph whose vertices may be identified with $\mathbb{N}$, while there are 2^n edges joining $n-1$ to n. These edges are in parallel, so they may be replaced by a single edge whose conductance is their sum, $(2/\lambda)^n$. Now we have edges in series, so the effective resistance is the sum of the edge resistances, $\sum_n (\lambda/2)^n$. This is finite iff $\lambda < 2$. Thus, current flows in the infinite binary tree iff $\lambda < 2$. Note the slight difference to water flow: when $\lambda = 2$, water can still flow on the binary tree.

In general, there will be a critical value of λ below which current flows and above which it does not. It turns out that this critical value is the same as that for water flow (Lyons, 1990):

Theorem 1.6. *If $\lambda < \operatorname{br} T$, then electrical current flows, but if $\lambda > \operatorname{br} T$, then it does not.*

§1.3. Random Walks.

There is a well-known and easily established correspondence between electrical networks and random walks that holds for all graphs. Namely, given a finite connected graph G with conductances assigned to the edges, we consider the random walk that can go from a vertex only to an adjacent vertex and whose transition probabilities from a vertex are proportional to the conductances along the edges to be taken. That is, if x is a vertex with neighbors $y_1, \ldots, y_d$ and the conductance of the edge (x, y_i) is c_i, then $p(x, y_j) = c_j / \sum_{i=1}^{d} c_i$. Now consider two fixed vertices a_0 and a_1 of G. Suppose that a battery is connected across them so that the voltage at a_i equals i $(i = 0, 1)$. Then certain currents will flow along the edges and establish certain voltages at the other vertices in accordance with Kirchhoff's law and Ohm's law. The following proposition provides the basic connection between random walks and electrical networks:

Proposition 1.7. (Voltage as Probability) *For any vertex x, the voltage at x equals the probability that the corresponding random walk visits a_1 before it visits a_0 when it starts at x.*

In fact, the proof of this proposition is simple: there is a discrete Laplacian (a difference operator) for which both the voltage and the probability mentioned are harmonic functions of x. The two functions clearly have the same values at a_i (the boundary points) and the uniqueness principle holds for this Laplacian, whence the functions agree at all vertices x. This is developed in detail in Section 2.1. A superb elementary exposition of this correspondence is given by Doyle and Snell (1984).

What does this say about our trees? Given N, identify all the vertices of level N, i.e., T_N, to one vertex, a_1 (see Figure 1.5). Use the root as a_0. Then according to Proposition 1.7, the voltage at x is the probability that the random walk visits level N before it visits the root when it starts from x. When $N \to \infty$, the limiting voltages are all 0 iff the limiting probabilities are all 0, which is the same thing as saying that on the infinite tree, the probability of visiting the root from any vertex is 1, i.e., the random walk is recurrent. Since no current flows across edges whose endpoints have the same voltage, we see that no electrical current flows iff the random walk is recurrent.

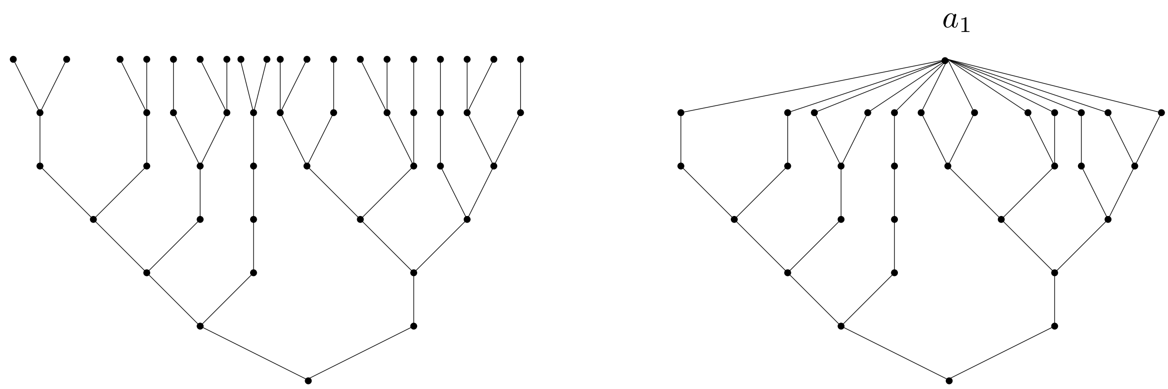


Figure 1.5. Identifying a level to a vertex, a_1.

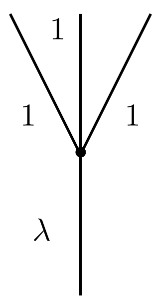


Figure 1.6. The relative weights at a vertex. The tree is growing upwards.

Now when the conductances decrease by a factor of λ as the distance increases, the relative weights at a vertex other than the root are as shown in Figure 1.6. That is, the edge leading back toward the root is λ times as likely to be taken as each other edge. Denoting the dependence of the random walk on the parameter λ by RW_λ, we may translate Theorem 1.6 into the following theorem (Lyons, 1990):

Theorem 3.5. *If $\lambda < \operatorname{br} T$, then RW_λ is transient, while if $\lambda > \operatorname{br} T$, then RW_λ is recurrent.*

Is this intuitive? Consider a vertex other than the root with, say, d children. If we consider only the distance from o, which increases or decreases at each step of the random walk, a balance between increasing and decreasing occurs when $\lambda = d$. If d were constant, it is easy to see that indeed d would be the critical value separating transience from recurrence. What Theorem 3.5 says is that this same heuristic can be used in the general case, provided we substitute the "average" $\operatorname{br} T$ for d.

We will also see how to use electrical networks to prove Pólya's wonderful theorem that simple random walk on the hypercubic lattice $\mathbb{Z}^d$ is recurrent for $d \le 2$ and transient for $d \ge 3$.

§1.4. Percolation.

Suppose that we remove edges at random from T. To be specific, keep each edge with some fixed probability p and make these decisions independently for different edges. This random process is called ***percolation***. By Kolmogorov's 0-1 law, the probability that an infinite connected component remains in the tree is either 0 or 1. On the other hand, this probability is monotonic in p, whence there is a critical value $p_{\mathrm{c}}(T)$ where it changes from 0 to 1. It is also clear that the "bigger" the tree, the more likely it is that there will be an infinite component for a given p. That is, the "bigger" the tree, the smaller the critical value p_{c}. Thus, p_{c} is vaguely inversely related to a notion of average branching number. Actually, this vague heuristic is precise (Lyons, 1990):

Theorem 5.15. *For any tree, $p_{\mathrm{c}}(T) = 1/\operatorname{br} T$.*

Let us look more closely at the intuition behind this. If a vertex x has d children, then the expected number of children after percolation is dp. If dp is "usually" less than 1, one would not expect that an infinite component would remain, while if dp is "usually" greater than 1, then one might guess that an infinite component would be present. Theorem 5.15 says that this intuition becomes correct when one replaces the "usual" d by $\operatorname{br} T$.

Note that there is an infinite component with probability 1 iff the component of the root is infinite with positive probability.

§1.5. Branching Processes.

Percolation on a fixed tree produces random trees by random pruning, but there is a way to grow trees randomly due to Bienaymé in 1845. Given probabilities p_k adding to 1 ($k = 0, 1, 2, \ldots$), we begin with one individual and let it reproduce according to these probabilities, i.e., it has k children with probability p_k. Each of these children (if there are any) then reproduce independently with the same law, and so on forever or until some generation goes extinct. The family trees produced by such a process are called ***(Bienaymé)-Galton-Watson trees***. A fundamental theorem in the subject is that extinction is a.s. iff $m \le 1$ and $p_1 < 1$, where $m := \sum_k kp_k$ is the mean number of offspring. This provides further justification for the intuition we sketched behind Theorem 5.15. It also raises a natural question: Given that a Galton-Watson tree is nonextinct (infinite), what is its branching number? All the intuition suggests that it is m a.s., and indeed it is. This was first proved by Hawkes (1981). But here is the idea of a very simple proof (Lyons, 1990).

According to Theorem 5.15, to determine $\operatorname{br} T$, we may determine $p_c(T)$. Thus, let T grow according to a Galton-Watson process, then perform percolation on T, i.e., keep edges with probability p. We are interested in the component of the root. Looked at as a random tree in itself, this component appears simply as some other Galton-Watson tree; its mean is mp by independence of the growing and the "pruning" (percolation). Hence, the component of the root is infinite w.p.p. iff $mp > 1$. This says that $p_c = 1/m$ a.s. on nonextinction, i.e., $\operatorname{br} T = m$.

Let Z_n be the size of the nth generation in a Galton-Watson process. How quickly does Z_n grow? It will be easy to calculate that $\mathbf{E}[Z_n] = m^n$. Moreover, a martingale argument will show that the limit $W := \lim_{n\to\infty} Z_n/m^n$ always exists (and is finite). When $1 < m < \infty$, do we have that $W > 0$ on the event of nonextinction? The answer is "yes", under a very mild hypothesis:

The Kesten-Stigum Theorem (1966). *The following are equivalent when $1 < m < \infty$:*

(i) $W > 0$ a.s. on the event of nonextinction;

(ii) $\sum_{k=1}^{\infty} p_k k \log k < \infty$.

This will be shown in Section 12.2. Although condition (ii) appears technical, we will enjoy a conceptual proof of the theorem that uses only the crudest estimates.

§1.6. Random Spanning Trees.

This fertile and fascinating field is one of the oldest areas to be studied in this book, but one of the newest to be explored in depth. A ***spanning tree*** of a (connected) graph is a subgraph that is connected, contains every vertex of the whole graph, and contains no cycle: see Figure 1.7 for an example. These trees are usually not rooted. The subject of random spanning trees of a graph goes back to Kirchhoff (1847), who showed its relation to electrical networks. (Actually, Kirchhoff did not think probabilistically, but, rather, he considered quotients of the number of spanning trees with a certain property divided by the total number of spanning trees.) One of these relations gives the probability that a uniformly chosen spanning tree will contain a given edge in terms of electrical current in the graph.

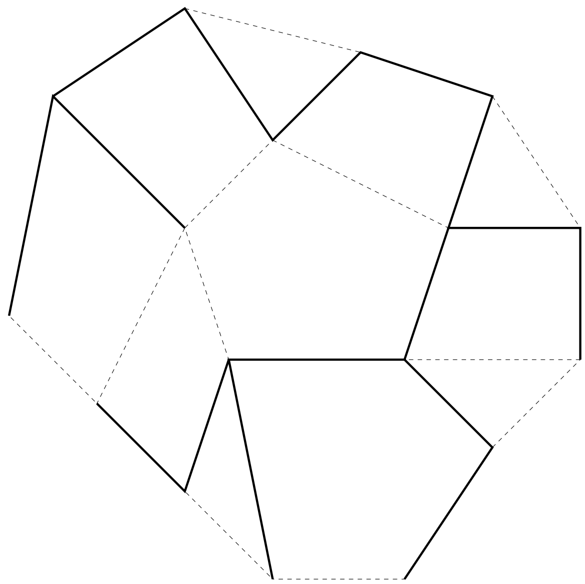


Figure 1.7. A spanning tree in a graph, where the edges of the graph not in the tree are dashed.

Let's begin with a very simple finite graph. Namely, consider the ladder graph of Figure 1.8. Among all spanning trees of this graph, what proportion contain the bottom rung (edge)? In other words, if we were to choose at random a spanning tree, what is the chance that it would contain the bottom rung? We have illustrated the entire probability spaces for the smallest ladder graphs in Figure 1.9.

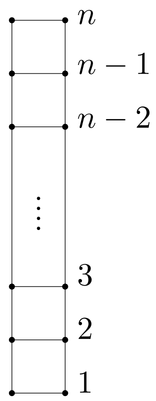


Figure 1.8. A ladder graph.

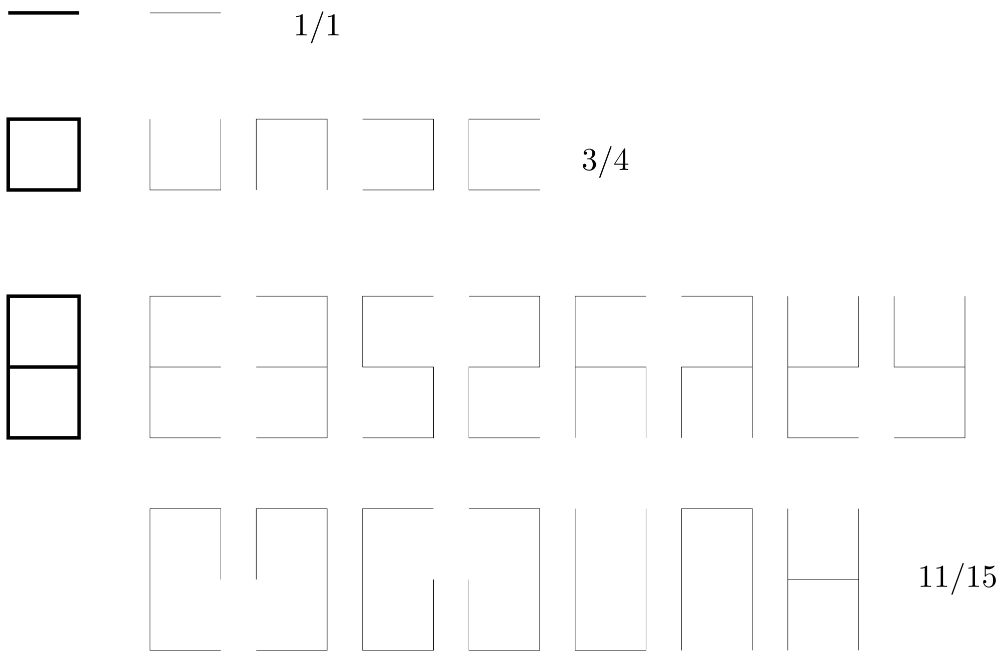


Figure 1.9. The ladder graphs of heights 0, 1, and 2, together with their spanning trees.

As shown, the probabilities in these cases are 1/1, 3/4, and 11/15. The next one is 41/56. Do you see any pattern? One thing that is fairly evident is that these numbers are decreasing, but hardly changing. It turns out that they come from every other term of the continued fraction expansion of $\sqrt{3}-1 = 0.73^{+}$ and, in particular, converge to $\sqrt{3}-1$. In the limit, then, the probability of using the bottom rung is $\sqrt{3}-1$ and, even before taking the limit, this gives an excellent approximation to the probability. How can we easily calculate such numbers? In this case, there is a rather easy recursion to set up and solve, but we will use this example to illustrate the more general theorem of Kirchhoff that we mentioned above. In fact, Kirchhoff's theorem will show us why these probabilities are decreasing even before we calculate them.

Suppose that each edge of our graph is an electric conductor of unit conductance. Hook up a battery between the endpoints of any edge e, say the bottom rung (Figure 1.10). Kirchhoff (1847) showed that the proportion of current that flows directly along e is then equal to the probability that e belongs to a randomly chosen spanning tree!

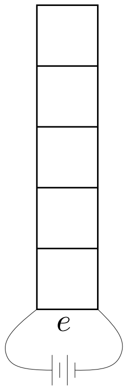


Figure 1.10. A battery is hooked up between the endpoints of e.

Coming back to the ladder graph and its bottom rung, e, we see that current flows in two ways: some flows directly across e and some flows through the rest of the network. It is intuitively clear (and justified by Rayleigh's monotonicity principle) that the higher the ladder, the greater the effective conductance of the ladder minus the bottom rung, hence, by Kirchhoff's theorem, the less the chance that a random spanning tree contains the bottom rung. This confirms our observations.

It turns out that generating spanning trees at random according to the uniform measure is of interest to computer scientists, who have developed various algorithms over the years for random generation of spanning trees. In particular, this is closely connected to generating a random state from any Markov chain. See Propp and Wilson (1998) for more on this issue.

Early algorithms for generating a random spanning tree used the Matrix-Tree Theorem, which counts the number of spanning trees in a graph via a determinant. A better algorithm than these early ones, especially for probabilists, was introduced by Aldous (1990) and Broder (1989). It says that if you start a simple random walk at *any* vertex of a finite (connected) graph G and draw every edge it traverses except when it would complete a cycle (i.e., except when it arrives at a previously-visited vertex), then when no more edges can be added without creating a cycle, what will be drawn is a uniformly chosen spanning tree of G. (To be precise: if X_n $(n \ge 0)$ is the path of the random walk, then the associated spanning tree is the set of edges $\big\{[X_n, X_{n+1}]\,;\; X_{n+1} \notin \{X_0, X_1, \ldots, X_n\}\big\}$.) This beautiful algorithm is quite efficient and useful for theoretical analysis, yet Wilson

(1996) found an even better one that we'll describe in Section 4.1.

Return for a moment to the ladder graphs. We saw that as the height of the ladder tends to infinity, there is a limiting probability that the bottom rung of the ladder graph belongs to a uniform spanning tree. This suggests looking at uniform spanning trees on general infinite graphs. So suppose that G is an infinite graph. Let G_n be finite subgraphs with $G_1 \subset G_2 \subset G_3 \subset \cdots$ and $\bigcup G_n = G$. Pemantle (1991) showed that the weak limit of the uniform spanning tree measures on G_n exists, as conjectured by Lyons. (In other words, if μ_n denotes the uniform spanning tree measure on G_n and B, B' are finite sets of edges, then $\lim_n \mu_n(B \subset T, B' \cap T = \varnothing)$ exists, where T denotes a random spanning tree.) This limit is now called the ***free uniform spanning forest**** on G, denoted FUSF. Considerations of electrical networks play the dominant role in Pemantle's proof. Pemantle (1991) discovered the amazing fact that on $\mathbb{Z}^d$, the uniform spanning forest is a single tree a.s. if $d \le 4$; but when $d \ge 5$, there are infinitely many trees a.s.!

§1.7. Hausdorff Dimension.

Consider again any finite connected graph with two distinguished vertices a and z. This time, the edges e have assigned positive numbers $c(e)$ that represent the maximum amount of water that can flow through the edge (in either direction). How much water can flow into a and out of z? Consider any set Π of edges that separates a from z, i.e., the removal of all edges in Π would leave a and z in different components. Such a set Π is called a ***cutset***. Since all the water must flow through Π, an upper bound for the maximum flow from a to z is $\sum_{e \in \Pi} c(e)$. The beautiful Max-Flow Min-Cut Theorem of Ford and Fulkerson (1962) says that these are the only constraints: the maximum flow equals $\min_{\Pi \text{ a cutset}} \sum_{e \in \Pi} c(e)$.

Applying this theorem to our tree situation where the amount of water that can flow through an edge at distance n from the root is limited to λ^{-n}, we see that the maximum flow from the root to infinity is

$$\inf \left\{ \sum_{x \in \Pi} \lambda^{-|x|} \,;\; \Pi \text{ cuts the root from infinity} \right\} .$$

Here, we identify a set of vertices Π with their preceding edges when considering cutsets.

By analogy with the leaves of a finite tree, we call the set of rays of T the ***boundary*** (at infinity) of T, denoted ∂T. (It does not include any leaves of T.) Now there is a natural

* In graph theory, "spanning forest" usually means a maximal subgraph without cycles, i.e., a spanning tree in each connected component. We mean, instead, a subgraph without cycles that contains every vertex.

metric on ∂T: if $\xi, \eta \in \partial T$ have exactly n edges in common, define their distance to be $d(\xi, \eta) := e^{-n}$. Thus, if $x \in T$ has more than one child with infinitely many descendants, the set of rays going through x,

$$B_x := \{\xi \in \partial T\,;\ \xi_{|x|} = x\}\,,$$

has diameter $\operatorname{diam} B_x = e^{-|x|}$. We call a collection $\mathscr{C}$ of subsets of ∂T a ***cover*** if

$$\bigcup_{B \in \mathscr{C}} B = \partial T\,.$$

Note that Π is a cutset iff $\{B_x\,;\ x \in \Pi\}$ is a cover. The ***Hausdorff dimension*** of ∂T is defined to be

$$\dim \partial T := \sup\left\{\alpha\,;\ \inf_{\mathscr{C} \text{ a countable cover}} \sum_{B \in \mathscr{C}} (\operatorname{diam} B)^\alpha > 0\right\}.$$

This is, in fact, already familiar to us, since

$$\begin{aligned}
\operatorname{br} T &= \sup\{\lambda\,;\ \text{water can flow through pipe capacities } \lambda^{-|x|}\} \\
&= \sup\left\{\lambda\,;\ \inf_{\Pi \text{ a cutset}} \sum_{x \in \Pi} \lambda^{-|x|} > 0\right\} \\
&= \exp\sup\left\{\alpha\,;\ \inf_{\Pi \text{ a cutset}} \sum_{x \in \Pi} e^{-\alpha|x|} > 0\right\} \\
&= \exp\sup\left\{\alpha\,;\ \inf_{\mathscr{C} \text{ a cover}} \sum_{B \in \mathscr{C}} (\operatorname{diam} B)^\alpha > 0\right\} \\
&= \exp\dim \partial T\,.
\end{aligned}$$

§1.8. Capacity.

An important tool in analyzing electrical networks is that of energy. Thomson's principle says that, given a finite graph and two distinguished vertices a, z, the unit current flow from a to z is the unit flow from a to z that minimizes energy, where, if the conductances are $c(e)$, the ***energy*** of a flow θ is defined to be $\sum_{e \text{ an edge}} \theta(e)^2/c(e)$. (It turns out that the energy of the unit current flow is equal to the effective resistance from a to z.) Thus,

$$\begin{gathered}\text{electrical current flows from the root of an infinite tree}\\ \iff \\ \text{there is a flow with finite energy.}\end{gathered} \tag{1.2}$$

Now on a tree, a unit flow can be identified with a function θ on the vertices of T that is 1 at the root and has the property that for all vertices x,

$$\theta(x) = \sum_i \theta(y_i)\,,$$

where y_i are the children of x. The energy of a flow is then

$$\sum_{x \in T} \theta(x)^2 \lambda^{|x|}\,,$$

whence

$$\operatorname{br} T = \sup \left\{ \lambda \,;\ \text{there exists a unit flow } \theta \quad \sum_{x \in T} \theta(x)^2 \lambda^{|x|} < \infty \right\}. \tag{1.3}$$

We can also identify unit flows on T with Borel probability measures μ on ∂T via

$$\mu(B_x) = \theta(x)\,.$$

A bit of algebra shows that (1.3) is equivalent to

$$\operatorname{br} T = \exp \sup \left\{ \alpha \,;\ \exists \text{ a probability measure } \mu \text{ on } \partial T \quad \iint \frac{d\mu(\xi)\, d\mu(\eta)}{d(\xi, \eta)^\alpha} < \infty \right\}.$$

For $\alpha > 0$, define the ***α-capacity*** to be the reciprocal of the minimum e^α-energy:

$$\operatorname{cap}_\alpha(\partial T)^{-1} := \inf \left\{ \iint \frac{d\mu(\xi)\, d\mu(\eta)}{d(\xi, \eta)^\alpha} \,;\ \mu \text{ a probability measure on } \partial T \right\}.$$

Then statement (1.2) says that for $\alpha > 0$,

$$\text{random walk with parameter } \lambda = e^\alpha \text{ is transient} \iff \operatorname{cap}_\alpha(\partial T) > 0\,. \tag{1.4}$$

It follows from Theorem 3.5 that

$$\text{the critical value of } \alpha \text{ for positivity of } \operatorname{cap}_\alpha(\partial T) \text{ is } \dim \partial T. \tag{1.5}$$

A refinement of Theorem 5.15 is (Lyons, 1992):

Theorem 15.3. **(Tree Percolation and Capacity)** *For $\alpha > 0$, percolation with parameter $p = e^{-\alpha}$ yields an infinite component a.s. iff* $\mathrm{cap}_\alpha(\partial T) > 0$. *Moreover,*

$$\mathrm{cap}_\alpha(\partial T) \le \mathbf{P}[\textit{the component of the root is infinite}] \le 2\,\mathrm{cap}_\alpha(\partial T)\,.$$

When T is spherically symmetric and $p = e^{-\alpha}$, we have (Exercise 15.1)

$$\mathrm{cap}_\alpha(\partial T) = \left(1 + (1-p)\sum_{n=1}^{\infty}\frac{1}{p^n|T_n|}\right)^{-1}.$$

The case of the first part of this theorem where all the degrees are uniformly bounded was shown first by Fan (1989, 1990).

One way to use this theorem is to combine it with (1.4); this allows us to translate problems freely between the domains of random walks and percolation (Lyons, 1992). The theorem actually holds in a much wider context (on trees) than that mentioned here: Fan allowed the probabilities p to vary depending on the generation and Lyons allowed the probabilities as well as the tree to be completely arbitrary.

§1.9. Embedding Trees into Euclidean Space.

The results described above, especially those concerning percolation, can be translated to give results on closed sets in Euclidean space. We will describe only the simplest such correspondence here, which was the one that was part of Furstenberg's motivation in 1970. Namely, for a closed nonempty set $E \subseteq [0,1]$ and for any integer $b \ge 2$, consider the system of b-adic subintervals of $[0,1]$. Those whose intersection with E is non-empty will form the vertices of the associated tree. Two such intervals are connected by an edge iff one contains the other and the ratio of their lengths is b. The root of this tree is $[0,1]$. We denote the tree by $T_{[b]}(E)$. Were it not for the fact that certain numbers have two representations in base b, we could identify $\partial T_{[b]}(E)$ with E. Note that such an identification would be Hölder continuous in one direction (only).

Hausdorff dimension is defined for subsets of $[0,1]$ just as it was for ∂T:

$$\dim E := \sup\left\{\alpha\,;\ \inf_{\mathscr{C}\text{ a cover of }E}\ \sum_{B\in\mathscr{C}}(\mathrm{diam}\,B)^\alpha > 0\right\},$$

where $\mathrm{diam}\,B$ denotes the (Euclidean) diameter of E. Covers of $\partial T_{[b]}(E)$ by sets of the form B_x correspond to covers of E by b-adic intervals, but of diameter $b^{-|x|}$, rather than $e^{-|x|}$. It easy to show that restricting to such covers does not change the computation

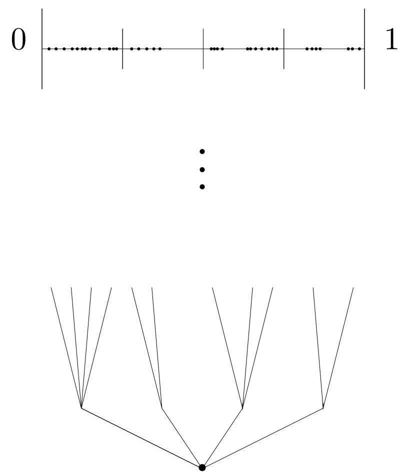


Figure 1.11. In this case $b = 4$.

of Hausdorff dimension, whence we may conclude (compare the calculation at the end of Section 1.7) that

$$\dim E = \frac{\dim \partial T_{[b]}(E)}{\log b} = \log_b(\operatorname{br} T_{[b]}(E)) .$$

For example, the Hausdorff dimension of the Cantor middle-thirds set is $\log 2/\log 3$, which we see by using the base $b = 3$. It is interesting that if we use a different base, we will still have $\operatorname{br} T_{[b]}(E) = 2$ when E is the Cantor set.

Capacity is also defined as it was on the boundary of a tree:

$$(\operatorname{cap}_\alpha E)^{-1} := \inf \left\{ \int\!\!\int \frac{d\mu(x)\, d\mu(y)}{|x-y|^\alpha} \,;\ \mu \text{ a probability measure on } E \right\} .$$

It was shown by Benjamini and Peres (1992) (see Section 15.3) that

$$(\operatorname{cap}_\alpha E)/3 \le \operatorname{cap}_{\alpha \log b}(\partial T_{[b]}(E)) \le b \operatorname{cap}_\alpha E . \tag{1.6}$$

This means that the percolation criterion Theorem 15.3 can be used in Euclidean space. In fact, inequalities such as (1.6) also hold for more general kernels than distance to a power and for higher-dimensional Euclidean spaces. Nice applications to the trace of Brownian motion were found by Peres (1996) by replacing the path by an "intersection-equivalent" random fractal that is much easier to analyze, being an embedding of a Galton-Watson tree. The point is that probability of an intersection of Brownian motion with another set can be estimated by a capacity in a fashion very similar to (1.6). This will allow us to prove some surprising things about Brownian motion in a very easy fashion.

Also, the fact (1.5) translates into the analogous statement about subsets $E \subseteq [0, 1]$; this is a classical theorem of Frostman (1935).

§1.10. Notes.

Other recent books that cover material related to the topics of this book include *Probability on Graphs* by Geoffrey Grimmett (forthcoming), *Reversible Markov Chains and Random Walks on Graphs* by David Aldous and Jim Fill (forthcoming), *Markov Chains and Mixing Times* by David A. Levin, Yuval Peres, and Elizabeth L. Wilmer, *Random Trees: An Interplay Between Combinatorics and Probability* by Michael Drmota, *A Course on the Web Graph* by Anthony Bonato, *Random Graph Dynamics* by Rick Durrett, *Complex Graphs and Networks* by Fan Chung and Linyuan Lu, *The Random-Cluster Model* by Geoffrey Grimmett, *Superfractals* by Michael Fielding Barnsley, *Introduction to Mathematical Methods in Bioinformatics* by Alexander Isaev, *Gaussian Markov Random Fields* by Håvard Rue and Leonhard Held, *Conformally Invariant Processes in the Plane* by Gregory F. Lawler, *Random Networks in Communication* by Massimo Franceschetti and Ronald Meester, *Percolation* by Bela Bollobás and Oliver Riordan, *Probability and Real Trees* by Steven Evans, *Random Trees, Lévy Processes and Spatial Branching Processes* by Thomas Duquesne and Jean-François Le Gall, *Combinatorial Stochastic Processes* by Jim Pitman, *Random Geometric Graphs* by Mathew Penrose, *Random Graphs* by Béla Bollobás, *Random Graphs* by Svante Janson, Tomasz Łuczak, and Andrzej Ruciński, *Phylogenetics* by Charles Semple and Mike Steel, *Stochastic Networks and Queues* by Philippe Robert, *Random Walks on Infinite Graphs and Groups* by Wolfgang Woess, *Percolation* by Geoffrey Grimmett, *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes* by Thomas M. Liggett, and *Discrete Groups, Expanding Graphs and Invariant Measures* by Alexander Lubotzky.

§1.11. Collected In-Text Exercises.

1.1. Prove (1.1).

1.2. Show that $\operatorname{br} T = \underline{\operatorname{gr}}\, T$ when T is spherically symmetric.

1.3. Verify that if $\langle n_k \rangle$ increases sufficiently rapidly, then $\underline{\operatorname{gr}}(T^{(1)} \vee T^{(2)}) = \sqrt{2}$. Furthermore, show that the set of possible values of $\underline{\operatorname{gr}}(T^{(1)} \vee T^{(2)})$ over *all* sequences $\langle n_k \rangle$ is $[\sqrt{2}, 2]$.

Chapter 2

Random Walks and Electric Networks

The two topics of the title of this chapter do not sound related to each other, but, in fact, they are intimately connected in several useful ways. This is a discrete version of more general profound and detailed connections between potential theory and probability; see, e.g., Chapter II of Bass (1995) or Doob (1984). A superb elementary introduction to the ideas of the first five sections of this chapter is given by Doyle and Snell (1984).

§2.1. Circuit Basics and Harmonic Functions.

Our principal interest in this chapter centers around transience and recurrence of irreducible Markov chains. If the chain starts at a state x, then we want to know whether the chance that it ever visits a state a is 1 or not.

In fact, we are interested only in reversible Markov chains, where we call a Markov chain ***reversible*** if there is a positive function $x \mapsto \pi(x)$ on the state space such that the transition probabilities satisfy $\pi(x)p_{xy} = \pi(y)p_{yx}$ for all pairs of states x, y. (Such a function $\pi(\bullet)$ will then provide a stationary measure: see Exercise 2.1. *Note that $\pi(\bullet)$ is not generally a probability measure.*) In this case, make a graph G by taking the states of the Markov chain for vertices and joining two vertices x, y by an edge when $p_{xy} > 0$. Assign weight

$$c(x, y) := \pi(x)p_{xy} \tag{2.1}$$

to that edge; note that the condition of reversibility ensures that this weight is the same no matter in what order we take the endpoints of the edge. With this network in hand, the Markov chain may be described as a random walk on G: when the walk is at a vertex x, it chooses randomly among the vertices adjacent to x with transition probabilities proportional to the weights of the edges. Conversely, every connected graph with weights on the edges such that the sum of the weights incident to every vertex is finite gives rise to a random walk with transition probabilities proportional to the weights. Such a random walk is an irreducible reversible Markov chain: define $\pi(x)$ to be the sum of the weights incident to x.*

The most well-known example is gambler's ruin. A gambler needs \$$n$ but has only \$$k$ $(1 \le k \le n-1)$. He plays games that give him chance p of winning \$1 and $q := 1-p$ of losing \$1 each time. When his fortune is either \$$n$ or 0, he stops. What is his chance of ruin (i.e., reaching 0 before n)? We will answer this in Example 2.4 by using the following weighted graph. The vertices are $\{0, 1, 2, \ldots, n\}$, the edges are between consecutive integers, and the weights are $c(i, i+1) = c(i+1, i) = (p/q)^i$.

▷ **Exercise 2.1.**

(Reversible Markov Chains) This exercise contains some background information and facts that we will use about reversible Markov chains.

(a) Show that if a Markov chain is reversible, then $\forall x_1, x_2, \ldots, x_n$,

$$\pi(x_1) \prod_{i=1}^{n-1} p_{x_i x_{i+1}} = \pi(x_n) \prod_{i=1}^{n-1} p_{x_{n+1-i} x_{n-i}} ,$$

whence $\prod_{i=1}^{n-1} p_{x_i x_{i+1}} = \prod_{i=1}^{n-1} p_{x_{n+1-i} x_{n-i}}$ if $x_1 = x_n$. This last equation also characterizes reversibility.

(b) Let $\langle X_n \rangle$ be a random walk on G and let x and y be two vertices in G. Let $\mathcal{P}$ be a path from x to y and $\mathcal{P}'$ its reversal, a path from y to x. Show that

$$\mathbf{P}_x\big[\langle X_n \,;\, n \le \tau_y \rangle = \mathcal{P} \bigm| \tau_y < \tau_x^+\big] = \mathbf{P}_y\big[\langle X_n \,;\, n \le \tau_x \rangle = \mathcal{P}' \bigm| \tau_x < \tau_y^+\big] ,$$

where τ_w denotes the first time the random walk visits w, τ_w^+ denotes the first time after 0 that the random walk visits w, and $\mathbf{P}_u$ denotes the law of random walk started at u. In words, paths between two states that don't return to the starting point and stop at the first visit to the endpoint have the same distribution in both directions of time.

(c) Consider a random walk on G that is either transient or is stopped on the first visit to a set of vertices Z. Let $\mathcal{G}(x, y)$ be the expected number of visits to y for a random walk started at x; if the walk is stopped at Z, we count only those visits that occur strictly before visiting Z. Show that for every pair of vertices x and y,

$$\pi(x)\mathcal{G}(x, y) = \pi(y)\mathcal{G}(y, x) .$$

---

* Suppose that we consider an edge e of G to have length $c(e)^{-1}$. Run a Brownian motion on G and observe it only when it reaches a vertex different from the previous one. Then we see the random walk on G just described. There are various equivalent ways to define rigorously Brownian motion on G. The essential point is that for Brownian motion on $\mathbb{R}$ started at 0 and for $a, b > 0$, if X is the first of $\{-a, b\}$ visited, then $\mathbf{P}[X = -a] = a/(a+b)$.

(d) Show that random walk on a connected weighted graph G is positive recurrent (i.e., has a stationary probability distribution) iff $\sum_{x,y} c(x,y) < \infty$, in which case the stationary probability distribution is proportional to $\pi(\bullet)$. Show that if the random walk is not positive recurrent, then $\pi(\bullet)$ is a stationary infinite measure.

We begin by studying random walks on finite networks. Let G be a finite connected network, x a vertex of G, and A, Z disjoint subsets of vertices of G. Let τ_A be the first time that the random walk visits ("hits") some vertex in A; if the random walk happens to start in A, then this is 0. Occasionally, we will use τ_A^+, which is the first time after 0 that the walk visits A; this is different from τ_A only when the walk starts in A. Usually A and Z will be singletons. Often, all the edge weights are equal; we call this case ***simple random walk***.

Consider the probability that the random walk visits A before it visits Z as a function of its starting point x:

$$F(x) := \mathbf{P}_x[\tau_A < \tau_Z] \,. \tag{2.2}$$

Recall that $\restriction$ indicates the restriction of a function to a set. Clearly $F{\restriction}A \equiv 1$, $F{\restriction}Z \equiv 0$, and for $x \notin A \cup Z$,

$$\begin{aligned} F(x) &= \sum_y \mathbf{P}_x[\text{first step is to } y]\mathbf{P}_x[\tau_A < \tau_Z \mid \text{first step is to } y] \\ &= \sum_{x \sim y} p_{xy} F(y) = \frac{1}{\pi(x)} \sum_{x \sim y} c(x,y) F(y) \,, \end{aligned}$$

where $x \sim y$ indicates that x, y are adjacent in G. In the special case of simple random walk, this equation becomes

$$F(x) = \frac{1}{\deg x} \sum_{x \sim y} F(y) \,,$$

where $\deg x$ is the degree of x, i.e., the number of edges incident to x. That is, $F(x)$ is the average of the values of F at the neighbors of x. In general, this is still true, but the average is taken with weights.

We say that a function f is ***harmonic*** at x when

$$f(x) = \frac{1}{\pi(x)} \sum_{x \sim y} c(x,y) f(y) \,.$$

If f is harmonic at each point of a set W, then we say that f is harmonic on W. Harmonic functions satisfy a maximum principle. To state it, we use the following notation: For $W \subseteq \mathsf{V}(G)$, write $\overline{W}$ for the set of vertices that are either in W or are adjacent to some vertex in W.

Maximum Principle. *Let G be a finite or infinite network. If $H \subseteq G$, H is connected and finite, $f : \mathsf{V}(G) \to \mathbb{R}$, f is harmonic on $\mathsf{V}(H)$, and $\max f{\restriction}\mathsf{V}(H) = \sup f$, then f is constant on $\overline{\mathsf{V}(H)}$.*

Proof. Let $K := \{y \in \mathsf{V}(G)\,;\; f(y) = \sup f\}$. Note that if $x \in \mathsf{V}(H) \cap K$, then $\overline{\{x\}} \subseteq K$ by harmonicity of f at x. Since H is connected and $\mathsf{V}(H) \cap K \neq \varnothing$, it follows that $K \supseteq \mathsf{V}(H)$ and therefore that $K \supseteq \overline{\mathsf{V}(H)}$. ◀

This leads to the

Uniqueness Principle. *Let $G = (\mathsf{V}, \mathsf{E})$ be a finite or infinite connected network. Let W be a finite proper subset of V. If $f, g : \mathsf{V} \to \mathbb{R}$, f, g are harmonic on W, and $f{\restriction}(\mathsf{V} \setminus W) = g{\restriction}(\mathsf{V} \setminus W)$, then $f = g$.*

Proof. Let $h := f - g$. We claim that $h \le 0$. This suffices to establish the corollary since then $h \ge 0$ by symmetry, whence $h \equiv 0$.

Now $h = 0$ off W, so if $h \not\le 0$ on W, then h is positive somewhere on W, whence $\max h{\restriction}W = \max h$. Let H be a connected component of $\big(W, \mathsf{E} \cap (W \times W)\big)$ where h achieves its maximum. According to the maximum principle, h is a positive constant on $\overline{\mathsf{V}(H)}$. In particular, $h > 0$ on the non-empty set $\overline{\mathsf{V}(H)} \setminus \mathsf{V}(H)$. However, $\overline{\mathsf{V}(H)} \setminus \mathsf{V}(H) \subseteq \mathsf{V} \setminus W$, whence $h = 0$ on $\overline{\mathsf{V}(H)} \setminus \mathsf{V}(H)$. This is a contradiction. ◀

Thus, the harmonicity of the function $x \mapsto \mathbf{P}_x[\tau_A < \tau_Z]$ (together with its values where it is not harmonic) characterizes it.

If f, f_1, and f_2 are harmonic on W and $a_1, a_2 \in \mathbb{R}$ with $f = a_1 f_1 + a_2 f_2$ on $\mathsf{V} \setminus W$, then $f = a_1 f_1 + a_2 f_2$ everywhere by the uniqueness principle. This is one form of the ***superposition principle***.

Given a function defined on a subset of vertices of a network, the Dirichlet problem asks whether the given function can be extended to all vertices of the network so as to be harmonic wherever it was not originally defined. The answer is often yes:

Existence Principle. *Let $G = (\mathsf{V}, \mathsf{E})$ be a finite or infinite network. If $W \subsetneq \mathsf{V}$ and $f_0 : \mathsf{V} \setminus W \to \mathbb{R}$ is bounded, then $\exists f : \mathsf{V} \to \mathbb{R}$ such that $f{\restriction}(\mathsf{V} \setminus W) = f_0$ and f is harmonic on W.*

Proof. For any starting point x of the network random walk, let X be the first vertex in $\mathsf{V} \setminus W$ visited by the random walk if $\mathsf{V} \setminus W$ is indeed visited. Let $Y := f_0(X)$ when $\mathsf{V} \setminus W$ is visited and $Y := 0$ otherwise. It is easily checked that $f(x) := \mathbf{E}_x[Y]$ works, by using the same method as we used to see that the function F of (2.2) is harmonic. ◀

The function F of (2.2) is the particular case of the Existence Principle where $W = \mathsf{V} \setminus (A \cup Z)$, $f_0 \restriction A \equiv 1$, and $f_0 \restriction Z \equiv 0$.

In fact, for finite networks, we could have immediately deduced existence from uniqueness: The Dirichlet problem on a finite network consists of a finite number of linear equations, one for each vertex in W. Since the number of unknowns is equal to the number of equations, the uniqueness principle implies the existence principle. An example is shown in Figure 2.1, where the function was specified to be 1 at two vertices, 0.5 at another, and 0 at a fourth; the function is harmonic elsewhere.

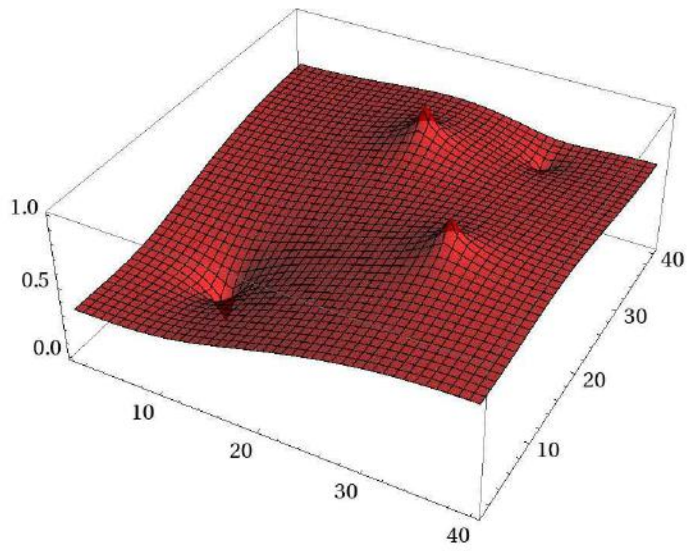


Figure 2.1. A harmonic function on a 40×40 square grid with 4 specified values where it is not harmonic.

In order to study the solution to the Dirichlet problem, especially for a sequence of subgraphs of an infinite graph, we will discover that electrical networks are useful. Electrical networks, of course, have a physical meaning whose intuition is useful to us, but also they can be used as a rigorous mathematical tool.

Mathematically, an electrical network is just a weighted graph. But now we call the weights of the edges ***conductances***; their reciprocals are called ***resistances***. (Note that later, we will encounter effective conductances and resistances; these are not the same.) We hook up a battery or batteries (this is just intuition) between A and Z so that the ***voltage*** at every vertex in A is 1 and in Z is 0 (more generally, so that the voltages on $\mathsf{V} \setminus W$ are given by f_0). (Sometimes, voltages are called ***potentials*** or potential differences.) ***Voltages*** v are then established at every vertex and ***current*** i runs through the edges.

These functions are implicitly defined and uniquely determined on finite networks, as we will see, by two "laws":

Ohm's Law: If $x \sim y$, the current flowing from x to y satisfies

$$v(x) - v(y) = i(x, y)r(x, y) .$$

Kirchhoff's Node Law: The sum of all currents flowing out of a given vertex is 0, provided the vertex is not connected to a battery.

Physically, Ohm's law, which is usually stated as $v = ir$ in engineering, is an empirical statement about linear response to voltage differences—certain components obey this law over a wide range of voltage differences. Notice also that current flows in the direction of decreasing voltage: $i(x, y) > 0$ iff $v(x) > v(y)$. Kirchhoff's node law expresses the fact that charge does not build up at a node (current being the passage rate of charge per unit time). If we add wires corresponding to the batteries, then the proviso in Kirchhoff's node law is unnecessary.

Mathematically, we'll take Ohm's law to be the definition of current in terms of voltage. In particular, $i(x, y) = -i(y, x)$. Then Kirchhoff's node law presents a constraint on what kind of function v can be. Indeed, it determines v uniquely: Current flows into G at A and out at Z. Thus, we may combine the two laws on $\mathsf{V} \setminus (A \cup Z)$ to obtain

$$\forall x \notin A \cup Z \qquad 0 = \sum_{x \sim y} i(x, y) = \sum_{x \sim y} [v(x) - v(y)]c(x, y) ,$$

or

$$v(x) = \frac{1}{\pi(x)} \sum_{x \sim y} c(x, y)v(y) .$$

That is, $v(\bullet)$ is harmonic on $\mathsf{V} \setminus (A \cup Z)$. Since $v \restriction A \equiv 1$ and $v \restriction Z \equiv 0$, it follows that if G is finite, then $v = F$ (defined in (2.2)); in particular, we have uniqueness and existence of voltages. The voltage function is just the solution to the Dirichlet problem.

Now if we sum the differences of a function, such as the voltage v, on the edges of a cycle, we get 0. Thus, by Ohm's law, we deduce:

Kirchhoff's Cycle Law: If $x_1 \sim x_2 \sim \cdots \sim x_n \sim x_{n+1} = x_1$ is a cycle, then

$$\sum_{i=1}^{n} i(x_i, x_{i+1})r(x_i, x_{i+1}) = 0 .$$

One can also deduce Ohm's law from Kirchhoff's two laws. A somewhat more general statement is in the following exercise.

▷ **Exercise 2.2.**

Suppose that an antisymmetric function j (meaning that $j(x, y) = -j(y, x)$) on the edges of a finite connected network satisfies Kirchhoff's cycle law and Kirchhoff's node law in the form $\sum_{x \sim y} j(x, y) = 0$ for all $x \in W$. Show that j is the current determined by imposing voltages off W and that the voltage function is unique up to an additive constant.

§2.2. More Probabilistic Interpretations.

Suppose that $A = \{a\}$ is a singleton. What is the chance that a random walk starting at a will hit Z before it returns to a? Write this as

$$\mathbf{P}[a \to Z] := \mathbf{P}_a[\tau_Z < \tau^+_{\{a\}}] .$$

Impose a voltage of $v(a)$ at a and 0 on Z. Since $v(\bullet)$ is linear in $v(a)$ by the superposition principle, we have that $\mathbf{P}_x[\tau_{\{a\}} < \tau_Z] = v(x)/v(a)$, whence

$$\begin{aligned} \mathbf{P}[a \to Z] &= \sum_x p_{ax}\Big(1 - \mathbf{P}_x[\tau_{\{a\}} < \tau_Z]\Big) = \sum_x \frac{c(a, x)}{\pi(a)}\left[1 - \frac{v(x)}{v(a)}\right] \\ &= \frac{1}{v(a)\pi(a)} \sum_x c(a, x)[v(a) - v(x)] = \frac{1}{v(a)\pi(a)} \sum_x i(a, x) . \end{aligned}$$

In other words,

$$v(a) = \frac{\sum_x i(a, x)}{\pi(a)\mathbf{P}[a \to Z]} . \tag{2.3}$$

Since $\sum_x i(a, x)$ is the ***total amount of current flowing into the circuit at*** a, we may regard the entire circuit between a and Z as a single conductor of ***effective conductance***

$$C_{\text{eff}} := \pi(a)\mathbf{P}[a \to Z] =: \mathscr{C}(a \leftrightarrow Z) , \tag{2.4}$$

where the last notation indicates the dependence on a and Z. (If we need to indicate the dependence on G, we will write $\mathscr{C}(a \leftrightarrow Z; G)$.) The similarity to (2.1) can provide a good mnemonic, but the analogy should not be pushed too far. We define the ***effective resistance*** $\mathscr{R}(a \leftrightarrow Z)$ to be its reciprocal; in case $a \in Z$, then we also define $\mathscr{R}(a \leftrightarrow Z) := 0$. One answer to our question above is thus $\mathbf{P}[a \to Z] = \mathscr{C}(a \leftrightarrow Z)/\pi(a)$. In Sections 2.3 and 2.4, we will see some ways to compute effective conductance.

Now the number of visits to a before hitting Z is a geometric random variable with mean $\mathbf{P}[a \to Z]^{-1} = \pi(a)\mathscr{R}(a \leftrightarrow Z)$. According to (2.3), this can also be expressed as $\pi(a)v(a)$ when there is unit current flowing from a to Z and the voltage is 0 on Z. This generalizes as follows. Let $\mathcal{G}(a,x)$ be the expected number of visits to x strictly before hitting Z by a random walk started at a. Thus, $\mathcal{G}(a,x)=0$ for $x \in Z$ and

$$\mathcal{G}(a,a)=\mathbf{P}[a \to Z]^{-1}=\pi(a)\mathscr{R}(a \leftrightarrow Z)\,. \tag{2.5}$$

The function $\mathcal{G}(\bullet,\bullet)$ is the ***Green*** function for the random walk absorbed (or "killed") on Z.

Proposition 2.1. (Green Function as Voltage) *Let G be a finite connected network. When a voltage is imposed on $\{a\}\cup Z$ so that a unit current flows from a to Z and the voltage is 0 on Z, then $v(x)=\mathcal{G}(a,x)/\pi(x)$ for all x.*

Proof. We have just shown that this is true for $x \in \{a\}\cup Z$, so it suffices to establish that $\mathcal{G}(a,x)/\pi(x)$ is harmonic elsewhere. But by Exercise 2.1, we have that $\mathcal{G}(a,x)/\pi(x)=\mathcal{G}(x,a)/\pi(a)$ and the harmonicity of $\mathcal{G}(x,a)$ is established just as for the function of (2.2). ◄

Given that we have two probabilistic interpretations of voltage, we naturally wonder whether current has a probabilistic interpretation. We might guess one by the following unrealistic but simple model of electricity: positive particles enter the circuit at a, they do Brownian motion on G (being less likely to pass through small conductors) and, when they hit Z, they are removed. The net flow rate of particles across an edge would then be the current on that edge. It turns out that in our mathematical model, this is correct:

Proposition 2.2. (Current as Edge Crossings) *Let G be a finite connected network. Start a random walk at a and absorb it when it first visits Z. For $x \sim y$, let S_{xy} be the number of transitions from x to y. Then $\mathbf{E}[S_{xy}]=\mathcal{G}(a,x)p_{xy}$ and $\mathbf{E}[S_{xy}-S_{yx}]=i(x,y)$, where i is the current in G when a potential is applied between a and Z in such an amount that unit current flows in at a.*

Note that we count a transition from y to x when $y \notin Z$ but $x \in Z$, although we do not count this as a visit to x in computing $\mathcal{G}(a,x)$.

Proof. We have

$$\begin{aligned}\mathbf{E}[S_{xy}]&=\mathbf{E}\left[\sum_{k=0}^{\infty}\mathbf{1}_{\{X_k=x\}}\mathbf{1}_{\{X_{k+1}=y\}}\right]=\sum_{k=0}^{\infty}\mathbf{P}[X_k=x,\,X_{k+1}=y]\\&=\sum_{k=0}^{\infty}\mathbf{P}[X_k=x]p_{xy}=\mathbf{E}\left[\sum_{k=0}^{\infty}\mathbf{1}_{\{X_k=x\}}\right]p_{xy}=\mathcal{G}(a,x)p_{xy}\,.\end{aligned}$$

Hence by Proposition 2.1, we have

$$\forall x, y \qquad \mathbf{E}[S_{xy} - S_{yx}] = \mathcal{G}(a, x)p_{xy} - \mathcal{G}(a, y)p_{yx}$$
$$= v(x)\pi(x)p_{xy} - v(y)\pi(y)p_{yx} = [v(x) - v(y)]c(x, y) = i(x, y)\,. \blacktriangleleft$$

Effective conductance is a key quantity because of the following relationship to the question of transience and recurrence when G is infinite. Recall that for an infinite network G, we assume that

$$\forall x \qquad \sum_{x \sim y} c(x, y) < \infty\,, \tag{2.6}$$

so that the associated random walk is well defined. (Of course, this is true when G is ***locally finite***—i.e., the number of edges incident to every given vertex is finite.) We allow more than one edge between a given pair of vertices: each such edge has its own conductance. Loops are also allowed (edges with only one endpoint), but these may be ignored for our present purposes since they only delay the random walk. Strictly speaking, then, G may be a ***multigraph***, not a graph. When a random walk moves from x to y in a multigraph that has several edges connecting x to y, then we think of the walk as moving along one of those edges, chosen with probability proportional to its conductance. Thus, the multigraph form of Proposition 2.2 is $\mathbf{E}[S_e] = \mathcal{G}(a, e^-)p_e$ and $\mathbf{E}[S_e - S_{-e}] = i(e)$. However, we will usually ignore the extra notational complications that arise for multigraphs. In fact, we have not yet used anywhere that G has only finitely many edges:

▷ **Exercise 2.3.**

Verify that Propositions 2.1 and 2.2 are valid when the number of edges is infinite but the number of vertices is finite.

Let $\langle G_n \rangle$ be any sequence of finite subgraphs of G that ***exhaust*** G, i.e., $G_n \subseteq G_{n+1}$ and $G = \bigcup G_n$. Let Z_n be the set of vertices in $G \setminus G_n$. (Note that if Z_n is identified to a point, the graph will have finitely many vertices but may have infinitely many edges even when loops are deleted.) Then for every $a \in G$, the events $\{a \to Z_n\}$ are decreasing in n, so the limit $\lim_n \mathbf{P}[a \to Z_n]$ is the probability of never returning to a in G—the ***escape*** probability from a. This is positive iff the random walk on G is transient. Hence by (2.4), $\lim_{n\to\infty} \mathscr{C}(a \leftrightarrow Z_n) > 0$ iff the random walk on G is transient. We call $\lim_{n\to\infty} \mathscr{C}(a \leftrightarrow Z_n)$ the ***effective conductance*** from a to ∞ in G and denote it by $\mathscr{C}(a \leftrightarrow \infty)$ or, if a is understood, by C_{eff}. Its reciprocal, ***effective resistance***, is denoted R_{eff}. We have shown:

Theorem 2.3. (Transience and Effective Conductance) *Random walk on an infinite connected network is transient iff the effective conductance from any of its vertices to infinity is positive.*

▷ **Exercise 2.4.**

For a fixed vertex a in G, show that $\lim_n \mathscr{C}(a \leftrightarrow Z_n)$ is the same for every sequence $\langle G_n \rangle$ that exhausts G.

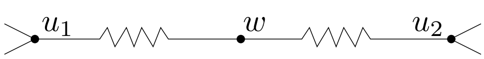
▷ **Exercise 2.5.**

When G is finite but A is not a singleton, define $\mathscr{C}(A \leftrightarrow Z)$ to be $\mathscr{C}(a \leftrightarrow Z)$ if all the vertices in A were to be identified to a single vertex, a. Show that if voltages are applied at the vertices of $A \cup Z$ so that $v\restriction A$ and $v\restriction Z$ are constants, then $v\restriction A - v\restriction Z = \mathcal{I}_{AZ}\mathscr{R}(A \leftrightarrow Z)$, where $\mathcal{I}_{AZ} := \sum_{x\in A}\sum_y i(x,y)$ is the total amount of current flowing from A to Z.

§2.3. Network Reduction.

How do we calculate effective conductance of a network? Since we want to replace a network by an equivalent single conductor, it is natural to attempt this by replacing more and more of G through simple transformations. There are, in fact, three such simple transformations, series, parallel, and star-triangle, and it turns out that they suffice to reduce all finite planar networks by a theorem of Epifanov; see Truemper (1989).

I. Series. Two resistors r_1 and r_2 in series are equivalent to a single resistor $r_1 + r_2$. In other words, if $w \in \mathsf{V}(G) \setminus (A \cup Z)$ is a node of degree 2 with neighbors u_1, u_2 and we replace the edges (u_i, w) by a single edge (u_1, u_2) having resistance $r(u_1, w) + r(w, u_2)$, then all potentials and currents in $G \setminus \{w\}$ are unchanged and the current that flows from u_1 to u_2 equals $i(u_1, w)$.



Proof. It suffices to check that Ohm's and Kirchhoff's laws are satisfied on the new network for the voltages and currents given. This is easy. ◀

▷ **Exercise 2.6.**

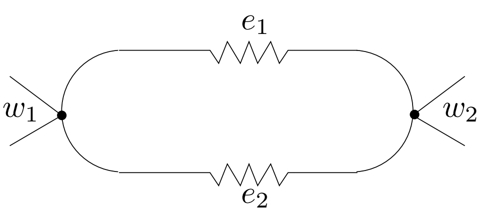
Give two harder but instructive proofs of the series equivalence: Since voltages determine currents, it suffices to check that the voltages are as claimed on the new network G'.

(1) Show that $v(x)$ $(x \in \mathsf{V}(G) \setminus \{w\})$ is harmonic on $\mathsf{V}(G') \setminus (A \cup Z)$. (2) Use the "craps principle" (Pitman (1993), p. 210) to show that $\mathbf{P}_x[\tau_A < \tau_Z]$ is unchanged for $x \in \mathsf{V}(G) \setminus \{w\}$.

Example 2.4. Consider simple random walk on $\mathbb{Z}$. Let $0 \le k \le n$. What is $\mathbf{P}_k[\tau_0 < \tau_n]$? It is the voltage at k when there is a unit voltage imposed at 0 and zero voltage at n. If we replace the resistors in series from 0 to k by a single resistor with resistance k and the resistors from k to n by a single resistor of resistance $n-k$, then the voltage at k does not change. But now this voltage is simply the probability of taking a step to 0, which is thus $(n-k)/n$.

For gambler's ruin, rather than simple random walk, we have the conductances $c(i, i+1) = (p/q)^i$. The replacement of edges in series by single edges now gives one edge from 0 to k of resistance $\sum_{i=0}^{k-1} (q/p)^i$ and one edge from k to n of resistance $\sum_{i=k}^{n-1} (q/p)^i$. The probability of ruin is therefore $\sum_{i=k}^{n-1} (q/p)^i / \sum_{i=0}^{n-1} (q/p)^i = [(p/q)^{n-k} - 1]/[(p/q)^n - 1]$.

II. Parallel. Two conductors c_1 and c_2 in parallel are equivalent to one conductor $c_1 + c_2$. In other words, if two edges e_1 and e_2 that both join vertices $w_1, w_2 \in \mathsf{V}(G)$ are replaced by a single edge e joining w_1, w_2 of conductance $c(e) := c(e_1) + c(e_2)$, then all voltages and currents in $G \setminus \{e_1, e_2\}$ are unchanged and the current $i(e)$ equals $i(e_1) + i(e_2)$ (if e_1 and e_2 have the "same" orientations, i.e., same tail and head). The same is true for an infinite number of edges in parallel.



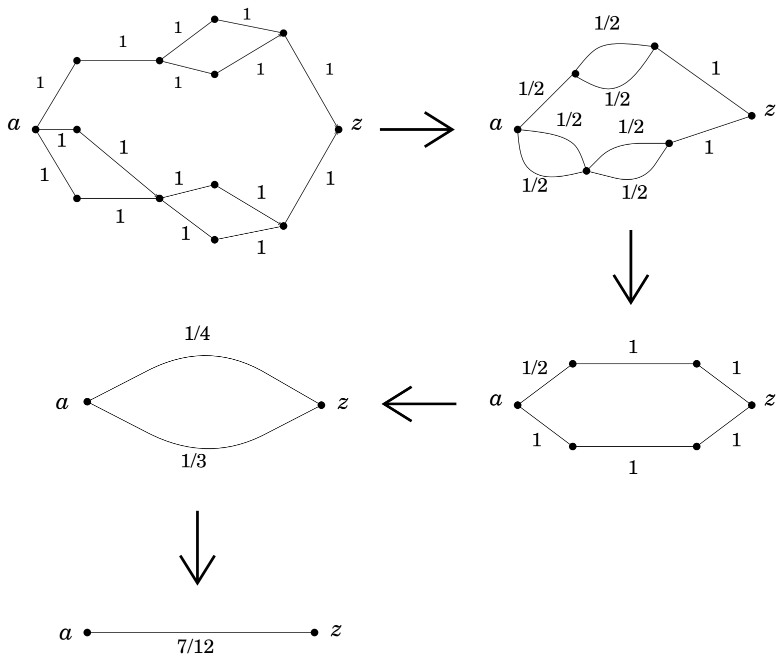
Proof. Check Ohm's and Kirchhoff's laws with $i(e) := i(e_1) + i(e_2)$. ◀

▷ **Exercise 2.7.**

Give two more proofs of the parallel equivalence as in Exercise 2.6.

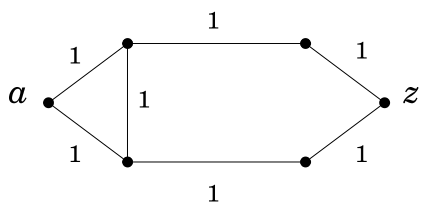
Example 2.5. Suppose that each edge in the following network has equal conductance. What is $\mathbf{P}[a \to z]$? Following the transformations indicated in the figure, we obtain $\mathscr{C}(a \leftrightarrow z) = 7/12$, so that

$$\mathbf{P}[a \to z] = \frac{\mathscr{C}(a \leftrightarrow z)}{\pi(a)} = \frac{7/12}{3} = \frac{7}{36}\,.$$



Note that in any network G with voltage applied from a to z, if it happens that $v(x) = v(y)$, then we may identify x and y to a single vertex, obtaining a new network $G/\{x, y\}$ in which the voltages at all vertices are the same as in G.

Example 2.6. What is $\mathbf{P}[a \to z]$ in the following network?



There are 2 ways to deal with the vertical edge:

(1) Remove it: by symmetry, the voltages at its endpoints are equal, whence no current flows on it.

(2) Contract it, i.e., remove it but combine its endpoints into one vertex (we could also combine the other two unlabelled vertices with each other): the voltages are the same,

so they may be combined.

In either case, we get $\mathscr{C}(a \leftrightarrow z) = 2/3$, whence $\mathbf{P}[a \to z] = 1/3$.

▷ **Exercise 2.8.**

Let (G, c) be a network. A ***network automorphism*** of (G, c) is a map $\phi : G \to G$ that is a bijection of the vertex set with itself and a bijection of the edge set with itself such that if x and e are incident, then so are $\phi(x)$ and $\phi(e)$ and such that $c(e) = c(\phi(e))$ for all edges e. Suppose that (G, c) is ***spherically symmetric*** about o, meaning that if x and y are any two vertices at the same distance from o, then there is an automorphism of (G, c) that leaves o fixed and that takes x to y. Let C_n be the sum of $c(e)$ over all edges e with $d(e^-, o) = n$ and $d(e^+, o) = n + 1$. Show that

$$\mathscr{R}(o \leftrightarrow \infty) = \sum_n \frac{1}{C_n},$$

whence the network random walk on G is transient iff

$$\sum_n \frac{1}{C_n} < \infty.$$

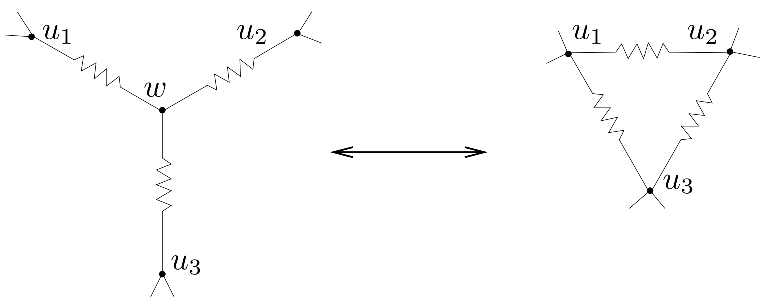
III. Star-triangle. The configurations below are equivalent when

$$\forall i \in \{1, 2, 3\} \qquad c(w, u_i)c(u_{i-1}, u_{i+1}) = \gamma,$$

where indices are taken mod 3 and

$$\gamma := \frac{\prod_i c(w, u_i)}{\sum_i c(w, u_i)} = \frac{\sum_i r(u_{i-1}, u_{i+1})}{\prod_i r(u_{i-1}, u_{i+1})}.$$

We won't use this equivalence except in Example 2.7 and the exercises. This is also called the "Y-Δ" or "Wye-Delta" transformation.



▷ **Exercise 2.9.**

Give at least one proof of star-triangle equivalence.

Actually, there is a fourth trivial transformation: we may prune (or add) vertices of degree 1 (and attendant edges) as well as loops.

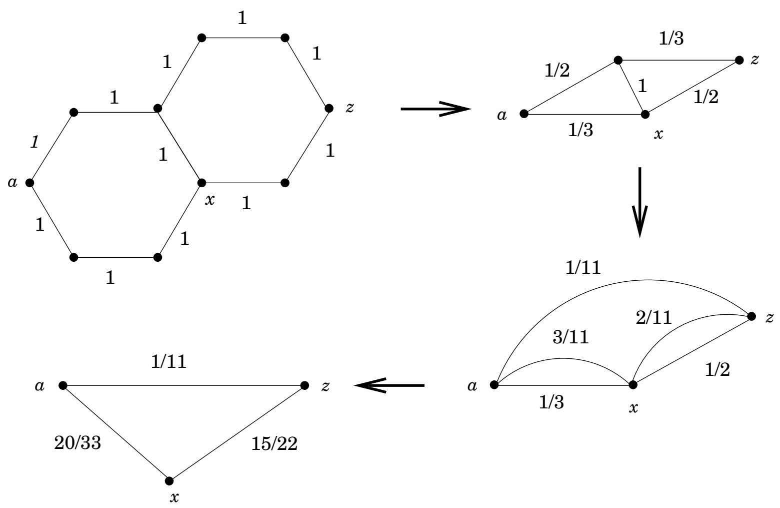
▷ **Exercise 2.10.**

Find a (finite) graph that can't be reduced to a single edge by the four transformations pruning, series, parallel, and star-triangle.

Either of the transformations star-triangle or triangle-star can also be used to reduce the network in Example 2.6.

Example 2.7. What is $\mathbf{P}_x[\tau_a < \tau_z]$ in the following network? Following the transformations indicated in the figure, we obtain

$$\mathbf{P}_x[\tau_a < \tau_z] = \frac{20/33}{20/33 + 15/22} = \frac{8}{17}\,.$$



§2.4. Energy.

We now come to another extremely useful concept, energy. (Although the term "energy" is used for mathematical reasons, the physical concept is actually power dissipation.) We will begin with some convenient notation and some facts about this notation. Unfortunately, there is actually a fair bit of notation. But once we have it all in place, we will be able to quickly reap some powerful consequences.

We will often contract some vertices in a graph, which may produce a multigraph. When we say that a graph is ***finite***, we mean that V and E are finite. Define $\ell^2(\mathsf{V})$ to be the usual real Hilbert space of functions on V with inner product

$$(f, g) := \sum_{x \in \mathsf{V}} f(x)g(x)\,.$$

Since we are interested in flows on E, it is natural to consider that what flows one way is the negative of what flows the other. ***From now on, each edge occurs with both orientations.*** Thus, define $\ell^2_-(\mathsf{E})$ to be the space of ***antisymmetric*** functions θ on E (i.e., $\theta(-e) = -\theta(e)$ for each edge e) with the inner product

$$(\theta, \theta') := \frac{1}{2} \sum_{e \in \mathsf{E}} \theta(e)\theta'(e) = \sum_{e \in \mathsf{E}_{1/2}} \theta(e)\theta'(e)\,,$$

where $\mathsf{E}_{1/2} \subset \mathsf{E}$ is a set of edges containing exactly one of each pair $e, -e$. Since voltage differences across edges lead to currents, define the ***coboundary operator*** $d : \ell^2(\mathsf{V}) \to \ell^2_-(\mathsf{E})$ by

$$(df)(e) := f(e^-) - f(e^+)\,.$$

(Note that this is the negative of the more natural definition; since current flows from greater to lesser voltage, however, it is the more useful definition for us.) This operator is clearly linear. Conversely, given an antisymmetric function on the edges, we are interested in the net flow out of a vertex, whence we define the ***boundary operator*** $d^* : \ell^2_-(\mathsf{E}) \to \ell^2(\mathsf{V})$ by

$$(d^*\theta)(x) := \sum_{e^- = x} \theta(e)\,.$$

This operator is also clearly linear. We use the superscript * because these two operators are adjoints of each other:

$$\forall f \in \ell^2(\mathsf{V}) \ \ \forall \theta \in \ell^2_-(\mathsf{E}) \qquad (\theta, df) = (d^*\theta, f)\,.$$

▷ **Exercise 2.11.**

Prove that d and d^* are adjoints of each other.

One use of this notation is that the calculation left here for Exercise 2.11 need not be repeated every time it arises—and it arises a lot. Another use is the following compact forms of the network laws. Let i be a current.

Ohm's Law: $dv = ir$, *i.e.*, $\forall e \in \mathsf{E}\ dv(e) = i(e)r(e)$.

Kirchhoff's Node Law: $d^*i(x) = 0$ if no battery is connected at x.

It will be useful to study flows other than current in order to discover a special property of the current flow. We can imagine water flowing through a network of pipes. The amount of water flowing into the network at a vertex a is $d^*\theta(a)$. Thus, we call $\theta \in \ell^2_-(\mathsf{E})$ a ***flow from*** A to Z if $d^*\theta$ is 0 off of A and Z, is nonnegative on A, and nonpositive on Z. The ***total amount flowing into the network*** is then $\sum_{a\in A} d^*\theta(a)$; as the next lemma shows, this is also the total amount flowing out of the network. We call

$$\mathsf{Strength}(\theta) := \sum_{a\in A} d^*\theta(a)$$

the ***strength*** of the flow θ. A flow of strength 1 is called a ***unit flow***.

Lemma 2.8. (Flow Conservation) *Let G be a finite graph and A and Z be two disjoint subsets of its vertices. If θ is a flow from A to Z, then*

$$\sum_{a\in A} d^*\theta(a) = -\sum_{z\in Z} d^*\theta(z)\,.$$

Proof. We have

$$\sum_{x\in A} d^*\theta(x) + \sum_{x\in Z} d^*\theta(x) = \sum_{x\in A\cup Z} d^*\theta(x) = (d^*\theta, \mathbf{1}) = (\theta, d\mathbf{1}) = (\theta, \mathbf{0}) = 0$$

since $d^*\theta(x) = 0$ for $x \notin A \cup Z$. ◀

The following consequence will be useful in a moment.

Lemma 2.9. *Let G be a finite graph and A and Z be two disjoint subsets of its vertices. If θ is a flow from A to Z and $f{\restriction}A$, $f{\restriction}Z$ are constants α and ζ, then*

$$(\theta, df) = \mathsf{Strength}(\theta)(\alpha - \zeta)\,.$$

Proof. We have $(\theta, df) = (d^*\theta, f) = \sum_{a\in A} d^*\theta(a)\alpha + \sum_{z\in Z} d^*\theta(z)\zeta$. Now apply Lemma 2.8. ◀

When a current i flows through a resistor of resistance r and voltage difference v, energy is dissipated at rate $P = iv = i^2 r = i^2/c = v^2 c = v^2/r$. We are interested in the total power (= energy per unit time) dissipated.

Notation. Write

$$(f, g)_h := (fh, g) = (f, gh)$$

and

$$\|f\|_h := \sqrt{(f, f)_h}\,.$$

Definition. For an antisymmetric function θ, define its ***energy*** to be

$$\mathscr{E}(\theta) := \|\theta\|_r^2\,,$$

where r is the collection of resistances.

Thus $\mathscr{E}(i) = (i, i)_r = (i, dv)$. If i is a unit current flow from A to Z with voltages v_A and v_Z constant on A and on Z, respectively, then by Lemma 2.9 and Exercise 2.5,

$$\mathscr{E}(i) = v_A - v_Z = \mathscr{R}(A \leftrightarrow Z)\,. \tag{2.7}$$

The inner product $(\bullet, \bullet)_r$ is important not only for its squared norm $\mathscr{E}(\bullet)$. For example, we may express Kirchhoff's laws as follows. Let $\chi^e := \mathbf{1}_{\{e\}} - \mathbf{1}_{\{-e\}}$ denote the unit flow along e represented as an antisymmetric function in $\ell^2_-(\mathsf{E})$. Note that for every antisymmetric function θ and every e, we have

$$(\chi^e, \theta)_r = \theta(e) r(e)\,,$$

so that

$$\Big(\sum_{e^- = x} c(e)\chi^e, \theta\Big)_r = d^*\theta(x)\,. \tag{2.8}$$

Let i be any current.

Kirchhoff's Node Law: For every vertex x, we have

$$\Big(\sum_{e^- = x} c(e)\chi^e, i\Big)_r = 0$$

except if a battery is connected at x.

Kirchhoff's Cycle Law: If $e_1, e_2, \ldots, e_n$ is an oriented cycle in G, then

$$\Big(\sum_{k=1}^{n} \chi^{e_k}, i\Big)_r = 0\,.$$

Now for our last bit of notation before everything comes into focus, let $\bigstar$ denote the subspace in $\ell^2_-(\mathsf{E})$ spanned by the ***stars*** $\sum_{e^-=x} c(e)\chi^e$ and let $\diamondsuit$ denote the subspace spanned by the ***cycles*** $\sum_{k=1}^n \chi^{e_k}$, where $e_1, e_2, \ldots, e_n$ forms an oriented cycle. We call these subspaces the ***star space*** and the ***cycle space*** of G. These subspaces are clearly orthogonal; here and subsequently, orthogonality refers to the inner product $(\bullet, \bullet)_r$. Moreover, the sum of $\bigstar$ and $\diamondsuit$ is all of $\ell^2_-(\mathsf{E})$, which is the same as saying that only the zero vector is orthogonal to both $\bigstar$ and $\diamondsuit$. To see that this is the case, suppose that $\theta \in \ell^2_-(\mathsf{E})$ is orthogonal to both $\bigstar$ and $\diamondsuit$. Since θ is orthogonal to $\diamondsuit$, there is a function F such that $\theta = c\, dF$ by Exercise 2.2. Since θ is orthogonal to $\bigstar$, the function F is harmonic. Since G is finite, the uniqueness principle implies that F is constant on each component of G, whence $\theta = \mathbf{0}$, as desired.

Thus, Kirchhoff's Cycle Law says that i, being orthogonal to $\diamondsuit$, is in $\bigstar$. Furthermore, any $i \in \bigstar$ is a current by Exercise 2.2 (let $W := \{x\,;\ (d^*i)(x) = 0\}$). Now if θ is any flow with the same sources and sinks as i, more precisely, if θ is any antisymmetric function such that $d^*\theta = d^*i$, then $\theta - i$ is a sourceless flow, i.e., by (2.8), is orthogonal to $\bigstar$ and thus is an element of $\diamondsuit$. Therefore, the expression

$$\theta = i + (\theta - i)$$

is the orthogonal decomposition of θ relative to $\ell^2_-(\mathsf{E}) = \bigstar \oplus \diamondsuit$. This shows that the orthogonal projection $P_\bigstar : \ell^2_-(\mathsf{E}) \to \bigstar$ plays a crucial role in network theory. In particular,

$$i = P_\bigstar \theta \tag{2.9}$$

and

$$\|\theta\|_r^2 = \|i\|_r^2 + \|\theta - i\|_r^2\,. \tag{2.10}$$

Thomson's Principle. *Let G be a finite network and A and Z be two disjoint subsets of its vertices. Let θ be a flow from A to Z and i be the current flow from A to Z with $d^*i = d^*\theta$. Then $\mathscr{E}(\theta) > \mathscr{E}(i)$ unless $\theta = i$.*

Proof. The result is an immediate consequence of (2.10). ◀

Note that given θ, the corresponding current i such that $d^*i = d^*\theta$ is unique (and given by (2.9)).

Recall that $\mathsf{E}_{1/2} \subset \mathsf{E}$ is a set of edges containing exactly one of each pair $e, -e$. What is the matrix of $P_\bigstar$ in the orthogonal basis $\{\chi^e\,;\ e \in \mathsf{E}_{1/2}\}$? We have

$$(P_\bigstar \chi^e, \chi^{e'})_r = (i^e, \chi^{e'})_r = i^e(e')r(e')\,, \tag{2.11}$$

where i^e is the unit current from e^- to e^+. Therefore, the matrix coefficient at (e', e) equals $(P_\bigstar \chi^e, \chi^{e'})_r/(\chi^{e'}, \chi^{e'})_r = i^e(e') =: Y(e, e')$, the current that flows across e' when a unit current is imposed between the endpoints of e. This matrix is called the ***transfer current matrix***. This matrix will be extremely useful for our study of random spanning trees and forests in Chapters 4 and 10. Since $\mathbf{P}_\bigstar$, being an orthogonal projection, is self-adjoint, we have $(P_\bigstar \chi^e, \chi^{e'})_r = (\chi^e, P_\bigstar \chi^{e'})_r$, whence

$$Y(e, e')r(e') = Y(e', e)r(e)\,. \tag{2.12}$$

This is called the ***reciprocity law***.

Consider $\mathbf{P}[a \to Z]$. How does this change when an edge is removed from G? when an edge is added? when the conductance of an edge is changed? These questions are not easy to answer probabilistically, but yield to the ideas we have developed. Since $\mathbf{P}[a \to Z] = \mathscr{C}(a \leftrightarrow Z)/\pi(a)$, if no edge incident to a is affected, then we need analyze only the change in effective conductance.

▷ **Exercise 2.12.**

Show that $\mathbf{P}[a \to Z]$ can increase in some situations and decrease in others when an edge incident to a is removed.

Effective conductance changes as follows. We use subscripts to indicate the edge conductances used.

Rayleigh's Monotonicity Principle. *Let G be a finite graph and A and Z two disjoint subsets of its vertices. If c and c' are two assignments of conductances on G with $c \le c'$, then $\mathscr{C}_c(A \leftrightarrow Z) \le \mathscr{C}_{c'}(A \leftrightarrow Z)$.*

Proof. By (2.7), we have $\mathscr{C}(A \leftrightarrow Z) = 1/\mathscr{E}(i)$ for a unit current flow i from A to Z. Now

$$\mathscr{E}_c(i_c) \ge \mathscr{E}_{c'}(i_c) \ge \mathscr{E}_{c'}(i_{c'})\,,$$

where the first inequality follows from the definition of energy and the second from Thomson's principle. Taking reciprocals gives the result. ◀

In particular, removing an edge decreases effective conductance, so if the edge is not incident to a, then its removal decreases $\mathbf{P}[a \to Z]$. In addition, contracting an edge (called "shorting" in electrical network theory), i.e., identifying its two endpoints and removing the resulting loop, increases the effective conductance between any sets of vertices. This is intuitive from thinking of increasing to infinity the conductance on the edge to be contracted, so we will still refer to it as part of Rayleigh's Monotonicity Principle. To prove it rigorously, let i be the unit current flow from A to Z. If the graph G with the edge e contracted is denoted $G/\{e\}$, then the edge set of $G/\{e\}$ may be identified with $\mathsf{E}(G) \setminus \{e\}$. If e does not connect A to Z, then the restriction θ of i to the edges of $G/\{e\}$ is a unit flow from A to Z, whence the effective resistance between A and Z in $G/\{e\}$ is at most $\mathscr{E}(\theta)$, which is at most $\mathscr{E}(i)$, which equals the effective resistance in G.

▷ **Exercise 2.13.**

Given disjoint vertex sets A, Z in a finite network, we may express the effective resistance between A and Z by Thomson's Principle as

$$\mathscr{R}(A \leftrightarrow Z) = \min\left\{ \sum_{e \in \mathsf{E}_{1/2}} r(e)\theta(e)^2\,;\ \theta \text{ is a unit flow from } A \text{ to } Z \right\}.$$

Prove the following dual expression for the effective conductance, known as ***Dirichlet's principle***:

$$\mathscr{C}(A \leftrightarrow Z) = \min\left\{ \sum_{e \in \mathsf{E}_{1/2}} c(e)dF(e)^2 \right\},$$

where F is a function that is 1 on A and 0 on Z.

§2.5. Transience and Recurrence.

We have seen that effective conductance from any vertex is positive iff the random walk is transient. We will formulate this as an energy criterion.

If $G = (\mathsf{V}, \mathsf{E})$ is a denumerable network, let

$$\ell^2(\mathsf{V}) := \{f : \mathsf{V} \to \mathbb{R}\,;\ \sum_{x \in \mathsf{V}} f(x)^2 < \infty\}$$

with the inner product $(f, g) := \sum_{x \in \mathsf{V}} f(x)g(x)$. Define the Hilbert space

$$\ell^2_-(\mathsf{E}, r) := \{\theta : \mathsf{E} \to \mathbb{R}\,;\ \forall e \quad \theta(-e) = -\theta(e) \text{ and } \sum_{e \in \mathsf{E}} \theta(e)^2 r(e) < \infty\}$$

with the inner product $(\theta, \theta')_r := \sum_{e \in \mathsf{E}_{1/2}} \theta(e)\theta'(e)r(e)$ and $\mathscr{E}(\theta) := (\theta, \theta)_r$. Define $df(e) := f(e^-) - f(e^+)$ as before. If $\sum_{e^- = x} |\theta(e)| < \infty$, then we also define $(d^*\theta)(x) := \sum_{e^- = x} \theta(e)$.

Suppose now that V is finite and $\sum_e |\theta(e)| < \infty$. Then the calculation of Exercise 2.11 shows that we still have $(\theta, df) = (d^*\theta, f)$ for all f. Likewise, under these hypotheses, we have Lemma 2.8 and Lemma 2.9 still holding. The remainder of Section 2.4 also then holds because of the following consequence of the Cauchy-Schwarz inequality:

$$\forall x \in \mathsf{V} \quad \sum_{e^- = x} |\theta(e)| \le \sqrt{\sum_{e^- = x} \theta(e)^2/c(e) \cdot \sum_{e^- = x} c(e)} \le \sqrt{\mathscr{E}(\theta)\pi(x)}\,. \tag{2.13}$$

In particular, if $\mathscr{E}(\theta) < \infty$, then $d^*\theta$ is defined.

▷ **Exercise 2.14.**

Let $G = (\mathsf{V}, \mathsf{E})$ be denumerable and $\theta_n \in \ell^2_-(\mathsf{E}, r)$ be such that $\mathscr{E}(\theta_n) \le M < \infty$ and $\forall e \in \mathsf{E}\ \theta_n(e) \to \theta(e)$. Show that θ is antisymmetric, $\mathscr{E}(\theta) \le \liminf_n \mathscr{E}(\theta_n) \le M$, and $\forall x \in \mathsf{V}\ \ d^*\theta_n(x) \to d^*\theta(x)$.

Call an antisymmetric function θ on E a ***unit flow*** from $a \in \mathsf{V}$ to ∞ if

$$\forall x \in \mathsf{V} \sum_{e^- = x} |\theta(e)| < \infty$$

and $(d^*\theta)(x) = \mathbf{1}_{\{a\}}(x)$. Our main theorem is the following criterion for transience due to T. Lyons (1983).

Theorem 2.10. (Energy and Transience) *Let G be a denumerable connected network. Random walk on G is transient iff there is a unit flow on G of finite energy from some (every) vertex to ∞.*

Proof. Let G_n be finite subgraphs that exhaust G. Let G_n^{W} be the graph obtained from G by identifying the vertices outside G_n to a single vertex, z_n. In other words, identify these vertices and throw away resulting loops (but keep multiple edges). Fix any vertex $a \in G$, which, without loss of generality, belongs to each G_n. We have, by definition, $\mathscr{R}(a \leftrightarrow \infty) = \lim \mathscr{R}(a \leftrightarrow z_n)$. Let i_n be the unit current flow in G_n^{W} from a to z_n. Then $\mathscr{E}(i_n) = \mathscr{R}(a \leftrightarrow z_n)$, so $\mathscr{R}(a \leftrightarrow \infty) < \infty \Leftrightarrow \lim \mathscr{E}(i_n) < \infty$.

Note that each edge of G_n^{W} comes from an edge in G and may be identified with it, even though one endpoint may be different.

If θ is a unit flow on G from a to ∞ and of finite energy, then the restriction $\theta\restriction G_n^{\mathrm{W}}$ of θ to G_n^{W} is a unit flow from a to z_n, whence Thomson's principle gives $\mathscr{E}(i_n) \le \mathscr{E}\big(\theta\restriction G_n^{\mathrm{W}}\big) \le \mathscr{E}(\theta) < \infty$. In particular, $\lim \mathscr{E}(i_n) < \infty$ and so the random walk is transient.

Conversely, suppose that G is transient. Then there is some $M < \infty$ such that $\mathscr{E}(i_n) \le M$ for all n. Start a random walk at a. Let $Y_n(x)$ be the number of visits to x before hitting $G \setminus G_n$ and $Y(x)$ be the total number of visits to x. Then $Y_n(x)$ increases to $Y(x)$, whence the Monotone Convergence Theorem implies that $\mathbf{E}[Y(x)] = \lim_{n\to\infty} \mathbf{E}[Y_n(x)] = \lim_{n\to\infty} \pi(x)v_n(x) =: \pi(x)v(x)$. By transience, we know that $\mathbf{E}[Y(x)] < \infty$, whence $v(x) < \infty$. Hence $i := c \cdot dv = \lim_{n\to\infty} c \cdot dv_n = \lim_{n\to\infty} i_n$ exists and is a unit flow from a to infinity of energy at most M by Exercise 2.14. ◀

This allows us to carry over the remainder of the electrical apparatus to infinite networks:

Proposition 2.11. *Let G be a transient connected network and G_n be finite subnetworks containing a vertex a that exhaust G. Identify the vertices outside G_n to z_n, forming G_n^{W}. Let i_n be the unit current flow in G_n^{W} from a to z_n. Then $\langle i_n \rangle$ has a pointwise limit i on G, which is the unique unit flow on G from a to ∞ of minimum energy. Let v_n be the voltages on G_n^{W} corresponding to i_n and with $v_n(z_n) := 0$. Then $v := \lim v_n$ exists on G and has the following properties:*

$$dv = ir\,,$$
$$v(a) = \mathscr{E}(i) = \mathscr{R}(a \leftrightarrow \infty)\,,$$
$$\forall x \qquad v(x)/v(a) = \mathbf{P}_x[\tau_a < \infty]\,.$$

Start a random walk at a. For all vertices x, the expected number of visits to x is $\mathcal{G}(a,x) = \pi(x)v(x)$. For all edges e, the expected signed number of crossings of e is $i(e)$.

Proof. We saw in the proof of Theorem 2.10 that v and i exist, that $dv = ir$, and that $\mathcal{G}(a,x) = \pi(x)v(x)$. The proof of Proposition 2.2 now applies as written for the last claim of the proposition. Since the events $\{\tau_a < \tau_{G\setminus G_n}\}$ are increasing in n with union $\{\tau_a < \infty\}$, we have (with superscript indicating on which network the random walk takes place)

$$v(x)/v(a) = \lim v_n(x)/v_n(a) = \lim_n \mathbf{P}_x^{G_n^{\mathrm{W}}}[\tau_a < \tau_{z_n}] = \lim_n \mathbf{P}_x^G[\tau_a < \tau_{G\setminus G_n}] = \mathbf{P}_x^G[\tau_a < \infty]\,.$$

Now $v(a) = \lim v_n(a) = \lim \mathscr{E}(i_n) = \lim \mathscr{R}(a \leftrightarrow z_n) = \mathscr{R}(a \leftrightarrow \infty)$. By Exercise 2.14, $\mathscr{E}(i) \le \liminf \mathscr{E}(i_n)$. Since $\mathscr{E}(i_n) \le \mathscr{E}(i)$ as in the proof of the theorem, we have $\mathscr{E}(i) = \lim \mathscr{E}(i_n) = v(a)$. Likewise, $\mathscr{E}(i_n) \le \mathscr{E}(\theta)$ for every unit flow from a to infinity, whence i has minimum energy.

Finally, we establish uniqueness of a unit flow (from a to ∞) with minimum energy. Note that $\forall \theta, \theta'$

$$\frac{\mathscr{E}(\theta) + \mathscr{E}(\theta')}{2} = \mathscr{E}\left(\frac{\theta+\theta'}{2}\right) + \mathscr{E}\left(\frac{\theta-\theta'}{2}\right)\,. \tag{2.14}$$

Therefore, if θ and θ' both have minimum energy, so does $(\theta+\theta')/2$ and hence $\mathscr{E}\big((\theta-\theta')/2\big)=0$, which gives $\theta=\theta'$. ◀

Thus, we may call i the unit current flow and v the voltage on G. We may regard G as grounded (i.e., has 0 voltage) at infinity.

By Theorem 2.10 and Rayleigh's monotonicity law, the ***type*** of a random walk, i.e., whether it is transient or recurrent, does not change when the conductances are changed by bounded factors. An extensive generalization of this is given in Theorem 2.16.

How do we determine whether there is a flow from a to ∞ of finite energy? There is no recipe, but there is a very useful necessary condition due to Nash-Williams (1959) and some useful techniques. A set Π of edges ***separates*** a ***and*** ∞ if every infinite simple path from a must include an edge in Π; we also call Π a ***cutset***.

The Nash-Williams Criterion. *If $\langle \Pi_n \rangle$ is a sequence of pairwise disjoint finite cutsets in a locally finite network G that separate a from ∞, then*

$$\mathscr{R}(a\leftrightarrow\infty)\ge\sum_n\left(\sum_{e\in\Pi_n}c(e)\right)^{-1}. \tag{2.15}$$

In particular, if the right-hand side is infinite, then G is recurrent.

Proof. By Proposition 2.11, it suffices to show that every unit flow θ from a to ∞ has energy at least the right-hand side. The Cauchy-Schwarz inequality gives

$$\sum_{e\in\Pi_n}\theta(e)^2r(e)\sum_{e\in\Pi_n}c(e)\ge\left(\sum_{e\in\Pi_n}|\theta(e)|\right)^2\ge 1\,.$$

The last inequality is intuitive and is proved as follows. Let K_n denote the set of vertices that are *not* separated from a by Π_n. Let F_n denote the edges that have their tail in K_n and their head not in K_n; obviously $F_n\subseteq\Pi_n$. Then cancellations show that

$$1=\sum_{x\in K_n}(d^*\theta)(x)=\sum_{e\in F_n}\theta(e)\le\sum_{e\in F_n}|\theta(e)|\le\sum_{e\in\Pi_n}|\theta(e)|\,.$$

Hence

$$\mathscr{E}(\theta)\ge\sum_n\sum_{e\in\Pi_n}\theta(e)^2r(e)\ge\sum_n\left(\sum_{e\in\Pi_n}c(e)\right)^{-1}.$$ ◀

Remark 2.12. If the cutsets can be ordered so that Π_1 separates a from Π_2 and for $n>1$, Π_n separates Π_{n-1} from Π_{n+1}, then the sum appearing in the statement of this Criterion has a natural interpretation: Short together (i.e., join by edges of infinite conductance, or, in other words, identify) all the vertices between Π_n and Π_{n+1} into one vertex U_n. Short all the vertices that Π_1 separates from ∞ into one vertex U_0. Then only parallel edges of Π_n join U_{n-1} to U_n. Replace these edges by a single edge of resistance $\left(\sum_{e\in\Pi_n} c(e)\right)^{-1}$. This new network is a series network with effective resistance from U_0 to ∞ equal to the right-hand side of (2.15). Thus, Rayleigh's monotonicity law shows that the effective resistance from a to ∞ in G is at least the right-hand side of (2.15).

The criterion of Nash-Williams is very useful for proving recurrence. In order to prove transience, there is a very useful way to define flows via random paths. Suppose that $\mathbf{P}$ is a probability measure on paths $\langle e_n \rangle$ from a to z on a finite graph or from a to ∞ on an infinite graph. (An infinite path is said to ***go to*** ∞ when no vertex is visited infinitely many times.) Define

$$\theta(e) := \sum_{n\geq 0} \big(\mathbf{P}[e_n = e] - \mathbf{P}[e_n = -e]\big) \tag{2.16}$$

provided

$$\sum_{n\geq 0} \big(\mathbf{P}[e_n = e] + \mathbf{P}[e_n = -e]\big) < \infty \,.$$

For example, the summability condition holds when the paths are edge-simple. Each path $\langle e_n \rangle$ determines a unit flow ψ from a to z (or to ∞) by sending 1 along each edge in the path:

$$\psi := \sum_{n\geq 0} \chi^{e_n} \,.$$

Since θ is an expectation of a random unit flow, we get that θ is a unit flow itself. We saw in Propositions 2.2 and 2.11 that this is precisely how network random walks and unit electric current are related (where the walk $\langle X_n \rangle$ gives rise to the path $\langle e_n \rangle$ with $e_n := \langle X_n, X_{n+1} \rangle$). However, there are other useful pairs of random paths and their expected flows as well.

We now illustrate the preceding techniques. First, we prove Pólya's famous theorem concerning random walk on the integer lattices. This proof is a slight modification of T. Lyons's 1983 proof.

Pólya's Theorem. *Simple random walk on the nearest-neighbor graph of $\mathbb{Z}^d$ is recurrent for $d = 1, 2$ and transient for all $d \geq 3$.*

Proof. For $d = 1, 2$, we can use the Nash-Williams Criterion with cutsets

$$\Pi_n := \{e\,;\ d(o, e^-) = n,\ |d(o, e^-) - d(o, e^+)| = 1\} \,,$$

where o is the origin and $d(\bullet, \bullet)$ is the graph distance.

For the other cases, by Rayleigh's Monotonicity Principle, it suffices to do $d = 3$. Let L be a random uniformly distributed ray from the origin o of $\mathbb{R}^3$ to ∞ (i.e., a straight line with uniform intersection on the unit sphere). Let $\mathcal{P}(L)$ be a simple path in $\mathbb{Z}^3$ from o to ∞ that stays within distance 4 of L; choose $\mathcal{P}(L)$ measurably, such as the (almost surely unique) closest path to L in the Hausdorff metric. Define the flow θ from the law of $\mathcal{P}(L)$ via (2.16). Then θ is a unit flow from o to ∞; we claim it has finite energy. There is some constant A such that if e is an edge whose midpoint is at Euclidean distance R from o, then $\mathbf{P}[e \in \mathcal{P}(L)] \le A/R^2$. Since all edge centers are separated from each other by Euclidean distance at least 1, there is also a constant B such that there are at most Bn^2 edge centers whose distance from the origin is between n and $n+1$. It follows that the energy of θ is at most $\sum_n A^2 B n^2 n^{-4}$, which is finite. Now transience follows from Theorem 2.10. ◀

Remark 2.13. The continuous case, i.e., Brownian motion in $\mathbb{R}^3$, is easier to handle (after establishing a similar relationship to an electrical framework) because of the spherical symmetry; see Section 2.8, the notes to this chapter. Here, we are approximating this continuous case in our solution. One can in fact use the transience of the continuous case to deduce that of the discrete case (or vice versa); see Theorem 2.24 in the notes.

The difference between 2 and 3 dimensions is illustrated in Figure 2.2. For information on the asymptotic behavior of these figures in dimension 2, see Dembo, Peres, Rosen, and Zeitouni (2001).

Since simple random walk on $\mathbb{Z}^2$ is recurrent, the effective resistance from the origin to distance n tends to infinity—but how quickly? Our techniques are good enough to answer this within a constant factor. Note first that the proof of (2.15) shows that if a and z are separated by pairwise disjoint cutsets $\Pi_1, \ldots, \Pi_n$, then

$$\mathscr{R}(a \leftrightarrow z) \ge \sum_{k=1}^{n} \left(\sum_{e \in \Pi_k} c(e) \right)^{-1} . \tag{2.17}$$

Proposition 2.14. *There are positive constants C_1, C_2 such that if one identifies to a single vertex z_n all vertices of $\mathbb{Z}^2$ that are at distance more than n from o, then*

$$C_1 \log n \le \mathscr{R}(o \leftrightarrow z_n) \le C_2 \log n .$$

Proof. The lower bound is an immediate consequence of (2.17) applied to the cutsets Π_k used in our proof of Pólya's theorem. The upper bound follows from the estimate of the energy of the unit flow analogous to that used for the transience of $\mathbb{Z}^3$. That is, $\theta(e)$ is defined via (2.16) from a uniform ray emanating from the origin. Then θ defines a unit flow from o to z_n and its energy is bounded by $C_2 \log n$. ◀

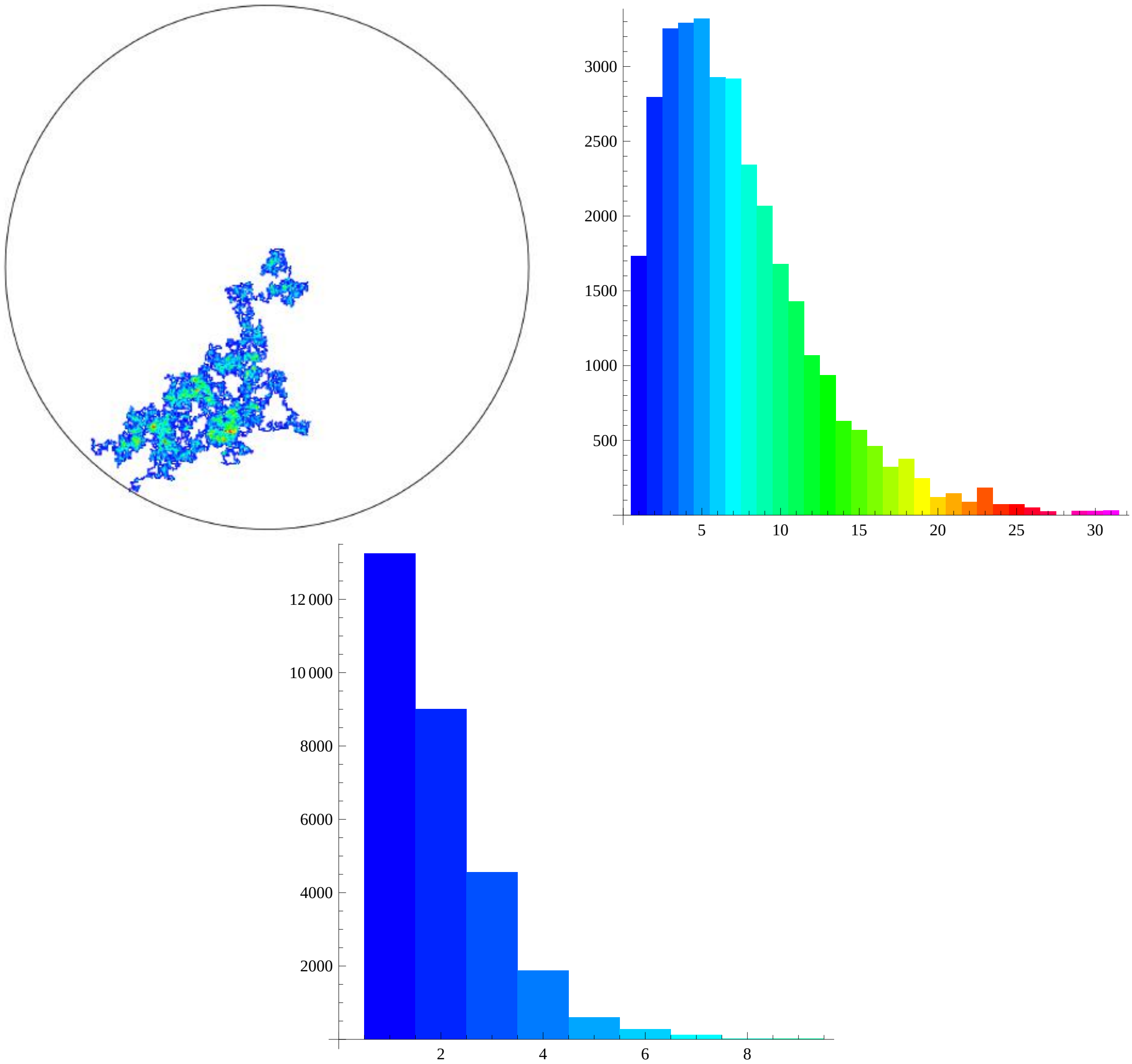


Figure 2.2. Random walk until it goes distance 200 from its starting point, colored according to the number of visits at a vertex. The histogram shows the time spent at vertices that were visited n times for each $n \geq 1$, with the same color coding. For 3 dimensions, only the histogram is shown.

We can extend Proposition 2.14 as follows.

Proposition 2.15. *For $d \geq 2$, there is a positive constant C_d such that if G_n is the subnetwork of $\mathbb{Z}^d$ induced on the vertices in a box of side length n, then for any pair of vertices x, y in G_n at mutual distance k,*

$$\mathscr{R}(x \leftrightarrow y; G_n) \in \begin{cases} (C_d^{-1} \log k, C_d \log k) & \text{if } d = 2, \\ (C_d^{-1}, C_d) & \text{if } d \geq 3. \end{cases}$$

Proof. The lower bounds follow from (2.17). For the upper bounds, we give the details for $d = 2$ only. There is a straight-line segment L of length k inside the portion of $\mathbb{R}^2$ that corresponds to G_n such that L meets the straight line M joining x and y at the midpoint of M in a right angle. Let Q be a random uniform point on L. Write $L(Q)$ for the union of the straight-line segments between x, y, and Q. Let $\mathcal{P}(Q)$ be a path in G_n from x to y that is closest to $L(Q)$. Use the law $\mathbf{P}$ of $\mathcal{P}(Q)$ to define the unit flow θ as in (2.16). Then $\mathscr{E}(\theta) \le C_2 \log k$ for some C_2, as in the proof of Proposition 2.14. ◀

Since the harmonic series, which arises in the recurrence of $\mathbb{Z}^2$, just barely diverges, it seems that the change from recurrence to transience occurs "just after" dimension 2, rather than somewhere else in $[2, 3]$. One way to make sense of this is to ask about the type of spaces intermediate between $\mathbb{Z}^2$ and $\mathbb{Z}^3$. For example, consider the wedge

$$W_f := \big\{(x, y, z)\,;\ |z| \le f\big(|x|\big)\big\}\,,$$

where $f : \mathbb{N} \to \mathbb{N}$ is an increasing function. The number of edges that leave $W_f \cap \{(x, y, z)\,;\ |x| \vee |y| \le n\}$ is of the order $n\big(f(n)+1\big)$, so that according to the Nash-Williams Criterion,

$$\sum_{n\ge 1} \frac{1}{n\big(f(n)+1\big)} = \infty \tag{2.18}$$

is sufficient for recurrence.

▷ **Exercise 2.15.**
Show that (2.18) is also necessary for recurrence if $f(n+1) \le f(n) + 1$ for all n.

The most direct proof of Pólya's theorem goes by calculation of the Green function and is not hard; see Exercise 2.81. However, that calculation depends on the precise structure of the graph. The proof here begins to show that the type doesn't change when fairly drastic changes are made in the lattice graph. Suppose, for example, that diagonal edges are added to the square lattice. Then clearly we can still use Nash-Williams' Criterion to show recurrence. Of course, a similar addition of edges in higher dimensions preserves transience simply by Rayleigh's Monotonicity Principle. But suppose that in $\mathbb{Z}^3$, we remove each edge $[(x, y, z), (x, y, z + 1)]$ with $x + y$ odd. Is the resulting graph still transient? If so, by how much has the effective resistance to infinity changed?

Notice that graph distances haven't changed much after these edges are removed. In a general network, if we think of the resistances r as the lengths of edges, then we are led to the following definition.

Given two networks G and G' with resistances r and r', we say that a map ϕ from the vertices of G to the vertices of G' is a ***rough embedding*** if there are constants $\alpha, \beta < \infty$ and a map Φ defined on the edges of G such that

(i) for every edge $\langle x, y\rangle \in G$, $\Phi\big(\langle x, y\rangle\big)$ is a non-empty simple oriented path of edges in G' joining $\phi(x)$ and $\phi(y)$ with

$$\sum_{e' \in \Phi(\langle x,y\rangle)} r'(e') \le \alpha r(x, y)$$

and $\Phi\big(\langle y, x\rangle\big)$ is the reverse of $\Phi\big(\langle x, y\rangle\big)$;

(ii) for every edge $e' \in G'$, there are no more than β edges in G whose image under Φ contains e'.

If we need to refer to the constants, we call the map (α, β)-***rough***. We call two networks ***roughly equivalent*** if there are rough embeddings in both directions. For example, every two Euclidean lattices of the same dimension are roughly equivalent. Kanai (1986) showed that rough embeddings preserve transience:

Theorem 2.16. (Rough Embeddings and Transience) *If G and G' are roughly equivalent connected networks, then G is transient iff G' is transient. In fact, if there is a rough embedding from G to G' and G is transient, then G' is transient.*

Proof. Suppose that G is transient and ϕ is an (α, β)-rough embedding from G to G'. Let θ be a unit flow on G of finite energy from a to infinity. We will use Φ to carry the flow θ to a unit flow θ' on G' that will have finite energy. Namely, define

$$\theta'(e') := \sum_{e' \in \Phi(e)} \theta(e)\,.$$

(The sum goes over all edges, not merely those in $\mathsf{E}_{1/2}$.) It is easy to see that θ is antisymmetric and $d^*\theta'(y) = \sum_{x \in \phi^{-1}(\{y\})} d^*\theta(x)$ for all $y \in G'$. Thus, θ' is a unit flow from $\phi(a)$ to infinity.

Now

$$\theta'(e')^2 \le \beta \sum_{e' \in \Phi(e)} \theta(e)^2$$

by the Cauchy-Schwarz inequality and the condition (ii). Therefore,

$$\begin{aligned}\sum_{e' \in \mathsf{E}'} \theta'(e')^2 r'(e') &\le \beta \sum_{e' \in \mathsf{E}'} \sum_{e' \in \Phi(e)} \theta(e)^2 r'(e') = \beta \sum_{e \in \mathsf{E}} \sum_{e' \in \Phi(e)} \theta(e)^2 r'(e') \\ &\le \alpha\beta \sum_{e \in \mathsf{E}} \theta(e)^2 r(e) < \infty\,.\end{aligned}$$ ◀

▷ **Exercise 2.16.**

Show that if we remove each edge $[(x, y, z), (x, y, z + 1)]$ with $x + y$ odd in $\mathbb{Z}^3$, then we obtain a transient graph with effective resistance to infinity at most 6 times what it was before removal.

A closely related notion is that of rough isometry, also called quasi-isometry. Given two graphs $G = (\mathsf{V}, \mathsf{E})$ and $G' = (\mathsf{V}', \mathsf{E}')$, call a function $\phi : \mathsf{V} \to \mathsf{V}'$ a ***rough isometry*** if there are positive constants α and β such that for all $x, y \in \mathsf{V}$,

$$\alpha^{-1} d(x, y) - \beta \le d'(\phi(x), \phi(y)) \le \alpha d(x, y) + \beta \tag{2.19}$$

and such that every vertex in G' is within distance β of the image of V. Here, d and d' denote the usual graph distances on G and G'. In fact, the same definition applies to metric spaces, with "vertex" replaced by "point".

▷ **Exercise 2.17.**

Show that being roughly isometric is an equivalence relation.

Proposition 2.17. (Rough Isometry and Rough Equivalence) *Let G and G' be two infinite roughly isometric networks with conductances c and c'. If c, c', c^{-1}, c'^{-1} are all bounded and the degrees in G and G' are all bounded, then G is roughly equivalent to G'.*

▷ **Exercise 2.18.**

Prove Proposition 2.17.

We can also use rough isometries and Theorem 2.16 to give a very simple proof of Pólya's theorem. First, consider simple random walk in one dimension. The probability of return to the origin after $2n$ steps is exactly $\binom{2n}{n} 2^{-2n}$. Stirling's formula shows that this is asymptotic to $1/\sqrt{\pi n}$. Since this series is not summable, the random walk is recurrent. If we consider random walk in d dimensions where each coordinate is independent of the other coordinates and does simple random walk in 1 dimension, then the return probability after $2n$ steps is $\left(\binom{2n}{n} 2^{-2n}\right)^d \sim (\pi n)^{-d/2}$. This is summable precisely when $d \ge 3$. On the other hand, this independent-coordinate walk is simple random walk on another graph whose vertices are a subset of $\mathbb{Z}^d$, and this other graph is clearly roughly isometric to the usual graph. Thus, we deduce Pólya's theorem.

We now consider graphs that are roughly isometric to hyperbolic spaces. Let $\mathbb{H}^d$ denote the standard hyperbolic space of dimension $d \ge 2$; it has scalar curvature -1 everywhere. See Figure 2.3 for one such graph, drawn by a program created by Don Hatch. This drawing uses the Poincaré disc model of $\mathbb{H}^2$, in which the unit disc $\{z \in \mathbb{C}\,;\ |z| < 1\}$ is given the arc-length metric $2|dz|/\big(1 - |z|^2\big)$. The corresponding ball model of $\mathbb{H}^d$ uses

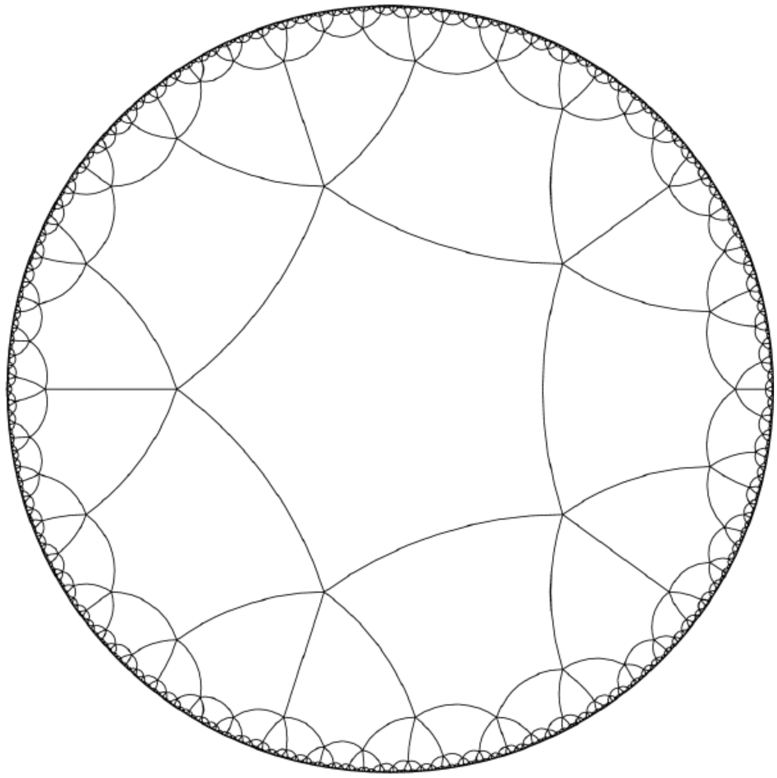


Figure 2.3. A graph in the hyperbolic disc formed from congruent regular hyperbolic pentagons of interior angle $2\pi/5$.

the unit ball $\{x \in \mathbb{R}^d ;\ |x| < 1\}$ with the arc-length metric $2|dx|/\big(1-|x|^2\big)$. For each point a in the ball, there is a hyperbolic isometry that takes a to the origin, namely,

$$x \mapsto a^* + \frac{|a^*|^2 - 1}{|x - a^*|^2}(x - a^*)\,,$$

where $a^* := a/|a|^2$; see, e.g., Matsuzaki and Taniguchi (1998) for the calculation. It follows easily from this that for each point $o \in \mathbb{H}^d$, the sphere of hyperbolic radius r about o has surface area asymptotic to $\alpha e^{r(d-1)}$ for some positive constant α depending on d. Indeed, if $|x| = R$, then the distance between the origin and x is

$$r = \int_0^R \frac{2ds}{1-s^2} = \log\frac{1+R}{1-R}\,,$$

so that

$$R = \frac{e^r - 1}{e^r + 1}\,.$$

The surface area of the sphere centered at the origin is therefore

$$\int_{|x|=R} \frac{2^{d-1} dS}{(1-|x|^2)^{d-1}}\,,$$

where dS is the element of surface area in $\mathbb{R}^d$. Integrating gives the result

$$C\left(\frac{R}{1-R^2}\right)^{d-1} = C(e^r - e^{-r})^{d-1}$$

for some constant C. Therefore there is a positive constant A such that the following hold for any fixed point $o \in \mathbb{H}^d$:

(1) the volume of the shell of points whose distance from o is between r and $r+1$ is at most $Ae^{r(d-1)}$;

(2) the solid angle subtended at o by a spherical cap of area δ on the sphere centered at o of radius r is at most $A\delta e^{-r(d-1)}$.

For more background on hyperbolic space, see, e.g., Ratcliffe (2006) or Benedetti and Petronio (1992). Graphs that are roughly isometric to $\mathbb{H}^d$ often arise as Cayley graphs of groups (see Section 3.4) or, more generally, as nets. A graph G is called an ***ϵ-net*** of a metric space M if the vertices of G form a maximal ϵ-separated subset of M and edges join distinct vertices iff their distance in M is at most 3ϵ.

Theorem 2.18. (Transience of Hyperbolic Space) *If G is roughly isometric to a hyperbolic space $\mathbb{H}^d$, then simple random walk on G is transient.*

Proof. By Theorem 2.16, given $d \ge 2$, it suffices to show transience for one such G. Let G be a 1-net of $\mathbb{H}^d$. Let L be a random uniformly distributed geodesic ray from some point $o \in G$ to ∞. Let $\mathcal{P}(L)$ be a simple path in G from o to ∞ that stays within distance 1 of L; choose $\mathcal{P}(L)$ measurably. (By choice of G, for all $p \in L$, there is a vertex $x \in G$ within distance 1 of p.) Define the flow θ from the law of $\mathcal{P}(L)$ via (2.16). Then θ is a unit flow from o to ∞; we claim it has finite energy. There is some constant C such that if e is an edge whose midpoint is at hyperbolic distance r from o, then $\mathbf{P}[e \in \mathcal{P}(L)] \le Ce^{-r(d-1)}$. Since all edge centers are separated from each other by hyperbolic distance at least 1, there is also a constant D such that there are at most $De^{n(d-1)}$ edge centers whose distance from the origin is between n and $n+1$. It follows that the energy of θ is at most $\sum_n C^2 De^{-2n(d-1)}e^{n(d-1)}$, which is finite. Now transience follows from Theorem 2.10. ◀

§2.6. Hitting and Cover Times.

How can we calculate the expected time it takes for a random walk to hit some set of vertices? The following answer is due to Tetali (1991).

Proposition 2.19. (Hitting-Time Identity) *Given a finite network with a vertex a and a disjoint subset of vertices Z, let $v(\bullet)$ be the voltage when a unit current flows from a to Z. We have $\mathbf{E}_a[\tau_Z] = \sum_{x \in \mathsf{V}} \pi(x)v(x)$.*

Proof. By Proposition 2.1, we have

$$\mathbf{E}_a[\tau_Z] = \sum_x \mathcal{G}(a, x) = \sum_x \pi(x)v(x)\,. \qquad \blacktriangleleft$$

The expected time for a random walk started at a to visit z and then return to a, i.e., $\mathbf{E}_a[\tau_z] + \mathbf{E}_z[\tau_a]$, is called the ***commute time*** between a and z. This turns out to have a particularly pleasant expression, as shown by Chandra, Raghavan, Ruzzo, Smolensky, and Tiwari (1996/97):

Corollary 2.20. (Commute-Time Identity) *Let G be a finite network and $\gamma := \sum_{e \in \mathsf{E}_{1/2}} c(e)$. Let a and z be two vertices of G. The commute time between a and z is $2\gamma\mathscr{R}(a \leftrightarrow z)$.*

Proof. The time $\mathbf{E}_a[\tau_z]$ is expressed via Proposition 2.19 using voltages $v(x)$. Now the voltage at x for a unit-current flow from z to a is equal to $v(a) - v(x) = \mathscr{R}(a \leftrightarrow z) - v(x)$. Thus, we may express $\mathbf{E}_z[\tau_a]$ similarly. Adding these, we get the desired formula. ◀

The ***cover time*** Cov of a finite Markov chain is the first time the process visits all states, i.e.,

$$\mathsf{Cov} := \min\{t\,;\ \forall x \in \mathsf{V} \quad \exists j \le t \quad Y_j = x\}\,.$$

For the complete graph, this is studied in the coupon-collector problem. It turns out that it can be estimated by hitting times (even without reversibility), as shown by Matthews (1988).

Theorem 2.21. (Cover-Time Upper Bound) *Given an irreducible finite Markov chain whose state space V has size n, we have*

$$\mathbf{E}[\mathsf{Cov}] \le \left(\max_{a,b \in \mathsf{V}} \mathbf{E}_a \tau_b\right)\left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right)\,.$$

Proof. It takes no time to visit the starting state, so order all of V except for the starting state according to a random permutation, $\langle j_1, \ldots, j_{n-1}\rangle$. Let t_k be the first time by which all $\{j_1, \ldots, j_k\}$ were visited, and let $L_k := Y_{t_k}$ be the state the chain is in at time t_k. In other words, L_k is the last state visited among the states $\{j_1, \ldots, j_k\}$. In particular, $\mathbf{P}[L_k = j_k] = 1/k$. Then

$$\mathbf{E}[t_k - t_{k-1} \mid Y_0, \ldots, Y_{t_{k-1}}, j_1, \ldots, j_k] = \mathbf{E}_{L_{k-1}}[\tau_{j_k} \mid j_1, \ldots, j_k] \mathbf{1}_{\{L_k = j_k\}} .$$

Taking unconditional expectations, we conclude that

$$\mathbf{E}[t_k - t_{k-1}] \le \left(\max_{a,b \in \mathsf{V}} \mathbf{E}_a \tau_b \right) \frac{1}{k} ,$$

and summing over k yields the result. ◀

The same technique can lead to computing lower bounds as well. For $A \subseteq \{1, \ldots, n\}$, consider the cover time of A, denoted by Cov_A. Clearly $\mathbf{E}[\mathsf{Cov}] \ge \mathbf{E}[\mathsf{Cov}_A]$. Let $t_{\min}^A := \min_{a,b \in A, a \ne b} \mathbf{E}_a \tau_b$. Then similarly to the last proof, we have

$$\mathbf{E}[\mathsf{Cov}_A] \ge t_{\min}^A \left(1 + \frac{1}{2} + \cdots + \frac{1}{|A| - 1} \right) ,$$

which gives the following result of Matthews (1988):

Theorem 2.22. (Cover-Time Lower Bound) *For any irreducible finite Markov chain on a state space* V,

$$\mathbf{E}[\mathsf{Cov}] \ge \max_{A \subseteq \mathsf{V}} \quad t_{\min}^A \left(1 + \frac{1}{2} + \cdots + \frac{1}{|A| - 1} \right) .$$

▷ **Exercise 2.19.**

Prove Theorem 2.22.

§2.7. The Gaussian Network.

This section concerns a model known variously as the ***Gaussian network***, the ***discrete Gaussian free (or massless) field***, ***random network potentials***, or in some contexts, the ***harmonic crystal***. More information on it is in Exercises 2.96, 2.97, 4.17, 10.24, and 10.25. Given a network, define the ***gradient*** of a function f on V to be the antisymmetric function

$$\nabla f := c\,df$$

on E. Thinking of resistance of an edge as its length makes this a natural name.

Suppose we want to measure the altitudes at a finite set of locations V. Assume we know the altitude at some location $o \in \mathsf{V}$. We find the other altitudes by measuring the differences in altitudes between certain pairs E of them. However, each measurement X has an error that is normally distributed. To be precise, if $X(e)$ is a measurement of the difference in the altitudes from $x \in \mathsf{V}$ to $y \in \mathsf{V}$, then $X(e) \sim \mathrm{N}\big(\alpha(x) - \alpha(y), \sigma_e^2\big)$, where $\alpha(x)$ is the true altitude at x and the variances σ_e^2 are assumed known. We assume all measurements are independent. Let $G = (\mathsf{V}, \mathsf{E})$ be the multigraph associated to the measurements. (There are multiple edges between vertices when multiple measurements are made of the same difference, but there are no loops.) Assume G is connected. Make this into a network by assigning the resistances $r(e) := \sigma_e^2$. The ***maximum likelihood estimate*** of the altitudes given these measurements is the function $\widehat{\alpha} : \mathsf{V} \to \mathbb{R}$ with $\widehat{\alpha}(o) = \alpha(o)$ that maximizes the ***likelihood***

$$\frac{1}{\prod_{e \in \mathsf{E}_{1/2}} \sqrt{2\pi r(e)}} \exp\Big\{ -\frac{1}{2} \sum_{e \in \mathsf{E}_{1/2}} \big(X(e) - (d\widehat{\alpha})(e)\big)^2 / r(e) \Big\},$$

which is what the joint density would be at the observed values X if the true altitudes were $\widehat{\alpha}$. The random variables $\widehat{\alpha}$ form the ***Gaussian network***.

One of the striking properties of the Gaussian network is that

$$\mathrm{Var}\big(\widehat{\alpha}(x) - \widehat{\alpha}(y)\big) = \mathscr{R}(x \leftrightarrow y) \tag{2.20}$$

for all $x, y \in \mathsf{V}$. Proving this will be relatively easy once we calculate the joint distribution of $d\widehat{\alpha}$, which we proceed to do.

Now maximizing the likelihood is the same as minimizing the sum of squares in the exponent, so that $\widehat{\alpha}$ is the β with $\beta(o) = \alpha(o)$ that minimizes $\|X/r - \nabla\beta\|_r$. Since $\nabla \mathbf{1}_{\{x\}}$ is the star at $x \in \mathsf{V}$, it follows that the set of all $\nabla\beta$ equals $\bigstar$, whence we are looking for the element of $\bigstar$ closest to X/r. Such a minimization is achieved through orthogonal projection, so that $\nabla\widehat{\alpha} = P_\bigstar(X/r)$. Since $X/r = \sum_{e \in \mathsf{E}_{1/2}} \chi^e X(e)/r(e)$, applying $P_\bigstar$ to both sides yields $\nabla\widehat{\alpha} = \sum_{e \in \mathsf{E}_{1/2}} i^e X(e)/r(e)$. In particular, the random variables $\nabla\widehat{\alpha}$ are linear combinations of independent normal random variables, so themselves are jointly normal. This explains the name "Gaussian".

Let's calculate the joint distribution of $\nabla\widehat{\alpha}(e)$ $(e \in \mathsf{E}_{1/2})$. Being joint normal, this is determined by the means and covariances. Since $\nabla\alpha \in \bigstar$, linearity of expectation gives $\mathbf{E}[\nabla\widehat{\alpha}] = \mathbf{E}\big[P_\bigstar(X/r)\big] = P_\bigstar\big(\mathbf{E}[X]/r\big) = P_\bigstar(\nabla\alpha) = \nabla\alpha$. Now, in the orthonormal basis $\langle \chi^e/\sqrt{r(e)}\rangle$, the vector X/r has coordinates $X(e)/\sqrt{r(e)}$. Write $[Z]$ for the matrix of the vector or linear map Z in this basis. To calculate the covariance, since $\nabla\alpha \in \bigstar$, we may simplify our notation by assuming that $\alpha = 0$. Then the covariance matrix of $[\nabla\widehat{\alpha}]$ is

$$\begin{aligned}\mathbf{E}\big[[\nabla\widehat{\alpha}][\nabla\widehat{\alpha}]^T\big] &= \mathbf{E}\big[[P_\bigstar][X/r][X/r]^T[P_\bigstar]^T\big] = [P_\bigstar]\mathbf{E}\big[[X/r][X/r]^T\big][P_\bigstar] \\ &= [P_\bigstar][P_\bigstar] = [P_\bigstar]\,,\end{aligned}$$

whose entry at (e, f) is $Y(e, f)\sqrt{r(f)/r(e)}$ by (2.11) and (2.12), where Y is the transfer current matrix. This means that $\mathrm{Cov}\big(\nabla\widehat{\alpha}(e), \nabla\widehat{\alpha}(f)\big) = Y(e, f)/r(e)$.

We can also write these facts as $d\widehat{\alpha}$ being jointly normal with $\mathbf{E}\big[d\widehat{\alpha}(e)\big] = d\alpha(e)$ and $\mathrm{Cov}\big(d\widehat{\alpha}(e), d\widehat{\alpha}(f)\big) = Y(e, f)r(f)$. This is the same as the voltage difference across f when unit current flows from e^- to e^+. (This matrix is called the ***transfer impedance matrix***.)

One can also say that $d\widehat{\alpha}$ has the distribution of X conditioned to sum to zero along every cycle; see Janson (1997), Section 9.4.

▷ **Exercise 2.20.**

Let $\widehat{\alpha}$ be the Gaussian network above.

(a) Show that the random variables $\widehat{\alpha}$ are jointly normal with

$$\widehat{\alpha}(x) - \widehat{\alpha}(y) \sim \mathrm{N}\big(\alpha(x) - \alpha(y), \mathscr{R}(x \leftrightarrow y)\big)$$

for $x \neq y \in \mathsf{V}$.

(b) Show that the covariance of $\widehat{\alpha}(x) - \widehat{\alpha}(y)$ and $\widehat{\alpha}(z) - \widehat{\alpha}(w)$ equals $v(z) - v(w)$ when v is the voltage associated to a unit current flow from x to y (with $x \neq y$).

(c) Show that the joint density of the random variables $\langle \widehat{\alpha}(x)\,;\ x \neq o\rangle$ is

$$C \exp\Big\{-\frac{1}{2}\big\|d(\widehat{\alpha} - \alpha)\big\|_c^2\Big\}$$

for some constant C.

(d) Show that if b_x are constants such that $\sum_{x \in \mathsf{V}} b_x \widehat{\alpha}(x)$ is constant a.s., then $b_x = 0$ for all $x \neq o$.

§2.8. Notes.

The continuous classical analogue of harmonic functions, the Dirichlet problem, and its solution via Brownian motion are as follows. Let D be an open subset of $\mathbb{R}^d$. If $f : D \to \mathbb{R}$ is Lebesgue integrable on each ball contained in D and for all x in D, $f(x)$ is equal to the average value of f on each ball in D centered at x, then f is called ***harmonic*** in D. If f is locally bounded, then this is equivalent to $f(x)$ being equal to its average on each sphere in D centered at x. Harmonic functions are always infinitely differentiable and satisfy $\Delta f = 0$. Conversely, if f has two continuous partial deriviatives and $\Delta f = 0$ in D, then f is harmonic in D. If D is bounded and connected and f is harmonic on D and continuous on its closure, $\overline{D}$, then $\max_{\overline{D}} f = \max_{\partial D} f$. The Dirichlet problem is the following. Given a bounded connected open set D and a continuous function f on ∂D, is there a continuous extension of f to $\overline{D}$ that is harmonic in D? The answer is yes when D satisfies certain regularity conditions and the solution can be given via Brownian motion X_t in D as $f(x) := \mathbf{E}_x[f(X_\tau)]$, where $\tau := \inf\{t \ge 0 ;\ X_t \notin D\}$. See, e.g., Bass (1995), pp. 83–90 for details.

Brownian motion in $\mathbb{R}^d$ is analogous to simple random walk in $\mathbb{Z}^d$. The electrical analogue to a discrete network is a uniformly conducting material. There is a similar relationship to an electrical framework for reversible diffusions, even on Riemannian manifolds: Let M be a complete Riemannian manifold. Given a function $\sigma(x)$ which is Borel-measurable, locally bounded and locally bounded below, called the ***(scalar) conductivity***, we associate the diffusion whose generator is $\left(2\sigma(x)\sqrt{g(x)}\right)^{-1} \sum \partial_i \sigma(x)\sqrt{g(x)} g^{ij}(x)\partial_j$ in coordinates, where the metric is g_{ij} with inverse g^{ij} and determinant g. In coordinate-free notation, this is $(1/2)\Delta + (1/2)\nabla \log \sigma$. In other words, the diffusion is Brownian motion with drift of half the gradient of the log of the conductivity. The main result of Ichihara (1978) [see also the exposition by Durrett (1986), p. 75; Fukushima (1980), Theorem 1.5.1, and Fukushima (1985); or Grigor'yan (1985)] gives the following test for transience, an analogue to Exercise 2.73.

Theorem 2.23. *On a complete Riemannian manifold, the diffusion corresponding to the scalar conductivity $\sigma(x)$ is transient iff*

$$\inf\left\{\int |\nabla u(x)|^2 \sigma(x)\, dx \,;\ u \in C_0^\infty(M),\ u \restriction B_1(0) \equiv 1\right\} > 0\,,$$

where dx is the volume form.

Recall that a graph G is called an ϵ-net of M if the vertices of G form a maximal ϵ-separated subset of M and edges join distinct vertices iff their distance in M is at most 3ϵ. When a conductivity σ is given on M, we assign conductances c to the edges of G by

$$c(u, w) := \int_{B_\epsilon(u)} \sigma(x)\, dx + \int_{B_\epsilon(w)} \sigma(x)\, dx\,.$$

An evident modification of the proof of Theorem 2 of Kanai (1986) shows the following analogue to Theorem 2.16. A manifold M is said to have ***bounded geometry*** if its Ricci curvature is bounded below and the injectivity radius is positive. If the Ricci curvature is bounded below, then nets have bounded degree (Kanai (1985), Lemma 2.3). We shall say that σ is ***ϵ-slowly varying*** if

$$\sup\{\sigma(x)/\sigma(y) \,;\ \mathrm{dist}(x, y) \le \epsilon\} < \infty\,.$$

Theorem 2.24. *Suppose that M is a complete Riemannian manifold of bounded geometry, that ϵ is at most half the injectivity radius of M, that σ is an ϵ-slowly varying Borel-measurable conductivity on M, and that G is an ϵ-net in M. Then the associated diffusion on M is transient iff the associated random walk on G is transient.*

The transformations of a network described in Section 2.3 can be used for several other purposes as well. As we will see (Chapter 4), spanning trees are intimately connected to electrical networks, so it will not be surprising that such network reductions can be used to count the number of spanning trees of a graph. See Colbourn, Provan, and Vertigan (1995) for this, as well as for applications to the Ising model and perfect matchings (also known as domino tilings). For a connection to knot theory, see Goldman and Kauffman (1993).

The statistical model of Section 2.7 and its connection to random walks (which is equivalent to electrical networks) is due to Borre and Meissl (1974); see Tjur (1991). The maximum likelihood estimate is also the best linear unbiased estimate, a general fact about linear Gaussian models. See Constantine (2003) for some additional information on unbiased estimates in this model. This model is similar to Dynkin's isomorphism, which is for continuous-time Markov processes; see Dynkin (1980). A different connection to Gaussian fields, obtained by using the network Laplacian (defined in Exercise 2.53) as covariance matrix, is due to Diaconis and Evans (2002). Scaling limits of Gaussian networks and other models sometimes give what's known as the (continuum) Gaussian free field; e.g., see Kenyon (2001, 2008), Rider and Virág (2007), and Sheffield (2007). The model was also introduced in molecular biology by Bahar, Atilgan, and Erman (1997), where it facilitates very useful computation. Here, the edges are regarded as springs. We remark that because the correlations are positive, the random variables $U(x)$ of Exercise 2.97 are positively associated (see Section 7.2 for the definition) by the main result of Pitt (1982); see Joag-Dev, Perlman, and Pitt (1983) for a simpler proof of this implication for normal random variables.

§2.9. Collected In-Text Exercises.

2.1. (Reversible Markov Chains) This exercise contains some background information and facts that we will use about reversible Markov chains.

(a) Show that if a Markov chain is reversible, then $\forall x_1, x_2, \ldots, x_n$,

$$\pi(x_1) \prod_{i=1}^{n-1} p_{x_i x_{i+1}} = \pi(x_n) \prod_{i=1}^{n-1} p_{x_{n+1-i} x_{n-i}} ,$$

whence $\prod_{i=1}^{n-1} p_{x_i x_{i+1}} = \prod_{i=1}^{n-1} p_{x_{n+1-i} x_{n-i}}$ if $x_1 = x_n$. This last equation also characterizes reversibility.

(b) Let $\langle X_n \rangle$ be a random walk on G and let x and y be two vertices in G. Let $\mathcal{P}$ be a path from x to y and $\mathcal{P}'$ its reversal, a path from y to x. Show that

$$\mathbf{P}_x[\langle X_n \,;\, n \le \tau_y \rangle = \mathcal{P} \mid \tau_y < \tau_x^+] = \mathbf{P}_y[\langle X_n \,;\, n \le \tau_x \rangle = \mathcal{P}' \mid \tau_x < \tau_y^+] ,$$

where τ_w denotes the first time the random walk visits w, τ_w^+ denotes the first time after 0 that the random walk visits w, and $\mathbf{P}_u$ denotes the law of random walk started at u. In words, paths between two states that don't return to the starting point and stop at the first visit to the endpoint have the same distribution in both directions of time.

(c) Consider a random walk on G that is either transient or is stopped on the first visit to a set of vertices Z. Let $\mathcal{G}(x, y)$ be the expected number of visits to y for a random walk started at x; if the walk is stopped at Z, we count only those visits that occur strictly before visiting Z. Show that for every pair of vertices x and y,

$$\pi(x)\mathcal{G}(x, y) = \pi(y)\mathcal{G}(y, x) .$$

(d) Show that random walk on a connected weighted graph G is positive recurrent (i.e., has a stationary probability distribution) iff $\sum_{x,y} c(x,y) < \infty$, in which case the stationary probability distribution is proportional to $\pi(\bullet)$. Show that if the random walk is not positive recurrent, then $\pi(\bullet)$ is a stationary infinite measure.

2.2. Suppose that an antisymmetric function j (meaning that $j(x,y) = -j(y,x)$) on the edges of a finite connected network satisfies Kirchhoff's cycle law and Kirchhoff's node law in the form $\sum_{x \sim y} j(x,y) = 0$ for all $x \in W$. Show that j is the current determined by imposing voltages off W and that the voltage function is unique up to an additive constant.

2.3. Verify that Propositions 2.1 and 2.2 are valid when the number of edges is infinite but the number of vertices is finite.

2.4. For a fixed vertex a in G, show that $\lim_n \mathscr{C}(a \leftrightarrow Z_n)$ is the same for every sequence $\langle G_n \rangle$ that exhausts G.

2.5. When G is finite but A is not a singleton, define $\mathscr{C}(A \leftrightarrow Z)$ to be $\mathscr{C}(a \leftrightarrow Z)$ if all the vertices in A were to be identified to a single vertex, a. Show that if voltages are applied at the vertices of $A \cup Z$ so that $v{\restriction}A$ and $v{\restriction}Z$ are constants, then $v{\restriction}A - v{\restriction}Z = \mathcal{I}_{AZ}\mathscr{R}(A \leftrightarrow Z)$, where $\mathcal{I}_{AZ} := \sum_{x \in A} \sum_y i(x,y)$ is the total amount of current flowing from A to Z.

2.6. Give two harder but instructive proofs of the series equivalence: Since voltages determine currents, it suffices to check that the voltages are as claimed on the new network G'. (1) Show that $v(x)$ ($x \in \mathsf{V}(G) \setminus \{w\}$) is harmonic on $\mathsf{V}(G') \setminus (A \cup Z)$. (2) Use the "craps principle" (Pitman (1993), p. 210) to show that $\mathbf{P}_x[\tau_A < \tau_Z]$ is unchanged for $x \in \mathsf{V}(G) \setminus \{w\}$.

2.7. Give two more proofs of the parallel equivalence as in Exercise 2.6.

2.8. Let (G,c) be a network. A ***network automorphism*** of (G,c) is a map $\phi : G \to G$ that is a bijection of the vertex set with itself and a bijection of the edge set with itself such that if x and e are incident, then so are $\phi(x)$ and $\phi(e)$ and such that $c(e) = c(\phi(e))$ for all edges e. Suppose that (G,c) is ***spherically symmetric*** about o, meaning that if x and y are any two vertices at the same distance from o, then there is an automorphism of (G,c) that leaves o fixed and that takes x to y. Let C_n be the sum of $c(e)$ over all edges e with $d(e^-, o) = n$ and $d(e^+, o) = n+1$. Show that

$$\mathscr{R}(o \leftrightarrow \infty) = \sum_n \frac{1}{C_n},$$

whence the network random walk on G is transient iff

$$\sum_n \frac{1}{C_n} < \infty.$$

2.9. Give at least one proof of star-triangle equivalence.

2.10. Find a (finite) graph that can't be reduced to a single edge by the four transformations pruning, series, parallel, and star-triangle.

2.11. Prove that d and d^* are adjoints of each other.

2.12. Show that $\mathbf{P}[a \to Z]$ can increase in some situations and decrease in others when an edge incident to a is removed.

2.13. Given disjoint vertex sets A, Z in a finite network, we may express the effective resistance between A and Z by Thomson's Principle as

$$\mathscr{R}(A \leftrightarrow Z) = \min \left\{ \sum_{e \in \mathsf{E}_{1/2}} r(e)\theta(e)^2 \,;\ \theta \text{ is a unit flow from } A \text{ to } Z \right\}.$$

Prove the following dual expression for the effective conductance, known as ***Dirichlet's principle***:

$$\mathscr{C}(A \leftrightarrow Z) = \min \left\{ \sum_{e \in \mathsf{E}_{1/2}} c(e) dF(e)^2 \right\},$$

where F is a function that is 1 on A and 0 on Z.

2.14. Let $G = (\mathsf{V}, \mathsf{E})$ be denumerable and $\theta_n \in \ell^2_-(\mathsf{E}, r)$ be such that $\mathscr{E}(\theta_n) \le M < \infty$ and $\forall e \in \mathsf{E}\ \theta_n(e) \to \theta(e)$. Show that θ is antisymmetric, $\mathscr{E}(\theta) \le \liminf_n \mathscr{E}(\theta_n) \le M$, and $\forall x \in \mathsf{V}\ \ d^*\theta_n(x) \to d^*\theta(x)$.

2.15. Show that (2.18) is also necessary for recurrence if $f(n+1) \le f(n) + 1$ for all n.

2.16. Show that if we remove each edge $[(x, y, z), (x, y, z + 1)]$ with $x + y$ odd in $\mathbb{Z}^3$, then we obtain a transient graph with effective resistance to infinity at most 6 times what it was before removal.

2.17. Show that being roughly isometric is an equivalence relation.

2.18. Prove Proposition 2.17.

2.19. Prove Theorem 2.22.

2.20. Let $\widehat{\alpha}$ be the Gaussian network above.

(a) Show that the random variables $\widehat{\alpha}$ are jointly normal with

$$\widehat{\alpha}(x) - \widehat{\alpha}(y) \sim \mathrm{N}(\alpha(x) - \alpha(y), \mathscr{R}(x \leftrightarrow y))$$

for $x \ne y \in \mathsf{V}$.

(b) Show that the covariance of $\widehat{\alpha}(x) - \widehat{\alpha}(y)$ and $\widehat{\alpha}(z) - \widehat{\alpha}(w)$ equals $v(z) - v(w)$ when v is the voltage associated to a unit current flow from x to y (with $x \ne y$).

(c) Show that the joint density of the random variables $\langle \widehat{\alpha}(x)\,;\ x \ne o \rangle$ is

$$C \exp \left\{ -\frac{1}{2} \| d(\widehat{\alpha} - \alpha) \|_c^2 \right\}$$

for some constant C.

(d) Show that if b_x are constants such that $\sum_{x \in \mathsf{V}} b_x \widehat{\alpha}(x)$ is constant a.s., then $b_x = 0$ for all $x \ne o$.

§2.10. Additional Exercises.

In all the exercises, assume the networks are connected.

2.21. Let G be a transient network and $x, y \in \mathsf{V}$. Show that

$$\pi(x)\mathbf{P}_x[\tau_y < \infty]\mathcal{G}(y,y) = \pi(y)\mathbf{P}_y[\tau_x < \infty]\mathcal{G}(x,x)\,.$$

2.22. Let G be a transient network and $f : \mathsf{V} \to \mathbb{R}$ satisfy $\sum_y \mathcal{G}(x,y)|f(y)| < \infty$ for every x. Define $(\mathcal{G}f)(x) := \sum_y \mathcal{G}(x,y)f(y)$. Let I be the identity operator and P be the ***transition operator*** (i.e., $(Pg)(x) := \sum_y p(x,y)g(y)$). Show that $(I-P)(\mathcal{G}f) = f$.

2.23. A function f on the vertices of a network is called ***subharmonic*** at x if

$$f(x) \le \pi(x)^{-1} \sum_{y \sim x} c(x,y)f(y)$$

and ***superharmonic*** if the opposite inequality holds. Show that the Maximum Principle extends to subharmonic functions and that there is a corresponding Minimum Principle for superharmonic functions.

2.24. Let G be transient and u be a nonnegative superharmonic function. Show that there exist unique functions f and h such that $f \ge 0$, h is harmonic, and $u = \mathcal{G}f + h$, where $\mathcal{G}f$ is defined in Exercise 2.22. Show that also $f = (I-P)u$, $h \ge 0$, and $h \ge g$ whenever $g \le u$ is harmonic, where I and P are defined in Exercise 2.22. *Hint:* Define $h := \lim_n P^n u$ and $f := (I-P)u$.

2.25. Give another proof of the Existence Principle along the following lines. Given f_0 off W, let $f(x) := \inf g(x)$ over all functions g that are superharmonic on W, that agree with f_0 off W, and such that $g \ge \inf f_0$. (See Exercise 2.23 for the definition of "superharmonic".) Then f is harmonic and agrees with f_0 off W.

2.26. Given a finite graph G and two of its vertices a and z, let $i_c(\bullet)$ be the unit current flow from a to z when conductances $c(\bullet)$ are assigned to the edges. Show that i_c is a continuous function of $c(\bullet)$.

2.27. Let G be a network. If $h : \mathsf{V} \to (0,\infty)$ is harmonic at every vertex of $W \subseteq \mathsf{V}$, there is another Markov chain associated to h called ***Doob's h-transform***; its transition probabilities are defined to be $p^h_{xy} := p_{xy}h(y)/h(x)$ and it is stopped when it reaches a vertex outside W. Check that these are indeed transition probabilities for a reversible Markov chain. Find corresponding conductances.

2.28. Let G be a finite network and $W \subsetneq \mathsf{V}$. At every visit to a vertex $x \in W$, a random walker collects a payment of $g(x)$. When reaching a vertex $y \notin W$, the walker receives a final retirement package of $h(y)$ and stops moving. Let $f(x)$ denote the expected total payment the walker receives starting from x.

(a) Write a set of linear equations that the values $f(x)$ for $x \in W$, must satisfy (one equation for each such vertex x).

(b) Uniqueness: Show that these equations specify f.

(c) Existence: Without using the probabilistic interpretation, prove there is a solution to this set of equations.

(d) Let i be the current associated to the voltage function f, that is, $i(x,y) := c(x,y)[f(x)-f(y)]$. (Batteries are connected to each vertex $x \in W$ where $g(x) \neq 0$, as well as to all vertices not in W.) Show that the amount of current flowing into the network at x, i.e., $\sum_y i(x,y)$, equals $\pi(x)g(x)$ for $x \in W$. Thus, currents can be specified by giving voltages h on one set of vertices and giving flow amounts $\pi(x)g(x)$ on the complementary set of vertices.

2.29. If a battery is connected between a and z of a finite network, must the voltages of the vertices be monotone along every shortest path between a and z?

2.30. Let A and Z be two sets of vertices in a finite network. Show that for any vertex $x \notin A \cup Z$, we have

$$\mathbf{P}_x[\tau_A < \tau_Z] \le \frac{\mathscr{C}(x \leftrightarrow A)}{\mathscr{C}(x \leftrightarrow A \cup Z)} .$$

2.31. Show that on every finite network $|\mathbf{E}[S_{xy}] - \mathbf{E}[S_{yx}]| \le 1$ for all x, y, where S_{xy} is defined as in Proposition 2.2.

2.32. When a voltage is imposed so that a unit current flows from a to Z in a finite network and $v \restriction Z \equiv 0$, show that the expected total number of times an edge $[x, y]$ is crossed by a random walk starting at a and absorbed at Z equals $c(x, y)[v(x) + v(y)]$.

2.33. Let G be a network such that $\gamma := \sum_{e \in \mathsf{E}_{1/2}} c(e) < \infty$ (for example, G could be finite). For every vertex $a \in G$, show that the expected time for a random walk started at a to return to a is $2\gamma/\pi(a)$.

2.34. Let $\langle X_n \rangle$ be a positive recurrent irreducible Markov chain with stationary probability measure $\pi(\bullet)$, not necessarily reversible. Let $\tau > 0$ be a stopping time such that $\mathbf{P}_a[X_\tau = a] = 1$ for some state a. Let $\mathcal{G}_\tau(x, y) := \mathbf{E}_x\Big[\sum_{0 \le n < \tau} \mathbf{1}_{\{X_n = y\}}\Big]$. Show that for all states x, we have $\pi(x) = \mathcal{G}_\tau(a, x)/\mathbf{E}_a[\tau]$. In particular, $\mathbf{E}_a[\tau_a^+] = 1/\pi(a)$. Give another proof of the formula of Exercise 2.33 from this.

2.35. Let G be a network such that $\sum_{e \in \mathsf{E}_{1/2}} c(e) = \infty$. For every vertex $a \in G$, show that the expected time for a random walk started at a to return to a is ∞.

2.36. Let G be a finite network and A and Z be two disjoint subsets of vertices in G. Show that

$$\mathscr{C}(A \leftrightarrow Z) = \sum_{x \in A} \pi(x) \mathbf{P}_x[\tau_Z < \tau_A^+] .$$

2.37. Let $\langle X_n \rangle$ be an irreducible Markov chain. A function f on the state space is called ***harmonic*** if $f(x) = \sum p(x, y) f(y)$ for every state x. Suppose that the Markov chain is recurrent.
(a) Show that there are no bounded harmonic functions other than the constants.
(b) Show that there are no nonnegative harmonic functions other than the constants.
(c) Show that the stationary measure is unique up to a multiplicative constant.

2.38. Let $\langle X_n \rangle$ be an irreducible Markov chain. A function f on the state space is called ***superharmonic*** if $f(x) \le \sum p(x, y) f(y)$ for every state x. Show that the Markov chain is recurrent iff every nonnegative superharmonic function is constant.

2.39. Let $\langle X_n \rangle$ be a positive recurrent irreducible Markov chain with stationary probability measure $\pi(\bullet)$, not necessarily reversible. Show that the expected hitting time of a random π-distributed target does not depend on the starting state. That is, show that if

$$f(x) := \sum_y \pi(y) \mathbf{E}_x[\tau_y] ,$$

then $f(x)$ is the same for all x.

2.40. Suppose that a tree T is transient for simple random walk $\langle X_n \rangle$. If we iteratively erase backtracking from the path of the walk, then we obtain a.s. a ray $\xi \in \partial T$ that intersects the path infinitely often. We say that $\langle X_n \rangle$ ***converges*** to ξ. Prove that if $\langle X_n \rangle$ and $\langle X'_n \rangle$ are independent simple random walks on T, then a.s. they converge to distinct rays.

2.41. Let $\langle X_n \rangle$ be a Markov chain and A be a set of states such that $\tau_A < \infty$ a.s. The distribution of X_{τ_A} is called ***harmonic measure*** on A. In the case that $\langle X_n \rangle$ is a random walk on a network $G = (\mathsf{V}, \mathsf{E})$ starting from a vertex z and $A \subseteq \mathsf{V}$ is finite, let μ be harmonic measure and define

$$\nu(x) := \mathbf{P}_x[\tau_z < \tau_A^+]$$

for $x \in A$.

(a) Show that

$$\mu(x) = \mathbf{P}_z[\tau_x < \tau_{A \setminus \{x\}} \mid \tau_A < \tau_z^+]$$

for $x \in A$.

(b) Show that

$$\mu(x) = \mathscr{R}(A \leftrightarrow z)\pi(x)\nu(x)$$

for all $x \in A$.

(c) Let G be a transient network and $\langle G_n \rangle$ be an exhaustion of G by finite subnetworks. Let G_n^{W} be the network obtained from G by identifying the vertices outside G_n to a single vertex, z_n. Fix a finite set $A \subset \mathsf{V}$. Let μ_n be harmonic measure on A for the network G_n^{W} from z_n. Show that ***wired harmonic measure from infinity*** $\mu := \lim_{n\to\infty} \mu_n$ exists and satisfies

$$\mu(x) = \mathscr{R}(A \leftrightarrow \infty)\pi(x)\mathbf{P}_x[\tau_A^+ = \infty]$$

for all $x \in A$.

2.42. Let G be a finite network with a fixed vertex, a. Fix $s \in (0,1)$. Add a new vertex, Δ, which is joined to each vertex x with an edge of conductance $w(x)$ chosen so that at x, the probability of taking a step to Δ is equal to $1 - s$. Call the new network G'. Prove that

$$\sum_{k \geq 0} p_k s^k = \pi(a)\mathscr{R}(a \leftrightarrow \Delta; G')/s\,,$$

where p_k is the probability that the network random walk on G (not on G') starting from a is back at a at time k. The series above is the generating function for the return probabilities and is sometimes called the "Green function", despite the other notion of Green function defined in this chapter.

2.43. Let G be a transient network with a fixed vertex, a. Fix $s \in (0,1)$. Add a new vertex, Δ, which is joined to each vertex x with an edge of conductance $w(x)$ chosen so that at x, the probability of taking a step to Δ is equal to s. Call the new network G'. Define the effective resistance between a and $\{\Delta, \infty\}$ to be the limit of the effective resistance between a and $G' \setminus G_n$, where G_n is an exhaustion of G (not of G'). Prove that the limit defining the effective resistance between a and $\{\Delta, \infty\}$ exists and

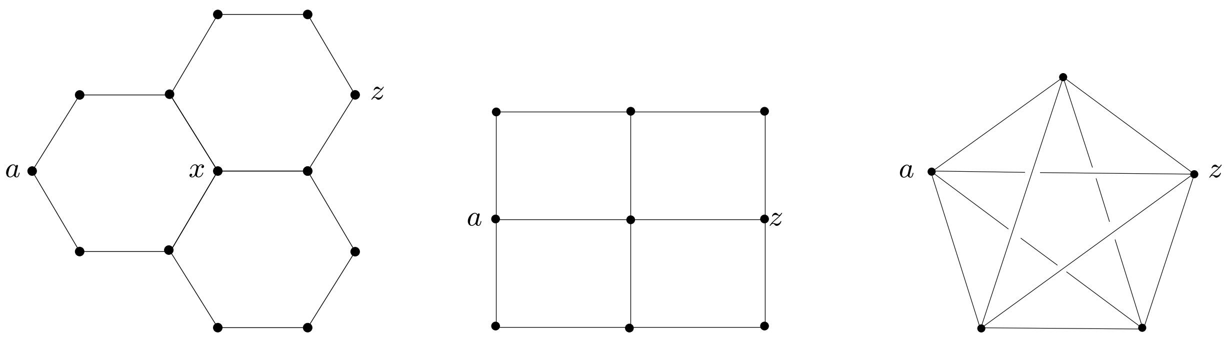
$$\sum_{k \geq 0} p_k s^k = \pi(a)\mathscr{R}(a \leftrightarrow \Delta, \infty; G')/s\,,$$

where p_k is the probability that the network random walk on G (not on G') starting from a is back at a at time k.

2.44. Give an example of two graphs $G_i = (\mathsf{V}, \mathsf{E}_i)$ on the same vertex set ($i = 1, 2$) such that both graphs are connected and recurrent, yet their union $(\mathsf{V}, \mathsf{E}_1 \cup \mathsf{E}_2)$ is transient.

2.45. Consider random walk on $\mathbb{N}$ that steps $+1$ with probability $3/4$ and -1 with probability $1/4$ unless the walker is at a multiple of 3, in which case the transition probabilities are $1/10$ and $9/10$, respectively. (Of course, at 0, the walker always moves to 1.) Show that the walk is recurrent. On the other hand, show that if before taking a step, a fair coin is tossed and one uses the transition probabilities of this biased walk when the coin shows heads and moves right or left with equal probability when the coin shows tails, then the walk is transient. In this latter case, show that the walk tends to infinity at a positive linear rate.

2.46. In the following networks, each edge has unit conductance. (More such exercises can be found in Doyle and Snell (1984).)



(a) (b) (c)

(a) What are $\mathbf{P}_x[\tau_a < \tau_z]$, $\mathbf{P}_a[\tau_x < \tau_z]$, and $\mathbf{P}_z[\tau_x < \tau_a]$?
(b) What is $\mathscr{C}(a \leftrightarrow z)$? (Or: show a sequence of transformations that could be used to calculate $\mathscr{C}(a \leftrightarrow z)$.)
(c) What is $\mathscr{C}(a \leftrightarrow z)$? (Or: show a sequence of transformations that could be used to calculate $\mathscr{C}(a \leftrightarrow z)$.)

2.47. The star-triangle equivalence can be extended as follows. Suppose that (G, c) and (G', c') are two finite networks with a common subset W of vertices that has the property that for all $x, y \in W$, the effective resistance between x and y is the same in each network. Then call G and G' ***W-equivalent***.
(a) Let G and G' be W-equivalent. Show that specifying voltages on W leads to the same current flows out of W in each of the two networks. More precisely, let $f_0 : W \to \mathbb{R}$ and let f, f' be the extensions of f_0 to G and G', respectively, that are harmonic off W. Show that $d^*cdf = d^*c'df'$ on W.
(b) Let G and G' be W-equivalent. Suppose that H is another network with subset W of vertices, but otherwise disjoint from G and G'. Form two new networks $G \cup H$ and $G' \cup H$ by identifying the copies of W. Show that if the same voltages are established at some vertices of H in each of these two networks, then the same voltages and currents will be present in each of these two copies of H.
(c) Given G and a vertex subset W with $|W| = 3$, show that there is a 4-vertex network G' with underlying graph a tree that is W-equivalent to G.
(d) For $x, y \in W$, let $p_W(x, y)$ be the probability that the network random walk on G starting at x is at y when it first returns to W; possibly $x = y$. Define the network $G' := (W, c')$ with

$c'(x,y) := \pi(x)p_W(x,y)$ for all $x \neq y \in W$. Show that $c'(x,y) = c'(y,x)$ for all $x \neq y \in W$ and that G and G' are W-equivalent. *Hint:* Consider adding loops to G'.

(e) **(The star-clique transformation)** Let $z \in \mathsf{V}(G)$ and N be the set of the neighbors of z. Form the network G' from G by deleting z and adding an edge between each pair of distinct vertices $x, y \in N$ of conductance $c(z,x)c(z,y)/\pi(z)$. Show that G and G' are $\mathsf{V}(G')$-equivalent. Note that when $|N| = 1$, this is the same as pruning vertices of degree 1; when $|N| = 2$, this is the same as the series transformation; and when $|N| = 3$, this is the same as the star-triangle transformation.

2.48. Suppose that G is a finite network and a battery of unit voltage is attached at vertices a and z. Let x and y be two other vertices of G and let G' be the graph obtained by shorting x and y, i.e., identifying them. Show that the voltage at the shorted vertex in G' lies between the original voltages at x and y in G.

2.49. Let W be a set of vertices in a finite network G. Let $j \in \ell^2_-(\mathsf{E})$ satisfy $\sum_{i=1}^n j(e_i)r(e_i) = 0$ whenever $\langle e_1, e_2, \ldots, e_n \rangle$ is a cycle; and $d^*j{\restriction}(\mathsf{V} \setminus W) = 0$. According to Exercise 2.2, the values of $d^*j{\restriction}W$ determine j uniquely. Show that the map $d^*j{\restriction}W \mapsto j$ is linear. This is another form of the superposition principle.

2.50. Let A and Z be subsets of vertices in a finite network. Show that

$$\frac{2}{|A||Z|}\sum_{a\in A, z\in Z}\mathscr{R}(a \leftrightarrow z) \geq \frac{1}{|A|^2}\sum_{a,b\in A}\mathscr{R}(a \leftrightarrow b) + \frac{1}{|Z|^2}\sum_{y,z\in Z}\mathscr{R}(y \leftrightarrow z)\,.$$

2.51. Let G be a finite network and $f : \mathsf{V} \to \mathbb{R}$ satisfy $\sum_x f(x) = 0$. Pick $z \in \mathsf{V}$. Let $\mathscr{G}(\bullet,\bullet)$ be the Green function for the random walk on G absorbed at z. Consider the voltage function $v(x) := \sum_y \mathscr{G}(x,y)f(y)/\pi(y)$. Show that the current $i = c \cdot dv$ satisfies $d^*i = f$ and $\mathscr{E}(i) = \sum_{x,y}\mathscr{G}(x,y)f(x)f(y)/\pi(y)$.

2.52. A ***cut*** in a graph G is a set of edges of the form $\{(x,y)\,;\ x \in A,\ y \notin A\}$ for some proper non-empty vertex set A of G. Show that for every finite network G, the linear span of $\big\{\sum_{e\in\Pi} c(e)\chi^e\,;\ \Pi \text{ is a cut of } G\big\}$ equals the star space.

2.53. Let G be a finite connected network. The ***network Laplacian*** is the $\mathsf{V} \times \mathsf{V}$ matrix Δ_G whose (x,y) entry is $-c(x,y)$ if $x \neq y$ and is $\pi(x)$ if $x = y$. Thus, Δ_G is symmetric and all its row sums are 0. Write $\Delta_G[a]$ for the matrix obtained from Δ_G by deleting the row and column indexed by a. Let $v_a(x,y)$ be the voltage at x when a unit current $i_{y,a}$ flows from y to a (so that the voltage at a is 0) if $y \neq a$ and be 0 otherwise. Prove the following statements:

(a) if $x, y \neq a$, then $v_a(x,y)$ is the (x,y)-entry of $\Delta_G[a]^{-1}$;
(b) $v_a(x,y) = v_a(y,x)$;
(c) $v_a(x,x) = v_x(a,a)$;
(d) for all $a, x, y \in \mathsf{V}$, we have $\mathscr{R}(x \leftrightarrow y) = v_a(x,x) - 2v_a(x,y) + v_a(y,y)$;
(e) if $x, y \neq a$, then $v_a(x,y) = (i_{x,a}, i_{y,a})_r$;
(f) for all $f \in \ell^2(\mathsf{V})$, we have $(f, \Delta_G f) = \|df\|_c^2$.

2.54. Let G be a finite connected network. Show that $\langle \mathscr{R}(x \leftrightarrow y)\,;\ x, y \in \mathsf{V}(G)\rangle$ determines $\langle c(x,y)\,;\ x, y \in \mathsf{V}(G)\rangle$, even if one does not know $\mathsf{E}(G)$.

2.55. Let G be a finite network. Show that if a unit-voltage battery is connected between a and z, then the current flow from a to z is the projection of the star at a on the orthocomplement of the span of all the other stars except that at z.

2.56. Show that R_{eff} is a concave function of the collection of resistances $\langle r(e)\rangle$.

2.57. Show that C_{eff} is a concave function of the collection of conductances $\langle c(e) \rangle$.

2.58. Show that in every finite network, for every three vertices u, x and w, we have

$$\mathscr{R}(u \leftrightarrow x) + \mathscr{R}(x \leftrightarrow w) \ge \mathscr{R}(u \leftrightarrow w) .$$

2.59. Show that in every finite network, for every three vertices a, x, z, we have

$$\mathbf{P}_x[\tau_z < \tau_a] = \frac{\mathscr{R}(a \leftrightarrow x) - \mathscr{R}(x \leftrightarrow z) + \mathscr{R}(a \leftrightarrow z)}{2\mathscr{R}(a \leftrightarrow z)} .$$

2.60. Show the following quantitative forms of Rayleigh's Monotonicity Principle in every finite network:
(a) if $r(e)$ denotes the resistance of the edge e and i is the unit current flow from a to z, then

$$\frac{\partial}{\partial r(e)} \mathscr{R}(a \leftrightarrow z) = i(e)^2 .$$

(b) if $c(e)$ denotes the resistance of the edge e and v is the unit voltage from a to z, then

$$\frac{\partial}{\partial c(e)} \mathscr{C}(a \leftrightarrow z) = \big(dv(e)\big)^2 .$$

2.61. Show that if a unit voltage is imposed between two vertices of a finite network, then for each fixed edge e, we have that $|dv(e)|$ is a decreasing function of $c(e)$.

2.62. Give another proof of Rayleigh's Monotonicity Principle using Exercise 2.13.

2.63. Let G be a recurrent network with an exhaustion $\langle G_n \rangle$. Suppose that $a, z \in \mathsf{V}(G_n)$ for all n. Let v_n be the voltage function on G_n that arises from a unit voltage at a and 0 voltage at z. Let i_n be the unit current flow on G_n from a to z. Let $\mathcal{G}(\bullet, \bullet; G_n)$ be the Green function on G_n for random walk absorbed at z.
(a) Show that $v := \lim_n v_n$ exists pointwise and that $v(x) = \mathbf{P}_x[\tau_a < \tau_z]$ for all $x \in \mathsf{V}$.
(b) Show that $i := \lim_n i_n$ exists pointwise.
(c) Show that $\mathscr{E}(i) dv = ir$.
(d) Show that the effective resistance between a and z in G_n is monotone decreasing with limit $\mathscr{E}(i)$. We define $\mathscr{R}(a \leftrightarrow z) := \mathscr{E}(i)$.
(e) Show that $\mathcal{G}(a, x; G_n) \to \mathcal{G}(a, x) = \mathscr{E}(i)\pi(x)v(x)$ for all $x \in \mathsf{V}$.
(f) Show that $i(e)$ is the expected number of signed crossings of e.

2.64. With all the notation of Exercise 2.63, also let G_n^{W} be the graph obtained from G by identifying the vertices outside G_n to a single vertex, z_n. Let v_n^{W} and i_n^{W} be the associated unit voltage and unit current.
(a) Show that v_n^{W} and i_n^{W} have the same limits v and i as v_n and i_n.
(b) Show that the effective resistance between a and z in G_n^{W} is monotone increasing with limit $\mathscr{E}(i)$.

2.65. Let G be a recurrent network. Define $\bigstar$ to be the closure of the linear span of the stars and $\diamondsuit$ to be the closure of the linear span of the cycles. Show that $\ell^2_-(\mathsf{E}, r) = \bigstar \oplus \diamondsuit$.

2.66. (Extremal length) Given disjoint vertex sets A, Z in a finite network, prove that

$$\mathscr{C}(A \leftrightarrow Z) = \min \left\{ \sum_{e \in \mathsf{E}_{1/2}} c(e)\ell(e)^2 \right\},$$

where ℓ is an assignment of nonnegative lengths so that the minimum distance from every point in A to every point in Z is 1.

2.67. Extend Exercise 2.13 to the full form of Dirichlet's principle in the finite setting: Let $A \subset \mathsf{V}$ and let $F_0 : A \to \mathbb{R}$ be given. Let $F : \mathsf{V} \to \mathbb{R}$ be the extension of F_0 that is harmonic at each vertex not in A. Then F is the unique extension of F_0 that minimizes $\mathscr{E}(c\,dF)$.

2.68. Show that if $\theta \in \ell^2_-(\mathsf{E}, r)$, then $\sum_x \pi(x)^{-1} d^*\theta(x)^2 \le \mathscr{E}(\theta)$.

2.69. It follows from (2.14) that the function $\theta \mapsto \mathscr{E}(\theta)$ is convex. Show the stronger statement that $\theta \mapsto \mathscr{E}(\theta)^{1/2}$ is a convex function. Why is this a stronger inequality?

2.70. Let i be the unit current flow from a to z on a finite network or from a to ∞ on a transient infinite network. Consider the random walk started at a and, if the network is finite, stopped at z. Let S_e be the number of times the edge e is traversed.

(a) Show that $i(e) = \mathbf{E}[S_e - S_{-e} \mid \tau_a^+ < \infty]$.

(b) Show that if $e^- = a$, then $i(e)$ is the probability that e is traversed following the last visit to a.

2.71. Show that the current i of Exercise 2.63 is the unique unit flow from a to z of minimum energy.

2.72. Let G be a transient network and $\langle X_n \rangle$ the corresponding random walk. Show that if v is the unit voltage between a and ∞ (with $v(a) = 1$), then $v(X_n) \to 0$ a.s.

2.73. Show that if G is infinite and A is a finite subset of vertices, then

$$\inf \left\{ \sum_{e \in \mathsf{E}_{1/2}} dF(e)^2 c(e)\,;\ F{\restriction}A \equiv 1 \quad \text{and } F \text{ has finite support} \right\} = \mathscr{C}(A \leftrightarrow \infty)\,.$$

2.74. Suppose that G is a graph with random resistances $R(e)$ having finite means $r(e) := \mathbf{E}[R(e)]$. Show that if (G, r) is transient, then (G, R) is a.s. transient.

2.75. Suppose that G is a graph with random conductances $C(e)$ having finite means $c(e) := \mathbf{E}[C(e)]$. Show that if (G, c) is recurrent, then (G, C) is a.s. recurrent.

2.76. Let (G, c) be a transient network and $o \in \mathsf{V}(G)$. Consider the voltage function v when the voltage is 1 at o and 0 at infinity. For $t \in (0, 1)$, let $A_t := \{x \in \mathsf{V}\,;\ v(x) > t\}$. Normally, A_t is finite. Show that even if A_t is infinite, the subnetwork it induces is recurrent.

2.77. Prove that (2.15) holds even when the network is not locally finite and the cutsets Π_n may be infinite.

2.78. Find a counterexample to the converse of the Nash-Williams Criterion. More specifically, find a tree of bounded degree on which simple random walk is recurrent, yet every sequence $\langle \Pi_n \rangle$ of pairwise disjoint cutsets separating the root and ∞ satisfies $\sum_n |\Pi_n|^{-1} \le C$ for some constant $C < \infty$.

2.79. Show that if $\langle \Pi_n \rangle$ is a sequence of pairwise disjoint cutsets separating the root and ∞ of a tree T without leaves, then $\sum_n |\Pi_n|^{-1} \le \sum_n |T_n|^{-1}$.

2.80. Give a probabilistic proof as follows of the Nash-Williams Criterion in case the cutsets are nested as in Remark 2.12.

(a) Show that it suffices to prove the Criterion for all networks but only for cutsets that consist of all edges incident to some set of vertices.

(b) Let Π_n be the set of edges incident to the set of vertices W_n. Let A_n be the event that a random walk starting at a visits W_n exactly once before returning to a. Let μ_n be the distribution of the first vertex of W_n visited by a random walk started at a. Show that

$$\mathbf{P}[A_n] \ge \mathcal{G}(a,a)^{-1} \sum_{x \in W_n} \pi(x)^{-1} \mu_n(x)^2 \ge \mathcal{G}(a,a)^{-1} \left(\sum_{x \in W_n} \pi(x) \right)^{-1} = \mathcal{G}(a,a)^{-1} \left(\sum_{e \in \Pi_n} c(e) \right)^{-1}.$$

(c) Conclude by Borel-Cantelli.

2.81. Complete the following outline of a calculational proof of Pólya's theorem. Define $\psi(\alpha) := d^{-1} \sum_{k=1}^{d} \cos 2\pi\alpha_k$ for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$. For all $n \in \mathbb{N}$, we have $p_n(\mathbf{0}, \mathbf{0}) = \int_{\mathbb{T}^d} \psi(\alpha)^n \, d\alpha$. Therefore $\sum_n p_n(\mathbf{0}, \mathbf{0}) = \int_{\mathbb{T}^d} 1/(1 - \psi(\alpha)) \, d\alpha < \infty$ iff $d \ge 3$. Use the same method for a random walk on $\mathbb{Z}^d$ that has mean-0 bounded jumps.

2.82. Give another proof of Theorem 2.16 by using Exercise 2.73.

2.83. Let (G, r) and (G', r') be two finite networks. Let $\phi : (G, r) \to (G', r')$ be an (α, β)-rough embedding. Show that for all vertices $x, y \in G$, we have $\mathscr{R}(\phi(x) \leftrightarrow \phi(y)) \le \alpha\beta \mathscr{R}(x \leftrightarrow y)$.

2.84. Show that if G is a graph that is roughly isometric to hyperbolic space, then the number of vertices within distance n of a fixed vertex of G grows exponentially fast in n.

2.85. Show that in every finite network,

$$\mathbf{E}_a[\tau_z] = \frac{1}{2} \sum_{x \in \mathsf{V}} \pi(x) [\mathscr{R}(a \leftrightarrow z) + \mathscr{R}(z \leftrightarrow x) - \mathscr{R}(x \leftrightarrow a)].$$

2.86. Let G be a network. Suppose that x and y are two vertices such that there is an automorphism of G that takes x to y (though it might not take y to x). Show that for every k, we have $\mathbf{P}_x[\tau_y = k] = \mathbf{P}_y[\tau_x = k]$. *Hint:* Show the equality with "$\le k$" in place of "$= k$".

2.87. Let G be a network such that $\gamma := \sum_{e \in \mathsf{E}_{1/2}} c(e) < \infty$ (for example, G could be finite). Let x, y, z be three distinct vertices in G. Write $\tau_{x,y,z}$ for the first time that the network random walk trajectory contains the vertices x, y, z in that order. Show that

$$\mathbf{P}_x[\tau_{y,z,x} \le \tau_x^+] = \mathbf{P}_x[\tau_{z,y,x} \le \tau_x^+] \tag{2.21}$$

and

$$\mathbf{E}_x[\tau_{y,z,x}] = \mathbf{E}_x[\tau_{z,y,x}] = \gamma[\mathscr{R}(x \leftrightarrow y) + \mathscr{R}(y \leftrightarrow z) + \mathscr{R}(z \leftrightarrow x)]. \tag{2.22}$$

2.88. Let G be a network and x, y, z be three distinct vertices in G. Write $\tau_{x,y,z}$ for the first time that the network random walk trajectory contains the vertices x, y, z in that order. Strengthen and generalize the first equality of (2.22) by showing that

$$\mathbf{P}_x[\tau_{y,z,x} = k] = \mathbf{P}_x[\tau_{z,y,x} = k]$$

for all k.

2.89. Consider a Markov chain that is not necessarily reversible. Let a, x, and z be three of its states. Show that

$$\mathbf{P}_x[\tau_z < \tau_a] = \frac{\mathbf{E}_x[\tau_a] + \mathbf{E}_a[\tau_z] - \mathbf{E}_x[\tau_z]}{\mathbf{E}_z[\tau_a] + \mathbf{E}_a[\tau_z]} \tag{2.23}$$

Use this in combination with (2.22) and Corollary 2.20 to give another solution to Exercise 2.59. *Hint:* Consider whether the chain visits z on the way to a or not.

2.90. Let G be a finite graph and x and y be two of its vertices such that there is a unique simple path in G that joins these vertices; e.g., G could be a tree and x and y any of its vertices. Show that $\mathbf{E}_x[\tau_y] \in \mathbb{N}$.

2.91. Let G be a transient network and $R_n := |\{X_0, X_1, \ldots, X_n\}|$ be the number of vertices visited by time n. Show that for all n and all $o \in \mathsf{V}(G)$,

$$\frac{\mathbf{E}_o[R_n]}{n+1} \ge \inf_{x \in \mathsf{V}(G)} \mathscr{C}(x \leftrightarrow \infty)/\pi(x)\,.$$

2.92. Let G be a finite network and let a and z be two vertices of G. Let $x \sim y$ in G. Show that the expected number of transitions from x to y for a random walk started at a and stopped at the first return to a that occurs after visiting z is $c(x,y)\mathscr{R}(a \leftrightarrow z)$. This is, of course, invariant under multiplication of the edge conductances by a constant. Give another proof of Corollary 2.20 by using this formula.

2.93. Show that Corollary 2.20 and Exercises 2.92, 2.59, and 2.85 hold for all recurrent networks.

2.94. Given two vertices a and z of a finite network (G,c), show that the commute time between a and z is at least twice the square of the graph distance between a and z. *Hint:* Consider the cutsets between a and z that are determined by spherical shells.

2.95. The ***hypercube*** of dimension n is the subgraph induced on the set $\{0,1\}^n$ in the usual nearest-neighbor graph on $\mathbb{Z}^n$. Find the first-order asymptotics of the effective resistance between opposite corners $(0,0,\ldots,0)$ and $(1,1,\ldots,1)$ and the first-order asymptotics of the commute time between them. Also find the first-order asymptotics of the cover time.

2.96. Let (G,c) be a finite network. Fix a non-empty proper subset $W \subset \mathsf{V}$ and a function $u : W \to \mathbb{R}$. Let $U : \mathsf{V} \to \mathbb{R}$ be the jointly normal random variables such that $U = u$ on W and the joint density of $U{\restriction}(\mathsf{V} \setminus W)$ is

$$C \exp\left\{ -\frac{1}{2}\|dU\|_c^2 \right\}$$

for some constant C. These random variables are called the ***Gaussian network pinned on*** W. Let v be the harmonic extension of u to V.

(a) Show that $\|dU\|_c^2 = \|d(U-v)\|_c^2 + \|dv\|_c^2$.

(b) Show that

$$U(x) - U(y) \sim \mathrm{N}\big(v(x) - v(y), \mathscr{R}(x \leftrightarrow y; G/W)\big)$$

for $x \ne y \in \mathsf{V}$ with not both x and y in W, where G/W is the network obtained from G by identifying W to a single vertex.

(c) Show that the covariance of $U(x) - U(y)$ and $U(z) - U(w)$ equals $v'(z) - v'(w)$ when v' is the voltage associated to a unit current flow from x to y in the network G/W (with $x \ne y$).

2.97. Let (G, c) be a finite network and $m > 0$. Fix a non-empty proper subset $W \subset \mathsf{V}$ and a function $u : W \to \mathbb{R}$. Let $U : \mathsf{V} \to \mathbb{R}$ be the jointly normal random variables such that $U = u$ on W and the joint density is

$$C \exp\left\{ -\frac{1}{2}\left(\|dU\|_c^2 + m \sum_{x \in \mathsf{V}} U(x)^2 \right) \right\}$$

for some constant C. These random variables are called the ***Gaussian network pinned on*** W ***with mass*** m. Calculate $\mathbf{E}\big[U(x) - U(y)\big]$ and $\mathrm{Cov}\big(U(x) - U(y), U(z) - U(w)\big)$ for $x, y, z, w \in \mathsf{V}$.

Chapter 3

Special Networks

In this chapter, we will apply the results of Chapter 2 to trees and to Cayley graphs of groups. This will require some preliminaries. First, we study flows that are not necessarily current flows. This involves some tools that are very general and useful.

§3.1. Flows, Cutsets, and Random Paths.

Notice that if there is a flow from a to ∞ of finite energy on some network and if i is the unit current flow, then $|i| = |c \cdot dv| \le v(a)c$. In particular, there is a non-0 flow bounded on each edge by c (*viz.*, $i/v(a)$).* The existence of flows that are bounded by some given numbers on the edges is an interesting and valuable property in itself. To determine whether there is a non-0 flow bounded by c, we turn to the Max-Flow Min-Cut Theorem of Ford and Fulkerson (1962). For finite networks, the theorem reads as follows. We call a set Π of edges a ***cutset*** separating A and Z if every path joining a vertex in A to a vertex in Z includes an edge in Π. We call $c(e)$ the ***capacity*** of e in this context. Think of water flowing through pipes. Recall that the strength of a flow from A to Z, i.e., the total amount flowing into the network at vertices in A (and out at Z) is $\mathsf{Strength}(\theta) = \sum_{a \in A} d^*\theta(a)$. Since all the water must flow through every cutset Π, it is intuitively clear that the strength of every flow bounded by c is at most $\inf_\Pi \sum_{e \in \Pi} c(e)$. Remarkably, this upper bound is always achieved.

It will be useful for the proof, as well as later, to establish a more general statement about flows in *directed* networks. In a directed network, we require flows to satisfy $\theta(e) \ge 0$ for every (directed) edge e; in particular, they are *not* antisymmetric functions. Also, the capacity function of edges is not necessarily symmetric, even if both orientations of an edge occur. Define the vertex-edge incidence function

$$\phi(x, e) := \mathbf{1}_{\{e^- = x\}} - \mathbf{1}_{\{e^+ = x\}} .$$

Other than the source and sink vertices, we require for all x that $\sum_e \phi(x, e)\theta(e) = 0$. The ***strength*** of a flow with source set A is $\sum_{x \in A} \sum_e \phi(x, e)\theta(e)$. ***Cutsets*** separating a source

* This also follows from the Nash-Williams Criterion and the Max-Flow Min-Cut Theorem below.

set A from a sink set Z are required to intersect every directed path from A to Z. To reduce the study of undirected networks to that of directed ones, we simply replace each undirected edge by a pair of parallel directed edges with opposite orientations and the same capacity. A flow on the undirected network is replaced by a flow on the directed network that is nonzero on only one edge of each new parallel pair, while a flow on the resulting directed network yields a flow on the undirected network by subtracting the values on each parallel pair of edges.

The Max-Flow Min-Cut Theorem. *Let A and Z be disjoint set of vertices in a (directed or undirected) finite network G. The maximum possible flow from A to Z equals the minimum cutset sum of the capacities: In the directed case,*

$$\max\Big\{\mathsf{Strength}(\theta)\,;\ \theta \textit{ flows from } A \textit{ to } Z \textit{ satisfying } \forall e\ \ 0 \le \theta(e) \le c(e)\Big\}$$
$$= \min\Big\{\sum_{e\in\Pi} c(e)\,;\ \Pi \textit{ separates } A \textit{ and } Z\Big\},$$

while in the undirected case,

$$\max\Big\{\mathsf{Strength}(\theta)\,;\ \theta \textit{ flows from } A \textit{ to } Z \textit{ satisfying } \forall e\ \ |\theta(e)| \le c(e)\Big\}$$
$$= \min\Big\{\sum_{e\in\Pi} c(e)\,;\ \Pi \textit{ separates } A \textit{ and } Z\Big\}.$$

Proof. We assume that the network is directed. Because the network is finite, the set of flows from A to Z bounded by c on each edge is a compact set in $\mathbb{R}^{\mathsf{E}}$, whence there is a flow of maximum strength. Let θ be a flow of maximum strength. If Π is any cutset separating A from Z, let A' denote the set of vertices that are not separated from A by Π. Since $A \subseteq A'$ and $A' \cap Z = \varnothing$, we have

$$\begin{aligned}\mathsf{Strength}(\theta) &= \sum_{x\in A}\sum_{e\in\mathsf{E}}\phi(x,e)\theta(e) = \sum_{x\in A'}\sum_{e\in\mathsf{E}}\phi(x,e)\theta(e)\\ &= \sum_{e\in\mathsf{E}}\theta(e)\sum_{x\in A'}\phi(x,e) \le \sum_{e\in\Pi}\theta(e) \le \sum_{e\in\Pi}c(e)\,,\end{aligned} \tag{3.1}$$

since $\sum_{x\in A'}\phi(x,e)$ is 0 when e joins two vertices in A', is 1 when e leads out of A', and is -1 when e leads into A'. This proves the intuitive half of the desired equality.

For the reverse inequality, call a sequence of vertices $x_0, x_1, \ldots, x_k$ an ***augmentable path*** if $x_0 \in A$ and for all $i = 1, \ldots, k$, either there is an edge e from x_{i-1} to x_i with $\theta(e) < c(e)$ or there is an edge e' from x_i to x_{i-1} with $\theta(e') > 0$. Let B denote the set of vertices x such that there exists an augmentable path (possibly just one vertex) from a vertex of A to x. If there were an augmentable path $x_0, x_1, \ldots, x_k$ with $x_k \in Z$, then we could obtain from θ a stronger flow bounded by c as follows: For each $i = 1, \ldots, k$ where there is an edge e from x_{i-1} to x_i, let $\theta^*(e) = \theta(e) + \epsilon$, while if there is an edge e' from x_i to x_{i-1} let $\theta^*(e') = \theta(e') - \epsilon$. By taking $\epsilon > 0$ sufficiently small, we would contradict maximality of θ. Therefore $Z \subset B^c$. Let Π be the set of edges connecting B to B^c. Then Π is a cutset separating A from Z. For every edge e leading from B to B^c, necessarily $\theta(e) = c(e)$, while θ must vanish on every edge from B^c to B. Therefore a calculation as in (3.1) shows that

$$\mathsf{Strength}(\theta) = \sum_{e \in \mathsf{E}} \theta(e) \sum_{x \in B} \phi(x, e) = \sum_{e \in \Pi} \theta(e) = \sum_{e \in \Pi} c(e)\,.$$

In conjunction with (3.1), this completes the proof. ◀

Suppose now that $G = (\mathsf{V}, \mathsf{E})$ is a countable directed or undirected network and a is one of its vertices. As usual, we assume that $\forall x \ \sum_{e \sim x} c(e) < \infty$. We want to extend the Max-Flow Min-Cut Theorem to G for flows from a to ∞. Recall that a cutset Π separates a and ∞ if every infinite simple path from a must include an edge in Π. A flow of maximum strength exists since a maximizing sequence of flows has an edgewise limit point which is a maximizing flow bounded by c in light of the dominated convergence theorem. We claim that this maximum strength is equal to the infimum of the cutset sums:

Theorem 3.1. *If a is a vertex in a countable directed network G, then*

$$\max\Big\{\mathsf{Strength}(\theta)\,;\ \theta \textit{ flows from } a \textit{ to } \infty \textit{ satisfying } \forall e \ \ 0 \le \theta(e) \le c(e)\Big\}$$
$$= \inf\Big\{\sum_{e \in \Pi} c(e)\,;\ \Pi \textit{ separates } a \textit{ and } \infty\Big\}.$$

Proof. In the proof, "cutset" will always mean "cutset separating a and ∞". Let θ be a flow of maximum strength among flows from a to ∞ that are bounded by $c(\bullet)$. Given $\epsilon > 0$, let D be a possibly empty set of edges such that $\sum_{e \in D} c(e) < \epsilon$ and $G' := (\mathsf{V}, \mathsf{E} \setminus D)$ is locally finite. The important aspect of this is that the set $\mathscr{P}$ of simple paths in G' from a to ∞ is compact in the product topology. If we associate to an edge e the set of paths in $\mathscr{P}$ that pass through e, then a cutset becomes associated to a cover of $\mathscr{P}$. Compactness of $\mathscr{P}$ therefore means that for any cutset Π in G, there is a finite cutset $\Pi' \subseteq \Pi$ in G'. Also, Π' separates only finitely many vertices A' from ∞ in G'. Therefore,

$$\mathsf{Strength}(\theta) = \sum_{e \in \mathsf{E}} \phi(a, e)\theta(e) = \sum_{x \in A'} \sum_{e \in \mathsf{E}} \phi(x, e)\theta(e)$$

$$
\begin{aligned}
&= \sum_{x\in A'} \sum_{e\in \mathsf{E}\setminus D} \phi(x,e)\theta(e) + \sum_{x\in A'} \sum_{e\in D} \phi(x,e)\theta(e) \\
&= \sum_{e\in \mathsf{E}\setminus D} \theta(e) \sum_{x\in A'} \phi(x,e) + \sum_{x\in A'} \sum_{e\in D} \phi(x,e)\theta(e) \\
&\qquad\qquad\qquad\qquad \text{[since the first sum is finite]} \\
&\le \sum_{e\in\Pi'} \theta(e) + \epsilon \le \sum_{e\in\Pi'} c(e) + \epsilon \le \sum_{e\in\Pi} c(e) + \epsilon\,.
\end{aligned}
$$

Since this holds for all $\epsilon > 0$, we obtain one inequality of the desired equality.

For the inequality in the other direction, let $C(H)$ denote the infimum cutset sum in any network H. Given $\epsilon > 0$, let D and G' be as before. Then $C(G) \le C(G') + \epsilon$ since we may add D to any cutset of G' to obtain a cutset of G. Let $\langle G'_n \rangle$ be an exhaustion of G' by finite connected networks with $a \in G'_n$ for all n. Identify the vertices outside G'_n to a single vertex z_n and remove loops at z_n to form the finite network G_n^{W} from G. Then $C(G') = \inf_n C(G_n^{\mathrm{W}})$ since every minimal cutset of G' is finite (it separates only finitely many vertices from ∞), where minimal means with respect to inclusion. Let θ_n be a flow on G_n^{W} of maximum strength among flows from a to z_n that are bounded by $c{\restriction}G_n^{\mathrm{W}}$. Then $\mathsf{Strength}(\theta_n) = C(G_n^{\mathrm{W}}) \ge C(G')$. Let θ be a limit point of $\langle \theta_n \rangle$. Then θ is a flow on G' with $\mathsf{Strength}(\theta) \ge C(G') \ge C(G) - \epsilon$. ◀

In Section 2.5, we constructed a unit flow from a random path. Now we do the reverse. We return to undirected graphs for this, so that a flow θ satisfies $\theta(-e) = -\theta(e)$ for all edges e. Suppose that

> θ is a unit flow from a to z on a finite graph or from a to ∞ on an infinite graph such that if $e^- = a$, then $\theta(e) \ge 0$ and, in the finite case, if $e^+ = z$, then $\theta(e) \ge 0$. (3.2)

Use θ to define a random path as the trajectory of a Markov chain $\langle Y_n \rangle$ as follows. The initial state is $Y_0 := a$ and z is an absorbing state. For a vertex $x \ne z$, set

$$
\theta_{\mathrm{out}}(x) := \sum_{\substack{e^- = x \\ \theta(e) > 0}} \theta(e)\,,
$$

the amount flowing out of x. The transition probability from x to w is then $(\theta(x,w) \vee 0)/\theta_{\mathrm{out}}(x)$. Define

$$
\theta'(e) := \sum_{n\ge 0} \big\{ \mathbf{P}[\langle Y_n, Y_{n+1} \rangle = e] - \mathbf{P}[\langle Y_{n+1}, Y_n \rangle = e] \big\}\,.
$$

As in Section 2.5, θ' is also a unit flow. We call θ ***acyclic*** if there is no cycle of oriented edges on each of which $\theta > 0$. Current flows are acyclic because they minimize energy (or because they equal $c\,dv$).

Proposition 3.2. *Suppose that θ is an acyclic unit flow satisfying the above conditions (3.2). With the above notation, for every edge e with $\theta(e) > 0$, we have*

$$0 \le \theta'(e) \le \theta(e) \tag{3.3}$$

with equality on the right if G is finite or if θ is the unit current flow from a to ∞.

Proof. Since the Markov chain only travels in the direction of θ, it clear that $\theta'(e) \ge 0$ when $\theta(e) > 0$.

For edges e with $\theta(e) > 0$, set

$$p_N(e) := \mathbf{P}[\exists n \le N \ \langle Y_n, Y_{n+1}\rangle = e]\,.$$

Because θ is acyclic, $p_N(e) \to \theta'(e)$. Thus, in order to show that $\theta'(e) \le \theta(e)$, it suffices to show that $p_N(e) \le \theta(e)$ for all N. We proceed by induction on N. This is clear for $N = 0$. For vertices x, define $p_N(x) := \mathbf{P}[\exists n \le N \ Y_n = x]$. Suppose that $p_N(e) \le \theta(e)$ for all edges e with $\theta(e) > 0$. Then also $p_{N+1}(x) \le \sum_{e^+=x, \theta(e)>0} \theta(e) = \theta_{\text{out}}(x)$ for all vertices $x \ne a$. Hence $p_{N+1}(x) \le \theta_{\text{out}}(x)$ for all vertices x. Therefore, for every edge e with $\theta(e) > 0$, if we put $x := e^-$, then we get $p_{N+1}(e) = p_{N+1}(x)\theta(e)/\theta_{\text{out}}(x) \le \theta(e)$. This completes the induction and proves (3.3).

If G is finite, then $\theta'' := \theta - \theta'$ is a sourceless acyclic flow since it is positive only where θ is positive. If there were an edge e_1 where $\theta''(e_1) > 0$, then there would exist an edge e_2 whose head is the tail of e_1 where also $\theta''(e_2) > 0$, and so on. Eventually, this would close a cycle, which is impossible. Thus, $\theta' = \theta$.

If θ is the unit current flow from a to ∞, then θ has minimum energy among all unit flows from a to ∞. Thus, (3.3) implies that $\theta' = \theta$. ◀

Remark. Of course, other random paths or other rules for transporting mass through the network according to the flow θ will lead to the same result.

▷ **Exercise 3.1.**

Show that if equality holds on the right-hand side of (3.3), then for all x, we have

$$\sum_{\substack{e^+=x,\\ \theta(e)>0}} \theta(e) \le 1\,.$$

▷ **Exercise 3.2.**

Suppose that simple random walk is transient on G and $a \in \mathsf{V}$. Show that there is a random edge-simple path from a to ∞ such that the expected number of edges common to two such independent paths is equal to $\mathscr{R}(a \leftrightarrow \infty)$ (for unit conductances).

Corollary 3.3. (Monotone-Voltage Paths) *Let G be a transient connected network and v the voltage function from the unit current flow i from a vertex a to ∞ with $v(\infty) = 0$. For every vertex x, there is a path of vertices from a to x along which v is monotone. If x is incident to some edge e with $i(e) > 0$, then there is such a path along which v is strictly monotone.*

Proof. Let W be the set of vertices incident to some edge e with $i(e) > 0$. By Proposition 3.2, if $i(e) > 0$, then a path $\langle Y_n \rangle$ chosen at random (as defined above) will cross e with positive probability. Thus, each vertex in W has a positive chance of being visited by $\langle Y_n \rangle$. Clearly v is strictly monotone along every path $\langle Y_n \rangle$. Thus, there is a path from a to any vertex in W along which v is strictly monotone. Now any vertex x not in W can be connected to some $w \in W$ by edges along which $i = 0$. Extending the path from a to w by such a path from w to x gives the required path from a to x. ◀

Curiously, we do not know a deterministic construction of a path satisfying the conclusion of Corollary 3.3.

§3.2. Trees.

Flows and electrical networks on trees can be analyzed with greater precision than on general graphs. One easy reason for this is the following. Fix a root o in a tree T and denote by $|e|$ the distance from an edge $e \in T$ to o, i.e., the number of edges between e and o. Choose a unique orientation for each edge, namely, the one leading away from o. Given any network on T, we claim that there is a flow of maximal strength from the root to infinity that does not have negative flow on any edge (with this orientation). Indeed, it suffices to prove this for flows on finite trees from the root to the boundary (when the boundary is identified to a single vertex). In such a case, consider a flow of maximal strength that has the minimum number of edges with negative flow. If there is an edge with negative flow, then by "following the flow", we can find either a path from the root to the boundary along which the flow is negative or a path from one boundary vertex to another along which the flow goes in the direction of the path. In the first case, we can easily increase the strength of the flow, while in the second case, we can easily reduce the number of edges with negative flow. Both cases therefore lead to a contradiction, which establishes our claim. Likewise, if the tree network is transient, then the unit current flow does not have negative flow on any edge. The proof is similar; in both cases of the preceding proof, we may obtain a flow of strength at least one whose energy is reduced, leading to a contradiction. For this reason, in our considerations, we may restrict to flows that are nonnegative.

▷ **Exercise 3.3.**
Let T be a locally finite tree and Π be a minimal finite cutset separating o from ∞. Let θ be a flow from o to ∞. Show that

$$\mathsf{Strength}(\theta) = \sum_{e \in \Pi} \theta(e) .$$

A partial converse to the Nash-Williams criterion for *trees* can be stated as follows.

Proposition 3.4. *Let c be conductances on a locally finite infinite tree T and w_n be positive numbers with $\sum_{n \geq 0} w_n < \infty$. Any flow θ on T satisfying $0 \leq \theta(e) \leq w_{|e|} c(e)$ for all edges e has finite energy.*

Proof. Apply Exercise 3.3 to the cutset formed by the edges at distance n from o to obtain

$$\sum_{e \in T} \theta(e)^2 r(e) = \sum_{n \geq 0} \sum_{|e|=n} \theta(e)[\theta(e) r(e)] \leq \sum_{n \geq 0} w_n \sum_{|e|=n} \theta(e) = \sum_{n \geq 0} w_n \mathsf{Strength}(\theta) < \infty . \quad \blacktriangleleft$$

Let's consider some particular conductances. Since trees tend to grow exponentially, let the conductance of an edge decrease exponentially with distance from o, say, $c(e) := \lambda^{-|e|}$, where $\lambda > 1$. Let $\lambda_c = \lambda_c(T)$ denote the critical λ for water flow from o to ∞, in other words, water can flow for $\lambda < \lambda_c$ but not for $\lambda > \lambda_c$. What is the critical λ for current flow? We saw at the start of Section 3.1 that if current flows for a certain value of λ (i.e., the associated random walk is transient), then so does water, whence $\lambda \leq \lambda_c$. Conversely, for $\lambda < \lambda_c$, we claim that current flows: choose $\lambda' \in (\lambda, \lambda_c)$ and set $w_n := (\lambda/\lambda')^n$. Of course, $\sum_n w_n < \infty$; by definition of λ_c, there is a non-zero flow θ satisfying $0 \leq \theta(e) \leq (\lambda')^{-|e|} = w_{|e|} \lambda^{-|e|}$, whence Proposition 3.4 shows that this flow has finite energy and so current indeed flows. Thus, the same λ_c is the critical value for current flow. Since λ_c balances the growth of T while taking into account the structure of the tree, we call it the ***branching number*** of T:

$$\operatorname{br} T := \sup \left\{ \lambda ;\ \exists \text{ a non-zero flow } \theta \text{ on } T \text{ with } \forall e \in T \quad 0 \leq \theta(e) \leq \lambda^{-|e|} \right\} .$$

Of course, the Max-Flow Min-Cut Theorem gives an equivalent formulation as

$$\operatorname{br} T = \sup \left\{ \lambda ;\ \inf_{\Pi} \sum_{e \in \Pi} \lambda^{-|e|} > 0 \right\} , \tag{3.4}$$

where the inf is over cutsets Π separating o from ∞. Denote by RW_λ the random walk associated to the conductances $e \mapsto \lambda^{-|e|}$. We may summarize some of our work in the following theorem of Lyons (1990).

Theorem 3.5. (Branching Number and Random Walk) *If T is a locally finite infinite tree, then $\lambda < \operatorname{br} T \Rightarrow \mathsf{RW}_\lambda$ is transient and $\lambda > \operatorname{br} T \Rightarrow \mathsf{RW}_\lambda$ is recurrent.*

In particular, we see that for simple random walk to be transient, it is sufficient that $\operatorname{br} T > 1$.

▷ **Exercise 3.4.**

For simple random walk on T to be transient, is it necessary that $\operatorname{br} T > 1$?

We might call RW_λ ***homesick random walk*** for $\lambda > 1$ since the random walker has a tendency to walk towards its starting place, the root.

▷ **Exercise 3.5.**

Find an example where $\mathsf{RW}_{\operatorname{br} T}$ is transient and an example where it is recurrent.

▷ **Exercise 3.6.**

Show that $\operatorname{br} T$ is independent of which vertex in T is the root.

Let's try to understand the significance of $\operatorname{br} T$. If T is an n-ary tree (i.e., every vertex has n children), then the distance of RW_λ from o is simply a biased random walk on $\mathbb{N}$. It follows that $\operatorname{br} T = n$.

▷ **Exercise 3.7.**

Show directly from the definition that $\operatorname{br} T = n$ for an n-ary tree T. Show that if every vertex of T has between n_1 and n_2 children, then $\operatorname{br} T$ is between n_1 and n_2.

Since λ_c balances the number of edges leading away from a vertex over all of T, it is reasonable to think of $\operatorname{br} T$ as an average number of branches per vertex. For a vertex $x \in T$, let $|x|$ denote its distance from o. Occasionally, we will use $|x - y|$ to denote the distance between x and y.

§3.3. Growth of Trees.

In this section, we consider only locally finite infinite trees. Define the ***lower (exponential) growth rate*** of a tree T by

$$\underline{\mathrm{gr}}\, T := \liminf_{n\to\infty} |T_n|^{1/n},$$

where $T_n := \{x \in T\,;\ |x| = n\}$ is ***level*** n of T. Similarly, the ***upper (exponential) growth rate*** of T is $\overline{\mathrm{gr}}\, T := \limsup |T_n|^{1/n}$. If $\underline{\mathrm{gr}}\, T = \overline{\mathrm{gr}}\, T$, then the common value is called the ***(exponential) growth rate*** of T and denoted $\mathrm{gr}\, T$.

In most of the examples of trees so far, the branching number was equal to the lower growth rate. In general, we have the inequality

$$\mathrm{br}\, T \le \underline{\mathrm{gr}}\, T\,,$$

since

$$\underline{\mathrm{gr}}\, T = \sup\{\lambda\,;\ \inf_n \sum_{x\in T_n} \lambda^{-|x|} > 0\} \tag{3.5}$$

and $\{e(x)\,;\ x \in T_n\}$ are cutsets, where $e(x)$ is the edge from the parent of x to x. Of course, the ***parent*** of a vertex $x \ne o$ is the neighbor of x that is closer to o.

▷ **Exercise 3.8.**
Show (3.5).

There are various ways to construct a tree whose branching number is different from its growth: see Section 1.1.

▷ **Exercise 3.9.**
We have seen that if $\mathrm{br}\, T > 1$, then simple random walk on T is transient. Is $\underline{\mathrm{gr}}\, T > 1$ sufficient for transience?

We call T ***spherically symmetric*** if $\deg x$ depends only on $|x|$. Recall from Exercise 1.2 that $\mathrm{br}\, T = \underline{\mathrm{gr}}\, T$ when T is spherically symmetric.

Notation. Write $x \le y$ if x is on the shortest path from o to y; $x < y$ if $x \le y$ and $x \ne y$; $x \to y$ if $x \le y$ and $|y| = |x| + 1$ (i.e., y is a child of x); and T^x for the subtree of T containing the vertices $y \ge x$.

There is an important class of trees whose structure is "periodic" in a certain sense. To exhibit these trees, we review some elementary notions from combinatorial topology. Let G be a finite connected graph and x_0 be any vertex in G. Define the tree T to have as vertices the finite paths $\langle x_0, x_1, x_2, \ldots, x_n \rangle$ that never backtrack, i.e., $x_i \neq x_{i+2}$ for $0 \le i \le n-2$. Join two vertices in T by an edge when one path is an extension by one vertex of the other. The tree T is called the ***universal cover*** (based at x_0) of G. See Figure 3.1 for an example.

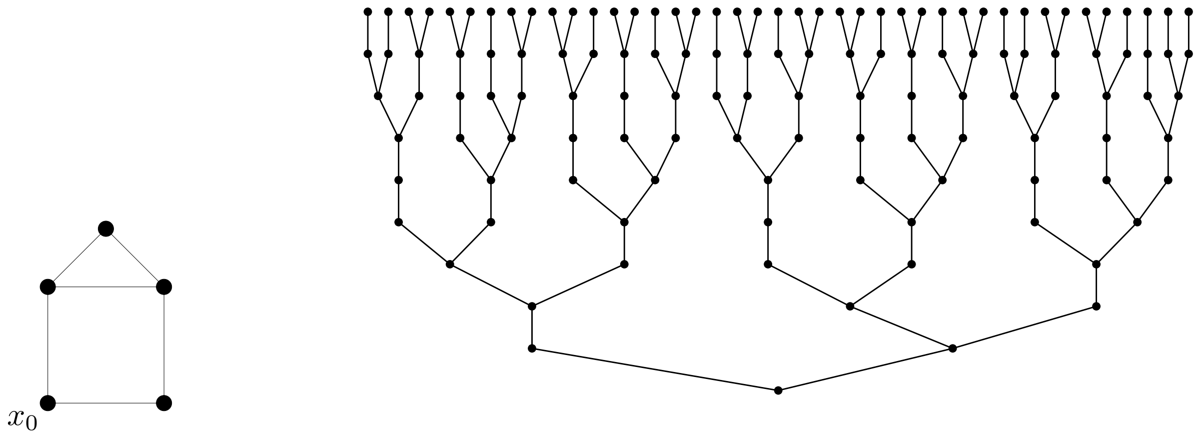


Figure 3.1. A graph and part of its universal cover.

In fact, this idea can be extended. Suppose that G is a finite directed multigraph and x_0 be any vertex in G. That is, edges are not required to appear with both orientations and two vertices can have many edges joining them. Loops are also allowed. Define the **directed cover** (based at x_0) of G to be the tree T whose vertices are the finite paths $\langle x_0, x_1, x_2, \ldots, x_n \rangle$ such that $\langle x_i, x_{i+1} \rangle$ is an edge of G and $n \ge 0$. We join two vertices in T as we did before. See Figure 3.2 for an example.

The periodic aspect of these trees can be formalized as follows.

Definition. Let $N \ge 0$. An infinite tree T is called N***-periodic*** (resp., N***-subperiodic***) if $\forall x \in T$ there exists an adjacency-preserving bijection (resp., injection) $f : T^x \to T^{f(x)}$ with $|f(x)| \le N$. A tree is ***periodic*** (resp., ***subperiodic***) if there is some N for which it is N-periodic (resp., N-subperiodic).

All universal and directed covers are periodic. Conversely, every periodic tree is a directed cover of a finite graph, G: If T is an N-periodic tree, take $\{x \in T\,;\ |x| \le N\}$ for the vertex set of G. For $|x| \le N$, let f_x be the identity map and for $x \in T_{N+1}$, let f_x be a bijection as guaranteed by the definition. Let the edges of G be $\{\langle x, f_y(y) \rangle\,;\ |x| \le N$ and y is a child of $x\}$. Then T is the directed cover of G based at the root.

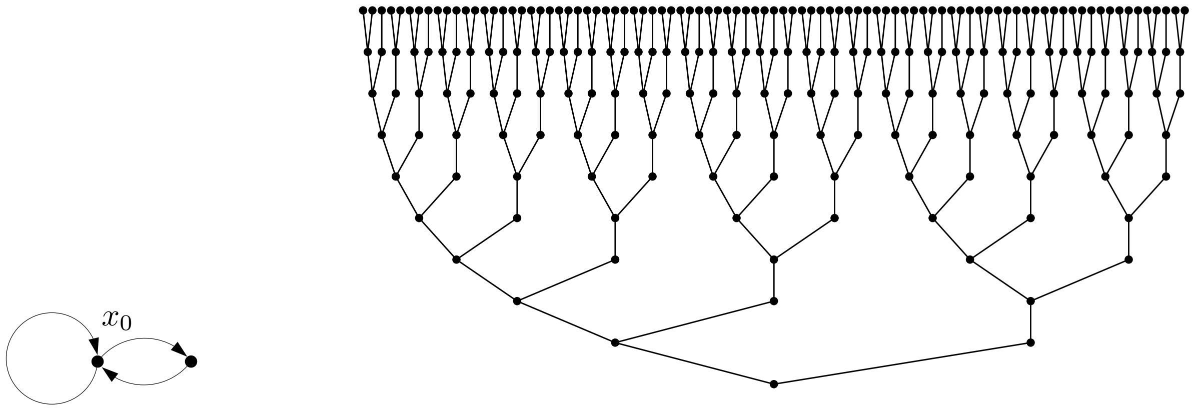


Figure 3.2. A graph and part of its directed cover. This tree is also called the ***Fibonacci tree***.

If two (sub)periodic trees are joined as in Example 1.3, then clearly the resulting tree is also (sub)periodic.

Example 3.6. Consider the finite paths in the lattice $\mathbb{Z}^2$ starting at the origin that go through no vertex more than once. These paths are called self-avoiding and are of substantial interest to mathematical physicists. Form a tree T whose vertices are the finite self-avoiding paths and with two such vertices joined when one path is an extension by one step of the other. Then T is 0-subperiodic and has infinitely many leaves. Its growth rate has been estimated at about 2.64; see Madras and Slade (1993).

Example 3.7. Suppose that E is a closed set in $[0,1]$ and $T_{[b]}(E)$ is the associated tree that represents E in base b, as in Section 1.9. Then $T_{[b]}(E)$ is 0-subperiodic iff E is invariant under the map $x \mapsto bx \pmod 1$, i.e., iff the fractional part of bx lies in E for every $x \in E$.

How can we calculate the growth rate of a periodic tree? If T is a periodic tree, let G be a finite directed graph of which T is the directed cover based at x_0. We may assume that G contains only vertices that can be reached from x_0, since the others do not contribute to T. Let A be the directed adjacency matrix of G, i.e., A is a square matrix indexed by the vertices of G with the (x, y) entry equal to the number of edges from x to y. Since all entries of A are nonnegative, the Perron-Frobenius Theorem (Minc (1988), Theorem 4.2) says that the spectral radius of A is equal to its largest positive eigenvalue, λ_*, and that there is a λ_*-eigenvector v_* all of whose entries are nonnegative. We call λ_* the ***Perron eigenvalue*** of A and v_* a ***Perron eigenvector*** of A. Let $\mathbf{1}$ denote a column vector all of whose entries are 1, $\mathbf{1}_x$ denote the column vector that is 1 at x and 0 elsewhere, and $\mathbf{1}'_x$ denote its transpose. Then the number of paths in G with n edges, which is $|T_n|$, is $\mathbf{1}'_{x_0} A^n \mathbf{1}$. Since the spectral radius of A equals $\lim_{n\to\infty} \|A^n\|^{1/n}$, it follows that $\limsup_{n\to\infty} |T_n|^{1/n} \le \lambda_*$. On the other hand, let x be any vertex such that $v_*(x) > 0$, let j be such that $A^j(x_0, x) > 0$, and let $c > 0$ be such that $\mathbf{1} \ge cv_*$. Then

$$|T_{j+n}| \ge \mathbf{1}_x' A^n \mathbf{1} \ge c\mathbf{1}_x' A^n v_* = c\lambda_*^n \mathbf{1}_x' v_* = cv_*(x)\lambda_*^n \,.$$

Therefore $\liminf_{n\to\infty} |T_n|^{1/n} \ge \lambda_*$. We conclude that $\operatorname{gr} T = \lambda_*$.

It turns out that subperiodicity is enough for equality of $\operatorname{br} T$ and $\operatorname{gr} T$. This is analogous to a classical fact about sequences of reals, known as Fekete's Lemma (see Exercise 3.10). A sequence $\langle a_n \rangle$ of real numbers is called ***subadditive*** if

$$\forall m, n \ge 1 \qquad a_{m+n} \le a_m + a_n \,.$$

A simple example is $a_n := \lceil \beta n \rceil$ for some real $\beta > 0$.

▷ **Exercise 3.10.**

(a) **(Fekete's Lemma)** Show that for every subadditive sequence $\langle a_n \rangle$, the sequence $\langle a_n/n \rangle$ converges to its infimum:

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf \frac{a_n}{n} \,.$$

(b) Show that Fekete's Lemma holds even if a finite number of the a_n are infinite.

(c) Show that for every 0-subperiodic tree T, the limit $\lim_{n\to\infty} |T_n|^{1/n}$ exists.

The following theorem is due to Furstenberg (1967). Given $\lambda > 0$ and a cutset Π in a tree T, denote

$$\|\Pi\|_\lambda := \sum_{e(x)\in\Pi} \lambda^{-|x|} \,.$$

Theorem 3.8. (Subperiodicity and Branching Number) *For every subperiodic infinite tree T, the growth rate exists and $\operatorname{br} T = \operatorname{gr} T$. Moreover,*

$$\inf_{\Pi} \|\Pi\|_{\operatorname{br} T} > 0 \,.$$

Proof. First, suppose that T has no leaves and is 0-subperiodic. We shall show that if for some cutset Π and some $\lambda_1 > 0$, we have

$$\|\Pi\|_{\lambda_1} < 1 \,, \tag{3.6}$$

then

$$\limsup_{n\to\infty} |T_n|^{1/n} < \lambda_1 \,. \tag{3.7}$$

Since $\inf_\Pi \|\Pi\|_{\lambda_1} = 0$ for all $\lambda_1 > \operatorname{br} T$, this implies that

$$\operatorname{br} T = \operatorname{gr} T \quad \text{and} \quad \inf_\Pi \|\Pi\|_{\operatorname{br} T} \ge 1\,. \tag{3.8}$$

So suppose that (3.6) holds. We may assume that Π is finite and minimal with respect to inclusion by Exercise 3.23. Now there certainly exists $\lambda \in (0, \lambda_1)$ such that

$$\|\Pi\|_\lambda < 1\,. \tag{3.9}$$

Let $d := \max_{e(x)\in\Pi} |x|$ denote the maximal level of Π. By 0-subperiodicity, for each $e(x)$ in Π, there is a cutset $\Pi(x)$ of T^x such that $\sum_{e(w)\in\Pi(x)} \lambda^{-|w-x|} \le \|\Pi\|_\lambda < 1$ and $\max_{e(w)\in\Pi(x)} |w-x| \le d$. Thus, $\|\Pi(x)\|_\lambda = \sum_{e(w)\in\Pi(x)} \lambda^{-|w|} < \lambda^{-|x|}$. (Note that $|w|$ always denotes the distance from w to the root of T.) Thus, if we replace those edges $e(x)$ in Π by the edges of the corresponding $\Pi(x)$ for $e(x) \in A \subseteq \Pi$, we obtain a cutset $\tilde\Pi := \bigcup_{e(x)\in A} \Pi(x) \cup (\Pi \setminus A)$ that satisfies $\|\tilde\Pi\|_\lambda = \sum_{e(x)\in A} \|\Pi(x)\|_\lambda + \sum_{e(x)\in\Pi\setminus A} \lambda^{-|x|} \le \|\Pi\|_\lambda < 1$.

Given $n > d$, we may iterate this process for all edges $e(x)$ in the cutset with $|x| < n$ until we obtain a cutset Π^* lying between levels n and $n+d$ with $\|\Pi^*\|_\lambda < 1$. Therefore $|T_n|\lambda^{-(n+d)} \le \|\Pi^*\|_\lambda < 1$, so that $\limsup |T_n|^{1/n} \le \limsup \lambda^{1+d/n} = \lambda < \lambda_1$. This establishes (3.7).

Now let T be N-subperiodic. Let Γ be the union of disjoint copies of the descendant trees $\{T^x : |x| \le N\}$ with their roots identified (which is not exactly the same as Example 1.3). It is easy to check that Γ is 0-subperiodic and $\operatorname{gr}\Gamma \ge \overline{\operatorname{gr}}\, T$. Moreover, for every cutset Π of T with $\min_{e(x)\in\Pi} |x| \ge N$, there is a corresponding cutset Π' of Γ such that

$$\forall \lambda > 0 \quad \|\Pi'\|_\lambda \le (1 + \lambda + \ldots + \lambda^N)\|\Pi\|_\lambda\,,$$

whence $\operatorname{br}\Gamma = \operatorname{br} T$. In conjunction with (3.8) for Γ, this completes the proof.

Finally, if T has leaves, consider the tree T' obtained from T by adding to each leaf of T an infinite ray. Then T' is subperiodic, so

$$\limsup |T_n|^{1/n} \ge \operatorname{br} T = \operatorname{br} T' = \limsup |T'_n|^{1/n} \ge \limsup |T_n|^{1/n}$$

and every cutset Π of T can be extended to a cutset Π' of T' with $\|\Pi'\|_{\operatorname{br} T}$ arbitrarily close to $\|\Pi\|_{\operatorname{br} T}$. ◀

For another proof of Theorem 3.8, see Section 14.5.

Next, we consider a notion dual to subperiodicity.

Definition. Let $N \ge 0$. A tree T is called N-***superperiodic*** if $\forall x \in T$ there is an adjacency-preserving injection $f : T \to T^{f(o)}$ with $f(o) \in T^x$ and $|f(o)| - |x| \le N$.

For example, every 0-periodic tree is 0-superperiodic, although 1-periodic trees are not necessarily 1-superperiodic. For another example, consider the finite paths in the lattice $\mathbb{Z}^2$ starting at the origin that stay in the right half-plane. Form the tree T whose vertices are the finite paths of this type and with two such vertices joined when one path is an extension by one step of the other. Then T is 0-superperiodic. For more examples, see Exercise 3.33.

Theorem 3.9. *Let $N \ge 0$. Any N-superperiodic tree T with $\overline{\mathrm{gr}}\, T < \infty$ satisfies $\mathrm{br}\, T = \mathrm{gr}\, T$ and $|T_n| \le (\mathrm{gr}\, T)^{n+N}$ for all n.*

Proof. Consider the case $N = 0$. In this case, $|T_{n+m}| \ge |T_n| \cdot |T_m|$. By Exercise 3.10, $\mathrm{gr}\, T$ exists and $|T_n| \le (\mathrm{gr}\, T)^n$ for all n. Fix any positive integer k. Let θ be the unit flow from o to T_k that is uniform on T_k. By 0-superperiodicity, we can extend θ in a periodic fashion to a flow from o to infinity that satisfies $\theta\big(e(x)\big) \le |T_k|^{-\lfloor |x|/k \rfloor}$ for all vertices x. Consequently, $\mathrm{br}\, T \ge |T_k|^{1/k}$. Letting $k \to \infty$ completes the proof for $N = 0$. ◀

▷ **Exercise 3.11.**
Prove the case $N > 0$ of Theorem 3.9.

§3.4. Cayley Graphs.

Suppose we investigate RW_λ on graphs other than trees. What does RW_λ mean in this context? Fix a vertex o in a graph G. If e is an edge at distance n from o, let the conductance of e be λ^{-n}. Again, there is a critical value of λ, denoted $\lambda_c(G)$, that separates the transient regime from the recurrent regime. In order to understand what $\lambda_c(G)$ measures, consider the class of spherically symmetric graphs, where we call G ***spherically symmetric*** about o if for all pairs of vertices x, y at the same distance from o, there is an automorphism of G fixing o that takes x to y. (Here, a graph automorphism is a network automorphism for the network in which all edges have the same conductances.) Let $\tilde{M}_n$ be the number of edges that lead from a vertex at distance $n-1$ from o to a vertex at distance n. Then the critical value of λ is the growth rate of G:

$$\lambda_c(G) = \liminf_{n\to\infty} \tilde{M}_n^{1/n}\,.$$

In fact, we have the following more precise criterion for transience:

▷ **Exercise 3.12.**
Show that if G is spherically symmetric about o, then RW_λ is transient iff $\sum_n \lambda^n/\tilde{M}_n < \infty$.

Next, consider the Cayley graphs of finitely generated groups: We say that a group Γ is ***generated*** by a subset S of its elements if the smallest subgroup containing S is all of Γ. In other words, every element of Γ can be written as a product of elements of the form s or s^{-1} with $s \in S$. If Γ is generated by S, then we form the associated ***Cayley graph*** G with vertices Γ and (unoriented) edges $\{[x, xs]\,;\ x \in G,\ s \in S\} = \{(x, y) \in \Gamma^2\,;\ x^{-1}y \in S \cup S^{-1}\}$. Because S generates Γ, the graph is connected. Cayley graphs are highly symmetric: they look the same from every vertex since left multiplication by yx^{-1} is an automorphism of G that carries x to y.

▷ **Exercise 3.13.**
Show that different Cayley graphs of the same finitely generated group are roughly isometric.

The Euclidean lattices are the most well-known Cayley graphs. It is useful to keep in mind other Cayley graphs as well, so we will look at some constructions of groups. First recall the ***cartesian*** or ***direct product*** of two groups Γ and Γ', where the multiplication on $\Gamma \times \Gamma'$ is defined coordinatewise: $(\gamma_1, \gamma_1')(\gamma_2, \gamma_2') := (\gamma_1\gamma_2, \gamma_1'\gamma_2')$. A similar definition is made for the direct product of any set of groups. It is convenient to rephrase this definition in terms of presentations.

First, recall that the ***free group*** generated by a set S is the set of all finite words in g and g^{-1} for $g \in S$ with the empty word as the identity and concatenation as multiplication, with the further stipulation that if a word contains either gg^{-1} or $g^{-1}g$, then the pair is eliminated from the word. The group defined by the ***presentation*** $\langle S \mid R\rangle$ is the quotient of the free group generated by the set S by the normal subgroup generated by the set R, where R consists of finite words, called ***relators***, in the elements of S. (We think of R as giving a list of products that must equal the identity; other identities are consequences of these ones and of the definition of a group.) For example, the free group on two letters $\mathbb{F}_2$ is $\langle \{a, b\} \mid \varnothing\rangle$, usually written $\langle a, b \mid\ \rangle$, while $\mathbb{Z}^2$ is (isomorphic to) $\langle \{a, b\} \mid \{aba^{-1}b^{-1}\}\rangle$, usually written $\langle a, b \mid aba^{-1}b^{-1}\rangle$, also known as the ***free abelian group*** on two letters (or of ***rank*** 2). In this notation, if $\Gamma = \langle S \mid R\rangle$ and $\Gamma' = \langle S' \mid R'\rangle$ with $S \cap S' = \varnothing$, then $\Gamma \times \Gamma' = \langle S \cup S' \mid R \cup R' \cup [S, S']\rangle$, where $[S, S'] := \{\gamma\gamma'\gamma^{-1}\gamma'^{-1}\,;\ \gamma \in S, \gamma' \in S'\}$. On the other hand, the ***free product*** of Γ and Γ' is $\Gamma * \Gamma' := \langle S \cup S' \mid R \cup R'\rangle$. A similar definition is made for the free product of any set of groups. Interesting free products to keep in mind as we examine various phenomena are $\mathbb{Z} * \mathbb{Z}$ (the free group on two letters), $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, $\mathbb{Z} * \mathbb{Z}_2$ (whose Cayley graph is isomorphic to that of $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, i.e., a 3-regular tree, when the usual generators are used), $\mathbb{Z}_2 * \mathbb{Z}_3$, $\mathbb{Z}^2 * \mathbb{Z}$, and $\mathbb{Z}^2 * \mathbb{Z}_2$. Write $\mathbb{T}_{b+1}$ for the regular tree of degree $b+1$ (so it has branching number b). It is a Cayley graph of the free product of $b+1$ copies of $\mathbb{Z}_2$. Its cartesian product with $\mathbb{Z}^d$ is another interesting graph. Some examples of Cayley graphs with respect to natural generating sets appear in Figure 3.3.

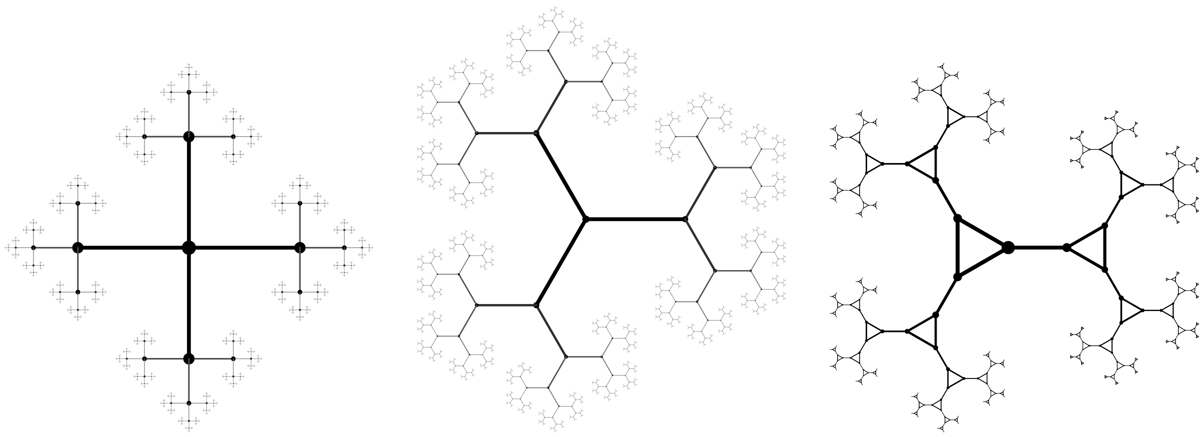


Figure 3.3. The Cayley graphs of the free group on 2 letters; the free product of $\mathbb{Z}_2$ with itself 3 times, $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$; and the free product $\mathbb{Z}_2 * \mathbb{Z}_3$.

A presentation is called ***finite*** when it uses only finite sets of generators and relators. For example, $\mathbb{Z}_2 * \mathbb{Z}_3$ has the presentation $\langle a, b \mid a^2, b^3 \rangle$. The graph of Figure 2.3 is the Cayley graph of another group corresponding to the presentation $\langle a, b, c, d, e \mid a^2, b^2, c^2, d^2, e^2, abcde \rangle$ (see Chaboud and Kenyon (1996)).

Fundamental groups of compact topological manifolds are finitely presentable, and each finitely presentable group is the fundamental group of a compact 4-manifold (see, e.g., Massey (1991), pp. 114–115). Thus, finitely presentable groups arise often "in practice". The fundamental group of a compact manifold is roughly isometric to the universal cover of the manifold.

Despite the symmetry of Cayley graphs, they are rarely spherically symmetric. Still, if M_n denotes the number of vertices at distance n from the identity, then $\lim M_n^{1/n}$ exists since $M_{m+n} \le M_m M_n$; thus, we may apply Fekete's lemma (Exercise 3.10) to $\langle \log M_n \rangle$. Note that this also implies that the exponential growth rate of the balls in G equals $\lim M_n^{1/n}$; we refer to this common number as the ***(exponential) growth rate of*** G. When the growth rate is 1, we say that the Cayley graph has ***subexponential growth***, and otherwise that it has ***exponential growth***. Is $\lambda_c(G) = \lim M_n^{1/n}$?

First of all, if $\lambda > \lim M_n^{1/n}$, then RW_λ is positive recurrent since for such λ,

$$\sum_{e \in \mathsf{E}_{1/2}} c(e) < 2k \sum_{x \in G} \lambda^{-|x|} = 2k \sum_{n \geq 0} M_n \lambda^{-n} < \infty \,,$$

where k is the degree of G and $|x|$ denotes the distance of x to the identity. (One could also use the Nash-Williams criterion to get merely recurrence.) To prove transience for a given λ, it suffices, by Rayleigh's monotonicity principle, to prove that a subgraph is transient. The easiest subgraph to analyze would be a subtree of G, while in order to have the greatest likelihood of being transient, it should be as big as possible, i.e., a ***spanning tree*** (one that includes every vertex). Here is one: Assume that the inverse of each generator is also in the generating set. Order the generating set $S = \{s_1, s_2, \ldots, s_k\}$. For each $x \in G$, there is a unique word $(s_{i_1}, s_{i_2}, \ldots, s_{i_n})$ in the generators such that $x = s_{i_1} s_{i_2} \cdots s_{i_n}$, $n = |x|$, and $(s_{i_1}, \ldots, s_{i_n})$ is ***lexicographically minimal*** with these properties, i.e., if $(s_{i'_1}, \ldots, s_{i'_n})$ is another and m is the first j such that $i_j \neq i'_j$, then $i_m < i'_m$. Call this word w_x. Let T be the subgraph of G containing all vertices and with u adjacent to x when either $|u| + 1 = |x|$ and w_u is an initial segment of w_x or vice versa.

▷ **Exercise 3.14.**
Show that T is a subperiodic tree when rooted at the identity.

Since T is spanning and since distances to the identity in T are the same as in G, we have $\operatorname{gr} T = \lim M_n^{1/n}$. Since T is subperiodic, we have $\operatorname{br} T = \lim M_n^{1/n}$. Hence RW_λ is transient on T for $\lambda < \lim M_n^{1/n}$, whence on G as well. We have proved the following theorem of Lyons (1995):

Theorem 3.10. (Group Growth and Random Walk) RW_λ *on an infinite Cayley graph has critical value λ_c equal to the exponential growth rate of the graph.*

Of course, this theorem makes Cayley graphs look spherically symmetric from a probabilistic point of view. Such a conclusion, however, should not be pushed too far, for there are Cayley graphs with the following very surprising properties; the lamplighter group defined in Section 7.1 is one such. Define the ***speed*** (or rate of escape) of RW_λ as the limit of the distance from the identity at time n divided by n as $n \to \infty$, if the limit exists. The speed is monotonically decreasing in λ on spherically symmetric graphs and is positive for any positive λ less than the growth rate. However, there are groups of exponential growth for which the speed of simple random walk is 0. This already shows that the Cayley graph is far from spherically symmetric. Even more surprisingly, on the lamplighter group, which has growth rate $(1+\sqrt{5})/2$, the speed is 0 at $\lambda = 1$, yet is strictly positive when $1 < \lambda < (1+\sqrt{5})/2$ (Lyons, Pemantle, and Peres, 1996b). Perhaps this is part of a general phenomenon:

Question 3.11. If G is a Cayley graph of growth rate b and $1 < \lambda < b$, must the speed of RW_λ exist and be positive?

Question 3.12. If T is a spanning tree of a graph G rooted at some vertex o, we call T ***geodesic*** if $\mathrm{dist}(o, x)$ is the same in T as in G for all vertices x. If T is a geodesic spanning tree of the Cayley graph of a finitely generated group G of growth rate b, is $\mathrm{br}\, T = b$?

§3.5. Notes.

Denote the growth rate of a group Γ with respect to a finite generating set F by $\mathrm{gr}_S\,\Gamma$. It is easy to show that if $\mathrm{gr}_S\,\Gamma > 1$ for some generating set, then $\mathrm{gr}_S\,\Gamma > 1$ for every generating set. In this case, is $\inf_S \mathrm{gr}_S\,\Gamma > 1$? For a long time, this question was open. It is known to hold for certain classes of groups, but recently a counterexample was found by Wilson (2004b); see also Bartholdi (2003) and Wilson (2004c).

§3.6. Collected In-Text Exercises.

3.1. Show that if equality holds on the right-hand side of (3.3), then for all x, we have

$$\sum_{\substack{e^+=x,\\ \theta(e)>0}} \theta(e) \le 1\,.$$

3.2. Suppose that simple random walk is transient on G and $a \in \mathsf{V}$. Show that there is a random edge-simple path from a to ∞ such that the expected number of edges common to two such independent paths is equal to $\mathscr{R}(a \leftrightarrow \infty)$ (for unit conductances).

3.3. Let T be a locally finite tree and Π be a minimal finite cutset separating o from ∞. Let θ be a flow from o to ∞. Show that

$$\mathsf{Strength}(\theta) = \sum_{e\in\Pi} \theta(e)\,.$$

3.4. For simple random walk on T to be transient, is it necessary that $\mathrm{br}\, T > 1$?

3.5. Find an example where $\mathsf{RW}_{\mathrm{br}\, T}$ is transient and an example where it is recurrent.

3.6. Show that $\mathrm{br}\, T$ is independent of which vertex in T is the root.

3.7. Show directly from the definition that $\mathrm{br}\, T = n$ for an n-ary tree T. Show that if every vertex of T has between n_1 and n_2 children, then $\mathrm{br}\, T$ is between n_1 and n_2.

3.8. Show (3.5).

3.9. We have seen that if $\operatorname{br} T > 1$, then simple random walk on T is transient. Is $\underline{\operatorname{gr}}\, T > 1$ sufficient for transience?

3.10. **(a)** **(Fekete's Lemma)** Show that for every subadditive sequence $\langle a_n \rangle$, the sequence $\langle a_n/n \rangle$ converges to its infimum:

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf \frac{a_n}{n} .$$

(b) Show that Fekete's Lemma holds even if a finite number of the a_n are infinite.

(c) Show that for every 0-subperiodic tree T, the limit $\lim_{n\to\infty} |T_n|^{1/n}$ exists.

3.11. Prove the case $N > 0$ of Theorem 3.9.

3.12. Show that if G is spherically symmetric about o, then RW_λ is transient iff $\sum_n \lambda^n/\tilde{M}_n < \infty$.

3.13. Show that different Cayley graphs of the same finitely generated group are roughly isometric.

3.14. Show that T is a subperiodic tree when rooted at the identity.

§3.7. Additional Exercises.

3.15. There are two other versions of the Max-Flow Min-Cut Theorem that are useful. We state them for directed finite networks using the notation of our proof of the Max-Flow Min-Cut Theorem.

(a) Suppose that each vertex x is given a capacity $c(x)$, meaning that a flow θ is required to satisfy $\theta(e) \ge 0$ for all edges e and for all x other than the sources A and sinks Z, $\sum_{e\in\mathsf{E}} \phi(x,e)\theta(e) = 0$ and $\sum_{e^+=x} \theta(e) \le c(x)$. A cutset now consists of vertices that intersect every directed path from A to Z. Show that the maximum possible flow from A to Z equals the minimum cutset sum of the capacities.

(b) Suppose that each edge and each vertex has a capacity, with the restrictions that both of these imply. A cutset now consists of both vertices and edges. Again, show that the maximum possible flow from A to Z equals the minimum cutset sum of the capacities.

3.16. Show that if all the edge capacities $c(e)$ in a directed finite network are integers, then among the flows θ of maximal strength, there is one such that all $\theta(e)$ are also integers. Show the same for networks with capacities assigned to the vertices or to both edges and vertices, as in Exercise 3.15.

3.17. **(Menger's Theorem)** Let a and z be vertices in a graph that are not adjacent. Show that the maximum number paths from a to z that are pairwise disjoint (except at a and z) is equal to the minimum cardinality of a set W of vertices such that every path from a to z passes through W.

3.18. Show that the maximum possible flow from A to Z (in a finite undirected network) also equals

$$\min \left\{ \sum_{e\in\mathsf{E}_{1/2}} c(e)\ell(e) \right\} ,$$

where ℓ is an assignment of nonnegative lengths to the edges so that the minimum distance from every point in A to every point in Z is 1.

3.19. Suppose that θ is a flow from A to Z in a finite undirected network. Show that if Π is a cutset separating A from Z that is minimal with respect to inclusion, then $\mathsf{Strength}(\theta) = \sum_{e\in\Pi}\theta(e)$.

3.20. Let G be a finite network and a and z be two of its vertices. Show that $\mathscr{R}(a \leftrightarrow z)$ is the minimum of $\sum_{e\in\mathsf{E}_{1/2}} r(e)\mathbf{P}[e \in \mathcal{P} \text{ or } -e \in \mathcal{P}]^2$ over all probability measures $\mathbf{P}$ on paths $\mathcal{P}$ from a to z.

3.21. Let G be an undirected graph and $o \in \mathsf{V}$. Recall that E consists of both orientations of each edge. Suppose that $q : \mathsf{E} \to [0,\infty)$ satisfies the following conditions:

(i) For every vertex $x \ne o$, we have

$$\sum_{u:(u,x)\in\mathsf{E}} q(u,x) \le \sum_{w:(x,w)\in\mathsf{E}} q(x,w)\,;$$

(ii)

$$\sum_{u:(u,o)\in\mathsf{E}} q(u,o) = 0 \quad \text{and} \quad \sum_{w:(o,w)\in\mathsf{E}} q(o,w) > 0\,; \text{ and}$$

(iii) there exists $K < \infty$ such that for every directed path $(u_0, u_1, \ldots)$ in G, starting at $u_0 = o$, we have

$$\sum_{i=0}^{\infty} q(u_i, u_{i+1}) \le K\,.$$

Show that simple random walk on (the undirected graph) G starting at o is transient.

3.22. Let G be a finite graph and a, z be two vertices of G. Let the edges be labelled by positive resistances $r(\bullet)$. Two players, a passenger and a troll, simultaneously pick edge-simple paths from a to z. The passenger then pays the troll the sum of $\pm r(e)$ for all the edges e common to both paths; if e is traversed in the same direction by the two paths, then the $+$ sign is used, otherwise the $-$ sign is used. Show that the troll has a strategy of picking a random path in such a way that no matter what path is picked by the passenger, the troll's gain has expectation equal to the effective resistance between a to z. Show further that the passenger has a similar strategy that has expected loss equal to $\mathscr{R}(a \leftrightarrow z)$ no matter what the troll does.

3.23. Let T be a locally finite tree and Π be a cutset separating o from ∞. Show that there is a finite cutset $\Pi' \subseteq \Pi$ separating o from ∞ that does not properly contain any other cutset.

3.24. Show that a network (T, c) on a tree is transient iff there exists a function F on the vertices of T such that $F \ge 0$, $\forall e\ \ dF(e) \ge 0$, and $\inf_\Pi \sum_{e\in\Pi} dF(e)c(e) > 0$.

3.25. Given a tree T and $k \ge 1$, form the tree $T^{[k]}$ by taking the vertices x of T for which $|x|$ is a multiple of k and joining x and y by an edge in $T^{[k]}$ when their distance is k in T. Show that $\operatorname{br} T^{[k]} = (\operatorname{br} T)^k$.

3.26. Show that RW_λ is positive recurrent on a tree T if $\lambda > \overline{\operatorname{gr}}\, T$.

3.27. Let $U(T)$ be the set of unit flows on a tree T (from o to ∞). For $\theta \in U(T)$, define its ***Frostman exponent*** to be

$$\operatorname{Frost}(\theta) := \liminf_{|x|\to\infty} \theta(x)^{-1/|x|}\,.$$

Show that

$$\operatorname{br} T = \sup_{\theta\in U(T)} \operatorname{Frost}(\theta)\,.$$

3.28. Let $k \ge 1$. Show that if T is a 0-periodic (resp., 0-subperiodic) tree, then for all vertices x with $|x| \ge k$, there is an adjacency-preserving bijection (resp., injection) $f : T^x \to T^{f(x)}$ with $|f(x)| = k$.

3.29. Given a finite *directed* multigraph G, one can also define another covering tree by using as vertices all directed paths of the form $\langle x_0, x_1, \ldots, x_n \rangle$ or $\langle x_{-n}, \ldots, x_{-1}, x_0 \rangle$, with the former a child of $\langle x_0, x_1, \ldots, x_{n-1} \rangle$ and the latter a child of $\langle x_{-(n-1)}, \ldots, x_{-1}, x_0 \rangle$. Show that this tree is also periodic.

3.30. Given an integer $k \ge 0$, construct a periodic tree T with $|T_n|$ approximately equal to n^k for all n.

3.31. Show that critical homesick random walk (i.e., $\mathsf{RW}_{\operatorname{br} T}$) is recurrent on each periodic tree.

3.32. Construct a subperiodic tree for which critical homesick random walk (i.e., $\mathsf{RW}_{\operatorname{br} T}$) is transient.

3.33. Let $N \ge 0$ and $0 < \alpha < 1$. Identify the binary tree with the set of all finite sequences of 0's and 1's. Let T be the subtree of the binary tree which contains the vertex corresponding to $(x_1, \ldots, x_n)$ iff $\forall k \le n \ \sum_{i=1}^k x_i \le \alpha(k + N)$. Show that T is N-superperiodic but not $(N-1)$-superperiodic. Also, determine $\operatorname{br} T$.

3.34. We have defined the right Cayley graph of a finitely generated group and noted that left multiplication is a graph automorphism. The left Cayley graph is defined similarly. Show that the right and left Cayley graphs are isomorphic.

Chapter 4

Uniform Spanning Trees

From what we learned in Chapter 2, in many ways we would be justified in thinking of electrical networks and random walks as two faces of the same underlying object. Here we discover a third face, which, again, sounds at first to be completely unrelated.

Every connected graph has a ***spanning tree***, i.e., a subgraph that is a tree and that includes every vertex. Special spanning trees of Cayley graphs were used in Section 3.4. Here, we consider finite and, more generally, recurrent graphs and properties of their spanning trees when such trees are chosen randomly.

Let G be a finite graph and e and f be edges of G. If T is a random spanning tree chosen uniformly from all spanning trees of G, then it is intuitively clear that the events $\{e \in T\}$ and $\{f \in T\}$ are negatively correlated, i.e., the probability that both happen is at most the product of the probabilities that each happens: Intuitively, the presence of f would make e less needed for connecting everything *and* more likely to create a cycle. Furthermore, since the number of edges in a spanning tree is constant, namely, one fewer than the number of vertices in the graph, the negative correlation is certainly true on average. We will indeed prove this negative correlation by a method that relies upon random walks and electrical networks. Interestingly, no direct proof is known.

Notation. In an undirected graph, a spanning tree is also composed of undirected edges. However, we will be using the ideas and notations of Chapter 2 concerning random walks and electrical networks, so that we will be making use of directed edges as well. Sometimes, e will even denote an undirected edge on one side of an equation and a directed edge on the other side; see, e.g., Corollary 4.4. This abuse of notation, we hope, will be easier for the reader than would the use of different notations for directed and undirected edges.

§4.1. Generating Uniform Spanning Trees.

A graph typically has an enormous number of spanning trees. Because of this, it is not obvious how to choose one uniformly at random in a reasonable amount of time. We are going to present an algorithm that works quickly by exploiting some hidden independence in Markov chains. This algorithm is of enormous theoretical importance for us. Although we are interested in spanning trees of undirected graphs, it turns out that for this algorithm, it is just as easy to work with directed graphs coming from Markov chains.

Let $p(\bullet, \bullet)$ be the transition probability of a finite-state irreducible Markov chain. The directed graph associated to this chain has for vertices the states and for edges all (x, y) for which $p(x, y) > 0$. Edges e are oriented from tail e^- to head e^+. We call a connected subgraph a ***spanning tree**** if it includes every vertex and there is one vertex, the ***root***, such that every vertex other than the root is the tail of exactly one edge in the tree and there is no cycle. Thus, the edges in a spanning tree point towards its root. For any vertex r, there is at least one spanning tree rooted at r: Pick some vertex other than r and draw a path from it to r that does not contain any cycles. Such a path exists by irreducibility. This starts the tree. Then continue with another vertex not on the part of the tree already drawn, draw a cycle-free path from it to the partial tree, and so on. Remarkably, with a little care, this naive method of drawing spanning trees leads to a very powerful algorithm.

We are going to choose spanning trees at random according not only to uniform measure, but, in general, proportional to their weights, where, for a spanning tree T, we define its ***weight*** to be

$$\alpha(T) := \prod_{e \in T} p(e) \,.$$

In case the original Markov chain $\langle X_n \rangle$ is reversible, let us see what the weights $\alpha(T)$ are. Given conductances $c(e)$ with $\pi(x) = \sum_{e^- = x} c(e)$, the transition probabilities are $p(e) := c(e)/\pi(e^-)$, so the weight of a spanning tree T is

$$\alpha(T) = \prod_{e \in T} p(e) = \prod_{e \in T} c(e) \bigg/ \prod_{\substack{x \in G \\ x \neq \mathrm{root}(T)}} \pi(x) \,.$$

Since the root is fixed, a tree T is also picked with probability proportional to $\alpha(T)/\pi\big(\mathrm{root}(T)\big)$, which is proportional to

$$\beta(T) := \prod_{e \in T} c(e) \,.$$

* For directed graphs, these are usually called ***spanning arborescences***.

Note that $\beta(T)$ is independent of the root of T. This new expression, $\beta(T)$, has a nice interpretation. If $c = \mathbf{1}$, then all spanning trees are equally likely. If all the weights $c(e)$ are positive integers, then we could replace each edge e by $c(e)$ parallel copies of e and interpret the uniform spanning tree measure in the resulting multigraph as the probability measure above with the probability of T proportional to $\beta(T)$. If all weights are divided by the same constant, then the probability measures does not change, so the case of rational weights can still be thought of as corresponding to a uniform spanning tree. Since the case of general weights is a limit of rational weights, we use the term ***weighted uniform spanning tree*** for these probability measures. Similar comments apply to the non-reversible case.

Now suppose we have some method of choosing a spanning tree at random proportionally to the weights $\alpha(\bullet)$. Consider any vertex u on a weighted undirected graph. If we choose a random spanning tree rooted at u proportionally to the weights $\alpha(\bullet)$ and forget about the orientation of its edges and also about the root, we obtain an unrooted spanning tree of the undirected graph, chosen proportionally to the weights $\beta(\bullet)$. In particular, if the conductances are all equal, corresponding to the Markov chain being simple random walk, we get a uniformly chosen spanning tree.

The method we now describe for generating random spanning trees is the fastest method known. It is due to Wilson (1996) (see also Propp and Wilson (1998)).

To describe Wilson's method, we define the important idea of loop erasure* of a path, due to Lawler (1980). If $\mathcal{P}$ is any finite path $\langle x_0, x_1, \ldots, x_l \rangle$ in G, we define the ***loop erasure*** of $\mathcal{P}$, denoted $\mathsf{LE}(\mathcal{P}) = \langle u_0, u_1, \ldots, u_m \rangle$, by erasing cycles in $\mathcal{P}$ in the order they appear. More precisely, set $u_0 := x_0$. If $x_l = x_0$, we set $m = 0$ and terminate; otherwise, let u_1 be the first vertex in $\mathcal{P}$ after the last visit to x_0, i.e., $u_1 := x_{i+1}$, where $i := \max\{j\,;\, x_j = x_0\}$. If $x_l = u_1$, then we set $m = 1$ and terminate; otherwise, let u_2 be the first vertex in $\mathcal{P}$ after the last visit to u_1, and so on. In the case of a multigraph, one cannot notate a path merely by the vertices it visits. However, the notion of loop erasure should still be clear. For example, the loop erasure of the planar path shown in Figure 2.2 appears in Figure 4.1.

Now in order to generate a random spanning tree with a given root r, create a growing sequence of trees $T(i)$ $(i \ge 0)$ as follows. Choose any ordering of the vertices $\mathsf{V} \setminus \{r\}$. Let $T(0) := \{r\}$. Suppose that $T(i)$ is known. If $T(i)$ spans G, we are done. Otherwise, pick the first vertex x in our ordering of V that is not in $T(i)$ and take an independent sample from the Markov chain beginning at x until it hits $T(i)$. Now create $T(i+1)$ by adding to $T(i)$ the loop erasure of this path from x to $T(i)$. We claim that the final tree

* This ought to be called "cycle erasure", but we will keep to the name already given to this concept.

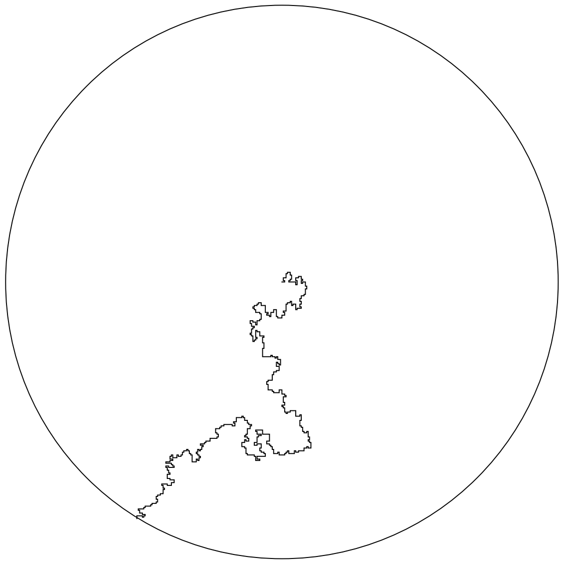


Figure 4.1. A loop-erased simple random walk in $\mathbb{Z}^2$ until it reaches distance 200 from its starting point.

in this growing sequence has the desired distribution. We call this ***Wilson's method*** of generating random spanning trees.

Theorem 4.1. *Given any finite-state irreducible Markov chain and any state r, Wilson's method yields a random spanning tree rooted at r with distribution proportional to $\alpha(\bullet)$. Therefore, for any finite connected undirected graph, Wilson's method yields a random spanning tree that, when the orientation and root are forgotten, has distribution proportional to $\beta(\bullet)$.*

In particular, this says that the distribution of the spanning tree resulting from Wilson's method does not depend on the choice made in ordering V. In fact, we'll see that in some sense, neither does the tree itself.

In order to state this precisely, we construct the Markov chain in a particular way. Every time we are at a state x, the next state will have a given probability distribution; and the choices of which states follow the visits to x are independent of each other. This, of course, is just part of the definition of a Markov chain. Constructively, however, we implement this as follows. Let $\langle S_i^x\,;\ x \text{ is a state and } i \geq 1\rangle$ be independent with each S_i^x having the transition probability distribution from x. For each x, think of $\langle S_i^x\,;\ i \geq 1\rangle$ as a stack lying under the state x. To run the Markov chain starting from a distribution π, we simply pick the first state according to π and then "pop off" the top state of the stack lying under the current state as long as we want to run the chain. In other words, we begin with the current state picked according to π. Then, from the current state at any time, the next state is the first state in the stack under the current state. This state is then removed from that stack and we repeat with the next state as the current state.

In order to generate a random spanning tree rooted at r, however, we make one small variation: give r an empty stack. Of course, our aim is not to generate the Markov chain, but a uniform spanning tree. We use the stacks as follows for this purpose. Observe that at any time, the top items of the stacks determine a directed graph, namely, the directed graph whose vertices are the states and whose edges are the pairs (x, y) where y is the top item of the stack under x. Call this the ***visible graph*** at that time. If it happens that the visible graph contains no (directed) cycles, then it is a spanning tree rooted at r. Otherwise, we ***pop*** a cycle, meaning that we remove the top items of the stacks under the vertices of a cycle. Then we pop a remaining cycle, if any, and so on. We claim that this process will stop with probability 1 at a spanning tree and that this spanning tree has the desired distribution conditional on being rooted at r. Note that we do not pop the top of a stack unless it belongs to a cycle. We will also show that this way of generating a random spanning tree is the same as Wilson's method.

Say that an edge (x, S_i^x) has ***color*** i. A ***colored cycle*** is simply a cycle all of whose edges are colored like this (the colors of the edges in a cycle do not have to be the same as each other). Thus, the initial visible graph has all edges colored 1, whereas later visible graphs will not generally have all their edges the same color. While a cycle of vertices might be popped many times, a colored cycle can be popped at most once. See Figure 4.2.

1	2	3	4	5	r
2	1	4	5	3	
4	3	2	2	1	
r	3	1	1	r	
2	1	2	5	1	
4	r	r	1	3	

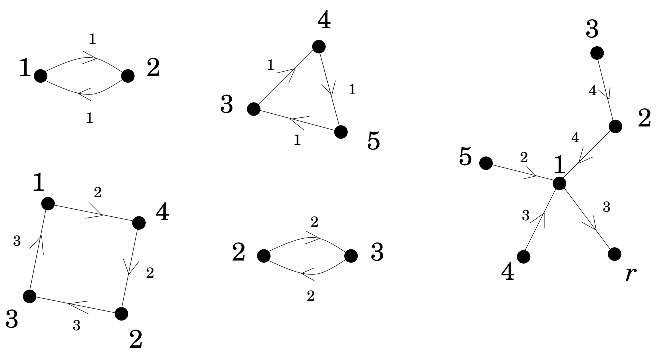


Figure 4.2. This Markov chain has 6 states, one called r, which is the root. The first 5 elements of each stack are listed under the corresponding states. Colored cycles are popped as shown clockwise, leaving the colored spanning tree shown.

Lemma 4.2. *Given any stacks under the states, the order in which cycles are popped is irrelevant in the sense that every order pops an infinite number of cycles or every order pops the same (finite number of) colored cycles, thus leaving the same colored spanning* *tree.*

Proof. We will show that if C is any colored cycle that can be popped, i.e., there is some sequence $C_1, C_2, \ldots, C_n = C$ that may be popped in that order, but some colored cycle $C' \neq C_1$ happens to be the first colored cycle popped, then $C = C'$ or else C can still be popped from the graph visible after C' is popped. Once we show this, we are done, since if there are an infinite number of colored cycles that can be popped, the popping can never stop; while in the alternative case, every colored cycle that can be popped will be popped.

Now if all the vertices of C' are disjoint from those of $C_1, C_2, \ldots, C_n$, then of course C can still be popped. Otherwise, let C_k be the first cycle that has a vertex in common with C'. Now, all the edges of C' have color 1. Consider any vertex x in $C' \cap C_k$. Since $x \notin C_1 \cup C_2 \cup \cdots \cup C_{k-1}$, the edge in C_k leading out of x also has color 1, so it leads to the same vertex as it does in C'. We can repeat the argument for this successor vertex of x, then for its successor, and so on, until we arrive at the conclusion that $C' = C_k$. Thus, $C' = C$ or we can pop C in the order $C', C_1, C_2, \ldots, C_{k-1}, C_{k+1}, \ldots, C_n$. ◀

Proof of Theorem 4.1. Wilson's method certainly stops with probability 1 at a spanning tree. Using stacks to run the Markov chain and noting that loop erasure in order of cycle creation is the same as cycle popping, we see that Wilson's method pops all the cycles lying over a spanning tree. Because of Lemma 4.2, popping cycles in any other manner also stops with probability 1 and with the same distribution. Furthermore, if we think of the stacks as given in advance, then we see that all our choices inherent in Wilson's method have no effect at all on the resulting spanning tree.

Now to show that the distribution is the desired one, think of a given set of stacks as defining a set O of colored cycles lying over a noncolored spanning tree T. We don't need to keep track of the colors in the spanning tree since they are easily recovered from the colors in the cycles over it. Let X be the set of all pairs (O, T) that can arise from stacks corresponding to our given Markov chain. If $(O, T) \in X$, then also $(O, T') \in X$ for any other spanning tree T': indeed, anything at all can be in the stacks under any set O of colored cycles. That is, $X = X_1 \times X_2$, where X_1 is a certain collection of sets of colored cycles and X_2 is the set of all noncolored spanning trees. Extend our definition of $\alpha(\bullet)$ to colored cycles by $\alpha(C) := \prod_{e \in C} p(e)$ and to sets of colored cycles by $\alpha(O) := \prod_{C \in O} \alpha(C)$. What is the chance of seeing a given set of colored cycles O lying over a given spanning tree T? It is simply the probability of seeing all the arrows in $\bigcup O \cup T$ in their respective places, which is simply the product of $p(e)$ for all $e \in \bigcup O \cup T$, i.e., $\alpha(O)\alpha(T)$. Letting $\mathbf{P}$ be the law of (O, T), we get $\mathbf{P} = \mu_1 \times \mu_2$, where μ_i are probability measures proportional to $\alpha(\bullet)$ on X_i. Therefore, the set of colored cycles seen is independent of the colored spanning tree and the probability of seeing the tree T is proportional to $\alpha(T)$. This shows that Wilson's method does what we claimed it does. ◀

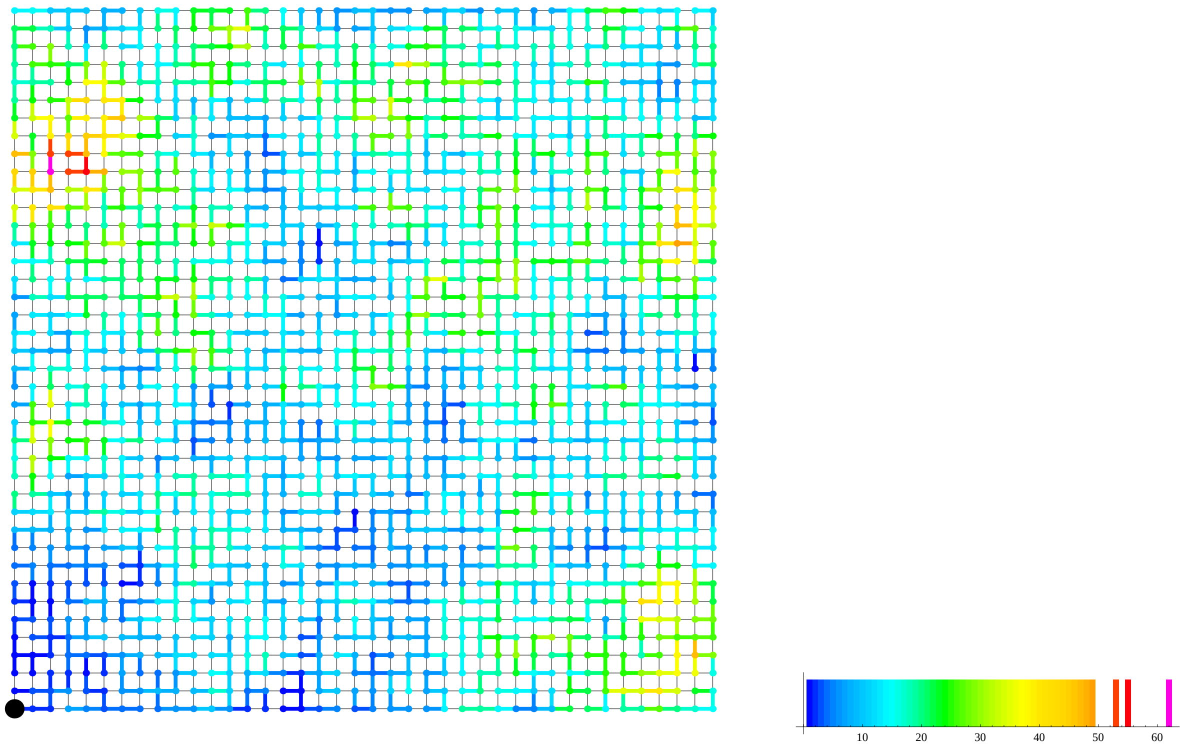


Figure 4.3. A colored uniform spanning tree in a 40×40 grid on the left, with a key on the right showing the correspondence of visual colors to numbered colors.

An actual example of Wilson's algorithm showing the colored spanning tree on a 40×40 grid is shown in Figure 4.3. Since the colors of the vertices and the spanning tree determine the colors of the edges, and since the colors of the vertices are determined by the popped cycles, which are independent of the spanning tree, it follows that the colors of the vertices are independent of the spanning tree. Just the vertex colors are shown for a uniform spanning tree on a 200×200 grid in Figure 4.4. There appears to be an interesting fractal nature in the limit of decreasing mesh size. In Figure 4.5, we show the distances in the tree to the lower left vertex, together with the path from the upper right vertex. This seems to be the best way of viewing a large spanning tree. Although the distances do not determine the tree, all spanning trees consistent with the given distances are, of course, equally likely. Furthermore, given the distances, one can easily sample from the consistent spanning trees by working one's way out from the root: The vertices at distance 1 from the root must be attached to the root, while the vertices at distance 2 can be attached uniformly at random to their neighbors at distance 1, etc. The distance in the tree from the root to the opposite corner, say, grows like $n^{5/4}$ in an $n \times n$ square: This was first calculated by Duplantier (1992) and Majumdar (1992) using nonrigorous conformal field theory. Kenyon (2000) proved a form of this using domino tilings associated to spanning trees. It was extended to other planar lattices by Masson (2009). An alternative view is given in Figure 4.6, where the distances in the tree to the path between the corners is shown.

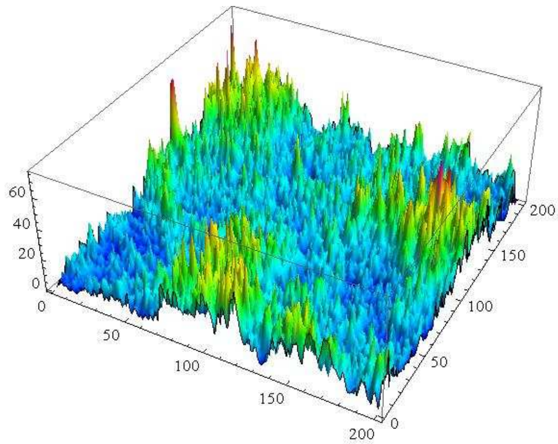


Figure 4.4. The colors of a uniform spanning tree in a 200×200 grid.

To illustrate one use of Wilson's method, we give a new proof of Cayley's formula for the number of spanning trees on a ***complete graph***, i.e., a graph in which every pair of distinct vertices is joined by an edge. A number of proofs are known of this result; for a collection of them, see Moon (1967). The following proof is inspired by the one of Aldous (1990), Prop. 19.

Corollary 4.3. (Cayley, 1889) *The number of labelled unrooted trees with n vertices is n^{n-2}. Here, a labelled tree is one whose vertices are labelled 1 through n.*

To prove this, we use the result of the following exercise.

▷ **Exercise 4.1.**

Suppose that Z is a set of states in a Markov chain and that x_0 is a state not in Z. Assume that when the Markov chain is started in x_0, then it visits Z with probability 1. Define the random path $Y_0, Y_1, \ldots$ by $Y_0 := x_0$ and then recursively by letting Y_{n+1} have the distribution of one step of the Markov chain starting from Y_n given that the chain will visit Z before visiting any of $Y_0, Y_1, \ldots, Y_n$ again. However, if $Y_n \in Z$, then the path is stopped and Y_{n+1} is not defined. Show that $\langle Y_n \rangle$ has the same distribution as loop-erasing a sample of the Markov chain started from x_0 and stopped when it reaches Z. In the case

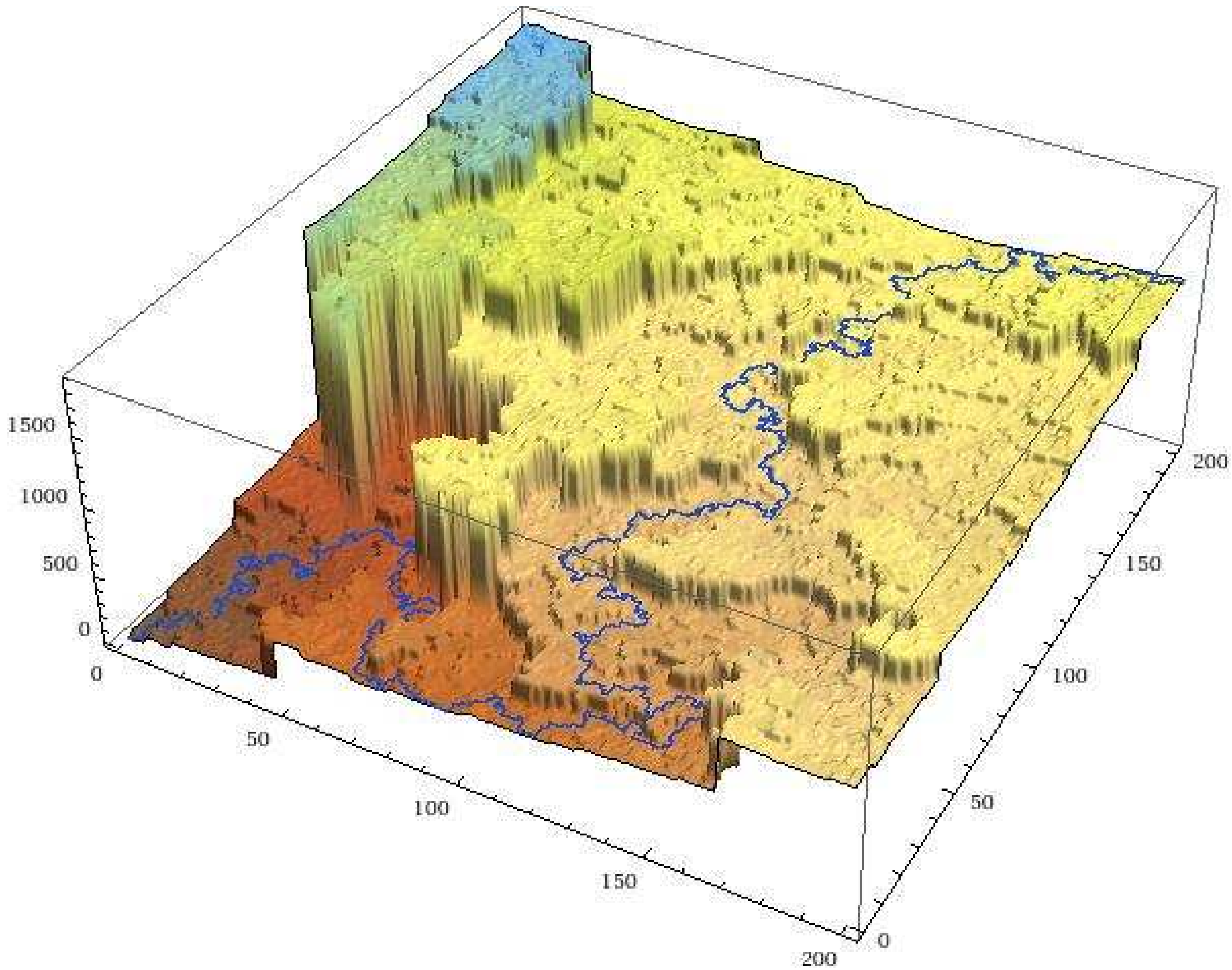


Figure 4.5. The distances to the root in a uniform spanning tree in a 200×200 grid, together with the path from the opposite corner.

of a random walk, the conditioned path $\langle Y_n \rangle$ is called the ***Laplacian random walk*** from x_0 to Z.

Proof of Corollary 4.3. We show that the uniform probability of a specific spanning tree of the complete graph on $\{1, 2, \ldots, n\}$ is $1/n^{n-2}$. Take the tree to be the path $\langle 1, 2, 3, \ldots, n \rangle$. We will calculate the probability of this tree by using Wilson's algorithm started at 1 and rooted at n. Since the root is n and the tree is a path from 1 to n, this tree probability is just the chance that loop-erased random walk from 1 to n is this particular path. By Exercise 4.1, we must show that the chance that the Laplacian random walk $\langle Y_n \rangle$ from 1 to n is equal to this path is $1/n^{n-2}$. Recall the following notation from Chapter 2: $\mathbf{P}_i$

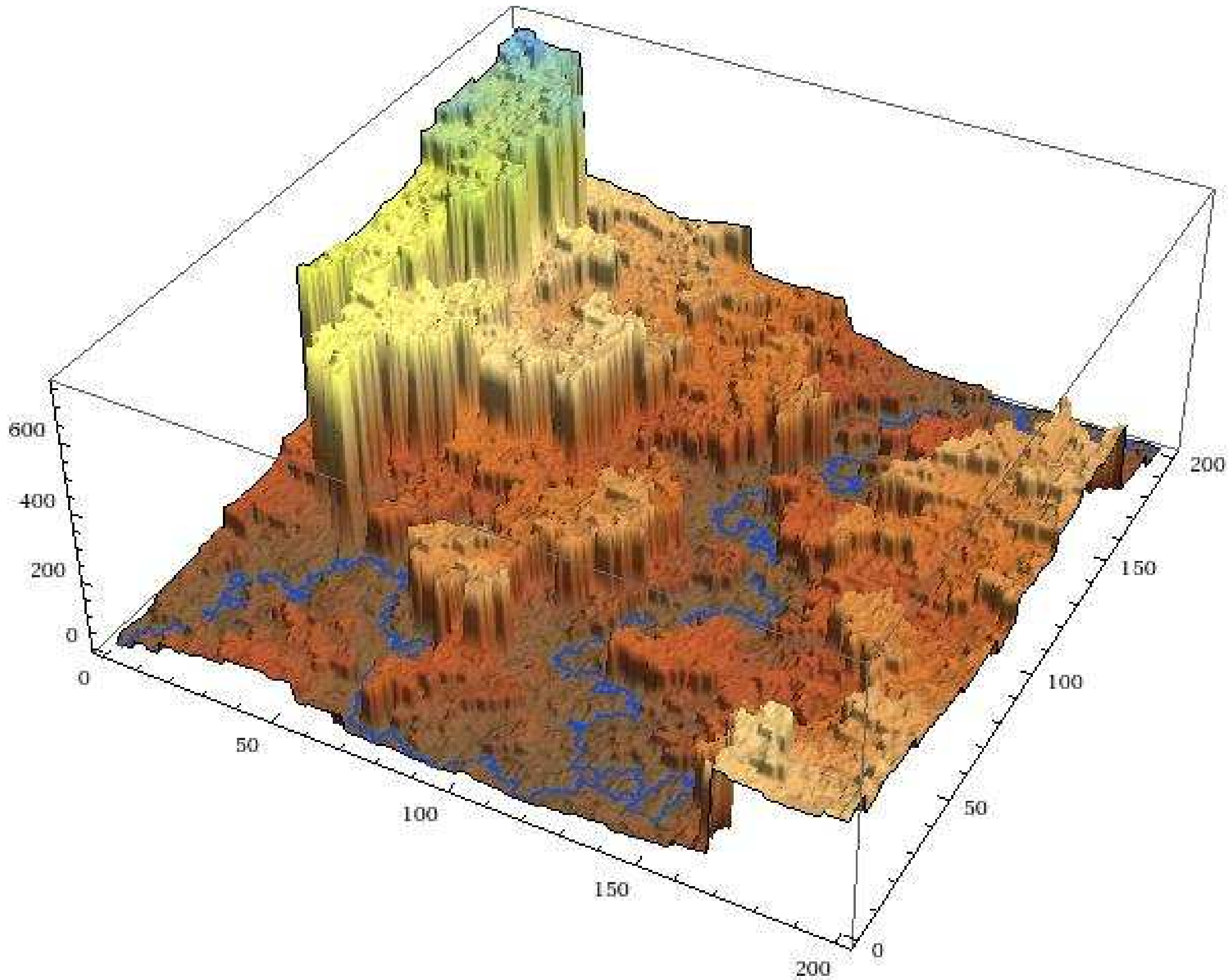


Figure 4.6. The distances to the path between opposite corners in a uniform spanning tree in a 200×200 grid.

denotes simple random walk started at state i; the first time ≥ 0 that the walk visits state k is denoted τ_k; and the first time ≥ 1 that the walk visits state k is denoted τ_k^+. Let $\langle X_n \rangle$ be the usual simple random walk.

Consider the distribution of Y_1. We have, for all $i \in [2, n]$,

$$\mathbf{P}[Y_1 = i] = \mathbf{P}_1[X_1 = i \mid \tau_n < \tau_1^+] = \frac{\mathbf{P}_1[X_1 = i, \tau_n < \tau_1^+]}{\mathbf{P}_1[\tau_n < \tau_1^+]}$$
$$= \frac{\mathbf{P}_1[X_1 = i]\mathbf{P}_i[\tau_n < \tau_1]}{\mathbf{P}_1[\tau_n < \tau_1^+]} = \frac{\mathbf{P}_i[\tau_n < \tau_1]}{(n-1)\mathbf{P}_1[\tau_n < \tau_1^+]} .$$

Now

$$\mathbf{P}_i[\tau_n < \tau_1] = \begin{cases} 1/2 & \text{if } i \neq n, \\ 1 & \text{if } i = n. \end{cases}$$

Since the probabilities for Y_1 add to 1, it follows that $\mathbf{P}_1[\tau_n < \tau_1^+] = n/[2(n-1)]$, whence $\mathbf{P}[Y_1 = i] = 1/n$ for $1 < i < n$. Similarly, for $j \in [1, n-2]$ and $i \in [j+1, n]$, we have

$$\begin{aligned}\mathbf{P}[Y_j = i \mid Y_1 = 2, \ldots, Y_{j-1} = j] &= \mathbf{P}_j[X_1 = i \mid \tau_n < \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_{j-1} \wedge \tau_j^+] \\ &= \frac{\mathbf{P}_j[X_1 = i, \tau_n < \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_{j-1} \wedge \tau_j^+]}{\mathbf{P}_j[\tau_n < \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_{j-1} \wedge \tau_j^+]} \\ &= \frac{\mathbf{P}_j[X_1 = i]\mathbf{P}_i[\tau_n < \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_{j-1} \wedge \tau_j]}{\mathbf{P}_j[\tau_n < \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_{j-1} \wedge \tau_j^+]}\,.\end{aligned}$$

Now the minimum of $\tau_1, \ldots, \tau_j, \tau_n$ for simple random walk starting at $i \in (j, n)$ is equally likely to be any one of these. Therefore,

$$\mathbf{P}_i[\tau_n < \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_{j-1} \wedge \tau_j] = \begin{cases} 1/(j+1) & \text{if } j < i < n, \\ 1 & \text{if } i = n. \end{cases}$$

Since $\mathbf{P}_j[X_1 = i] = 1/(n-1)$, we obtain $\mathbf{P}_j[\tau_n < \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_{j-1} \wedge \tau_j^+] = n/[(j+1)(n-1)]$ and thus

$$\mathbf{P}[Y_j = j+1 \mid Y_1 = 2, \ldots, Y_{j-1} = j] = 1/n$$

for all $j \in [1, n-2]$. Of course,

$$\mathbf{P}[Y_{n-1} = n \mid Y_1 = 2, \ldots, Y_{n-2} = n-1] = 1\,.$$

Multiplying together these conditional probabilities gives the result. ◀

§4.2. Electrical Interpretations.

We return now to undirected graphs and networks, except for occasional parenthetical remarks about more general Markov chains. Wilson's method for generating spanning trees will also give a random spanning tree (a.s.) on any recurrent network (or for any recurrent irreducible Markov chain). How should we interpret it? Suppose that G' is a finite subnetwork of G and consider T_G and $T_{G'}$, random spanning trees generated by Wilson's method on G and G', respectively. After describing a connection to electrical networks, we will show that for any event $\mathscr{B}$ depending on only finitely many edges, we can make $|\mathbf{P}[T_G \in \mathscr{B}] - \mathbf{P}[T_{G'} \in \mathscr{B}]|$ arbitrarily small by choosing G' sufficiently large*. Thus, the random spanning tree of G looks locally like a tree chosen with probability proportional to $\beta(\bullet)$. Also, these local probabilities determine the distribution of T_G uniquely. In

* This can also be proved by coupling the constructions using Wilson's method.

particular, when simple random walk is recurrent, such as on $\mathbb{Z}^2$, we may regard T_G as a "uniform" random spanning tree of G. Also, this will show that Wilson's method on a recurrent network generates a random spanning tree whose distribution again does not depend on the choice of root nor on the ordering of vertices**.

We will study uniform spanning trees on recurrent networks further and also "uniform" spanning forests on transient networks in Chapter 10. For now, though, we will deduce some important theoretical consequences of the connection between random walk and spanning trees which hold for finite as well as for recurrent graphs. For recurrent networks, the definitions and relations among random walks and electrical networks appear in Exercises 2.63, 2.64 and 2.65; some are also covered in Section 9.1 and Corollary 9.6, but we won't need any material from Chapter 9 here.

Corollary 4.4. (Single-Edge Marginals) *Let T be an unrooted weighted uniform spanning tree of a recurrent network G and e an edge of G. Then*

$$\mathbf{P}[e \in T] = \mathbf{P}_{e^-}[\textit{1st hit } e^+ \textit{ via traveling along } e] = i(e) = c(e)\mathscr{R}(e^- \leftrightarrow e^+)\,,$$

where i is the unit current from e^- to e^+.

Remark. That $\mathbf{P}[e \in T] = i(e)$ in finite networks is due to Kirchhoff (1847).

Proof. The first equality follows by taking the vertex e^+ as the root of T and then starting the construction of Wilson's method at e^-. The second equality then follows from the probabilistic interpretation, Exercise 2.63, of i as the expected number of crossings of e minus the expected number of crossings of the reversed edge $-e$ for a random walk started at e^- and stopped at e^+: e is crossed once or not at all and $-e$ is never crossed. The third equality comes from the definition of effective resistance. ◀

▷ **Exercise 4.2.**

Consider the ladder graph L_n of height n shown in Figure 1.8. Choose a spanning tree $T(n)$ of L_n uniformly. Use Corollary 4.4 to determine $\mathbf{P}[\text{rung 1 is in } T(n)]$ and its limiting behavior as $n \to \infty$.

We can use Corollary 4.4 to compute the chance that certain edges are in T and certain others are not. To see this, denote the dependence of T on G by T_G. The ***contraction*** G/e of a graph G along an edge e is obtained by removing the edge e and identifying its endpoints. Note that this may give a multigraph even if G is a simple graph. Deleting

** This can also be shown directly for any recurrent irreducible Markov chain.

e without identifying its endpoints gives the graph denoted $G\backslash e$. In both cases, we may identify the edges of G other than e with the edges of G/e and of $G\backslash e$. We also think of a spanning tree primarily as a set of edges. Now the distribution of T_G/e (the contraction of T_G along e) given $e \in T_G$ is the same as that of $T_{G/e}$ and the distribution of T_G given $e \notin T$ is the same as that of $T_{G\backslash e}$. This gives a recursive method to compute $\mathbf{P}[e_1, \ldots, e_k \in T, e_{k+1}, \ldots, e_l \notin T]$: for example, if $e \neq f$, then

$$\mathbf{P}[e, f \in T_G] = \mathbf{P}[e \in T_G]\mathbf{P}[f \in T_G \mid e \in T_G] = \mathbf{P}[e \in T_G]\mathbf{P}[f \in T_{G/e}]$$

and

$$\mathbf{P}[e \notin T_G, f \in T_G] = \mathbf{P}[e \notin T_G]\mathbf{P}[f \in T_{G\backslash e}]\,.$$

Thus, we may deduce that the events $e \in T$ and $f \in T$ are negatively correlated:

▷ **Exercise 4.3.**
By using Corollary 4.4, show that if $e \neq f$, then the events $e \in T$ and $f \in T$ are negatively correlated.

We can now also establish our claim at the beginning of this section that on a recurrent graph, the random spanning tree looks locally like that of large finite subnetworks. For example, given an edge e and a subnetwork G' of G, the current in G' flowing along e arising from a unit current between the endpoints of e will be very close to the corresponding current along e in G, provided G' is sufficiently large, by Exercise 2.63. That means that $\mathbf{P}[e \in T_{G'}]$ will be very close to $\mathbf{P}[e \in T_G]$.

▷ **Exercise 4.4.**
Write out the rest of the proof that for any event $\mathscr{B}$ depending on only finitely many edges, $|\mathbf{P}[T_{G'} \in \mathscr{B}] - \mathbf{P}[T_G \in \mathscr{B}]|$ is arbitrarily small for sufficiently large G'.

What does contraction of some edges do to a current in the setting of the inner-product space $\ell^2_-(\mathsf{E}, r)$? Let i^e denote the unit current from the tail of e to the head of e in a finite network G. Contract the edges $f \in F$ to obtain the graph G/F and let $\widehat{i}^e$ be the unit current flowing in G/F, where $e \notin F$ and, moreover, e does not form any undirected cycle with the edges of F, so that e does not become a loop when the edges of F are contracted. Note that the restriction of any $\theta \in \ell^2_-(\mathsf{E}, r)$ to $\mathsf{E} \setminus F$ yields an antisymmetric function on the edges of the contracted graph G/F. Let Z be the linear span of $\{i^f\,;\ f \in F\}$. We claim that

$$\widehat{i}^e = (P_Z^\perp i^e)\restriction(\mathsf{E} \setminus F) \tag{4.1}$$

and

$$(P_Z^\perp i^e)\restriction F = \mathbf{0}\,, \tag{4.2}$$

where $P_Z^\perp$ denotes the orthogonal projection onto the orthocomplement of Z.

To prove this, note that since $Z \subseteq \bigstar$ and $i^e \in \bigstar$, also $P_Z^\perp i^e = i^e - P_Z i^e \in \bigstar$. Recall from (2.9) or (2.11) that $P_\bigstar \chi^f = i^f$. Therefore, for $f \in F$,

$$(P_Z^\perp i^e, \chi^f)_r = (P_\bigstar P_Z^\perp i^e, \chi^f)_r = (P_Z^\perp i^e, P_\bigstar \chi^f)_r = (P_Z^\perp i^e, i^f)_r = 0\,.$$

That is, there is no flow across any edge in F for $P_Z^\perp i^e$, which is (4.2). Since $P_Z^\perp i^e \in \bigstar$ satisfies the cycle law in G, it follows from this that $(P_Z^\perp i^e)\restriction(\mathsf{E} \setminus F)$ satisfies the cycle law in G/F. To verify Kirchhoff's Node Law and finish the proof, write

$$P_Z^\perp i^e = i^e - \sum_{f \in F} \alpha_f i^f$$

for some constants α_f. Note that the stars ψ in G/F are sums θ of stars in G such that $\theta(f) = 0$ for all $f \in F$. Since i^e is orthogonal to all the stars in G except those at the endpoints of e, it follows that $i^e\restriction(\mathsf{E}\setminus F) \perp \psi$ if ψ is a star in G/F other than at an endpoint of e. Likewise, the restrictions to $\mathsf{E} \setminus F$ of i^f for all $f \in F$ are orthogonal to *all* the stars in G/F. Therefore, $i^e - \sum_{f \in F} \alpha_f i^f$ is orthogonal to all the stars in G/F except those at the endpoints of e, where the inner products are ± 1. This proves (4.1).

Although we have indicated that successive contractions can be used for computing $\mathbf{P}[e_1, \ldots, e_k \in T]$, this requires computations of effective resistance on k different graphs. In order to stick with the same graph, we may use a wonderful theorem of Burton and Pemantle (1993):

The Transfer-Current Theorem. *For any distinct edges $e_1, \ldots, e_k \in G$,*

$$\mathbf{P}[e_1, \ldots, e_k \in T] = \det[Y(e_i, e_j)]_{1 \le i,j \le k}\,. \tag{4.3}$$

Recall that $Y(e, f) = i^e(f)$. Note that, in particular, we get a quantitative version of the negative correlation between $\{e \in T\}$ and $\{f \in T\}$: for distinct edges e, f, we have

$$\mathbf{P}[e, f \in T] - \mathbf{P}[e \in T]\mathbf{P}[f \in T] = -Y(e,f)Y(f,e) = -c(e)r(f)Y(e,f)^2$$

by the reciprocity law (2.12).

Proof. It suffices to show the result for finite G, since taking limits of this result implies it holds for infinite recurrent G by Exercise 2.63.

Note that if some cycle can be formed from the edges $e_1, \ldots, e_k$, then a linear combination of the corresponding columns of $[Y(e_i, e_j)]$ is zero: suppose that such a cycle is $\sum_j a_j \chi^{e_j}$, where $a_j \in \{-1, 0, 1\}$. Then for $1 \le m \le k$,

$$\sum_j a_j r(e_j) Y(e_m, e_j) = \sum_j a_j r(e_j) i^{e_m}(e_j) = 0$$

by the cycle law applied to the current i^{e_m}. Therefore, both sides of (4.3) are 0. For the remainder of the proof, then, we may assume that there are no such cycles.

We will proceed by induction. When $k = 1$, (4.3) is the same as Corollary 4.4. For $1 \le m \le k$, let

$$Y_m := [Y(e_i, e_j)]_{1 \le i,j \le m} . \tag{4.4}$$

To carry the induction from $m = k - 1$ to $m = k$, we must show that

$$\det Y_k = \mathbf{P}[e_k \in T \mid e_1, \ldots, e_{k-1} \in T] \det Y_{k-1} . \tag{4.5}$$

Now we know that

$$\mathbf{P}[e_k \in T \mid e_1, \ldots, e_{k-1} \in T] = \widehat{i}^{e_k}(e_k) \tag{4.6}$$

for the current $\widehat{i}^{e_k}$ in the graph $G/\{e_1, \ldots, e_{k-1}\}$. In addition,

$$P_Z^{\perp} i^{e_k} = i^{e_k} - \sum_{m=1}^{k-1} a_m i^{e_m}$$

for some constants a_m. Subtracting these same multiples of the first $k - 1$ rows from the last row of Y_k leads to a matrix $\widehat{Y}$ whose (m, j)-entry is that of Y_{k-1} for $m, j < k$ and whose (k, j)-entry is

$$i^{e_k}(e_j) - \sum_{m=1}^{k-1} a_m i^{e_m}(e_j) = \left(P_Z^{\perp} i^{e_k}\right)(e_j) = \begin{cases} 0 & \text{if } j < k, \\ \widehat{i}^{e_k}(e_k) & \text{if } j = k \end{cases}$$

by (4.2) and (4.1). Therefore expansion of $\det \widehat{Y}$ along the kth row is very simple and gives that

$$\det Y_k = \det \widehat{Y} = \widehat{i}^{e_k}(e_k) \det Y_{k-1} .$$

Combining this with (4.6), we obtain (4.5). [At bottom, we are using (or proving) the fact that the determinant of a Gram matrix is the square of the volume of the parallelepiped determined by the vectors whose inner products give the entries.] ◀

It turns out that there is a more general negative correlation than that between the presence of two given edges. Regard a spanning tree as simply a set of edges. We may extend our probability measure $\mathbf{P}$ on the set of spanning trees to the product σ-field on $2^{\mathsf{E}(G)}$ by defining the probability to be 0 of the event that the set of edges do not form a spanning tree. Surprisingly, this is useful. Call an event $\mathscr{A} \subseteq 2^{\mathsf{E}(G)}$ ***increasing*** (also called ***upwardly closed***) if the addition of any edge to any set in $\mathscr{A}$ results in another set in $\mathscr{A}$, that is, $A \cup \{e\} \in \mathscr{A}$ for all $A \in \mathscr{A}$ and all $e \in \mathsf{E}$. For example, $\mathscr{A}$ could be the collection of all subsets of $\mathsf{E}(G)$ that contain at least two of the edges $\{e_1, e_2, e_3\}$. We say that an event $\mathscr{A}$ ***ignores an edge*** an edge e if $A \cup \{e\} \in \mathscr{A}$ and $A \setminus \{e\} \in \mathscr{A}$ for all $A \in \mathscr{A}$. In the prior example, e is ignored provided $e \notin \{e_1, e_2, e_3\}$. We also say that $\mathscr{A}$ ***depends on a set*** a set $F \subseteq \mathsf{E}(G)$ when $\mathscr{A}$ ignores every $e \in \mathsf{E}(G) \setminus F$.

▷ **Exercise 4.5.**

Suppose that $\mathscr{A}$ is an increasing event on a graph G and $e \in \mathsf{E}$. Note that $\mathsf{E}(G/e) = \mathsf{E}(G\backslash e) = \mathsf{E}(G) \setminus \{e\}$. Define $\mathscr{A}/e := \{F \subseteq \mathsf{E}(G/e)\,;\ F \cup \{e\} \in \mathscr{A}\}$ and $\mathscr{A}\backslash e := \{F \subseteq \mathsf{E}(G\backslash e)\,;\ F \in \mathscr{A}\}$. Show that these are increasing events on G/e and $G\backslash e$, respectively.

Pemantle conjectured (personal communication, 1990) that $\mathscr{A}$ and $e \in T$ are negatively correlated when $\mathscr{A}$ is an increasing event that ignores e. Though unaware that Pemantle had conjectured this, Feder and Mihail (1992) proved it:

Theorem 4.5. *Let G be a finite network. If $\mathscr{A}$ is an increasing event that ignores some edge e, then $\mathbf{P}[\mathscr{A} \mid e \in T] \le \mathbf{P}[\mathscr{A}]$.*

Proof. We induct on the number of edges in G. The case of exactly one (undirected) edge is trivial. Now assume that the number of (undirected) edges is $m \ge 2$ and that we know the result for graphs with $m-1$ edges. Let G have m edges. If $\mathbf{P}[f \in T] = 1$ for some $f \in \mathsf{E}$, then we could contract f and reduce to the case of $m-1$ edges by Exercise 4.5, so assume this is not the case. If $|\mathsf{V}| = 2$ and G has parallel edges, then the result also follows without using induction, so assume that $|\mathsf{V}| \ge 3$. Fix an increasing event $\mathscr{A}$ and an edge e ignored by $\mathscr{A}$. We may assume that $\mathbf{P}[\mathscr{A} \mid e \in T] > 0$ in order to prove our inequality. The graph G/e has only $m-1$ edges and every spanning tree of G/e has $|\mathsf{V}| - 2$ edges. Thus, we have

$$\sum_{f \in \mathsf{E}\backslash e} \mathbf{P}[\mathscr{A}, f \in T \mid e \in T] = (|\mathsf{V}| - 2)\mathbf{P}[\mathscr{A} \mid e \in T] = \mathbf{P}[\mathscr{A} \mid e \in T] \sum_{f \in \mathsf{E}\backslash e} \mathbf{P}[f \in T \mid e \in T]\,.$$

Therefore there is some $f \in \mathsf{E} \setminus e$ with $\mathbf{P}[f \in T \mid e \in T] > 0$ such that

$$\mathbf{P}[\mathscr{A}, f \in T \mid e \in T] \ge \mathbf{P}[\mathscr{A} \mid e \in T]\mathbf{P}[f \in T \mid e \in T]\,,$$

which is the same as $\mathbf{P}[\mathscr{A} \mid f, e \in T] \geq \mathbf{P}[\mathscr{A} \mid e \in T]$. This also means that

$$\mathbf{P}[\mathscr{A} \mid f, e \in T] \geq \mathbf{P}[\mathscr{A} \mid f \notin T, e \in T]\,; \tag{4.7}$$

in case this is not evident, one can deduce it from

$$\mathbf{P}[\mathscr{A} \mid e \in T] = \mathbf{P}[f \in T \mid e \in T]\mathbf{P}[\mathscr{A} \mid f, e \in T] + \mathbf{P}[f \notin T \mid e \in T]\mathbf{P}[\mathscr{A} \mid f \notin T, e \in T]\,. \tag{4.8}$$

Now we also have

$$\mathbf{P}[f \in T \mid e \in T] \leq \mathbf{P}[f \in T]$$

by Exercise 4.3. Because of (4.7), it follows that

$$\mathbf{P}[\mathscr{A} \mid e \in T] \leq \mathbf{P}[f \in T]\mathbf{P}[\mathscr{A} \mid f, e \in T] + \mathbf{P}[f \notin T]\mathbf{P}[\mathscr{A} \mid f \notin T, e \in T]\ : \tag{4.9}$$

we have replaced a convex combination in (4.8) by another in (4.9) that puts more weight on the larger term. We also have

$$\mathbf{P}[\mathscr{A} \mid f, e \in T] \leq \mathbf{P}[\mathscr{A} \mid f \in T] \tag{4.10}$$

by the induction hypothesis applied to the event $\mathscr{A}/f$ on the network G/f (see Exercise 4.5), and

$$\mathbf{P}[\mathscr{A} \mid f \notin T, e \in T] \leq \mathbf{P}[\mathscr{A} \mid f \notin T] \tag{4.11}$$

by the induction hypothesis applied to the event $\mathscr{A}\backslash f$ on the network $G\backslash f$. By (4.10) and (4.11), we have that the right-hand side of (4.9) is

$$\leq \mathbf{P}[f \in T]\mathbf{P}[\mathscr{A} \mid f \in T] + \mathbf{P}[f \notin T]\mathbf{P}[\mathscr{A} \mid f \notin T] = \mathbf{P}[\mathscr{A}]\,. \qquad \blacktriangleleft$$

▷ **Exercise 4.6.**

(Negative Association) Let G be a finite network. Extend Theorem 4.5 to show that if $\mathscr{A}$ and $\mathscr{B}$ are both increasing events and they depend on disjoint sets of edges, then they are negatively correlated. Still more generally, show the following. Say that a random variable X ***depends on a set*** a set $F \subseteq \mathsf{E}(G)$ if X is measurable with respect to the σ-field consisting of events that depend on F. Say also that X is ***increasing*** if $X(H) \leq X(H')$ whenever $H \subset H'$. If X and Y are increasing random variables with finite second moments that depend on disjoint sets of edges, then $\mathbf{E}[XY] \leq \mathbf{E}[X]\mathbf{E}[Y]$. This property of $\mathbf{P}$ is called ***negative association***.

As we shall see in Chapter 10, the negative correlation result of Theorem 4.5 is quite useful. A related type of negative correlation that is unknown is as follows. It was conjectured by Benjamini, Lyons, Peres, and Schramm (2001), hereinafter referred to as BLPS (2001). We say that $\mathscr{A}, \mathscr{B} \subset 2^{\mathsf{E}}$ ***occur disjointly*** for $F \subset \mathsf{E}$ if there are disjoint sets $F_1, F_2 \subset \mathsf{E}$ such that $F' \in \mathscr{A}$ for every F' with $F' \cap F_1 = F \cap F_1$ and $F' \in \mathscr{B}$ for every F' with $F' \cap F_2 = F \cap F_2$. For example, $\mathscr{A}$ may be the event that x and y are joined by a path of length at most 5, while $\mathscr{B}$ may be the event that z and w are joined by a path of length at most 6. If there are disjoint paths of lengths at most 5 and 6 joining the first and second pair of vertices, respectively, then $\mathscr{A}$ and $\mathscr{B}$ occur disjointly.

Conjecture 4.6. *Let $\mathscr{A}, \mathscr{B} \subset 2^{\mathsf{E}}$ be increasing. Then the probability that $\mathscr{A}$ and $\mathscr{B}$ occur disjointly for the weighted uniform spanning tree T is at most $\mathbf{P}[T \in \mathscr{A}]\mathbf{P}[T \in \mathscr{B}]$.*

The BK inequality of van den Berg and Kesten (1985) says that this inequality holds when T is a random subset of E chosen according to any product measure on 2^{E}; it was extended by Reimer (2000) to allow $\mathscr{A}$ and $\mathscr{B}$ to be any events, confirming a conjecture of van den Berg and Kesten (1985). However, we cannot allow arbitrary events for uniform spanning trees: consider the case where $\mathscr{A} := \{e \in T\}$ and $\mathscr{B} := \{f \notin T\}$, where $e \neq f$.

§4.3. The Square Lattice $\mathbb{Z}^2$.

Uniform spanning trees on the nearest-neighbor graph on the square lattice $\mathbb{Z}^2$ are particularly interesting. A portion of one is shown in Figure 4.7. This can be thought of as an infinite maze. In Section 10.6, we show that that there is exactly one way to get from any square to any other square without backtracking and exactly one way to get from any square to infinity without backtracking.

What is the distribution of the degree of a vertex with respect to a uniform spanning tree in $\mathbb{Z}^2$? It turns out that the expected degree is easy to calculate and is a quite general result. This uses the amenability of $\mathbb{Z}^2$. This is the following notion. If G is a graph and $K \subset \mathsf{V}$, the ***edge boundary*** of K is the set $\partial_{\mathsf{E}} K$ of (unoriented) edges that connect K to its complement. We say that G is ***edge amenable*** if there are finite $\mathsf{V}_n \subset \mathsf{V}$ with

$$\lim_{n\to\infty} |\partial_{\mathsf{E}} \mathsf{V}_n| / |\mathsf{V}_n| = 0 \,.$$

▷ **Exercise 4.7.**

Let G be an edge-amenable infinite graph as witnessed by the sequence $\langle \mathsf{V}_n \rangle$. Show that the average degree of vertices in any spanning tree of G is 2. That is, if $\deg_T(x)$ denotes

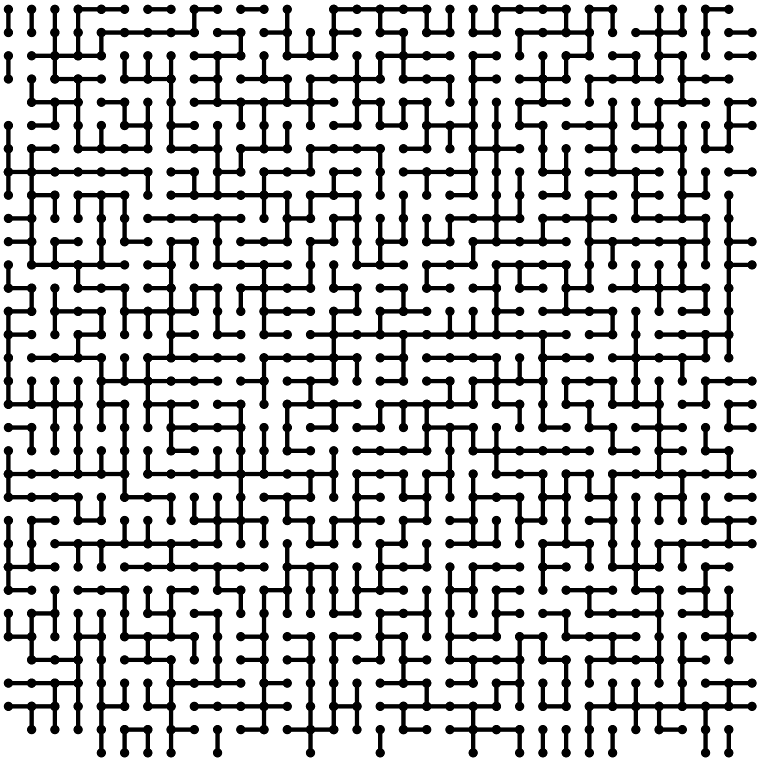


Figure 4.7. A portion of a uniformly chosen spanning tree on $\mathbb{Z}^2$, drawn by David Wilson.

the degree of x in a spanning tree T of G, then

$$\lim_{n\to\infty} |\mathsf{V}_n|^{-1} \sum_{x\in\mathsf{V}_n} \deg_T(x) = 2\,.$$

Every infinite recurrent graph can be shown to be edge-amenable by various results from Chapter 6 that we'll look at later, such as Theorems 6.5, 6.7, or 6.18. Deduce that for the uniform spanning tree measure on a recurrent graph,

$$\lim_{n\to\infty} |\mathsf{V}_n|^{-1} \sum_{x\in\mathsf{V}_n} \mathbf{E}[\deg_T(x)] = 2\,.$$

In particular, if G is also transitive, such as $\mathbb{Z}^2$, meaning that for every pair of vertices x and y, there is a bijection of V with itself that preserves adjacency and takes x to y, then every vertex has expected degree 2.

By symmetry, each edge of $\mathbb{Z}^2$ has the same probability to be in a uniform spanning tree of $\mathbb{Z}^2$. Since the expected degree of a vertex is 2 by Exercise 4.7, it follows that

$$\mathbf{P}[e \in T] = 1/2 \tag{4.12}$$

for each $e \in \mathbb{Z}^2$. By Corollary 4.4, this means that if unit current flows from the tail to the head of e, then 1/2 of the current flows directly across e and that the effective resistance between two adjacent vertices is 1/2. These electrical facts are classic engineering puzzles.

We will use the Transfer-Current Theorem to calculate the distribution of the degree, whose appearance is rather surprising. Namely, it has the following distribution:

Degree	Probability	
1	$\dfrac{8}{\pi^2}\left(1-\dfrac{2}{\pi}\right)$	$= .294^+$
2	$\dfrac{4}{\pi}\left(2-\dfrac{9}{\pi}+\dfrac{12}{\pi^2}\right)$	$= .447^-$
3	$2\left(1-\dfrac{2}{\pi}\right)\left(1-\dfrac{6}{\pi}+\dfrac{12}{\pi^2}\right)$	$= .222^+$
4	$\left(\dfrac{4}{\pi}-1\right)\left(1-\dfrac{2}{\pi}\right)^2$	$= .036^+$

(4.13)

In order to find the transfer currents $Y(e, f)$, we will first find voltages, then use $i = dv$. (We assume unit conductances on the edges.) When i is a unit flow from x to y, we have $d^* i = \mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}$. Hence the voltages satisfy $\Delta v := d^* dv = \mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}$; here, Δ is called the ***graph Laplacian***. We are interested in solving this equation when $x := e^-$, $y := e^+$ and then computing $v(f^-) - v(f^+)$. Our method is to use Fourier analysis. We begin with a formal (i.e., heuristic) derivation of the solution, then prove that our formula is correct.

Let $\mathbb{T}^2 := (\mathbb{R}/\mathbb{Z})^2$ be the 2-dimensional torus. For $(x_1, x_2) \in \mathbb{Z}^2$ and $(\alpha_1, \alpha_2) \in \mathbb{T}^2$, write $(x_1, x_2) \cdot (\alpha_1, \alpha_2) := x_1\alpha_1 + x_2\alpha_2 \in \mathbb{R}/\mathbb{Z}$. For a function f on $\mathbb{Z}^2$, define the function $\widehat{f}$ on $\mathbb{T}^2$ by

$$\widehat{f}(\alpha) := \sum_{x \in \mathbb{Z}^2} f(x) e^{-2\pi i x \cdot \alpha} .$$

We are not worrying here about whether this converges in any sense, but certainly $\widehat{\mathbf{1}_{\{x\}}}(\alpha) = e^{-2\pi i x \cdot \alpha}$. Now a formal calculation shows that for a function f on $\mathbb{Z}^2$, we have

$$\widehat{\Delta f}(\alpha) = \varphi(\alpha) \widehat{f}(\alpha) ,$$

where

$$\varphi\big((\alpha_1, \alpha_2)\big) := 4 - \left(e^{2\pi i \alpha_1} + e^{-2\pi i \alpha_1} + e^{2\pi i \alpha_2} + e^{-2\pi i \alpha_2}\right) = 4 - 2\left(\cos 2\pi\alpha_1 + \cos 2\pi\alpha_2\right) .$$

Hence, to solve $\Delta f = g$, we may try to solve $\widehat{\Delta f} = \widehat{g}$ by using $\widehat{f} := \widehat{g}/\varphi$ and then finding f. In fact, a formal calculation shows that we may recover f from $\widehat{f}$ by the formula

$$f(x) = \int_{\mathbb{T}^2} \widehat{f}(\alpha) e^{2\pi i x \cdot \alpha} \, d\alpha \,,$$

where the integration is with respect to Lebesgue measure. This is the approach we will follow. Note that we need to be careful about the nonuniqueness of solutions to $\Delta f = g$ since there are non-zero functions f with $\Delta f = 0$.

▷ **Exercise 4.8.**

Show that $(\widehat{\mathbf{1}_{\{x\}}} - \widehat{\mathbf{1}_{\{y\}}})/\varphi \in L^1(\mathbb{T}^2)$ for any $x, y \in \mathbb{Z}^2$.

▷ **Exercise 4.9.**

Show that if $F \in L^1(\mathbb{T}^2)$ and $f(x) = \int_{\mathbb{T}^2} F(\alpha) e^{2\pi i x \cdot \alpha} \, d\alpha$, then

$$(\Delta f)(x) = \int_{\mathbb{T}^2} F(\alpha) e^{2\pi i x \cdot \alpha} \varphi(\alpha) \, d\alpha \,.$$

Proposition 4.7. (Voltage on $\mathbb{Z}^2$) *The voltage at u when a unit current flows from x to y in $\mathbb{Z}^2$ and when $v(y) = 0$ is*

$$v(u) = v'(u) - v'(y) \,,$$

where

$$v'(u) := \int_{\mathbb{T}^2} \frac{e^{-2\pi i x \cdot \alpha} - e^{-2\pi i y \cdot \alpha}}{\varphi(\alpha)} e^{2\pi i u \cdot \alpha} \, d\alpha \,.$$

Proof. By Exercises 4.8 and 4.9, we have

$$\Delta v'(u) = \int_{\mathbb{T}^2} \left(e^{-2\pi i x \cdot \alpha} - e^{-2\pi i y \cdot \alpha}\right) e^{2\pi i u \cdot \alpha} \, d\alpha = \mathbf{1}_{\{x\}}(u) - \mathbf{1}_{\{y\}}(u) \,.$$

That is, $\Delta v' = \mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}$. Since v satisfies the same equation, we have $\Delta(v' - v) = 0$. In other words, $v' - v$ is harmonic at every point in $\mathbb{Z}^2$. Furthermore, v' is bounded in absolute value by the L^1 norm of $(\widehat{\mathbf{1}_{\{x\}}} - \widehat{\mathbf{1}_{\{y\}}})/\varphi$. Since v is also bounded (by $v(x)$), it follows that $v' - v$ is bounded. Since the only bounded harmonic functions on $\mathbb{Z}^2$ are the constants (by, say, Exercise 2.37), this means that $v' - v$ is constant. Since $v(y) = 0$, we obtain $v = v' - v'(y)$, as desired. ◀

We now need to find a good method to compute the integral v'. Set

$$H(u) := 4 \int_{\mathbb{T}^2} \frac{1 - e^{2\pi i u \cdot \alpha}}{\varphi(\alpha)}\, d\alpha \,.$$

Note that the integrand is integrable by Exercise 4.8 applied to $x := (0,0)$ and $y := -u$. The integral H is useful because $v'(u) = \big[H(u-y) - H(u-x)\big]/4$. (The factor of 4 is introduced in H in order to conform to the usage of other authors.) Putting $x := e^-$ and $y := e^+$, we get

$$\begin{aligned} Y(e,f) &= v(f^-) - v(f^+) \\ &= \frac{1}{4}\big[H(f^- - e^+) - H(f^- - e^-) - H(f^+ - e^+) + H(f^+ - e^-)\big] \,. \end{aligned} \tag{4.14}$$

Thus, we concentrate on calculating H. Now $H(0,0) = 0$ and a direct calculation as in Exercise 4.9 shows that $\Delta H = -4 \cdot \mathbf{1}_{\{(0,0)\}}$. Furthermore, the symmetries of φ show that H is invariant under reflection in the axes and in the 45° line. Therefore, all the values of H can be computed from those on the 45° line by computing values at gradually increasing distance from the origin and from the 45° line. (For example, we first compute $H(1,0) = 1$ from the equations $H(0,0) = 0$ and $(\Delta H)(0,0) = 4H(0,0) - H(0,1) - H(1,0) - H(0,-1) - H(-1,0) = -4$, then $H(2,1)$ from the value of $H(1,1)$ and the equation $(\Delta H)(1,1) = 4H(1,1) - H(1,0) - H(0,1) - H(1,2) - H(2,1) = 0$, then $H(2,0)$ from $(\Delta H)(1,0) = 0$, then $H(3,2)$, etc.)

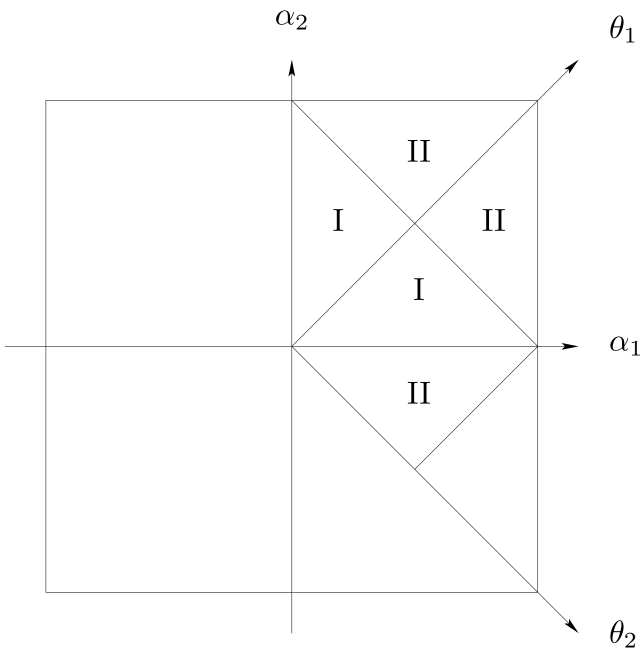


Figure 4.8. The integrals over the regions labeled I are all equal, as are those labeled II.

The reflection symmetries we observed for H imply that $H(u) = H(-u)$, whence H is real. Thus, for $n \ge 1$, we can write

$$H(n,n) = 4\int_{\mathbb{T}^2} \frac{1-\cos 2\pi n(\alpha_1+\alpha_2)}{\varphi(\alpha)}\,d\alpha$$
$$= \int_0^1\int_0^1 \frac{1-\cos 2\pi n(\alpha_1+\alpha_2)}{1-\cos\big(\pi(\alpha_1+\alpha_2)\big)\cos\big(\pi(\alpha_1-\alpha_2)\big)}\,d\alpha_1\,d\alpha_2\,.$$

The integrand has various symmetries shown in Figure 4.8. This implies that if we change variables to $\theta_1 := \pi(\alpha_1+\alpha_2)$ and $\theta_2 := \pi(\alpha_1-\alpha_2)$, we obtain

$$H(n,n) = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \frac{1-\cos 2n\theta_1}{1-\cos\theta_1\cos\theta_2}\,d\theta_1\,d\theta_2\,.$$

Now for $0 < a < 1$, we have

$$\int_0^\pi \frac{d\theta}{1-a\cos\theta} = \frac{2}{\sqrt{1-a^2}}\tan^{-1}\frac{\sqrt{1-a^2}\tan(\theta/2)}{1-a}\bigg|_0^\pi = \frac{\pi}{\sqrt{1-a^2}}\,.$$

Therefore integration on θ_2 gives

$$H(n,n) = \frac{1}{\pi}\int_0^\pi \frac{1-\cos 2n\theta_1}{\sin\theta_1}\,d\theta_1\,.$$

Now $(1-\cos 2n\theta_1)/\sin\theta_1 = 2\sum_{k=1}^n \sin(2k-1)\theta_1$, as can be seen by using complex notation. Therefore

$$H(n,n) = \frac{2}{\pi}\int_0^\pi \sum_{k=1}^n \sin(2k-1)\theta_1\,d\theta_1 = \frac{4}{\pi}\sum_{k=1}^n \frac{1}{2k-1}\,.$$

▷ **Exercise 4.10.**

Deduce the distribution of the degree of a vertex in the uniform spanning tree of $\mathbb{Z}^2$, i.e., the table (4.13).

We may also make use of the above work, without the actual values being needed, to prove the following remarkable fact. Edges of the uniform spanning tree in $\mathbb{Z}^2$ along diagonals are like fair coin flips! We have seen in (4.12) that each edge has 50% chance to be in the tree. The independence we are now asserting is the following theorem.

Theorem 4.8. (Independence on Diagonals) *Let e be any edge of $\mathbb{Z}^2$. For $n \in \mathbb{Z}$, let X_n be the indicator that $e + (n,n)$ lies in the spanning tree. Then X_n are i.i.d. Likewise for $e + (n,-n)$.*

Proof. By symmetry, it suffices to prove the first part and under the assumption that e is the edge from the origin to $(1,0)$. By the Transfer-Current Theorem, it suffices to show that $Y\big(e, e+(n,n)\big) = 0$ for all $n \ne 0$. The formula (4.14) shows that

$$4Y\big(e, e+(n,n)\big) = H(n+1,n) - H(n,n) - H(n,n) + H(n-1,n)\,.$$

Now the symmetries we have noted already of the function $H(\bullet,\bullet)$ show that $H(n,n+1) = H(n+1,n)$ and $H(n,n-1) = H(n-1,n)$. Since $H(n,n)$ is the average of these four numbers for $n \neq 0$, it follows that $H(n+1,n) - H(n,n) = H(n,n) - H(n-1,n)$. This proves the result. ◀

§4.4. Notes.

There is another important connection of spanning trees to Markov chains:

The Markov Chain Tree Theorem. *The stationary distribution of a finite-state irreducible Markov chain is proportional to the measure that assigns the state x the measure*

$$\sum_{\mathrm{root}(T)=x} \alpha(T)\,.$$

It is for this reason that generating spanning trees at random is very closely tied to generating a state of a Markov chain at random according to its stationary distribution. This latter topic is especially interesting in computer science. See Propp and Wilson (1998) for more details. See Anantharam and Tsoucas (1989) for some of the history of the Markov Chain Tree Theorem.

To prove the Markov Chain Tree Theorem, one can associate to the original Markov chain a new Markov chain on spanning trees. Given a spanning tree T and an edge e with $e^- = \mathrm{root}(T)$, define two new spanning trees:

"forward procedure" This creates the new spanning tree $F(T,e)$. First, add e to T. This creates a cycle. Delete the edge $f \in T$ out of e^+ that breaks the cycle. See Figure 4.9.

"backward procedure" This creates the new spanning tree $B(T,e)$. Again, first add e to T. This creates a cycle. Break it by removing the appropriate edge $g \in T$ that leads into e^-. See Figure 4.9.

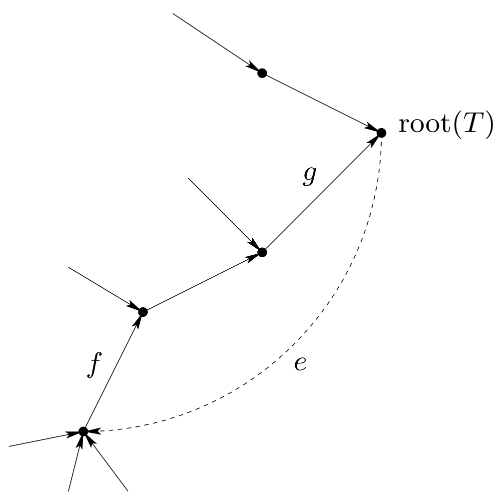


Figure 4.9.

Note that in both procedures, it is possible that $f = -e$ or $g = -e$. Also, note that

$$B(F(T, e), f) = F(B(T, e), g) = T\,,$$

where f and g are as specified in the definitions of the forward and backward procedures.

Now define transition probabilities on the set of spanning trees by

$$p(T, F(T, e)) := p(e) = p\big(\mathrm{root}(T), \mathrm{root}(F(T, e))\big). \tag{4.15}$$

Thus $p(T, \widetilde{T}) > 0 \iff \exists e \;\; \widetilde{T} = F(T, e) \iff \exists g \;\; T = B(\widetilde{T}, g)$.

▷ **Exercise 4.11.**
Prove that the Markov chain on trees given by (4.15) is irreducible.

▷ **Exercise 4.12.**
(a) Show that the weight $\alpha(\bullet)$ is a stationary measure for the Markov chain on trees given by (4.15).
(b) Prove the Markov Chain Tree Theorem.

Another way to express the relationship between the original Markov chain and this associated one on trees is as follows. Recall that one can create a stationary Markov chain $\langle X_n \rangle_{-\infty}^{\infty}$ indexed by $\mathbb{Z}$ with the original transition probabilities $p(\bullet, \bullet)$ by, say, Kolmogorov's existence theorem. Set

$$L_n(w) := \max\{m < n\,;\; X_m = w\}\,.$$

This is well defined a.s. by recurrence. Let Y_n be the tree formed by the edges

$$\big\{\langle w, X_{L_n(w)+1} \rangle\,;\; w \in \mathsf{V} \setminus \{X_n\}\big\}$$

Then Y_n is rooted at X_n and $\langle Y_n \rangle$ is a stationary Markov chain with the transition probabilities (4.15).

An older method due to Aldous (1990) and Broder (1989) of generating weighted uniform spanning trees comes from reversing these Markov chains; related ideas were in the air at that time and both these authors thank Persi Diaconis for discussions. Let $\langle X_n \rangle_{-\infty}^{\infty}$ be a stationary Markov chain on a finite state space. Then so is the ***reversed*** process $\langle X_{-n} \rangle$: the definition of the Markov property via independence of the past and the future given the present shows this immediately. We can also find the transition probabilities $\tilde{p}$ for the reversed chain: Let π be the stationary probability $\pi(a) := \mathbf{P}[X_0 = a]$. Then clearly π is also the stationary probability for the reversed chain. Comparing the chance of seeing state a followed by state b for the forward chain with the equal probability of seeing state b followed by state a for the reversed chain, we see that

$$\pi(a)p(a, b) = \pi(b)\tilde{p}(b, a)\,,$$

whence

$$\tilde{p}(b, a) = \frac{\pi(a)}{\pi(b)} p(a, b)\,.$$

As discussed in Section 2.1, the chain is reversible iff $\tilde{p} = p$.

If we reverse the above chains $\langle X_n \rangle$ and $\langle Y_n \rangle$, then we find that Y_{-n} can be expressed in terms of X_{-n} as follows: let

$$H_n(w) := \min\{m > n\,;\; X_{-m} = w\}.$$

Then Y_{-n} has edges $\big\{\langle w, X_{-[H_n(w)-1]} \rangle\,;\; w \in \mathsf{V} \setminus \{X_{-n}\}\big\}$.

▷ **Exercise 4.13.**
Prove that the transition probabilities of Y_{-n} are

$$\tilde{p}(T, B(T, e)) = p(e)\,.$$

Since the stationary measure of $\langle Y_n \rangle$ is still proportional to $\alpha(\bullet)$, we get the following algorithm for generating a random spanning tree with distribution proportional to $\alpha(\bullet)$: find the stationary probability π of the chain giving rise to the weights $\alpha(\bullet)$, run the reversed chain starting at a state chosen according to π, and construct the tree via H_0. That is, we draw an edge from u to w the first time ≥ 1 that the reversed chain hits u, where w is the state preceding the visit to u.

In case the chain is reversible, this construction simplifies. From the discussion in Section 4.1, we have:

Corollary 4.9. (Aldous/Broder Algorithm) *Let $\langle X_n \rangle_0^\infty$ be a random walk on a finite connected graph G with X_0 arbitrary (not necessarily random). Let $H(u) := \min\{m > 0\,;\ X_m = u\}$ and let T be the unrooted tree with edges $\{(u, X_{H(u)-1})\,;\ X_0 \neq u \in G\}$. Then the distribution of T is proportional to $\beta(\bullet)$. In particular, simple random walk on a finite connected graph gives a uniform unrooted random spanning tree.*

This method of generating uniform spanning trees can be and was used in place of Wilson's for the purposes of this chapter. However, Wilson's method is much better suited to the study of uniform spanning forests, the topic of Chapter 10.

▷ **Exercise 4.14.**
Let G be a cycle and $x \in \mathsf{V}$. Start simple random walk at x and stop when all edges but one have been traversed at least once. Show that the edge that has not been traversed is equally likely to be any edge.

▷ **Exercise 4.15.**
Suppose that the graph G has a Hamiltonian path, $\langle x_k\,;\ 1 \leq k \leq n \rangle$, i.e., a path that is a spanning tree. Let q_k be $\mathbf{P}_{x_k}[\tau_{x_k}^+ < \tau_{x_{k+1},\ldots,x_n}]$ for simple random walk on G. Show that the number of spanning trees of G equals $\prod_{k<n} q_k \deg_G x_k$.

▷ **Exercise 4.16.**
Let $\langle X_n \rangle$ and $\langle Y_n \rangle$ be the Markov chains defined above, so that Y_n is a spanning tree rooted at X_n. Show directly that Y_0 has the same distribution as Wilson's method produces, given that $\mathrm{root}(Y_0) = X_0$.

The use of Wilson's method for infinite recurrent networks was first made in BLPS (2001). The Transfer-Current Theorem was shown for the case of two edges in Brooks, Smith, Stone, and Tutte (1940). The proof here is due to BLPS (2001).

We are grateful to David Wilson for permission to include Figure 4.7. It was created using the linear algebraic techniques of Wilson (1997) for generating domino tilings; the needed matrix inversion was accomplished using the formulas of Kenyon (1997). The resulting tiling gives dual spanning trees by the bijection of Temperley (see Kenyon, Propp, and Wilson (2000)). One of the trees is Figure 4.7.

Theorem 4.8 is due to R. Lyons and is published here for the first time. Another way to state this result is that if independent fair coin flips are used to decide which of the edges $\{e + (n, n)\,;\ n \in \mathbb{Z}\}$ will be present, for some fixed edge e, then there exists a percolation on the remaining edges of $\mathbb{Z}^2$ that will lead in the end to a percolation on all of $\mathbb{Z}^2$ with the distribution of the uniform spanning tree. A related surprising result of Lyons and Steif (2003) says that we can independently determine some of the horizontal edges and then decide the remaining edges to get a uniform spanning tree. To be precise, fix a horizontal edge, e. Suppose that $\langle U(e+x)\,;\ x \in \mathbb{Z}^2\rangle$ are i.i.d. uniform $[0, 1]$ random variables. Let $K_0 := \{e+x\,;\ U(e+x) \le e^{-4\mathrm{G}/\pi}\}$ and $K_1 := \{e + x\,;\ U(e + x) \ge 1 - e^{-4\mathrm{G}/\pi}\}$, where

$$\mathrm{G} := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)^2}$$

is ***Catalan's constant***. (We have that $e^{-4\mathrm{G}/\pi} = 0.3115^+$.) Then there exists a percolation ω on $\mathbb{Z}^2$ such that $\omega \cup K_1 \setminus K_0$ has the distribution of the uniform spanning tree.

Additional information on loop-erased random walk and another proof of Wilson's algorithm can be found in Marchal (2000).

Enumeration of spanning trees in graphs is an old topic. There are many proofs of Cayley's formula, Corollary 4.3. The shortest proof is due to Joyal (1981) and goes as follows. Denote $[n] := \{1, \ldots, n\}$. First note that because every permutation can be represented as a product of disjoint directed cycles, it follows that for any finite set S the number of sets of cycles of elements of S (each element appearing exactly once in some cycle) is equal to the number of linear arrangements of the elements of S. The number of functions from $[n]$ to $[n]$ is clearly n^n. To each such function f we may associate its functional digraph, which has a directed edge from i to $f(i)$ for each i in $[n]$. Every weakly connected component of the functional digraph can be represented by a cycle of rooted trees. So n^n is also the number of linear arrangements of rooted trees on $[n]$. We claim now that $n^n = n^2 t_n$, where t_n is the number of trees on $[n]$.

It is clear that $n^2 t_n$ is the number of triples (x, y, T), where $x, y \in [n]$ and T is a tree on $[n]$. Given such a triple, we obtain a linear arrangement of rooted trees by removing all directed edges on the unique path from x to y and taking the nodes on this path to be the roots of the trees that remain. This correspondence is bijective, and thus $t_n = n^{n-2}$. ◀

The Matrix-Tree Theorem of graph theory says that the number of spanning trees of a graph G equals $\det \Delta_G[x]$ for each $x \in \mathsf{V}$, where Δ_G is the graph Laplacian defined in Exercise 2.53 and $[x]$ indicates striking the row and column indexed by x. More generally, the sum $\tau(G)$ of the weights $\prod_{e\in T} c(e)$ over spanning trees T in a network equals $\det \Delta_G[x]$.

▷ **Exercise 4.17.**
Show that the constant C in Exercise 2.20(c) equals $\sqrt{\tau(G)/(2\pi)^{|\mathsf{V}|}}$.

Asymptotics of the number of spanning trees is connected to mathematical physics. For example, if one combines the entropy result for domino tilings proved by Montroll (1964) with the Temperley (1974) bijection, then one gets that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n^2} & \log\big(\text{number of spanning trees of } [1, n]^2 \text{ in } \mathbb{Z}^2\big) \\
&= \int_0^1 \int_0^1 \log(4 - 2\cos 2\pi x - 2\cos 2\pi y)\, dx\, dy \\
&= \frac{4\mathrm{G}}{\pi} = 1.166^+\,,
\end{aligned}$$

where G is Catalan's constant, as above (see Kasteleyn (1961) or Montroll (1964) for the evaluation of the integral). This result first appeared explicitly in Burton and Pemantle (1993). Thus, $e^{1.166^+} = 3.21^-$ can be thought of as the average number of independent choices per vertex to make a spanning tree of $\mathbb{Z}^2$. See, e.g., Burton and Pemantle (1993) and Shrock and Wu (2000) and the references therein for this and several other such examples. Very general methods of calculating and comparing asymptotics were given by Lyons (2005, 2010).

▷ **Exercise 4.18.**
Consider simple random walk on $\mathbb{Z}^2$. Let $A := \{(x,y) \in \mathbb{Z}^2\,;\ y < 0 \text{ or } (y = 0 \text{ and } x < 0)\}$. Show that $\mathbf{P}_{(0,0)}[\tau^+_{(0,0)} > \tau_A] = e^{4\mathrm{G}/\pi}/4$, where G is Catalan's constant.

One may consider the uniform spanning tree on $\mathbb{Z}^2$ embedded in $\mathbb{R}^2$. In fact, consider it on $\epsilon\mathbb{Z}^2$ in $\mathbb{R}^2$ and let $\epsilon \to 0$. In appropriate senses, one can describe the limit and show that it has a conformal invariance property. This is proved by Lawler, Schramm, and Werner (2004). The stochastic Loewner evolution, SLE, introduced by Schramm (2000) initially for this very purpose, plays the central role. For the uniform spanning tree, there are two ways SLE enters the analysis: One is the scaling limit of loop-erased random walk, which is the path between two vertices and fundamental to this chapter. The second is less obvious. If we draw a curve around the spanning tree in a bounded region, as in Figure 4.10, we obtain a cycle in another graph. That cycle visits every vertex and is very reminiscent of Peano's space-filling curve. It is called the UST Peano curve and it, too, has a scaling limit described by SLE. Figure 4.11 shows the curve from a uniform spanning tree in a 99 × 99 square grid, where the hue represents progress along the curve. SLE is also central to the study of scaling limits of other planar processes, including percolation.

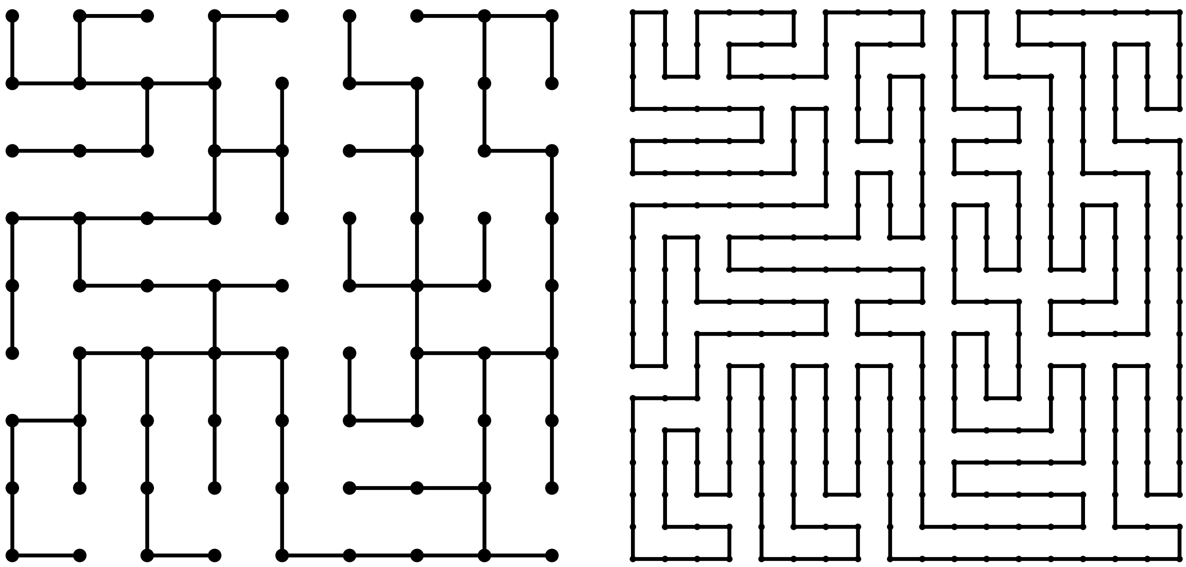


Figure 4.10. A uniform spanning tree in a 9 × 9 grid on the left, with its surrounding Peano-like curve in an 18 × 18 grid on the right.

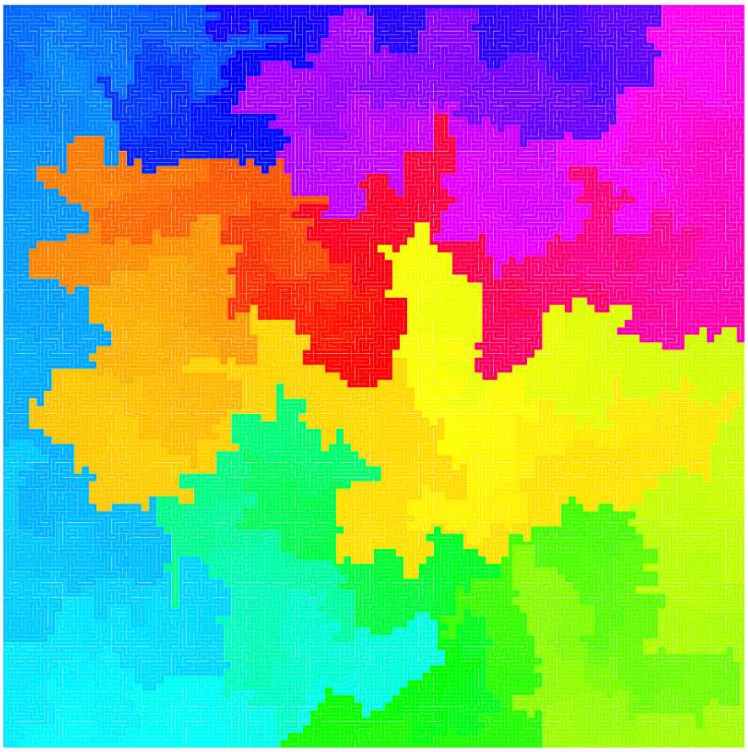


Figure 4.11. The Peano-like curve surrounding a uniform spanning tree on a 99×99 grid.

§4.5. Collected In-Text Exercises.

4.1. Suppose that Z is a set of states in a Markov chain and that x_0 is a state not in Z. Assume that when the Markov chain is started in x_0, then it visits Z with probability 1. Define the random path $Y_0, Y_1, \ldots$ by $Y_0 := x_0$ and then recursively by letting Y_{n+1} have the distribution of one step of the Markov chain starting from Y_n given that the chain will visit Z before visiting any of $Y_0, Y_1, \ldots, Y_n$ again. However, if $Y_n \in Z$, then the path is stopped and Y_{n+1} is not defined. Show that $\langle Y_n \rangle$ has the same distribution as loop-erasing a sample of the Markov chain started from x_0 and stopped when it reaches Z. In the case of a random walk, the conditioned path $\langle Y_n \rangle$ is called the ***Laplacian random walk*** from x_0 to Z.

4.2. Consider the ladder graph L_n of height n shown in Figure 1.8. Choose a spanning tree $T(n)$ of L_n uniformly. Use Corollary 4.4 to determine $\mathbf{P}[\text{rung 1 is in } T(n)]$ and its limiting behavior as $n \to \infty$.

4.3. By using Corollary 4.4, show that if $e \neq f$, then the events $e \in T$ and $f \in T$ are negatively correlated.

4.4. Write out the rest of the proof that for any event $\mathscr{B}$ depending on only finitely many edges, $|\mathbf{P}[T_{G'} \in \mathscr{B}] - \mathbf{P}[T_G \in \mathscr{B}]|$ is arbitrarily small for sufficiently large G'.

4.5. Suppose that $\mathscr{A}$ is an increasing event on a graph G and $e \in \mathsf{E}$. Note that $\mathsf{E}(G/e) = \mathsf{E}(G \backslash e) = \mathsf{E}(G) \backslash \{e\}$. Define $\mathscr{A}/e := \{F \subseteq \mathsf{E}(G/e)\,;\ F \cup \{e\} \in \mathscr{A}\}$ and $\mathscr{A} \backslash e := \{F \subseteq \mathsf{E}(G \backslash e)\,;\ F \in \mathscr{A}\}$. Show that these are increasing events on G/e and $G \backslash e$, respectively.

4.6. **(Negative Association)** Let G be a finite network. Extend Theorem 4.5 to show that if $\mathscr{A}$ and $\mathscr{B}$ are both increasing events and they depend on disjoint sets of edges, then they are negatively correlated. Still more generally, show the following. Say that a random variable X ***depends on a set*** a set $F \subseteq \mathsf{E}(G)$ if X is measurable with respect to the σ-field consisting of events that depend on F. Say also that X is ***increasing*** if $X(H) \le X(H')$ whenever $H \subset H'$. If X and Y are increasing random variables with finite second moments that depend on disjoint sets of edges, then $\mathbf{E}[XY] \le \mathbf{E}[X]\mathbf{E}[Y]$. This property of $\mathbf{P}$ is called ***negative association***.

4.7. Let G be an edge-amenable infinite graph as witnessed by the sequence $\langle \mathsf{V}_n \rangle$. Show that the average degree of vertices in any spanning tree of G is 2. That is, if $\deg_T(x)$ denotes the degree of x in a spanning tree T of G, then

$$\lim_{n\to\infty} |\mathsf{V}_n|^{-1} \sum_{x \in \mathsf{V}_n} \deg_T(x) = 2\,.$$

Every infinite recurrent graph can be shown to be edge-amenable by various results from Chapter 6 that we'll look at later, such as Theorems 6.5, 6.7, or 6.18. Deduce that for the uniform spanning tree measure on a recurrent graph,

$$\lim_{n\to\infty} |\mathsf{V}_n|^{-1} \sum_{x \in \mathsf{V}_n} \mathbf{E}[\deg_T(x)] = 2\,.$$

In particular, if G is also transitive, such as $\mathbb{Z}^2$, meaning that for every pair of vertices x and y, there is a bijection of V with itself that preserves adjacency and takes x to y, then every vertex has expected degree 2.

4.8. Show that $(\widehat{\mathbf{1}_{\{x\}}} - \widehat{\mathbf{1}_{\{y\}}})/\varphi \in L^1(\mathbb{T}^2)$ for any $x, y \in \mathbb{Z}^2$.

4.9. Show that if $F \in L^1(\mathbb{T}^2)$ and $f(x) = \int_{\mathbb{T}^2} F(\alpha) e^{2\pi i x\cdot\alpha}\, d\alpha$, then

$$(\Delta f)(x) = \int_{\mathbb{T}^2} F(\alpha) e^{2\pi i x\cdot\alpha} \varphi(\alpha)\, d\alpha\,.$$

4.10. Deduce the distribution of the degree of a vertex in the uniform spanning tree of $\mathbb{Z}^2$, i.e., the table (4.13).

4.11. Prove that the Markov chain on trees given by (4.15) is irreducible.

4.12. **(a)** Show that the weight $\alpha(\bullet)$ is a stationary measure for the Markov chain on trees given by (4.15).

(b) Prove the Markov Chain Tree Theorem.

4.13. Let $\langle Y_n \rangle$ be a stationary Markov chain with transition probabilities (4.15) and consider its reversal. Prove that the transition probabilities of Y_{-n} are

$$\tilde{p}\big(T, B(T, e)\big) = p(e)\,.$$

4.14. Let G be a cycle and $x \in \mathsf{V}$. Start simple random walk at x and stop when all edges but one have been traversed at least once. Show that the edge that has not been traversed is equally likely to be any edge.

4.15. Suppose that the graph G has a Hamiltonian path, $\langle x_k\,;\ 1 \le k \le n \rangle$, i.e., a path that is a spanning tree. Let q_k be $\mathbf{P}_{x_k}[\tau^+_{x_k} < \tau_{x_{k+1},\ldots,x_n}]$ for simple random walk on G. Show that the number of spanning trees of G equals $\prod_{k<n} q_k \deg_G x_k$.

4.16. Let $\langle X_n \rangle$ and $\langle Y_n \rangle$ be the Markov chains defined in the proof of the Markov Chain Tree Theorem, so that Y_n is a spanning tree rooted at X_n. Show directly that Y_0 has the same distribution as Wilson's method produces, given that $\operatorname{root}(Y_0) = X_0$.

4.17. Show that the constant C in Exercise 2.20(c) equals $\sqrt{\tau(G)/(2\pi)^{|\mathsf{V}|}}$.

4.18. Consider simple random walk on $\mathbb{Z}^2$. Let $A := \big\{(x, y) \in \mathbb{Z}^2 \,;\; y < 0 \text{ or } (y = 0 \text{ and } x < 0)\big\}$. Show that $\mathbf{P}_{(0,0)}[\tau^+_{(0,0)} > \tau_A] = e^{4\mathrm{G}/\pi}/4$, where G is Catalan's constant.

§4.6. Additional Exercises.

4.19. Show that not every probability distribution on spanning trees of an undirected graph is proportional to a weight distribution, where the weight of a tree equals the product of the weights of its edges.

4.20. Given a probability measure $\mathbf{P}$ on spanning trees of a finite graph G, there is the vector of marginal edge probabilities, $\mu(e) := \mathbf{P}[e \in T]$ $(e \in \mathsf{E})$. The set of such vectors forms a polytope, called the ***spanning tree polytope***. Show that this polytope consists of vectors μ that satisfy

(i) $\mu(e) \ge 0$ for all $e \in \mathsf{E}(G)$,
(ii) $\sum_{e \in \mathsf{E}(G)} \mu(e) = |\mathsf{V}(G)| - 1$, and
(iii) $\sum_{e \in \mathsf{E}(G \restriction K)} \mu(e) \le |K| - 1$ for all $K \subseteq \mathsf{V}(G)$.

Show in addition that if G has no ***cut-vertices*** (vertices whose removal disconnects G), then the relative interior of this polytope (i.e., the interior as a subset of the affine span of the polytope) equals the set of such μ that satisfy strict inequality in every instance of (i) and (iii).

4.21. Given a probability measure $\mathbf{P}$ on spanning trees of a finite graph G, there is the vector of marginal edge probabilities, $\mu_{\mathbf{P}}(e) := \mathbf{P}[e \in T]$ $(e \in \mathsf{E})$. The ***entropy*** of $\mathbf{P}$ is defined to be $H(\mathbf{P}) := -\sum_T \mathbf{P}(T) \log \mathbf{P}(T)$, where $0 \log 0 := 0$.

(a) Show that if $\mathbf{P}$ is a weighted uniform spanning tree measure and Q is any probability measure on spanning trees with the same edge marginals $\mu_{\mathbf{P}} = \mu_Q$, then $H(\mathbf{P}) > H(Q)$ unless $Q = \mathbf{P}$.

(b) Suppose that G has no cut-vertices. Show that if μ lies in the relative interior of the spanning tree polytope (see Exercise 4.20), then there is a unique weighted spanning tree measure whose edge marginal equals μ.

4.22. Let G be a finite or recurrent network and $a \ne z$ be two of its vertices. Let i be the unit current flow from a to z. Show that for every edge e, the probability that loop-erased random walk from a to z crosses e minus the probability that it crosses $-e$ equals $i(e)$.

4.23. Show that the following procedure also gives a.s. a random spanning tree rooted at r with distribution proportional to $\alpha(\bullet)$. Let $G_0 := \{r\}$. Given G_i, if G_i spans G, stop. Otherwise, choose any vertex $x \ne r$ that does not have an edge in G_i that leads out of x and add a (directed) edge from x picked according to the transition probability $p(x, \bullet)$ independently of the past. Add this edge to G_i and remove any cycle it creates to make G_{i+1}.

4.24. How efficient is Wilson's method? What takes time is to generate a random successor state of a given state. Call this a step of the algorithm. Show that the expected number of steps to generate a random spanning tree rooted at r for a finite-state irreducible Markov chain is

$$\sum_{x \text{ a state}} \pi(x) \big(\mathbf{E}_x[\tau_r] + \mathbf{E}_r[\tau_x]\big) ,$$

where π is the stationary probability distribution for the Markov chain. In the case of a random walk on a network (V, E), this is

$$\sum_{e \in \mathsf{E}_{1/2}} c(e)\big(\mathscr{R}(e^- \leftrightarrow r) + \mathscr{R}(e^+ \leftrightarrow r)\big)\,,$$

where edge e has conductance $c(e)$ and endpoints e^- and e^+, and $\mathscr{R}$ denotes effective resistance.

4.25. Let $\langle X_n \rangle$ be a transient Markov chain. Then its loop erasure $\langle Y_n \rangle$ is well defined a.s. Show that $\mathbf{P}_{x_0}[Y_1 = x_1] = p(x_0, x_1)\mathbf{P}_{x_1}[\tau_{x_0} = \infty]/\mathbf{P}_{x_0}[\tau_{x_0}^+ = \infty]$.

4.26. Suppose that x and y are two vertices in the complete graph K_n. Show that the probability that the distance between x and y is k in a uniform spanning tree of K_n is

$$\frac{k+1}{n^k} \prod_{i=1}^{k-1} (n-i-1)\,.$$

4.27. Prove Cayley's formula another way as follows: let t_{n-1} be a spanning tree of the complete graph on n vertices and $t_1 \subset t_2 \subset \cdots \subset t_{n-2} \subset t_{n-1}$ be subtrees such that t_i has i edges. Then

$$\mathbf{P}[T = t_{n-1}] = \mathbf{P}[t_1 \subseteq T] \cdot \prod_{i=1}^{n-2} \mathbf{P}[t_{i+1} \subseteq T \mid t_i \subseteq T]\,.$$

Show that $\mathbf{P}[t_1 \subseteq T] = 2/n$ and

$$\mathbf{P}[t_{i+1} \subseteq T \mid t_i \subseteq T] = \frac{i+2}{n(i+1)}\,.$$

4.28. Show that if G has n vertices, then $\sum_{e \in \mathsf{E}_{1/2}} c(e)\mathscr{R}(e^- \leftrightarrow e^+) = n - 1$.

4.29. Let G be a vertex with n vertices and consider two of its vertices, a and z. Consider a random walk $\langle X_k\,;\, 0 \le k \le \tau \rangle$ that starts at a, visits z, and is then stopped at its first return to a after visiting z. Show that $\mathbf{E}\big[\sum_{k=0}^{\tau-1} \mathscr{R}(X_k \leftrightarrow X_{k+1})\big] = 2(n-1)\mathscr{R}(a \leftrightarrow z)$.

4.30. Kirchhoff (1847) generalized Corollary 4.4 in two ways. To express them, denote the sum of $\beta(T)$ over all spanning trees of G by $\beta(G)$.

(a) Given a finite network G and vertices $a \ne z \in G$, show that the effective conductance from a to z is given by

$$\mathscr{C}(a \leftrightarrow z) = \frac{\beta(G)}{\beta(G/\{a, z\})}\,, \tag{4.16}$$

where $G/\{a, z\}$ indicates the network G with a and z identified.

(b) Given a finite network G and $a \ne z \in G$, let $\beta_a^z(e)$ denote the sum of the weights $\beta(T)$ of trees T such that the path in T from a to z passes along e. Show that the unit current flow from a to z is

$$i : e \mapsto \frac{\beta_a^z(e) - \beta_a^z(-e)}{\beta(G)}\,.$$

(*Hint:* Use Wilson's method and Proposition 2.2.)

4.31. Let (G, c) be a finite network. Denote the sum of $\beta(T)$ over all spanning trees of G by $\beta(G)$. Show that $\mathbf{P}[e \in T] = d \log \beta(G)/dc(e)$.

4.32. Suppose that G is a graph with two sets of conductances, c and c'. Show that if for every edge e, we have $c(e)\mathscr{R}(e^- \leftrightarrow e^+; c) = c'(e)\mathscr{R}(e^- \leftrightarrow e^+; c')$, then c/c' is constant.

4.33. Let (G, c) be a finite network. Recall from Exercise 2.58 that $(x, y) \mapsto \mathscr{R}(x \leftrightarrow y)$ is a metric on V. Show that V with this effective-resistance metric can be embedded isometrically into some ℓ^1 space.

4.34. Consider the doubly infinite ladder graph, the Cayley graph G of $\mathbb{Z} \times \mathbb{Z}_2$ with respect to its natural generators. Show that the uniform spanning tree T on G has the following description: the "rungs" $[(n, 0), (n, 1)]$ in T form a stationary renewal process with inter-rung distance being k with probability $2k(2 - \sqrt{3})^k$ $(k \geq 1)$. Given two successive rungs in T, all the other edges of the form $[(n, 0), (n+1, 0)]$ and $[(n, 1), (n+1, 1)]$ lie in T with one exception chosen uniformly and independently for different pairs of successive rungs.

4.35. Let G be a finite network. Let e and f be two edges not sharing the same two endpoints. Let $\widehat{i}^e$ be the unit current in G/f between the endpoints of e and let $\widetilde{i}^e$ be the unit current in $G \backslash f$ between the endpoints of e. Let

$$i^e_{\mathrm{c}}(g) := \begin{cases} \widehat{i}^e(g) & \text{if } g \neq f, \\ 0 & \text{if } g = f, \end{cases}$$

and

$$i^e_{\mathrm{d}}(g) := \begin{cases} \widetilde{i}^e(g) & \text{if } g \neq f, \\ 0 & \text{if } g = f. \end{cases}$$

(a) From (4.1), we have $i^e_{\mathrm{c}} = P^{\perp}_{i^f} i^e$. Show that

$$i^e = i^e_{\mathrm{c}} + \frac{Y(e, f)}{Y(f, f)} i^f .$$

(b) Show that $\chi^e - i^e_{\mathrm{d}} = P^{\perp}_{\chi^f - i^f}(\chi^e - i^e)$ and that

$$i^e = i^e_{\mathrm{d}} + \frac{Y(e, f)}{1 - Y(f, f)} (\chi^f - i^f) .$$

(c) Show that

$$i^e = Y(f, f) i^e_{\mathrm{c}} + [1 - Y(f, f)] i^e_{\mathrm{d}} + Y(e, f) \chi^f$$

and that the three terms on the right-hand side are pairwise orthogonal.

4.36. Let G be a finite network and i a current on G. If all the conductances are the same except that on edge f, which is changed to $c'(f)$, let i' be the current with the same sources and sinks, i.e., so that $d^* i' = d^* i$. Show that

$$i = i' + \frac{[c(f) - c'(f)] i(f)}{c(f)[1 - Y(f, f)] + c'(f) Y(f, f)} (\chi^f - i^f) ,$$

where i^f is the unit current with the original conductances from f^- to f^+ as defined after (2.11), and deduce that

$$\frac{di}{dc(f)} = i(f) r(f) (\chi^f - i^f) .$$

4.37. Given any numbers x_i $(i = 1, \ldots, k)$, let X be the diagonal matrix with entries $x_1, \ldots, x_k$. Show that $\det(Y_k + X) = \mathbf{E}[\prod_i (\mathbf{1}_{\{e_i \in T\}} + x_i)]$, where Y_k is as in (4.4). Deduce that $\mathbf{P}[e_1, \ldots, e_m \notin T,\ e_{m+1}, \ldots, e_k \in T] = \det Z_m$, where

$$Z_m(i,j) := \begin{cases} 1 - Y(e_i, e_j) & \text{if } j = i \le m, \\ -Y(e_i, e_j) & \text{if } j \ne i \text{ and } i \le m, \\ Y(e_i, e_j) & \text{if } i > m. \end{cases}$$

4.38. Give another proof of Cayley's formula (Corollary 4.3) by using the Transfer-Current Theorem.

4.39. Consider the weighted uniform spanning tree measure on an infinite recurrent network G. Let X and Y be increasing random variables with finite second moments that depend on disjoint sets of edges. Show that $\mathbf{E}[XY] \le \mathbf{E}[X]\mathbf{E}[Y]$.

4.40. Let (G, c) be a finite network. Let e a fixed edge in G and $\mathscr{A}$ an increasing event that ignores e. Suppose that a new network is formed from (G, c) by increasing the conductance on e while leaving unchanged all other conductances. Show that in the new network, the chance of $\mathscr{A}$ under the weighted spanning tree measure is no larger than it was in the original network.

4.41. Let E be a finite set and $k < |E|$. Let $\mathbf{P}$ be uniform measure on subsets of E of size k. Show that if X and Y are increasing random variables with finite second moments that depend on disjoint subsets of E, as defined in Exercise 4.6, then $\mathbf{E}[XY] \le \mathbf{E}[X]\mathbf{E}[Y]$.

4.42. Given two probability measures μ_1 and μ_2 on $\mathbb{R}$, we say that μ_1 ***stochastically dominates*** μ_2 if for all $r \in \mathbb{R}$, we have $\mu_1(r, \infty) \ge \mu_2(r, \infty)$. Let E be a finite set and $k < |E|$. Let X be a uniform random subset of E of size k. Show that if $\mathscr{A}$ is an increasing event that depends only on $F \subset E$, then the conditional distribution of $|X \cap F|$ given $\mathscr{A}$ stochastically dominates the unconditional distribution of $|X \cap F|$.

4.43. Show that for $n \ge 1$, the probability that simple random walk on $\mathbb{Z}^2$ starting at $(0,0)$ visits (n,n) before returning to $(0,0)$ equals

$$\frac{\pi}{8}\left(\sum_{k=1}^{n} \frac{1}{2k-1}\right)^{-1}.$$

4.44. For a function $f \in L^1(\mathbb{T}^2)$ and integers x, y, define

$$\widehat{f}(x,y) := \int_{\mathbb{T}^2} f(\alpha) e^{-2\pi i (x\alpha_1 + y\alpha_2)}\, d\alpha\,.$$

Let Y be the transfer-current matrix for the square lattice $\mathbb{Z}^2$. Let $e^{\mathrm{h}}_{x,y} := [(x,y),(x+1,y)]$ and $e^{\mathrm{v}}_{x,y} := [(x,y),(x,y+1)]$. Show that $Y(e^{\mathrm{h}}_{0,0}, e^{\mathrm{h}}_{x,y}) = \widehat{f}(x,y)$ and $Y(e^{\mathrm{h}}_{0,0}, e^{\mathrm{v}}_{x,y}) = \widehat{g}(x,y)$, where

$$f(\alpha_1, \alpha_2) := \frac{\sin^2 \pi\alpha_1}{\sin^2 \pi\alpha_1 + \sin^2 \pi\alpha_2}$$

and

$$g(\alpha_1, \alpha_2) := \frac{(1 - e^{2\pi i \alpha_1})(1 - e^{-2\pi i \alpha_2})}{4(\sin^2 \pi\alpha_1 + \sin^2 \pi\alpha_2)}\,.$$

4.45. Consider the ladder graph on $\mathbb{Z} \times \mathbb{Z}_2$ that is the doubly infinite limit of the ladder graphs shown in Figure 1.8. Calculate its transfer-current matrix.

Chapter 5

Percolation on Trees

Consider groundwater percolating down through soil and rock. How can we model the effects of the irregularities of the medium through which the water percolates? One common approach is to use a model in which the medium is random. More specifically, the pathways by which the water can travel are randomly chosen out of some regular set of possible pathways. For example, one may treat the ground as a half-space in which possible pathways are the rectangular lattice lines. Thus, we consider the nearest-neighbor graph on $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^-$ and each edge is independently chosen to be open (allowing water to flow) or closed. Commonly, the marginal probability that an edge is open, p, is the same for all edges. In this case, the only parameter in the model is p and one studies how p affects large-scale behavior of possible water flow.

In fact, this model of percolation is used in many other contexts in order to have a simple model that nevertheless captures some important aspects of an irregular situation. In particular, it has an interesting phase transition. In this chapter, we will consider percolation on trees, rather than on lattices. This turns out to be interesting and also useful for other seemingly unrelated probabilistic processes and questions. In Chapter 7, we will look at percolation on transitive graphs, especially, on non-amenable graphs.

§5.1. Galton-Watson Branching Processes.

Percolation on a tree breaks up the tree into random subtrees. Historically, the first random trees to be considered were a model of genealogical (family) trees. Since such trees will be an important source of examples and an important tool in later work, we too will consider their basic theory before turning to percolation.

Galton-Watson branching processes are generally defined as Markov chains $\langle Z_n\,;\ n \geq 0\rangle$ on the nonnegative integers, but we will be interested as well in the family trees that they produce. Given numbers $p_k \in [0,1]$ with $\sum_{k\geq 0} p_k = 1$, the process is defined as follows. We start with one particle, $Z_0 \equiv 1$, unless specified otherwise. It has k children with probability p_k. Then each of these children (should there be any) also have children with the same progeny (or "offspring") distribution, independently of each other and of their parent. This continues forever or until there are no more children. To be formal, let L be a random variable with $\mathbf{P}[L = k] = p_k$ and let $\langle L_i^{(n)}\,;\; n,\, i \geq 1\rangle$ be independent copies of L. The generation sizes of the branching process are then defined inductively by

$$Z_{n+1} := \sum_{i=1}^{Z_n} L_i^{(n+1)}\,. \tag{5.1}$$

The ***probability generating function (p.g.f.)*** of L is very useful and is denoted

$$f(s) := \mathbf{E}[s^L] = \sum_{k \geq 0} p_k s^k\,.$$

We call the event $\{\exists n\ Z_n = 0\}$ $(= \{Z_n \to 0\})$ ***extinction***. We will often omit the superscripts on L when not needed. The family (or genealogical) tree associated to a branching process is obtained simply by having one vertex for each particle ever produced and joining two by an edge if one is the parent of the other. See Figure 5.1 for an example.

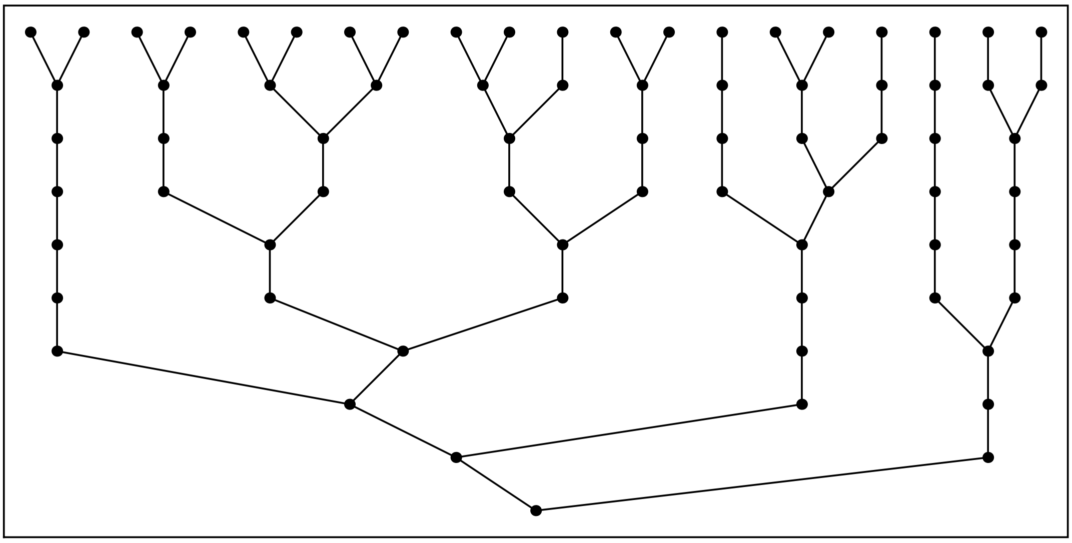


Figure 5.1. Generations 0 to 9 of a typical Galton-Watson tree for $f(s) = (s + s^2)/2$.

Proposition 5.1. *On the event of nonextinction, $Z_n \to \infty$ a.s. provided $p_1 \neq 1$.*

Proof. We want to see that 0 is the only nontransient state. If $p_0 = 0$, this is clear, while if $p_0 > 0$, then from any state $n \geq 1$, eventually returning to n requires not immediately becoming extinct, whence has probability $\leq 1 - p_0^n < 1$. ◀

What is $q := \mathbf{P}[\text{extinction}]$? To find out, we need:

Proposition 5.2. $\mathbf{E}[s^{Z_n}] = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}(s) =: f^{(n)}(s)$.

Proof. We have

$$\begin{aligned}\mathbf{E}[s^{Z_n}] &= \mathbf{E}\Big[\mathbf{E}\Big[s^{\sum_{i=1}^{Z_{n-1}} L_i} \;\Big|\; Z_{n-1}\Big]\Big] = \mathbf{E}\left[\mathbf{E}\left[\prod_{i=1}^{Z_{n-1}} s^{L_i} \;\middle|\; Z_{n-1}\right]\right] \\ &= \mathbf{E}\left[\prod_{i=1}^{Z_{n-1}} \mathbf{E}\big[s^{L_i}\big]\right] = \mathbf{E}\big[\mathbf{E}[s^L]^{Z_{n-1}}\big] = \mathbf{E}\big[f(s)^{Z_{n-1}}\big]\end{aligned}$$

since the L_i are independent of each other and of Z_{n-1} and have the same distribution as L. Iterate this equation n times. ◀

Note that within this proof is the result that

$$\mathbf{E}[s^{Z_n} \mid Z_0, Z_1, \ldots, Z_{n-1}] = f(s)^{Z_{n-1}}\,. \tag{5.2}$$

Corollary 5.3. (Extinction Probability) $q = \lim_n f^{(n)}(0)$.

Proof. Since extinction is the increasing union of the events $\{Z_n = 0\}$, we have $q = \lim_n \mathbf{P}[Z_n = 0] = \lim_n f^{(n)}(0)$. ◀

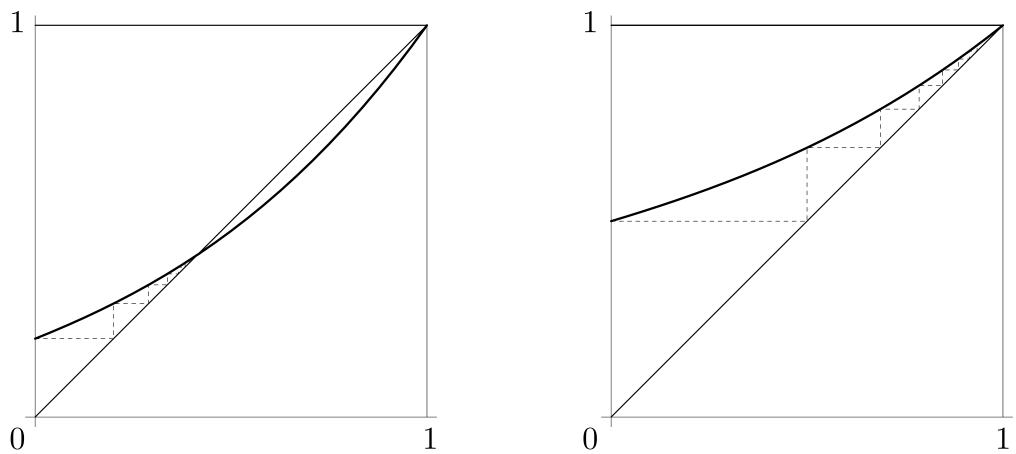


Figure 5.2. Typical graphs of f when $m > 1$ and $m \leq 1$.

Looking at a graph of the increasing convex function f (Figure 5.2) we discover the most-used result in the field and the answer to our question:

Proposition 5.4. (Extinction Criterion) *Provided $p_1 \neq 1$, we have*

(i) $q = 1 \Leftrightarrow f'(1) \le 1$;

(ii) *q is the smallest root of $f(s) = s$ in $[0, 1]$ — the only other possible root being 1.*

When we differentiate f at 1, we mean the left-hand derivative. Note that

$$f'(1) = \mathbf{E}[L] =: m = \sum kp_k \,, \tag{5.3}$$

the mean number of offspring. We call m simply the ***mean*** of the branching process.

▷ **Exercise 5.1.**

Justify the differentiation in (5.3). Show too that $\lim_{s \uparrow 1} f'(s) = m$.

Because of Proposition 5.4, a branching process is called ***subcritical*** if $m < 1$, ***critical*** if $m = 1$, and ***supercritical*** if $m > 1$.

How quickly does $Z_n \to \infty$ on the event of nonextinction? We begin this analysis by dividing Z_n by m^n, which, it results, is its mean:

Proposition 5.5. *The sequence $\langle Z_n/m^n \rangle$ is a martingale when $0 < m < \infty$.*

In particular, $\mathbf{E}[Z_n] = m^n$ since $\mathbf{E}[Z_0] = \mathbf{E}[\mathbf{1}] = 1$.

Proof. We have

$$\begin{aligned}\mathbf{E}\big[Z_{n+1}/m^{n+1} \mid Z_n\big] &= \mathbf{E}\left[\frac{1}{m^{n+1}} \sum_{i=1}^{Z_n} L_i^{(n+1)} \;\middle|\; Z_n\right] \\ &= \frac{1}{m^{n+1}} \sum_{i=1}^{Z_n} \mathbf{E}[L_i^{(n+1)} \mid Z_n] = \frac{1}{m^{n+1}} \sum_{i=1}^{Z_n} m = Z_n/m^n \,. \end{aligned}$$ ◀

Since this martingale $\langle Z_n/m^n \rangle$ is nonnegative, it has a finite limit a.s., called W. Thus, when $W > 0$, the generation sizes Z_n grow as expected, i.e., like m^n up to a random factor. Otherwise, they grow more slowly. Our attention is thus focussed on the following two questions.

Question 1. When is $W > 0$?

Question 2. When $W = 0$ and the process does not become extinct, what is the rate at which $Z_n \to \infty$?

We first note a general zero-one property of Galton-Watson branching processes. Call a property of trees ***inherited*** if whenever a tree has this property, so do *all* the descendant trees of the children of the root, and if every finite tree has this property.

Proposition 5.6. *Every inherited property has probability either 0 or 1 given nonextinction.*

Proof. Let A be the set of trees possessing a given inherited property. For a tree T with k children of the root, let $T^{(1)}, \ldots, T^{(k)}$ be the descendant trees of these children. Then

$$\mathbf{P}[A] = \mathbf{E}\big[\mathbf{P}[T \in A \mid Z_1]\big] \le \mathbf{E}\big[\mathbf{P}[T^{(1)} \in A, \ldots, T^{(Z_1)} \in A \mid Z_1]\big]$$

by definition of inherited. Since $T^{(1)}, \ldots, T^{(Z_1)}$ are i.i.d. given Z_1, the last quantity above is equal to $\mathbf{E}\big[\mathbf{P}[A]^{Z_1}\big] = f(\mathbf{P}[A])$. Thus, $\mathbf{P}[A] \le f(\mathbf{P}[A])$. On the other hand, $\mathbf{P}[A] \ge q$ since every finite tree is in A. It follows upon inspection of a graph of f that $\mathbf{P}[A] \in \{q, 1\}$, from which the desired conclusion follows. ◀

Corollary 5.7. *Suppose that $0 < m < \infty$. Either $W = 0$ a.s. or $W > 0$ a.s. on nonextinction. In other words, $\mathbf{P}[W = 0] \in \{q, 1\}$.*

Proof. The property that $W = 0$ is clearly inherited, whence this is an immediate consequence of Proposition 5.6. ◀

In answer to the above two questions, we have the following two theorems.

The Kesten-Stigum Theorem (1966). *The following are equivalent when $1 < m < \infty$:*

(i) $\mathbf{P}[W = 0] = q$;
(ii) $\mathbf{E}[W] = 1$;
(iii) $\mathbf{E}[L \log^+ L] < \infty$.

This will be shown in Section 12.2. Since (iii) requires barely more than the existence of a mean, generation sizes "typically" do grow as expected. When (iii) fails, however, the means m^n overestimate the rate of growth. Yet there is still a deterministic rate of growth, as shown by Seneta (1968) and Heyde (1970):

The Seneta-Heyde Theorem. *If $1 < m < \infty$, then there exist constants c_n such that*

(i) $\lim Z_n/c_n$ *exists a.s. in* $[0, \infty)$;
(ii) $\mathbf{P}[\lim Z_n/c_n = 0] = q$;
(iii) $c_{n+1}/c_n \to m$.

Proof. Choose $s_0 \in (q, 1)$ and set $s_{n+1} := f^{-1}(s_n)$ for $n \ge 0$. Then $s_n \uparrow 1$. By (5.2), we have that $\langle s_n^{Z_n} \rangle$ is a martingale. Being positive and bounded, it converges a.s. and in L^1 to a limit $Y \in [0,1]$ such that $\mathbf{E}[Y] = \mathbf{E}[s_0^{Z_0}] = s_0$. Set $c_n := -1/\log s_n$. Then $s_n^{Z_n} = e^{-Z_n/c_n}$, so that $\lim Z_n/c_n$ exists a.s. in $[0,\infty]$. By l'Hôpital's rule and Exercise 5.1,

$$\lim_{s\to 1} \frac{-\log f(s)}{-\log s} = \lim_{s\to 1} \frac{f'(s)s}{f(s)} = m\,.$$

Considering this limit along the sequence $\langle s_n \rangle$, we get (iii). It follows from (iii) that the property that $\lim Z_n/c_n = 0$ is inherited, whence by Proposition 5.6 and the fact that $\mathbf{E}[Y] = s_0 < 1$, we deduce (ii). Likewise, the property that $\lim Z_n/c_n < \infty$ is inherited and has probability 1 since $\mathbf{E}[Y] > q$. This implies (i). ◀

The proof of the Seneta-Heyde theorem gives a prescription for calculating the constants c_n, but does not immediately provide estimates for them. Another approach gives a different prescription that leads sometimes to an explicit estimate; see Asmussen and Hering (1983), pp. 45–49.

We will often want to consider random trees produced by a Galton-Watson branching process. Up to now, we have avoided that by giving theorems just about the random variables Z_n. One approach is as follows. A ***rooted labelled tree*** T is a non-empty collection of finite sequences of positive integers such that if $\langle i_1, i_2, \ldots, i_n \rangle \in T$, then for every $k \in [0,n]$, also the initial segment $\langle i_1, i_2, \ldots, i_k \rangle \in T$, where the case $i = 0$ means the empty sequence; and for every $j \in [1, i_n]$, also the sequence $\langle i_1, i_2, \ldots, i_{n-1}, j \rangle \in T$. The root of the tree is the empty sequence, $\varnothing$. Thus, $\langle i_1, \ldots, i_n \rangle$ is the i_nth child of the i_{n-1}th child of $\ldots$ of the i_1th child of the root. If $\langle i \rangle \in T$, then we define $T^{(i)} := \{\langle i_1, i_2, \ldots, i_n \rangle\,;\ \langle i, i_1, i_2, \ldots, i_n \rangle \in T\}$ to be the ***descendant tree*** of the vertex i in T. The ***height*** of a tree is the supremum of the lengths of the sequences in the tree. If T is a tree and $n \in \mathbb{N}$, write the ***truncation*** of T to its first n levels as $T{\restriction}n := \{\langle i_1, i_2, \ldots, i_k \rangle \in T\,;\ k \le n\}$. This is a tree of height at most n. A tree is called ***locally finite*** if its truncation to every finite level is finite. Let $\mathscr{T}$ be the space of rooted labelled locally finite trees. We define a metric on $\mathscr{T}$ by setting $d(T,T') := (1 + \sup\{n\,;\ T{\restriction}n = T'{\restriction}n\})^{-1}$.

▷ Exercise 5.2.

Verify that d is a metric and that $\mathscr{T}$ is complete and separable.

▷ Exercise 5.3.

Define the measure **GW** formally on the space $\mathscr{T}$ of Exercise 5.2; your measure should be the law of a random tree produced by a Galton-Watson process with arbitrary given offspring distribution.

We can now use this formalism to give meaning to statements such as that in the following exercise.

▷ **Exercise 5.4.**
Show that for any Galton-Watson process with mean $m > 1$, the family tree T has growth rate $\operatorname{gr} T = m$ a.s. given nonextinction. (Don't use the Kesten-Stigum Theorem to show this, as we have not yet proved that theorem.)

§5.2. The First-Moment Method.

Let G be a countable, possibly unconnected, graph. The most common percolation on G is ***Bernoulli bond percolation with constant survival parameter*** p, or ***Bernoulli(p) percolation*** for short; here, for fixed $p \in [0, 1]$, each edge is kept with probability p and removed otherwise, independently of the other edges. Denote the random subgraph of G that remains by ω. The connected components of ω are called ***clusters***. Given a vertex x of G, one is often interested in the cluster of x in ω, written $K(x)$, and especially in knowing the chance that the diameter of $K(x)$ is infinite. The first-moment method explained in this section gives a simple upper bound on this probability. In fact, this method is so simple that it works in complete generality: Suppose that ω is *any* random subgraph of G. The only measurability needed is that for each vertex x and each edge e, the sets $\{\omega\,;\ x \in \omega\}$ and $\{\omega\,;\ e \in \omega\}$ are measurable. We will call such a random subgraph a ***general percolation*** on G. We will say that a set Π of edges of G ***separates*** x ***from infinity*** if the removal of Π leaves x in a component of finite diameter. Denote by $\{x \leftrightarrow e\}$ the event that e is in the cluster of x and by $\{x \leftrightarrow \infty\}$ that x is in a cluster of infinite diameter.

▷ **Exercise 5.5.**
Show that for a general percolation, the events $\{x \leftrightarrow e\}$ and $\{x \leftrightarrow \infty\}$ are indeed measurable.

Proposition 5.8. *Given a general percolation on G,*

$$\mathbf{P}[x \leftrightarrow \infty] \le \inf\left\{\sum_{e\in\Pi} \mathbf{P}[x \leftrightarrow e]\,;\ \Pi \textit{ separates } x \textit{ from infinity}\right\}. \tag{5.4}$$

Proof. For any Π separating x from infinity, we have

$$\{x \leftrightarrow \infty\} \subseteq \bigcup_{e\in\Pi} \{x \leftrightarrow e\}$$

by definition. Therefore $\mathbf{P}[x \leftrightarrow \infty] \le \sum_{e\in\Pi} \mathbf{P}[x \leftrightarrow e]$. ◀

Returning to Bernoulli percolation with constant survival parameter p, denote the law of ω by $\mathbf{P}_p$. By Kolmogorov's zero-one law,

$$\mathbf{P}_p[\omega \text{ has a cluster of infinite diameter}] \in \{0,1\}\,.$$

It is intuitively clear that this probability is increasing in p. For a rigorous proof of this, we couple all the percolation processes at once as follows. Let $U(e)$ be i.i.d. uniform $[0,1]$ random variables indexed by the edges of G. If ω_p is the graph containing all the vertices of G and exactly those edges e with $U(e) < p$, then the law of ω_p will be precisely $\mathbf{P}_p$. This coupling is referred to as the ***standard coupling*** of Bernoulli percolation. But now when $p \le q$, the event that ω_p has a cluster of infinite diameter is contained in the event that ω_q has a cluster of infinite diameter. Hence the probability of the first event is at most the probability of the second. This leads us to define the ***critical probability***

$$p_{\mathrm{c}}(G) := \sup\{p\,;\ \mathbf{P}_p[\exists \text{ infinite-diameter cluster}] = 0\}\,.$$

If G is connected and x is any given vertex of G, then

$$p_{\mathrm{c}}(G) = \sup\{p\,;\ \mathbf{P}_p[x \leftrightarrow \infty] = 0\}\,. \tag{5.5}$$

▷ **Exercise 5.6.**
Prove this.

Again, the coupling provides a rigorous proof that $\mathbf{P}_p[x \leftrightarrow \infty]$ is increasing in p.

Generally, $p_{\mathrm{c}}(G)$ is extremely difficult to calculate. Clearly $p_{\mathrm{c}}(\mathbb{Z}) = 1$. After long efforts, it was shown that $p_{\mathrm{c}}(\mathbb{Z}^2) = 1/2$ (Kesten, 1980). There is not even a conjecture for the value of $p_{\mathrm{c}}(\mathbb{Z}^d)$ for any $d \ge 3$. Now Proposition 5.8 provides a lower bound for p_{c} provided we can estimate $\mathbf{P}[o \leftrightarrow e]$. If G is a tree T, this is easy: $\mathbf{P}[o \leftrightarrow e] = p^{|e|+1}$. Hence,

$$p_{\mathrm{c}}(T) \ge 1/\operatorname{br} T\,. \tag{5.6}$$

In fact, we have equality here (Theorem 5.15), but this requires the second-moment method. Nevertheless, there are some cases among non-lattice graphs where it is easy to determine p_{c} exactly, even without the first-moment method. One example is given in the next exercise:

▷ **Exercise 5.7.**

Show that for $p \ge p_c(G)$, we have $p_c(\omega_p) = p_c(G)/p$ $\mathbf{P}_p$-a.s. Physicists often refer to p as the "density" of edges in ω and this helps the intuition.

For another example, if T is an n-ary tree, then the cluster of the root under percolation is a Galton-Watson tree with progeny distribution $\mathrm{Bin}(n, p)$. Thus, this cluster is infinite w.p.p. iff $np > 1$, whence $p_c(T) = 1/n$. This reasoning may be extended to all Galton-Watson trees (in which case, $p_c(T)$ is a random variable):

Proposition 5.9. (Lyons, 1990) *Let T be the family tree of a Galton-Watson process with mean $m > 1$. Then $p_c(T) = 1/m$ a.s. given nonextinction.*

Proof. Let T be a given tree and write K for the cluster of the root of T after percolation on T with survival parameter p. When T has the law of a Galton-Watson tree with mean m, we claim that K has the law of another Galton-Watson tree having mean mp: if Y_i represent i.i.d. $\mathrm{Bin}(1, p)$ random variables that are also independent of L, then

$$\mathbf{E}\left[\sum_{i=1}^{L} Y_i\right] = \mathbf{E}\left[\mathbf{E}\left[\sum_{i=1}^{L} Y_i \;\middle|\; L\right]\right] = \mathbf{E}\left[\sum_{i=1}^{L} \mathbf{E}[Y_i]\right] = \mathbf{E}\left[\sum_{i=1}^{L} p\right] = pm\,.$$

Hence K is finite a.s. iff $mp \le 1$. Since

$$\mathbf{E}[\mathbf{P}[|K| < \infty \mid T]] = \mathbf{P}[|K| < \infty]\,, \tag{5.7}$$

this means that for almost every Galton-Watson tree* T, the cluster of its root is finite a.s. if $p \le 1/m$. On the other hand, for fixed p, the property $\{T\,;\ \mathbf{P}_p[|K| < \infty] = 1\}$ is inherited, so has probability q or 1. If it has probability 1, then (5.7) shows that $mp \le 1$. That is, if $mp > 1$, this property has probability q, so that the cluster of the root of T will be infinite with positive probability a.s. on the event of nonextinction. Considering a sequence $p_n \downarrow 1/m$, we see that this holds a.s. on the event of nonextinction for all $p > 1/m$ at once, not just for a fixed p. We conclude that $p_c(T) = 1/m$ a.s. on nonextinction. ◀

We may easily deduce the branching number of Galton-Watson trees, as shown by Lyons, 1990:

Corollary 5.10. (Branching Number of Galton-Watson Trees) *If T is a Galton-Watson tree with mean $m > 1$, then $\mathrm{br}\, T = m$ a.s. given nonextinction.*

Proof. By Proposition 5.9 and (5.6), we have $\mathrm{br}\, T \ge m$ a.s. given nonextinction. On the other hand, $\mathrm{br}\, T \le \mathrm{gr}\, T = m$ a.s. given nonextinction by Exercise 5.4. ◀

---

* This typical abuse of language means "for almost every tree with respect to Galton-Watson measure".

This corollary was shown in the language of Hausdorff dimension (which will be explained in Chapter 14) by Hawkes (1981) under the assumption that $\mathbf{E}[L(\log L)^2] < \infty$.

§5.3. The Weighted Second-Moment Method.

A lower bound on $\mathbf{P}[o \leftrightarrow \infty]$ for a general percolation on a graph G can be obtained by the second-moment method. Fix $o \in G$ and let Π be a minimal set of edges that separates o from ∞. Assume for simplicity that $\mathbf{P}[e \in \omega] > 0$ for each $e \in \mathsf{E}$. Let $\mathcal{P}(\Pi)$ be the set of probability measures μ on Π. The second moment method consists in calculating the second moment of the number of edges in Π that are connected to o. However, we will see that it can be much better to use a weighted count, rather than a pure count, of the number of connected edges. We will use $\mu \in \mathcal{P}$ to make such weights.

Namely, if, as before, we write $K(o)$ for the cluster of o in the percolation graph ω and if we set

$$X(\mu) := \sum_{e \in \Pi} \mu(e) \mathbf{1}_{\{e \in K(o)\}} / \mathbf{P}[e \in K(o)]\,,$$

then

$$\mathbf{E}[X(\mu)] = 1\,.$$

Write $o \leftrightarrow \Pi$ for the event that $o \leftrightarrow e$ for some $e \in \Pi$. This event is implied by the event that $X(\mu) > 0$. We are looking for a lower bound on the probability of $o \leftrightarrow \infty$. Since

$$\mathbf{P}[o \leftrightarrow \infty] = \inf\{\mathbf{P}[o \leftrightarrow \Pi]\,;\ \Pi \text{ separates } o \text{ from } \infty\}\,, \tag{5.8}$$

we therefore seek a lower bound on $\mathbf{P}[X(\mu) > 0]$. This will be a consequence of an upper bound on the second moment of $X(\mu)$ as follows.

Proposition 5.11. *Given a general percolation on G,*

$$\mathbf{P}[o \leftrightarrow \Pi] \ge 1/\mathbf{E}[X(\mu)^2]$$

for every $\mu \in \mathcal{P}(\Pi)$.

Proof. Given $\mu \in \mathcal{P}(\Pi)$, the Cauchy-Schwarz inequality yields

$$\begin{aligned} 1 = \mathbf{E}[X(\mu)]^2 = \mathbf{E}[X(\mu)\mathbf{1}_{\{X(\mu)>0\}}]^2 &\le \mathbf{E}[X(\mu)^2]\mathbf{E}[\mathbf{1}^2_{\{X(\mu)>0\}}] \\ = \mathbf{E}[X(\mu)^2]\mathbf{P}[X(\mu) > 0] &\le \mathbf{E}[X(\mu)^2]\mathbf{P}[o \leftrightarrow \Pi] \end{aligned}$$

since $X(\mu) > 0$ implies $o \leftrightarrow \Pi$. Therefore, $\mathbf{P}[o \leftrightarrow \Pi] \ge 1/\mathbf{E}[X(\mu)^2]$. ◀

Clearly, we want to choose the weight function $\mu \in \mathcal{P}(\Pi)$ that optimizes this lower bound. Now

$$\mathbf{E}[X(\mu)^2] = \sum_{e_1, e_2 \in \Pi} \mu(e_1)\mu(e_2) \frac{\mathbf{P}[e_1, e_2 \in K(o)]}{\mathbf{P}[e_1 \in K(o)]\mathbf{P}[e_2 \in K(o)]} \,. \tag{5.9}$$

We denote this quantity by $\mathscr{E}(\mu)$ and call it the ***energy*** of μ.

Why is this called an "energy"? In general, like the energy of a flow in Chapter 2, energy is a quadratic form, usually positive definite. In electrostatics, if μ is a charge distribution confined to a conducting region Ω in space, then μ will minimize the energy

$$\int_\Omega \int_\Omega \frac{d\mu(x)\, d\mu(y)}{|x-y|^2} \,.$$

One could also write (5.9) as a double integral to put it in closer form to this. We too are interested in minimizing the energy. Thus, from Proposition 5.11, we obtain

Proposition 5.12. *Given a general percolation on G,*

$$\mathbf{P}[o \leftrightarrow \infty] \ge \inf \left\{ \sup_{\mu \in \mathcal{P}(\Pi)} 1/\mathscr{E}(\mu) \,;\; \Pi \textit{ separates } o \textit{ from infinity} \right\} .$$

Proof. We have shown that $\mathbf{P}[o \leftrightarrow \Pi] \ge 1/\mathscr{E}(\mu)$ for every $\mu \in \mathcal{P}(\Pi)$. Hence, the same holds when we take the sup of the right-hand side over μ. Then the result follows from (5.8). ◀

Of course, for Proposition 5.12 to be useful, one has to find a way to estimate such energies. As for the first-moment method, the case of trees is most amenable to such analysis.

Consider first the case of ***independent percolation***, i.e., with $\{e \in T(\omega)\}$ mutually independent events for all edges e. If $\mu \in \mathcal{P}(\Pi)$, write $\mu(x)$ for $\mu\big(e(x)\big)$. Then we have

$$\mathscr{E}(\mu) = \sum_{e(x), e(y) \in \Pi} \mu(x)\mu(y) \frac{\mathbf{P}[o \leftrightarrow x, o \leftrightarrow y]}{\mathbf{P}[o \leftrightarrow x]\mathbf{P}[o \leftrightarrow y]} = \sum_{e(x), e(y) \in \Pi} \mu(x)\mu(y)/\mathbf{P}[o \leftrightarrow x \wedge y] \,, \tag{5.10}$$

where $x \wedge y$ denotes the furthest vertex from o that is a common ancestor of both x and y (we say that z is an ***ancestor*** of w if z lies on the shortest path between o and w, so that $z = w$ is not excluded). This looks suspiciously similar to the result of the following calculation: Consider conductances on a finite tree T and some flow θ from o to the leaves, which we'll write as $\partial_{\mathrm{L}} T$, of T. Write $\theta(x)$ for $\theta\big(e(x)\big)$.

Lemma 5.13. *Let θ be a flow on a finite tree T from o to $\partial_{\mathrm{L}} T$. Then*

$$\mathscr{E}(\theta) = \sum_{x,y \in \partial_{\mathrm{L}} T} \theta(x)\theta(y)\mathscr{R}(o \leftrightarrow x \wedge y)\,.$$

Proof. We use the fact that $\sum_{\{x \in \partial_{\mathrm{L}} T\,;\, e \le x\}} \theta(x) = \theta(e)$ for any edge e (see Exercise 3.3). Thus, we have

$$\begin{aligned}\sum_{x,y \in \partial_{\mathrm{L}} T} \theta(x)\theta(y)\mathscr{R}(o \leftrightarrow x \wedge y) &= \sum_{x,y \in \partial_{\mathrm{L}} T} \theta(x)\theta(y) \sum_{e \le x \wedge y} r(e) \\ &= \sum_{e \in T} r(e) \sum_{\substack{x,y \in \partial_{\mathrm{L}} T \\ x,y \ge e}} \theta(x)\theta(y) = \sum_{e \in T} r(e)\theta(e)^2 = \mathscr{E}(\theta)\,. \quad \blacktriangleleft\end{aligned}$$

Write $\overline{\Pi} := \{x\,;\; e(x) \in \Pi\}$. Let Π be a minimal set of edges that separates o from ∞. If we happen to have $\mathbf{P}[o \leftrightarrow x] = \mathscr{C}(o \leftrightarrow x)$ and if θ is the flow induced by μ from o to $\overline{\Pi}$, i.e.,

$$\theta(e) := \sum_{e \le x \in \overline{\Pi}} \mu(x)\,,$$

then we see from (5.10) and Lemma 5.13 that $\mathscr{E}(\mu) = \mathscr{E}(\theta)$ and we can hope to profit from our understanding of electrical networks and random walks. However, if $x = o$, then the desired equation cannot hold since $\mathbf{P}[o \leftrightarrow o] = 1$ and $\mathscr{C}(o \leftrightarrow o) = \infty$. But suppose we have, instead, that

$$1/\mathbf{P}[o \leftrightarrow x] = 1 + \mathscr{R}(o \leftrightarrow x) \tag{5.11}$$

for all vertices x. Then we find that

$$\mathscr{E}(\mu) = 1 + \mathscr{E}(\theta)\,,$$

which is hardly worse. This suggests that given a percolation problem, we choose conductances so that (5.11) holds in order to use our knowledge of electrical networks. Let us say that conductances are ***adapted*** to a percolation, and vice versa, if (5.11) holds.

Let the survival probability of $e(x)$ be p_x. To solve (5.11) explicitly for the conductances in terms of the survival parameters, write $\overleftarrow{x}$ for the parent of x. Note that the right-hand side of (5.11) is a resistance of a series, whence

$$\begin{aligned}r\big(e(x)\big) &= [1 + \mathscr{R}(o \leftrightarrow x)] - [1 + \mathscr{R}(o \leftrightarrow \overleftarrow{x})] = 1/\mathbf{P}[o \leftrightarrow x] - 1/\mathbf{P}[o \leftrightarrow \overleftarrow{x}] \\ &= (1 - p_x)/\mathbf{P}[o \leftrightarrow x]\,,\end{aligned}$$

or, in other words,

$$c\big(e(x)\big) = \frac{\mathbf{P}[o \leftrightarrow x]}{1 - p_x} = \frac{1}{1 - p_x} \prod_{o < u \le x} p_u\,. \tag{5.12}$$

In particular, for a given $p \in (0, 1)$, we have that $p_x \equiv p$ iff $c\big(e(x)\big) = (1-p)^{-1} p^{|x|}$; these conductances correspond to the random walk $\mathsf{RW}_{1/p}$ defined in Section 3.2.

▷ **Exercise 5.8.**

Show that, conversely, the survival parameters adapted to given edge resistances are

$$p_x = \frac{1 + \sum_{o<u<x} r\big(e(u)\big)}{1 + \sum_{o<u\le x} r\big(e(u)\big)}\,.$$

For example, simple random walk ($c \equiv 1$) is adapted to $p_x = |x|/(|x|+1)$.

Theorem 5.14. (Lyons, 1992) *For an independent percolation and adapted conductances on the same tree, we have*

$$\frac{\mathscr{C}(o \leftrightarrow \infty)}{1 + \mathscr{C}(o \leftrightarrow \infty)} \le \mathbf{P}[o \leftrightarrow \infty]\,.$$

Proof. We first estimate the sup in Proposition 5.12: given a minimal set Π of edges that separates o from ∞, let $\mu \in \mathcal{P}(\Pi)$ be the measure in $\mathcal{P}(\Pi)$ that has minimum energy and let θ be the flow induced by μ. We have

$$\mathscr{E}(\mu) = 1 + \mathscr{E}(\theta) = 1 + \mathscr{R}(o \leftrightarrow \overline{\Pi})\,.$$

Therefore,

$$\mathbf{P}[o \leftrightarrow \infty] \ge \inf_{\Pi} 1/[1 + \mathscr{R}(o \leftrightarrow \overline{\Pi})] = 1/[1 + \mathscr{R}(o \leftrightarrow \infty)]\,,$$

as desired. ◀

An immediate corollary of this combined with (5.6) and Theorems 2.3 and 3.5 is:

Theorem 5.15. (Lyons, 1990) *For any locally finite infinite tree T,*

$$p_c(T) = \frac{1}{\operatorname{br} T}\,.$$

This reinforces the idea of $\operatorname{br} T$ as an average number of branches per vertex.

Question 5.16. This result shows that the first-moment method correctly identifies the critical value for Bernoulli percolation on trees. Does it in general? In other words, if the right-hand side of (5.4) is strictly positive for Bernoulli(p) percolation on a connected graph G, then must it be the case that $p \ge p_c(G)$? This is known to hold on $\mathbb{Z}^d$ and on "tree-like" graphs; see Lyons (1989). However, Kahn (2003) gave a counterexample and suggested the following modification of the question: Write $A(x, e, \Pi)$ for the event that there is an open path from x to e that is disjoint from $\Pi \setminus \{e\}$. If

$$\inf\left\{\sum_{e\in\Pi} \mathbf{P}_p[A(x,e,\Pi)]\,;\ \Pi \text{ separates } x \text{ from infinity}\right\} > 0\,,$$

then is $p \geq p_c(G)$?

It turns out that the inequality in Theorem 5.14 can be reversed up to a factor of 2. We will show this by a stopping-time method in the next section.

We will now consider other applications of the second-moment method to percolation on trees. Suppose that we label the edges e of a tree T by independent random variables $Z(e)$ that take the values ± 1 with probability $1/2$ each. Fix an integer $N > 0$. Define $S(x) := \sum_{e \leq x} Z(e)$. Consider the percolation

$$\omega_N := \big\{[x, y]\,;\; S(x) \in [0, N],\; S(y) \in [0, N]\big\}\,. \tag{5.13}$$

Obviously the component of the root in ω_1 is the same as the component of the root in Bernoulli(1/2) percolation on T. In particular, percolation occurs for $\operatorname{br} T > 2$ but not for $\operatorname{br} T < 2$. The next case, ω_2, is almost as simple: the edges at even distance from the root are kept with probability $1/2$, while the others are kept with probability 1, so by Exercise 3.25, percolation occurs for $\operatorname{br} T > \sqrt{2}$ but not for $\operatorname{br} T < \sqrt{2}$. However, the succeeding cases ω_N for $N \geq 3$ are more complicated as there is dependency in the percolation that was not there before. Luckily, the dependency is not very large; we will show that it is an example of the following kind of percolation.

We call a percolation ***quasi-independent**** if $\exists M < \infty \;\; \forall x, y$ with $\mathbf{P}[o \leftrightarrow x \wedge y] > 0$,

$$\mathbf{P}[o \leftrightarrow x,\; o \leftrightarrow y \mid o \leftrightarrow x \wedge y] \leq M\mathbf{P}[o \leftrightarrow x \mid o \leftrightarrow x \wedge y]\mathbf{P}[o \leftrightarrow y \mid o \leftrightarrow x \wedge y]\,, \tag{5.14}$$

or, what is the same, if $\mathbf{P}[o \leftrightarrow x]\mathbf{P}[o \leftrightarrow y] > 0$, then

$$\frac{\mathbf{P}[o \leftrightarrow x, o \leftrightarrow y]}{\mathbf{P}[o \leftrightarrow x]\mathbf{P}[o \leftrightarrow y]} \leq M/\mathbf{P}[o \leftrightarrow x \wedge y]\,.$$

Example 5.17. Sometimes, this condition holds for easy reasons: If

$$\inf_{x \neq o} \mathbf{P}[o \leftrightarrow x \mid o \leftrightarrow \overleftarrow{x}] > 0$$

and

$$\mathbf{P}[o \leftrightarrow x,\; o \leftrightarrow y \mid o \leftrightarrow x \wedge y] = \mathbf{P}[o \leftrightarrow x \mid o \leftrightarrow \overleftarrow{x}]\mathbf{P}[o \leftrightarrow \overleftarrow{x},\; o \leftrightarrow y \mid o \leftrightarrow \overleftarrow{x} \wedge y]$$

whenever $\overleftarrow{x} \neq x \wedge y$, then the percolation is quasi-independent.

* This was called "quasi-Bernoulli" in Lyons (1989, 1992).

▷ **Exercise 5.9.**
Verify the assertion of Example 5.17.

Example 5.18. Now we verify that the percolation ω_N of (5.13) is also quasi-independent. Indeed, let $\langle S_n \rangle$ be simple random walk on $\mathbb{Z}$. There is clearly a constant M such that for all $n \ge 0$ and all $k, k' \in [0, N]$, we have

$$\mathbf{P}\big[S_n \in [0,N] \bigm| S_0 = k\big] \le M\mathbf{P}\big[S_n \in [0,N] \bigm| S_0 = k'\big] .$$

We claim that this M works in (5.14). To see this, let $\langle S'_n \rangle$ be another simple random walk on $\mathbb{Z}$, independent of $\langle S_n \rangle$. Then given x and y, put $r := |x \wedge y|$, $m := |x| - r$, and $n := |y| - r$. Also, write $p_k := \mathbf{P}\big[S_r = k \bigm| S_r \in [0,N]\big]$. We have

$$\begin{aligned}
&\mathbf{P}[o \leftrightarrow x,\ o \leftrightarrow y \mid o \leftrightarrow x \wedge y] \\
&\qquad = \sum_{k=0}^{N} \mathbf{P}\big[S_m \in [0,N], S'_n \in [0,N] \bigm| S_0 = S'_0 = k\big] p_k \\
&\qquad = \sum_{k=0}^{N} \mathbf{P}\big[S_m \in [0,N] \bigm| S_0 = k\big] \mathbf{P}\big[S'_n \in [0,N] \bigm| S'_0 = k\big] p_k \\
&\qquad \le M \min_k \mathbf{P}\big[S_n \in [0,N] \bigm| S_0 = k\big] \cdot \sum_{k=0}^{N} \mathbf{P}\big[S_m \in [0,N] \bigm| S_0 = k\big] p_k \\
&\qquad \le M \sum_{k=0}^{N} \mathbf{P}\big[S_n \in [0,N] \bigm| S_0 = k\big] p_k \sum_{k=0}^{N} \mathbf{P}\big[S_m \in [0,N] \bigm| S_0 = k\big] p_k \\
&\qquad = M\mathbf{P}[o \leftrightarrow x \mid o \leftrightarrow x \wedge y]\mathbf{P}[o \leftrightarrow y \mid o \leftrightarrow x \wedge y] .
\end{aligned}$$

This shows that our percolation is indeed quasi-independent.

Theorem 5.19. (Lyons, 1989) *For a quasi-independent percolation with constant M and adapted conductances, we have*

$$\frac{1}{M} \frac{\mathscr{C}(o \leftrightarrow \infty)}{1 + \mathscr{C}(o \leftrightarrow \infty)} \le \mathbf{P}[o \leftrightarrow \infty] .$$

Proof. For $\mu \in \mathcal{P}(\Pi)$, write

$$\mathscr{E}'(\mu) := \sum_{e(x), e(y) \in \Pi} \mu(x)\mu(y)/\mathbf{P}[o \leftrightarrow x \wedge y] .$$

Then the definition of quasi-independent gives $\mathscr{E}(\mu) \le M\mathscr{E}'(\mu)$. Also, if θ is the flow induced by μ, then $\mathscr{E}'(\mu) = 1 + \mathscr{E}(\theta)$. Hence

$$\mathbf{E}[X(\mu)^2] = \mathscr{E}(\mu) \le M\mathscr{E}'(\mu) = M[1 + \mathscr{E}(\theta)] ,$$

and the rest of the proof of Theorem 5.14 can be followed to the desired conclusion. ◀

Example 5.20. Let's apply this to Example 5.18. If we consider simple random walk on $[0, N]$ killed on exiting the interval, the corresponding substochastic transition matrix P is symmetric and so real diagonalizable. Let λ_k be its eigenvalues and v_k be the corresponding eigenvectors with $\|v_k\| = 1$. Thus,

$$P^n(i,j) = \sum_k \lambda_k^n v_k(i) v_k(j)\,.$$

By the Perron-Frobenius theorem, $|\lambda_k| \le \rho$, where ρ is the largest positive eigenvalue and the corresponding eigenvector has positive entries. Since this Markov chain has period 2, it follows that $P^n(i,j) \sim 2v_k(i)v_k(j)\rho^n$ when n and $i - j$ have the same parity; otherwise $P^n(i,j) = 0$. Now in our case, the top eigenvalue equals $\cos\frac{\pi}{N+2}$ (e.g., see Spitzer (1976), Chap. 21, Proposition 1), whence $\mathbf{P}[o \leftrightarrow x] \sim a_{|x|}\left(\cos\frac{\pi}{N+2}\right)^{|x|}$ as $|x| \to \infty$ for some constants $a_m > 0$, where a_m depends only on the parity of m. This means that for the conductances $c(e)$ adapted to this percolation, there are constants a_1' and a_2' such that $a_1'\left(\cos\frac{\pi}{N+2}\right)^{|e|} \le c(e) \le a_2'\left(\cos\frac{\pi}{N+2}\right)^{|e|}$. Thus, Theorem 5.19 yields that $\mathbf{P}[o \leftrightarrow \infty] > 0$ if RW_λ is transient on T for $\lambda := 1/\cos\frac{\pi}{N+2}$, which holds, in particular, if $\operatorname{br} T > 1/\cos\frac{\pi}{N+2}$. This example is due to Benjamini and Peres (1994).

▷ **Exercise 5.10.**
Show that if $\operatorname{br} T < 1/\cos\frac{\pi}{N+2}$, then percolation does not occur in ω_N.

§5.4. Reversing the Second-Moment Inequality.

The first- and second-moment methods give inequalities in very general situations. These two inequalities usually give fairly close estimates of a probability, but do not agree up to a constant factor. Thus, one must search for additional information. Usually, the estimate that the second-moment method gives is sharper than the one provided by the first moment. A method for showing the sharpness of the estimate given by the second-moment method will be described here in the context of percolation; it depends on a Markov-like structure (see also Exercise 15.12).

This method seems to be due to Hawkes (1970/71) and Shepp (1972). It was applied to trees by Lyons (1992) and to Markov chains (with a slight improvement) by Benjamini, Pemantle, and Peres (1995) (see Exercise 15.12). Consider independent percolation on a tree. Embed the tree in the upper half-plane with its root at the origin. Given a minimal cutset Π separating o from ∞, order Π clockwise. Call this linear ordering $\preccurlyeq$. This has the property that for all $e \in \Pi$, the events $\{o \leftrightarrow e'\}$ for $e' \preccurlyeq e$ are conditionally independent of the events $\{o \leftrightarrow e''\}$ for $e'' \succcurlyeq e$ given that $o \leftrightarrow e$. On the event $o \leftrightarrow \Pi$, define e^* to be the least edge in Π that is in the cluster of o; on $o \nleftrightarrow \Pi$, define e^* to take some value not in Π. Note that e^* is a random variable. Let σ be the (possibly defective) hitting measure

$$\sigma(e) := \mathbf{P}[e^* = e] \qquad (e \in \Pi)\,,$$

so that

$$\sigma(\Pi) = \mathbf{P}[o \leftrightarrow \Pi]\,.$$

Provided $\mathbf{P}[o \leftrightarrow \Pi] > 0$, we may define

$$\mu := \sigma/\mathbf{P}[o \leftrightarrow \Pi] \in \mathcal{P}(\Pi)\,.$$

For all $e \in \Pi$, we have

$$\begin{aligned}
\sum_{e' \preccurlyeq e} \sigma(e') \frac{\mathbf{P}[o \leftrightarrow e', o \leftrightarrow e]}{\mathbf{P}[o \leftrightarrow e']} &= \sum_{e' \preccurlyeq e} \mathbf{P}[e^* = e']\mathbf{P}[o \leftrightarrow e \mid o \leftrightarrow e'] \\
&= \sum_{e' \preccurlyeq e} \mathbf{P}[e^* = e']\mathbf{P}[o \leftrightarrow e \mid e^* = e'] \\
&= \sum_{e' \in \Pi} \mathbf{P}[e^* = e']\mathbf{P}[o \leftrightarrow e \mid e^* = e'] = \mathbf{P}[o \leftrightarrow e]\,.
\end{aligned}$$

Thus

$$\sum_{e' \preccurlyeq e} \mu(e') \frac{\mathbf{P}[o \leftrightarrow e', o \leftrightarrow e]}{\mathbf{P}[o \leftrightarrow e']\mathbf{P}[o \leftrightarrow e]} = 1/\mathbf{P}[o \leftrightarrow \Pi]\,.$$

By symmetry, it follows that

$$\mathscr{E}(\mu) \le 2 \sum_{e \in \Pi} \sum_{e' \preccurlyeq e} \mu(e)\mu(e') \frac{\mathbf{P}[o \leftrightarrow e', o \leftrightarrow e]}{\mathbf{P}[o \leftrightarrow e']\mathbf{P}[o \leftrightarrow e]} = 2 \sum_{e \in \Pi} \mu(e)/\mathbf{P}[o \leftrightarrow \Pi] = 2/\mathbf{P}[o \leftrightarrow \Pi]\,.$$

Therefore,

$$\mathbf{P}[o \leftrightarrow \Pi] \le 2/\mathscr{E}(\mu) \le 2 \sup_{\nu \in \mathcal{P}(\Pi)} 1/\mathscr{E}(\nu)\,.$$

To sum up, provided such orderings on cutsets Π exist, we are able to reverse the inequality of Proposition 5.12 up to a factor of 2 (Lyons, 1992).

Theorem 5.21. (Tree Percolation and Conductance) *For an independent percolation* $\mathbf{P}$ *on a tree with adapted conductances (i.e., such that (5.11) holds), we have*

$$\frac{\mathscr{C}(o \leftrightarrow \infty)}{1+\mathscr{C}(o \leftrightarrow \infty)} \le \mathbf{P}[o \leftrightarrow \infty] \le 2\frac{\mathscr{C}(o \leftrightarrow \infty)}{1+\mathscr{C}(o \leftrightarrow \infty)}\,, \tag{5.15}$$

which is the same as

$$\frac{\mathbf{P}[o \leftrightarrow \infty]}{2-\mathbf{P}[o \leftrightarrow \infty]} \le \mathscr{C}(o \leftrightarrow \infty) \le \frac{\mathbf{P}[o \leftrightarrow \infty]}{1-\mathbf{P}[o \leftrightarrow \infty]}\,.$$

Consequently, we obtain a sharp refinement of Theorem 5.15:

Corollary 5.22. *Percolation occurs (i.e.,* $\mathbf{P}[o \leftrightarrow \infty] > 0$*) iff random walk on* T *is transient for corresponding adapted conductances (satisfying (5.11)).*

Sometimes it is useful to consider percolation on a finite portion of T (which is, in fact, how our proofs have proceeded). We have shown that for finite trees,

$$\frac{\mathscr{C}(o \leftrightarrow \partial_{\mathrm{L}} T)}{1+\mathscr{C}(o \leftrightarrow \partial_{\mathrm{L}} T)} \le \mathbf{P}[o \leftrightarrow \partial_{\mathrm{L}} T] \le 2\frac{\mathscr{C}(o \leftrightarrow \partial_{\mathrm{L}} T)}{1+\mathscr{C}(o \leftrightarrow \partial_{\mathrm{L}} T)}\,, \tag{5.16}$$

which is the same as

$$\frac{\mathbf{P}[o \leftrightarrow \partial_{\mathrm{L}} T]}{2-\mathbf{P}[o \leftrightarrow \partial_{\mathrm{L}} T]} \le \mathscr{C}(o \leftrightarrow \partial_{\mathrm{L}} T) \le \frac{\mathbf{P}[o \leftrightarrow \partial_{\mathrm{L}} T]}{1-\mathbf{P}[o \leftrightarrow \partial_{\mathrm{L}} T]}\,.$$

Remark. Add a vertex Δ to T joined to o by an edge of conductance 1. Then doing random walk on $T \cup \{\Delta\}$, we have

$$\frac{\mathscr{C}(o \leftrightarrow \partial_{\mathrm{L}} T)}{1+\mathscr{C}(o \leftrightarrow \partial_{\mathrm{L}} T)} = \mathbf{P}_o[\tau_{\partial_{\mathrm{L}} T} \le \tau_\Delta]\,,$$

so (5.16) is equivalent to

$$\mathbf{P}_o[\tau_{\partial_{\mathrm{L}} T} \le \tau_\Delta] \le \mathbf{P}[o \leftrightarrow \partial_{\mathrm{L}} T] \le 2\mathbf{P}_o[\tau_{\partial_{\mathrm{L}} T} \le \tau_\Delta]\,.$$

Likewise, on an infinite tree, (5.15) is equivalent to

$$\mathbf{P}_o[\tau_\Delta = \infty] \le \mathbf{P}[o \leftrightarrow \partial T] \le 2\mathbf{P}_o[\tau_\Delta = \infty]\,.$$

Note also that the condition of being adapted, (5.11), becomes

$$\mathbf{P}[o \leftrightarrow x] = \mathscr{C}(\Delta \leftrightarrow x)\,.$$

▷ **Exercise 5.11.**

Give a tree for which percolation does and a tree for which percolation does not occur at criticality.

▷ **Exercise 5.12.**

Show that critical homesick random walk on supercritical Galton-Watson trees is a.s. recurrent in two ways: (1) by using this corollary; (2) by using the Nash-Williams criterion.

As in Exercise 5.43, Theorem 5.21 can be used to solve problems about random walk on deterministic or random trees. Indeed, sometimes percolation is crucial to such solutions (see, e.g., Lyons (1992)).

§5.5. Surviving Galton-Watson Trees.

What does the cluster of a vertex look like in percolation given that the cluster is infinite? This question is easy to answer on regular trees. In fact, the analogous question for Galton-Watson trees is easy to answer. We actually give two kinds of answers. The first describes the tree given survival in terms of other Galton-Watson processes. The second tells us how large we can find regular subtrees.

Let Z_n^* be the number of particles in generation n that have an infinite line of descent. A little thought shows that $\langle Z_n^* \rangle$ is a Galton-Watson process where each individual has k children with probability $\sum_{j \ge k} p_j \binom{j}{k} (1-q)^{k-1} q^{j-k}$. In order to calculate its p.g.f. in a different way and get further information, we will also show a general decomposition. A little thought shows that $\langle Z_n \rangle$ is also a Galton-Watson process given extinction. A little more thought shows that the family tree of $\langle Z_n \rangle$ has the same law as a tree grown first by $\langle Z_n^* \rangle$, then adding "bushes" independently and in the appropriate number to each node. We calculate (and prove) all this as follows.

Let T denote the genealogical tree of a Galton-Watson process with p.g.f. f and T' denote the ***reduced*** subtree of particles with an infinite line of descent. (It may be that $T' = \varnothing$.) Let $Y_{i,j}^{(n)}$ be the indicator of the jth child of the ith particle in generation n having an infinite line of descent. Write

$$\overline{q} := 1 - q\,,$$

$$L_i^{*(n+1)} := \sum_{j=1}^{L_i^{(n)}} Y_{i,j}^{(n)}\,,$$

$$Z_{n+1}^* := \sum_{i=1}^{Z_n} L_i^{*(n+1)}\,.$$

Thus $Z_0^* = \mathbf{1}_{\text{nonextinction}}$. Parts (iii) and (iv) of the following proposition are due to Lyons (1992). Parts (i) and (ii) are illustrated in Figure 5.3.

Proposition 5.23. (Decomposition) *Suppose that $0 < q < 1$.*

(i) The law of T' given nonextinction is the same as that of a Galton-Watson process with p.g.f.

$$f^*(s) := [f(q + \overline{q}s) - q]/\overline{q}\,.$$

(ii) The law of T given extinction is the same as that of a Galton-Watson process with p.g.f.

$$\tilde{f}(s) := f(qs)/q\,.$$

(iii) The joint p.g.f. of $L - L^*$ and L^* is

$$\mathbf{E}[s^{L-L^*} t^{L^*}] = f(qs + \overline{q}t).$$

The joint p.g.f. of $Z_n - Z_n^*$ and Z_n^* is

$$\mathbf{E}[s^{Z_n - Z_n^*} t^{Z_n^*}] = f^{(n)}(qs + \overline{q}t).$$

(iv) The law of T given nonextinction is the same as that of a tree $\overline{T}$ generated as follows: Let T^* be the tree of a Galton-Watson process with p.g.f. f^* as in (i). To each vertex σ of T^* having d_σ children, attach U_σ independent copies of a Galton-Watson tree with p.g.f. $\tilde{f}$ as in (ii), where U_σ has the p.g.f.

$$\mathbf{E}[s^{U_\sigma}] = \frac{(D^{d_\sigma} f)(qs)}{(D^{d_\sigma} f)(q)},$$

where d_σ derivatives of f are indicated; all U_σ and all trees added are mutually independent given T^*. The resultant tree is $\overline{T}$.

Remark. Part (iv) is not so interesting in terms of growth (Z_n, etc.) alone.

Proof. We begin with (iii). We have

$$\begin{aligned}\mathbf{E}[s^{L-L^*} t^{L^*}] &= \mathbf{E}[\mathbf{E}[s^{L-L^*} t^{L^*} \mid L]] = \mathbf{E}\Big[\mathbf{E}\Big[s^{\sum_{j=1}^{L}(1-Y_j)} t^{\sum_{j=1}^{L} Y_j} \;\Big|\; L\Big]\Big] \\ &= \mathbf{E}[\mathbf{E}[s^{1-Y_1} t^{Y_1}]^L] = \mathbf{E}[(qs + \overline{q}t)^L] = f(qs + \overline{q}t)\,.\end{aligned}$$

The other part of (iii) follows by a precisely parallel calculation.

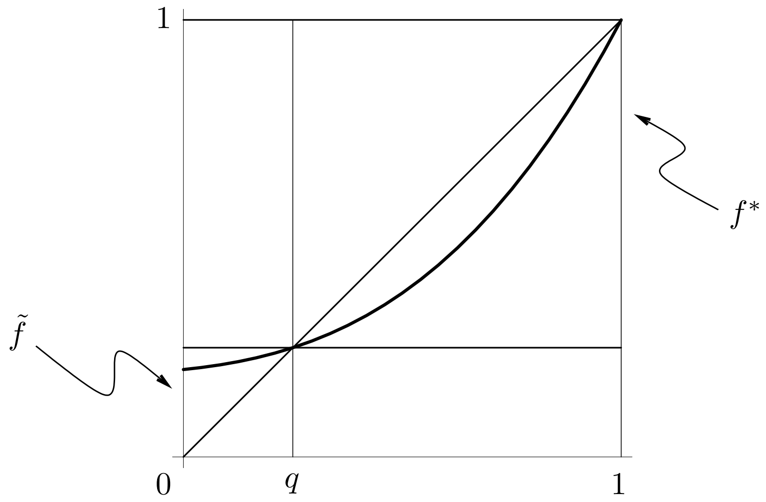


Figure 5.3. The graph of f embeds scaled versions of the graphs of $\tilde{f}$ and f^*.

In order to show (i), we need to show that given $L_i^{*(n)} \neq 0$ and given the σ-field $\mathcal{F}_{n,i}$ generated by $L_k^{*(n)}$ $(k \neq i)$ and $L_k^{*(m)}$ $(m < n,\ k \geq 1)$, the p.g.f. of $L_i^{*(n)}$ is f^*. Indeed, on the event $L_i^{*(n)} \neq 0$, we have

$$\begin{aligned}
\mathbf{E}[s^{L_i^{*(n)}} \mid \mathcal{F}_{n,i}] &= \mathbf{E}[s^{L_i^{*(n)}} \mid L_i^{*(n)} \neq 0] \\
&= \mathbf{E}[s^{L^*}\mathbf{1}_{\{L^* \neq 0\}}]/\overline{q} = \mathbf{E}[s^{L^*}(\mathbf{1} - \mathbf{1}_{\{L^*=0\}})]/\overline{q} \\
&= \{\mathbf{E}[1^{L-L^*}s^{L^*}] - \mathbf{P}[L^* = 0]\}/\overline{q} \\
&= [f(q1 + \overline{q}s) - q]/\overline{q} = f^*(s)
\end{aligned}$$

by (iii).

Similarly, (ii) comes from the fact that on the event of extinction,

$$\begin{aligned}
\mathbf{E}[s^{L_i^{(n)}} \mid \mathcal{F}_{n,i}] &= \mathbf{E}[s^{L_i^{(n)}} \mid L_i^{*(n)} = 0] = \mathbf{E}[s^L\mathbf{1}_{\{L^*=0\}}]/q \\
&= \mathbf{E}[s^{L-L^*}0^{L^*}]/q = f(qs + \overline{q}0)/q = \tilde{f}(s)
\end{aligned}$$

since $0^x = \mathbf{1}_{\{x=0\}}$.

Finally, (iv) follows once we show that the function claimed to be the p.g.f. of U_σ is the same as the p.g.f. of $L - L^*$ given $L^* = d_\sigma$. Once again, by (iii), we have that for some constants c and c',

$$\mathbf{E}[s^{L-L^*} \Bigm| L^* = d] = c\left(\frac{\partial}{\partial t}\right)^d f(qs + \overline{q}t)\Big|_{t=0} = c'(D^d f)(qs)\,.$$

Substitution of $s = 1$ yields $c' = 1/(D^d f)(q)$. ◀

Remark. Part (iii), which led to all the other parts of the proposition, is an instance of the following general calculation. Suppose that we are given a nonnegative random number X of particles with X having p.g.f. F. One interpretation of F is that if we color each particle red with probability r, then the probability that all particles are colored red is $F(r)$. Now suppose that we first categorize each particle independently as small or big, with the probability of a particle being categorized small being q. Then independently color each small particle red with probability s and color each big particle red with probability t. Since the chance that a given particle is colored red is then $qs + \overline{q}t$, we have (by our interpretation) that the probability that all particles are colored red is $F(qs + \overline{q}t)$. On the other hand, if Y is the number of big particles, then by conditioning on Y, we see that this is also equal to the joint p.g.f. of $X - Y$ and Y.

▷ **Exercise 5.13.**

A ***multi-type Galton-Watson branching process*** has J types of individuals, with an individual of type i generating k_j individuals of type j for $j = 1, \ldots, J$ with probability $p^{(i)}_{\mathbf{k}}$, where $\mathbf{k} := (k_1, \ldots, k_J)$. As in the single-type case, all individuals generate their children independently of each other. Show that a supercritical Galton-Watson tree given survival has the following alternative description as a 2-type Galton-Watson tree. Let $\langle p_k \rangle$ be the offspring distribution and q be the extinction probability. Let type 1 have offspring distribution obtained as follows: Begin with k children with probability $p_k(1 - q^k)/(1 - q)$. Then make each child type 1 with probability $1 - q$ and type 2 with probability q, independently but conditional on having at least one type-1 child. The type-2 offspring distribution is simpler: it has k children of type 2 (only) with probability $p_k q^{k-1}$. Let the root be type 1.

Let $\tau(d)$ be the probability that a Galton-Watson tree contains a d-ary subtree beginning at the initial individual. Thus, $\tau(1)$ is the survival probability, $1 - q$. Pakes and Dekking (1991) found a method to determine $\tau(d)$, after special cases done by Chayes, Chayes, and Durrett (1988) and Dekking and Meester (1990):

Theorem 5.24. (Regular Subtrees) *Let f be the p.g.f. of a supercritical Galton-Watson process. Set*

$$G_d(s) := \sum_{j=0}^{d-1} (1 - s)^j (D^j f)(s)/j!\,.$$

Then $1 - \tau(d)$ is the smallest fixed point of G_d in $[0, 1]$.

Proof. Let $g_d(s)$ be the probability that the root has at most $d - 1$ marked children when each child is marked with probability $1 - s$. This function is clearly monotonic increasing in s. By considering how many children the root has in total, we see that

$$g_d(s) = \sum_{k=0}^{\infty} p_k \sum_{j=0}^{d-1} \binom{k}{j} (1-s)^j s^{k-j} . \tag{5.17}$$

After changing the order of summation in (5.17), we obtain

$$\begin{aligned} g_d(s) &= \sum_{j=0}^{d-1} \frac{(1-s)^j}{j!} \sum_{k \ge j} p_k \, k(k-1) \cdots (k-j+1) s^{k-j} \\ &= \sum_{j=0}^{d-1} \frac{(1-s)^j (D^j f)(s)}{j!} = G_d(s) , \end{aligned}$$

that is, $g_d = G_d$.

Let q_n be the chance the Galton-Watson tree does not contain a d-ary subtree of height n at the initial individual; notice that $1 - q_n \downarrow \tau(d)$. By marking a child of the root when it has a d-ary subtree of height $n-1$, we see that $q_n = G_d(q_{n-1})$. Since $q_0 = 0$, letting $n \to \infty$ shows that $\lim_n q_n = 1 - \tau(d)$ is the smallest fixed point of G_d. ◀

▷ **Exercise 5.14.**

Show that for a Galton-Watson tree with offspring distribution $\mathrm{Bin}(d+1, p)$, we have $\tau(d) > 0$ for p sufficiently close to 1.

Pakes and Dekking (1991) used this theorem to show, e.g., that the critical mean value for a geometric offspring distribution to produce an d-ary subtree in the Galton-Watson tree with positive probability is asymptotic to ed as $d \to \infty$, and it is asymptotically d for a Poisson offspring. An interesting feature of the phase transitions from $\tau(d) = 0$ to $\tau(d) > 0$ as the parameter is varied in geometric or Poisson offspring distributions is that unlike the case of usual percolation $d = 1$, for $d \ge 2$ the probability of having the d-ary subtree is already positive at criticality. For a binomial offspring distribution, the critical mean value is asymptotically d, as shown by the following result. This was proved by Balogh, Peres, and Pete (2006), where it was applied to bootstrap percolation, a model we will discuss in Section 7.8.

For integers $n \ge 2$ and $1 \le d \le n$, define $\pi(n, d)$ to be the infimum of probabilities p such that $\tau(d) > 0$ in a Galton-Watson tree with offspring distribution $\mathrm{Bin}(n, p)$.

Proposition 5.25. (Regular Subtrees in Binomial Galton-Watson Trees) *The critical probability $\pi(n, d)$ is the infimum of all p for which the equation*

$$\mathbf{P}\big[\mathrm{Bin}(n, (1-s)p) \le d-1\big] = s \tag{5.18}$$

has a real root $s \in [0,1)$. For any constant $\gamma \in [0,1]$ and sequence of integers d_n with $\lim_{n\to\infty} d_n/n = \gamma$, *we have*

$$\lim_{n\to\infty} \pi(n, d_n) = \gamma. \tag{5.19}$$

Proof. In the proof of Theorem 5.24, we saw that $G_d = g_d$; in the present case, it is clear that $g_d(s) = \mathbf{P}\big[\mathrm{Bin}(n,(1-s)p) \le d-1\big] =: B_{n,d,p}(s)$. If the probability of not having the required subtree is denoted by $y = y(p)$, then by Theorem 5.24, y is the smallest fixed point of the function $B_{n,d,p}(s)$ in $s \in [0,1]$. One fixed point is $s = 1$, and $\pi(n,d)$ is the infimum of the p values for which there is a fixed point $s \in [0,1)$. It is easy to see that

$$\frac{\partial}{\partial s} B_{n,d,p}(s) = np\,\mathbf{P}\big[\mathrm{Bin}(n-1,(1-s)p) = d-1\big]\,,$$

which is positive for $s \in [0,1)$, with at most one extremal point (a maximum) in $(0,1)$. Thus $B_{n,d,p}(s)$ is a monotone increasing function, with $B_{n,d,p}(0) > 0$ when $p < 1$, and with at most one inflection point in $(0,1)$.

If $\lim_{n\to\infty} d_n/n = \gamma$, then for any fixed p and s, by the Weak Law of Large Numbers,

$$B_{n,d_n,p}(s) = \mathbf{P}\left[\frac{\mathrm{Binom}(n,(1-s)p)}{n} \le \frac{d_n - 1}{n}\right] \to \begin{cases} 1 & \text{if } (1-s)p < \gamma, \\ 0 & \text{if } (1-s)p > \gamma \end{cases}$$

as $n \to \infty$. Solving the equation $(1-s)p = \gamma$ for s gives a critical value $s_c = 1 - \gamma/p$. Thus for $p < \gamma$ we have $\lim_{n\to\infty} B_{n,d_n,p}(s) \to 1$ for all $s \in [0,1]$, while for large enough n, $B_{n,d,p}(s)$ is concave in $[0,1]$, so there is no positive root $s < 1$ of $B_{n,d_n,p}(s) = s$. On the other hand, for $p > \gamma$ there must be a root $s = s(n)$ for large enough n. These prove (5.19). ◀

§5.6. Galton-Watson Networks.

In the preceding sections, we have seen some examples of the application of percolation to questions that do not appear to involve percolation (Corollary 5.10, Proposition 13.3, and Exercise 5.43). We will give another example in this section. Consider the following ***Galton-Watson networks*** generated by a random variable $\mathcal{L} := (L, A_1, \ldots, A_L)$, $L \in \mathbb{N}$, $A_i \in (0,1]$. First, use L as an offspring random variable to generate a Galton-Watson tree. Then give each individual x of the tree an i.i.d. copy $\mathcal{L}_x$ of $\mathcal{L}$; thus, $\mathcal{L}_x = (L_x, A_{y_1}, \ldots, A_{y_{L_x}})$, where $y_1, \ldots, y_{L_x}$ are the children of x. Use these random variables to assign edge capacities (or weights or conductances)

$$c\big(e(x)\big) := \prod_{w \le x} A_w\,.$$

We will see the usefulness of such networks for the study of random fractals in Section 14.3. When can water flow to ∞? Note that if $A_i \equiv 1$, then water can flow to ∞ iff the tree is infinite. Let

$$\gamma := \mathbf{E}\left[\sum_{i=1}^{L} A_i\right].$$

The following theorem of Falconer (1986) shows that the condition for positive flow to infinity with positive probability on these Galton-Watson networks is $\gamma > 1$. Of course, when $A_i \equiv 1$, this reduces to the usual survival criterion for Galton-Watson processes. In general, this theorem confirms the intuition that in order for flow to infinity to be possible, more water must be able to flow from parent to children, on average, than from grandparent to parent.

Theorem 5.26. (Flow in Galton-Watson Networks) *If $\gamma \le 1$, then a.s. no flow is possible unless $\sum_1^L A_i = 1$ a.s. If $\gamma > 1$, then flow is possible a.s. on nonextinction.*

Proof. (Lyons and Peres) As usual, let T_1 denote the set of individuals (or vertices) of the first generation. Let F be the maximum strength of an admissible flow, i.e., one that satisfies the capacity constraints on the edges.

The proof of the first part of the theorem uses the same idea as the proof of Lemma 4.4(b) of Falconer (1987). For $x \in T_1$, let F_x be the maximum strength of an admissible flow from x to infinity through the subtree T^x with capacities $e \mapsto c(e)/A_x$. Thus, F_x has the same law as F has. It is easily seen that

$$F = \sum_{x \in T_1} \big(A_x \wedge (A_x F_x)\big) = \sum_{x \in T_1} A_x (1 \wedge F_x).$$

Now suppose that $\gamma \le 1$. Taking expectations in the preceding equation yields

$$\begin{aligned}\mathbf{E}[F] = \mathbf{E}[\mathbf{E}[F \mid \mathcal{L}_o]] &= \mathbf{E}\left[\mathbf{E}\left[\sum_{x \in T_1} A_x (1 \wedge F_x) \middle| \mathcal{L}_o\right]\right] = \mathbf{E}\left[\sum_{x \in T_1} \mathbf{E}[A_x(1 \wedge F_x) \mid \mathcal{L}_o]\right] \\ &= \mathbf{E}\left[\sum_{x \in T_1} A_x \mathbf{E}[1 \wedge F_x \mid \mathcal{L}_o]\right] = \mathbf{E}\left[\sum_{x \in T_1} A_x \mathbf{E}[1 \wedge F]\right] = \gamma \mathbf{E}[1 \wedge F] \le \mathbf{E}[1 \wedge F].\end{aligned}$$

Therefore $F \le 1$ almost surely and $\mathbf{P}[F > 0] > 0$ only if $\gamma = 1$. In addition, we have, by independence,

$$\| F \|_\infty = \| \sum_{x \in T_1} A_x \|_\infty \cdot \| F \|_\infty .$$

If $\| F \|_\infty > 0$, it follows that $\| \sum_{x \in T_1} A_x \|_\infty = 1$. In combination with $\gamma = 1$, this implies that $\sum_{x \in T_1} A_x = 1$ a.s.

For the second part, we introduce percolation as in the proof of Corollary 5.10 via Proposition 5.9. Namely, augment the probability space so that for each vertex $u \neq o$, there is a random variable X_u taking the value 1 with probability A_u and 0 otherwise, with $\langle X_u \rangle$ conditionally independent given all $\mathcal{L}_u$. Denote by $\mathcal{F}$ the σ-field generated by the random variables $\mathcal{L}_u$. Consider the subtree of the Galton-Watson tree consisting of the initial individual o together with those individuals u such that $\prod_{w \le u} X_w = 1$. This subtree has, unconditionally, the law of a Galton-Watson branching process with progeny distribution the unconditional law of

$$\sum_{u \in T_1} X_u \,.$$

Let $\mathcal{Q}$ be the probability that this subtree is infinite conditional on $\mathcal{F}$. Now the unconditional mean of the new process is

$$\mathbf{E}\Bigg[\sum_{u \in T_1} X_u\Bigg] = \mathbf{E}\Bigg[\mathbf{E}\Bigg[\sum_{u \in T_1} X_u \;\Bigg|\; \mathcal{L}_o\Bigg]\Bigg] = \mathbf{E}\Bigg[\sum_{u \in T_1} A_u\Bigg] = \gamma,$$

whence if $\gamma > 1$, then this new Galton-Watson branching process survives with positive probability, say, Q. Of course, $Q = \mathbf{E}[\mathcal{Q}]$. On the other hand, for any cutset Π of the original Galton-Watson tree, $\sum_{e \in \Pi} c(e)$ is the expected number, given $\mathcal{F}$, of edges in Π that are also in the subtree. This expectation is at least the probability that the number of such edges is at least one, which, in turn, is at least $\mathcal{Q}$:

$$F = \inf_{\Pi} \sum_{e \in \Pi} c(e) \ge \mathcal{Q}.$$

Hence $F > 0$ on the event $\mathcal{Q} > 0$. The event $\mathcal{Q} > 0$ has positive probability since $Q > 0$, whence $\mathbf{P}[F > 0] > 0$. Since the event that $F = 0$ is clearly inherited, it follows that $\mathbf{P}[F = 0] = q$. ◀

We will return to percolation on trees in Chapter 15.

§5.7. Notes.

Galton-Watson processes are sometimes called Bienaymé-Galton-Watson processes since Bienaymé, in 1845, was the first to give the fundamental theorem, Proposition 5.4. However, while he stated the correct result, he gave barely a hint of his proof. See Heyde and Seneta (1977), pp. 116–120 for some of the history. The first published proof of Bienaymé's theorem appears on pp. 83–86 of Cournot (1847), where the context is gambling: An urn contains tickets marked with nonnegative integers, a proportion p_k of them being marked k. Pierre begins with 1 écu, which he gives to Paul for the right of drawing a ticket from the urn. If the ticket is marked k, then Paul gives Pierre k écus. The ticket is returned to the urn. This is the end of the first round. For the second and succeeding rounds, if Pierre is not broke, then for each of the écus he has, he repeats the procedure of the first round. The problem was to determine the probability, for each n, that Pierre is broke at the end of the nth round. If one keeps track of Pierre's fortune at all times, not merely at the ends of rounds, then one obtains a coding of the associated tree by a random walk as in Exercise 5.27.

For other codings of the trees as random walks, as well as various uses, see Geiger (1995), Pitman (1998), Bennies and Kersting (2000), Dwass (1969), Harris (1952), Le Gall and Le Jan (1998), Duquesne and Le Gall (2002), Marckert and Mokkadem (2003), Marckert (2008), Lamperti (1967), and Kendall (1951).

For additional material on branching processes, see Chapter 12, the books by Athreya and Ney (1972) and Asmussen and Hering (1983), and the review articles by Vatutin and Zubkov (1985, 1993).

A more subtle example of a quasi-independent percolation than the one of Example 5.18 is obtained by replacing the requirement $S(x) \in [0, N]$ by $S(x) \geq 0$. This is proved by Benjamini and Peres (1994) in the course of their proof of Theorem 1.1. For more on this particular case, see Pemantle and Peres (1995a).

A precursor to Theorem 5.24 can be found in Lemma 5 of Pemantle (1988).

§5.8. Collected In-Text Exercises.

5.1. Justify the differentiation in (5.3). Show too that $\lim_{s\uparrow 1} f'(s) = m$.

5.2. Let $\mathscr{T}$ be the space of rooted labelled locally finite trees. Verify that the function $d(T, T') := \left(1 + \sup\{n\,;\ T\restriction n = T'\restriction n\}\right)^{-1}$ is a metric on $\mathscr{T}$ and that $\mathscr{T}$ is complete and separable.

5.3. Define the measure $\mathbf{GW}$ formally on the space $\mathscr{T}$ of Exercise 5.2; your measure should be the law of a random tree produced by a Galton-Watson process with arbitrary given offspring distribution.

5.4. Show that for any Galton-Watson process with mean $m > 1$, the family tree T has growth rate $\operatorname{gr} T = m$ a.s. given nonextinction. (Don't use the Kesten-Stigum Theorem to show this, as we have not yet proved that theorem.)

5.5. Show that for a general percolation, the events $\{x \leftrightarrow e\}$ and $\{x \leftrightarrow \infty\}$ are indeed measurable.

5.6. Prove (5.5).

5.7. Show that for $p \geq p_c(G)$, we have $p_c(\omega_p) = p_c(G)/p$ $\mathbf{P}_p$-a.s. Physicists often refer to p as the "density" of edges in ω and this helps the intuition.

5.8. Show that, conversely to (5.12), the survival parameters adapted to given edge resistances are
$$p_x = \frac{1 + \sum_{o<u<x} r(e(u))}{1 + \sum_{o<u\le x} r(e(u))} \,.$$
For example, simple random walk ($c \equiv 1$) is adapted to $p_x = |x|/(|x|+1)$.

5.9. Verify the assertion of Example 5.17.

5.10. Show that if $\operatorname{br} T < 1/\cos\dfrac{\pi}{N+2}$, then percolation does not occur in ω_N of (5.13).

5.11. Give a tree for which percolation does and a tree for which percolation does not occur at criticality.

5.12. Show that critical homesick random walk on supercritical Galton-Watson trees is a.s. recurrent in two ways: (1) by using this corollary; (2) by using the Nash-Williams criterion.

5.13. A ***multi-type Galton-Watson branching process*** has J types of individuals, with an individual of type i generating k_j individuals of type j for $j = 1, \ldots, J$ with probability $p^{(i)}_{\mathbf{k}}$, where $\mathbf{k} := (k_1, \ldots, k_J)$. As in the single-type case, all individuals generate their children independently of each other. Show that a supercritical Galton-Watson tree given survival has the following alternative description as a 2-type Galton-Watson tree. Let $\langle p_k \rangle$ be the offspring distribution and q be the extinction probability. Let type 1 have offspring distribution obtained as follows: Begin with k children with probability $p_k(1-q^k)/(1-q)$. Then make each child type 1 with probability $1-q$ and type 2 with probability q, independently but conditional on having at least one type-1 child. The type-2 offspring distribution is simpler: it has k children of type 2 (only) with probability $p_k q^{k-1}$. Let the root be type 1.

5.14. Show that for a Galton-Watson tree with offspring distribution $\mathrm{Bin}(d+1, p)$, we have $\tau(d) > 0$ for p sufficiently close to 1, where $\tau(d)$ is the probability that the tree contains a d-ary subtree beginning at the initial individual.

§5.9. Additional Exercises.

5.15. Consider a rooted Galton-Watson tree (T, o) whose offspring distribution is Poisson(c) for some $c > 0$. If the total number of vertices of T is $k < \infty$, then label the vertices of T uniformly with the integers $1, \ldots, k$. Show that every rooted labeled tree on k vertices arises with probability $e^{-ck}c^{k-1}/k!$. Consequently, if we condition that $|\mathsf{V}(T)| = k$, then the rooted labeled tree is uniformly distributed among all rooted labeled trees on k vertices, i.e., it has the distribution of a uniformly rooted uniform spanning tree on the complete graph on $[1, k]$.

5.16. Consider Bernoulli(c/n) percolation on the complete graph K_n with fixed $c > 0$. Fix a vertex o of K_n. Let $C(o)$ be the cluster of o.

(a) Show that as $n \to \infty$, the distribution of $C(o)$ tends to that of a rooted Galton-Watson tree (T, o) whose offspring distribution is Poisson(c) in the sense that for every r, the ball of radius r about o in $C(o)$ has the same limiting distribution as the ball about the root in a Galton-Watson tree.

(b) Let C_{L} be the result of labeling the vertices of $C(o)$ uniformly by the integers $1, \ldots, |C(o)|$. Show that for $k < \infty$, if we condition that $|C(o)| = k$, then the distribution of the labeled cluster C_{L} tends to that of a uniformly labeled Galton-Watson-Poisson(c) tree conditioned to have size k.

(c) Show that for a rooted labeled tree T of size $k < \infty$,

$$\lim_{n\to\infty} \mathbf{P}[C_{\mathrm{L}} = T] = \frac{e^{-ck}c^{k-1}}{k!}\,.$$

This gives another solution of Exercise 5.15.

5.17. Suppose that L_n are offspring random variables that converge in law to an offspring random variable $L \not\equiv 1$. Let the corresponding extinction probabilities be q_n and q. Show that $q_n \to q$ as $n \to \infty$.

5.18. Give another proof that $m \le 1 \Rightarrow$ a.s. extinction unless $p_1 = 1$ as follows. Let Z_1^* be the number of particles of the first generation with an infinite line of descent. Let $Z_1^{(i)*}$ be the number of children of the ith particle of the first generation with an infinite line of descent. Show that $Z_1^* = \sum_{i=1}^{Z_1} 1 \wedge Z_1^{(i)*}$. Take the L^1 and L^∞ norms of this equation to get the desired conclusion.

5.19. Give another proof of Proposition 5.1 and Proposition 5.4 as follows. Write

$$Z_{n+1} = Z_n + \sum_{i=1}^{Z_n} \left(L_i^{(n+1)} - 1\right).$$

This is a randomly sampled random walk, i.e., a random subsequence of the locations of a random walk on $\mathbb{Z}$ whose steps have the same distribution as $L-1$. Apply the strong law of large numbers or the Chung-Fuchs theorem (Durrett (2005), p. 188), as appropriate.

5.20. Show that $\langle Z_n \rangle$ is a nonnegative supermartingale when $m \le 1$ to give another proof of a.s. extinction when $m \le 1$ and $p_1 \ne 1$.

5.21. Show that if $0 < r < 1$ satisfies $f(r) = r$, then $\langle r^{Z_n} \rangle$ is a martingale. Use this to give another proof of Proposition 5.1 and Proposition 5.4 in the case $m > 1$.

5.22. (Grey, 1980) Let $\langle Z_n \rangle$, $\langle Z_n' \rangle$ be independent Galton-Watson processes with identical offspring distribution and arbitrary, possibly random, Z_0, Z_0' with $Z_0 + Z_0' \ge 1$ a.s., and set

$$Y_n := \begin{cases} Z_n/(Z_n + Z_n') & \text{if } Z_n + Z_n' \ne 0; \\ Y_{n-1} & \text{if } Z_n + Z_n' = 0. \end{cases}$$

(a) Fix n. Let A be the event

$$A := \{Z_{n+1} + Z_{n+1}' \ne 0\}$$

and $\mathscr{F}_n$ be the σ-field generated by $Z_0, \ldots, Z_n, Z_0', \ldots, Z_n'$. Use symmetry to show that $\mathbf{E}[L_i^{(n+1)}/(Z_{n+1} + Z_{n+1}') \mid \mathscr{F}_n] = 1/(Z_n + Z_n')$ on the event A.

(b) Show that $\langle Y_n \rangle$ is a martingale with a.s. limit Y.

(c) If $1 < m < \infty$, show that $0 < Y < 1$ a.s. on the event $Z_n \not\to 0$ and $Z_n' \not\to 0$. *Hint:* Let $Y^{(k)} := \lim_{n\to\infty} Z_n/(Z_n + Z_{k+n}')$. Then $\mathbf{E}[Y^{(k)} \mid Z_0,\ Z_k'] = Z_0/(Z_0 + Z_k')$ and $\mathbf{P}[Y = 1,\ Z_n \not\to 0,\ Z_n' \not\to 0] = \mathbf{P}[Y^{(k)} = 1,\ Z_n \not\to 0,\ Z_n' \not\to 0] \le \mathbf{E}[Y^{(k)}\mathbf{1}_{\{Z_k' > 0\}}] \to 0$ as $k \to \infty$.

5.23. In the notation of Exercise 5.22, show that if $Z_0 \equiv Z_0' \equiv 1$ and $1 < m < \infty$, then

$$\mathbf{P}[Y \in (0,1)] = (1-q)^2 + p_0^2 + \sum_{n\ge 1} \left[f^{(n+1)}(0) - f^{(n)}(0)\right]^2.$$

5.24. Deduce the Seneta-Heyde theorem from Grey's theorem (Exercise 5.22).

5.25. Show that $Z_{n+1}/Z_n \to m$ a.s. on nonextinction if $m < \infty$.

5.26. Let f be the p.g.f. of a Galton-Watson process. Show that

$$\mathbf{E}\Big[s^{\sum_{n=0}^{N} Z_n}\Big] = g_N(s)\,,$$

where $g_0(s) := s$ and $g_{n+1}(s) := sf(g_n(s))$ for $n \ge 1$. Define $s_0 := \sup\{t/f(t)\,;\ t \ge 1\} \ge 1$. Show that

$$\mathbf{E}\Big[s^{\sum_{n\ge 0} Z_n}\Big] = g_\infty(s)\,,$$

where $g_\infty(s) := \lim_{n\to\infty} g_n(s) = sf(g_\infty(s))$ is finite for $s < s_0$. Show that if the process is subcritical and $f(s) < \infty$ for some $s > 1$, then $s_0 > 1$.

5.27. Let L be the offspring random variable of a Galton-Watson process. There are various useful ways to encode by a random walk the family tree when it is finite. We consider one such way here. Suppose that the process starts with k individuals, i.e., $Z_0 = k$. Let $Z_{\mathrm{tot}} := \sum_{n\ge 0} Z_n$ be the total size of the process. Let $S_n := \sum_{j=1}^{n}(L_j - 1)$, where L_j are independent copies of L.

(a) Show that $\mathbf{P}[Z_{\mathrm{tot}} = n] = \mathbf{P}[S_n = -k,\ \forall i < n\ \ S_i > -k]$.

(b) **(Otter-Dwass formula)** Show that $\mathbf{P}[Z_{\mathrm{tot}} = n] = \frac{k}{n}\mathbf{P}[S_n = -k]$.

(c) Show that in the non-critical case, the expected number of visits to $-k$ of the random walk $\langle S_n\,;\ n \ge 0\rangle$ is $q^k/(1 - f'(q))$, where q is the extinction probability of the Galton-Watson process and f is the probability generating function of L. Show that in the critical case, this expectation is infinite.

(d) Let τ_n be the time of the nth visit to $-k$ of the random walk $\langle S_n\,;\ n \ge 0\rangle$, where $\tau_n := \infty$ if fewer than n visits are made to $-k$. Show that $\mathbf{P}[\tau_1 < \infty] = q^k$ and $\mathbf{E}[\sum_n 1/\tau_n] = q^k/k$.

5.28. Consider a Galton-Watson process with offspring distribution equal to Poisson(1) and total size Z_{tot}.

(a) Show that $\mathbf{P}[Z_{\mathrm{tot}} = n] = n^{n-1}e^{-n}/n!$. *Hint:* Use the Otter-Dwass formula from Exercise 5.27(b).

(b) By comparing with Exercise 5.15, derive Cayley's formula that the number of trees on n vertices is n^{n-2}.

5.29. Consider a Galton-Watson process with offspring distribution equal to Poisson(1). Let o be the root and X be a random uniform vertex of the tree. Show that $\mathbf{P}[X = o] = 1/2$.

5.30. Let T be a tree. Show that for $p < 1/\overline{\mathrm{gr}}\,T$, the expected size of the cluster of a vertex is finite for Bernoulli(p) percolation, while it is infinite for $p > 1/\overline{\mathrm{gr}}\,T$.

5.31. Deduce Corollary 5.10 from Hawkes's earlier result (that is, from the special case where it is assumed that $\mathbf{E}[L(\log L)^2] < \infty$) by considering truncation of the progeny random variable, L.

5.32. Let $\theta_n(p)$ denote the probability that the root of an n-ary tree has an infinite cluster under Bernoulli(p) percolation. Thus, $\theta_n(p) = 0$ iff $p \le 1/n$. Calculate and graph $\theta_2(p)$ and $\theta_3(p)$. Show that for all $n \ge 2$, the left-hand derivative of θ_n at 1 is 0 and the right-hand derivative of θ_n at $1/n$ is $2n^2/(n-1)$.

5.33. Show that for $\mu_1, \mu_2 \in \mathcal{P}(\Pi)$,

$$\mathscr{E}\Big(\frac{\mu_1 + \mu_2}{2}\Big) + \mathscr{E}\Big(\frac{\mu_1 - \mu_2}{2}\Big) = \frac{\mathscr{E}(\mu_1) + \mathscr{E}(\mu_2)}{2}\,,$$

where $\mathscr{E}(\bullet)$ is extended in the obvious way to non-probability measures.

5.34. Show that if G is locally finite and $\mathbf{P}[e_1, e_2 \subseteq K(o)] \neq 0$ for every pair $e_1, e_2 \in \Pi$, then $\mathscr{E}(\bullet)$ has a unique minimum on $\mathcal{P}(\Pi)$.

5.35. Let T be the family tree of a supercritical Galton-Watson process. Show that a.s. on the event of nonextinction, simple random walk on T is transient.

5.36. Let T be the family tree of a supercritical Galton-Watson process without extinction and with mean m. For $0 < \lambda < m$, consider RW_λ with the conductances $c(e) = \lambda^{-|e|}$.
(a) Show that $\mathbf{E}[\mathscr{C}(o \leftrightarrow \infty; T)] \le m - \lambda$.
(b) Show that $\mathbf{E}\big[\mathbf{P}[\tau_o^+ = \infty]\big] \le 1 - \lambda/m$.

5.37. Given a quasi-independent percolation on a locally finite tree T with $\mathbf{P}[o \leftrightarrow u \mid o \leftrightarrow \overleftarrow{u}] \equiv p$, show that if $p < (\operatorname{br} T)^{-1}$, then $\mathbf{P}[o \leftrightarrow \infty] = 0$ while if $p > (\operatorname{br} T)^{-1}$, then $\mathbf{P}[o \leftrightarrow \infty] > 0$.

5.38. Let T be any tree on which simple random walk is transient. Let U be a random variable uniform on $[0, 1]$. Define the percolation on T by taking the subtree of all edges at distance at most $1/U$. Show that the inequality of Theorem 5.19 fails if conductances are adapted to this percolation.

5.39. Let T be any infinite tree. Let $U(x)$ be i.i.d. random variables uniform on $[0, 1]$ indexed by the vertices of T. Define the percolation on T by taking the subgraph spanned by all vertices x such that $U(x) \le U(o)$. Show that percolation occurs iff $\operatorname{br} T > 1$. Show that this percolation is not quasi-independent.

5.40. Consider Bernoulli(1/2) percolation on a tree T. Let $p_n := \mathbf{P}_{1/2}[o \leftrightarrow T_n]$.
(a) Show that if there is a constant c such that for every n we have $|T_n| \le c2^n$, then $p_n \le 4c/(n + 2c)$ for all n.
(b) Show that if there are constants c_1 and c_2 such that for every n we have $|T_n| \ge c_1 2^n$ and also for every vertex x we have $|T_n^x| \le c_2 2^n$, then $p_n \ge c_3/(n + c_3)$ for all n, where $c_3 := 2c_1^2/c_2^3$.

5.41. Let $T(n)$ be a binary tree of height n and consider Bernoulli(1/2) percolation. For large n, which inequalities in (5.16) are closest to equalities?

5.42. Let T be a binary tree and consider Bernoulli(p) percolation with $p \in (1/2, 1)$. Which inequalities in Theorem 5.21 have the ratios of the two sides closest to 1?

5.43. Randomly stretch a tree T by adding vertices to ("subdividing") the edges to make the edge $e(x)$ a path of length $L_x(\omega)$ with L_x i.i.d. Call the resulting tree $T(\omega)$. Calculate $\operatorname{br} T(\omega)$ in terms of $\operatorname{br} T$. Show that if $\operatorname{br} T > 1$, then $\operatorname{br} T(\omega) > 1$ a.s.

5.44. Suppose that $\operatorname{br} T = \overline{\operatorname{gr}}\, T$. Consider the stretched tree $T(\omega)$ from Exercise 5.43. Show that $\operatorname{br} T(\omega) = \overline{\operatorname{gr}}\, T(\omega)$ a.s.

5.45. Consider the stretched tree $T(\omega)$ from Exercise 5.43. If we assume only that simple random walk on T is transient, is simple random walk on $T(\omega)$ transient a.s.?

5.46. Consider a percolation on T such that there is some $M' > 0$ such that, for all $x, y \in T$ and $A \subseteq T$ with the property that the removal of $x \wedge y$ would disconnect y from every vertex in A,

$$\mathbf{P}[o \leftrightarrow y \mid o \leftrightarrow x,\ o \nleftrightarrow A] \ge M' \mathbf{P}[o \leftrightarrow x \mid o \leftrightarrow x \wedge y]\,.$$

Show that the adapted conductances satisfy

$$\mathbf{P}[o \leftrightarrow \infty] \le \frac{2}{M'} \frac{\mathscr{C}(o \leftrightarrow \infty)}{1 + \mathscr{C}(o \leftrightarrow \infty)}\,.$$

5.47. Consider the percolation of Example 5.20. Sharpen Exercise 5.10 by showing that if percolation occurs, then RW_λ is transient on T for $\lambda := 1/\cos\dfrac{\pi}{N+2}$.

5.48. The inequalities (5.16) can also be proved by entirely elementary means.

(a) Prove that if $0 < x_n \le 1$, then

$$\sum \frac{1-x_n}{x_n} \le \frac{1-\prod x_n}{\prod x_n} \quad \text{and} \quad \sum \frac{1-x_n}{1+x_n} \ge \frac{1-\prod x_n}{1+\prod x_n}\,.$$

(b) Use induction to deduce (5.16) from the inequalities of part (a).

(c) Prove that for $C \ge 0$ and $C/(1+C) < p \le 1$,

$$p\left(1-\exp\left(\frac{-2C}{p(1+C)-C}\right)\right) \le 1-e^{-2C}\,.$$

(d) Use induction to deduce the following sharper form of the right-hand inequality of (5.16) from part (c):

$$\mathbf{P}[o \leftrightarrow \partial_{\mathrm{L}} T] \le 1-\exp\bigl(-2\mathscr{C}(o \leftrightarrow \partial_{\mathrm{L}} T)\bigr)\,.$$

5.49. Show that if f is the p.g.f. of a supercritical Galton-Watson process, then the p.g.f. of Z_n given survival is

$$[f^{(n)}(s) - f^{(n)}(qs)]/\overline{q}\,.$$

5.50. Show that the extinction probability $y(p)$ introduced in the proof of Proposition 5.25 satisfies $y(p) \to 0$ as $p \to 1$.

5.51. Show that

$$\pi(n, n-1) = \frac{(n-1)^{2n-3}}{n^{n-1}(n-2)^{n-2}}$$

for the critical probability of Section 5.5. What is the probability of having a $(n-1)$-ary subtree exactly at this value of p?

5.52. Let X and Y be real-valued random variables. Say that X is at least Y in the ***increasing convex order*** if $\mathbf{E}[h(X)] \ge \mathbf{E}[h(Y)]$ for all nonnegative increasing convex functions $h : \mathbb{R} \to \mathbb{R}$. Show that this is equivalent to $\int_a^\infty \mathbf{P}[X > t]\,dt \ge \int_a^\infty \mathbf{P}[Y > t]\,dt$ for all $a \in \mathbb{R}$. When the means of X and Y are the same, one also says that X is ***stochastically more variable*** than Y.

5.53. Suppose that X_i are nonnegative independent identically distributed random variables, that Y_i are nonnegative independent identically distributed random variables, and that X_i is stochastically more variable than Y_i for each $i \ge 1$.

(a) Show that $\sum_{i=1}^n X_i$ is at least $\sum_{i=1}^n Y_i$ in the increasing convex order for each $n \ge 1$.

(b) Let M and N be nonnegative integer-valued random variables independent of all X_i and Y_i. Suppose that M is at least N in the increasing convex order. Show that $\sum_{i=1}^M X_i$ is at least $\sum_{i=1}^N X_i$ in the increasing convex order, which, in turn, is at least $\sum_{i=1}^N Y_i$ in the increasing convex order.

5.54. Suppose that $L^{(1)}$ is an offspring random variable that is at least $L^{(2)}$ in the increasing convex order and that $Z_n^{(i)}$ are the corresponding generation sizes of Galton-Watson processes beginning with 1 individual each.

(a) Show that $Z_n^{(1)}$ is at least $Z_n^{(2)}$ in the increasing convex order for each n.

(b) Show that $\mathbf{P}[Z_n^{(1)} = 0] \ge \mathbf{P}[Z_n^{(2)} = 0]$ for each n if the means of $L^{(1)}$ and $L^{(2)}$ are the same.

(c) Show that the conditional distribution of $Z_n^{(1)}$ given $Z_n^{(1)} > 0$ is at least the conditional distribution of $Z_n^{(2)}$ given $Z_n^{(2)} > 0$ in the increasing convex order for each n.

(d) Show that the following is an example where $L^{(1)}$ is at least $L^{(2)}$ in the increasing convex order. Write $p_k^{(i)}$ for $\mathbf{P}[L^{(i)} = k]$. Suppose that $p_0^{(1)} > 0$, that $a := \min\{k \ge 1\,;\ p_k^{(1)} > 0\}$, that $p_k^{(2)} = 0$ for $k > K$, that $p_k^{(2)} = p_k^{(1)}$ for $a < k \le K$, and that $\mathbf{E}[L^{(1)}] = \mathbf{E}[L^{(2)}]$.

Chapter 6

Isoperimetric Inequalities

Just as the branching number of a tree is for most purposes more important than the growth rate, there is a number for a general graph that is more important for many purposes than its growth rate. In the present chapter, we consider this number, or, rather, several variants of it, called isoperimetric or expansion constants. This is not an extension of the branching number, however; for that, the reader can see Section 13.3. Also, see Virág (2000b) for a more powerful extension of branching number to graphs and even to networks.

§6.1. Flows and Submodularity.

A common illegal scheme for making money, known as a pyramid scheme or Ponzi scheme, goes essentially as follows. You convince 10 people to send you \$100 each and to ask 10 others in turn to send them \$100. Everyone who manages to do this will profit \$900 (and you will profit \$1000). Of course, some people will lose \$100 in the end. But suppose that we had an infinite number of people. Then no one need lose money and indeed everyone can profit \$900. But what if people can ask only people they know and people are at the vertices of the square lattice and know only their 4 nearest neighbors? Is it now possible for everyone to profit \$900 (we now do not assume that only \$100 can be passed from one person to another)? Well, if the amount of money allowed to change hands (i.e., the amount crossing any edge) is unbounded, then certainly this is possible. But suppose the amount is bounded by, say, \$1,000,000? The answer now is "no" (although it is still possible for everyone to profit, but the profit cannot be bounded away from 0).

Why is this? Consider first the case that there are only a finite number of people. If we simply add up the total net gains, we obtain 0, whence it is impossible for everyone to gain a strictly positive amount. For the lattice case, consider all the people lying within distance n of the origin. What is the *average* net gain of these people? The only reason this average might not be 0 is that money can cross the boundary. However, because of our assumption that the money crossing any edge is bounded, it follows that for the average net gain, this boundary crossing is negligible in the limit as $n \to \infty$. Hence the average net gain of everyone is 0 and it cannot be that everyone profits \$900 (or even just one cent).

But what if the neighbor graph was that of the hyperbolic tessellation of Figure 2.3? The figure suggests that the preceding argument, which depended on finding finite subsets of vertices with relatively few edges leading out of them, will not work. Does that mean everyone could profit \$900? What is the maximum profit everyone could make? The picture also suggests that the graph is somewhat like a tree, which suggests that significant profit is possible. Indeed, this is true, as we now show. We will generalize this problem a little by restricting the amount that can flow over edge e to be at most $c(e)$ and by supposing that the person at location x has as a goal to profit $D(x)$. We ask what the maximum α is such that each person x can profit $\alpha D(x)$. Thus, on the Euclidean lattice, we saw that $\alpha = 0$ when c is bounded above and D is bounded below.

Let now G be a connected graph. Weight the edges by positive numbers $c(e)$ and the vertices by positive numbers $D(x)$. We call (G, c, D) a ***network***. We assume that G is locally finite, or, more generally, that for each x,

$$\sum_{e^- = x} c(e) < \infty \,. \tag{6.1}$$

For $K \subset \mathsf{V}$, we define the ***edge boundary*** $\partial_{\mathsf{E}} K$ to be the set of (unoriented) edges that connect K to its complement. Write

$$|K|_D := \sum_{x \in K} D(x)$$

and

$$|\partial_{\mathsf{E}} K|_c := \sum_{e \in \partial_{\mathsf{E}} K} c(e) \,.$$

Define the ***edge-expansion constant*** or ***edge-isoperimetric constant*** of (G, c, D) by

$$\Phi_{\mathsf{E}}(G) := \Phi_{\mathsf{E}}(G, c, D) := \inf \left\{ \frac{|\partial_{\mathsf{E}} K|_c}{|K|_D} \,;\ \varnothing \neq K \subset \mathsf{V} \text{ is finite} \right\} .$$

This is a measure of how small we can make the boundary effects in our argument above. The most common choices for (c, D) are $(\mathbf{1}, \mathbf{1})$ and $(\mathbf{1}, \deg)$. For a network with conductances c, one often chooses (c, π) (as in Section 6.2 below). When $\Phi_{\mathsf{E}}(G) = 0$, we say the network is ***edge amenable*** (see the definition of "vertex amenable" below for the origin of the name). Thus, the square lattice is edge amenable; but note that there are large sets in $\mathbb{Z}^2$ that also have large boundary. The point is that there exist sets that have relatively small boundary. This is impossible for a regular tree of degree at least 3.

▷ **Exercise 6.1.**

Show that $\Phi_{\mathsf{E}}(\mathbb{T}_{b+1}, \mathbf{1}, \mathbf{1}) = b - 1$ for all $b \ge 1$, where $\mathbb{T}_{b+1}$ is the regular tree of degree $b + 1$.

The opposite of "amenable" is ***non-amenable***; one also says that a non-amenable network satisfies a ***strong isoperimetric inequality***.

The following theorem shows that $\Phi_{\mathsf{E}}(G)$ is precisely the maximum proportional profit α that we seek (actually, because of our choice of d^*, we need to take the negative of the function θ guaranteed by Theorem 6.1). The theorem thus dualizes the inf in $\Phi_{\mathsf{E}}(G)$ not only to a sup, but to a max. It is due to Benjamini, Lyons, Peres, and Schramm (1999b), hereinafter referred to as BLPS (1999b).

Theorem 6.1. (Duality for Edge Expansion) *For any network (G, c, D), we have*

$$\Phi_{\mathsf{E}}(G, c, D) = \max\{\alpha \ge 0\,;\ \exists\theta\ \forall e\ |\theta(e)| \le c(e)\ \text{and}\ \forall x\ d^*\theta(x) = \alpha D(x)\}\,,$$

where θ runs over the antisymmetric functions on E.

Proof. Denote by A the set of α of which the maximum is taken on the right-hand side of the desired equation. Thus, we want to show that $\max A = \Phi_{\mathsf{E}}(G)$. In fact, we will show that $A = [0, \Phi_{\mathsf{E}}(G)]$.

Given $\alpha \ge 0$ and any finite nonempty $K \subset \mathsf{V}$, define the network $G_K = G_{K,\alpha}$ with vertices K and two extra vertices, a and z. The edges are those of G with both endpoints in K, those in $\partial_{\mathsf{E}} K$ where the endpoint not in K is replaced by z, and an edge between a and each point in K. Give all edges e with both endpoints in K the capacity $c(e)$. Give the edges $[x, z]$ the capacity of the corresponding edge in $\partial_{\mathsf{E}} K$. Give the other edges capacity $c(a, x) := \alpha D(x)$.

In view of the cutset consisting of all edges incident to z, it is clear that any flow from a to z in G_K has strength at most $|\partial_{\mathsf{E}} K|_c$. Suppose that $\alpha \in A$, in other words, there is a function θ satisfying $\forall e\ |\theta(e)| \le c(e)$ and $\forall x\ d^*\theta(x) = \alpha D(x)$. This function θ induces a flow on G_K from a to z meeting the capacity constraints and of strength $\alpha |K|_D$, whence $\alpha \le |\partial_{\mathsf{E}} K|_c / |K|_D$. Since this holds for all K, we get $\alpha \le \Phi_{\mathsf{E}}(G)$.

In the other direction, if $\alpha \le \Phi_{\mathsf{E}}(G)$, then for all finite K, we claim that there is a flow from a to z in G_K of strength $\alpha |K|_D$. Consider any cutset Π separating a and z. Let K' be the vertices in K that Π separates from z. Then $\partial_{\mathsf{E}} K' \subseteq \Pi$ and $[a, x] \in \Pi$ for all $x \in K \setminus K'$. Therefore,

$$\begin{aligned}\sum_{e \in \Pi} c(e) &\ge |\partial_{\mathsf{E}} K'|_c + \alpha |K \setminus K'|_D \ge \Phi_{\mathsf{E}}(G) |K'|_D + \alpha |K \setminus K'|_D \\ &\ge \alpha |K'|_D + \alpha |K \setminus K'|_D = \alpha |K|_D\,.\end{aligned}$$

Thus, our claim follows from the Max-Flow Min-Cut Theorem. Note that the flow along $[a, x]$ is $\alpha D(x)$ for every $x \in K$ since its strength is $\alpha|K|_D$.

Now let K_n be finite sets increasing to V and let θ_n be flows on G_{K_n} with strength $\alpha|K_n|_D$. There is a subsequence $\langle n_i \rangle$ such that for all edges $e \in \mathsf{E}$, the limit $\theta(e) := \theta_{n_i}(e)$ exists; clearly, $\forall e\ |\theta(e)| \le c(e)$ and, by (6.1) and the dominated convergence theorem, $\forall x\ d^*\theta(x) = \alpha D(x)$. Thus, $\alpha \in A$. ◀

Is the infimum in the definition of $\Phi_{\mathsf{E}}(G)$ a minimum? Certainly not in the edge amenable case. The reader should find an example where it is a minimum, however. It turns out that in the transitive case, it is never a minimum. Here, we say that a network (G, c, D) is ***transitive*** if for every pair $x, y \in \mathsf{V}(G)$, there is an automorphism of G that takes x to y and that preserves the edge weights c and vertex weights D. In order to prove that transitive networks do not have minimizing sets, we will use the following concept.

A function b on finite subsets of V is called ***submodular*** if

$$\forall K, K' \quad b(K \cup K') + b(K \cap K') \le b(K) + b(K') \,. \tag{6.2}$$

For example, $K \mapsto |K|_D$ is obviously submodular with equality holding in (6.2). The identity

$$|\partial_{\mathsf{E}}(K \cup K')|_c + |\partial_{\mathsf{E}}(K \cap K')|_c + |\partial_{\mathsf{E}}(K \setminus K') \cap \partial_{\mathsf{E}}(K' \setminus K)|_c = |\partial_{\mathsf{E}} K|_c + |\partial_{\mathsf{E}} K'|_c \,, \tag{6.3}$$

is easy, though tedious, to check. It shows that $K \mapsto |\partial_{\mathsf{E}} K|_c$ is submodular, with equality holding in (6.2) iff $\partial_{\mathsf{E}}(K \setminus K') \cap \partial_{\mathsf{E}}(K' \setminus K) = \varnothing$, which is the same as $K \setminus K'$ not adjacent to $K' \setminus K$.

Theorem 6.2. (BLPS (1999b)) *If (G, c, D) is an infinite transitive network, then for all finite nonempty $K \subset \mathsf{V}$, we have $|\partial_{\mathsf{E}} K|_c / |K|_D > \Phi_{\mathsf{E}}(G)$.*

Proof. At first, we do not need to suppose that G is transitive. By the submodularity of $b(K) := |\partial_{\mathsf{E}} K|_c$ and the fact that

$$\forall K, K' \quad |K \cup K'|_D + |K \cap K'|_D = |K|_D + |K'|_D \,,$$

we have for any finite K and K',

$$\frac{b(K \cup K') + b(K \cap K')}{|K \cup K'|_D + |K \cap K'|_D} \le \frac{b(K) + b(K')}{|K|_D + |K'|_D} \,,$$

with equality iff $K \setminus K'$ is not adjacent to $K' \setminus K$. Now when a, b, c, d are positive numbers, we have

$$\min\{a/b, c/d\} \le (a + c)/(b + d) \le \max\{a/b, c/d\} \,.$$

Therefore

$$\min\left\{\frac{b(K\cup K')}{|K\cup K'|_D}, \frac{b(K\cap K')}{|K\cap K'|_D}\right\} \le \max\left\{\frac{b(K)}{|K|_D}, \frac{b(K')}{|K'|_D}\right\}, \tag{6.4}$$

with equality iff $K \setminus K'$ is not adjacent to $K' \setminus K$ and all four quotients appearing in (6.4) are equal. (In case $K \cap K'$ is empty, omit it on the left-hand side.)

Now suppose that G is transitive. Suppose for a contradiction that there is some finite set K with $b(K)/|K|_D = \Phi_{\mathsf{E}}(G)$. Choose some such set K of minimal cardinality. Let $o \in K$ and choose an automorphism γ such that γo is outside K but adjacent to some vertex in K. Define $K' := \gamma K$. Note that $b(K')/|K'|_D = b(K)/|K|_D$. If $K \cap K' \ne \varnothing$, then our choice of K implies that equality cannot hold in (6.4), whence $K \cup K'$ has a strictly smaller quotient, a contradiction. But if $K \cap K' = \varnothing$, then $K \setminus K'$ and $K' \setminus K$ are adjacent, whence (6.4) shows again that $K \cup K'$ has a strictly smaller quotient, a contradiction. ◀

Sometimes it is useful to look at boundary vertices rather than boundary edges. Thus, given positive numbers $D(x)$ on the vertices of a graph G, define the ***(external) vertex boundary***

$$\partial_{\mathsf{V}} K := \{x \notin K\,;\ \exists y \in K,\ y \sim x\}$$

and

$$\Phi_{\mathsf{V}}(G) := \Phi_{\mathsf{V}}(G, D) := \inf\left\{\frac{|\partial_{\mathsf{V}} K|_D}{|K|_D}\,;\ \varnothing \ne K \subset \mathsf{V} \text{ is finite}\right\},$$

the ***vertex-expansion constant*** or ***vertex-isoperimetric constant*** of G. The most common choice is $D = \mathbf{1}$. We call G ***vertex amenable*** if its vertex-expansion constant is 0. (This is equivalent to the definition in Section 4.3; see Exercise 6.24.) For two functions f_1 and f_2 on the same domain, write $f_1 \asymp f_2$ to mean that the ratio f_1/f_2 is bounded and bounded away from 0 on their domain. Note that if (G, c, D) satisfies $c \asymp \mathbf{1}$ and $D \asymp \deg$, then (G, c, D) is edge amenable iff (G, D) is vertex amenable. We'll call (G, c, D) simply ***amenable*** if it is both edge amenable and vertex amenable.

▷ **Exercise 6.2.**

Suppose that G is a graph such that for some $o \in \mathsf{V}$, we have subexponential growth of balls: $\liminf_{n\to\infty} |\{x \in \mathsf{V}\,;\ d(o, x) \le n\}|^{1/n} = 1$, where $d(\bullet, \bullet)$ denotes the graph distance in G. Show that G is vertex amenable.

▷ **Exercise 6.3.**

Show that every Cayley graph of a finitely generated abelian group is amenable.

▷ **Exercise 6.4.**

Suppose that G_1 and G_2 are roughly isometric graphs with bounded degrees and having both edge and vertex weights $\asymp \mathbf{1}$. Show that G_1 is amenable iff G_2 is.

Because of Exercises 6.4 and 3.13, either all Cayley graphs of a group are amenable or none are; that is, amenability is a property of the group. In fact, the concept of amenability comes from groups, not graphs. This origin also explains the name "amenable" in the following way. Let Γ be any countable group and $\ell^\infty(\Gamma)$ be the Banach space of bounded real-valued functions on Γ. A linear functional on $\ell^\infty(\Gamma)$ is called a ***mean*** if it maps the constant function $\mathbf{1}$ to 1 and nonnegative functions to nonnegative numbers. If $f \in \ell^\infty(\Gamma)$ and $\gamma \in \Gamma$, we write $R_\gamma f(\gamma') := f(\gamma'\gamma)$. We call a mean μ ***invariant*** if $\mu(R_\gamma f) = \mu(f)$ for all $f \in \ell^\infty(\Gamma)$ and $\gamma \in \Gamma$. Finally, we say that Γ is ***amenable*** if there is an invariant mean on $\ell^\infty(\Gamma)$. Thus, "amenable" was introduced as a play on words that evoked the word "mean". How is this related to the definitions we have given? Suppose that Γ is finitely generated and that G is one of its Cayley graphs. If G is amenable, then there is a sequence of finite sets K_n with $|\partial_\mathsf{V} K_n|/|K_n| \to 0$. Now consider the sequence of means

$$f \mapsto \mu_n(f) := \frac{1}{|K_n|} \sum_{x \in K_n} f(x) \,.$$

Then for every generator γ of Γ, we see that $|\mu_n(f) - \mu_n(R_\gamma f)| \to 0$ as $n \to \infty$, whence the same holds for all $\gamma \in \Gamma$. One can use a weak* limit point of the means μ_n to obtain an invariant mean and therefore show that Γ is amenable. The converse was established by Følner (1955) and is usually stated in the form that for every nonempty finite $B \subset \Gamma$ and $\epsilon > 0$, there is a nonempty finite set $A \subset \Gamma$ such that $|BA \triangle A| \le \epsilon |A|$; see Paterson (1988), Theorem 4.13, for a proof. In this case, one often refers informally to A as a Følner set. In conclusion, a finitely generated group Γ is amenable iff any of its Cayley graphs is.

The analogue for vertex amenability of Theorem 6.1 is due to Benjamini and Schramm (1997). It involves the amount flowing along edges into vertices,

$$\mathsf{flow}_+(\theta, x) := \sum_{e^+ = x} (\theta(e) \vee 0) \,.$$

Theorem 6.3. (Duality for Vertex Expansion) *For any graph G with vertex weights D, we have*

$$\Phi_\mathsf{V}(G, D) = \max\{\alpha \ge 0\,;\ \exists \theta\ \forall x\ \mathsf{flow}_+(\theta, x) \le D(x)\ \textit{and}\ d^*\theta(x) = \alpha D(x)\} \,,$$

where θ runs over the antisymmetric functions on E.

Proof. Given $\alpha \ge 0$ and any finite nonempty $K \subset \mathsf{V}$, define the network G_K with vertices $K \cup \partial_\mathsf{V} K$ and two extra vertices, a and z. The edges are those of G that have both endpoints in $K \cup \partial_\mathsf{V} K$, an edge between a and each point in K, and an edge between z

and each point in $\partial_{\mathsf{V}} K$. Give all edges incident to a capacity $c(a,x) := \alpha D(x)$. Let the capacity of the vertices in K be $c(x) := (\alpha+1)D(x)$ and those in $\partial_{\mathsf{V}} K$ be $D(x)$. The remaining edges and vertices are given infinite capacity.

In view of the cutset consisting of all vertices in $\partial_{\mathsf{V}} K$, it is clear that any flow from a to z in G_K has strength at most $|\partial_{\mathsf{V}} K|_D$. Now a function θ satisfying $\forall x\ \mathsf{flow}_+(\theta, x) \le D(x)$ and $d^*\theta(x) = \alpha D(x)$ induces a flow on G_K from a to z meeting the capacity constraints and of strength $\alpha |K|_D$, whence $\alpha \le |\partial_{\mathsf{V}} K|_D / |K|_D$. Since this holds for all K, we get $\alpha \le \Phi_{\mathsf{V}}(G)$.

In the other direction, if $\alpha \le \Phi_{\mathsf{V}}(G)$, then for all nonempty K, we claim that there is a flow from a to z in G_K of strength $\alpha|K|_D$. Consider any cutset Π of edges and vertices separating a and z. Let K' be the vertices in K that Π separates from z. Then $\partial_{\mathsf{V}} K' \subset \Pi$ and $[a,x] \in \Pi$ for all $x \in K \setminus K'$. Therefore,

$$\sum_{e \in \Pi \cap \mathsf{E}} c(e) + \sum_{x \in \Pi \cap \mathsf{V}} c(x) \ge \alpha |K \setminus K'|_D + |\partial_{\mathsf{V}} K'|_D \ge \alpha |K \setminus K'|_D + \Phi_{\mathsf{V}}(G) |K'|_D \ge \alpha |K|_D \,.$$

Thus, the claim follows from the Max-Flow Min-Cut Theorem (the version in Exercise 3.15). Note that the flow along $[a,x]$ is $\alpha D(x)$ for every $x \in K$.

Now let K_n be finite nonempty sets increasing to V and let θ_n be the corresponding flows on G_{K_n}. There is a subsequence $\langle n_i \rangle$ such that for all edges $e \in \mathsf{E}$, the limit $\theta(e) := \theta_{n_i}(e)$ exists; clearly, $\forall x\ \mathsf{flow}_+(\theta, x) \le D(x)$ and $d^*\theta(x) = \alpha D(x)$. ◀

It is easy to check that the function $K \mapsto |\partial_{\mathsf{V}} K|_D$ is submodular. In fact, the following identity holds, where $\overline{K} := K \cup \partial_{\mathsf{V}} K$:

$$|\partial_{\mathsf{V}}(K \cup K')|_D + |\partial_{\mathsf{V}}(K \cap K')|_D + |(\overline{K} \cap \overline{K'}) \setminus \overline{K \cap K'}|_D = |\partial_{\mathsf{V}} K|_D + |\partial_{\mathsf{V}} K'|_D \,. \tag{6.5}$$

Theorem 6.4. (BLPS (1999b)) *If G is an infinite transitive graph, then for all finite nonempty $K \subset \mathsf{V}$, we have $|\partial_{\mathsf{V}} K|/|K| > \Phi_{\mathsf{V}}(G)$.*

Proof. By the submodularity of $b(K) := |\partial_{\mathsf{V}} K|$, we have for any finite K and K', as in the proof of Theorem 6.2,

$$\min\left\{ \frac{b(K \cup K')}{|K \cup K'|}, \frac{b(K \cap K')}{|K \cap K'|} \right\} \le \frac{b(K) + b(K')}{|K| + |K'|} \,, \tag{6.6}$$

with equality iff both terms on the left-hand side are equal to the right-hand side. (In case $K \cap K'$ is empty, omit it on the left-hand side.)

Now suppose that G is transitive and that K is a finite set minimizing $|K|$ among those K with $b(K)/|K| = \Phi_{\mathsf{V}}(G)$. Let γ be any automorphism such that $\gamma K \cap K \ne \varnothing$. Define $K' := \gamma K$. Then (6.6) shows that $K' = K$. In other words, if, instead, we choose γ so that $\gamma K \cap \partial_{\mathsf{V}} K \ne \varnothing$, then $\gamma K \cap K = \varnothing$, whence (6.5) shows that $K'' := K \cup \gamma K$ satisfies $b(K'')/|K''| < b(K)/|K|$, a contradiction. ◀

An interesting aspect of Theorem 6.3 is that it enables us to find (virtually) regular subtrees in many non-amenable graphs, as shown by Benjamini and Schramm (1997):

Theorem 6.5. (Regular Subtrees in Non-Amenable Graphs) *Let G be any graph with $n := \lfloor \Phi_{\mathsf{V}}(G, \mathbf{1}) \rfloor \ge 1$. Then G has a spanning forest in which every tree has one vertex of degree n and all others of degree $n+2$.*

Proof. The proof of Theorem 6.3 in combination with Exercise 3.16 shows that there is an integer-valued θ satisfying

$$\forall x \ \mathsf{flow}_+(\theta, x) \le 1 \text{ and } d^*\theta(x) = n\,. \tag{6.7}$$

If there is an oriented cycle along which $\theta = 1$, then we may change the values of θ to be 0 on the edges of this cycle without changing the validity of (6.7). Thus, we may assume that there is no such oriented cycle. After this, there is no (unoriented) cycle in the support of θ since this would force a flow $\mathsf{flow}_+(\theta, x) \ge 2$ into some vertex x on the cycle. We may similarly assume that there is no oriented bi-infinite path along which $\theta = 1$. Thus, (6.7) shows that for every x with $\mathsf{flow}_+(\theta, x) = 1$, there are exactly $n+1$ edges leaving x with flow out of x equal to 1. Furthermore, the lack of an oriented bi-infinite path of flows shows that each component of the support of θ contains a vertex into which there is no flow, and (6.7) implies that such a vertex has n edges leading out with flow 1 each. Thus, the support of θ is the desired spanning forest. ◀

This result can be extended to graphs with $\Phi_{\mathsf{V}} > 0$, but it is more complicated; see Benjamini and Schramm (1997).

§6.2. Spectral Radius.

As in Chapter 2, write

$$(f, g)_h := (fh, g) = (f, gh)$$

and

$$\|f\|_h := \sqrt{(f, f)_h}\,.$$

Also, let $\mathbf{D}_{00}$ denote the collection of functions on V with finite support.

Suppose that $\langle X_n \rangle$ is a Markov chain on a countable state space V with a stationary measure π. We define the ***transition operator***

$$(Pf)(x) := \mathbf{E}_x[f(X_1)] = \sum_{y \in \mathsf{V}} p(x, y) f(y)\,.$$

Then P maps $\ell^2(\mathsf{V}, \pi)$ to itself with norm

$$\|P\|_\pi := \|P\|_{\ell^2(\mathsf{V},\pi)} := \sup\left\{\frac{\|Pf\|_\pi}{\|f\|_\pi}\,;\ f \neq \mathbf{0}\right\}$$

at most 1.

▷ **Exercise 6.5.**
Prove that $\|P\|_\pi \le 1$.

As the reader should recall, we have that

$$(P^n f)(x) = \sum_{y \in \mathsf{V}} p_n(x,y) f(y)$$

when $p_n(x,y) := \mathbf{P}_x[X_n = y]$.

Let G be a (connected) graph with conductances $c(e) > 0$ on the edges and $\pi(x)$ be the sum of the conductances incident to a vertex x. The operator P that we defined above is self-adjoint: for functions $f, g \in \mathbf{D}_{00}$, we have

$$\begin{aligned}(Pf, g)_\pi &= \sum_{x \in \mathsf{V}} \pi(x)(Pf)(x)g(x) = \sum_{x \in \mathsf{V}} \pi(x)\left[\sum_{y \in \mathsf{V}} p(x,y)f(y)\right] g(x)\\ &= \sum_{x \in \mathsf{V}} \sum_{y \in \mathsf{V}} c(x,y) f(y) g(x)\,.\end{aligned}$$

Since this is symmetric in f and g, it follows that $(Pf, g)_\pi = (f, Pg)_\pi$. Since the functions with finite support are dense in $\ell^2(\mathsf{V}, \pi)$, we get this identity for all $f, g \in \ell^2(\mathsf{V}, \pi)$.

▷ **Exercise 6.6.**
Show that

$$\|P\|_\pi = \sup\left\{\frac{|(Pf,f)_\pi|}{(f,f)_\pi}\,;\ f \in \mathbf{D}_{00} \setminus \{0\}\right\} = \sup\left\{\frac{(Pf,f)_\pi}{(f,f)_\pi}\,;\ f \in \mathbf{D}_{00} \setminus \{0\}\right\}.$$

The norm $\|P\|_\pi$ is usually called the spectral radius* due to the following fact:

* For a general operator on a Banach space, its spectral radius is defined to be $\max |z|$ for z in the spectrum of the operator. This agrees with the usage in probability theory, but we shall not need this representation explicitly.

▷ **Exercise 6.7.**
Show that for any two vertices $x, y \in \mathsf{V}$, we have

$$\|P\|_\pi = \limsup_{n\to\infty} \sup_z \left(p_n(x,z)/\sqrt{\pi(z)}\right)^{1/n} = \limsup_{n\to\infty} p_n(x,y)^{1/n}\,.$$

Moreover,

$$\forall n \;\; p_n(x,y) \le \sqrt{\pi(y)/\pi(x)}\|P\|_\pi^n\,.$$

In probability theory, $\limsup_{n\to\infty} p_n(x,y)^{1/n}$ is referred to as the ***spectral radius*** of the Markov chain, denoted $\rho(G)$.

In Section 13.3, we will give some necessary conditions for random walk on a network to have positive speed. Here, we give some sufficient conditions based on the following simple observation. Define the ***upper (exponential) growth rate*** of a graph to be $\limsup_{n\to\infty} |B(o,n)|^{1/n}$, where $B(o,n)$ is the ***ball*** of radius n centered at o, i.e., the set of vertices whose distance from o is at most n.

Proposition 6.6. (Speed and Spectral Radius) *Let G be a graph with upper exponential growth rate $b \in (1,\infty)$ and fix some vertex $o \in G$. Suppose that the edges are weighted so that the spectral radius $\rho(G) < 1$ and π is bounded. Consider the network random walk $\langle X_n \rangle$ on G. Then the liminf speed is positive:*

$$\liminf_{n\to\infty} \operatorname{dist}_G(o,X_n)/n \ge -\log\rho(G)/\log b \qquad a.s.$$

Note that $\operatorname{dist}_G(\bullet,\bullet)$ denotes the graph distance, so that the quotient whose limit we are taking is distance divided by time. This is the reason we call it "speed".

Proof. Without loss of generality, assume $X_0 = o$. Let $\alpha < -\log\rho(G)/\log b$, so that $\rho(G)b^\alpha < 1$. Choose $\lambda > b$ so that $\rho(G)\lambda^\alpha < 1$. By Exercise 6.7 and our assumption that π is bounded, there is some $c < \infty$ so that for all n,

$$\forall u \;\; p_n(o,u) \le c\rho(G)^n$$

and

$$|\{u\,;\ \operatorname{dist}_G(o,u) \le n\}| \le c\lambda^n\,,$$

whence

$$\mathbf{P}_o[\operatorname{dist}_G(o,X_n) \le \alpha n] = \sum_{\operatorname{dist}_G(o,u)\le\alpha n} p_n(o,u) \le c^2\rho(G)^n\lambda^{\alpha n} = c^2(\rho(G)\lambda^\alpha)^n\,.$$

Since this is summable in n, it follows by the Borel-Cantelli lemma that $\operatorname{dist}_G(o,X_n) \le \alpha n$ only finitely often a.s. ◀

Thus, we are interested to know when the spectral radius is less than 1. We shall show that this is equivalent to non-amenability and, in fact, there are inequalities relating the spectral radius to the edge-expansion constant $\Phi_{\mathsf{E}}(G, c, \pi)$.

Theorem 6.7. (Expansion and Spectral Radius) *For any network random walk, we have*

$$\Phi_{\mathsf{E}}(G)^2/2 \le 1 - \sqrt{1 - \Phi_{\mathsf{E}}(G)^2} \le 1 - \rho(G) \le \Phi_{\mathsf{E}}(G)\,. \tag{6.8}$$

To show this, we will use the following easy calculation:

▷ **Exercise 6.8.**

Show that for $f \in \mathbf{D}_{00}$, we have $d^*(c\,df) = \pi(f - Pf)$.

The meaning of the equation in this exercise is as follows. Define the ***gradient*** of a function f on V to be the antisymmetric function

$$\nabla f := c\,df$$

on E. (Thinking of resistance of an edge as its length makes this a natural name.) If we define the divergence by $\operatorname{div}\theta := \pi^{-1} d^*\theta$, then Exercise 6.8 says that $\operatorname{div}\nabla = I - P$, where I is the identity. This is the discrete probabilistic Laplace operator. (For example, consider the definition of "harmonic".)

By Exercise 6.8, we have that for $f \in \mathbf{D}_{00}$,

$$(df, df)_c = (c\,df, df) = (d^*(c\,df), f) = (\pi(f - Pf), f) = (f, f)_\pi - (Pf, f)_\pi\,. \tag{6.9}$$

We will also use the following lemma.

Lemma 6.8. *For any $f \in \mathbf{D}_{00}$, we have*

$$\Phi_{\mathsf{E}}(G, c, \pi) \sum_{x \in \mathsf{V}} f(x)\pi(x) \le \sum_{e \in \mathsf{E}_{1/2}} |df(e)| c(e)\,.$$

Proof. For $t > 0$, we may use $K := \{x\,;\ f(x) > t\}$ in the definition of Φ_{E} to see that

$$\Phi_{\mathsf{E}} |\{x\,;\ f(x) > t\}|_\pi \le \sum_{x,y \in \mathsf{V}} c(x, y) \mathbf{1}_{\{f(x) > t \ge f(y)\}}\,.$$

Now

$$\int_0^\infty |\{x\,;\ f(x) > t\}|_\pi\, dt = \sum_{x \in \mathsf{V}} f(x)\pi(x)$$

and

$$\int_0^\infty \mathbf{1}_{\{f(x) > t \ge f(y)\}}\, dt = f(x) - f(y)\,.$$

Therefore, integrating our inequality on $t \in (0, \infty)$ gives the desired result. ◀

Proof of Theorem 6.7. Use Exercises 6.6 and 6.7, together with (6.9), to write

$$\begin{aligned}\rho(G) &= \sup\left\{\frac{(Pf,f)_\pi}{(f,f)_\pi}\,;\ f \in \mathbf{D}_{00}\setminus\{0\}\right\}\\ &= \sup\left\{\frac{(f,f)_\pi - (df,df)_c}{(f,f)_\pi}\,;\ f \in \mathbf{D}_{00}\setminus\{0\}\right\}\\ &= 1 - \inf\{(df,df)_c/(f,f)_\pi\,;\ f \in \mathbf{D}_{00}\setminus\{0\}\}\,. \end{aligned}\tag{6.10}$$

Choosing $f := \mathbf{1}_K \in \mathbf{D}_{00}$ shows the last inequality in (6.8).

The first inequality in (6.8) comes from the elementary inequality $1 - x/2 \ge \sqrt{1-x}$. To prove the crucial middle inequality, let $f \in \mathbf{D}_{00}$. Apply Lemma 6.8 to f^2 to get

$$\begin{aligned}(f,f)_\pi^2 &\le \Phi_{\mathsf{E}}(G)^{-2}\left(\sum_{e\in\mathsf{E}_{1/2}} c(e)|f(e^+)^2 - f(e^-)^2|\right)^2 \qquad (6.11)\\ &= \Phi_{\mathsf{E}}(G)^{-2}\left(\sum_e c(e)\big|f(e^+) - f(e^-)\big|\cdot\big|f(e^+) + f(e^-)\big|\right)^2\\ &\le \Phi_{\mathsf{E}}(G)^{-2}\left(\sum_e c(e)df(e)^2\right)\left(\sum_e c(e)(f(e^+)+f(e^-))^2\right)\end{aligned}$$

by the Cauchy-Schwarz inequality. The last factor is

$$\begin{aligned}\sum_e c(e)\big[f(e^+)^2 + f(e^-)^2 + 2f(e^+)f(e^-)\big] &= \sum_x \pi(x)f(x)^2 + \sum_{x\sim y} c(x,y)f(x)f(y)\\ &= (f,f)_\pi + \sum_x f(x)\pi(x)\sum_{y\sim x} p(x,y)f(y)\\ &= (f,f)_\pi + (f,Pf)_\pi = 2(f,f)_\pi - (df,df)_c\end{aligned}$$

by (6.9). Therefore,

$$(f,f)_\pi^2 \le \Phi_{\mathsf{E}}(G)^{-2}[2(f,f)_\pi - (df,df)_c](df,df)_c\,.$$

After a little algebra, this is transformed to

$$\left(1 - \frac{(df,df)_c}{(f,f)_\pi}\right)^2 \le 1 - \Phi_{\mathsf{E}}(G)^2\,.$$

This gives the middle inequality of (6.8) when combined with (6.10). ◀

▷ **Exercise 6.9.**

Show that for simple random walk on $\mathbb{T}_{b+1}$, we have $\rho(\mathbb{T}_{b+1}) = 2\sqrt{b}/(b+1)$.

§6.3. Mixing Time.

In this section we investigate the rate at which random walk on a finite network converges to the stationary distribution. This is an analogue to speed of random walk on infinite networks (how quickly does the random walk leave or forget its starting point?) and there is an analogous inequality that relates it to an expansion constant.

There are many ways to measure convergence to the stationary distribution. We'll consider just one. Namely, for any vertex x, we'll consider the walk started at x to be close to the stationary distribution at time t if all the numbers

$$\left|\frac{p_t(x,y)-\pi(y)}{\pi(y)}\right|$$

are small.

In this section, all inner products are with respect to the stationary probability measure π, i.e.,

$$\langle f,g\rangle := (f,g)_\pi = \sum_{x\in\mathsf{V}} \pi(x)f(x)g(x)\,.$$

As we have seen in the preceding section, P is self-adjoint with respect to this inner product. Now the norm of P is at most 1 by Exercise 6.5 and, in fact, has $\mathbf{1}$ as an eigenvector with eigenvalue 1. Thus, we may write the eigenvalues of P as $-1 \le \lambda_n \le \cdots \le \lambda_1 = 1$, where $n := |\mathsf{V}|$.

We will show that when the Markov chain is aperiodic and exhibits what is known as a spectral gap, i.e., the second largest eigenvalue of P is strictly smaller than 1, the chain converges to the stationary distribution at an exponential rate in t. The idea is that any function can be expanded in a basis of eigenfunctions, which shows clearly how P^t acts on the given function. Those parts of the function multiplied by small eigenvalues go quickly to 0 as $t \to \infty$.

▷ **Exercise 6.10.**

Show that $\lambda_2 < 1$ iff the Markov chain is irreducible and that $\lambda_n > -1$ iff the Markov chain is aperiodic.

Theorem 6.9. (Mixing Time and Spectral Gap) *Consider random walk on a finite connected network that is aperiodic. Let $\pi_* := \min_x \pi(x)$ and $\lambda_* := \max_{i\ge 2} |\lambda_i|$. Write $g := 1-\lambda_*$. Then for any vertices x and y,*

$$\left|\frac{p_t(x,y)-\pi(y)}{\pi(y)}\right| \le \frac{e^{-gt}}{\pi_*}\,.$$

Proof. Consider the normalized indicator functions $\psi_x := \mathbf{1}_{\{x\}}/\pi(x)$ on the vertices, indexed by vertices x. Since for any x and y we have $(P^t\psi_y)(x) = p_t(x,y)/\pi(y)$, we get that

$$\frac{p_t(x,y) - \pi(y)}{\pi(y)} = \langle \psi_x, P^t\psi_y - \mathbf{1} \rangle \, .$$

Since $P\mathbf{1} = \mathbf{1}$, we have

$$\langle \psi_x, P^t\psi_y - \mathbf{1} \rangle = \langle \psi_x, P^t(\psi_y - \mathbf{1}) \rangle \le \|\psi_x\|_\pi \|P^t(\psi_y - \mathbf{1})\|_\pi \, .$$

Let $\langle f_i \rangle_{i=1}^n$ be a basis of orthogonal eigenvectors of P, where f_i is an eigenvector of eigenvalue λ_i. Since $\psi_y - \mathbf{1}$ is orthogonal to $\mathbf{1}$, there exist constants $\langle a_i \rangle_{i=2}^n$ such that $\psi_y - \mathbf{1} = \sum_{i=2}^n a_i f_i$, whence

$$\begin{aligned}
\|P^t(\psi_y - \mathbf{1})\|_\pi^2 &= \Big\| \sum_{i=2}^n \lambda_i^t a_i f_i \Big\|_\pi^2 = \sum_{i=2}^n |\lambda_i^t a_i|^2 \|f_i\|_\pi^2 \\
&\le \sum_{i=2}^n |\lambda_*^t a_i|^2 \|f_i\|_\pi^2 = \lambda_*^{2t} \Big\| \sum_{i=2}^n a_i f_i \Big\|_\pi^2 \\
&= \lambda_*^{2t} \|\psi_y - \mathbf{1}\|_\pi^2 \, .
\end{aligned}$$

Since $(\psi_y - \mathbf{1}) \perp \mathbf{1}$, we have $\|\psi_y - \mathbf{1}\|_\pi \le \|\psi_y\|_\pi$, and thus

$$\left| \frac{p_t(x,y) - \pi(y)}{\pi(y)} \right| \le \lambda_*^t \|\psi_x\|_\pi \|\psi_y\|_\pi = \frac{\lambda_*^t}{\sqrt{\pi(x)\pi(y)}} \le \frac{e^{-gt}}{\pi_*} \, . \qquad ◀$$

Because of this bound, we'd like to know how we can estimate the spectral gap g. Intuitively, if a Markov chain has a "bottleneck", that is, a large set of states with large complement that is difficult to transition into or out of, then it will take it more time to mix. To formulate this intuition and relate it to the spectral gap, we define what is known as the expansion constant.

For any two subsets of vertices A and B, let $c(A,B) := \sum_{a \in A, b \in B} c(a,b)$ and $\pi(A) := \sum_{a \in A} \pi(a)$ (which we have also denoted $|A|_\pi$).

Definition 6.10. The ***expansion constant*** of a stationary Markov chain is

$$\Phi_* := \min_{S\,;\, 0 < \pi(S) \le 1/2} \Phi_S \, ,$$

where

$$\Phi_S := \frac{c(S, S^c)}{\pi(S)} \, .$$

Note that $0 \le \Phi_S \le 1$. (The standard notation Φ may suggest a graph cut in two.) Some people prefer the definition

$$\Phi_* := \min_{S\,;\,0<\pi(S)<1} \frac{c(S, S^c)}{\pi(S)\pi(S^c)},$$

while others prefer

$$\Phi_* := \min_{S\,;\,0<\pi(S)<1} \frac{c(S, S^c)}{\min\{\pi(S), \pi(S^c)\}}.$$

These are all equal up to a factor of at most 2.

The following theorem combined with the previous one connects the expansion properties of a Markov chain with its mixing time via its spectral gap. We assume that the chain is "lazy", i.e., for any state x we have that $p(x, x) \ge 1/2$. In that case, $P = (I + \tilde{P})/2$ where $\tilde{P}$ is another stochastic matrix, and hence all the eigenvalues of P are in $[0, 1]$, so $\lambda_* = \lambda_2$. Note that laziness implies aperiodicity. If the chain is not lazy, then we can always consider the new chain with transition matrix $(I + P)/2$.

Theorem 6.11. (Expansion and Spectral Gap) *Let λ_2 be the second eigenvalue of a reversible and lazy Markov chain and $g := 1 - \lambda_2$. Then*

$$\frac{\Phi_*^2}{2} \le g \le 2\Phi_*.$$

We will use the following lemma in the proof of the lower bound. The proof is the same as that for Lemma 6.8.

Lemma 6.12. *Let $\psi \ge 0$ be a function on a state space V of a stationary Markov chain with $\pi\{\psi > 0\} \le 1/2$. Then*

$$\Phi_* \sum_x \psi(x)\pi(x) \le \frac{1}{2} \sum_{x,y} |\psi(x) - \psi(y)| c(x, y).$$

We also need the following analogue of Exercise 6.6:

▷ **Exercise 6.11.**

Show that

$$\lambda_2 = \max_{f \perp \mathbf{1}} \frac{\langle Pf, f\rangle}{\langle f, f\rangle}.$$

Proof of Theorem 6.11. The upper bound is easier. By Exercise 6.11, we have

$$g = \min_{f \perp \mathbf{1}} \frac{\langle (I-P)f, f\rangle}{\langle f, f\rangle} \,. \tag{6.12}$$

As we have seen before in (6.9), expanding the numerator gives

$$\langle (I-P)f, f\rangle = \frac{1}{2} \sum_{x,y} \pi(x)p(x,y)[f(y)-f(x)]^2 \,. \tag{6.13}$$

To get $g \le 2\Phi_*$, given any S with $\pi(S) \le 1/2$, define a function f by $f(x) := \pi(S^c)$ for $x \in S$ and $f(x) = \pi(S)$ for $x \notin S$. Then $\sum_x f(x)\pi(x) = 0$, so $f \perp \mathbf{1}$. Using this f in (6.13), we get that

$$g \le \frac{2c(S,S^c)}{2\pi(S)\pi(S^c)} \le \frac{2c(S,S^c)}{\pi(S)} = 2\Phi_S \,,$$

and so $g \le 2\Phi_*$.

To prove the lower bound, take an eigenfunction f_2 such that $Pf_2 = \lambda_2 f_2$ and $\pi\{f_2 > 0\} \le 1/2$ (if this does not hold, take $-f_2$). Define a new function $f := \max(f_2, 0)$. Observe that

$$\forall x \;\; [(I-P)f](x) \le gf(x) \,.$$

This is because if $f(x) = 0$, this translates to $-(Pf)(x) \le 0$, which is true since $f \ge 0$, while if $f(x) > 0$, then $[(I-P)f](x) \le [(I-P)f_2](x) = gf_2(x) \le gf(x)$. Since $f \ge 0$, we get

$$\langle (I-P)f, f\rangle \le g\langle f, f\rangle \,,$$

or equivalently,

$$g \ge \frac{\langle (I-P)f, f\rangle}{\langle f, f\rangle} =: R \,.$$

(This looks like a contradiction to (6.12), but it is not since f is not orthogonal to $\mathbf{1}$.) Then just as in the proof of Theorem 6.7, we obtain

$$1 - \frac{\Phi_*^2}{2} \ge \sqrt{1-\Phi_*^2} \ge 1 - R \ge 1 - g \,. \qquad \blacktriangleleft$$

A family of d-regular graphs $\{G_n\}$ is said to be a (d,c)***-expander*** family if the expansion constant of the simple random walk on G_n satisfies $\Phi_*(G_n) \ge c$ for all n.

We now construct a 3-regular family of expander multi-graphs. This was the first construction of an expander family and it is due to Pinsker (1973). Let $G_n = (\mathsf{V}, \mathsf{E})$ be a bipartite graph with parts A and B, each with n vertices. Although A and B are distinct, we will denote them both by $\{1, \ldots, n\}$. Draw uniformly at random two permutations σ_1, σ_2 of $\{1, \ldots, n\}$ and take the edge set to be $\mathsf{E} = \{(i,i), (i, \sigma_1(i)), (i, \sigma_2(i))\,;\; 1 \le i \le n\}$.

Theorem 6.13. *There exists $\delta > 0$ such that for all n, with positive probability $\forall S \subset \mathsf{V}$ with $0 < |S| \le n$, we have*

$$\frac{|\partial_{\mathsf{E}} S|}{|S|} > \delta\,.$$

Proof. It is enough to prove that for some $\delta > 0$, with positive probability every non-empty $S \subset A$ of size $k \le \lfloor n/2 \rfloor$ has at least $\lceil (1+\delta)k \rceil$ neighbors $\partial_{\mathsf{V}} S$ in B. To see this, consider any non-empty $S \subset \mathsf{V}$. Write $S = S_1 \cup S_2$ with $S_1 \subseteq A$ and $S_2 \subseteq B$. We may assume that $|S_1| \ge |S_2|$. If $|S_1| > n/2$, then let S' be a subset of S_1 of cardinality $\lfloor n/2 \rfloor$; otherwise, let $S' := S_1$. In either case, we have $|S'| \ge |S_2|$ and so $|\partial_{\mathsf{E}} S| \ge |\partial_{\mathsf{V}} S'| - |S_2| \ge \lceil (1+\delta)|S'| \rceil - |S'| \ge \delta |S'| \ge \delta |S|/2$ if our condition holds.

So let $S \subset A$ be a set of size $0 < k \le \lfloor n/2 \rfloor$. We wish to bound the probability that $|\partial_{\mathsf{V}} S| \le \lfloor (1+\delta)k \rfloor$. Since (i,i) is an edge for every $1 \le i \le n$, we always have that $|\partial_{\mathsf{V}} S| \ge k$. Consider therefore the possible sets of size $\lfloor (1+\delta)k \rfloor$ vertices that could contain $\partial_{\mathsf{V}} S$ and calculate the probability that both $\sigma_1(S)$ and $\sigma_2(S)$ fall within that set. This first-moment argument gives

$$\mathbf{P}\Big[|\partial_{\mathsf{V}} S| \le \lfloor (1+\delta)k \rfloor\Big] \le \frac{\binom{n}{\lfloor \delta k \rfloor}\binom{\lfloor (1+\delta)k \rfloor}{k}^2}{\binom{n}{k}^2}\,.$$

Considering now all possible S, we obtain

$$\mathbf{P}\Big[\exists S \subset A \quad 0 < |S| \le \lfloor \frac{n}{2} \rfloor\,, |\partial_{\mathsf{V}} S| \le \lfloor (1+\delta)k \rfloor\Big] \le \sum_{k=1}^{\frac{n}{2}} \binom{n}{k} \frac{\binom{n}{\lfloor \delta k \rfloor}\binom{\lfloor (1+\delta)k \rfloor}{\lfloor \delta k \rfloor}^2}{\binom{n}{k}^2}\,,$$

which is strictly less than 1 for $\delta > 0$ small enough by the following calculation.

We bound $\binom{n}{\lfloor \delta k \rfloor} \le \frac{n^{\lfloor \delta k \rfloor}}{\lfloor \delta k \rfloor !}$, and similarly bound $\binom{\lfloor (1+\delta)k \rfloor}{\lfloor \delta k \rfloor}$, while $\binom{n}{k} \ge \frac{n^k}{k^k}$. This gives

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \frac{\binom{n}{\lfloor \delta k \rfloor}\binom{\lfloor (1+\delta)k \rfloor}{\lfloor \delta k \rfloor}^2}{\binom{n}{k}} \le \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{n^{\delta k}((1+\delta)k)^{2\delta k} k^k}{\lfloor \delta k \rfloor !^3 n^k}\,.$$

Recall that for any integer ℓ we have $\ell! > (\ell/e)^\ell$, and bound $\lfloor \delta k \rfloor !$ by this. We get

$$\mathbf{P}\Big[\exists S \subset A \quad 0 < |S| \le \lfloor \frac{n}{2} \rfloor\,, |\partial_{\mathsf{V}} S| \le \lfloor (1+\delta)k \rfloor\Big] \le \sum_{k=1}^{\lfloor n/2 \rfloor} \Big(\frac{k}{n}\Big)^{(1-\delta)k} \Big[\frac{e^3(1+\delta)^2}{\delta^3}\Big]^{\delta k}\,.$$

Each term clearly tends to 0 as n tends to ∞, for any $\delta \in (0,1)$, and since $\frac{k}{n} \le \frac{1}{2}$ and $\left(\frac{1}{2}\right)^{(1-\delta)} \left[\frac{e^3(1+\delta)^2}{\delta^3}\right]^{\delta} < 1$ for $\delta < 0.05$, for any such δ the whole sum tends to 0 as n tends to ∞ by the dominated convergence theorem. ◀

§6.4. Planar Graphs.

Planar graphs are generally the only ones we can draw in a nice way and gaze at. See Figures 6.1 and 6.2 for some examples drawn by a program created by Don Hatch. They also make great art; see Figure 6.3 for a transformation by Doug Dunham of a print by Escher.

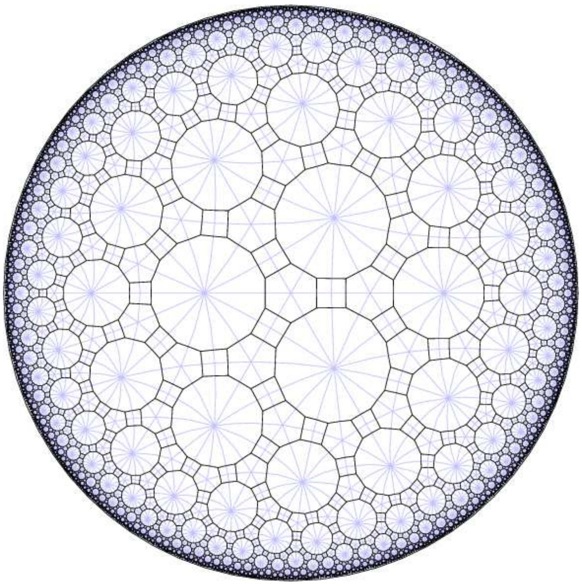


Figure 6.1. A Cayley graph in the hyperbolic disc and its dual in light blue, whose faces are triangles of interior angles $\pi/2$, $\pi/3$, and $\pi/7$.

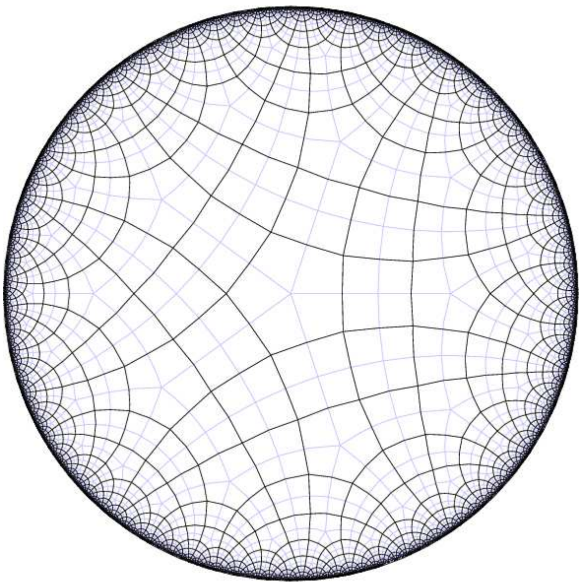


Figure 6.2. Dual tessellations in the hyperbolic disc.

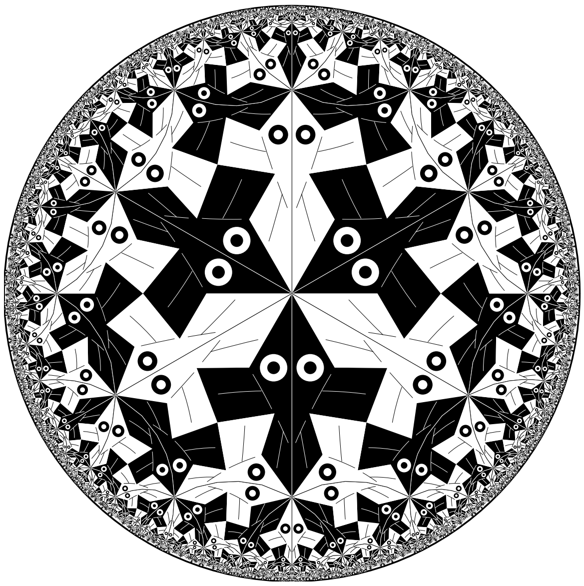


Figure 6.3. A transformed Escher print, *Circle Limit I*, based on a (6,4)-tessellation of the hyperbolic disc.

These are good reasons for studying planar graphs separately. But a more important reason is that they often exhibit special behavior, as we will see several times in this book, and there are special techniques available as well. Planar duality is a prototype for more general types of duality, which is yet another reason to study it.

A ***planar*** graph is one that can be drawn in the plane in such a way that edges do not cross; an actual such embedding is called a ***plane*** graph. If G is a plane graph such that each bounded set in the plane contains only finitely many vertices of G, then G is said to be ***properly embedded*** in the plane. We will always assume without further mention that plane graphs are properly embedded. A ***face*** of a plane graph is a connected component of the complement of the graph in the plane. If G is a plane (multi)graph, then the ***plane dual*** $G^\dagger$ of G is the (multi)graph formed as follows: The vertices of $G^\dagger$ are the faces formed by G. Two faces of G are joined by an edge in $G^\dagger$ precisely when they share an edge in G. Thus, $\mathsf{E}(G)$ and $\mathsf{E}(G^\dagger)$ are in a natural one-to-one correspondence. Furthermore, if one draws each vertex of $G^\dagger$ in the interior of the corresponding face of G and each edge of $G^\dagger$ so that it crosses the corresponding edge of G, then the dual of $G^\dagger$ is G.

In this section, drawn from Häggström, Jonasson, and Lyons (2002), we will calculate the edge expansion constants of certain regular planar graphs that arise from tessellations of the hyperbolic plane. The same results were found independently (with different proofs) by Higuchi and Shirai (2003). Of course, edge graphs of Euclidean tessellations are amenable, whence their expansion constants are 0. During this section, we assume that G and its plane dual $G^\dagger$ are locally finite, whence each graph has one end, which means that the deletion of any finite set of vertices leaves exactly one infinite component. We first examine the combinatorial difference between Euclidean and hyperbolic tessellations.

A regular Euclidean polygon of $d^\dagger$ sides has interior angles $\pi(1-2/d^\dagger)$. In order for such polygons to form a tessellation of the plane with d polygons meeting at each vertex, we must have $\pi(1-2/d^\dagger) = 2\pi/d$, i.e., $1/d + 1/d^\dagger = 1/2$, or, equivalently, $(d-2)(d^\dagger - 2) = 4$. There are three such cases, and in all three, tessellations have been well known since antiquity. In the hyperbolic plane, the interior angles can take any value in $\big(0, \pi(1-2/d^\dagger)\big)$, whence a tessellation exists only if $1/d + 1/d^\dagger < 1/2$, or, equivalently, $(d-2)(d^\dagger - 2) > 4$; again, this condition is also sufficient for the existence of a hyperbolic tessellation, as has been known since the 19th century. Furthermore, in the hyperbolic plane, any two d-gons with interior angles all equal to some number α are congruent. (There are no homotheties of the hyperbolic plane.) The edges of a tessellation form the associated ***edge graph***. Clearly, when the tessellation is any of the above ones, the associated edge graph is regular and its dual is regular as well. An example is drawn in Figure 2.3. (We remark that the cases $(d-2)(d^\dagger - 2) < 4$ correspond to the spherical tessellations that arise from the five regular solids.)

Moreover, we claim that if G is a plane regular graph with regular dual, then G is transitive, as is $G^\dagger$, and G is the edge graph of a tessellation by congruent regular polygons. The proof of this claim is a nice application of geometry to graph theory.

First, suppose we are given the edge graphs of any two tessellations by congruent regular polygons (in the Euclidean or hyperbolic plane, as necessary) of the same type $(d, d^\dagger)$ and one fixed vertex in each edge graph. Then there is an isomorphism of the two edge graphs that takes one fixed vertex to the other. This is easy to see by going out ring by ring around a starting polygon. Thus, such edge graphs are transitive.

To prove the general statement, we claim that any (proper) tessellation of a plane with degree d and codegree $d^\dagger$ has an edge graph that is isomorphic to the edge graph of the corresponding tessellation above. In case $(d-2)(d^\dagger - 2) = 4$, replace each face by a congruent copy of a flat polygon; in case $(d-2)(d^\dagger - 2) > 4$, replace it by a congruent copy of a regular hyperbolic polygon (with curvature -1) of $d^\dagger$ sides and interior angles $2\pi/d$; while if $(d-2)(d^\dagger - 2) < 4$, replace it by a congruent copy of a regular spherical polygon (with curvature $+1$) of $d^\dagger$ sides and interior angles $2\pi/d$. Glue these together along the edges. We get a metrically complete Riemannian surface of curvature 0, -1, or $+1$, correspondingly, that is homeomorphic to the plane since our assumption is that the plane is the union of the faces, edges, and vertices of the tessellation, without needing any limit points. A theorem of Riemann says that the surface is isometric to either the

Euclidean plane or the hyperbolic plane (the spherical case is impossible). That is, we now have a tessellation by congruent polygons. (One could also prove the existence of such tessellations, referred to above, in a similar manner. We also remark that either the graph or its dual is a Cayley graph; see Chaboud and Kenyon (1996).)

We write d_G for the degree of vertices in G when G is regular. Our main result will be the following calculation of the edge-expansion constant $\Phi_{\mathsf{E}}(G, \mathbf{1}, \mathbf{1})$:

Theorem 6.14. *If G is a plane regular graph with regular dual $G^\dagger$, then*

$$\Phi_{\mathsf{E}}(G, \mathbf{1}, \mathbf{1}) = (d_G - 2)\sqrt{1 - \frac{4}{(d_G - 2)(d_{G^\dagger} - 2)}} .$$

Compare this to the regular tree of degree d, where the left-hand side is equal to $d-2$ (Exercise 6.1). (Note that in the preceding section, $\Phi_{\mathsf{E}}(G)$ denoted $\Phi_{\mathsf{E}}(G, \mathbf{1}, \pi)$, which differs in the regular case by a factor of d_G from $\Phi_{\mathsf{E}}(G, \mathbf{1}, \mathbf{1})$ used here.)

To prove Theorem 6.14, we need to introduce a few more expansion constants. For $K \subseteq \mathsf{V}$, recall that $E(K) := \{[x, y] \in \mathsf{E};\ x, y \in K\}$ and set $E^*(K) := \{[x, y] \in \mathsf{E};\ x \in K \text{ or } y \in K\}$. Thus, $\partial_{\mathsf{E}} K = E^*(K) \setminus E(K)$. Define $G(K) := \big(K, E(K)\big)$. Write

$$\Phi'_{\mathsf{E}}(G) := \lim_{N\to\infty} \inf \left\{ \frac{|\partial_{\mathsf{E}} K|}{|K|} ;\ K \subset \mathsf{V},\ G(K) \text{ connected},\ N \le |K| < \infty \right\} ,$$

$$\beta(G) := \lim_{N\to\infty} \inf \left\{ \frac{|K|}{|E(K)|} ;\ K \subset \mathsf{V},\ G(K) \text{ connected},\ N \le |K| < \infty \right\} ,$$

$$\delta(G) := \lim_{N\to\infty} \sup \left\{ \frac{|K|}{|E^*(K)|} ;\ K \subset \mathsf{V},\ G(K) \text{ connected},\ N \le |K| < \infty \right\} .$$

When G is regular, we have for all finite K that $2|E(K)| = d_G|K| - |\partial_{\mathsf{E}} K|$ and $2|E^*(K)| = d_G|K| + |\partial_{\mathsf{E}} K|$, whence

$$\beta(G) = \frac{2}{d_G - \Phi'_{\mathsf{E}}(G)} \tag{6.14}$$

and

$$\delta(G) = \frac{2}{d_G + \Phi'_{\mathsf{E}}(G)} . \tag{6.15}$$

Combining this with Theorem 6.2, we see that when G is transitive,

$$\delta(G) = \sup \left\{ \frac{|K|}{|E^*(K)|} ;\ K \subset \mathsf{V} \text{ finite and nonempty} \right\} . \tag{6.16}$$

A short bit of algebra shows that Theorem 6.14 follows from applying the following identity to G, as well as to $G^\dagger$, then solving the resulting two equations using (6.14) and (6.15):

Theorem 6.15. *For any plane regular graph G with regular dual $G^\dagger$, we have*

$$\beta(G) + \delta(G^\dagger) = 1\,.$$

Proof. We begin with a sketch of the proof in the simplest case. Suppose that $K \subset \mathsf{V}(G)$ and that the graph $G(K)$ looks like Figure 6.4 in the following sense. Let K^f be the vertices of $G^\dagger$ that are faces of $G(K)$. Then K^f consists of all the faces, other than the outer face, of $G(K)$. Further, $|E(K)| = |E^*(K^f)|$. In this nice case, Euler's formula applied to the graph $G(K)$ gives

$$|K| - |E(K)| + [|K^f| + 1] = 2\,,$$

which is equivalent to

$$|K|/|E(K)| + |K^f|/|E^*(K^f)| = 1 + 1/|E(K)|\,.$$

Now if we also assume that K can be chosen so that the first term above is arbitrarily close to $\beta(G)$ and the second term is arbitrarily close to $\delta(G^\dagger)$ with $|E(K)|$ arbitrarily large, then the desired formula follows at once. Thus, our work will consist in reducing or comparing things to such a nice situation.

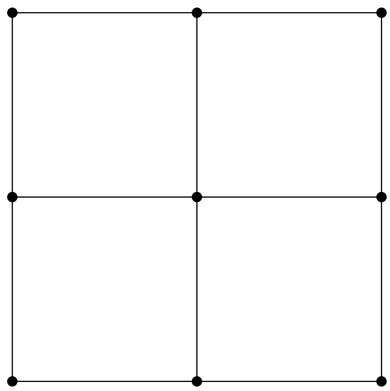


Figure 6.4. The simplest case.

Let $\epsilon > 0$ and let K be a finite set in $\mathsf{V}(G^\dagger)$ such that $G(K)$ is connected, $|K|/|E^*(K)| \geq \delta(G^\dagger) - \epsilon$, and $|E^*(K)| > 1/\epsilon$. Regarding each element of K as a face of G, let $K' \subset \mathsf{V}(G)$ be the set of vertices bounding these faces and let $E' \subset \mathsf{E}(G)$ be the set of edges bounding these faces. Then $|E(K')| \geq |E'| = |E^*(K)|$. Since the number of faces F of the graph (K', E') is at least $|K| + 1$, we have

$$\begin{aligned} |K'|/|E(K')| + |K|/|E^*(K)| &\leq |K'|/|E'| + |K|/|E'| \\ &\leq |K'|/|E'| + (F-1)/|E'| \\ &= 1 + 1/|E'| < 1 + \epsilon\,, \end{aligned} \tag{6.17}$$

where the identity comes from Euler's formula applied to the graph (K', E'). Our choice of K then implies that

$$|K'|/|E(K')| + \delta(G^\dagger) \le 1 + 2\epsilon\,.$$

Since $G(K')$ is connected and $|K'| \to \infty$ when $\epsilon \to 0$, it follows that $\beta(G) + \delta(G^\dagger) \le 1$.

To prove that $\beta(G) + \delta(G^\dagger) \ge 1$, note that the constant $\Phi'_{\mathsf{E}}(G)$ is unchanged if, in its definition, we require K to be connected and simply connected when K is regarded as a union of closed faces of $G^\dagger$ in the plane. This is because filling in holes increases $|K|$ and decreases $|\partial_{\mathsf{E}} K|$. Since G is regular, the same holds for $\beta(G)$ by (6.14). Now let $\epsilon > 0$. Let $K \subset \mathsf{V}(G)$ be connected and simply connected (when regarded as a union of closed faces of $G^\dagger$ in the plane) such that $|K|/|E(K)| \le \beta(G) + \epsilon$. Let K^f be the set of vertices in $G^\dagger$ that correspond to faces of $G(K)$. Since $|E^*(K^f)| \le |E(K)|$ and the number of faces of the graph $G(K)$ is precisely $|K^f| + 1$, we have

$$|K|/|E(K)| + |K^f|/|E^*(K^f)| \ge |K|/|E(K)| + |K^f|/|E(K)| = 1 + 1/|E(K)| \ge 1$$

by Euler's formula applied to the graph $G(K)$. (In case K^f is empty, a comparable calculation shows that $|K|/|E(K)| \ge 1$.) Because $G^\dagger$ is transitive, (6.16) allows us to conclude that

$$\beta(G) + \delta(G^\dagger) + \epsilon \ge \beta(G) + \epsilon + |K^f|/|E^*(K^f)| \ge |K|/|E(K)| + |K^f|/|E^*(K^f)| \ge 1\,.$$

Since ϵ is arbitrary, the desired inequality follows. ◀

Question 6.16. What is $\Phi_{\mathsf{V}}(G)$ for regular co-regular plane graphs G? What are $\Phi_{\mathsf{E}}(G)$ and $\Phi_{\mathsf{V}}(G)$ for more general transitive plane graphs G?

Question 6.17. Suppose that G is a planar graph with all degrees in $[d_1, d_2]$ and all co-degrees in $[d_1^\dagger, d_2^\dagger]$. Do we have

$$(d_1 - 2)\sqrt{1 - \frac{4}{(d_1-2)(d_1^\dagger - 2)}} \le \Phi_{\mathsf{E}}(G, \mathbf{1}, \mathbf{1}) \le (d_2 - 2)\sqrt{1 - \frac{4}{(d_2-2)(d_2^\dagger - 2)}} \quad ?$$

For partial results, see Lawrencenko, Plummer, and Zha (2002) and the references therein.

§6.5. Profiles and Transience.

Write

$$\psi(G,t) := \inf\big\{|\partial_{\mathsf{E}} K|_c\,;\; t \le |K|_\pi < \infty\big\}\,.$$

This is a functional elaboration of the expansion constant. We will relate effective resistance to this expansion "profile". The following result of Lyons, Morris, and Schramm (2008) refines Thomassen (1992) and is adapted from a similar result of He and Schramm (1995). It is very similar to an independent result of Benjamini and Kozma (2005). Since we will consider effective resistance from a set to infinity, we will use the still more refined function

$$\psi(G,A,t) := \inf\big\{|\partial_{\mathsf{E}} K|_c\,;\; A \subseteq K,\; K/A \text{ is connected}, \; t \le |K|_\pi < \infty\big\} \tag{6.18}$$

for $A \subset \mathsf{V}(G)$. Here, we say that K/A is connected to mean that the graph induced by K in G/A is connected, where we have identified all of A to a single vertex. If $A = \{a\}$, then we'll write more simply $\psi(G,a,t)$ for $\psi(G,\{a\},t)$.

Theorem 6.18. *Let A be a finite set of vertices in a network G with $|\mathsf{V}(G)|_\pi = \infty$. Let $\psi(t) := \psi(G,A,t)$. Define $s_0 := |A|_\pi$ and $s_{k+1} := s_k + \psi(s_k)/2$ inductively for $k \ge 0$. Then*

$$\mathscr{R}(A \leftrightarrow \infty) \le \sum_{k\ge 0} \frac{2}{\psi(s_k)}\,.$$

This follows immediately from the following analogue for finite networks and the following exercise.

▷ **Exercise 6.12.**

Let G be a network without loops and $A \subset \mathsf{V}(G)$. Let G' be the network obtained from G by identifying A to a single vertex a and removing any resulting loops. Show that

$$\psi\big(G',a,\pi(a)+t\big) = \psi(G,A,|A|_\pi + t) \ge \psi(G,|A|_\pi + t) \ge \psi(G,\pi(a)+t)$$

for all $t \ge 0$.

Lemma 6.19. *Let a and z be two distinct vertices in a finite connected network G. Define*

$$\psi(t) := \min\big\{|\partial_{\mathsf{E}} W|_c\,;\; a \in W,\; z \notin W,\; W \textit{ is connected}, \; t \le |W|_\pi\big\}$$

for $t \le |\mathsf{V}(G) \setminus \{z\}|_\pi$ and $\psi(t) := \infty$ for $t > |\mathsf{V}(G) \setminus \{z\}|_\pi$. Define $s_0 := \pi(a)$ and $s_{k+1} := s_k + \psi(s_k)/2$ recursively for $k \ge 0$. Then

$$\mathscr{R}(a \leftrightarrow z) \le \sum_{k=0}^{\infty} \frac{2}{\psi(s_k)}\,.$$

Proof. Let $v(\bullet)$ be the voltage corresponding to the unit current flow i from z to a, with $v(a) = 0$.

For $t \ge 0$, let $W(t) := \big\{x \in \mathsf{V}\,;\; v(x) \le t\big\}$, and for $0 \le t < t'$, let $E(t, t')$ be the set of directed edges from $W(t)$ to $\big\{x \in \mathsf{V}\,;\; v(x) \ge t'\big\}$. Define $t_0 := 0$ and inductively,

$$t_{k+1} := \sup\big\{t > t_k\,;\; |E(t_k, t)|_c \ge |\partial_{\mathsf{E}} W(t_k)|_c/2\big\}.$$

Set $\bar{k} := \min\{j\,;\; z \in W(t_j)\} = \min\{j\,;\; t_{j+1} = \infty\}$. Fix some $k < \bar{k}$. Note that $i(e) \le 0$ for every $e \in \partial_{\mathsf{E}} W(t_k)$ (where edges in $\partial_{\mathsf{E}} W(t_k)$ are oriented away from $W(t_k)$). Now

$$\begin{aligned} 1 = \sum_{e \in \partial_{\mathsf{E}} W(t_k)} |i(e)| &\ge \sum_{e \in E(t_k, t_{k+1})} c(e)\, \big(v(e^+) - v(e^-)\big) \\ &\ge \sum_{e \in E(t_k, t_{k+1})} c(e)\, (t_{k+1} - t_k) \ge (t_{k+1} - t_k)\, \frac{|\partial_{\mathsf{E}} W(t_k)|_c}{2}\,, \end{aligned}$$

where the last inequality follows from the definition of t_{k+1}.

Thus

$$t_{k+1} - t_k \le 2/\psi\big(|W_k|_\pi\big)\,, \tag{6.19}$$

where we abbreviate $W_k := W(t_k)$. Clearly,

$$\begin{aligned} |W_{k+1}|_\pi &= |W_k|_\pi + |W_{k+1} \setminus W_k|_\pi \\ &\ge |W_k|_\pi + \frac{1}{2}\, |\partial_{\mathsf{E}} W_k|_c \ge |W_k|_\pi + \frac{1}{2}\, \psi\big(|W_k|_\pi\big)\,. \end{aligned}$$

Since ψ is a non-decreasing function, it follows by induction that $\pi(W_k) \ge s_k$ for $k < \bar{k}$ and (6.19) gives

$$\mathscr{R}(a \leftrightarrow z) = v(z) = t_{\bar{k}} - t_0 \le \sum_{k=0}^{\bar{k}-1} \frac{2}{\psi\big(\pi(W_k)\big)} \le \sum_{k=0}^{\bar{k}-1} \frac{2}{\psi(s_k)}\,. \qquad \blacktriangleleft$$

It is commonly the case that $\psi(t) = \psi(G, A, t) \ge f(t)$ for some increasing function f on $\big[|A|_\pi, \infty\big)$ that satisfies $0 < f(t) \le t$ and $f(2t) \le \alpha f(t)$ for some α. In this case, define $t_0 := |A|_\pi$ and $t_{k+1} := t_k + f(t_k)/2$ inductively. We have that $s_k \ge t_k$ and $t_k \le t_{k+1} \le 2t_k$, whence for $t_k \le t \le t_{k+1}$, we have $f(t) \le f(2t_k) \le \alpha f(t_k)$, so that

$$\begin{aligned} \int_{|A|_\pi}^{\infty} \frac{4\alpha^2}{f(t)^2} dt &= \sum_{k \ge 0} \int_{t_k}^{t_{k+1}} \frac{4\alpha^2}{f(t)^2} dt \ge \sum_{k \ge 0} \int_{t_k}^{t_{k+1}} \frac{4}{f(t_k)^2} dt \\ &= \sum_{k \ge 0} \frac{4(t_{k+1} - t_k)}{f(t_k)^2} = \sum_{k \ge 0} \frac{2f(t_k)}{f(t_k)^2} \\ &\ge \sum_{k \ge 0} \frac{2}{\psi(t_k)} \ge \sum_{k \ge 0} \frac{2}{\psi(s_k)} \\ &\ge \mathscr{R}(A \leftrightarrow \infty)\,. \end{aligned}$$

This bound on the effective resistance is usually easier to estimate.

The following theorem is due to Coulhon and Saloff-Coste (1993). Our proof is modelled on the one presented by Gromov (1999), p. 348. Define the ***internal vertex boundary*** of a set K as $\partial_{\mathsf{V}}^{\mathrm{int}} K := \{x \in K \,;\, \exists y \notin K \;\; y \sim x\}$.

Theorem 6.20. (Expansion of Cayley Graphs) *Let G be a Cayley graph. Let $\rho(m)$ be the smallest radius of a ball in G that contains at least m vertices. Then for all finite $K \subset \mathsf{V}$, we have*

$$\frac{|\partial_{\mathsf{V}}^{\mathrm{int}} K|}{|K|} \geq \frac{1}{2\rho(2|K|)}\,.$$

Proof. Let s be a generator of the group Γ used for the right Cayley graph G. The bijection $x \mapsto xs$ moves x to a neighbor of x. Thus, it moves at most $|\partial_{\mathsf{V}}^{\mathrm{int}} K|$ vertices of K to the complement of K. If γ is the product of r generators, then the map $x \mapsto x\gamma$ is a composition of r moves of distance 1, each of which moves at most $|\partial_{\mathsf{V}}^{\mathrm{int}} K|$ points of K out of K, whence it itself moves at most $r|\partial_{\mathsf{V}}^{\mathrm{int}} K|$ points of K out of K. Let $\rho := \rho(2|K|)$. Now by choice of ρ, a random γ in the ball of radius ρ about the identity has chance at least $1/2$ of moving any given $x \in K$ out of K, so that a random γ moves at least $|K|/2$ points of K out of K in expectation. Hence there is some γ that moves this many points, whence $\rho|\partial_{\mathsf{V}}^{\mathrm{int}} K| \geq |K|/2$. This is the desired inequality. ◀

The same proof shows the same bound for the external vertex boundary, $\partial_{\mathsf{V}} K$. An extension to transitive graphs is given in Lemma 10.43 (see also Proposition 8.12).

§6.6. Anchored Expansion and Percolation.

Recall the probability measure $\mathbf{P}_p$ defining Bernoulli percolation from Section 5.2 and the critical probability p_{c}. There, we took a random subset of edges, but an alternative is to take a random subset of vertices and all edges both of whose endpoints are kept. This alternative is called ***site percolation***. The adjective Bernoulli applies when the presence of vertices are i.i.d. When we need different notation for these two processes, we use $\mathbf{P}_p^{\mathrm{site}}$ and $\mathbf{P}_p^{\mathrm{bond}}$ for the two product measures on 2^{V} and 2^{E}, respectively, and $p_{\mathrm{c}}^{\mathrm{site}}$ and $p_{\mathrm{c}}^{\mathrm{bond}}$ for the two critical probabilities. If we don't indicate whether the percolation is bond or site and both make sense in context, then results we state should be taken to apply to both types of percolation. Grimmett, Kesten, and Zhang (1993) showed that simple random walk on the infinite cluster of Bernoulli percolation in $\mathbb{Z}^d$ (when $p > p_{\mathrm{c}}$) is transient for $d \geq 3$; in other words, in Euclidean lattices, transience is preserved when the whole lattice is replaced by an infinite percolation cluster.

Conjecture 6.21. (Percolation and Transience) *If G is a transient Cayley graph, then a.s. every infinite cluster of Bernoulli percolation on G is transient.*

Conjecture 6.22. (Percolation and Speed) *Suppose that G is a Cayley graph on which simple random walk has positive speed. Then simple random walk on infinite clusters of Bernoulli percolation also has positive speed a.s. and conversely.*

These conjectures were made by Benjamini, Lyons, and Schramm (1999), who proved that simple random walk on an infinite cluster of any non-amenable Cayley graph has positive speed. One might hope to use Proposition 6.6 to establish this result, but, in fact, the infinite clusters are amenable:

▷ **Exercise 6.13.**

For any $p < 1$, every infinite cluster K of Bernoulli(p) percolation on any graph G of bounded degree has $\Phi_{\mathsf{E}}(K) = 0$ a.s.

On the other hand, the infinite clusters might satisfy the following weaker "anchored expansion" property, which is known to imply positive speed (Theorem 6.37).

Fix $o \in \mathsf{V}(G)$. The ***anchored expansion constants*** of G are

$$\Phi^*_{\mathsf{E}}(G) := \lim_{n\to\infty} \inf \left\{ \frac{|\partial_{\mathsf{E}} K|}{|K|} \,;\, o \in K \subset \mathsf{V},\, G(K) \text{ is connected},\, n \le |K| < \infty \right\}$$

and

$$\Phi^*_{\mathsf{V}}(G) := \lim_{n\to\infty} \inf \left\{ \frac{|\partial_{\mathsf{V}} K|}{|K|} \,;\, o \in K \subset \mathsf{V},\, G(K) \text{ is connected},\, n \le |K| < \infty \right\}.$$

These are closely related to the number $\psi(G, o, 0)$, but have the advantage that $\Phi^*_{\mathsf{E}}(G)$ and $\Phi^*_{\mathsf{V}}(G)$ do not depend on the choice of the basepoint o. We say that a graph G has ***anchored expansion*** if $\Phi^*_{\mathsf{E}}(G) > 0$.

▷ **Exercise 6.14.**

For any transitive graph G, show that $\Phi_{\mathsf{E}}(G) > 0$ is equivalent to $\Phi^*_{\mathsf{E}}(G) > 0$.

An important feature of anchored expansion is that several probabilistic implications of non-amenability remain true with this weaker assumption. Furthermore, anchored expansion is quite stable under percolation.

A simple relationship between anchored expansion and percolation is the following upper bound on p_c due to Benjamini and Schramm (1996b):

Theorem 6.23. (Percolation and Anchored Expansion) *For any graph G, we have* $p_c^{\mathrm{bond}}(G) \le 1/\big(1 + \Phi_{\mathsf{E}}^*(G)\big)$ *and* $p_c^{\mathrm{site}}(G) \le 1/\big(1 + \Phi_{\mathsf{V}}^*(G)\big)$.

Note that equality holds in both inequalities when G is a regular tree. For the proof, as well as later, we will identify a subset $\omega \subseteq \mathsf{E}$ with its indicator function, so that $\omega(e)$ takes the value 0 or 1 depending on whether e lies in the subset or not.

Proof. The proofs of both inequalities are completely analogous, so we prove only the first. In fact, we prove it with $\Phi_{\mathsf{E}}(G)$ in place of $\Phi_{\mathsf{E}}^*(G)$, leaving the improvement to Exercise 6.15.

Choose any ordering of $\mathsf{E} = \langle e_1, e_2, \ldots \rangle$ so that o is an endpoint of e_1. Fix $p > 1/\big(1 + \Phi_{\mathsf{E}}(G)\big)$ and let $\langle Y_k \rangle$ and $\langle Y_k' \rangle$ be independent $\{0,1\}$-valued Bernoulli(p) random variables. If A is the event that

$$\frac{1}{n}\sum_{k=1}^{n} Y_k > \frac{1}{1 + \Phi_{\mathsf{E}}(G)}$$

for all $n \ge 1$, then A has positive probability.

Define $E_0 := \varnothing$. We will look at a finite or infinite subsequence of edges $\langle e_{n_j} \rangle$ via a recursive procedure and define a percolation ω as we go. Suppose that the edges $E_k := \langle e_{n_1}, \ldots, e_{n_k} \rangle$ have been selected and that $\omega(e_{n_j}) = Y_j$ for $j \le k$. Let V_k be the union of $\{o\}$ and the endpoints of the open edges of E_k. Let n_{k+1} be the smallest index of an edge in $\mathsf{E} \setminus E_k$ that has exactly one endpoint in V_k, if any. If there are none, then stop; $K(o)$ is finite and we set $\omega(e_j) := Y_j'$ for the remaining edges $e_j \in \mathsf{E} \setminus E_k$. Otherwise, let $\omega(e_{n_{k+1}}) := Y_{k+1}$.

If this procedure never ends, then $K(o)$ is infinite; assign $\omega(e_j) := Y_j'$ for any remaining edges $e_j \in \mathsf{E} \setminus E_k$.

In both cases (whether $K(o)$ is finite or infinite), ω is a fair sample of Bernoulli(p) percolation on G.

We claim that $K(o)$ is infinite on the event A. This would mean that $p \ge p_c^{\mathrm{bond}}(G)$ and would complete the proof.

For suppose that $K(o)$ is finite and contains m vertices. Let E_n be the final set of selected edges. Note that E_n contains $\partial_{\mathsf{E}} K(o)$ (all edges of which are closed) and a spanning tree of $K(o)$ (all edges of which are open). This means that $n \ge |\partial_{\mathsf{E}} K(o)| + m - 1$ and $\sum_{k=1}^n Y_k = m - 1$. Since $|\partial_{\mathsf{E}} K(o)|/m \ge \Phi_{\mathsf{E}}(G)$, we have

$$\frac{1}{n}\sum_{k=1}^{n} Y_k \le \frac{m-1}{|\partial_{\mathsf{E}} K(o)| + m - 1} = \frac{1}{1 + |\partial_{\mathsf{E}} K(o)|/(m-1)} < \frac{1}{1 + |\partial_{\mathsf{E}} K(o)|/m} \le \frac{1}{1 + \Phi_{\mathsf{E}}(G)}$$

and the event A does not occur. ◀

▷ **Exercise 6.15.**
Prove the first inequality of Theorem 6.23 as written with anchored expansion.

Similar ideas show the following general property. Given two graphs $G = (\mathsf{V}, \mathsf{E})$ and $G' = (\mathsf{V}', \mathsf{E}')$, call a surjection $\phi : \mathsf{V} \to \mathsf{V}'$ a ***weak covering map*** if for every vertex $x \in \mathsf{V}$ and every neighbor y' of $\phi(x)$, there is some neighbor y of x such that $\phi(y) = y'$. For example, if $\phi : \Gamma \to \Gamma'$ is a group homomorphism that maps a generating set S for Γ onto a generating set S' for Γ', and if G, G' are the corresponding Cayley graphs, then ϕ is also a weak covering map. In the case $|S| = |S'|$, we get a stronger notion than weak covering map, one which is closer to the topological notion of covering map; we define it for networks. Given two graphs $G = (\mathsf{V}, \mathsf{E})$ and $G' = (\mathsf{V}', \mathsf{E}')$ with edges weighted by c, c', respectively, and vertices weighted by D, D', respectively, call a surjection $\phi : \mathsf{V} \to \mathsf{V}'$ a ***covering map*** if for every vertex $x \in \mathsf{V}$, the restriction $\phi : T(x) \to T\big(\phi(x)\big)$ is a network isomorphism, where $T(x)$ denotes the star at x, i.e., the network induced on the edges incident to x. (A network isomorphism is a graph isomorphism that preserves vertex and edge weights.) If there is such a covering map, then we call G a ***covering network*** of G'. For example, the nearest-neighbor graph on $\{-1, 0, 1\}$ with unit weights can be mapped to the edge between 0 and 1 by mapping -1 to 1. This provides a weak covering map, but not a covering map. The following result is due to Campanino (1985), but our proof is modelled on that of Benjamini and Schramm (1996b).

Theorem 6.24. (Covering and Percolation) *Suppose that ϕ is a weak covering map from G to G'. Then for any $x \in \mathsf{V}$ and $p \in (0, 1)$, we have*

$$\mathbf{P}_p[x \leftrightarrow \infty] \ge \mathbf{P}_p[\phi(x) \leftrightarrow \infty]\,.$$

Therefore $p_c(G) \le p_c(G')$.

Proof. We prove this for bond percolation, the proof for site percolation being almost identical. We will construct a coupling of the percolation measures on the two graphs. That is, given $\omega' \in 2^{\mathsf{E}'}$, we will define $\omega \in 2^{\mathsf{E}}$ in such a way that, first, if ω' has distribution $\mathbf{P}_p$ on G', then ω has distribution $\mathbf{P}_p$ on G; and, second, if $K\big(\phi(x)\big)$ is infinite, then so is $K(x)$.

Choose any ordering of $\mathsf{E}' = \langle e'_1, e'_2, \ldots \rangle$ so that $\phi(x)$ is an endpoint of e'_1 and so that for each $k > 1$, one endpoint of e'_k is also an endpoint of some e'_j with $j < k$.

Let e_1 be any edge that ϕ maps to e'_1. Define $\omega(e_1) := \omega'(e'_1)$ and set $n_1 := 1$. We will select a subsequence of edges $\langle e'_{n_j} \rangle$ via a recursive procedure. Suppose that $E'_k := \{e'_{n_1}, \ldots, e'_{n_k}\}$ have been selected and edges e_j that ϕ maps onto e'_{n_j} $(j \le k)$ have been chosen. Let n_{k+1} be the smallest index of an edge in $\mathsf{E}' \setminus E'_k$ that shares an endpoint with at least one of the open edges in E'_k, if any. If there are none, then stop; $K\big(\phi(x)\big)$ is finite and $\omega(e)$ for the remaining edges $e \in \mathsf{E}$ may be assigned independently in any order. Otherwise, if $e'_{n_{k+1}}$ is incident with e'_{n_j}, then let e_{k+1} be any edge that ϕ maps to $e'_{n_{k+1}}$ and that is incident with e_j; such an edge exists because ϕ is a weak covering map. Set $\omega(e_{k+1}) := \omega'(e'_{n_{k+1}})$.

If this procedure never ends, then $K\big(\phi(x)\big)$ is infinite; assigning $\omega(e)$ for any remaining edges $e \in \mathsf{E}$ independently in any order gives a fair sample of Bernoulli percolation on G that has $K(x)$ infinite. This proves the theorem. ◀

The following combinatorial consequence of satisfying an anchored isoperimetric inequality is quite useful. The result is from Chen and Peres (2004), but the idea of the proof is from Kesten (1982).

Proposition 6.25. *Let*

$$\mathcal{A}_n := \{K \subset \mathsf{V}(G)\,;\ o \in K,\ K \text{ is connected and finite},\ |\partial_{\mathsf{E}} K| = n\} \tag{6.20}$$

and

$$h_n := \inf\left\{\frac{|\partial_{\mathsf{E}} K|}{|K|}\,;\ K \in \mathcal{A}_n\right\}. \tag{6.21}$$

Then

$$|\mathcal{A}_n| \le [\Psi(h_n)]^n\,,$$

where $\Psi(\bullet)$ is the monotone decreasing function

$$\Psi(h) := (1+h)^{1+\frac{1}{h}}/h, \qquad \Psi(0) := \infty\,.$$

Proof. Consider Bernoulli(p) bond percolation in G. Let $K(o)$ be the open cluster containing o. For any $K \in \mathcal{A}_n$, we have $|\mathsf{E}(K)| \ge |K| - 1$ since a spanning tree on K has $|K| - 1$ edges; also, $|\partial_{\mathsf{E}} K| \ge h_n |K|$. Therefore,

$$\mathbf{P}[\mathsf{V}(K(o)) = K] \ge p^{|K|-1}(1-p)^{|\partial_{\mathsf{E}} K|} \ge p^{n/h_n}(1-p)^n\,,$$

whence

$$1 \ge \mathbf{P}[\mathsf{V}(K(o)) \in \mathcal{A}_n] = \sum_{K \in \mathcal{A}_n} \mathbf{P}[\mathsf{V}(K(o)) = K] \ge |\mathcal{A}_n| p^{n/h_n}(1-p)^n.$$

Thus,

$$|\mathcal{A}_n| \le \left(\frac{1}{p}\right)^{n/h_n}\left(\frac{1}{1-p}\right)^n$$

for every $p \in (0,1)$. Letting $p := 1/(1+h_n)$ concludes the proof. ◀

In the appendix to Chen and Peres (2004), G. Pete noted the following strengthening of the conclusion of Theorem 6.23:

Theorem 6.26. (Anchored Expansion of Clusters) *Consider Bernoulli(p) bond percolation on a graph G with $\Phi^*_{\mathsf{E}}(G) > 0$. If $p > 1/\big(1+\Phi^*_{\mathsf{E}}(G)\big)$, then almost surely on the event that the open cluster K containing o is infinite, it satisfies $\Phi^*_{\mathsf{E}}(K) > 0$. Likewise, for $p > 1/\big(1+\Phi^*_{\mathsf{V}}(G)\big)$, we have $\mathbf{P}_p\big[\Phi^*_{\mathsf{V}}(K) > 0 \bigm| |K| = \infty\big] = 1$.*

Proof. We prove only the first assertion, as the second is similar. Let $\mathcal{A}_n$ be as in (6.20). We will consider edge boundaries with respect to both K and G, so we denote them by ∂^K_{E} and ∂^G_{E}, respectively. Note that in Bernoulli(p) bond percolation, for any $0 < \alpha < p$ and $S \in \mathcal{A}_n$, we can estimate the conditional probability

$$\mathbf{P}\left[\frac{|\partial^K_{\mathsf{E}} S|}{|\partial^G_{\mathsf{E}} S|} \le \alpha \,\middle|\, S \subseteq K\right] = \mathbf{P}\big[\mathrm{Bin}\big(n,p\big) \le \alpha n\big] \le e^{-nI_p(\alpha)}\,, \tag{6.22}$$

where the large deviation rate function

$$I_p(\alpha) := \alpha \log \frac{\alpha}{p} + (1-\alpha)\log\frac{1-\alpha}{1-p}\,, \tag{6.23}$$

is continuous and $-\log(1-p) = I_p(0) > I_p(\alpha) > 0$ for $0 < \alpha < p$ (see Billingsley (1995), p. 151, or Dembo and Zeitouni (1998), Theorem 2.1.14). Therefore,

$$\begin{aligned}
\mathbf{P}\left[\exists S \in \mathcal{A}_n,\, S \subseteq K\,;\, \frac{|\partial^K_{\mathsf{E}} S|}{|\partial^G_{\mathsf{E}} S|} \le \alpha\right] &\le \sum_{S\in\mathcal{A}_n} \mathbf{P}\left[S \subseteq K,\, \frac{|\partial^K_{\mathsf{E}} S|}{|\partial^G_{\mathsf{E}} S|} \le \alpha\right] \\
&\le \sum_{S\in\mathcal{A}_n} e^{-nI_p(\alpha)}\mathbf{P}[S \subseteq K] \\
&= e^{n(I_p(0)-I_p(\alpha))} \sum_{S\in\mathcal{A}_n} (1-p)^n \mathbf{P}[S \subseteq K] \\
&= e^{n(I_p(0)-I_p(\alpha))} \sum_{S\in\mathcal{A}_n} \mathbf{P}[K = S] \\
&= e^{n(I_p(0)-I_p(\alpha))}\mathbf{P}\big[|K| < \infty,\ |\partial^G_{\mathsf{E}} K| = n\big]\,.
\end{aligned} \tag{6.24}$$

To estimate $\mathbf{P}\big[|K| < \infty,\ |\partial^G_{\mathsf{E}}(K)| = n\big]$ for $p > 1/\big(1+\Phi^*_{\mathsf{E}}(G)\big)$, recall the argument of Theorem 6.23. Choose $h < \Phi^*_{\mathsf{E}}(G)$ such that $p > 1/(1+h)$. Then there exists $n_h < \infty$ such that $|\partial^G_{\mathsf{E}} S|/|S| > h$ for all $S \in \mathcal{A}_n$ with $n > n_h$. We showed that

$$\big\{|K| < \infty,\ |\partial^G_{\mathsf{E}}(K)| = n\big\} \subset \bigcup_{N=n}^{\infty} B_N\,,$$

where

$$B_N := \left\{\sum_{j=1}^{N} Y_j \le \frac{N}{1+h}\right\}$$

and $\langle Y_j \rangle$ is an i.i.d. sequence of Bernoulli(p) random variables.

As above, $\mathbf{P}[B_N] \le e^{-N\delta_p}$ where $\delta_p := I_p\left(\frac{1}{1+h}\right) > 0$, since $p > 1/(1+h)$. Thus for some constant $C_p < \infty$,

$$\mathbf{P}\big[|K| < \infty,\ |\partial_{\mathsf{E}}^G(K)| = n\big] \le \sum_{N=n}^{\infty} e^{-N\delta_p} \le C_p e^{-n\delta_p}\,. \tag{6.25}$$

Taking $\alpha > 0$ in (6.24) so small that $I_p(0) - I_p(\alpha) < \delta_p$, we deduce that (6.24) is summable in n. By the Borel-Cantelli Lemma,

$$\lim_{n\to\infty} \inf\left\{\frac{|\partial_{\mathsf{E}}^K S|}{|\partial_{\mathsf{E}}^G S|}\,;\ o \in S \subset \mathsf{V}(K),\ S \text{ is connected},\ n \le |\partial_{\mathsf{E}}^G S|\right\} \ge \alpha \quad \text{a.s.},$$

whence

$$\Phi_{\mathsf{E}}^*(K) \ge \alpha \Phi_{\mathsf{E}}^*(G) > 0$$

almost surely on the event that K is infinite. ◀

The following possible extension is open:

Question 6.27. If $\Phi_{\mathsf{E}}^*(G) > 0$, does every infinite cluster K in a Bernoulli percolation satisfy $\Phi_{\mathsf{E}}^*(K) > 0$?

A converse is known, namely, that if G is a transitive amenable graph, then for every invariant percolation on G, a.s. each cluster has 0 anchored expansion constant; see Corollary 8.35.

Percolation is one way of randomly thinning a graph. Another way is to replace an edge by a random path of edges. What happens to expansion then? We will use the following notation.

Let G be an infinite graph of bounded degree and pick a probability distribution ν on the positive integers. Replace each edge $e \in \mathsf{E}(G)$ by a path of $L_e \ge 1$ edges, where the random variables $\{L_e\}_{e\in\mathsf{E}(G)}$ are independent with law ν. Let G^ν denote the random graph obtained in this way. We call G^ν a ***random subdivision*** of G. Say that ν has an ***exponential tail*** if for some $\epsilon > 0$ and all sufficiently large ℓ, we have $\nu[\ell, \infty) < e^{-\epsilon\ell}$. This is equivalent to the condition that if $X \sim \nu$, then $\mathbf{E}[s^X] < \infty$ for some $s > 1$.

▷ **Exercise 6.16.**

If the support of ν is unbounded, then $\Phi_{\mathsf{E}}(G^\nu) = 0$ a.s.

Define

$$\Phi^{**}_{\mathsf{E}}(G) := \lim_{n\to\infty} \inf\left\{ \frac{|\partial_{\mathsf{E}} K|}{|E(K)|} \,;\, o \in K \subset \mathsf{V},\, G(K) \text{ is connected, } n \le |K| < \infty \right\}.$$

Since $|E(K)| \ge |K| - 1$, we have

$$\Phi^{**}_{\mathsf{E}}(G) \le \Phi^{*}_{\mathsf{E}}(G)$$

for every graph G. Conversely, if the maximum degree of G is D, then

$$\Phi^{**}_{\mathsf{E}}(G) \ge \Phi^{*}_{\mathsf{E}}(G)/D$$

since $|E(K)| \le D|K|$. On the other hand, for trees G, we have $|E(K)| = |K| - 1$, so $\Phi^{**}_{\mathsf{E}}(G) = \Phi^{*}_{\mathsf{E}}(G)$.

Theorem 6.28. (Anchored Expansion and Subdivision) *Suppose that $\Phi^{**}_{\mathsf{E}}(G) > 0$. If ν has an exponential tail, then the random subdivision satisfies $\Phi^{**}_{\mathsf{E}}(G^{\nu}) > 0$ a.s. In particular, if G has bounded degree and $\Phi^{*}_{\mathsf{E}}(G) > 0$, then $\Phi^{*}_{\mathsf{E}}(G^{\nu}) > 0$ a.s.*

Proof. Let $L_1, L_2, \ldots$ be i.i.d. random variables with distribution ν. Since ν has an exponential tail, there is an increasing convex rate function $I(\bullet)$ such that $I(c) > 0$ for $c > \mathbf{E}L_i$ and $\mathbf{P}\big[\sum_{i=1}^{n} L_i > cn\big] \le \exp\big(-nI(c)\big)$ for all n (see Dembo and Zeitouni (1998), Theorem 2.2.3). Fix $h < \Phi^{*}_{\mathsf{E}}(G)$. Choose c large enough that $I(c) > \log \Psi(h)$. For $S \in \mathcal{A}_n$ (defined in (6.20)),

$$\mathbf{P}\left[\frac{\sum_{e\in E^*(S)} L_e}{|E^*(S)|} > c\right] \le \exp\big(-|E^*(S)|I(c)\big) \le \exp\big(-|\partial_{\mathsf{E}} S|I(c)\big)$$

since $\partial_{\mathsf{E}} S \subseteq E^*(S)$. Therefore for all n,

$$\mathbf{P}\left[\exists S \in \mathcal{A}_n \,;\, \frac{\sum_{e\in E^*(S)} L_e}{|E^*(S)|} > c\right] \le |\mathcal{A}_n| e^{-I(c)n},$$

which is summable since h_n (defined in (6.21)) is strictly larger than h for all large n and we can apply Proposition 6.25. By the Borel-Cantelli Lemma, with probability one, for any sequence of sets $\langle S_n \rangle$ such that $S_n \in \mathcal{A}_n$ for each n, we have

$$\limsup_{n\to\infty} \frac{\sum_{e\in E^*(S_n)} L_e}{|E^*(S_n)|} \le c \quad \text{a.s.}$$

Therefore

$$\lim_{n\to\infty} \inf\left\{ \frac{|\partial_{\mathsf{E}} S|}{\sum_{e\in E^*(S)} L_e} \,;\, o \in S \subset \mathsf{V}(G),\, S \text{ is connected, } n \le |\partial_{\mathsf{E}} S| \right\} \ge \frac{\Phi^{**}_{\mathsf{E}}(G)}{c(1+\Phi^{**}_{\mathsf{E}}(G))} \quad \text{a.s.}$$

since $E^*(S) = E(S) \cup \partial_{\mathsf{E}} S$.

Since G^ν is obtained from G by adding new vertices, $\mathsf{V}(G)$ can be embedded into $\mathsf{V}(G^\nu)$ as a subset. In particular, we can choose the same basepoint o in G^ν and in G. For S connected in G such that $o \in S \subset \mathsf{V}(G)$, there is a unique *maximal* connected $\widetilde{S} \subset \mathsf{V}(G^\nu)$ such that $\widetilde{S} \cap \mathsf{V}(G) = S$; it satisfies $|E(\widetilde{S})| \le \sum_{e \in E^*(S)} L_e$. In computing $\Phi^*_{\mathsf{E}}(G^\nu)$, it suffices to consider only such maximal $\widetilde{S}$'s, so we conclude that $\Phi^{**}_{\mathsf{E}}(G^\nu) \ge \Phi^{**}_{\mathsf{E}}(G)/\big(c(1+\Phi^{**}_{\mathsf{E}}(G))\big) > 0$. ◀

The exponential tail condition is necessary to ensure the positivity of $\Phi^*_{\mathsf{E}}(G^\nu)$; see Exercise 6.57.

Do Galton-Watson trees have anchored expansion? Clearly they do when the offspring distribution $\langle p_k \rangle$ satisfies $p_0 = p_1 = 0$. On the other hand, when $p_1 \in (0, 1)$, the tree can be obtained from a different Galton-Watson tree with $p_1 = 0$ by randomly subdividing the edges. This will allow us to establish the case when $p_0 = 0$, and another argument will cover the case $p_0 > 0$. This result is due to Chen and Peres (2004), but the original proof was incomplete.

Theorem 6.29. (Anchored Expansion of Galton-Watson Trees) *For a supercritical Galton-Watson tree T, given non-extinction we have $\Phi^*_{\mathsf{E}}(T) > 0$ a.s.*

Proof. Case (i): $p_0 = p_1 = 0$. For any finite $S \subset \mathsf{V}(T)$, we have

$$|S| \le |\partial_{\mathsf{E}} S|\Big(\frac{1}{2} + \frac{1}{2^2} + \cdots\Big) \le |\partial_{\mathsf{E}} S|\,.$$

So $\Phi^*_{\mathsf{E}}(T) \ge \Phi_{\mathsf{E}}(T) \ge 1$.

Case (ii): $p_0 = 0$, $p_1 > 0$. In this case, let x be the vertex closest to the root that has at least 2 children. Then T^x has the law of a random subdivision G^ν of another Galton-Watson tree G and T differs from T^x by a finite path. Here, G is generated according to the offspring distribution $\langle p'_k \rangle$, where $p'_k := p_k/(1-p_1)$ for $k = 2, 3, \ldots$ and $p'_0 := p'_1 := 0$ and ν is the geometric distribution with parameter $1 - p_1$. By Theorem 6.28, $\Phi^*_{\mathsf{E}}(T) = \Phi^*_{\mathsf{E}}(T^x) = \Phi^*_{\mathsf{E}}(G^\nu) > 0$ a.s.

Case (iii): $p_0 > 0$. Let $A(n, h)$ be the event that there is a subtree $S \subset T$ with n vertices that includes the root of T and with $|\partial_{\mathsf{E}} S| \le hn$.

We claim that

$$\mathbf{P}\big[A(n,h)\big] \le e^{nf(h)} \mathbf{P}\big[n \le |\mathsf{V}(T)| < \infty\big] \tag{6.26}$$

for some function $f : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfies $\lim_{h \to 0} f(h) = 0$. The idea is that the event $A(n, h)$ is "close" to the event that $\mathsf{V}(T)$ is finite but at least n. That is, it could have happened that the hn leaves of the growing Galton-Watson tree had no children after it already had n vertices, and for $T \in A(n,h)$, this alternative scenario isn't too unlikely compared to what actually happened.

For the proof, we can map any tree T in $A(n,h)$ to a finite tree $\phi(T)$ with at least n vertices as follows: Given $x \in \mathsf{V}(T)$, label its children from 1 to the number of children of x. Use this to place a canonical total order on all finite subtrees of T that include the root. (This can be done in a manner similar to the lexicographic order of finite strings.) Choose the first n-vertex S in this order such that the edge boundary of S in T has at most hn edges. Define $\phi(T)$ from T by retaining all edges in S and its edge boundary in T, while deleting all other edges. Note that for each vertex $x \in T$, the tree $\phi(T)$ contains either all children of x or none.

Any finite tree t with m vertices arises as $\phi(T)$ from at most $\sum_{k \le hm} \binom{m}{k}$ choices of S because for $n \le m$, there are at most $hn \le hm$ edges in $t \setminus S$, while for $n > m$, there are no choices of S. Now $\sum_{k \le hm} \binom{m}{k} \le \exp\{m f_1(h)\}$, where

$$f_1(\alpha) := -\alpha \log \alpha - (1-\alpha)\log(1-\alpha)$$

by (6.22). Given S and a possible tree t in the image of ϕ on $A(n,h)$, we have

$$\mathbf{P}[\phi(T) = t] \le p_0^{-hn} \mathbf{P}[T = t] .$$

Indeed, let $L(t)$ denote the leaves of t and $J(t) := \mathsf{V}(t) \setminus L(t)$. Let $d(x)$ be the number of children in t of $x \in J(t)$. Then

$$\mathbf{P}[\phi(T) = t] = \prod_{x \in J(t)} p_{d(x)} = p_0^{-|L(t)|} \mathbf{P}[T = t] \le p_0^{-hn} \mathbf{P}[T = t] .$$

Thus, if we let $f(\alpha) := f_1(\alpha) - \alpha \log p_0$, we obtain (6.26).

Now a supercritical Galton-Watson process conditioned on extinction is subcritical with mean $f'(q)$ by Proposition 5.23(ii). The total size of a subcritical Galton-Watson process decays exponentially by Exercise 5.26. Therefore, the last term of (6.26) decays exponentially in n. By choosing h small enough, we can ensure that also the left-hand side $\mathbf{P}\big[A(n,h)\big]$ also decays exponentially in n. ◀

§6.7. Euclidean Lattices.

Euclidean space is amenable, so does not satisfy the kind of strong isoperimetric inequality that we studied in the earlier sections of this chapter. However, it is the origin of isoperimetric inequalities, which continue to be useful today. The main result of this section is the following analogue of the classical isoperimetric inequality for balls in space.

Theorem 6.30. (Discrete Isoperimetric Inequality) *Let $A \subset \mathbb{Z}^d$ be a finite set, then*

$$|\partial_{\mathsf{E}} A| \geq 2d|A|^{\frac{d-1}{d}}\,.$$

Observe that the $2d$ constant in the inequality is the best possible as the example of the d-dimensional cube shows: If $A = [0, n)^d \cap \mathbb{Z}^d$, then $|A| = n^d$ and $|\partial_{\mathsf{E}} A| = 2dn^{d-1}$. The same inequality without the sharp constant follows from Theorem 6.20.

To prove this inequality, we will develop other very useful tools.

For every $1 \leq i \leq d$, define the projection $\mathcal{P}_i : \mathbb{Z}^d \to \mathbb{Z}^{d-1}$ simply as the function dropping the ith coordinate, i.e., $\mathcal{P}_i(x_1, \ldots, x_d) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$. Theorem 6.30 follows easily from the following beautiful inequality of Loomis and Whitney (1949).

Lemma 6.31. (Discrete Loomis and Whitney Inequality) *For any finite $A \subset \mathbb{Z}^d$,*

$$|A|^{d-1} \leq \prod_{i=1}^{d} |\mathcal{P}_i(A)|\,.$$

Before proving Lemma 6.31, we show how it gives our isoperimetric inequality.

Proof of Theorem 6.30. The important observation is that $|\partial_{\mathsf{E}} A| \geq 2\sum_{i=1}^{d} |\mathcal{P}_i A|$. To see this, observe that any vertex in $\mathcal{P}_i(A)$ matches to a straight line in the ith coordinate direction which intersects A. Thus, since A is finite, to any vertex in $\mathcal{P}_i(A)$, we can always match two distinct edges in $\partial_{\mathsf{E}} A$: the first and last edges on the straight line that intersects A. Using this and the arithmetic-geometric mean inequality, we get

$$|A|^{d-1} \leq \prod_{i=1}^{d} |\mathcal{P}_i(A)| \leq \left(\frac{1}{d}\sum_{i=1}^{d} |\mathcal{P}_i(A)|\right)^d \leq \left(\frac{|\partial_{\mathsf{E}} A|}{2d}\right)^d,$$

as required. ◀

To prove Lemma 6.31, we introduce the powerful notion of entropy. Let X be a random variable taking values $x_1, \ldots, x_n$. Denote $p(x) := \mathbf{P}[X = x]$, and define the ***entropy*** of X to be

$$H(X) := \sum_{i=1}^{n} p(x_i) \log \frac{1}{p(x_i)} = -\sum_{i=1}^{n} p(x_i) \log p(x_i)\,.$$

By concavity of the logarithm function and Jensen's inequality, we have

$$H(X) \le \log n\,. \tag{6.27}$$

Clearly this depends only on the law μ_X of X and it will be convenient to write this functional as $H[\mu_X]$.

Given another random variable, Y, define the ***conditional entropy*** $H(X \mid Y)$ ***of*** X ***given*** Y as

$$H(X \mid Y) := H(X,Y) - H(Y)\,.$$

The name "conditional entropy" comes from the following expression for it. Write $\mu_{X|Y}$ for the conditional distribution of X given Y; this is a random variable, a function of Y, with $\mathbf{E}[\mu_{X|Y}] = \mu_X$.

Proposition 6.32. *Given two discrete random variables X and Y on the same space, $H(X \mid Y) = \mathbf{E}\big[H\big[\mu_{X|Y}\big]\big]$.*

Proof. We have

$$\begin{aligned}\mathbf{E}\big[H\big[\mu_{X|Y}\big]\big] &= -\sum_y \mathbf{P}[Y=y]\sum_x \mathbf{P}[X=x \mid Y=y]\log \mathbf{P}[X=x \mid Y=y]\\ &= -\sum_{x,y}\mathbf{P}[X=x,Y=y]\big(\log\mathbf{P}[X=x,Y=y]-\log\mathbf{P}[Y=y]\big)\\ &= H(X,Y)-H(Y)\,.\end{aligned}$$ ◀

Corollary 6.33. (Shannon's Inequalities) *For any discrete random variables X, Y, and Z, we have*

$$0 \le H(X \mid Y) \le H(X)\,, \tag{6.28}$$

$$H(X,Y) \le H(X) + H(Y)\,, \tag{6.29}$$

and

$$H(X,Y \mid Z) \le H(X \mid Z) + H(Y \mid Z)\,. \tag{6.30}$$

Proof. Since entropy is nonnegative, so is conditional entropy by Proposition 6.32. Since the function $t \mapsto -t\log t$ is concave, so is the functional $\mu \mapsto H[\mu]$, whence Jensen's inequality gives

$$H(X \mid Y) = \mathbf{E}\big[H\big[\mu_{X|Y}\big]\big] \le H\big[\mathbf{E}\big[\mu_{X|Y}\big]\big] = H[\mu_X] = H(X)\,.$$

This proves (6.28). Combined with the definition of $H(X \mid Y)$, (6.28) gives (6.29). Because of Proposition 6.32, (6.29) gives (6.30). ◀

Our last step before proving Lemma 6.31 is the following inequality of Han (1978):

Theorem 6.34. (Han's Inequality) *For any discrete random variables $X_1, \ldots, X_k$, we have*

$$(k-1)H(X_1, ..., X_k) \le \sum_{i=1}^{k} H(X_1, ..., X_{i-1}, X_{i+1}, ..., X_k)\,.$$

Proof. Write X_i^* for the vector $(X_1, ..., X_{i-1}, X_{i+1}, ..., X_k)$. Then a telescoping sum and (6.28) give

$$\begin{aligned} H(X_1, ..., X_k) &= \sum_{i=1}^{k} H(X_i | X_1, ..., X_{i-1}) \\ &\ge \sum_{i=1}^{k} H(X_i | X_i^*) = \sum_{i=1}^{k} [H(X_1, ..., X_k) - H(X_i^*)]\,. \end{aligned}$$

Rearranging the terms gives Han's inequality. ◀

We now prove the Loomis-Whitney inequality.

Proof of Lemma 6.31. Take random variables $X_1, \ldots, X_d$ such that $(X_1, \ldots, X_d)$ is distributed uniformly on A. Clearly $H(X_1, \ldots, X_d) = \log |A|$, and by (6.27),

$$H(X_1, \ldots X_{i-1}, X_{i+1}, \ldots, X_d) \le \log |\mathcal{P}_i(A)|\,.$$

Now use Theorem 6.34 on $X_1, \ldots, X_d$ to find that

$$(d-1)\log |A| \le \sum_{i=1}^{d} \log |\mathcal{P}_i(A)|\,,$$

as required. ◀

The proof of Han's inequality, while short, leaves some mystery why the inequality is true. In fact, a more general and very beautiful inequality due to Shearer, discovered in the same year but not published until later by Chung, Graham, Frankl, and Shearer (1986), has a proof that shows not only why the inequality holds, but also why it must hold. We present this now. Shearer's inequality has many applications in combinatorics.

Corollary 6.35. *Given random variables $X_1, \ldots, X_k$ and $S \subseteq [1, k]$, write X_S for the random variable $\langle X_i\,;\ i \in S\rangle$. The function $S \mapsto H(X_S)$ is submodular.*

Proof. Given $S, T \subseteq [1, k]$, we wish to prove that

$$H(X_{S\cup T}) + H(X_{S\cap T}) \le H(X_S) + H(X_T)\,.$$

Subtracting $2H(X_{S\cap T})$ from both sides, we see that this is equivalent to

$$H(X_{S\cup T} \mid X_{S\cap T}) \le H(X_S \mid X_{S\cap T}) + H(X_T \mid X_{S\cap T})\,,$$

which is (6.30). ◀

Theorem 6.36. (Shearer's Inequality) *In the notation of Corollary 6.35, let $\mathscr{S}$ be a collection of subsets of $[1, k]$ such that each integer in $[1, k]$ appears in exactly r of the sets in $\mathscr{S}$. Then*

$$rH(X_1, \ldots, X_k) \le \sum_{S \in \mathscr{S}} H(X_S)\,.$$

Proof. Apply submodularity to the right-hand side by combining summands in pairs as much as possible and iteratively: When we apply submodularity to the right-hand side, we take a pair of index sets and replace them by their union and their intersection. We get a new sum that is smaller. It does not change the number of times that any element appears in the collection of sets. We repeat on any pair we wish. This won't change anything if one of the pair is a subset of the other, but it will otherwise. So we keep going until we can't change anything, that is, until each remaining pair has the property that one index set is contained in the other. Since each element appears exactly r times in $\mathscr{S}$, it follows that we are left with r copies of $[1, k]$ (and some copies of $\varnothing$, which may be ignored). ◀

▷ **Exercise 6.17.**

Prove the following generalization of the Loomis-Whitney inequality. Let $A \subset \mathbb{Z}^d$ be finite and $\mathscr{S}$ be a collection of subsets of $\{1, \ldots, d\}$ such that each integer in $[1, d]$ appears in exactly r of the sets in $\mathscr{S}$. Write $\mathcal{P}_S$ for the projection of $\mathbb{Z}^d \to \mathbb{Z}^S$ onto the coordinates in S. Then

$$|A|^r \le \prod_{S \in \mathscr{S}} |\mathcal{P}_S A|\,.$$

§6.8. Notes.

Kesten (1959a, 1959b) proved the qualitative statement that a countable group G is amenable iff some (or every) symmetric random walk with support generating G has spectral radius less than 1. Making this quantitative, as in Theorem 6.7, was accomplished by Cheeger (1970) in the continuous setting; he dealt with the bottom of the spectrum of the Laplacian, rather than the spectral radius, but this is equivalent: in the discrete case, the Laplacian is $I - P$. Cheeger's inequality states the following: Let M be a closed n-dimensional Riemannian manifold. Let $\lambda_1(M)$ denote the smallest positive eigenvalue of the Laplace-Beltrami operator on M. Let $h(M)$ be the infimum of $V_{n-1}(E)/\min\{V_n(A), V_n(B)\}$ when M is divided into two pieces A and B by an $(n-1)$-submanifold E and V_k denotes k-dimensional volume. Then

$$\lambda_1(M) \ge h(M)^2/4\,.$$

An inequality in the opposite direction was proved later by Buser (1982), whose showed that in this context, if the Ricci curvature of M is always at least $-(n-1)a^2$, then

$$\lambda_1(M) \le 2a(n-1)h(M) + 10h(M)^2 .$$

In the discrete case, the direction of Buser's inequality is the easy one. Cheeger's result was transferred to the discrete setting in various contexts of infinite graphs by Dodziuk (1984), Dodziuk and Kendall (1986), Varopoulos (1985a), Ancona (1988), Gerl (1988), Biggs, Mohar, and Shawe-Taylor (1988), and Kaimanovich (1992). Cheeger's method of proof is used in all of these. We have incorporated an improvement due to Mohar (1988). Similar inequalities were proved independently for finite graphs, again inspired by Cheeger (1970). The first results were by Alon and Milman (1985) and Alon (1986), and the final form was given by Jerrum and Sinclair (1989) and Lawler and Sokal (1988) independently.

Kesten (1959b) also showed that if G is a Cayley graph of degree d and $\rho(G) = 2\sqrt{d-1}/d$, then G is a tree (compare Exercise 6.9).

The fact proved in Section 6.4 that proper tessellations of the same type are isomorphic and transitive is folklore.

Benjamini, Lyons, and Schramm (1999) initiated a systematic study of the properties of a transitive graph G that are preserved for infinite percolation clusters.

The notion of anchored expansion was implicit in Thomassen (1992), and made explicit in Benjamini, Lyons, and Schramm (1999). The relevance of anchored expansion to random walks is exhibited by the following theorem of Virág (2000a), the first part of which was conjectured by Benjamini, Lyons, and Schramm (1999).

Theorem 6.37. *Let G be a bounded degree graph with $\Phi^*_{\mathsf{E}}(G) > 0$. For a vertex x, denote by $|x|$ the distance from x to the basepoint o in G. Then the simple random walk $\langle X_n \rangle$ in G, started at o, satisfies $\liminf_{n\to\infty} |X_n|/n > 0$ a.s., and there exists $C > 0$ such that $\mathbf{P}[X_n = o] \le \exp(-Cn^{1/3})$ for all $n \ge 1$.*

Note that this theorem, combined with Theorem 6.26, implies positive speed on the infinite clusters of Bernoulli(p) percolation on any G with $\Phi^*_{\mathsf{E}}(G) > 0$, provided $p > 1/(1 + \Phi^*_{\mathsf{E}}(G))$. This partially answers Question 6.27. Furthermore, in conjunction with Theorem 6.29, we get that the speed of simple random walk on supercritical Galton-Watson trees is positive, a result first proved in Lyons, Pemantle, and Peres (1995b); see Exercise 13.13, where a formula for the speed is given.

▷ **Exercise 6.18.**
Show that the bound of Virág (2000a) on the return probabilities is sharp by giving an example of a graph with anchored expansion that has $\mathbf{P}[X_n = o] \ge \exp(-Cn^{1/3})$ for some $C < \infty$. *Hint:* Take a Galton-Watson tree T with offspring distribution $p_1 = p_2 = 1/2$, rooted at o. Look for a long pipe (of length $n^{1/3}$) starting at level $n^{1/3}$ of T.

Loomis and Whitney (1949) proved an inequality analogous to Lemma 6.31 for bodies in $\mathbb{R}^d$. It implies our inequality by taking a cube in $\mathbb{R}^d$ centered at each point of A.

§6.9. Collected In-Text Exercises.

6.1. Show that $\Phi_{\mathsf{E}}(\mathbb{T}_{b+1}, \mathbf{1}, \mathbf{1}) = b - 1$ for all $b \ge 1$, where $\mathbb{T}_{b+1}$ is the regular tree of degree $b+1$.

6.2. Suppose that G is a graph such that for some $o \in \mathsf{V}$, we have subexponential growth of balls: $\liminf_{n\to\infty} |\{x \in \mathsf{V}\,;\ d(o,x) \le n\}|^{1/n} = 1$, where $d(\bullet,\bullet)$ denotes the graph distance in G. Show that G is vertex amenable.

6.3. Show that every Cayley graph of a finitely generated abelian group is amenable.

6.4. Suppose that G_1 and G_2 are roughly isometric graphs with bounded degrees and having both edge and vertex weights $\asymp \mathbf{1}$. Show that G_1 is amenable iff G_2 is.

6.5. Prove that $\|P\|_\pi \le 1$.

6.6. Show that

$$\|P\|_\pi = \sup\left\{\frac{|(Pf,f)_\pi|}{(f,f)_\pi}\,;\ f \in \mathbf{D}_{00}\setminus\{0\}\right\} = \sup\left\{\frac{(Pf,f)_\pi}{(f,f)_\pi}\,;\ f \in \mathbf{D}_{00}\setminus\{0\}\right\}.$$

6.7. Show that for any two vertices $x, y \in \mathsf{V}$, we have

$$\|P\|_\pi = \limsup_{n\to\infty} \sup_z \left(p_n(x,z)/\sqrt{\pi(z)}\right)^{1/n} = \limsup_{n\to\infty} p_n(x,y)^{1/n}.$$

Moreover,

$$\forall n\ \ p_n(x,y) \le \sqrt{\pi(y)/\pi(x)}\,\|P\|_\pi^n.$$

In probability theory, $\limsup_{n\to\infty} p_n(x,y)^{1/n}$ is referred to as the ***spectral radius*** of the Markov chain, denoted $\rho(G)$.

6.8. Show that for $f \in \mathbf{D}_{00}$, we have $d^*(c\,df) = \pi(f - Pf)$.

6.9. Show that for simple random walk on $\mathbb{T}_{b+1}$, we have $\rho(\mathbb{T}_{b+1}) = 2\sqrt{b}/(b+1)$.

6.10. Show that $\lambda_2 < 1$ iff the Markov chain is irreducible and that $\lambda_n > -1$ iff the Markov chain is aperiodic.

6.11. Show that

$$\lambda_2 = \max_{f \perp \mathbf{1}} \frac{\langle Pf, f\rangle}{\langle f, f\rangle}.$$

6.12. Let G be a network without loops and $A \subset \mathsf{V}(G)$. Let G' be the network obtained from G by identifying A to a single vertex a and removing any resulting loops. Show that

$$\psi(G', a, \pi(a) + t) = \psi(G, A, |A|_\pi + t) \ge \psi(G, |A|_\pi + t) \ge \psi(G, \pi(a) + t)$$

for all $t \ge 0$.

6.13. For any $p < 1$, every infinite cluster K of Bernoulli(p) percolation on any graph G of bounded degree has $\Phi_{\mathsf{E}}(K) = 0$ a.s.

6.14. For any transitive graph G, show that $\Phi_{\mathsf{E}}(G) > 0$ is equivalent to $\Phi^*_{\mathsf{E}}(G) > 0$.

6.15. Prove the first inequality of Theorem 6.23 as written with anchored expansion.

6.16. If the support of ν is unbounded, then $\Phi_{\mathsf{E}}(G^\nu) = 0$ a.s.

6.17. Prove the following generalization of the Loomis-Whitney inequality. Let $A \subset \mathbb{Z}^d$ be finite and $\mathscr{S}$ be a collection of subsets of $\{1, \ldots, d\}$ such that each integer in $[1, d]$ appears in exactly r of the sets in $\mathscr{S}$. Write $\mathcal{P}_S$ for the projection of $\mathbb{Z}^d \to \mathbb{Z}^S$ onto the coordinates in S. Then

$$|A|^r \le \prod_{S \in \mathscr{S}} |\mathcal{P}_S A| \,.$$

6.18. Show that the bound of Virág (2000a) on the return probabilities is sharp by giving an example of a graph with anchored expansion that has $\mathbf{P}[X_n = o] \ge \exp(-Cn^{1/3})$ for some $C < \infty$. *Hint:* Take a Galton-Watson tree T with offspring distribution $p_1 = p_2 = 1/2$, rooted at o. Look for a long pipe (of length $n^{1/3}$) starting at level $n^{1/3}$ of T.

§6.10. Additional Exercises.

6.19. If (G_1, c_1, D_1) and (G_2, c_2, D_2) are networks, consider the ***cartesian product*** graph $G_1 \times G_2 = (\mathsf{V}, \mathsf{E})$ defined by $\mathsf{V} := \mathsf{V}_1 \times \mathsf{V}_2$,

$$\mathsf{E} := \{\big((x_1, x_2), (y_1, y_2)\big) \,;\; (x_1 = y_1,\ (x_2, y_2) \in \mathsf{E}_2) \text{ or } ((x_1, y_1) \in \mathsf{E}_1,\ x_2 = y_2)\}$$

with the weights $D\big((x_1, x_2)\big) := D_1(x_1) D_2(x_2)$ on the vertices and

$$c\big([(x_1, x_2), (x_1, y_2)]\big) := D_1(x_1) c_2([x_2, y_2]) \quad \text{and} \quad c\big([(x_1, x_2), (y_1, x_2)]\big) := D_2(x_2) c_1([x_1, y_1])$$

on the edges. Show that with these weights, $\Phi_{\mathsf{E}}(G_1 \times G_2) = \Phi_{\mathsf{E}}(G_1) + \Phi_{\mathsf{E}}(G_2)$.

6.20. Use Theorem 6.1 and its proof to give another proof of Theorem 6.2.

6.21. Refine Theorem 6.2 to show that if K is any finite vertex set in a transitive infinite graph G, then $|\partial_{\mathsf{E}} K|/|K| \ge \Phi_{\mathsf{E}}(G) + 1/|K|$.

6.22. Show that if G is a transitive graph of degree d and the edge-expansion constant $\Phi_{\mathsf{E}}(G, \mathbf{1}, \mathbf{1}) = d - 2$, then G is a tree.

6.23. Show that if G is a finite transitive network, then the minimum of $|\partial_{\mathsf{E}} K|_c / |K|$ over all K of size at most $|\mathsf{V}|/2$ occurs only for $|K| > |\mathsf{V}|/4$.

6.24. Suppose that we had used the ***internal vertex boundary*** of sets K, defined as $\partial_{\mathsf{V}}^{\mathrm{int}} K := \{x \in K \,;\; \exists y \notin K \;\; y \sim x\}$, in place of the external vertex boundary, to define vertex amenability. Show that this would not change the set of networks that are vertex amenable.

6.25. Let G and $G^\dagger$ be plane dual graphs such that $G^\dagger$ has bounded degrees. Show that if G is amenable, then so is $G^\dagger$.

6.26. Show that every finitely generated subgroup of an amenable finitely generated group is itself amenable.

6.27. Use Theorem 6.3 and its proof to give another proof of Theorem 6.4.

6.28. Refine Theorem 6.4 to show that if K is any finite vertex set in a transitive infinite graph G, then $|\partial_{\mathsf{V}} K|/|K| \ge \Phi_{\mathsf{V}}(G) + 1/|K|$.

6.29. Let G be a transitive graph and b be a submodular function that is invariant under the automorphisms of G and is such that if K and K' are disjoint but adjacent, then strict inequality holds in (6.2). Show that there is no finite set K that minimizes $b(K)/|K|$.

6.30. Show that a transitive graph G is non-amenable iff there exists a function $f : \mathsf{V}(G) \to \mathsf{V}(G)$ such that $\sup_{x \in \mathsf{V}(G)} \mathrm{dist}_G(x, f(x)) < \infty$ and for all $x \in \mathsf{V}(G)$, the cardinality of $f^{-1}(x)$ is at least 2.

6.31. Consider a random walk on a graph with spectral radius ρ. Suppose that we introduce a delay so that each step goes nowhere with probability p_{delay}, and otherwise chooses a neighbor with the same distribution as before. Show that the new spectral radius equals $p_{\mathrm{delay}} + (1 - p_{\mathrm{delay}})\rho$.

6.32. Show that if G is a covering network of G', then $\Phi_{\mathsf{E}}(G) \ge \Phi_{\mathsf{E}}(G')$ and $\Phi_{\mathsf{V}}(G) \ge \Phi_{\mathsf{V}}(G')$. Also, $\rho(G) \le \rho(G')$.

6.33. Show that if G is a graph of maximum degree d, then the edge-expansion constant $\Phi_{\mathsf{E}}(G, \mathbf{1}, \mathbf{1}) \le d - 2$.

6.34. Let T be a tree and $\mathbb{T}_{b+1}$ be the regular tree of degree $b + 1$.

(a) Show that if the degree of each vertex in T is at least $b + 1$, then $\rho(T) \le \rho(\mathbb{T}_{b+1})$.

(b) Show that if for every r, the ball of radius r in $\mathbb{T}_{b+1}$ is isomorphic to some ball in T, then $\rho(T) \ge \rho(\mathbb{T}_{b+1})$.

6.35. Let G be a graph and H be a transitive graph. Which of the following extensions of Exercise 6.34 are valid?

(a) If each vertex in G is contained in a subgraph of G that is isomorphic to H, then $\rho(G) \le \rho(H)$.

(b) If for every r, the ball of radius r in H is isomorphic to some ball in G, then $\rho(G) \ge \rho(H)$.

6.36. Give another proof of (6.11) by using Theorem 6.1.

6.37. Show that for a network G with weights c, D, we have

$$\Phi_{\mathsf{E}}(G, c, D) = \inf \left\{ \frac{\|df\|_{\ell_1(c)}}{\|f\|_{\ell_1(D)}} \,;\ 0 < \|f\|_{\ell_1(D)} < \infty \right\} .$$

6.38. For a network (G, c, D) with $0 < |\mathsf{V}(G)|_D < \infty$, one of the alternative definitions of the ***expansion constant*** (also known, unfortunately, as the ***conductance***) is

$$\Phi_{c,D}(G) := \inf \left\{ \frac{|\partial_{\mathsf{E}} K|_c}{\min\{|K|_D, |\mathsf{V} \setminus K|_D\}} \,;\ K \subset \mathsf{V},\ 0 < |K|_D < |\mathsf{V}|_D \right\} .$$

Show that

$$\Phi_{c,D}(G) = \inf \left\{ \frac{\|df\|_{\ell_1(c)}}{\inf_{a \in \mathbb{R}} \|f - a\|_{\ell_1(D)}} \,;\ 0 < \|f\|_{\ell_1(D)} < \infty \right\} .$$

6.39. Let G be a network with spectral radius $\rho(G)$ and let A be a set of vertices in G. Show that for any $x \in \mathsf{V}$, we have $\mathbf{P}_x[X_n \in A] \le \rho(G)^n \sqrt{|A|_\pi / \pi(x)}$.

6.40. Suppose that G is a network with bounded π. Improve Proposition 6.6 to show, in the notation there, that

$$\liminf_{n \to \infty} |X_n|/n \ge -2 \log \rho(G) / \log b \qquad \text{a.s.}$$

6.41. Let (G, c) be a network. For a finite non-empty set of vertices A, let $\pi_A(\bullet) = \pi(\bullet)/\pi(A)$ be the normalized restriction of π to A. Write $\mathbf{P}_{\pi_A}$ for the network random walk $\langle X_n \rangle$ started at a point $x \in A$ with probability $\pi(x)/\pi(A)$. Show that (G, c) is non-amenable iff there is some function $f : \mathbb{N} \to [0, 1]$ that tends to 0 and that has the following property: For all finite A, we have $\mathbf{P}_{\pi_A}[X_n \in A] \le f(n)$.

6.42. Let (G, c) be a network with spectral radius $\rho < 1$. Let $v(\bullet)$ be the voltage function from a fixed vertex o to infinity. Show that $\sum_{x\in\mathsf{V}} \pi(x)v(x)^2 < \infty$.

6.43. Let (G, c) be a network with spectral radius $\rho < 1$. Let $A \subset \mathsf{V}$ be a nonempty set of states with $\pi(A) < \infty$ and let $\pi_A(\bullet) = \pi(\bullet)/\pi(A)$ be the normalized restriction of π to A. Show that when the chain is started according to π_A, the chance that it never returns to A is at least $1 - \rho$:

$$\mathbf{P}_{\pi_A}[X_n \text{ never returns to } A] \ge 1 - \rho\,.$$

Hint: Consider the function $f(x)$ defined as the chance that starting from x, the set A will ever be visited. Use Exercise 6.6.

6.44. Let G be the Cayley graph of a group Γ with respect to a finite generating set S. Without assuming that S is closed under inverses, let A be the associated averaging operator including the identity, i.e., $(Af)(x) := \Big(f(x) + \sum_{s\in S} f(xs)\Big)/(|S| + 1)$ for $f \in \ell^2(\Gamma)$. Show that $\|A\| < 1$ iff Γ is non-amenable.

6.45. We give an interpretation of the expansion constant Φ_S in terms of return times. We begin with some general results on return times.

(a) Suppose that $\langle X_n\rangle$ is a stationary ergodic sequence with values in some measurable space. Let the distribution of X_0 be μ. Fix a measurable set A of possible values and let $\tau_A^+ := \inf\{n \ge 1\,;\ X_n \in A\}$. Let the distribution of X_0 given that $X_0 \in A$ be μ_A. Show that the conditional distribution of $X_{\tau_A^+}$ given that $X_0 \in A$ is also μ_A. Show that $\mathbf{E}[\tau_A^+ \mid X_0 \in A] = 1/\mu(A)$ (this formula is known as the ***Kac lemma***).

(b) For a general irreducible positive recurrent Markov chain with stationary probability measure π, prove that for any set S of states, we have

$$\sum_{x\in S} \pi(x)\mathbf{E}_x[\tau_S^+] = 1$$

and

$$\sum_{x\in S}\sum_{y\in S^c} \pi(x)p(x,y)\mathbf{E}_y[\tau_S] = \pi(S^c)\,.$$

This shows that starting at the stationary measure conditioned on just having made a transition from S to S^c, the expected time to hit S again is $1/\Phi_{S^c}$. *Hint:* Write $\mathbf{E}_x[\tau_S^+] = 1 + \sum_y p(x,y)\mathbf{E}_y[\tau_S]$ and observe that in the last sum only $y \in S^c$ contribute.

6.46. Let $\langle X_n\rangle$ be a reversible Markov chain with a stationary probability distribution π on the state space V. Let A be a set of states with $|A| \ge 2$. Consider the chain $\langle Y_n\rangle$ induced on A, i.e., $\mathbf{P}[Y_0 = x] = \pi(x)/\pi(A)$ and $\mathbf{P}[Y_{n+1} = y \mid Y_n = x] = \mathbf{P}[X_{\tau_A^+} = y \mid X_0 = x]$. Show that the spectral gap for the chain $\langle Y_n\rangle$ is at least that for the chain $\langle X_n\rangle$.

6.47. Show that if G is a plane regular graph with regular dual, then $\Phi_{\mathsf{E}}(G)$ is either 0 or irrational.

6.48. Show that if G is a plane regular graph with non-amenable regular dual $G^\dagger$, then

$$\beta(G) + \beta(G^\dagger) > 1\,.$$

6.49. Let G be a plane regular graph with regular dual $G^\dagger$. Write K' for the set of vertices incident to the faces corresponding to K, for both $K \subset \mathsf{V}$ and for $K \subset \mathsf{V}^\dagger$. Likewise, let $\widehat{K}$ denote the faces inside the outermost cycle of $E(K')$. Let $K_0 \subset \mathsf{V}$ be an arbitrary finite connected set and recursively define $L_n := (\widehat{K}_n)' \subset \mathsf{V}^\dagger$ and $K_{n+1} := (\widehat{L}_n)' \subset \mathsf{V}$. Show that $|\partial_{\mathsf{E}} K_n|/|K_n| \to \Phi'_{\mathsf{E}}(G)$ and $|\partial_{\mathsf{E}} L_n|/|L_n| \to \Phi'_{\mathsf{E}}(G^\dagger)$.

6.50. The upper bound on effective resistance given in Lemma 6.19, while useful for applications to resistance to infinity as in Theorem 6.18, is very weak for finite networks. Prove the following better version and some consequences.

(a) Let a and z be two distinct vertices in a finite connected network G. Define

$$\psi(t) := \min\left\{|\partial_{\mathsf{E}} W|_c\,;\ a \in W,\ z \notin W,\ W \text{ is connected},\ t \le \min\left\{|W|_\pi, |\mathsf{V}(G)\setminus W|_\pi\right\}\right\}$$

when this set is non-empty and $\psi(t) := \infty$ otherwise. Define $s_0 := \pi(a)$ and $s_{k+1} := s_k + \psi(s_k)/2$ recursively for $k \ge 0$. Then

$$\mathscr{R}(a \leftrightarrow z) \le \sum_{k=0}^{\infty} \frac{4}{\psi(s_k)}\,.$$

(b) Let a and z be two distinct vertices in a finite connected network G with $c(e) \ge 1$ for all edges e. Show that

$$\mathscr{R}(a \leftrightarrow z) \le \frac{12}{\Phi_*} + 4\,,$$

where Φ_* is the expansion constant of Definition 6.10.

(c) Give another proof of the upper bound of Proposition 2.14 that uses (a).

6.51. Show that there is a function $C : (1,\infty)\times(0,\infty) \to (0,1)$ such that every graph G with the property that the cardinality of each of its balls of radius r lies in $[a^r/c, ca^r]$ satisfies

$$\frac{|\partial_{\mathsf{V}} K|}{|K|} \ge \frac{C(c,a)}{\log(1+|K|)}$$

for each finite non-empty $K \subset \mathsf{V}(G)$. You may wish to complete the following outline of its proof. Define $f(x,y) := a^{-d(x,y)}$, where the distance is measured in G. Fix K. Set $Z := \sum_{x\in K}\sum_{y\in\partial_{\mathsf{V}} K} f(x,y)$. Estimate Z in two ways, depending on the order of summation.

(a) Fix $x \in K$. Choose R so that $|B(x,R)| \ge 2|K|$ and let $W := B(x,R)\setminus K$. For $w \in W$, fix a geodesic path from x to w and let w' be the first vertex in $\partial_{\mathsf{V}} K$ on this path. Let $B := |\{(w,w')\,;\ w \in W\}|$. Show that $B \ge Ca^R$.

(b) Show that $B \le C' a^R \sum_{y\in\partial_{\mathsf{V}} K} f(x,y)$.

(c) Deduce that $Z \ge C|K|$.

(d) Fix $y \in \partial_{\mathsf{V}} K$. Show that $\sum_{x\in K} f(x,y) \le C'\log(1+|K|)$.

(e) Deduce that $Z \le C'|\partial_{\mathsf{V}} K|\log(1+|K|)$.

(f) Deduce the result.

(g) Find a tree with bounded degree and such that every ball of radius r has cardinality in $[2^{\lfloor r/2\rfloor}, 3\cdot 2^r]$, yet there are arbitrarily large finite subsets with only one boundary vertex.

(h) Show that if a tree satisfies $|\partial_{\mathsf{V}} K| \ge 3$ for every vertex set K of size at least m, where m is fixed, then the tree is non-amenable.

6.52. Let G be a Cayley graph of growth rate b. Show that a.s., every infinite cluster of Bernoulli(p) percolation on G is transient when $p > 1/b$.

6.53. Write out the proof of the second inequality of Theorem 6.23.

6.54. Show that $\Phi^*_{\mathsf{E}}(G) > 0$ implies that for p sufficiently close to 1, in Bernoulli(p) percolation the open cluster $K(o)$ of any vertex $o \in \mathsf{V}(G)$ satisfies $\mathbf{P}\big[|\mathsf{V}(K(o))| < \infty,\ |\partial_{\mathsf{E}}\mathsf{V}(K(o))| = n\big] < Cq^n$ for some $q < 1$ and $C < \infty$.

6.55. Suppose that G satisfies an ***anchored (at-least-) two-dimensional isoperimetric inequality***, i.e., $h_n > c/n$ for a fixed $c > 0$ and all n, where h_n is as in (6.21). Show that $|\mathcal{A}_n| \le e^{Cn \log n}$ for some $C < \infty$. Give an example of a graph G that satisfies an anchored two-dimensional isoperimetric inequality, has $p_c(G) = 1$, and $|\mathcal{A}_n| \ge e^{c_1 n \log n}$ for some $c_1 > 0$.

6.56. Write out the proof of the second inequality of Theorem 6.26.

6.57. Let G be a binary tree. Show that if ν has a tail that decays slower than exponentially, then $\Phi^*_{\mathsf{E}}(G) > 0$ yet $\Phi^*_{\mathsf{E}}(G^\nu) = 0$ a.s.

6.58. Show that equality holds in (6.27) iff X is uniform on n values.

6.59. Show that equality holds on the left in (6.28) iff X is a function of Y, whereas equality holds on the right iff X and Y are independent. Show that equality holds in (6.30) iff X and Y are independent given Z.

6.60. Use concavity of the entropy functional $H[\,\bullet\,]$ to prove (6.27).

6.61. Show that there exists a constant $C_d > 0$ such that if A is a subgraph of the box $\{0, \ldots, n-1\}^d$ and $|A| < \frac{n^d}{2}$, then

$$|\partial_{\mathsf{E}} A| \ge C_d |A|^{\frac{d-1}{d}} .$$

Here, $\partial_{\mathsf{E}} A$ refers to the edge boundary within the box, so this is not a special case of Theorem 6.30.

6.62. Consider a stationary Markov chain $\langle X_n \rangle$ on a finite state space V with stationary measure $\pi(x)$ and transition probabilities $p(x, y)$. Show that for $n \ge 0$, we have

$$H(X_0, \ldots, X_n) = H[\pi] - n \sum_{x,y \in \mathsf{V}} \pi(x) p(x, y) \log p(x, y) .$$

Chapter 7

Percolation on Transitive Graphs

In this chapter, all graphs are assumed to be locally finite without explicit mention. Many natural graphs look the same from every vertex. To make this notion precise, define an ***automorphism*** of a graph $G = (\mathsf{V}, \mathsf{E})$ to be a bijection $\phi : \mathsf{V} \to \mathsf{V}$ such that $[\phi(x), \phi(y)] \in \mathsf{E}$ iff $[x, y] \in \mathsf{E}$. We write $\mathrm{Aut}(G)$ for the group of automorphisms of G. If G has the property that for every pair of vertices x, y, there is an automorphism of G that takes x to y, then G is called ***(vertex-) transitive***. The most prominent examples of transitive graphs are Cayley graphs, defined in Section 3.4. Of course, these include the usual Euclidean lattices $\mathbb{Z}^d$, on which the classical theory of percolation has been built.

A slight generalization of transitive graphs is the class of ***quasi-transitive*** graphs, which are those that have only finitely many equivalence classes of vertices under the equivalence relation that makes two vertices equivalent if there is an automorphism that takes one vertex to the other. Most results concerning quasi-transitive graphs can be deduced from corresponding results for transitive graphs or can be deduced in a similar fashion but with some additional attention to details. In this context, the following construction is useful: Suppose that $\Gamma \subseteq \mathrm{Aut}(G)$ acts ***quasi-transitively***, i.e., V/Γ is finite. Let o be a vertex in G. Let r be such that every vertex in G is within distance r of some vertex in the orbit, Γo. Form the graph G' from the vertices Γo by joining two vertices by an edge if their distance in G is at most $2r + 1$. It is easy to see that G' is connected: if $f : \mathsf{V} \to \Gamma o$ is a map such that $d_G\big(x, f(x)\big) \le r$ for all x, then any path $x_0, x_1, \ldots, x_n$ in G between two vertices of Γo maps to a path $x_0, f(x_1), f(x_2), \ldots, f(x_{n-1}), x_n$ in G'. Also, restriction of the elements of Γ to G' yields a subgroup $\Gamma' \subseteq \mathrm{Aut}(G')$ that acts transitively on G', i.e., V' is a single orbit. We call G' a ***transitive representation*** of G.

Our purpose is not to develop the classical theory of percolation, for which Grimmett (1999) is an excellent source, but we will now briefly state some of the important facts from that theory that motivate some of the questions that we will treat. Recall from Section 6.6 the probability measures $\mathbf{P}_p^{\mathrm{site}}$ and $\mathbf{P}_p^{\mathrm{bond}}$ for the two product measures on 2^{V} and 2^{E} defining Bernoulli percolation and the associated critical probabilities $p_{\mathrm{c}}^{\mathrm{bond}}$ and $p_{\mathrm{c}}^{\mathrm{site}}$. As we said there, if we don't indicate whether the percolation is bond or site and both make sense in context, then results we state should be taken to apply to both types of percolation. It is known (and we will prove below) that for all $d \ge 2$, we have $0 < p_c(\mathbb{Z}^d) < 1$. It was conjectured about 1955 that $p_c^{\text{bond}}(\mathbb{Z}^2) = 1/2$, but this was not proved until Kesten (1980). See Figure 7.1 for an illustration. How many infinite clusters are there when $p \ge p_c$? We will see in Theorem 7.5 that for each p, this number is a random variable that is constant a.s. More precisely, Aizenman, Kesten, and Newman (1987) showed that there is only one infinite cluster when $p > p_c(\mathbb{Z}^d)$, and one of the central conjectures in the field is that there is no infinite cluster when $p = p_c(\mathbb{Z}^d)$ $(d \ge 2)$. This was proved for $d = 2$ by Harris (1960) and Kesten (1980) and for $d \ge 19$ (and for $d \ge 7$ when bonds between all pairs of vertices within distance L of each other are added, for some L) by Hara and Slade (1990, 1994).

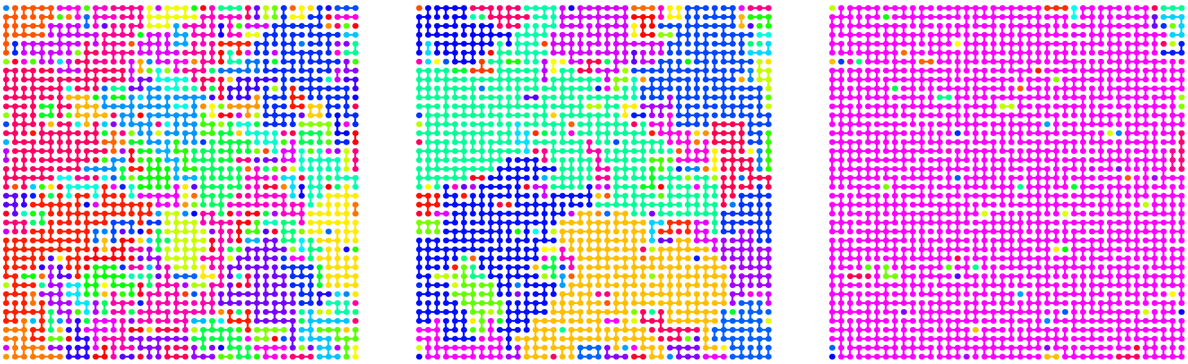


Figure 7.1. Bernoulli bond percolation on a 40×40 square grid graph at levels $p = 0.4, 0.5, 0.6$.

The conventional notation for $\mathbf{P}_p[x \text{ belongs to an infinite cluster}]$ is $\theta_x(p)$, not to be confused with the notation for a flow, used in other chapters. For a transitive graph, it is clear that this probability does not depend on x, so the subscript is usually omitted. For all $d \ge 2$, van den Berg and Keane (1984) showed that $\theta(p)$ is continuous for all $p \ne p_c$ and is continuous at p_c iff $\theta(p_c) = 0$. Thus, θ is a continuous function iff the conjecture above [that $\theta(p_c) = 0$] holds. Results that lend support to this conjecture are that for all $d \ge 2$,

$$\lim_{k\to\infty} p_c(\mathbb{Z}^2 \times [0,k]^{d-2}) = p_c(\mathbb{Z}^d) = p_c(\mathbb{Z}^{d-1} \times \mathbb{Z}^+)$$

(Grimmett and Marstrand, 1990) and $\theta(p_c) = 0$ on the graph $\mathbb{Z}^{d-1} \times \mathbb{Z}^+$ (Barsky, Grimmett, and Newman, 1991).

§7.1. Groups and Amenability.

In Section 3.4, we looked at some basic constructions of groups. Another useful but more complex construction is that of amalgamation. Suppose that $\Gamma_1 = \langle S_1 \mid R_1 \rangle$ and $\Gamma_2 = \langle S_2 \mid R_2 \rangle$ are groups that both have a subgroup isomorphic to Γ'. More precisely, suppose that $\phi_i : \Gamma' \to \Gamma_i$ are monomorphisms for $i = 1, 2$ and that $S_1 \cap S_2 = \varnothing$. Let $R := \{\phi_1(\gamma)\phi_2^{-1}(\gamma)\,;\ \gamma \in \Gamma'\}$. Then the group $\langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup R \rangle$ is called the ***amalgamation of*** Γ_1 ***and*** Γ_2 ***over*** Γ' and denoted $\Gamma_1 *_{\Gamma'} \Gamma_2$. The name and notation do not reflect the role of the maps ϕ_i, even though they are crucial. For example, $\mathbb{Z} *_{2\mathbb{Z}} \mathbb{Z} = \langle a, b \mid a^2 b^{-2} \rangle$ has a Cayley graph that, if its edges are not labelled, looks just like the usual square lattice; see Figure 7.2. However, $\mathbb{Z} *_{3\mathbb{Z}} \mathbb{Z} = \langle a, b \mid a^3 b^{-3} \rangle$ is quite different: it is non-amenable by Exercise 6.32, as it has the quotient $\langle a, b \mid a^3 b^{-3}, a^3, b^3 \rangle = \mathbb{Z}_3 * \mathbb{Z}_3$. Of course, both $2\mathbb{Z}$ and $3\mathbb{Z}$ are isomorphic to $\mathbb{Z}$; our notation evokes inclusion as the appropriate maps ϕ_i.

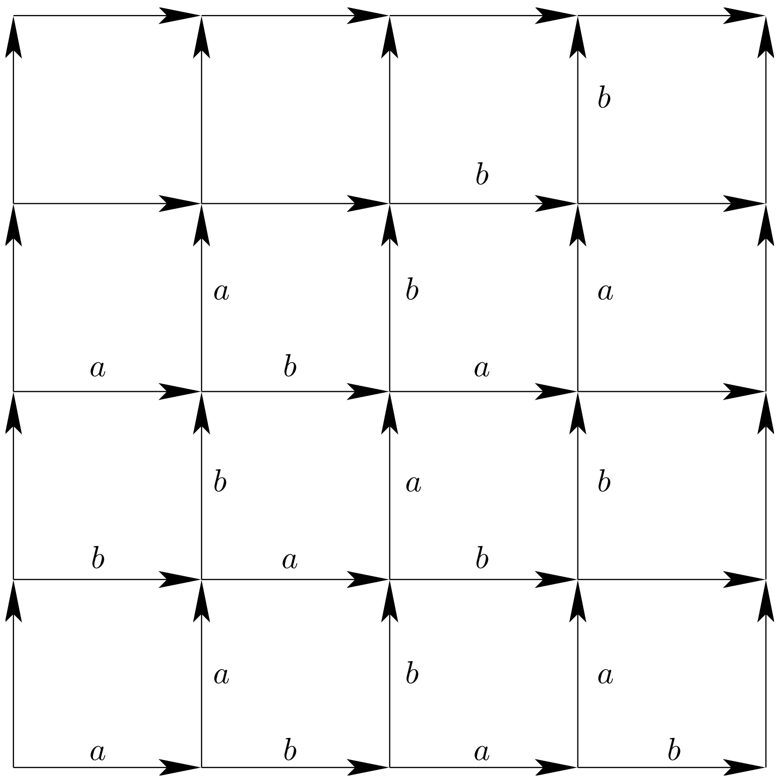


Figure 7.2. A portion of the edge-labelled Cayley graph of $\mathbb{Z} *_{2\mathbb{Z}} \mathbb{Z}$.

In this chapter, we will use $\Phi_\mathsf{E}(G)$ to mean always $\Phi_\mathsf{E}(G, \mathbf{1}, \mathbf{1})$-expansion constant and $\Phi_\mathsf{V}(G)$ to mean always $\Phi_\mathsf{V}(G, \mathbf{1})$. Since $|\partial_\mathsf{E} K| \ge |\partial_\mathsf{V} K|$, we have $\Phi_\mathsf{E}(G) \ge \Phi_\mathsf{V}(G)$. In the other direction, for any graph of bounded degree, if $\Phi_\mathsf{E}(G) > 0$, then also $\Phi_\mathsf{V}(G) > 0$. For a tree T, it is clear that $\Phi_\mathsf{E}(T) = \Phi_\mathsf{V}(T)$. See Section 6.4 for the calculation of $\Phi_\mathsf{E}(G)$

when G arises from certain hyperbolic tessellations; e.g., $\Phi_{\mathsf{E}}(G) = \sqrt{5}$ for the graph in Figure 2.3.

▷ **Exercise 7.1.**

Let T be a tree that has no vertices of degree 1. Let B be the set of vertices of degree at least 3. Let K be a finite nonempty subset of vertices of T. Show that $|\partial_{\mathsf{E}} K| \ge |K \cap B| + 2$.

We claim that for any graph G, the balls $B_G(o, n)$ in G about a fixed point o of radius n satisfy

$$\liminf_{n\to\infty} |B_G(o,n)|^{1/n} \ge 1 + \Phi_{\mathsf{V}}^*(G)\,. \tag{7.1}$$

Indeed, for every $c < \Phi_{\mathsf{V}}^*(G)$ and for all sufficiently large n, we have that $|\partial_{\mathsf{V}} B(o,n)| > c|B(o,n)|$, i.e., $|B(o,n+1)| > (1+c)|B(o,n)|$. This implies (7.1). In particular, if a group Γ is non-amenable, then all its Cayley graphs G have exponential growth.

It is not obvious that the converse fails, but various classes of counterexamples are known. One counterexample is the group G_1, also known as the ***lamplighter group***. It is defined as a wreath product, which is a special kind of semidirect product. First, the group $\sum_{x\in\mathbb{Z}} \mathbb{Z}_2$, the direct sum of copies of $\mathbb{Z}_2$ indexed by $\mathbb{Z}$, is the group of maps $\eta : \mathbb{Z} \to \mathbb{Z}_2$ with $\eta^{-1}\big(\{1\}\big)$ finite and with componentwise addition mod 2, which we denote $\oplus$: that is, $(\eta\oplus\eta')(j) := \eta(j)+\eta'(j) \pmod 2$. Let $\mathcal{S}$ be the left shift, $\mathcal{S}(\eta)(j) := \eta(j+1)$. Now define $G_1 := \mathbb{Z} \ltimes \sum_{x\in\mathbb{Z}} \mathbb{Z}_2$, which is the set $\mathbb{Z} \times \sum_{x\in\mathbb{Z}} \mathbb{Z}_2$ with the following group operation: for $m, m' \in \mathbb{Z}$ and $\eta, \eta' \in \sum_{x\in\mathbb{Z}} \mathbb{Z}_2$, the group operation is

$$(m,\eta)(m',\eta') := (m+m', \eta \oplus \mathcal{S}^{-m}\eta')\,.$$

We call an element $\eta \in \sum_{m\in\mathbb{Z}} \mathbb{Z}_2$ a ***configuration*** and call $\eta(k)$ the ***bit*** at k. We identify $\mathbb{Z}_2$ with $\{0,1\}$. The first component of an element $x = (m,\eta) \in G_1$ is called the ***position of the marker in the state*** x. One nice set of generators of G_1 is $\big\{(1,\mathbf{0}), (-1,\mathbf{0}), (0,\mathbf{1}_{\{0\}})\big\}$. The reason for the name of this group is that we may think of a street lamp at each integer with the configuration η representing which lights are on, namely, those where $\eta = 1$. We also may imagine a lamplighter at the position of the marker. The first two generators of G_1 correspond (for right multiplication) to the lamplighter taking a step either to the right or to the left (leaving the lights unchanged); the third generator corresponds to flipping the light at the position of the lamplighter. See Figure 7.3.

To see that G_1 has exponential growth, consider the subset T_n of group elements at distance n from the identity and that can be arrived at from the identity by the lamplighter never moving to the left. Since T_n is a disjoint union of $(1,\mathbf{0})T_{n-1}$ and $(0,\mathbf{1}_{\{0\}})(1,\mathbf{0})T_{n-2}$,

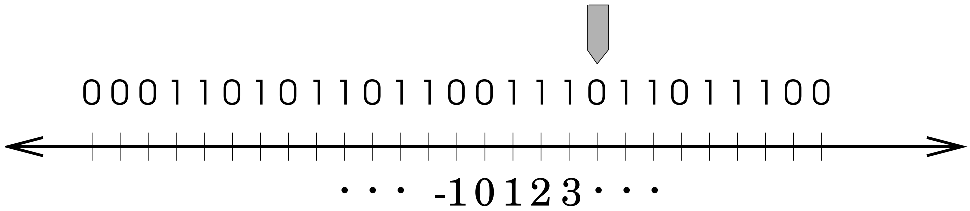


Figure 7.3. A typical element of G_1.

we have $|T_n| = |T_{n-1}| + |T_{n-2}|$. Thus, $\langle |T_n| \rangle$ is the sequence of Fibonacci numbers. Therefore the growth rate of $|T_n|$ equals the golden mean, $(1+\sqrt{5})/2$. In fact, it is easy to check that this is the growth rate of balls in G_1. On the other hand, to see that G_1 is amenable, consider the "boxes"

$$K_n := \big\{(m,\eta)\,;\ m \in [-n,n],\ \eta^{-1}\big(\{1\}\big) \subseteq [-n,n]\big\}\,.$$

Then $|K_n| = (2n+1)2^{2n+1}$, while $|\partial_{\mathsf{E}} K_n| = 2^{2n+2}$.

Other examples of non-amenable Cayley graphs arise from hyperbolic tessellations. In fact, many of these graphs are not Cayley graphs, but are still transitive graphs. The graph in Figure 2.3 is a Cayley graph; see Chaboud and Kenyon (1996) for an analysis of which regular tessellations are Cayley graphs. Another Cayley graph is shown in Figure 6.1.

Some other transitive graphs that are not Cayley graphs can be constructed as follows.

Example 7.1. (Grandparent Graph) Let ξ be a fixed end of a regular tree T of degree at least 3. Ends in graphs will be defined more generally in Section 7.3, but for trees, the definition is simpler. Namely, an ***end*** is an equivalence class of rays (i.e., infinite simple paths), where rays may start from any vertex and two rays are equivalent if they share all but finitely many vertices. Thus, given an end ξ, for every vertex x in T, there is a unique ray $x_\xi := \langle x_0 = x, x_1, x_2, \ldots \rangle$ starting at x such that x_ξ and y_ξ differ by only finitely many vertices for any pair x, y. Call x_2 the ***ξ-grandparent*** of x. Let G be the graph obtained from T by adding the edges $[x, x_2]$ between each x and its ξ-grandparent. Then G is a transitive graph that is not the Cayley graph of any group. A proof that this is not a Cayley graph is given in Section 8.2. To see that G is transitive, let x and y be two of its vertices. Let $\langle x_n\,;\ n \in \mathbb{Z} \rangle$ be a bi-infinite simple path in T that extends x_ξ to negative integer indices. There is some k such that $x_k \in y_\xi$. We claim that there is an automorphism of G that takes x to x_k; this implies transitivity since there is then also an automorphism that takes x_k to y. Indeed, the graph $G \setminus \{x_n\,;\ n \in \mathbb{Z}\}$ consists of components that are isomorphic, one for each $n \in \mathbb{Z}$. Seeing this, we also see that it is easy to shift along the path $\langle x_n\,;\ n \in \mathbb{Z} \rangle$ by any integer amount, in particular, by k, via an automorphism of G. This proves the claim. These examples were described by Trofimov (1985).

Example 7.2. (Diestel-Leader Graph) Let $T(1)$ be a 3-regular tree with edges oriented towards some distinguished end, and let $T(2)$ be a 4-regular tree with edges oriented towards some distinguished end. (An edge $\langle x, y \rangle$ is oriented towards an end in a tree if y is on the ray starting from x that belongs to that end.) Let V be the cartesian product of the vertices of $T(1)$ and the vertices of $T(2)$. Join $(x_1, x_2), (y_1, y_2) \in \mathsf{V}$ by an edge if $[x_1, y_1]$ is an edge of $T(1)$ and $[x_2, y_2]$ is an edge of $T(2)$, and precisely one of these edges goes against the orientation. The resulting graph G has infinitely many components, any two components being isomorphic. Each component is a transitive graph, but not a Cayley graph for the same (still-to-be-explained) reason as in Example 7.1. To see that each component of G is transitive, note that if ϕ_i is an automorphism of $T(i)$ that preserves its distinguished end, then $\phi_1 \times \phi_2 : (x_1, x_2) \mapsto \big(\phi_1(x_1), \phi_2(x_2)\big)$ is an automorphism of G. We saw in Example 7.1 that such automorphisms ϕ_i act transitively. Therefore, G is transitive, whence so is each of its components (and all components are isomorphic). One way to describe this graph is as a family graph: Suppose that there is an infinity of individuals, each of which has 2 parents and 3 children. The children are shared by the parents, as is the case in the real world. If an edge is drawn between each individual and his parent, then one obtains this graph for certain parenthood relations (with one component if each individual is related to every other individual). This graph is not a tree: If, say, John and Jane are both parents of Alice, Betty, and Carl, then one cycle in the family graph is from John to Alice to Jane to Betty to John. This example of a transitive graph was first discovered by Diestel and Leader (2001) for the purpose of providing a potential example of a transitive graph that is not roughly isometric to any Cayley graph. This question is due to Woess (1991); Eskin, Fisher, and Whyte (2009) showed that indeed the Diestel-Leader graph is not roughly isometric to any Cayley graph.

§7.2. Tolerance, Ergodicity, and Harris's Inequality.

This section is devoted to some quite general properties of the measures $\mathbf{P}_p$. Notation we will use throughout is that 2^A denotes the collection of subsets of A. It also denotes $\{0,1\}^A$, the set of functions from A to $\{0,1\}$. These spaces are identified by identifying a subset with its indicator function.

The first probabilistic property that we treat is that $\mathbf{P}_p$ is insertion tolerant, which means the following. Given a configuration $\omega \in 2^{\mathsf{E}}$ and an edge $e \in \mathsf{E}$, write

$$\Pi_e \omega := \omega \cup \{e\}\,,$$

where we regard ω as the subset of open edges. Extend this notation to events A by

$$\Pi_e A := \{\Pi_e \omega\,;\ \omega \in A\}\,.$$

Also, for any set F of edges, write

$$\Pi_F \omega := \omega \cup F$$

and extend to events as above. A bond percolation process $\mathbf{P}$ on G is ***insertion tolerant*** if $\mathbf{P}(\Pi_e A) > 0$ for every $e \in \mathsf{E}$ and every measurable $A \subset 2^{\mathsf{E}}$ satisfying $\mathbf{P}(A) > 0$. Here, we use the Borel σ-field generated by the product topology on 2^{E}. For Bernoulli(p) percolation, we have

$$\mathbf{P}_p(\Pi_e A) \ge p\mathbf{P}_p(A)\,. \tag{7.2}$$

▷ **Exercise 7.2.**
Prove (7.2).

Likewise, a bond percolation process $\mathbf{P}$ on G is ***deletion tolerant*** if $\mathbf{P}(\Pi_{\neg e} A) > 0$ whenever $e \in \mathsf{E}$ and $\mathbf{P}(A) > 0$, where $\Pi_{\neg e}\omega := \omega \setminus \{e\}$. We extend this notation to sets F by $\Pi_{\neg F}\omega := \omega \setminus F$. By symmetry, Bernoulli percolation is also deletion tolerant, with

$$\mathbf{P}_p(\Pi_{\neg e} A) \ge (1-p)\mathbf{P}_p(A)\,.$$

Similar definitions hold for site percolation processes.

▷ **Exercise 7.3.**
Prove that if $\theta_y(p) > 0$ for some y, then for any x, also $\theta_x(p) > 0$.

We will use insertion and deletion tolerance often. Another crucial pair of properties is invariance and ergodicity. Suppose that Γ is a group of automorphisms of a graph G. A measure $\mathbf{P}$ on 2^{E}, on 2^{V}, or on $2^{\mathsf{E}\cup\mathsf{V}}$ is called a Γ-***invariant*** percolation if $\mathbf{P}(\gamma A) = \mathbf{P}(A)$ for all $\gamma \in \Gamma$ and all events A; also, in the case of a measure on $2^{\mathsf{E}\cup\mathsf{V}}$, we assume that $\mathbf{P}$ is concentrated on subgraphs of G, i.e., whenever an edge lies in a configuration, so do both of its endpoints. Let $\mathscr{B}_\Gamma$ denote the σ-field of events that are invariant under all elements of Γ. The measure $\mathbf{P}$ is called Γ-***ergodic*** if for each $A \in \mathscr{B}_\Gamma$, we have either $\mathbf{P}(A) = 0$ or $\mathbf{P}(\neg A) = 0$. Bernoulli percolation on any Cayley graph is both invariant and ergodic with respect to translations. The invariance is obvious, while ergodicity is proved in the following proposition.

Proposition 7.3. (Ergodicity of Bernoulli Percolation) *If Γ acts on a connected locally finite graph G in such a way that each vertex has an infinite orbit, then $\mathbf{P}_p$ is ergodic.*

We note that if some vertex in G has an infinite Γ-orbit, then every vertex has an infinite Γ-orbit. Recall that we identify a subset $\omega \subseteq \mathsf{E}$ with its indicator function, so that $\omega(e)$ takes the value 0 or 1 depending on whether e lies in the subset or not.

Proof. Our notation will be for bond percolation. The proof for site percolation is identical. Let $A \in \mathscr{B}_\Gamma$ and $\epsilon > 0$. Because A is measurable, there is an event B that depends only on $\omega(e)$ for those e in some *finite* set F such that $\mathbf{P}_p(A \triangle B) < \epsilon$. For all $\gamma \in \Gamma$, we have $\mathbf{P}_p(\gamma A \triangle \gamma B) = \mathbf{P}_p[\gamma(A \triangle B)] < \epsilon$. By the assumption that vertices have infinite orbits, there is some γ such that F and γF are disjoint (because their graph distance can be made arbitrarily large). Since γB depends only on γF, it follows that B and γB are independent. Now for any events C_1, C_2, D, we have

$$|\mathbf{P}_p(C_1 \cap D) - \mathbf{P}_p(C_2 \cap D)| \le \mathbf{P}_p[(C_1 \cap D) \triangle (C_2 \cap D)] \le \mathbf{P}_p(C_1 \triangle C_2)\,.$$

Therefore,

$$\begin{aligned}
|\mathbf{P}_p(A) - \mathbf{P}_p(A)^2| &= |\mathbf{P}_p(A \cap \gamma A) - \mathbf{P}_p(A)^2| \\
&\le |\mathbf{P}_p(A \cap \gamma A) - \mathbf{P}_p(B \cap \gamma A)| + |\mathbf{P}_p(B \cap \gamma A) - \mathbf{P}_p(B \cap \gamma B)| \\
&\qquad + |\mathbf{P}_p(B \cap \gamma B) - \mathbf{P}_p(B)^2| + |\mathbf{P}_p(B)^2 - \mathbf{P}_p(A)^2| \\
&\le \mathbf{P}_p(A \triangle B) + \mathbf{P}_p(\gamma A \triangle \gamma B) + |\mathbf{P}_p(B)\mathbf{P}_p(\gamma B) - \mathbf{P}_p(B)^2| \\
&\qquad + |\mathbf{P}_p(B) - \mathbf{P}_p(A)|(\mathbf{P}_p(B) + \mathbf{P}_p(A)) \\
&< \epsilon + \epsilon + 0 + 2\epsilon\,.
\end{aligned}$$

It follows that $\mathbf{P}_p(A) \in \{0, 1\}$, as desired. ◀

Another property that is valid for Bernoulli percolation on any graph is an inequality due to Harris (1960), though it is nowadays usually called the FKG inequality due to an extension by Fortuin, Kasteleyn, and Ginibre (1971). We will hardly use it, but it is a crucial tool in proving other results in percolation theory.

Harris's inequality permits us to conclude such things as the positive correlation of the events $\{x \leftrightarrow y\}$ and $\{u \leftrightarrow v\}$ for any vertices x, y, u, and v. The special property that these two events have is that they are increasing, where an event $A \subseteq 2^{\mathsf{E}}$ is called ***increasing*** if whenever $\omega \in A$ and $\omega \subseteq \omega'$, then $\omega' \in A$. As a natural extension, we call a random variable X on 2^{E} ***increasing*** if $X(\omega) \le X(\omega')$ whenever $\omega \subseteq \omega'$. Thus, $\mathbf{1}_A$ is increasing iff A is increasing. Similar definitions apply for site processes.

Harris's Inequality. (a) *If A and B are both increasing events, then A and B are positively correlated:* $\mathbf{P}_p(AB) \ge \mathbf{P}_p(A)\mathbf{P}_p(B)$.

(b) *If X and Y are both increasing random variables with finite second moments, then* $\mathbf{E}_p[XY] \ge \mathbf{E}_p[X]\mathbf{E}_p[Y]$.

Property (a) of Harris's inequality says, by definition, that the measure $\mathbf{P}_p$ has ***positive associations*** (in contrast to the negative associations of the uniform spanning tree measure in Section 4.2).

▷ **Exercise 7.4.**

Suppose that X is an increasing random variable with finite expectation and $\mathscr{F}$ is the σ-field generated by the "coordinate functions" $e \mapsto \omega(e)$ ($\omega \in 2^{\mathsf{E}}$) for the edges e belonging to some finite subset of E. Show that there is a version of $\mathbf{E}_p[X \mid \mathscr{F}]$ that is an increasing random variable.

Harris's inequality is essentially the following lemma, where the case $d = 1$ is due to Chebyshev.

Lemma 7.4. *Suppose that $\mu_1, \ldots, \mu_d$ are probability measures on $\mathbb{R}$ and $\mu := \mu_1 \times \cdots \times \mu_d$. Let $f, g \in L^2(\mathbb{R}^d, \mu)$ be functions that are increasing in each coordinate. Then*

$$\int fg\,d\mu \ge \int f\,d\mu \int g\,d\mu\,.$$

Proof. We proceed by induction on d. Suppose first that $d = 1$. Note that

$$[f(x) - f(y)][g(x) - g(y)] \ge 0$$

for all $x, y \in \mathbb{R}$. Therefore

$$0 \le \iint [f(x) - f(y)][g(x) - g(y)]\,d\mu(x)\,d\mu(y) = 2\int fg\,d\mu - 2\int f\,d\mu \int g\,d\mu\,,$$

which gives the desired inequality.

Now suppose that the inequality holds for $d = k$ and let us prove the case $d = k + 1$. Define

$$f_1(x_1) := \int_{\mathbb{R}^k} f(x_1, x_2, \ldots, x_{k+1})\, d\mu_2(x_2) \cdots d\mu_{k+1}(x_{k+1})$$

and similarly define g_1. Clearly, f_1 and g_1 are increasing functions, whence

$$\int f_1 g_1\, d\mu_1 \geq \int f_1\, d\mu_1 \int g_1\, d\mu_1 = \int f\, d\mu \int g\, d\mu\,. \tag{7.3}$$

On the other hand, the inductive hypothesis tells us that for each fixed x_1,

$$f_1(x_1) g_1(x_1) \leq \int_{\mathbb{R}^k} f(x_1, x_2, \ldots, x_{k+1}) g(x_1, x_2, \ldots, x_{k+1})\, d\mu_2(x_2) \cdots d\mu_{k+1}(x_{k+1})\,,$$

whence $\int f_1 g_1\, d\mu_1 \leq \int fg\, d\mu$. In combination with (7.3), this proves the inequality for $d = k + 1$ and completes the proof. ◀

Proof of Harris's Inequality. The proofs for bond and site processes are identical; our notation will be for bond processes.

Since (a) derives from (b) by using indicator random variables, it suffices to prove (b). If X and Y depend on only finitely many edges, then the inequality is a consequence of Lemma 7.4 since $\omega(e)$ are mutually independent random variables for all e. To prove it in general, write $\mathsf{E} = \{e_1, e_2, \ldots\}$. Let X_n and Y_n be the expectations of X and Y conditioned on $\omega(e_1), \ldots, \omega(e_n)$. According to Exercise 7.4, the random variables X_n and Y_n are increasing, whence $\mathbf{E}_p[X_n Y_n] \geq \mathbf{E}_p[X_n]\mathbf{E}_p[Y_n]$. Since $X_n \to X$ and $Y_n \to Y$ in L^2 by the martingale convergence theorem, which implies that $X_n Y_n \to XY$ in L^1, we may deduce the desired inequality for X and Y. ◀

▷ **Exercise 7.5.**

Show that if $A_1, \ldots, A_n$ are increasing events, then for all p, $\mathbf{P}_p\big(\bigcap A_i\big) \geq \prod \mathbf{P}_p(A_i)$ and $\mathbf{P}_p\big(\bigcap A_i^c\big) \geq \prod \mathbf{P}_p(A_i^c)$.

▷ **Exercise 7.6.**

Use Harris's inequality to do Exercise 7.3.

§7.3. The Number of Infinite Clusters.

Newman and Schulman (1981) proved the following theorem:

Theorem 7.5. *If G is a transitive connected graph, then the number of infinite clusters is constant a.s. and equal to either 0, 1, or ∞. In fact, it suffices that vertices have infinite orbits under the automorphism group of G.*

Proof. Let N_∞ denote the number of infinite clusters and Γ denote the automorphism group of G. The action of any element of Γ on a configuration does not change N_∞. That is, N_∞ is measurable with respect to the σ-field $\mathscr{B}_\Gamma$ of events that are invariant under all elements of Γ. By ergodicity (Proposition 7.3), this means that N_∞ is constant a.s.

Now suppose that $\mathsf{N}_\infty \in [2, \infty)$ a.s. Then there must exist two vertices x and y that belong to distinct infinite clusters with positive probability. For bond percolation, let $e_1, e_2, \ldots, e_n$ be a path of edges from x to y. If A denotes the event that x and y belong to distinct infinite clusters and B denotes the event $\Pi_{\{e_1, e_2, \cdots, e_n\}} A$, then by insertion tolerance, we have $\mathbf{P}_p(B) > 0$. Yet N_∞ takes a strictly smaller value on B than on A, which contradicts the constancy of N_∞. (The proof for site percolation is parallel.) ◀

The result mentioned in the introduction that there is at most one infinite cluster for percolation on $\mathbb{Z}^d$, due to Aizenman, Kesten, and Newman (1987), was extended and simplified until the following result appeared, due to Burton and Keane (1989) and Gandolfi, Keane, and Newman (1992). Let $E_R(x)$ denote the set of edges that have at least one endpoint at distance at most $R - 1$ from x. This is the edge-interior of the ball of radius R about x. We will denote the component of x in a percolation configuration by $K(x)$. Call a vertex x a ***furcation*** of a configuration ω if closing all edges incident to x would split $K(x)$ into at least 3 infinite clusters*.

Theorem 7.6. *If G is a connected transitive amenable graph, then $\mathbf{P}_p$-a.s. there is at most one infinite cluster.*

A major open conjecture is that the converse of Theorem 7.6 holds; see Conjecture 7.28.

Proof. We will do the proof for bond percolation; the case of site percolation is exactly analogous. Let $0 < p < 1$. Let $\Lambda = \Lambda(\omega)$ denote the set of furcations of ω. We claim that if there is more than one infinite cluster with positive probability, then there are sufficiently many furcations as to create tree-like structures that force G to be non-amenable. To be

---

* In Burton and Keane (1989), these vertices were called ***encounter points***. When $K(x)$ is split into exactly 3 infinite clusters, then x is called a ***trifurcation*** by Grimmett (1999).

precise, we claim first that

$$c := \mathbf{P}_p[o \in \Lambda] > 0$$

and, second, that for each finite set $K \subset \mathsf{V}$,

$$|\partial_{\mathsf{E}} K| \ge c|K|\,. \tag{7.4}$$

These two claims together imply that G is non-amenable.

Now our assumption that there is more than one infinite cluster with positive probability implies that there are, in fact, infinitely many infinite clusters a.s. by Theorem 7.5. Choose some $R > 0$ such that the ball of radius R about o intersects at least 3 infinite clusters with positive probability. When this event occurs and then $E_R(o)$ is closed, this event still occurs. Thus, we may choose x, y, z at the same distance, R, from o such that $\mathbf{P}_p(A) > 0$, where A is the event that x, y, z belong to distinct infinite clusters even when $E_R(o)$ is closed. Join o to x by a path $\mathcal{P}_x$ of length R. There is a path $\mathcal{P}_y$ of length R that joins o to y and such that $\mathcal{P}_x \cup \mathcal{P}_y$ does not contain a cycle. Finally, there is also a path $\mathcal{P}_z$ of length R that joins o to z and such that $T := \mathcal{P}_x \cup \mathcal{P}_y \cup \mathcal{P}_z$ does not contain a cycle. Necessarily there is a vertex u such that x, y, z are in distinct components of $T \setminus \{u\}$. By insertion and deletion tolerance, $\mathbf{P}_p(A') > 0$, where $A' := \Pi_T \Pi_{\neg E_R(o)} A$. Furthermore, u is a furcation on the event A'. Hence $c = \mathbf{P}_p[o \in \Lambda] = \mathbf{P}_p[u \in \Lambda] \ge \mathbf{P}_p(A') > 0$.

It follows that for every finite $K \subset \mathsf{V}$,

$$\mathbf{E}_p\big[|K \cap \Lambda|\big] = \sum_{x \in K} \mathbf{P}_p[x \in \Lambda] = c|K|\,. \tag{7.5}$$

We next claim that

$$|\partial_{\mathsf{E}} K| \ge |K \cap \Lambda|\,. \tag{7.6}$$

Taking the expectation of (7.6) and using (7.5) shows (7.4).

To see why (7.6) is true, let T be a spanning tree of an infinite component η of ω. Remove all vertices of degree 1 from T and iterate until there are no more vertices of degree 1; call the resulting tree $T(\eta)$. Note that every furcation in η has degree ≥ 3 in $T(\eta)$. Apply Exercise 7.1 to conclude that

$$\big|(\partial_{\mathsf{E}} K) \cap T(\eta)\big| = \big|\partial_{\mathsf{E}(T(\eta))}\big(K \cap T(\eta)\big)\big| \ge \big|K \cap \Lambda \cap T(\eta)\big| = \big|K \cap \Lambda \cap \eta\big|\,.$$

If we sum this over all possible η, we arrive at (7.6). ◀

There is an extension of Theorem 7.6 to invariant insertion-tolerant percolation that was proved in the original papers. Since this theorem is so important, we use the opportunity to prove the extension by a technique from BLPS (1999b). This technique will itself be important in Chapter 8. The method uses the important notion of ends of graphs. Let $G = (\mathsf{V}, \mathsf{E})$ be any graph. The ***number of ends*** of G is defined to be the supremum of the number of infinite components of $G \setminus K$ over all finite subsets K of G. In particular, G has no ends iff G is finite and only 1 end iff the complement of each finite set in G has exactly 1 infinite component. This definition suffices for our purposes, but nevertheless, what is an end? To define an end, we make two preliminary definitions. First, an infinite set of vertices $V_0 \subset \mathsf{V}$ is ***end convergent*** if for every finite $K \subset \mathsf{V}$, there is a component of $G \setminus K$ that contains all but finitely many vertices of V_0. Second, two end-convergent sets V_0, V_1 are ***equivalent*** if $V_0 \cup V_1$ is end convergent. Now, an ***end*** of G is an equivalence class of end-convergent sets. For example, $\mathbb{Z}$ has two ends, which we could call $+\infty$ and $-\infty$. For a tree, the ends are in natural one-to-one correspondence with the boundary.

▷ **Exercise 7.7.**

Show that for any finitely generated group, the number of ends is the same for all of its Cayley graphs. In fact, if two graphs are roughly isometric, then they have the same number of ends.

▷ **Exercise 7.8.**

Show that if G and G' are any two infinite connected graphs, then the cartesian product graph $G \times G'$ (defined in Exercise 6.19) has only one end.

▷ **Exercise 7.9.**

Show that if Γ and Γ' are any two finitely generated groups with $|\Gamma| \ge 2$ and $|\Gamma'| \ge 3$, then $\Gamma * \Gamma'$ has infinitely many ends.

Lemma 7.7. *Let* $\mathbf{P}$ *be a* Γ*-invariant percolation process on a graph* G*. If with positive probability there is a component of* ω *with at least three ends, then (on a larger probability space) there is a random forest* $\mathfrak{F} \subset \omega$ *such that the distribution of the pair* $(\mathfrak{F}, \omega)$ *is* Γ*-invariant and with positive probability, there is a component of* $\mathfrak{F}$ *that has at least three ends.*

Proof. Assign independent uniform $[0, 1]$ random variables to the edges (independently of ω) to define the free minimal spanning forest $\mathfrak{F}$ of ω. By definition, this means that an edge $e \in \omega$ is present in $\mathfrak{F}$ iff there is no cycle in ω containing e in which e is assigned the maximum value. There is no cycle in $\mathfrak{F}$ since the edge with the largest label in any cycle could not belong to $\mathfrak{F}$. (Minimal spanning forests will be studied in greater depth in Chapter 11.)

Suppose that $K(x)$ has at least 3 ends with positive probability. Choose any finite tree T containing x so that with positive probability, $T \subset K(x)$ and $K(x) \setminus \mathsf{V}(T)$ has at least 3 infinite components. Then with positive probability, all of the following 4 events occur: (1) $T \subset K(x)$; (2) $K(x) \setminus \mathsf{V}(T)$ has at least 3 infinite components; (3) all edges in T are assigned values less than $1/2$; and (4) all edges incident to $\mathsf{V}(T)$ but not in T are assigned values greater than $1/2$. On this event, $\mathfrak{F}$ contains T and T is part of a spanning tree in $\mathfrak{F}$ with at least 3 ends. ◀

Theorem 7.8. *If Γ acts transitively on a connected amenable graph G and $\mathbf{P}$ is a Γ-invariant insertion-tolerant percolation process on G, then $\mathbf{P}$-a.s. there is at most one infinite cluster.*

Proof. It suffices to prove the case of bond percolation, since graphs induced by the components of site percolation form a bond percolation process. Let $\mathfrak{F}$ be as in Lemma 7.7. Since $\mathfrak{F}$ has a component with at least 3 ends with positive probability, there is a furcation of $\mathfrak{F}$ with positive probability. This shows (7.5) for some $c > 0$ with Λ equal to the furcations of $\mathfrak{F}$. Now apply the rest of the proof of Theorem 7.6 to $\mathfrak{F}$ to deduce the result. ◀

§7.4. Inequalities for p_c.

We have already given two inequalities for p_c in Section 6.6. Here we give several more.

Sometimes it is useful to compare site with bond percolation:

Proposition 7.9. (Hammersley, 1961a) *For any graph G, any $p \in (0,1)$, and any vertex $o \in G$, we have $\theta_o^{\mathrm{site}}(p) \le \theta_o^{\mathrm{bond}}(p)$. Therefore $p_c^{\mathrm{site}}(G) \ge p_c^{\mathrm{bond}}(G)$.*

Proof. We will construct a coupling of the two percolation measures. That is, given $\xi \in 2^{\mathsf{V}}$, we will define $\omega \in 2^{\mathsf{E}}$ in such a way that, first, if ξ has distribution $\mathbf{P}_p^{\mathrm{site}}$ on G, then ω has distribution $\mathbf{P}_p^{\mathrm{bond}}$ on G; and, second, if $K_\xi(o)$ is infinite, then so is $K_\omega(o)$, where the subscript indicates which configuration determines the cluster of o.

Choose any ordering of $\mathsf{V} = \langle x_1, x_2, \ldots \rangle$ with $x_1 = o$. Let $\langle Y_e \rangle_{e \in \mathsf{E}}$ be $\{0,1\}$-valued Bernoulli(p) random variables.

We will look at a finite or infinite subsequence of vertices $\langle x_{n_j} \rangle$ via a recursive procedure. If $\xi(o) = 0$, then stop. Otherwise, let $V_1 := \{o\}$, $W_1 := \varnothing$, and set $n_1 := 1$.

Suppose that V_k and W_k have been selected. Let n_{k+1} be the smallest index of a vertex in $\mathsf{V} \setminus (V_k \cup W_k)$ that neighbors some vertex in V_k, if any. In this case, let x'_{k+1} be the vertex in V_k that neighbors $x_{n_{k+1}}$ and that has smallest index, and set $\omega([x'_{k+1}, x_{n_{k+1}}]) := \xi(x_{n_{k+1}})$. Also, put $V_{k+1} := V_k \cup \{x_{n_{k+1}}\}$ if $\xi(x_{n_{k+1}}) = 1$ and $W_{k+1} := W_k$, and otherwise put $W_{k+1} := W_k \cup \{x_{n_{k+1}}\}$ and $V_{k+1} := V_k$. When n_{k+1} is not defined, stop; $K_\xi(o)$ is finite and we set $\omega(e) := Y_e$ for the remaining edges $e \in \mathsf{E}$ for which we have not yet specified $\omega(e)$.

If this procedure never ends, then both $K_\xi(o)$ and $K_\omega(o)$ are infinite; assigning $\omega(e) := Y_e$ for any remaining edges $e \in \mathsf{E}$ gives a fair sample of Bernoulli(p) bond percolation on G when $\xi \sim \mathbf{P}_p^{\text{site}}$. This gives the desired inequality. ◀

In the preceding proof, we constructed a certain coupling or two percolation measures. Another kind of coupling that is important is the following. Given two percolation measures $\mathbf{P}$ and $\mathbf{P}'$ on G, we say that $\mathbf{P}$ ***stochastically dominates*** $\mathbf{P}'$, written $\mathbf{P} \succcurlyeq \mathbf{P}'$, if there are random variables ω and ω' with laws $\mathbf{P}$ and $\mathbf{P}'$, respectively, such that $\omega \geq \omega'$ a.s.* We now use this to prove inequalities in the other direction to those in Proposition 7.9.

Proposition 7.10. (Grimmett and Stacey, 1998) *For any graph G of maximal degree d, any $p \in (0,1)$, and any vertex $o \in G$ of degree d_o, we have*

$$\theta_o^{\text{site}}\big(1-(1-p)^{d-1}\big) \geq \big[1-(1-p)^{d_o}\big]\theta_o^{\text{bond}}(p)\,.$$

Therefore

$$p_c^{\text{site}}(G) \leq 1-\big(1-p_c^{\text{bond}}(G)\big)^{d-1}\,.$$

Proof. We will again construct a coupling of the two percolation measures. This time, given $\omega \in 2^{\mathsf{E}}$, we will define $\xi \in 2^{\mathsf{V}}$ in such a way that, first, if ω has distribution $\mathbf{P}_p^{\text{bond}}$ on G, then ξ has a distribution that is stochastically dominated by $\mathbf{P}_q^{\text{site}}$ on G conditioned on $\xi(o) = 1$, where $q := 1-(1-p)^{d-1}$; and, second, if $K_\omega(o)$ is infinite, then so is $K_\xi(o)$.

Choose any ordering of $\mathsf{V} = \langle x_1, x_2, \ldots \rangle$ with $x_1 = o$. Let $\langle Y_x \rangle_{x \in \mathsf{V}}$ be $\{0,1\}$-valued Bernoulli(q) random variables.

We will look at a finite or infinite subsequence of vertices $\langle x_{n_j} \rangle$ via a recursive procedure. If $\omega(e) = 0$ for all edges e incident to o, then stop. Otherwise, let $V_1 := \{o\}$, set $\xi(o) = 1$, and set $n_1 := 1$. Note that the probability that some edge incident to o is open is $1-(1-p)^{d_o}$. Also set $W_1 := \varnothing$.

Suppose that V_k and W_k have been selected. Let n_{k+1} be the smallest index of a vertex in $\mathsf{V} \setminus (V_k \cup W_k)$ that neighbors some vertex in V_k, if any. Define $\xi(x_{n_{k+1}})$ to be the

* See Section 10.2 for more on this concept.

indicator that there is some vertex x *not* in $V_k \cup W_k$ for which $\omega([x_{n_{k+1}}, x]) = 1$. Note that the conditional probability that $\xi(x_{n_{k+1}}) = 1$ is $1 - (1-p)^r \le q$, where r is the degree of $x_{n_{k+1}}$ in $G \setminus (V_k \cup W_k)$. Put $V_{k+1} := V_k \cup \{x_{n_{k+1}}\}$ if $\xi(x_{n_{k+1}}) = 1$ and $W_{k+1} := W_k$, and otherwise put $W_{k+1} := W_k \cup \{x_{n_{k+1}}\}$ and $V_{k+1} := V_k$. When n_{k+1} is not defined, stop; $K_\omega(o)$ is finite and we set $\xi(x) := Y_x$ for the remaining vertices $x \in \mathsf{V} \setminus V_k$.

If this procedure never ends, then $K_\xi(o)$ (though perhaps not $K_\omega(o)$) is infinite; assigning $\xi(x) := Y_x$ for any remaining vertices $x \in \mathsf{V}$ gives a law of ξ that is dominated by Bernoulli(q) site percolation on G conditioned to have $\xi(o) = 1$. This gives the desired inequality. ◀

In Example 3.6, we described a tree formed from the self-avoiding walks in $\mathbb{Z}^2$. If G is any graph, not necessarily transitive, we may again form the tree T^{SAW} of self-avoiding walks of G, where all walks begin at some fixed base point o. Its lower growth rate, $\mu(G) := \underline{\mathrm{gr}}\, T^{\mathrm{SAW}}$, is called the ***connective constant*** of G. This does not depend on choice of o, although that will not matter for us. If G is transitive, then T^{SAW} is 0-subperiodic, whence $\mu(G) = \mathrm{br}\, T^{\mathrm{SAW}}$ in such a case by Theorem 3.8.

Proposition 7.11. *For any connected graph G, we have $p_c(G) \ge 1/\mu(G)$. In particular, if G has bounded degree, then $p_c(G) > 0$.*

Proof. In view of Proposition 7.9, it suffices to prove the inequality for bond percolation.

Write $K_n(o)$ for the self-avoiding walks of length n within $K(o)$. Thus, $K_n(o) \subseteq T_n^{\mathrm{SAW}}$. Suppose $K(o)$ is infinite. Then for each n, we have $K_n(o) \ne \varnothing$. Since $K(o)$ is infinite with positive probability whenever $p > p_c^{\mathrm{bond}}$, it follows that $\inf_n \mathbf{P}_p[K_n(o) \ne \varnothing] > 0$ for $p > p_c^{\mathrm{bond}}$. Now $\mathbf{P}_p[K_n(o) \ne \varnothing] \le \mathbf{E}_p\big[|K_n(o)|\big] = |T_n^{\mathrm{SAW}}| p^n$. Thus, $p \ge \limsup_{n\to\infty} |T_n^{\mathrm{SAW}}|^{-1/n} = 1/\mu(G)$ for $p > p_c^{\mathrm{bond}}$. ◀

This lower bound improves on the one that follows from Theorem 6.24.

Upper bounds for p_c are more difficult. In fact, we have seen examples of graphs (trees) that have exponential growth, bounded degree, and $p_c = 1$. The transitive case is better behaved according to the following conjecture:

Conjecture 7.12. (Benjamini and Schramm, 1996b) *If G is a transitive graph with at least quadratic growth (i.e., the ball of size n grows at least as fast as some quadratic function of n), then $p_c(G) < 1$.*

Most cases of this conjecture have been established, as we will see. Now given a group, certainly the value of p_c depends on which generators are used. Nevertheless, whether $p_c < 1$ does not depend on the generating set chosen. To prove this, consider two generating sets S_1, S_2 of a group Γ. Let G_1, G_2 be the corresponding Cayley graphs.

We will transfer Bernoulli percolation on G_1 to a dependent percolation on G_2 by making an edge of G_2 open iff a corresponding path in G_1 is open. We then want to compare this dependent percolation to Bernoulli percolation on G_2. To do this, we first establish a weak form of a result of Liggett, Schonmann, and Stacey (1997). This form, due to Lyons and Schramm (1999), Remark 6.2, is much easier to prove. We'll use the following general principle:

▷ **Exercise 7.10.**

Suppose that $\mathbf{P}_i$ $(i = 1, 2, 3)$ are three percolation measures on 2^A such that $\mathbf{P}_1 \succcurlyeq \mathbf{P}_2$ and $\mathbf{P}_2 \succcurlyeq \mathbf{P}_3$. Show that there exist random variables ω_i on a common probability space such that $\omega_i \sim \mathbf{P}_i$ for all i and $\omega_1 \ge \omega_2 \ge \omega_3$ a.s.

Proposition 7.13. *Write $\mathbf{P}_p^A$ for the Bernoulli(p) product measure on a set 2^A. Let A and B be two sets and $D \subseteq A \times B$. Write $D_a := \big(\{a\} \times B\big) \cap D$ and $D^b := \big(A \times \{b\}\big) \cap D$. Suppose that $m := \sup_{a \in A} |D_a| < \infty$ and that $n := \sup_{b \in B} |D^b| < \infty$. Given $0 < p < 1$, let $q := \big(1 - (1-p)^{1/m}\big)^n$. Given $\omega \in 2^A$, define $\omega' \in 2^B$ by*

$$\omega'(b) := \begin{cases} \min\{\omega(a)\,;\ (a,b) \in D^b\} & \text{if } D^b \ne \varnothing, \\ 1 & \text{if } D^b = \varnothing. \end{cases}$$

Let $\mathbf{P}$ be the law of ω' when ω has the law of $\mathbf{P}_p^A$. Then $\mathbf{P}$ stochastically dominates $\mathbf{P}_q^B$.

Proof. Let η have law $\mathbf{P}_{q^{1/n}}^{A \times B}$. Define

$$\zeta(a) := \begin{cases} \max \eta\restriction D_a & \text{if } D_a \ne \varnothing, \\ 0 & \text{if } D_a = \varnothing \end{cases} \qquad \text{and} \qquad \zeta'(b) := \begin{cases} \min \eta\restriction D^b & \text{if } D^b \ne \varnothing, \\ 1 & \text{if } D^b = \varnothing. \end{cases}$$

Then the collection ζ' has a law μ' that dominates $\mathbf{P}_q^B$ since $\zeta'(b)$ are mutually independent for $b \in B$ and

$$\mathbf{P}_{q^{1/n}}^{A \times B}[\zeta'(b) = 1] = (q^{1/n})^{|D^b|} \ge q\,.$$

Similarly, $\mathbf{P}_p^A$ dominates the law μ of ζ since $\zeta(a)$ are mutually independent for $a \in A$ and

$$\mathbf{P}_{q^{1/n}}^{A \times B}[\zeta(a) = 0] = (1 - q^{1/n})^{|D_a|} = (1-p)^{|D_a|/m} \ge 1 - p\,.$$

Therefore, if $\omega \sim \mathbf{P}_p^A$, then we may couple ω and ζ so that $\omega \ge \zeta$. Since $\zeta(a) \ge \eta(a,b)$, we have that for each fixed b with $D^b \ne \varnothing$, under our coupling (as extended by Exercise 7.10),

$$\omega'(b) = \min\{\omega(a)\,;\ (a,b) \in D^b\} \ge \min\{\zeta(a)\,;\ (a,b) \in D^b\} \ge \min \eta\restriction D^b = \zeta'(b)\,,$$

while $\omega'(b) = 1 = \zeta'(b)$ if $D^b = \varnothing$. It follows that $\mathbf{P} \succcurlyeq \mu' \succcurlyeq \mathbf{P}_q^B$, as desired. ◀

Theorem 7.14. *Let S_1 and S_2 be two finite generating sets for a countable group Γ, yielding corresponding Cayley graphs G_1 and G_2. Then $p_c(G_1) < 1$ iff $p_c(G_2) < 1$.*

Proof. Left and right Cayley graphs with respect to a given set of generators are isomorphic via $x \mapsto x^{-1}$, so we consider only right Cayley graphs. We also give the proof only for bond percolation, as site percolation is treated analogously.

Express each element $s \in S_2$ in terms of a minimal-length word $\varphi(s)$ using letters from S_1. Let ω_1 be Bernoulli(p) percolation on G_1 and define ω_2 on G_2 by letting $[x, xs] \in \omega_2$ (for $s \in S_2$) iff the path from x to xs in G_1 determined by $\varphi(s)$ lies in ω_1. Then we may apply Proposition 7.13 with A the set of edges of G_1, B the set of edges of G_2, and D the set of pairs (e, e') such that e is used in a path between the endpoints of e' determined by some $\varphi(s)$ for the appropriate $s \in S_2$. Clearly $n = \max\{|\varphi(s)|\,;\ s \in S_2\}$ and m is finite, whence we may conclude that ω_2 stochastically dominates Bernoulli(q) percolation on G_2. Since o lies in an infinite cluster with respect to ω_1 if it lies in an infinite cluster with respect to ω_2, it follows that if $q > p_c(G_2)$, then $p \geq p_c(G_1)$, showing that $p_c(G_2) < 1$ implies $p_c(G_1) < 1$. ◀

It is easy to see that the proof extends beyond Cayley graphs to cover the case of two graphs that are roughly isometric and have bounded degrees.

We now show that the Euclidean lattices $\mathbb{Z}^d$ ($d \geq 2$) have a true phase transition for Bernoulli percolation in that $p_c < 1$.

Theorem 7.15. *For all $d \geq 2$, we have $p_c(\mathbb{Z}^d) < 1$.*

Proof. We use the standard generators of $\mathbb{Z}^d$. In view of Proposition 7.10, it suffices to consider bond percolation. Also, since $\mathbb{Z}^d$ contains a copy of $\mathbb{Z}^2$, it suffices to prove this for $d = 2$. Consider the plane dual lattice $(\mathbb{Z}^2)^\dagger$. (See Section 6.4 for the definition.) To each configuration ω in 2^{E}, we associate the dual configuration $\omega^\times$ in $2^{\mathsf{E}^\dagger}$ by $\omega^\times(e^\dagger) := 1 - \omega(e)$. Those edges of $(\mathbb{Z}^2)^\dagger$ that lie in $\omega^\times$ we call "open". If $K(o)$ is finite, then $\partial_{\mathsf{E}} K(o)^\dagger$ contains a simple cycle of open edges in $(\mathbb{Z}^2)^\dagger$ that surrounds o. Now each edge in $(\mathbb{Z}^2)^\dagger$ is open with probability $1 - p$. Furthermore, the number of simple cycles in $(\mathbb{Z}^2)^\dagger$ of length n surrounding o is at most $n3^n$ since each one must intersect the x-axis somewhere in $(0, n)$. Let $M(n)$ be the total number of open simple cycles in $(\mathbb{Z}^2)^\dagger$ of length n surrounding o. Then

$$
\begin{aligned}
1 - \theta(p) = \mathbf{P}_p\big[|K(o)| < \infty\big] &= \mathbf{P}_p\Big[\sum_n M(n) \geq 1\Big] \\
&\leq \mathbf{E}_p\Big[\sum_n M(n)\Big] = \sum_n \mathbf{E}_p\big[M(n)\big] \leq \sum_n (n3^n)(1-p)^n\,.
\end{aligned}
$$

By choosing p sufficiently close to 1, we may make this last sum less than 1. ◀

Although we will not prove the full theorem that $p_c^{\rm bond}(\mathbb{Z}^2) = 1/2$ and $\theta(p_c, \mathbb{Z}^2) = 0$, we will give a simple proof due to Zhang, taken from Grimmett (1999), of Harris's theorem that $\theta(1/2, \mathbb{Z}^2) = 0$, which implies that $p_c^{\rm bond}(\mathbb{Z}^2) \ge 1/2$.

Theorem 7.16. *For bond percolation on $\mathbb{Z}^2$, we have $\theta(1/2) = 0$.*

Proof. Assume that $\theta(1/2) > 0$. By Theorem 7.6, there is a unique infinite cluster a.s. Let B be a square box with sides parallel to the axes in $\mathbb{Z}^2$. Let A be the event that B intersects the infinite cluster. Then $\mathbf{P}_{1/2}(A)$ is arbitrarily close to 1 provided we take B large enough. Let A_i $(1 \le i \le 4)$ be the events that the ith side of B intersects the infinite cluster when the interior of B is deleted. These events are increasing and have equal probability. Since $A^c = \bigcap_{i=1}^4 A_i^c$, it follows from Exercise 7.5 that $\mathbf{P}_{1/2}(A_i)$ are also arbitrarily close to 1 when B is large. As in the proof of Theorem 7.15, to each configuration ω in 2^{E}, we associate the dual configuration $\omega^\times$ in $2^{\mathsf{E}^\dagger}$ by $\omega^\times(e^\dagger) := 1 - \omega(e)$. Let B' be the smallest square box in the dual lattice $(\mathbb{Z}^2)^\dagger$ that contains B in its interior. Then similar statements apply to the sides of B' with respect to $\omega^\times$. In particular, the probability that the left and right sides of B intersect the infinite cluster when the interior of B is deleted in ω and that also the top and bottom of B' intersect the infinite cluster of the dual when the interior of B' is deleted in $\omega^\times$ is close to 1 when B is large. However, on the event that these four things occur simultaneously, there cannot be a unique infinite cluster in both $\mathbb{Z}^2$ and its dual, which is the contradiction we sought. ◀

In Corollary 7.18 and Theorem 7.19, we establish Conjecture 7.12 for the "majority" of groups. We say that a group ***almost*** has a property is it has a subgroup of finite index that has the property in question. We will use the following result:

Theorem 7.17. *If Γ is a finitely generated group of at most polynomial growth, then either Γ is almost (isomorphic to) $\mathbb{Z}$ or Γ contains a subgroup isomorphic to $\mathbb{Z}^2$.*

See Section 7.9 for the proof. Combining this with Theorems 7.15 and 7.14, we may immediately conclude

Corollary 7.18. *If G is a Cayley graph of a group Γ with at most polynomial growth, then either Γ is almost (isomorphic to) $\mathbb{Z}$ or $p_c(G) < 1$.*

Now we continue with groups of exponential growth.

Theorem 7.19. (Lyons, 1995) *If G is a Cayley graph of a group with exponential growth, then $p_c(G) < 1$.*

Proof. Let b be the exponential growth rate of balls in G and let $T^{\rm lexmin}(G)$ be the lexicographically-minimal spanning tree constructed in Section 3.4. This subperiodic tree is isomorphic to a subgraph of G, whence

$$p_c(G) \le p_c(T^{\mathrm{lexmin}}) = 1/\operatorname{br}(T^{\mathrm{lexmin}}) = 1/\operatorname{gr}(T^{\mathrm{lexmin}}) = 1/b < 1 \tag{7.7}$$

by Theorems 5.15 and 3.8. ◀

▷ **Exercise 7.11.**

Extend Theorem 7.19 to quasi-transitive graphs of upper exponential growth rate larger than 1.

There exist groups that have superpolynomial growth but subexponential growth, the groups of so-called "intermediate growth". The examples constructed by Grigorchuk (1983) also have $p_c < 1$ (Muchnik and Pak, 2001). It is not proved that other groups of intermediate growth have $p_c < 1$. These are the only cases of Conjecture 7.12 that remain unresolved.

§7.5. Merging Infinite Clusters and Invasion Percolation.

As p is increased, Bernoulli(p) percolation adds more edges, whence if there is an infinite cluster a.s. at p_0, then there is also an infinite cluster a.s. at each $p > p_0$. This intuition was made precise by using the standard coupling defined in Section 5.2. How does the number of infinite clusters vary as p is increased through the range where there is at least one infinite cluster? There are two competing features of increasing p: the fact that finite clusters can join means that new infinite clusters could form, while the fact that infinite clusters can join or can absorb finite clusters means that the number of infinite clusters could decrease. For general graphs, either one of these two features could be dominant. For example, if the Cayley graph of $\mathbb{Z}^2$ is joined by an edge to a tree T with branching number in $\big(1, 1/p_c(\mathbb{Z}^2)\big)$, then for $p_c(\mathbb{Z}^2) < p < 1/\operatorname{br} T$, there is a unique infinite cluster, while for $1/\operatorname{br} T < p < 1$, there are infinitely many infinite clusters. On the other hand, we will see in Theorem 8.21 that for the graph in Figure 2.3, the number of infinite clusters is first 0 in an interval of p, then ∞, and then 1. By combining examples such as these, one can obtain more complicated behavior of the number of infinite clusters. However, on quasi-transitive graphs, it turns out that once there is a unique infinite cluster, then that remains the case for all larger values of p. To prove this, we use the standard coupling and prove something stronger, due to Häggström, Peres, and Schonmann (1999).

Theorem 7.20. (Merging Infinite Clusters) *Let G be a quasi-transitive graph. If $p_1 \in \big(p_c(G), 1\big)$ is such that there exists a unique infinite cluster $\mathbf{P}_{p_1}$-a.s., then for all*

$p_2 > p_1$, there is a unique infinite cluster $\mathbf{P}_{p_2}$-a.s. Furthermore, in the standard coupling of Bernoulli percolation processes, a.s. for all $p_1, p_2 \in \big(p_c(G), 1\big)$ with $p_2 > p_1$, every infinite p_2-cluster contains an infinite p_1-cluster.

If we define

$$p_u(G) := \inf\big\{p\,;\ \text{there is a.s. a unique infinite cluster in Bernoulli}(p)\text{ percolation}\big\}\,, \tag{7.8}$$

then it follows from Theorem 7.20 that when G is a quasi-transitive graph,

$$p_u(G) = \sup\{p\,;\ \text{there is a.s. not a unique infinite cluster in Bernoulli}(p)\text{ percolation}\}\,.$$

Of course, two new questions immediately arise: When is $p_u < 1$? When is $p_c < p_u$? We will address these questions in Sections 7.6 and 7.7.

Our first order of business, however, is to prove Theorem 7.20. For this, we introduce invasion percolation, which will also be important in our study of minimal spanning forests in Chapter 11. We describe first invasion bond percolation. Fix distinct labels $U(e) \in [0, 1]$ for $e \in \mathsf{E}$, usually a sample of independent uniform random variables. Fix a vertex x. Imagine $U(e)$ as the height of e and the height of x as 0. If we pour water gradually on x, when the water reaches the height of the lowest edge incident to x, then water will flow along that edge. As we keep pouring, the water will "invade" the lowest edge that it touches. This is invasion percolation. More precisely, let $I_1(x)$ be the lowest edge incident to x and define $I_n(x)$ recursively thereafter by letting $I_{n+1}(x) := I_n(x) \cup \{e\}$, where e is the lowest among the edges that are not in $I_n(x)$ but are adjacent to (some edge of) $I_n(x)$. Finally, define the ***invasion basin*** of x to be $I(x) := \bigcup_n I_n(x)$. Invasion site percolation is similar, but the vertices are labelled rather than the edges and the invasion basin is a set of vertices, rather than a set of edges. For invasion site percolation, we start with $I_1(x) := \{x\}$.

Given the labels $U(e)$, we have the usual subgraphs $\omega_p := \{e\,;\ U(e) < p\}$ that we use for the standard coupling of Bernoulli percolation. The first connection of invasion percolation to Bernoulli percolation is that if x belongs to an infinite cluster η of ω_p, then $I(x) \subseteq \eta$. Also, if some edge $e \notin I(x)$ satisfies $U(e) < p$ and is also adjacent to some edge of $I(x)$, then $|I(x) \cap \eta| = \infty$ for some infinite cluster η of ω_p. A deeper connection is the following result of Häggström, Peres, and Schonmann (1999):

Theorem 7.21. (Invasion of Infinite Clusters) *Let G be an infinite quasi-transitive graph. Then in the standard coupling of Bernoulli percolation processes, a.s. for all $p > p_c(G)$ and all $x \in \mathsf{V}$, there is some infinite p-cluster that intersects $I(x)$.*

Before proving this, we show how it implies Theorem 7.20.

Proof of Theorem 7.20. We give the proof for bond percolation, as site percolation is treated in an identical manner. Let A be the event of probability 1 that all edge labels are distinct and that for all $p > p_c(G)$ and all $x \in \mathsf{V}$, there is some infinite p-cluster that intersects $I(x)$. On A, for each infinite cluster η_2 of ω_{p_2} and each $x \in \eta_2$, there is some infinite cluster η_1 of ω_{p_1} that intersects $I(x)$. Since $I(x) \subseteq \eta_2$, it follows that η_1 intersects η_2, whence $\eta_1 \subseteq \eta_2$ on A, as desired. ◀

The proof of Theorem 7.21 is rather tricky. The main steps will be to show that $I(x)$ contains arbitrarily large balls, that it comes infinitely often within distance 1 of an infinite p-cluster, and finally that it actually invades some infinite p-cluster. We present the proof for invasion bond percolation, as the site case is similar.

Lemma 7.22. *Let G be any infinite graph with bounded degrees, $x \in \mathsf{V}$, and $R \in \mathbb{N}$. Then* $\mathbf{P}$*-a.s. there is some $y \in \mathsf{V}$ such that $E_R(y) \subset I(x)$.*

Proof. The idea is that as the invasion from x proceeds, it will encounter balls of radius R for the first time infinitely often. Each time, the ball might have all its labels fairly small, in which case that ball would eventually be contained in the invasion basin. Since the encountered balls can be chosen far apart, these events are independent and therefore one of them will eventually happen.

To make this precise, fix an enumeration of V. Let d be the maximum degree in G,

$$S_r := \{y\,;\ \mathrm{dist}_G(x, y) = r\}\,,$$

and

$$\tau_n := \inf\Big\{k\,;\ d\big(I_k(x), S_{2nR}\big) = R\Big\}\,.$$

Since $I(x)$ is infinite, $\tau_n < \infty$. Let Y_n be the first vertex in the enumeration of V such that $Y_n \in S_{2nR}$ and $d\big(I_{\tau_n}(x), Y_n\big) = R$. Let A be the event of probability 1 that there is no infinite p-cluster for any $p < p_c(G)$ and consider the events

$$A_n := \big\{\forall e \in E_R(Y_n) \quad U(e) < p_c(G)\big\}\,.$$

On the event $A \cap A_n$, we have $E_R(Y_n) \subset I(x)$, so that it suffices to show that a.s. some A_n occurs. The sets $E_R(Y_n)$ are disjoint for different n. Also, the invasion process up to time τ_n gives no information about the labels of $E_R(Y_n)$. Since $|E_R(y)| \le d^R$ for all y, it follows that $\mathbf{P}(A_n \mid Y_n, A_1, A_2, \ldots, A_{n-1}) = p_c(G)^{|E_R(Y_n)|} \ge p_c(G)^{d^R}$, so that $\mathbf{P}(A_n \mid A_1, A_2, \ldots, A_{n-1}) \ge p_c(G)^{d^R}$. Since $p_c(G) > 0$ by Proposition 7.11, it follows that a.s. some A_n occurs. ◀

We now need an extension of insertion and deletion tolerance that holds for the edge labels.

Lemma 7.23. *Let X be a denumerable set and A be an event in $[0,1]^X$. Let λ be the product of Lebesgue measure on $[0,1]^X$. Given a finite $Y \subset X$ and a continuously differentiable injective map $\varphi : [0,1]^Y \to [0,1]^Y$ with strictly positive Jacobian determinant, let*

$$A' := \big\{\omega\,;\ \exists \eta \in A\ \ \omega\restriction(X \setminus Y) = \eta\restriction(X \setminus Y) \text{ and } \omega\restriction Y = \varphi(\eta\restriction Y)\big\}\,.$$

Then $\lambda(A') > 0$ if $\lambda(A) > 0$.

Proof. Regard $[0,1]^X$ as $[0,1]^Y \times [0,1]^{X \setminus Y}$. Apply Fubini's theorem to this product to see that it suffices to prove the case where $X = Y$. Then it follows from the usual change-of-variable formula

$$\lambda(A') = \int_A |J|\, d\lambda\,,$$

where J is the Jacobian determinant of φ. ◀

Let $\xi_p(x)$ be the number of edges $[y, z]$ with $y \in I(x)$ and z in some infinite p-cluster. (It is irrelevant to the definition whether $z \in I(x)$, but if $z \in I(x)$ and in some infinite p-cluster, then $\xi_p(x) = \infty$.)

Lemma 7.24. *If G is a quasi-transitive graph, $p > p_c(G)$, and $x \in \mathsf{V}$, then $\xi_p(x) = \infty$ a.s.*

Proof. Let $p > p_c(G)$ and $\epsilon > 0$. Because G is quasi-transitive, there is some R so that

$$\forall y \in \mathsf{V}\quad \mathbf{P}\big[\text{some infinite } p\text{-cluster comes within distance } R \text{ of } y\big] \ge 1 - \epsilon\,. \tag{7.9}$$

We need to reformulate this. Given a set of edges F, write $V(F)$ for the set of its endpoints. By (7.9), if $F \subset \mathsf{E}$ and contains some set $E_R(y)$, then with probability at least $1 - \epsilon$, the set F is adjacent to some infinite p-cluster. Let $A(F)$ be the event that there is an infinite path in ω_p that starts at distance 1 from $V(F)$ and that does not use any vertex in $V(F)$. Our reformulation of (7.9) is that if $F \subset \mathsf{E}$ is *finite* and contains some set $E_R(y)$, then $\mathbf{P}[A(F)] \ge 1 - \epsilon$.

By Lemma 7.22, there is a.s. a smallest finite k for which $I_k(x)$ contains some $E_R(y)$. Call this smallest time τ. Since the invasion process up to time τ gives no information about the labels of edges that are not adjacent to $I_\tau(x)$, we have $\mathbf{P}\big[A\big(I_\tau(x)\big) \bigm| I_\tau(x)\big] \ge 1 - \epsilon$, whence $\mathbf{P}\big[A\big(I_\tau(x)\big)\big] \ge 1 - \epsilon$. Since $\xi_p(x) \ge 1$ on the event $A\big(I_\tau(x)\big)$ and since ϵ was arbitrary, we obtain $\mathbf{P}[\xi_p(x) \ge 1] = 1$.

Now suppose that $\mathbf{P}[\xi_p(x) = n] > 0$ for some finite n. Then there is a set F_1 of n edges such that $\mathbf{P}(A_1) > 0$ for the event A_1 that F_1 is precisely the set of edges joining $I(x)$ to an infinite p-cluster. Among the edges adjacent to F_1, there is a set F_2 of edges such that $\mathbf{P}(A_1 \cap A_2) > 0$ for the event A_2 that F_2 is precisely the set of edges adjacent to (some edge of) F_1 that belong to an infinite p-cluster. Now changing the labels $U(e)$ for $e \in F_2$ to be $p + (1-p)U(e)$ changes $A_1 \cap A_2$ to an event A_3 where $\xi_p(x) = 0$, since $I(x)$ on $A_1 \cap A_2$ is the same as $I(x)$ on the corresponding configuration in A_3. By Lemma 7.23, $\mathbf{P}(A_3) > 0$, so $\mathbf{P}[\xi_p(x) = 0] > 0$. This contradicts the preceding conclusion. Therefore, $\xi_p(x) = \infty$ a.s. ◀

Proof of Theorem 7.21. First fix p_1, p_2 with $p_2 > p_1 > p_c(G)$ and fix $x \in \mathsf{V}$. Given a labelling of the edges, color an edge blue if it lies in an infinite p_1-cluster. Color an edge red if it is adjacent to some blue edge but is not itself blue. Observe that for red edges e, the labels $U(e)$ are independent and uniform on $[p_1, 1]$.

Now consider invasion from x given this coloring information. We claim that a.s., some edge of $I(x)$ is adjacent to some colored edge e with label $U(e) < p_2$. By Lemma 7.24, $\xi_{p_1}(x) = \infty$ a.s., so that infinitely many edges of $I(x)$ are adjacent to colored edges. When invasion from x first becomes adjacent to a colored edge, e, the distribution of $U(e)$, conditional on the colors of *all* edges and on the invasion process so far, is still concentrated on $[0, p_1)$ if e is blue or is uniform on $[p_1, 1]$ if e is red. Hence $U(e) < p_2$ with conditional probability at least $(p_2 - p_1)/(1 - p_1) > 0$. Since this is the same *every* time a colored edge is encountered, there must be one colored edge with label less than p_2 a.s. that is adjacent to some edge of $I(x)$. This proves the claim.

Now the claim we have just established implies that $I(x)$ a.s. intersects some infinite p_2-cluster by the observation preceding the statement of Theorem 7.21.

Apply this result to a sequence of values of p_2 approaching $p_c(G)$ to get the theorem. ◀

§7.6. Upper Bounds for p_u.

If G is a regular tree, then it is easy to see that $p_u(G) = 1$. (In fact, this is true for any tree; see Exercise 7.34.) What about the Cayley graph of a free group with respect to a non-free generating set? A special case of the following exercise (combined with Exercise 7.7) shows that this still has $p_u(G) = 1$.

▷ **Exercise 7.12.**

(a) Show that if ω is a $\mathbf{P}$-a.s. non-empty $\mathrm{Aut}(G)$-invariant percolation on a quasi-transitive graph G, then $\mathbf{P}$-a.s. every end of G contains an end-convergent subset of ω.

(b) Show that if G is a quasi-transitive graph with infinitely many ends, then $p_u(G) = 1$.

In fact, one can show that the property $p_u(G) < 1$ does not depend on the generating set for any group (Lyons and Schramm, 1999), but this would be subsumed by the following conjecture, suggested by a question of Benjamini and Schramm (1996b):

Conjecture 7.25. *If G is a quasi-transitive graph with one end, then $p_u(G) < 1$.*

This conjecture has been verified in many cases, most notably, when G is the Cayley graph of a finitely *presented* group. A key ingredient in that case is the following combinatorial fact.

Lemma 7.26. *Suppose that G is a graph that has a set K of cycles of length at most t such that every cycle belongs to the linear span (in $\ell^2_-(\mathsf{E})$) of K. Let Π be a (possibly infinite) cutset of edges that separates two given vertices and that is minimal with respect to inclusion. Then for each nontrivial partition $\Pi = \Pi_1 \cup \Pi_2$, there exist vertices $x_i \in \mathsf{V}(\Pi_i)$ $(i = 1, 2)$ such that the distance between x_1 and x_2 is at most $t/2$.*

This lemma is due to Babson and Benjamini (1999), who used topological definitions and proofs. We give a proof due to Timár (2007). If instead of hypothesizing that K spans the cycles in $\ell^2_-(\mathsf{E};\mathbb{R})$, one assumes that K spans the cycles in $\ell^2_-(\mathsf{E};\mathbf{F}_2)$, where $\mathbf{F}_2$ is the field of two elements, then the same statement is true with a very similar proof. Note that when a presentation $\langle S \mid R \rangle$ has relators (elements of R) with length at most t, then the associated Cayley graph satisfies the hypothesis of Lemma 7.26.

Proof. It suffices to show that there is some cycle of K that intersects both Π_i. Let x and y be two vertices separated by Π. By the assumption of minimality, for $i = 1, 2$, there is a path $\mathcal{P}_i$ from x to y that does not intersect Π_{3-i}. Considering $\mathcal{P}_i$ as elements of $\ell^2_-(\mathsf{E})$, we may write the sum of cycles $\mathcal{P}_1 - \mathcal{P}_2$ as a linear combination $\sum_{c \in K'} a_c c$ for some finite set

$K' \subset K$ with $a_c \in \mathbb{R}$. Let K'_1 denote the cycles of K' that intersect Π_1 and $K'_2 := K' \setminus K'_1$. We have

$$\theta := \mathcal{P}_1 - \sum_{c \in K'_1} a_c c = \mathcal{P}_2 + \sum_{c \in K'_2} a_c c \,.$$

The right-hand side is composed of paths and cycles that do not intersect Π_1, whence the inner product of θ with χ^e is 0 for all $e \in \Pi_1$. Since θ is a unit flow from x to y, it follows that θ includes an edge from Π_2. Since $\mathcal{P}_1$ does not, we deduce that some cycle in K'_1 does. This provides the sort of cycle we sought. ◀

Write $\mathsf{V}_E := \big\{(e, f) \in \mathsf{E} \times \mathsf{E}\,;\ e, f \text{ are incident}\big\}$ in order to define the ***line graph**** of G as $G_E := (\mathsf{E}, \mathsf{V}_E)$. Write $\mathsf{E}^t := \big\{(x, y) \in \mathsf{V} \times \mathsf{V}\,;\ \mathrm{dist}_G(x, y) \le t\big\}$ in order to define the ***t*-fuzz** of G as $G^t := (\mathsf{V}, \mathsf{E}^t)$. Finally, write $G^t_E := (G_E)^t$. If every cycle can be written as a (finite) linear combination of cycles with length $< 2t$, then according to Lemma 7.26,

$$\text{every minimal cutset in } G \text{ is the vertex set of a connected subgraph of } G^t_E. \tag{7.10}$$

Theorem 7.27. (Babson and Benjamini, 1999) *If G is the Cayley graph of a non-amenable finitely presented group with one end, then $p_{\mathrm{u}}(G) < 1$. In fact, the same holds for any non-amenable quasi-transitive graph G with one end such that there is a set of cycles with bounded length whose linear span contains all cycles.*

An extension to the amenable case (where $p_{\mathrm{u}} = p_{\mathrm{c}}$) is given in Exercise 7.39.

Proof. Let $2t$ be strictly larger than the lengths of cycles of some generating set. We do the case of bond percolation and prove that

$$p^{\mathrm{bond}}_{\mathrm{u}}(G) \le \max\{p^{\mathrm{bond}}_{\mathrm{c}}(G), 1 - p^{\mathrm{site}}_{\mathrm{c}}(G^t_E)\} \,.$$

The case of site percolation is left as Exercise 7.35. Choose $p < 1$ so that $p > p^{\mathrm{bond}}_{\mathrm{c}}(G)$ and $p > 1 - p^{\mathrm{site}}_{\mathrm{c}}(G^t_E)$. This is possible by Theorem 7.19, (7.1), and Proposition 7.11. Because G has only one end, any two open infinite clusters must be separated by a closed infinite minimal cutset (indeed, one within the edge boundary of one of the open infinite clusters). Combining this with (7.10), we obtain

$$\begin{aligned}\mathbf{P}^{\mathrm{bond}}_{p,G}[\exists \text{ at least 2 open infinite clusters}] &\le \mathbf{P}^{\mathrm{bond}}_{p,G}[\exists \text{ a closed infinite minimal cutset in } G] \\ &\le \mathbf{P}^{\mathrm{site}}_{p,G^t_E}[\exists \text{ a closed infinite cluster in } G^t_E] \\ &= \mathbf{P}^{\mathrm{site}}_{1-p,G^t_E}[\exists \text{ an open infinite cluster in } G^t_E] = 0 \,.\end{aligned}$$

---

* Since site percolation on G_E is equivalent to bond percolation on G, one might wonder why we study bond percolation separately. There are two principal reasons for this: One is that other sorts of bond percolation processes have no natural site analogue. Another is that planar duality is quite different for bonds than for sites.

On the other hand, $\mathbf{P}^{\mathrm{bond}}_{p,G}[\exists$ at least 1 open infinite cluster$] = 1$ since $p > p_c^{\mathrm{bond}}(G)$. This proves the result. ◀

§7.7. Lower Bounds for p_u.

According to Theorem 7.6, when G is amenable and transitive, there can never be infinitely many infinite clusters, whence $p_c(G) = p_u(G)$. Behavior that is truly different from the amenable case arises when there *are* infinitely many infinite clusters. This has been conjectured always to be the case on non-amenable transitive graphs for some interval of p:

Conjecture 7.28. (Benjamini and Schramm, 1996b) *If G is a quasi-transitive non-amenable graph, then $p_c(G) < p_u(G)$.*

In order to give some examples where $0 < p_c < p_u < 1$, so that we have three phases in Bernoulli percolation, we will use the following lower bound for p_u. Recall that a ***simple cycle*** is a cycle that does not use any vertex or edge more than once. Let $a_n(G)$ be the number of simple cycles of length n in G that contain o and

$$\gamma(G) := \limsup_{n\to\infty} a_n(G)^{1/n}\,, \tag{7.11}$$

where a cycle is counted only as a set without regard to ordering or orientation. (We do not know whether $\lim_{n\to\infty} a_n(G)^{1/n}$ exists for every transitive, or even every Cayley, graph.)

The first part of the following theorem is due to Schramm; his proof is published by Lyons (2000). The second part strengthens a result in the proof of Theorem 4 of Benjamini and Schramm (1996b).

Theorem 7.29. *Let G be a transitive graph. Then*

$$p_u(G) \ge 1/\gamma(G)\,. \tag{7.12}$$

Moreover, if $p < 1/\gamma(G)$, then $\lim_{x\to\infty} \mathbf{P}_p[o \leftrightarrow x] = 0$.

Proof. When there is a unique infinite cluster, Harris's inequality implies that $\mathbf{P}_p[o \leftrightarrow x] \ge \theta(p)^2$. Thus, the second part of the theorem implies the first.

Consider first site percolation. Let $p < p_+ < 1/\gamma(G)$. We use the standard coupling of Bernoulli percolation. A vertex z is called an (x, y)-***cutpoint*** if x and y belong to the same p_+-open cluster, but would not if z were closed. Let $u_k(r)$ denote the maximum over x outside the ball $B(o,r)$ of the probability that o and x are connected via a p_+-open path with at most k (o,x)-cutpoints (for the p_+-open vertices). When there are no (o,x)-cutpoints, there are two disjoint paths joining o to x by Menger's theorem (Exercise 3.17), i.e., there is a simple cycle containing o and x. Thus, $u_0(r) \le \sum_{n>2r} a_n(G)p_+^n \to 0$ as $r \to \infty$. For $k \ge 0$ and $r \le R$, consider $x \notin B(o,R)$. If there is a p_+-open path from o to x that has at most $k+1$ (o,x)-cutpoints, then either this path contains no cutpoints in $B(o,r)$ or it has at most k cutpoints in $B(o,R) \setminus B(o,r)$. In the latter case, it intersects the sphere of radius r in one of at most d^r points, where d is the degree of G. Therefore, we obtain the bound

$$u_{k+1}(R) \le u_0(r) + d^r u_k(R-r)\,. \tag{7.13}$$

The inequality (7.13) implies inductively that (for any fixed k) $u_k(R)$ tends to zero as $R \to \infty$. [Given ϵ, first choose r such that $u_0(r) < \epsilon$ and then choose R' so that $d^r u_k(R-r) < \epsilon$ for all $R \ge R'$.]

Next, let $\tau_p(R)$ denote the maximum over $x \notin B(o,R)$ of the probability that o and x are in the same p-cluster. Then for every k, we have

$$\tau_p(R) \le (p/p_+)^{k+1} + u_k(R) \tag{7.14}$$

by considering whether there is a p_+-open path from o to x with at most k cutpoints or not; in the latter case, there are $k+1$ cutpoints that lie on every p_+-open path from o to x and they must also be p-open. Finally (7.14) implies that $\tau_p(R)$ tends to zero as $R \to \infty$. [Given ϵ, first choose k such that $(p/p_+)^{k+1} < \epsilon$, then choose R' so that $u_k(R) < \epsilon$ for all $R \ge R'$.]

The proof for bond percolation is similar, but needs just the following modification: replace $(p/p_+)^{k+1}$ by $(1-(1-p/p_+)^d)^{(k+1)/2}$ and note that if there is a no p_+-open path from o to x with at most k cutpoints, then at least one edge incident to each cutpoint must also be p-open. Furthermore, the cutpoints must appear in a fixed order in every p_+-open path from o to x, and two cutpoints can share at most one edge. ◀

It seems difficult to calculate $\gamma(G)$, although Hammersley (1961b) showed that $\gamma(\mathbb{Z}^d) = \mu(\mathbb{Z}^d)$ for all $d \ge 2$, where μ is the connective constant defined in Section 7.4; in particular, $\gamma(\mathbb{Z}^2) \approx 2.64$ (Madras and Slade, 1993). However, although it is crude, quite a useful estimate of $\gamma(G)$ arises from the spectral radius of G, which is much easier to calculate. For any graph G, recall that the ***spectral radius*** of (simple random walk on) G is defined to be

$$\rho(G) := \limsup_{n\to\infty} \mathbf{P}_o[X_n = o]^{1/n}\,,$$

where $\big(\langle X_n\rangle, \mathbf{P}_o\big)$ denotes simple random walk on G starting at o.

▷ **Exercise 7.13.**

Show that for any graph G, $\rho(G) = \lim_{n\to\infty} \mathbf{P}_o[X_{2n} = o]^{1/2n}$ and $\mathbf{P}_o[X_n = o] \le \rho(G)^n$ for all n.

Lemma 7.30. *For any graph G of maximum degree d, we have $\gamma(G) \le \rho(G)d$.*

Proof. For any simple cycle $o = x_0, x_1, \ldots, x_{n-1}, x_n = o$, one way that simple random walk could return to o at time n is by following this cycle. That event has probability at least $1/d^n$. Therefore $\mathbf{P}_o[X_n = o] \ge a_n(G)/d^n$, which gives the result. ◀

Proposition 7.31. (Spectral Radius of Products) *Let G and G' be transitive graphs of degrees d_G and $d_{G'}$. We have*

$$\rho(G \times G') = \frac{d_G}{d_G + d_{G'}}\rho(G) + \frac{d_{G'}}{d_G + d_{G'}}\rho(G')\,.$$

Proof. We first sketch the calculation, then show how to make it rigorous. We will use superscripts on $\mathbf{P}_o$ to denote on which graph the walk is taking place. The degree in $G \times G'$ is $d_G + d_{G'}$, whence each step of simple random walk has chance $d_G/(d_G + d_{G'})$ to be along an edge coming from G. Take o in $G \times G'$ to be the product of the basepoints in the two graphs. Then a walk of n steps from o in $G \times G'$ that takes k steps along edges from G and the rest from G' is back at o iff the k steps in G lead back to o in G and likewise for the $n-k$ steps in G'. Thus,

$$\begin{aligned}
\mathbf{P}_o^{G\times G'}[X_n = o] &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{d_G}{d_G + d_{G'}}\right)^k \left(\frac{d_{G'}}{d_G + d_{G'}}\right)^{n-k} \mathbf{P}_o^G[X_k = o]\mathbf{P}_o^{G'}[X_{n-k} = o] \\
&\approx \sum_{k=0}^{n} \binom{n}{k} \left(\frac{d_G}{d_G + d_{G'}}\right)^k \left(\frac{d_{G'}}{d_G + d_{G'}}\right)^{n-k} \rho(G)^k \rho(G')^{n-k} \\
&= \left(\frac{d_G\rho(G) + d_{G'}\rho(G')}{d_G + d_{G'}}\right)^n .
\end{aligned}$$

Taking nth roots gives the result.

To make this argument rigorous, we will replace the vague approximation by inequalities in both directions.

For an upper bound, we may use the inequality of Exercise 7.13.

For a lower bound, we note that according to Exercise 7.13, it suffices to consider only even n, which we now do. Then we may sum over even k only (for a lower bound) and use $\mathbf{P}_o^G[X_k = o] \ge C(\rho(G) - \epsilon)^k$ and $\mathbf{P}_o^{G'}[X_{n-k} = o] \ge C(\rho(G') - \epsilon)^{n-k}$ for some constant C and arbitrarily small ϵ. Note that for any $0 < p < 1$, we have

$$\lim_{n\to\infty} \sum_{k=0}^{n} \binom{2n}{2k} p^{2k}(1-p)^{2n-2k} = 1/2\,.$$

Putting these ingredients together completes the proof. ◀

Since $\rho(\mathbb{Z}) = 1$, it follows that $\rho(\mathbb{Z}^d) = 1$ as well, a fact that is also easy to verify directly. Another particular value of ρ that is useful for us is that of trees:

Proposition 7.32. (Spectral Radius of Trees) *For all $b \ge 1$, we have*

$$\rho(\mathbb{T}_{b+1}) = \frac{2\sqrt{b}}{b+1}\,.$$

Proof. If X_n is simple random walk on $\mathbb{T}_{b+1}$, then $|X_n|$ is a biased random walk on $\mathbb{N}$, with probability $p := b/(b+1)$ to increase by 1 and $q := 1/(b+1)$ to decrease by 1 when $X_n \ne 0$. Let $\tau_0^+ := \inf\{n \ge 1\,;\ |X_n| = 0\}$. Now the number of paths of length $2n$ from 0 to 0 in $\mathbb{N}$ that do not visit 0 in between is equal to the number of paths of length $2n-2$ from 1 to 1 in $\mathbb{Z}$ minus the number of the latter that visit 0. The former number is clearly $\binom{2n-2}{n-1}$. The latter number is $\binom{2n-2}{n}$, since reflection after the first visit to 0 of a path from 1 to 1 that visits 0 yields a bijection to the set of paths from 1 to -1 of length $2n-2$. Hence, the difference is

$$\binom{2n-2}{n-1} - \binom{2n-2}{n} = \frac{1}{n}\binom{2n-2}{n-1}\,.$$

(This is the sequence of ***Catalan numbers***, which appear in many combinatorial problems.) Each path of length $2n$ from 0 to 0 in $\mathbb{N}$ that does not visit 0 has probability $p^{n-1}q^n$, whence we have

$$\mathbf{P}[\tau_0^+ = 2n] = \frac{1}{n}\binom{2n-2}{n-1}p^{n-1}q^n = \frac{-1}{2}(-4)^n\binom{1/2}{n}p^{n-1}q^n\,.$$

Therefore, provided $|z|$ is sufficiently small,

$$g(z) := \mathbf{E}[z^{\tau_0^+}] = \sum_{n\ge 1}\frac{-1}{2}(-4)^n\binom{1/2}{n}p^{n-1}q^n z^{2n} = (1-\sqrt{1-4pqz^2})/(2p)\,.$$

The radius of convergence of the power series for g is therefore $1/(2\sqrt{pq})$. Since

$$G(z) := \sum_{n\ge 0}\mathbf{P}[|X_n| = 0]z^n = \frac{1}{1-g(z)}\,,$$

it follows that the radius of convergence of G is also $1/(2\sqrt{pq})$, which shows that the spectral radius is $2\sqrt{pq}$, as desired.

(See Exercise 6.9 for another proof.) ◀

Combining this with Proposition 7.31, we get:

Corollary 7.33. *For all $b \geq 1$, we have*

$$\rho(\mathbb{T}_{b+1} \times \mathbb{Z}) = \frac{2\sqrt{b}+2}{b+3}\,. \qquad \blacktriangleleft$$

We now have enough tools at our disposal to give our first example of a graph with two phase transitions; this was also historically the first example ever given.

Theorem 7.34. (Grimmett and Newman, 1990) *For all $b \geq 8$, we have*

$$0 < p_{\mathrm{c}}(\mathbb{T}_{b+1} \times \mathbb{Z}) < p_{\mathrm{u}}(\mathbb{T}_{b+1} \times \mathbb{Z}) < 1\,.$$

Proof. The first inequality follows from Proposition 7.11. Since $\mathbb{T}_{b+1}$ and $\mathbb{Z}$ are both finitely presented, so is their product, whence the last inequality is a consequence of Theorem 7.27. To prove the middle inequality, note that by Theorem 7.29,

$$p_{\mathrm{c}}(\mathbb{T}_{b+1} \times \mathbb{Z}) \leq p_{\mathrm{c}}(\mathbb{T}_{b+1}) = 1/b < 1/\gamma(\mathbb{T}_{b+1} \times \mathbb{Z}) \leq p_{\mathrm{u}}(\mathbb{T}_{b+1} \times \mathbb{Z})$$

if $\gamma(\mathbb{T}_{b+1} \times \mathbb{Z}) < b$. Combining Lemma 7.30 and Corollary 7.33, we obtain $\gamma(\mathbb{T}_{b+1} \times \mathbb{Z}) \leq \rho(\mathbb{T}_{b+1} \times \mathbb{Z})(b+3) = 2\sqrt{b}+2$. This last quantity is less than b for $b \geq 8$. ◀

To establish some more cases of Conjecture 7.28, we use the following consequence of (6.8). (Recall that in this chapter, $\Phi_{\mathsf{E}}(G)$ means $\Phi_{\mathsf{E}}(G, \mathbf{1}, \mathbf{1})$, while in (6.8), it meant $\Phi_{\mathsf{E}}(G, c, \pi)$.)

Theorem 7.35. *If G is a regular graph of degree d, then*

$$\rho(G)^2 + \left(\frac{\Phi_{\mathsf{E}}(G)}{d}\right)^2 \leq 1 \tag{7.15}$$

and

$$\rho(G) + \frac{\Phi_{\mathsf{E}}(G)}{d} \geq 1\,. \tag{7.16}$$

◀

The following corollary is due to Schonmann (2001) (parts (i) and (ii)) and Pak and Smirnova-Nagnibeda (2000) (part (iii)).

Corollary 7.36. *Let G be a transitive graph.*

(i) *If $\Phi_{\mathsf{E}}(G)/d_G \geq 1/\sqrt{2}$, then $p_{\mathrm{c}}^{\mathrm{bond}}(G) < p_{\mathrm{u}}^{\mathrm{bond}}(G)$.*
(ii) *If $\Phi_{\mathsf{V}}(G)/d_G \geq 1/\sqrt{2}$, then $p_{\mathrm{c}}^{\mathrm{site}}(G) < p_{\mathrm{u}}^{\mathrm{site}}(G)$.*
(iii) *If $\rho(G) \leq 1/2$, then $p_{\mathrm{c}}^{\mathrm{bond}}(G) < p_{\mathrm{u}}^{\mathrm{bond}}(G)$.*

Proof. We will use the following implications:

$$\rho(G)d_G \le \Phi_{\mathsf{E}}(G) \implies p_c^{\mathrm{bond}}(G) < p_u^{\mathrm{bond}}(G)\,, \tag{7.17}$$

$$\rho(G)d_G \le \Phi_{\mathsf{V}}(G) \implies p_c^{\mathrm{site}}(G) < p_u^{\mathrm{site}}(G)\,. \tag{7.18}$$

To prove (7.17), use Theorem 6.23, Lemma 7.30, and Theorem 7.29 to see that when $\rho(G)d_G \le \Phi_{\mathsf{E}}(G)$, we have

$$p_c^{\mathrm{bond}}(G) \le \frac{1}{1+\Phi_{\mathsf{E}}(G)} < \frac{1}{\Phi_{\mathsf{E}}(G)} \le \frac{1}{\rho(G)d_G} \le \frac{1}{\gamma(G)} \le p_u^{\mathrm{bond}}(G)\,.$$

To prove (7.18), use Theorem 6.23, Lemma 7.30, and Theorem 7.29 to see that when $\rho(G)d_G \le \Phi_{\mathsf{V}}(G)$, we have

$$p_c^{\mathrm{site}}(G) \le \frac{1}{1+\Phi_{\mathsf{V}}(G)} < \frac{1}{\Phi_{\mathsf{V}}(G)} \le \frac{1}{\rho(G)d_G} \le \frac{1}{\gamma(G)} \le p_u^{\mathrm{site}}(G)\,.$$

Part (i) is now an immediate consequence of (7.15) and (7.17). Part (ii) follows similarly from (7.15) and (7.18), with the observation that $\Phi_{\mathsf{E}}(G) \ge \Phi_{\mathsf{V}}(G)$, so that if $\Phi_{\mathsf{V}}(G)/d_G \ge 1/\sqrt{2}$, then also $\Phi_{\mathsf{E}}(G)/d_G \ge 1/\sqrt{2}$, so that $\rho(G)d_G \le d_G/\sqrt{2} \le \Phi_{\mathsf{V}}(G)$. Part (iii) follows from (7.16) and (7.17). ◀

To get a sense of the strength of the hypotheses of Corollary 7.36, note that by Exercise 6.1, we have $\Phi(\mathbb{T}_{b+1})/d_{\mathbb{T}_{b+1}} = (b-1)/(b+1)$. Also, for a given degree, the regular tree maximizes Φ over all graphs by Exercise 6.33.

It turns out that any non-amenable group has a generating set that gives a Cayley graph satisfying the hypotheses of Corollary 7.36(iii). To show this, consider the following construction. Let G be a graph and $k \ge 1$. Define a new multigraph $G^{[k]}$ to have vertex set $\mathsf{V}(G)$ and to have one edge joining $x, y \in \mathsf{V}(G)$ for every path in G of length exactly k that joins x and y. Thus $G^{[1]} = G$. Further, if G is the Cayley graph of Γ corresponding to a generating set S that is closed under inverses, then for any odd k, we have that $G^{[k]}$ is the Cayley graph of the same group Γ, but presented as $\langle S^{[k]} \mid R^{[k]}\rangle$, where

$$S^{[k]} := \{x_{w_1,w_2,\ldots,w_k}\,;\ w_i \in S\}$$

and $R^{[k]}$ is the set of products $x_{w_{1,1},w_{1,2},\ldots,w_{1,k}}x_{w_{2,1},w_{2,2},\ldots,w_{2,k}}\cdots x_{w_{n,1},w_{n,2},\ldots,w_{n,k}}$ of elements from $S^{[k]}$ such that $w_{1,1}w_{1,2}\cdots w_{1,k}w_{2,1}w_{2,2}\cdots w_{2,k}\cdots w_{n,1}w_{n,2}\cdots w_{n,k} = \mathbf{1}_\Gamma$. (If G is not bipartite, then the same statement holds for even k.)

Corollary 7.37. (Pak and Smirnova-Nagnibeda (2000)) *If G is any non-amenable transitive graph, then $p_c^{\text{bond}}(G^{[k]}) < p_u^{\text{bond}}(G^{[k]})$ for all odd $k \ge -\log 2/\log\rho(G)$. In particular, every finitely generated non-amenable group has a generating set whose Cayley graph satisfies $p_c^{\text{bond}} < p_u^{\text{bond}}$.*

Proof. Let k be as given. Then $\rho(G)^k \le 1/2$. Now simple random walk on $G^{[k]}$ is the same as simple random walk on G sampled every k steps. Therefore $\rho(G^{[k]}) = \rho(G)^k$ by Exercise 7.13. This means that $\rho(G^{[k]}) \le 1/2$, whence the result follows from Corollary 7.36. ◀

Corollary 7.37 would confirm Conjecture 7.28 if the following conjecture were established:

Conjecture 7.38. (Benjamini and Schramm, 1996b) *If G and G' are roughly isometric quasi-transitive graphs and $p_c(G) < p_u(G)$, then $p_c(G') < p_u(G')$.*

More detailed information about percolation requires additional tools. Unfortunately, these tools are not available for all quasi-transitive graphs. The next chapter is devoted to them.

§7.8. Bootstrap Percolation on Infinite Trees.

Bootstrap percolation on an infinite graph G has a random initial configuration, where each vertex is occupied with probability p, independently of each other, and a deterministic spreading rule with a fixed parameter k: if a vacant site has at least k occupied neighbors at a certain time step, then it becomes occupied in the next step. The critical probability $p(G, k)$ is the infimum of the initial probabilities p that make $\mathbf{P}_p[\text{complete occupation}] > 0$. If $k = 1$, then clearly everything becomes occupied when G is connected and $p > 0$, while if k is the maximum degree of G, then clearly not everything becomes occupied when $p < 1$.

▷ **Exercise 7.14.**
Show that $p(\mathbb{Z}^2, 2) = 0$ for the usual square lattice graph.

This process is well studied on $\mathbb{Z}^d$ and on finite boxes (see, e.g., Aizenman and Lebowitz (1988), Schonmann (1992), Holroyd (2003), Adler and Lev (2003)). Balogh, Peres, and Pete (2006) investigated it on regular and general infinite trees and graphs with anchored expansion (such as non-amenable Cayley graphs).

Balogh, Peres, and Pete (2006) showed that if Γ is a finitely generated group with a free subgroup on two elements, then Γ has a Cayley graph G such that $0 < p(G,k) < 1$ for some k. In contrast, Schonmann (1992) proved that $p(\mathbb{Z}^d, k) = 0$ if $k \le d$ and $= 1$ if $k > d$. These results raise the following question:

Question 7.39. Is a group amenable if and only if for any finite generating set, the resulting Cayley graph G has $p(G,k) \in \{0,1\}$ for any k-neighbor rule?

It turns out that finding the critical probability $p(\mathbb{T}_{d+1}, k)$ on a $(d+1)$-regular tree is equivalent to the problem of finding certain regular subtrees in a Bernoulli percolation process, so the results of Section 5.5 are directly applicable to this case.

First we have to introduce the following simple notion:

Definition 7.40. A finite or infinite connected subset $F \subseteq V$ of vertices is called a ***k-fort*** if each $v \in F$ has external degree $\deg_{V \setminus F}(v) \le k$. Here $\deg_H(v) = |\{w \in H : (v,w) \in E\}|$, for any $H \subseteq V$.

A key observation is that the failure of complete occupation by the k-neighbor rule for a given initial configuration ω is equivalent to the existence of a vacant $(k-1)$-fort in ω. (The unoccupied sites in the final configuration are a $(k-1)$-fort and unoccupied in ω, while any vacant $(k-1)$-fort remains vacant.)

For integers $d \ge 2$ and $1 \le k \le d$, define $\pi(d,k)$ to be the infimum of probabilities p for which the cluster of a fixed vertex of a $(d+1)$-regular tree in Bernoulli(p) bond percolation contains a $(k+1)$-regular subtree with positive probability. By a simple use of Harris's inequality, this critical probability is the same as the one for having a k-ary subtree at the root in a Galton-Watson tree with offspring distribution Bin(d,p), which we analyzed in Section 5.5. Ergodicity also shows that the probability that there is a $(k+1)$-regular subtree in Bernoulli(p) percolation on $\mathbb{T}_{d+1}$ is either 0 or 1; this probability is monotonic in p and changes at $\pi(d,k)$.

Proposition 7.41. (Balogh, Peres, and Pete, 2006) *Let $1 \le k \le d$, and consider k-neighbor bootstrap percolation on the $(d+1)$-regular tree $\mathbb{T}_{d+1}$. We have the following equality of critical probabilities:*

$$p(\mathbb{T}_{d+1}, k) = 1 - \pi(d, d+1-k)\,. \tag{7.19}$$

In particular, for any constant $\gamma \in [0,1]$ and sequence of integers k_d with $\lim_{d\to\infty} k_d/d = \gamma$, we have

$$\lim_{d\to\infty} p(\mathbb{T}_{d+1}, k_d) = \gamma. \tag{7.20}$$

Proof. The tree $\mathbb{T}_{d+1}$ has no finite $(k-1)$-forts, and it is easy to see that any infinite $(k-1)$-fort of $\mathbb{T}_{d+1}$ contains a complete $(d+2-k)$-regular subtree. Hence, unsuccessful complete occupation for the k-rule is equivalent to the existence of a $(d+2-k)$-regular vacant subtree in the initial configuration. Furthermore, the set of initial configurations that lead to complete occupation on $\mathbb{T}_{d+1}$ is invariant under the automorphism group of $\mathbb{T}_{d+1}$, hence has probability 0 or 1: see Proposition 7.3. So incomplete occupation has probability 1 if and only if a fixed origin is contained in a $(d+2-k)$-regular vacant subtree with positive probability. Since the vacant vertices in $\mathbb{T}_{d+1}$ form a Bernoulli$(1-p)$ percolation process, we find that (7.19) holds.

Note that if $\lim_{d\to\infty} k_d/d = \gamma$, then $\lim_{d\to\infty}(d+1-k_d)/d = 1-\gamma$, so (5.19) of Proposition 5.25 implies (7.20). ◀

As noted in Section 5.5, Pakes and Dekking (1991) proved that for $k \geq 2$ we already have a k-ary subtree at the critical probability $\pi(d,k)$. For bootstrap percolation, this means that the probability of complete occupation is still 0 at $p = p(\mathbb{T}_{d+1}, k)$ if $k < d$.

§7.9. Notes.

In the non-amenable case, two Cayley graphs are roughly isometric iff there is a bijective rough isometry between them, which is the same as a bi-Lipschitz map: see Whyte (1999). However, in the amenable case, this is not so and there are lamplighter groups that are roughly isometric but not bi-Lipschitz equivalent: see Dymarz (2009). These lamplighter groups are of the form $\mathbb{Z} \ltimes \sum_{x\in\mathbb{Z}} \mathbb{Z}_m$; other lamplighter groups of interest in probability replace the base space $\mathbb{Z}$ by $\mathbb{Z}^d$; see Kaĭmanovich and Vershik (1983).

Proposition 7.11 was first observed for bond percolation on $\mathbb{Z}^2$ by Broadbent and Hammersley (1957) and Hammersley (1959).

Special cases of Proposition 7.9 were proved by Fisher (1961).

Inequality (7.7) also follows from the following result of Aizenman and Barsky (1987):

Theorem 7.42. (Expected Cluster Size) *If G is any transitive graph and $p < p_c(G)$, then $\mathbf{E}_p[|K(o)|] < \infty$.*

Aizenman and Barsky (1987) worked only on $\mathbb{Z}^d$, but their proof works in greater generality: in their notation, one simply has to use the inequality $\mathrm{Prob}_L(C_{\Lambda_L\setminus A}(y)\cap G \neq \varnothing) \leq \mathrm{Prob}(C(y)\cap G \neq \varnothing) = M$, so that one does not need periodic finite approximations.

To prove Theorem 7.17, we will use the following result:

Proposition 7.43. *A subgroup of finite index in a finitely generated group is itself finitely generated.*

Proof. Let Γ be generated by the set S. For a subgroup Γ', let $A \subset \Gamma$ be such that A intersects each coset of Γ' exactly once. We may assume that $1 \in A$, where 1 is the identity of Γ. Thus, for each $\gamma \in \Gamma$, there is a unique $k(\gamma) \in A$ and a unique $h(\gamma) \in \Gamma'$ such that $\gamma = k(\gamma)h(\gamma)$. Now if $\gamma = ah(\gamma)$ and $\gamma' = a'h(\gamma')$, then $\gamma\gamma' = \gamma a'h(\gamma') = a''h(\gamma a')h(\gamma')$ for some $a'' \in A$, whence $h(\gamma\gamma') = h(\gamma k(\gamma'))h(\gamma')$. It follows by induction that

$$h(\gamma_1 \cdots \gamma_m) = h(\gamma_1 k(\gamma_2 \cdots \gamma_m))h(\gamma_2 k(\gamma_3 \cdots \gamma_m)) \cdots h(\gamma_{m-1} k(\gamma_m))h(\gamma_m)\,.$$

Therefore, given any choice of $s_i \in S \cup S^{-1}$, the element $h(s_1 \cdots s_m)$ is a product of elements from $T := \{h(sa)\,;\ s \in S \cup S^{-1},\ a \in A\}$. Since $1 \in A$, $h(\gamma) = \gamma$ for all $\gamma \in \Gamma'$, whence if $s_1 \cdots s_m \in \Gamma'$, then $h(s_1 \cdots s_m) = s_1 \cdots s_m$. This means that Γ' is generated by T. ◀

Proof of Theorem 7.17. The principal fact we need is the difficult theorem of Gromov (1981), who showed that Γ is almost a nilpotent group. (The converse is true as well; see Wolf (1968).) Now every finitely generated nilpotent group is almost torsion-free*, while the upper central series of a torsion-free finitely generated nilpotent group has all factors isomorphic to free abelian groups (Kargapolov and Merzljakov (1979), Theorem 17.2.2 and its proof). Thus, Γ has a subgroup Γ' of finite index that is torsion-free and either Γ' equals its center, C, or there is a subgroup C' of Γ' such that C'/C is the center of Γ'/C and C'/C is free abelian of rank at least 1. If the rank of C is at least 2, then C already contains $\mathbb{Z}^2$, so suppose that $C \approx \mathbb{Z}$. If $\Gamma' = C$, then Γ is almost $\mathbb{Z}$. If not, then let C'' be a subgroup of C' such that C''/C is isomorphic to $\mathbb{Z}$. We claim that C'' is isomorphic to $\mathbb{Z}^2$, and thus that C'' provides the subgroup we seek. Choose $\gamma \in C''$ such that γC generates C''/C. Let D be the group generated by γ. Then clearly $D \approx \mathbb{Z}$. Since $C'' = DC$ and C lies in the center of C'', it follows that $C'' \approx D \times C \approx \mathbb{Z}^2$. ◀

The first part of Theorem 7.20 and a weakening of the second part, when p_1 and p_2 are fixed in advance, was shown first by Häggström and Peres (1999) for Cayley graphs and other unimodular transitive graphs, then by Schonmann (1999b) in general. This answered affirmatively a question of Benjamini and Schramm (1996b).

Conjecture 7.25 on whether $p_{\mathrm{u}} < 1$ has been shown not only for finitely presented groups, as in Theorem 7.27, but also for planar quasi-transitive graphs, which will be shown in Section 8.5 and which also follows from Theorem 7.27. In addition, other techniques have established the conjecture for the cartesian product of two infinite graphs (Häggström, Peres, and Schonmann, 1999) and for Cayley graphs of Kazhdan groups, i.e., groups with Kazhdan's property T (Lyons and Schramm, 1999). The definition of Kazhdan's property is the following. Let Γ be a countable group and S a finite subset of Γ. Let $\mathscr{U}(\mathscr{H})$ denote the set of unitary representations of Γ on a Hilbert space $\mathscr{H}$ that have no invariant vectors except 0. Set

$$\kappa(\Gamma, S) := \max\left\{\epsilon\,;\ \forall \mathscr{H}\ \forall \pi \in \mathscr{U}(\mathscr{H})\ \forall v \in \mathscr{H}\ \exists s \in S \quad \|\pi(s)v - v\| \ge \epsilon\|v\|\right\}.$$

Then Γ is called ***Kazhdan*** (or has Kazhdan's property (T)) if $\kappa(\Gamma, S) > 0$ for all finite S. The only amenable Kazhdan groups are the finite ones. Examples of Kazhdan groups include $\mathrm{SL}(n, \mathbb{Z})$ for $n \ge 3$. See de la Harpe and Valette (1989) for background; in particular, every Kazhdan group is finitely generated (p. 11), but not necessarily finitely presentable (as shown by examples of Gromov; see p. 43). Every infinite Kazhdan group has only one end. See Żuk (1996) for examples of Kazhdan groups arising as fundamental groups of finite simplicial complexes. There is also a beautiful probabilistic characterization of Kazhdan groups. Let $\mathbf{P}_*$ be the probability measure on subsets of Γ that is the empty set half the time and all of Γ half the time. Recall that Γ acts by translation on the probability measures on 2^Γ.

---

* A group is torsion-free if all its elements have infinite order, other than the identity element.

Theorem 7.44. (Glasner and Weiss, 1997) *A countable infinite group* Γ *is Kazhdan iff* $\mathbf{P}_*$ *is not in the weak* closure of the* Γ*-invariant ergodic probability measures on* 2^Γ.

The inequality $p_u(G) \ge 1/(\rho(G)d)$ for quasi-transitive graphs of maximum degree d, which follows from Theorem 7.29 combined with Lemma 7.30, was proved earlier by Benjamini and Schramm (1996b).

According to Schonmann (1999a), techniques of that paper combined with those of Stacey (1996) can be used to show the same inequality as that of Theorem 7.34 for all $b \ge 2$, but the proof "is quite technical".

The only groups for which it is known that all their Cayley graphs satisfy $p_c < p_u$ are the groups of "cost" larger than 1. This includes, first, free groups of rank at least 2 and fundamental groups of compact surfaces of genus larger than 1. Second, let Γ_1 and Γ_2 be two groups of finite cost with Γ_1 having cost larger than 1. Then every amalgamation of Γ_1 and Γ_2 over an amenable group has cost larger than 1. Third, every HNN extension of Γ_1 over an amenable group has cost larger than 1. Also, every HNN extension of an infinite group over a finite group has cost larger than 1. Finally, every group with strictly positive first ℓ^2-Betti number has cost larger than 1. For the definition of cost and proofs that these groups have cost larger than 1, see Gaboriau (1998, 2000, 2002). The proof that $p_c(G) < p_u(G)$ follows fairly easily from Theorem 8.19 in the next chapter, as noted by Lyons (2000). If Γ is a group with a Cayley graph G and the free and wired uniform spanning forest measures on G differ, then Γ has cost larger than 1; possibly these are exactly the groups of cost larger than 1. See Questions 10.11 and 10.12.

The following results concerning the uniqueness phase of Bernoulli percolation are also known. (The definition of "unimodular" is in the next chapter.)

Theorem 7.45. *Let* G *be a transitive graph.*

(i) **(Schonmann, 1999b)**

$$p_u(G) = \inf\{p\,;\ \sup_R \inf_x \mathbf{P}_p[B(o,R) \leftrightarrow B(x,R)] = 1\}\,. \tag{7.21}$$

(ii) **(Lyons and Schramm, 1999)** *If* G *is unimodular and* $\inf_x \tau_p(o,x) > 0$, *then there is a unique infinite cluster* $\mathbf{P}_p$*-a.s. Therefore,*

$$p_u(G) = \inf\{p\,;\ \inf_x \tau_p(o,x) > 0\}\,. \tag{7.22}$$

Equation (7.22) implies (7.21), but it is unknown whether (7.22) holds in the nonunimodular case.

▷ **Exercise 7.15.**
Show that for every transitive graph G, $p_u^{\text{site}}(G) \ge p_u^{\text{bond}}(G)$.

▷ **Exercise 7.16.**
Show that if G and G' are roughly isometric quasi-transitive graphs, then $p_u(G) < 1$ iff $p_u(G') < 1$.

§7.10. Collected In-Text Exercises.

7.1. Let T be a tree that has no vertices of degree 1. Let B be the set of vertices of degree at least 3. Let K be a finite nonempty subset of vertices of T. Show that $|\partial_{\mathsf{E}} K| \ge |K \cap B| + 2$.

7.2. Prove (7.2).

7.3. Prove that if $\theta_y(p) > 0$ for some y, then for any x, also $\theta_x(p) > 0$.

7.4. Suppose that X is an increasing random variable with finite expectation and $\mathscr{F}$ is the σ-field generated by the "coordinate functions" $e \mapsto \omega(e)$ ($\omega \in 2^{\mathsf{E}}$) for the edges e belonging to some finite subset of E. Show that there is a version of $\mathbf{E}_p[X \mid \mathscr{F}]$ that is an increasing random variable.

7.5. Show that if $A_1, \ldots, A_n$ are increasing events, then for all p, $\mathbf{P}_p\big(\bigcap A_i\big) \ge \prod \mathbf{P}_p(A_i)$ and $\mathbf{P}_p\big(\bigcap A_i^c\big) \ge \prod \mathbf{P}_p(A_i^c)$.

7.6. Use Harris's inequality to do Exercise 7.3.

7.7. Show that for any finitely generated group, the number of ends is the same for all of its Cayley graphs. In fact, if two graphs are roughly isometric, then they have the same number of ends.

7.8. Show that if G and G' are any two infinite connected graphs, then the cartesian product graph $G \times G'$ (defined in Exercise 6.19) has only one end.

7.9. Show that if Γ and Γ' are any two finitely generated groups with $|\Gamma| \ge 2$ and $|\Gamma'| \ge 3$, then $\Gamma * \Gamma'$ has infinitely many ends.

7.10. Suppose that $\mathbf{P}_i$ ($i = 1, 2, 3$) are three percolation measures on 2^A such that $\mathbf{P}_1 \succcurlyeq \mathbf{P}_2$ and $\mathbf{P}_2 \succcurlyeq \mathbf{P}_3$. Show that there exist random variables ω_i on a common probability space such that $\omega_i \sim \mathbf{P}_i$ for all i and $\omega_1 \ge \omega_2 \ge \omega_3$ a.s.

7.11. Extend Theorem 7.19 to quasi-transitive graphs of upper exponential growth rate larger than 1.

7.12. **(a)** Show that if ω is a $\mathbf{P}$-a.s. non-empty $\mathrm{Aut}(G)$-invariant percolation on a quasi-transitive graph G, then $\mathbf{P}$-a.s. every end of G contains an end-convergent subset of ω.
(b) Show that if G is a quasi-transitive graph with infinitely many ends, then $p_{\mathrm{u}}(G) = 1$.

7.13. Show that for any graph G, $\rho(G) = \lim_{n\to\infty} \mathbf{P}_o[X_{2n} = o]^{1/2n}$ and $\mathbf{P}_o[X_n = o] \le \rho(G)^n$ for all n.

7.14. Show that $p(\mathbb{Z}^2, 2) = 0$ for the usual square lattice graph.

7.15. Show that for every transitive graph G, $p_{\mathrm{u}}^{\mathrm{site}}(G) \ge p_{\mathrm{u}}^{\mathrm{bond}}(G)$.

7.16. Show that if G and G' are roughly isometric quasi-transitive graphs, then $p_{\mathrm{u}}(G) < 1$ iff $p_{\mathrm{u}}(G') < 1$.

§7.11. Additional Exercises.

7.17. Show that if G' is a transitive representation of G, then G is amenable iff G' is amenable.

7.18. Show that a tree is quasi-transitive iff it is a universal cover of a finite undirected (multi-) graph (as in Section 3.3).

7.19. Let T be a finite tree. Call a vertex x a ***k*-branch point** of T if $T \backslash x$ has at least 3 components that each have at least k vertices. Let B_k be the set of k-branch points of T. Show that if $|B_k| \geq 1$, then $|T \setminus B_k| \geq k(|B_k| + 2)$.

7.20. Find the Cayley graph of the lamplighter group G_1 with respect to the 4 generators $(1, \mathbf{0})$, $(-1, \mathbf{0})$, $(1, \mathbf{1}_{\{0\}})$, and $(-1, \mathbf{1}_{\{0\}})$.

7.21. Show that if b_n is the size of a ball of radius n in the graph $\mathbb{T}_{b+1} \times \mathbb{Z}^d$, then $\lim_{n\to\infty} b_n^{1/n} = b$ for all $b \geq 2$ and $d \geq 0$.

7.22. Suppose that ω is an invariant percolation on a transitive graph G. Let Λ be the set of vertices x that belong to an infinite component η of ω such that $\eta \setminus \{x\}$ has at least 3 infinite components. Show that $\Phi_{\mathsf{E}}(G) \geq \mathbf{P}[o \in \Lambda]$.

7.23. Suppose that $\mathbf{P}$ is an invariant percolation on an amenable transitive graph. Show that all infinite clusters have at most 2 ends a.s.

7.24. Let $\mathfrak{F}$ be a random invariant forest on a transitive graph G. Show that

$$\mathbf{E}[\deg_{\mathfrak{F}} o] \leq \Phi_{\mathsf{V}}(G) + 2\,.$$

7.25. Show that if G is a quasi-transitive graph with at least 3 ends, then G has infinitely many ends.

7.26. Extend Theorem 7.8 to quasi-transitive amenable graphs.

7.27. Refine Proposition 7.9 to prove that $\theta_o^{\mathrm{site}}(p) \leq p\theta_o^{\mathrm{bond}}(p)$.

7.28. Extend Proposition 7.9 to show that there is a coupling of site percolation ξ and bond percolation ω such that every ξ-cluster is contained in some ω-cluster.

7.29. Strengthen Proposition 7.11 to show that for any graph G, we have $p_c(G) \geq 1/\operatorname{br} T^{\mathrm{SAW}}$.

7.30. Prove that $p_c^{\mathrm{bond}}(\mathbb{Z}^2) \leq 1 - 1/\mu(\mathbb{Z}^2)$.

7.31. Suppose that G is a graph such that for some constant $a < \infty$ and all $n \geq 1$, we have

$$\left|\left\{K \subset V\,;\, o \in K,\ K \text{ is finite},\ (K, \mathsf{E} \cap K \times K) \text{ is connected},\ |\partial_{\mathsf{E}} K| = n\right\}\right| \leq ae^{an}\,.$$

Show that $p_c^{\mathrm{bond}}(G) < 1$.

7.32. Let $U(e)$ be distinct labels on an infinite graph. Show that if some edge $e \in \partial_{\mathsf{E}} I(x)$ belongs to an infinite cluster of ω_p, then $|I(x) \setminus \eta| < \infty$ for some infinite cluster η of ω_p.

7.33. Give an example of a graph for which with positive probability there is some $p > p_c$ such that invasion percolation from a given vertex does not meet any infinite p-cluster.

7.34. Show that if T is any tree and $p < 1$, then Bernoulli(p) percolation on T has either no infinite clusters a.s. or infinitely many infinite clusters a.s.

7.35. Prove Theorem 7.27 for site percolation.

7.36. Let $G = (\mathsf{V}, \mathsf{E})$ be any graph all of whose vertices have degrees at most d. For any fixed $o \in \mathsf{V}$, let a_n be the number of connected subgraphs of G that contain o and have exactly n vertices. Show that $\limsup_{n\to\infty} a_n^{1/n} < ed$, where e is the base of natural logarithms.

7.37. Let $G = (\mathsf{V}, \mathsf{E})$ be any graph all of whose vertices have degrees at most d. For any fixed $o \in \mathsf{V}$, let b_n be the number of connected subgraphs of G that contain o and have exactly n edges. Show that $\limsup_{n\to\infty} b_n^{1/n} < e(d-1)$, where e is the base of natural logarithms.

7.38. A sequence of vertices $\langle x_k \rangle_{k\in\mathbb{Z}}$ is called a ***bi-infinite geodesic*** if $\mathrm{dist}_G(x_j, x_k) = |j-k|$ for all $j, k \in \mathbb{Z}$. Show that a transitive graph contains a bi-infinite geodesic passing through o.

7.39. Show that if G is a transitive graph G such that there is a set of cycles with bounded length whose linear span contains all cycles, then $p_c(G) < 1$.

7.40. Let G be any graph and let b_n be the number of paths of length n from o to o. Show that if $p > 1/\limsup_{n\to\infty} b_n^{1/n}$, then there is a unique infinite cluster $\mathbf{P}_p$-a.s.

7.41. Let G and G' be transitive networks with conductance sums at vertices π, π', respectively. (These are constants by transitivity.) If the cartesian product network is defined as in Exercise 6.19, then show that

$$\rho(G \times G') = \frac{\pi}{\pi + \pi'}\rho(G) + \frac{\pi'}{\pi + \pi'}\rho(G').$$

7.42. Prove that $p_c^{\mathrm{bond}}(\mathbb{T}_{b+1} \times \mathbb{Z}) < 1/b - 1/(2b^2)$ for $b \geq 3$.

7.43. Prove that for every transitive graph G, if b is sufficiently large, then $p_c(\mathbb{T}_{b+1} \times G) < p_u(\mathbb{T}_{b+1} \times G)$.

7.44. Show that Corollary 7.36 holds for quasi-transitive graphs when d_G is replaced by the maximum degree in G.

7.45. Show that if G is a $2k$-regular graph with $\Phi^*_{\mathsf{E}}(G) > 0$, then $p(G,k) > 0$. *Hint:* Suppose not. For small p, find a large finite set K such that K becomes occupied even if the outside of K were to be made vacant. Count the increase of the boundary throughout the evolution of the process. Use Exercise 7.36.

7.46. In the notation of Proposition 7.41, show that for the extreme values of the parameter k,

$$p(\mathbb{T}_{d+1}, d) = 1 - \frac{1}{d} \qquad \text{and} \qquad p(\mathbb{T}_{d+1}, 2) = 1 - \frac{(d-1)^{2d-3}}{d^{d-1}(d-2)^{d-2}} \sim \frac{1}{2d^2}. \tag{7.23}$$

7.47. For bootstrap percolation, define another critical probability, $b(G,k)$, as the infimum of initial probabilities for which, following the k-neighbor rule on G, there will be an infinite connected component of occupied vertices in the final configuration with positive probability. Clearly, $b(G,k) \leq p(G,k)$. Show that for any integers $d, k \geq 2$, if T is an infinite tree with maximum degree $d+1$, then $p(T,k) \geq b(\mathbb{T}_{d+1}, k) > 0$.

7.48. Let T_ξ be the family tree of a Galton-Watson process with offspring distribution ξ.
(a) Show that $p(T_\xi, k)$ is a constant almost surely, given non-extinction.
(b) Prove that $p(T_\xi, k) \ge p(T_\eta, k)$ if η stochastically dominates ξ.

7.49. Consider the Galton-Watson tree T_ξ with offspring distribution $\mathbf{P}(\xi = 2) = \mathbf{P}(\xi = 4) = 1/2$. Then $\mathrm{br}(T_\xi) = \mathbf{E}\xi = 3$ a.s., there are no finite 1-forts in T_ξ, and $0 < p(T_\xi, 2) < 1$ is an almost sure constant by Exercise 7.48. Prove that $p(T_\xi, 2) < p(\mathbb{T}_{3+1}, 2) = 1/9$.

Chapter 8

The Mass-Transport Technique and Percolation

In this chapter, all graphs are assumed to be locally finite without explicit mention. We will develop the Mass-Transport Principle in the context of percolation, where it has become indispensable, but we will find it quite useful in Chapters 10 and 13 as well. Recall that for us, ***percolation*** means simply a measure on subgraphs of a given graph. There are two main types, ***bond percolation***, wherein each subgraph has all the vertices, and ***site percolation***, where each subgraph is the graph induced by some of the vertices.

§8.1. The Mass-Transport Principle for Cayley Graphs.

Early forms of the mass-transport technique were used by Liggett (1987), Adams (1990) and van den Berg and Meester (1991). It was introduced in the study of percolation by Häggström (1997) and developed further in BLPS (1999b). This method is useful far beyond Bernoulli percolation. The principle on which it depends is best stated in a form that does not mention percolation or even probability at all. However, to motivate it, we first consider processes that are invariant with respect to a *countable* group Γ, meaning, in its greatest generality, a probability measure $\mathbf{P}$ on a space Ω on which Γ acts in such a way as to preserve $\mathbf{P}$. For example, we could take $\mathbf{P} := \mathbf{P}_p$ and $\Omega := 2^{\mathsf{E}}$, where E is the edge set of a Cayley graph of Γ. Let $F(x, y; \omega) \in [0, \infty]$ be a function of $x, y \in \Gamma$ and $\omega \in \Omega$. Suppose that F is invariant under the diagonal action of Γ; that is, $F(\gamma x, \gamma y; \gamma\omega) = F(x, y, \omega)$ for all $\gamma \in \Gamma$. We think of giving each element $x \in \Gamma$ some initial mass, possibly depending on ω, then redistributing it so that x sends y the mass $F(x, y; \omega)$. With this terminology, one hopes for "conservation" of mass, at least in expectation. Of course, the total amount of mass is usually infinite. Nevertheless, it turns out that there is a sense in which mass is conserved: the expected mass at an element before transport equals the expected mass at an element afterwards. Since F enters this equation only in expectation, it is convenient to set $f(x, y) := \mathbf{E}F(x, y; \omega)$. Then f is also diagonally invariant, i.e., $f(\gamma x, \gamma y) = f(x, y)$ for all $\gamma, x, y \in \Gamma$, because $\mathbf{P}$ is invariant.

The Mass-Transport Principle for Countable Groups. *Let Γ be a countable group. If $f : \Gamma \times \Gamma \to [0, \infty]$ is diagonally invariant, then*

$$\sum_{x \in \Gamma} f(o, x) = \sum_{x \in \Gamma} f(x, o) \,.$$

Proof. Just note that $f(o, x) = f(x^{-1}o, x^{-1}x) = f(x^{-1}, o)$ and that summation of $f(x^{-1}, o)$ over all x^{-1} is the same as $\sum_{x \in \Gamma} f(x, o)$ since inversion is a bijection of Γ. ◀

Before we use the Mass-Transport Principle in a significant way, we examine a few simple questions to illustrate where it is needed and how it is different from simpler principles. Let G be a Cayley graph of the group Γ and let $\mathbf{P}$ be an invariant percolation, i.e., an invariant measure on 2^{V}, on 2^{E}, or even on $2^{\mathsf{V} \cup \mathsf{E}}$. Let ω be a configuration with distribution $\mathbf{P}$.

Example 8.1. Could it be that ω is a single vertex a.s.? I.e., is there an invariant way to pick a vertex at random?

No: If there were, the assumptions would imply that the probability p of picking x is the same for all x, whence an infinite sum of p would equal 1, an impossibility.

Example 8.2. Could it be that ω is a finite nonempty vertex set a.s.? I.e., is there an invariant way to pick a finite set of vertices at random?

No: If there were, then we could pick one of the vertices of the finite set at random (uniformly), thereby obtaining an invariant probability measure on singletons.

Recall that ***cluster*** means connected component of the percolation subgraph.

Example 8.3. The number of *finite* clusters is $\mathbf{P}$-a.s. 0 or ∞. For if not, then we could condition on the number of finite clusters being finite and positive, then take the set of their vertices and arrive at an invariant probability measure on 2^{V} that is concentrated on finite sets.

Recall that a vertex x is a ***furcation*** of a configuration ω if removing x would split the cluster containing x into at least 3 infinite clusters.

Example 8.4. The number of furcations is $\mathbf{P}$-a.s. 0 or ∞. For the set of furcations has an invariant distribution on 2^{V}.

Example 8.5. $\mathbf{P}$-a.s. each cluster has 0 or ∞ furcations.

This does not follow from elementary considerations as the previous examples do, but requires the Mass-Transport Principle. (See Exercise 8.15 for a proof that elementary considerations do not suffice.) Namely, given the configuration ω, define $F(x, y; \omega)$ to be

0 if $K(x)$ has 0 or ∞ furcations, but to be $1/N$ if y is one of N furcations of $K(x)$ and $1 \le N < \infty$. Then F is diagonally invariant, whence the Mass-Transport Principle applies to $f(x,y) := \mathbf{E}F(x,y;\omega)$. Since $\sum_y F(x,y;\omega) \le 1$, we have

$$\sum_x f(o,x) \le 1\,. \tag{8.1}$$

If any cluster has a finite positive number of furcations, then each of them receives infinite mass. More precisely, if o is one of a finite number of furcations of $K(o)$, then $\sum_x F(x,o;\omega) = \infty$. Therefore, if with positive probability some cluster has a finite positive number of furcations, then with positive probability o is one of a finite number of furcations of $K(o)$, and therefore $\mathbf{E}\Big[\sum_x F(x,o;\omega)\Big] = \infty$. That is, $\sum_x f(x,o) = \infty$, which contradicts the Mass-Transport Principle and (8.1).

This is a typical application of the Mass-Transport Principle. Most applications are qualitative, like this one. For example, a combination of Examples 8.2 and 8.5 is:

Example 8.6. If there are infinite clusters with positive probability, then there is no invariant way to pick a finite nonempty subset from one or more infinite clusters, whether deterministically or with additional randomness. More precisely, there is no invariant measure on pairs (ω, ϕ) whose marginal on the first coordinate is $\mathbf{P}$ and with the properties that $\phi : \mathsf{V} \to 2^{\mathsf{V}}$ is a function such that $\phi(x) \subseteq K(x)$ is finite for all x, nonempty for at least one x belonging to an infinite ω-cluster, and such that whenever x and y lie in the same ω-cluster, $\phi(x) = \phi(y)$.

To illustrate a deeper use of the Mass-Transport Principle, we now give another proof that in the standard coupling of Bernoulli bond percolation on a Cayley graph, G, for all $p_c(G) < p_1 < p_2$, a.s. every infinite p_2-cluster contains some infinite p_1-cluster.

Let $\omega_1 \subseteq \omega_2$ be the configurations in the standard coupling of the two Bernoulli percolations. Note that (ω_1, ω_2) is invariant. Write $K_2(x)$ for the infinite cluster of x in ω_2.

Let η denote the union of all infinite clusters of ω_1. Define $F\big(x,y;(\omega_1,\omega_2)\big)$ to be 1 if x and y belong to the same ω_2-cluster and y is the unique vertex in η that is closest in ω_2 to x; otherwise, define $F\big(x,y;(\omega_1,\omega_2)\big) := 0$. Note that if there is not a unique such y, then x does not send any mass anywhere. Then F is diagonally invariant and $\sum_y F\big(x,y;(\omega_1,\omega_2)\big) \le 1$.

Suppose that with positive probability there is an infinite cluster of ω_2 that is disjoint from η. Let $A(z,y,e_1,e_2,\ldots,e_n)$ be the event that $K_2(z)$ is infinite and disjoint from η, that $y \in \eta$, and that $e_1, e_2, \ldots, e_n$ form a path of edges from z to y that lies outside

$K_2(z) \cup \eta$. Whenever there is an infinite cluster of ω_2 that is disjoint from η, there must exist two vertices z and y and some edges $e_1, e_2, \ldots, e_n$ for which $A(z, y, e_1, e_2, \ldots, e_n)$ holds. Hence, there exists $z, y, e_1, \ldots, e_n$ such that $\mathbf{P}\big(A(z, y, e_1, \ldots, e_n)\big) > 0$. Let $h : [0,1] \to [p_1, p_2]$ be affine and surjective. If B denotes the event obtained by replacing each label $U(e_k)$ by $h\big(U(e_k)\big)$ on each configuration in $A(z, y, e_1, \ldots, e_n)$, then $\mathbf{P}(B) > 0$ by Lemma 7.23. On the event B, we have $F\big(x, y; (\omega_1, \omega_2)\big) = 1$ for all $x \in K_2(z)$, whence $\sum_x \mathbf{E}F\big(x, y; (\omega_1, \omega_2)\big) = \infty$, which contradicts the Mass-Transport Principle.

§8.2. Beyond Cayley Graphs: Unimodularity.

The Mass-Transport Principle does not hold for all transitive graphs in the form given in Section 8.1. For example, if G is the graph of Example 7.1 with T having degree 3, then consider the function $f(x, y)$ that is the indicator of y being the ξ-grandparent of x.

▷ **Exercise 8.1.**
Show that every automorphism of the graph G (in the preceding sentence) fixes the end ξ and therefore that f is diagonally invariant under $\mathrm{Aut}(G)$.

Although f is diagonally invariant, we have that $\sum_{x \in T} f(o, x) = 1$, yet $\sum_{x \in T} f(x, o) = 4$. In particular, G is not a Cayley graph.

▷ **Exercise 8.2.**
Show that the graph of Example 7.2 is not a Cayley graph.

Nevertheless, there are many transitive graphs for which the Mass-Transport Principle does hold, the so-called unimodular graphs, and there is a generalization of the Mass-Transport Principle that holds for all graphs. The case of unimodular (quasi-transitive) graphs is the most important case and the one that we will focus on in the rest of this chapter.

Let G be a locally finite graph and Γ be a group of automorphisms of G. Let $S(x) := \{\gamma \in \Gamma ;\ \gamma x = x\}$ denote the ***stabilizer*** of x. Since all points in $S(x)y$ are at the same distance from x and G is locally finite, the set $S(x)y$ is finite for all x and y.

Theorem 8.7. (Mass-Transport Principle) *If Γ is a group of automorphisms of a graph $G = (\mathsf{V}, \mathsf{E})$, $f : \mathsf{V} \times \mathsf{V} \to [0, \infty]$ is invariant under the diagonal action of Γ, and $u, w \in \mathsf{V}$, then*

$$\sum_{z \in \Gamma w} f(u, z) = \sum_{y \in \Gamma u} f(y, w) \frac{|S(y)w|}{|S(w)y|} . \tag{8.2}$$

This formula is too complicated to remember, but note the form it takes when Γ is transitive:

Corollary 8.8. *If Γ is a transitive group of automorphisms of a graph $G = (\mathsf{V}, \mathsf{E})$, $f : \mathsf{V} \times \mathsf{V} \to [0, \infty]$ is invariant under the diagonal action of Γ, and $o \in \mathsf{V}$, then*

$$\sum_{x \in \mathsf{V}} f(o, x) = \sum_{x \in \mathsf{V}} f(x, o) \frac{|S(x)o|}{|S(o)x|} . \tag{8.3}$$

Note how this works to restore "conservation of mass" for the graph of Example 7.1: Suppose that the graph is based on a tree of degree 3. If o is the ξ-grandparent of x, then $|S(x)o| = 1$ and $|S(o)x| = 4$, so that the left-hand side of (8.3) is the sum of 1 term equal to 1, while the right-hand side is the sum of 4 terms, each equal to $1/4$.

We say that Γ is ***unimodular*** if $|S(x)y| = |S(y)x|$ for all pairs (x, y) such that $y \in \Gamma x$. We also say that a graph is ***unimodular*** when its (full) automorphism group is. If Γ is unimodular and also transitive, then (8.3) simplifies to

$$\sum_{x \in \mathsf{V}} f(o, x) = \sum_{x \in \mathsf{V}} f(x, o) . \tag{8.4}$$

Thus, all the applications of the Mass-Transport Principle in Section 8.1, which were qualitative, apply (with the same proofs) to transitive unimodular graphs. In fact, they apply as well to all quasi-transitive unimodular graphs since for such graphs, $|S(x)y|/|S(y)x|$ is bounded over all pairs (x, y) by Theorem 8.10 below, and we may sum (8.2) over all pairs (u, v) chosen from a complete set of orbit representatives.

To prove Theorem 8.7, let

$$\Gamma_{x,y} := \{\gamma \in \Gamma ;\ \gamma x = y\} .$$

Note that for any $\gamma \in \Gamma_{x_1,x_2}$, we have

$$\Gamma_{x_1,x_2} = \gamma S(x_1) = S(x_2)\gamma .$$

Therefore, for all x_1, x_2, y_1 and any $\gamma \in \Gamma_{x_1,x_2}$,

$$|\Gamma_{x_1,x_2} y_1| = |\gamma S(x_1) y_1| = |S(x_1) y_1| \tag{8.5}$$

and, writing $y_2 := \gamma y_1$, we have

$$|\Gamma_{x_1,x_2} y_1| = |S(x_2)\gamma y_1| = |S(x_2) y_2| . \tag{8.6}$$

Proof of Theorem 8.7. Let $z \in \Gamma w$, so that $w = \gamma z$ for some $\gamma \in \Gamma_{z,w}$. If $y := \gamma u$, then $f(u, z) = f(\gamma u, \gamma z) = f(y, w)$. That is, $f(u, z) = f(y, w)$ whenever $y \in \Gamma_{z,w} u$. Therefore,

$$\sum_{z \in \Gamma w} f(u, z) = \sum_{z \in \Gamma w} \frac{1}{|\Gamma_{z,w} u|} \sum_{y \in \Gamma_{z,w} u} f(y, w) = \sum_{y \in \Gamma u} f(y, w) \sum_{z \in \Gamma_{y,u} w} \frac{1}{|\Gamma_{z,w} u|}$$

because

$$\begin{aligned}\big\{(z, y)\,;\ z \in \Gamma w,\ y \in \Gamma_{z,w} u\big\} &= \big\{(z, y)\,;\ \exists \gamma \in \Gamma\ \ y = \gamma u,\ w = \gamma z\big\} \\ &= \big\{(z, y)\,;\ \exists \gamma \in \Gamma\ \ u = \gamma^{-1} y,\ z = \gamma^{-1} w\big\} \\ &= \big\{(z, y)\,;\ y \in \Gamma u,\ z \in \Gamma_{y,u} w\big\}\,.\end{aligned}$$

Using (8.6) and then (8.5), we may rewrite this as

$$\sum_{y \in \Gamma u} f(y, w) \frac{|\Gamma_{y,u} w|}{|S(w) y|} = \sum_{y \in \Gamma u} f(y, w) \frac{|S(y) w|}{|S(w) y|}\,. \qquad ◀$$

▷ **Exercise 8.3.**

Let Γ be a transitive group of automorphisms of a graph that satisfies (8.4) for all Γ-invariant f. Show that Γ is unimodular.

▷ **Exercise 8.4.**

Show that if Γ is a transitive unimodular group of automorphisms and Γ' is a larger group of automorphisms of the same graph, then Γ' is also transitive and unimodular.

By Exercise 8.3, every Cayley graph is unimodular. We will generalize this in Proposition 8.9. Call a group Γ of automorphisms ***discrete*** if all stabilizers are finite. Recall that $[\Gamma : \Gamma']$ denotes the index of a subgroup Γ' in a group Γ, i.e., the number of cosets of Γ' in Γ.

▷ **Exercise 8.5.**

Show that for all x and y, we have $|S(x)y| = [S(x) : S(x) \cap S(y)]$.

Proposition 8.9. *If Γ is a discrete group of automorphisms, then Γ is unimodular.*

Proof. Suppose that $y = \gamma x$. Then $S(x)$ and $S(y)$ are conjugate subgroups since $\gamma_1 \mapsto \gamma\gamma_1\gamma^{-1}$ is a bijection of $S(x)$ to $S(y)$, so $|S(x)| = |S(y)|$, whence

$$\frac{|S(x)y|}{|S(y)x|} = \frac{[S(x) : S(x) \cap S(y)]}{[S(y) : S(x) \cap S(y)]} = \frac{|S(x)|/|S(x) \cap S(y)|}{|S(y)|/|S(x) \cap S(y)|} = \frac{|S(x)|}{|S(y)|} = 1\,. \qquad ◀$$

Sometimes we can make additional use of the Mass-Transport Principle from the following fact.

Theorem 8.10. *If Γ is a group of automorphisms of any graph G, then there are numbers μ_x $(x \in \mathsf{V})$ that are unique up to a constant multiple such that for all x and y,*

$$\frac{\mu_x}{\mu_y} = \frac{|S(x)y|}{|S(y)x|} \,. \tag{8.7}$$

Proof. Recall the following standard extension of Lagrange's theorem: if Γ_3 is a subgroup of Γ_2, which in turn is a subgroup of Γ_1, then

$$[\Gamma_1 : \Gamma_2][\Gamma_2 : \Gamma_3] = [\Gamma_1 : \Gamma_3] \,. \tag{8.8}$$

Applying (8.8) to $\Gamma_3 := S(x) \cap S(y) \cap S(z)$, $\Gamma_2 := S(x) \cap S(y)$, and Γ_1 equal to either $S(x)$ or $S(y)$, we get

$$\frac{[S(x) : S(x) \cap S(y)]}{[S(y) : S(x) \cap S(y)]} = \frac{[S(x) : S(x) \cap S(y) \cap S(z)]}{[S(y) : S(x) \cap S(y) \cap S(z)]} \,.$$

Combining this in the three forms arising from the three cyclic permutations of the ordered triple (x, y, z) together with Exercise 8.5, we obtain the "cocycle" identity

$$\frac{|S(x)y|}{|S(y)x|} \frac{|S(y)z|}{|S(z)y|} = \frac{|S(x)z|}{|S(z)x|} \tag{8.9}$$

for all x, y, z.

Thus, if we fix $o \in \mathsf{V}$, choose $\mu_o \in \mathbb{R} \setminus \{0\}$, and define $\mu_x := \mu_o |S(x)o| / |S(o)x|$, then for all x, y, we have

$$\frac{\mu_x}{\mu_y} = \frac{\mu_o |S(x)o| / |S(o)x|}{\mu_o |S(y)o| / |S(o)y|} = \frac{|S(x)y|}{|S(y)x|}$$

by (8.9). On the other hand, if μ'_x $(x \in \mathsf{V})$ also satisfies (8.7), then

$$\frac{\mu_x}{\mu'_x} = \frac{\mu_o |S(x)o| / |S(o)x|}{\mu'_o |S(x)o| / |S(o)x|} = \frac{\mu_o}{\mu'_o}$$

for all x. ◀

Note that Γ is unimodular iff $\mu_y = \mu_x$ whenever $y \in \Gamma x$.

The following exercise makes it easier to check the definition of unimodularity.

▷ **Exercise 8.6.**

Show that if Γ acts transitively, then Γ is unimodular iff $|S(x)y| = |S(y)x|$ for all edges $[x, y]$.

▷ **Exercise 8.7.**

Show that if Γ acts transitively and for all edges $[x, y]$, there is some $\gamma \in \Gamma$ such that $\gamma x = y$ and $\gamma y = x$, then Γ is unimodular.

Using the numbers μ_x, we may write (8.2) as

$$\sum_{z \in \Gamma w} f(u, z)\mu_w = \sum_{y \in \Gamma u} f(y, w)\mu_y \,. \tag{8.10}$$

If one knows about Haar measure, it is not hard to translate the proof of Proposition 8.9 to show that μ_x is the left-invariant Haar measure of $S(x)$ (in the closure of Γ if Γ is not already closed, where the topology is defined in Exercise 8.17). Unimodularity is equivalent to the existence of a Borel measure on Γ that is both left and right invariant; this is, in fact, the usual definition of unimodularity for a locally compact group. One can also use Haar measure to give a simple proof of Theorem 8.7; see BLPS (1999b) for such a proof (due to Woess). In the amenable quasi-transitive case, Exercise 8.23 gives another interpretation of the weights μ_x.

Corollary 8.11. *Let Γ be a quasi-transitive group of automorphisms of a graph, G. Choose a complete set $\{o_1, \ldots, o_L\}$ of representatives in V of the orbits of Γ. Let μ_i be the weight of o_i as given by Theorem 8.10. If Γ is unimodular, then whenever $f : \mathsf{V} \times \mathsf{V} \to [0, \infty]$ is invariant under the diagonal action of Γ, we have*

$$\sum_{i=1}^{L} \mu_i^{-1} \sum_{z \in \mathsf{V}} f(o_i, z) = \sum_{j=1}^{L} \mu_j^{-1} \sum_{y \in \mathsf{V}} f(y, o_j) \,. \tag{8.11}$$

Conversely, if there exist numbers $\nu_x \ge 0$ for $x \in \mathsf{V}(G)$ such that $0 < \sum_x \nu_x < \infty$ and

$$\sum_{x \in \mathsf{V}} \nu_x \sum_{z \in \mathsf{V}} f(x, z) = \sum_{x \in \mathsf{V}} \nu_x \sum_{y \in \mathsf{V}} f(y, x) \tag{8.12}$$

whenever $f : \mathsf{V} \times \mathsf{V} \to [0, \infty]$ is invariant under the diagonal action of Γ, then Γ is unimodular and

$$\mu_i^{-1} = \sum_{x \in \Gamma o_i} \nu_x \tag{8.13}$$

for $1 \le i \le L$.

Note that (8.11) is the special case of (8.12) where $\nu_x = \mu_i^{-1}$ if $x = o_i$ and $\nu_x = 0$ otherwise.

Proof. Assume first that Γ is unimodular. Then $\mu_y = \mu_i$ for $y \in \Gamma o_i$. Thus, for each i and j, (8.10) gives

$$\sum_{z \in \Gamma o_j} f(o_i, z)\mu_j = \sum_{y \in \Gamma o_i} f(y, o_j)\mu_i\,,$$

i.e.,

$$\mu_i^{-1} \sum_{z \in \Gamma o_j} f(o_i, z) = \mu_j^{-1} \sum_{y \in \Gamma o_i} f(y, o_j)\,.$$

Adding these equations over all i and j gives the desired result.

Conversely, assume that ν_x are as stated. Define

$$a_i := \sum_{x \in \Gamma o_i} \nu_x\,.$$

We first show that $a_i > 0$. Let each vertex x send mass 1 to each vertex in Γo_i that is nearest to x. Since the left-hand side of (8.12) is positive, so is the right-hand side. Since only vertices in Γo_i receive mass, it follows that $a_i > 0$, as desired.

To see that Γ is unimodular, consider any j, k and any $u \in \Gamma o_j$ and $v \in \Gamma o_k$. Let $f(x, y) := \mathbf{1}_{\Gamma_{u,x} v}(y)$. It is straightforward to check that f is diagonally invariant under Γ. Note that

$$|S(x)z|\mathbf{1}_{\Gamma x}(y) = |\Gamma_{x,y} z|$$

for all $x, y, z \in \mathsf{V}(G)$ and that

$$z \in \Gamma_{u,x} v \iff x \in \Gamma_{v,z} u\,. \tag{8.14}$$

Therefore, we have

$$\begin{aligned}
|S(u)v|a_j &= \sum_y \nu_y |\Gamma_{u,y} v| = \sum_y \nu_y \sum_x \mathbf{1}_{\Gamma_{u,y} v}(x) \\
&= \sum_y \nu_y \sum_x f(y, x) = \sum_z \nu_z \sum_x f(x, z) \qquad [\text{by } (8.12)] \\
&= \sum_z \nu_z \sum_x \mathbf{1}_{\Gamma_{u,x} v}(z) = \sum_z \nu_z \sum_x \mathbf{1}_{\Gamma_{v,z} u}(x) \qquad [\text{by } (8.14)] \\
&= \sum_z \nu_z |\Gamma_{v,z} u| = |S(v)u|a_k\,.
\end{aligned}$$

That is,

$$|S(u)v|a_j = |S(v)u|a_k\,. \tag{8.15}$$

If we take $j = k$, then we see that G is unimodular. In general, comparison of (8.15) with (8.7) shows (8.13). ◀

Because of this result, in the quasi-transitive unimodular case, we will always assume that the weights are chosen so that $\sum_i \mu_i^{-1} = 1$. It then makes sense to think of o_i being picked randomly with probability μ_i^{-1}. If we denote such a random root by $\hat{o}$, then (8.11) assumes a very simple form:

$$\mathbf{E}\Big[\sum_x f(\hat{o}, x)\Big] = \mathbf{E}\Big[\sum_x f(x, \hat{o})\Big] . \tag{8.16}$$

This will be the usual way we apply the Mass-Transport Principle on quasi-transitive unimodular graphs. We will call such a random root $\hat{o}$ ***normalized***.

▷ **Exercise 8.8.**

Extend Exercise 8.4 to the quasi-transitive case: Show that if Γ is a quasi-transitive unimodular group of automorphisms and Γ' is a larger group of automorphisms of the same graph, then Γ' is also quasi-transitive and unimodular.

It is nice to know that when one is in the amenable setting, unimodularity is automatic, as shown by Soardi and Woess (1990):

Proposition 8.12. (Amenability Implies Unimodularity) *Any transitive group Γ of automorphisms of an amenable graph G is unimodular.*

Proof. Let $\langle F_n \rangle$ be a sequence of finite sets of vertices in G such that $|\overline{F_n}|/|F_n| \to 1$ as $n \to \infty$, where $\overline{F_n} := F_n \cup \partial_{\mathsf{V}} F_n$ is the union of F_n with its exterior vertex boundary.

Fix neighboring vertices $x \sim y$. We count the number of pairs (z, w) such that $z \in F_n$ and $w \in \Gamma_{x,z} y$ (or equivalently, $z \in \Gamma_{y,w} x$) in two ways: by summing over z first or over w first. In view of (8.5), this gives

$$|F_n||S(x)y| = \sum_{z \in F_n} \sum_{w \in \Gamma_{x,z} y} 1 \le \sum_{w \in \overline{F_n}} \sum_{z \in \Gamma_{y,w} x} 1 = |\overline{F_n}||S(y)x| .$$

Dividing both sides by F_n and taking a limit, we get $|S(x)y| \le |S(y)x|$. By symmetry and Exercise 8.6, we are done. ◀

Likewise, amenable quasi-transitive graphs are unimodular: see Exercise 8.22. For example, the weights of the two types of vertices in the graph of Figure 8.1 are $1/5$ and $4/5$.

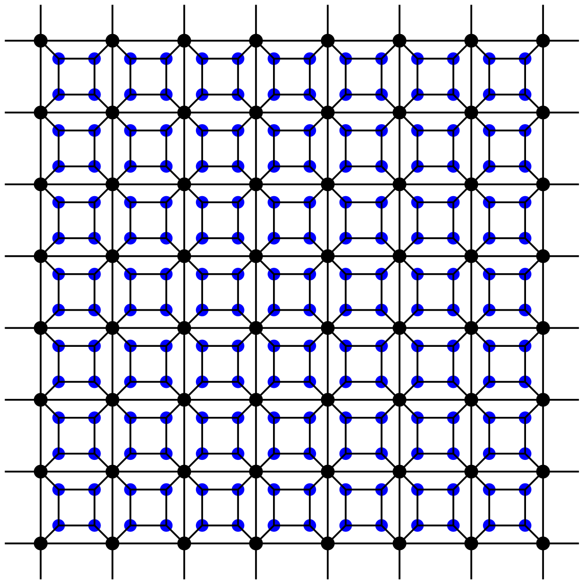


Figure 8.1.

§8.3. Infinite Clusters in Invariant Percolation.

What happens in Bernoulli percolation at p_c itself, i.e., is there an infinite cluster a.s.? Extending the classical conjecture for Euclidean lattices, Benjamini and Schramm (1996b) made the following conjecture:

Conjecture 8.13. *If G is any quasi-transitive graph with $p_c(G) < 1$, then $\theta\big(p_c(G)\big) = 0$.*

In the next section, we will establish this conjecture under the additional hypotheses that G is non-amenable and unimodular. This will utilize in a crucial way more general invariant percolation processes, which is the topic of the present section. What is the advantage conferred by nonamenability that allows a resolution of this conjecture? In essence, it is that *all* finite sets have a comparatively large boundary; since clusters are chosen at random in percolation, this is an advantage compared to knowing, in the amenable case, that *certain* large sets have comparatively small boundary.

We first present a simple but powerful result on a threshold for having infinite clusters and then discuss the number of ends in infinite clusters when the percolation gives a forest. Write $\deg_K(x)$ for the degree of x as a vertex in a subgraph K.

Theorem 8.14. (Thresholds for Finite Clusters) *Let G be a transitive unimodular graph of degree d_G and $\mathbf{P}$ be an automorphism-invariant probability measure on 2^{E}. If $\mathbf{P}$-a.s. all clusters are finite, then*

$$\mathbf{E}[\deg_\omega o] \le d_G - \Phi_{\mathsf{E}}(G)\,.$$

This was first proved for regular trees by Häggström (1997) and then extended by BLPS (1999b). Häggström (1997) showed that the threshold is sharp for regular trees. Of course, it is useless when G is amenable. For Bernoulli percolation, the bound it gives on p_c is worse than Theorem 6.23. But it is extremely useful for a wide variety of other percolation processes.

We could have stated it more generally with a unimodular group $\Gamma \subseteq \mathrm{Aut}(G)$ acting transitively on G and $\mathbf{P}$ being assumed invariant under Γ. In fact, all such results that we present have similar generalizations, but for simplicity of language, we will refer just to invariant percolation on a transitive or quasi-transitive graph.

To prove Theorem 8.14, define for finite subgraphs $K = \big(V(K), E(K)\big) \subset G$

$$\alpha_K := \frac{1}{|V(K)|} \sum_{x \in \mathsf{V}(K)} \deg_K(x)\,,$$

the average (internal) degree of K. Then set*

$$\alpha(G) := \sup\{\alpha_K\,;\ K \subset G \text{ is finite}\}\,.$$

If G is a regular graph of degree d_G, then

$$\alpha(G) + \Phi_{\mathsf{E}}(G) = d_G\,. \tag{8.17}$$

This is because we may restrict attention to induced subgraphs, i.e., those subgraphs K where $E(K) = \mathsf{V}(K) \times \mathsf{V}(K) \cap \mathsf{E}$, and for such subgraphs, $\sum_{x \in \mathsf{V}(K)} \deg_K(x) + |\partial_{\mathsf{E}} K|$ counts each edge in $E(K)$ twice and each edge in $\partial_{\mathsf{E}} K$ once, as does $d_G|V(K)|$. Dividing by $|V(K)|$ gives (8.17).

In view of (8.17), we may write the conclusion of Theorem 8.14 as

$$\mathbf{E}[\deg_\omega o] \le \alpha(G)\,. \tag{8.18}$$

This inequality is actually rather intuitive: It says that random finite clusters have average degree no more than the supremum average degree of arbitrary finite subgraphs. Of course, the first sense of "average" is "expectation", while the second is "arithmetic mean". This is reminiscent of the ergodic theorem, which says that a spatial average (expectation) is the limit of time averages (means).

---

* We remark that $\beta(G) = 2/\alpha(G)$, with $\beta(G)$ defined as in Section 6.4, except that β was defined with a liminf, rather than an infimum.

Proof of Theorem 8.14. We use the Mass-Transport Principle (8.4). Start with mass $\deg_\omega x$ at each vertex x and redistribute it equally among the vertices in its cluster $K(x)$ (including x itself). After transport, the mass at x is $\alpha_{K(x)}$. This is an invariant transport, so that if $f(x, y)$ denotes the expected mass taken from x and transported to y, then we have

$$\mathbf{E}[\deg_\omega o] = \sum_x f(o, x) = \sum_x f(x, o) = \mathbf{E}[\alpha_{K(o)}] .$$

By definition, $\alpha_{K(o)} \le \alpha(G)$, whence (8.18) follows. ◀

Corollary 8.15. *Let G be a quasi-transitive non-amenable unimodular graph. There is some $\epsilon > 0$ such that if $\mathbf{P}$ is an automorphism-invariant probability measure on 2^{E} with all clusters finite $\mathbf{P}$-a.s., then for some $x \in \mathsf{V}$,*

$$\mathbf{E}[\deg_\omega x] \le \deg_G x - \epsilon .$$

Proof. Let G' be a transitive representation of G. Then $\mathbf{P}$ induces an invariant percolation $\mathbf{P}'$ on G' by letting an edge $[x, y]$ in G' be open iff x and y are joined by a path of open edges in G of length at most $2r + 1$ (where r is as in the definition of transitive representation). If $\mathbf{P}$ has a high marginal at all x, then so does $\mathbf{P}'$, whence the conclusion follows easily from Theorem 8.14. ◀

Some variations on Theorem 8.14 are contained in the following exercises, as well as in others at the end of the chapter.

▷ **Exercise 8.9.**

Show that for any invariant percolation $\mathbf{P}$ on subgraphs of a transitive unimodular graph that has only finite clusters a.s., $\mathbf{E}[\deg_\omega o \mid o \in \omega] \le \alpha(G)$. More generally, let G be a quasi-transitive unimodular graph with a normalized random root $\hat{o}$. Show that if $\mathbf{P}$ is an invariant percolation on subgraphs of G such that all clusters are finite a.s., then $\mathbf{E}[\deg_\omega \hat{o} \mid \hat{o} \in \omega] \le \alpha(G)$.

▷ **Exercise 8.10.**

Let $\mathbf{P}$ be an invariant percolation on subgraphs of a transitive unimodular graph such that all clusters are finite trees a.s. Show that $\mathbf{E}[\deg_\omega o \mid o \in \omega] < 2$. More generally, let G be a quasi-transitive unimodular graph with a normalized random root $\hat{o}$. Show that if $\mathbf{P}$ is an invariant percolation on subgraphs of G such that all clusters are finite trees a.s., then $\mathbf{E}[\deg_\omega \hat{o} \mid \hat{o} \in \omega] < 2$.

We now take a look at forest percolations with infinite trees.

Proposition 8.16. *Let G be a quasi-transitive unimodular graph with a normalized random root $\hat{o}$. Let $\mathfrak{F}$ be the configuration of an invariant percolation on G such that $\mathfrak{F}$ is a forest, all of whose trees are infinite a.s. Then*

(i) *if each tree in $\mathfrak{F}$ has 1 or 2 ends a.s., then $\mathbf{E}\big[\deg_{\mathfrak{F}} \hat{o} \bigm| \hat{o} \in \mathfrak{F}\big] = 2$;*

(ii) *if each tree in $\mathfrak{F}$ has at least 3 ends a.s., then $\mathbf{E}\big[\deg_{\mathfrak{F}} \hat{o} \bigm| \hat{o} \in \mathfrak{F}\big] > 2$.*

Proof. Let $\xi(x, y; \mathfrak{F})$ be the indicator that there is a ray in $\mathfrak{F}$ starting at x whose first vertex after x is y. Let

$$F(x, y; \mathfrak{F}) := \begin{cases} 2\xi(x, y; \mathfrak{F}) & \text{if } \xi(y, x; \mathfrak{F}) = 0, \\ \xi(x, y; \mathfrak{F}) & \text{otherwise.} \end{cases}$$

and $f(x, y) := \mathbf{E}[F(x, y; \mathfrak{F})]$. Now $\sum_x \big(F(o, x; \mathfrak{F}) + F(x, o; \mathfrak{F})\big) = 2\big(\deg_{\mathfrak{F}} o\big)\mathbf{1}_{\{o \in \mathfrak{F}\}}$, so that $\sum_x \big(f(o, x) + f(x, o)\big) = 2\mathbf{E}[\deg_{\mathfrak{F}} o\,;\ o \in \mathfrak{F}]$ for all $o \in \mathfrak{F}$. By (8.16), we obtain that

$$\mathbf{E}\big[\deg_{\mathfrak{F}} \hat{o}\,;\ \hat{o} \in \mathfrak{F}\big] = \frac{1}{2}\mathbf{E}\Big[\sum_x \big(f(\hat{o}, x) + f(x, \hat{o})\big)\Big] = \mathbf{E}\Big[\sum_x f(\hat{o}, x)\Big]. \tag{8.19}$$

Now in case (i), $\sum_x F(o, x; \mathfrak{F}) = 2 \cdot \mathbf{1}_{\{o \in \mathfrak{F}\}}$, whence by (8.19), $\mathbf{E}\big[\deg_{\mathfrak{F}} \hat{o}\,;\ \hat{o} \in \mathfrak{F}\big] = 2\mathbf{P}[\hat{o} \in \mathfrak{F}]$. This gives the desired result.

In case (ii), $\sum_x F(o, x; \mathfrak{F}) \geq 2$ for all $o \in \mathfrak{F}$ and $\sum_x F(o, x; \mathfrak{F}) \geq 3$ for all furcations $o \in \mathfrak{F}$; note that $\hat{o}$ is a furcation with positive probability since $\mathfrak{F}$ has trees with at least 3 ends. Therefore, (8.19) yields $\mathbf{E}[\deg_{\mathfrak{F}} \hat{o}\,;\ \hat{o} \in \mathfrak{F}] > 2\mathbf{P}[\hat{o} \in \mathfrak{F}]$. ◀

As a consequence of this, the number of ends is related to the critical value for trees in invariant percolation:

Theorem 8.17. *Let G be a quasi-transitive unimodular graph and $\hat{o}$ a normalized random root. Let $\mathfrak{F}$ be the configuration of an invariant percolation on G such that $\mathfrak{F}$ is a forest a.s. Then the following are equivalent:*

(i) *some component of $\mathfrak{F}$ has at least 3 ends with positive probability;*

(ii) *some component of $\mathfrak{F}$ has $p_c < 1$ with positive probability;*

(iii) $\mathbf{E}\big[\deg_{\mathfrak{F}} \hat{o} \bigm| |K(\hat{o})| = \infty\big] > 2$.

Of course, by Theorem 5.15, (ii) is equivalent to saying that some component of $\mathfrak{F}$ has branching number > 1 with positive probability.

Proof. The implication (i) implies (iii) is immediate from Proposition 8.16.

Now assume (iii). Let $\mathfrak{F}'$ be the mixed (site and bond) percolation obtained from $\mathfrak{F}$ by retaining only those vertices and edges that belong to an infinite cluster. Then we may rewrite (iii) as $\mathbf{E}[\deg_{\mathfrak{F}'} \hat{o} \mid \hat{o} \in \mathfrak{F}'] > 2$. Let p be sufficiently close to 1 that independent Bernoulli(p) bond percolation on $\mathfrak{F}'$ yields a configuration $\mathfrak{F}''$ with $\mathbf{E}[\deg_{\mathfrak{F}''} \hat{o} \mid \hat{o} \in \mathfrak{F}''] > 2$. (Note that $\mathsf{V}(\mathfrak{F}'') = \mathsf{V}(\mathfrak{F}')$, so that $\hat{o} \in \mathfrak{F}''$ iff $\hat{o} \in \mathfrak{F}'$.) According to Exercise 8.10, we have that $\mathfrak{F}''$ contains infinite clusters with positive probability, whence (ii) follows.

Finally, (ii) implies (i) trivially. ◀

Corollary 8.18. (BLPS (1999b)) *Let G be a quasi-transitive unimodular graph. Let $\mathfrak{F}$ be the configuration of an invariant percolation on G such that $\mathfrak{F}$ is a forest a.s. Then almost surely every component that has at least 3 ends has $p_c < 1$.*

Proof. If not, condition on having some component with at least 3 ends and $p_c = 1$. Then the collection of all such components gives an invariant percolation that contradicts Theorem 8.17. ◀

§8.4. Critical Percolation on Nonamenable Transitive Unimodular Graphs.

Here we establish Conjecture 8.13 for non-amenable quasi-transitive unimodular graphs. This was shown by BLPS (1999b), with a more direct proof given in Benjamini, Lyons, Peres, and Schramm (1999a). The proof here is a mixture of the two proofs.

Theorem 8.19. *If G is a non-amenable quasi-transitive unimodular graph, then $\theta\big(p_c(G)\big) = 0$.*

We remark that this also implies that $p_c(G) < 1$.

Proof. The proof for site percolation is similar to that for bond, so we treat only bond percolation.

In light of Theorem 7.5, we must rule out the possibilities that the number of infinite clusters is 1 or ∞ at criticality. We do these separately and use the standard coupling of percolation arising from i.i.d. uniform $[0, 1]$-valued random variables $U(e)$ on the edges. As usual, let ω_p consist of the edges e with $U(e) < p$.

First suppose that there is a unique infinite cluster in ω_{p_c}. Let ω' be that infinite cluster. For each $\epsilon > 0$, define a new bond percolation ξ_ϵ consisting of those edges $[x, y] \in \mathsf{E}$ such that

(a) $\mathrm{dist}_G(x, \omega') < 1/\epsilon$;
(b) $\mathrm{dist}_G(y, \omega') < 1/\epsilon$; and
(c) if x' is any of the points in ω' of minimal distance (in G) to x and y' is any of the points in ω' of minimal distance to y, then x' and y' are in the same cluster in $\omega_{p_c-\epsilon}$.

Then $\xi_\epsilon \subseteq \xi_{\epsilon'}$ for $\epsilon > \epsilon'$ and $\bigcup_\epsilon \xi_\epsilon = \mathsf{E}$, whence

$$\lim_{\epsilon \to 0} \mathbf{P}\big[[x, y] \in \xi_\epsilon\big] = 1 .$$

By Corollary 8.15, it follows that for small enough ϵ, there is an infinite cluster in ξ_ϵ with positive probability. But whenever ξ_ϵ contains an infinite cluster, so must $\omega_{p_c-\epsilon}$: if $\langle x_n \rangle$ are the vertices of an infinite simple path in ξ_ϵ, then choose a nearest point $x'_n \in \omega_{p_c-\epsilon}$ to x_n. We have that x'_n is connected to x'_{n+1} in $\omega_{p_c-\epsilon}$ for each n, so all the points x'_n lie in the same cluster of $\omega_{p_c-\epsilon}$. By (a), there are infinitely many distinct points among the x'_n, so there is indeed an infinite cluster in $\omega_{p_c-\epsilon}$. This contradicts the definition of p_c and proves that a unique infinite cluster is impossible at p_c.

Now suppose that there are infinitely many infinite clusters in ω_{p_c}. Let $\mathfrak{F}$ consist of those edges $[x, y]$ such that $U([x, y]) < p_c$ and such that there is no path from x to y that uses only edges e with $U(e) < U([x, y])$. (This is the free minimal spanning forest of ω_{p_c}, a topic to be studied in Chapter 11.) We claim that $\mathfrak{F}$ contains a spanning tree in each infinite cluster of ω_{p_c} (a.s.). First, it is clear that $\mathfrak{F}$ contains no cycles, as the edge with the largest label in any cycle cannot be in $\mathfrak{F}$. Second, if the edges from $\mathfrak{F}$ yield more than one tree (including the possibility of an isolated vertex) in some infinite cluster of ω_{p_c}, then there is some edge $[x, y] \in \omega_{p_c}$ whose endpoints belong to different trees in $\mathfrak{F}$. Then some path in ω_{p_c} from x to y uses only edges e with $U(e) < U([x, y])$, and every such path contains an edge e' that itself can be replaced by a path joining its endpoints and using only edges with values smaller than $U(e')$. This implies that there is an infinite cluster of edges e with $U(e) < U([x, y])$, which contradicts the fact that $U([x, y]) < p_c$. This proves our claim.

Now by insertion tolerance, as in the proof of Theorem 7.6, some infinite cluster has at least 3 ends, whence so does some tree in $\mathfrak{F}$. According to Theorem 8.17, this implies that some tree in $\mathfrak{F}$ has $p_c < 1$, whence the same is true of its containing cluster in ω_{p_c}. But this contradicts the definition of p_c. ◀

▷ **Exercise 8.11.**

Suppose that $\mathbf{P}$ is an invariant percolation on a non-amenable quasi-transitive unimodular graph that has a unique infinite cluster ω' a.s. Show that $p_c(\omega') < 1$ a.s.

Sometimes the following generalization of Theorem 8.19 is useful. Given a family of probability measures μ_p $(0 \le p \le 1)$ on, say, 2^{E}, we say that the family is ***smoothly parameterized*** if for each edge e, the function $p \mapsto \mu_p[\omega(e) = 1]$ is continuous from the left. A ***monotone coupling*** of the family, if it exists, is a random field $Z : \mathsf{E} \to [0, 1]$

such that for each p, the law of $\{e\,;\ Z(e) < p\}$ is μ_p. (Here, a random field is just a random variable whose values are functions from E to $[0,1]$. Equivalently, it is a collection of random variables $Z(e)$ for $e \in \mathsf{E}$.) The coupling is called ***injective*** if a.s. $Z(e) \neq Z(e')$ for $e \neq e'$. Also, let us call a probability measure $\mathbf{P}$ ***weakly insertion tolerant*** if there is a function $f : \mathsf{E} \times 2^{\mathsf{E}} \to 2^{\mathsf{E}}$ such that

(i) for all e and all ω, we have $\omega \cup \{e\} \subseteq f(e,\omega)$;

(ii) for all e and all ω, the difference $f(e,\omega) \setminus [\omega \cup \{e\}]$ is finite; and

(iii) for all e and each event A of positive probability, the image of A under $f(e,\bullet)$ is an event of positive probability.

Of course, an insertion tolerant probability measure satisfies this definition with $f(e,\omega) := \omega \cup \{e\}$. ***Weak deletion tolerance*** has a similar definition.

Theorem 8.20. *Let G be a non-amenable quasi-transitive unimodular graph. Let μ_p be a smoothly parameterized family of ergodic weakly insertion-tolerant probability measures on 2^{E}. Suppose that the family has an injective monotone coupling by a random field Z whose law is invariant under automorphisms. Let $\mathscr{A}$ be the event that all clusters are finite. If $p_c := \sup\{p\,;\ \mu_p[\mathscr{A}] = 1\} > 0$, then $\mu_{p_c}[\mathscr{A}] = 1$.*

▷ **Exercise 8.12.**

Prove Theorem 8.20.

§8.5. Percolation on Planar Quasi-Transitive Graphs.

For planar non-amenable Cayley graphs with one end, we can answer all the most basic questions about percolation. It should not be hard to do the same for planar non-amenable Cayley graphs with infinitely many ends, but no one has done this yet. Of course, the case of two ends is trivial. This entire section is adapted from Benjamini and Schramm (2001a).

▷ **Exercise 8.13.**

Show that a plane (properly embedded locally finite) quasi-transitive graph with one end has no face with an infinite number of sides.

Planarity is used in order to exploit properties of percolation on the plane dual of the original graph. However, this dual is not necessarily a Cayley graph; indeed, it is not necessarily even transitive. For this reason, we will need to go beyond Cayley graphs.

In the setting of Exercise 8.13, the dual of a Cayley graph is always locally finite, quasi-transitive and unimodular, as we will see. Also, recall from Exercise 6.25 that if G is non-amenable, then so is $G^\dagger$. Thus, the natural setting preserved under duality is that of non-amenable plane quasi-transitive graphs. The main theorem of this section is the following.

Theorem 8.21. *Let G be a non-amenable plane quasi-transitive graph with one end. Then $0 < p_c(G) < p_u(G) < 1$ and Bernoulli(p_u) percolation on G has a unique infinite cluster a.s.*

This result was proved for certain planar Cayley graphs earlier by Lalley (1998). It is not known which Cayley graphs have the property that there is a unique infinite cluster $\mathbf{P}_{p_u}$-a.s. For example, Schonmann (1999a) proved that this does *not* happen on $\mathbb{T}_{b+1} \times \mathbb{Z}$ with $b \geq 2$, which Peres (2000) extended to all non-amenable cartesian products of infinite transitive graphs.

We need the following fundamental fact whose proof is given in the notes to this chapter.

Theorem 8.22. *If G is a planar quasi-transitive graph with one end, then* $\mathrm{Aut}(G)$ *is unimodular. Furthermore, there is some plane embedding of G such that $G^\dagger$ is quasi-transitive.*

We will assume for the rest of this section that G ***is a non-amenable plane quasi-transitive graph with one end, embedded in such a way that*** $G^\dagger$ ***is quasi-transitive, and*** ω ***is the configuration of an invariant percolation on*** G. When more restrictions are needed, we will be explicit about them. We define the dual configuration $\omega^\times$ on the plane dual graph $G^\dagger$ by

$$\omega^\times(e^\dagger) := 1 - \omega(e)\,. \tag{8.20}$$

For a set A of edges, we let $A^\dagger$ denote the set of edges $e^\dagger$ for $e \in A$. Write N_∞ for the number of infinite clusters of ω and $\mathsf{N}_\infty^\times$ for the number of infinite clusters of $\omega^\times$.

Lemma 8.23. $\mathsf{N}_\infty + \mathsf{N}_\infty^\times \geq 1$ *a.s.*

Proof. Suppose that $\mathsf{N}_\infty + \mathsf{N}_\infty^\times = 0$ with positive probability. Then by conditioning on that event, we may assume that it holds surely. In this case, for each (open) cluster K of ω, there is a unique infinite component of $G \backslash K$ since K is finite and G has only one end. This means that there is a unique open component of $(\partial_{\mathsf{E}} K)^\dagger$ that "surrounds" K; this component of $(\partial_{\mathsf{E}} K)^\dagger$ is contained in some cluster, K', of $\omega^\times$. Since our assumption implies that K' is also finite, this same procedure yields a cluster K'' of ω that surrounds K' and hence surrounds K.

Let $\mathscr{K}_0$ denote the collection of all clusters of ω. Define inductively $\mathscr{K}_{j+1} := \{K'' ;\ K \in \mathscr{K}_j\}$ for $j \ge 0$. Since $\bigcap_j \mathscr{K}_j = \varnothing$, we may define

$$r(K) := \max\{j ;\ K \in \mathscr{K}_j\}$$

for all clusters $K \in \mathscr{K}_0$. Given $N \ge 0$, let ω^N be those edges whose endpoints belong to (possibly different) clusters K with $r(K) \le N$, i.e., the "interiors" of the clusters in $\mathscr{K}_{N+1}$. Then $\omega^N \subseteq \omega^{N+1}$ for all N and $\bigcup_N \omega^N = \mathsf{E}$. Also ω^N is an invariant percolation for each N. Since $\deg_{\omega^N} o \to \deg_G o$, it follows from Corollary 8.15 that for sufficiently large N, with positive probability ω^N has infinite clusters. Yet the interiors of the clusters of $\mathscr{K}_{N+1}$ are finite and disjoint. This is a contradiction. ◀

Lemma 8.24. (BLPS (1999b)) $\mathsf{N}_\infty \in \{0, 1, \infty\}$ *a.s.*

Proof. If not, we may condition on $2 \le \mathsf{N}_\infty < \infty$. In this case, we may pick uniformly at random two distinct infinite clusters of ω, call them K_1 and K_2. Let τ consist of those edges $[x, y]$ such that $x \in K_1$ and y belongs to the component of $G \setminus K_1$ that contains K_2. Then $\tau^\times$ is a bi-infinite path in $G^\times$ and is an invariant percolation on $G^\times$. But the fact that $p_c(\tau^\times) = 1$ contradicts Exercise 8.11. ◀

Corollary 8.25. $\mathsf{N}_\infty + \mathsf{N}_\infty^\times \in \{0, 1, \infty\}$ *a.s.*

Proof. Draw G and $G^\dagger$ in the plane in such a way that every edge e intersects $e^\dagger$ in one point, v_e, and there is no other intersection of G and $G^\dagger$. For $e \in G$, write $\hat{e}$ for the pair of edges that result from the division of e by v_e, and likewise for $\hat{e^\dagger}$. This defines a new graph $\hat{G}$, whose vertices are $\mathsf{V}(G) \cup \mathsf{V}(G^\dagger) \cup \Big\{v_e ;\ e \in \mathsf{E}(G)\Big\}$ and whose edges are $\bigcup_{e \in \mathsf{E}(G)} (\hat{e} \cup \hat{e^\dagger})$. The proof of Theorem 8.22 shows that $\hat{G}$ is quasi-transitive.

Consider the percolation

$$\omega' := \bigcup_{e \in \omega} \hat{e} \cup \bigcup_{e^\dagger \in \omega^\times} \hat{e^\dagger}$$

on $\hat{G}$. This percolation is invariant under $\mathrm{Aut}(\hat{G})$. The number of infinite components of ω' is $\mathsf{N}_\infty + \mathsf{N}_\infty^\times$. Applying Lemma 8.24 to ω', we obtain our desired conclusion. ◀

Theorem 8.26. $(\mathsf{N}_\infty, \mathsf{N}_\infty^\times) \in \Big\{(1, 0), (0, 1), (1, \infty), (\infty, 1), (\infty, \infty)\Big\}$ *a.s.*

Proof. Lemma 8.24 gives $\mathsf{N}_\infty, \mathsf{N}_\infty^\times \in \{0, 1, \infty\}$. Lemma 8.23 rules out $(\mathsf{N}_\infty, \mathsf{N}_\infty^\times) = (0, 0)$. Since every two infinite clusters of ω must be separated by at least 1 infinite cluster of $\omega^\times$ (*viz.*, the one containing the path τ in the proof of Lemma 8.24), the case $(\mathsf{N}_\infty, \mathsf{N}_\infty^\times) = (\infty, 0)$ is impossible. Dual reasoning shows that $(\mathsf{N}_\infty, \mathsf{N}_\infty^\times) = (0, \infty)$ cannot happen. Corollary 8.25 rules out $(\mathsf{N}_\infty, \mathsf{N}_\infty^\times) = (1, 1)$. This leaves the cases mentioned. ◀

We now come to the place where more assumptions on the percolation are needed. In particular, insertion and deletion tolerance become crucial. In order to treat site percolation as well as bond percolation, we will use the bond percolation ω^ξ associated to a site percolation ξ as follows:

$$\omega^\xi := \Big\{[x,y]\,;\; x,y \in \xi\Big\}\,. \tag{8.21}$$

The ω^N used in the proof of Lemma 8.23 are examples of such associated bond percolations. Note that even when ξ is Bernoulli percolation, ω^ξ is neither insertion tolerant nor deletion tolerant. However, ω^ξ is still weakly insertion tolerant. More generally, it is clear that when ξ is insertion [resp., deletion] tolerant, then ω^ξ is weakly insertion [resp., deletion] tolerant. It is also clear that when ξ is Bernoulli percolation, ω^ξ is ergodic.

Theorem 8.27. *Assume that ω is ergodic, weakly insertion tolerant, and weakly deletion tolerant. Then a.s.*

$$(\mathsf{N}_\infty, \mathsf{N}_\infty^\times) \in \Big\{(1,0), (0,1), (\infty,\infty)\Big\}\,.$$

Proof. By Theorem 8.26, it is enough to rule out the cases $(1,\infty)$ and $(\infty,1)$. Let K be a finite connected subgraph of G. If K intersects distinct infinite clusters of ω, then $\omega^\times \setminus \{e^\dagger\,;\; e \in \mathsf{E}(K)\}$ must have at least 2 infinite clusters by planarity. Define $A(K)$ to be the event that K intersects distinct infinite clusters of ω. If $\mathsf{N}_\infty = \infty$ a.s., then there is some finite subgraph K such that $A(K)$ has positive probability. Fix some such K and write its edge set as $\{e_1, \ldots, e_n\}$. Let $f(e,\omega)$ be a function witnessing the weak insertion tolerance. Define $f(e,A) := \{f(e,\omega)\,;\; \omega \in A\}$. Let $B(K) := f(e_1, f(e_2, \cdots f(e_n, A(K))\cdots))$. Then $B(K)$ has positive probability and we have $\mathsf{N}_\infty^\times > 1$ on $B(K)$. But ergodicity implies that $(\mathsf{N}_\infty, \mathsf{N}_\infty^\times)$ is an a.s. constant. Hence it is (∞,∞). A dual argument rules out $(1,\infty)$. ◀

Theorem 8.27 allows us to deduce precisely the value of N_∞ from the value of $\mathsf{N}_\infty^\times$ and thus critical values on G from those on $G^\dagger$.

Theorem 8.28. $p_{\mathrm{c}}^{\mathrm{bond}}(G^\dagger) + p_{\mathrm{u}}^{\mathrm{bond}}(G) = 1$ *and* $\mathsf{N}_\infty = 1$ $\mathbf{P}_{p_{\mathrm{u}}}$*-a.s.*

Proof. Let ω_p be Bernoulli(p) bond percolation on G. Then $\omega_p^\times$ is Bernoulli($1-p$) bond percolation on $G^\dagger$. It follows from Theorem 8.27 that

$$p > p_{\mathrm{u}}^{\mathrm{bond}}(G) \quad\Longrightarrow\quad \mathsf{N}_\infty = 1 \quad\Longrightarrow\quad \mathsf{N}_\infty^\times = 0 \quad\Longrightarrow\quad 1-p \le p_{\mathrm{c}}^{\mathrm{bond}}(G^\dagger)$$

and

$$p < p_{\mathrm{u}}^{\mathrm{bond}}(G) \quad\Longrightarrow\quad \mathsf{N}_\infty \ne 1 \quad\Longrightarrow\quad \mathsf{N}_\infty^\times \ne 0 \quad\Longrightarrow\quad 1-p \ge p_{\mathrm{c}}^{\mathrm{bond}}(G^\dagger)\,.$$

This gives us $p_c^{\rm bond}(G^\dagger)+p_u^{\rm bond}(G)=1$. Furthermore, according to Theorem 8.19, we have $\mathsf{N}_\infty^\times = 0$ $\mathbf{P}^{G^\dagger}_{p_c^{\rm bond}(G^\dagger)}$-a.s., whence Theorem 8.27 tells us that $\mathsf{N}_\infty = 1$ $\mathbf{P}^{G}_{p_u^{\rm bond}(G)}$-a.s.

For site percolation, we must prove that $\mathsf{N}_\infty = 1$ $\mathbf{P}^{\rm site}_{p_u}$-a.s. Let ξ_p be the standard coupling of site percolation and ω^{ξ_p} the corresponding bond percolation processes. As above, we may conclude that for $p > p_u^{\rm site}(G)$, we have that $\left(\omega^{\xi_p}\right)^\dagger$ has no infinite clusters a.s., while for $p < p_u^{\rm site}(G)$, we have that $\left(\omega^{\xi_p}\right)^\dagger$ has infinite clusters a.s. Because $\left(\omega^{\xi_{1-p}}\right)^\dagger$ satisfies all the hypotheses of Theorem 8.20, it follows that for $p = p_u^{\rm site}(G)$, we have that $\left(\omega^{\xi_p}\right)^\dagger$ has no infinite clusters a.s. This gives us the result we want. ◀

Proof of Theorem 8.21. We already know that $p_c > 0$ from Proposition 7.11. Applying this to $G^\dagger$ and using Theorem 8.28, we obtain that $p_u^{\rm bond} < 1$. In light of Proposition 7.13, given $\epsilon > 0$, for all $q \in (1-\epsilon, 1)$, there is some $\delta > 0$ such that if $p \in (1-\delta, 1)$ and $\xi \sim \mathbf{P}_p^{\rm site}$, then ω^ξ stochastically dominates $\mathbf{P}_q^{\rm bond}$, whence for such p, $\left(\omega^\xi\right)^\dagger$ has no infinite clusters a.s. and thus ω^ξ has a unique infinite cluster a.s. This proves that $p_u^{\rm site} < 1$ too. (The result that $p_u < 1$ also follows from Theorem 7.27.) Comparing Theorem 8.19 and Theorem 8.28, we see that it is impossible that $p_c = p_u$, whence $p_c < p_u$. ◀

§8.6. Properties of Infinite Clusters.

In the proof of Theorem 7.6, we saw that if Bernoulli(p) percolation produces infinitely many infinite clusters a.s., then a.s. at least one of them has a furcation, whence at least 3 ends. This was a relatively simple consequence of insertion and deletion tolerance. Likewise, deletion tolerance implies that if there is a unique infinite cluster (and $p < 1$), then that cluster has a unique end a.s.

As the reader might suspect from Example 8.5, one can derive stronger results from the Mass-Transport Principle. By using only the properties of invariance and weak insertion tolerance, we will prove the following theorem.

Theorem 8.29. *Let* $\mathbf{P}$ *be an invariant weakly insertion-tolerant percolation process on a non-amenable quasi-transitive unimodular graph* G*. If there are infinitely many infinite clusters a.s., then a.s. every infinite cluster has continuum many ends, no isolated end, and is transient for simple random walk.*

Here, we are using a topology on the set of ends of a graph, defined as follows. Let G be a graph with fixed base point o. Let B_n be the ball of radius n about o. Let Ends be the set of ends of G. Define a metric on Ends by putting

$$d(\xi,\zeta) := \inf\{1/n\,;\ n=1 \text{ or } \forall A\in\xi\ \ \forall B\in\zeta\ \ \exists \text{ a component } C \text{ of } G\setminus B_n \\ |A\setminus C| + |B\setminus C| < \infty\}\,.$$

It is easy to verify that this is a metric; in fact, it is an ultrametric, i.e., for any $\xi_1, \xi_2, \xi_3 \in$ Ends, we have $d(\xi_1, \xi_3) \le \max\{d(\xi_1, \xi_2), d(\xi_2, \xi_3)\}$. Since G is locally finite, it is easy to check that Ends is compact in this metric. Finally, it is easy to see that the topology on Ends does not depend on choice of base point. A set of vertices that, for some n, contains the component of $G \setminus B_n$ that has an infinite intersection with every set in ξ will be called a ***vertex-neighborhood*** of ξ.

The set of isolated points in any compact metric space is countable; the nonisolated points form a perfect subset (the Cantor-Bendixson Theorem), whence, if there are any, they have the cardinality of the continuum (see, e.g., Kuratowski, 1966).

In the case of site percolation ξ, by transience of ξ we mean transience of ω^ξ, as defined in (8.21).

The history of Theorem 8.29 is as follows. Benjamini and Schramm (1996b) conjectured that for Bernoulli percolation on any quasi-transitive graph, if there are infinitely many infinite clusters, then a.s. every infinite cluster has continuum many ends. This was proved by Häggström and Peres (1999) for transitive unimodular graphs and then by Häggström, Peres, and Schonmann (1999) in general. Our proof is from Lyons and Schramm (1999), which also proved the statement about transience. It is also true that for any invariant insertion-tolerant percolation process on a non-amenable quasi-transitive unimodular graph with a unique infinite cluster a.s., that cluster is transient, but this is more difficult; see Benjamini, Lyons, and Schramm (1999) for a proof. (Of course, if Conjecture 7.28 were proven, then this would also follow from Theorem 8.29 and the Rayleigh Monotonicity Law.) For amenable transient quasi-transitive graphs, Benjamini, Lyons, and Schramm (1999) conjectured that infinite clusters are transient a.s. for Bernoulli percolation; in $\mathbb{Z}^d$, $d > 2$, this was established by Grimmett, Kesten, and Zhang (1993) (see Benjamini, Pemantle, and Peres (1998) for a different proof).

We begin with the following proposition.

Proposition 8.30. *Let* $\mathbf{P}$ *be an invariant percolation process on a non-amenable quasi-transitive unimodular graph* G*. Almost surely each infinite cluster that has at least* 3 *ends has no isolated ends.*

Of course, it follows that each infinite cluster has 1, 2, or ∞ ends, a fact proved in the same manner in BLPS (1999b).

Proof. For each $n = 1, 2, \ldots$, let A_n be the (random) union of all vertex sets A such that there is some percolation cluster K with the properties that $K \supset A$, the diameter of A is at most n in the metric of K, and $K \setminus A$ has at least 3 infinite components. Note that if ξ is an isolated end of a percolation cluster K, then for each n, some vertex-neighborhood of ξ in K is disjoint from A_n. Also observe that if K is a cluster with at least 3 ends, then K intersects A_n for some n.

Fix some $n \ge 1$. Consider the mass transport that sends one unit of mass from each vertex x in a percolation cluster that intersects A_n and distributes it equally among the vertices in A_n that are closest to x in the metric of $K(x)$. In other words, let $C(x)$ be the set of vertices in $K(x) \cap A_n$ that are closest to x in the metric of ω, and set $F(x, y; \omega) := |C(x)|^{-1}$ if $y \in C(x)$ and otherwise $F(x, y; \omega) := 0$. Then $F(x, y; \omega)$ is invariant under the diagonal action. If ξ is an isolated end of an infinite cluster K that intersects A_n, then there is a finite set of vertices B that gets all the mass from all the vertices in a vertex-neighborhood of ξ. But the Mass-Transport Principle tells us that the expected mass transported to a vertex is finite. Hence, a.s. clusters that intersect A_n do not have isolated ends. Since this holds for all n, we gather that a.s. infinite clusters with isolated ends do not intersect $\bigcup_n A_n$, whence they have at most two ends. ◀

Proposition 8.31. *Let* $\mathbf{P}$ *be an invariant weakly insertion-tolerant bond percolation process on a non-amenable quasi-transitive unimodular graph* G*. If there are infinitely many infinite clusters a.s., then a.s. every infinite cluster has continuum many ends and no isolated end.*

Proof. It suffices to prove that there are no isolated ends of clusters. To prove this in turn, observe that if some cluster has an isolated end with positive probability, then because of weak insertion tolerance, with positive probability, some cluster will have at least 3 ends with one of them being isolated. Hence Proposition 8.31 follows from Proposition 8.30. ◀

We proved a weak form of the following principle in Lemma 7.7 and used a special case of it in our proof of Theorem 8.19.

Lemma 8.32. (BLPS (1999b)) *Let* $\mathbf{P}$ *be an invariant bond percolation process on a non-amenable quasi-transitive unimodular graph* G*. If a.s. there is a component of* ω *with at least three ends, then (on a larger probability space) there is a random forest* $\mathfrak{F} \subset \omega$ *such that the distribution of the pair* $(\mathfrak{F}, \omega)$ *is* Γ*-invariant and a.s. whenever a component* K *of* ω *has at least three ends, there is a component of* $K \cap \mathfrak{F}$ *that has infinitely many ends.*

Proof. We begin as in the proof of Lemma 7.7. We use independent uniform $[0, 1]$ random variables assigned to the edges (independently of ω) to define the free minimal spanning forest $\mathfrak{F}$ of ω. This means that an edge $e \in \omega$ is present in $\mathfrak{F}$ iff there is no cycle in ω containing e in which e is assigned the maximum value. As before, we have (a.s.) that $\mathfrak{F}$ is a forest with each tree having the same vertex-cardinality as the cluster of ω in which it lies.

Suppose that $K(x)$ has at least 3 ends with positive probability. Choose any finite tree T containing x with edge set $E(T)$ so that with positive probability, $T \subset K(x)$ and $K(x) \setminus E(T)$ has at least 3 infinite components. Then with positive probability, $T \subset K(x)$, $K(x) \setminus E(T)$ has at least 3 infinite components, all edges in T are assigned values less than 1/2, and all edges in $\partial_{\mathsf{E}} T$ are assigned values greater than 1/2. On this event, $\mathfrak{F}$ contains T and T is part of a spanning tree in $\mathfrak{F}$ with at least 3 ends.

To convert this event of positive probability to an event of probability 1, let $r_x = r(x, \omega)$ be the least cardinality r of a tree T in G such that $K(x) \setminus T$ has at least 3 infinite components, if such an r exists. If not, set $r(x, \omega) := \infty$. Note that $r(x, \omega) < \infty$ iff $K(x)$ has at least 3 ends. By Example 8.6, if r_x is finite, then there are a.s. infinitely many such trees T in $K(x)$. Therefore, given that $K(x)$ has at least 3 ends, there are a.s. infinitely many trees $T \subset K(x)$ at pairwise distance at least 2 from each other such that $K(x) \setminus T$ has at least 3 infinite components. For such T, the events that all edges in T are assigned values less than 1/2 and all edges in $\partial_{\mathsf{E}} T$ are assigned values greater than 1/2 are independent and have probability bounded below, whence an infinite number of these events occur a.s. Therefore, there is a component of $K(x) \cap \mathfrak{F}$ that has at least 3 ends a.s., whence, by Proposition 8.30, has infinitely many ends a.s. ◀

Proposition 8.33. *Let* $\mathbf{P}$ *be an invariant weakly insertion-tolerant percolation process on a non-amenable quasi-transitive unimodular graph* G*. If there are infinitely many infinite clusters a.s., then a.s. each infinite cluster is transient.*

Proof. It suffices to consider bond percolation. By Proposition 8.31, every infinite cluster of ω has infinitely many ends. Consequently, there is an invariant random forest $\mathfrak{F} \subset \omega$ such that a.s. each infinite cluster K of ω contains a tree of $\mathfrak{F}$ with infinitely many ends by Lemma 8.32. By Corollary 8.18, we know that any such tree has $p_c < 1$. Since it has branching number > 1, it follows that such a tree is transient by Theorem 3.5. The Rayleigh Monotonicity Law then implies that K is transient. ◀

Theorem 8.29 now follows from Propositions 8.31 and 8.33.

§8.7. Percolation on Amenable Graphs.

The Mass-Transport Principle is most useful in the non-amenable setting. In this section, we give two results that show that non-amenability is in fact necessary for some of our previous work. Our first result is from BLPS (1999b). There is also a bond version, but the site version we prove is a little cleaner, both in statement and in proof.

We will use Haar measure for the proofs since it would be artificial and somewhat cumbersome to avoid it. In particular, the set $\Gamma_{x,y}$ of automorphisms that take x to y is compact in $\mathrm{Aut}(G)$, whence it has finite Haar measure. This means that we can choose one of its elements at random (via normalized Haar measure).

▷ **Exercise 8.14.**
Let G be a transitive unimodular graph and $o \in \mathsf{V}$. For each $x \in \mathsf{V}$, choose a Haar-random $\gamma_x \in \mathrm{Aut}(G)$ that takes o to x. Show that for every finite set $L \subset \mathsf{V}$, we have

$$\mathbf{E}\big|\{x \in \mathsf{V}\,;\ o \in \gamma_x L\}\big| = |L|\,.$$

Theorem 8.34. (Amenability and Finite Percolation) *Let G be a quasi-transitive unimodular graph. Then G is amenable iff for all $\alpha < 1$, there is a site percolation on G with no infinite clusters and such that $\mathbf{P}[x \in \omega] > \alpha$ for all $x \in \mathsf{V}(G)$.*

Proof. We assume that G is transitive and leave the quasi-transitive case to the reader. One direction follows from the site version of Theorem 8.14, which is Exercise 8.28.

Now we prove the converse. Suppose that G is amenable and fix $o \in \mathsf{V}(G)$.

Fix a finite set $K \subset \mathsf{V}$ and consider the following percolation. For each $x \in \mathsf{V}$, choose a random $\gamma_x \in \mathrm{Aut}(G)$ that takes o to x and let Z_x be a random bit that equals 1 with probability $1/|K|$. Choose all γ_x and Z_x independently. Remove the vertices

$$\bigcup_{x \in \mathsf{V}, Z_x = 1} \partial_{\mathsf{V}}(\gamma_x K)\,,$$

that is, consider the percolation subgraph

$$\omega := \omega_K := \mathsf{V} - \bigcup\{\partial_{\mathsf{V}}(\gamma_x K)\,;\ Z_x = 1\}\,.$$

Then the distribution of ω is an invariant percolation on G.

We claim that

$$\mathbf{P}[o \notin \omega] \le |\partial_{\mathsf{V}} K|/|K| \tag{8.22}$$

and

$$\mathbf{P}[|K(o)| < \infty] \ge 1 - 1/e \tag{8.23}$$

(where e is the base of natural logarithms). To prove (8.22), note that the probability that $o \notin \omega$ is at most the expected number of x such that $Z_x = 1$ and $o \in \partial_{\mathsf{V}}(\gamma_x K)$. But this expectation is exactly the right-hand side of (8.22) by Exercise 8.14 applied to $L := \partial_{\mathsf{V}} K$. To prove (8.23), use the independence to calculate that

$$\begin{aligned}
\mathbf{P}[|K(o)| < \infty] &\ge \mathbf{P}[\exists x\ o \in \gamma_x K \text{ and } Z_x = 1] \\
&= 1 - \mathbf{P}[\forall x\ \neg\big(o \in \gamma_x K \text{ and } Z_x = 1\big)] \\
&= 1 - \prod_{x \in \mathsf{V}} (1 - \mathbf{P}[o \in \gamma_x K]\mathbf{P}[Z_x = 1]) \\
&= 1 - \prod_{x \in \mathsf{V}} (1 - \mathbf{P}[o \in \gamma_x K]/|K|) \\
&\ge 1 - \exp\left\{-\sum_{x \in \mathsf{V}} \mathbf{P}[o \in \gamma_x K]/|K|\right\} \\
&= 1 - \exp\left\{-\mathbf{E}\big|\{x \in \mathsf{V}\,;\ o \in \gamma_x K\}\big|/|K|\right\} \\
&= 1 - 1/e
\end{aligned}$$

by Exercise 8.14 again, applied to $L := K$.

Now, since G is amenable, there is a sequence K_n of finite sets of vertices with $\sum_n |\partial_{\mathsf{V}} K_n|/|K_n| < 1 - \alpha$. For each n, let ω_n be the random subgraph from the percolation just described based on the set K_n. Choose ω_n to be independent and consider the percolation with configuration $\omega := \bigcap \omega_n$. By (8.22), we have $\mathbf{P}[o \in \omega] > \alpha$ and by (8.23), we have $\mathbf{P}[|K(o)| < \infty] = 1$. ◀

We now use Theorem 8.34 to establish a converse to Theorem 6.26 on anchored expansion constants in percolation. This result is due to Häggström, Schonmann, and Steif (2000), Theorem 2.4(ii), with a rather indirect proof. Our proof is due to O. Schramm (personal communication).

Corollary 8.35. *Let G be a quasi-transitive amenable graph and ω be an invariant bond percolation on G. Then a.s. every component of ω has anchored expansion constant equal to 0.*

Proof. Again, we do the transitive case and leave the quasi-transitive case to the reader. Recall that amenability guarantees unimodularity by Proposition 8.12. For simplicity, we use the anchored expansion constant defined using the inner vertex boundary, where we define the ***internal vertex boundary*** of a set K as $\partial_{\mathsf{V}}^{\mathrm{int}} K := \{x \in K\,;\ \exists y \notin K\ \ y \sim x\}$. Choose a sequence $\alpha_n < 1$ with $\sum_n (1-\alpha_n) < \infty$. Choose a sequence of independent invariant percolations ω_n, also independent of ω, with no infinite component and such that $\mathbf{P}[x \in \omega_n] > \alpha_n$ for all $x \in \mathsf{V}(G)$. This exists by Theorem 8.34. Let $K(x)$ denote the component of x in ω and $K_n(x)$ the component of x in ω_n. It suffices to prove that a.s.,

$$\lim_{n\to\infty} \frac{|\partial_{\mathsf{V}}^{\mathrm{int}} K_n(o) \setminus \partial_{\mathsf{V}}^{\mathrm{int}} K(o)|}{|K(o) \cap K_n(o)|} = 0\,.$$

Now

$$\begin{aligned}\mathbf{E}\left[\frac{|\partial_{\mathsf{V}}^{\mathrm{int}} K_n(o) \setminus \partial_{\mathsf{V}}^{\mathrm{int}} K(o)|}{|K(o) \cap K_n(o)|}\right] &= \mathbf{P}[o \in \partial_{\mathsf{V}}^{\mathrm{int}} K_n(o) \setminus \partial_{\mathsf{V}}^{\mathrm{int}} K(o)] \\ &\le \deg_G(o)\mathbf{P}[o \notin \omega_n] < \deg_G(o)(1-\alpha_n)\end{aligned}$$

by the Mass-Transport Principle. (For the equality, every point $x \in \partial_{\mathsf{V}}^{\mathrm{int}} K_n(x) \setminus \partial_{\mathsf{V}}^{\mathrm{int}} K(x)$ sends mass 1 split equally among the vertices of its component $K_n(x) \cap K(x)$. For the first inequality, x sends mass 1 to y when $x \sim y$ and $y \notin K_n(x)$.) Therefore

$$\sum_n \mathbf{E}\left[\frac{|\partial_{\mathsf{V}}^{\mathrm{int}} K_n(o) \setminus \partial_{\mathsf{V}}^{\mathrm{int}} K(o)|}{|K(o) \cap K_n(o)|}\right] < \infty\,,$$

whence

$$\sum_n \frac{|\partial_{\mathsf{V}}^{\mathrm{int}} K_n(o) \setminus \partial_{\mathsf{V}}^{\mathrm{int}} K(o)|}{|K(o) \cap K_n(o)|} < \infty \text{ a.s.},$$

which gives the result. ◀

§8.8. Notes.

The proof of uniqueness monotonicity given in Section 8.1 is essentially that of Häggström and Peres (1999).

Theorem 8.17 is from BLPS (1999b) and Aldous and Lyons (2007).

Theorem 8.21 was proved in the transitive case by Benjamini and Schramm (2001a). The quasi-transitive case is new, but not essentially different.

Theorem 8.22 was known in the transitive case (Benjamini and Schramm, 2001a), but the quasi-transitive case is due to R. Lyons and is published here for the first time. We now give its proof.

We need several lemmas. Our approach is based on some ideas we heard from O. Schramm. The transitive case is easier to prove, as this proof simplifies to show that the graph is 3-connected.

A graph $G = (\mathsf{V}, \mathsf{E})$ is called ***k-connected*** if $|\mathsf{V}| \ge k+1$ and whenever at most $k-1$ vertices are removed from G (together with their incident edges), the resulting graph is connected. For the next lemmas, let $K(x)$ denote the set of vertices that lie in finite components of $G \setminus \{x\}$, and let $K(x, y)$ denote the set of vertices, *other than* $K(x) \cup K(y)$, that lie in finite components of $G \setminus \{x, y\}$. Note that $x \notin K(x)$ and $x \notin K(x, y)$.

Lemma 8.36. *If G is a quasi-transitive graph, then* $\sup_{x \in \mathsf{V}} |K(x)| < \infty$.

Proof. Since x has finite degree, $G \setminus \{x\}$ has a finite number of components, whence $K(x)$ is finite. If x and y are in the same orbit, then $|K(x)| = |K(y)|$. Since there are only finitely many orbits, the result follows. ◀

Lemma 8.37. *If G is a quasi-transitive graph with one end, then* $\sup_{x,y \in \mathsf{V}} |K(x, y)| < \infty$.

Proof. Suppose not. Since there are only a finite number of orbits, there must be some x and y_n such that $d(x, y_n) \to \infty$ and $K(x, y_n) \neq \varnothing$ for all n. Because of Lemma 8.36, for all large n, we have $x \notin K(y_n)$, whence for all large n, there are neighbors a_n, b_n of x that cannot be joined by a path in $G \setminus \{x, y_n\}$, but such that a_n lies in an infinite component of $G \setminus \{x, y_n\}$ and b_n can be joined to y_n using only vertices in $K(x, y_n) \cup \{y_n\}$. There is some pair of neighbors a, b of x for which $a = a_n$ and $b = b_n$ for infinitely many n. But then a and b lie in distinct infinite components of $G \setminus \{x\}$, contradicting the assumption that G has one end. ◀

We partially order the collection of sets $K(x, y)$ by inclusion.

Lemma 8.38. *Let G be any graph. If $K(x, y)$ and $K(z, w)$ are maximal and non-empty, then either $\{x, y\} = \{z, w\}$ or $K(x, y) \cap K(z, w) = \varnothing$. Also,*

$$\{x, y, z, w\} \cap \big(K(x, y) \cup K(z, w)\big) = \varnothing\,. \tag{8.24}$$

Proof. Suppose that $z \in K(x, y)$. Then we cannot have $w \in K(x, y)$ since that would imply $K(z, w) \subsetneq K(x, y)$. Now there is a path from z to w using only vertices in $K(z, w) \cup \{z, w\}$. Since $z \in K(x, y)$ and $w \notin K(x, y)$, this path must include either x or y, say, y. Therefore $y \in K(z, w) \cup \{w\}$. We cannot have $y = w$ since that would imply $K(z, w) \subsetneq K(x, y)$.

Thus $y \in K(z, w)$. Consider any infinite simple path starting from any vertex in $K(x, y) \cup K(z, w)$. We claim it must visit x or w. For if not, then it must visit z or y. If the last vertex among $\{z, y\}$ that it visits is z, then it must visit x since $z \in K(x, y)$, a contradiction. If the last vertex among $\{z, y\}$ that it visits is y, then it must visit w since $y \in K(z, w)$, a contradiction. This proves our claim, whence $K(x, w) \supsetneq K(x, y) \cup K(z, w)$. But this contradicts maximality of $K(x, y)$.

Thus, we have proved that $z \notin K(x, y)$. By symmetry, we deduce (8.24).

Now suppose that $K(x, y) \cap K(z, w) \neq \varnothing$. Choose some $a \in K(x, y) \cap K(z, w)$. Then every infinite simple path from a must pass through $\{x, y\}$ and through $\{z, w\}$. Consider an infinite simple path from a. Without loss of generality, suppose that the first point among $\{x, y, z, w\}$ that it visits is z. Then $z \in K(x, y) \cup \{x, y\}$. By the above, z is equal to x or y, say, x. Now there is also an infinite simple path from a such that the first point among $\{x, y, w\}$ is not x. Say it is w. Then $w \in K(x, y) \cup \{y\}$. By the above, $w = y$ and this proves the lemma. ◀

Every embedding ϕ of a planar graph into the plane induces a cyclic ordering $[\phi(x)]$ of the edges incident to any vertex x by looking at the clockwise ordering of these edges after embedding. Two cyclic orderings are considered the same if they differ only by a cyclic permutation. Two cyclic orderings are inverses if they can be written in opposite order from each other.

The following extends a theorem of Whitney (1932):

Lemma 8.39. (Imrich, 1975) *If G is a planar 3-connected graph and ϕ and ψ are two embeddings of G in the plane, then either for all x, we have $[\phi(x)] = [\psi(x)]$ or for all x, we have that $[\phi(x)]$ and $[\psi(x)]$ are inverses.*

Proof. Imrich (1975) gives a proof that is valid for graphs that are not necessarily properly embedded nor locally finite. In order to simplify the proof, we will assume that the graph is not only properly embedded and locally finite, but also quasi-transitive and has only one end. This is the only case we will use.

Given any vertices $x \neq y$, Menger's theorem shows that we can find three paths joining x and y that are disjoint except at x and y. Comparison of the first edges e_1, e_2, e_3 and the last edges f_1, f_2, f_3 of these paths shows that the cyclic ordering of $\phi(e_i)$ is opposite to that of $\phi(f_i)$, and likewise for ψ. Therefore, it suffices to prove that for each x separately, we have either $[\phi(x)] = [\psi(x)]$ or $[\phi(x)]$ and $[\psi(x)]$ are inverses. For this, it suffices to show that if e_1 and e_2 are two edges incident to x and $\phi(e_1)$ is adjacent to $\phi(e_2)$ in the cyclic ordering $[\phi(x)]$, then $\psi(e_1)$ is adjacent to $\psi(e_2)$ in the cyclic ordering $[\psi(x)]$.

So let $\phi([x, y])$ and $\phi([x, z])$ be adjacent in $[\phi(x)]$. Assume that $\psi([x, y])$ and $\psi([x, z])$ are not adjacent in $[\psi(x)]$. Let C be the cycle such that $\phi(C)$ is the border of the (unique) face having sides that include both $\phi([x, y])$ and $\phi([x, z])$. (This exists by Exercise 8.13.)

Now by our assumption, there are two neighbors v and w of x such that the cyclic ordering $[\psi(x)]$ induces the cyclic order $\psi(y), \psi(v), \psi(z), \psi(w)$ on these latter 4 points. The Jordan curve theorem tells us that the ψ-image of every path from v to w must intersect $\psi(C)$, i.e., that every path from v to w must intersect C.

However, if we return to the picture provided by ϕ, then since there are 3 paths joining v to w that are disjoint except at their endpoints, at least 2 of these paths do not contain x and hence at least one, $\mathcal{P}'$, is mapped by ϕ to a curve that does not intersect $\phi(C)$. But this means $\mathcal{P}'$ does not intersect C, a contradiction. ◀

Lemma 8.40. *If G is a planar 3-connected graph, then* $\mathrm{Aut}(G)$ *is discrete.*

Proof. Let x be any vertex. Note that the degree of x is at least 3. By Exercise 8.16, it suffices to show that only the identity fixes x and all its neighbors. Let ϕ be an embedding of G in the plane and let $\gamma \in \mathrm{Aut}(G)$ fix x and all its neighbors. Then $\phi \circ \gamma$ is an embedding of G in the plane that induces the same cyclic ordering of the edges of x as does ϕ. By Lemma 8.39, it follows that $[(\phi \circ \gamma)(y)] = [\phi(y)]$ for all y. By induction on the distance of y to x, it is easy to deduce that $\gamma(y) = y$ for all y, which proves the claim. ◀

Because of this and the next corollary, every automorphism of a planar 3-connected graph can be characterized as either orientation preserving or orientation reversing.

Corollary 8.41. *If G is a planar 3-connected graph, $\gamma \in \mathrm{Aut}(G)$, and ϕ is an embedding of G in the plane, then either for all x, we have $[\phi(x)] = [\phi(\gamma x)]$ or for all x, we have that $[\phi(x)]$ and $[\phi(\gamma x)]$ are inverses.*

Proof. Define another embedding ψ of G as follows: Let $\psi(x) := \phi(\gamma x)$ for $x \in \mathsf{V}(G)$ and $\psi(e) := \phi(\gamma e)$ for $e \in \mathsf{E}(G)$. The conclusion follows from Lemma 8.39. ◀

The next corollary implies that every $\gamma \in \mathrm{Aut}(G)$ induces an element of $\mathrm{Aut}(G^\dagger)$.

Corollary 8.42. *If G is a plane 3-connected graph and $\gamma \in \mathrm{Aut}(G)$, then γ maps every facial cycle to a facial cycle.*

Proof. Let $x_0, x_1, \ldots, x_n = x_0$ be the vertices in counter-clockwise order of a facial cycle. Then for each $i \in [0, n-1]$, the edge $[x_i, x_{i+1}]$ is the edge following $[x_{i-1}, x_i]$ in the cyclic order $[x_i]$. Consider the cycle C formed by $\gamma x_0, \gamma x_1, \ldots, \gamma x_n$. If γ is orientation preserving, then these vertices traverse C in counter-clockwise order, so for each $i \in [0, n-1]$, the edge $[\gamma x_i, \gamma x_{i+1}]$ is the edge following $[\gamma x_{i-1}, \gamma x_i]$ in the cyclic order $[\gamma x_i]$. This means that C is facial. The argument is similar when γ is orientation reversing. ◀

Proof of Theorem 8.22. Fix an embedding of G. If G is 3-connected, then let $G' := G$. Otherwise, let G' be the graph formed from G by removing all vertices in $W := \bigcup_{x,y \in \mathsf{V}} \big(K(x) \cup K(x,y)\big)$ and by adding new edges $[x,y]$ between each pair of vertices $x, y \notin W$ for which $K(x,y)$ is maximal and non-empty. Because of Lemma 8.36 and Lemma 8.37, the graph G' is not empty and has one end. Since each new edge may be placed along the trace of a path of edges in G, Lemma 8.38 guarantees that G' is planar. Furthermore, by construction, G' is 3-connected (and quasi-transitive with only one end). Lemma 8.40 shows that $\mathrm{Aut}(G')$ is discrete, whence every subgroup of $\mathrm{Aut}(G')$ is unimodular by Proposition 8.9. Now the restriction to $\mathsf{V}(G')$ of an automorphism Γ of G induces an automorphism of G'. Let Γ be the subgroup of $\mathrm{Aut}(G')$ given by such restrictions. Since Γ acts quasi-transitively, it is unimodular, whence $|S(x)y| = |S(y)x|$ for $x \in \mathsf{V}(G')$ and $y \in \Gamma x$, where S denotes the stabilizer in Γ. But in addition, for any pair $x, y \in \mathsf{V}(G')$, the set $S(x)y$ is the same for the stabilizer in Γ as for the stabilizer in $\mathrm{Aut}(G)$, and $\Gamma x = \mathrm{Aut}(G)x$. Therefore, $\mathrm{Aut}(G)$ is unimodular by Exercise 8.20.

We claim next that $(G')^\dagger$ is quasi-transitive. Fix one vertex x_i from each orbit of $\mathrm{Aut}(G)$. Given any face, let x be one of its vertices. Then for some i, there is a $\gamma \in \mathrm{Aut}(G)$ that maps x to x_i, whence, by Corollary 8.42, maps the face to a face containing x on its boundary. That is, there is an induced $\gamma' \in \mathrm{Aut}\big((G')^\dagger\big)$ that maps the face to one of a finite set of faces. This proves our claim.

Finally, if G is not 3-connected, then temporarily add a vertex to each face of G' and connect such a vertex to each of the vertices on the boundary of that face. Call the new temporary graph G''. This graph G'' produces a triangulation of the region spanned by all the faces. There is a triangulation of either the euclidean plane or the hyperbolic plane using geodesic line segments that is isomorphic to G''; one way to get this is to use circle packing: see Beardon and Stephenson (1990), He and Schramm (1995), or Babai (1997). Now use this new embedding of G' (and forget G''). We can extend each automorphism of G' to an isometry of the (euclidean or hyperbolic) plane. By Selberg's lemma, there is a torsion-free finite-index normal subgroup Γ of $\mathrm{Aut}(G')$ (see, e.g., Corollary 7.6.4 of Ratcliffe (2006)). Since Γ has finite index in $\mathrm{Aut}(G')$, it also acts quasi-transitively on G'. The quotient of the plane by Γ is a compact orientable surface inheriting a finite graph H from G'. Now replace each edge in H by a corresponding $K(x,y)$ from G and add $K(x)$ to each vertex in H that is an image of $x \in \mathsf{V}(G)$. We may do this in a way that results in a graph L embedded in the surface. Finally, we lift L to the plane by taking the universal cover of the surface; this is an embedding of G. Furthermore, a subgroup of isometries acts quasi-transitively on this embedding, whence the dual of G is quasi-transitive. ◀

Remark 8.43. If G is transitive and has one end, then the construction of G' in the proof of Theorem 8.22 must yield all of G (because if x is omitted, then so is the entire orbit of x). Thus, G is 3-connected. Additional results on connectivity were proved by Mader (1970) and Watkins (1970).

The last part of the proof of Theorem 8.22 is not self-contained, which means that our proofs of several results from Section 8.5 are also not self-contained. One can avoid that last part by using an induced percolation on the graph G' that occurs in the proof of Theorem 8.22 and by appealing more to Theorem 8.20.

Theorem 8.34 was proved earlier by Ornstein and Weiss (1987) in a more specialized form. Theorem 5.1 of BLPS (1999b) is somewhat more general than our form of it here, Theorem 8.34.

§8.9. Collected In-Text Exercises.

8.1. Let G be the graph of Example 7.1 with T having degree 3 and $f(x, y)$ be the indicator that y is the ξ-grandparent of x. Show that every automorphism of G fixes the end ξ and therefore that f is diagonally invariant under $\mathrm{Aut}(G)$.

8.2. Show that the graph of Example 7.2 is not a Cayley graph.

8.3. Let Γ be a transitive group of automorphisms of a graph that satisfies (8.4) for all Γ-invariant f. Show that Γ is unimodular.

8.4. Show that if Γ is a transitive unimodular group of automorphisms and Γ' is a larger group of automorphisms of the same graph, then Γ' is also transitive and unimodular.

8.5. Show that for all x and y, we have $|S(x)y| = [S(x) : S(x) \cap S(y)]$.

8.6. Show that if Γ acts transitively, then Γ is unimodular iff $|S(x)y| = |S(y)x|$ for all edges $[x, y]$.

8.7. Show that if Γ acts transitively and for all edges $[x, y]$, there is some $\gamma \in \Gamma$ such that $\gamma x = y$ and $\gamma y = x$, then Γ is unimodular.

8.8. Extend Exercise 8.4 to the quasi-transitive case: Show that if Γ is a quasi-transitive unimodular group of automorphisms and Γ' is a larger group of automorphisms of the same graph, then Γ' is also quasi-transitive and unimodular.

8.9. Show that for any invariant percolation $\mathbf{P}$ on subgraphs of a transitive unimodular graph that has only finite clusters a.s., $\mathbf{E}[\deg_\omega o \mid o \in \omega] \le \alpha(G)$. More generally, let G be a quasi-transitive unimodular graph with a normalized random root $\hat{o}$. Show that if $\mathbf{P}$ is an invariant percolation on subgraphs of G such that all clusters are finite a.s., then $\mathbf{E}[\deg_\omega \hat{o} \mid \hat{o} \in \omega] \le \alpha(G)$.

8.10. Let $\mathbf{P}$ be an invariant percolation on subgraphs of a transitive unimodular graph such that all clusters are finite trees a.s. Show that $\mathbf{E}[\deg_\omega o \mid o \in \omega] < 2$. More generally, let G be a quasi-transitive unimodular graph with a normalized random root $\hat{o}$. Show that if $\mathbf{P}$ is an invariant percolation on subgraphs of G such that all clusters are finite trees a.s., then $\mathbf{E}[\deg_\omega \hat{o} \mid \hat{o} \in \omega] < 2$.

8.11. Suppose that $\mathbf{P}$ is an invariant percolation on a non-amenable quasi-transitive unimodular graph that has a unique infinite cluster ω' a.s. Show that $p_c(\omega') < 1$ a.s.

8.12. Prove Theorem 8.20.

8.13. Show that a plane (properly embedded locally finite) quasi-transitive graph with one end has no face with an infinite number of sides.

8.14. Let G be a transitive unimodular graph and $o \in \mathsf{V}$. For each $x \in \mathsf{V}$, choose a Haar-random $\gamma_x \in \mathrm{Aut}(G)$ that takes o to x. Show that for every finite set $L \subset \mathsf{V}$, we have

$$\mathbf{E}\big|\{x \in \mathsf{V}\,;\ o \in \gamma_x L\}\big| = |L|\,.$$

§8.10. Additional Exercises.

8.15. Prove that the result of Example 8.5 cannot be proved by elementary considerations. Do this by giving an example of an invariant percolation on a transitive graph such that the number of furcations in some cluster is a.s. finite and positive.

8.16. Show that if Γ is a group of automorphisms of a connected graph and $S(x_1) \cap S(x_2) \cap \cdots \cap S(x_n)$ is finite for some $x_1, x_2, \ldots, x_n$, then Γ is discrete.

8.17. Give $\mathrm{Aut}(G)$ the weak topology generated by its action on G, in other words, a base of open sets at $\gamma \in \mathrm{Aut}(G)$ consists of the sets $\{\gamma' \in \mathrm{Aut}(G)\,;\ \gamma' \restriction K = \gamma \restriction K\}$ for finite $K \subset \mathsf{V}$. Show that this topology is metrizable, that a subgroup is discrete in this topology iff it is discrete in the sense of Section 8.2, and that if Γ is a closed countable subgroup of $\mathrm{Aut}(G)$, then Γ is discrete.

8.18. Give another proof of Proposition 8.12 by using Exercise 8.3.

8.19. Show that for every d, we have $\inf \Phi_{\mathsf{V}}(G) > 0$, where the infimum is taken over transitive, non-unimodular G with degree at most d.

8.20. Show that if Γ is not unimodular and μ_x are the weights of Theorem 8.10, then $\sup_x \mu_x = \infty$ and $\inf_x \mu_x = 0$. In fact, the supremum and infimum may each be taken over any single orbit.

8.21. Let G' be a transitive representation of a quasi-transitive graph G.
(a) Show that $\mathrm{Aut}(G')$ is unimodular iff $\mathrm{Aut}(G)$ is unimodular.
(b) Let Γ act quasi-transitively on G. Show that Γ is unimodular iff $\mathrm{Aut}(G')$ is unimodular.

8.22. Show that Proposition 8.12 is also valid for quasi-transitive automorphism groups.

8.23. Let G be an amenable graph and $\Gamma \subseteq \mathrm{Aut}(G)$ be a quasi-transitive subgroup. Choose a complete set $\{o_1, \ldots, o_L\}$ of representatives in V of the orbits of Γ. Choose the weights μ_{o_i} of Theorem 8.10 so that $\sum_i \mu_{o_i}^{-1} = 1$. Show that if K_n is any sequence of finite subsets of vertices such that $|\partial_{\mathsf{V}} K_n|/|K_n| \to 0$, then for all i,

$$\lim_{n\to\infty} \frac{|\Gamma o_i \cap K_n|}{|K_n|} = \mu_{o_i}^{-1}\,.$$

8.24. Show that if Γ is a compact group of automorphisms of a graph, then Γ is unimodular.

8.25. Sharpen Theorem 8.14 to conclude that $\mathbf{E}[\deg_\omega o] < d_G - \Phi_{\mathsf{E}}(G)$ by showing that there is some invariant $\mathbf{P}'$ with all clusters finite and with $\mathbf{E}[\deg_\omega o] < \mathbf{E}'[\deg_{\omega'} o]$.

8.26. A subset K of the vertices of a graph is called ***dominating*** if every vertex is in K or is adjacent to some vertex of K. Suppose that an invariant site percolation on a transitive unimodular graph of degree d is a dominating set a.s. Show that o belongs to the percolation with probability at least $1/(d+1)$.

8.27. Show that for any invariant percolation $\mathbf{P}$ on a transitive unimodular graph that has finite clusters with positive probability, $\mathbf{E}\big[\deg_\omega o \mid |K(o)| < \infty\big] \le \alpha(G)$.

8.28. Let $\mathbf{P}$ be an invariant site percolation on a transitive unimodular graph such that all clusters are finite a.s. Show that $\mathbf{P}[o \in \omega] < d_G/(d_G + \Phi_{\mathsf{V}}(G))$.

8.29. Let G be a quasi-transitive unimodular graph and $\hat{o}$ a normalized random root. Let $\mathfrak{F}$ be the configuration of an invariant random spanning forest on G such that a.s. each tree has one end. ("Spanning" means that the forest includes all vertices of G.) For a vertex x, denote by $\xi(x) = \langle \xi_n(x)\,;\ n \ge 0 \rangle$ the unique infinite simple path starting at x. If $y \in \xi(x)$, call x a ***descendant*** of y. Let $D(x)$ be the set of all descendants of x.

(a) Show that $\mathbf{E}\big[|D(\hat{o})|\big] = \infty$.

(b) Show that $\mathbf{E}\big[|\{y\,;\ \hat{o} = \xi_n(y)\}|\big] = 1$ for each $n \ge 0$.

(c) Show that $\mathbf{E}\Big[\sum_{n\ge 0} 1/|D(\xi_n(\hat{o}))|\Big] = 1$.

(d) Show that $\mathbf{E}\big[|D(\xi_n(\hat{o})) \setminus D(\xi_{n-1}(\hat{o}))|\big] = \infty$ for each $n \ge 1$.

(e) Show that $\mathbf{E}\big[|D(\hat{o})|(\deg_{\mathfrak{F}} \hat{o} - 2)\big] = \infty$.

8.30. Let x and y be two vertices of any graph and define $\tau_p(x, y) := \mathbf{P}_p[x \leftrightarrow y]$. Show that τ is continuous from the left as a function of p.

8.31. Let G be the 1-skeleton of the triangle tessellation of Figure 6.1, i.e., the plane dual of the Cayley graph there. It is quasi-transitive with 3 vertex-orbits. By Theorem 8.22 and Remark 8.43, it is unimodular. Find the weight of each vertex that is given by Theorem 8.10.

8.32. Let G be a plane transitive graph with one end. By Theorem 8.22 and Remark 8.43, its dual $G^\dagger$ is unimodular. Find the distribution of a normalized random root of $G^\dagger$.

8.33. Let G be a plane transitive graph with one end. Every edge $e \in \mathsf{E}(G)$ intersects $e^\dagger \in \mathsf{E}(G^\dagger)$ in one point, v_e. (These are the only intersections of G and $G^\dagger$.) For $e \in \mathsf{E}(G)$, write $\widehat{e}$ for the pair of edges that result from the subdivision of e by v_e, and likewise for $\widehat{e^\dagger}$. This defines a new graph $\widehat{G}$, whose vertices are $\mathsf{V}(G) \cup \mathsf{V}(G^\dagger) \cup \big\{v_e\,;\ e \in \mathsf{E}(G)\big\}$ and whose edges are $\bigcup_{e\in\mathsf{E}(G)} \big(\widehat{e} \cup \widehat{e^\dagger}\big)$. Show that $\widehat{G}$ is unimodular and find the distribution of a normalized random root of $\widehat{G}$.

8.34. Let G be the usual Cayley graph of the (p, q, r)-triangle group and $G^\dagger$ be its dual, where $1/p + 1/q + 1/r \le 1$. Let $\mathfrak{F}$ be an invariant random spanning forest of $G^\dagger$ such that all of its trees are infinite and have at most 2 ends a.s. ("Spanning" means that the forest includes all vertices of G.) Let the edges of $G^\dagger$ opposite to the angles of measure $\pi/p, \pi/q, \pi/r$ have probabilities $\alpha_p, \alpha_q, \alpha_r$ of belonging to $\mathfrak{F}$, respectively. Show that $\alpha_p + \alpha_q + \alpha_r = 1/p + 1/q + 1/r$. Show also that if $\mathfrak{F}^\times$ is defined on G as in (8.20), then $\mathbf{E}[\deg_o \mathfrak{F}^\times] = 3 - 1/p - 1/q - 1/r$.

8.35. Let G be a plane transitive graph with one end. Show that

$$\frac{1}{\mu(G^\dagger)} + \frac{1}{\gamma(G)} \le 1\,,$$

where μ is the connective constant and γ is defined as in (7.11). Deduce that if $G^\dagger$ is regular of degree $d^\dagger$, then

$$\gamma(G) \ge \frac{d^\dagger - 1}{d^\dagger - 2}\,.$$

8.36. Let $\mathbf{P}$ be an invariant bond percolation on a transitive unimodular graph such that all clusters are infinite a.s. Show that $\mathbf{E}[\deg_\omega o] \ge 2$.

8.37. Let ω be the configuration of an invariant percolation on a transitive unimodular graph G. Show that if

(i) some component of ω has at least 3 ends with positive probability,

then

(ii) a.s. every component of ω with at least 3 ends has $p_c < 1$ and

(iii) $\mathbf{E}\big[\deg_\omega o \mid |K(o)| = \infty\big] > 2$.

8.38. Let $\mathbf{P}$ be an invariant percolation on a transitive unimodular graph G. Let F_o be the event that $K(o)$ is an infinite tree with finitely many ends, and let F'_o be the event that $K(o)$ is a finite tree. Show the following.

(i) If $\mathbf{P}[F_o] > 0$, then $\mathbf{E}\big[D(o) \mid F_o\big] = 2$.

(ii) If $\mathbf{P}[F'_o] > 0$, then $\mathbf{E}\big[D(o) \mid F'_o\big] < 2$.

8.39. Give an example of an invariant random forest on a transitive graph where each component has 3 ends, but the expected degree of each vertex is smaller than 2.

8.40. Give an example of an invariant random forest on a transitive graph where each tree has one end and the expected degree of each vertex is greater than 2.

Chapter 9

Infinite Electrical Networks and Dirichlet Functions

§9.1. Free and Wired Electrical Currents.

We have considered currents from one vertex to another on finite networks and from one vertex to infinity on infinite networks. We now take a look at currents from one vertex to another on infinite networks. It turns out that there are two natural ways of defining such currents that correspond to two ways of taking limits on finite networks. In some sense, these two ways may differ due to the possibility of current "passing via infinity" in more than one way. These two currents will be important to our study of random spanning forests in Chapter 10. Our approach in this section will be to give definitions of both these currents using Hilbert space; then we will show how they correspond to limits of currents on finite graphs.

Let G be a connected network whose conductances c satisfy the usual condition that $\sum_{e^-=x} c(e) < \infty$ for each vertex $x \in G$. Note that this condition guarantees that the stars $\sum_{e^-=x} c(e)\chi^e$ of G have finite energy. We assume this condition is satisfied for all networks in this chapter. We also assume that networks are connected. Let $\bigstar$ denote the closure of the linear span of the stars and $\diamondsuit$ the closure of the linear span of the cycles of a graph $G = (\mathsf{V}, \mathsf{E})$, both of these closures taking place in the space

$$\ell^2_-(\mathsf{E}, r) := \left\{\theta : \mathsf{E} \to \mathbb{R}\,;\ \forall e \quad \theta(-e) = -\theta(e) \text{ and } \sum_{e \in \mathsf{E}} \theta(e)^2 r(e) < \infty\right\}.$$

Recall that by (2.13), we have $\sum_{e^-=x} |\theta(e)| < \infty$ for all $x \in \mathsf{V}$ and all $\theta \in \ell^2_-(\mathsf{E}, r)$. For a finite subnetwork H of G, recall that H^{W} is formed by identifying the complement of H to a single vertex. The star space of H^{W} lies in the star space of G, whence the unit current flow i_a in G from a vertex a to infinity (defined in Proposition 2.11) lies in the star space of G.

Since every star and every cycle are orthogonal, it is still true (as for finite networks) that $\bigstar \perp \diamondsuit$. However, it is no longer necessarily the case that $\ell^2_-(\mathsf{E}, r) = \bigstar \oplus \diamondsuit$. Thus, we are led to define two possibly different currents,

$$i_{\mathrm{F}}^{e} := P_{\diamondsuit}^{\perp} \chi^{e} , \tag{9.1}$$

the ***unit free current*** between the endpoints of e (also called the ***limit current***), and

$$i_{\mathrm{W}}^{e} := P_{\bigstar} \chi^{e} , \tag{9.2}$$

the ***unit wired current*** between the endpoints of e (also called the ***minimal current***).

▷ **Exercise 9.1.**

Calculate i_{F}^{e} and i_{W}^{e} in a regular tree.

The names for these currents are explained by the following two propositions. Recall that for a subnetwork $G_n \subset G$, we identify $\mathsf{E}(G_n)$ as a subset of $\mathsf{E}(G)$, and also identify $\mathsf{E}(G_n^{\mathrm{W}})$ as a subset of $\mathsf{E}(G)$.

Proposition 9.1. *Let G be an infinite network exhausted by finite subnetworks $\langle G_n \rangle$. Let e be an edge in G_1 and i_n be the unit current flow in G_n from e^- to e^+. Then $\|i_n - i_{\mathrm{F}}^{e}\|_r \to 0$ as $n \to \infty$ and $\mathscr{E}(i_{\mathrm{F}}^{e}) = i_{\mathrm{F}}^{e}(e) r(e)$.*

Proof. Decompose $\ell^2_-(\mathsf{E}_n, r) = \bigstar_n \oplus \diamondsuit_n$ on G_n into the spaces spanned by the stars and cycles in G_n and recall that $i_n = \chi^e - P_{\diamondsuit_n} \chi^e$. We may regard the spaces $\diamondsuit_n$ as lying in $\ell^2_-(\mathsf{E}, r)$, where they form an increasing sequence. (Each cycle in G_n lies in G_{n+1}, but the same is not true of the stars.) The closure of $\bigcup_n \diamondsuit_n$ is $\diamondsuit$. Note that the projection on $\diamondsuit_n$ of any element of $\ell^2_-(\mathsf{E}_n, r)$ is the same as its projection on $\diamondsuit_n$ in $\ell^2_-(\mathsf{E}, r)$ since for $\theta \in \ell^2_-(\mathsf{E}, r)$, if $\theta \perp \diamondsuit_n$ in $\ell^2_-(\mathsf{E}_n, r)$, then also $\theta \perp \diamondsuit_n$ in $\ell^2_-(\mathsf{E}, r)$. Thus, the fact that $\|i_n - i_{\mathrm{F}}^{e}\|_r \to 0$ follows from the standard result given below in Exercise 9.2. Also, we have that

$$i_{\mathrm{F}}^{e}(e) r(e) = (i_{\mathrm{F}}^{e}, \chi^e)_r = (i_{\mathrm{F}}^{e}, P_{\diamondsuit}^{\perp} \chi^e)_r = \mathscr{E}(i_{\mathrm{F}}^{e}) . \qquad \blacktriangleleft$$

▷ **Exercise 9.2.**

Let H_n be increasing closed subspaces of a Hilbert space H and P_n be the orthogonal projection on H_n. Let P be the orthogonal projection on the closure of $\bigcup H_n$. Show that for all $u \in H$, we have $\|P_n u - P u\| \to 0$ as $n \to \infty$.

Proposition 9.2. *Let G be an infinite network exhausted by finite subnetworks $\langle G_n \rangle$. Form G_n^{W} by identifying the complement of G_n to a single vertex. Let e be an edge in G_1 and i_n be the unit current flow in G_n^{W} from e^- to e^+. Then $\|i_n - i_{\mathrm{W}}^{e}\|_r \to 0$ as $n \to \infty$ and $\mathscr{E}(i_{\mathrm{W}}^{e}) = i_{\mathrm{W}}^{e}(e) r(e)$, which is the minimum energy among all unit flows from e^- to e^+.*

▷ **Exercise 9.3.**

Prove Proposition 9.2.

Since $\bigstar \subseteq \diamondsuit^\perp$, we have $\mathscr{E}(i^e_{\rm W}) \le \mathscr{E}(i^e_{\rm F})$ with equality iff $i^e_{\rm W} = i^e_{\rm F}$. Therefore,

$$i^e_{\rm W}(e) \le i^e_{\rm F}(e) \text{ with equality iff } i^e_{\rm W} = i^e_{\rm F}\,. \tag{9.3}$$

Proposition 9.3. *For any network, $i^e_{\rm W} = i^e_{\rm F}$ for every edge e iff $\ell^2_-(\mathsf{E}, r) = \bigstar \oplus \diamondsuit$.*

Proof. Note that $\ell^2_-(\mathsf{E}, r) = \bigstar \oplus \diamondsuit$ is equivalent to $P_\bigstar = P^\perp_\diamondsuit$. Since $\{\chi^e\,;\ e \in \mathsf{E}_{1/2}\}$ is a basis for $\ell^2_-(\mathsf{E}, r)$, this is also equivalent to $P_\bigstar \chi^e = P^\perp_\diamondsuit \chi^e$ for all edges e, as desired. ◀

Given two vertices a and z, define the ***free*** and ***wired unit currents*** from a to z as $i^{a,z}_{\rm F} := \sum_{k=1}^n i^{e_k}_{\rm F}$ and $i^{a,z}_{\rm W} := \sum_{k=1}^n i^{e_k}_{\rm W}$, where $e_1, e_2, \ldots, e_n$ is an oriented path from a to z.

▷ **Exercise 9.4.**

Show that the choice of path in the definition of the free and wired currents from a to z does not influence their values.

We call $\mathscr{R}^{\rm F}(a \leftrightarrow z) := \mathscr{E}\big(i^{a,z}_{\rm F}\big)$ and $\mathscr{R}^{\rm W}(a \leftrightarrow z) := \mathscr{E}\big(i^{a,z}_{\rm W}\big)$ the ***free*** and ***wired effective resistance***, respectively, between a and z. Note that these are equal to the limits of $\mathscr{E}(i_n)$, where i_n is the unit current flow from a to z on G_n or $G^{\rm W}_n$, respectively, since for any sequence of vectors u_n converging in norm to a vector u, we have $\|u_n\| \to \|u\|$; indeed, $\big|\|u_n\| - \|u\|\big| \le \|u_n - u\|$ by the triangle inequality. Of course, the reciprocals of the effective resistances are the free and wired effective conductances.

§9.2. Planar Duality.

In this section, we recall from Section 6.4 the basic notions of duality for planar graphs and show how the dual graphs give related electrical networks.

A ***planar*** graph is one that can be drawn in the plane in such a way that edges do not cross; an actual such embedding is called a ***plane*** graph. If G is a plane graph such that each bounded set in the plane contains only finitely many vertices of G, then G is said to be ***properly embedded*** in the plane. We will always assume without further mention that plane graphs are properly embedded. A ***face*** of a plane graph is a connected component of the complement of the graph in the plane. If G is a plane (multi)graph, then the ***plane dual*** $G^\dagger$ of G is the (multi)graph formed as follows: The vertices of $G^\dagger$ are the faces formed by G. Two faces of G are joined by an edge in $G^\dagger$ precisely when they share an edge in

G. Thus, $\mathsf{E}(G)$ and $\mathsf{E}(G^\dagger)$ are in a natural one-to-one correspondence. Furthermore, if one draws each vertex of $G^\dagger$ in the interior of the corresponding face of G and each edge of $G^\dagger$ so that it crosses the corresponding edge of G, then the dual of $G^\dagger$ is G.

We choose orientations of the edges so that for $e \in \mathsf{E}$, the corresponding edge $e^\dagger$ of the dual crosses e from right to left as viewed from the direction of e. Thus, the orientation of the pair $(e, e^\dagger)$ is the same as the usual counter-clockwise orientation of $\mathbb{R}^2$.

If conductances $c(e)$ are assigned to the edges e of G, then we define the conductance $c(e^\dagger)$ of $e^\dagger$ to be the *resistance* $r(e)$. In this case, we shall assume without mention that G is such that $\sum_{(e^\dagger)^- = u} c(e^\dagger) < \infty$ for each vertex $u \in G^\dagger$, so that the stars of $G^\dagger$ have finite energy.

The bijection $e \mapsto e^\dagger$ provides a natural isometric isomorphism $^\dagger : \ell^2_-\big(\mathsf{E}(G), r\big) \to \ell^2_-\big(\mathsf{E}(G^\dagger), r\big)$ via $f^\dagger(e^\dagger) := r(e)f(e)$. That is, $f \mapsto f^\dagger$ is a surjective linear map such that for all $f, g \in \ell^2_-\big(\mathsf{E}(G), r\big)$, we have $f^\dagger, g^\dagger \in \ell^2_-\big(\mathsf{E}(G^\dagger), r\big)$ and

$$(f, g)_r = \sum_{e \in \mathsf{E}(G)} f(e)g(e)r(e) = \sum_{e \in \mathsf{E}(G^\dagger)} f^\dagger(e)g^\dagger(e)r(e) = (f^\dagger, g^\dagger)_r \,.$$

Note that $(f^\dagger)^\dagger = -f$.

It is clear that if f is a star in $\ell^2_-\big(\mathsf{E}(G), r\big)$, then $f^\dagger$ is a cycle in $\ell^2_-\big(\mathsf{E}(G^\dagger), r\big)$. Moreover, it is easy to see that † induces an isomorphism from the star space on G to the cycle space on $G^\dagger$ and from the cycle space on G to the star space on $G^\dagger$. Let's see what this means for currents. For an edge $e \in G$, consider the orthogonal decomposition

$$\chi^e = i^e_{\mathrm{W}} + f \,,$$

where $i^e_{\mathrm{W}} \in \bigstar(G)$ and $f \in \bigstar(G)^\perp$. Applying the map †, we obtain

$$r(e)\chi^{e^\dagger} = (\chi^e)^\dagger = (i^e_{\mathrm{W}})^\dagger + f^\dagger \,,$$

whence

$$\chi^{e^\dagger} = c(e)(i^e_{\mathrm{W}})^\dagger + c(e)f^\dagger \,,$$

where the first term on the right is a vector in $\diamondsuit(G^\dagger)$ and the second is in $\diamondsuit(G^\dagger)^\perp$. It follows from this and the definition (9.1) that

$$i^{e^\dagger}_{\mathrm{F}} = c(e)f^\dagger = \chi^{e^\dagger} - c(e)(i^e_{\mathrm{W}})^\dagger \,.$$

Likewise, one can check that

$$i^{e^\dagger}_{\mathrm{W}} = \chi^{e^\dagger} - c(e)(i^e_{\mathrm{F}})^\dagger \,.$$

In particular, we obtain

$$i_{\mathrm{F}}^{e^\dagger}(e^\dagger) = \chi^{e^\dagger}(e^\dagger) - c(e)(i_{\mathrm{W}}^e)^\dagger(e^\dagger) = 1 - c(e)r(e)i_{\mathrm{W}}^e(e) = 1 - i_{\mathrm{W}}^e(e) \tag{9.4}$$

and

$$i_{\mathrm{W}}^{e^\dagger}(e^\dagger) = 1 - i_{\mathrm{F}}^e(e)\,.$$

This has the following interesting consequence:

Proposition 9.4. *Let G be a plane network and $[a,z]$ be an edge of G. Let the dual edge be $[b,y]$. Suppose that the graph G' obtained by deleting the edge $[a,z]$ from G is connected and that the graph $(G^\dagger)'$ obtained by deleting the edge $[b,y]$ is connected. Then the free effective resistance between a and z in G' equals the wired effective conductance between b and y in $(G^\dagger)'$.*

▷ **Exercise 9.5.**
Prove Proposition 9.4.

§9.3. Harmonic Dirichlet Functions.

Here we give some conditions for equality of wired and free current.

Recall that the ***gradient*** of a function f on V is the antisymmetric function

$$\nabla f := c\, df$$

on E. Ohm's law is then $\nabla v = i$. Define the space of ***Dirichlet*** functions

$$\mathbf{D} := \big\{f\,;\ \nabla f \in \ell^2_-(\mathsf{E}, r)\big\}\,.$$

Given a vertex $o \in \mathsf{V}$, we use the inner product on $\mathbf{D}$

$$\langle f, g\rangle := f(o)g(o) + (\nabla f, \nabla g)_r = f(o)g(o) + (df, dg)_c\,.$$

This makes $\mathbf{D}$ a Hilbert space, whose norm we denote by $\|\bullet\|_{\mathbf{D}}$. The choice of o does not matter in the sense that changing it leads to an equivalent norm: for any x, take a path of edges $\langle e_j\,;\ 1 \le j \le n\rangle$ leading from x to o and note that by the Cauchy-Schwarz inequality,

$$\begin{aligned} f(x)^2 = \big[f(o) + \sum_j df(e_j)\big]^2 &\le \big[1 + \sum_j r(e_j)\big]\big[f(o)^2 + \sum_j c(e_j)df(e_j)^2\big] \\ &\le \big[1 + \sum_j r(e_j)\big]\langle f, f\rangle\,. \end{aligned}$$

The quantity $\mathscr{D}(f) := \|\nabla f\|_r^2 = \|df\|_c^2$ is called the ***Dirichlet energy*** of f.* Of course, the constant functions, which we identify as $\mathbb{R}$, lie in $\mathbf{D}$. Since it is the gradient of a function that matters most here, we usually work with $\mathbf{D}/\mathbb{R}$ using the inner product

$$\langle f + \mathbb{R}, g + \mathbb{R} \rangle := (df, dg)_c\,.$$

Then $\mathbf{D}/\mathbb{R}$ is a Hilbert space.

If $\phi : \mathbb{R} \to \mathbb{R}$ is a contraction (i.e., $|\phi(x) - \phi(y)| \le |x - y|$ for $x, y \in \mathbb{R}$), then $f \mapsto \phi \circ f$ maps $\mathbf{D}$ to $\mathbf{D}$ by decreasing the energy: $\mathscr{D}(\phi \circ f) \le \mathscr{D}(f)$. Useful examples include $\phi(s) := |s|$ and the truncation maps

$$\phi_N(s) := \begin{cases} s & \text{if } |s| \le N, \\ sN/|s| & \text{if } |s| > N. \end{cases}$$

Thus, for $f \in \mathbf{D}$, there is a sequence $\langle \phi_N \circ f \rangle$ of bounded functions in $\mathbf{D}$ that converge to f in norm by Lebesgue's dominated convergence theorem. If $f, g \in \mathbf{D}$ are both bounded functions, then

$$\|d(fg)\|_c \le \|f\|_\infty \|dg\|_c + \|df\|_c \|g\|_\infty \tag{9.5}$$

since

$$\begin{aligned} \big|(fg)(x) - (fg)(y)\big| &= \big|f(x)[g(x) - g(y)] + [f(x) - f(y)]g(y)\big| \\ &\le \big\|f\|_\infty |g(x) - g(y)| + \|g\|_\infty |f(x) - f(y)| \\ &= |g'(x) - g'(y)| + |f'(x) - f'(y)|\,, \end{aligned}$$

where $g' := \|f\|_\infty g$ and $f' := \|g\|_\infty f$, whence

$$\|d(fg)\|_c \le \||dg'| + |df'|\|_c \le \|dg'\|_c + \|df'\|_c = \|f\|_\infty \|dg\|_c + \|df\|_c \|g\|_\infty\,.$$

Recall that $\bigstar$ denotes the closed span of the stars in $\ell^2_-(\mathsf{E}, r)$ and $\diamondsuit$ the closed span of the cycles in $\ell^2_-(\mathsf{E}, r)$. The gradient map $\nabla : \mathbf{D}/\mathbb{R} \to \diamondsuit^\perp$ is an isometric isomorphism (since G is connected). Just as we reasoned in Section 2.4, an element $\theta \in (\bigstar \oplus \diamondsuit)^\perp$ is the gradient of a harmonic function $f \in \mathbf{D}$. Thus, if $\mathbf{HD}$ denotes the set of $f \in \mathbf{D}$ that are harmonic, we have the orthogonal decomposition

$$\ell^2_-(\mathsf{E}, r) = \bigstar \oplus \diamondsuit \oplus \nabla\mathbf{HD}\,. \tag{9.6}$$

Since $\ell^2_-(\mathsf{E}, r) = \bigstar \oplus \diamondsuit$ iff $\nabla\mathbf{HD} = \mathbf{0}$ iff $\mathbf{HD} = \mathbb{R}$, we may add the condition that there are no nonconstant harmonic Dirichlet functions to those in Proposition 9.3:

* The classical Dirichlet energy of a smooth function f in a domain Ω is $\int_\Omega |\nabla f|^2 \, dA$, where dA is area measure.

Theorem 9.5. (Doyle, 1988) *Let G be a denumerable network. We have* $\mathbf{HD} = \mathbb{R}$ *iff* $i^e_{\mathrm{W}} = i^e_{\mathrm{F}}$ *for each* $e \in \mathsf{E}$.

Doyle's theorem is often stated another way as a criterion for uniqueness of currents. We call the elements i of $\diamondsuit^\perp$ ***currents*** (with sources where $d^*i > 0$ and sinks where $d^*i < 0$). We say that ***currents are unique*** if whenever i, i' are currents with $d^*i = d^*i'$, we have $i = i'$. Observe that by subtraction and by (2.8), this is the same as saying that $\mathbf{HD} = \mathbb{R}$. For this reason, the class of networks with unique currents is often denoted $\mathcal{O}_{\mathrm{HD}}$.

Let $\mathbf{D}_0$ be the closure in $\mathbf{D}$ of the set of $f \in \mathbf{D}$ with finite support.

▷ **Exercise 9.6.**

(a) Show that $\nabla \mathbf{D}_0 = \bigstar$.

(b) Show that $\mathbf{D}/\mathbb{R} = \widetilde{\mathbf{D}_0}/\mathbb{R} \oplus \mathbf{HD}/\mathbb{R}$, where $\widetilde{\mathbf{D}_0} := \mathbf{D}_0 + \mathbb{R}$.

(c) Show that currents are unique iff $\mathbf{D}/\mathbb{R} = \widetilde{\mathbf{D}_0}/\mathbb{R}$.

(d) Show that $\|\mathbf{1} - \mathbf{D}_0\|^2_{\mathbf{D}} = \mathscr{C}(o \leftrightarrow \infty)/[1 + \mathscr{C}(o \leftrightarrow \infty)]$, where o is the vertex used to define the inner product on $\mathbf{D}$.

(e) Show that G is recurrent iff $\mathbf{1} \in \mathbf{D}_0$.

(f) **(Royden Decomposition)** Show that if G is transient, then every $f \in \mathbf{D}$ has a unique decomposition $f = g + h$ with $g \in \mathbf{D}_0$ and $h \in \mathbf{HD}$. Note that this is not an orthogonal decomposition.

(g) With the assumptions and notation of part (f), show that $g(x) = (\nabla f, i_x)_r$ and that $g(x)^2 \le \mathscr{D}(f)\mathcal{G}(x,x)/\pi(x)$, where i_x is the unit current flow from x to infinity (from Proposition 2.11) and $\mathcal{G}(\bullet,\bullet)$ is the Green function.

(h) Show that if G is transient, then $\nabla : \mathbf{D}_0 \to \bigstar$ is invertible with bounded inverse.

This allows us to show that recurrent networks have unique currents. Since current cannot go to infinity at all in recurrent networks, this fits with our intuition that non-unique currents require the ability of current to pass to infinity in more than one way.

Corollary 9.6. *A network G is recurrent iff $\mathbf{D} = \mathbf{D}_0$, in which case currents are unique.*

Proof. If $\mathbf{D} = \mathbf{D}_0$, then $\mathbf{1} \in \mathbf{D}_0$, whence G is recurrent by Exercise 9.6(e). Conversely, suppose that G is recurrent. By Exercise 9.6(e), we have $\mathbf{1} \in \mathbf{D}_0$. Thus, there exist $g_n \to \mathbf{1}$ with g_n having finite support and $\mathbf{0} \le g_n \le \mathbf{1}$. (We use the contraction $\phi(s) := s\mathbf{1}_{[0,1]}(s) + \mathbf{1}_{(1,\infty)}(s)$ if necessary to get the values of g_n to be in $[0,1]$.) Let $f \in \mathbf{D}$ be bounded. Then $fg_n \in \mathbf{D}$ by (9.5) and $fg_n \to f$ in $\mathbf{D}$ by the dominated convergence theorem. Hence $f \in \mathbf{D}_0$, i.e., $\mathbf{D}_0$ contains all bounded Dirichlet functions. Since these functions are dense in $\mathbf{D}$, we get $\mathbf{D}_0 = \mathbf{D}$. Finally, when this happens, currents are unique by Exercise 9.6(c). ◀

In order to exhibit transient networks that have unique currents, we shall use the following criterion, which generalizes a result of Thomassen (1989). It is analogous to the Nash-Williams criterion. It shows that high connectedness of cutsets, rather than their small size, can force currents to be unique. Let $\mathscr{R}(x \leftrightarrow y\,;\ A)$ denote the effective resistance between vertices x and y in a finite network A. We will use this when A is a subnetwork of G; in this case, the effective resistance is computed purely in terms of the network A. Let

$$\mathsf{RD}(A) := \sup\{\mathscr{R}(x \leftrightarrow y\,;\ A)\,;\ x, y \in A\}$$

be the "effective-resistance diameter" of A. Note that in the case of unit conductances on the edges, $\mathsf{RD}(A)$ is at most the graph diameter of A. We say that a subnetwork W ***separates*** x ***from*** ∞ if every simple infinite path starting at x intersects W in some vertex.

Theorem 9.7. (Unique Currents from Internal Connectivity) *If $\langle W_n \rangle$ is a sequence of pairwise edge-disjoint finite subnetworks of a locally finite network G such that each [equivalently, some] vertex is separated from ∞ by all but finitely many W_n and such that*

$$\sum_n 1/\mathsf{RD}(W_n) = \infty\,,$$

then G has unique currents.

Proof. Let f be any nonconstant harmonic function. Take an edge e_0 whose endpoints have different values for f, i.e., $df(e_0) \neq 0$. Suppose that W_n separates e_0 from infinity for $n \geq n_0$. Let H_n be the set of vertices that W_n separates from infinity, including the vertices of W_n. Because G is locally finite, H_n is finite. Let x_n and y_n be points of H_n where f takes its maximum and minimum, respectively, on H_n. By the maximum principle, we may assume that $x_n, y_n \in W_n$. Thus, for $n \geq n_0$, we have $f(x_n) - f(y_n) \geq |df(e_0)|$. Define F_n on V to be the function $F_n := \big(f - f(y_n)\big)/\big(f(x_n) - f(y_n)\big)$. Then $|dF_n| \leq |df|/|df(e_0)|$. By Dirichlet's principle (Exercise 2.13), we have

$$1/\mathsf{RD}(W_n) \leq \mathscr{C}(x_n \leftrightarrow y_n\,;\ W_n) \leq \sum_{e \in W_n} c(e) dF_n(e)^2 \leq \sum_{e \in W_n} c(e) df(e)^2/df(e_0)^2\,.$$

Since edges of the networks W_n are disjoint, it follows that

$$\sum_{n \geq n_0} 1/\mathsf{RD}(W_n) \leq \sum_{n \geq n_0} \sum_{e \in W_n} c(e) df(e)^2/df(e_0)^2 \leq \langle f, f \rangle/df(e_0)^2\,.$$

Therefore our hypothesis implies that f is not Dirichlet. Thus, $\mathbf{HD} = \mathbb{R}$, so the conclusion follows from Theorem 9.5. ◀

▷ **Exercise 9.7.**

One can define the product of two networks in various ways. For example, given two networks $G_i = (\mathsf{V}_i, \mathsf{E}_i)$ with conductances c_i $(i = 1, 2)$, define the ***cartesian product*** $G = (\mathsf{V}, \mathsf{E})$ with conductances c by $\mathsf{V} := \mathsf{V}_1 \times \mathsf{V}_2$,

$$\mathsf{E} := \{\big((x_1, x_2), (y_1, y_2)\big) \,;\, (x_1 = y_1,\ (x_2, y_2) \in \mathsf{E}_2) \text{ or } ((x_1, y_1) \in \mathsf{E}_1,\ x_2 = y_2)\}\,,$$

and

$$c\big((x_1, x_2), (y_1, y_2)\big) := \begin{cases} c(x_2, y_2) & \text{if } x_1 = y_1, \\ c(x_1, y_1) & \text{if } x_2 = y_2. \end{cases}$$

Show that if G_i are infinite locally finite graphs with unit conductances, then G has unique currents.

It follows that the usual nearest-neighbor graph on $\mathbb{Z}^d$ has unique currents.

If one network is "similar" to another, must they both have unique currents or both not? One such case that is easy to decide is a graph with two assignments on conductances, c and c'. If $c \asymp c'$, meaning that the ratio c/c' is bounded and bounded away from 0, then c has unique currents iff c' does. Indeed, from Exercise 9.6(c), currents are unique iff the functions with finite support span a dense subspace of $\mathbf{D}/\mathbb{R}$. Since the two norms on $\mathbf{D}/\mathbb{R}$ for the different conductances c and c' are equivalent, density is the same for both. This has a very useful generalization due to Soardi (1993), analogous to Proposition 2.17.

Theorem 9.8. (Rough Isometry Preserves Unique Currents) *Let G and G' be two infinite roughly isometric networks with conductances c and c'. If c, c', c^{-1}, c'^{-1} are all bounded and the degrees in G and G' are all bounded, then G has unique currents iff G' does.*

Proof. (due to O. Schramm) Since $c \asymp \mathbf{1}$ and $c' \asymp \mathbf{1}$, we may assume that actually $c = \mathbf{1}$ and $c' = \mathbf{1}$.

Let $\phi : \mathsf{V} \to \mathsf{V}'$ be a rough isometry. Suppose first that ϕ is a bijection. We claim that the map $\Phi : f + \mathbb{R} \mapsto f \circ \phi^{-1} + \mathbb{R}$ from $\mathbf{D}/\mathbb{R}$ to $\mathbf{D}'/\mathbb{R}$ is an isomorphism of Banach spaces, where $\mathbf{D}'$ is the space of Dirichlet functions on G'. Since the minimum distance between distinct vertices is 1, the fact that ϕ is a bijective rough isometry implies that ϕ is actually bi-Lipschitz: for some constant γ_1, we have

$$\gamma_1^{-1} d(x, y) \le d'\big(\phi(x), \phi(y)\big) \le \gamma_1 d(x, y)$$

for $x, y \in \mathsf{V}$. For $e' = \big(\phi(x), \phi(y)\big) \in \mathsf{E}'$, let $\mathcal{P}(e')$ be a path of $d(x, y) \le \gamma_1$ edges in E joining x to y. Given $e \in \mathsf{E}$ and $e' \in \mathsf{E}'$ with $e \in \mathcal{P}(e')$, we know that the endpoints of e' are the ϕ-images of vertices that are within distance γ_1 of the endpoints of e. Since the degrees of G are bounded, there are no more than some constant γ_2 choices for such pairs of endpoints of e'. Therefore no edge in E appears in more than γ_2 paths of the form $\mathcal{P}(e')$ for $e' \in \mathsf{E}'$. Thus, for $f \in \mathbf{D}$, we have by the Cauchy-Schwarz inequality

$$\begin{aligned}\|f \circ \phi^{-1} + \mathbb{R}\|^2 &= \frac{1}{2} \sum_{e' \in \mathsf{E}'} \nabla(f \circ \phi^{-1})(e')^2 = \frac{1}{2} \sum_{e' \in \mathsf{E}'} \left(\sum_{e \in \mathcal{P}(e')} \nabla f(e) \right)^2 \\ &\le \frac{\gamma_1}{2} \sum_{e' \in \mathsf{E}'} \sum_{e \in \mathcal{P}(e')} \nabla f(e)^2 \le \frac{\gamma_1 \gamma_2}{2} \sum_{e \in \mathsf{E}} \nabla f(e)^2 = \gamma_1 \gamma_2 \|f + \mathbb{R}\|^2 .\end{aligned}$$

This shows that Φ is a bounded map; symmetry gives the boundedness of Φ^{-1}, establishing our claim.

Now clearly Φ is a bijection between the subspaces of functions with finite support. Hence Φ also gives an isomorphism between $\mathbf{D}_0/\mathbb{R}$ and $\mathbf{D}_0'/\mathbb{R}$. Therefore, the result follows from Exercise 9.6(c).

Now consider the case that ϕ is not a bijection. We will "fluff up" the graphs G and G' to extend ϕ to a bijection so as to use the result we have just established. Because the image of V comes within some fixed distance β of every vertex in G' and because G' has bounded degrees, V' can be partitioned into subsets $N'\big(\phi(x)\big)$ ($x \in \mathsf{V}$) of bounded cardinality in such a way that every vertex in $N'\big(\phi(x)\big)$ lies within distance β of $\phi(x)$. Also, because ϕ does not shrink distances too much and the degrees in G are bounded, the cardinalities of the preimages $\phi^{-1}(x)$ ($x \in \mathsf{V}'$) are bounded. For each $x \in \phi(\mathsf{V})$, let $\psi(x)$ denote some vertex in $\phi^{-1}(x)$. Create a new graph G_* by joining each vertex $\psi(x)$ ($x \in \mathsf{V}'$) to new vertices $v_1(x), \ldots, v_{|N'(x)|-1}(x)$ by new edges. Also, create a new graph G'_* by joining each vertex $\phi(x)$ ($x \in \mathsf{V}$) to new vertices $w_1(x), \ldots, w_{|\phi^{-1}(\phi(x))|-1}(x)$ by new edges. Then G_* and G'_* have bounded degrees. Define $\phi_* : G_* \to G'_*$ as follows: For $x \in \mathsf{V}'$, let $\phi_*\big(\psi(x)\big) := x$ and let ϕ_* be a bijection from $v_1(x), \ldots, v_{|N'(x)|-1}(x)$ to $N'(x) \setminus \{x\}$. For $x \in \mathsf{V}$, let $\phi_*\big(\psi\big(\phi(x)\big)\big) := \phi(x)$ and let ϕ_* be a bijection from $\phi^{-1}\big(\phi(x)\big) \setminus \big\{\psi\big(\phi(x)\big)\big\}$ to $w_1(x), \ldots, w_{|\phi^{-1}(\phi(x))|-1}(x)$. Then ϕ_* is a bijective rough isometry. By the first part of the proof, we have unique currents on G_* iff we do on G'_*. Since every harmonic function on G_* has the same value on $v_i(x)$ as on $\psi(x)$ ($x \in \mathsf{V}'$), it follows that G_* has unique currents iff G does. The same holds for G'_* and G', which proves the theorem. ◀

Every finitely generated abelian group is isomorphic to $\mathbb{Z}^d \times \Gamma$ for some d and some finite abelian group Γ. Therefore, each of its Cayley graphs is roughly isometric to the usual graph on $\mathbb{Z}^d$, whence has unique currents. We record this conclusion:

Corollary 9.9. *Every Cayley graph of a finitely generated abelian group has unique currents. More generally, every bounded-degree graph roughly isometric to a Euclidean space has unique currents.*

We'll see in Exercise 10.10 that amenable transitive graphs also have unique currents.

Our next goal is to look at graphs roughly isometric to hyperbolic spaces. In the following section, we will use the fact that $\langle f(X_n)\rangle$ converges a.s. for all $f \in \mathbf{D}$ on planar transient networks. This is true not only on planar transient networks, but on all transient networks, as we show next. We will use the result of the following exercise, which illustrates a basic technique in the theory of harmonic functions.

▷ **Exercise 9.8.**

Let G be transient and let $f \in \mathbf{D}_0$. Show that there is a unique $g \in \mathbf{D}_0$ having minimal energy such that $g \geq |f|$. Show that this g is ***superharmonic***, meaning that for all vertices x,

$$g(x) \geq \frac{1}{\pi(x)} \sum_{y \sim x} c(x,y) g(y) \,.$$

Theorem 9.10. (Dirichlet Functions Along Random Walks) *If G is a transient network, $\langle X_n\rangle$ the corresponding random walk, and $f \in \mathbf{D}$, then $\langle f(X_n)\rangle$ has a finite limit a.s. and in L^2. If $f = f_{\mathbf{D}_0} + f_{\mathbf{HD}}$ is the Royden decomposition of f, then $\lim f(X_n) = \lim f_{\mathbf{HD}}(X_n)$ a.s.*

This is due to Ancona, Lyons, and Peres (1999).

Proof. The following notation will be handy:

$$\mathscr{E}_f(x) := \sum_y p(x,y)[f(y) - f(x)]^2 = \mathbf{E}_x[|f(X_1) - f(X_0)|^2] \,.$$

Thus, we have

$$\mathscr{D}(f) = \frac{1}{2} \sum_{x \in \mathsf{V}} \pi(x) \mathscr{E}_f(x) \,.$$

Since the vertex o used in defining the norm on $\mathbf{D}$ is arbitrary, we may take $o = X_0$. (We assume that X_0 is non-random, without loss of generality.) We first observe that for any $f \in \mathbf{D}$, it is easy to bound the sum of squared increments along the random walk: For any Markov chain, we have

$$\begin{aligned}\mathcal{G}(y, o) &= \sum_{n\ge 0} \mathbf{P}_y[\tau_o = n]\mathbf{E}_y\Big[\sum_{k\ge 0} \mathbf{1}_{\{X_{n+k}=o\}} \;\Big|\; \tau_o = n\Big] \\ &= \mathbf{P}_y[\tau_o < \infty]\mathcal{G}(o, o) \le \mathcal{G}(o, o)\,.\end{aligned}$$

In our reversible case, Exercise 2.1(c) tells us that $\pi(o)\mathcal{G}(o, y) = \pi(y)\mathcal{G}(y, o) \le \pi(y)\mathcal{G}(o, o)$, whence

$$\begin{aligned}\sum_{k=1}^{\infty} \mathbf{E}\big[|f(X_k) - f(X_{k-1})|^2\big] &= \sum_{y\in\mathsf{V}}\sum_{k=1}^{\infty} \mathbf{E}\big[|f(X_k) - f(X_{k-1})|^2 \mid X_{k-1} = y\big]\mathbf{P}[X_{k-1} = y] \\ &= \sum_{y\in\mathsf{V}} \mathcal{G}(o, y)\mathscr{E}_f(y) \le \frac{\mathcal{G}(o, o)}{\pi(o)} \sum_{y\in\mathsf{V}} \pi(y)\mathscr{E}_f(y) \\ &= 2\frac{\mathcal{G}(o, o)}{\pi(o)}\mathscr{D}(f)\,. \end{aligned} \tag{9.7}$$

Also, we have

$$\begin{aligned}\mathbf{E}\big[f(X_k)^2 - f(X_{k-1})^2\big] &= \mathbf{E}\big[|f(X_k) - f(X_{k-1})|^2\big] + 2\mathbf{E}\big[\big(f(X_k) - f(X_{k-1})\big)f(X_{k-1})\big] \\ &\le \mathbf{E}\big[|f(X_k) - f(X_{k-1})|^2\big]\end{aligned}$$

in case f is harmonic (since then $\langle f(X_n)\rangle$ is a martingale) or f is superharmonic and nonnegative (since then $\langle f(X_n)\rangle$ is a supermartingale). In either of these two cases, we obtain by summing these inequalities for $k = 1, \ldots, n$ that

$$\mathbf{E}\big[f(X_n)^2 - f(X_0)^2\big] \le \sum_{k=1}^{n} \mathbf{E}\big[|f(X_k) - f(X_{k-1})|^2\big] \le 2\frac{\mathcal{G}(o, o)}{\pi(o)}\mathscr{D}(f)$$

by (9.7). That is,

$$\mathbf{E}\big[f(X_n)^2\big] \le 2\frac{\mathcal{G}(o, o)}{\pi(o)}\mathscr{D}(f) + f(o)^2 \tag{9.8}$$

in case f is harmonic or f is superharmonic and nonnegative.

It follows from (9.8) applied to $f_{\mathbf{HD}}$ that $\langle f_{\mathbf{HD}}(X_n)\rangle$ is a martingale bounded in L^2, whence by Doob's theorem it converges a.s. and in L^2. It remains to show that $f_{\mathbf{D}_0}$ converges to 0 a.s. and in L^2. Given $\epsilon > 0$, write $f_{\mathbf{D}_0} = f_1 + f_2$, where f_1 is finitely supported and $\mathscr{D}(f_2) < \pi(o)\epsilon/\big(3\mathcal{G}(o, o)\big)$. Exercise 9.8 applied to $f_2 \in \mathbf{D}_0$ yields a superharmonic function $g \in \mathbf{D}_0$ that satisfies $g \ge |f_2|$ and $\mathscr{D}(g) \le \mathscr{D}\big(|f_2|\big) \le \mathscr{D}(f_2)$. Also, $g(o)^2 \le \mathcal{G}(o, o)\mathscr{D}(g)/\pi(o)$ by Exercise 9.6(g) applied to $g \in \mathbf{D}_0$ (that is, $g = g + \mathbf{0}$ is its

Royden decomposition). Combining this with (9.8) applied to g, we get that $\mathbf{E}\big[g(X_n)^2\big] \le \epsilon$ for all n. Since $\langle g(X_n)\rangle$ is a nonnegative supermartingale, it converges a.s. and in L^2 to a limit whose second moment is at most ϵ. Since $|f_2(X_n)| \le g(X_n)$, it follows that both $\mathbf{E}\big[\limsup_{n\to\infty} f_2(X_n)^2\big]$ and $\mathbf{E}\big[f_2(X_n)^2\big]$ are at most ϵ. Now transience implies that both $f_1(X_n) \to 0$ a.s. and (by the bounded convergence theorem) $\mathbf{E}\big[f_1(X_n)^2\big] \to 0$ as $n \to \infty$. Therefore, it follows that $\mathbf{E}\big[\limsup_{n\to\infty} f_{\mathbf{D}_0}(X_n)^2\big] \le \epsilon$ and $\limsup_{n\to\infty} \mathbf{E}\big[f_{\mathbf{D}_0}(X_n)^2\big] \le \epsilon$. Since ϵ was arbitrary, $\langle f_{\mathbf{D}_0}(X_n)\rangle$ must tend to 0 a.s. and in L^2. This completes the proof. ◀

§9.4. Planar Graphs and Hyperbolic Graphs.

There is an interesting phase transition between dimensions 2 and 3 in hyperbolic space $\mathbb{H}^d$: currents are not unique for graphs that are roughly isometric to $\mathbb{H}^2$, but for $d \ge 3$, currents are unique for graphs that are roughly isometric to $\mathbb{H}^d$. We first treat the case of $\mathbb{H}^2$. This is, in fact, a general result for transient planar graphs, due to Benjamini and Schramm (1996a, 1996c):

Theorem 9.11. *Suppose that G is a transient planar network. Let $\pi(x)$ denote the sum of the conductances of the edges incident to x. If $\pi(\bullet)$ is bounded, then currents are not unique.*

For simplicity, we will assume from now on that G ***is a proper simple plane transient network all of whose faces have a finite number of sides***.

To prove Theorem 9.11, we show that in some sense, random walk on G is like Brownian motion in the hyperbolic disc. Benjamini and Schramm showed this in two geometric senses: one used circle packing (1996a) and the other used square tiling (1996c). We will show this in a combinatorial sense that is essentially the same as the approach with square tiling.

Our first goal is to establish a (polar) coordinate system on V. Pick a vertex $o \in \mathsf{V}$. Let i_o be the unit current flow on G from o to ∞ and let v be the voltage function which is 0 at o and 1 at ∞, i.e., $v(x)$ is the probability that a random walk started at x will never visit o. Note that the voltage function corresponding to i_o is not v but rather $\mathscr{E}(i_o)(\mathbf{1}-v)$.

Define $i_o^\times(e^\dagger) := i_o(e)$. Now any face of $G^\dagger$ contains a vertex of G in its interior. If $i_o^\times$ is summed counterclockwise around a face of $G^\dagger$ surrounding x, then we obtain $d^* i_o(x)$, which is 0 unless $x = o$, in which case it is 1. Since any cycle in $G^\dagger$ can be written as a sum of cycles surrounding faces, it follows that the sum of $i_o^\times$ along any cycle is an integer. Therefore, we may define $v^\times : G^\dagger \to \mathbb{R}/\mathbb{Z}$ by picking any vertex $o^\dagger \in G^\dagger$ and, for

$x^\dagger \in \mathsf{V}(G^\dagger)$, setting $v^\times(x^\dagger)$ to be the sum (mod 1) of $i_o^\times$ along any path in $G^\dagger$ from $o^\dagger$ to $x^\dagger$.

We now have the essence of the polar coordinates on V, with v giving the radial distance and $v^\times$ giving the angle; however, $v^\times$ is not yet defined on V. In fact, we prefer to assign to each $x \in \mathsf{V}$ an arc $J(x)$ of angles in order to get all angles. To do this, let $\mathsf{Out}(x) := \{e\,;\; e^- = x,\ i_o(e) > 0\}$ and $\mathsf{In}(x) := \{e\,;\; e^+ = x,\ i_o(e) > 0\}$. For example, $\mathsf{In}(o)$ is empty.

Lemma 9.12. *For every $x \in \mathsf{V}$, the sets $\mathsf{Out}(x)$ and $\mathsf{In}(x)$ do not interleave, i.e., their union can be ordered counterclockwise so that no edge of $\mathsf{In}(x)$ precedes any edge of $\mathsf{Out}(x)$.*

Proof. Consider any x and any two edges $\langle y, x\rangle, \langle z, x\rangle \in \mathsf{In}(x)$. We have $v(y) < v(x)$ and $v(z) < v(x)$ by definition of $\mathsf{In}(x)$. By Corollary 3.3, there are paths from o to y and o to z using only vertices with $v < v(x)$. Extend these paths to x by adjoining the edges $\langle y, x\rangle$ and $\langle z, x\rangle$, respectively. These paths from o to x bound one or more regions in the plane. By the maximum principle, any vertices in these regions also have $v < v(x)$. In particular, there can be none that are endpoints of edges in $\mathsf{Out}(x)$. ◀

If $J(e)$ denotes the counterclockwise arc on $\mathbb{R}/\mathbb{Z}$ from $v^\times\big((e^\dagger)^-\big)$ to $v^\times\big((e^\dagger)^+\big)$, then we let

$$J(x) := \bigcup_{e \in \mathsf{Out}(x)} J(e)\,.$$

By Lemma 9.12, we can also write $J(x) := \bigcup_{e \in \mathsf{In}(x)} J(e)$. Consider some ϑ not in the range of $v^\times$. If $\vartheta \in J(e)$ for some $e \in \mathsf{In}(x)$, then there is exactly one $e \in \mathsf{Out}(x)$ with $\vartheta \in J(e)$ by Exercise 3.1. Thus, there is a unique infinite path $\mathcal{P}_\vartheta$ in G starting at o and containing only vertices x with $\vartheta \in J(x)$. Such a path corresponds to a radial line in the hyperbolic disc. We claim that for every edge e with $i_o(e) > 0$, the Lebesgue measure of $\{\vartheta\,;\; e \in \mathcal{P}_\vartheta\}$ is the length of $J(e)$, i.e., $i_o(e)$. Indeed, consider the flow ϕ defined to be the expectation of the path $\mathcal{P}_\vartheta$ when ϑ is chosen uniformly at random in $\mathbb{R}/\mathbb{Z}$, just as we created flows from random paths in Section 2.5. Then ϕ is a unit flow from o and $0 \le \phi(e) \le i_o(e)$ when $i_o(e) > 0$. Therefore, $\mathscr{E}(\phi) \le \mathscr{E}(i_o)$, yet i_0 has the minimum energy among all unit flows form o (Proposition 2.11). Therefore $\phi = i_o$, which implies our claim. Figure 9.1 shows this embedding for the (2,3,7)-triangle tessellation of the hyperbolic plane. Each edge corresponds to an annular region and vertices correspond to collections of adjacent arcs. To make it look like a standard graph, we can place a vertex in the middle of the arc collection to which it corresponds; this gives the right-hand part of Figure 9.1. Although the square lattice is recurrent, if we make the conductance of each edge $\langle x, y\rangle$ equal to the

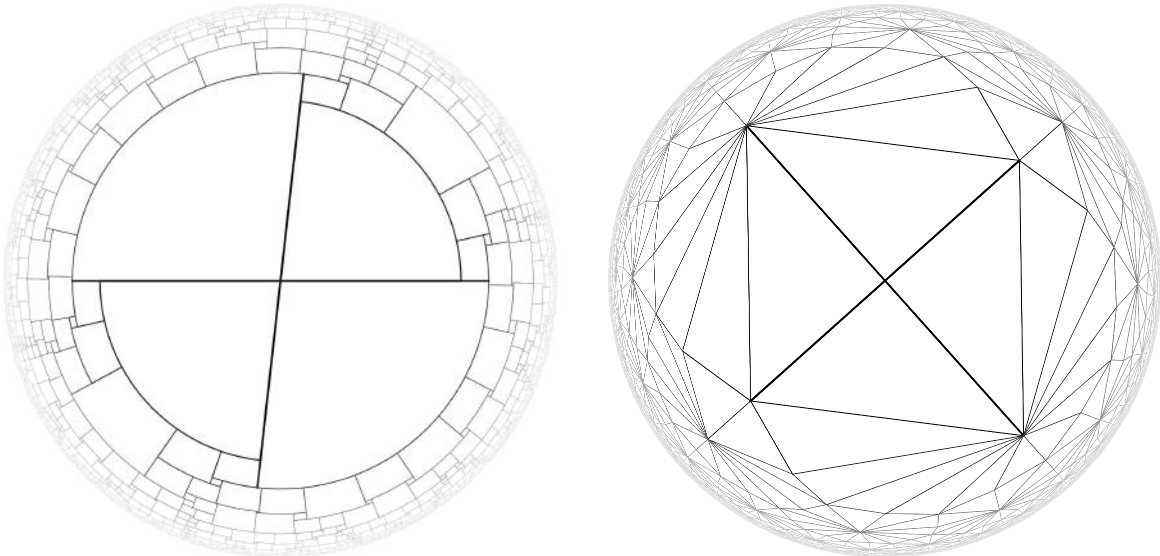


Figure 9.1. The polar embeddings of the (2,3,7)-triangle tessellation of the hyperbolic plane.

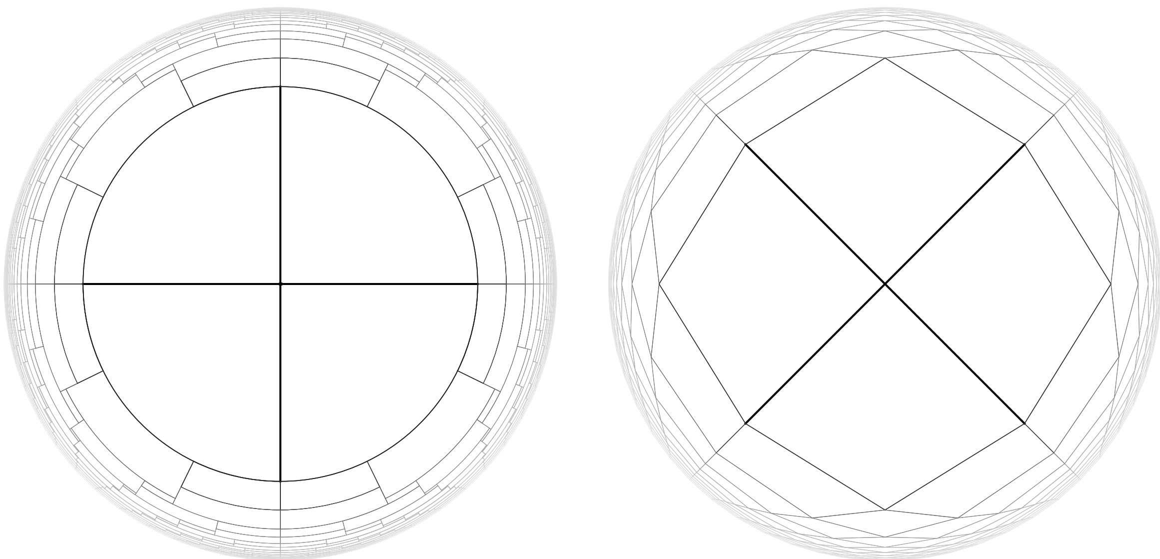


Figure 9.2. The polar embeddings of the distance-weighted square lattice.

maximum of the distance of x and y to the origin, then is becomes transient. The polar embeddings of this network are shown in Figure 9.2.

Lemma 9.13. *For almost every* $\vartheta \in \mathbb{R}/\mathbb{Z}$ *in the sense of Lebesgue measure,* $\sup\{v(x)\,;\, x \in \mathcal{P}_\vartheta\} = 1$.

Proof. Let $h(\vartheta) := \sup\{v(x)\,;\ x \in \mathcal{P}_\vartheta\}$. Then $h \le 1$ everywhere. Also,

$$\int_{\mathbb{R}/\mathbb{Z}} h(\vartheta)\, d\vartheta = \int_{\mathbb{R}/\mathbb{Z}} \sum_{e \in \mathcal{P}_\vartheta} |dv(e)|\, d\vartheta = \sum_{i_o(e)>0} |dv(e)| i_o(e) = 1$$

since $|dv(e)| = i_o(e)r(e)/\mathscr{E}(i_o)$ (recall that v is not exactly the voltage function corresponding to i_o). Therefore, $h = 1$ a.e. ◀

For $0 \le \rho < 1$ and for a.e. ϑ, Lemma 9.13 allows us to define $x(\rho, \vartheta)$ as the vertex $x \in \mathcal{P}_\vartheta$ where $v(x) \le \rho$ and $v(x)$ is maximum. We now come to the key calculation made possible by our coordinate system.

Lemma 9.14. *For any $f \in \mathbf{D}$,*

$$\overline{f}(\vartheta) := \lim_{\rho \uparrow 1} f\big(x(\rho, \vartheta)\big)$$

exists for almost every ϑ and satisfies

$$\|\overline{f}\|_{L^1} \le \sqrt{1 + \mathscr{E}(i_o)}\|f\|_{\mathbf{D}}\,.$$

Proof. The Cauchy-Schwarz inequality yields the bound

$$\int_{\mathbb{R}/\mathbb{Z}} \sum_{e \in \mathcal{P}_\vartheta} |df(e)|\, d\vartheta = \sum_{i_o(e)>0} |df(e)| i_o(e) \le \|df\|_c \|i_o\|_r < \infty\,,$$

whence the integrand is finite a.e. This proves that

$$\lim_{\rho \uparrow 1} f\big(x(\rho, \vartheta)\big) = f(o) - \lim_{\rho \uparrow 1} \sum_{\substack{e \in \mathcal{P}_\vartheta,\\ v(e^+) \le \rho}} df(e)$$

exists a.e. and has L^1 norm at most

$$|f(o)| + \|df\|_c \|i_o\|_r \le \sqrt{1 + \mathscr{E}(i_o)}\|f\|_{\mathbf{D}}$$

by the Cauchy-Schwarz inequality. ◀

▷ **Exercise 9.9.**

Show that $\|f\big(x(\rho, \bullet)\big) - \overline{f}\|_{L^1} \to 0$ as $\rho \uparrow 1$.

If f has finite support, then of course $\overline{f} \equiv 0$. Since the map $f \mapsto \overline{f}$ from $\mathbf{D}$ to L^1 is continuous by Lemma 9.14, it follows that $\overline{f} = 0$ a.e. for $f \in \mathbf{D}_0$. Therefore, if $f = f_{\mathbf{D}_0} + f_{\mathbf{HD}}$ is the Royden decomposition of f (see Exercise 9.6(f)), we have $\overline{f} = \overline{f}_{\mathbf{HD}}$ a.e. To show that $\mathbf{HD} \neq \mathbb{R}$, then, it suffices to show that there is a Dirichlet function f with $\overline{f}$ not an a.e. constant. Such an f is given by the angle function $v^\times$. More precisely, since $v^\times$ is defined on $G^\dagger$ rather than on G and since, moreover, $v^\times$ takes values in $\mathbb{R}/\mathbb{Z}$, we make the following modifications. Let F_x be any face of G with x as one of its vertices. For $\vartheta \in \mathbb{R}/\mathbb{Z}$, let $|\vartheta|$ denote the distance of (any representative of) ϑ to the integers. Set $\psi(x) := |v^\times(F_x)|$.

Lemma 9.15. *If $\pi(\bullet)$ is bounded, then $\psi \in \mathbf{D}$.*

Proof. Let $\pi \le M$. Given adjacent vertices x, y in G, there is a path of edges $e_1^\dagger, \ldots, e_j^\dagger$ in $G^\dagger$ from F_x to F_y with each e_k incident to either x or y (see Figure 9.3). Therefore,

$$d\psi(x,y)^2 \le |v^\times(F_x) - v^\times(F_y)|^2 \le \left(\sum_{k=1}^{j} |i_o(e_k)|\right)^2$$
$$\le \sum_{k=1}^{j} i_o(e_k)^2 r(e_k) \sum_{k=1}^{j} c(e_k) \le 2M \sum_{e^- \in \{x,y\}} i_o(e)^2 r(e)$$

by the Cauchy-Schwarz inequality. Rewrite this as

$$d\psi(e)^2 \le M \sum_{e' \sim e} i_o(e')^2 r(e')$$

for every edge e, where $e' \sim e$ denotes that e' and e are incident. (Note that both orientations of e' are included, which is why we lost a factor of 2.) It follows that

$$\mathscr{D}(\psi) = \frac{1}{2} \sum_{e \in \mathsf{E}} d\psi(e)^2 c(e) \le \frac{M}{2} \sum_{e \in \mathsf{E}} \sum_{e' \sim e} i_o(e')^2 r(e') c(e)$$
$$= \frac{M}{2} \sum_{e' \in \mathsf{E}} i_o(e')^2 r(e') \sum_{e \sim e'} c(e) \le 4M^2 \mathscr{E}(i_o) < \infty\,. \qquad \blacktriangleleft$$

Since $v^\times(F_x) \in J(x)$ and $J(x)$ has length tending to zero as the distance from x to o tends to infinity (by the first inequality of (2.13)), we have that $\lim_{\rho \uparrow 1} \psi\big(x(\rho, \vartheta)\big) = |\vartheta|$ for every ϑ for which $\sup\{v(x)\,;\ x \in \mathcal{P}_\vartheta\} = 1$, i.e., for a.e. ϑ. Thus, ψ is the sought-for Dirichlet function with $\overline{\psi}$ not an a.e. constant. This proves Theorem 9.11.

By Theorem 9.10, $\psi(X_n) = |v^\times(F_{X_n})|$ converges a.s. when $\pi(\bullet)$ is bounded. Since the length of $J(X_n) \to 0$ a.s. and $J(X_n) \cap J(X_{n+1}) \neq \varnothing$, it follows that $v^\times(F_{X_n})$ also

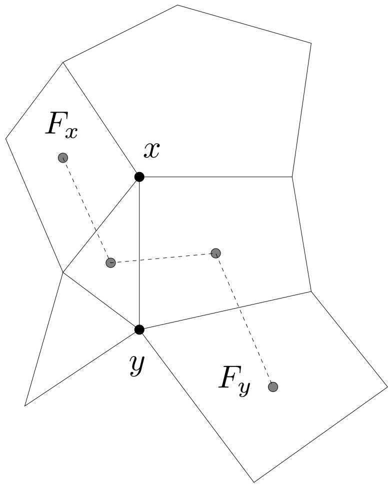


Figure 9.3.

converges a.s. Benjamini and Schramm (1996c) showed that its limiting distribution is Lebesgue measure. Of course, the choice of faces F_x has no effect since the lengths of the intervals $J(X_n)$ tend to 0 a.s. This angle convergence is what we meant by saying that random walk on G is similar to Brownian motion in the disc.

Theorem 9.16. (Benjamini and Schramm, 1996c) *If G is a transient simple plane network with $\pi(\bullet)$ bounded, then $J(X_n)$ tends to a point on the circle $\mathbb{R}/\mathbb{Z}$ a.s. The distribution of the limiting point is Lebesgue measure when $X_0 = o$.*

Proof. That $J(X_n)$ tends to a point a.s. follows from the a.s. convergence of $v^\times(F_{X_n})$. To say that the limiting distribution is Lebesgue measure is to say that for any arc A,

$$\mathbf{P}_o[\lim v^\times(F_{X_n}) \in A] = \int \mathbf{1}_A(\vartheta)\, d\vartheta\,. \tag{9.9}$$

This is implied by the statement

$$\mathbf{E}_o[\lim f(X_n)] = \int \overline{f}(\vartheta)\, d\vartheta \tag{9.10}$$

for every Lipschitz function $\overline{f}$ on $\mathbb{R}/\mathbb{Z}$, where f is defined by

$$f(x) := \overline{f}\big(v^\times(F_x)\big)\,.$$

The reason is that if (9.10) holds for all Lipschitz f, then it also holds for $\mathbf{1}_A$ since we can sandwich $\mathbf{1}_A$ by Lipschitz functions. This gives

$$\mathbf{P}_o[\lim \mathbf{1}_A\big(v^\times(F_{X_n})\big)] = \int \mathbf{1}_A(\vartheta)\, d\vartheta\,.$$

In particular, the chance that $\lim v^\times(F_{X_n})$ is an endpoint of A is 0, so that (9.9) holds.

To show (9.10), note that the same arguments as we used to prove Lemma 9.15 and Theorem 9.11 yield that all such f are Dirichlet functions and

$$\lim_{\rho\uparrow 1} f\big(x(\rho,\vartheta)\big) = \overline{f}(\vartheta)$$

for all ϑ for which $\sup\{v(x)\,;\ x \in \mathcal{P}_\vartheta\} = 1$. Now for any $f \in \mathbf{D}$, Theorem 9.10 gives that

$$\mathbf{E}_o[\lim f(X_n)] = \mathbf{E}_o[\lim f_{\mathbf{HD}}(X_n)] = \lim \mathbf{E}_o[f_{\mathbf{HD}}(X_n)] = f_{\mathbf{HD}}(o)\,,$$

where we use the convergence in L^1 in the middle step. On the other hand, Exercise 9.9 gives

$$\begin{aligned}\int \overline{f}(\vartheta)\,d\vartheta &= \lim_{\rho\uparrow 1}\int f\big(x(\rho,\vartheta)\big)\,d\vartheta = \lim_{\rho\uparrow 1}\int \Big[f(o) - \sum_{\substack{e\in\mathcal{P}_\vartheta,\\ v(e^+)\le\rho}} df(e)\Big]\,d\vartheta \\ &= f(o) - (\nabla f, i_o)_r = f(o) - f_{\mathbf{D}_0}(o) = f_{\mathbf{HD}}(o)\end{aligned}$$

by Exercise 9.6(g). ◀

We may now completely analyze which graphs roughly isometric to a hyperbolic space have unique currents.

Theorem 9.17. *If G is a bounded-degree graph that is roughly isometric to $\mathbb{H}^d$ for some $d \ge 2$, then currents are unique on G iff $d \ge 3$.*

Proof. That graphs roughly isometric to $\mathbb{H}^d$ are transient was established in Theorem 2.18; an example appeared in Figure 2.3. Thus, by Theorems 9.11 and 9.8, they do not have unique currents when $d = 2$.

Suppose now that $d \ge 3$. By Theorem 9.8, it suffices to prove unique currents for a particular graph. Choose an origin $o \in \mathbb{H}^d$. For each $n \ge 1$, choose a maximal 1-separated set A_n in the sphere S_n of radius n centered at o. Let $G := (\mathsf{V}, \mathsf{E})$ with $\mathsf{V} := \bigcup_n A_n$ and E the set of pairs of vertices with mutual distance at most 3. Clearly G is roughly isometric to $\mathbb{H}^d$. Consider the subgraphs W_n of G induced on A_n. We claim that they satisfy the hypotheses of Theorem 9.7.

Elementary hyperbolic geometry shows that S_n is isometric to a Euclidean sphere of radius $r_n := \alpha_n e^{\beta_n n}$ for some numbers α_n and β_n (depending on d) that have positive finite limits as $n \to \infty$. Let $\mathsf{RD}(W_n)$ be the effective resistance diameter of W_n. The random path method shows, just as in the proof of Proposition 2.15, that $\mathsf{RD}(W_n)$ is comparable to $\log r_n$ if $d = 3$ and to a constant if $d \ge 4$. Therefore, we have $\mathsf{RD}(W_n) \le Cn$ for some constant C. Theorem 9.7 completes the proof. ◀

§9.5. Random Walk Traces.

Consider the network random walk on a transient network (G, c) when it starts from some fixed vertex o. The ***trace*** of the random walk is the random set of edges traversed by the random walk. How big can the trace be? We show that it cannot be very large in that the trace forms a.s. a recurrent graph (for simple random walk). This result is due to Benjamini, Gurel-Gurevich, and Lyons (2007), from which this section is taken.

Our proof will demonstrate the following stronger results. Let $N(x, y)$ denote the number of traversals of the edge (x, y).

Theorem 9.18. (Recurrence of Traces) *The network* $\big(G, \mathbf{E}[N]\big)$ *is recurrent. The networks* (G, N) *and* $(G, \mathbf{1}_{\{N>0\}})$ *are a.s. recurrent.*

Recall from Proposition 2.11 that the effective resistance from o to infinity in the network (G, c) equals

$$\alpha := \mathcal{G}(o, o)/\pi(o)\,. \tag{9.11}$$

By Exercise 2.75, if $\big(G, \mathbf{E}[N]\big)$ is recurrent, then so a.s. is (G, N). Furthermore, Rayleigh's monotonicity principle implies that when (G, N) is recurrent, so is $(G, \mathbf{1}_{\{N>0\}})$.

Thus, it remains to prove that $\big(G, \mathbf{E}[N]\big)$ is recurrent.

The effective conductance to infinity from an infinite set A of vertices is defined to be the supremum of the effective conductance from B to infinity over all finite subsets $B \subset A$. Let the voltage function be $v(\bullet)$ throughout this section, where $v(o) = 1$ and $v(\bullet)$ is 0 at ∞. Then $v(x)$ is the probability of ever visiting o for a random walk starting at x.

Note that

$$\mathbf{E}[N(x, y)] = \mathcal{G}(o, x)p(x, y) + \mathcal{G}(o, y)p(y, x) = \big(\mathcal{G}(o, x)/\pi(x) + \mathcal{G}(o, y)/\pi(y)\big)c(x, y)$$

and, by Exercise 2.1 and Proposition 2.11,

$$\pi(o)\mathcal{G}(o, x) = \pi(x)\mathcal{G}(x, o) = \pi(x)v(x)\mathcal{G}(o, o)\,.$$

Thus, we have (from the definition (9.11))

$$\mathbf{E}[N(x, y)] = \alpha c(x, y)[v(x) + v(y)] \tag{9.12}$$

$$\le 2\alpha \max\big\{v(x), v(y)\big\}c(x, y)\,. \tag{9.13}$$

In a finite network (H, c), we write $\mathscr{C}(A \leftrightarrow z; H, c)$ for the effective conductance between a subset A of its vertices and a vertex z. Clearly, $A \subset B \subset \mathsf{V}$ implies that $\mathscr{C}(A \leftrightarrow z; H, c) \le \mathscr{C}(B \leftrightarrow z; H, c)$.

Lemma 9.19. *Let (H, c) be a finite network and $a, z \in \mathsf{V}(H)$. Let v_H be the voltage function that is 1 at a and 0 at z. For $0 < t < 1$, let A_t be the set of vertices x with $v_H(x) \ge t$. Then $\mathscr{C}(A_t \leftrightarrow z; H, c) \le \mathscr{C}(a \leftrightarrow z; H, c)/t$. Thus, for every $A \subset \mathsf{V}(H) \setminus \{z\}$, we have*

$$\mathscr{C}(A \leftrightarrow z; H, c) \le \frac{\mathscr{C}(a \leftrightarrow z; H, c)}{\min \left(v_H \restriction A\right)}.$$

Proof. We subdivide edges as follows. If any edge (x, y) is such that $v_H(x) > t$ and $v_H(y) < t$, then subdividing the edge (x, y) with a vertex z by giving resistances

$$r(x, z) := \frac{v_H(x) - t}{v_H(x) - v_H(y)} r(x, y)$$

and

$$r(z, y) := \frac{t - v_H(y)}{v_H(x) - v_H(y)} r(x, y)$$

will result in a network such that $v_H(z) = t$ while no other voltages change (this is the series law). Doing this for all such edges gives a possibly new graph H' and a new set of vertices $A'_t \supseteq A_t$ whose internal vertex boundary is a set W'_t on which the voltage is identically equal to t. We have $\mathscr{C}(A_t \leftrightarrow z; H, c) = \mathscr{C}(A_t \leftrightarrow z; H', c) \le \mathscr{C}(A'_t \leftrightarrow z; H', c)$. Now $\mathscr{C}(A'_t \leftrightarrow z; H', c) = \mathscr{C}(a \leftrightarrow z; H, c)/t$ since the current flowing in (H', c) from a to z induces a current from A'_t to z with strength $\mathscr{C}(a \leftrightarrow z; H, c)$ and voltage difference t. Therefore, $\mathscr{C}(A_t \leftrightarrow z; H, c) \le \mathscr{C}(a \leftrightarrow z; H, c)/t$, as desired.

For a general A, let $t := \min v_H \restriction A$. Since $A \subset A_t$, we have $\mathscr{C}(A \leftrightarrow z; H, c) \le \mathscr{C}(A_t \leftrightarrow z; H, c)$. Combined with the inequality just reached, this yields the final conclusion. ◀

For $t \in (0, 1)$, let $\mathsf{V}_t := \{x \in \mathsf{V};\ v(x) < t\}$. Let W_t be the external vertex boundary of V_t, i.e., the set of vertices outside V_t that have a neighbor in V_t. Write G_t for the subgraph of G induced by $\mathsf{V}_t \cup W_t$.

We will refer to the conductances c as the ***original*** ones and the conductances $\mathbf{E}[N]$ as the ***new*** ones for convenience.

Lemma 9.20. *The effective conductance from W_t to ∞ in the network $\big(G_t, \mathbf{E}[N]\big)$ is at most 2.*

Proof. If any edge (x, y) is such that $v(x) > t$ and $v(y) < t$, then subdividing the edge (x, y) with a vertex z as in the proof of Lemma 9.19 and consequently adding z to W_t has the effect of replacing the edge (x, y) by an edge (z, y) with conductance $c(z, y) = c(x, y)[v(x) - v(y)]/[t - v(y)] > c(x, y)$ in the original network and, by (9.12), with larger conductance in the new network:

$$\alpha c(z, y)[t + v(y)] = \alpha c(z, y)[t - v(y) + 2v(y)] = \alpha c(x, y)[v(x) - v(y)] + 2\alpha c(z, y)v(y)$$
$$> \alpha c(x, y)[v(x) - v(y)] + 2\alpha c(x, y)v(y) = \mathbf{E}[N(x, y)].$$

Since raising edge conductances clearly raises effective conductance, it suffices to prove the lemma in the case that $v(x) = t$ for all $x \in W_t$. Thus, we assume this case for the remainder of the proof.

Suppose that $\langle (H_n, c)\,;\ n \ge 1\rangle$ is an increasing exhaustion of (G, c) by finite networks that include o. Identify the boundary (in G) of H_n to a single vertex, z_n. Let v_n be the corresponding voltage functions with $v_n(o) = 1$ and $v_n(z_n) = 0$. Then $\mathscr{C}(o \leftrightarrow z_n; H_n, c) \downarrow 1/\alpha$ and $v_n(x) \uparrow v(x)$ as $n \to \infty$ for all $x \in \mathsf{V}(G)$. Let A be a finite subset of W_t. By Lemma 9.19, as soon as $A \subset \mathsf{V}(H_n)$, we have that the effective conductance from A to z_n in H_n is at most $\mathscr{C}(o \leftrightarrow z_n; H_n, c)/\min\{v_n(x)\,;\ x \in A\}$. Therefore by Rayleigh's monotonicity principle, $\mathscr{C}(A \leftrightarrow \infty; G_t, c) \le \mathscr{C}(A \leftrightarrow \infty; G, c) = \lim_{n\to\infty} \mathscr{C}(A \leftrightarrow z_n; H_n, c) \le 1/(\alpha t)$. Since this holds for all such A, we have

$$\mathscr{C}(W_t \leftrightarrow \infty; G_t, c) \le 1/(\alpha t)\,. \tag{9.14}$$

By (9.13), the new conductances on G_t are obtained by multiplying the original conductances by factors that are at most $2\alpha t$. Combining this with (9.14), we obtain that the new effective conductance from W_t to infinity in G_t is at most 2. ◀

When the complement of V_t is finite for all t, which is the case for "most" networks, this completes the proof by the following lemma (and by the fact that $\bigcap_{t>0} \mathsf{V}_t = \varnothing$):

Lemma 9.21. *If H is a transient network, then for all $m > 0$, there exists a finite subset $K \subset \mathsf{V}(H)$ such that for all finite $K' \supseteq K$, the effective conductance from K' to infinity is more than m.*

▷ **Exercise 9.10.**
Prove this lemma.

Even when the complement of V_t is not finite for all t, this is enough to show that the network (G, N) is a.s. recurrent: If X_n denotes the position of the random walk on (G, c) at time n, then $v(X_n) \to 0$ a.s.: it converges a.s. since it is a non-negative supermartingale, and its expectation tends to the probability that the random walk visits o infinitely often, i.e., to 0. Thus, the path is a.s. contained in V_t after some time, no matter the value of $t > 0$. Let B_n be the ball of radius n about o. By Lemma 9.21, if (G, N) is transient with probability $p > 0$, then $\mathscr{C}(B_n \leftrightarrow \infty; G, N)$ tends in probability, as $n \to \infty$, to a random variable that is infinite with probability p. In particular, this effective conductance is at least $6/p$ with probability at least $p/2$ for all large n. Fix n with this property. Let $t > 0$ be such that $\mathsf{V}_t \cap B_n = \varnothing$. Write D for the (random finite) set of vertices in G

incident to some edge $e \notin G_t$ with $N(e) > 0$. Then $\mathscr{C}(W_t \leftrightarrow \infty; G_t, N) = \mathscr{C}(W_t \cup D \leftrightarrow \infty; G, N) \ge \mathscr{C}(B_n \leftrightarrow \infty; G, N)$. However, in combination with Exercise 2.57, this implies that $\mathscr{C}\big(W_t \leftrightarrow \infty; G_t, \mathbf{E}[N]\big) \ge \mathbf{E}\big[\mathscr{C}(W_t \leftrightarrow \infty; G_t, N)\big] \ge (p/2)(6/p) = 3$, which contradicts Lemma 9.20.

We now complete the proof that $\big(G, \mathbf{E}[N]\big)$ is recurrent in general.

Proof of Theorem 9.18. The function $x \mapsto v(x)$ has finite Dirichlet energy for the original network, hence for the new (since conductances are multiplied by a bounded factor). Assume (for a contradiction) that the new random walk is transient. Then by Theorem 9.10, $\langle v(X_n)\rangle$ converges a.s., where v is the old voltage function and $\langle X_n\rangle$ is the new random walk. Consider $t > 0$. By Exercise 2.76, the complement of V_t induces a recurrent network for the original conductances; since the new conductances are at most 2α times the original ones by (9.13), this recurrence holds also for the new conductances. Therefore, $\langle v(X_n)\rangle$ a.s. cannot have a limit $> t$. Thus, it converges to 0 a.s.

This means that the unit current flow i for the new network (which is the expected number of signed crossings of edges) has total flow 1 through W_t into G_t for all $t > 0$. Thus, we may choose a finite subset A_t of W_t through which at least 1/2 of the new current enters. With the notation $(d_t^* i)(x) := \sum_{y \in \mathsf{V}(G_t)} i(x, y)$, this means that $\sum_{x \in A_t} d_t^* i(x) \ge 1/2$. By Lemma 9.20 and Exercise 2.73, there is a function $F_t : \mathsf{V}_t \cup W_t \to [0, 1]$ with finite support and with $F_t \equiv 1$ on A_t whose Dirichlet energy on the network $\big(G_t, \mathbf{E}[N]\big)$ is at most 3. By the Cauchy-Schwarz inequality, we have (since each edge is counted twice)

$$\begin{aligned}\Bigg[\sum_{x \ne y \in \mathsf{V}(G_t)} i(x, y) dF_t(x, y)\Bigg]^2 &\le \sum_{x \ne y \in \mathsf{V}(G_t)} i(x, y)^2/\mathbf{E}N(x, y) \sum_{x \ne y \in \mathsf{V}(G_t)} \mathbf{E}N(x, y) dF_t(x, y)^2 \\ &\le 6 \sum_{x \ne y \in \mathsf{V}(G_t)} i(x, y)^2/\mathbf{E}N(x, y)\,.\end{aligned}$$

On the other hand, summation by parts yields that

$$\sum_{x \ne y \in \mathsf{V}(G_t)} i(x, y) dF_t(x, y) = \sum_{x \in \mathsf{V}(G_t)} d_t^* i(x) F_t(x) \ge \sum_{x \in A_t} d_t^* i(x) \ge 1/2\,.$$

Therefore, $\sum_{x \ne y \in \mathsf{V}(G_t)} i(x, y)^2/\mathbf{E}N(x, y) \ge 1/24$, which contradicts $\bigcap_t \mathsf{V}(G_t) = \varnothing$ and the fact that i has finite energy. ◀

§9.6. Notes.

Propositions 9.1 and 9.2 go back in some form to Flanders (1971) and Zemanian (1976).

Equation (9.6) is an elementary form of the Hodge decomposition of L^2 1-cochains. To see this, add an oriented 2-cell with boundary θ for each cycle θ in some spanning set of cycles.

Proposition 9.4 is classical in the case of a finite plane network and its dual. The finite case can also be proved using Kirchhoff's laws and Ohm's law, combined with the Max-Flow Min-Cut Theorem: The cycle law for i implies the node law for $i^\dagger$, while the node law for i implies the cycle law for $i^\dagger$.

Theorem 9.7 is due to the authors and is published here for the first time. Thomassen (1989) proved the weaker result where $\mathsf{RD}(A)$ is replaced by the diameter of A.

The proof we have given of Theorem 9.8 was communicated to us by O. Schramm.

Cayley graphs of infinite Kazhdan groups have unique currents: see Bekka and Valette (1997).

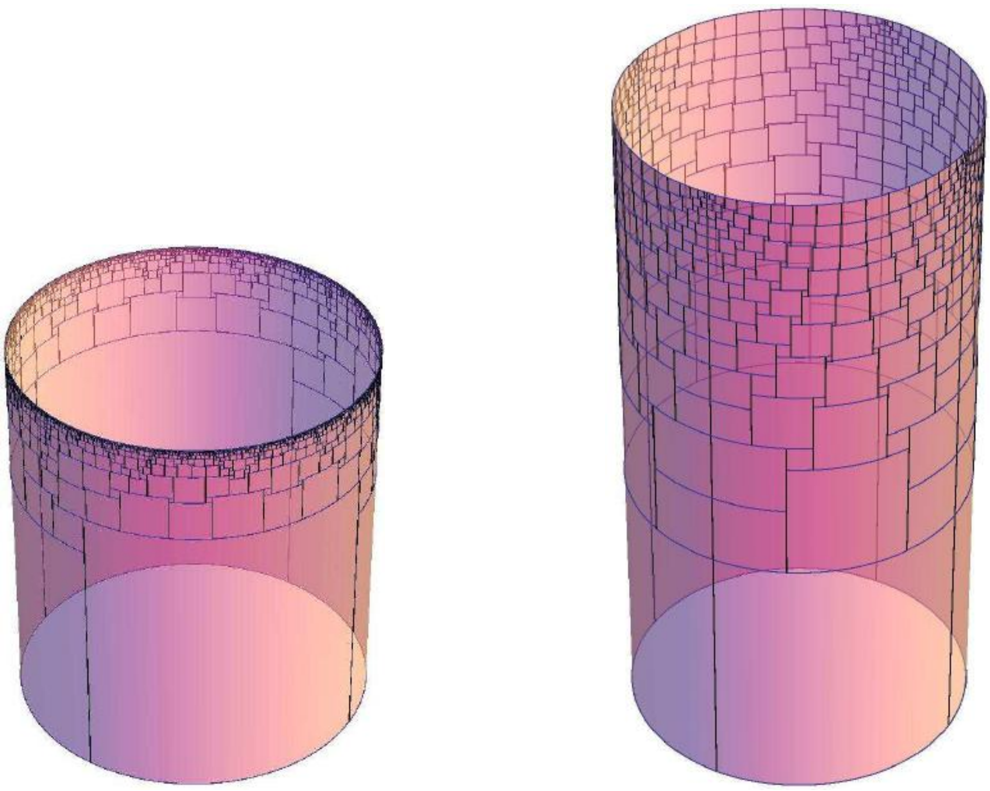


Figure 9.4. Tilings of cylinders by squares corresponding to the (2,3,7)-triangle tessellation of the hyperbolic plane on the left and the 21×21 grid with current from its center vertex to its boundary vertices on the right.

Our proof of Theorem 9.11 was influenced by the proof of a related result by Kenyon (1998). The tiling associated by Benjamini and Schramm (1996c) to a transient plane network is the following. We use the notation at the beginning of Section 9.4. Let $R := \mathscr{R}(o \leftrightarrow \infty)$. If $i_o(e) > 0$, then let $S(e) := J(e) \times [Rv(e^-), Rv(e^+)]$ in the cylinder $\mathbb{R}/\mathbb{Z} \times [0, R]$. Each such $S(e)$ is a square and the set of all such squares tiles $\mathbb{R}/\mathbb{Z} \times [0, R]$. For the (2,3,7)-triangle tessellation of the hyperbolic place, the result is shown in Figure 9.4. This works on finite networks too, of course.

For example, a square tiling of a cylinder arising from a 21×21 grid in the plane is shown in Figure 9.4. When current flows from one vertex to another on the same face (such as the outer face), then one can unroll the cylinder to a rectangle, as in Figure 9.5. The polar embeddings of large pieces of the square lattice have an interesting structure, as shown in Figure 9.6. See Section II.2 of Bollobás (1998) for more on square tilings, following the original work of Brooks, Smith, Stone, and Tutte (1940). There are also connections to Riemann's mapping theorem; see Cannon, Floyd, and Parry (1994).

▷ **Exercise 9.11.**
We have seen that in square tilings of cylinders corresponding to planar graphs, the squares correspond to edges and horizontal segments correspond to vertices. What corresponds to the faces of the planar graphs?

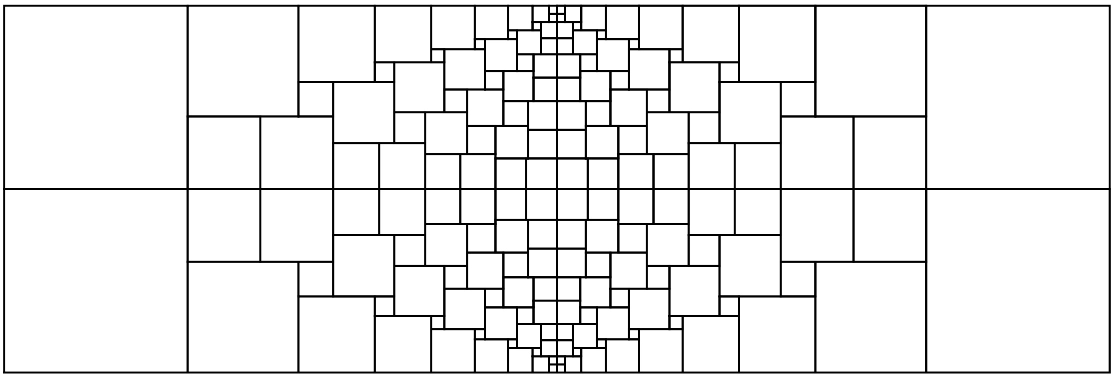


Figure 9.5. A tiling of a rectangle by squares corresponding to the 10×10 grid with current from one corner vertex to its opposite corner vertex.

Theorem 9.17 also follows immediately from a theorem of Holopainen and Soardi (1997), which says that the property $\mathbf{HD} = \mathbb{R}$ is preserved under rough isometries between graphs and manifolds, together with a theorem of Dodziuk (1979), which says that $\mathbb{H}^d$ satisfies $\mathbf{HD} = \mathbb{R}$.

The question of existence of nonconstant bounded harmonic functions is very important, but we only touch on it. The following is due to Blackwell (1955) and later generalized by Choquet and Deny (1960) and many others. Among the significant papers on the topic are Avez (1974), Derriennic (1980), Kaĭmanovich and Vershik (1983), and Varopoulos (1985b). See, e.g., the survey Kaimanovich (2007). We say a function f is ***μ-harmonic*** if for all x, we have $f(x) = \sum_{\mu(g)>0} \mu(g) f(xg)$.

Theorem 9.22. *If G is an abelian group and μ is a probability measure on G with countable support that generates G, then there are no nonconstant bounded μ-harmonic functions.*

Proof. (Due to Dynkin and Maljutov (1961).) Let f be a harmonic function. For any element g of the support of μ, the function $w_g(x) := f(x) - f(xg)$ is also harmonic because G is abelian. If f is not constant, then for some g, the function w_g is not identically 0, whence it takes, say, a positive value. Let $M := \sup w_g$. If $M = \infty$, then f is not bounded. Otherwise, for any x, we have

$$w_g(x) = \sum_{\mu(h)>0} \mu(h) w_g(xh) \le M(1 - \mu(g)) + \mu(g) w_g(xg) \,,$$

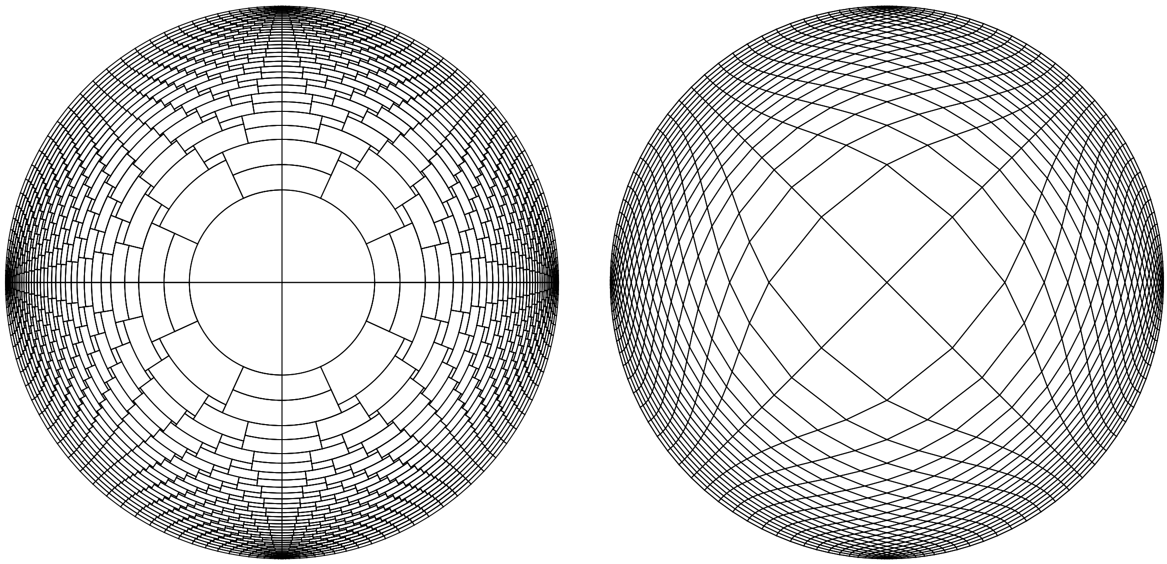


Figure 9.6. The polar embeddings of a 41×41 square grid graph from its center vertex to its boundary vertices.

which is to say that

$$M - w_g(xg) \le \big(M - w_g(x)\big)/\mu(g) .$$

Iterating this inequality gives for all $n \ge 1$,

$$M - w_g(xg^n) \le \big(M - w_g(x)\big)/\mu(g)^n .$$

Choose x so that $M - w_g(x) < M\mu(g)^n/2$. Then $w_g(xg^k) > M/2$ for $k = 1, \ldots, n$, whence $w_g(x) + w_g(xg) + \cdots + w_g(xg^n) = f(x) - f(xg^{n+1}) > M(n+1)/2$. Since n is arbitrary, f is not bounded. ◀

By Exercise 9.40, if a graph has no non-constant bounded harmonic functions, then it also has no non-constant Dirichlet harmonic functions. Thus, Theorem 9.22 strengthens Corollary 9.9 for abelian groups.

▷ **Exercise 9.12.**
Give another proof of Theorem 9.22 using the Kreĭn-Milman theorem.

▷ **Exercise 9.13.**
Give another proof of Theorem 9.22 using the Hewitt-Savage theorem.

Benjamini, Gurel-Gurevich, and Lyons (2007) suggest that a Brownian analogue of Theorem 9.18 may be true, that is, given Brownian motion on a transient Riemannian manifold, the 1-neighborhood of its trace is recurrent for Brownian motion. For background on recurrence in the Riemannian context, see, e.g., Section 2.8. It would be interesting to prove similar theorems for other processes. For example, consider the trace of a branching random walk on a graph G. Then Benjamini, Gurel-Gurevich, and Lyons (2007) conjecture that almost surely the trace is recurrent for branching random walk with the same branching law.

§9.7. Collected In-Text Exercises.

9.1. Calculate i_{F}^{e} and i_{W}^{e} in a regular tree.

9.2. Let H_n be increasing closed subspaces of a Hilbert space H and P_n be the orthogonal projection on H_n. Let P be the orthogonal projection on the closure of $\bigcup H_n$. Show that for all $u \in H$, we have $\|P_n u - Pu\| \to 0$ as $n \to \infty$.

9.3. Prove Proposition 9.2.

9.4. Show that the choice of path in the definition of the free and wired currents from a to z does not influence their values.

9.5. Prove Proposition 9.4.

9.6. **(a)** Show that $\nabla \mathbf{D}_0 = \bigstar$.

(b) Show that $\mathbf{D}/\mathbb{R} = \widetilde{\mathbf{D}_0}/\mathbb{R} \oplus \mathbf{HD}/\mathbb{R}$, where $\widetilde{\mathbf{D}_0} := \mathbf{D}_0 + \mathbb{R}$.

(c) Show that currents are unique iff $\mathbf{D}/\mathbb{R} = \widetilde{\mathbf{D}_0}/\mathbb{R}$.

(d) Show that $\|\mathbf{1} - \mathbf{D}_0\|_{\mathbf{D}}^2 = \mathscr{C}(o \leftrightarrow \infty)/[1 + \mathscr{C}(o \leftrightarrow \infty)]$, where o is the vertex used to define the inner product on $\mathbf{D}$.

(e) Show that G is recurrent iff $\mathbf{1} \in \mathbf{D}_0$.

(f) **(Royden Decomposition)** Show that if G is transient, then every $f \in \mathbf{D}$ has a unique decomposition $f = g + h$ with $g \in \mathbf{D}_0$ and $h \in \mathbf{HD}$. Note that this is not an orthogonal decomposition.

(g) With the assumptions and notation of part (f), show that $g(x) = (\nabla f, i_x)_r$ and that $g(x)^2 \le \mathscr{D}(f)\mathcal{G}(x,x)/\pi(x)$, where i_x is the unit current flow from x to infinity (from Proposition 2.11) and $\mathcal{G}(\bullet,\bullet)$ is the Green function.

(h) Show that if G is transient, then $\nabla : \mathbf{D}_0 \to \bigstar$ is invertible with bounded inverse.

9.7. One can define the product of two networks in various ways. For example, given two networks $G_i = (\mathsf{V}_i, \mathsf{E}_i)$ with conductances c_i ($i = 1, 2$), define the ***cartesian product*** $G = (\mathsf{V}, \mathsf{E})$ with conductances c by $\mathsf{V} := \mathsf{V}_1 \times \mathsf{V}_2$,

$$\mathsf{E} := \left\{\big((x_1,x_2),(y_1,y_2)\big)\, ;\ \big(x_1 = y_1,\ (x_2,y_2) \in \mathsf{E}_2\big) \text{ or } \big((x_1,y_1) \in \mathsf{E}_1,\ x_2 = y_2\big)\right\},$$

and

$$c\big((x_1,x_2),(y_1,y_2)\big) := \begin{cases} c(x_2,y_2) & \text{if } x_1 = y_1, \\ c(x_1,y_1) & \text{if } x_2 = y_2. \end{cases}$$

Show that if G_i are infinite locally finite graphs with unit conductances, then G has unique currents.

9.8. Let G be transient and let $f \in \mathbf{D}_0$. Show that there is a unique $g \in \mathbf{D}_0$ having minimal energy such that $g \ge |f|$. Show that this g is ***superharmonic***, meaning that for all vertices x,

$$g(x) \ge \frac{1}{\pi(x)} \sum_{y \sim x} c(x,y) g(y)\,.$$

9.9. Show that $\|f\big(x(\rho,\bullet)\big) - \overline{f}\|_{L^1} \to 0$ as $\rho \uparrow 1$.

9.10. Prove Lemma 9.21.

9.11. We have seen that in square tilings of cylinders corresponding to planar graphs, the squares correspond to edges and horizontal segments correspond to vertices. What corresponds to the faces of the planar graphs?

9.12. Give another proof of Theorem 9.22 using the Kreĭn-Milman theorem.

9.13. Give another proof of Theorem 9.22 using the Hewitt-Savage theorem.

§9.8. Additional Exercises.

9.14. Let G be an infinite network exhausted by finite subnetworks $\langle G_n \rangle$. Form G_n^{W} by identifying the complement of G_n to a single vertex.

(a) Given $\theta \in \ell^2_-(\mathsf{E}, r)$, define $f := d^*\theta$. Let i_n be the current on G_n^{W} such that $d^* i_n \restriction \mathsf{V}(G_n) = f \restriction \mathsf{V}(G_n)$. Show that $i_n \to P_{\bigstar}\theta$ in $\ell^2_-(\mathsf{E}, r)$.

(b) Let $f : \mathsf{V} \to \mathbb{R}$ and i_n be the current on G_n^{W} such that $d^* i_n \restriction \mathsf{V}(G_n) = f \restriction \mathsf{V}(G_n)$. Show that $\sup \mathscr{E}(i_n) < \infty$ iff there is some $\theta \in \ell^2_-(\mathsf{E}, r)$ such that $f = d^*\theta$.

9.15. Let G be a network and, if G is recurrent, $z \in \mathsf{V}$. Let H be the Hilbert space of functions f on V with $\sum_{x,y} \pi(x)\mathcal{G}(x,y)f(x)f(y) < \infty$ and inner product $\langle f, g \rangle := \sum_{x,y} \pi(x)\mathcal{G}(x,y)f(x)g(y)$, where $\mathcal{G}(\bullet, \bullet)$ is the Green function for random walk, absorbed at z if G is recurrent. Define the ***divergence operator*** by $\operatorname{div}\theta := \pi^{-1} d^*\theta$. Show that $\operatorname{div} : \bigstar \to H$ is an isometric isomorphism.

9.16. Let G be a transient network. Define the space H as in Exercise 9.15 and $\mathcal{G}$, I, and P as in Exercise 2.22. Show that $I - P$ is a bounded operator from $\mathbf{D}_0$ to H with bounded inverse $\mathcal{G}$.

9.17. Let G be a transient network. Show that if $u \in \mathbf{D}_0$ is superharmonic, then $u \ge 0$.

9.18. Let G be a transient network and $f \in \mathbf{HD}$. Show that there exist nonnegative $u_1, u_2 \in \mathbf{HD}$ such that $f = u_1 - u_2$.

9.19. Let $u \in \mathbf{D}$. Show that u is superharmonic iff $\mathscr{D}(u) \le \mathscr{D}(u + f)$ for all nonnegative $f \in \mathbf{D}_0$.

9.20. Show that if G is recurrent, then the only superharmonic Dirichlet functions are the constants. *Hint:* If u is superharmonic, then use $f := u - Pu$ in Exercise 9.14, where P is the transition operator defined in Exercise 2.22. Use Exercise 2.51 to show that $f = \mathbf{0}$.

9.21. Suppose that there is a finite set K of vertices such that $G \backslash K$ has at least two transient components, where this notation indicates that K and all edges incident to K are deleted from G. Show that G has a non-constant harmonic Dirichlet function.

9.22. Find the free and wired effective resistances between arbitrary pairs of vertices in regular trees.

9.23. Let a and z be distinct vertices in a network. Show that the wired effective resistance between a and z equals

$$\min \{\mathscr{E}(\theta)\,;\ \theta \text{ is a unit flow from } a \text{ to } z\}\ ,$$

while the free effective resistance between a and z equals

$$\min \left\{ \mathscr{E}(\theta)\,;\ \theta - \sum_{j=1}^{k} \chi^{e_j} \in \diamondsuit \right\}$$

for any oriented path $e_1, \ldots, e_k$ from a to z.

9.24. Let G be a network with an exhaustion $\langle G_n \rangle$. Suppose that $a, z \in \mathsf{V}(G_n)$ for all n. Show that the effective resistance between a and z in G_n is monotone decreasing with limit the free effective resistance in G, while the effective resistance between a and z in G_n^{W} is monotone increasing with limit the wired effective resistance in G.

9.25. Let G be an infinite network and $x, y \in \mathsf{V}(G)$. Show that $\mathscr{R}^{\mathrm{W}}(x \leftrightarrow y) \le \mathscr{R}(x \leftrightarrow \infty) + \mathscr{R}(y \leftrightarrow \infty)$.

9.26. Let G be a finite plane network. In the notation of Proposition 9.4, show that the maximum flow from a to z in G' when the conductances are regarded as capacities is equal to the distance between b and y in $(G^\dagger)'$ when the resistances are regarded as edge lengths.

9.27. Suppose that G is a plane graph with bounded degree and bounded number of sides of its faces. Show that G is transient iff $G^\dagger$ is transient.

9.28. Let G be transient, $a \in \mathsf{V}$, and $f(x) := \mathcal{G}(x, a)/\pi(a)$. Show that $f \in \mathbf{D}_0$ and $\nabla f = i_a$.

9.29. Let $\mathbf{BD}$ denote the space of bounded Dirichlet functions with the norm $\|f\| := \|f\|_\infty + \|df\|_c$. Show that $\mathbf{BD}$ is a commutative Banach algebra (with respect to the pointwise product) and that $\mathbf{BD} \cap \mathbf{D}_0$ is a closed ideal.

9.30. **(a)** Show that G is recurrent iff every [or some] star lies in the closed span of the other stars.

(b) Show that if G is transient, then the current flow from any vertex a to infinity corresponding to unit voltage at a and zero voltage at infinity is the orthogonal projection of the star at a on the orthocomplement of the other stars. (Here, the current flow is $i_a/\mathscr{E}(i_a)$.)

9.31. Show that no transient tree has unique currents; use Exercise 2.40 instead of Theorem 9.11.

9.32. Let G be a recurrent network. Show that if $\theta \in \ell^2_-(\mathsf{E}, r)$ satisfies $\sum_x |d^*\theta(x)| < \infty$, then $\sum_x d^*\theta(x) = 0$.

9.33. Show that $\|dPf\|_c \le \|df\|_c$ for all $f \in \mathbf{D}$.

9.34. Show that (G, c, π) is edge non-amenable iff $\ell^2(\mathsf{V}, \pi) = \mathbf{D}_0$.

9.35. Let G be a transient network and i_x be the unit current flow from x to infinity. Given $\theta \in \ell^2_-(\mathsf{E}, r)$, define $F(x) := (\theta, i_x)_r$. Show that $F \in \mathbf{D}_0$ and $\nabla F = P_\bigstar \theta$.

9.36. Let G be a transient network and i_x be the unit current flow from x to infinity. Show that $i_x - i_y = i_{\mathrm{W}}^{x,y}$ for all $x \ne y \in \mathsf{V}$.

9.37. Let $a \ne z \in \mathsf{V}$.

(a) Show that if $F \in \mathbf{D}$, then $(\nabla F, i_{\mathrm{F}}^{a,z})_r = F(a) - F(z)$.

(b) Show that if $F \in \mathbf{D}_0$, then $(\nabla F, i_{\mathrm{W}}^{a,z})_r = F(a) - F(z)$.

9.38. Let a and z be distinct vertices in a network. Show that the free effective conductance between a and z equals

$$\min\left\{\mathscr{D}(F)\,;\ F \in \mathbf{D},\ F(a) = 1,\ F(z) = 0\right\},$$

while the wired effective conductance between a and z equals

$$\min\left\{\mathscr{D}(F)\,;\ F \in \mathbf{D}_0,\ F(a) = 1,\ F(z) = 0\right\}.$$

9.39. Let a and z be distinct vertices in a network. Show that the free effective conductance between a and z equals

$$\min\left\{\sum_{e \in \mathsf{E}_{1/2}} c(e)\ell(e)^2\right\},$$

where ℓ is an assignment of nonnegative lengths so that the distance from a to z is 1, while the wired effective conductance between a and z equals

$$\min \left\{ \sum_{e \in \mathsf{E}_{1/2}} c(e)\ell(e)^2 \right\},$$

where ℓ is an assignment of nonnegative lengths so that the distance from a to z is 1 and so that all but finitely many vertices are connected to z by paths of edges of length 0.

9.40. Show that the bounded harmonic Dirichlet functions are dense in $\mathbf{HD}$.

9.41. Let G be an infinite graph and H be a finite graph. Consider the cartesian product graph $G \times H$. Show that every $f \in \mathbf{HD}(G \times H)$ has the property that it does not depend on the second coordinate, i.e., $f(x, y) = f(x, z)$ for all $x \in \mathsf{V}(G)$ and all $y, z \in \mathsf{V}(H)$.

9.42. Let $W \subseteq \mathsf{V}$ be such that for all $x \in \mathsf{V}$, random walk started at x eventually visits $\mathsf{V} \setminus W$, i.e., $\mathbf{P}_x[\tau_{\mathsf{V} \setminus W} < \infty] = 1$. Show that if $f \in \mathbf{D}$ is supported on W and is harmonic at all vertices in W, then $f \equiv 0$.

9.43. Give an example of a graph with unique currents and with non-constant bounded harmonic functions.

9.44. Extend Theorem 9.8 to show that under the same assumptions, $\mathbf{HD}$ and $\mathbf{HD}'$ have the same dimensions.

9.45. Suppose that there is a rough embedding from a network G to a network G' such that each vertex of G' is within some constant distance of the image of $\mathsf{V}(G)$. Show that if G has unique currents, then so does G'.

9.46. Show that the result of Exercise 9.8 also holds when G is recurrent.

9.47. Let G be a transient network. Suppose that $f \in \mathbf{D}_0$, h is harmonic, and $|h| \le |f|$. Prove that $h = \mathbf{0}$.

9.48. Complete an alternative proof of Theorem 9.16 as follows. Show that by subdividing (adding vertices to) edges as necessary, we may assume that for each k, there is a set of vertices Π_k where $v = 1 - 1/k$ and such that the random walk visits Π_k a.s. Show that the harmonic measure on Π_k "converges" to Lebesgue measure on the circle.

9.49. Give an example of two recurrent graphs $(\mathsf{V}, \mathsf{E}_i)$ $(i = 1, 2)$ on the same vertex set V whose union $(\mathsf{V}, \mathsf{E}_1 \cup \mathsf{E}_2)$ is transient. On the other hand, show that on any transient network, the union of finitely many traces is a.s. recurrent, even if the random walks that produce the traces are dependent.

Chapter 10

Uniform Spanning Forests

There is more than one way to extend the ideas of Chapter 4 to connected infinite graphs. Sometimes we end up with spanning trees, but other times, with spanning forests, all of whose trees are infinite. (A ***forest*** is a graph all of whose connected components are trees.) Most unattributed results in this chapter are from BLPS (2001).

We will discover several remarkable things. Besides the intimate connections between spanning trees, random walks, and electric networks exposed in Chapter 4 and deepened here, we will also find an intimate connection to harmonic Dirichlet functions. We will see some remarkable phase transitions of uniform spanning forests in Euclidean space as the dimension increases, and then again in hyperbolic space. Many interesting open questions remain. Some are collected in the last section.

§10.1. Limits Over Exhaustions.

How can one define a "uniform" spanning tree on an infinite graph? One natural way to try is to take the uniform spanning tree on each of a sequence of finite subgraphs that grow larger and larger so as to exhaust the whole infinite graph, hoping all the while that the measures on spanning trees have some sort of limit. We saw in Section 4.2 that this works for recurrent graphs. Luckily for us, this works also on transient graphs. In fact, there will be two ways to make it work. Here are the details.

Let G be an infinite connected locally finite network; in fact, ***throughout this chapter, assume our graphs are locally finite***. Let $G_n = (\mathsf{V}_n, \mathsf{E}_n)$ be finite connected subgraphs that ***exhaust*** G, i.e., $G_n \subseteq G_{n+1}$ and $G = \bigcup G_n$. Let μ_n^{F} be the uniform spanning tree probability measure on G_n that is described in Chapter 4 (the superscript F stands for "free" and will be explained below). Given a finite set B of edges, we have $B \subseteq \mathsf{E}_n$ for large enough n and, for such n, we claim that $\mu_n^{\mathrm{F}}[B \subseteq T]$ is decreasing in n, where T denotes the random spanning tree. To see this, let $i(e; H)$ be the current that flows along e when a unit current is imposed in the network H from the tail to the head of e. Write $B = \{e_1, \ldots, e_m\}$. By Corollary 4.4, we have that

$$\begin{aligned}\mu_n^{\mathrm{F}}[B \subseteq T] &= \prod_{k=1}^{m} \mu_n^{\mathrm{F}}[e_k \in T \mid \forall j < k \quad e_j \in T] = \prod_{k=1}^{m} i\big(e_k; G_n/\{e_j\,;\ j<k\}\big)\\ &\ge \prod_{k=1}^{m} i\big(e_k; G_{n+1}/\{e_j\,;\ j<k\}\big)\\ &= \mu_{n+1}^{\mathrm{F}}[B \subseteq T]\,,\end{aligned} \tag{10.1}$$

where the inequality is from Rayleigh's monotonicity principle. In particular, $\mu^{\mathrm{F}}[B \subseteq \mathfrak{F}] := \lim_{n\to\infty} \mu_n^{\mathrm{F}}[B \subseteq T]$ exists. (Here, we denote a random forest by $\mathfrak{F}$ and a random tree by T in order to avoid prejudice about whether the forest is a tree or not. We say "forest" since if B contains a cycle, then by our definition, $\mu^{\mathrm{F}}[B \subseteq \mathfrak{F}] = 0$.) It follows that we may define μ^{F} on all ***elementary cylinder*** sets (i.e., sets of the form $\{\mathfrak{F}\,;\ B_1 \subseteq \mathfrak{F}, B_2 \cap \mathfrak{F} = \varnothing\}$ for finite disjoint sets $B_1, B_2 \subset \mathsf{E}$) by the inclusion-exclusion formula:

$$\begin{aligned}\mu^{\mathrm{F}}[B_1 \subseteq \mathfrak{F}, B_2 \cap \mathfrak{F} = \varnothing] &:= \sum_{S \subset B_2} \mu^{\mathrm{F}}[B_1 \cup S \subseteq \mathfrak{F}](-1)^{|S|}\\ &= \sum_{S \subset B_2} \lim_{n\to\infty} \mu_n^{\mathrm{F}}[B_1 \cup S \subseteq \mathfrak{F}](-1)^{|S|}\\ &= \lim_{n\to\infty} \mu_n^{\mathrm{F}}[B_1 \subseteq T, B_2 \cap T = \varnothing]\,.\end{aligned}$$

This lets us define μ^{F} on ***cylinder*** sets, i.e., finite (disjoint) unions of elementary cylinder sets. Again, $\mu^{\mathrm{F}}(\mathscr{A}) = \lim_{n\to\infty} \mu_n^{\mathrm{F}}(\mathscr{A})$ for cylinder sets $\mathscr{A}$, so these probabilities are consistent and hence uniquely define a probability measure μ^{F} on subgraphs of G. We call μ^{F} the ***free (uniform) spanning forest*** measure on G, denoted FSF, since clearly it is carried by the set of spanning forests of G. We say that μ_n^{F} ***converges weakly*** to FSF. The term "free" will be explained in a moment.

How can it be that the limit FSF not be concentrated on the set of spanning trees of G? This happens if, given $x, y \in \mathsf{V}$, the μ_n^{F}-distributions of the distance in T between x and y do not form a tight family. For example, if G is the lattice graph $\mathbb{Z}^d$, Pemantle (1991) proved the remarkable theorem that FSF is concentrated on spanning trees iff $d \le 4$ (see Theorem 10.28).

Now there is another possibility for taking similar limits. In disregarding the complement of G_n, we are (temporarily) disregarding the possibility that a spanning tree or forest of G may connect the boundary vertices of G_n in ways that would affect the possible connections within G_n itself. An alternative approach takes the opposite view and forces all connections outside of G_n: As in the proof of Theorem 2.10 and in Chapter 9, let G_n^{W}

be the graph obtained from G_n by identifying all the vertices of $G \setminus G_n$ to a single vertex, z_n. (The superscript W stands for "wired" since we think of G_n^{W} as having its boundary "wired" together.) Let μ_n^{W} be the random spanning tree measure on G_n^{W}. Note that for any given n, as long as r is large enough, G_r contains the subgraph induced by $\mathsf{V}(G_n)$. For any finite $B \subset \mathsf{E}$ and any n with $B \subset \mathsf{E}(G_n)$, there is some N such that for all $r \geq N$, we have $\mu_n^{\mathrm{W}}[B \subseteq T] \leq \mu_r^{\mathrm{W}}[B \subseteq T]$: This is proved just like the inequality (10.1), with the key difference that

$$\prod_{k=1}^{m} i[e_k; G_n^{\mathrm{W}}/\{e_j\,;\ j<k\}] \leq \prod_{k=1}^{m} i[e_k; G_r^{\mathrm{W}}/\{e_j\,;\ j<k\}]$$

when G_r contains the subgraph induced by $\mathsf{V}(G_n)$, since G_n^{W} may be obtained from G_r^{W} by contracting still more edges (and removing loops). Thus, we may again define the limiting probability measure μ^{W}, called the ***wired (uniform) spanning forest*** and denoted WSF. In the case where G is itself a tree, the free spanning forest is trivially concentrated on just $\{G\}$, while the wired spanning forest is usually more interesting (see Exercise 10.5). In statistical mechanics, measures on infinite configurations are also defined using limiting procedures similar to those employed here to define FSF and WSF. One needs to specify boundary conditions, just as we did. The terms "free" and "wired" have analogous uses there.

As we will see, the wired spanning forest is much better understood than the free case. Indeed, there is a direct construction of it that avoids exhaustions: Let G be a transient network. Define $\mathfrak{F}_0 = \varnothing$. Inductively, for each $n = 1, 2, \ldots$, pick a vertex x_n and run a network random walk starting at x_n. Stop the walk when it hits $\mathfrak{F}_{n-1}$, if it does, but otherwise let it run indefinitely. Let $\mathcal{P}_n$ denote this walk. Since G is transient, with probability 1, $\mathcal{P}_n$ visits no vertex infinitely often, so its loop-erasure $\mathsf{LE}(\mathcal{P}_n)$ is well defined. Set $\mathfrak{F}_n := \mathfrak{F}_{n-1} \cup \mathsf{LE}(\mathcal{P}_n)$ and $\mathfrak{F} := \bigcup_n \mathfrak{F}_n$. Assume that the choices of the vertices x_n are made in such a way that $\{x_1, x_2, \ldots\} = \mathsf{V}$. The same reasoning as in Section 4.1 shows that the resulting distribution on forests is independent of the order in which we choose starting vertices. This also follows from the result we are about to prove. We will refer to this method of generating a random spanning forest as ***Wilson's method rooted at infinity*** (though it was introduced by BLPS (2001)).

Proposition 10.1. *The wired spanning forest on any transient network G is the same as the random spanning forest generated by Wilson's method rooted at infinity.*

Proof. For any path $\langle x_k \rangle$ that visits no vertex infinitely often, $\mathsf{LE}(\langle x_k\,;\ k \leq K \rangle) \to \mathsf{LE}(\langle x_k\,;\ k \geq 0 \rangle)$ as $K \to \infty$. That is, if $\mathsf{LE}(\langle x_k\,;\ k \leq K \rangle) = \langle u_i^K\,;\ i \leq m_K \rangle$ and

$\mathsf{LE}(\langle x_k\,;\ k \ge 0\rangle) = \langle u_i\,;\ i \ge 0\rangle$, then for each i and all large K, we have $u_i^K = u_i$; this follows from the definition of loop-erasure. Since G is transient, it follows that $\mathsf{LE}(\langle X_k\,;\ k \le K\rangle) \to \mathsf{LE}(\langle X_k\,;\ k \ge 0\rangle)$ as $K \to \infty$ a.s., where $\langle X_k\rangle$ is a random walk starting from any fixed vertex.

Let G_n be an exhaustion of G and G_n^{W} the graph formed by contracting the vertices outside G_n to a vertex z_n. Let $T(n)$ be a random spanning tree on G_n^{W} and $\mathfrak{F}$ the limit of $T(n)$ in law. Given $e_1, \ldots, e_M \in \mathsf{E}$, let $\langle X_k(u_i)\rangle$ be independent random walks starting from the endpoints $u_1, \ldots, u_L$ of $e_1, \ldots, e_M$. Run Wilson's method rooted at z_n from the vertices $u_1, \ldots, u_L$ in that order; let τ_j^n be the time that $\langle X_k(u_j)\rangle$ reaches the portion of the spanning tree created by the preceding random walks $\langle X_k(u_l)\rangle$ $(l < j)$. Then

$$\mathbf{P}[e_i \in T(n) \text{ for } 1 \le i \le M] = \mathbf{P}\left[e_i \in \bigcup_{j=1}^{L} \mathsf{LE}\big(\langle X_k(u_j)\,;\ k \le \tau_j^n\rangle\big) \text{ for } 1 \le i \le M\right].$$

Let τ_j be the stopping times corresponding to Wilson's method rooted at infinity. By induction on j, we see that $\tau_j^n \to \tau_j$ as $n \to \infty$, so that

$$\mathbf{P}[e_i \in \mathfrak{F} \text{ for } 1 \le i \le M] = \mathbf{P}\left[e_i \in \bigcup_{j=1}^{L} \mathsf{LE}\big(\langle X_k(u_j)\,;\ k \le \tau_j\rangle\big) \text{ for } 1 \le i \le M\right].$$

That is, $\mathfrak{F}$ has the same law as the random spanning forest generated by Wilson's method rooted at infinity. ◀

We will see that in many important cases, such as $\mathbb{Z}^d$, the free and wired spanning forests agree.

▷ **Exercise 10.1.**

The choice of exhaustion $\langle G_n\rangle$ does not change the resulting measure WSF by Proposition 10.1. Show that the choice also does not change the resulting measure FSF.

An ***automorphism*** of a network is an automorphism of the underlying graph that preserves edge weights.

▷ **Exercise 10.2.**

Show that FSF and WSF are invariant under any automorphisms that the network may have.

▷ **Exercise 10.3.**

Show that if G is an infinite recurrent network, then the wired spanning forest on G is the same as the free spanning forest, i.e., the random spanning tree of Section 4.2.

▷ **Exercise 10.4.**

Let G be a network such that there is a finite subset of edges whose removal from G leaves at least 2 transient components. Show that the free and wired spanning forests are different on G.

▷ **Exercise 10.5.**

Let G be a tree with unit conductances. Show that $\mathsf{FSF} = \mathsf{WSF}$ iff G is recurrent.

Proposition 10.2. *Let G be a locally finite network. For both* FSF *and* WSF*, all trees are a.s. infinite.*

Proof. A finite tree, if it occurs, must occur with positive probability at some specific location, meaning that certain specific edges are present and certain other specific edges surrounding the edges of the tree are absent. But every such event has probability 0 for the approximations μ_n^{F} and μ_n^{W} and there are only countably many such events. ◀

▷ **Exercise 10.6.**

Let G be an edge-amenable infinite graph as witnessed by the vertex sets $\langle \mathsf{V}_n \rangle$ (see Section 4.3). Let G_n be the subgraph induced by V_n.

(a) Let F be any spanning forest all of whose components (trees) are infinite. Show that if k_n denotes the number of trees of $F \cap G_n$, then $k_n = o(|\mathsf{V}_n|)$.

(b) Show that the average degree, in two senses, of vertices in both the free spanning forest and the wired spanning forest is 2:

$$\lim_{n\to\infty} |\mathsf{V}_n|^{-1} \sum_{x \in \mathsf{V}_n} \deg_{\mathfrak{F}}(x) = 2 \quad \text{a.s.}$$

and

$$\lim_{n\to\infty} |\mathsf{V}_n|^{-1} \sum_{x \in \mathsf{V}_n} \mathbf{E}[\deg_{\mathfrak{F}}(x)] = 2\,.$$

In particular, if G is a transitive graph such as $\mathbb{Z}^d$, then every vertex has expected degree 2 in both the free spanning forest and the wired spanning forest.

§10.2. Coupling and Equality.

Often $\mathsf{FSF} = \mathsf{WSF}$ and it will turn out to be quite interesting to see when this happens. In all cases, though, there is a simple inequality between these two probability measures, namely,

$$\forall e \in \mathsf{E} \quad \mathsf{FSF}[e \in \mathfrak{F}] \ge \mathsf{WSF}[e \in \mathfrak{F}] \tag{10.2}$$

since, by Rayleigh's monotonicity principle, this is true for μ_n^{F} and μ_n^{W} as soon as n is large enough that $e \in \mathsf{E}(G_n)$. Alternatively, we can write

$$\mathsf{FSF}[e \in \mathfrak{F}] = i_{\mathrm{F}}^e(e) \text{ and } \mathsf{WSF}[e \in \mathfrak{F}] = i_{\mathrm{W}}^e(e) \tag{10.3}$$

by Propositions 9.1 and 9.2 combined with Corollary 4.4. In (9.3), we saw the inequality (10.2) for these currents.

More generally, we claim that for every increasing cylinder set $\mathscr{A}$, we have

$$\mathsf{FSF}(\mathscr{A}) \ge \mathsf{WSF}(\mathscr{A})\,. \tag{10.4}$$

We therefore say that FSF ***stochastically dominates*** WSF and we write $\mathsf{FSF} \succcurlyeq \mathsf{WSF}$. To show (10.4), it suffices to show that for each n we have $\mu_n^{\mathrm{F}}(\mathscr{A}) \ge \mu_n^{\mathrm{W}}(\mathscr{A})$ for $\mathscr{A}$ an increasing event in the edge set E_n. Note that G_n is a subgraph of G_n^{W}. Thus, what we want is a consequence of the following more general result:

Lemma 10.3. *Let G be a connected subgraph of a finite connected graph H. Let μ_G and μ_H be the corresponding uniform spanning tree measures. Then $\mu_G(\mathscr{A}) \ge \mu_H(\mathscr{A})$ for every increasing event $\mathscr{A}$ in the edge set $\mathsf{E}(G)$.*

Proof. By induction, it suffices to prove this when H has only one more edge, e, than G. Now

$$\mu_H(\mathscr{A}) = \mu_H[e \in T]\mu_H[\mathscr{A} \mid e \in T] + \mu_H[e \notin T]\mu_H[\mathscr{A} \mid e \notin T]\,.$$

If $\mu_H[e \notin T] = 0$, then $\mu_H(\mathscr{A}) = \mu_G(\mathscr{A})$, while, if $\mu_H[e \notin T] > 0$, then

$$\mu_H[\mathscr{A} \mid e \in T] \le \mu_H[\mathscr{A} \mid e \notin T] = \mu_G(\mathscr{A})$$

by Theorem 4.5. This gives the result. ◀

▷ **Exercise 10.7.**

Let G be a graph obtained by identifying some vertices of a finite connected graph H. Let μ_G and μ_H be the corresponding uniform spanning tree measures. Show that $\mu_G(\mathscr{A}) \le \mu_H(\mathscr{A})$ for every increasing event $\mathscr{A}$ depending on the edges of G.

The stochastic inequality $\mathsf{FSF} \succcurlyeq \mathsf{WSF}$ implies that the two measures, FSF and WSF, can be ***monotonically coupled***. What this means is that there is a probability measure on the set

$$\Big\{(\mathfrak{F}_1, \mathfrak{F}_2)\,;\ \mathfrak{F}_i \text{ is a spanning forest of } G \text{ and } \mathfrak{F}_1 \subseteq \mathfrak{F}_2\Big\}$$

that projects in the first coordinate to WSF and in the second to FSF. It is easy to see that the existence of a monotonic coupling implies the stochastic domination inequality. This equivalence between existence of a monotonic coupling and stochastic domination is quite a general result. In fact, we've encountered the notion of stochastic domination before in Exercise 4.42 and Section 7.4. They can all be unified by considering probability measures on partially ordered sets. Two equivalent definitions of stochastic domination on finite sets are presented in the following theorem. Extension to infinite sets is often straightforward.

Theorem 10.4. (Strassen, 1965) *Let $(X, \preccurlyeq)$ be a partially ordered finite set with two probability measures, μ_1 and μ_2. The following are equivalent:*

(i) There is a probability measure ν on $\{(x, y) \in X \times X\,;\ x \preccurlyeq y\}$ whose coordinate projections are μ_i.

(ii) For each subset $A \subseteq X$ such that if $x \in A$ and $x \preccurlyeq y$, then $y \in A$, we have $\mu_1(A) \le \mu_2(A)$.

In case these properties hold, we write $\mu_1 \preccurlyeq \mu_2$ and we say that μ_1 is ***stochastically dominated*** by μ_2. We are interested here in the case where X consists of the subsets of edges of a finite graph, ordered by $\subseteq$.

Proof. The coupling of (i) is a way to distribute the μ_1-mass of each point $x \in X$ among the points $y \succcurlyeq x$ in such a way as to obtain the distribution of μ_2. This is just a more graphic way of expressing the requirements of (i) that

$$\forall x \quad \sum_{y \succcurlyeq x} \nu(x, y) = \mu_1(x)\,,$$
$$\forall y \quad \sum_{x \preccurlyeq y} \nu(x, y) = \mu_2(y)\,.$$

If we make a directed graph whose vertices are $(X \times \{1\}) \cup (X \times \{2\}) \cup \{\Delta_1, \Delta_2\}$ with edges from Δ_1 to each vertex of $X \times \{1\}$, from each vertex of $X \times \{2\}$ to Δ_2, and from $(x, 1)$ to $(y, 2)$ whenever $x \preccurlyeq y$, then we can think of ν as a flow from Δ_1 to Δ_2 by letting $\nu(x, y)$ be the amount of flow from $(x, 1)$ to $(y, 2)$. To put this into the framework of the Max-Flow Min-Cut Theorem, let the capacity of the edge joining Δ_i with (x, i) be $\mu_i(x)$ for $i = 1, 2$ and the capacity of all other edges be 2. It is evident that the condition (i) is that the maximum flow from Δ_1 to Δ_2 is 1. We claim that the condition (ii) is that the minimum cutset sum is 1, from which the theorem follows. To see this, note that any cutset of minimum sum does not use any edges of the form $\langle (x,1),(y,2)\rangle$ since these all have capacity 2. Thus, given a minimum cutset sum, let $B \times \{1\}$ be the set of vertices in $X \times \{1\}$ that are *not* separated from Δ_1 by the cutset. By minimality, we have that the set of vertices in $X \times \{2\}$ that *are* separated from Δ_2 is $A \times \{2\}$, where

$$A := \{y \in X\,;\ \exists x \in B\ \ x \preccurlyeq y\}\,.$$

Thus, the cutset sum is $\mu_2(A) + 1 - \mu_1(B)$. Now minimality again shows that, without loss of generality, $B = A$, whence the cutset sum is $\mu_2(A) + 1 - \mu_1(A)$. Conversely, every A as in (ii) yields a cutset sum equal to $\mu_2(A) + 1 - \mu_1(A)$. Thus, the minimum cutset sum equals 1 iff (ii) holds, as claimed. ◀

Corollary 10.5. *On every infinite network G, we have* $\mathsf{FSF} \succcurlyeq \mathsf{WSF}$ *and there is a monotone coupling* $(\mathfrak{F}_1, \mathfrak{F}_2) \in 2^{\mathsf{E}(G)} \times 2^{\mathsf{E}(G)}$ *with* $\mathfrak{F}_1 \sim \mathsf{FSF}$*,* $\mathfrak{F}_2 \sim \mathsf{WSF}$*, and* $\mathfrak{F}_1 \supseteq \mathfrak{F}_2$ *a.s.*

Proof. By Theorem 10.4, we may monotonically couple the measures induced by FSF and WSF on any finite subgraph of G. By taking an exhaustion of G and a weak limit point of these couplings, we obtain a monotone coupling of FSF and WSF on all of G. ◀

Therefore, $\mathsf{FSF}(\mathscr{A}) \ge \mathsf{WSF}(\mathscr{A})$ not only for all increasing cylinder events $\mathscr{A}$, but for all increasing events $\mathscr{A}$.

Question 10.6. Is there a "natural" monotone coupling of FSF and WSF? In particular, is there a monotone coupling that is invariant under all graph automorphisms? As we will see soon, $\mathsf{FSF} = \mathsf{WSF}$ on amenable Cayley graphs, so in that case, there is nothing to do. Bowen (2004) has proved there is an invariant monotone coupling for all so-called residually amenable groups.

▷ **Exercise 10.8.**

Show that the number of trees in the free spanning forest on a network is stochastically dominated by the number in the wired spanning forest on the network. If the number of trees in the free spanning forest is a.s. finite, then, in distribution, it equals the number in the wired spanning forest iff $\mathsf{FSF} = \mathsf{WSF}$.

So when are the free and wired spanning forests the same? Here is one test.

Proposition 10.7. *If* $\mathbf{E}[\deg_{\mathfrak{F}}(x)]$ *is the same under* FSF *and* WSF *for every* $x \in \mathsf{V}$*, then* $\mathsf{FSF} = \mathsf{WSF}$*.*

Proof. In the monotone coupling described above, the set of edges adjacent to a vertex x in the WSF is a subset of those adjacent to x in the FSF. The hypothesis implies that for each x, these two sets coincide a.s. ◀

Note that this proof works for any pair of measures where one stochastically dominates the other.

Remark 10.8. It follows that if FSF and WSF agree on single-edge probabilities, i.e., if equality holds in (10.2) for all $e \in \mathsf{E}$, then $\mathsf{FSF} = \mathsf{WSF}$. This is due to Häggström (1995).

We may now deduce that $\mathsf{FSF} = \mathsf{WSF}$ for many graphs, including Cayley graphs of abelian groups such as $\mathbb{Z}^d$. Call a graph or network ***transitive*** if for every pair of vertices x and y, there is an automorphism of the graph or network that takes x to y.

The following is essentially due to Häggström (1995).

Corollary 10.9. *On any transitive amenable network,* $\mathsf{FSF} = \mathsf{WSF}$*.*

Proof. By transitivity and Exercise 10.6, $\mathbf{E}[\deg_{\mathfrak{F}}(x)] = 2$ for both FSF and WSF. Apply Proposition 10.7. ◀

▷ **Exercise 10.9.**
Give an amenable graph on which $\mathsf{FSF} \neq \mathsf{WSF}$.

The amenability assumption is not needed to determine the expected degree in the WSF on a transitive network:

Proposition 10.10. *If G is a transitive network, the* WSF*-expected degree of every vertex is* 2*.*

Proof. If G is recurrent, then it is amenable by Theorem 6.7 and the result follows from Corollary 10.9. So assume that G is transient. Think of the wired spanning forest as oriented toward infinity from Wilson's method rooted at infinity; i.e., orient each edge of the forest in the direction it is last traversed. We claim that the law of the orientation does not depend on the choices in Wilson's method rooted at infinity. Indeed, since this obviously holds for finite graphs when orienting the tree towards a fixed root, it follows by taking an exhaustion of G and using the proof of Proposition 10.1. Alternatively, one can modify the proof of Theorem 4.1 to prove it directly.

Now the out-degree of every vertex in this orientation is 1. Fix a vertex o. We need to show that the expected in-degree of o is 1. For this, it suffices to prove that for every neighbor x of o, the probability of the edge $\langle x, o\rangle$ is the same as the probability of the edge $\langle o, x\rangle$.

Now the probability of the edge $\langle o, x\rangle$ is $(\pi(o)\mathbf{P}_o[\tau_o^+ = \infty])^{-1}c(o,x)\mathbf{P}_x[\tau_o = \infty]$ by Exercise 4.25. We have

$$\mathbf{P}_x[\tau_o < \infty] = \mathbf{P}_o[\tau_x < \infty] \tag{10.5}$$

since $\pi(x) = \pi(o)$ and $\mathcal{G}(x,x) = \mathcal{G}(o,o)$ by transitivity; see Exercise 2.21. This gives the result. ◀

Now what about the expected degree of a vertex in the FSF? This turns out to be even more interesting than in the WSF. One can show (Lyons, 2009) from (10.3) and Definition 2.9 of Gaboriau (2005) that the expected degree in a transitive graph G equals $2 + 2\beta_1(G)$, where $\beta_1(G)$ is the so-called ***first ℓ^2-Betti number*** of G. This is a nonnegative real number whose definition was made originally by Atiyah (1976), was considerably extended by Cheeger and Gromov (1986), and was made in this context by Gaboriau (2005). Consider the special case where G is a Cayley graph of a group, Γ. It turns out that $\beta_1(G)$ is the same for all Cayley graphs of Γ, so we normally write $\beta_1(\Gamma)$ instead. Here are some known values (see, e.g., Gaboriau (2002), Cheeger and Gromov (1986), and Lück (2009)):

- $\beta_1(\Gamma) = 0$ if Γ is finite or amenable
- $\beta_1(\Gamma_1 * \Gamma_2) = \beta_1(\Gamma_1) + \beta_1(\Gamma_2) + 1 - \frac{1}{|\Gamma_1|} - \frac{1}{|\Gamma_2|}$
- $\beta_1(\Gamma_1 *_{\Gamma_3} \Gamma_2) = \beta_1(\Gamma_1) + \beta_1(\Gamma_2)$ if Γ_3 is amenable and infinite
- $\beta_1(\Gamma_2) = [\Gamma_1 : \Gamma_2]\beta_1(\Gamma_1)$ if Γ_2 has finite index in Γ_1
- $\beta_1(\Gamma) = 2g - 2$ if Γ is the fundamental group of an orientable surface of genus g
- $\beta_1(\Gamma) = s - 2$ if Γ is torsion free and can be presented with $s \ge 2$ generators and 1 non-trivial relation

Question 10.11. If G is a Cayley graph of a finitely presented group Γ, is the FSF-expected degree of o a rational number? Atiyah (1976) has asked whether all the ℓ^2-Betti numbers are rational if Γ is finitely presented.

An important open question is whether $\beta_1(\Gamma)$ is equal to one less than the ***cost*** of Γ, which is one-half of the infimum of the expected degree in any random connected spanning graph of Γ whose law is Γ-invariant (see Gaboriau (2000)). Gaboriau (2002) proved that $\beta_1(\Gamma)$ is always at most one less than the cost of Γ. To establish equality, it would therefore suffice to show that for every $\epsilon > 0$, there is some probability space with two 2^{E}-valued random variables, $\mathfrak{F}$ and $\mathfrak{G}$, such that the law of $\mathfrak{F}$ is FSF, the union $\mathfrak{F} \cup \mathfrak{G}$ is connected and has a Γ-invariant law, and the expected degree of a vertex in $\mathfrak{G}$ is less than ϵ. In Chapter 11, we will see that an analogue of this does hold for the free minimal spanning forest.

Question 10.12. If G is a Cayley graph, $\mathfrak{F}$ is the FSF on G, and $\epsilon > 0$, is there an invariant connected percolation ω containing $\mathfrak{F}$ such that for every edge e, the probability that $e \in \omega \setminus \mathfrak{F}$ is less than ϵ? This question was asked by D. Gaboriau (personal communication, 2001).

We turn now to an electrical criterion for the equality of FSF and WSF. As we saw in Chapter 9, there are two natural ways of defining currents between vertices of a graph, corresponding to the two ways of defining spanning forests.

We may add to Proposition 9.3 and Theorem 9.5 as follows:

Proposition 10.13. *For any network, the following are equivalent:*

(i) $\mathsf{FSF} = \mathsf{WSF}$*;*

(ii) $i^e_{\mathrm{W}} = i^e_{\mathrm{F}}$ *for every edge* e*;*

(iii) $\ell^2_-(\mathsf{E}) = \bigstar \oplus \diamondsuit$*;*

(iv) $\mathbf{HD} = \mathbb{R}$*.*

Proof. Use Remark 10.8, (10.3), and (9.3) to deduce that (i) and (ii) are equivalent. The other equivalences were proved already. ◀

Combining this with Theorem 9.7, we get another proof that $\mathsf{FSF} = \mathsf{WSF}$ on $\mathbb{Z}^d$.

▷ **Exercise 10.10.**

Show that every transitive amenable network has unique currents.

Let $Y_{\mathrm{F}}(e, f) := i^e_{\mathrm{F}}(f)$ and $Y_{\mathrm{W}}(e, f) := i^e_{\mathrm{W}}(f)$ be the free and wired transfer-current matrices.

Theorem 10.14. *Given any network G and any distinct edges $e_1, \ldots, e_k \in G$, we have*

$$\mathsf{FSF}[e_1, \ldots, e_k \in \mathfrak{F}] = \det[Y_{\mathrm{F}}(e_i, e_j)]_{1 \le i,j \le k}$$

and

$$\mathsf{WSF}[e_1, \ldots, e_k \in \mathfrak{F}] = \det[Y_{\mathrm{W}}(e_i, e_j)]_{1 \le i,j \le k} \,.$$

Proof. This is immediate from the Transfer-Current Theorem of Section 4.2 and Propositions 9.1 and 9.2. ◀

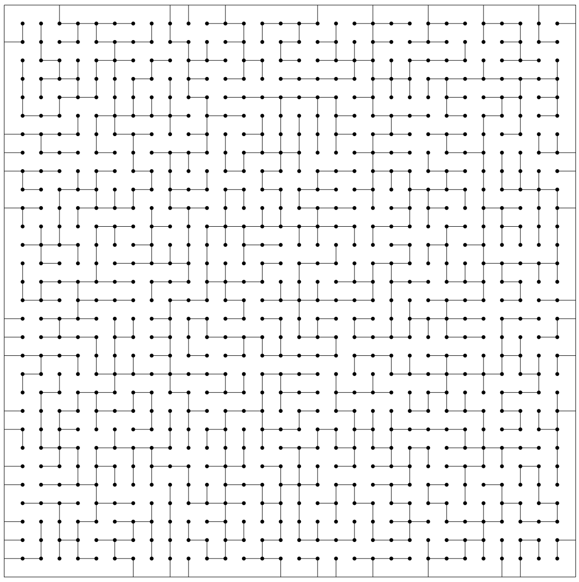


Figure 10.1. A uniformly chosen wired spanning tree on a subgraph of $\mathbb{Z}^2$, drawn by Wilson (see Propp and Wilson (1998)).

§10.3. Planar Networks and Euclidean Lattices.

Consider now plane graphs. Examination of Figure 10.1 reveals two spanning trees: one in white, the other in black on the plane dual graph. (See Section 9.2 for the definition of dual.) Note that in the dual, the outer boundary of the grid is identified to a single vertex. In general, suppose that G is a simple plane network whose plane dual $G^\dagger$ is locally finite. Given a finite connected subnetwork G_n of G, let $G_n^\dagger$ be its plane dual. Note that $G_n^\dagger$ can be regarded as a finite subnetwork of $G^\dagger$, but with the outer boundary vertices identified to a single vertex. Spanning trees T of G_n are in one-to-one correspondence with spanning trees $T^\times$ of $G_n^\dagger$ in the same way as in Figure 10.1:

$$e \in T \iff e^\dagger \notin T^\times . \tag{10.6}$$

Furthermore, this correspondence preserves the relative weights: we have

$$\alpha(T) = \prod_{e \in T} c(e) = \prod_{e^\dagger \notin T^\times} r(e^\dagger) = \frac{\prod_{e^\dagger \in T^\times} c(e^\dagger)}{\prod_{e^\dagger \in G_n^\dagger} c(e^\dagger)} = \frac{\alpha(T^\times)}{\prod_{e^\dagger \in G_n^\dagger} c(e^\dagger)} .$$

Therefore, the FSF of G is "dual" to the WSF of $G^\dagger$; that is, the relation (10.6) transforms one to the other. As a consequence, (10.3) and the definition (10.6) explain (9.4). Let's look at this a little more closely for the graph $\mathbb{Z}^2$. By recurrence and Exercise 10.3, the free and wired spanning forests are the same as the uniform spanning tree. In particular,

$\mathbf{P}[e \in T] = \mathbf{P}[e^\dagger \in T^\times]$. Since these add to 1, they are equal to 1/2, as we derived in another fashion in (4.12).

Let's see what Theorem 10.14 says more explicitly for unit conductances on the hypercubic lattice $\mathbb{Z}^d$. The case $d = 2$ was treated already in Section 4.3 and Exercise 4.44. The transient case $d \ge 3$ is actually easier, but the approach is the same as in Section 4.3. In order to find the transfer currents $Y(e, f)$, we will first find voltages, then use $i = dv$. When i is a unit flow from x to y, we have $d^* i = \mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}$. Hence the voltages satisfy $\Delta v := d^* dv = \mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}$. We are interested in solving this equation when $x := e^-$, $y := e^+$ and then computing $v(f^-) - v(f^+)$. Now, however, we can solve $\Delta v = \mathbf{1}_{\{x\}}$, i.e., the voltage v for a current from x to infinity. Again, we begin with a formal (i.e., heuristic) derivation of the solution, then prove that our formula is correct.

Let $\mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$ be the d-dimensional torus. For $(x_1, \ldots, x_d) \in \mathbb{Z}^d$ and $(\alpha_1, \ldots, \alpha_d) \in \mathbb{T}^d$, write $(x_1, \ldots, x_d) \cdot (\alpha_1, \ldots, \alpha_d) := x_1\alpha_1 + \cdots + x_d\alpha_d \in \mathbb{R}/\mathbb{Z}$. For a function f on $\mathbb{Z}^d$, define the function $\widehat{f}$ on $\mathbb{T}^d$ by

$$\widehat{f}(\alpha) := \sum_{x \in \mathbb{Z}^d} f(x) e^{-2\pi i x \cdot \alpha}\,.$$

For example, $\widehat{\mathbf{1}_{\{x\}}}(\alpha) = e^{-2\pi i x \cdot \alpha}$. Now a formal calculation shows that

$$\widehat{\Delta f}(\alpha) = \widehat{f}(\alpha)\varphi(\alpha)\,,$$

where

$$\varphi\big((\alpha_1, \ldots, \alpha_d)\big) := 2d - \sum_{k=1}^{d} (e^{2\pi i \alpha_k} + e^{-2\pi i \alpha_k}) = 2d - 2\sum_{k=1}^{d} \cos 2\pi\alpha_k\,. \tag{10.7}$$

Hence, to solve $\Delta f = g$, we may try to solve $\widehat{\Delta f} = \widehat{g}$ by using $\widehat{f} := \widehat{g}/\varphi$ and then finding f. In fact, a formal calculation shows that we may recover f from $\widehat{f}$ by the formula

$$f(x) = \int_{\mathbb{T}^d} \widehat{f}(\alpha) e^{2\pi i x \cdot \alpha}\, d\alpha\,,$$

where the integration is with respect to Lebesgue measure. What makes the transient case easier is that $1/\varphi \in L^1(\mathbb{T}^d)$, since

$$\varphi(\alpha) = (2\pi)^2 |\alpha|^2 + O(|\alpha|^4) \tag{10.8}$$

as $|\alpha| \to 0$ and $r \mapsto 1/r^2$ is integrable in $\mathbb{R}^d$ for $d \ge 3$; here, $|\alpha|$ is the distance in $\mathbb{R}^d$ from any element of the coset α to $\mathbb{Z}^d$.

▷ **Exercise 10.11.**

(The Riemann-Lebesgue Lemma) Show that if $F \in L^1(\mathbb{T}^d)$ and $f(x) = \int_{\mathbb{T}^d} F(\alpha) e^{2\pi i x \cdot \alpha}\, d\alpha$, then $\lim_{|x| \to \infty} f(x) = 0$. *Hint:* This is obvious if F is a ***trigonometric polynomial***, i.e., a finite linear combination of functions $\alpha \mapsto e^{2\pi i x \cdot \alpha}$. The Stone-Weierstrass theorem implies that such functions are dense in $L^1(\mathbb{T}^d)$.

Proposition 10.15. (Voltages on $\mathbb{Z}^d$) *Let $d \ge 3$. The voltage at x when a unit current flows from $\mathbf{0}$ to infinity in $\mathbb{Z}^d$ and the voltage is 0 at infinity is*

$$v(x) = \int_{\mathbb{T}^d} \frac{e^{2\pi i x \cdot \alpha}}{\varphi(\alpha)}\, d\alpha\,. \tag{10.9}$$

Proof. Define $v'(x)$ to be the integral. By the analogue of Exercise 4.9 for $d \ge 3$, we have

$$\Delta v'(x) = \int_{\mathbb{T}^d} \frac{e^{2\pi i x \cdot \alpha}}{\varphi(\alpha)} \varphi(\alpha)\, d\alpha = \int_{\mathbb{T}^d} e^{2\pi i x \cdot \alpha}\, d\alpha = \mathbf{1}_{\{\mathbf{0}\}}(x)\,.$$

That is, $\Delta v' = \mathbf{1}_{\{\mathbf{0}\}}$. Since v satisfies the same equation, we have $\Delta(v' - v) = 0$. In other words, $v' - v$ is harmonic at every point in $\mathbb{Z}^d$. Furthermore, v' is bounded in absolute value by the L^1 norm of $1/\varphi$, whence $v' - v$ is bounded. Since the only bounded harmonic functions on $\mathbb{Z}^d$ are the constants (by, say, Theorem 9.22), this means that $v' - v$ is constant. Now the Riemann-Lebesgue Lemma* (Exercise 10.11) implies that v' vanishes at infinity. Since v also vanishes at infinity, this constant is 0. Therefore, $v' = v$, as desired. ◀

Now one can compute

$$Y(e, f) = v(f^- - e^-) - v(f^+ - e^-) - v(f^- - e^+) + v(f^+ - e^+)$$

using (10.9).

* One could avoid the Riemann-Lebesgue Lemma by solving the more complicated equation $\Delta v = \mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}$ instead, as in Section 4.3.

§10.4. Tail Triviality.

How much does the configuration of a uniform spanning forest in one region influence the configuration in a far away region? Are they asymptotically independent? It turns out that they are indeed, in several senses.

For a set of edges $K \subseteq \mathsf{E}$, let $\mathscr{F}(K)$ denote the σ-field of events depending only on K. Define the ***tail σ-field*** to be the intersection of $\mathscr{F}(\mathsf{E} \setminus K)$ over all finite K. Triviality of the tail σ-field means that every tail-measurable event has probability 0 or 1. This is a strong form of asymptotic independence:

Proposition 10.16. *For any probability measure* $\mathbf{P}$, *tail triviality is equivalent to*

$$\forall \mathscr{A}_1 \in \mathscr{F}(\mathsf{E}) \;\; \forall \epsilon > 0 \;\; \exists K \text{ finite } \; \forall \mathscr{A}_2 \in \mathscr{F}(\mathsf{E} \setminus K) \qquad \big|\mathbf{P}(\mathscr{A}_1 \cap \mathscr{A}_2) - \mathbf{P}(\mathscr{A}_1)\mathbf{P}(\mathscr{A}_2)\big| < \epsilon\,. \tag{10.10}$$

Proof. Suppose first that (10.10) holds. Let $\mathscr{A}$ be a tail event. Then we may take $\mathscr{A}_1 := \mathscr{A}_2 := \mathscr{A}$ in (10.10), no matter what ϵ is. This shows that $\mathscr{A}$ is independent of itself, i.e., is trivial.

For the converse, let $\mathbf{P}$ be tail trivial. Let $\langle K_n \rangle$ be an exhaustion of E. Then $\mathcal{T} := \bigcap_n \mathscr{F}(\mathsf{E} \setminus K_n)$ is the tail σ-field. The reversed-martingale convergence theorem shows that for every $\mathscr{A}_1 \in \mathscr{F}(\mathsf{E})$, we have

$$Z_n := \mathbf{P}\big(\mathscr{A}_1 \bigm| \mathscr{F}(\mathsf{E} \setminus K_n)\big) \to \mathbf{P}(\mathscr{A}_1 \mid \mathcal{T})$$

not only a.s., but in $L^1(\mathbf{P})$. Since the tail is trivial, the limit equals $\mathbf{P}(\mathscr{A}_1)$ a.s. Thus, given $\epsilon > 0$, there is some n such that $\mathbf{E}\big[|Z_n - \mathbf{P}(\mathscr{A}_1)|\big] < \epsilon$. Hence, for all $\mathscr{A}_2 \in \mathscr{F}(\mathsf{E} \setminus K_n)$, we have

$$\big|\mathbf{P}(\mathscr{A}_1 \cap \mathscr{A}_2) - \mathbf{P}(\mathscr{A}_1)\mathbf{P}(\mathscr{A}_2)\big| = \big|\mathbf{E}\big[\big(Z_n - \mathbf{P}(\mathscr{A}_1)\big)\mathbf{1}_{\mathscr{A}_2}\big]\big| \le \mathbf{E}\big[\big|(Z_n - \mathbf{P}(\mathscr{A}_1))\mathbf{1}_{\mathscr{A}_2}\big|\big] < \epsilon\,. \quad ◀$$

This shows why the following theorem is interesting.

Theorem 10.17. *The* WSF *and* FSF *have trivial tail on every network.*

Proof. Let G be an infinite network exhausted by finite connected subnetworks $\langle G_n \rangle$. Write $G_n = (\mathsf{V}_n, \mathsf{E}_n)$. Recall from Section 10.1 that μ_n^{F} denotes the uniform spanning tree measure on G_n and μ_n^{W} denotes the uniform spanning tree measure on the "wired" graph G_n^{W}. Let ν_n be any "partially wired" measure, i.e., the uniform spanning tree measure on a graph G_n^* obtained from a finite network G_n' satisfying $G_n \subset G_n' \subset G$ by contracting some of the edges in G'_n that are not in G_n. Lemma 10.3 and Exercise 10.7 give that if $\mathscr{B} \in \mathscr{F}(\mathsf{E}_n)$ is increasing, then

$$\mu_n^{\mathrm{W}}(\mathscr{B}) \le \nu_n(\mathscr{B}) \le \mu_n^{\mathrm{F}}(\mathscr{B})\,. \tag{10.11}$$

Let $M > n$ and let $\mathscr{A} \in \mathscr{F}(\mathsf{E}_M \setminus \mathsf{E}_n)$ be a cylinder event such that $\mu_M^{\mathrm{W}}(\mathscr{A}) > 0$. For each increasing $\mathscr{B} \in \mathscr{F}(\mathsf{E}_n)$, we have

$$\mu_n^{\mathrm{W}}(\mathscr{B}) \le \mu_M^{\mathrm{W}}(\mathscr{B} \mid \mathscr{A})\,. \tag{10.12}$$

To see this, condition separately on each possible configuration of edges of $G_M \setminus G_n$ that is in $\mathscr{A}$, and use (10.11). Fixing $\mathscr{A}$ and letting $M \to \infty$ in (10.12) gives

$$\mu_n^{\mathrm{W}}(\mathscr{B}) \le \mathsf{WSF}(\mathscr{B} \mid \mathscr{A})\,. \tag{10.13}$$

This applies to all cylinder events $\mathscr{A} \in \mathscr{F}(\mathsf{E} \setminus \mathsf{E}_n)$ with $\mathsf{WSF}(\mathscr{A}) > 0$, and therefore the assumption that $\mathscr{A}$ is a cylinder event can be dropped. Thus (10.13) holds for all tail events $\mathscr{A}$ of positive probability. Taking $n \to \infty$ there gives

$$\mathsf{WSF}(\mathscr{B}) \le \mathsf{WSF}(\mathscr{B} \mid \mathscr{A})\,, \tag{10.14}$$

where $\mathscr{B}$ is any increasing cylinder event and $\mathscr{A}$ is any tail event. Thus, (10.14) also applies to the complement $\mathscr{A}^c$. Since $\mathsf{WSF}(\mathscr{B}) = \mathsf{WSF}(\mathscr{A})\mathsf{WSF}(\mathscr{B} \mid \mathscr{A}) + \mathsf{WSF}(\mathscr{A}^c)\mathsf{WSF}(\mathscr{B} \mid \mathscr{A}^c)$, it follows that $\mathsf{WSF}(\mathscr{B}) = \mathsf{WSF}(\mathscr{B} \mid \mathscr{A})$. Therefore, every tail event $\mathscr{A}$ is independent of every increasing cylinder event. By inclusion-exclusion, such $\mathscr{A}$ is also independent of every elementary cylinder event, whence of every cylinder event, whence of every event. That is, $\mathscr{A}$ is trivial. The argument for the FSF is similar. ◀

If Γ is a group of automorphisms of a network or graph G, we say that Γ ***acts*** on G. Any action extends in the obvious way to an action on the collection of subnetworks or subgraphs of G. An element of Γ acts in a "large way" on G if it moves points far. To be precise, for an action of Γ and a real-valued function f on Γ, we write $\lim_{\gamma\to\infty} f(\gamma) = a$ to mean that for every $\epsilon > 0$ and every $x \in \mathsf{V}$, there is some N such that whenever $\mathrm{dist}(x, \gamma x) > N$, we have $|f(\gamma) - a| < \epsilon$. Similarly, $\limsup_{\gamma\to\infty} f(\gamma) \le a$ means that for every $\epsilon > 0$ and every $x \in \mathsf{V}$, there is some N such that whenever $\mathrm{dist}(x, \gamma x) > N$, we have $f(\gamma) < a + \epsilon$. We say that the action is ***mixing*** for a Γ-invariant probability measure $\mathbf{P}$ on subnetworks or subgraphs of G if for any pair of events $\mathscr{A}$ and $\mathscr{B}$,

$$\lim_{\gamma\to\infty} \mathbf{P}(\mathscr{A}, \gamma\mathscr{B}) = \mathbf{P}(\mathscr{A})\mathbf{P}(\mathscr{B})\,. \tag{10.15}$$

We call an action ***ergodic*** for $\mathbf{P}$ if the only $(\Gamma, \mathbf{P})$-invariant events are trivial. Here, an event $\mathscr{A}$ is called $(\Gamma, \mathbf{P})$***-invariant*** if $\mathbf{P}(\mathscr{A} \triangle \gamma \mathscr{A}) = 0$ for all $\gamma \in \Gamma$. As usual, as long as there is some infinite orbit, mixing implies ergodicity: if $\mathscr{A}$ is an invariant event, then just set $\mathscr{B} = \mathscr{A}$ in (10.15). In addition, tail triviality implies mixing: Let $\mathscr{A}$ and $\mathscr{B}$ be two events. The equation (10.15) follows immediately from (10.10) in case $\mathscr{B}$ is a cylinder. For general $\mathscr{B}$, let $\epsilon > 0$ and let $\mathscr{D}$ be a cylinder such that $\mathbf{P}(\mathscr{B} \triangle \mathscr{D}) < \epsilon$. Since $\mathbf{P}$ is invariant under Γ, we have

$$
\begin{aligned}
|\mathbf{P}(\mathscr{A}, \gamma\mathscr{B}) - \mathbf{P}(\mathscr{A})\mathbf{P}(\mathscr{B})| &\le |\mathbf{P}(\mathscr{A}, \gamma\mathscr{D}) - \mathbf{P}(\mathscr{A})\mathbf{P}(\mathscr{D})| + \mathbf{P}\big(\mathscr{A}, \gamma(\mathscr{B} \triangle \mathscr{D})\big) + \mathbf{P}(\mathscr{A})\mathbf{P}(\mathscr{B} \triangle \mathscr{D}) \\
&< |\mathbf{P}(\mathscr{A}, \gamma\mathscr{D}) - \mathbf{P}(\mathscr{A})\mathbf{P}(\mathscr{D})| + 2\epsilon\,.
\end{aligned}
$$

Taking γ to infinity, we get that $\limsup_{\gamma \to \infty} |\mathbf{P}(\mathscr{A}, \gamma\mathscr{B}) - \mathbf{P}(\mathscr{A})\mathbf{P}(\mathscr{B})| \le 2\epsilon$. Since ϵ is arbitrary, the action is mixing.

Lastly, distinct ergodic invariant measures under any group action are always singular: Suppose that μ_1 and μ_2 are both invariant and ergodic probability measures for an action of a group Γ. According to the Lebesgue decomposition theorem, there is a unique pair of measures ν_1, ν_2 such that $\mu_1 = \nu_1 + \nu_2$ with $\nu_1 \ll \mu_2$ and $\nu_2 \perp \mu_2$. Applying any element of Γ, we see that ν_1 and ν_2 are both Γ-invariant. Choose an event $\mathscr{A}$ with $\nu_1(\mathscr{A}^c) = 0$ and $\nu_2(\mathscr{A}) = 0$. Then $\mathscr{A}$ is (Γ, ν_1)-invariant and (Γ, ν_2)-invariant, whence (Γ, μ_1)-invariant, whence μ_1-trivial. If $\mu_1(\mathscr{A}) = 0$, then $\mu_1 = \nu_2 \perp \mu_2$. On the other hand, if $\mu_1(\mathscr{A}^c) = 0$, then $\mu_1 = \nu_1 \ll \mu_2$. Let f be the Radon-Nikodým derivative of μ_1 with respect to μ_2. Then f is measurable with respect to the σ-field of (Γ, μ_2)-invariant events, which is trivial, so f is constant μ_2-a.e. That is, $\mu_1 = \mu_2$.

Thus, by Theorem 10.17 and Exercise 10.2, we obtain the following consequences:

Corollary 10.18. *Let Γ be a group acting on a network G so that every vertex has an infinite orbit. Then the action is mixing and ergodic for* FSF *and for* WSF*. If* WSF *and* FSF *on G are distinct, then they are singular measures on the space* 2^{E}*.*

We do not know if the singularity assertion above holds without the hypothesis that every vertex has an infinite orbit under $\mathrm{Aut}(G)$; see Question 10.55.

§10.5. The Number of Trees.

When is the free spanning forest or the wired spanning forest of a network a.s. a single tree, as in the case of recurrent networks? The following answer for the wired spanning forest is due to Pemantle (1991).

Proposition 10.19. *Let G be any network. The wired spanning forest is a single tree a.s. iff from every (or some) vertex, random walk and independent loop-erased random walk intersect infinitely often a.s. Moreover, the chance that two given vertices x and y belong to the same tree equals the probability that random walk from x intersects independent loop-erased random walk from y.*

This is obvious from Proposition 10.1 (which wasn't available to Pemantle at the time). The case of the free spanning forest (when it differs from the wired) is largely unknown. See Theorem 10.50 for one case that is known.

How, then, do we decide whether a random walk and a loop-erased random walk intersect a.s.? Pemantle (1991) used results of Lawler (1986, 1988) in order to answer this for simple random walk in $\mathbb{Z}^d$. However, Lyons, Peres, and Schramm (2003) later showed that for any transient Markov chain, two independent paths started at any pair of states intersect infinitely often (i.o.) with probability 1 iff the loop erasure of one intersects the other i.o. with probability 1. In fact, more is true:

Theorem 10.20. *Let $\langle X_m \rangle$ and $\langle Y_n \rangle$ be independent transient Markov chains on the same state space V that have the same transition probabilities, but possibly different initial states. Then given the event that $|\{X_m\} \cap \{Y_n\}| = \infty$, almost surely $|\mathsf{LE}\langle X_m \rangle \cap \{Y_n\}| = \infty$.*

This makes it considerably easier to decide whether the wired spanning forest is a single tree. Thus:

Theorem 10.21. *Let G be any network. The wired spanning forest is a single tree a.s. iff two independent random walks started at any different states intersect with probability 1.*

And how do we decide whether two random walks intersect a.s.? We will give a useful test in the transitive case. In fact, this will work for Markov chains that may not be reversible, so we should say what we mean by transitive Markov chain:

Definition 10.22. Let $p(\bullet, \bullet)$ be a transition kernel on a state space V. Suppose that there is a group Γ of permutations of V that acts transitively (i.e., with a single orbit) and satisfies $p(\gamma x, \gamma y) = p(x, y)$ for all $\gamma \in \Gamma$ and $x, y \in \mathsf{V}$. Then we call the Markov chain with transition kernel $p(\bullet, \bullet)$ ***transitive***.

We will use $\mathbf{P}_{x_0,y_0}$ and $\mathbf{E}_{x_0,y_0}$ to denote probability and expectation when independent Markov chains $\langle X_n \rangle$ and $\langle Y_n \rangle$ start at x_0 and y_0, respectively.

In the transitive case, we may use the Green function for our test:

Theorem 10.23. *Let $p(\bullet, \bullet)$ be the transition kernel of a transient transitive Markov chain on the countable state space V. Let X and Y be two independent copies of the Markov chain with initial states x_0 and y_0. Let o be a fixed element of V. If*

$$\sum_{z \in \mathsf{V}} \mathcal{G}(o, z)^2 = \infty\,, \tag{10.16}$$

then $\mathbf{P}_{x_0,y_0}[|\{X_m\} \cap \{Y_n\}| = \infty] = \mathbf{P}_{x_0,y_0}[|\mathsf{LE}\langle X_m \rangle \cap \{Y_n\}| = \infty] = 1$, while if (10.16) fails, then $\mathbf{P}_{x_0,y_0}[|\{X_m\} \cap \{Y_n\}| < \infty] = 1$.

If we specialize still further, we obtain:

Corollary 10.24. *Let G be an infinite, locally finite, vertex-transitive graph. Denote by V_n the number of vertices in G at distance at most n from a fixed vertex o.*

(i) *If $\sup_n V_n/n^4 = \infty$, then two independent sample paths of simple random walk in G have only finitely many intersections a.s.*

(ii) *Conversely, if $\sup_n V_n/n^4 < \infty$, then two independent sample paths of simple random walk in G intersect infinitely often a.s.*

How many trees are in the wired spanning forest when there is more than one? Usually there are infinitely many a.s., but there can be only finitely many:

▷ **Exercise 10.12.**

Join two copies of the usual nearest-neighbor graph of $\mathbb{Z}^3$ by an edge at their origins. How many trees does the free uniform spanning forest have? How many does the wired uniform spanning forest have?

To give the general answer, we use the following quantity: let $\alpha(w_1, \ldots, w_K)$ be the probability that independent random walks started at $w_1, \ldots, w_K$ have no pairwise intersections.

Theorem 10.25. *Let G be a connected network. The number of trees of the* WSF *is a.s.*

$$\sup\bigl\{K\,;\ \exists w_1, \ldots, w_K \quad \alpha(w_1, \ldots, w_K) > 0\bigr\}\,. \tag{10.17}$$

Moreover, if the probability is 0 that two independent random walks from every (or some) vertex x intersect infinitely often, then the number of trees of the WSF *is a.s. infinite.*

In particular, the number of trees of the WSF is equal a.s. to a constant. The case of the free spanning forest (when it differs from the wired) is largely mysterious. In particular, we do not know whether the number of components is deterministic or random (Question 10.27). If $\mathrm{Aut}(G)$ has an infinite orbit, then the number is deterministic since the number is invariant under automorphisms and the invariant σ-field is trivial by Corollary 10.18. See Theorem 10.50 for one case that is understood completely.

The following question was suggested by O. Häggström:

Question 10.26. Let G be a transitive network. By ergodicity, the number of trees of the FSF is a.s. constant. Is it 1 or ∞ a.s.?

One motivation for that question is Theorem 7.5. A similar question is:

Question 10.27. Let G be an infinite network. Is the number of trees of the FSF a.s. constant?

Returning to the WSF, we deduce the following wonderful result of Pemantle (1991), which is stunning without the understanding of the approach using random walks:

Theorem 10.28. *The uniform spanning forest on $\mathbb{Z}^d$ has one tree a.s. for $d \le 4$ and infinitely many trees a.s. for $d \ge 5$.*

(Recall that by Corollary 10.9, FSF = WSF on $\mathbb{Z}^d$.)

We now prove all the above claims, though sometimes only in special cases. The special cases always include simple random walk on $\mathbb{Z}^d$. In particular, we prove Theorem 10.23, but not Theorem 10.20.

Nevertheless, we begin with a heuristic argument for Theorem 10.20. On the event that $X_m = Y_n$, the continuation paths $X' := \langle X_j \rangle_{j \ge m}$ and $Y' := \langle Y_k \rangle_{k \ge n}$ have the same distribution, whence the chance is at least $1/2$ that Y' intersects $L := \mathsf{LE}\langle X_0, \ldots, X_m \rangle$ at an earlier point than X' ever does, where "earlier" means in the clock of L. On this event, the earliest intersection point of Y' and L will remain in $\mathsf{LE}\langle X_j \rangle_{j \ge 0} \bigcap \{Y_k\}_{k \ge 0}$. The difficulty in making this heuristic precise lies in selecting a pair (m, n) such that $X_m = Y_n$, given that such pairs exist. The natural rules for selecting such a pair (e.g., lexicographic ordering) affect the law of at least one of the continuation paths, and invalidate the argument above; R. Pemantle (private communication, 1996) showed that this holds for *all* selection rules. Our solution to this difficulty is based on applying a second moment argument to a count of intersections. In the cases we will prove here (Theorem 10.23), we will show that there is a second moment bound for intersections of X and Y. The general case, Theorem 10.20, in fact also has a similar second moment bound, as shown by Lyons, Peres, and Schramm

(2003), but this is a little too long to prove here. We will then transfer the second moment bound for intersections of X and Y to one for intersections of $\mathsf{LE}\langle X\rangle$ and Y.

Ultimately, the second moment argument relies on the following widely used inequality. It allows one to deduce that a random variable has a reasonable chance to be large from knowing that its first moment is large.

Lemma 10.29. *Let Z be a nonnegative random variable and $0 < \epsilon < 1$. Then*

$$\mathbf{P}\Big[Z \ge \epsilon \mathbf{E}[Z]\Big] > (1-\epsilon)^2 \frac{\mathbf{E}[Z]^2}{\mathbf{E}[Z^2]} .$$

Proof. Let $\mathscr{A}$ be the event that $Z \ge \epsilon \mathbf{E}[Z]$. The Cauchy-Schwarz inequality gives

$$\mathbf{E}[Z^2]\mathbf{P}(\mathscr{A}) = \mathbf{E}[Z^2]\mathbf{E}[\mathbf{1}_{\mathscr{A}}^2] \ge \mathbf{E}[Z\mathbf{1}_{\mathscr{A}}]^2 = \big(\mathbf{E}[Z] - \mathbf{E}[Z\mathbf{1}_{\mathscr{A}^c}]\big)^2 > \big(\mathbf{E}[Z] - \epsilon\mathbf{E}[Z]\big)^2 . \quad \blacktriangleleft$$

We begin with a second moment bound on intersections of two Markov chains that start at the same state. The calculation leading to (10.19) follows Le Gall and Rosen (1991), Lemma 3.1. Denote $\mathcal{G}_N(o,x) := \sum_{m=0}^N \mathbf{P}_o[X_m = x]$. Let

$$I_N := \sum_{m=0}^N \sum_{n=0}^N \mathbf{1}_{\{X_m = Y_n\}} \tag{10.18}$$

be the number of intersections of X and Y by time N. We'll be interested in whether $\mathbf{E}[I_N] \to \infty$, and Lemma 10.29 will be used to show that $I_N \to \infty$ with reasonable probability when $\mathbf{E}[I_N] \to \infty$.

Lemma 10.30. *Let $p(\bullet,\bullet)$ be the transition kernel of a transitive Markov chain on a countable state space V. Start both Markov chains X, Y at o. Then*

$$\frac{\mathbf{E}[I_N]^2}{\mathbf{E}[I_N^2]} \ge \frac{1}{4} . \tag{10.19}$$

Proof. By transitivity,

$$\sum_{w\in\mathsf{V}} \mathcal{G}_N(z,w)^2 = \sum_{w\in\mathsf{V}} \mathcal{G}_N(o,w)^2 \tag{10.20}$$

for all $z \in \mathsf{V}$. We have

$$I_N = \sum_{z\in\mathsf{V}} \sum_{m,n=0}^N \mathbf{1}_{\{X_m = z = Y_n\}} .$$

Thus,

$$\begin{aligned}
\mathbf{E}[I_N] &= \sum_{z \in \mathsf{V}} \sum_{m=0}^{N} \sum_{n=0}^{N} \mathbf{P}[X_m = z = Y_n] \\
&= \sum_{z \in \mathsf{V}} \sum_{m=0}^{N} \mathbf{P}[X_m = z] \cdot \sum_{n=0}^{N} \mathbf{P}[Y_n = z] \\
&= \sum_{z \in \mathsf{V}} \mathcal{G}_N(o,z)^2 \,. \qquad (10.21)
\end{aligned}$$

To estimate the second moment of I_N, observe that

$$\begin{aligned}
\sum_{m,j=0}^{N} \mathbf{P}[X_m = z, X_j = w] &= \sum_{m=0}^{N} \sum_{j=m}^{N} \mathbf{P}[X_m = z]\mathbf{P}[X_j = w \mid X_m = z] \\
&\quad + \sum_{j=0}^{N} \sum_{m=j+1}^{N} \mathbf{P}[X_j = w]\mathbf{P}[X_m = z \mid X_j = w] \\
&\le \mathcal{G}_N(o,z)\,\mathcal{G}_N(z,w) + \mathcal{G}_N(o,w)\,\mathcal{G}_N(w,z) \,.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbf{E}[I_N^2] &= \sum_{z,w \in \mathsf{V}} \sum_{m,n=0}^{N} \sum_{j,k=0}^{N} \mathbf{P}[X_m = z = Y_n, X_j = w = Y_k] \\
&= \sum_{z,w \in \mathsf{V}} \sum_{m,j=0}^{N} \mathbf{P}[X_m = z, X_j = w] \cdot \sum_{n,k=0}^{N} \mathbf{P}[Y_n = z, Y_k = w] \\
&\le \sum_{z,w \in \mathsf{V}} [\mathcal{G}_N(o,z)\,\mathcal{G}_N(z,w) + \mathcal{G}_N(o,w)\,\mathcal{G}_N(w,z)]^2 \\
&\le \sum_{z,w \in \mathsf{V}} 2[\mathcal{G}_N(o,z)^2\,\mathcal{G}_N(z,w)^2 + \mathcal{G}_N(o,w)^2\,\mathcal{G}_N(w,z)^2] \\
&= 4 \sum_{z,w \in \mathsf{V}} \mathcal{G}_N(o,z)^2\,\mathcal{G}_N(z,w)^2 \,. \qquad (10.22)
\end{aligned}$$

Summing first over w and using (10.20), then (10.21), we deduce that

$$\mathbf{E}[I_N^2] \le 4\Big(\sum_{z \in \mathsf{V}} \mathcal{G}_N(o,z)^2\Big)^2 = 4\,\mathbf{E}[I_N]^2 \,. \qquad ◀$$

We now extend this to any pair of starting points when (10.16) holds.

Corollary 10.31. *Let $p(\bullet, \bullet)$ be the transition kernel of a transitive Markov chain on a countable state space* V. *Start the Markov chains X, Y at o, o'. If (10.16) holds, then*

$$\liminf_{N \to \infty} \frac{\mathbf{E}_{o,o'}[I_N]^2}{\mathbf{E}_{o,o'}[I_N^2]} \ge \frac{1}{4} . \tag{10.23}$$

Proof. Let $a_N := \mathbf{E}_{o,o}[I_N]$. As in the proof of Lemma 10.30, we have any pair o, o',

$$\begin{aligned}
\mathbf{E}_{o,o'}[I_N^2] &= \sum_{z,w \in \mathsf{V}} \sum_{m,j=0}^{N} \mathbf{P}_o[X_m = z, X_j = w] \cdot \sum_{n,k=0}^{N} \mathbf{P}_{o'}[Y_n = z, Y_k = w] \\
&\le \sum_{z,w \in \mathsf{V}} [\mathcal{G}_N(o,z)\, \mathcal{G}_N(z,w) + \mathcal{G}_N(o,w)\, \mathcal{G}_N(w,z)] \\
&\qquad \cdot [\mathcal{G}_N(o',z)\, \mathcal{G}_N(z,w) + \mathcal{G}_N(o',w)\, \mathcal{G}_N(w,z)] \\
&\le \sum_{z,w \in \mathsf{V}} [\mathcal{G}_N(o,z)^2\, \mathcal{G}_N(z,w)^2 + \mathcal{G}_N(o,w)^2\, \mathcal{G}_N(w,z)^2 \\
&\qquad + \mathcal{G}_N(o',z)^2\, \mathcal{G}_N(z,w)^2 + \mathcal{G}_N(o',w)^2\, \mathcal{G}_N(w,z)^2] \\
&= 4a_N^2 ,
\end{aligned} \tag{10.24}$$

by the elementary inequality

$$(a+b)(c+d) \le a^2 + b^2 + c^2 + d^2 .$$

We also have

$$\mathbf{E}_{o,o'}[I_N] = \sum_{m,n \le N} \sum_{z \in \mathsf{V}} p_m(o,z) p_n(o',z) = \sum_{z \in \mathsf{V}} \mathcal{G}_N(o,z) \mathcal{G}_N(o',z) \le a_N \tag{10.25}$$

by the Cauchy-Schwarz inequality, transitivity, and (10.21).

By (10.16), we have $a_N \to \infty$. Let $u_N(x,y) := \mathbf{E}_{x,y}[I_N]/a_N$. Thus, $u_N(x,x) = 1$ for all x and $u_N(x,y) \le 1$ for all x, y. Fix r such that $p_r(o,o') > 0$. Now

$$\begin{aligned}
a_{r+N} = a_{r+N} u_{r+N}(o,o) &= \sum_{m,n \le r+N} \mathbf{P}_{o,o}[X_m = Y_n] \\
&= \sum_{m,n \le N} \mathbf{P}_{o,o}[X_m = Y_{r+n}] \\
&\quad + \sum_{m \le r+N} \sum_{n < r} \mathbf{P}_{o,o}[X_m = Y_n] \\
&\quad + \sum_{1 \le m \le r} \sum_{n \le N} \mathbf{P}_{o,o}[X_{N+m} = Y_{r+n}] .
\end{aligned} \tag{10.26}$$

The Markov property and (10.25) show that the first of the 3 latter sums is equal to

$$\mathbf{E}_{o,o}[a_N u_N(o, Y_r)] \le a_N[p_r(o,o')u_N(o,o') + 1 - p_r(o,o')] .$$

The second sum in (10.26) is

$$\sum_{z \in \mathsf{V}} \mathcal{G}_{r+N}(o,z)\mathcal{G}_{r-1}(o,z) \le \sqrt{a_{r+N}a_{r-1}}$$

by the Cauchy-Schwarz inequality. The third sum in (10.26) is

$$\mathbf{E}_{o,o}\Big[\sum_{z \in \mathsf{V}} \mathcal{G}_{r-1}(X_{N+1}, z)\mathcal{G}_N(Y_r, z)\Big] \le \sqrt{a_{r-1}a_N} .$$

Substituting these bounds in (10.26) yields

$$\begin{aligned} 1 &\le (a_N/a_{r+N})[p_r(o,o')u_N(o,o') + 1 - p_r(o,o')] + \sqrt{a_{r-1}/a_{r+N}} + \sqrt{a_{r-1}a_N}/a_{r+N} \\ &\le p_r(o,o')u_N(o,o') + 1 - p_r(o,o') + o(1) \end{aligned}$$

as $N \to \infty$. This implies that $\liminf_{N\to\infty} u_N(o,o') \ge 1$. Combining this with (10.24) gives the result. ◀

We now give the crucial step that converts intersections of random walks to intersections when one of the paths is loop erased.

Lemma 10.32. *Let $p(\bullet,\bullet)$ be a transition kernel on a countable state space V that gives a transient transitive Markov chain. Fix $k \ge 0$ and a path $\langle x_j\rangle_{j=-k}^{0}$ in V. Let $\langle X_m\rangle_{m\ge 0}$ and $\langle Y_n\rangle_{n\ge 0}$ be independent Markov chains on V with initial states x_0 and y_0. Set $X_j := x_j$ for $-k \le j \le -1$. If (10.16) holds, then the probability that $\big|\mathsf{LE}\langle X_m\rangle_{m\ge -k} \cap \{Y_n\}\big| = \infty$ is at least $1/16$.*

Proof. Denote

$$\langle L_j^m\rangle_{j=0}^{J(m)} := \mathsf{LE}\langle X_{-k}, X_{-k+1}, \ldots, X_m\rangle .$$

When it happens that $X_m = Y_n$, we want to see which of the continuations of X and Y intersect $\langle L_j^m\rangle$ earlier. On the event $\{X_m = Y_n\}$, define

$$j(m,n) := \min\big\{j \ge 0 ;\ L_j^m \in \{X_m, X_{m+1}, X_{m+2}, \ldots\}\big\}, \tag{10.27}$$

$$i(m,n) := \min\big\{i \ge 0 ;\ L_i^m \in \{Y_n, Y_{n+1}, Y_{n+2}, \ldots\}\big\} . \tag{10.28}$$

Note that the sets on the right-hand sides of (10.27) and (10.28) both contain $J(m)$ if $X_m = Y_n$. Define $j(m,n) := i(m,n) := 0$ on the event $\{X_m \ne Y_n\}$. When the continuation of Y has an earlier intersection than the continuation of X does, then that intersection will be an intersection of the loop-erasure of X with Y. Thus, let $\chi(m,n) := \mathbf{1}_{\{i(m,n)\le j(m,n)\}}$. Given $\{X_m = Y_n = z\}$, the continuations $\langle X_m, X_{m+1}, X_{m+2}, \ldots\rangle$ and $\langle Y_n, Y_{n+1}, Y_{n+2}, \ldots\rangle$ are exchangeable with each other, so for every $z \in \mathsf{V}$,

$$\mathbf{E}\big[\chi(m,n) \mid X_m = Y_n = z\big] = \mathbf{P}\big[i(m,n) \le j(m,n) \mid X_m = Y_n = z\big] \ge \frac{1}{2}\,. \tag{10.29}$$

As we said, if $X_m = Y_n$ and $i(m,n) \le j(m,n)$, then $L^m_{i(m,n)}$ is in $\mathsf{LE}\langle X_r\rangle_{r=-k}^\infty \bigcap \{Y_\ell\}_{\ell=0}^\infty$.

Consider the random variables

$$\Upsilon_N := \sum_{m=0}^N \sum_{n=0}^N \mathbf{1}_{\{X_m=Y_n\}}\,\chi(m,n)$$

for $N \ge 1$. Obviously $\Upsilon_N \le I_N$ everywhere, so

$$\frac{1}{\mathbf{E}[\Upsilon_N^2]} \ge \frac{1}{\mathbf{E}[I_N^2]}\,. \tag{10.30}$$

On the other hand, by conditioning on X_m and Y_n and by applying (10.29), we see that

$$\mathbf{E}[\Upsilon_N] = \sum_{m=0}^N \sum_{n=0}^N \mathbf{E}\Big[\mathbf{1}_{\{X_m=Y_n\}}\,\mathbf{E}\big[\chi(m,n) \mid X_m, Y_n\big]\Big] \ge \frac{1}{2}\mathbf{E}[I_N]\,. \tag{10.31}$$

By Lemma 10.29, we have for every $\epsilon > 0$,

$$\mathbf{P}\Big[\Upsilon_N \ge \epsilon\mathbf{E}[\Upsilon_N]\Big] \ge (1-\epsilon)^2\frac{\mathbf{E}[\Upsilon_N]^2}{\mathbf{E}[\Upsilon_N^2]}\,.$$

By (10.30), (10.31), and Corollary 10.31, we conclude that for every $\epsilon > 0$, we have for large enough N that

$$\mathbf{P}\Big[\Upsilon_N \ge \epsilon\mathbf{E}[\Upsilon_N]\Big] \ge (1-\epsilon)^2\frac{\mathbf{E}[I_N]^2}{4\mathbf{E}[I_N^2]} \ge \frac{(1-\epsilon)^2}{16} - \epsilon\,.$$

Since $\mathbf{E}[\Upsilon_N] \to \infty$ by (10.31) and (10.16), it follows that $\Upsilon_N \to \infty$ with probability at least $1/16$. On the event that $\Upsilon_N \to \infty$, we have $\big|\mathsf{LE}\langle X_m\rangle_{m\ge -k} \cap \{Y_n\}\big| = \infty$ by the observation following (10.29). This finishes the proof. ◀

Proof of Theorem 10.23. Suppose that (10.16) holds. We will show that $\mathbf{P}\big[|\mathsf{LE}\langle X_m\rangle \cap \{Y_n\}| = \infty\big] = 1$.

Denote by Λ the event $|\mathsf{LE}\langle X_m\rangle \cap \{Y_n\}| = \infty$. By Lévy's 0-1 law, we have that $\lim_{n\to\infty} \mathbf{P}_{x_0,y_0}(\Lambda \mid X_1, \ldots, X_n, Y_1, \ldots, Y_n) = \mathbf{1}_\Lambda$ a.s. On the other hand,

$$\mathbf{P}_{x_0,y_0}(\Lambda \mid X_1, \ldots, X_n, Y_1, \ldots, Y_n) = \mathbf{P}_{x_0,Y_n}(\Lambda \mid X_1, \ldots, X_n) \ge 1/16$$

by the Markov property and by Lemma 10.32. Thus, $\mathbf{1}_\Lambda \ge 1/16$ a.s., which means that $\mathbf{P}_{x_0,y_0}(\Lambda) = 1$, as desired.

Now suppose that (10.16) fails. Then the expected number of pairs (m,n) such that $X_m = Y_n$ is (by the monotone convergence theorem) $\lim_N \mathbf{E}[I_N] = \sum_z \mathcal{G}(o,z)^2 < \infty$. ◀

▷ **Exercise 10.13.**

(Parseval's Identity) Suppose that $F \in L^1(\mathbb{T}^d)$ and that $f(x) := \int_{\mathbb{T}^d} F(\alpha)e^{2\pi i x\cdot\alpha}\,d\alpha$ for $x \in \mathbb{Z}^d$. Show that $\int_{\mathbb{T}^d} |F(\alpha)|^2\,d\alpha = \sum_{\mathbb{Z}^d} |f(x)|^2$. *Hint:* Prove that the functions $G_x(\alpha) := e^{-2\pi i x\cdot\alpha}$ for $x \in \mathbb{Z}^d$ form an orthonormal basis of $L^2(\mathbb{T}^d)$.

Proof of Corollary 10.24. For $\mathbb{Z}^d$, the result is easy: We need treat only the transient case. By Proposition 2.1, (10.16) is equivalent to the voltage function v of Proposition 10.15 not being square summable. By Exercise 10.13, this in turn is equivalent to $1/\varphi \notin L^2(\mathbb{T}^d)$, where φ is defined in (10.7). By (10.8), this holds iff $d \le 4$.

The general statement requires certain facts that are beyond the scope of this book, but otherwise is not hard: For independent simple random walks, reversibility and regularity of G imply that

$$\begin{aligned}\sum_z \mathcal{G}(o,z)^2 &= \lim_{n\to\infty} \mathbf{E}_{o,o}[I_N] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathbf{P}_{o,o}[X_m = Y_n] \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathbf{P}_o[X_{m+n} = o] = \sum_{n=0}^{\infty} (n+1)\mathbf{P}_o[X_n = o]\,. \end{aligned} \tag{10.32}$$

(i) The assumption that $\sup_n V_n/n^4 = \infty$ implies that $V_n \ge cn^5$ for some $c > 0$ and all n: see Theorem 5.11 in Woess (2000). Corollary 14.5 in the same reference yields $\mathbf{P}_x[X_n = x] \le Cn^{-5/2}$. Thus the sum in (10.32) converges. (In particular, simple random walk is transient.)

(ii) Combining the results (14.5), (14.12) and (14.19) in Woess (2000), we infer that the assumption $V_n = O(n^4)$ implies that $P_x[X_{2n} = x] \ge cn^{-2}$ for some $c > 0$ and all $n \ge 1$. Thus the series (10.32) diverges.

Thus, both assertions follow from Theorem 10.23. ◀

Proof of Theorem 10.25. Of course, the recurrent case is a consequence of our work in Chapter 4, so we restrict ourselves to the transient case.

Let $\langle X_n(u)\,;\ n \ge 0\rangle$ be a collection of independent random walks starting at u for $u \in \mathsf{V}$. Denote the event that $X_n(w_i) \ne X_m(w_k)$ for all $i \ne k$ and $n, m \ge j$ by $\mathscr{B}_j(w_1, \ldots, w_K)$. Thus, $\alpha(w_1, \ldots, w_K) = \mathbf{P}[\mathscr{B}_0(w_1, \ldots, w_K)]$ and $\mathscr{B}(w_1, \ldots, w_K) := \bigcup_j \mathscr{B}_j(w_1, \ldots, w_K)$ is the event that there are only finitely many pairwise intersections among the random walks starting at $w_1, \ldots, w_K$. For every $j \ge 0$, we have

$$\begin{aligned}\liminf_{n\to\infty} \alpha\big(X_n(w_1), \ldots, X_n(w_K)\big) &\ge \lim_{n\to\infty} \mathbf{P}\big[\mathscr{B}_j(w_1, \ldots, w_K) \bigm| \langle X_m(w_i)\,;\ m \le n,\ i \le K\rangle\big] \\ &= \mathbf{1}_{\mathscr{B}_j(w_1,\ldots,w_K)} \text{ a.s.}\end{aligned}$$

by Lévy's 0-1 law. It follows that

$$\liminf_{n\to\infty} \alpha\big(X_n(w_1), \ldots, X_n(w_K)\big) \ge \mathbf{1}_{\mathscr{B}(w_1,\ldots,w_K)} \text{ a.s.} \tag{10.33}$$

First suppose that $\alpha(w_1, \ldots, w_K) > 0$ for some $w_1, \ldots, w_K$. Then by (10.33), for every $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $\alpha\big(X_n(w_1), \ldots, X_n(w_K)\big) > 1 - \epsilon$ with positive probability. In particular, there are $w_1', \ldots, w_K'$ such that $\alpha(w_1', \ldots, w_K') > 1 - \epsilon$. Using Wilson's method rooted at infinity starting with the vertices $w_1', \ldots, w_K'$, this implies that with probability greater than $1 - \epsilon$, the number of trees for WSF is at least K. As $\epsilon > 0$ was arbitrary, this implies that the number of trees is WSF-a.s. at least (10.17).

For the converse, suppose that $\alpha(w_1, \ldots, w_K) = 0$ for all $w_1, \ldots, w_K$. By (10.33), we find that there exist a.s. $i \ne j$ with infinitely many intersections between $\langle X_n(w_i)\rangle$ and $\langle X_n(w_j)\rangle$, whence also between $\mathsf{LE}\langle X_n(w_i)\rangle$ and $\langle X_n(w_j)\rangle$ by Theorem 10.20 in general and by Theorem 10.23 in the transitive case. By Wilson's method rooted at infinity, the probability that all $w_1, \ldots, w_K$ belong to different trees is 0. Since this holds for all $w_1, \ldots, w_K$, it follows that the number of trees is WSF-a.s. at most (10.17).

Moreover, if the probability is zero that two independent random walks X^1, X^2 intersect i.o. starting at some $w \in \mathsf{V}$, then $\lim_{n\to\infty} \alpha\big(X_n^1, X_n^2\big) = 1$ a.s. by (10.33). Therefore

$$\lim_{n\to\infty} \alpha\big(X_n^1, \ldots, X_n^k\big) = 1$$

a.s. for any independent random walks $X^1, \ldots, X^k$. This implies that the number of components of WSF is a.s. infinite. ◀

§10.6. The Size of the Trees.

In the preceding section, we discovered how many trees there are in the wired spanning forest. Now we ask how big these trees are. Of course, there is more than one way to define "big". The most obvious probabilistic notion of "big" is "transient". In this sense, all the trees are small:

Theorem 10.33. (Morris, 2003) *Let (G, c) be a network. For WSF-a.e. $\mathfrak{F}$, all trees T in $\mathfrak{F}$ have the property that the wired spanning forest of (T, c) equals T. In particular, if $c(\bullet)$ is bounded, then (T, c) is recurrent.*

We will use the following observation:

Lemma 10.34. *Let (G, c) be a transient network and $\langle e_n \rangle$ an enumeration of its edges. Write G_n for the network obtained by contracting the edges $e_1, e_2, \ldots, e_n$. For any vertex x, we have $\lim_{n\to\infty} \mathscr{R}(x \leftrightarrow \infty; G_n) = 0$.*

Proof. Let θ be a unit flow in G from x to ∞ of finite energy. The restriction of θ to $\langle e_k\,;\ k > n \rangle$ is a unit flow in G_n from x to infinity with energy tending to 0 as $n \to \infty$. Since the energy of any unit flow is an upper bound on the effective resistance, the result follows. ◀

We will also use the result of the following exercise:

▷ **Exercise 10.14.**

Let (T, c) be a transient network on a tree with bounded conductances $c(\bullet)$. Show that there is some $e \in T$ such that both components of $T\backslash e$ are transient (with respect to c).

Proof of Theorem 10.33. Consider any edge e of G. Let $\mathscr{A}_e$ be the event that both endpoints of e lie in transient components of $\mathfrak{F}\backslash e$ (with respect to the conductances $c(\bullet)$). Enumerate the edges of $G\backslash e$ as $\langle e_n \rangle$. Let $\mathscr{F}_n$ be the σ-field generated by the events $\{e_k \in \mathfrak{F}\}$ for $k < n$, where $n \le \infty$. Recall that conditioning on an edge being in the spanning forest is equivalent to contracting the edge from the original network, while conditioning on the edge being absent is equation to deleting the edge from the original network. Thus, let G_n be the random network obtained from G by contracting e_k when $e_k \in \mathfrak{F}$ and deleting e_k otherwise, where we do this for all $k < n$. Let i_n be the wired unit current flow in G_n from the tail to the head of e. Then $\mathsf{WSF}[e \in \mathfrak{F} \mid \mathscr{F}_n] = i_n(e)$ by (10.3). This is $c(e)$ times the wired effective resistance between e^- and e^+ in G_n. By Lemma 10.34 and Exercise 9.25, this tends to 0 on the event $\mathscr{A}_e$. By Lévy's martingale convergence theorem, we obtain $\mathsf{WSF}[e \in \mathfrak{F} \mid \mathscr{F}_\infty]\mathbf{1}_{\mathscr{A}_e} = 0$ a.s. Taking the expectation of this equation gives $\mathsf{WSF}[e \in \mathfrak{F}, \mathscr{A}_e] = 0$ since $\mathscr{A}_e$ is $\mathscr{F}_\infty$-measurable.

Let $\mathfrak{G}$ be the union of the wired spanning forests of $\mathfrak{F}$, where each forest is generated independently of the others given $\mathfrak{F}$. For $e \in \mathfrak{F}$, let T_e be the component of $\mathfrak{F}$ containing e. If $e \in \mathfrak{F}$ and $\mathscr{A}_e$ does not hold, then Wilson's method rooted at infinity shows that e is in the wired spanning forest of T_e. Thus, by what we have shown, e is in the wired spanning forest of T_e a.s. given that $e \in \mathfrak{F}$: $\mathbf{P}[e \in \mathfrak{F}\backslash\mathfrak{G}] = 0$. Since there are only countably many edges, this holds for all e at once, i.e., $\mathfrak{F} = \mathfrak{G}$ a.s., which proves the first part of the theorem. The last part follows from Exercise 10.14. ◀

We now look at another notion of size of trees. Call an infinite path in a tree that starts at any vertex and does not backtrack a ***ray***. Call two rays ***equivalent*** if they have infinitely many vertices in common. An equivalence class of rays is called an ***end***. How many ends do the trees of a uniform spanning forest have?

Let's begin by thinking about the case of the wired uniform spanning forest on a regular tree of degree $d+1$. Choose a vertex, o. Begin Wilson's method rooted at infinity from o. We obtain a ray ξ from o to start our forest. Now o has d neighbors not in ξ, $x_1, \ldots, x_d$. By beginning random walks at each of them in turn, we see that the events $\mathscr{A}_i := \{x_i \text{ connected to } o\}$ are independent. Furthermore, the resistance from x_i to ∞ in the descendant subtree of x_i (we think of o as the parent of x_i) is $1/d + 1/d^2 + 1/d^3 + \cdots = 1/(d-1)$, whence the probability that random walk started at x_i ever hits o is $1/d$. This is the probability of $\mathscr{A}_i$. On the event $\mathscr{A}_i$, we add only the edge $[o, x_i]$ to the forest and then we repeat the analysis from x_i. Thus, the tree containing o includes, apart from the ray ξ, a critical Galton-Watson tree with offspring distribution Binomial$(d, 1/d)$. In addition, each vertex on ξ has another random subtree attached to it; its first generation has distribution Binomial$(d-1, 1/d)$, but subsequent generations yield Galton-Watson trees with distribution Binomial$(d, 1/d)$. In particular, a.s. every tree added to ξ is finite. This means that the tree containing o has only one end, the equivalence class of ξ. This analysis is easily extended to form a complete description of the entire wired spanning forest. In fact, if we work on the d-ary tree instead of the $(d+1)$-regular tree, the description is slightly simpler: each tree in the forest is a size-biased Galton-Watson tree grown from its lowest vertex, generated from the offspring distribution Binomial$(d, 1/d)$ (see Section 12.1). The resulting description is due to Häggström (1998), whose work predates Wilson's algorithm.

Now in general, we see that if we begin Wilson's algorithm at a vertex o in a graph, it immediately generates one end of the tree containing o. In order for this tree to have more than one end, however, we need a succession of "coincidences" to occur, building up other ends by gradually adding on finite pieces. This is possible (see Exercises 10.15, 10.38, or 10.39), but it suggests that usually, the wired spanning forest has trees with only one end each. Indeed, we will show that this is very often the case.

▷ **Exercise 10.15.**

Give a transient graph such that the wired uniform spanning forest has a single tree with more than one end.

First, we consider the planar recurrent case, which has an amazingly simple solution:

Theorem 10.35. *Suppose that G is a simple plane recurrent network and $G^\dagger$ its plane dual. Assume that $G^\dagger$ is locally finite and recurrent. Then the uniform spanning tree on G has only one end a.s.*

Proof. Because both networks are recurrent, their free and wired spanning forests coincide and are a single tree a.s. We observed in Section 10.3 that the uniform spanning tree T of G is "dual" to that of $G^\dagger$. If T had at least two ends, the bi-infinite path in T joining two of them would separate $T^\times$ into at least two trees. Since we know that $T^\times$ is a single tree, this is impossible. ◀

In particular, we have verified the claim in Section 4.3 that the maze in $\mathbb{Z}^2$ is connected and has exactly one way to get to infinity from any square.

Similar reasoning shows:

Proposition 10.36. *Suppose that G is a simple plane network and $G^\dagger$ its plane dual. Assume that $G^\dagger$ is locally finite. If each tree of the* WSF *of G has only one end a.s., then the* FSF *of $G^\dagger$ has only one tree a.s. If, in addition, the* WSF *of G has infinitely many trees a.s., then the tree of the* FSF *of $G^\dagger$ has infinitely many ends a.s.*

This is illustrated for the Cayley graph of Figure 6.1 and its dual in Figure 10.2.

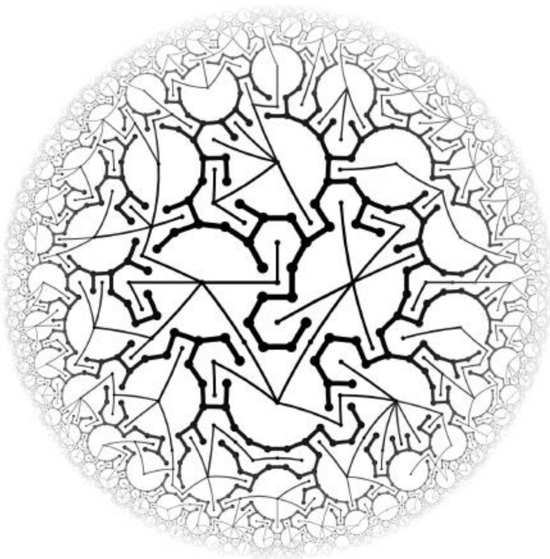


Figure 10.2. The FSF of a Cayley graph in the hyperbolic disc, which is a tree, and its thinner dual WSF, each of whose trees has one end.

By Proposition 10.10 and Theorem 8.17, it follows that if G is a transient transitive unimodular network, almost surely each tree of the wired spanning forest has one or two ends. In fact, each tree of the WSF has only one end in every quasi-transitive transient network, as well as in a host of other natural networks. The first result of this kind was due to Pemantle (1991), who proved it for $\mathbb{Z}^d$ with $d = 3, 4$ and also showed that there are at most two ends per tree for $d \geq 5$. BLPS (2001) extended and completed this for all unimodular transitive networks, showing that each tree has only one end. This was then extended to all quasi-transitive transient networks and more by Lyons, Morris, and Schramm (2008), who found a simpler method of proof that worked in greater generality. This is the proof we present here.

Lemma 10.37. *Let $\mathscr{A}$ be an event of positive probability in a probability space and $\mathscr{F}$ be a σ-field. Then $\mathbf{P}(\mathscr{A} \mid \mathscr{F}) > 0$ a.s. given $\mathscr{A}$.*

Proof. Let $\mathscr{B}$ be the event where $\mathbf{P}(\mathscr{A} \mid \mathscr{F}) = 0$. We want to show that $\mathbf{P}(\mathscr{B} \cap \mathscr{A}) = 0$. Since $\mathscr{B} \in \mathscr{F}$, we have by definition of conditional probability that

$$\mathbf{P}(\mathscr{B} \cap \mathscr{A}) = \mathbf{E}[\mathbf{1}_{\mathscr{B}} \mathbf{P}(\mathscr{A} \mid \mathscr{F})] = 0 .$$ ◀

Lemma 10.38. *Let $B(n)$ be a box in $\mathbb{Z}^d$ whose sides are parallel to the axes and of length n. Then*

$$\lim_{N \to \infty} \inf \left\{ \mathscr{C}(K \leftrightarrow \infty; \mathbb{Z}^d \backslash B(n)) ;\ K \subset \mathbb{Z}^d \backslash B(n),\ |K| = N,\ n \ge 0 \right\} = \infty .$$

Proof. Since the effective conductance is monotone increasing in K, it suffices to bound it from below when all points in K lie on one side of $B(n)$ and are at pairwise distance at least N. Since the effective conductance is also monotone increasing in the rest of the network (i.e., Rayleigh's monotonicity principle), it suffices to bound from below the conductance from K to ∞ in $G := \mathbb{N} \times \mathbb{Z}^{d-1}$. Let θ be a unit flow of finite energy from the origin to ∞ in G. Write θ_x for the image of θ under the translation of the origin to $x \in G$. Then $\theta_K := \sum_{x \in K} \theta_x / |K|$ is a unit flow from K to ∞, so it suffices to bound its energy from above. We claim that the inner product (θ_x, θ_y) is small when x and y are far from each other. Indeed, given $\epsilon > 0$, choose a finite set $F \subset \mathsf{E}(G)$ such that $\sum_{e \notin F} \theta(e)^2 < \epsilon^2$. Write $\theta_x = \theta_x^{(1)} + \theta_x^{(2)}$, where $\theta_x^{(1)} := \theta_x \restriction (F + x)$ and $\theta_x^{(2)} := \theta_x \restriction (F + x)^c$. Then if the distance between x and y is so large that $(F + x) \cap (F + y) = \varnothing$, we have

$$(\theta_x, \theta_y) = (\theta_x^{(1)}, \theta_y^{(2)}) + (\theta_x^{(2)}, \theta_y^{(1)}) + (\theta_x^{(2)}, \theta_y^{(2)}) \le 2\epsilon \mathscr{E}(\theta)^{1/2} + \epsilon^2$$

by the Cauchy-Schwarz inequality. Therefore, we have

$$\mathscr{E}(\theta_K) = \sum_{x \in K} \mathscr{E}(\theta_x / |K|) + \sum_{x \ne y \in K} (\theta_x, \theta_y) / |K|^2 = \mathscr{E}(\theta)/|K| + o(1) = o(1) ,$$

as desired. ◀

Lemma 10.39. *Let G be a transient network and $F \subset \mathsf{E}(G)$ be such that $G\backslash F$ has no finite components. For $e \in F$, write $r_e := \mathscr{R}^{\mathrm{W}}(e^- \leftrightarrow e^+; G\backslash F)$. Then*

$$\mathsf{WSF}[F \cap \mathfrak{F} = \varnothing] \ge \prod_{e \in F} \frac{1}{1 + c(e) r_e}\,.$$

Proof. It suffices to prove the analogous statement for finite networks, so we write T for the random spanning tree, rather then $\mathfrak{F}$. For $e \in F$, we have

$$\mathbf{P}[e \in T] = c(e)\mathscr{R}(e^- \leftrightarrow e^+; G) = \frac{c(e)}{c(e) + \mathscr{C}(e^- \leftrightarrow e^+; G\backslash e)} \le \frac{c(e)}{c(e) + \mathscr{C}(e^- \leftrightarrow e^+; G\backslash F)}$$

by Corollary 4.4 and Rayleigh's monotonicity principle. Thus,

$$\mathbf{P}[e \notin T] \ge \frac{1}{1 + c(e) r_e}\,.$$

If we order F and use this bound one at a time, conditioning each time that the prior edges of F are not in T and deleting them from G, we get the desired bound. ◀

Define the ***internal vertex boundary*** of a set K as

$$\partial_{\mathsf{V}}^{\mathrm{int}} K := \{x \in K\,;\, \exists y \notin K \;\; y \sim x\}\,.$$

Theorem 10.40. *In $\mathbb{Z}^d$ with $d \ge 2$, WSF-a.s. every tree has only one end.*

Proof. The case $d = 2$ is part of Theorem 10.35, so assume that $d \ge 3$ and, thus, that the graph is transient by Pólya's theorem. Let $\mathscr{A}_e$ be the event that $\mathfrak{F}\backslash e$ has a finite component. We will show that every edge $e \in \mathsf{E}(G)$ has the property that $\mathsf{WSF}[\mathscr{A}_e \mid e \in \mathfrak{F}] = 1$. Fix e. Let $\langle G_n \rangle$ be an exhaustion by boxes that contain e. Let $\mathscr{F}_n$ be the σ-field generated by the events $\{f \in \mathfrak{F}\}$ for $f \in \mathsf{E}(G_n) \setminus \{e\}$. Let G'_n be $\mathbb{Z}^d$ after deleting the edges of $G_n \setminus (\mathfrak{F} \cup \{e\})$ and contracting each edge of $G_n \cap (\mathfrak{F} \setminus \{e\})$. Note that $e \in G'_n$. As in the proof of Theorem 10.33,

$$\mathsf{WSF}[e \in \mathfrak{F} \mid \mathscr{F}_n] = \mathscr{R}^{\mathrm{W}}(e^- \leftrightarrow e^+; G'_n) = \frac{1}{1 + \mathscr{C}^{\mathrm{W}}(e^- \leftrightarrow e^+; G'_n\backslash e)}\,.$$

Since $\mathsf{WSF}[e \in \mathfrak{F} \mid \mathscr{F}_n]$ has a non-0 limit a.s. given $e \in \mathfrak{F}$ by the martingale convergence theorem and Lemma 10.37, it follows that $\mathscr{C}^{\mathrm{W}}(e^- \leftrightarrow e^+; G'_n\backslash e)$ is bounded a.s. given $e \in \mathfrak{F}$. Let T be the component of $\mathfrak{F}$ that contains e when $e \in \mathfrak{F}$ and let $T := \varnothing$ otherwise. By combining the previous bound on the wired effective conductance with Lemma 10.38 and Exercise 9.25, it follows that at least one of the two components of $T\backslash e$, given that $e \in \mathfrak{F}$, has at most N vertices on the internal vertex boundary of G_n for some finite random N

that does not depend on n. Suppose that this is the component T' of e^-, so that e^- has degree at most $2dN$ in $G'_n \backslash e$. By Lemma 10.39, it follows that on the event $e \in \mathfrak{F}$, the conditional probability that $T' = T' \cap G_n$ is the component of e^- in $\mathfrak{F} \backslash e$, given $\mathscr{F}_n$, is at least ϵ^{2dN} for some $\epsilon > 0$ and all large n. In particular, $\mathsf{WSF}[\mathscr{A}_e \mid \mathscr{F}_n] \ge \epsilon^{2dN}$ on the event that $e \in \mathfrak{F}$. Since $\mathscr{A}_e$ lies in the σ-field generated by $\bigcup \mathscr{F}_n$, the limit of these conditional probabilities as $n \to \infty$ is a.s. the indicator of $\mathscr{A}_e$, whence $\mathbf{1}_{\mathscr{A}_e} \ge \epsilon^{2dN} \mathbf{1}_{\{e \in \mathfrak{F}\}}$ a.s., whence $\mathbf{1}_{\mathscr{A}_e} \ge \mathbf{1}_{\{e \in \mathfrak{F}\}}$ a.s., as desired. ◀

We now prove the same for networks with a "reasonable expansion profile". Recall that $|F|_c := \sum_{e \in F} c(e)$ for $F \subseteq \mathsf{E}$ and $|K|_\pi := \sum_{x \in K} \pi(x)$ for $K \subseteq \mathsf{V}$. Write

$$\psi(G, t) := \inf \left\{ |\partial_{\mathsf{E}} K|_c \,;\ t \le |K|_\pi < \infty \right\}.$$

Lemma 10.41. *Let G be an infinite connected network such that $\psi(0^+) > 0$. Then G has an exhaustion $\langle G_n \rangle$ by connected subgraphs with $|\mathsf{V}(G_n)|_\pi < \infty$ such that*

$$|\partial_{\mathsf{E}} U \setminus \partial_{\mathsf{E}} \mathsf{V}(G_n)|_c \ge |\partial_{\mathsf{E}} U|_c / 2 \tag{10.34}$$

for all n and all finite $U \subset \mathsf{V}(G) \setminus \mathsf{V}(G_n)$ and

$$\psi\big(G \backslash \mathsf{V}(G_n), t\big) \ge \psi(G, t)/2 \tag{10.35}$$

for all n and $t > 0$.

Proof. Given $K \subset \mathsf{V}(G)$ such that its induced subgraph $G \restriction K$ is connected, let $W(K)$ minimize $|\partial_{\mathsf{E}} L|_c$ over all sets L with $|L|_\pi < \infty$ that contain $K \cup \partial_{\mathsf{V}} K$; such a set $W(K)$ exists by our assumption on G. Furthermore, its induced subgraph $G \restriction W(K)$ is connected. Let $G' := G \backslash W(K)$ and write ∂'_{E} for the edge-boundary operator in G'. If U is a finite subset of vertices in G', then $|\partial'_{\mathsf{E}} U|_c \ge |\partial_{\mathsf{E}} U|_c / 2$, which is the same as $|\partial_{\mathsf{E}} U \setminus \partial_{\mathsf{E}} W(K)|_c \ge |\partial_{\mathsf{E}} U|_c / 2$, since if not, $W(K) \cup U$ would have a smaller edge boundary but larger size than $W(K)$, contradicting the definition of $W(K)$. Thus, $\psi(G', t) \ge \psi(G, t)/2$ for all $t > 0$. It follows that we may define an exhaustion having the desired properties inductively by $K_1 := W\big(\{o\}\big)$ and $K_{n+1} := W(K_n)$, where o is a fixed vertex of G, and $G_n := G \restriction K_n$. ◀

Theorem 10.42. *Suppose that G is an infinite network. Let $\psi(t) := \psi(G, t)$. Define $s_1 := \psi(0^+)/2$ and $s_{k+1} := s_k + \psi(s_k)/2$ inductively for $k \ge 1$. Suppose that $\psi(0^+) > 0$ and*

$$\sum_{k \ge 1} \frac{1}{\psi(s_k)} < \infty \,.$$

Then WSF*-a.s. every tree has only one end.*

Proof. Let $\mathscr{A}_e$ be the event that $\mathfrak{F}\backslash e$ has a finite component. We will show that every edge $e \in \mathsf{E}(G)$ has the property that $\mathsf{WSF}[\mathscr{A}_e \mid e \in \mathfrak{F}] = 1$. Let $\langle G_n \rangle$ be an exhaustion by finite connected subgraphs that contain e and satisfy (10.34) and (10.35). Let G'_n be G after deleting $\mathsf{E}(G_n) \setminus (\mathfrak{F} \cup \{e\})$, contracting $\mathsf{E}(G_n) \cap (\mathfrak{F} \setminus \{e\})$, and removing any resulting loops. Note that $e \in G'_n$. Let $H_n := G'_n \backslash e$ and write π_n for the corresponding vertex weights in H_n. As in the proof of Theorem 10.40,

$$\mathsf{WSF}[e \in \mathfrak{F} \mid G'_n] = c(e)\mathscr{R}^{\mathrm{W}}(e^- \leftrightarrow e^+; G'_n) = \frac{c(e)}{c(e) + \mathscr{C}^{\mathrm{W}}(e^- \leftrightarrow e^+; H_n)}\,.$$

Since $\mathsf{WSF}[e \in \mathfrak{F} \mid G'_n]$ has a non-0 limit a.s. given $e \in \mathfrak{F}$ by the martingale convergence theorem and Lemma 10.37, it follows that

$$\sup_n \mathscr{C}^{\mathrm{W}}(e^- \leftrightarrow e^+; H_n) < \infty \text{ a.s.} \tag{10.36}$$

given $e \in \mathfrak{F}$.

If either of the endpoints of e is isolated in H_n, then $\mathscr{A}_e$ occurs. If not, then H_n is obtained from $L_n := G \backslash \mathsf{V}(G_n)$ by adding new vertices and edges. Write $\pi'_n(x)$ for the sum of edge weights incident to x in L_n. We claim that for some fixed $R < \infty$, all $n \ge 1$, and all $x, y \in \mathsf{V}(L_n)$, we have

$$\mathscr{R}^{\mathrm{W}}(x \leftrightarrow y; L_n) \le R\,. \tag{10.37}$$

To see this, given $x \in \mathsf{V}(L_n)$, define $r_0 := \pi'_n(x)$ and $r_{k+1} := r_k + \psi(L_n, x, r_k)/2$ inductively. Here, as in (6.18),

$$\psi(L_n, x, t) := \inf\left\{|\partial_{\mathsf{E}} K|_c\,;\ x \in K \subset L_n,\ K \text{ is connected, } t \le |K|_\pi < \infty\right\}.$$

We claim that $r_{2k} \ge s_k$ for all k. We prove this by induction on k. Obviously $r_0 \ge s_0$. Assume that $r_{2k} \ge s_k$ for some $k \ge 0$. By (10.35), we have

$$r_{2k+2} \ge r_{2k+1} + \psi(G, r_{2k+1})/4 \ge r_{2k} + \psi(G, r_{2k})/4 + \psi(G, r_{2k})/4 \ge s_k + \psi(G, s_k)/2 = s_{k+1}\,,$$

which completes the proof. Since $r_{2k+1} \ge s_k$ as well, the bound of Theorem 6.18 (which applies since $\psi(0^+) > 0$ implies $|\mathsf{V}(G)|_\pi = \infty$) shows that

$$\mathscr{R}(x \leftrightarrow \infty; L_n) \le \sum_k \frac{2}{\psi(L_n, x, r_k)} \le \sum_k \frac{4}{\psi(G, r_k)} \le \sum_k \frac{8}{\psi(G, s_k)}\,.$$

In combination with Exercise 9.25, this proves (10.37) with $R = \sum_k 16/\psi(G, s_k)$. In addition, the same proof shows that if it happens that $\pi'_n(x) \ge s_m$ for some m, then $\mathscr{R}(x \leftrightarrow \infty; L_n) \le \sum_{k \ge m} 8/\psi(G, s_k)$. This tends to 0 as $m \to \infty$.

Let x_n and y_n be the endpoints of e in H_n. Consider now $\mathscr{R}^{\mathrm{W}}(x_n \leftrightarrow \infty; H_n) \le \mathscr{R}^{\mathrm{W}}(x_n \leftrightarrow \infty; J_n)$, where J_n is the network formed by adding to L_n the vertex x_n together with the edges joining it to its neighbors in H_n. By (10.34), $\psi(J_n, x_n, t) = \pi_n(x_n)$ for $t = \pi_n(x_n)$. We claim that

$$\psi(J_n, x_n, t) \ge \psi(L_n, t/3) \tag{10.38}$$

for all $t \ge (3/2)\pi_n(x_n)$. Let ∂'_{E} denote the edge-boundary operator in L_n. Let π''_n denote the vertex weights in J_n. Consider a finite connected set K of vertices in J_n that strictly contains x_n and with $|K|_{\pi''_n} \ge (3/2)\pi_n(x_n)$. Let $K' := K \setminus \{x_n\}$. Then $|K'|_{\pi'_n} \ge |K|_{\pi''_n}/3$, while $|\partial'_{\mathsf{E}} K'|_c \le |\partial_{\mathsf{E}} K|_c$. This proves the claim. An argument similar to the above shows, therefore, that $\mathscr{R}(x_n \leftrightarrow \infty; J_n)$ is small if $\pi_n(x_n)$ is large because the first two terms in the series of Theorem 6.18 are $2/\pi_n(x_n)$ and $2/\psi\big(J_n, x_n, (3/2)\pi_n(x_n)\big)$, and the latter is at most $2/\psi\big(L_n, \pi_n(x_n)/2\big)$ by (10.38). The same holds for y_n. If both resistances were small, then $\mathscr{R}^{\mathrm{W}}(x_n \leftrightarrow y_n; H_n)$ would also be small by Exercise 9.25, which would contradict (10.36). Therefore, $\langle \min\{\pi_n(x_n), \pi_n(y_n)\}\rangle$ is bounded a.s. For simplicity, let's say that $\langle \pi_n(x_n)\rangle$ is bounded a.s.

Form G''_n from G'_n by contracting the edges incident to y_n (other than e) that lie in $\mathfrak{F}$ and by deleting the others. If none lie in $\mathfrak{F}$, then $\mathscr{A}_e$ occurs, so assume at least one does belong to $\mathfrak{F}$. If F denotes the set of edges of H_n incident to x_n, then by Lemma 10.39, (10.37), and Rayleigh's monotonicity principle, we have

$$\mathsf{WSF}[F \cap \mathfrak{F} = \varnothing \mid G''_n] \ge \exp\Big\{-\sum_{f \in F} c(f)\big(r(e) + R\big)\Big\} = \exp\Big\{-\pi_n(x_n)\big(r(e) + R\big)\Big\}$$

if neither endpoint of e is isolated in H_n. This is bounded away from 0. In particular, $\mathsf{WSF}[\mathscr{A}_e \mid G''_n]$ is bounded away from 0 a.s. on the event that $e \in \mathfrak{F}$. Since the limit of these conditional probabilities as $n \to \infty$ is a.s. the indicator of $\mathscr{A}_e$, the proof is completed as before. ◀

Which graphs satisfy the hypothesis? Of course, all non-amenable graphs do. Although it is not obvious, so do all quasi-transitive transient graphs. To show this, we begin with the following lemma due to Coulhon and Saloff-Coste (1993), Saloff-Coste (1995), and O. Schramm (personal communication, 2005). It extends Theorem 6.20.

Lemma 10.43. *Let G be a unimodular transitive graph. Let $\rho(m)$ be the smallest radius* *of a ball in G that contains at least m vertices. Then for all finite $K \subset \mathsf{V}$, we have*

$$\frac{|\partial_{\mathsf{V}}^{\mathrm{int}} K|}{|K|} \ge \frac{1}{2\rho(2|K|)}\,.$$

Proof. Fix a finite set K and let $\rho := \rho(2|K|)$. Let $B'(x,r)$ be the ball of radius r about x except for x itself and let $b := |B'(x,\rho)|$. For $x, y, z \in \mathsf{V}(G)$, define $f_k(x,y,z)$ as the proportion of geodesic paths from x to z whose kth vertex is y. Let $S(x,r)$ be the sphere of radius r about x. Write $q_r := |S(x,r)|$. Let $F_{r,k}(x,y) := \sum_{z \in S(x,r)} f_k(x,y,z)$. Clearly, $\sum_y F_{r,k}(x,y) = q_r$ for every $x \in \mathsf{V}(G)$ and $r \ge 1$. Since $F_{r,k}$ is invariant under the diagonal action of the automorphism group of G, the Mass-Transport Principle gives $\sum_x F_{r,k}(x,y) = q_r$ for every $y \in \mathsf{V}(G)$ and $r \ge 1$. Now we consider the sum

$$Z_r := \sum_{x \in K} \sum_{z \in S(x,r) \setminus K} \sum_{y \in \partial_{\mathsf{V}}^{\mathrm{int}} K} \sum_{k=0}^{r-1} f_k(x,y,z)\,.$$

If we fix $x \in K$ and $z \in S(x,r) \setminus K$, then the inner double sum is at least 1, since if we fix any geodesic path from x to z, it must pass through $\partial_{\mathsf{V}}^{\mathrm{int}} K$. It follows that

$$Z_r \ge \sum_{x \in K} |S(x,r) \setminus K|\,,$$

whence

$$Z := \sum_{r=1}^{\rho} Z_r \ge \sum_{x \in K} |B'(x,\rho) \setminus K| \ge \sum_{x \in K} |B'(x,\rho)|/2 = |K|b/2\,.$$

On the other hand, if we do the summation in another order, we find

$$\begin{aligned} Z_r &= \sum_{y \in \partial_{\mathsf{V}}^{\mathrm{int}} K} \sum_{k=0}^{r-1} \sum_{x \in K} \sum_{z \in S(x,r) \setminus K} f_k(x,y,z) \\ &\le \sum_{y \in \partial_{\mathsf{V}}^{\mathrm{int}} K} \sum_{k=0}^{r-1} \sum_{x \in \mathsf{V}(G)} \sum_{z \in S(x,r)} f_k(x,y,z) \\ &= \sum_{y \in \partial_{\mathsf{V}}^{\mathrm{int}} K} \sum_{k=0}^{r-1} \sum_{x \in \mathsf{V}(G)} F_{r,k}(x,y) \\ &= \sum_{y \in \partial_{\mathsf{V}}^{\mathrm{int}} K} \sum_{k=0}^{r-1} q_r = |\partial_{\mathsf{V}}^{\mathrm{int}} K| r q_r\,. \end{aligned}$$

Therefore,

$$Z \le \sum_{r=1}^{\rho} |\partial_{\mathsf{V}}^{\mathrm{int}} K| r q_r \le |\partial_{\mathsf{V}}^{\mathrm{int}} K| \rho b\,.$$

Comparing these upper and lower bounds for Z, we get the desired result. ◀

An immediate consequence (relying on Proposition 8.12) is the following bound:

Corollary 10.44. *If G is a transitive graph with balls of radius n having at least cn^3 vertices for some constant c, then*

$$\psi(G,t) \ge c' t^{2/3}$$

for some constant c' and all $t \ge 1$.

Since all quasi-transitive transient graphs have at least cubic volume growth by a theorem of Gromov (1981) and Trofimov (1985) (see, e.g., Theorem 5.11 of Woess (2000)), we obtain:

Theorem 10.45. *If G is a transient transitive network or a non-amenable network, then* WSF*-a.s. every tree has only one end.*

▷ **Exercise 10.16.**
Show that Theorem 10.45 holds in the quasi-transitive case as well.

Question 10.46. If G and G' are roughly isometric graphs and the wired spanning forest in G has only one end in each tree a.s., then is the same true in G'?

In contrast, we have:

Proposition 10.47. *If G is a unimodular transitive network with* $\mathsf{WSF} \ne \mathsf{FSF}$*, then* FSF*-a.s., there is a tree with infinitely many ends—in fact, with branching number > 1.*

Proof. We have that $\mathbf{E}_{\mathsf{WSF}}[\deg_{\mathfrak{F}} x] = 2$ for all x, whence $\mathbf{E}_{\mathsf{FSF}}[\deg_{\mathfrak{F}} x] > 2$ for all x. Apply Theorem 8.17 and ergodicity (Corollary 10.18). ◀

Question 10.48. Let G be a transitive network with $\mathsf{WSF} \ne \mathsf{FSF}$. Must *all* components of the FSF have infinitely many ends a.s.? Presumably, the answer is "yes".

In view of Proposition 10.47, this would follow in the unimodular case from a proof of the following conjecture:

Conjecture 10.49. *The components of the* FSF *on a unimodular transitive graph are indistinguishable in the sense that for every automorphism-invariant property $\mathcal{A}$ of subgraphs, either a.s. all components satisfy $\mathcal{A}$ or a.s. they all do not. The same holds for the* WSF*.*

This fails in the nonunimodular setting, as an example in Lyons and Schramm (1999) shows.

Taking stock, we arrive at the following surprising results:

Theorem 10.50. *Let G be a transitive graph. If G is a plane graph such that both it and its plane dual are roughly isometric to $\mathbb{H}^2$, then the* WSF *of G has infinitely many trees a.s., each having one end a.s., while the* FSF *of G has one tree a.s. with infinitely many ends a.s. If G is a graph roughly isometric to $\mathbb{H}^d$ for $d \ge 3$, then the* $\mathsf{WSF} = \mathsf{FSF}$ *of G has infinitely many trees a.s., each having one end a.s.*

Proof. Rough isometry preserves exponential volume growth. Use Theorems 9.17 and 10.45 and Propositions 10.13 and 10.36. In the planar case, it suffices merely to have both G and its dual locally finite and non-amenable. ◀

An example of a self-dual plane Cayley graph roughly isometric to $\mathbb{H}^2$ was shown in Fig. 2.3.

Let G be a proper transient plane graph with bounded degree and a bounded number of sides to its faces. Recall that Theorem 9.11 and Proposition 10.13 imply that $\mathsf{WSF} \ne \mathsf{FSF}$. By Theorem 10.25 and Theorem 9.16, WSF has infinitely many trees. If the wired spanning forest has only one end in each tree, then the end containing any given vertex has a limiting direction that is distributed according to Lebesgue measure in the parametrization of Section 9.4. This raises the following question:

Question 10.51. Let G be a proper transient plane graph with bounded degree and a bounded number of sides to its faces. Does each tree have only one end WSF-a.s.? Equivalently, is the free spanning forest a single tree a.s.?

Of course, the recurrent case has a positive answer by Theorem 10.35, the fact that $G^\dagger$ is roughly isometric to G, and thus that $G^\dagger$ is also recurrent by Theorem 2.16.

Here is a summary of the phase transitions. It is quite surprising that the free spanning forest undergoes 3 phase transitions as the dimension increases. (We think of hyperbolic space as having infinitely many Euclidean dimensions, since volume grows exponentially in hyperbolic space, but only polynomially in Euclidean space.)

	$\mathbb{Z}^d$		$\mathbb{H}^d$	
d	$2-4$	≥ 5	2	≥ 3
FSF: trees	1	∞	1	∞
ends	1	1	∞	1
WSF: trees	1	∞	∞	∞
ends	1	1	1	1

Finally, we mention the following beautiful extension of Pemantle's Theorem 10.28; see Benjamini, Kesten, Peres, and Schramm (2004).

Theorem 10.52. *Identify each tree in the uniform spanning forest on $\mathbb{Z}^d$ to a single point. In the induced metric, the diameter of the resulting (locally infinite) graph is a.s. $\lfloor (d-1)/4 \rfloor$.*

§10.7. Loop-Erased Random Walk and Harmonic Measure From Infinity.

Infinite loop-erased random walk is defined in any transient network by chronologically erasing cycles from the random walk path. On a recurrent network, the natural substitute is to run random walk until it first reaches distance n from its starting point, erase cycles, and take a weak limit as $n \to \infty$.

▷ **Exercise 10.17.**
Show that on a general recurrent network, such a weak limit need not exist.

In $\mathbb{Z}^2$, weak convergence was established by Lawler (1988) using Harnack inequalities (see Lawler (1991), Prop. 7.4.2). Lawler's approach yields explicit estimates of the rate of convergence, but is difficult to extend to other networks. Using spanning trees, however, we can often prove that the limit exists, as shown in the following exercise.

▷ **Exercise 10.18.**
Let $\langle G_n \rangle$ be an exhaustion of a recurrent network G. Consider the network random walk $\langle X_o(k) \rangle$ started from $o \in G$. Denote by τ_n the first exit time of G_n, and let L_n be the loop erasure of the path $\langle X_o(k)\,;\ 0 \le k \le \tau_n \rangle$. Show that if the uniform spanning tree T_G in G has one end a.s., then the random paths L_n converge weakly to the law of the unique ray from o in T_G. In particular, show that this applies if G is a proper plane network with a locally finite recurrent dual.

Let A be a finite set of vertices in a recurrent network G. Denote by τ_A the hitting time of A, and by h_u^A the harmonic measure from u on A:

$$\forall B \subseteq A \quad h_u^A(B) := \mathbf{P}[X_u(\tau_A) \in B]\,.$$

If the measures h_u^A converge when $\mathrm{dist}(u, A) \to \infty$, then it is natural to refer to the limit as ***harmonic measure from*** ∞ ***on*** A. This convergence fails in some recurrent networks (e.g., in $\mathbb{Z}$), but it does hold in $\mathbb{Z}^2$; see Lawler (1991), Thm. 2.1.3. On transient networks, wired harmonic measure from infinity always exists: see Exercise 2.41. Here, we show that on recurrent networks with one end in their uniform spanning tree, harmonic measure from infinity also exists:

Theorem 10.53. *Let G be a recurrent network and A be a finite set of vertices. Suppose that the random spanning tree T_G in G has one end a.s. Then the harmonic measures h_u^A converge as $dist(u, A) \to \infty$.*

Proof. Essentially, we would like to run Wilson's algorithm with all of A as root. To do this, add a finite set F of edges to G to form a graph G' in which the subgraph (A, F) is connected, and suppose that F is minimal with respect to this property. Thus, (A, F) is a tree. Assign unit conductance to the edges of F. Note that having at most one end in T is a tail event. Now $T_{G'}$ conditioned on $T_{G'} \cap F = \varnothing$ has the same distribution as T_G. Therefore, tail triviality (Theorem 10.17) tell us that a.s. $T_{G'}$ has one end. Similarly, because $T_{G'}$ conditioned on $T_{G'} \cap F = F$ has the same distribution as $T_{G'/F}$, also $T_{G'/F}$ has one end a.s.

The path from u to A in $T_{G'/F}$ is constructed by running a random walk from u until it hits A and then loop erasing. Thus, when $\mathrm{dist}(u, A) \to \infty$, the measures h_u^A must tend to the conditional distribution, given $T_{G'} \cap F = F$, of the point in A that is closest (in $T_{G'}$) to the unique end of $T_{G'}$. ◀

§10.8. Open Questions.

There are many open questions related to spanning forests. Besides the ones we have already given, here are a few more.

Question 10.54. Does each component in the wired spanning forest on an infinite supercritical Galton-Watson tree have one end a.s.? This is answered positively by Aldous and Lyons (2007) when the offspring distribution is bounded. If the probability of 0 or 1 offspring is 0, then of course the result is true by Theorem 10.45.

Question 10.55. Let G be an infinite network such that $\mathsf{WSF} \neq \mathsf{FSF}$ on G. Does it follow that WSF and FSF are mutually singular measures?

This question has a positive answer for trees (there is exactly one component FSF-a.s. on a tree, while the number of components is a constant WSF-a.s. by Theorem 10.25) and for networks G where $\mathrm{Aut}(G)$ has an infinite orbit (Corollary 10.18).

Question 10.56. Is the probability that each tree has only one end equal to either 0 or 1 for both WSF and FSF? This is true on trees by Theorem 11.1 of BLPS (2001).

Conjecture 10.57. *Let T_o be the component of the identity o in the* WSF *on a Cayley graph, and let $\xi = \langle x_n\,;\ n \ge 0 \rangle$ be the unique ray from o in T_o. The sequence of "bushes"*

$\langle b_n \rangle$ observed along ξ converges in distribution. (Formally, b_n is the connected component of x_n in $T \setminus \{x_{n-1}, x_{n+1}\}$, multiplied on the left by x_n^{-1}.)

§10.9. Notes.

The free spanning forest was first suggested by Lyons, but its existence was proved by Pemantle (1991). Pemantle also implicitly proved the existence of the wired spanning forest and showed that the free and wired uniform spanning forests are the same on Euclidean lattices $\mathbb{Z}^d$. The first explicit construction of the wired spanning forest is due to Häggström (1995).

The main results in this chapter that come from BLPS (2001) are Theorems 10.14, 10.17, 10.25, 10.35, 10.45, 10.50, and 10.53 and Propositions 10.1, 10.10, 10.13, 10.36, and 10.47. Theorems 10.45 and 10.50 were proved there in more restricted versions. The extended versions proved here are from Lyons, Morris, and Schramm (2008).

The version we present of Strassen's theorem, Theorem 10.4, is only a special case of what he proved.

Theorems 10.20 and 10.23, Corollary 10.24 and Lemma 10.30 are from Lyons, Peres, and Schramm (2003). We have modified the more general results of Lyons, Peres, and Schramm (2003) and simplified their proofs for the special cases presented here in Theorem 10.23, Corollary 10.31, and Lemma 10.32.

The last part of Theorem 10.33 was conjectured by BLPS (2001).

Theorem 10.35 was first stated without proof for $\mathbb{Z}^2$ by Pemantle (1991). It is shown in BLPS (2001) that the same is true on any recurrent transitive network.

§10.10. Collected In-Text Exercises.

10.1. The choice of exhaustion $\langle G_n \rangle$ does not change the resulting measure WSF by Proposition 10.1. Show that the choice also does not change the resulting measure FSF.

10.2. Show that FSF and WSF are invariant under any automorphisms that the network may have.

10.3. Show that if G is an infinite recurrent network, then the wired spanning forest on G is the same as the free spanning forest, i.e., the random spanning tree of Section 4.2.

10.4. Let G be a network such that there is a finite subset of edges whose removal from G leaves at least 2 transient components. Show that the free and wired spanning forests are different on G.

10.5. Let G be a tree with unit conductances. Show that $\mathsf{FSF} = \mathsf{WSF}$ iff G is recurrent.

10.6. Let G be an edge-amenable infinite graph as witnessed by the vertex sets $\langle \mathsf{V}_n \rangle$ (see Section 4.3). Let G_n be the subgraph induced by V_n.

(a) Let F be any spanning forest all of whose components (trees) are infinite. Show that if k_n denotes the number of trees of $F \cap G_n$, then $k_n = o(|\mathsf{V}_n|)$.

(b) Show that the average degree, in two senses, of vertices in both the free spanning forest and the wired spanning forest is 2:

$$\lim_{n\to\infty} |\mathsf{V}_n|^{-1} \sum_{x\in\mathsf{V}_n} \deg_{\mathfrak{F}}(x) = 2 \quad \text{a.s.}$$

and

$$\lim_{n\to\infty} |\mathsf{V}_n|^{-1} \sum_{x\in\mathsf{V}_n} \mathbf{E}[\deg_{\mathfrak{F}}(x)] = 2\,.$$

In particular, if G is a transitive graph such as $\mathbb{Z}^d$, then every vertex has expected degree 2 in both the free spanning forest and the wired spanning forest.

10.7. Let G be a graph obtained by identifying some vertices of a finite connected graph H. Let μ_G and μ_H be the corresponding uniform spanning tree measures. Show that $\mu_G(\mathscr{A}) \le \mu_H(\mathscr{A})$ for every increasing event $\mathscr{A}$ depending on the edges of G.

10.8. Show that the number of trees in the free spanning forest on a network is stochastically dominated by the number in the wired spanning forest on the network. If the number of trees in the free spanning forest is a.s. finite, then, in distribution, it equals the number in the wired spanning forest iff $\mathsf{FSF} = \mathsf{WSF}$.

10.9. Give an amenable graph on which $\mathsf{FSF} \ne \mathsf{WSF}$.

10.10. Show that every transitive amenable network has unique currents.

10.11. **(The Riemann-Lebesgue Lemma)** Show that if $F \in L^1(\mathbb{T}^d)$ and $f(x) = \int_{\mathbb{T}^d} F(\alpha)e^{2\pi i x\cdot\alpha}\,d\alpha$, then $\lim_{|x|\to\infty} f(x) = 0$. *Hint:* This is obvious if F is a ***trigonometric polynomial***, i.e., a finite linear combination of functions $\alpha \mapsto e^{2\pi i x\cdot\alpha}$. The Stone-Weierstrass theorem implies that such functions are dense in $L^1(\mathbb{T}^d)$.

10.12. Join two copies of the usual nearest-neighbor graph of $\mathbb{Z}^3$ by an edge at their origins. How many trees does the free uniform spanning forest have? How many does the wired uniform spanning forest have?

10.13. **(Parseval's Identity)** Suppose that $F \in L^1(\mathbb{T}^d)$ and that $f(x) := \int_{\mathbb{T}^d} F(\alpha)e^{2\pi i x\cdot\alpha}\,d\alpha$ for $x \in \mathbb{Z}^d$. Show that $\int_{\mathbb{T}^d} |F(\alpha)|^2\,d\alpha = \sum_{\mathbb{Z}^d} |f(x)|^2$. *Hint:* Prove that the functions $G_x(\alpha) := e^{-2\pi i x\cdot\alpha}$ for $x \in \mathbb{Z}^d$ form an orthonormal basis of $L^2(\mathbb{T}^d)$.

10.14. Let (T, c) be a transient network on a tree with bounded conductances $c(\bullet)$. Show that there is some $e \in T$ such that both components of $T\backslash e$ are transient (with respect to c).

10.15. Give a transient graph such that the wired uniform spanning forest has a single tree with more than one end.

10.16. Show that Theorem 10.45 holds in the quasi-transitive case as well.

10.17. Consider a random walk until it first reaches distance n from its starting point and erase cycles. Show that on a general recurrent network, a weak limit of these random paths need not exist.

10.18. Let $\langle G_n\rangle$ be an exhaustion of a recurrent network G. Consider the network random walk $\langle X_o(k)\rangle$ started from $o \in G$. Denote by τ_n the first exit time of G_n, and let L_n be the loop erasure of the path $\langle X_o(k)\,;\ 0 \le k \le \tau_n\rangle$. Show that if the uniform spanning tree T_G in G has one end a.s., then the random paths L_n converge weakly to the law of the unique ray from o in T_G. In particular, show that this applies if G is a proper plane network with a locally finite recurrent dual.

§10.11. Additional Exercises.

10.19. Let $\langle G_n \rangle$ be an exhaustion of G. Write F_n for the set of spanning forests of G_n such that each component tree includes exactly one vertex of the internal vertex boundary of G_n. Show that WSF is the limit as $n \to \infty$ of the uniform measure on F_n.

10.20. Show that the FSF of the usual Cayley graph of $\mathbb{Z}_2 * \mathbb{Z}_3$ (shown in Figure 3.3) is a tree whose branching number equals 1.35^+.

10.21. Show that for each n, we have $\mu_n^{\mathrm{F}} \succcurlyeq \mu_{n+1}^{\mathrm{F}} \restriction 2^{\mathsf{E}(G_n)}$ and $\mu_n^{\mathrm{W}} \preccurlyeq \mu_{n+1}^{\mathrm{W}} \restriction 2^{\mathsf{E}(G_n^{\mathrm{W}})}$.

10.22. Consider the free or wired uniform spanning tree measure on an infinite transient network G. Let X and Y be increasing random variables with finite second moments that depend on disjoint sets of edges. Show that $\mathbf{E}[XY] \le \mathbf{E}[X]\mathbf{E}[Y]$.

10.23. Let G be a planar Cayley graph. Show that simple random walk on G is transient iff G is non-amenable iff G has exponential growth.

10.24. Let (G, c) be a denumerable network with an exhaustion by finite subnetworks G_n. Fix $o \in \mathsf{V}(G_1)$. Let U_n be the Gaussian network on G_n from Exercise 2.96 (with $W = \{o\}$ and $u(o) = 0$ there). Show that the weak limit of U_n exists; it is called the ***free Gaussian network***. Show also that if U_n^{W} denotes the Gaussian network on the wired network G_n^{W}, then the weak limit of U_n^{W} also exists, called the ***wired Gaussian network***. Finally, show that the wired and free Gaussian networks coincide in distribution iff $\mathbf{HD} = \mathbb{R}$.

10.25. Consider the Gaussian network U on $\mathbb{Z}^d$ with $U(\mathbf{0}) = 0$. Show that $\{U(x)\,;\ x \in \mathbb{Z}^d\}$ is tight iff $d \ge 3$.

10.26. Let G be a plane regular graph of degree d with regular dual of degree $d^\dagger$. Show that the FSF-expected degree of each vertex in G is $d(1 - 2/d^\dagger)$.

10.27. Let G be the usual Cayley graph of the (p, q, r)-triangle group, where $1/p + 1/q + 1/r \le 1$, shown in Figure 6.1 for $(2, 3, 7)$. It has 3 generators, which are reflections in the sides of a fundamental triangle. Show that the expected degree of the FSF of G is $3 - 1/p - 1/q - 1/r$.

10.28. Complete the following outline of an alternative proof of Proposition 10.15. Let $\psi(\alpha) := 1 - \varphi(\alpha)/(2d)$, where φ is as defined in (10.7). For all $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$, we have $p_n(\mathbf{0}, x) = \int_{\mathbb{T}^d} \psi(\alpha)^n e^{2\pi i x\cdot\alpha}\, d\alpha$. Therefore, $\mathcal{G}(\mathbf{0}, x)/(2d)$ equals the right-hand side of (10.9). Now apply Proposition 2.1.

10.29. For a function $f \in L^1(\mathbb{T}^d)$ and $x \in \mathbb{Z}^d$, define

$$\widehat{f}(x) := \int_{\mathbb{T}^d} f(\alpha) e^{-2\pi i x\cdot\alpha}\, d\alpha\,.$$

Let Y be the transfer-current matrix for the hypercubic lattice $\mathbb{Z}^d$. Write $u_k := \mathbf{1}_{\{k\}}$ for the vector with a 1 in the kth place and 0s elsewhere. Let $e_x^k := [x, x + u_k]$. Show that $Y(e_{\mathbf{0}}^1, e_x^k) = \widehat{f}_k(x)$, where

$$f_1(\alpha_1, \ldots, \alpha_d) := \frac{\sin^2 \pi\alpha_1}{\sum_{j=1}^d \sin^2 \pi\alpha_j}$$

and

$$f_k(\alpha_1, \ldots, \alpha_d) := \frac{(1 - e^{-2\pi i\alpha_1})(1 - e^{-2\pi i\alpha_k})}{4\sum_{j=1}^d \sin^2 \pi\alpha_j}$$

for $k \ge 1$.

10.30. Let $d \geq 3$ and let e be any edge of $\mathbb{Z}^d$. For $n \in \mathbb{Z}$, let X_n be the indicator that $e + (n, n, \ldots, n)$ lies in the uniform spanning forest of $\mathbb{Z}^d$. Show that X_n are i.i.d. The same holds for the other 2^{d-1} collections of translates given by possibly changing the signs of the last $d-1$ coordinates.

10.31. Show that if $\mathsf{FSF} = \mathsf{WSF}$, then for all cylinder events $\mathscr{A}$ and $\epsilon > 0$, there is a finite set of edges K such that

$$\sup\{|\mathsf{WSF}[\mathscr{A} \mid \mathscr{B}] - \mathsf{WSF}[\mathscr{A}]|\,;\ \ \mathscr{B} \in \mathscr{F}(\mathsf{E} \setminus K)\} < \epsilon\,.$$

10.32. For $k \geq 1$, we say that an action is ***mixing of order*** k (or k***-mixing***) if for any events $A_1, \ldots, A_{k+1}$, we have

$$\lim_{g_1,\ldots,g_k \to \infty} \mathbf{P}(A_1, g_1 A_2, g_2 g_1 A_3, \ldots, g_k g_{k-1} \cdots g_2 g_1 A_{k+1}) = \mathbf{P}(A_1)\mathbf{P}(A_2) \cdots \mathbf{P}(A_{k+1})\,.$$

Show that tail triviality and an infinite orbit imply mixing of all orders.

10.33. Let (G, c) be a network with spectral radius $\rho < 1$ and $\sup_{x \in \mathsf{V}} \pi(x) < \infty$. Show that there are infinitely many trees in the WSF a.s.

10.34. Define I_N as in (10.18). Show that $\mathbf{E}[I_N^k] \leq (k!)^2 (\mathbf{E} I_N)^k$ for every $k \geq 1$.

10.35. Let $\alpha(w_1, \ldots, w_K)$ be the probability that independent random walks started at $w_1, \ldots, w_K$ have no pairwise intersections. Let $\mathscr{B}(w_1, \ldots, w_K)$ be the event that the number of pairwise intersections among the random walks $\langle X_n(w_i) \rangle$ is finite. Show that

$$\lim_{n \to \infty} \alpha(X_n(w_1), \ldots, X_n(w_K)) = \mathbf{1}_{\mathscr{B}(w_1,\ldots,w_K)} \quad \text{a.s.}$$

10.36. Suppose that there are k components in the WSF and that $w_1, \ldots, w_k$ are vertices satisfying $\alpha(w_1, \ldots, w_k) > 0$. Let $\{X(w_i)\,;\ 1 \leq i \leq k\}$ be independent random walks indexed by their initial states. Consider the random functions

$$h_i(w) := \mathbf{P}[Y(w) \text{ intersects } X(w_i) \text{ i.o.} \mid X(w_1), \ldots, X(w_k)]\,,$$

where the random walk $Y(w)$ starts at w and is independent of all $X(w_i)$. Show that a.s. on the event that $X(w_1), \ldots, X(w_k)$ have pairwise disjoint paths, the functions $\{h_1, \ldots, h_k\}$ form a basis for $\mathbf{BH}(G)$, the vector space $\mathbf{BH}(G)$ of bounded harmonic functions on G. Deduce that if the number of components of the WSF is finite a.s., then it a.s. equals the dimension of $\mathbf{BH}(G)$.

10.37. It follows from Corollary 10.24 that if two transitive graphs are roughly isometric, then the a.s. number of trees in the wired spanning forests are the same. Show that this is false without the assumption of transitivity.

10.38. Consider a d-ary tree with the conductances corresponding to the random walk RW_λ. Show that a.s. all the trees in the wired spanning forest have branching number λ for $1 \leq \lambda \leq d$.

10.39. Let T be an arbitrary tree with arbitrary conductances and root o. For a vertex $x \in T$, let $\alpha(x)$ denote the effective conductance of T^x (from x to infinity). Consider the independent percolation on T that keeps the edge preceding x with probability $1/(1+\alpha(x))$. Show that the WSF on T has a tree with more than one end with positive probability iff percolation on T occurs. In the case of WSF on a spherically symmetric tree, show that this is equivalent to

$$\sum_n \frac{1}{|T_n|^2 \rho_n} \prod_{k=1}^{n-1} \left(1 + \frac{1}{|T_k| \rho_k}\right) < \infty\,,$$

where $\rho_n := \sum_{k>n} 1/|T_k|$. In particular, show that this holds if $|T_n| \approx n^a$ for $a > 1$.

10.40. Show that if G is a transitive graph such that the balls of radius r have cardinality asymptotic to αr^d for some positive finite α and d, then the lower bound of Lemma 10.43 is optimal up to a constant factor. In other words, show that in this case,

$$\liminf_{|K|\to\infty} \rho(2|K|)\frac{|\partial_{\mathsf{V}}^{\mathrm{int}} K|}{|K|} < \infty\,.$$

10.41. Let G_0 be formed by two copies of $\mathbb{Z}^3$ joined by an edge $[x, y]$; put $G := G_0 \times \mathbb{Z}$. Show that $\mathsf{FSF}(G) = \mathsf{WSF}(G)$ and that the spanning forests have exactly two trees a.s.

10.42. Let G be the Cayley graph of the free product $\mathbb{Z}^d * \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the group with two elements, with the obvious generating set. Show that the FSF is connected iff $d \le 4$ and $\mathsf{FSF} \ne \mathsf{WSF}$ for all $d > 0$.

Chapter 11

Minimal Spanning Forests

In Chapters 4 and 10, we looked at spanning trees chosen uniformly at random and their analogues for infinite graphs. We saw that they are intimately connected to random walks. Another measure on spanning trees and forests has been studied a great deal, especially in optimization theory. This measure, minimal spanning trees and forests, turns out to be connected to percolation, rather than to random walks. In fact, one of the measures on forests is closely tied to critical bond percolation and invasion percolation, while the other is related to percolation at the uniqueness threshold. Our treatment is drawn from Lyons, Peres, and Schramm (2006), as are most of the results.

On the whole, minimal spanning forests share many similarities with uniform spanning forests. In some cases, we know a result for one measure whose analogue is only conjectured for the other. Occasionally, there are striking differences between the two settings.

The standard coupling of Bernoulli bond percolation will be ubiquitous in this chapter, so much that we need some special notation for it. Namely, given labels $U : \mathsf{E}(G) \to \mathbb{R}$, we'll write $G[p]$ for the subgraph formed by the edges $\{e\,;\ U(e) < p\}$. (In previous chapters, we denoted $G[p]$ by ω_p, but in this chapter, both G and p often assume more complicated expressions.)

§11.1. Minimal Spanning Trees.

Let $G = (\mathsf{V}, \mathsf{E})$ be a finite connected graph. Given an injective function, $U : \mathsf{E} \to \mathbb{R}$, we'll refer to $U(e)$ as the ***label*** of e. The labeling U induces a total ordering on E, where $e < e'$ if $U(e) < U(e')$. We'll say that e is ***lower*** than e' and that e' is ***higher*** than e when $e < e'$.

Define T_U to be the subgraph whose vertex set is V and whose edge set consists of all edges $e \in \mathsf{E}$ whose endpoints cannot be joined by a path whose edges are strictly lower than e. We claim that T_U is a spanning tree. First, the largest edge in any cycle of G is not in T_U, whence T_U is a forest. Second, if $\varnothing \neq A \subsetneq \mathsf{V}$, then the least edge of G connecting A with $\mathsf{V} \setminus A$ must belong to T_U, which shows that T_U is connected. Thus, it is a spanning tree.

▷ **Exercise 11.1.**
Show that among all spanning trees, T_U has minimal edge-label sum, $\sum_{e \in T} U(e)$.

When $\langle U(e)\,;\ e \in \mathsf{E} \rangle$ are independent uniform $[0,1]$ random variables, the law of the corresponding spanning tree T_U is called simply the ***minimal spanning tree (measure)***. It is a probability measure on 2^{E}.

There is an easy monotonicity principle for the minimal spanning tree measure, which is analogous to a similar principle, Lemma 10.3 and Exercise 10.7, for uniform spanning trees:

Proposition 11.1. *Let G be a connected subgraph of the finite connected graph H. Let T_G and T_H be the corresponding minimal spanning trees. Then T_G stochastically dominates $T_H \cap G$. On the other hand, if G is obtained by identifying some vertices in H, then T_G is stochastically dominated by $T_H \cap G$.*

Proof. We'll prove the first part, as the second part is virtually identical. Let $U(e)$ be i.i.d. uniform [0, 1] random variables for $e \in \mathsf{E}(H)$. We use these labels to construct both T_G and T_H. In this coupling, if $[x, y] \in \mathsf{E}(G)$ is contained in T_H, i.e., there is no path in $\mathsf{E}(H)$ joining x and y that uses only lower edges, then *a fortiori* there is no such path in $\mathsf{E}(G)$, i.e., $[x, y]$ is also contained in T_G. That is, $T_H \cap G \subseteq T_G$, which proves the result. ◀

On the other hand, contrary to what happens for uniform spanning trees, the presence of two edges can be positively correlated. To see this, we first present the following formula for computing probabilities of spanning trees. Let MST denote the minimal spanning tree measure on a finite connected graph.

Proposition 11.2. *Let G be a finite connected graph. Given a set F of edges, let $N(F)$ be the number of edges of G that do not become loops when each edge in F is contracted. Note that $N(\varnothing)$ is the number of edges of G that are not loops. Let $N'(e_1, \ldots, e_k) := \prod_{j=0}^{k-1} N(\{e_1, \ldots, e_j\})$. Let $T = \{e_1, \ldots, e_n\}$ be a spanning tree of G. Then*

$$\mathsf{MST}(T) = \sum_{\sigma \in S_n} N'(e_{\sigma(1)}, \ldots, e_{\sigma(n)})^{-1},$$

where S_n is the group of permutations of $\{1, 2, \ldots, n\}$.

Proof. To make the dependence on G explicit, we write $N(F) = N(G; F)$. Note that $N(G/F; \varnothing) = N(G; F)$, where G/F is the graph G with each edge in F contracted. Given an edge e that is not a loop, the chance that e is the least edge in the minimal spanning tree of G equals $N(G; \varnothing)^{-1}$. Furthermore, given that this is the case, the ordering on the non-loops of the edge set of $G/\{e\}$ is uniform. Thus, if f is not a loop in $G/\{e\}$, then the chance that f is the next least edge in the minimal spanning tree of G given that e is the least edge in the minimal spanning tree of G equals $N(G/\{e\};\varnothing)^{-1} = N(G;\{e\})^{-1}$. Thus, we may easily condition, contract, and repeat.

Thus, the probability that the minimal spanning tree is T *and* that $e_{\sigma(1)} < \cdots < e_{\sigma(n)}$ is equal to

$$\prod_{j=0}^{n-1} N(G/\{e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(j)}\};\varnothing)^{-1} = N'(e_{\sigma(1)}, \ldots, e_{\sigma(n)})^{-1}.$$

Summing this over all possible induced orderings of T gives $\mathsf{MST}(T)$. ◀

An example of a graph where MST has positive correlations is provided by the following exercise.

▷ **Exercise 11.2.**

Construct G as follows. Begin with the complete graph, K_4. Let e and f be two of its edges that do not share endpoints. Replace e by three edges in parallel, e_1, e_2, and e_3, that have the same endpoints as e. Likewise, replace f by three parallel edges f_i. Show that $\mathsf{MST}[e_1, f_1 \in T] > \mathsf{MST}[e_1 \in T]\mathsf{MST}[f_1 \in T]$.

▷ **Exercise 11.3.**

The following difference from the uniform spanning tree must be kept in mind: Show that given an edge, e, the minimal spanning tree measure on G conditioned on the event not to contain e need not be the same as the minimal spanning tree measure on $G \setminus e$, the graph G with e deleted.

In Figure 11.1, we show the distances to the lower left vertex in a minimal spanning tree on a 100×100 square grid. However, unlike the case of uniform spanning trees (Figure 4.5), it does not seem simple to sample from the set of trees that are consistent with a given distance profile. Also, the distances seem to be generally lower than for the uniform spanning tree: see the value for "D_f" reported by Wieland and Wilson (2003), Table III.

▷ **Exercise 11.4.**

Show that there is a finite rooted graph with a given function f on the vertices such that there are two spanning trees for each of which the distance from x to the root in the tree is $f(x)$, yet those trees are not equally likely under the minimal spanning tree measure.

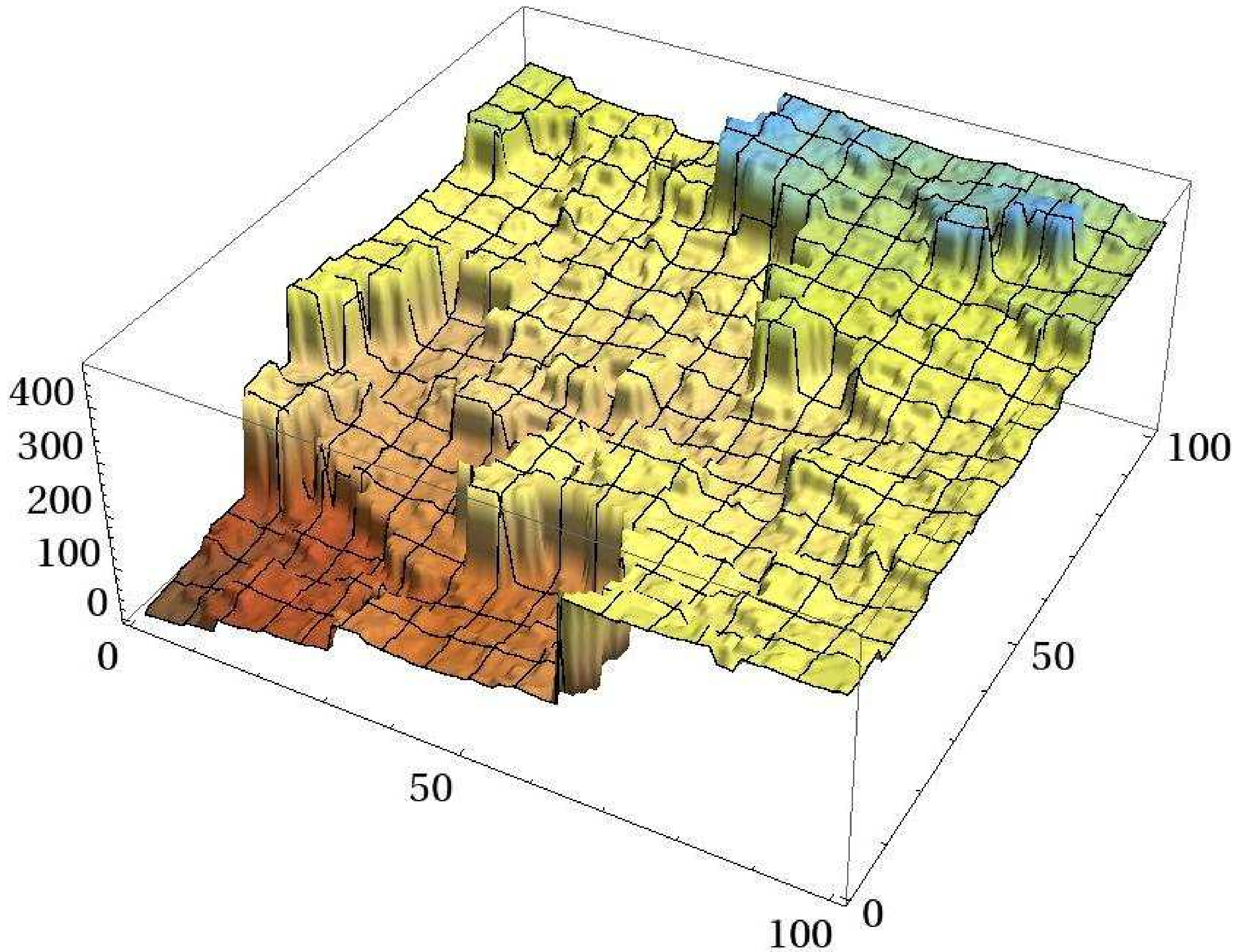


Figure 11.1. The distances to a vertex in a minimal spanning tree in a 100×100 grid.

This is all the theory of minimal spanning trees that we'll need! We can move directly to infinite graphs.

§11.2. Deterministic Results.

Just as for the uniform spanning trees, there are free and wired extensions to infinite graphs. Let $G = (\mathsf{V}, \mathsf{E})$ be an infinite connected locally finite graph and $U : \mathsf{E} \to \mathbb{R}$ be an injective labeling of the edges. Let $\mathfrak{F}_{\mathrm{f}} = \mathfrak{F}_{\mathrm{f}}(U) = \mathfrak{F}_{\mathrm{f}}(U, G)$ be the set of edges $e \in \mathsf{E}$ such that in every path in G connecting the endpoints of e there is at least one edge e' with $U(e') \geq U(e)$. When $\langle U(e) ;\ e \in \mathsf{E} \rangle$ are independent uniform random variables in $[0, 1]$, the law of $\mathfrak{F}_{\mathrm{f}}$ (or sometimes, $\mathfrak{F}_{\mathrm{f}}$ itself) is called the ***free minimal spanning forest*** on G and is denoted by FMSF or $\mathsf{FMSF}(G)$.

An ***extended path*** joining two vertices $x, y \in \mathsf{V}$ is either a simple path in G joining them, or the union of a simple infinite path starting at x and a disjoint simple infinite path starting at y. (The latter possibility may be considered as a simple path connecting x and y through ∞.) Let $\mathfrak{F}_{\mathrm{w}} = \mathfrak{F}_{\mathrm{w}}(U) = \mathfrak{F}_{\mathrm{w}}(U, G)$ be the set of edges $e \in \mathsf{E}$ such that in every extended path joining the endpoints of e there is at least one edge e' with $U(e') \ge U(e)$. Again, when U is chosen according to the product measure on $[0,1]^{\mathsf{E}}$, we call $\mathfrak{F}_{\mathrm{w}}$ the ***wired minimal spanning forest*** on G. The law of $\mathfrak{F}_{\mathrm{w}}$ is denoted WMSF or $\mathsf{WMSF}(G)$.

▷ **Exercise 11.5.**
Show that $\mathfrak{F}_{\mathrm{w}}(U)$ consists of those edges e such that there is a finite set of vertices $W \subset \mathsf{V}$ such that e is the least edge joining W to $\mathsf{V} \setminus W$.

Clearly, $\mathfrak{F}_{\mathrm{w}}(U) \subset \mathfrak{F}_{\mathrm{f}}(U)$. Note that $\mathfrak{F}_{\mathrm{w}}(U)$ and $\mathfrak{F}_{\mathrm{f}}(U)$ are indeed forests, since in every simple cycle of G the edge e with $U(e)$ maximal is not present in either $\mathfrak{F}_{\mathrm{f}}(U)$ nor in $\mathfrak{F}_{\mathrm{w}}(U)$. In addition, all the connected components in $\mathfrak{F}_{\mathrm{f}}(U)$ and in $\mathfrak{F}_{\mathrm{w}}(U)$ are infinite. Indeed, the lowest edge joining any finite vertex set to its complement belongs to both forests.

One of the nice properties that minimal spanning forests have is that there are these direct definitions on infinite graphs. Although we won't need this property, one can also describe them as weak limits:

▷ **Exercise 11.6.**
Consider an increasing sequence of finite, nonempty, connected subgraphs $G_n \subset G$, $n \in \mathbb{N}$, such that $\bigcup_n G_n = G$. For $n \in \mathbb{N}$, let G_n^{W} be the graph obtained from G by identifying the vertices outside of G_n to a single vertex, then removing all resulting loops based at that vertex. Let $T_n(U)$ and $T_n^{\mathrm{W}}(U)$ denote the minimal spanning trees on G_n and G_n^{W}, respectively, that are induced by the labels U. Show that $\mathfrak{F}_{\mathrm{f}}(U) = \lim_{n\to\infty} T_n(U)$ and, provided G_n is the subgraph of G induced by $\mathsf{V}(G_n)$, that $\mathfrak{F}_{\mathrm{w}}(U) = \lim_{n\to\infty} T_n^{\mathrm{W}}(U)$ in the sense that for every $e \in \mathfrak{F}_{\mathrm{f}}(U)$, we have $e \in T_n(U)$ for every sufficiently large n, for every $e \notin \mathfrak{F}_{\mathrm{f}}(U)$ we have $e \notin T_n(U)$ for every sufficiently large n, and similarly for $\mathfrak{F}_{\mathrm{w}}(U)$. Deduce that for any G_n (not necessarily induced), $T_n(U) \Rightarrow \mathsf{FMSF}$ and $T_n^{\mathrm{W}}(U) \Rightarrow \mathsf{WMSF}$.

It will be useful to make more explicit the comparisons that determine which edges belong to the two spanning forests. Define

$$Z_{\mathrm{f}}(e) = Z_{\mathrm{f}}^U(e) := \inf_{\mathcal{P}} \max \left\{ U(e') \, ; \; e' \in \mathcal{P} \right\},$$

where the infimum is over simple paths $\mathcal{P}$ in $G \backslash e$ that connect the endpoints of e; if there are none, the infimum is defined to be ∞. Thus, $\mathfrak{F}_{\mathrm{f}}(U) = \{e \, ; \; U(e) \le Z_{\mathrm{f}}(e)\}$. Since

$Z_f(e)$ is independent of $U(e)$ and $U(e)$ is a continuous random variable, we can also write $\mathfrak{F}_f(U) = \{e\,;\, U(e) < Z_f(e)\}$ a.s. Similarly, define

$$Z_w(e) = Z_w^U(e) := \inf_{\mathcal{P}} \sup \big\{U(e')\,;\, e' \in \mathcal{P}\big\}\,,$$

where the infimum is over extended paths $\mathcal{P}$ in $G\backslash e$ that join the endpoints of e. Again, if there are no such extended paths, then the infimum is defined to be ∞. Thus, $\{e\,;\, U(e) < Z_w(e)\} \subseteq \mathfrak{F}_w(U) \subseteq \{e\,;\, U(e) \le Z_w(e)\}$ and $\mathfrak{F}_w(U) = \{e\,;\, U(e) < Z_w(e)\}$ a.s.

It turns out that there are also dual definitions for Z_f and Z_w. In order to state these, recall that if $W \subseteq \mathsf{V}$, then the set of edges $\partial_{\mathsf{E}} W$ joining W to $\mathsf{V} \setminus W$ is a ***cut***.

Lemma 11.3. (Dual Criteria) *For any injection $U : \mathsf{E} \to \mathbb{R}$ on any graph G, we have*

$$Z_f(e) = \sup_{\Pi} \inf \big\{U(e')\,;\, e' \in \Pi \setminus \{e\}\big\}\,, \tag{11.1}$$

where the supremum is over all cuts Π that contain e. Similarly,

$$Z_w(e) = \sup_{\Pi} \min \big\{U(e')\,;\, e' \in \Pi \setminus \{e\}\big\}\,, \tag{11.2}$$

where now the supremum is over all cuts Π containing e such that $\Pi = \partial_{\mathsf{E}} W$ for some finite $W \subset \mathsf{V}$.

Proof. We first verify (11.1). If $\mathcal{P}$ is a simple path in $G\backslash e$ that connects the endpoints of e and Π is a cut that contains e, then $\mathcal{P} \cap \Pi \ne \varnothing$, so $\max \big\{U(e')\,;\, e' \in \mathcal{P}\big\} \ge \inf \big\{U(e')\,;\, e' \in \Pi \setminus \{e\}\big\}$. This proves one inequality ($\ge$) in (11.1). To prove the reverse inequality, fix one endpoint x of e, and let W be the vertex set of the component of x in $(G\backslash e)[Z_f(e)]$. Then $\Pi := \partial_{\mathsf{E}} W$ is a cut that contains e by definition of Z_f. Using Π in the right-hand side of (11.1) yields the inequality $\le$ in (11.1), and shows that the supremum there is achieved.

The inequality $\ge$ in (11.2) is proved in the same way as in (11.1). For the other direction, we dualize the above proof. Let Z denote the right hand side in (11.2), and let W be the vertex set of the connected component of one of the endpoints of e in the set of edges $e' \ne e$ such that $U(e') \le Z$. We clearly have $U(e') > Z$ for each $e' \in \partial_{\mathsf{E}} W \setminus \{e\}$. Thus, by the definition of Z, the other endpoint of e is in W if W is finite, in which case there is a path in $G\backslash e$ connecting the endpoints of e that uses only edges with labels at most Z. The same argument applies with the roles of the endpoints of e switched. If the sets W corresponding to both endpoints of e are infinite, then there is an extended path $\mathcal{P}$ connecting the endpoints of e in $G\backslash e$ with $\sup\{U(e')\,;\, e' \in \mathcal{P}\} \le Z$. This completes the proof of (11.2), and also shows that the infimum in the definition of $Z_w(e)$ is attained. ◀

The ***invasion tree*** $T(x) = T_U(x)$ of a vertex x is defined as the increasing union of the trees Γ_n, where $\Gamma_0 := \{x\}$ and Γ_{n+1} is Γ_n together with the least edge joining Γ_n to a vertex not in Γ_n. (If G is finite, we stop when Γ_n contains V.) Invasion trees play a role in the wired minimal spanning forest similar to the role played by Wilson's method rooted at infinity in the wired uniform spanning forest:

▷ **Exercise 11.7.**
Let $U : \mathsf{E} \to \mathbb{R}$ be an injective labeling of the edges of a locally finite graph $G = (\mathsf{V}, \mathsf{E})$. Show that the union $\bigcup_{x \in \mathsf{V}} T_U(x)$ of all the invasion trees is equal to $\mathfrak{F}_{\mathrm{w}}(U)$.

Recall from Section 7.5 that the ***invasion basin*** $I(x)$ of a vertex x is defined as the union of the subgraphs G_n, where $G_0 := \{x\}$ and G_{n+1} is G_n together with the lowest edge not in G_n but incident to some vertex in G_n. Note that $I(x)$ has the same vertices as $T(x)$, but may have additional edges.

Proposition 11.4. *Let $U : \mathsf{E} \to \mathbb{R}$ be an injective labeling of the edges of a locally finite graph $G = (\mathsf{V}, \mathsf{E})$. If x and y are vertices in the same component of $\mathfrak{F}_{\mathrm{w}}(U)$, then the symmetric differences $I(x) \triangle I(y)$ and $T_U(x) \triangle T_U(y)$ are finite.*

Proof. We prove only that $|I(x) \triangle I(y)| < \infty$, since the proof for $T_U(x) \triangle T_U(y)$ is essentially the same. It suffices to prove this when $e := [x, y] \in \mathfrak{F}_{\mathrm{w}}(U)$. Consider the connected components $C(x)$ and $C(y)$ of x and y in $G[U(e)]$. Not both $C(x)$ and $C(y)$ can be infinite since $e \in \mathfrak{F}_{\mathrm{w}}(U)$. If both are finite, then invasion from each x and y will fill $C(x) \cup C(y) \cup \{e\}$ before invading elsewhere, and therefore $I(x) = I(y)$ in this case. Finally, if, say, $C(x)$ is finite and $C(y)$ is infinite, then $I(x) = C(x) \cup \{e\} \cup I(y)$. ◀

Lemma 11.5. *Let G be any infinite locally finite graph with distinct fixed labels $U(e)$ on its edges. Let $\mathfrak{F}$ be the corresponding free or wired minimal spanning forest. If the label $U(e)$ is changed at a single edge e, then the forest changes at most at e and at one other edge (an edge f with $U(f) = Z_{\mathrm{f}}(e)$ or $Z_{\mathrm{w}}(e)$, respectively). More generally, if $\mathfrak{F}'$ is the forest when labels only in $K \subset \mathsf{E}$ are changed, then $|(\mathfrak{F} \triangle \mathfrak{F}') \setminus K| \le |K|$.*

Proof. Consider first the free minimal spanning forest. Suppose the two values of $U(e)$ are u_1 and u_2 with $u_1 < u_2$. Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be the corresponding free minimal spanning forests. Then $\mathfrak{F}_1 \setminus \mathfrak{F}_2 \subseteq \{e\}$. Suppose that $f \in \mathfrak{F}_2 \setminus \mathfrak{F}_1$. Then there must be a path $\mathcal{P} \subset G \backslash e$ joining the endpoints of e and containing f such that $U(f) = \max_{\mathcal{P}} U > u_1$. Suppose that there were a path $\mathcal{P}' \subset G \backslash e$ joining the endpoints of e such that $\max_{\mathcal{P}'} U < U(f)$. Then $\mathcal{P} \cup \mathcal{P}'$ would contain a cycle containing f but not e on which f has the maximum label.

This contradicts $f \in \mathfrak{F}_2$. Therefore, $Z_{\mathrm{f}}(e) = U(f)$. Since the labels are distinct, there is at most one such f.

For the WMSF, the proof is the same, only with "extended path" replacing "path" and "$Z_{\mathrm{w}}(e)$" replacing "$Z_{\mathrm{f}}(e)$".

The second conclusion in the lemma follows by induction from the first. ◀

§11.3. Basic Probabilistic Results.

Here are some of the easier analogues of several results on uniform spanning forests:

▷ **Exercise 11.8.**

Let G be a connected locally finite graph. Prove the following.

- **(a)** If G is edge-amenable, then the average degree of vertices in both the free and wired minimal spanning forests on G is a.s. 2.
- **(b)** The free and wired minimal spanning forests on G are the same if they have a.s. the same finite number of trees, or if the expected degree of every vertex is the same for both measures.
- **(c)** The free and wired minimal spanning forests on G are the same on any transitive amenable graph.
- **(d)** If $\mathfrak{F}_{\mathrm{w}}$ is connected a.s. or if each component of $\mathfrak{F}_{\mathrm{f}}$ has a.s. one end, then $\mathsf{WMSF}(G) = \mathsf{FMSF}(G)$.
- **(e)** If G is unimodular and transitive with $\mathsf{WMSF}(G) \neq \mathsf{FMSF}(G)$, then a.s. the FMSF has a component with uncountably many ends, in fact, with $p_{\mathrm{c}} < 1$.

When are the free and wired minimal spanning forests the same? Say that a graph G has ***almost everywhere uniqueness*** (of the infinite cluster) if for almost every $p \in (0, 1)$ in the sense of Lebesgue measure, there is a.s. at most one infinite cluster for Bernoulli(p) percolation on G. This is the analogue for minimal spanning forests of uniqueness of currents for uniform spanning forests (Proposition 10.13):

Proposition 11.6. *On any connected graph G, we have* $\mathsf{FMSF} = \mathsf{WMSF}$ *iff G has almost everywhere uniqueness.*

Proof. Since $\mathfrak{F}_{\mathrm{w}} \subset \mathfrak{F}_{\mathrm{f}}$ and E is countable, $\mathsf{FMSF} \neq \mathsf{WMSF}$ is equivalent to the existence of an edge e such that $\mathbf{P}[Z_{\mathrm{w}}(e) < U(e) \le Z_{\mathrm{f}}(e)] > 0$. Let $A(e)$ be the event that the two endpoints of e are in distinct infinite components of $(G \backslash e)[U(e)]$. Then $\{Z_{\mathrm{w}}(e) <$

$U(e) \le Z_{\mathrm{f}}(e)\} \subset A(e) \subset \{Z_{\mathrm{w}}(e) \le U(e) \le Z_{\mathrm{f}}(e)\}$. Consequently, $\mathbf{P}[A(e)] = \mathbf{P}[Z_{\mathrm{w}}(e) < U(e) < Z_{\mathrm{f}}(e)]$. Hence, $\mathsf{FMSF} \ne \mathsf{WMSF}$ is equivalent to the existence of an $e \in \mathsf{E}$ such that $\mathbf{P}[A(e)] > 0$. It is easy to see that almost everywhere uniqueness fails iff there is some $e \in \mathsf{E}$ with $\mathbf{P}[A(e)] > 0$. ◀

Corollary 11.7. *On any graph G, if almost everywhere uniqueness fails, then a.s.* WMSF *is not a tree.*

Proof. By Exercise 11.8(b), if WMSF is a tree a.s., then WMSF = FMSF. ◀

It is not easy to give a graph on which the FMSF is not a tree a.s., especially if the graph is transitive. See the discussion in Section 1 of Lyons, Peres, and Schramm (2006) for a transitive example and Example 6.1 there for a non-transitive example.

Another corollary of Proposition 11.6 is the following result of Häggström (1998).

Corollary 11.8. *If G is a tree, then the free and wired minimal spanning forests are the same iff $p_{\mathrm{c}}(G) = 1$.*

Proof. By Exercise 7.34, only at $p = 1$ can $G[p]$ have a unique infinite cluster a.s. ◀

Proposition 11.6 and Theorem 7.20 show that for quasi-transitive G, FMSF = WMSF iff $p_{\mathrm{c}} = p_{\mathrm{u}}$, which conjecturally holds iff G is amenable (Conjecture 7.28). In fact, for a quasi-transitive amenable G and every $p \in [0, 1]$, there is a.s. at most one infinite cluster in $G[p]$; see Theorem 7.6. This is slightly stronger than $p_{\mathrm{c}} = p_{\mathrm{u}}$, and gives another proof that for quasi-transitive amenable graphs, FMSF = WMSF (cf. Exercise 11.8(a),(b)).

Tail triviality holds for minimial spanning forests, just as for uniform spanning forests (Theorem 10.17):

Theorem 11.9. *Both measures* WMSF *and* FMSF *have a trivial tail σ-field on any graph.*

Proof. This is really a non-probabilistic result. Let $\mathscr{F}(K)$ be the σ-field generated by $U(e)$ for $e \in K$. We will show that the tail σ-field is contained in the tail σ-field of the labels of the edges, $\bigcap_{K \text{ finite}} \mathscr{F}(\mathsf{E} \setminus K)$. This implies the desired result by Kolmogorov's 0-1 Law.

Let $\phi : [0, 1]^{\mathsf{E}} \to 2^{\mathsf{E}}$ be the map that assigns the (free or wired) minimal spanning forest to a configuration of labels. (Actually, ϕ is defined only on the configurations of distinct labels.) Let A be a tail event of 2^{E}. We claim that $\phi^{-1}(A)$ lies in the tail σ-field $\bigcap_{K \text{ finite}} \mathscr{F}(\mathsf{E} \setminus K)$. Indeed, for any finite set K of edges and any two labellings ω_1, ω_2 that differ only on K, we know by Lemma 11.5 that $\phi(\omega_1)$ and $\phi(\omega_2)$ differ at most on $2|K|$ edges, whence either both ω_i are in $\phi^{-1}(A)$ or neither are. In other words, $\phi^{-1}(A) \in \mathscr{F}(\mathsf{E} \setminus K)$. ◀

§11.4. Tree Sizes.

Here we prove analogues of results from Section 10.6. There, we gave very general sufficient conditions for each tree in the wired spanning forest to have one end a.s. We do not have such a general theorem for the wired minimal spanning forest. Even in the transitive case, we do not know how to prove this without assuming unimodularity and $\theta(p_c) = 0$. On the other hand, we will be able to answer the analogue of Question 10.48 in the unimodular case and to prove analogues of Theorem 10.33 for both the wired and free minimal spanning forests and in great generality.

Theorem 11.10. (One End) *Let G be a unimodular transitive graph. Then the* WMSF*-expected degree of each vertex is 2. If $\theta(p_c, G) = 0$, then a.s. each component of the* WMSF *has one end.*

Proof. Fix a vertex x. Let $e_1, e_2, \ldots$ be the edges in the invasion tree of x, in the order they are added. Suppose that $\theta(p_c) = 0$. Then $\sup_{n \ge k} U(e_n) > p_c$ for any k. By Theorem 7.21, $\limsup U(e_n) = p_c$. For each k such that $U(e_k) = \sup_{n \ge k} U(e_n)$, the edge e_k separates x from ∞ in the invasion tree of x. It follows that the invasion tree of x a.s. has one end.

Thus, all invasion trees have one end a.s. Since each pair of invasion trees is either disjoint or shares all but finitely many vertices by Proposition 11.4, there is a well-defined special end for each component of $\mathfrak{F}_\mathrm{w}$, namely, the end of any invasion tree contained in the component by Exercise 11.7.

Orient each edge in $\mathfrak{F}_\mathrm{w}$ towards the special end of the component containing that edge. Then each vertex has precisely one outgoing edge. By the Mass-Transport Principle, it follows that the WMSF-expected degree of a vertex is 2. Since $\theta(p_c) = 0$ when G is non-amenable by Theorem 8.19, this conclusion holds for all such G. It also holds for amenable G by Exercise 11.8(a).

Combining the fact that the expected degree is 2 with Theorem 8.17, we deduce that a.s. each component of $\mathfrak{F}_\mathrm{w}$ has 1 or 2 ends.

Suppose that with positive probability some component had 2 ends. Let the ***trunk*** of a component with two ends be the unique bi-infinite path that it would contain. Label the vertices of the trunk x_n ($n \in \mathbb{Z}$), with $\langle x_n, x_{n+1} \rangle$ being the oriented edges of the trunk. Since $\theta(p_c) = 0$, there would be an $\epsilon > 0$ such that with positive probability $\sup_{n \in \mathbb{Z}} U\big([x_n, x_{n+1}]\big) > p_c + \epsilon$. By Theorem 7.21, $\limsup_{n \to \infty} U\big([x_n, x_{n+1}]\big) = p_c$. Thus, with positive probability there would be a largest $m \in \mathbb{Z}$ such that $U\big([x_m, x_{m+1}]\big) > p_c + \epsilon$. When this event occurs, we could then transport mass 1 from each vertex in such a component to the vertex x_m. The vertex x_m would then receive infinite mass, contradicting the Mass-Transport Principle. Therefore, all components have only 1 end a.s. ◀

Question 11.11. Let G be a transitive graph whose automorphism group is not unimodular. Does every tree of the WMSF on G have one end a.s.? Are there reasonably general conditions to guarantee 1 end in the WMSF without any homogeneity of the graph?

Now we answer the analogue of Question 10.48 in the unimodular case. This result is due to Timár (2006).

Theorem 11.12. (Infinitely Many Ends) *If G is a quasi-transitive unimodular graph and* WMSF $\neq$ FMSF*, then* FMSF*-a.s. every tree has infinitely many ends.*

Proof. Suppose that WMSF $\neq$ FMSF. Then G is non-amenable by Exercise 11.8. Since $\mathfrak{F}_{\mathrm{w}} \subseteq \mathfrak{F}_{\mathrm{f}}$, each tree of $\mathfrak{F}_{\mathrm{f}}$ consists of trees of $\mathfrak{F}_{\mathrm{w}}$ together with edges joining them. By Example 8.6, the number of edges in a tree of $\mathfrak{F}_{\mathrm{f}}$ that do not belong to $\mathfrak{F}_{\mathrm{w}}$ is either 0 or ∞. By Theorems 11.10 and 8.19, each tree of $\mathfrak{F}_{\mathrm{w}}$ has one end, so it remains to show that no tree of $\mathfrak{F}_{\mathrm{f}}$ is a tree of $\mathfrak{F}_{\mathrm{w}}$. Call a tree of $\mathfrak{F}_{\mathrm{f}}$ that is a tree of $\mathfrak{F}_{\mathrm{w}}$ ***lonely***. All other trees of $\mathfrak{F}_{\mathrm{f}}$ have infinitely many ends.

By the discussion in Section 11.3, we have $p_{\mathrm{c}} < p_{\mathrm{u}}$, so we may choose $p \in (p_{\mathrm{c}}, p_{\mathrm{u}})$. We may therefore choose a finite path $\mathcal{P}$ with vertices $\langle x_1, x_2, \ldots, x_n \rangle$ in order such that $\mathbf{P}(A) > 0$ for the event A that x_1 and x_n belong to distinct infinite clusters of $G[p]$, that x_1 belongs to a lonely tree, T, and that T does not intersect $\mathcal{P}$ except at x_1. Let $F \subset \mathsf{E}$ be the set of edges not in $\mathcal{P}$ that have an endpoint in $\{x_2, \ldots, x_{n-1}\}$.

Define B to be the event that results from A by changing the labels $U(e)$ for $e \in F$ to $p + (1-p)U(e)$. This increases all the labels in F and also makes them larger than p. Let $\mathfrak{F}'_{\mathrm{f}}$ be the new free minimal spanning forest and T' be the component of $\mathfrak{F}'_{\mathrm{f}}$ that contains x_1. By Lemma 11.5, $\mathfrak{F}'_{\mathrm{f}} \triangle \mathfrak{F}_{\mathrm{f}}$ is finite. Further, $T' \supseteq (T \setminus F)$. Therefore, if T' is not lonely on B, it contains infinitely many ends and an isolated end (the end of T). Since $\mathbf{P}(B) > 0$ by Lemma 7.23, this is impossible by Proposition 8.30. Hence, T' is lonely on B.

Define C to be the event that results from B by changing the labels $U(e)$ for $e \in \mathcal{P}$ to $pU(e)$. This decreases all the labels in $\mathcal{P}$ and also makes them smaller than p. Let $\mathfrak{F}''_{\mathrm{f}}$ be the new free minimal spanning forest and T'' be the component of $\mathfrak{F}''_{\mathrm{f}}$ that contains x_1. By Lemma 11.5, $\mathfrak{F}''_{\mathrm{f}} \triangle \mathfrak{F}_{\mathrm{f}}$ is finite. On C, the path $\mathcal{P}$ belongs to a single infinite cluster of $G[p]$. Further, $\mathcal{P} \subset \mathfrak{F}''_{\mathrm{f}}$ on C since all cycles containing an edge of $\mathcal{P}$ must contain an edge with label larger than p. In addition, $T' \subseteq T''$ for the same reason (this would not apply to the trees of the wired spanning forest). Likewise, the component of $\mathfrak{F}'_{\mathrm{f}}$ that contains x_n has all its edges in $\mathfrak{F}''_{\mathrm{f}}$, which means that T'' has infinitely many ends and an isolated end (the end of T). Since $\mathbf{P}(C) > 0$ by Lemma 7.23, this is again impossible by Proposition 8.30. This contradiction proves the theorem. ◀

Conjecture 11.13. *If G is a quasi-transitive graph and* $\mathsf{WMSF} \neq \mathsf{FMSF}$*, then* FMSF*-a.s. every tree has infinitely many ends.*

Theorem 11.10 gives one relation between the WMSF and critical Bernoulli percolation. Also, according to Theorem 7.21 and Exercise 11.7, every component of $\mathsf{WMSF}(G)$ intersects some infinite cluster of $G[p]$ for every $p > p_c(G)$, provided G is quasi-transitive. This is not true for general graphs:

▷ **Exercise 11.9.**
Give an infinite connected graph G such that for some $p > p_c(G)$, with positive probability there is some component of the WMSF that intersects no infinite cluster of $G[p]$.

The next result gives a relation between the FMSF and Bernoulli(p_u) percolation.

Proposition 11.14. *Under the standard coupling, a.s. each component of* $\mathsf{FMSF}(G)$ *intersects at most one infinite cluster of* $G[p_u]$*. Thus, the number of trees in* $\mathsf{FMSF}(G)$ *is a.s. at least the number of infinite clusters in* $G[p_u]$*. If G is quasi-transitive with* $p_u(G) > p_c(G)$*, then a.s. each component of* $\mathsf{FMSF}(G)$ *intersects exactly one infinite cluster of* $G[p_u]$*.*

Proof. Let $\langle p_j \rangle$ be a sequence satisfying $\lim_{j\to\infty} p_j = p_u$ that is contained in the set of $p \in [p_u, 1]$ such that there is a.s. a unique infinite cluster in $G[p]$. Let $\mathcal{P}$ be a finite simple path in G, and let $\mathscr{A}$ be the event that $\mathcal{P} \subset \mathfrak{F}_f$ and the endpoints of $\mathcal{P}$ are in distinct infinite p_u-clusters. Since a.s. for every $j = 1, 2, \ldots$ there is a unique infinite cluster in $G[p_j]$, a.s. on $\mathscr{A}$ there is a path joining the endpoints of $\mathcal{P}$ in $G[p_j]$. Because $\mathcal{P} \subset \mathfrak{F}_f$ on $\mathscr{A}$, a.s. on $\mathscr{A}$ we have $\max_{\mathcal{P}} U \le p_j$. Thus, $\max_{\mathcal{P}} U \le p_u$ a.s. on $\mathscr{A}$. On the other hand, $\max_{\mathcal{P}} U \ge p_u$ a.s. on $\mathscr{A}$ since on $\mathscr{A}$, the endpoints of $\mathcal{P}$ are in distinct p_u components. This implies $\mathbf{P}[\mathscr{A}] \le \mathbf{P}[\max_{\mathcal{P}} U = p_u] = 0$, and the first statement follows.

The second sentence follows from the fact that every vertex belongs to some component of $\mathfrak{F}_f$. Finally, the third sentence follows from Theorem 7.21 and the fact that invasion trees are contained in the wired minimal spanning forest, which, in turn, is contained in the free minimal spanning forest. ◀

There is a related conjecture of Benjamini and Schramm (personal communication, 1998):

Conjecture 11.15. *Let G be a quasi-transitive non-amenable graph. Then* FMSF *is a single tree a.s. iff there is a unique infinite cluster in* $G[p_u]$ *a.s.*

We can strengthen this conjecture to say that the number of trees in the FMSF equals the number of infinite clusters at p_u. An even stronger conjecture would be that in the natural coupling of Bernoulli percolation and the FMSF, each infinite cluster at p_u intersects exactly one component of the FMSF and each component of the FMSF intersects exactly one infinite cluster at p_u.

Question 11.16. Must the number of trees in the FMSF and the WMSF in a quasi-transitive graph be either 1 or ∞ a.s.? This question for $\mathbb{Z}^d$ is due to Newman (1997).

We now prove an analogue of Theorem 10.33; recurrence for the wired spanning forest there is replaced by $p_c = 1$ here. We show this, in fact, for something even larger than the trees of the wired minimal spanning forest. Namely, define the ***invasion basin of infinity*** as the set of edges $[x, y]$ such that there do not exist disjoint infinite simple paths from x and y consisting only of edges e satisfying $U(e) < U\big([x, y]\big)$, and denote the invasion basin of infinity by $I(\infty) = I^U(\infty)$. Thus, we have

$$I(\infty) \supset \bigcup_{x \in \mathsf{V}} I(x) \supset \mathfrak{F}_{\mathrm{w}}(U)\,.$$

For an edge e, define

$$Z^U_\infty(e) := Z_\infty(e) := \inf_{\mathcal{P}} \sup \big\{U(f)\,;\; f \in \mathcal{P} \setminus \{e\}\big\}\,,$$

where the infimum is over bi-infinite simple paths that contain e; if there is no such path $\mathcal{P}$, define $Z_\infty(e) := 1$. Similarly to the expression for $\mathfrak{F}_{\mathrm{w}}$ in terms of Z_{w}, we have $\{e\,;\; U(e) < Z_\infty(e)\} \subseteq I(\infty) \subseteq \{e\,;\; U(e) \le Z_\infty(e)\}$ and a.s. $I(\infty) = \{e\,;\; U(e) < Z_\infty(e)\}$.

Theorem 11.17. *Let $G = (\mathsf{V}, \mathsf{E})$ be a graph of bounded degree. Then $p_c\big(I(\infty)\big) = 1$ a.s. Therefore $p_c(\mathfrak{F}_{\mathrm{w}}) = 1$ a.s.*

To prove this, we begin with the following lemma that will provide a coupling between percolation and invasion that is different from the usual one we have been working with.

Lemma 11.18. *Let $G = (\mathsf{V}, \mathsf{E})$ be a locally finite infinite graph and $\langle U(e)\,;\; e \in \mathsf{E}\rangle$ be i.i.d. uniform $[0, 1]$ random variables. Let $A \subset \mathsf{E}$ be finite. Conditioned on $A \subset I(\infty)$, the random variables*

$$\frac{U(e)}{Z_\infty(e)} \qquad (e \in A)$$

are i.i.d. uniform $[0, 1]$.

We will use the result of the following exercise:

▷ **Exercise 11.10.**

Let $\langle U_i\,;\; 1 \le i \le k\rangle$ be a random vector distributed uniformly in $[0, 1]^k$, and let $\langle Z_i\,;\; 1 \le i \le k\rangle$ be an independent random vector with an arbitrary distribution in $(0, 1]^k$. Then given $U_i < Z_i$ for all $1 \le i \le k$, the conditional law of the vector $\langle U_i/Z_i\,;\; 1 \le i \le k\rangle$ is uniform in $[0, 1]^k$.

Proof of Lemma 11.18. The essence of the proof is the intuitive fact that $U\restriction I(\infty)$ and $Z_\infty \restriction I(\infty)$ are independent. This is reasonable since no edge in $I(\infty)$ can be the highest edge in any bi-infinite simple path. Let $A \subset \mathsf{E}$ be finite. Define $\widetilde{U}(e) := 0$ for $e \in A$ and $\widetilde{U}(e) := U(e)$ for $e \notin A$, and let $Z_A^U := Z_\infty^U \restriction A$ denote the restriction of Z_∞^U to A. Certainly $Z_A^{\widetilde{U}}$ is independent of $U\restriction A$.

We claim that on the event $\left[A \subset I^U(\infty)\right]$, we have $Z_A^U = Z_A^{\widetilde{U}}$. Indeed, consider any bi-infinite simple path $\mathcal{P}$. If $e \in I^U(\infty) \cap \mathcal{P}$, then $U(e) < \sup\{U(e')\,;\ e \neq e' \in \mathcal{P}\}$. Hence, for every such $\mathcal{P}$,

$$\sup_{\mathcal{P}} U = \sup_{\mathcal{P}\setminus A} U = \sup_{\mathcal{P}\setminus A} \widetilde{U} = \sup_{\mathcal{P}} \widetilde{U}$$

on the event $\left[A \subset I^U(\infty)\right]$. This proves the claim.

One consequence is that conditioning on the event $\left[A \subset I^U(\infty)\right]$ is a.s. the same as conditioning on $\left[U\restriction A < Z_A^{\widetilde{U}}\right]$. Indeed, the second event is contained in the first event because $Z_A^{\widetilde{U}}(e) \le Z_A^U(e)$ for all $e \in A$. For the converse, $A \subset I^U(\infty)$ implies that $U\restriction A < Z_A^U = Z_A^{\widetilde{U}}$ a.s.

Thus, the distribution of $\langle U(e)/Z_A^U(e)\,;\ e \in A\rangle$ conditional on $A \subset I^U(\infty)$ is a.s. the same as the distribution of $\langle U(e)/Z_A^{\widetilde{U}}(e)\,;\ e \in A\rangle$ conditional on $U\restriction A < Z_A^{\widetilde{U}}$. By Exercise 11.10, this distribution is uniform on $[0,1]^A$, as desired. ◀

We also need the following fact.

Lemma 11.19. *If a graph H of bounded degree does not contain a simple bi-infinite path, then $p_c^{\text{bond}}(H) = 1$.*

Proof. If x is a vertex in H, then each pair of infinite paths from x must intersect at some point other than x. Fix an infinite path $\mathcal{P}$ from x. If there were no vertex $y \neq x$ on $\mathcal{P}$ such that every other infinite path from x goes through y, then by taking a limit of a sequence of paths that intersect $\mathcal{P}$ at further and further locations, we would obtain an infinite path from x that did not intersect $\mathcal{P}$ at any point other than x. Thus, there is a vertex $y \neq x$ on $\mathcal{P}$ that every infinite path from x includes. That is, removal of y leaves x in a finite component. We can then repeat the argument with paths starting at y and eventually find infinitely many vertices z such that x is in a finite component of $H \setminus \{z\}$. Since H has bounded degree, it follows that $p_c^{\text{bond}}(H) = 1$. (Even without bounded degree, we get that $p_c^{\text{site}}(H) = 1$.) ◀

Proof of Theorem 11.17. A random subset ω of E is Bernoulli(p) percolation on $I(\infty)$ iff $\omega \subseteq I(\infty)$ a.s. and for all finite $A \subset \mathsf{E}$, the probability that $A \subseteq \omega$ given that $A \subset I(\infty)$ is $p^{|A|}$. Let ω_p be the set of edges e satisfying $U(e) < p\,Z_\infty(e)$. Lemma 11.18 implies that

ω_p has the law of Bernoulli(p) percolation on $I(\infty)$. Thus, by Lemma 11.19, it suffices to show that ω_p does not contain any simple bi-infinite path. Let $\mathcal{P}$ be a simple bi-infinite path and $\alpha := \sup_{e\in\mathcal{P}} U(e)$. If $\mathcal{P} \subset \omega_p$, then we would have, for $e \in \mathcal{P}$,

$$U(e) < pZ_\infty(e) \le p \sup_{\mathcal{P}} U = p\,\alpha\,.$$

This would imply that $\alpha \le p\,\alpha$, whence $\alpha = 0$, which is clearly impossible. So $\mathcal{P}$ is not a subset of ω_p, as desired. ◀

A dual argument shows that the FMSF is almost connected in the following sense.

Theorem 11.20. *Let G be any locally finite connected graph and $\epsilon \in (0,1)$. Let $\mathfrak{F}_\mathrm{f}$ be a configuration of the* FMSF *and ω be an independent copy of $G[\epsilon]$. Then $\mathfrak{F}_\mathrm{f} \cup \omega$ is connected a.s.*

For this, we use a lemma dual to Lemma 11.18; it will provide a coupling of $\mathfrak{F}_\mathrm{f}$ and $\mathfrak{F}_\mathrm{f} \cup \omega$.

Lemma 11.21. *Let $G = (\mathsf{V}, \mathsf{E})$ be a locally finite infinite graph and $\langle U(e)\,;\ e \in \mathsf{E}\rangle$ be i.i.d. uniform $[0,1]$ random variables. Let $A \subset \mathsf{E}$ be a finite set such that $\mathbf{P}[A \cap \mathfrak{F}_\mathrm{f} = \varnothing] > 0$. Conditioned on $A \cap \mathfrak{F}_\mathrm{f} = \varnothing$, the random variables*

$$\frac{1 - U(e)}{1 - Z_\mathrm{f}(e)} \qquad (e \in A)$$

are i.i.d. uniform $[0,1]$.

Proof. Let $A \subset \mathsf{E}$ be a finite set such that $\mathbf{P}[A \cap \mathfrak{F}_\mathrm{f} = \varnothing] > 0$. Let $\widetilde{U}(e) := 1$ for $e \in A$ and $\widetilde{U}(e) := U(e)$ for $e \notin A$, and let Z_A^U denote the restriction of Z_f^U to A. Consider any cut Π. If $e \in A \setminus \mathfrak{F}_\mathrm{f}(U)$, then $U(e) > Z_\mathrm{f}^U(e) \ge \inf\big\{U(e')\,;\ e' \in \Pi \setminus \{e\}\big\}$ by (11.1). Hence, if $A \cap \mathfrak{F}_\mathrm{f} = \varnothing$, then for every cut Π,

$$\inf_{\Pi} U = \inf_{\Pi\setminus A} U = \inf_{\Pi\setminus A} \widetilde{U} = \inf_{\Pi} \widetilde{U}\,,$$

and therefore (still assuming that $A \cap \mathfrak{F}_\mathrm{f} = \varnothing$) $Z_A^U = Z_A^{\widetilde{U}}$. Hence $A \cap \mathfrak{F}_\mathrm{f}(U) = \varnothing$ implies $U > Z_A^{\widetilde{U}}$ on A. In fact, $A \cap \mathfrak{F}_\mathrm{f}(U) = \varnothing$ is equivalent to $U > Z_A^{\widetilde{U}}$ on A, because $Z_A^{\widetilde{U}} \ge Z_A^U$. Thus (by Exercise 11.10), conditioned on $A \cap \mathfrak{F}_\mathrm{f}(U) = \varnothing$, the random variables $\langle (1 - U(e))/(1 - Z_\mathrm{f}(e))\,;\ e \in A\rangle = \langle (1 - U(e))/(1 - Z_A^{\widetilde{U}}(e))\,;\ e \in A\rangle$ are i.i.d. uniform in $[0,1]$. ◀

Proof of Theorem 11.20. According to Lemma 11.21, $\mathfrak{F}_{\mathrm{f}} \cup \omega$ has the same law as

$$\xi := \big\{e\,;\ 1 - U(e) \ge (1-\epsilon)[1 - Z_{\mathrm{f}}(e)]\big\}\,. \tag{11.3}$$

Thus, it suffices to show that ξ is connected a.s. Consider any nonempty cut Π in G, and let $\alpha := \inf_{e\in\Pi} U(e)$. Then $1 - \alpha = \sup_\Pi(1-U)$, so we may choose $e \in \Pi$ to satisfy $1 - U(e) \ge (1-\epsilon)(1-\alpha)$. By (11.1), $Z_{\mathrm{f}}(e) \ge \inf_{\Pi\setminus\{e\}} U \ge \alpha$, whence $e \in \xi$. Since ξ intersects every nonempty cut, it is connected. ◀

§11.5. Planar Graphs.

When we add planar duality to our tools, it will be easy to deduce all the major properties of both the free and wired minimal spanning forests on planar quasi-transitive graphs.

Recall the definition (10.6)

$$e \in T \iff e^\dagger \notin T^\times\,.$$

Proposition 11.22. *Let G and $G^\dagger$ be proper locally finite dual plane graphs. For any injection $U : \mathsf{E} \to \mathbb{R}$, let $U^\dagger(e^\dagger) := 1 - U(e)$. We have*

$$\big(\mathfrak{F}_{\mathrm{f}}(U,G)\big)^\times = \big\{e^\dagger\,;\ U^\dagger(e^\dagger) < Z_{\mathrm{w}}^{U^\dagger}(e^\dagger)\big\}\,,$$

whence $\big(\mathfrak{F}_{\mathrm{f}}(U,G)\big)^\times = \mathfrak{F}_{\mathrm{w}}(U^\dagger, G^\dagger)$ if $U^\dagger(e^\dagger) \ne Z_{\mathrm{w}}^{U^\dagger}(e^\dagger)$ for all $e^\dagger \in \mathsf{E}^\dagger$.

Proof. The Jordan curve theorem implies that a set $\mathcal{P} \subset \mathsf{E}\setminus\{e\}$ is a simple path between the endpoints of e iff the set $\Pi := \{f^\dagger\,;\ f \in \mathcal{P}\} \cup \{e^\dagger\}$ is a cut of a finite set. Thus $Z_{\mathrm{f}}^U(e) = 1 - Z_{\mathrm{w}}^{U^\dagger}(e^\dagger)$ by (11.2). This means that $e^\dagger \in \big(\mathfrak{F}_{\mathrm{f}}(U,G)\big)^\times$ iff $e \notin \mathfrak{F}_{\mathrm{f}}(U,G)$ iff $U(e) > Z_{\mathrm{f}}^U(e)$ iff $U^\dagger(e^\dagger) < Z_{\mathrm{w}}^{U^\dagger}(e^\dagger)$. ◀

The following corollary is proved in the same way that Proposition 10.36 is proved.

Corollary 11.23. *Let G be a proper plane graph with $G^\dagger$ locally finite. If each tree of the* WMSF *of G has only one end a.s., then the* FMSF *of $G^\dagger$ has only one tree a.s. If, in addition, the* WMSF *of G has infinitely many trees a.s., then the tree of the* FMSF *of $G^\dagger$ has infinitely many ends a.s.*

The following result is due to Alexander and Molchanov (1994).

Corollary 11.24. *The minimal spanning forest of $\mathbb{Z}^2$ is a.s. a tree with one end.*

Proof. The hypothesis $\theta(p_c) = 0$ of Theorem 11.10 applies by Harris (1960) and Kesten (1980). Therefore, each tree in the WMSF has one end. By Corollary 11.23, this means that the FMSF has one tree. On the other hand, the wired and free measures are the same by Exercise 11.8.

However, we can get by in our proof with less than Kesten's theorem, namely, with only Harris's Theorem 7.16. To see this, consider the labels $U^\dagger(e^\dagger) := 1 - U(e)$ on $(\mathbb{Z}^2)^\dagger$. Let U be an injective $[0,1]$-labeling where all the $(1/2)$-clusters in $\mathbb{Z}^2$ and $(\mathbb{Z}^2)^\dagger$ are finite and $U(e) \neq Z_{\mathrm{w}}^U(e)$ for all $e \in \mathsf{E}$, which happens a.s. for the standard labeling. Suppose that $U(e) \le 1/2$. We claim that the endpoints of e belong to the same tree in $\mathfrak{F}_{\mathrm{w}}(U, \mathbb{Z}^2)$. Indeed, the invasion basin of e^- contains e by our assumption, whence the invasion tree of e^- contains e^+.

Therefore, if $\mathfrak{F}_{\mathrm{w}}(U, \mathbb{Z}^2)$ contains more than one tree, all edges joining two of its components have labels larger than $1/2$. If F is the edge boundary of one of the components, then $F^\dagger$ contains an infinite path with labels all less than $1/2$, which contradicts our assumption. This proves that $\mathfrak{F}_{\mathrm{w}}(U, \mathbb{Z}^2)$ is one tree. So is $\mathfrak{F}_{\mathrm{f}}\big(U^\dagger, (\mathbb{Z}^2)^\dagger\big)$, whence $\mathfrak{F}_{\mathrm{w}}(U, \mathbb{Z}^2) = \big(\mathfrak{F}_{\mathrm{f}}(U^\dagger, G^\dagger)\big)^\times$ has just one end. ◀

The same reasoning shows:

Proposition 11.25. *Let G be a connected non-amenable quasi-transitive planar graph with one end. Then the FMSF on G is a.s. a tree.*

The nonamenability assumption can be replaced by the assumption that the planar dual of G satisfies $\theta(p_c) = 0$. The latter assumption is known to hold in many amenable cases (see Kesten (1982)).

Proof. Let G be such a graph. By Theorem 8.22, $\mathrm{Aut}(G)$ is unimodular and we may embed G so that $\mathrm{Aut}(G^\dagger)$ is also unimodular. By Exercise 6.25, the graph $G^\dagger$ is also non-amenable. Thus, we may apply Theorem 8.19 to $G^\dagger$ to see that $\theta(p_c, G^\dagger) = 0$. Theorem 11.10 and Corollary 11.23 now yield the desired conclusion. ◀

We may also use similar reasoning to give another proof of the bond percolation part of Theorem 8.21.

Corollary 11.26. *If G is a connected non-amenable quasi-transitive planar graph with one end, then for bond percolation, $p_c(G) < p_u(G)$. In addition, there is a unique infinite cluster in Bernoulli($p_u(G)$) bond percolation.*

Proof. Again, by Theorem 8.22, $\mathrm{Aut}(G)$ is unimodular and we may embed G so that $\mathrm{Aut}(G^\dagger)$ is also unimodular. By Theorem 7.20 and Proposition 11.6, it suffices to show that $\mathsf{WMSF} \neq \mathsf{FMSF}$ on G. Indeed, if they were the same, then they would also be the same on $G^\dagger$, so that each would be one tree with one end, as in the proof of Proposition 11.25. But this is impossible by Exercise 8.11.

Furthermore, by Proposition 11.25, the FMSF is a tree on G, whence by Proposition 11.14, there is a unique infinite cluster in Bernoulli(p_{u}) percolation on G. ◀

§11.6. Non-Treeable Groups.

We don't know any good way to tell when the free minimal spanning forest is a.s. a single tree, even for most $\mathbb{Z}^d$. So far in this chapter, we have not presented a single example of a Cayley graph where it is not a tree. In fact, we don't know of any direct way to do this, but here, we will show that some groups don't admit *any* invariant random spanning tree!

Theorem 11.27. (Non-Treeable Products) *Let Γ and Δ be infinite countable groups and G be a Cayley graph of $\Gamma \times \Delta$. If there is a random invariant spanning tree of G, then G is amenable.*

Proof. Let H be a Cayley graph of Γ. We'll use the spanning tree to create an invariant percolation on H with finite clusters and with expected degree arbitrarily close to the degree of H; amenability of H (and hence of Γ) then follows from Theorem 8.14. By symmetry, Δ is also amenable, whence so is G by Exercise 6.19. To create this percolation on H, we first define an equivalence relation on Γ.

Let T be a spanning tree of G. Write o for the identities of Γ and Δ. Write Γ_n for the points in G that lie within distance n of $\Gamma \times \{o\}$ and similarly for Δ_n. For $n \geq 1$, let $\delta_n \in \Delta$ be any element such that the distance in G from (o, o) to (o, δ_n) is at least $2n + 1$. Given n and r, let $C(T, n, r) \in \Gamma \times \Gamma$ consist of the pairs (γ, γ') such that the following four properties hold:

(i) the path in T from (γ, o) to (γ', o) lies in Γ_n;
(ii) the path in T from (γ, δ_n) to (γ', δ_n) lies in $\delta_n \Gamma_n$;
(iii) the path in T from (γ, o) to (γ, δ_n) lies in $\gamma \Delta_r$; and
(iv) the path in T from (γ', o) to (γ', δ_n) lies in $\gamma' \Delta_r$.

It is easy to see that $C(T, n, r)$ is an equivalence relation for each n and r.

Now let T be random with a $\Gamma \times \Delta$-invariant law. Since $C(\gamma T, n, r) = \gamma C(T, n, r)$ for every $\gamma \in \Gamma$, the law of $C(T, n, r)$ is Γ-invariant. The probabilities of the events in (i) and (ii), which are the same, tend to 1 as $n \to \infty$. Given n, the probabilities of the events in (iii) and (iv) tend to 1 as $r \to \infty$. Thus, given any γ and γ', we may choose n and r large enough that $\mathbf{P}[(\gamma, \gamma') \in C(T, n, r)]$ is as close as desired to 1. On the other hand, when $(\gamma, \gamma') \in C(T, n, r)$, we may concatenate the paths in T from (γ, o) to (γ', o) to (γ', δ_n) to (γ, δ_n) to (γ, o). Since T contains no cycles, this means that $\gamma\Delta_r \cap \gamma'\Delta_r \neq \varnothing$, whence γ and γ' lie within distance $2r$ of each other. Thus, the equivalence classes of $C(T, n, r)$ are finite. To make a percolation out of them, just take $C(T, n, r) \cap \mathsf{E}(H)$. ◀

§11.7. Open Questions.

Conjecture 11.28. *The components of the* FMSF *on a unimodular transitive graph are indistinguishable in the sense that for every automorphism-invariant property $\mathcal{A}$ of subgraphs, either a.s. all components satisfy $\mathcal{A}$ or a.s. all do not. The same holds for the* WMSF.

This fails in the nonunimodular setting, as the example in Lyons and Schramm (1999) shows.

Conjecture 11.29. *Let T_o be the component of the identity o in the* WMSF *on a Cayley graph, and let $\xi = \langle v_n \,;\, n \geq 0 \rangle$ be the unique ray from o in T_o. The sequence of "bushes" $\langle b_n \rangle$ observed along ξ converges in distribution. (Formally, b_n is the connected component of v_n in $T \setminus \{v_{n-1}, v_{n+1}\}$, multiplied on the left by v_n^{-1}.)*

Question 11.30. For which d is the minimal spanning forest of $\mathbb{Z}^d$ a.s. a tree? This question is due to Newman and Stein (1996), who conjecture that the answer is $d < 8$ or $d \leq 8$. Jackson and Read (2009a, 2009b) suggest instead that the answer is $d < 6$ or $d \leq 6$.

Question 11.31. One may consider the minimal spanning tree on $\epsilon\mathbb{Z}^2 \subset \mathbb{R}^2$ and let $\epsilon \to 0$. It would be interesting to show that the limit exists in various senses. Aizenman, Burchard, Newman, and Wilson (1999) have shown that a subsequential limit exists. According to simulations of Wilson (2004a), the scaling limit in a simply connected domain with free or wired boundary conditions does not have the conformal invariance property one might expect. This contrasts with the situation of the uniform spanning forest, where the limit exists and is conformally invariant, as was proved by Lawler, Schramm, and Werner (2004).

Question 11.32. If G is a graph that is roughly isometric to a tree, then is the free minimal spanning forest on G a.s. a tree?

§11.8. Notes.

Proposition 11.4 was first proved by Chayes, Chayes, and Newman (1985) for $\mathbb{Z}^2$, then by Alexander (1995) for all $\mathbb{Z}^d$, and finally by Lyons, Peres, and Schramm (2006) for general graphs.

Lemma 11.5 is a strengthening due to Lyons, Peres, and Schramm (2006) of Theorem 5.1(i) of Alexander (1995).

All other results in this chapter that are not explicitly attributed are due to Lyons, Peres, and Schramm (2006).

A curious comparison of FSF and FMSF was found by Lyons, Peres, and Schramm (2006), extending work of Lyons (2000) and Gaboriau (2005): Let $\overline{\deg}(\mu)$ denote the expected degree of a vertex under an automorphism-invariant percolation μ on a transitive graph (so that it is the same for all vertices).

Proposition 11.33. *Let $G = (\mathsf{V}, \mathsf{E})$ be a transitive unimodular connected infinite graph of degree d. Then*

$$\overline{\deg}(\mathsf{FSF}) \le \overline{\deg}(\mathsf{FMSF}) \le 2 + d \int_{p_c}^{p_u} \theta(p)^2 \, dp \,.$$

It is not known whether the same holds on all transitive graphs.

Some examples of particular behaviors of minimal spanning forests are given by Lyons, Peres, and Schramm (2006); they are somewhat hard to prove, but worth recounting here. Namely, there are examples of the following: a planar graph whose free and wired minimal spanning forests are equal and have two components; a planar graph such that the number of trees in the wired spanning forest is not an a.s. constant; a planar graph such that the number of trees in the free spanning forest is not an a.s. constant; and a graph for which $\mathsf{WSF} \neq \mathsf{FSF}$ and $\mathsf{WMSF} = \mathsf{FMSF}$ (unlike the situation in Proposition 11.33).

Theorem 11.27 is essentially taken from Pemantle and Peres (2000), which contains generalizations. A precursor is in Adams (1988). Gaboriau (2000) (Cor. VI.22) shows that groups of cost 1 are not treeable; this implies Theorem 11.27. Cayley graphs of Kazhdan groups also are not treeable; a version of this result appears in Adams and Spatzier (1990).

▷ **Exercise 11.11.**
Call a graph G ***almost treeable*** if there exists a sequence of $\mathrm{Aut}(G)$-invariant spanning forests $\mathfrak{F}_n$ on G with the property that for all $x, y \in \mathsf{V}(G)$, we have $\lim_{n\to\infty} \mathbf{P}[x \leftrightarrow y \text{ in } \mathfrak{F}_n] = 1$. Use Theorem 7.44 to show that Cayley graphs of Kazhdan groups are not almost treeable.

§11.9. Collected In-Text Exercises.

11.1. Show that among all spanning trees, T_U has minimal edge-label sum, $\sum_{e\in T} U(e)$.

11.2. Construct G as follows. Begin with the complete graph, K_4. Let e and f be two of its edges that do not share endpoints. Replace e by three edges in parallel, e_1, e_2, and e_3, that have the same endpoints as e. Likewise, replace f by three parallel edges f_i. Show that $\mathsf{MST}[e_1, f_1 \in T] > \mathsf{MST}[e_1 \in T]\mathsf{MST}[f_1 \in T]$.

11.3. The following difference from the uniform spanning tree must be kept in mind: Show that given an edge, e, the minimal spanning tree measure on G conditioned on the event not to contain e need not be the same as the minimal spanning tree measure on $G \setminus e$, the graph G with e deleted.

11.4. Show that there is a finite rooted graph with a given function f on the vertices such that there are two spanning trees for each of which the distance from x to the root in the tree is $f(x)$, yet those trees are not equally likely under the minimal spanning tree measure.

11.5. Show that $\mathfrak{F}_{\mathrm{w}}(U)$ consists of those edges e such that there is a finite set of vertices $W \subset \mathsf{V}$ such that e is the least edge joining W to $\mathsf{V} \setminus W$.

11.6. Consider an increasing sequence of finite, nonempty, connected subgraphs $G_n \subset G$, $n \in \mathbb{N}$, such that $\bigcup_n G_n = G$. For $n \in \mathbb{N}$, let G_n^{W} be the graph obtained from G by identifying the vertices outside of G_n to a single vertex, then removing all resulting loops based at that vertex. Let $T_n(U)$ and $T_n^{\mathrm{W}}(U)$ denote the minimal spanning trees on G_n and G_n^{W}, respectively, that are induced by the labels U. Show that $\mathfrak{F}_{\mathrm{f}}(U) = \lim_{n\to\infty} T_n(U)$ and, provided G_n is the subgraph of G induced by $\mathsf{V}(G_n)$, that $\mathfrak{F}_{\mathrm{w}}(U) = \lim_{n\to\infty} T_n^{\mathrm{W}}(U)$ in the sense that for every $e \in \mathfrak{F}_{\mathrm{f}}(U)$, we have $e \in T_n(U)$ for every sufficiently large n, for every $e \notin \mathfrak{F}_{\mathrm{f}}(U)$ we have $e \notin T_n(U)$ for every sufficiently large n, and similarly for $\mathfrak{F}_{\mathrm{w}}(U)$. Deduce that for any G_n (not necessarily induced), $T_n(U) \Rightarrow \mathsf{FMSF}$ and $T_n^{\mathrm{W}}(U) \Rightarrow \mathsf{WMSF}$.

11.7. Let $U : \mathsf{E} \to \mathbb{R}$ be an injective labeling of the edges of a locally finite graph $G = (\mathsf{V}, \mathsf{E})$. Show that the union $\bigcup_{x\in\mathsf{V}} T_U(x)$ of all the invasion trees is equal to $\mathfrak{F}_{\mathrm{w}}(U)$.

11.8. Let G be a connected locally finite graph. Prove the following.

(a) If G is edge-amenable, then the average degree of vertices in both the free and wired minimal spanning forests on G is a.s. 2.

(b) The free and wired minimal spanning forests on G are the same if they have a.s. the same finite number of trees, or if the expected degree of every vertex is the same for both measures.

(c) The free and wired minimal spanning forests on G are the same on any transitive amenable graph.

(d) If $\mathfrak{F}_{\mathrm{w}}$ is connected a.s. or if each component of $\mathfrak{F}_{\mathrm{f}}$ has a.s. one end, then $\mathsf{WMSF}(G) = \mathsf{FMSF}(G)$.

(e) If G is unimodular and transitive with $\mathsf{WMSF}(G) \neq \mathsf{FMSF}(G)$, then a.s. the FMSF has a component with uncountably many ends, in fact, with $p_{\mathrm{c}} < 1$.

11.9. Give an infinite connected graph G such that for some $p > p_{\mathrm{c}}(G)$, with positive probability there is some component of the WMSF that intersects no infinite cluster of $G[p]$.

11.10. Let $\langle U_i\,;\ 1 \le i \le k\rangle$ be a random vector distributed uniformly in $[0,1]^k$, and let $\langle Z_i\,;\ 1 \le i \le k\rangle$ be an independent random vector with an arbitrary distribution in $(0,1]^k$. Then given $U_i < Z_i$ for all $1 \le i \le k$, the conditional law of the vector $\langle U_i/Z_i\,;\ 1 \le i \le k\rangle$ is uniform in $[0,1]^k$.

11.11. Call a graph G ***almost treeable*** if there exists a sequence of $\mathrm{Aut}(G)$-invariant spanning forests $\mathfrak{F}_n$ on G with the property that for all $x, y \in \mathsf{V}(G)$, we have $\lim_{n\to\infty} \mathbf{P}[x \leftrightarrow y \text{ in } \mathfrak{F}_n] = 1$. Use Theorem 7.44 to show that Cayley graphs of Kazhdan groups are not almost treeable.

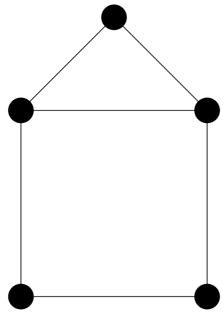


Figure 11.2.

§11.10. Additional Exercises.

11.12. Let G be the graph in Figure 11.2. There are 11 spanning trees of G. Show that under the minimal spanning tree measure, they are not all equally likely and calculate their probabilities. Show, however, that there are conductances such that the corresponding uniform spanning tree measure equals the minimal spanning tree measure.

11.13. Let G be a complete graph on 4 vertices (i.e., all pairs of vertices are joined by an edge). Calculate the minimal spanning tree measure and show that there are no conductances that give the minimal spanning tree measure.

11.14. Show that the FMSF of the usual Cayley graph of $\mathbb{Z}_2 * \mathbb{Z}_3$ is a tree whose branching number equals 1.35^+.

11.15. Given a finitely generated group Γ, does the expected degree of a vertex in the FMSF of a Cayley graph of Γ depend on which Cayley graph is used? As discussed in Section 10.2, the analogous result is true for the FSF.

11.16. Let T be a 3-regular tree. Calculate the chance that a given vertex is a leaf in the wired minimal spanning forest on T.

11.17. Show that for the ladder graph of Exercise 4.2, the minimal spanning forest is a tree and calculate the chance that the bottom rung of the ladder is in the minimal spanning tree.

11.18. Let $f(p)$ be the probability that two given neighbors in $\mathbb{Z}^d$ are in different components in Bernoulli(p) percolation. Show that

$$\int_0^1 \frac{f(p)}{1-p}\, dp = \frac{1}{d}\,.$$

11.19. Let G be a connected graph. Let $\alpha(x_1, \ldots, x_K)$ be the probability that $I(x_1), \ldots, I(x_K)$ are pairwise vertex-disjoint. Show that the WMSF-essential supremum of the number of trees is

$$\sup\{K\,;\ \exists x_1, \ldots, x_K \in \mathsf{V} \quad \alpha(x_1, \ldots, x_K) > 0\}\,.$$

11.20. Show that the total number of ends of all trees in either minimal spanning forest is an a.s. constant.

11.21. Show that if G is a unimodular quasi-transitive graph and $\theta(p_c, G) = 0$, then a.s. each component of the WMSF has one end.

11.22. Theorem 11.17 was stated for bounded degree graphs. Prove that if $G = (\mathsf{V}, \mathsf{E})$ is an infinite graph, then the WMSF $\mathfrak{F}_{\mathrm{w}}$ satisfies $p_{\mathrm{c}}(\mathfrak{F}_{\mathrm{w}}) = 1$ a.s., and moreover, $\bigcup_{v \in \mathsf{V}} I(v)$ has $p_{\mathrm{c}} = 1$.

11.23. Show that if G is a unimodular transitive locally finite connected graph, then $p_{\mathrm{c}}(G) < p_{\mathrm{u}}(G)$ iff $p_{\mathrm{c}}(\mathfrak{F}_{\mathrm{f}}) < 1$ a.s.

11.24. Let G be a plane regular graph of degree d with regular dual of degree $d^\dagger$. Show that the FMSF-expected degree of each vertex in G is $d(1 - 2/d^\dagger)$.

11.25. Let G be the usual Cayley graph of the (p, q, r)-triangle group, where $1/p + 1/q + 1/r \le 1$, shown in Figure 6.1 for $(2, 3, 7)$. It has 3 generators, which are reflections in the sides of a fundamental triangle. Show that the expected degree of the FMSF of G is $3 - 1/p - 1/q - 1/r$.

11.26. Consider the Cayley graph corresponding to the presentation $\langle a, b, c, d \mid a^2, b^2, c^2, abd^{-1} \rangle$. Show that the expected degree of a vertex in the FMSF is 17/5.

Chapter 12

Limit Theorems for Galton-Watson Processes

Our first section contains a conceptual tool that will be used for proving some classical limit theorems for Galton-Watson branching processes.

§12.1. Size-biased Trees and Immigration.

This section and the next two are adapted from Lyons, Pemantle, and Peres (1995a). Recall from Section 5.1

The Kesten-Stigum Theorem (1966). *Let L be the offspring random variable of a Galton-Watson process with mean $m \in (1, \infty)$ and martingale limit W. The following are equivalent:*
(i) $\mathbf{P}[W = 0] = q$*;*
(ii) $\mathbf{E}[W] = 1$*;*
(iii) $\mathbf{E}[L \log^+ L] < \infty$*.*

Although condition (iii) appears technical, there is a conceptual proof of the theorem that uses only the crudest estimates. The dichotomy of Corollary 5.7 as expanded in the Kesten-Stigum Theorem turns out to arise from the following elementary dichotomy:

Lemma 12.1. *Let $X, X_1, X_2, \ldots$ be nonnegative i.i.d. random variables. Then a.s.*

$$\limsup_{n\to\infty} \frac{1}{n} X_n = \begin{cases} 0 & \textit{if } \mathbf{E}[X] < \infty\,, \\ \infty & \textit{if } \mathbf{E}[X] = \infty\,. \end{cases}$$

▷ **Exercise 12.1.**
Prove this by using the Borel-Cantelli lemma.

▷ **Exercise 12.2.**
Given X, X_n as in Lemma 12.1, show that if $\mathbf{E}[X] < \infty$, then $\sum_n e^{X_n} c^n < \infty$ for all $c \in (0, 1)$, while if $\mathbf{E}[X] = \infty$, then $\sum_n e^{X_n} c^n = \infty$ for all $c \in (0, 1)$.

This dichotomy will be applied to an auxiliary random variable. Let $\widehat{L}$ be a random variable whose distribution is that of ***size-biased*** L; i.e.,

$$\mathbf{P}[\widehat{L} = k] = \frac{kp_k}{m}\,.$$

Then

$$\mathbf{E}[\log^+ \widehat{L}] = \frac{1}{m}\mathbf{E}[L \log^+ L]\,.$$

Lemma 12.1 will be applied to $\log^+ \widehat{L}$.

▷ **Exercise 12.3.**

Let X be a nonnegative random variable with $0 < \mathbf{E}[X] < \infty$. We say that $\widehat{X}$ has the ***size-biased distribution*** of X if $\mathbf{P}[\widehat{X} \in A] = \mathbf{E}[X \mathbf{1}_A(X)]/\mathbf{E}[X]$ for intervals $A \subseteq [0, \infty)$. Show that this is equivalent to $\mathbf{E}[f(\widehat{X})] = \mathbf{E}[Xf(X)]/\mathbf{E}[X]$ for all Borel $f : [0, \infty) \to [0, \infty)$.

▷ **Exercise 12.4.**

Suppose that X_n are nonnegative random variables such that $\mathbf{P}[X_n > 0]/\mathbf{E}[X_n] \to 0$. Show that the size-biased random variables $\widehat{X}_n$ tend to infinity in probability.

We will now define certain "size-biased" random trees, called ***size-biased Galton-Watson*** trees. Note that this, as well as the usual Galton-Watson process, will be a way of putting a measure on the space of trees, which we think of as rooted and labelled. We will show that the Kesten-Stigum dichotomy is equivalent to the following: these two measures on the space of trees are either mutually absolutely continuous or mutually singular. The law of this random tree will be denoted $\widehat{\mathbf{GW}}$, whereas the law of an ordinary Galton-Watson tree is denoted $\mathbf{GW}$.

How can we size bias in a probabilistic manner? Suppose that we have an urn of balls such that the probability of picking a ball numbered k is q_k and $\sum kq_k < \infty$. If, for each k, we replace each ball numbered k with k balls numbered k, then the new probability of picking a ball numbered k is the size-biased probability. Thus, a probabilistic way of biasing $\mathbf{GW}$ according to Z_n is to choose uniformly a random vertex in the nth generation; the resulting joint distribution will be called $\widehat{\mathbf{GW}}_*$. This motivates the following definitions.

For a tree t with Z_n vertices at level n, write $W_n(t) := Z_n/m^n$. For any rooted tree t and any $n \geq 0$, denote by $[t]_n$ the set of rooted trees whose first n levels agree with those of t. (In particular, if the height of t is less than n, then $[t]_n = \{t\}$.) If v is a vertex at the nth level of t, then let $[t; v]_n$ denote the set of ***trees with distinguished paths*** such that the tree is in $[t]_n$ and the path starts from the root, does not backtrack, and goes through v.

In order to construct $\widehat{\mathbf{GW}}$, we will construct a measure $\widehat{\mathbf{GW}}_*$ on the set of infinite trees with infinite distinguished paths; this measure will satisfy

$$\widehat{\mathbf{GW}}_*[t;v]_n = \frac{1}{m^n}\mathbf{GW}[t]_n \tag{12.1}$$

for all n and all $[t;v]_n$ as above. By using the branching property and the fact that the expected number of children of v is m, it is easy to verify consistency of these finite-height distributions. Kolmogorov's existence theorem thus provides such a measure $\widehat{\mathbf{GW}}_*$. However, this verification may be skipped, as we will give a more useful direct construction of a measure with these marginals in a moment.

Note that *if* a measure $\widehat{\mathbf{GW}}_*$ satisfying (12.1) exists, then its projection to the space of trees, which is denoted simply by $\widehat{\mathbf{GW}}$, automatically satisfies

$$\widehat{\mathbf{GW}}[t]_n = W_n(t)\mathbf{GW}[t]_n \tag{12.2}$$

for all n and all trees t. It is for this reason that we call $\widehat{\mathbf{GW}}$ "size-biased".

How do we define $\widehat{\mathbf{GW}}_*$? Assuming still that (12.1) holds, note that the recursive structure of Galton-Watson trees yields a recursion for $\widehat{\mathbf{GW}}_*$. Assume that t is a tree of height at least $n+1$ and that the root of t has k children with descendant trees $t^{(1)}, t^{(2)}, \ldots, t^{(k)}$. Any vertex v in level $n+1$ of t is in one of these, say $t^{(i)}$. Now

$$\mathbf{GW}[t]_{n+1} = p_k \prod_{j=1}^{k} \mathbf{GW}[t^{(j)}]_n = kp_k \cdot \frac{1}{k} \cdot \mathbf{GW}[t^{(i)}]_n \cdot \prod_{j \neq i} \mathbf{GW}[t^{(j)}]_n \,.$$

Thus any measure $\widehat{\mathbf{GW}}_*$ that satisfies (12.1) must satisfy the recursion

$$\widehat{\mathbf{GW}}_*[t;v]_{n+1} = \frac{kp_k}{m} \cdot \frac{1}{k} \cdot \widehat{\mathbf{GW}}_*[t^{(i)};v]_n \cdot \prod_{j \neq i} \mathbf{GW}[t^{(j)}]_n \,. \tag{12.3}$$

Conversely, if a probability measure $\widehat{\mathbf{GW}}_*$ on the set of trees with distinguished paths satisfies this recursion, then by induction it satisfies (12.1); this observation leads to the following direct construction of $\widehat{\mathbf{GW}}_*$.

Recall that $\widehat{L}$ is a random variable whose distribution is that of size-biased L, i.e., $\mathbf{P}[\widehat{L} = k] = kp_k/m$. To construct a size-biased Galton-Watson tree $\widehat{T}$, start with an initial particle v_0. Give it a random number $\widehat{L}_1$ of children, where $\widehat{L}_1$ has the law of $\widehat{L}$. Pick one of these children at random, v_1. Give the *other* children independently ordinary Galton-Watson descendant trees and give v_1 an independent size-biased number $\widehat{L}_2$ of children. Again, pick one of the children of v_1 at random, call it v_2, and give the others ordinary

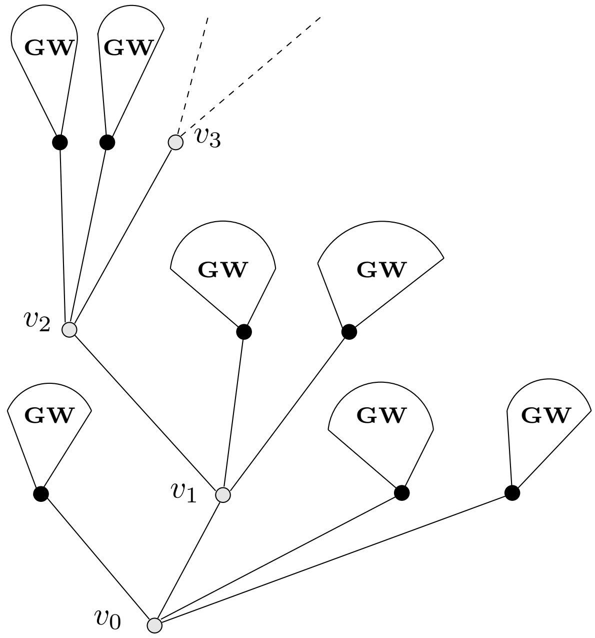


Figure 12.1. Schematic representation of size-biased Galton-Watson trees.

Galton-Watson descendant trees. Continue in this way indefinitely. (See Figure 12.1.) Note that since $\widehat{L} \ge 1$, size-biased Galton-Watson trees are always infinite (there is no extinction).

Now we can finally ***define*** the measure $\widehat{\mathbf{GW}}_*$ as the joint distribution of the random tree $\widehat{T}$ and the random path $\langle v_0, v_1, v_2, \ldots \rangle$. This measure clearly satisfies the recursion (12.3), and hence also (12.1).

Note that, given the first n levels of the tree $\widehat{T}$, the measure $\widehat{\mathbf{GW}}_*$ makes the vertex v_n in the random path $\langle v_0, v_1, \ldots \rangle$ uniformly distributed on the nth level of $\widehat{T}$; this is not obvious from the explicit construction of this random path, but it is immediate from the formula (12.1) in which the right-hand side does not depend on v.

▷ **Exercise 12.5.**

Define $\widehat{\mathbf{GW}}_*$ formally on a space analogous to the space $\mathscr{T}$ of Exercise 5.2 and define $\widehat{\mathbf{GW}}$ formally on $\mathscr{T}$.

The vertices off the distinguished path $\langle v_0, v_1, \ldots \rangle$ of the size-biased tree form a ***branching process with immigration***. In general, such a process is defined by two distributions, an offspring distribution and an immigration distribution. The process starts with no particles, say, and at every generation $n \ge 1$, there is an immigration of Y_n particles, where Y_n are i.i.d. with the given immigration law. Meanwhile, each particle has, independently, an ordinary Galton-Watson descendant tree with the given offspring distribution.

Thus, the $\widehat{\mathbf{GW}}$-law of $Z_n - 1$ is the same as that of the generation sizes of an immigration process with $Y_n = \widehat{L}_n - 1$. The probabilistic content of the assumption $\mathbf{E}[L \log^+ L] < \infty$ will arise in applying Lemma 12.1 to the variables $\langle \log^+ Y_n \rangle$, since $\mathbf{E}[\log^+(\widehat{L} - 1)] = m^{-1}\mathbf{E}[L \log^+(L-1)]$.

§12.2. Supercritical Processes: Proof of the Kesten-Stigum Theorem.

The following lemma is more or less standard.

Lemma 12.2. *Let μ be a finite measure and ν a probability measure on a σ-field $\mathscr{F}$. Suppose that $\mathscr{F}_n$ are increasing sub-σ-fields whose union generates $\mathscr{F}$ and that $(\mu{\restriction}\mathscr{F}_n)$ is absolutely continuous with respect to $(\nu{\restriction}\mathscr{F}_n)$ with Radon-Nikodým derivative X_n. Set $X := \limsup_{n\to\infty} X_n$. Then*

$$\mu \ll \nu \iff X < \infty \quad \mu\text{-a.e.} \iff \int X\, d\nu = \int d\mu$$

and

$$\mu \perp \nu \iff X = \infty \quad \mu\text{-a.e.} \iff \int X\, d\nu = 0\,.$$

Proof. As $\langle (X_n, \mathscr{F}_n) \rangle$ is a nonnegative martingale with respect to ν (see Exercise 12.15), it converges to X ν-a.s. and $X < \infty$ ν-a.s. We claim that the lemma follows from the following decomposition of μ into a ν-absolutely continuous part and a ν-singular part:

$$\forall A \in \mathscr{F} \qquad \mu(A) = \int_A X\, d\nu + \mu(A \cap \{X = \infty\})\,. \tag{12.4}$$

For if $\mu \ll \nu$, then $X < \infty$ μ-a.e.; if $X < \infty$ μ-a.e., then by (12.4), $\int X\, d\nu = \int d\mu$; and if $\int X\, d\nu = \int d\mu$, then by (12.4), $X < \infty$ μ-a.e. and $\mu \ll \nu$. On the other hand, if $\mu \perp \nu$, then there is a set A with $\mu(A^c) = 0 = \nu(A)$, which, by (12.4), implies $\mu(A \cap \{X = \infty\}) = \mu(A)$, whence $X = \infty$ μ-a.e.; if $X = \infty$ μ-a.e., then by (12.4), $\int X\, d\nu = 0$; and if $\int X\, d\nu = 0$, then by (12.4), $X = \infty$ μ-a.e., whence $\mu \perp \nu$.

To establish (12.4), suppose first that $\mu \ll \nu$ with Radon-Nikodým derivative $\widetilde{X}$. Then X_n is (a version of) the conditional expectation of $\widetilde{X}$ given $\mathscr{F}_n$ (with respect to ν), whence $X_n \to \widetilde{X}$ ν-a.s. by the martingale convergence theorem. In particular, $X = \widetilde{X}$ ν-a.s., so the decomposition is simply the definition of Radon-Nikodým derivative.

In order to treat the general case, we use a common trick: define the probability measure $\rho := (\mu+\nu)/C$, where $C := \int d(\mu+\nu)$. Then $\mu, \nu \ll \rho$, so that we may apply what we have just shown to the variables $U_n := d(\mu{\restriction}\mathscr{F}_n)/d(\rho{\restriction}\mathscr{F}_n)$ and $V_n := d(\nu{\restriction}\mathscr{F}_n)/d(\rho{\restriction}\mathscr{F}_n)$.

Let $U := \limsup U_n$ and $V := \limsup V_n$. Since $U_n + V_n = C$ ρ-a.s., it follows that $\rho(\{U = V = 0\}) = 0$ and thus

$$U/V = \lim U_n / \lim V_n = \lim(U_n/V_n) = \lim X_n = X \quad \rho\text{-a.s.}$$

Therefore, for $A \in \mathscr{F}$,

$$\mu(A) = \int_A U \, d\rho = \int_A XV \, d\rho + \int_A \mathbf{1}_{\{V=0\}} U \, d\rho \,,$$

which is the same as (12.4). ◀

The Kesten-Stigum Theorem will be an immediate consequence of the following theorem on the growth rate of immigration processes.

Theorem 12.3. (Seneta, 1970) *Let Z_n be the generation sizes of a Galton-Watson process with immigration Y_n. Let $m := \mathbf{E}[L] > 1$ be the mean of the offspring law and let Y have the same law as Y_n. If $\mathbf{E}[\log^+ Y] < \infty$, then $\lim Z_n/m^n$ exists and is finite a.s., while if $\mathbf{E}[\log^+ Y] = \infty$, then $\limsup Z_n/c^n = \infty$ a.s. for every constant $c > 0$.*

Proof. (Asmussen and Hering (1983), pp. 50–51) Assume first that $\mathbf{E}[\log^+ Y] = \infty$. By Lemma 12.1, $\limsup Y_n/c^n = \infty$ a.s. Since $Z_n \ge Y_n$, the result follows.

Now assume that $\mathbf{E}[\log^+ Y] < \infty$. Let $\mathcal{G}$ be the σ-field generated by $\{Y_k \,;\ k \ge 1\}$. Let $Z_{n,k}$ be the number of descendants at level n of the particles that immigrated in generation k. Thus, the total number of vertices at level n is $\sum_{k=1}^n Z_{n,k}$. Note that conditioning on $\mathcal{G}$ is just fixing values for all Y_k; since Y_k are independent of all other random variables, conditioning on $\mathcal{G}$ amounts to using Fubini's theorem. With our new notation, we have

$$\mathbf{E}[Z_n/m^n \mid \mathcal{G}] = \mathbf{E}\left[\frac{1}{m^n}\sum_{k=1}^n Z_{n,k} \,\middle|\, \mathcal{G}\right] = \sum_{k=1}^n \frac{1}{m^k}\mathbf{E}\left[\frac{Z_{n,k}}{m^{n-k}} \,\middle|\, \mathcal{G}\right].$$

Now for $k \le n$, the random variable $Z_{n,k}/m^{n-k}$ is the $(n-k)$th element of the ordinary Galton-Watson martingale sequence starting, however, with Y_k particles. Therefore, its conditional expectation is just Y_k and so

$$\mathbf{E}[Z_n/m^n \mid \mathcal{G}] = \sum_{k=1}^n \frac{Y_k}{m^k}\,.$$

Our assumption gives, by Exercise 12.2, that this series converges a.s. This implies by Fatou's lemma that $\liminf Z_n/m^n < \infty$ a.s. Finally, since $\langle Z_n/m^n \rangle$ is a submartingale when conditioned on $\mathcal{G}$ with bounded expectation (given $\mathcal{G}$), it converges a.s. ◀

Proof of the Kesten-Stigum Theorem. (Lyons, Pemantle, and Peres, 1995a) Rewrite (12.2) as follows. Let $\mathscr{F}_n$ be the σ-field generated by the first n levels of trees. Then (12.2) is the same as

$$\frac{d(\widehat{\mathbf{GW}}\restriction\mathscr{F}_n)}{d(\mathbf{GW}\restriction\mathscr{F}_n)}(t) = W_n(t)\,. \tag{12.5}$$

In order to define W for every infinite tree t, set

$$W(t) := \limsup_{n\to\infty} W_n(t)\,.$$

From (12.5) and Lemma 12.2 follows the key dichotomy:

$$\int W\,d\mathbf{GW} = 1 \iff \widehat{\mathbf{GW}} \ll \mathbf{GW} \iff W < \infty \quad \widehat{\mathbf{GW}}\text{-a.s.} \tag{12.6}$$

while

$$W = 0 \quad \mathbf{GW}\text{-a.s.} \iff \mathbf{GW} \perp \widehat{\mathbf{GW}} \iff W = \infty \quad \widehat{\mathbf{GW}}\text{-a.s.} \tag{12.7}$$

This is key because it allows us to change the problem from one about the $\mathbf{GW}$-behavior of W to one about the $\widehat{\mathbf{GW}}$-behavior of W. Indeed, since the $\widehat{\mathbf{GW}}$-behavior of W is described by Theorem 12.3, the theorem is immediate: if $\mathbf{E}[L\log^+ L] < \infty$, i.e., $\mathbf{E}[\log^+ \widehat{L}] < \infty$, then $W < \infty$ $\widehat{\mathbf{GW}}$-a.s. by Theorem 12.3, whence $\int W\,d\mathbf{GW} = 1$ by (12.6); while if $\mathbf{E}[L\log^+ L] = \infty$, then $W = \infty$ $\widehat{\mathbf{GW}}$-a.s. by Theorem 12.3, whence $W = 0$ $\mathbf{GW}$-a.s. by (12.7). ◀

§12.3. Subcritical Processes.

When a Galton-Watson process is subcritical or critical, the questions we asked in Section 5.1 about rate of growth are inappropriate. Other questions come to mind, however, such as: How quickly does the process die out? One way to make this question precise is to ask for the decay rate of $\mathbf{P}[Z_n > 0]$. An easy estimate in the subcritical case is

$$\mathbf{P}[Z_n > 0] \le \mathbf{E}[Z_n] = m^n\,. \tag{12.8}$$

We will determine in this section when m^n is the right decay rate (up to some factor), while in the next section, we will treat the critical case.

Theorem 12.4. (Heathcote, Seneta, and Vere-Jones, 1967) *For any Galton-Watson process with $0 < m < \infty$, the sequence $\langle \mathbf{P}[Z_n > 0]/m^n \rangle$ is decreasing. If $m < 1$, then the following are equivalent:*

(i) $\lim_{n\to\infty} \mathbf{P}[Z_n > 0]/m^n > 0$;
(ii) $\sup \mathbf{E}[Z_n \mid Z_n > 0] < \infty$;
(iii) $\mathbf{E}[L\log^+ L] < \infty$.

The fact that (i) holds when $\mathbf{E}[L^2] < \infty$ was proved by Kolmogorov (1938).

In order to prove Theorem 12.4, we use an approach analogous to that in the preceding section: We combine a general lemma with a result on immigration.

Lemma 12.5. *Let $\langle \nu_n \rangle$ be a sequence of probability measures on the positive integers with finite means a_n. Let $\widehat{\nu}_n$ be size-biased, i.e., $\widehat{\nu}_n(k) = k\nu_n(k)/a_n$. If $\{\widehat{\nu}_n\}$ is tight, then* $\sup a_n < \infty$, *while if $\widehat{\nu}_n \to \infty$ in distribution, then $a_n \to \infty$.*

▷ **Exercise 12.6.**

Prove Lemma 12.5.

Theorem 12.6. (Heathcote, 1966) *Let Z_n be the generation sizes of a Galton-Watson process with immigration Y_n. Let Y have the same law as Y_n. Suppose that the mean m of the offspring random variable L is less than 1. If $\mathbf{E}[\log^+ Y] < \infty$, then Z_n converges in distribution to a proper* random variable, while if $\mathbf{E}[\log^+ Y] = \infty$, then Z_n converges in probability to infinity.*

The following proof is a slight improvement on Asmussen and Hering (1983), pp. 52–53.

Proof. Let $\mathcal{G}$ be the σ-field generated by $\{Y_k\,;\ k \ge 1\}$. For any n, let $Z_{n,k}$ be the number of descendants at level n of the vertices that immigrated in generation k. Thus, the total number of vertices at level n is $\sum_{k=1}^n Z_{n,k}$. Since the distribution of $Z_{n,k}$ depends only on $n-k$, this total Z_n has the same distribution as $\sum_{k=1}^n Z_{2k-1,k}$. This latter sum increases in n to some limit Z'_∞. By Kolmogorov's zero-one law, Z'_∞ is a.s. finite or a.s. infinite. Hence, we need only to show that $Z'_\infty < \infty$ a.s. iff $\mathbf{E}[\log^+ Y] < \infty$.

Assume that $\mathbf{E}[\log^+ Y] < \infty$. Now $\mathbf{E}[Z'_\infty \mid \mathcal{G}] = \sum_{k=1}^\infty Y_k m^{k-1}$. Since $\langle Y_k \rangle$ is almost surely subexponential in k by Lemma 12.1, this sum converges a.s. (see Exercise 12.2). Therefore, Z'_∞ is finite a.s.

Now assume that $Z'_\infty < \infty$ a.s. Writing $Z_{2k-1,k} = \sum_{i=1}^{Y_k} \zeta_k(i)$, where $\zeta_k(i)$ are the sizes of generation $k-1$ of i.i.d. Galton-Watson branching processes with one initial particle, we have $Z'_\infty = \sum_{k=1}^\infty \sum_{i=1}^{Y_k} \zeta_k(i)$ written as a random sum of independent random variables. Only a finite number of the $\zeta_k(i)$ are at least one, whence by the Borel-Cantelli lemma conditioned on $\mathcal{G}$, we get $\sum_{k=1}^\infty Y_k \mathbf{GW}(Z_{k-1} \ge 1) < \infty$ a.s. Since $\mathbf{GW}(Z_{k-1} \ge 1) \ge \mathbf{P}[L > 0]^{k-1}$, it follows by Lemma 12.1 (see Exercise 12.2) that $\mathbf{E}[\log^+ Y] < \infty$. ◀

* i.e., finite a.s.

Proof of Theorem 12.4. (Lyons, Pemantle, and Peres, 1995a) Let μ_n be the law of Z_n conditioned on $Z_n > 0$. For any tree t with $Z_n > 0$, let $\xi_n(t)$ be the leftmost child of the root that has at least one descendant in generation n. If $Z_n > 0$, let $H_n(t)$ be the number of descendants of $\xi_n(t)$ in generation n; otherwise, let $H_n(t) := 0$. It is easy to see that

$$\mathbf{P}[H_n = k \mid Z_n > 0] = \mathbf{P}[H_n = k \mid Z_n > 0, \, \xi_n = x] = \mathbf{P}[Z_{n-1} = k \mid Z_{n-1} > 0]$$

for all children x of the root. Since $H_n \le Z_n$, this shows that $\langle \mu_n \rangle$ increases stochastically as n increases. Now

$$\mathbf{P}[Z_n > 0] = \frac{\mathbf{E}[Z_n]}{\mathbf{E}[Z_n \mid Z_n > 0]} = \frac{m^n}{\int x \, d\mu_n(x)} \,.$$

Therefore, $\langle \mathbf{P}[Z_n > 0]/m^n \rangle$ is decreasing and (i) $\Leftrightarrow$ (ii).

Now $\widehat{\mu}_n$ is not only the size-biased version of μ_n, but also the law of the size-biased random variable $\widehat{Z}_n$. Thus, from Section 12.1, we know that $\langle \widehat{\mu}_n \rangle$ describes the generation sizes (plus 1) of a process with immigration $\widehat{L} - 1$. Suppose that $m < 1$. If (ii) holds, i.e., the means of μ_n are bounded, then by Lemma 12.5, the laws $\widehat{\mu}_n$ do not tend to infinity. Applying Theorem 12.6 to the associated immigration process, we see that (iii) holds. Conversely, if (iii) holds, then by Theorem 12.6, $\langle \widehat{\mu}_n \rangle$ converges, whence is tight. In light of Lemma 12.5, (ii) follows. ◀

§12.4. Critical Processes.

In the critical case, the easy estimate (12.8) is useless. What then is the rate of decay?

Theorem 12.7. (Kesten, Ney, and Spitzer, 1966) *Suppose that $m = 1$ and let $\sigma^2 := \mathrm{Var}(L) = \mathbf{E}[L^2] - 1$. Then we have*

(i) *Kolmogorov's estimate:*

$$\lim_{n \to \infty} n\mathbf{P}[Z_n > 0] = \frac{2}{\sigma^2} \,;$$

(ii) *Yaglom's limit law: If $\sigma < \infty$, then the conditional distribution of Z_n/n given $Z_n > 0$ converges as $n \to \infty$ to an exponential law with mean $\sigma^2/2$.*

Under a third moment assumption, parts (i) and (ii) of this theorem are due to Kolmogorov (1938) and Yaglom (1947), respectively. The case where $\sigma = \infty$ in (ii) appears to be open. We give a proof that uses ideas of Lyons, Pemantle, and Peres (1995a) and Geiger (1999). The exponential limit law in part (ii) will arise from the following characterization of exponential random variables due to Pakes and Khattree (1992):

▷ **Exercise 12.7.**

Let A be a nonnegative random variable with a positive finite mean and let $\widehat{A}$ have the corresponding size-biased distribution. Denote by U a uniform random variable in $[0, 1]$ that is independent of $\widehat{A}$. Prove that $U \cdot \widehat{A}$ and A have the same distribution iff A is exponential.

We will also use the results of the following exercises.

▷ **Exercise 12.8.**

Suppose that A, A_n are nonnegative random variables with positive finite means such that $A_n \to A$ in law and $\widehat{A}_n \to B$ in law. Show that if B is a proper random variable, then B has the law of $\widehat{A}$.

▷ **Exercise 12.9.**

Suppose that $0 \le A_i \le B_i$ are random variables, that $A_i \to 0$ in probability, and that B_i are identically distributed with finite mean. Show that $\sum_{i=1}^n A_i/n \to 0$ in probability. Show that if in addition, $C_{i,j}$ are random variables with $\mathbf{E}\big[|C_{i,j}| \bigm| A_i\big] \le 1$ and A_i takes integer values, then $\sum_{i=1}^n \sum_{j=1}^{A_i} C_{i,j}/n \to 0$ in probability.

▷ **Exercise 12.10.**

Let A be a random variable independent of the random variables B and C. Suppose that the function $x \mapsto \mathbf{P}[C \le x]/\mathbf{P}[B \le x]$ is increasing, that $\mathbf{P}[A \ge B] > 0$, and that $\mathbf{P}[A \ge C] > 0$. Show that the law of A given that $A \ge B$ is stochastically dominated by the law of A given that $A \ge C$. Show that the hypothesis on B and C is satisfied when they are geometric random variables with B having a larger parameter than C.

Proof of Theorem 12.7. Since each child of the initial progenitor independently has a descendant in generation n with probability $\mathbf{P}[Z_{n-1} > 0]$, we have that the law of Z_1 given that $Z_n > 0$ is the law of Z_1 given that $Z_1 \ge D_n$, where D_n is a geometric random variable independent of Z_1 with mean $1/\mathbf{P}[Z_{n-1} > 0]$. Since $\mathbf{P}[Z_{n-1} > 0] > \mathbf{P}[Z_n > 0]$, Exercise 12.10 implies that the conditional distribution of Z_1 given $Z_n > 0$ stochastically increases with n. Let Y_n be the number of individuals of the first generation that have a descendant in generation n. Since $\mathbf{P}[Z_n > 0] \to 0$, it follows that $\mathbf{P}[Y_n = 1 \mid Z_n > 0] \to 1$, that the conditional distribution of Z_1 given $Z_n > 0$ tends to the distribution of the size-biased $\widehat{L}$, and that the conditional distribution of the left-most individual of the first generation that has a descendant in generation n tends to a uniform pick among the individuals of the first generation. Since $\mathbf{P}[Z_1 \ge k \mid Z_n > 0]$ increases with n, the tail formula for expectation yields $\mathbf{E}[Z_1 \mid Z_n > 0] \to \mathbf{E}[\widehat{L}] = \sigma^2 + 1$.

Let u_n^n be the left-most individual in generation n when $Z_n > 0$. Let its ancestors back to the initial progenitor be $u_{n-1}^n, \ldots, u_0^n$, where u_i^n is in generation i. Let X_i' denote the number of descendants of u_i^n in generation n that are not descendants of u_{i+1}^n. Let X_i be the number of children of u_i^n that are to the right of u_{i+1}^n. Then $Z_n = 1 + \sum_{i=0}^{n-1} X_i'$ and $\mathbf{E}[X_i' \mid Z_n > 0] = \mathbf{E}[X_i \mid Z_n > 0]$ since each of these X_i individuals generates an independent critical Galton-Watson descendant tree (with offspring law the same as that of the original process). Therefore,

$$\frac{1}{n\mathbf{P}[Z_n > 0]} = \frac{1}{n}\mathbf{E}[Z_n \mid Z_n > 0] = \frac{1}{n} + \frac{1}{n}\sum_{i=0}^{n-1} \mathbf{E}[X_i \mid Z_n > 0]\,.$$

We have seen that the conditional distribution of X_i given that $Z_n > 0$ tends to that of $\lfloor U \cdot \widehat{L} \rfloor$, where U denotes a uniform $[0,1]$-random variable that is independent of $\widehat{L}$. Thus, $\lim_{n\to\infty} \mathbf{E}[X_i \mid Z_n > 0] = \mathbf{E}\big[\lfloor U \cdot \widehat{L} \rfloor\big] = \mathbf{E}\big[\widehat{L} - 1\big]/2 = \sigma^2/2$, which gives Kolmogorov's estimate.

Now suppose that $\sigma < \infty$. We are going to compare the conditional distribution of Z_n/n given $Z_n > 0$ with the law of R_n/n, where R_n is the number of individuals in generation n to the right of v_n in the size-biased tree. Recall that X_i denotes the number of children of u_i^n to the right of u_{i+1}^n. Since we are interested in its distribution as $n \to \infty$, we will be explicit and write $X_i^{(n)} := X_i$. For $1 \le j \le X_i^{(n)}$, let $S_{i,j}^{(n)}$ be the number of descendants in generation n of the jth child of u_i^n to the right of u_{i+1}^n. On the other hand, in the size-biased tree, let Y_i be the number of children of v_i to the right of v_{i+1} and let $V_{i,j}^{(n)}$ be the number of descendants in generation n of the jth child of v_{i-1} to the right of v_i. We may couple all these random variables so that $S_{i,j}^{(n)}$ and $V_{i,j}^{(n)}$ are i.i.d. with mean 1 (since they pertain to independent critical Galton-Watson trees) and $|X_i - Y_i| \le \widehat{L}_{i+1}$ with $X_i^{(n)} - Y_i \to 0$ in measure as $n \to \infty$ (in virtue of the first paragraph). Since

$$Z_n = 1 + \sum_{i=0}^{n-1} \sum_{j=1}^{X_i^{(n)}} S_{i,j}^{(n)} \qquad \text{and} \qquad R_n = 1 + \sum_{i=0}^{n-1} \sum_{j=1}^{Y_i} V_{i,j}^{(n)}\,,$$

it follows from Exercise 12.9 that $Z_n/n - R_n/n \to 0$ in measure as $n \to \infty$.

Now we prove that the limit of R_n/n exists in law and identify it. The $\widehat{\mathbf{GW}}$ laws of Z_n/n have uniformly bounded means and so are tight. This implies the tightness of μ_n, the $\mathbf{GW}$-conditional distribution of Z_n/n given that $Z_n > 0$, and the $\widehat{\mathbf{GW}}_*$ laws of R_n/n. Therefore, we can find $n_k \to \infty$ so that μ_{n_k} and the $\widehat{\mathbf{GW}}_*$ law of R_{n_k}/n_k converge to the law of a (proper) random variable A and the $\widehat{\mathbf{GW}}$ laws of Z_{n_k}/n_k converge to the law of a (proper) random variable B. Note that the $\widehat{\mathbf{GW}}$ law of Z_n/n can also be gotten by size-biasing μ_n. By virtue of Exercise 12.8, therefore, the variables $\widehat{A}$ and B are identically distributed. As R_n is a uniform pick from $\{1, 2, \ldots, Z_n\}$, we also have that A has the same law as $U \cdot B$, i.e., as $U \cdot \widehat{A}$. By Exercise 12.7, it follows that A is an exponential random variable with mean $\sigma^2/2$. In particular, the limit of μ_{n_k} is independent of the sequence $\langle n_k \rangle$, and hence we actually have convergence in law of the whole sequence μ_n to A, as desired. ◀

§12.5. Notes.

Ideas related to size-biased Galton-Watson trees occur in Hawkes (1981), Joffe and Waugh (1982), Waymire and Williams (1994), and Chauvin, Rouault, and Wakolbinger (1991). There have been very many uses since then of these ideas. The generation sizes of size-biased Galton-Watson trees are known as a Q-process in the case $m \le 1$; see Athreya and Ney (1972), pp. 56–60.

The proof that the law of Z_1 given $Z_n > 0$ stochastically increases in n that appears at the beginning of the proof of Theorem 12.7 is due to Matthias Birkner (personal communication, 2000). A rate of convergence in Yaglom's limit law is given by Peköz and Röllin (2009), using Stein's method and ideas from the proof of Theorem 12.7 given in Lyons, Pemantle, and Peres (1995a).

§12.6. Collected In-Text Exercises.

12.1. Prove Lemma 12.1 by using the Borel-Cantelli lemma.

12.2. Given X, X_n as in Lemma 12.1, show that if $\mathbf{E}[X] < \infty$, then $\sum_n e^{X_n} c^n < \infty$ for all $c \in (0, 1)$, while if $\mathbf{E}[X] = \infty$, then $\sum_n e^{X_n} c^n = \infty$ for all $c \in (0, 1)$.

12.3. Let X be a nonnegative random variable with $0 < \mathbf{E}[X] < \infty$. We say that $\widehat{X}$ has the ***size-biased distribution*** of X if $\mathbf{P}[\widehat{X} \in A] = \mathbf{E}[X \mathbf{1}_A(X)]/\mathbf{E}[X]$ for intervals $A \subseteq [0, \infty)$. Show that this is equivalent to $\mathbf{E}[f(\widehat{X})] = \mathbf{E}[X f(X)]/\mathbf{E}[X]$ for all Borel $f : [0, \infty) \to [0, \infty)$.

12.4. Suppose that X_n are nonnegative random variables such that $\mathbf{P}[X_n > 0]/\mathbf{E}[X_n] \to 0$. Show that the size-biased random variables $\widehat{X}_n$ tend to infinity in probability.

12.5. Define $\widehat{\mathbf{GW}}_*$ formally on a space analogous to the space $\mathscr{T}$ of Exercise 5.2 and define $\widehat{\mathbf{GW}}$ formally on $\mathscr{T}$.

12.6. Prove Lemma 12.5.

12.7. Let A be a nonnegative random variable with a positive finite mean and let $\widehat{A}$ have the corresponding size-biased distribution. Denote by U a uniform random variable in $[0, 1]$ that is independent of $\widehat{A}$. Prove that $U \cdot \widehat{A}$ and A have the same distribution iff A is exponential.

12.8. Suppose that A, A_n are nonnegative random variables with positive finite means such that $A_n \to A$ in law and $\widehat{A}_n \to B$ in law. Show that if B is a proper random variable, then B has the law of $\widehat{A}$.

12.9. Suppose that $0 \le A_i \le B_i$ are random variables, that $A_i \to 0$ in probability, and that B_i are identically distributed with finite mean. Show that $\sum_{i=1}^n A_i/n \to 0$ in probability. Show that if in addition, $C_{i,j}$ are random variables with $\mathbf{E}\big[|C_{i,j}| \mid A_i\big] \le 1$ and A_i takes integer values, then $\sum_{i=1}^n \sum_{j=1}^{A_i} C_{i,j}/n \to 0$ in probability.

12.10. Let A be a random variable independent of the random variables B and C. Suppose that the function $x \mapsto \mathbf{P}[C \le x]/\mathbf{P}[B \le x]$ is increasing, that $\mathbf{P}[A \ge B] > 0$, and that $\mathbf{P}[A \ge C] > 0$. Show that the law of A given that $A \ge B$ is stochastically dominated by the law of A given that $A \ge C$. Show that the hypothesis on B and C is satisfied when they are geometric random variables with B having a larger parameter than C.

§12.7. Additional Exercises.

12.11. Show that if $X \sim \text{Bin}(n, p)$, then $\widehat{X} \sim 1 + \text{Bin}(n-1, p)$, while if $X \sim \text{Pois}(\lambda)$, then $\widehat{X} \sim 1 + \text{Pois}(\lambda)$.

12.12. Let X be a mixed binomial random variable, i.e., there are independent events $A_1, \ldots, A_n$ such that $X = \sum_{i=1}^n \mathbf{1}_{A_i}$. Let I be a random variable independent of A_i such that $I = i$ with probability $\mathbf{P}[A_i]/\sum_{j=1}^n \mathbf{P}[A_j]$. Show that $\widehat{X} \sim 1 + \sum_{i=1}^n \mathbf{1}_{A_i}\mathbf{1}_{\{I \ne i\}}$.

12.13. Let $X, X_1, \ldots, X_k$ be i.i.d. nonnegative random variables for some $k \ge 0$ and let $\widehat{X}$ be an independent random variable with the size-biased distribution of X. Show that

$$\mathbf{E}\left[\frac{k+1}{\widehat{X} + X_1 + \cdots + X_k}\right] = \frac{1}{\mathbf{E}[X]}\,.$$

12.14. Show that if $X \ge 0$, then X is stochastically dominated by $\widehat{X}$. Deduce the arithmetic mean-quadratic mean inequality, that $\mathbf{E}[X]^2 \le \mathbf{E}[X^2]$, and determine when equality occurs. Deduce the Cauchy-Schwarz inequality from this.

12.15. In the notation of Lemma 12.2, show that $\langle (X_n, \mathscr{F}_n) \rangle$ is a martingale with respect to ν. Deduce that if μ is a probability measure, then $\langle (X_n, \mathscr{F}_n) \rangle$ is a submartingale with respect to μ.

12.16. The simplest proof of the Kesten-Stigum Theorem along traditional lines is due to Tanny (1988). Complete the following outline of this proof. A branching process in varying environments (BPVE) is one in which the offspring distribution depends on the generation. Namely, if $\langle L_i^{(n)}\,;\ n, i \ge 1\rangle$ are independent random variables with values in $\mathbb{N}$ such that for each n, the variables $L_i^{(n)}$ are identically distributed, then set $Z_0 := 1$ and, inductively, $Z_{n+1} := \sum_{i=1}^{Z_n} L_i^{(n+1)}$. Let $m_n := \mathbf{E}[L_i^{(n)}]$ and $M_n := \prod_{k=1}^n m_k$. Show that $M_n = \mathbf{E}[Z_n]$ and that Z_n/M_n is a martingale. Its limit is denoted W.

Given a Galton-Watson branching process $\langle Z_n \rangle$ and a number $A > 0$, define a BPVE $\langle Z_n(A) \rangle$ by letting $\langle L_i^{(n)}(A) \rangle$ have the distribution of $L\mathbf{1}_{\{L < Am^n\}}$. Use the fact that $W < \infty$ a.s. to show that for any $\epsilon > 0$, one can choose A sufficiently large that $\mathbf{P}[\forall n \quad Z_n = Z_n(A)] > 1 - \epsilon$. Show that when $Z_n = Z_n(A)$ for all n, we have

$$W = W(A) \prod_{n \geq 1} (1 - \mathbf{E}[L\,;\ L \geq m^n]/m)\ .$$

Show that this product is 0 iff $\mathbf{E}[L \log^+ L] = \infty$. Conclude that if $\mathbf{E}[L \log^+ L] = \infty$, then $W = 0$ a.s.

For the converse, define a BPVE $\langle Z_n(B) \rangle$ by letting $\langle L_i^{(n)}(B) \rangle$ have the distribution of $L\mathbf{1}_{\{L<Bm^{3n/4}\}}$. Choose B large enough so that $\mathbf{E}[Z_n(B)] > 0$ for all n. Show that $Z_n(B)/M_n(B)$ is bounded in L^2, whence its limit $W(B)$ has expectation 1. From $Z_n \geq Z_n(B)$, conclude that $\mathbf{E}[W] \geq \lim M_n(B)/m^n$. Show that by appropriate choice of B, if $\mathbf{E}[L \log^+ L] < \infty$, then this last limit can be made arbitrarily close to 1.

12.17. Let $G_n := \sup\{|u|\,;\ T_n \subseteq T^u\}$ be the generation of the most recent common ancestor of all individuals in generation n. Show that if $m = 1$ and $\mathrm{Var}(L) < \infty$, then the conditional distribution of G_n/n given $Z_n > 0$ tends to the uniform distribution on $[0, 1]$.

A traditional proof of Theorem 12.7 observes that $\mathbf{P}[Z_n > 0] = 1 - f^{(n)}(0)$ *and analyzes the rate at which the iterates of* f *tend to 1. The following exercises outline such a proof.*

12.18. Show that if $m = 1$, then $\lim_{s\uparrow 1} f''(s) = \sigma^2$.

12.19. Suppose that $m = 1$, $p_1 \neq 1$, and $\sigma < \infty$.
(a) Define $\delta(s) := [1 - f(s)]^{-1} - [1 - s]^{-1}$. Show that $\lim_{s\uparrow 1} \delta(s) = \sigma^2/2$.
(b) Let $s_n \in [0, 1)$ be such that $n(1 - s_n) \to \alpha \in [0, \infty]$. Show that

$$\lim_{n\to\infty} n[1 - f^{(n)}(s_n)] = \frac{1}{\sigma^2/2 + \alpha^{-1}}\ .$$

12.20. Use Exercise 12.19 and Laplace transforms to prove Theorem 12.7.

Chapter 13

Speed of Random Walks

If a random walk on a network is transient, how quickly does the walk increase its distance from its starting point? We will be particularly interested in this chapter when the rate is linear. Since this rate is then the limit of the distance divided by the time, we call it the speed of the random walk. We'll look at this on trees, on general graphs, and on random trees. We'll also see some applications to problems of embedding metric spaces in Euclidean space.

§13.1. Basic Examples.

If $\langle S_n \rangle$ is a sum of i.i.d. real-valued random variables, then $\lim_n S_n/n$ could be called its speed when it exists. Of course, the Strong Law of Large Numbers (SLLN) says that this limit does exist a.s. and equals the mean increment when this mean is well defined. Actually, the independence of the increments is not needed if we have some other control of the increments. This is a useful tool in other situations. We present two such general results. Recall that X and Y are called ***uncorrelated*** if they have finite variance and $\mathbf{E}\big[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])\big] = 0$.

Theorem 13.1. (SLLN for Uncorrelated Random Variables) *Let $\langle X_n \rangle$ be a sequence of uncorrelated random variables with $\sup_n \operatorname{Var}(X_n) < \infty$. Then*

$$\frac{1}{n}\sum_{k=1}^{n}(X_k - \mathbf{E}[X_k]) \to 0$$

a.s. as $n \to \infty$.

Proof. We may clearly assume that $\mathbf{E}[X_n] = 0$ and $\mathbf{E}[X_n^2] \le 1$ for all n. Write $S_n := \sum_{k=1}^{n} X_k$.

We begin with the simple observation that if $\langle Y_n \rangle$ is a sequence of random variables such that

$$\sum_n \mathbf{E}\big[|Y_n|^2\big] < \infty\,,$$

then $\mathbf{E}\big[\sum_n |Y_n|^2\big] < \infty$, whence $\sum_n |Y_n|^2 < \infty$ a.s. and $Y_n \to 0$ a.s.

Using this, it is easy to verify the SLLN for $n \to \infty$ along the sequence of squares. Namely,

$$\mathbf{E}[(S_n/n)^2] = \frac{1}{n^2}\mathbf{E}\big[|S_n|^2\big] = \frac{1}{n^2}\sum_{k=1}^{n}\mathbf{E}\big[|X_k|^2\big] \le 1/n\,.$$

Therefore, if we set $Y_n := S_{n^2}/n^2$, we have

$$\mathbf{E}\big[|S_{n^2}|^2/n^2\big] \le 1/n^2\,.$$

Since the right hand side is summable, the observation above implies $Y_n \to 0$ a.s. This is the same as $S_{n^2}/n^2 \to 0$ a.s.

To deal with limit over all the integers, take $m^2 \le n < (m+1)^2$ and set $m(n) := \lfloor\sqrt{n}\rfloor$. Then

$$\mathbf{E}\left[\left|\frac{S_n}{m^2} - \frac{S_{m^2}}{m^2}\right|^2\right] = \frac{1}{m^4}\mathbf{E}\left[\Big|\sum_{k=m^2+1}^{n} X_k\Big|^2\right] = \frac{1}{m^4}\mathbf{E}\left[\sum_{k=m^2+1}^{n}|X_k|^2\right] \le \frac{2}{m^3}\,,$$

since the sum has at most $2m$ terms, each of size at most 1. Put

$$Z_n := \frac{S_n}{m(n)^2} - \frac{S_{m(n)^2}}{m(n)^2}\,.$$

Then since each $m = m(n)$ is associated to at most $2m+1$ different values of n, we get

$$\sum_{n=1}^{\infty}\mathbf{E}[|Y_n|^2] \le \sum_{n=1}^{\infty} 2/m(n)^3 \le \sum_m (2m+1)\frac{2}{m^3} < \infty\,,$$

so by the initial observation, $Z_n \to 0$ a.s. This implies $S_n/m(n)^2 \to 0$ a.s., which in turn implies $S_n/n \to 0$ a.s., which is what we wanted. ◀

As an example, note that martingale increments (i.e., the differences between successive terms of a martingale) are uncorrelated.

More refined information for sums of i.i.d. real-valued random variables is given, of course, by the Central Limit Theorem, or by Chernoff-Cramér's theorem on large deviations. For the case of simple random walk on $\mathbb{Z}$, the latter implies that given $0 < s < 1$, the chance that the distance at time n is at least sn is at most $2e^{-nI(s)}$, where

$$I(s) := \frac{(1+s)\log(1+s) + (1-s)\log(1-s)}{2}$$

(see Billingsley (1995), p. 151, or Dembo and Zeitouni (1998), Theorem 2.1.14). Note that for small $|s|$,

$$I(s) = \frac{s^2}{2} + O(s^4)\,. \tag{13.1}$$

In other situations, one does not have i.i.d. random variables. An extension (though not as sharp) of Chernoff-Cramér's theorem is a large deviation inequality due to Hoeffding (1963) and rediscovered by Azuma (1967). As an upper bound, the Hoeffding-Azuma inequality is just as sharp to the first two orders in the exponent as the Chernoff-Cramér theorem for simple random walk on $\mathbb{Z}$. We follow the exposition by Steele (1997).

Theorem 13.2. (Hoeffding-Azuma Inequality) *Let $\{X_1, \ldots, X_n\}$ be bounded random variables such that*

$$\mathbf{E}\Big[X_{i_1} \cdots X_{i_k}\Big] = 0 \qquad \forall \quad 1 \le i_1 < \ldots < i_k\,,$$

(for instance, independent variables with zero mean or martingale differences). Then

$$\mathbf{P}\left[\sum_{i=1}^n X_i \ge L\right] \le e^{-L^2/\left(2\sum_{i=1}^n \|X_i\|_\infty^2\right)}\,.$$

Proof. For any sequences of constants $\{a_i\}$ and $\{b_i\}$, we have

$$\mathbf{E}\Big[\prod_{i=1}^n (a_i + b_i X_i)\Big] = \prod_{i=1}^n a_i\,. \tag{13.2}$$

Since the function $f(x) = e^{ax}$ is convex on the interval $[-1, 1]$, it follows that for any $x \in [-1, 1]$

$$e^{ax} = f(x) \le \frac{1-x}{2} f(-1) + \frac{x+1}{2} f(1) = \cosh a + x \sinh a\,.$$

If we now let $x = X_i/\|X_i\|_\infty$ and $a = t\|X_i\|_\infty$ we find

$$\exp\left(t\sum_{i=1}^n X_i\right) \le \prod_{i=1}^n \left(\cosh(t\|X_i\|_\infty) + \frac{X_i}{\|X_i\|_\infty} \sinh(t\|X_i\|_\infty)\right)\,.$$

When we take expectations and use (13.2), we find

$$\mathbf{E}\exp\left(t\sum_{i=1}^n X_i\right) \le \prod_{i=1}^n \cosh(t\|X_i\|_\infty)\,,$$

so, by the elementary bound

$$\cosh x = \sum_{k=0}^\infty \frac{x^{2k}}{(2k)!} \le \sum_{k=0}^\infty \frac{x^{2k}}{2^k k!} = e^{x^2/2}\,,$$

we have

$$\mathbf{E} \exp\left(t \sum_{i=1}^n X_i\right) \le \exp\left(\frac{1}{2} t^2 \sum_{i=1}^n \|X_i\|_\infty^2\right).$$

By Markov's inequality and the above we have that for any $t > 0$

$$\mathbf{P}\left[\sum_{i=1}^n X_i \ge L\right] = \mathbf{P}\left[\exp\left(t \sum_{i=1}^n X_i\right) \ge e^{Lt}\right] \le e^{-Lt} \exp\left(\frac{t^2}{2} \sum_{i=1}^n \|X_i\|_\infty^2\right),$$

so by making the choice $t := L(\sum_{i=1}^n \|X_i\|_\infty^2)^{-1}$, we obtain the required result. ◀

For example, consider simple random walk $\langle X_n \rangle$ on an infinite tree T starting at its root, o. When the walk is at x, it has a push away from the root equal to

$$f(x) := \begin{cases} (\deg x - 2)/\deg x & \text{if } x \ne o \\ 1 & \text{if } x = o. \end{cases}$$

That is, $\langle |X_n| - |X_{n-1}| - f(X_n) \rangle$ is a martingale-difference sequence, whence by either of the above theorems, it obeys the SLLN,

$$\lim_{n\to\infty} \frac{1}{n}\big(|X_n| - \sum_{k=1}^n f(X_k)\big) = 0$$

a.s. Now the density of times at which the walk visits the root is 0 since the tree is infinite and has a uniform stationary measure, whence we may write the above equation as

$$\lim_{n\to\infty} \frac{1}{n}\big(|X_n| - \sum_{k=1}^n (1 - 2/\deg X_k)\big) = 0 \tag{13.3}$$

a.s. For example, if T is regular of degree d, then the random walk has a speed of $1 - 2/d$ a.s. For a more interesting example, suppose that T is the universal cover of the finite connected graph G with at least one cycle (so that T is infinite). Call $\varphi : T \to G$ the covering map. Then $\langle \varphi(X_n) \rangle$ is simple random walk on G, whence the density of times at which $\varphi(X_k) = y \in \mathsf{V}(G)$ equals $\deg_G y/D(G)$ a.s., where $D(G) := 2|\mathsf{E}(G)|$ is the sum of all the degrees. Of course, $\deg_G \varphi(x) = \deg_T x$, whence the speed on T is a.s.

$$1 - \sum_{y\in\mathsf{V}(G)} (\deg y)(2/\deg y)/D(G) = 1 - |\mathsf{V}(G)|/|\mathsf{E}(G)| = 1 - 2/\bar{d}(G), \tag{13.4}$$

where $\bar{d}(G) = D(G)/|\mathsf{V}(G)|$ is the average degree in G.

How does this compare to $1 - 2/(\operatorname{br} T + 1)$, which is the speed when T is regular? To answer this, we need an estimate of $\operatorname{br} T$. We want to compare $\bar{d}(G)$ to $\operatorname{br} T + 1$. Let H be the graph obtained from G by iteratively removing all vertices (if any) of degree 1 and let T' be its universal cover. Then clearly $\operatorname{br} T' = \operatorname{br} T$ while $\bar{d}(H) \ge \bar{d}(G)$. By Theorem 3.8, we know that $\operatorname{br} T' = \operatorname{gr} T'$. Now $\operatorname{gr} T'$ equals the growth rate of the number $N(L)$ of non-backtracking paths in H of length L (from any starting point) as $L \to \infty$:

$$\operatorname{gr} T' = \lim_{L\to\infty} N(L)^{1/L} .$$

To estimate this, let B be the matrix indexed by the oriented edges of H such that $B\big((x,y),(y,z)\big) = 1$ when $(x,y),(y,z) \in \mathsf{E}(H)$ and $x \ne z$, with all other entries of B equal to 0. Consider a stationary Markov chain on the oriented edges $\mathsf{E}(H)$ with stationary probability measure σ and transition probabilities $p(e,f)$ such that $p(e,f) > 0$ only if $B(e,f) > 0$. Such a chain gives a probability measure on the paths of length L whose entropy is at most that of the uniform measure, i.e., at most $\log N(L)$ (see (6.27)). On the other hand, this path entropy equals $-\sum_{e,f} \sigma(e)\log\sigma(e) - L\sum_{e,f}\sigma(e)p(e,f)\log p(e,f)$ (see Exercise 6.62). Thus, $\log \operatorname{gr} T'$ is at least the Markov-chain entropy:

$$\log \operatorname{gr} T' \ge -\sum_{e,f} \sigma(e)p(e,f)\log p(e,f) . \tag{13.5}$$

Now choose $p\big((x,y),(y,z)\big) = 1/(\deg y - 1)$ when $B\big((x,y),(y,z)\big) > 0$. It is easy to verify that $\sigma(x,y) = 1/D(H)$ is a stationary probability measure. This chain has entropy $D(H)^{-1}\sum_{y\in\mathsf{V}(H)}(\deg y)\log(\deg y - 1)$. Since the function $t \mapsto t\log(t-1)$ is convex for $t \ge 2$, it follows that the entropy is at least $\log(\bar{d}(H) - 1)$. Therefore,

$$\operatorname{br} T = \operatorname{gr} T' \ge \bar{d}(H) - 1 \ge \bar{d}(G) - 1 .$$

This result is due to David Wilson (personal communication, 1993), but was first published by Alon, Hoory, and Linial (2002). Substitute this in (13.4) to obtain that the speed on T is at most $1 - 2/(\operatorname{br} T + 1)$. If the branching number is an integer, then this shows that the regular tree of that branching number has the greatest speed among all covering trees of the same branching number.

For more general trees, for positive speed of simple random walk is it sufficient that the tree have growth rate or branching number larger than 1? Is it necessary?

If every vertex has at least 2 children, then by (13.3), the liminf speed is positive a.s. A more general sufficient condition is given in the following exercise.

▷ **Exercise 13.1.**

Let T be a tree without leaves such that for every vertex u, there is some vertex $x \ge u$ with at least two children and with $|x| < |u| + N$. Show that if $\langle X_n \rangle$ is simple random walk on T, then

$$\liminf_{n\to\infty} \frac{|X_n|}{n} \geq \frac{1}{3N} \quad \text{a.s.}$$

We'll call this the ***liminf speed*** of the random walk.

▷ **Exercise 13.2.**

However, it does not suffice that $\operatorname{br} T > 1$ for the speed of simple random walk on T to be positive. Show this for the tree T, which is a binary tree to every vertex of which is joined a unary tree; see Figure 13.1.

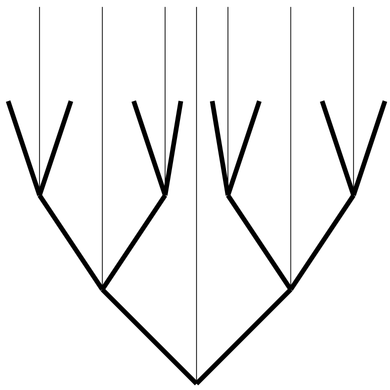


Figure 13.1.

In the other direction, we have the following condition necessary for positive speed. It was proved by Peres (1999), Theorem 5.4; see Section 13.8 for a better result. Note that $s \mapsto I(s)/s$ is monotonic increasing on $[0,1]$ since its derivative is $-(2s^2)^{-1}\log(1-s^2)$.

Proposition 13.3. *If simple random walk on T escapes at a linear rate, then* $\operatorname{br} T > 1$. *More precisely, if $\langle X_n \rangle$ is simple random walk on T and*

$$\liminf_{n\to\infty} \frac{|X_n|}{n} \geq s$$

with positive probability, then $\operatorname{br} T \geq e^{I(s)/s}$.

Proof. We may assume that T has no leaves since leaves only slow the random walk and do not change the branching number. (The slowing effect of leaves can be proved rigorously by coupling a random walk $\langle X_n \rangle$ on T with a random walk $\langle X'_n \rangle$ on T', where T' is the result of iteratively removing the leaves from T. They can be coupled so that $\liminf |X'_n|/n \geq \liminf |X_n|/n$ by letting $\langle X'_n \rangle$ take only the moves of $\langle X_n \rangle$ that do not lead to vertices with no infinite line of descent.) Given $0 < s' < s$, there is some L such that

$$q := \mathbf{P}[\forall n \ge L \quad |X_n| > s'n] > 0\,. \tag{13.6}$$

Define the general percolation on T by keeping those edges $e(x)$ such that $|x| \le s'L$ or $\exists n < |x|/s' \quad X_n = x$. According to (13.6), the component of the root in this percolation is infinite with probability at least q. On the other hand, if $|x| > s'L$, then $\mathbf{P}[o \leftrightarrow x]$ is bounded above by the probability that simple random walk $\langle S_k \rangle$ on $\mathbb{Z}$ moves distance at least $|x|$ in $|x|/s'$ steps:

$$\mathbf{P}[o \leftrightarrow x] \le \mathbf{P}[\max_{n<|x|/s'} |S_n| \ge |x|]\,. \tag{13.7}$$

(This is proved rigorously by coupling the random walk on T to a random walk $\langle Y_n \rangle$ on $\mathbb{N}$ by letting $\langle Y_n \rangle$ take only the moves of $\langle X_n \rangle$ that lie on the shortest path between o and x.) Now by the reflection principle,

$$\mathbf{P}[\max_{n\le N} |S_n| \ge j] \le 2\mathbf{P}[\max_{n\le N} S_n \ge j] \le 4\mathbf{P}[S_N \ge j]\,.$$

We have seen that $\mathbf{P}[S_N \ge j] \le e^{-NI(j/N)}$. Therefore, (13.7) gives $\mathbf{P}[o \leftrightarrow x] \le 4e^{-|x|I(s')/s'}$. In light of Proposition 5.8, this means that for any cutset Π with all edges at level $> s'L$, we have

$$q \le \sum_{e(x)\in\Pi} 4e^{-|x|I(s')/s'}\,.$$

Therefore, $\operatorname{br} T \ge e^{I(s')/s'}$. Since this holds for all $s' < s$, the result follows. ◀

§13.2. The Varopoulos-Carne Bound.

Recall from Section 6.2 that the transition operator

$$(Pf)(x) := \sum_y p(x,y)f(y)$$

is a bounded self-adjoint operator on $\ell^2(\mathsf{V}, \pi)$.

Carne (1985) gave the following generalization and improvement of a fundamental result of Varopoulos (1985b), which we have improved a bit more.

Theorem 13.4. (Varopoulos-Carne Bound) *For any reversible random walk, we have*

$$p_n(x,y) \le 2\sqrt{\pi(y)/\pi(x)}\ \|P\|_\pi^n\ e^{-|x-y|^2/(2n)}\,.$$

The exponential part is sharp, as we can see from the formula (13.1) for simple random walk on $\mathbb{Z}$. In fact, we will use the result for simple random walk and some notation will be helpful. Let $q_n(k)$ denote the probability that simple random walk on $\mathbb{Z}$ starting at 0 is at k after n steps. Then a special case of the Hoeffding-Azuma inequality says that

$$\sum_{|k|\ge d} q_n(k) \le 2e^{-d^2/(2n)} . \tag{13.8}$$

To prove Theorem 13.4, we need some standard facts from analysis. These are detailed in the following two exercises.

▷ **Exercise 13.3.**

Let T be a bounded self-adjoint operator on a Hilbert space and Q be a polynomial with real coefficients. Show that

$$\|Q(T)\| \le \max_{|s|\le\|T\|} |Q(s)| .$$

▷ **Exercise 13.4.**

Show that for every $k \in \mathbb{Z}$, there are unique polynomials Q_k of degree $|k|$ such that $Q_k(\cos\theta) = \cos k\theta$. Show that $|Q_k(s)| \le 1$ whenever $|s| \le 1$. These polynomials are called the ***Chebyshev polynomials***.

The following formula is a modification of that proved by Carne (1985). It relates any reversible random walk to simple random walk on $\mathbb{Z}$.

Lemma 13.5. *Let Q_k be the Chebyshev polynomials. For any reversible random walk, we have*

$$P^n = \|P\|_\pi^n \sum_{k\in\mathbb{Z}} q_n(k) Q_k(P/\|P\|_\pi) . \tag{13.9}$$

Furthermore, $\|Q_k(P/\|P\|_\pi)\|_\pi \le 1$ for all k.

Proof. Let w be such that $z = (w + w^{-1})/2$. Then

$$Q_k(z) = Q_k\big((w + w^{-1})/2\big) = (w^k + w^{-k})/2$$

since it holds for all z of modulus 1. By the binomial theorem,

$$z^n = [(w + w^{-1})/2]^n = \sum_{k\in\mathbb{Z}} q_n(k) w^k = \sum_{k\in\mathbb{Z}} q_n(k)(w^k + w^{-k})/2 = \sum_{k\in\mathbb{Z}} q_n(k) Q_k(z) .$$

Since this is an identity between polynomials, we may apply it to $z := P/\|P\|_\pi$ to get (13.9). The final estimate derives from Exercise 13.3 (for $T := P/\|P\|_\pi$) and Exercise 13.4. ◀

Proof of Theorem 13.4. Fix $x, y \in \mathsf{V}$ and write $d := |x - y|$. Set $e_x := \mathbf{1}_{\{x\}}/\sqrt{\pi(x)}$. Then

$$p_n(x, y) = \sqrt{\pi(y)/\pi(x)}(e_x, P^n e_y)_\pi \,. \tag{13.10}$$

When we substitute (13.9) for P^n here, we may exploit the fact that $(Q_k(P/\|P\|_\pi)e_y, e_x)_\pi = 0$ for $|k| < d$ since Q_k has degree $|k|$ and $p_i(x, y) = 0$ for $i < d$. Furthermore, we may use the bound $|(Q_k(P/\|P\|_\pi)e_y, e_x)_\pi| \le \|Q_k(P/\|P\|_\pi)\|_\pi \|e_y\|_\pi \|e_x\|_\pi \le 1$. We obtain

$$p_n(x, y) \le \sqrt{\pi(y)/\pi(x)}\; \|P\|_\pi^n \sum_{|k| \ge d} q_n(k) \,.$$

Now use (13.8) to complete the proof. ◀

§13.3. Branching Number of a Graph.

For a connected locally finite graph G, choose some vertex $o \in G$ and define the ***branching number*** $\operatorname{br} G$ of G as the supremum of those $\lambda \ge 1$ such that there is a positive flow from o to infinity when the edges have capacities $\lambda^{-|e|}$, where distance is measured from o. By the Max-Flow Min-Cut Theorem, this is the same as

$$\operatorname{br} G = \inf \left\{ \lambda \ge 1 \,;\ \inf \left\{ \sum_{e \in \Pi} \lambda^{-|e|} \,;\ \Pi \text{ separates } o \text{ from } \infty \right\} = 0 \right\} .$$

▷ **Exercise 13.5.**

Show that $\operatorname{br} G$ does not depend on the choice of vertex o.

▷ **Exercise 13.6.**

Show that if G' is a subgraph of G, then $\operatorname{br} G' \le \operatorname{br} G$.

Recall that $\lambda_c(G)$ is the critical value of λ separating transience from recurrence for RW_λ on G.

▷ **Exercise 13.7.**

Show that $\lambda_c(G) \le \operatorname{br} G$.

Recall from Section 3.4 that we call a subtree T of G rooted at o ***geodesic*** if for every vertex $x \in T$, the distance from x to o is the same in T as in G. Note that for such trees, $\operatorname{br} T \le \operatorname{br} G$ since any cutset of G restricts to one in T. It follows from Section 3.4 that when G is a Cayley graph of a finitely generated group of growth rate b, there is a geodesic spanning tree T of G with $\operatorname{br} T = b = \lambda_c(G)$. Hence, we also have $\operatorname{br} G = b = \lambda_c(G)$. Most of this also holds for many planar graphs, as we show now. The proof will show how planar graphs are like trees. (For other resemblances of planar graphs to trees, see Theorems 9.11 and 10.50 and Proposition 11.25.)

Theorem 13.6. **(Lyons, 1996)** *Let G be an infinite connected plane graph of bounded degree such that only finitely many vertices lie in any bounded region of the plane. Suppose that G has a geodesic spanning tree T with no leaves. Then $\lambda_c(G) = \operatorname{br} G = \operatorname{br} T$.*

Proof. We first prove that $\operatorname{br} T = \operatorname{br} G$. It suffices to show that for $\lambda > \operatorname{br} T$, we have $\lambda \geq \operatorname{br} G$. Given a cutset Π of T, we will define a cutset Π^* of G whose corresponding cutset sum is not much larger than that of Π. We may assume that o is at the origin of the plane and that all vertices in T_n are on the circle of radius n in the plane. Now every vertex $x \in T$ has a descendant subtree $T^x \subseteq T$. For $n \geq |x|$, this subtree cuts off an arc of the circle of radius n; in the clockwise order of $T^x \cap T_n$, there is a least element $\underline{x}_n$ and a greatest element $\overline{x}_n$. Each edge in Π has two endpoints; collect the ones farther from o in a set W. Define Π^* to be the collection of edges incident to the set of vertices

$$W^* := \{\underline{x}_n, \overline{x}_n \,;\ x \in W,\, n \geq |x|\} \,.$$

We claim that Π^* is a cutset of G. For if $o = y_1, y_2, \ldots$ is a path in G with an infinite number of distinct vertices, let y_k be the first vertex belonging to T^x for some $x \in W$. Planarity implies that $y_k \in W^*$, whence the path intersects Π^*, as desired.

Now let c be the maximum degree of vertices in G. We have

$$\begin{aligned}\sum_{e \in \Pi^*} \lambda^{-|e|} \leq c \sum_{x \in W^*} \lambda^{-|x|+1} &\leq c \sum_{x \in W} \sum_{n \geq |x|} 2\lambda^{-n+1} \\ &= \frac{2c\lambda}{\lambda - 1} \sum_{x \in W} \lambda^{-|x|+1} = \frac{2c\lambda}{\lambda-1} \sum_{e \in \Pi} \lambda^{-|e|} \,.\end{aligned}$$

Now the desired conclusion is evident. This also implies that $\lambda_c(G) \leq \operatorname{br} G$.

To finish, we need only show that $\lambda_c(G) \geq \operatorname{br} G$. Let $\lambda < \operatorname{br} G$. Since $\lambda < \operatorname{br} T$, it follows from Theorem 3.5 that RW_λ is transient on T. Therefore, RW_λ is also transient on G. ◀

Our bound for trees, Proposition 13.3, can be extended to general graphs as follows, but it is not as good as the one obtained for trees. See Section 13.8 for a better result.

Proposition 13.7. *If G is a connected locally finite graph with bounded degree on which simple random walk has positive speed, then $\operatorname{br} G > 1$. More precisely, if*

$$\liminf_{n \to \infty} |X_n|/n \geq s$$

with positive probability, then $\operatorname{br} G \geq e^{s/2}$.

Proof. Given $0 < s' < s'' < s$, there is some L such that

$$q := \mathbf{P}[\forall n \geq L \quad |X_n| > s''n] > 0 \,. \tag{13.11}$$

As in the proof of Proposition 13.3, define the general percolation on G by keeping those edges $e(x)$ such that $|x| \le s''L$ or $\exists n < |x|/s'' \quad X_n = x$. According to (13.11), the component of the root in this percolation is infinite with probability at least q. On the other hand, if $|x| > s''L$, then by the Varopoulos-Carne bound,

$$\begin{aligned}\mathbf{P}[o \leftrightarrow x] &\le \mathbf{P}[\exists n < |x|/s'' \quad X_n = x] \le \sum_{n=1}^{|x|/s''} p_n(o,x) \\ &\le \sum_{n=1}^{|x|/s''} 2\sqrt{\pi(x)/\pi(o)}e^{-|x|^2/(2n)} \le 2\frac{|x|}{s''}\sqrt{\frac{\pi(x)}{\pi(o)}}e^{-s''|x|/2} \\ &< Ce^{-s'|x|/2}\end{aligned}$$

for some constant C. In light of Proposition 5.8, this means that for any cutset Π, we have

$$q < \sum_{e(x)\in\Pi} Ce^{-s'|x|/2}\,.$$

Therefore, $\operatorname{br} G \ge e^{s'/2}$. Since this holds for all $s' < s$, the result follows. ◀

▷ **Exercise 13.8.**
Show that the hypothesis of bounded degree in Proposition 13.7 is necessary.

§13.4. Stationary Measures on Trees.

We consider here simple random walk on random trees, where the tree is chosen according to some probability measure that gives us a Markov chain which is stationary in an appropriate sense. In order to achieve a stationary Markov chain, we need to change our point of view from a random walk on a fixed (though random) tree to a random walk on the space of isomorphism classes of trees.

Actually, there is no good measurable space of unrooted trees. Think, for example, how one would define a distance between two trees. Instead, it is necessary to consider rooted trees. Then two rooted trees are close if they agree, up to isomorphism sending one root to the other, in a large ball around their roots. This gives a topology, and the topology generates a Borel σ-field. What one then does is walk on the space of rooted trees, changing the root to the location of the walker and keeping the underlying unrooted tree the same. However, in order to have a measure on rooted trees that is stationary with respect to this chain, we need to use isomorphism classes of rooted trees.

The formalism is as follows. Let $\mathscr{T}$ be the space of rooted trees in Exercise 5.2. Call two rooted trees ***(rooted) isomorphic*** if there is a bijection of their vertex sets preserving adjacency and mapping one root to the other. Since the roots will be changing with the walker, we will in this section write the root explicitly. Our notation for a rooted tree will be (T, x), where $x \in \mathsf{V}(T)$ designates the root. For $(T, o) \in \mathscr{T}$, let $[T, o]$ denote the set of trees that are isomorphic to (T, o). Let $[\mathscr{T}] := \big\{[T, o]\,;\ (T, o) \in \mathscr{T}\big\}$. Normally, we have a measure μ such as $\mathbf{GW}$ on rooted trees $\mathscr{T}$; such a measure induces a measure $[\mu]$ on isomorphism classes of rooted trees $[\mathscr{T}]$ in the obvious way.

Consider the Markov chain that moves from a rooted tree (T, x) to the rooted tree (T, y) for a random neighbor y of x. For fixed T, this chain is isomorphic to simple random walk on T. Write the transition probabilities as

$$p\big((T, x), (T, y)\big) = \begin{cases} 1/\deg_T(x) & \text{if } y \sim x; \\ 0 & \text{otherwise.} \end{cases}$$

Now we are really interested in the Markov chain induced by this chain on isomorphism classes of trees, so define

$$p\big([T, x], [T', y]\big) := \frac{1}{\deg_T(x)} \big|\big\{z \in T\,;\ z \sim x,\ [T', y] = [T, z]\big\}\big|\,.$$

Let P^* be the corresponding operator on measures on $[\mathscr{T}]$. We call a Borel measure μ on $[\mathscr{T}]$ ***stationary*** (for simple random walk) if $P^*\mu = \mu$. We will also call a Borel measure μ on $\mathscr{T}$ stationary if the induced measure $[\mu]$ on $[\mathscr{T}]$ is. Note that in such a case, the measure, say σ, on trajectories $\langle (T, X_k)\,;\ k \ge 0\rangle$ of rooted trees determined by the random walk is stationary for the left shift (which sends $\langle (T, X_k)\,;\ k \ge 0\rangle$ to $\langle (T, X_k)\,;\ k \ge 1\rangle$). We call μ ***ergodic*** if σ is, in other words, if the only shift-invariant events have σ-measure 0 or 1. To make explicit the statement that μ is stationary for simple random walk, we will write that μ is SRW-stationary.

▷ **Exercise 13.9.**

Let $G = (\mathsf{V}, \mathsf{E})$ be a finite connected graph with E (undirected) edges. For $x \in \mathsf{V}$, let T_x be the universal cover of G based at x (see Section 3.3). Define

$$\mu\big([T, o]\big) := \frac{1}{2E} \sum \big\{ \deg x\,;\ x \in \mathsf{V},\ [T_x, x] = [T, o]\big\}\,.$$

Show that μ is a stationary ergodic probability measure on $[\mathscr{T}]$.

For a tree T, write $T^\diamondsuit$ for the ***bi-infinitary part*** of T consisting of the vertices and edges of T that belong to some bi-infinite self-avoiding path. As long as T has at least two boundary points, its bi-infinitary part is non-empty. The parts of T that are not in its bi-infinitary part are finite trees that we call ***shrubs***. Since simple random walk on a shrub is recurrent, simple random walk on T visits $T^\diamondsuit$ infinitely often and, in fact, takes infinitely many steps on $T^\diamondsuit$. If we observe $\langle X_k \rangle$ only when it makes a transition along an edge of $T^\diamondsuit$, then we see simple random walk on $T^\diamondsuit$ (by the strong Markov property). In particular, if we begin simple random walk on $\mathscr{T}$ with an initial stationary probability measure μ, then $X_k \in T^\diamondsuit$ for some k a.s., whence $X_0 \in T^\diamondsuit$ with positive probability. Thus, the set of states $\mathscr{A}_\diamondsuit := \{(T, o)\,;\; o \in T^\diamondsuit\}$ has positive probability. Write $\mu^\diamondsuit$ for the measure induced on the bi-infinitary parts by μ, i.e.,

$$\text{for all events } \mathscr{B} \qquad \mu^\diamondsuit(\mathscr{B}) := \big(\mu(\mathscr{A}_\diamondsuit)\big)^{-1} \mu\big\{(T, o)\,;\; (T^\diamondsuit, o) \in \mathscr{B}\big\}\,.$$

Let $\mathscr{A}'_\diamondsuit$ be the event that $(T, X_0), (T, X_1) \in \mathscr{A}_\diamondsuit$. Certainly $\sigma(\mathscr{A}'_\diamondsuit) > 0$, so the sequence of returns to $\mathscr{A}'_\diamondsuit$ is also shift-stationary by Exercise 6.45, whence $\mu^\diamondsuit$ is stationary for the induced simple random walk on the bi-infinitary parts.

Let $\mathsf{SRW} \times \mu$ denote the probability measure on paths in trees given by choosing a tree according to μ and then independently running simple random walk on the tree starting at its root.

Theorem 13.8. *If μ is an ergodic SRW-stationary probability measure on the space of rooted trees $\mathscr{T}$ such that μ-a.e. tree is infinite, then the speed (rate of escape) of simple random walk is $\mathsf{SRW} \times \mu$-a.s.*

$$\lim_{n \to \infty} \frac{|X_n|}{n} = \sum_{k \ge 1} \mu[\deg_T o = k] \left(1 - \frac{2}{k}\right) \ge 0\,. \tag{13.12}$$

The speed is positive iff μ-a.e. tree has at least 3 boundary points, in which case μ-a.e. tree has uncountably many boundary points and branching number > 1.

Proof. Since the measure on rooted trees is stationary and ergodic for simple random walk, so is the degree: $\langle \deg_T X_k \rangle$ is a stationary ergodic sequence. Thus, (13.12) follows from (13.3) and the ergodic theorem.

Now the speed is of course 0 if the trees have at most 2 ends. In the opposite case, $\mu^\diamondsuit[\deg_T(o) \ge 3] > 0$ since otherwise the walk would be restricted to a copy of $\mathbb{Z}$. Since $\deg_T(o) \ge 2$ $\mu^\diamondsuit$-a.s., it follows from (13.12) that the speed is positive $\mu^\diamondsuit$-a.s.

Now let $\langle Y_k \rangle$ be the random walk induced on $T^\diamondsuit$ and Z_k be the number of steps that the random walk on T takes between the kth step on $T^\diamondsuit$ and the $(k+1)$th step on $T^\diamondsuit$.

Since the random walk returns to $T^\diamondsuit$ infinitely often a.s., its speed on T is

$$\lim_{k\to\infty} \frac{|Y_k|}{\sum_{j<k} Z_j}\,.$$

To show that this is positive a.s., it remains to show that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j<k} Z_j < \infty \tag{13.13}$$

a.s., since the $\mu^\diamondsuit$-speed on $T^\diamondsuit$ is $\lim |Y_k|/k$, which we have already shown is positive a.s. Now $\langle Z_j \rangle$ is a stationary nonnegative sequence if we condition on $\mathscr{A}_\diamondsuit$, so the limit in (13.13) equals the mean of Z_0 by the ergodic theorem. But Z_0 is the time it takes for the random walk to make a step along $T^\diamondsuit$, i.e., the time it takes to return to $\mathscr{A}'_\diamondsuit$. This has finite expectation by the Kac lemma (Exercise 6.45).

The fact that μ-a.e. tree has infinitely many boundary points is a consequence of the transience, and the stronger fact that the branching number is > 1 follows from Proposition 13.3. ◀

Sometimes, it is easier to find a measure that is stationary for delayed simple random walk, rather than for simple random walk. Here, we define ***delayed simple random walk*** on a graph with maximum degree at most D, abbreviated DSRW, to have the transition probabilities

$$p(x, y) := \begin{cases} 1/D & \text{if } x \sim y, \\ 1 - \deg x/D & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

(The choice of D will be left implicit, but will be clear from context.) Thus, any uniform measure on the vertices is an (infinite) stationary measure for delayed simple random walk. However, there is a simple relation between DSRW-stationary measures and SRW-stationary measures:

Lemma 13.9. *If μ is a* DSRW*-stationary probability measure on $\mathscr{T}$, then the degree-biased μ' is a* SRW*-stationary probability measure on $\mathscr{T}$; more precisely, μ' is defined as the measure $\mu' \ll \mu$ with*

$$\frac{d\mu'}{d\mu}(T, o) := \frac{\deg_T(o)}{\int \deg_t(o)\, d\mu(t, o)}\,.$$

If μ is ergodic, then so is μ'.

Proof. We are given that for all events $\mathscr{A}$,

$$\mu\big[[T, X_0] \in \mathscr{A}\big] = \mu\big[[T, X_1] \in \mathscr{A}\big]\,.$$

If we write this out, it becomes

$$\int \mathbf{1}_{\{[T,X_0 \in \mathscr{A}]\}}\, d\mu(T,X_0) = \int \left(\left(1 - \frac{\deg_T X_0}{D}\right) \mathbf{1}_{\{[T,X_0 \in \mathscr{A}]\}} + \frac{1}{D}\Big|\{x \sim X_0 \,;\ [T,x] \in \mathscr{A}\}\Big| \right) d\mu(T,X_0)\,.$$

Cancelling what can be obviously cancelled, we get

$$\int (\deg_T X_0)\mathbf{1}_{\{[T,X_0 \in \mathscr{A}]\}}\, d\mu(T,X_0) = \int \Big|\{x \sim X_0 \,;\ [T,x] \in \mathscr{A}\}\Big|\, d\mu(T,X_0)\,.$$

This is the same as

$$\mu'\big[[T,X_0] \in \mathscr{A}\big] = \mu'\big[[T,X_1] \in \mathscr{A}\big]\,.$$

The ergodicity claim is immediate from the definitions. ◀

▷ **Exercise 13.10.**

Show that if μ is an ergodic DSRW-stationary probability measure on the space of rooted trees $\mathscr{T}$ such that μ-a.e. tree has at least 3 boundary points and maximum degree at most D, then the rate of escape of simple random walk (not delayed) is SRW $\times$ μ-a.s.

$$\lim_{n\to\infty} \frac{|X_n|}{n} = 1 - \frac{2}{\int \deg_T(o)\, d\mu(T,o)}\,.$$

Where do invariant probability measures on rooted trees come from? One place is from certain probability measures on forests in Cayley graphs. Recall that an edge $[x,y]$ is present in a Cayley graph iff there is a generator s such that $xs = y$. (Cayley graphs defined by multiplication on the left instead of on the right are isomorphic by inversion.) Thus, for any γ in the group, multiplication by γ on the left is an automorphism of G. Given a percolation on a Cayley graph $G = (\mathsf{V}, \mathsf{E})$, i.e., a Borel probability measure $\mathbf{P}$ on the subsets of E, write $\omega \subseteq \mathsf{E}$ for the random subset given by the percolation. The action of multiplication by γ induces a map

$$\gamma\omega := \{[\gamma x, \gamma y]\,;\ [x,y] \in \omega\}\,.$$

Thus, γ acts on $\mathbf{P}$; we call $\mathbf{P}$ ***translation-invariant*** if $\gamma\mathbf{P} = \mathbf{P}$ for all $\gamma \in \mathsf{V}$. If $\{s_1, s_2, \ldots, s_D\}$ denotes the generating set for G, assumed to be closed under inverses, then clearly $\mathbf{P}$ is translation-invariant iff $s_i\mathbf{P} = \mathbf{P}$ for $i = 1, \ldots, D$. Call the percolation a ***random forest*** if each component is a tree. For example, the uniform spanning forests FUSF and WUSF are translation-invariant random forests by Exercise 10.2. It is obvious that the minimal spanning forests, FMSF and WMSF, are as well.

Example 13.10. Let G be the usual Cayley graph of $\mathbb{Z}$. Consider the 3 possible spanning forests that have only trees with 3 vertices. If these 3 are equally likely, then we get a translation-invariant random forest of G. This is easily seen *not* to be SRW-stationary.

Let μ denote the law of the component of the identity, o, of a translation-invariant random forest $\mathbf{P}$. Then μ is stationary for a random walk $\langle X_n \rangle$ starting at $X_0 = o$ iff $\mathbf{E}[X_1 \mu] = \mu$. The following is essentially due to Häggström (1997).

Theorem 13.11. (Invariance and Stationarity) *If μ is the component law of a translation-invariant random forest on a Cayley graph, then μ is stationary for delayed simple random walk. In fact, μ is globally reversible.*

Proof. Let $\langle X_k \rangle$ be DSRW on the component of the identity. Let T_x denote the component of x, so $\mu(\mathscr{A}) = \mathbf{P}\big[[T_o, o] \in \mathscr{A}\big]$. Global reversibility is the statement that for all Borel $\mathscr{A}, \mathscr{B} \subseteq \mathscr{T}$,

$$\mathbf{P}\big[[T_o, o] \in \mathscr{A},\ [T_{X_1}, X_1] \in \mathscr{B}\big] = \mathbf{P}\big[[T_o, o] \in \mathscr{B},\ [T_{X_1}, X_1] \in \mathscr{A}\big] .$$

We will show more generally that for Borel $\mathscr{A}, \mathscr{B} \subseteq \{0,1\}^{\mathsf{E}}$, we have

$$\mathbf{P}[\mathscr{A},\ X_1 \mathscr{B}] = \mathbf{P}[\mathscr{B},\ X_1 \mathscr{A}] .$$

Now we may write the first event as a disjoint union

$$\mathscr{A} \cap X_1 \mathscr{B} = \bigcup_{i=1}^{D} \big(\mathscr{A} \cap \mathscr{B} \cap \{X_1 = o\} \cap \{\deg o = i\}\big) \cup \bigcup_{i=1}^{D} \big(\mathscr{A} \cap s_i \mathscr{B} \cap \{X_1 = s_i\}\big) .$$

The first union is unchanged when we switch $\mathscr{A}$ and $\mathscr{B}$, so it suffices to show that the probability of the second union is also unchanged under switching. Now

$$\mathbf{P}\left[\bigcup_{i=1}^{D} \big(\mathscr{A} \cap s_i \mathscr{B} \cap \{X_1 = s_i\}\big)\right] = \sum_{i=1}^{D} \mathbf{P}[\mathscr{A},\ s_i \mathscr{B},\ [o, s_i] \in \omega]/D .$$

By translation invariance of $\mathbf{P}$, this equals

$$\begin{aligned} \sum_{i=1}^{D} \mathbf{P}[s_i^{-1}\big(\mathscr{A} \cap s_i \mathscr{B} \cap \{[o, s_i] \in \omega\}\big)]/D &= \sum_{i=1}^{D} \mathbf{P}[s_i^{-1}\mathscr{A},\ \mathscr{B},\ [s_i^{-1}, o] \in \omega]/D \\ &= \sum_{i=1}^{D} \mathbf{P}[\mathscr{B},\ s_i \mathscr{A}, [o, s_i] \in \omega]/D \end{aligned}$$

since inversion is a permutation of $\{s_1, \ldots, s_D\}$. But this amounts to switching $\mathscr{A}$ and $\mathscr{B}$, as desired. ◀

The following corollary generalizes results of Häggström (1997). It shows how the obvious property of Cayley graphs having 1, 2 or uncountably many ends generalizes to translation-invariant random forests. Here, an ***end*** is an equivalence class of infinite nonself-intersecting paths in G, with two paths being equivalent if for all finite $A \subset G$, the paths are eventually in the same connected component of $G \setminus A$. (Part of this was proved in Corollary 8.18.)

Corollary 13.12. *If μ is the component law of a translation-invariant random forest on a Cayley graph, then simple random walk on the infinite trees with at least 3 boundary points has positive speed a.s. Hence, these trees have* $\mathrm{br} > 1$. *In particular, there are a.s. no trees with a finite number of boundary points* ≥ 3.

Proof. We have seen that $[\mu]$ is DSRW-stationary on $[\mathscr{T}]$. If $\mathscr{A} \subseteq [\mathscr{T}]$ denotes the set of tree classes with at least 3 boundary points, then $\mathscr{A}$ is invariant under the random walk, so $[\mu]$ conditioned to $\mathscr{A}$ is also DSRW-stationary. From Lemma 13.9, we also get a SRW-stationary probability measure on $\mathscr{A}$, to which we may apply Theorem 13.8. ◀

Since a tree with branching number > 1 also has exponential growth, it follows that if G is a group of subexponential growth, then every tree in a translation-invariant random forest has at most 2 ends a.s. Actually, this holds for all amenable groups:

Corollary 13.13. *If G is an amenable group, then every tree in a translation-invariant random forest has at most 2 ends a.s.*

Proof. By reasoning exactly parallel to Exercise 10.6, the expected degree of every vertex in the forest is two. Therefore, the speed of delayed simple random walk is 0 a.s. by Exercise 13.10, whence the number of ends cannot be at least 3 with positive probability. ◀

In fact, this result holds still more generally:

Theorem 13.14. *If G is an amenable group, then every component in a translation-invariant percolation has at most 2 ends a.s.*

This is due to Burton and Keane (1989) in the case where $G = \mathbb{Z}^d$. It was proved in Exercise 7.23.

▷ **Exercise 13.11.**

Use Corollary 13.12 to prove that for a translation-invariant random forest on a Cayley graph, there are a.s. no isolated boundary points in trees with an infinite boundary.

§13.5. Galton-Watson Trees.

When we run random walks on Galton-Watson trees T, their asymptotic properties reveal information about the structure of T beyond the growth rate and branching number. This is the theme of the present section and of Chapter 16. All the results in this section are from the paper Lyons, Pemantle, and Peres (1995b).

We know by Theorem 3.5 and Corollary 5.10 that simple random walk on a Galton-Watson tree T is almost surely transient. Equivalently, by Theorem 2.3, the effective conductance of T from the root to infinity is a.s. finite when each edge has unit conductance. See Figure 13.2 for the distribution of the effective conductance when an individual has 1 or 2 children with equal probability and Section 16.7 for further discussion of this graph.

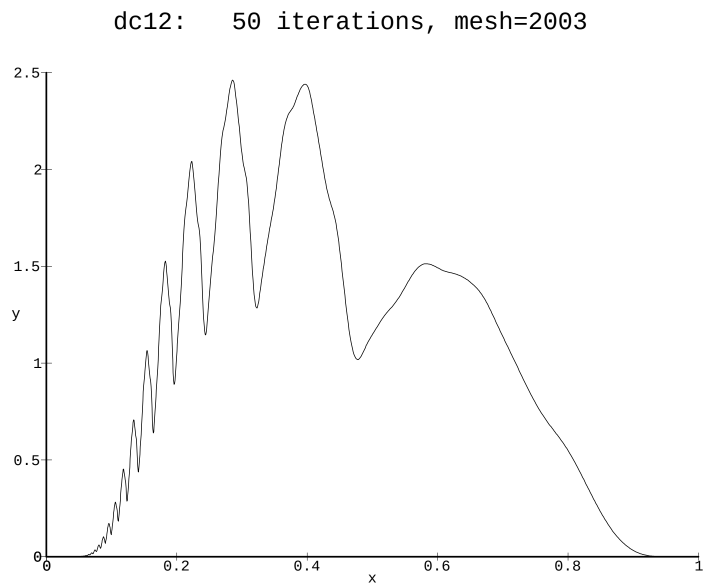


Figure 13.2. The density of the conductance for $p_1 = p_2 = 1/2$.

Transience means only that the distance of a random walker from the root of T tends to infinity a.s. Is the rate of escape positive? Can we calculate the rate? According to (13.3), it would suffice to know the proportion of the time the walk spends at vertices of degree $k+1$ for each k. As we demonstrate in Theorem 13.17, this proportion is p_k, so that the speed is a.s.

$$l := \sum_{k=1}^{\infty} p_k \frac{k-1}{k+1} \,.$$

In particular, the speed is positive.

The function $s \mapsto (s-1)/(s+1)$ is concave, whence by Jensen's inequality, unless $Z_1 = m$ a.s., $l < (m-1)/(m+1)$. Notice that the latter is the speed on the deterministic tree of the same growth rate when m is an integer.

Our plan is to identify a stationary ergodic measure for simple random walk on the space of trees and then use Theorem 13.8. In order to construct a stationary Markov chain on the space of trees, we will use unlabelled but rooted trees. The family tree of a Galton-Watson process is rooted at the initial individual. Note that since our trees are unlabelled, the chance, say, that the family tree has two children of the root, one having one child and the other having three, is

$$2p_2p_1p_3 \,. \tag{13.14}$$

Since various measures on the space of trees will need to be considered, we use **GW** to denote the standard measure on (family) trees given by a Galton-Watson process. We use **E** to refer only to the law of Z_1 in order to avoid confusion. We assume throughout this section, unless stated otherwise, that $p_k < 1$ for all k and that $p_0 = 0$.

In order to analyze the speed of simple random walk, we need to find a stationary measure for the environment process, i.e., the tree as seen from the current vertex. This will be a fundamental tool as well for our analysis of harmonic measure. Now the root of a Galton-Watson tree is different from the other vertices since it has stochastically one fewer neighbor. To remedy this defect, we consider ***augmented Galton-Watson measure***, **AGW**. This measure is defined just like **GW** except that the number of children of the root (only) has the law of $Z_1 + 1$; i.e., the root has $k+1$ children with probability p_k and these children all have independent standard Galton-Watson descendant trees.

Theorem 13.15. *The Markov chain with transition probabilities* $\mathbf{p}_{\mathsf{SRW}}$ *and initial distribution* **AGW** *is stationary.*

Proof. The measure **AGW** is stationary since when the walk takes a step to a neighbor of the root, it goes to a vertex with one neighbor (where it just came from) plus a **GW**-tree; and the neighbor it came from also has attached another independent **GW**-tree. This is the same as **AGW**. ◀

Remark. It is clear that the chain is locally reversible, i.e., that it is reversible on every communicating class. This, however, does not imply (global) reversibility. We will have no need for global reversibility, only for local reversibility (which is, indeed, a consequence of global reversibility). See Exercise 13.26 for the definitions. A proof of global reversibility may be found in Lyons, Pemantle, and Peres (1995b); this proof may also serve to convince the skeptic who finds insufficient the above reasoning for stationarity.

It now follows from Theorem 13.8 that simple random walk has positive speed with positive **AGW**-probability; we want to establish this a.s., so we will show ergodicity. This will also give us our formula for the speed.

We will find it convenient to work with random walks indexed by $\mathbb{Z}$ rather than by $\mathbb{N}$. We will denote such a bi-infinite path $\ldots, x_{-1}, x_0, x_1, \ldots$ by $\overleftrightarrow{x}$. Similarly, a path of vertices $x_0, x_1, \ldots$ in T will be denoted $\overrightarrow{x}$ and a path $\ldots, x_{-1}, x_0$ will be denoted $\overleftarrow{x}$. We will regard a ray as either a path $\overrightarrow{x}$ or $\overleftarrow{x}$ that starts at the root and doesn't backtrack. The path of simple random walk has the property that a.s., it ***converges*** to a ray in the sense that there is a unique ray with which it shares infinitely many vertices. If a path $\overrightarrow{x}$ converges to a ray ξ in this sense, then we will write $x_{+\infty} = \xi$. Similarly for a limit $x_{-\infty}$ of a path $\overleftarrow{x}$. The space of convergent paths $\overrightarrow{x}$ in T will be denoted $\overrightarrow{T}$; likewise, $\overleftarrow{T}$ denotes the convergent paths $\overleftarrow{x}$ and $\overleftrightarrow{T}$ denotes the paths $\overleftrightarrow{x}$ for which both $\overrightarrow{x}$ and $\overleftarrow{x}$ converge and have distinct limits.

Now since we actually want to work with a Markov chain on the space of trees rather than on a random tree, we will use the path space (actually, path bundle over the space of trees)

$$\mathsf{PathsInTrees} := \left\{ (\overleftrightarrow{x}, T)\,;\ \overleftrightarrow{x} \in \overleftrightarrow{T} \right\}.$$

The rooted tree corresponding to $(\overleftrightarrow{x}, T)$ is (T, x_0). Let S be the shift map:

$$(S\overleftrightarrow{x})_n := x_{n+1}\,,$$

$$S(\overleftrightarrow{x}, T) := (S\overleftrightarrow{x}, T).$$

Extend simple random walk to all integer times by letting $\overleftarrow{x}$ be an independent copy of $\overrightarrow{x}$. Let $\mathsf{SRW} \times \mathbf{AGW}$ denote the measure on $\mathsf{PathsInTrees}$ associated to this stationary Markov chain, although this is not a product measure.

▷ **Exercise 13.12.**
Show that this chain is indeed stationary.

When a random walk traverses an edge for the first and last time simultaneously, we say it ***regenerates*** since it will now remain in a previously unexplored tree. Thus, we define the set of ***regeneration points***

$$\mathsf{Regen} := \left\{ (\overleftrightarrow{x}, T) \in \mathsf{PathsInTrees}\,;\ \forall n < 0\ \ x_n \neq x_0,\ \ \forall n > 0\ x_n \neq x_{-1} \right\}.$$

Proposition 13.16. *For* $\mathsf{SRW} \times \mathbf{AGW}$*-a.e.* $(\overleftrightarrow{x}, T)$*, there are infinitely many* $n \geq 0$ *for which* $S^n(\overleftrightarrow{x}, T) \in \mathsf{Regen}$*.*

Proof. Define the set of ***fresh points***

$$\mathsf{Fresh} := \left\{(\overleftrightarrow{x}, T) \in \mathsf{PathsInTrees}\,;\ \forall n < 0\ \ x_n \neq x_0\right\}.$$

Note that by a.s. transience of simple random walk and the fact that independent simple random walks on a transient tree a.s. converge to distinct ends (Exercise 2.40), there are a.s. infinitely many $n \ge 0$ such $S^n(\overleftrightarrow{x}, T) \in \mathsf{Fresh}$. Let $\mathscr{F}_n$ be the σ-field generated by $\langle \ldots, x_{-1}, x_0, \ldots, x_n\rangle$ and their degrees. Then by a.s. transience of $\mathbf{GW}$ trees, $\alpha := (\mathsf{SRW} \times \mathbf{AGW})(\mathsf{Regen} \mid \mathsf{Fresh}) > 0$ and, in fact, $(\mathsf{SRW} \times \mathbf{AGW})(\mathsf{Regen} \mid \mathsf{Fresh},\ \mathscr{F}_{-1}) = \alpha$. Fix k_0. Let R be the event that there is at least one $k \ge k_0$ for which $S^k(\overleftrightarrow{x}, T) \in \mathsf{Regen}$. Then for $n \ge k_0$, the conditional probability $(\mathsf{SRW} \times \mathbf{AGW})(R \mid \mathscr{F}_n)$ is at least the conditional probability that the walk comes to a fresh vertex after time n, which is 1, times the conditional probability that at the first such time, the walk regenerates, which is a constant, α. On the other hand, by the martingale convergence theorem, $(\mathsf{SRW} \times \mathbf{AGW})(R \mid \mathscr{F}_n) \to \mathbf{1}_R$ a.s., whence the limit must be $\mathbf{1}$. That is, R occurs a.s., which completes the proof since k_0 was arbitrary. ◀

Given a rooted tree T and a vertex x in T, the subtree T^x rooted at x denotes the subgraph of T formed from those edges and vertices that become disconnected from the root of T when x is removed. This may be considered as the descendant tree of x. The sequence of fresh trees $T \setminus T^{x_{-1}}$ seen at regeneration points $(\overleftrightarrow{x}, T)$ is clearly stationary, but not i.i.d. However, the part of a tree between regeneration points, together with the path taken through this part of the tree, is independent of the rest of the tree and of the rest of the walk. We call this part a ***slab***. To define this notion precisely, we use the return time n_{Regen}, where $n_{\mathsf{Regen}}(\overleftrightarrow{x}, T) := \inf\{n > 0\,;\ S^n(\overleftrightarrow{x}, T) \in \mathsf{Regen}\}$. For $(\overleftrightarrow{x}, T) \in \mathsf{Regen}$, the associated ***slab*** (including the path taken through the slab) is

$$\mathsf{Slab}(\overleftrightarrow{x}, T) := \left(\langle x_0, x_1, \ldots, x_{n-1}\rangle,\ T \setminus \left(T^{x_{-1}} \cup T^{x_n}\right)\right)\,,$$

where $n := n_{\mathsf{Regen}}(\overleftrightarrow{x}, T)$. Let $S_{\mathsf{Regen}} := S^{n_{\mathsf{Regen}}}$ when $(\overleftrightarrow{x}, T) \in \mathsf{Regen}$. Then the random variables $\mathsf{Slab}(S^k_{\mathsf{Regen}}(\overleftrightarrow{x}, T))$ are i.i.d. Since the slabs generates the whole tree and the walk through the tree (except for the location of the root), it is easy to see that the system $(\mathsf{PathsInTrees}, \mathsf{SRW} \times \mathbf{AGW}, S)$ is mixing, hence ergodic.

Theorem 13.17. *The speed (rate of escape) of simple random walk is* $\mathsf{SRW} \times \mathbf{AGW}$*-a.s.*

$$l := \lim_{n\to\infty} \frac{|x_n|}{n} = \mathbf{E}\left[\frac{Z_1 - 1}{Z_1 + 1}\right]. \tag{13.15}$$

This is immediate now.

▷ **Exercise 13.13.**
Show that the same formula (13.15) holds for the speed of simple random walk on $\mathbf{GW}$-a.e. tree.

Now we consider the case when $p_0 > 0$. As usual, let q be the probability of extinction of the Galton-Watson process. Let $\neg\mathsf{Ext}$ be the event of nonextinction of an $\mathbf{AGW}$ tree. Since $\mathbf{AGW}$ is still invariant and $\neg\mathsf{Ext}$ is an invariant subset of trees, $\mathbf{AGW}_{\neg\mathsf{Ext}}$ is SRW-invariant. The $\mathbf{AGW}_{\neg\mathsf{Ext}}$-distribution of the degree of the root is

$$\begin{aligned}\mathbf{AGW}(\deg x_0 = k+1 \mid \neg\mathsf{Ext}) &= \frac{\mathbf{AGW}(\neg\mathsf{Ext} \mid \deg x_0 = k+1)}{\mathbf{AGW}(\neg\mathsf{Ext})} p_k \\ &= p_k \frac{1-q^{k+1}}{1-q^2}\,.\end{aligned} \tag{13.16}$$

▷ **Exercise 13.14.**
Prove this formula.

The proof of Theorem 13.17 on speed is valid when one conditions on nonextinction in the appropriate places. It gives the following formula:

$$\lim_{n\to\infty} \frac{|x_n|}{n} = \mathbf{E}\left[\frac{Z_1-1}{Z_1+1} \;\middle|\; \neg\mathsf{Ext}\right] = \sum_{k\ge 0} \frac{k-1}{k+1} p_k \frac{1-q^{k+1}}{1-q^2}\,.$$

For example, the speed when $p_1 = p_2 = 1/2$ is $1/6$, while for the offspring distribution of the same unconditional mean $p_0 = p_3 = 1/2$, the speed given nonextinction is only $(7-3\sqrt{5})/8 = 0.036^+$.

§13.6. Markov Type of Metric Spaces.

The remaining sections of this chapter are about sublinear rates of escape or escape rates that are linear only for a finite time.

We consider Markov chains in metric spaces and see how quickly they increase their squared distance in expectation. That is, given a metric space (X, d) and a finite-state reversible stationary Markov chain $\langle Y_t \rangle$ whose state space is a subset of X, how quickly can $\mathbf{E}\big[d(Y_t, Y_0)^2\big]$ increase in t? Since the chain is stationary, the proper normalization for the distance is $\mathbf{E}\big[d(Y_1, Y_0)^2\big]$. It turns out that this notion, invented by Ball (1992), is connected to many interesting questions in functional analysis. It is more convenient to relax the notion slightly from Markov chains to functions of Markov chains. Thus, with Ball, we make the following definition, where the "2" refers to the exponent.

Definition 13.18. Given a metric space (X, d), we say that X has ***Markov type*** 2 if there exists a constant $M < \infty$ such that for every positive integer n and every stationary reversible Markov chain $\langle Z_t \rangle_{t=0}^{\infty}$ on $\{1, \ldots, n\}$, every mapping $f : \{1, \ldots, n\} \to X$, and every time $t \in \mathbb{N}$,

$$\mathbf{E}\big[d(f(Z_t), f(Z_0))^2\big] \le Mt\mathbf{E}\big[d(f(Z_1), f(Z_0))^2\big] .$$

We will prove that the real line has Markov type 2 (see Exercise 13.15 for a space that does not have Markov type 2). Since adding the *squared* coordinates gives squared distance in higher dimensions, even in Hilbert space, this shows that Hilbert space also has Markov type 2. This result is due to Ball (1992).

Theorem 13.19. *$\mathbb{R}$ has Markov type 2.*

Proof. As in Section 6.2, let P be the transition operator of the Markov chain. We saw in that section that P is a self-adjoint operator in $L^2(\pi)$. This implies that $L^2(\pi)$ has an orthogonal basis of eigenfunctions of P with real eigenvalues. We also saw (Exercise 6.5) that $\|P\|_\pi \le 1$, whence all eigenvalues lie in $[-1, 1]$.

We prove the statement with constant $M = 1$. Let's re-express that quantity of interest in terms of the operator P. We have

$$\mathbf{E}d(f(Z_t), f(Z_0))^2 = \sum_{i,j} \pi_i p_{ij}^{(t)} [f(i) - f(j)]^2 = 2\big((I - P^t)f, f\big)_\pi .$$

In particular,

$$\mathbf{E}d(f(Z_1), f(Z_0))^2 = 2\big((I - P)f, f\big)_\pi .$$

Thus, we want to prove that

$$\big((I - P^t)f, f\big)_\pi \le t\big((I - P)f, f\big)_\pi$$

for all functions f. (Those comfortable with operator inequalities may note that this can be written as $I - P^t \le t(I - P)$; since $I - P^t = (I - P)\sum_{s=0}^{t-1} P^s$, this follows from $P \le I$.) Note that if f is an eigenfunction with eigenvalue λ, this reduces to the inequality $(1 - \lambda^t) \le t(1 - \lambda)$. Since $|\lambda| \le 1$, this reduces to

$$1 + \lambda + \cdots + \lambda^{t-1} \le t ,$$

which is obviously true.

The claim follows for all other f by taking $f = \sum_{j=1}^n a_j f_j$, where $\{f_j\}$ is an orthonormal basis of eigenfunctions:

$$\big((I-P^t)f, f\big)_\pi = \sum_{j=1}^n a_j^2 \big((I-P^t)f_j, f_j\big)_\pi \le \sum_{j=1}^n a_j^2 t\big((I-P)f_j, f_j\big)_\pi = t\big((I-P)f, f\big)_\pi . \blacktriangleleft$$

▷ **Exercise 13.15.**

A collection of metric spaces has ***uniform Markov type*** 2 if there exists a constant $M < \infty$ such that each space in the collection has Markov type 2 with constant M. Prove that the k-dimensional hypercube graphs $\{0,1\}^k$ do not have uniform Markov type 2. From this, construct a space that does not have Markov type 2.

§13.7. Embeddings of Finite Metric Spaces.

If we map one metric space into another, distances can change in various ways. For example, a homothety merely multiplies all distances by the same constant. Thus, a homothety does not change the "shape" of the domain space. We can measure changes in shape, or distortion, by how much some distances change compared to other distances. This motivates the following definition.

Definition 13.20. An invertible mapping $f : X \to Y$, where (X, d_X) and (Y, d_Y) are metric spaces, has ***distortion at most*** C if there exists a number $r > 0$ such that for all $x, y \in X$,

$$r d_X(x,y) \le d_Y\big(f(x), f(y)\big) \le C r d_X(x,y) . \tag{13.17}$$

The infimum of such numbers C is called the ***distortion of*** f.

For example, consider embedding a hypercube graph in Hilbert space. The distance in the graph is like an ℓ^1-metric, while in Hilbert space, it is an ℓ^2-metric. These are generally incomparable, suggesting that there must be a fair amount of distortion. This is established by the following result of Enflo (1969).

Proposition 13.21. (Distortion of Hypercubes) *There exists $c > 0$ such that for all k, every embedding of the hypercube $\{0,1\}^k$ in Hilbert space has distortion at least $c\sqrt{k}$.*

Proof. In the proof of Exercise 13.15, we showed that if $\{X_j\}$ is a simple random walk in the hypercube, then

$$\mathbf{E} d(X_0, X_j) \ge \frac{j}{2} \qquad \forall j \le k/4 .$$

Take $j := \lfloor k/4 \rfloor$. By the arithmetic mean-quadratic mean inequality, $\mathbf{E}d^2(X_0, X_j) \geq j^2/4$. Now let $f : \{0,1\}^k \to \ell^2(\mathbb{N})$ be a map. Assume that (13.17) holds; we may take $r = 1$ there. We saw that Hilbert space has Markov type 2 with constant $M = 1$. Therefore, if we take X_0 to be uniform, we obtain

$$\mathbf{E}d^2\big(f(X_0), f(X_j)\big) \leq j\mathbf{E}d^2\big(f(X_0), f(X_1)\big) \leq C^2 j\mathbf{E}d^2(X_0, X_1) = C^2 j\,.$$

We conclude

$$C^2 j \geq \mathbf{E}d^2\big(f(X_0), f(X_j)\big) \geq \mathbf{E}d^2(X_0, X_j) \geq j^2/4\,,$$

whence $C \geq \sqrt{j}/2$, which implies the result. ◀

Remark 13.22. Enflo's original proof gives $c = 1$. Furthermore, this is best possible. See Exercise 13.31 for the proof of this fact. However, his proof is algebraic and cannot handle perturbations of the hypercube, whereas our proof via Markov type can.

What if we know nothing about the finite metric space of the domain other than its cardinality: how little can we distort it by embedding in Hilbert space? Of course, if the space has n points, then its image spans an n-dimensional subspace of Hilbert space, so we may always embed it in $\mathbb{R}^n$ just as well as in Hilbert space. In fact, the dimension of the co-domain can always be made much smaller without increasing the distortion much, as we show next. This dimension reduction proposition is due to Johnson and Lindenstrauss (1984). The proposition is now widely used in computer science. It shows that once the original space of n points is embedded in $\mathbb{R}^n$, the reduction in dimension can be effectuated by a linear map.

Proposition 13.23. (Dimension Reduction) *For any $0 < \epsilon < 1/2$ and $v_1, \ldots, v_n \in \mathbb{R}^l$ with the Euclidean metric, there exists a linear map $A : \mathbb{R}^l \to \mathbb{R}^k$, where $k := \lceil 24 \log n/\epsilon^2 \rceil$, that has distortion at most $(1+\epsilon)/(1-\epsilon)$ when restricted to the n-point space $\{v_1, \ldots, v_n\}$.*

Proof. Let $A := \frac{1}{\sqrt{k}}\big(X_{i,j}\big)_{1 \leq i \leq k, 1 \leq j \leq l}$ be a $k \times l$ matrix where the entries $X_{i,j}$ are independent standard normal $N(0,1)$ random variables. We prove that with probability at least $1/n$, this map has distortion at most $1+\epsilon$. Since A is a linear map, the distortion of A on the pair v_p, v_q for some $v_p \neq v_q$ is measured by $\|Au\|$, where $u := (v_p - v_q)/\|v_p - v_q\|$ has norm 1. Fix $p \neq q$ and denote the coordinates of the associated vector u by $u = (u_1, \ldots, u_l)$. Clearly,

$$Au = \frac{1}{\sqrt{k}}\left(\sum_{j=1}^{l} u_j X_{1,j}, \cdots, \sum_{j=1}^{l} u_j X_{k,j}\right)\,,$$

so

$$\|Au\|^2 = \frac{1}{k}\sum_{i=1}^{k}\left(\sum_{j=1}^{l} u_j X_{i,j}\right)^2 .$$

Note that for any i the sum $\sum_{j=1}^{l} u_j X_{i,j}$ is distributed as a standard normal random variable. So $\|Au\|^2$ is distributed as $\frac{1}{k}\sum_{i=1}^{k} Y_i^2$, where $Y_1, \ldots, Y_k$ are independent standard normal random variables. We wish to show that Au is quite concentrated around its mean. To achieve that, we compute the moment generating function of Y^2, where $Y \sim N(0,1)$. For any real $\lambda < 1/2$, we have

$$\mathbf{E}e^{\lambda Y^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\lambda y^2} e^{-y^2/2} dy = \frac{1}{\sqrt{1-2\lambda}} ,$$

and using Taylor expansion, we get

$$\begin{aligned}\varphi(\lambda) &:= \left|\log \mathbf{E}e^{\lambda(Y^2-1)}\right| = \left| -\frac{1}{2}\log(1-2\lambda) - \lambda\right| \\ &= \sum_{k=2}^{\infty}\frac{2^{k-1}\lambda^k}{k} \le 2\lambda^2\big(1 + 2\lambda + (2\lambda)^2 + \cdots\big) = \frac{2\lambda^2}{1-2\lambda} .\end{aligned}$$

Now,

$$\mathbf{P}\big[\|Au\|^2 > 1+\epsilon\big] = \mathbf{P}\left[e^{\lambda\sum_{i=1}^{k}(Y_j^2-1)} > e^{\lambda\epsilon k}\right] \le e^{-\lambda\epsilon k}e^{k\varphi(\lambda)} \le \exp\left(-\lambda\epsilon k + \frac{2\lambda^2 k}{1-2\lambda}\right) .$$

Choose $\lambda := \epsilon/4$; this gives

$$-\lambda\epsilon k + \frac{2\lambda^2 k}{1-2\lambda} = -\frac{\epsilon^2 k}{4}\,\frac{1/2-\epsilon/2}{1-\epsilon/2} < -\frac{\epsilon^2 k}{12}$$

since $\epsilon < 1/2$. With our definition of $k := \lceil 24\log n/\epsilon^2\rceil$, this yields

$$\mathbf{P}\big[\|Au\|^2 > 1+\epsilon\big] \le \exp(-\epsilon^2 k/12) \le n^{-2} .$$

One can prove similarly that

$$\mathbf{P}\big[\|Au\|^2 < 1-\epsilon\big] \le n^{-2} .$$

Since we have $\binom{n}{2}$ pairs of vectors v_p, v_q, it follows that with probability at least $1/n$, for all $p \ne q$,

$$(1-\epsilon)\|v_p - v_q\| \le \|Av_p - Av_q\| \le (1+\epsilon)\|v_p - v_q\| ,$$

which implies that the distortion of A is no more than $(1+\epsilon)/(1-\epsilon)$. ◀

We now prove a theorem of Bourgain (1985) that any metric on n points can be embedded in Hilbert space with distortion $O(\log n)$. In Exercise 13.32, this is shown to be sharp up to constants.

Theorem 13.24. *Every n-point metric space (X, d) can be embedded in Hilbert space with distortion at most $24 \log n$.*

Proof. We follow Linial, London, and Rabinovich (1995), but the idea is the same as the original proof of Bourgain. We may obviously assume that $n \geq 4$. Let $\alpha > 40$. For each integer $k < n$ that is a power of 2, randomly pick $l := \lceil \alpha \log n \rceil$ sets $A \subset X$ independently, by including independently each $x \in X$ with probability $1/k$, i.e., each such set is a Bernoulli$(1/k)$ percolation on X. Write $m := \lfloor \log_2 n \rfloor$. Then altogether, we obtain lm random sets $A_1, \ldots, A_{lm}$. Define the mapping $f : X \to \mathbb{R}^{lm}$ by

$$f(x) := \big(d(x, A_1), d(x, A_2), \ldots, d(x, A_m)\big) . \tag{13.18}$$

We will show that f has distortion at most $24 \log n$ with probability at least $1 - n^{2-\alpha/20} \log n/2$.

By the triangle inequality, for any $x, y \in X$ and any $A_i \subset X$ we have $|d(x, A_i) - d(y, A_i)| \leq d(x, y)$, so

$$\|f(x) - f(y)\|_2^2 = \sum_{i=1}^{lm} |d(x, A_i) - d(y, A_i)|^2 \leq lmd(x, y)^2 .$$

For the lower bound, let $B(x, \rho) := \{z \in X \,;\; d(x, z) \leq \rho\}$ and $B^\circ(x, \rho) := \{z \in X \,;\; d(x, z) < \rho\}$ denote the closed and open balls of radius ρ centered at x. Consider two points $x \neq y \in X$. Let $\rho_0 := 0$. If there is some $\rho \leq d(x, y)/4$ such that both $|B(x, \rho)| \geq 2^t$ and $|B(y, \rho)| \geq 2^t$, then define ρ_t to be the least such ρ. Let $\hat{t} \in [0, m]$ be the largest such index. Also let $\rho_{\hat{t}+1} := d(x, y)/4$. Observe that $B(x, \rho_i)$ and $B(y, \rho_j)$ are always disjoint for $i, j \leq \hat{t} + 1$.

Let $1 \leq t \leq \hat{t} + 1$. By definition, either $|B^\circ(x, \rho_t)| < 2^t$ or $|B^\circ(y, \rho_t)| < 2^t$. Let us assume the former, without loss of generality. Notice that $A \cap B^\circ(x, \rho_t) = \varnothing \iff d(x, A) \geq \rho_t$, and $A \cap B(y, \rho_{t-1}) \neq \varnothing \iff d(y, A) \leq \rho_{t-1}$. Therefore, if both conditions hold, then $|d(x, A) - d(y, A)| \geq \rho_t - \rho_{t-1}$. Now, we also have $|B(y, \rho_{t-1})| \geq 2^{t-1}$. Note that these inequalities hold even if $\hat{t} = 0$ or $t = \hat{t} + 1$, unless $\rho_{\hat{t}} = d(x, y)/4$; but in this last case, the inequalities below that we derive will be trivially true. Suppose A is a Bernoulli(2^{-t}) percolation on X. Since $2 \leq k \leq n$, we have

$$\mathbf{P}[A \text{ misses } B^\circ(x, \rho_t)] \geq (1 - 2^{-t})^{2^t} \geq \frac{1}{4}$$

and

$$\mathbf{P}[A \text{ hits } B(y, \rho_{t-1})] \ge 1 - (1 - 2^{-t})^{2^{t-1}} \ge 1 - e^{-1/2} \ge \frac{1}{3}\,.$$

Since these balls are disjoint, these events are independent, whence such an A has probability at least $1/12$ both to intersect $B(y, \rho_{t-1})$ and to miss $B^\circ(x, \rho_t)$. Since for each k we choose l such sets, by (6.22), the probability that fewer than $l/16$ of them have the previous property is less than

$$e^{-lI_{1/12}(1/24)} < e^{-l/20} \le n^{-\alpha/20}$$

in the notation of (6.23). So with probability at least $1 - n^{2-\alpha/20} \log n/2$, for all $x, y \in X$ and k, we have at least $l/16$ sets which satisfy the condition. In this case,

$$\|f(x) - f(y)\|_2^2 \ge \sum_{t=1}^{\hat{t}+1} \frac{l}{16}(\rho_t - \rho_{t-1})^2\,.$$

Since $\sum_{t=1}^{\hat{t}+1}(\rho_t - \rho_{t-1}) = \rho_{\hat{t}+1} = d(x, y)/4$ and $2 \cdot 2^{\hat{t}} \le n$, we then have

$$\|f(x) - f(y)\|_2^2 \ge \frac{l}{16}\left(\frac{d(x, y)}{4(\hat{t}+1)}\right)^2 (\hat{t}+1) = \frac{ld(x, y)^2}{256(\hat{t}+1)} \ge \frac{ld(x, y)^2}{256m}\,.$$

In this case, the distortion of f is at most $16m < 24 \log n$, which proves our claim. ◀

§13.8. Notes.

The first statement of (13.4) was in Lyons, Pemantle, and Peres (1996b). The extension to the much harder case of directed covers was accomplished by Takacs (1997), while the extension to the biased random walks RW_λ was done by Takacs (1998).

Proposition 13.7 is due to Peres (1997), unpublished. The inequalities of Propositions 13.3 and 13.7 were sharpened in remarkable work of Virág (2000b, 2002). In the first paper, he proved that if RW_λ is considered instead of simple random walk, then we have the bound

$$\frac{\operatorname{br} G - \lambda}{\operatorname{br} G + \lambda} \ge s$$

for all graphs. In fact, he defined the branching number for networks and proved that

$$\frac{\operatorname{br} G - 1}{\operatorname{br} G + 1} \ge s$$

holds for all networks. In the second paper, he proved an analogous inequality relating limsup speed and upper growth for general networks.

In Section 13.5, we gave a short proof that simple random walk on infinite Galton-Watson trees is a.s. transient. This result was first proved by Grimmett and Kesten (1983), Lemma 2. See also Exercises 16.10 and 16.23. For another proof that is direct and short, see Collevecchio (2006).

Since random walk on a random spherically symmetric tree is essentially the same as a special case of random walk in a random environment (RWRE) on the nonnegative integers, we may compare the slowing of speed on Galton-Watson trees to the fact that randomness also slows down random walk for the general RWRE on the integers (Solomon, 1975).

Homesick random walks RW_λ on Galton-Watson trees have been studied as well, but are more difficult than simple random walks because no explicit stationary measure is known. Nevertheless, Lyons, Pemantle, and Peres (1996a) showed that the speed is a positive constant a.s. when $1 < \lambda < m$. See also Exercise 13.33 for the case $\lambda < 1$. The critical case $\lambda = m$ has been studied by Peres and Zeitouni (2008), who found a stationary measure.

Other works about random walks on Galton-Watson trees include Kesten (1986), Aldous (1991), Piau (1996), Chen (1997), Piau (1998), Pemantle and Stacey (2001), Dembo, Gantert, Peres, and Zeitouni (2002), Piau (2002), Dembo, Gantert, and Zeitouni (2004), Dai (2005), Collevecchio (2006), Chen and Zhang (2007), Aïdékon (2008b), Croydon and Kumagai (2008), Croydon (2008), Ben Arous, Fribergh, Gantert, and Hammond (2007), Aïdékon (2008a), and Faraud (2008). However, the following questions from Lyons, Pemantle, and Peres (1996a, 1997) are open:

Question 13.25. Is the speed of RW_λ on Galton-Watson trees monotonic nonincreasing in the parameter λ when $p_0 = 0$?

Question 13.26. Is the speed of RW_λ a real-analytic function of $\lambda \in (0, m)$ for Galton-Watson trees T?

Our proof of Proposition 13.23 is a small variant of the original proof; it was known to many people shortly after the original paper appeared. A version of it was published by Indyk and Motwani (1999). For more variants and history, see Matoušek (2008). The dependence of the smallest dimension on ϵ is not known, even up to constants. Alon (2003) shows that embedding a simplex on n points with distortion at most $1 + \epsilon$ requires a space of dimension at least $c \log n / (\epsilon^2 \log(1/\epsilon))$ for some constant c.

From an algorithmic perspective, it is important to achieve Proposition 13.23 using i.i.d., ± 1 with probability $1/2$, random variables as our $X_{i,j}$. That this is possible was first noted by Achlioptas and McSherry (2001, 2007). Actually, it is possible for any random variable X for which there exists a constant $C > 0$ such that $\mathbf{E} e^{\lambda X} \le e^{C\lambda^2}$ (for $X = \pm 1$ with probability $1/2$ each, we have $\mathbf{E} e^{\lambda X} = \cosh(\lambda) \le e^{\lambda^2/2}$) by the following argument due to Assaf Naor (personal communication, 2004):

Let X_i be i.i.d. with the same distribution as X and set $Y := \sum_{i=1}^k u_i X_i$ with $\sum_{j=1}^k u_j^2 = 1$. Take Z to be a standard normal random variable independent of $\{X_i\}$. Recall that for all real α we have $\mathbf{E} e^{\alpha Z} = e^{\alpha^2/2}$. Since Y and Z are independent, using Fubini's Theorem we get that for any $\lambda < C/4$,

$$\begin{aligned} \mathbf{E} e^{\lambda Y^2} &= \mathbf{E} e^{(\sqrt{2\lambda} Y)^2/2} = \mathbf{E} e^{\sqrt{2\lambda} Y Z} = \mathbf{E} e^{\sum_{i=1}^k \sqrt{2\lambda} u_i X_i Z} = \mathbf{E}\, \mathbf{E}\left[e^{\sum_{i=1}^k \sqrt{2\lambda} u_i X_i Z} \;\middle|\; Z \right] \\ &\le \mathbf{E} e^{C \sum_{i=1}^k 2\lambda u_i^2 Z^2} = \mathbf{E} e^{2C\lambda Z^2} = \frac{1}{\sqrt{1 - 4C\lambda}}\,, \end{aligned}$$

and the rest of the argument is the same as Proposition 13.23.

Embeddings of the sort (13.18) are known as ***Fréchet embeddings***.

§13.9. Collected In-Text Exercises.

13.1. Let T be a tree without leaves such that for every vertex u, there is some vertex $x \geq u$ with at least two children and with $|x| < |u| + N$. Show that if $\langle X_n \rangle$ is simple random walk on T, then

$$\liminf_{n \to \infty} \frac{|X_n|}{n} \geq \frac{1}{3N} \quad \text{a.s.}$$

13.2. It does not suffice that $\operatorname{br} T > 1$ for the speed of simple random walk on T to be positive. Show this for the tree T, which is a binary tree to every vertex of which is joined a unary tree; see Figure 13.1.

13.3. Let T be a bounded self-adjoint operator on a Hilbert space and Q be a polynomial with real coefficients. Show that

$$\|Q(T)\| \leq \max_{|s| \leq \|T\|} |Q(s)| \,.$$

13.4. Show that for every $k \in \mathbb{Z}$, there are unique polynomials Q_k of degree $|k|$ such that $Q_k(\cos\theta) = \cos k\theta$. Show that $|Q_k(s)| \leq 1$ whenever $|s| \leq 1$. These polynomials are called the ***Chebyshev polynomials***.

13.5. Show that $\operatorname{br} G$ does not depend on the choice of vertex o.

13.6. Show that if G' is a subgraph of G, then $\operatorname{br} G' \leq \operatorname{br} G$.

13.7. Show that $\lambda_c(G) \leq \operatorname{br} G$.

13.8. Show that the hypothesis of bounded degree in Proposition 13.7 is necessary.

13.9. Let $G = (\mathsf{V}, \mathsf{E})$ be a finite connected graph with E (undirected) edges. For $x \in \mathsf{V}$, let T_x be the universal cover of G based at x (see Section 3.3). Define

$$\mu([T, o]) := \frac{1}{2E} \sum \big\{ \deg x \,;\; x \in \mathsf{V},\ [T_x, x] = [T, o] \big\} \,.$$

Show that μ is a stationary ergodic probability measure on $[\mathscr{T}]$.

13.10. Show that if μ is an ergodic DSRW-stationary probability measure on the space of rooted trees $\mathscr{T}$ such that μ-a.e. tree has at least 3 boundary points and maximum degree at most D, then the rate of escape of simple random walk (not delayed) is $\mathsf{SRW} \times \mu$-a.s.

$$\lim_{n \to \infty} \frac{|X_n|}{n} = 1 - \frac{2}{\int \deg_T(o)\, d\mu(T, o)} \,.$$

13.11. Use Corollary 13.12 to prove that for a translation-invariant random forest on a Cayley graph, there are a.s. no isolated boundary points in trees with an infinite boundary.

13.12. Show that the Markov chain $\mathsf{SRW} \times \mathbf{AGW}$ is stationary.

13.13. Show that the same formula (13.15) holds for the speed of simple random walk on $\mathbf{GW}$-a.e. tree.

13.14. Prove (13.16).

13.15. A collection of metric spaces has ***uniform Markov type*** 2 if there exists a constant $M < \infty$ such that each space in the collection has Markov type 2 with constant M. Prove that the k-dimensional hypercube graphs $\{0, 1\}^k$ do not have uniform Markov type 2. From this, construct a space that does not have Markov type 2.

§13.10. Additional Exercises.

13.16. Show that any submartingale $\langle Y_n \rangle$ with bounded increments satisfies $\liminf_{n\to\infty} Y_n/n \ge 0$.

13.17. Extend Theorem 13.1 as follows.
(a) Show that if $a_n \ge 0$ satisfy $\sum_n a_n/n < \infty$, then there exists an increasing sequence of integers n_k such that $\sum_k a_{n_k}^2 < \infty$ and $\sum_k (n_{k+1}/n_k - 1)^2 < \infty$.
(b) Show that if X_n are random variables with $\mathbf{E}[X_n^2] \le 1$ and

$$\sum_n \frac{1}{n} \mathbf{E}\left[\left|\frac{1}{n}\sum_{k=1}^n X_k\right|^2\right]^{1/2} < \infty\,,$$

then $\sum_{k=1}^n X_k/n \to 0$ a.s.

13.18. Extend Theorem 13.1 as follows.
(a) Show that if $a_n \ge 0$ satisfy $\sum_n a_n/n < \infty$, then there exists an increasing sequence of integers n_k such that $\sum_k a_{n_k} < \infty$ and $\lim_{k\to\infty} n_{k+1}/n_k = 1$.
(b) Show that if X_n are random variables with $|X_n| \le 1$ a.s. and

$$\sum_n \frac{1}{n} \mathbf{E}\left[\left|\frac{1}{n}\sum_{k=1}^n X_k\right|^2\right] < \infty\,,$$

then $\sum_{k=1}^n X_k/n \to 0$ a.s.

13.19. Let G be a directed graph such that at every vertex x, the number of edges going out from x equals the number of edges coming into x, which we denote by $d(x)$. Let $b(G)$ denote the maximum growth rate of the directed covers of G (the maximum is taken over possible starting vertices of paths). Show that $b(G) \ge \sum_{x\in\mathsf{V}} d(x)/|\mathsf{V}|$.

13.20. Suppose that G is a finite directed graph such that there is a (directed) path from each vertex to each other vertex. Show that there is a stationary Markov chain on the vertices of G such that the transition probability from x to y is positive only if $(x, y) \in \mathsf{E}(G)$ and such that the entropy of the Markov chain equals the growth rate of the directed cover of G.

13.21. One might expect that the speed of RW_λ would be monotonic decreasing in λ. But this is not the case, even in simple examples: Let T be a binary tree to every vertex of which is joined a unary tree as in Figure 13.1. Show that for $1 \le \lambda \le 2$, the speed is

$$\frac{(2-\lambda)(\lambda-1)}{\lambda^2+3\lambda-2}\,,$$

which is maximized at $\lambda = 4/3$.

13.22. Find a finite connected graph who universal cover has the property that the speed of RW_λ is not monotonic.

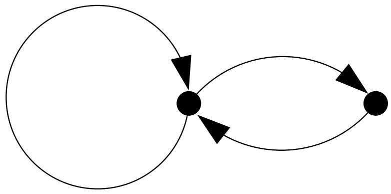


Figure 13.3. A directed graph whose cover is the Fibonacci tree.

13.23. Consider the directed graph shown in Figure 13.3. Let T be one of its directed covers. Why is T called the ***Fibonacci tree***? Show that the speed of RW_λ on T is

$$\frac{(\sqrt{\lambda+1}+2)(\sqrt{\lambda+1}-\lambda)}{\sqrt{\lambda+1}(2+\lambda+\sqrt{\lambda+1})}$$

for $0 \le \lambda < (\sqrt{5}+1)/2$.

13.24. Suppose that (G, c) is a network with π bounded above. Suppose that for all large r, the balls $B(o, r)$ have cardinality at most Ar^d for some finite constants A, d. Show that the network random walk $\langle X_n \rangle$ obeys

$$\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{n \log n}} \le \sqrt{d} \qquad \text{a.s.}$$

13.25. Suppose that (G, c) is a network with π bounded above. Suppose that for all large r, the balls $B(o, r)$ have cardinality at most $Ar^d e^{Br^\alpha}$ for some finite constants A, B, d, α, with $0 < \alpha \le 1$. Show that the network random walk $\langle X_n \rangle$ obeys

$$\limsup_{n\to\infty} \frac{|X_n|}{n^{1/(2-\alpha)}} \le (2B)^{1/(2-\alpha)} \qquad \text{a.s.}$$

13.26. Since simple random walk on a graph is a reversible Markov chain, the Markov chain on $[\mathscr{T}]$ induced by simple random walk is ***locally reversible*** in being reversible on each communicating class of states. However, a stationary measure μ is not necessarily ***globally reversible***, i.e., it is not necessarily the case that for Borel sets $A, B \subseteq [\mathscr{T}]$,

$$\int_A p((T, o), B)\, d\mu(T, o) = \int_B p((T, o), A)\, d\mu(T, o)\,.$$

Give such an example. On the other hand, prove that every globally reversible μ is stationary.

13.27. Show that if μ is a probability measure on rooted trees that is stationary for simple random walk, then $\int_{\mathscr{A}_\diamond} \deg_T(o)/\deg_{T^\diamond}(o)\, d\mu(T, o) < \infty$.

13.28. For a rooted tree (T, o), define T_Δ to be the tree obtained by adding an edge from o to a new vertex Δ and rooting at Δ. This new vertex Δ is thought of as representing the past. Let $\gamma(T)$ be the probability that simple random walk started at Δ never returns to Δ:

$$\gamma(T) := \mathsf{SRW}_{T_\Delta}(\forall n > 0 \quad x_n \ne \Delta)\,.$$

Let $\mathscr{C}(T)$ denote the effective conductance of T from its root to infinity when each edge has unit conductance. The series law gives us that

$$\gamma(T) = \frac{\mathscr{C}(T)}{1 + \mathscr{C}(T)} = \mathscr{C}(T_\Delta)\,.$$

The notation γ is intended to remind us of the word "conductance". Compare the remark following Corollary 5.22. For $(\overleftrightarrow{x}, T) \in \mathsf{PathsInTrees}$, write $N(\overleftrightarrow{x}, T) := |\{n;\ x_n = x_0\}|$ for the number of visits to the root of T. For $k \in \mathbb{N}$, let $D_k(\overleftrightarrow{x}, T) := \{j \in \mathbb{N};\ \deg x_j = k+1\}$.

(a) Show that

$$\lim_{n\to\infty} \frac{1}{2n+1} |D_k(\overleftrightarrow{x}, T) \cap [-n, n]| = p_k \quad \mathsf{SRW} \times \mathbf{AGW}\text{-a.s.}$$

This means that the proportion of time that simple random walk spends at vertices of degree $k+1$ is p_k.

(b) For $i \ge 0$, let γ_i be i.i.d. random variables with the distribution of the $\mathbf{GW}$-law of $\gamma(T)$. Let

$$\Gamma_k := \int (k+1)/(\gamma_0 + \cdots + \gamma_k)\, d\mathbf{GW}\,.$$

Show that

$$\int N(\overleftrightarrow{x}, T)\, d\mathsf{SRW} \times \mathbf{AGW}((\overleftrightarrow{x}, T) \mid \deg x_0 = k+1) = 2\Gamma_k - 1$$

and that Γ_k is decreasing in k. What is $\lim_{k\to\infty} \Gamma_k$?

(c) The result in (a) says that there is no biasing of visits to a vertex according to its degree, just as Theorem 13.15 says. Yet the result in (b) indicates that there is indeed a biasing. How can these results be compatible?

(d) What is

$$\int N(\overleftrightarrow{x}, T)\, d\mathsf{SRW} \times \mathbf{AGW}((\overleftrightarrow{x}, T) \mid \deg x_0 = k+1,\ \mathsf{Fresh})\,?$$

13.29. Show that a labelled tree chosen according to $\mathbf{GW}$ a.s. has no graph-automorphisms except for the identity map.

13.30. Show that a metric space (X, d) has Markov type 2 iff there exists a constant $M < \infty$ such that for every reversible Markov chain $\langle Z_t \rangle_{t=0}^\infty$ with a stationary probability measure on a countable state space W and every mapping $f : W \to X$, and every time $t \in \mathbb{N}$,

$$\mathbf{E}\big[d(f(Z_t), f(Z_0))^2\big] \le Mt\mathbf{E}\big[d(f(Z_1), f(Z_0))^2\big]\,.$$

13.31. Let $\Omega_d = \{-1, 1\}^d$ be the d-dimensional hypercube with ℓ_1 metric. Show that every $f : \Omega_d \to \ell^2(\mathbb{N})$ has distortion at least $\sqrt{d}$ and give an example for an f with $\sqrt{d}$ distortion. *Hint:* Show that

$$\sum_{x\in\Omega_d} \|f(x) - f(-x)\|^2 \le \sum_{x\sim y} \|f(x) - f(y)\|^2\,.$$

13.32. Use the proof that $\mathbb{R}$ has Markov type 2 to show that for an (n, d, λ)-expander family $\langle G_n \rangle$, any invertible mapping $f_n : \mathsf{V}_n \to \ell^2(\mathbb{N})$ of the vertices to a Hilbert space has distortion at least $C_{d,\lambda} \log n$, where $C_{d,\lambda}$ is constant.

13.33. Suppose that $p_0 > 0$. Let T be a Galton-Watson tree conditioned on nonextinction. Show that the speed of RW_λ is zero if $0 \le \lambda \le f'(q)$.

Chapter 14

Hausdorff Dimension

§14.1. Basics.

How do we capture the intrinsic dimension of a geometric object (such as a curve, a surface, or a body)? Assume that the object is bounded. One approach is via scaling: in its dimension, its measure would change by a factor r^d, where r is the linear scale change and d is the dimension. But this requires knowing the dimension d and requires moving outside the object itself. Another approach is to cover the object by small sets, leaving the object itself unchanged. If E is a bounded set in Euclidean space, let $N(E, \epsilon)$ be the number of balls of diameter at most ϵ required to cover E. Then when E is d-dimensional, $N(E, \epsilon) \approx C/\epsilon^d$. To get at d, then, look at $\log N(E, \epsilon)/\log(1/\epsilon)$ and take some kind of limit as $\epsilon \to 0$. This gives the ***upper*** and ***lower Minkowski dimensions***:

$$\dim^{\mathrm{M}} E := \limsup_{\epsilon \to 0} \frac{\log N(E, \epsilon)}{\log(1/\epsilon)},$$

$$\dim_{\mathrm{M}} E := \liminf_{\epsilon \to 0} \frac{\log N(E, \epsilon)}{\log(1/\epsilon)}.$$

The lower Minkowski dimension is often called simply the Minkowski dimension. Note that we could use cubes instead of balls, or even b-adic cubes

$$[a_1 b^{-n}, (a_1+1)b^{-n}] \times [a_2 b^{-n}, (a_2+1)b^{-n}] \times \cdots \times [a_d b^{-n}, (a_d+1)b^{-n}] \qquad (a_i \in \mathbb{Z},\ n \in \mathbb{N}),$$

and not change these dimensions. For future reference, we call n the ***order*** of a b-adic cube if the sides of the cube have length b^{-n}.

These definitions make sense in any metric space E. Thus they are intrinsic. It is clear that the Minkowski dimension of a singleton is 0.

Example 14.1. Let E be the standard middle-thirds Cantor set,

$$E := \left\{ \sum_{n \geq 1} x_n 3^{-n}\ ;\ x_n \in \{0, 2\} \right\}.$$

To calculate the Minkowski dimensions of E, it is convenient to use triadic intervals, of course. We have that $N(E, 3^{-n}) = 2^n$, so for $\epsilon = 3^{-n}$, we have

$$\frac{\log N(E,\epsilon)}{\log(1/\epsilon)} = \frac{\log 2}{\log 3}.$$

Thus, the upper and lower Minkowski dimensions are both $\log 2/\log 3$.

Example 14.2. Let $E := \{1/n;\ n \geq 1\}$. Given $\epsilon \in (0,1)$, let k be such that $1/k^2 \approx \epsilon$. Then it takes about $1/\sqrt{\epsilon}$ intervals of length ϵ to cover $E \cap [0, 1/k]$ and about $1/\sqrt{\epsilon}$ more to cover the k points in $E \cap [1/k, 1]$. It can be shown that, indeed, $N(E,\epsilon) \approx 2/\sqrt{\epsilon}$, so that $\dim_{\mathrm{M}} E = 1/2$.

This example shows that a countable union of sets of Minkowski dimension 0 may have positive Minkowski dimension, even when the union consists entirely of isolated points. For this reason, Minkowski dimensions are not entirely suitable.

So we consider yet another approach. In what dimension should we measure the *size* of E? Note that a surface has infinite 1-dimensional measure but zero 3-dimensional measure. How do we measure size in an arbitrary dimension? Define ***Hausdorff*** α***-dimensional (outer) measure*** by

$$\mathcal{H}_\alpha(E) := \lim_{\epsilon \to 0} \inf \Big\{ \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^\alpha\,;\ E \subseteq \bigcup_{i=1}^{\infty} E_i,\ \forall i \quad \operatorname{diam} E_i < \epsilon \Big\}.$$

(The limit as $\epsilon \to 0$ here exists because the infimum is taken over smaller sets as ϵ decreases.) Note that, up to a bounded factor, the restriction of $\mathcal{H}_d$ to Borel sets in $\mathbb{R}^d$ is d-dimensional Lebesgue measure, $\mathcal{L}_d$.

▷ **Exercise 14.1.**

Show that for any $E \subseteq \mathbb{R}^d$, there exists a real number α_0 such that $\alpha < \alpha_0 \Rightarrow \mathcal{H}_\alpha(E) = +\infty$ and $\alpha > \alpha_0 \Rightarrow \mathcal{H}_\alpha(E) = 0$.

The number α_0 of the preceding exercise is called the ***Hausdorff dimension*** of E, denoted $\dim_{\mathrm{H}} E$ or simply $\dim E$. We can also write

$$\dim E = \inf \left\{ \alpha\,;\ \inf \Big\{ \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^\alpha\,;\ E \subseteq \bigcup_i E_i \Big\} = 0 \right\}.$$

Again, these definitions make sense in any metric space (in some circumstances, you may need to recall that the infimum of the empty set is $+\infty$). In $\mathbb{R}^d$, we could restrict ourselves to covers by open sets, spheres or b-adic cubes; this would change $\mathcal{H}_\alpha$ by at most a bounded factor, and so would leave the dimension unchanged. Also, $\dim E \le \dim_{\mathrm{M}} E$ for any E: if $E \subseteq \bigcup_{i=1}^{N(E,\epsilon)} E_i$ with $\operatorname{diam} E_i \le \epsilon$, then

$$\sum_{i=1}^{N(E,\epsilon)} (\operatorname{diam} E_i)^\alpha \le \sum_{i=1}^{N(E,\epsilon)} \epsilon^\alpha = N(E,\epsilon)\epsilon^\alpha \,.$$

Thus, if $\alpha > \dim_{\mathrm{M}} E$, we get $\mathcal{H}_\alpha(E) = 0$.

Example 14.3. Let E be the standard middle-thirds Cantor set again. For an upper bound on its Hausdorff dimension, we use the Minkowski dimension, $\log 2/\log 3$. For a lower bound, let μ be the Cantor-Lebesgue measure, i.e., the law of $\sum_{n\ge1} X_n 3^{-n}$ when X_n are i.i.d. with $\mathbf{P}[X_n = 0] = \mathbf{P}[X_n = 2] = 1/2$. Let E_i be triadic intervals whose union covers E. Now every triadic interval either is disjoint from E or is one of those entering in the construction of E. When E_i is one of the latter and has diameter 3^{-n}, we have

$$(\operatorname{diam} E_i)^{\log 2/\log 3} = 2^{-n} = \mu(E_i)\,,$$

whence we get that

$$\sum_i (\operatorname{diam} E_i)^{\log 2/\log 3} \ge \sum_i \mu(E_i) \ge \mu(E) = 1\,.$$

(Except for wasteful covers, these inequalities are equalities.) This shows that $\dim E \ge \log 2/\log 3$. Therefore, the Hausdorff dimension is in fact equal to the Minkowski dimension in this case.

Example 14.4. If $E := \{1/n\,;\ n \ge 1\}$, then $\dim E = 0$: For any $\alpha, \epsilon > 0$, we may cover E by the sets $E_i := [1/i, 1/i + (\epsilon/2^i)^{1/\alpha}]$, showing that $\mathcal{H}_\alpha(E) < \epsilon$.

Example 14.5. Let E be the Cantor middle-thirds set. The set $E \times E$ is called the planar Cantor set. We have $\dim E \times E = \dim^{\mathrm{M}} E \times E = \log 4/\log 3$. To prove this, note that it requires 4^n triadic squares of order n to cover $E \times E$, whence $\dim^{\mathrm{M}} E \times E = \log 4/\log 3$. On the other hand, if μ is the Cantor-Lebesgue measure, then $\mu \times \mu$ is supported by $E \times E$. If E_i are triadic squares covering $E \times E$, then as in Example 14.3,

$$\sum_i (\operatorname{diam} E_i)^{\log 4/\log 3} \ge \sum_i (\mu \times \mu)(E_i) \ge (\mu \times \mu)(E \times E) = 1\,.$$

Hence $\dim E \times E \ge \log 4/\log 3$.

▷ **Exercise 14.2.**

The Sierpinski carpet is the set

$$E := \left\{ \left(\sum_n x_n 3^{-n}, \sum_n y_n 3^{-n} \right) \; ; \; x_n, y_n \in \{0, 1, 2\}, \; \forall n \; x_n \neq 1 \text{ or } y_n \neq 1 \right\} .$$

That is, the unit square is divided into its nine triadic subsquares of order 1 and (the interior of) the middle one is removed. This process is repeated on each of the remaining 8 squares, and so on. Show that $\dim E = \dim^{\mathrm{M}} E = \log 8 / \log 3$.

▷ **Exercise 14.3.**

The Sierpinski gasket is the set obtained by partitioning a triangle into four congruent pieces and removing (the interior of) the middle one, then repeating this process *ad infinitum* on the remaining pieces, as in Figures 14.1 and 14.2. Show that its Hausdorff dimension is $\log 3 / \log 2$.

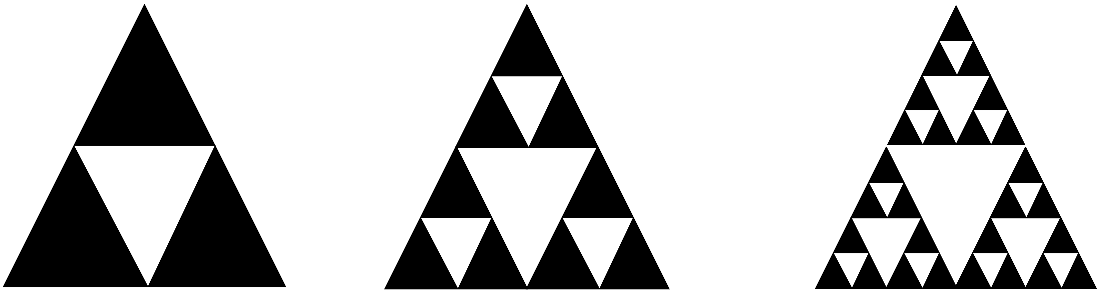


Figure 14.1. The first three stages of the construction of the Sierpinski gasket.

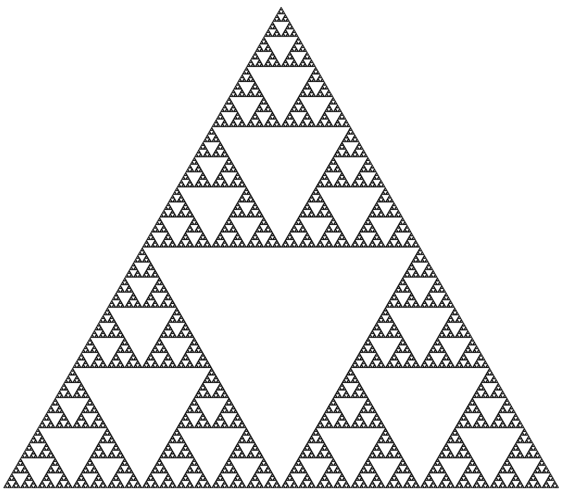


Figure 14.2. The Sierpinski gasket, drawn by O. Schramm.

▷ **Exercise 14.4.**

Show that for any sets E_n, $\dim(\bigcup_{n=1}^\infty E_n) = \sup \dim E_n$.

▷ **Exercise 14.5.**

Show that if $E_1 \supseteq E_2 \supseteq \cdots$, then $\dim \bigcap E_n \le \lim \dim E_n$.

One can extend the notion of Hausdorff measures to other ***gauge functions*** $h : \mathbb{R}^+ \to \mathbb{R}^+$ in place of $h(t) := t^\alpha$. We require merely that h be continuous and increasing and that $h(0^+) = 0$. This allows for finer examination of the size of E in cases when $\mathcal{H}_{\dim E}(E) \in \{0, \infty\}$.

§14.2. Coding by Trees.

Recall from Section 1.9 how, following Furstenberg (1970), we associate rooted trees to closed sets in $[0, 1]$ (later in this section, we will consider much more general associations to bounded sets in $\mathbb{R}^d$). Namely, for $E \subseteq [0, 1]$ and for any integer $b \ge 2$, consider the system of b-adic subintervals of $[0, 1]$. Those whose intersection with E is non-empty will form the vertices of the associated tree. Two such intervals are connected by an edge iff one contains the other and their orders differ by one. The root of this tree is $[0, 1]$ (provided $E \ne \varnothing$). We denote the tree by $T_{[b]}(E)$ and call it the ***b-adic coding of*** E.

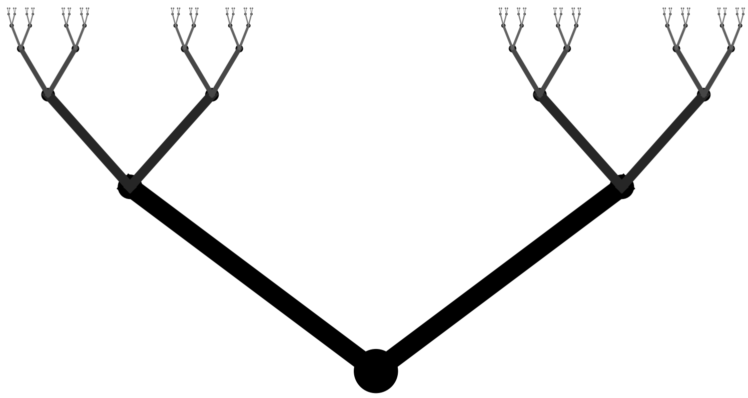


Figure 14.3. The Furstenberg coding of the Cantor middle-thirds set in base 3.

Example: For the Cantor middle-thirds set and for the base $b = 3$, the associated tree is the binary tree, shown in Figure 14.3. Note that if, instead of always taking out the middle third in the construction of the set, we took out the last third, we would still get the binary tree as its 3-adic coding. However, if we code in base 2, we get a different tree, as in Figures 14.4 and 14.5.

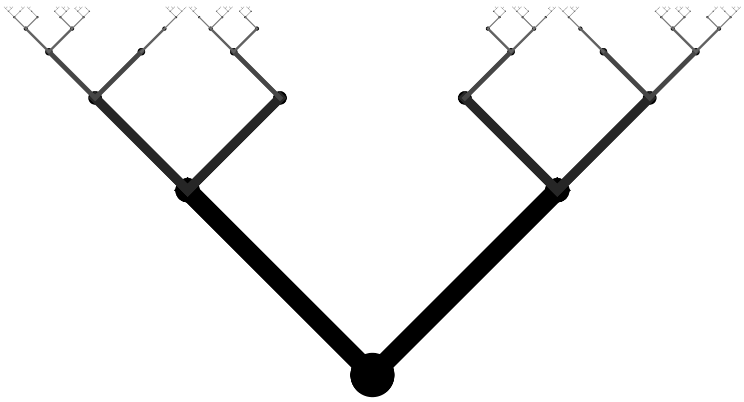


Figure 14.4. The Furstenberg coding of the Cantor middle-thirds set in base 2.

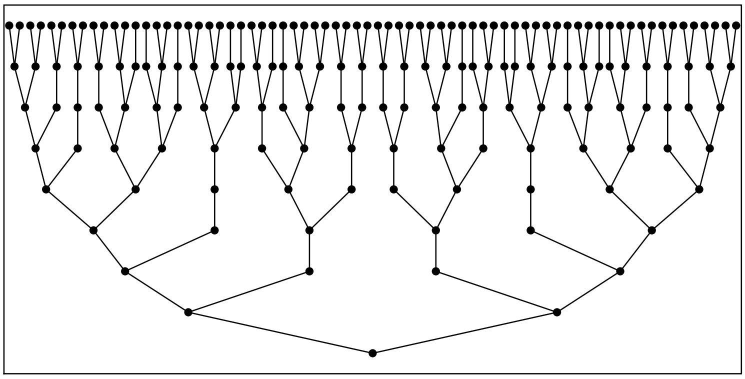


Figure 14.5. The tree of the Furstenberg coding of the Cantor middle-thirds set in base 2. The tree has branching number $2^{\log 2/\log 3} = 1.55^-$.

If $T_{[b]}(E)$ the b-adic coding of E, then we claim that

$$\operatorname{br} T_{[b]}(E) = b^{\dim E}, \tag{14.1}$$

$$\underline{\operatorname{gr}}\, T_{[b]}(E) = b^{\dim_{\mathrm{M}} E}, \tag{14.2}$$

$$\overline{\operatorname{gr}}\, T_{[b]}(E) = b^{\dim^{\mathrm{M}} E}. \tag{14.3}$$

For note that a cover of E by b-adic intervals is essentially the same as a cutset of $T_{[b]}(E)$. If the interval I_v corresponds to (actually, is) the vertex $v \in T_{[b]}(E)$, then $\operatorname{diam} I_v = b^{-|v|}$. Thus,

$$\dim E = \inf \left\{ \alpha \,;\ \inf_{\Pi} \sum_{e \in \Pi} b^{-\alpha|e|} = 0 \right\}.$$

Comparing this with the formula (3.4) for the branching number gives (14.1).

▷ **Exercise 14.6.**

Deduce (14.2) and (14.3) in the same way.

This helps us to see that the tree coding a set determines the dimension of the set, that the placement of particular digits in the coding doesn't influence the dimension. We may also say that $b^{\dim E}$ is an average number of b-adic subintervals intersecting E of order 1 more than a given b-adic interval intersecting E.

Example 14.6. We may easily rederive the Hausdorff and Minkowski dimensions of the Cantor middle-thirds set, E. Since $T_{[3]}(E)$ is a binary tree, it has branching number and growth 2. From (14.1) and (14.2), it follows that the Hausdorff and Minkowski dimensions are both $\log 2/\log 3$.

▷ **Exercise 14.7.**
Reinterpret Theorems 3.5 and 5.15 as theorems on random walks on b-adic intervals intersecting E and on random b-adic possible coverings of E.

For any tree T without leaves, define a metric on its boundary ∂T by $d(\xi,\eta) := e^{-|\xi\wedge\eta|}$ ($\xi \neq \eta$), where $\xi \wedge \eta$ is defined to be the vertex common to both ξ and η that is furthest from 0 if $\xi \neq \eta$ and $\xi \wedge \eta := \xi$ if $\xi = \eta$. With this metric, we can consider $\dim E$ for $E \subseteq \partial T$. The covers to consider here are by sets of the form

$$I_v := \{\xi \in \partial T\,;\ v \in \xi\} \tag{14.4}$$

since any subset of ∂T is contained in such a special set of the same diameter. (Namely, if $E \in \partial T$ and $v := \bigwedge_{\xi\in E} \xi$, in the obvious extension of the $\wedge$ notation, then $E \subseteq I_v$ and $\operatorname{diam} E = \operatorname{diam} I_v$.)

▷ **Exercise 14.8.**
What is $\operatorname{diam} I_v$?

In particular, we have

$$\dim \partial T = \log \operatorname{br} T$$

as in Section 1.7, so that

$$\dim \partial T_{[b]}(E) = (\dim E) \log b\,.$$

There is a more general way than b-adic coding to relate trees to certain types of sets. Denote the interior of a set I by $\operatorname{int} I$. Suppose that for each vertex $v \in T$, there is a compact set $\varnothing \neq I_v \subseteq \mathbb{R}^d$ with the following properties:

$$I_v = \overline{\operatorname{int} I_v}\,, \tag{14.5}$$

$$u \to v \quad \Longrightarrow \quad I_v \subseteq I_u\,, \tag{14.6}$$

$$\overleftarrow{u} = \overleftarrow{v} \text{ and } u \neq v \quad \Longrightarrow \quad \operatorname{int} I_u \cap \operatorname{int} I_v = \varnothing\,, \tag{14.7}$$

$$\forall \xi \in \partial T \quad \lim_{v \in \xi} \operatorname{diam} I_v = 0\,, \tag{14.8}$$

$$C_1 := \inf_{v \neq o} \frac{\operatorname{diam} I_v}{\operatorname{diam} I_{\overleftarrow{v}}} > 0\,, \tag{14.9}$$

$$C_2 := \inf_{v} \frac{\mathcal{L}_d(\operatorname{int} I_v)}{(\operatorname{diam} I_v)^d} > 0\,. \tag{14.10}$$

For example, if the sets I_v are b-adic cubes of order $|v|$, this is just a multidimensional version of the previous coding. For the Sierpinski gasket (Exercise 14.3), more natural sets I_v are equilateral triangles. Similarly, for the von Koch snowflake (Figure 14.6), natural sets to use are again equilateral triangles (see Figure 14.7).

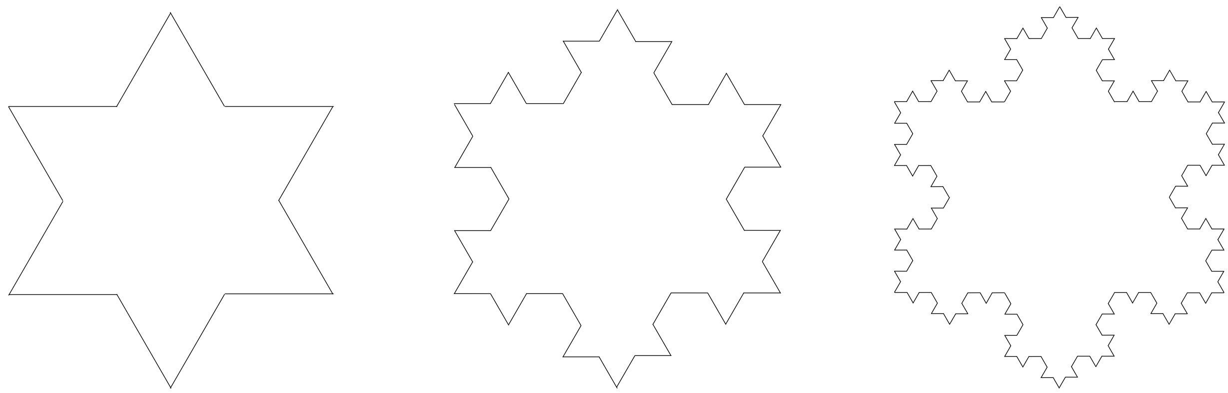


Figure 14.6. The first three stages of the construction of the von Koch snowflake.

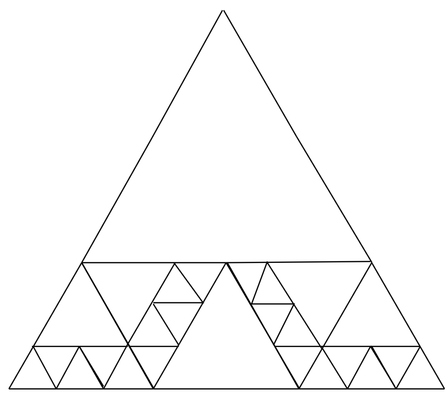


Figure 14.7. The sets I_v for the top side of the von Koch snowflake.

▷ **Exercise 14.9.**

Prove that under the conditions (14.5)–(14.10),

$$\lim_{n\to\infty} \max_{|v|=n} \operatorname{diam} I_v = 0\,.$$

We may associate the following set to T and $I_\bullet$:

$$I_T := \bigcup_{\xi\in\partial T} \bigcap_{v\in\xi} I_v\,.$$

If T is locally finite (as it must be when C_1 and C_2 are positive), then we also have

$$I_T = \bigcap_{n\ge 1} \bigcup_{|v|=n} I_v\,. \tag{14.11}$$

▷ **Exercise 14.10.**

Prove this equality (if T is locally finite).

For example, $I_{T_{[b]}(E)} = E$.

The sets $\{I_v\}_{v\in T}$ are actually the only ones we need consider in determining $\dim I_T$ or even, up to a bounded factor, $\mathcal{H}_\alpha(I_T)$:

Theorem 14.7. (Transference of Hausdorff Dimension and Measure) *If (14.5)–(14.10) hold, then*

$$\dim I_T = \inf\left\{\alpha\,;\ \inf_{\Pi} \sum_{e(v)\in\Pi} (\operatorname{diam} I_v)^\alpha = 0\right\}.$$

In fact, for $\alpha > 0$, we have

$$\mathcal{H}_\alpha(I_T) \le \liminf_{d(0,\Pi)\to\infty} \sum_{e(v)\in\Pi} (\operatorname{diam} I_v)^\alpha \le C\mathcal{H}_\alpha(I_T)\,, \tag{14.12}$$

where

$$C := \frac{4^d}{C_2 C_1^d}\,.$$

Corollary 14.8. *If the sets I_v are b-adic cubes of order $|v|$ in $\mathbb{R}^d$, then*

$$\dim I_T = \dim \partial T / \log b\,.$$

When this corollary is applied to $T = T_{[b]}(E)$, we obtain (14.1) again. To prove Theorem 14.7, we need a little geometric fact:

Lemma 14.9. *Let E and O_i $(1 \le i \le n)$ be subsets of $\mathbb{R}^d$ such that O_i are open and disjoint, $\overline{O}_i \cap E \neq \varnothing$, $\mathrm{diam} O_i \le \mathrm{diam} E$, and $\mathcal{L}_d(O_i) \ge C(\mathrm{diam} E)^d$. Then $n \le 4^d/C$.*

Proof. Fix $x_0 \in E$ and let B be the closed ball centered at x with radius $2{\cdot}\mathrm{diam}\, E$. Then $\overline{O_i} \subseteq B$, whence

$$(4\,\mathrm{diam} E)^d \ge \mathcal{L}_d(B) \ge \sum_{i=1}^{n} \mathcal{L}_d(O_i) \ge nC(\mathrm{diam} E)^d\,. \qquad ◀$$

▷ **Exercise 14.11.**

Prove the left-hand inequality of (14.12).

Proof of Theorem 14.7. For the right-hand inequality of (14.12), consider a cover by sets of positive diameter

$$I_T \subseteq \bigcup_{i=1}^{\infty} E_i\,.$$

For each i, put

$$\Pi_i := \{e(v)\,;\ \mathrm{diam}\, I_v \le \mathrm{diam}\, E_i < \mathrm{diam}\, I_{\overleftarrow{v}}\}\,.$$

This is a cutset, so

$$I_T \cap E_i \subseteq I_T \subseteq \bigcup_{e(v)\in\Pi_i} I_v\,.$$

Also, for $e(v) \in \Pi_i$, we have $\mathcal{L}_d(\mathrm{int}\, I_v) \ge C_2(\mathrm{diam}\, I_v)^d \ge C_2 C_1^d(\mathrm{diam}\, I_{\overleftarrow{v}})^d > C_2 C_1^d(\mathrm{diam}\, E_i)^d$, whence by Lemma 14.9,

$$V_i := \{e(v) \in \Pi_i\,;\ I_v \cap E_i \neq \varnothing\}$$

has at most $C = 4^d/(C_2 C_1^d)$ elements. Now

$$I_T \subseteq \bigcup_{i=1}^{\infty}\left(E_i \cap \bigcup_{e(v)\in V_i} I_v\right) \subseteq \bigcup_i \bigcup_{e(v)\in V_i} I_v$$

and the cutset $\Pi := \bigcup V_i$ satisfies

$$\sum_{e(v)\in\Pi} (\mathrm{diam}\, I_v)^\alpha \le \sum_{i,\, e(v)\in V_i} (\mathrm{diam}\, I_v)^\alpha \le \sum_{i,\, e(v)\in V_i} (\mathrm{diam}\, E_i)^\alpha \le c\sum_i (\mathrm{diam}\, E_i)^\alpha\,. \qquad ◀$$

§14.3. Galton-Watson Fractals.

We may combine Theorem 14.7 with Falconer's Theorem 5.26 on Galton-Watson networks to determine the dimension of random fractals. Thus, suppose that the sets I_v are randomly assigned to a Galton-Watson tree in such a way that the regularity conditions

$$I_v = \overline{\operatorname{int} I_v}\,,$$
$$u \to v \quad\Longrightarrow\quad I_v \subseteq I_u\,,$$
$$\overleftarrow{u} = \overleftarrow{v} \text{ and } u \neq v \quad\Longrightarrow\quad \operatorname{int} I_u \cap \operatorname{int} I_v = \varnothing\,,$$
$$\inf_v \mathcal{L}_d(\operatorname{int} I_v)/(\operatorname{diam} I_v)^d > 0$$

are satisfied a.s. and their normalized diameters $c(e(v)) := \operatorname{diam} I_v / \operatorname{diam} I_0$ are the capacities of a Galton-Watson network with generating random variable $\mathcal{L} := (L, A_1, \ldots, A_L)$, $0 < A_i < 1$ a.s. Because Galton-Watson trees can have leaves, we denote by T' the subtree of T that consists of vertices with infinite lines of descent. (You might want to glance at the examples following the proof of the theorem at this point.) The following result is due to Falconer (1986) and Mauldin and Williams (1986).

Theorem 14.10. (Dimension of Galton-Watson Fractals) *Almost surely on nonextinction,*

$$\dim I_{T'} = \min\left\{\alpha\,;\ \mathbf{E}\Big[\sum_{i=1}^{L} A_i^\alpha\Big] \le 1\right\}.$$

Proof. Since $I_o \supseteq \bigcup_{|v|=1} I_v$ and $\{\operatorname{int} I_v\}_{|v|=1}$ are disjoint, $\mathcal{L}_d(I_o) \ge \sum \mathcal{L}_d(\operatorname{int} I_v)$, whence $(\operatorname{diam} I_o)^d \ge \mathcal{L}_d(I_o) \ge C_2 \sum (\operatorname{diam} I_v)^d = C_2 (\operatorname{diam} I_o)^d \sum_{|v|=1} A_v^d$, whence $\mathbf{E}[\sum_1^L A_i^d] \le 1/C_2$. By the Lebesgue dominated convergence theorem, $\alpha \mapsto \mathbf{E}[\sum_1^L A_i^\alpha]$ is continuous on $[d, \infty]$ and has limit 0 at ∞, so there is some α such that $\mathbf{E}[\sum_1^L A_i^\alpha] \le 1$. Since $\alpha \mapsto \mathbf{E}[\sum_1^L A_i^\alpha]$ is continuous from the right on $[0, \infty)$ by the monotone convergence theorem, it follows that the minimum written exists.

Assume first that $\exists \epsilon > 0\ \forall i \quad \epsilon \le A_i \le 1 - \epsilon$ a.s. Then the last regularity conditions, (14.8) and (14.9), are satisfied a.s.: $\forall \xi \in \partial T' \quad \lim_{v \in \xi} \operatorname{diam} I_v = 0$,

$$\inf_{v \ne o} \frac{\operatorname{diam} I_v}{\operatorname{diam} I_{\overleftarrow{v}}} = \inf_{v \ne o} A_v \ge \epsilon.$$

Hence Theorem 14.7 assures us that

$$\begin{aligned}\dim I_{T'} &= \inf\left\{\alpha\,;\ \inf_\Pi \sum_{e(v)\in\Pi} (\operatorname{diam} I_v)^\alpha = 0\right\}\\ &= \inf\left\{\alpha\,;\ \inf_\Pi \sum_{e\in\Pi} c(e)^\alpha = 0\right\}.\end{aligned}$$

Now the capacities $\langle c(e)^\alpha \rangle$ come from a Galton-Watson network with $\mathcal{L}^{(\alpha)} := (L, A_1^\alpha, \ldots, A_L^\alpha)$, whence Falconer's Theorem 5.26 says that

$$\inf_\Pi \sum_{e\in\Pi} c(e)^\alpha = 0 \text{ a.s. if } \mathbf{E}\left[\sum_1^L A_i^\alpha\right] \le 1$$

while

$$\inf_\Pi \sum_{e\in\Pi} c(e)^\alpha > 0 \text{ a.s. if } \mathbf{E}\left[\sum_1^L A_i^\alpha\right] > 1 \,.$$

This gives the result.

For the general case, note that $\mathbf{E}[\sum_1^L A_i^\alpha] \le 1 \Rightarrow \inf_\Pi \sum c(e)^\alpha = 0$ a.s. $\Rightarrow \mathcal{H}_\alpha(I_{T'}) = 0$ a.s. For the other direction, consider the Galton-Watson subnetwork $T_{(\epsilon)}$ consisting of those branches with $\epsilon \le A_i \le 1-\epsilon$. Thus $I_{T'_{(\epsilon)}} \subseteq I_{T'}$. From the above, we have

$$\begin{aligned}\mathbf{E}\left[\sum_1^L A_i^\alpha \mathbf{1}_{\epsilon \le A_i \le 1-\epsilon}\right] > 1 \quad &\Longrightarrow \quad \mathcal{H}_\alpha(I_{T'_{(\epsilon)}}) > 0 \text{ a.s. on nonextinction of } T_{(\epsilon)} \\ &\Longrightarrow \quad \mathcal{H}_\alpha(I_{T'}) > 0 \text{ a.s. on nonextinction of } T_{(\epsilon)} \,.\end{aligned}$$

Since $\mathbf{P}[T'_{(\epsilon)} \ne \varnothing] \to \mathbf{P}[T' \ne \varnothing]$ as $\epsilon \to 0$ by Exercise 5.17 and

$$\mathbf{E}[\sum_1^L A_i^\alpha] = \lim_{\epsilon\to 0} \mathbf{E}[\sum_1^L A_i^\alpha \mathbf{1}_{\epsilon \le A_i \le 1-\epsilon}]$$

by the monotone convergence theorem, it follows that $\mathbf{E}[\sum A_i^\alpha] > 1 \Rightarrow \mathcal{H}_\alpha(I_{T'}) > 0$ a.s. on $T' \ne \varnothing$. ◀

Remark. By Falconer's theorem, $\mathcal{H}_{\dim I_{T'}}(I_{T'}) = 0$ a.s.

Example 14.11. Divide $[0, 1]$ into 3 equal parts and keep each independently with probability p. Repeat this with the remaining intervals *ad infinitum*. The Galton-Watson network comes from the random variable $\mathcal{L} = (L, A_1, \ldots, A_L)$, where L is a $\mathrm{Bin}(3, p)$-random variable, so that $\mathbf{E}[s^L] = (\overline{p} + ps)^3$, and $A_i \equiv 1/3$. Thus extinction is a.s. iff $p \le 1/3$, while for $p > 1/3$, the probability of extinction q satisfies $(\overline{p} + pq)^3 = q$ and a.s. on nonextinction,

$$\begin{aligned}\dim I_{T'} &= \min\left\{\alpha\,;\ \mathbf{E}\left[\sum_1^L A_i^\alpha\right] \le 1\right\} \\ &= \min\{\alpha\,;\ (1/3)^\alpha 3p \le 1\} \\ &= 1 + \log p / \log 3 \,.\end{aligned}$$

Example 14.12. Remove from $[0,1]$ a central portion leaving two intervals of random length $A_1 = A_2 \in (0, 1/2)$. Repeat on the remaining intervals *ad infinitum.* Here $L \equiv 2$. There is no extinction and a.s. $\dim I_T$ is the root of

$$1 = \mathbf{E}[A_1^\alpha + A_2^\alpha] = 2\mathbf{E}[A_1^\alpha]\,.$$

For example, if A_1 is uniform on $[0, 1/2]$, then

$$\mathbf{E}[A_1^\alpha] = 2\int_0^{1/2} t^\alpha\, dt = \frac{1}{2^\alpha(\alpha+1)}\,,$$

whence $\dim I_T \approx 0.457$.

Example 14.13. Suppose M and N are random integers with $M \ge 2$ and $0 \le N \le M^2$. Divide the unit square into M^2 equal squares and keep N of them (in some manner). Repeat on the remaining squares *ad infinitum.* The probability of extinction of the associated Galton-Watson network is a root of $\mathbf{E}[s^N] = s$ and a.s. on nonextinction,

$$\dim I_{T'} = \min\{\alpha\,;\ \mathbf{E}[N/M^{2\alpha}] \le 1\}\,.$$

By selecting the squares appropriately, one can have infinite connectivity of $I_{T'}$.

Example 14.14. Here, we consider the zero set E of Brownian motion B_t on $\mathbb{R}$. That is, $E := \{t\,;\ B_t = 0\}$. To find $\dim E$, we may restrict to an interval $[0, t_0]$ where $B_{t_0} = 0$. Of course, t_0 is random. By scale invariance of B_t and of Hausdorff dimension, the time t_0 makes no difference. So let's condition on $B_1 = 0$. This is known as ***Brownian bridge***. Calculation of finite-dimensional marginals shows B_t given $B_1 = 0$ to have the same law as $B_t^0 := B_t - tB_1$. One way to construct E is as follows. From $[0,1]$, remove the interval (τ_1, τ_2), where $\tau_1 := \max\{t < \frac{1}{2}\,;\ B_t^0 = 0\}$ and $t_2 := \min\{t > \frac{1}{2}\,;\ B_t^0 = 0\}$. Independence and scaling shows that E is obtained as the iteration of this process. That is, $E = I_T$, where T is the binary tree and $A_1 := \tau_1$ and $A_2 := 1 - \tau_2$ are the lengths of the surviving intervals. One can show that the joint density of A_1 and A_2 is

$$\mathbf{P}[A_1 \in da_1, A_2 \in da_2] = \frac{1}{2\pi}\frac{1}{\sqrt{a_1a_2(1-a_1-a_2)^3}}\,da_1\,da_2 \qquad (a_1, a_2 \in [0, 1/2])\,.$$

Straightforward calculus then shows that $\mathbf{E}[A_1^{1/2} + A_2^{1/2}] = 1$, so $\dim E = 1/2$ a.s. This result is due to Taylor (1955), but this method is due to Graf, Mauldin, and Williams (1988).

More examples of the use of Theorem 14.10 can be found in Mauldin and Williams (1986).

§14.4. Hölder Exponent.

Let T be an infinite locally finite rooted tree and μ be a Borel probability measure on ∂T. Even if there is no proper closed subset of ∂T of μ-measure 1, there may still be "much smaller" subsets of ∂T that carry μ. That is, there may be sets $E \subseteq \partial T$ with $\dim E < \dim \partial T$ and $\mu(E) = 1$. This suggests defining the ***dimension*** of μ as

$$\dim \mu := \min\{\dim E\,;\ \mu(E) = 1\}\,. \tag{14.13}$$

▷ **Exercise 14.12.**

Show that this minimum exists.

▷ **Exercise 14.13.**

Let T be an infinite locally finite tree. Show that there is a one-to-one correspondence between unit flows θ from o to ∂T and Borel probability measures μ on ∂T satisfying

$$\theta(v) = \mu(\{\xi\,;\ v \in \xi\})\,.$$

▷ **Exercise 14.14.**

(a) Suppose that μ is a probability measure on $\mathbb{R}^d$ such that the measure of every b-adic cube of order n is at most $Cb^{-n\alpha}$ for some constants C and α. Show that there is a constant C' such that the measure of every ball of radius r is at most $C'r^\alpha$.

(b) The ***Frostman exponent*** of a probability measure μ on $\mathbb{R}^d$ is

$$\mathrm{Frost}(\mu) := \sup\{\alpha \in \mathbb{R}\,;\ \sup_{x \in \mathbb{R}^d, r>0} \mu\big(B_r(x)\big)/r^\alpha < \infty\}\,,$$

where $B_r(x)$ is the ball of radius r centered at x. Show that the Hausdorff dimension of every compact set is the supremum of the Frostman exponents of the probability measures it supports.

The dimension of a measure is most often computed by calculating pointwise information, rather than by using the definition, and then using the following theorem of Billingsley (1965). For a ray $\xi = \langle \xi_1, \xi_2, \ldots \rangle \in \partial T$, identify μ with the unit flow corresponding to μ as in Exercise 14.13 and define the ***Hölder exponent*** of μ at ξ to be

$$\mathsf{H\ddot{o}}(\mu)(\xi) := \liminf_{n\to\infty} \frac{1}{n} \log \frac{1}{\mu(\xi_n)}\,.$$

For example, if T is the m-ary tree, $\theta(v) := m^{-|v|}$, and μ is the probability measure corresponding to the unit flow θ, then the Hölder exponent of μ is everywhere $\log m$.

Theorem 14.15. (Dimension and Hölder Exponent) *For any Borel probability measure μ on the boundary of a tree,*

$$\dim \mu = \mu\text{-}\operatorname{ess\,sup} \mathsf{H\ddot{o}}(\mu)\,.$$

Proof. Let $d := \mu\text{-}\operatorname{ess\,sup} \mathsf{H\ddot{o}}(\mu)$. We first exhibit a carrier of μ whose dimension is at most d. Then we show that every carrier of μ has dimension at least d.

Given $k \in \mathbb{N}$ and $\alpha > d$, let

$$\begin{aligned} E(k,\alpha) : &= \{\xi \in \partial T\,;\ \exists n \ge k \quad (1/n)\log(1/\mu(\xi_n)) \le \alpha\} \\ &= \{\xi \in \partial T\,;\ \exists n \ge k \quad e^{-n\alpha} \le \mu(\xi_n)\}\,. \end{aligned}$$

Then clearly $\bigcap_\alpha \bigcap_k E(k,\alpha)$ is a carrier of μ. To show that it has Hausdorff dimension at most d, it suffices, by Exercise 14.5, to show that $\dim \bigcap_k E(k,\alpha) \le \alpha$. Clearly $E(k,\alpha)$ is open, so is a countable union of disjoint sets of the form (14.4), call them I_{v_i}, with all $|v_i| \ge k$ and

$$(\operatorname{diam} I_{v_i})^\alpha \le e^{-|v_i|\alpha} \le \mu(I_{v_i})\,.$$

Since

$$\sum_i (\operatorname{diam} I_{v_i})^\alpha \le \sum_i \mu(I_{v_i}) \le 1$$

and the sets I_{v_i} also cover $\bigcap_k E(k,\alpha)$, it follows that $\mathcal{H}_\alpha(\bigcap_k E(k,\alpha)) \le 1$. This implies our desired inequality $\dim \bigcap_k E(k,\alpha) \le \alpha$.

For the other direction, suppose that F is a carrier of μ. For $k \in \mathbb{N}$ and $\alpha < d$, let

$$F(k,\alpha) := \{\xi \in F\,;\ \forall n \ge k \quad (1/n)\log(1/\mu(\xi_n)) \ge \alpha\}\,.$$

Then $\mu(F(k,\alpha)) > 0$ for sufficiently large k. Fix such a k. To show that $\dim F \ge d$, it suffices to show that $\dim F(k,\alpha) \ge \alpha$. Indeed, reasoning similar to that in the preceding paragraph shows that $\mathcal{H}_\alpha(F(k,\alpha)) \ge \mu(F(k,\alpha))$, which completes the proof. ◀

As an example of the calculation of Hölder exponent, consider the harmonic measure of ***simple forward random walk***, which is the random walk that starts at the root and chooses randomly (uniformly) among the children of the present vertex as the next vertex. The corresponding harmonic measure on ∂T is called ***visibility measure***, denoted VIS_T, and corresponds to the ***equally-splitting flow***. Suppose now that T is a Galton-Watson tree starting, as usual, with one particle and having Z_1 children. Identifying measure and flow, we have that VIS_T is a flow on the random tree T. Let **GW** denote the distribution of Galton-Watson trees. The probability measure that corresponds to choosing a Galton-Watson tree T at random followed by a ray of ∂T chosen according to VIS_T will be denoted $\mathsf{VIS} \times \mathbf{GW}$. Formally, this is

$$(\mathsf{VIS} \times \mathbf{GW})(F) := \iint \mathbf{1}_F(\xi, T)\, d\mathsf{VIS}_T(\xi)\, d\mathbf{GW}(T)\,.$$

Since

$$\frac{1}{n} \log \frac{1}{\mathsf{VIS}_T(\xi_n)} = \frac{1}{n} \sum_{k=0}^{n-1} \log \frac{\mathsf{VIS}_T(\xi_k)}{\mathsf{VIS}_T(\xi_{k+1})}$$

and the random variables $\mathsf{VIS}_T(\xi_{k-1})/\mathsf{VIS}_T(\xi_k)$ are $\mathsf{VIS} \times \mathbf{GW}$-i.i.d. with the same distribution as Z_1, the strong law of large numbers gives

$$\mathsf{H\ddot{o}}(\mathsf{VIS}_T)(\xi) = \mathbf{E}[\log Z_1] \quad \mathsf{VIS} \times \mathbf{GW}\text{-a.s. } (\xi, T)\,.$$

Thus $\dim \mathsf{VIS}_T = \mathbf{E}[\log Z_1]$ for $\mathbf{GW}$-a.e. tree T. Jensen's inequality (or the arithmetic mean-geometric mean inequality) shows that this dimension is less than $\log m$ except in the deterministic case $Z_1 = m$ a.s.

§14.5. Derived Trees.

This section is based on some unpublished ideas of Furstenberg. Most of the time, for the sake of compactness, we need to restrict our trees to have uniformly bounded degree. Thus, fix an integer r and let $\mathbf{T}$ be the r-ary tree, which we identify with $\{0, 1, \ldots, r-1\}^{\mathbb{Z}^+}$; we will consider only subtrees of $\mathbf{T}$ rooted at the root of $\mathbf{T}$ and that have no leaves. Recall that for a tree T and vertex $v \in T$, we denote by T^v the subtree of T formed from the descendants of v. We view T^v as a rooted subtree of $\mathbf{T}$ with v identified with the root of $\mathbf{T}$. Given a unit flow θ on T and a vertex $v \in T$, the conditional flow through v is defined to be the unit flow

$$\theta^v(x) := \theta(x)/\theta(v) \qquad \text{for } x \geq v$$

on T^v. The class of subtrees of $\mathbf{T}$ is given the natural topology as in Exercise 5.2. This class is compact. For a subtree T, let $\mathscr{D}(T)$ denote the closure of the set of its descendant trees, $\{T^v\,;\ v \in T\}$. We call these trees the ***derived trees*** of T. Thus, $\mathscr{D}(\mathbf{T}) = \{\mathbf{T}\}$.

▷ **Exercise 14.15.**

Show that if $T^* \in \mathscr{D}(T)$, then $\mathscr{D}(T^*) \subseteq \mathscr{D}(T)$.

Define

$$
\begin{aligned}
T_n^v &:= \{x \in T^v\,;\ |x| \le |v| + n\}\,,\\
M_n(v) &:= |\partial T_n^v| = |\{x \in T^v; |x| = |v| + n\}|\,,\\
\dim\sup \partial T &:= \lim_{n\to\infty} \max_{v\in T} \frac{1}{n} \log M_n(v)\,, && (14.14)\\
\dim\inf \partial T &:= \lim_{n\to\infty} \min_{v\in T} \frac{1}{n} \log M_n(v)\,. && (14.15)
\end{aligned}
$$

▷ **Exercise 14.16.**
Show that these limits exist.

Clearly,

$$\dim \partial T \le \dim_{\mathrm{M}} \partial T = \liminf_{n\to\infty} \frac{1}{n} \log M_n \le \dim\sup \partial T\,. \tag{14.16}$$

Also, if $\alpha < \dim\inf \partial T$, then there is some n such that

$$\min_{v\in T} \frac{1}{n} \log M_n(v) > \alpha\,,$$

i.e., $M_n(v) \ge \lceil e^{\alpha n}\rceil$ for all $v \in T$. Thus, in the notation of Exercise 3.25, $T^{[n]}$ contains a $\lceil e^{\alpha n}\rceil$-ary subtree. Therefore $\dim \partial T^{[n]} \ge \log\lceil e^{\alpha n}\rceil \ge \alpha n$, whence $\dim \partial T \ge \alpha$. Thus, we obtain

$$\dim \partial T \ge \dim\inf \partial T\,. \tag{14.17}$$

Since for any $T^* \in \mathscr{D}(T)$, $v \in T^*$, and $n \ge 0$, there is some $w \in T$ such that $T_n^w = (T^*)_n^v$, we have $\dim\inf \partial T \le \dim\inf \partial T^*$ and $\dim\sup \partial T^* \le \dim\sup \partial T$. Combining these inequalities with (14.16) and (14.17) as applied to T^*, we arrive at

Proposition 14.16. *For any $T^* \in \mathscr{D}(T)$, we have*

$$\dim\inf \partial T \le \dim \partial T^* \le \dim\sup \partial T\,.$$

▷ **Exercise 14.17.**
Show that if T is subperiodic, then $\dim \partial T = \dim\sup \partial T$, while if T is superperiodic, then $\dim \partial T = \dim\inf \partial T$.

Proposition 14.16 is sharp in a strong sense, as shown by the following theorem of Furstenberg.

Theorem 14.17. *If T is a tree of uniformly bounded degree, then there exist $T^* \in \mathscr{D}(T)$ and a unit flow θ^* on T^* such that*

$$\dim \partial T^* = \dim\sup \partial T\,, \tag{14.18}$$

$$\forall x \in T^* \qquad \frac{1}{|x|}\log\frac{1}{\theta^*(x)} \ge \dim\sup \partial T\,, \tag{14.19}$$

and for θ^*-a.e. $\xi \in \partial T^*$

$$\mathsf{H\ddot{o}}(\theta^*)(\xi) = \dim\sup \partial T\,. \tag{14.20}$$

Similarly, there exist $T^{**} \in \mathscr{D}(T)$ and a unit flow θ^{**} on T^{**} such that

$$\dim \partial T^{**} = \dim\inf \partial T \tag{14.21}$$

and

$$\forall x \in T^{**} \qquad \frac{1}{|x|}\log\frac{1}{\theta^{**}(x)} \le \dim\inf \partial T\,. \tag{14.22}$$

Proof. (Ledrappier and Peres) We concentrate first on (14.19). We claim that for any positive integer L and any $\alpha < \dim\sup \partial T$, there is some vertex $v \in T$ and some unit flow θ on T_L^v (from v to ∂T_L^v) such that

$$\forall x \in T_L^v \quad \frac{1}{|x|-|v|}\log\frac{1}{\theta(x)} \ge \alpha\,.$$

Suppose for a contradiction that this is not the case for some L and α. Choose v and n so that n is a multiple of L and

$$M_n(v)^{1/n} > e^{\alpha} r^{L/n}\,.$$

(Recall that r is the degree of the root of $\mathbf{T}$.) Let θ be the flow on T_n^v such that

$$\forall x \in \partial T_n^v \quad \theta(x) = 1/M_n(v)\,.$$

Then θ restricts to a flow on T_L^v, whence, by our assumption, $\exists x_1 \in T_L^v \quad \theta(x) > e^{-(|x_1|-|v|)\alpha}$. By similar reasoning, define a finite sequence $\langle x_k \rangle$ inductively as follows. Provided $|x_k| - |v| \le n - L$, choose $x_{k+1} \in T_L^{x_k}$ such that $\theta^{x_k}(x_{k+1}) > e^{-(|x_{k+1}|-|x_k|)\alpha}$. Let the last index thereby achieved on $\langle x_k \rangle$ be K. Then, using $x_0 := v$, we have

$$\theta(x_K) = \prod_{k=0}^{K-1} \theta^{x_k}(x_{k+1}) > e^{-(|x_K|-|v|)\alpha} \ge e^{-n\alpha} > \frac{r^L}{M_n(v)}\,.$$

On the other hand, since $|v| + n - |x_K| < L$, we have

$$\theta(x_K) = \frac{\left|\partial T^{x_K}_{|v|+n-|x_K|}\right|}{M_n(v)} < \frac{r^L}{M_n(v)} \,.$$

As these two inequalities contradict each other, our claim is established.

For $j \ge 1$, there is thus some unit flow θ_j on some $T_j^{v_j}$ such that

$$\forall x \in T_j^{v_j} \qquad \frac{1}{|x| - |v_j|} \log \frac{1}{\theta_j(x)} \ge \left(1 - \frac{1}{j}\right) \dim\sup \partial T \,. \tag{14.23}$$

Identifying $T_j^{v_j}$ with a rooted subtree of $\mathbf{T}$ identifies θ_j with a unit flow on $\mathbf{T}$. These flows have an edgewise limit point θ^*. Those edges where $\theta^* > 0$ form a tree $T^* \in \mathscr{D}(T)$. Because of (14.23), we obtain (14.19). By definition of Hölder exponent, this means that $\mathsf{H\ddot{o}}(\theta^*) \ge \dim\sup \partial T$. On the other hand, Theorem 14.15 gives $\mathsf{H\ddot{o}}(\theta^*) \le \dim\sup \partial T^* \le \dim\sup \partial T$, whence (14.18) and (14.20) follow.

The proof of (14.22) is parallel to that of (14.19) and we omit it. To deduce (14.21), let

$$M_n^{**} := |\{x \in T^{**} ; |x| = n\}|$$

and $\alpha := \dim\inf \partial T$. By (14.22), we have

$$1 = \theta^{**}(0) = \sum_{|v|=n} \theta^{**}(v) \ge \sum_{|v|=n} e^{-\alpha n} = M_n^{**} e^{-\alpha n} \,,$$

so that

$$\frac{1}{n} \log M_n^{**} \le \alpha \,.$$

Therefore, $\dim \partial T^{**} \le \alpha$. ◄

The first part of Theorem 14.17 allows us to give another proof of Furstenberg's important Theorem 3.8: If T is subperiodic, then every $T^* \in \mathscr{D}(T)$ is isomorphic to a subtree of a descendant subtree T^v. In particular,

$$\dim \partial T^* \le \dim \partial T \,.$$

On the other hand, if $T^* \in \mathscr{D}(T)$ satisfies (14.18), then

$$\dim \partial T \ge \dim \partial T^* = \dim\sup \partial T \ge \dim^{\mathrm{M}} \partial T \ge \dim \partial T \,.$$

Therefore, $\dim \partial T = \dim^{\mathrm{M}} \partial T$, as desired.

§14.6. Collected In-Text Exercises.

14.1. Show that for any $E \subseteq \mathbb{R}^d$, there exists a real number α_0 such that $\alpha < \alpha_0 \Rightarrow \mathcal{H}_\alpha(E) = +\infty$ and $\alpha > \alpha_0 \Rightarrow \mathcal{H}_\alpha(E) = 0$.

14.2. The Sierpinski carpet is the set

$$E := \left\{ \left(\sum_n x_n 3^{-n}, \sum_n y_n 3^{-n} \right) ;\ x_n, y_n \in \{0, 1, 2\},\ \forall n\ x_n \neq 1 \text{ or } y_n \neq 1 \right\} .$$

That is, the unit square is divided into its nine triadic subsquares of order 1 and (the interior of) the middle one is removed. This process is repeated on each of the remaining 8 squares, and so on. Show that $\dim E = \dim^{\mathrm{M}} E = \log 8/\log 3$.

14.3. The Sierpinski gasket is the set obtained by partitioning a triangle into four congruent pieces and removing (the interior of) the middle one, then repeating this process *ad infinitum* on the remaining pieces, as in Figures 14.1 and 14.2. Show that its Hausdorff dimension is $\log 3/\log 2$.

14.4. Show that for any sets E_n, $\dim(\bigcup_{n=1}^\infty E_n) = \sup \dim E_n$.

14.5. Show that if $E_1 \supseteq E_2 \supseteq \cdots$, then $\dim \bigcap E_n \leq \lim \dim E_n$.

14.6. Deduce (14.2) and (14.3) in the same way as we deduced (14.1).

14.7. Reinterpret Theorems 3.5 and 5.15 as theorems on random walks on b-adic intervals intersecting E and on random b-adic possible coverings of E.

14.8. What is $\operatorname{diam} I_v$ of (14.4)?

14.9. Prove that under the conditions (14.5)–(14.10),

$$\lim_{n\to\infty} \max_{|v|=n} \operatorname{diam} I_v = 0 .$$

14.10. Prove (14.11) (if T is locally finite).

14.11. Prove the left-hand inequality of (14.12).

14.12. Show that the minimum of (14.13) exists.

14.13. Let T be an infinite locally finite tree. Show that there is a one-to-one correspondence between unit flows θ from o to ∂T and Borel probability measures μ on ∂T satisfying

$$\theta(v) = \mu(\{\xi ;\ v \in \xi\}) .$$

14.14. **(a)** Suppose that μ is a probability measure on $\mathbb{R}^d$ such that the measure of every b-adic cube of order n is at most $Cb^{-n\alpha}$ for some constants C and α. Show that there is a constant C' such that the measure of every ball of radius r is at most $C' r^\alpha$.

(b) The ***Frostman exponent*** of a probability measure μ on $\mathbb{R}^d$ is

$$\operatorname{Frost}(\mu) := \sup\{\alpha \in \mathbb{R} ;\ \sup_{x \in \mathbb{R}^d, r > 0} \mu(B_r(x))/r^\alpha < \infty\} ,$$

where $B_r(x)$ is the ball of radius r centered at x. Show that the Hausdorff dimension of every compact set is the supremum of the Frostman exponents of the probability measures it supports.

14.15. Show that if $T^* \in \mathscr{D}(T)$, then $\mathscr{D}(T^*) \subseteq \mathscr{D}(T)$.

14.16. Show that the limits in (14.14) and (14.15) exist.

14.17. Show that if T is subperiodic, then $\dim \partial T = \dim\sup \partial T$, while if T is superperiodic, then $\dim \partial T = \dim\inf \partial T$.

§14.7. Additional Exercises.

14.18. Let $\Omega \subseteq \mathbb{R}^2$ be open and non-empty and $f : \Omega \to \mathbb{R}$ be Lipschitz. Show that the Hausdorff dimension of the graph of f is 2.

14.19. Let T be a Galton-Watson tree with offspring random variable L having mean $m > 1$. Show that the Hausdorff measure of ∂T in dimension $\log m$ is 0 a.s. except when L is constant.

14.20. Let $m > 1$. Show that for $\gamma < 1$,

$$\max \left\{ \dim I_{T'} \,;\ \|L\|_\infty < \infty,\ \mathbf{E}[L] = m,\ \mathbf{E}\left[\sum_1^L A_i\right] = \gamma \right\} = \left(1 - \frac{\log \gamma}{\log m}\right)^{-1} < 1\,,$$

where the maximum is over all Galton-Watson fractals $I_{T'}$ corresponding to random variables $(L, A_1, \ldots, A_L)$, while for $m > \gamma > 1$,

$$\min \left\{ \dim I_{T'} \,;\ \|L\|_\infty < \infty,\ \mathbf{E}[L] = m,\ \mathbf{E}\left[\sum_1^L A_i\right] = \gamma \right\} = \left(1 - \frac{\log \gamma}{\log m}\right)^{-1} > 1\,,$$

with equality in each case iff $\forall i \ \ A_i = \gamma/m$ a.s. What if $\gamma = 1$?

14.21. To create (the top side of) a random von Koch curve, begin with the unit interval. Replace the middle portion of random length $\in (0, 1/3)$ by the other two sides of an equilateral triangle, as in Figure 14.8. Repeat this proportionally on all the remaining 4 pieces, etc. Of course, here, the associated Galton-Watson network has no extinction. Show that if the length is uniform on $[0, 1/3]$, then $\dim I_T \approx 1.144$ a.s.

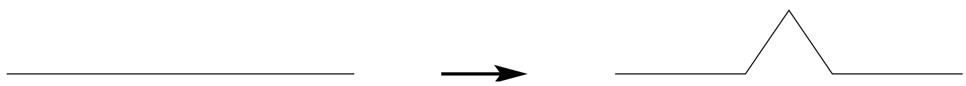


Figure 14.8.

14.22. Create a random Cantor set in $[0, 1]$ by removing a middle interval between two points chosen independently uniformly on $[0, 1]$. Repeat indefinitely proportionally on the remaining intervals. Show that $\dim I_T = (\sqrt{17} - 3)/2$ a.s.

14.23. Given a probability vector $\langle p_i \,;\ 1 \le i \le k \rangle$, the capacities of the Galton-Watson network generated by $(k, p_1, p_2, \ldots, p_k)$ define a unit flow θ, which corresponds to a probability measure μ on the boundary. Show that

$$\dim \mu = \sum_i p_i \log \frac{1}{p_i}\,.$$

14.24. Let $p(\bullet, \bullet)$ be the transition probabilities of a finite-state irreducible Markov chain. Let $\pi(\bullet)$ be the stationary probability distribution. The directed graph G associated to this chain has for vertices the states and for edges all (u, v) for which $p(u, v) > 0$. Let T be the directed cover of G (see Section 3.3). Define the unit flow

$$\theta(\langle v_1, \ldots, v_{n+1} \rangle) := \pi(v_1) \prod_{i=1}^n p(v_i, v_{i+1})\,.$$

Let μ be the corresponding probability measure on ∂T. Show that

$$\dim \mu = \sum_u \pi(u) \sum_v p(u,v) \log \frac{1}{p(u,v)} .$$

14.25. Let $p_k > 0$ $(1 \le k \le r)$ satisfy $\sum_{k=1}^r p_k = 1$. Let **GW** be the corresponding Galton-Watson measure on trees. Show that for **GW**-a.e. T, $\mathscr{D}(T)$ is equal to the set of *all* subtrees of $\mathbf{T}$.

14.26. Let T be the tree of Exercise 3.33 with $N = 0$ and $\alpha = 1/3$. Show that $\dim\inf T = \frac{1}{3}\log 3 + \frac{2}{3}\log\frac{3}{2}$ and $\dim\sup T = \log 2$.

14.27. Let $p_k \ge 0$ $(k \ge 1)$ satisfy $\sum_{k=1}^\infty p_k = 1$. Let **GW** be the corresponding Galton-Watson measure on trees. Show that for **GW**-a.e. T, $\dim\sup \partial T = \sup\{k\,;\ p_k > 0\}$ and $\dim\inf \partial T = \min\{k\,;\ p_k > 0\}$.

Chapter 15

Capacity

The notion of capacity will lead to important reformulations of our theorems concerning random walk and percolation on trees. As one consequence, we will be able to deduce a classical relationship of Hausdorff dimension and capacity in Euclidean space.

§15.1. Definitions.

We call a function Ψ on a rooted tree T such that $\Psi \ge 0$ and $x \to y \Rightarrow \Psi(x) \le \Psi(y)$ a ***gauge***. For a gauge Ψ, define $\Psi(\xi) := \lim_{x \in \xi} \Psi(x)$ and define the ***kernel*** $K := \partial T \times \partial T \to [0, \infty]$ by

$$K(\xi, \eta) := \begin{cases} \Psi(\xi \wedge \eta) & \text{if } \ \xi \ne \eta, \\ \Psi(\xi) & \text{if } \ \xi = \eta. \end{cases}$$

A common gauge is $\Psi(x) := \lambda^{|x|}$ or $\Psi(x) := \lambda^{|x|}/(\lambda - 1)$ for $\lambda > 1$. Using $c\big(e(x)\big) := \lambda^{1-|x|}(\lambda - 1)$ or $c\big(e(x)\big) := \lambda^{1-|x|}$, we see that these gauges are special cases of the following assignment of a gauge to conductances:

$$\Psi(x) := \Psi(o) + \sum_{o < u \le x} c\big(e(u)\big)^{-1} . \tag{15.1}$$

Furthermore, given a gauge Ψ, we can implicitly define conductances by (15.1).

Given a Borel probability μ on ∂T, its ***potential*** is the function

$$V_\mu(\xi) := \int_{\partial T} K(\xi, \eta) \, d\mu(\eta)$$

and its ***energy*** is the number

$$\mathscr{E}(\mu) := \int_{\partial T \times \partial T} K \, d(\mu \times \mu) = \int_{\partial T} V_\mu(\xi) \, d\mu(\xi) .$$

The ***capacity*** of $E \subseteq \partial T$ is

$$\operatorname{cap} E := \big[\inf\{\mathscr{E}(\mu) ; \ \mu \text{ a Borel probability and } \mu(\partial T \setminus E) = 0\}\big]^{-1} .$$

These definitions are made similarly for any topological space X in place of ∂T and any Borel function $K : X \times X \to [0, \infty]$. Recall from Exercise 14.13 that Borel probabilities μ on ∂T are in 1-1 correspondence with unit flows θ on T via

$$\theta\big(e(x)\big) = \theta(x) = \mu(\{\xi\,;\ x \in \xi\})\,.$$

We may express V_μ and $\mathscr{E}(\mu)$ in terms of θ as follows. Set

$$\Phi(x) := \Psi(x) - \Psi(\overleftarrow{x})\,, \qquad x \neq o\,.$$

(Recall that $\overleftarrow{x}$ is the parent of x.) Thus, $\Phi(u)$ is the resistance of $e(u)$ when (15.1) holds.

Proposition 15.1. (Lyons, 1990) *We have*

$$\forall \xi \in \partial T \qquad V_\mu(\xi) = \Psi(o) + \sum_{o < x \in \xi} \Phi(x)\theta(x)$$

and

$$\mathscr{E}(\mu) = \Psi(o) + \sum_{o \neq x \in T} \Phi(x)\theta(x)^2.$$

Proof. By linearity, we may assume that $\Psi(o) = 0$. Then $\Psi(x) = \sum_{o < u \le x} \Phi(u)$ and $\Psi(\xi) = \sum_{o < x \in \xi} \Phi(x)$. Therefore

$$\begin{aligned} V_\mu(\xi) &= \int_{\partial T} K(\xi, \eta)\, d\mu(\eta) = \int_{\partial T} \Psi(\xi \wedge \eta)\, d\mu(\eta) = \int_{\partial T} \sum_{o < x \le \xi \wedge \eta} \Phi(x)\, d\mu(\eta) \\ &= \int_{\partial T} \sum_{o < x \in \xi} \Phi(x) \mathbf{1}_{\{x \in \eta\}}\, d\mu(\eta) = \sum_{o < x \in \xi} \Phi(x) \int_{\partial T} \mathbf{1}_{\{x \in \eta\}}\, d\mu(\eta) = \sum_{o < x \in \xi} \Phi(x)\theta(x)\,. \end{aligned}$$

Now integrate to get

$$\begin{aligned} \mathscr{E}(\mu) &= \int_{\partial T} V_\mu\, d\mu = \int_{\partial T} \sum_{o < x \in \xi} \Phi(x)\theta(x)\, d\mu(\xi) \\ &= \sum_{o \neq x \in T} \Phi(x)\theta(x)\mu(\{\xi\,;\ x \in \xi\})\,. \end{aligned}$$ ◀

When conductances determine Ψ through (15.1) and the equation $\Psi(o) := 0$, we have $\Phi(x) = c\big(e(x)\big)^{-1}$ and thus $\mathscr{E}(\mu) = \mathscr{E}(\theta)$, as we defined energy of flows in Section 2.4. In particular, $\mathscr{E}(\mu)$ is minimum when $\mathscr{E}(\theta)$ is and, hence, $\mathscr{E}(\bullet)$ has a unique minimum corresponding to unit current flow when $\mathscr{C}_{\text{eff}} > 0$. If i denotes unit current flow and μ_i the corresponding measure on ∂T, then μ_i gives the distribution of current outflow on

∂T. Probabilistically, this is ***harmonic measure*** for the random walk, i.e., the "hitting" distribution on ∂T: by Proposition 2.11, given an edge $e(x)$, we have

$$\begin{aligned}\mu_i(\{\xi\,;\ x \in \xi\}) &= i\big(e(x)\big)\\ &= \mathbf{E}[\text{number of transitions from } \overleftarrow{x} \text{ to } x\\ &\qquad - \text{number of transitions from } x \text{ to } \overleftarrow{x}]\\ &= \mathbf{P}\Big[\text{random walk eventually stays in } T^x\Big]\,.\end{aligned}$$

We may reformulate Theorem 2.10 and Proposition 2.11 as follows for trees:

Theorem 15.2. (Random Walk and Capacity) *Given conductances on a tree T, the associated random walk is transient iff the capacity of ∂T is positive in the associated gauge. The unit flow corresponds to the harmonic measure on ∂T, which minimizes the energy.*

In this context, Theorem 3.5 becomes

$$\dim \partial T = \inf\{\lambda > 1\,;\ \operatorname{cap} \partial T = 0 \text{ in the gauge } \Psi(u) = \lambda^{|u|}\}\,.$$

Although $\dim \partial T$ has a similar definition via Hausdorff measures, it is capacity, not Hausdorff measure, in the critical gauge that determines the type of critical random walk, $\mathsf{RW}_{\operatorname{br} T}$.

Also, we have

$$\begin{aligned}V_{\mu_i}(\xi) &= \sum_{o<u\in\xi} i\big(e(u)\big)/c\big(e(u)\big) = \sum_{o<u\in\xi} [V(\overleftarrow{u}) - V(u)] = V(o) - \lim_{u\in\xi} V(u)\\ &= \mathscr{E}(i) - \lim_{u\in\xi} V(u) = \mathscr{E}(\mu_i) - \lim_{u\in\xi} V(u) = (\operatorname{cap} \partial T)^{-1} - \lim_{u\in\xi} V(u)\,.\end{aligned}$$

In particular, $\forall\xi\quad V_{\mu_i}(\xi) \le (\operatorname{cap}\partial T)^{-1}$. One can show that $V_{\mu_i}(\xi) = (\operatorname{cap}\partial T)^{-1}$, i.e., $\lim_{u\in\xi} V(u) = 0$, except for a set of ξ of capacity 0 (Lyons, 1990). This further justifies thinking of the electrical network T as grounded at ∞.

§15.2. Percolation on Trees.

Let p_x be survival probabilities for independent percolation, as in Section 5.3. Consider the gauge

$$\Psi(x) := \mathbf{P}[o \leftrightarrow x]^{-1} = \prod_{o<u\le x} p_u^{-1} \,.$$

Note that

$$\Psi(o) = 1 \,.$$

If we define conductances by

$$c\big(e(u)\big) := \Psi(u)^{-1} - \Psi(\overleftarrow{u})^{-1} \,,$$

then $\Psi(x)$ is one plus the resistance between o and x. Thus, (5.11) holds. Because $\Psi(o) = 1$, we have

$$\mathscr{E}(\mu_i) = 1 + \mathscr{E}(i) = 1 + \mathscr{C}(o \leftrightarrow \infty)^{-1} \,,$$

whence

$$\operatorname{cap} \partial T = \mathscr{E}(\mu_i)^{-1} = \frac{\mathscr{C}(o \leftrightarrow \infty)}{1 + \mathscr{C}(o \leftrightarrow \infty)} \,.$$

We may therefore rewrite Theorem 5.21 as

$$\operatorname{cap} \partial T \le \mathbf{P}[o \leftrightarrow \partial T] \le 2 \ \operatorname{cap} \partial T \,. \tag{15.2}$$

These inequalities are easily generalized to subsets of ∂T (Lyons, 1992):

Theorem 15.3. (Tree Percolation and Capacity) *For Borel $E \subseteq \partial T$, we have*

$$\operatorname{cap} E \le \mathbf{P}[o \leftrightarrow E] \le 2 \ \operatorname{cap} E.$$

Proof. If E is closed, let $T_{(E)} := \{u ;\ \exists \xi \in E \quad u \in \xi\}$. Then $T_{(E)}$ is a subtree of T with $\partial T_{(E)} = E$. Hence the result follows from (15.2). The general case follows from the theory of Choquet capacities, which we omit (see Lyons (1992)). ◀

▷ **Exercise 15.1.**

Show that if $p_x \equiv p \in (0, 1)$ and T is spherically symmetric, then

$$\operatorname{cap} \partial T = \left(1 + (1-p) \sum_{n=1}^{\infty} \frac{1}{p^n |T_n|}\right)^{-1} .$$

Thus, $\mathbf{P}[o \leftrightarrow \partial T] > 0$ iff $\sum_{n=1}^{\infty} \frac{1}{p^n |T_n|} < \infty$.

If we specialize to trees coding closed sets, as in Section 1.9 and Section 14.2, and take $p_u \equiv p$, the result that $\mathbf{P}[o \leftrightarrow \partial T] > 0$ iff $\operatorname{cap} \partial T > 0$ is due to Fan (1989, 1990).

§15.3. Euclidean Space.

Like Hausdorff dimension in Euclidean space, capacity in Euclidean space is also related to trees. We treat first the case of $\mathbb{R}^1$.

Consider a closed set $E \subseteq [0,1]$ and a kernel

$$K(x,y) := f\big(|x-y|\big)\,,$$

where $f \geq 0$ and f is decreasing. Here, f is called the ***gauge*** function. We denote the capacity of E with respect to this kernel by $\mathrm{cap}_f(E)$.

▷ **Exercise 15.2.**

Suppose that f and g are two gauge functions such that $f/c_1 - c_2 \leq g \leq c_1 f + c_2$ for some constants c_1 and c_2. Show that for all E, $\mathrm{cap}_f(E) > 0$ iff $\mathrm{cap}_g(E) > 0$.

The gauge functions $f(z) = z^{-\alpha}$ $(\alpha > 0)$ or $f(z) = \log 1/z$ are so frequently used that the corresponding capacities have their own notation, *viz.*, cap_α and cap_0. For any set E, there is a critical α_0 such that $\mathrm{cap}_\alpha(E) > 0$ for $\alpha < \alpha_0$ and $\mathrm{cap}_\alpha(E) = 0$ for $\alpha > \alpha_0$. In fact, $\alpha_0 = \dim E$. This result is due to Frostman (1935); we will deduce it from our work on trees. We will also show the following result.

Theorem 15.4. (Benjamini and Peres, 1992) *Critical homesick random walk on $T_{[b]}(E)$ has the same type for all b; it is transient iff $\mathrm{cap}_{\dim E}(E) > 0$.*

(Actually, Benjamini and Peres stated this only for simple random walk and $\mathrm{cap}_0 E$. That case bears a curious relation to a result of Kakutani (1944): $\mathrm{cap}_0 E > 0$ iff Brownian motion in $\mathbb{R}^2$ hits E a.s.; see Lemma 15.10.) We need the following important proposition that relates energy on trees to energy in Euclidean space. It is due to Benjamini and Peres (1992), as extended by Pemantle and Peres (1995b).

Proposition 15.5. (Energy and Capacity Transference) *Let $r_n > 0$ satisfy $\sum r_n = \infty$. Give $T_{[b]}(E)$ the conductances $c\big(e(u)\big) := r_{|u|}^{-1}$ and suppose that $f \geq 0$ is decreasing and satisfies*

$$f(b^{-n-1}) = \sum_{1 \leq k \leq n} r_k\,.$$

If $\nu \in \mathsf{Prob}(E)$ and $\mu \in \mathsf{Prob}\big(\partial T_{[b]}(E)\big)$ are continuous and satisfy $\theta\big(e(u)\big) = \nu(I_u)$, where θ is the flow corresponding to μ, then

$$b^{-1}\mathscr{E}_c(\theta) \leq \mathscr{E}_f(\nu) \leq 3\mathscr{E}_c(\theta)\,.$$

Hence $(T_{[b]}(E), c)$ *is transient iff* $\operatorname{cap}_f(E) > 0$.

Proof of Theorem 15.4. Critical random walk uses the conductances $c\big(e(u)\big) = \lambda_c^{-|u|}$, where $\lambda_c = \operatorname{br} T_{[b]}(E) = b^{\dim E}$. Hence, set $r_n := b^{n \dim E}$ in Proposition 15.5. According to the proposition, RW_{λ_c} is transient iff $\operatorname{cap}_f E > 0$. Since there are constants c_1 and c_2 such that $f(z)/c_1 - c_2 \le z^{\dim E} \le c_1 f(z) + c_2$ when $\dim E' > 0$ and $f(z)/c_1 - c_2 \le \log z \le c_1 f(z) + c_2$ when $\dim E = 0$, we have $\operatorname{cap}_f E > 0$ iff $\operatorname{cap}_{\dim E}(E) > 0$ by Exercise 15.2. ◀

Corollary 15.6. (Frostman, 1935) $\dim E = \inf\{\alpha\,;\ \operatorname{cap}_\alpha(E) = 0\}$.

▷ **Exercise 15.3.**
Prove this.

Proof of Proposition 15.5. If $t \le b^{-N}$, then $f(t) \ge f(b^{-N})$, whence

$$f(t) \ge \sum_{n \ge 2} r_{n-1} \mathbf{1}_{\{b^{-n} \ge t\}}\,,$$

so

$$\begin{aligned}\mathscr{E}(\nu) &= \iint f\left(|x-y|\right)\, d\nu(x)\, d\nu(y) \ge \iint \sum_{n\ge 2} r_{n-1} \mathbf{1}_{\{|x-y| \le b^{-n}\}}\, d\nu(x)\, d\nu(y) \\ &= \sum_{n \ge 2} r_{n-1} (\nu \times \nu)\{|x-y| \le b^{-n}\}\,.\end{aligned}$$

Now

$$\{|x-y| \le b^{-n}\} \supseteq \bigcup_{k=0}^{b^n - 1} I_k^n \times I_k^n\,, \tag{15.3}$$

where $I_k^n := \left[\frac{k}{b^n}, \frac{k+1}{b^n}\right]$, whence by the arithmetic-quadratic mean inequality (or the Cauchy-Schwarz inequality),

$$\begin{aligned}\mathscr{E}(\nu) &\ge \sum_{n\ge 2} r_{n-1} \sum_k (\nu I_k^n)^2 = \sum_{n \ge 2} r_{n-1} \sum_{|u|=n} \theta(u)^2 = \sum_{n\ge 2} r_{n-1} \sum_{|u|=n-1} \sum_{u \to x} \theta(x)^2 \\ &\ge \sum_{n \ge 2} r_{n-1} \sum_{|u|=n-1} \frac{1}{b} \left(\sum_{u \to x} \theta(x) \right)^2 = \frac{1}{b} \sum_{n\ge 2} r_{n-1} \sum_{|u|=n-1} \theta(u)^2 = \frac{1}{b} \mathscr{E}(\theta)\,.\end{aligned}$$

On the other hand, $f(t) \le f(b^{-N-1})$ when $t \ge b^{-N-1}$, i.e.,

$$f(t) \le \sum_{n \ge 1} r_n \mathbf{1}_{\{b^{-n} > t\}}\,.$$

Therefore

$$\begin{aligned}\mathscr{E}(\nu) &= \iint f\left(|x-y|\right)\,d\nu(x)\,d\nu(y) \le \iint \sum_{n\ge 1} r_n \mathbf{1}_{\{|x-y|<b^{-n}\}}\,d\nu(x)\,d\nu(y)\\ &= \sum_{n\ge 1} r_n(\nu\times\nu)\{|x-y|<b^{-n}\}\,.\end{aligned}$$

Now

$$\{|x-y|\le b^{-n}\} \subseteq \bigcup_{k=0}^{b^n-1}\left(I_k^n \times (I_k^n \cup I_{k-1}^n \cup I_{k+1}^n)\right). \tag{15.4}$$

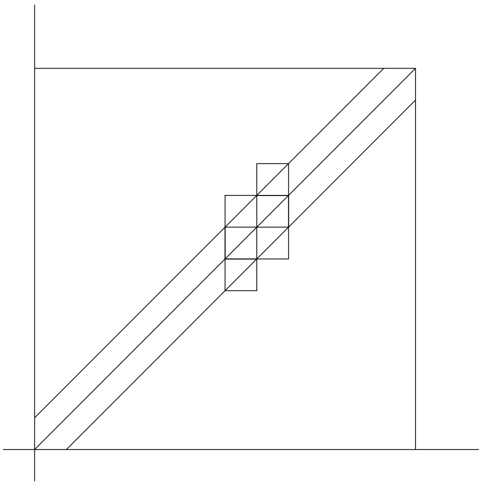


Figure 15.1.

(See Figure 15.1.) This gives the bound

$$(\nu\times\nu)\{|x-y|\le b^{-n}\} \le \sum_{k=0}^{b^n-1}(\nu I_k^n)(\nu I_k^n + \nu I_{k-1}^n + \nu I_{k+1}^n)\,.$$

The cross terms are estimated by the following inequality:

$$AB \le \frac{A^2+B^2}{2}\,.$$

Thus

$$\begin{aligned}(\nu\times\nu)\{|x-y|\le b^{-n}\} &\le \sum_{k=0}^{b^n-1}(\nu I_k^n)^2 + \sum_{k=0}^{b^n-1}(\nu I_k^n)^2 + \sum_{k=0}^{b^n-1}\frac{(\nu I_{k-1}^n)^2+(\nu I_{k+1}^n)^2}{2}\\ &\le 3\sum_{|u|=n}(\nu I_u)^2 = 3\sum_{|u|=n}\theta(u)^2\,,\end{aligned}$$

whence

$$\mathscr{E}(\nu) \le 3 \sum_{n \ge 1} r_n \sum_{|u|=n} \theta(u)^2 = 3\mathscr{E}(\theta)\,. \qquad \blacktriangleleft$$

Question 15.7. Let E be the Cantor middle-thirds set, μ_b harmonic measure of simple random walk on $T_{[b]}(E)$, and ν_b the corresponding measure on E. Thus ν_3 is Cantor-Lebesgue measure. Is $\nu_2 \perp \nu_3$?

The situation in higher dimensional Euclidean space is very similar. Denote the Euclidean distance between x and y by $|x-y|$. Consider a closed set $E \subseteq [0,1]^d$ and a kernel $K(x,y) := f(|x-y|)$, where $f \ge 0$ and f is decreasing. Again, f is called the ***gauge*** function and we denote the capacity of E with respect to f by $\operatorname{cap}_f(E)$. The tree $T_{[b]}(E)$ is now formed from the b-adic cubes in $[0,1]^d$.

Proposition 15.8. (Benjamini and Peres, 1992) *Let $r_n > 0$ satisfy $\sum r_n = \infty$. Give $T_{[b]}(E)$ the conductances $c\big(e(u)\big) := r_{|u|}^{-1}$ and suppose that f is decreasing and satisfies*

$$f(b^{-n-1}) = \sum_{1 \le k \le n} r_k\,.$$

If $\nu \in \mathsf{Prob}(E)$ and $\mu \in \mathsf{Prob}\big(\partial T_{[b]}(E)\big)$ are continuous and satisfy $\theta\big(e(u)\big) = \nu(I_u)$, where θ is the flow corresponding to μ, then

$$b^{-d(1+\ell)}\mathscr{E}_c(\theta) \le \mathscr{E}_f(\nu) \le 3^d \mathscr{E}_c(\theta)\,,$$

where $\ell := (1/2)\log_b d$.

▷ **Exercise 15.4.**

Prove Proposition 15.8.

§15.4. Fractal Percolation and Intersections.

In this section, we will consider a special case, known as fractal percolation, of the Galton-Watson fractals defined in Example 14.13. We will see how the use of capacity can help us understand intersection properties of Brownian motion traces in $\mathbb{R}^d$ by transferring the question to these much simpler fractal percolation sets. In the next section, we continue by studying connectivity properties of fractal percolation sets. **** Some small corrections are still needed for the case $d = 2$. ****

Given integers $d, b \geq 2$ and probabilities $\langle p_n \,;\; n \geq 1 \rangle$, consider the natural tiling of the unit cube $[0,1]^d$ by b^d closed cubes of side $1/b$. Let Z_1 be a random subcollection of these cubes, where each cube has probability p_1 of belonging to Z_1, and these events are mutually independent. (Thus the cardinality $|Z_1|$ of Z_1 is a binomial random variable.) In general, if Z_n is a collection of cubes of side b^{-n}, tile each cube $Q \in Z_n$ by b^d closed subcubes of side b^{-n-1} (with disjoint interiors) and include each of these subcubes in Z_{n+1} with probability p_{n+1} (independently). Finally, define

$$A_n := A_{n,d,b}(p_n) := \bigcup Z_n \qquad \text{and} \qquad Q_d\big(\langle p_n \rangle\big) := Q_{d,b}\big(\langle p_n \rangle\big) := \bigcap_{n=1}^{\infty} A_n \,.$$

In the construction of $Q_d\big(\langle p_n \rangle\big)$, the cardinalities $|Z_n|$ of Z_n form a branching process in a varying environment, where the offspring distribution at level n is $\mathrm{Bin}(b^d, p_n)$. When $p_n \equiv p$, the cardinalities form a Galton-Watson branching process. Alternatively, the successive subdivisions into b-ary subcubes define a natural mapping from b^d-ary tree to the unit cube; the construction of $Q_d\big(\langle p_n \rangle\big)$ corresponds to performing independent percolation with parameter p_n at level n on this tree and considering the set of infinite paths emanating from the root in its percolation cluster. The process $\langle A_n \,;\; n \geq 1 \rangle$ is called ***fractal percolation***, while $Q_d\big(\langle p_n \rangle\big)$ is the ***limit set***. When $p_n \equiv p$, we write $Q_d(p)$ for $Q_d\big(\langle p_n \rangle\big)$.

▷ **Exercise 15.5.**

(a) Show that $p_n \leq b^{-d}$ for all n implies that $Q_{d,b}\big(\langle p_n \rangle\big) = \varnothing$ a.s.

(b) Compute the almost sure Hausdorff dimension of $Q_d(p)$ given that it is non-empty.

(c) Characterize the sequences $\langle p_n \rangle$ for which $Q_d\big(\langle p_n \rangle\big)$ has positive volume with positive probability.

The main result in this section, from Peres (1996), is that the Brownian trace is "intersection equivalent" to the limit set of fractal percolation for an appropriate choice of parameters $\langle p_n \rangle$. We will obtain as easy corollaries facts about Brownian motion, for example, that two traces in 3-space intersect, but not three traces.

Definition 15.9. Two random (Borel) sets A and B in $\mathbb{R}^d$ are ***intersection equivalent*** in a set U if for every closed set $\Lambda \subseteq U$, we have

$$\mathbf{P}(A \cap \Lambda \neq \varnothing) \asymp \mathbf{P}(B \cap \Lambda \neq \varnothing)\,, \tag{15.5}$$

where the symbol $\asymp$ means that the ratio of the two sides is bounded above and below by positive constants that do not depend on Λ.

In fact, intersection equivalence of A and B implies that (15.5) holds for all Borel sets Λ by the Choquet Capacitability Theorem (see Carleson (1967), p. 3, or Dellacherie and Meyer (1978), III.28.) In the preceding definition, a random set A is a function $\omega \mapsto A(\omega)$ on a measurable space Ω whose values are sets; this function must satisfy the property that for every closed Λ, the outcomes ω where $A(\omega) \cap \Lambda \neq \varnothing$ form an event (i.e., a measurable subset of Ω).

If $f \geq 1$ is a non-increasing function and $\langle p_n \rangle$ are probabilities satisfying

$$p_1 \cdots p_n = f(b^{-n})^{-1}\,, \tag{15.6}$$

then the limit set of fractal percolation with retention probability p_n at level n will be denoted by $Q_d[f]$.

We will write $\mathrm{cap}(\,\bullet\,; f) := \mathrm{cap}_f(\bullet)$ in order to make it easier to see the changes we will consider in the gauge function f.

If g_d is the d-dimensional ***radial potential*** function,

$$g_d(r) := \begin{cases} \log^+(r^{-1}) & \text{if } d = 2, \\ r^{2-d} & \text{if } d \geq 3, \end{cases}$$

then the kernel $G_d(x, y) = c_d g_d\big(|x - y|\big)$, where $c_d > 0$ is a constant given in Exercise 15.6, is the ***Green kernel*** for Brownian motion. This is the continuous analogue of the Green function for Markov chains: As Exercise 15.6 shows, the expected 1-dimensional Lebesgue measure of the time that Brownian motion spends in a set turns out to be absolutely continuous with respect to d-dimensional Lebesgue measure, and its density at y when started at x is the Green kernel $G_d(x, y)$. Actually, in two dimensions, recurrence of Brownian motion means that we need to kill it at some finite time. If we kill it at a random time with an exponential distribution, then it remains a Markov process. It is not hard to compare properties of the exponentially-killed process to one that is killed at a fixed time: see Exercise 15.14. Note that the probabilities $\langle p_n \rangle$ satisfying (15.6) for $f = g_d$ are

$$\begin{cases} p_1 = 1/\log b,\ \ p_n = (n-1)/n \text{ for } n \geq 2 & \text{if } d = 2, \\ p_n = b^{2-d} & \text{if } d \geq 3. \end{cases}$$

For $d = b = 2$, this definition of p_1 is not a probability, so we need to take $f = g_2/(\log 2)$ and $p_1 = 1$. With a slight sloppiness of notation, we will understand this modified function g_2 whenever we write $Q_2[g_2]$ or $\mathrm{cap}(\Lambda; g_2)$.

The following exercise shows how the Green function arises in the theory of Brownian motion. For more information on Brownian motion, see the book by Mörters and Peres (2010), which is the text closest in spirit to the present book. Further references can also be found in the notes at the end of this chapter.

▷ **Exercise 15.6.**
Let $p_t(x, y) = (2\pi t)^{-d/2} \exp\big(-|x-y|^2/(2t)\big)$ be the Brownian transition density function, and for $d \geq 3$, define $G_d(x, y) := \int_0^\infty p_t(x, y)\,dt$ for $x, y \in \mathbb{R}^d$.

(a) Show that $G_d(x, y) = c_d g_d(|x - y|)$, with a constant $0 < c_d < \infty$.

(b) Define $F_B(x) = \int_B G_d(x, z)\,dz$ for any Borel set $B \subseteq \mathbb{R}^d$. Show that $F_B(x)$ is the expected time the Brownian motion started at x spends in B.

(c) Show that $x \mapsto G_d(0, x)$ is harmonic on $\mathbb{R}^d \setminus \{0\}$.

(d) Consider Brownian motion in $\mathbb{R}^2$, killed at a random time with an exponential distribution with parameter 1. Prove the "analogues" of the previous statements.

▷ **Exercise 15.7.**
Let $\Lambda \subset \mathbb{R}^d$ be a k-dimensional cube of side-length t, where $1 \leq k \leq d$. Find the capacity $\mathrm{cap}(\Lambda; g_d)$ up to constant factors depending on k and d only.

The following is a classical result relating the hitting probability of a set Λ by Brownian motion to the capacity of Λ in the Green kernel. A proof using the so-called Martin kernel and the second moment method was found by Benjamini, Pemantle, and Peres (1995); see Section 15.7.

Lemma 15.10. *If B is Brownian motion (killed at an exponential time for $d = 2$), and the initial distribution ν has a bounded density on $\mathbb{R}^d$, then*

$$\mathbf{P}_\nu(\exists t \geq 0 : B_t \in \Lambda) \asymp \mathrm{cap}(\Lambda; g_d) \tag{15.7}$$

for any Borel set $\Lambda \subset [0, 1]^d$.

Let B and B' be two independent Brownian motions, and write $[B]$ for $\{B_t\,;\ t \geq 0\}$. Paul Lévy asked the following question:

$$\text{For which } \Lambda \text{ is } \mathbf{P}\big[\Lambda \cap [B_1] \cap [B_2] \neq \varnothing\big] > 0\,?$$

Evans (1987) and Tongring (1988) gave a partial answer to this question:

$$\text{If } \operatorname{cap}(\Lambda; g_d^2) > 0, \text{ then } \mathbf{P}\big[\Lambda \cap [B_1] \cap [B_2] \neq \varnothing\big] > 0\,.$$

Later Fitzsimmons and Salisbury (1989) gave the full answer: $\operatorname{cap}(\Lambda; g_d^2) > 0$ is also necessary. Moreover, in dimension $d = 2$,

$$\operatorname{cap}(\Lambda; g_2^k) > 0 \text{ if and only if } \mathbf{P}\big[\Lambda \cap [B_1] \cap \cdots \cap [B_k] \neq \varnothing\big] > 0\,.$$

Chris Bishop then made the following conjecture in any $\mathbb{R}^d$:

$$\mathbf{P}\big[\operatorname{cap}(\Lambda \cap [B]; f) > 0\big] > 0 \text{ if and only if } \operatorname{cap}(\Lambda; fg_d) > 0\,. \tag{15.8}$$

We will establish all of the above.

Consider the canonical map F from the boundary ∂T of the b^d-ary tree T onto $[0,1]^d$. Observe that $a \in [0,1]^d$ is in $Q_d\big(\langle p_n\rangle\big)$ if and only if $F^{-1}(a) \subset \partial T$ is connected to the root in the percolation on T with the probability of retaining e equal to p_n when e joins vertices at level $n-1$ to vertices at level n. This fact, along with the percolation result Theorem 15.3 and the transfer result Proposition 15.5, are the ingredients of the following theorem.

Theorem 15.11. (Peres, 1996) *Let $f \geq 1$ be a non-increasing function with $f(0^+) = \infty$. Then for any closed set $\Lambda \subseteq [0,1]^d$,*

$$\operatorname{cap}(\Lambda; f) \asymp \mathbf{P}\big[\Lambda \cap Q_d[f] \neq \varnothing\big]\,. \tag{15.9}$$

For $f = g_d$ in particular, $Q_d[g_d]$ is intersection equivalent in $[0,1]^d$ to the Brownian trace: If B is Brownian motion with initial distribution ν that has a bounded density, stopped at an exponential time when $d = 2$, and $[B] := \{B_t\,;\ t \geq 0\}$, then for all $\Lambda \subseteq [0,1]^d$,

$$\mathbf{P}\big[\Lambda \cap Q_d[g_d] \neq \varnothing\big] \asymp \mathbf{P}\big[\Lambda \cap [B] \neq \varnothing\big]\,. \tag{15.10}$$

Proof. Let $\langle p_n\rangle$ be determined by (15.6). Let $\widetilde{\mathbf{P}}$ be the percolation on T with retention probability p_n for each edge joining level $n-1$ to level n. By Theorem 15.3,

$$\mathbf{P}\big[Q_d[f] \cap \Lambda \neq \varnothing\big] = \widetilde{\mathbf{P}}\big[o \leftrightarrow F^{-1}(\Lambda)\big] \asymp \operatorname{cap}\big(F^{-1}(\Lambda); f\big)\,, \tag{15.11}$$

where the constants in $\asymp$ on the right-hand side are 1 and 2. On the other hand, Proposition 15.5 says that

$$\operatorname{cap}\big(F^{-1}(\Lambda); f\big) \asymp \operatorname{cap}(\Lambda; f)\,. \tag{15.12}$$

Combining (15.11) and (15.12) yields (15.9). From (15.9) we get (15.10) using (15.7). ◀

Lemma 15.12. *Suppose that $A_1, \ldots, A_k, F_1, \ldots, F_k$ are independent random Borel sets, with A_i intersection equivalent to F_i for $1 \le i \le k$. Then $A_1 \cap A_2 \cap \cdots \cap A_k$ is intersection equivalent to $F_1 \cap F_2 \cap \cdots \cap F_k$.*

Proof. By induction, reduce to the case $k = 2$. It clearly suffices to show that $A_1 \cap A_2$ is intersection equivalent to $F_1 \cap A_2$, and this is done by conditioning on A_2:

$$\begin{aligned}\mathbf{P}[A_1 \cap A_2 \cap \Lambda \neq \varnothing] &= \mathbf{E}\big[\mathbf{P}[A_1 \cap A_2 \cap \Lambda \neq \varnothing \mid A_2]\big] \\ &\asymp \mathbf{E}\big[\mathbf{P}[F_1 \cap A_2 \cap \Lambda \neq \varnothing \mid A_2]\big] = P[F_1 \cap A_2 \cap \Lambda \neq \varnothing]\,.\end{aligned}$$ ◀

Lemma 15.13. *For any $0 < p, q < 1$, if $Q_d(p)$ and $Q'_d(q)$ are independent, then their intersection $Q_d(p) \cap Q'_d(q)$ has the same distribution as $Q_d(pq)$.*

Proof. This is immediate from the construction of $Q_d(p)$. ◀

Now we can start reaping the corollaries. In what follows, Brownian paths will be started either from arbitrary fixed points, or from initial distributions that have bounded densities in the unit cube $[0, 1]^d$. The proof of the corollary was first completed in Dvoretzky, Erdős, Kakutani, and Taylor (1957), following earlier work of Dvoretzky, Erdös, and Kakutani (1950). A proof using the renormalization group method was given by Aizenman (1985).

Corollary 15.14. (Dvoretzky, Erdős, Kakutani, Taylor)

(i) *For all $d \ge 4$, two independent Brownian traces in $\mathbb{R}^d$ are disjoint a.s. except, of course, at their starting point if they are identical.*

(ii) *In $\mathbb{R}^3$, two independent Brownian traces intersect a.s., but three traces a.s. have no points of mutual intersection (except, possibly, their starting point).*

(iii) *In $\mathbb{R}^2$, any finite number of independent Brownian traces have nonempty mutual intersection almost surely.*

Proof. (i) The distribution of B_ϵ has bounded density. By Theorem 15.11 and Lemma 15.12, the intersection of two independent copies of $\{B_t\,;\ t \ge \epsilon\}$ is intersection equivalent to the intersection of two independent copies of $Q_{4,b}(b^{-2})$; by Lemma 15.13, this latter intersection is intersection equivalent to $Q_{4,b}(b^{-4})$. But $Q_{4,b}(b^{-4})$ is a.s. empty because critical branching processes die out. Thus two independent copies of $\{B_t\,;\ t \ge \epsilon\}$ are a.s. disjoint. This argument actually worked for any cube of any size, whence two independent copies of $\{B_t\,;\ t \ge \epsilon\}$ are a.s. disjoint. Since ϵ is arbitrary, the claim follows.

(ii) Since $\{B_t\,;\ t \ge \epsilon\}$ is intersection equivalent to $Q_{3,b}(b^{-1})$ in the unit cube, the intersection of three independent Brownian traces (from any time $\epsilon > 0$ on) is intersection

equivalent in the cube to the intersection of three independent copies of $Q_3(b^{-1})$, which has the same distribution as $Q_3(b^{-3})$. Again, a critical branching process is obtained, and hence the triple intersection is a.s. empty.

On the other hand, the intersection of two independent copies of $\{B_t\,;\ t \ge \epsilon\}$ is intersection equivalent to $Q_{3,b}(b^{-2})$ in the unit cube for any positive ϵ. Since $Q_3(b^{-2})$ is defined by a supercritical branching process, we have $p(\epsilon, \infty) > 0$, where

$$p(u, v) := \mathbf{P}\big[\{B_t\,;\ u < t < v\} \cap \{B'_s\,;\ u < s < v\} \neq \varnothing\big]\,.$$

Here, B' is an independent copy of B. Suppose now first that the two Brownian motions are started from the same point; we will show that $p(1, \infty) = 1$. Note that $p(\epsilon, \infty) \uparrow p(0, \infty)$ as $\epsilon \downarrow 0$, hence $p(0, \infty) > 0$. Furthermore, $p(0, v) \to p(0, \infty)$ as $v \to \infty$. On the other hand, by Brownian scaling, $p(0, v)$ is independent of $v > 0$ and $p(u, \infty)$ is independent of $u > 0$. Therefore, $p(0, v) = p(u, \infty) = p(0, \infty) > 0$ for all $u, v > 0$, and

$$\lim_{v \downarrow 0} p(0, v) = \mathbf{P}\Big[0 = \inf\big\{v > 0\,;\ \{B_t\,;\ 0 < t < v\} \cap \{B'_s\,;\ 0 < s < v\} \neq \varnothing\big\}\Big] > 0\,.$$

This last positive probability, by Blumenthal's 0-1 law (see Durrett (2005), Section 7.2, or Mörters and Peres (2010), Section 2.1), has to be 1, thus $p(0, 1) = p(1, \infty) = p(0, \infty) = 1$.

When the Brownian motions are started at different points, or from any initial distributions, then at unit time, their joint distribution is absolutely continuous with respect to the joint distribution of Brownian motions at unit time, started from a joint initial point. For these latter Brownian motions, we know already that $p(1, \infty) = 1$, so we have this for the original motions as well. ◀

▷ **Exercise 15.8.**
Prove part (iii) of the above Corollary.

We now show how Theorem 15.11 leads to a proof of (15.8).

Corollary 15.15. *Let f and h be nonnegative and non-increasing functions. If a random closed set A in $[0, 1]^d$ satisfies*

$$\mathbf{P}[A \cap \Lambda \neq \varnothing] \asymp \operatorname{cap}(\Lambda; h) \tag{15.13}$$

for all closed $\Lambda \subseteq [0, 1]^d$, then

$$\mathbf{E}\big[\operatorname{cap}(A \cap \Lambda; f)\big] \asymp \operatorname{cap}(\Lambda; fh) \tag{15.14}$$

for all closed $\Lambda \subseteq [0,1]^d$*. In particular, for each* Λ*,*

$$\mathbf{P}\big[\mathrm{cap}(A\cap\Lambda; f) > 0\big] > 0 \ \textit{ if and only if } \ \mathrm{cap}(\Lambda; fh) > 0$$

and (15.8) follows with $A = [B]$ *and* $h = g_d$*.*

Proof. Enlarge the probability space on which A is defined to include two independent fractal percolations, $Q_d[f]$ and $Q_d[h]$. Because

$$\mathbf{P}\big[A\cap\Lambda\cap Q_d[f] \neq \varnothing\big] \ = \ \mathbf{E}\Big[\mathbf{P}\big[A\cap\Lambda\cap Q_d[f] \neq \varnothing \mid A\big]\Big]\,,$$

and by Theorem 15.11

$$\mathbf{P}\big[A\cap\Lambda\cap Q_d[f] \neq \varnothing \mid A\big] \asymp \mathrm{cap}(A\cap\Lambda; f)\,,$$

we have

$$\mathbf{E}\big[\mathrm{cap}(A\cap\Lambda; f)\big] \asymp \mathbf{P}\big[A\cap\Lambda\cap Q_d[f] \neq \varnothing\big]\,. \tag{15.15}$$

Conditioning on $Q_d[f]$, and then using (15.13) with $\Lambda\cap Q_d[f]$ in place of Λ gives

$$\mathbf{P}\big[A\cap\Lambda\cap Q_d[f] \neq \varnothing\big] \asymp \mathbf{E}\big[\mathrm{cap}(\Lambda\cap Q_d[f]; h)\big]\,. \tag{15.16}$$

Conditioning on $Q_d[f]$ and then applying Theorem 15.11 yields

$$\mathbf{E}\Big[\mathrm{cap}\big(\Lambda\cap Q_d[f]; h\big)\Big] \asymp \mathbf{P}\big[\Lambda\cap Q_d[f]\cap Q_d[h] \neq \varnothing\big]\,. \tag{15.17}$$

Note that $Q_d[f]\cap Q_d[h]$ has the same distribution as $Q_d[fh]$, and use again Theorem 15.11 to deduce that

$$\mathbf{P}\big[\Lambda\cap Q_d[f]\cap Q_d[h] \neq \varnothing\big] \asymp \mathrm{cap}(\Lambda; fh)\,. \tag{15.18}$$

Combining (15.15), (15.16), (15.17) and (15.18) proves (15.14). ◀

§15.5. Left-to-Right Crossing in Fractal Percolation.

We have seen in the last section how useful fractal percolation can be. In this section, we study it some more and prove the following theorem of Chayes, Chayes, and Durrett (1988): if the fixed probability $0 < p < 1$ is large enough, and $b \geq 2$, then the limit set of the planar fractal percolation $Q_2(p)$ contains a ***monotonic left-to-right crossing*** of the unit square, i.e., a continuous path $\big(x(t), y(t)\big) : [0,1] \to [0,1]^2$ such that $x(0) = 0$, $x(1) = 1$, and $x(s) \leq x(t)$ whenever $s < t$.

Lemma 15.16. *Consider fractal percolation with $b \geq 3$ in the plane. If each square retained at level n always contains at least $b^2 - 1$ surviving subsquares at level $n+1$, then there is a monotonic left-to-right crossing of squares (defined precisely below in the proof) at all levels n.*

Proof. We give the proof for $b = 3$; the other cases are similar. Suppose there is a ***monotonic crossing*** of squares at level n, that is, there is a sequence of squares $a_1, \ldots, a_r$ contained in Z_n so that a_1 shares a side with the left side of $[0,1]^2$, all successive squares a_j, a_{j+1} share a side other than the left side of a_j, and a_r shares a side with the right side of $[0,1]^2$. We say that a_j (for $j \geq 2$) is a ***horizontal segment*** of the path if a_{j-1} and a_j share a vertical side, and a ***vertical segment*** otherwise. See Figure 15.2 for a diagram of a monotonic crossing at level $n = 2$.

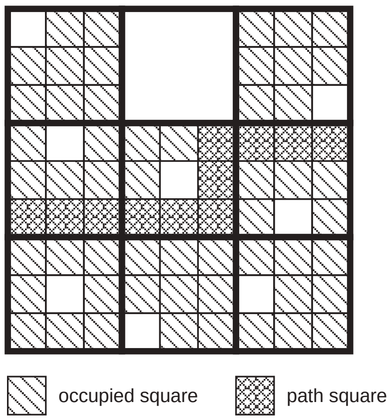


Figure 15.2. A monotonic crossing at level 2.

We now construct a path of squares in Z_{n+1} which also crosses $[0,1]^2$. We go by induction: It is obvious that a_1 contains a path in Z_{n+1} from its left side to its right side. Suppose that the left side of $[0,1]^2$ is connected to the right side of a_j by a path of squares in Z_{n+1} that are in $\bigcup_{i=1}^{j} a_i$. We now show that the left side of $[0,1]^2$ is connected to the right side of a_{j+1} by a path of squares in Z_{n+1} that are in $\bigcup_{i=1}^{j+1} a_i$.

Suppose first that a_{j+1} is a horizontal segment. There is always at least one row of squares in Z_{n+1} connecting the left side of a_j and the right side of a_{j+1}, and one column of squares in Z_{n+1} connecting the top and bottom of a_j—see Figure 15.3.

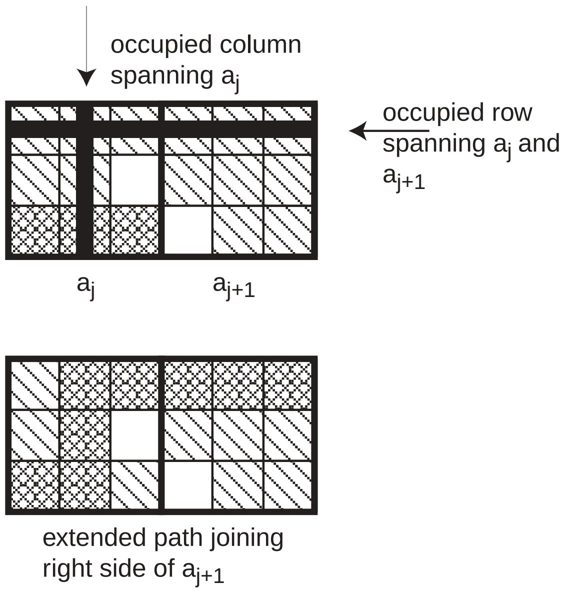


Figure 15.3. A horizontal crossing.

The path connecting the right side of a_j to the left side of $[0,1]^2$ by squares in Z_{n+1} must contain a square in this column spanning a_j. Thus using squares of Z_{n+1} contained in this column, the row spanning a_j and a_{j+1}, and the previous path, it is possible to create a path from the right side of a_{j+1} to the left side of $[0,1]^2$ using squares of Z_{n+1}.

Now suppose that a_{j+1} is a vertical segment. Analogously to the horizontal case just treated, there is a column of squares in Z_{n+1} spanning both a_j and a_{j+1}, and there is a row of squares in Z_{n+1} spanning a_{j+1}. The path of Z_{n+1}-squares joining the left side of $[0,1]^2$ to the right side of a_j must contain a square of this column. Using the column spanning a_j and a_{j+1}, the row in a_{j+1}, and the previous path, the right side of a_{j+1} can be connected to the left side of $[0,1]^2$ by a path of Z_{n+1}-squares—see Figure 15.4. ◀

▷ **Exercise 15.9.**
Is Lemma 15.16 true for $b = 2$?

Define $\theta_n(p)$ as the probability of a monotonic left-to-right crossing at level n, as defined in the proof of Lemma 15.16. The sequence θ_n is non-increasing in n, hence the limit $\theta_\infty(p)$, the chance of a monotonic crossing in A_∞, exists.

Theorem 5.24 and Lemma 15.16 can be combined to give an easy proof that there is a nontrivial phase where there exist monotonic left-to-right crossings in the limit set.

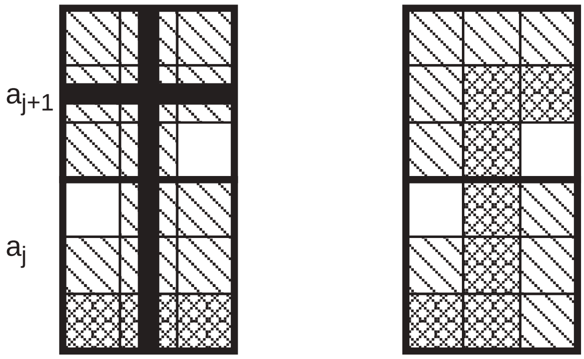


Figure 15.4. A vertical crossing.

Theorem 15.17. (Chayes, Chayes, and Durrett, 1988) *For p close enough to* 1*, the monotonic crossing probability $\theta_\infty(p)$ is positive.*

Proof. Assume first that $b \ge 3$. By Lemma 15.16, it suffices to show that with positive probability each square of Z_n contains at least b^2-1 subsquares in Z_{n+1}. This event occurs if and only if the tree associated with $\{Z_n\}$ contains a (b^2-1)-ary descendant subtree. Exercise 5.14 shows that such subtrees exist with positive probability provided p is large enough.

For the case $b = 2$, see the next exercise. ◀

▷ **Exercise 15.10.**

(a) Show that for $p \in (0,1)$, there exists $q \in (0,1)$ such that the first stage $A_{1,d,b^2}(p)$ of one fractal percolation is stochastically dominated (in the sense of Section 10.2 with respect to inclusion) by the second stage $A_{1,d,b}(q) \cap A_{2,d,b}(q)$ of another fractal percolation. *Hint:* Let $\mathrm{Bern}(q)$ denote a Bernoulli random variable with parameter q, i.e., a random variable that takes the value 1 with probability q and 0 otherwise. Take q so that $\mathrm{Bern}(q)$ dominates the maximum of b^d independent copies of a $\mathrm{Bern}(\sqrt{p})$ random variable.

(b) Deduce that Theorem 15.17 holds for $b = 2$ as well.

§15.6. Generalized Diameters and Average Meeting Height on Trees.

There are amusing alternative definitions of capacity that we present in some generality and then specialize to trees. Suppose that X is a compact Hausdorff* space and $K : X \times X \to [0, \infty]$ is continuous and symmetric. Let $\mathsf{Prob}(X)$ denote the set of Borel probability measures on X. For $\mu \in \mathsf{Prob}(X)$, set

$$V_\mu(x) := \int_X K(x, y)\, d\mu(y)\,, \qquad \mathscr{E}(\mu) := \int_{X \times X} K d(\mu \times \mu)\,,$$
$$\mathscr{E}(X) := \inf\{\mathscr{E}(\mu)\,;\ \mu \in \mathsf{Prob}(X)\}\,, \qquad \mathrm{cap}(X) := \mathscr{E}(X)^{-1}\,,$$

and define the ***Chebyshev constants***

$$M_n(X) := \max_{x_1, \ldots, x_n \in X} \min_{x \in X} \frac{1}{n} \sum_{k=1}^{n} K(x, x_k)\,,$$

and the ***generalized diameters***

$$D_n(X) := \min_{x_1, \ldots, x_n \in X} \frac{1}{\binom{n}{2}} \sum_{1 \le j < k \le n} K(x_j, x_k)\,.$$

[To see where generalized diameters get their name, consider the case $n = 2$ and $K(x, y) = 1/d(x, y)$ for a metric space. The original kernel, however, was $\log 1/d(x, y)$.]

Theorem 15.18. *In the preceding notation, we have:*

(a) $\mathscr{E}(X) = \inf\{\|V_\mu\|_{L^\infty(\mu)}\,;\ \mu \in \mathsf{Prob}(X)\}$. *If* $\mathscr{E}(X) < \infty$*, then* $\exists \mu \in \mathsf{Prob}(X)\quad V_\mu = \mathscr{E}(X)$ *μ-a.e.*

(b) (Fekete-Szegő) $D_n(X) \uparrow \mathscr{E}(X) = \lim_{n \to \infty} \inf\{M_n(\hat{X})\,;\ \hat{X} \subseteq X$ *is compact*$\}$. *Also,* $\mathscr{E}(X) = \lim_{n \to \infty} M_n(\hat{X})$ *for any compact* $\hat{X} \subseteq X$ *such that* $\exists \mu \in \mathsf{Prob}(\hat{X})\ \forall x \in \hat{X}\quad V_\mu(x) \le \mathscr{E}(X)$.

See the notes at the end of this chapter for a proof.

We have seen that in case X is the boundary of a tree T and $K(\xi, \eta) = \Psi(\xi \wedge \eta)$ as in Section 15.1, there is a measure, *viz.*, harmonic measure, such that $V_\mu \le \mathscr{E}(\partial T)$ everywhere. Hence

$$\mathscr{E}(\partial T) = \lim D_n(\partial T) = \lim M_n(\partial T)\,.$$

We transfer this from the boundary to the interior of T to obtain the following theorem.

* We shall have need only of metric spaces, but the proofs are no simpler than for Hausdorff spaces.

Theorem 15.19. (Benjamini and Peres, 1992) *If* $\forall \xi \in \partial T \quad \sum_{x \in \xi} c\big(e(x)\big)^{-1} = \infty$ *and T is locally finite, then the following are equivalent:*

(i) $\mathscr{C}(o \leftrightarrow \partial T) > 0$*;*

(ii) $\exists A < \infty \quad \forall n \geq 1 \quad \exists$ *distinct* $x_1, \ldots, x_n \in T$ *such that*

$$\binom{n}{2}^{-1} \sum_{1 \leq j < k \leq n} \mathscr{C}(o \leftrightarrow x_j \wedge x_k)^{-1} \leq A\,;$$

(iii) $\exists A < \infty \quad \forall n \geq 1 \quad \forall x_1, \ldots, x_n \in T \quad \exists u \in T \quad \forall k \quad u \not\leq x_k$ *and*

$$\frac{1}{n} \sum_{k=1}^{n} \mathscr{C}(o \leftrightarrow u \wedge x_k)^{-1} \leq A\,.$$

Remark. For simple random walk, i.e., $c \equiv 1$, we have $\mathcal{C}(o \to x \wedge u)^{-1} = |x \wedge u|$. Thus, the quantities in (ii) and (iii) are average meeting heights.

For example, consider simple random walk on the unary tree. This is recurrent, whence the three conditions in Theorem 15.19 are false. Indeed, it is not hard to see that given n distinct vertices, the average in (ii) is at least $(n+1)/2$, while even for $n = 1$, there is no bound on the quantity in (iii).

For a more interesting example, consider simple random walk on the binary tree, T. Now the three conditions are true. To illustrate (ii), suppose that n is a power of 2. Choose $x_1, \ldots, x_n$ to be the nth level of T, T_n. Then the quantity in (ii) is approximately equal to 1. To see that (iii) holds with $A = 1$, suppose that $H := \langle x_1, \ldots, x_n \rangle$ is given. Choose a child u_1 of the root of T such that T^{u_1} contains at most half of H. Then choose a child u_2 of u_1 such that T^{u_2} contains at most half of $H \cap T^{u_1}$. Continue in this way until we reach a vertex u_k with $H \cap T^{u_k} = \varnothing$. Let $u := u_k$. Since $u \wedge x = o$ for $x \in H \setminus T^{u_1}$, these vertices x contribute 0 to the left-hand side of (iii). Similarly, the vertices in $H \setminus T^{u_2}$ contribute at most $1/4$ to the left-hand side of (iii), and so on.

Remark. The proof will show that the smallest possible A in (ii), as well as in (iii), is $\mathscr{C}(o \leftrightarrow \partial T)^{-1}$.

Proof of Theorem 15.19. Assume (i). Use $K(\xi, t) := \mathscr{C}(o \leftrightarrow \xi \wedge \eta)^{-1}$. Then $D_n(\partial T) \leq \mathscr{E}(\partial T)$, so $\exists \xi_1, \ldots, \xi_n \in \partial T$ such that

$$\binom{n}{2}^{-1} \sum_{j<k} \mathscr{C}(o \leftrightarrow \xi_j \wedge \xi_k)^{-1} \leq \mathscr{E}(\partial T)\,.$$

From the hypothesis that $\forall \xi \in \partial T \;\; \mathscr{C}(o \leftrightarrow \xi)^{-1} = \infty$, the ξ_k are distinct. Hence $\exists x_k \in \xi_k$ distinct such that $x_j \wedge x_k = \xi_j \wedge \xi_k$, namely, pick $x_k \in \xi_k$ such that $|x_k| > \max_{i \neq j} |\xi_i \wedge \xi_j|$. This gives (ii) with $A = \mathscr{E}(\partial' T)$.

For the remainder of the proof of the theorem, we will need new trees T^* and T^{**}, created by adding a ray to each leaf or vertex of T, respectively, with all new edges having conductance 1. Since $\partial T^* \setminus \partial T$ and $\partial T^{**} \setminus \partial T$ are countable and $K(\xi, \xi) \equiv \infty$, we have $\mathrm{cap}(\partial T^*) = \mathrm{cap}(\partial T^{**}) = \mathrm{cap}(\partial T)$.

Assume (i) again. Given $x_1, \ldots, x_n \in T$, let $\xi_k \in \partial T^*$ such that $x_k \in \xi_k$. Since $M_n(\partial T^*) \le \mathscr{E}(\partial T^*) = \mathscr{E}(\partial T)$, there exists $\eta \in \partial T^*$ such that

$$\frac{1}{n} \sum_{k=1}^{n} \mathscr{C}(o \leftrightarrow \eta \wedge \xi_k)^{-1} \le \mathscr{E}(\partial T) \,.$$

Then $\forall k \;\; \eta \neq \xi_k$, so $\exists u \in \eta \cap T$ such that $\forall k \;\; u \wedge x_k \le \eta \wedge \xi_k$ and $u \not\le x_k$. This gives (iii).

Now assume (iii). Given $\xi_1, \ldots, \xi_n \in \partial T^*$, choose $x_k \in \xi_k \cap T$ so that in case $\xi_k \in \partial T^* \setminus \partial T$, then $x_k \in \partial T \setminus \partial T$, while if not, then $|x_k|$ is so large that $\mathscr{C}(o \leftrightarrow x_k)^{-1} > nA$. Let u be as asserted in (iii). Then $\forall k \;\; u \not\ge x_k$. Choose $\eta \in \partial T^*$ such that $u \in \eta$. Then $\forall k \;\; \eta \wedge \xi_k = u \wedge x_k$, so

$$\frac{1}{n} \sum_{k=1}^{n} \mathscr{C}(o \leftrightarrow \eta \wedge \xi_k)^{-1} \le A \,.$$

Therefore $\mathscr{E}(\partial T^*) \le A$, so $\mathrm{cap}(\partial T) > 0$.

Finally, assume (ii). Given n, let $x_1, \ldots, x_n$ be as in (ii). Choose $\xi_k \in \partial T^{**} \setminus \partial T$ such that $x_k \in \xi_k$. Then $\forall j < k \;\; \xi_j \wedge \xi_k = x_j \wedge x_k$, whence

$$\binom{n}{2}^{-1} \sum_{j<k} \mathscr{C}(o \leftrightarrow \xi_j \wedge \xi_k)^{-1} \le A \,.$$

Therefore $\mathscr{E}(\partial T^{**}) \le A$, so $\mathrm{cap}(\partial T) > 0$. ◀

§15.7. Notes.

For general background on Brownian motion, see Mörters and Peres (2010). Other nice references on potential theory are Bass (1995), Port and Stone (1978), and Sznitman (1998).

As noted by Benjamini, Pemantle, and Peres (1995), capacity in the ***Martin kernel*** $K_d(x, y) = G_d(x, y)/G_d(0, y)$ is better suited for studying Brownian hitting probabilities than the Green kernel $G_d(x, y)$. The reason is that while the Green kernel, and hence the corresponding capacity, are translation invariant, the hitting probability of a set Λ by standard d-dimensional Brownian motion is not translation invariant, but is invariant under scaling. This scale-invariance is shared by the Martin kernel $K_d(x, y)$. In particular, using the second moment method, Benjamini, Pemantle, and Peres (1995) proved the following result, where the bounds $1/2$ and 1 are the best possible. The case $d = 2$ is spelled out in Mörters and Peres (2010):

Theorem 15.20. *Let B be Brownian motion started at 0 in $\mathbb{R}^d$ for $d \ge 3$, or in $\mathbb{R}^2$ but killed at an exponential time, and let K_d be the Martin kernel. Then, for any closed set Λ in $\mathbb{R}^d$,*

$$\frac{1}{2} \le \frac{\mathbf{P}(\exists t \ge 0 : B_t \in \Lambda)}{\operatorname{cap}(\Lambda; K_d)} \le 1\,. \tag{15.19}$$

If the Brownian motion is started according to the measure ν, then (15.19) remains true with the Martin kernel $K_d(x, y) = G_d(x, y)/G_d(\nu, y)$, where

$$G_d(\nu, y) = \int G_d(x, y)\, d\nu(x)\,. \tag{15.20}$$

From this result, Lemma 15.10 is immediate:

Proof of Lemma 15.10. By Theorem 15.20, it is enough to show that the ratio of the Martin kernel to the Green kernel is bounded above and below. By definition of the Martin kernel, it suffices to check that the Greenian potential $G_d(\nu, y)$ defined in (15.20) is bounded. This is clearly the case when ν has a bounded density. ◀

Now we prove Theorem 15.18.

Proof of Theorem 15.18. (a) Since $\mathscr{E}(\mu) = \|V_\mu\|_{L^1(\mu)} \le \|V_\mu\|_{L^\infty(\mu)}$, we have $\mathscr{E}(X) \le \inf_\mu \|V_\mu\|_{L^\infty(\mu)}$. On the other hand, hand, suppose that $\mathscr{E}(X) < \infty$ (since otherwise certainly $\mathscr{E}(X) \ge \inf_\mu \|V_\mu\|_{L^\infty(\mu)}$). The space $\mathsf{Prob}(X)$ is Hausdorff. The definition of Hausdorff is equivalent to the diagonal $\Delta := \{(\mu, \mu) \in \mathsf{Prob}(X) \times \mathsf{Prob}(X)\}$ being closed in $\mathsf{Prob}(X) \times \mathsf{Prob}(X)$. Since $\forall t \in \mathbb{R}\ \ K \wedge t \in C(X \times X)$, we have

$$\Delta \cap \big\{(\mu, \nu) \in \mathsf{Prob}(X) \times \mathsf{Prob}(X)\,;\ \int K \wedge t\, d\mu \times \nu \le \mathscr{E}(X) + \epsilon\big\}$$

is compact and non-empty for each $\epsilon > 0$. Hence there is a measure μ in all these sets for $t < \infty$ and $\epsilon > 0$. This measure μ necessarily satisfies $\mathscr{E}(\mu) = \mathscr{E}(X)$.

We claim that $V_\mu \ge \mathscr{E}(X)$ μ-a.e.; this gives $V_\mu = \mathscr{E}(X)$ μ-a.e. since $\mathscr{E}(X) = \mathscr{E}(\mu) = \int V_\mu\, d\mu$. Suppose there were a set $F \subseteq X$ and $\delta > 0$ such that $\mu F > 0$ and $V_\mu \restriction F \le \mathscr{E}(X) - \delta$. Then move a little more of μ's mass to F: Let $\nu := \mu(\bullet \mid F)$ be the normalized restriction of μ to F and $\eta > 0$. We have $\mathscr{E}(\nu) < \infty$ and

$$\begin{aligned}\mathscr{E}((1-\eta)\mu + \eta\nu) &= (1-\eta)^2 \mathscr{E}(\mu) + \eta^2 \mathscr{E}(\nu) + 2(1-\eta)\eta \int V_\mu\, d\nu \\ &\le (1-\eta)^2 \mathscr{E}(X) + \eta^2 \mathscr{E}(\nu) + 2(1-\eta)\eta(\mathscr{E}(X) - \delta) \\ &\le \mathscr{E}(X) - 2\eta\delta + O(\eta^2)\,.\end{aligned}$$

Hence $\mathscr{E}((1-\eta)\mu + \eta\nu) < \mathscr{E}(X)$ for sufficiently small η, a contradiction.

(b) Regard $D_{n+1}(X)$ as an average of averages over all n-subsets of $\{x_1, \ldots, x_{n+1}\}$; each n-subset average is $\ge D_n(X)$, whence so is $D_{n+1}(X)$.

Claim: $D_{n+1}(X) \le M_n(X)$. Let $D_{n+1}(X) = \binom{n+1}{2}^{-1} \sum_{1 \le j < k \le n+1} K(x_j, x_k)$. For each j, write this as $f_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}) + n^{-1}(n+1)^{-1} \sum_{k \ne j} K(x_j, x_k)$. Only the second term depends on x_j, hence x_j is such that

$$\sum_{k \ne j} K(x_j, x_k) = \min_{x \in X} \sum_{k \ne j} K(x, x_k) \le n M_n(X)\,.$$

Therefore

$$D_{n+1}(X) = \frac{1}{n(n+1)} \sum_{j=1}^{n+1} \sum_{k \ne j} K(x_j, x_k) \le \frac{1}{n+1} \sum_{j=1}^{n+1} M_n(X) = M_n(X)\,.$$

Therefore for any compact $\hat{X} \subseteq X$, we have $D_{n+1}(X) \le D_{n+1}(\hat{X}) \le M_n(\hat{X})$, whence $D_{n+1}(X) \le \inf_{\hat{X}} M_n(\hat{X})$.

Claim: $\inf_{\hat{X}} M_n(\hat{X}) \le \mathscr{E}(X)$ and if $V_\mu \le \mathscr{E}(X)$ on $\hat{X}$ with $\mu\hat{X} = 1$, then $M_n(\hat{X}) \le \mathscr{E}(X)$. We may assume $\mathscr{E}(X) < \infty$. From part (a), $\exists \mu \in \mathsf{Prob}(X)$ $V_\mu = \mathscr{E}(X)$ μ-a.e. Since V_μ is lower semicontinuous by Fatou's lemma, the set $\hat{X}_0$ where $V_\mu \le \mathscr{E}(X)$ is compact and $\mu\hat{X}_0 = 1$. For any $\hat{X} \subseteq \hat{X}_0$ with $\mu\hat{X} = 1$ and any $x_k \in \hat{X}$, we have

$$\begin{aligned}\min_{x \in \hat{X}} \frac{1}{n} \sum_1^n K(x, x_k) &\le \int_{\hat{X}} \frac{1}{n} \sum_1^n K(x, x_k)\, d\mu(x) = \int_X \frac{1}{n} \sum_1^n K(x, x_k)\, d\mu(x) \\ &= \frac{1}{n} \sum_1^n V_\mu(x_k) \le \mathscr{E}(X)\,,\end{aligned}$$

i.e., $M_n(\hat{X}) \le \mathscr{E}(X)$.

Claim: $\mathscr{E}(X) \le \lim D_n(X)$. Let $D_n(X) = \binom{n}{2}^{-1} \sum_{1 \le j < k \le n} K(x_j, x_k)$ and $\mu_n := \sum_{j=1}^n \frac{1}{n} \delta(x_j)$. Let μ be a weak* limit point of $\{\mu_n\}$. We have for $t \in \mathbb{R}$,

$$\int K \wedge t\, d(\mu_n \times \mu_n) \le \frac{n-1}{n} D_n(X) + \frac{t}{n}\,,$$

whence

$$\int K \wedge t\, d(\mu \times \mu) \le \lim_{n \to \infty} D_n(X)\,,$$

whence

$$\mathscr{E}(\mu) \le \lim_{n \to \infty} D_n(X)\,.$$

Putting the claims together, we see that

$$D_{n+1}(X) \le \inf_{\hat{X}} M_n(\hat{X}) \le \mathscr{E}(X) \le \lim_{n \to \infty} D_n(X)\,,$$

from which the theorem follows. ◀

Remark. We have also seen that when $\mathscr{E}(X) < \infty$, a minimizing measure (called equilibrium measure) on X can be obtained as a weak* limit point of minimizing "sets" for $D_n(X)$.

§15.8. Collected In-Text Exercises.

15.1. Show that if $p_x \equiv p \in (0,1)$ and T is spherically symmetric, then

$$\operatorname{cap} \partial T = \left(1 + (1-p)\sum_{n=1}^{\infty} \frac{1}{p^n |T_n|}\right)^{-1} .$$

Thus, $\mathbf{P}[o \leftrightarrow \partial T] > 0$ iff $\sum_{n=1}^{\infty} \frac{1}{p^n |T_n|} < \infty$.

15.2. Suppose that f and g are two gauge functions such that $f/c_1 - c_2 \le g \le c_1 f + c_2$ for some constants c_1 and c_2. Show that for all E, $\operatorname{cap}_f(E) > 0$ iff $\operatorname{cap}_g(E) > 0$.

15.3. Prove Corollary 15.6.

15.4. Prove Proposition 15.8.

15.5. **(a)** Show that $p_n \le b^{-d}$ for all n implies that $Q_{d,b}(\langle p_n \rangle) = \varnothing$ a.s.
(b) Compute the almost sure Hausdorff dimension of $Q_d(p)$ given that it is non-empty.
(c) Characterize the sequences $\langle p_n \rangle$ for which $Q_d(\langle p_n \rangle)$ has positive volume with positive probability.

15.6. Let $p_t(x,y) = (2\pi t)^{-d/2} \exp\left(-|x-y|^2/(2t)\right)$ be the Brownian transition density function, and for $d \ge 3$, define $G_d(x,y) := \int_0^\infty p_t(x,y)\,dt$ for $x, y \in \mathbb{R}^d$.
(a) Show that $G_d(x,y) = c_d g_d(|x-y|)$, with a constant $0 < c_d < \infty$.
(b) Define $F_B(x) = \int_B G_d(x,z)\,dz$ for any Borel set $B \subseteq \mathbb{R}^d$. Show that $F_B(x)$ is the expected time the Brownian motion started at x spends in B.
(c) Show that $x \mapsto G_d(0,x)$ is harmonic on $\mathbb{R}^d \setminus \{0\}$.
(d) Consider Brownian motion in $\mathbb{R}^2$, killed at a random time with an exponential distribution with parameter 1. Prove the "analogues" of the previous statements.

15.7. Let $\Lambda \subset \mathbb{R}^d$ be a k-dimensional cube of side-length t, where $1 \le k \le d$. Find the capacity $\operatorname{cap}(\Lambda; g_d)$ up to constant factors depending on k and d only.

15.8. Prove part (iii) of Corollary 15.14.

15.9. Is Lemma 15.16 true for $b = 2$?

15.10. **(a)** Show that for $p \in (0,1)$, there exists $q \in (0,1)$ such that the first stage $A_{1,d,b^2}(p)$ of one fractal percolation is stochastically dominated (in the sense of Section 10.2 with respect to inclusion) by the second stage $A_{1,d,b}(q) \cap A_{2,d,b}(q)$ of another fractal percolation. *Hint:* Let $\mathrm{Bern}(q)$ denote a Bernoulli random variable with parameter q, i.e., a random variable that takes the value 1 with probability q and 0 otherwise. Take q so that $\mathrm{Bern}(q)$ dominates the maximum of b^d independent copies of a $\mathrm{Bern}(\sqrt{p})$ random variable.
(b) Deduce that Theorem 15.17 holds for $b = 2$ as well.

§15.9. Additional Exercises.

15.11. Consider a network on a tree T and a subset $E \subseteq \partial T$. Let μ_i be harmonic measure on ∂T. Show that $\operatorname{cap} E \ge \operatorname{cap}(\partial T) \cdot \mu_i(E)$, whence if random walk can hit E, so can percolation. Find T and $E \subseteq \partial T$ so that $\mu_i(E) = 0$ yet $\mathbf{P}[o \leftrightarrow E] > 0$ (so $\operatorname{cap} E > 0$).

15.12. Use a method similar to the proof of Theorem 5.21 to prove the following theorem of Benjamini, Pemantle, and Peres (1995). Let $\langle X_n \rangle$ be a transient Markov chain on a countable state space with initial distribution π and transition probabilities $p(x, y)$. Define the Green function $\mathcal{G}(x, y) := \sum_{n=1}^{\infty} p^{(n)}(x, y)$ and the Martin kernel $K(x, y) := \mathcal{G}(x, y)/\sum_z \pi(z)\mathcal{G}(z, y)$. Note that K may not be symmetric. Then with capacity defined using the kernel K, we have for any set S of states,

$$\frac{1}{2}\operatorname{cap}(S) \le \mathbf{P}[\exists n \ge 0\,;\; X_n \in S] \le \operatorname{cap}(S)\,.$$

See Benjamini, Pemantle, and Peres (1995) for various applications.

15.13. Show that if μ is a probability measure on $\mathbb{R}^d$ satisfying $\mu\big(B_r(x)\big) \le Cr^\alpha$ for some constants C and α and if $\beta < \alpha$, then the energy of μ in the gauge $z^{-\beta}$ is finite. By using this and previous exercises, give another proof of Corollary 15.6.

15.14. Let A be a set contained in the d-dimensional ball of radius a centered at the origin. Write $B(s, t)$ for the trace of d-dimensional Brownian motion during the time interval (s, t). Let τ be an exponential random variable with parameter 1 independent of B. Show that

$$e^{-1}\mathbf{P}_0[B(0, 1) \cap A \ne \varnothing] \le \mathbf{P}_0[B(0, \tau) \cap A \ne \varnothing] \le C\mathbf{P}_0[B(0, 1) \cap A \ne \varnothing]$$

for some constant C (depending on a and d). *Hint:* For the upper bound, first show the following general lemma: Let f_1, f_2 be densities on $[0, \infty)$. Suppose that the likelihood ratio $\psi(r) := \frac{f_2(r)}{f_1(r)}$ is increasing and $h : [0, \infty) \to [0, \infty)$ is decreasing on $[a, \infty)$. Then

$$\frac{\int_0^\infty h(r) f_2(r)\,dr}{\int_0^\infty h(r) f_1(r)\,dr} \le \psi(a) + \frac{\int_a^\infty f_2(r)\,dr}{\int_a^\infty f_1(r)\,dr}\,.$$

Second, use this by conditioning on $|B_{t_j}|$ for $j = 1, 2$ to get an upper bound on

$$\frac{\mathbf{P}_0[B(t_2, t_2 + s) \cap A \ne \varnothing]}{\mathbf{P}_0[B(t_1, t_1 + s) \cap A \ne \varnothing]}$$

for $t_1 \le t_2$ and $s \ge 0$. Third, bound $\mathbf{P}_0[B(0, \tau) \cap A \ne \varnothing]$ by summing over intersections in $[j/2, (j + 1)/2]$ for $j \in \mathbb{N}$.

15.15. Consider the following variant of the fractal percolation process, defined on all of $\mathbb{R}^3$. Tile space by unit cubes, all integer translations of the cube with center $\mathbf{0} = (0, 0, 0)$ and sides parallel to the coordinate axes. Generate a random collection Z_0 of these cubes by always including the cube containing 0, and including each of the remaining cubes independently with probability p. The collection Z_1 is generated by tiling each cube in Z_0 by 27 subcubes of side-length 1/3, and including each with probability p except the subcube containing 0, which is always retained. The collections Z_n for $n \ge 2$ are generated by repeating this process, appropriately rescaled.

Now tile $\mathbb{R}^3$ by cubes of side length 3, with one such cube centered at 0. Obtain the collection Z_{-1} by always including the cube centered at 0, and including each other cube independently with probability p. Tile $\mathbb{R}^3$ by cubes of side length 9, again with one such cube centered at 0. The collection Z_{-2} of cubes of side-length 9 always includes the cube centered at 0, and each other cube independently with probability p. Repeat this process to obtain Z_{-n} for $n \geq 2$.

As before, define

$$C_n := \bigcup Z_n \quad \text{and} \quad R(p) := \bigcap_{n \in \mathbb{Z}} C_n\,.$$

Show that for $p = 1/3$, $R(p)$ is intersection equivalent in all of $\mathbb{R}^3$ to the trace of Brownian motion started at 0. *Hint:* Use Theorem 15.20.

15.16. Let $A, B \subset [0,1]^d$ be two random Borel sets that are intersection equivalent in $[0,1]^d$, with α and β being the essential supremum of their Hausdorff dimensions. Prove that $\alpha = \beta$.

15.17. Remove the hypothesis from Theorem 15.19 that T be locally finite.

15.18. Give an example of a kernel on a space X such that $\mathscr{E}(X) < \infty$ yet for all $\mu \in \mathsf{Prob}(X)$, there is some $x \in X$ with $V_\mu(x) = \infty$.

15.19. Suppose that X is a compact Hausdorff space and $K : X \times X \to [0, \infty]$ is continuous and symmetric. Give $\mathsf{Prob}(X)$ the weak* topology. Show that $\mu \mapsto \mathscr{E}(\mu)$ is lower semicontinuous.

Chapter 16

Harmonic Measure on Galton-Watson Trees

This chapter is based on Lyons, Pemantle, and Peres (1995b, 1996a). Unless otherwise attributed, all results here are from those papers, especially the former. All trees in this chapter are rooted.

§16.1. Introduction.

There are two "first-order" aspects of the asymptotic behavior of a transient random walk that generally bear study: its rate of escape and its "direction" of escape. In Section 13.5, the former was studied for simple random walk on Galton-Watson trees. The latter, with "direction" interpreted as harmonic measure, will be studied in this chapter. That is, since the random walk on a Galton-Watson tree T is transient, it converges to a ray of T a.s. The law of that ray is called ***harmonic measure***, denoted $\mathsf{HARM}(T)$, which we identify with a unit flow on T from its root to infinity.

Of course, if the offspring distribution is concentrated on a single integer, then the direction is uniform. So throughout this chapter, we assume this is not the case, i.e., the tree is ***nondegenerate***. We also assume that $p_0 = 0$ unless otherwise specified, since if we condition on nonextinction, the random walk will always leave any finite descendant subtree, and therefore the harmonic measure lives on the subtree of vertices with infinite lines of descent. We will see that the random irregularities that recur in a nondegenerate Galton-Watson tree T direct or confine the random walk to an exponentially smaller subtree of T. One aspect of this, and the key tool in its proof, is the dimension of harmonic measure. Now, since $\operatorname{br} T = m$ a.s., the boundary ∂T has Hausdorff dimension $\log m$ a.s.

▷ **Exercise 16.1.**
Show that the Hausdorff dimension of harmonic measure is a.s. constant.

The main result of this chapter is that the Hausdorff dimension of harmonic measure is strictly less than that of the full boundary:

Theorem 16.1. (Dimension Drop of Harmonic Measure) *The Hausdorff dimension of harmonic measure on the boundary of a nondegenerate Galton-Watson tree T is a.s. a constant $d < \log m = \dim(\partial T)$, i.e., there is a Borel subset of ∂T of full harmonic measure and dimension d.*

This result is established in a sharper form in Theorem 16.16.

With some further work, Theorem 16.1 yields the following restriction on the range of random walk.

Corollary 16.2. (Confinement of Random Walk) *Fix a nondegenerate offspring distribution with mean m. Let d be as in Theorem 16.1. For any $\epsilon > 0$ and for almost every Galton-Watson tree T, there is a rooted subtree Γ of T of growth*

$$\lim_{n\to\infty} |\Gamma_n|^{\frac{1}{n}} = e^d < m$$

such that with probability $1-\epsilon$, the sample path of simple random walk on T is contained in Γ. (Here, $|\Gamma_n|$ is the cardinality of the nth level of Γ.)

See Corollary 16.19 for a restatement and proof.

This corollary gives a partial explanation for the "low" speed of simple random walk on a Galton-Watson tree: the walk is confined to a smaller subtree.

The setting and results of Section 13.5 will be fundamental to our work here. Certain Markov chains on the space of trees (inspired by Furstenberg (1970)) are discussed in Section 16.2 and used in Section 16.3 to compute the dimension of the limit uniform measure, extending a theorem of Hawkes (1981). A general condition for dimension drop is given in Section 16.4 and applied to harmonic measure in Section 16.5, where Theorem 16.1 is proved; its application to Corollary 16.2 is given in Section 16.6. In Section 16.7, we analyze the electrical conductance of a Galton-Watson tree using a functional equation for its distribution. This yields a numerical scheme for approximating the dimension of harmonic measure.

§16.2. Markov Chains on the Space of Trees.

These chains are inspired by Furstenberg (1970). In the rest of this chapter, a ***flow*** on a tree will mean a unit flow from its root to infinity. Given a flow θ on a tree T and a vertex $x \in T$ with $\theta(x) > 0$, we write θ^x for the (conditional) flow on T^x given by

$$\theta^x(y) := \theta(y)/\theta(x) \qquad (y \in T^x).$$

We call a Borel function $\Theta : \{\text{trees}\} \to \{\text{flows on trees}\}$ a ***(consistent) flow rule*** if $\Theta(T)$ is a flow on T such that

$$x \in T,\ |x| = 1,\ \Theta(T)(x) > 0 \quad \Longrightarrow \quad \Theta(T)^x = \Theta(T^x).$$

A consistent flow rule may also be thought of as a Borel function that assigns to a k-tuple $(T^{(1)}, \ldots, T^{(k)})$ of trees a k-tuple of nonnegative numbers adding to one representing the probabilities of choosing the corresponding trees $T^{(i)}$ in $\bigvee_{i=1}^k T^{(i)}$, which is the tree formed by joining the roots of $T^{(i)}$ by single edges to a new vertex, the new vertex being the root of the new tree. It follows from the definition that for all $x \in T$, not only those at distance 1 from the root, $\Theta(T)(x) > 0 \Rightarrow \Theta(T)^x = \Theta(T^x)$. We will usually write Θ_T for $\Theta(T)$.

Recall that trees in this chapter are unlabelled but rooted. This will be important for constructing stationary Markov chains on the space of trees, just as it was in Section 13.4. However, for a tree with no nontrivial graph-automorphisms, it still makes sense to refer to vertices of the tree. All of our trees will have no nontrivial graph-automorphisms.

▷ **Exercise 16.2.**

The formalism is as follows. Let $\mathscr{T}$ be the space of trees in Exercise 5.2. Call two trees ***(rooted) isomorphic*** if there is a bijection of their vertex sets preserving adjacency and mapping one root to the other. For $T \in \mathscr{T}$, let $[T]$ denote the set of trees that are isomorphic to T. Let $\mathscr{T}_0$ denote the set of trees whose only automorphism is the identity (it is not required that the root be fixed) and let $[\mathscr{T}_0] := \{[T];\ T \in \mathscr{T}_0\}$. Show that the metric on $\mathscr{T}$ induces an incomplete separable metric on $[\mathscr{T}_0]$. Show that **GW** is concentrated on $\mathscr{T}_0$ provided $p_k \neq 1$ for all k. We will also denote by **GW** the measure on $[\mathscr{T}_0]$ induced by **GW** on $\mathscr{T}_0$. When we make statements or construct processes on $[\mathscr{T}_0]$, we will implicitly choose representatives $T \in [T]$ and leave it as an exercise to check that the choice of representative has no effect. Note that if $[T] = [T'] \in [\mathscr{T}_0]$, then there is a *unique* isomorphism from T to T'.

▷ **Exercise 16.3.**

Define formally the space of flows on trees with a natural metric. Define "consistent flow rule" on a space of flows on $[\mathscr{T}_0]$.

The principal object of interest in this chapter, harmonic measure, comes from a flow rule, HARM. Another natural example is harmonic measure, HARM^λ, for homesick random walk RW_λ; this was studied by Lyons, Pemantle, and Peres (1996a). Visibility measure, encountered in Section 14.4, gives a flow rule VIS. A final example, UNIF, will be studied in Section 16.3.

Proposition 16.3. *If Θ and Θ' are two flow rules such that for* **GW***-a.e. tree T and all vertices $|x| = 1$, $\Theta_T(x) + \Theta'_T(x) > 0$, then $\mathbf{GW}(\Theta_T = \Theta'_T) \in \{0, 1\}$.*

Proof. By the hypothesis, if $\Theta_T = \Theta'_T$ and $|x| = 1$, then $\Theta_{T^x} = \Theta'_{T^x}$. Thus, the result follows from Proposition 5.6. ◀

Given a flow rule Θ, there is an associated Markov chain on the space of trees given by the transition probabilities

$$\forall T\ \forall x \in T\ |x| = 1 \quad \Longrightarrow \quad \mathbf{p}_\Theta(T, T^x) = \Theta_T(x)\,.$$

We say that a (possibly infinite) measure μ on the space of trees is Θ-***stationary*** if it is $\mathbf{p}_\Theta$-stationary, i.e.,

$$\mu\mathbf{p}_\Theta = \mu\,,$$

or, in other words, for any Borel set A of trees,

$$\begin{aligned}\mu(A) = (\mu\mathbf{p}_\Theta)(A) &= \int \sum_{T' \in A} \mathbf{p}_\Theta(T, T')\, d\mu(T) \\ &= \int \sum_{\substack{|x|=1 \\ T^x \in A}} \mathbf{p}_\Theta(T, T^x)\, d\mu(T) \\ &= \int \sum_{\substack{|x|=1 \\ T^x \in A}} \Theta_T(x)\, d\mu(T)\,.\end{aligned}$$

If we denote the vertices along a ray ξ by $\xi_0, \xi_1, \ldots$, then the path of such a Markov chain is a sequence $\langle T^{\xi_n} \rangle_{n=0}^\infty$ for some tree T and some ray $\xi \in \partial T$. Clearly, we may identify the space of such paths with the ray bundle

$$\mathsf{RaysInTrees} := \{(\xi, T)\,;\ \xi \in \partial T\}\ .$$

For the corresponding path measure on $\mathsf{RaysInTrees}$, write

$$(\Theta \times \mu)(F) := \iint \mathbf{1}_F(\xi, T)\, d\Theta_T(\xi)\, d\mu(T),$$

although this is not a product measure.

The ergodic theory of Markov chains on general state spaces is rarely discussed, so we review it here. Our development is based on Kifer (1986), pp. 19–22; for another approach, see Rosenblatt (1971), especially pp. 96–97.

We begin with a brief review of ergodic theory. A ***measure-preserving system*** $(X, \mathscr{F}, \mu, S)$ is a measure space $(X, \mathscr{F}, \mu)$ together with a measurable map S from X to itself such that for all $A \in \mathscr{F}$, $\mu(S^{-1}A) = \mu(A)$. Fix $A \in \mathscr{F}$ with $0 < \mu(A) < \infty$. We denote the ***induced measure*** on A by $\mu_A(C) := \mu(C)/\mu(A)$ for $C \subseteq A$. We also write $\mu(C \mid A)$ for $\mu_A(C)$ since it is a conditional measure. A set $A \in \mathscr{F}$ is called S***-invariant*** if $\mu(A \triangle S^{-1}A) = 0$. The σ-field generated by the invariant sets is called the ***invariant*** σ***-field***. The system is called ***ergodic*** if the invariant σ-field is trivial (i.e., consists only of sets of measure 0 and sets whose complement has measure 0). For a function f on X, we write Sf for the function $f \circ S^{-1}$. Suppose that μ is a probability measure. The ***ergodic theorem*** states that for $f \in L^1(X, \mu)$, the limit of the averages $\sum_{k=0}^{n-1} S^k f/n$ exists a.s. and equals the conditional expectation of f with respect to the invariant σ-field. In particular, if the system is ergodic, then the limit equals the expectation of f.

Now let $\mathscr{X}$ be a measurable state space and $\mathbf{p}$ be a transition probability function on $\mathscr{X}$ (i.e., $\mathbf{p}(x, A)$ is measurable in x for each measurable set A and is a probability measure for each fixed x). This gives the usual operator P on bounded* measurable functions f, where

$$(Pf)(x) := \int_{\mathscr{X}} f(y)\,\mathbf{p}(x, dy)\,,$$

and its adjoint P^* on probability measures, where

$$\int_{\mathscr{X}} f\,dP^*\nu := \int_{\mathscr{X}} Pf\,d\nu\,.$$

Let μ be a stationary probability measure, i.e., $P^*\mu = \mu$, and let $\mathscr{I}$ be the σ-field of ***invariant*** sets, i.e., those measurable sets A such that

$$\mathbf{p}(x, A) = \mathbf{1}_A(x) \quad \mu\text{-a.s.}$$

The Markov chain is called ***ergodic*** if $\mathscr{I}$ is trivial. A bounded measurable function f is called ***harmonic*** if $Pf = f$ μ-a.s. Let $(\mathscr{X}^\infty, \mathbf{p} \times \mu)$ be the space of (one-sided) sequences of states with the measure induced by choosing the initial state according to μ and making transitions via $\mathbf{p}$. That is, if $\langle X_n\,;\ n \ge 0\rangle$ is the Markov chain on $\mathscr{X}$, then $\mathbf{p} \times \mu$ is its law.

---

* Everything we will say about bounded functions applies equally to nonnegative functions.

Lemma 16.4. *If f is a bounded measurable function on $\mathscr{X}$, then the following are equivalent:*

(i) f is harmonic;

(ii) f is $\mathscr{I}$-measurable;

(iii) $f(X_0) = f(X_1) \quad \mathbf{p} \times \mu$-a.s.

The idea is that when there is a finite stationary measure, then, as in the case of a Markov chain with a denumerable number of states, there really aren't any nontrivial bounded harmonic functions.

▷ **Exercise 16.4.**

Show that Lemma 16.4 may not be true if μ is an infinite stationary measure.

Proof of Lemma 16.4. (i) ⇒ (ii): Fix $\alpha \in \mathbb{R}$. Since P is a positive operator (i.e., it maps nonnegative functions to nonnegative functions),

$$P(f \wedge \alpha) \le (Pf) \wedge \alpha = f \wedge \alpha .$$

Also

$$\int P(f \wedge \alpha)\, d\mu = \int f \wedge \alpha \, dP^* \mu = \int f \wedge \alpha \, d\mu ,$$

whence $f \wedge \alpha$ is harmonic. Therefore, for μ-a.e. $x \in \{f \ge \alpha\}$, we have $\mathbf{p}(x, \{f \ge \alpha\}) = 1$. Likewise, $\mathbf{p}(x, \{f \le \beta\}) = 1$ for every β, whence $\{f \ge \alpha\} \in \mathscr{I}$. Thus, f is $\mathscr{I}$-measurable.

(ii) ⇒ (iii): For any interval I, we have $\mathbf{p}(x, \{f \in I\}) = \mathbf{1}_{\{f \in I\}}(x)$ μ-a.s. Thus, if $f(X_0) \in I$, then $f(X_1) \in I$ a.s.

(iii) ⇒ (i): This is immediate from the definition of harmonic. ◀

We now show that functions on $\mathscr{X}^\infty$ that are (a.s.) shift invariant depend only on their first coordinate:

Proposition 16.5. *A bounded measurable function h on $\mathscr{X}^\infty$ is shift invariant iff there exists a bounded $\mathscr{I}$-measurable function f on $\mathscr{X}$ such that $h(X_0, X_1, \ldots) = f(X_0)$ $\mathbf{p} \times \mu$-a.s. Indeed, f can be determined from h by $f(X_0) = \mathbf{E}_{X_0}[h(X_0, X_1, \ldots)]$.*

Proof. If h has the form given in terms of f, then the lemma shows that h is shift invariant. Conversely, given h, define f (μ-a.e.) as indicated. Then we have

$$\begin{aligned} f(X_0) &= \mathbf{E}\big[\mathbf{E}[h(X_0, X_1, \ldots) \mid X_0, X_1] \bigm| X_0\big] \\ &= \mathbf{E}\big[\mathbf{E}[h(X_1, X_2, \ldots) \mid X_0, X_1] \bigm| X_0\big] \quad \text{by shift invariance} \\ &= \mathbf{E}[f(X_1) \mid X_0] \quad \text{by the Markov property and the definition of } f \\ &= (Pf)(X_0) \quad \text{by definition of } P . \end{aligned}$$

That is, f is harmonic, so is $\mathscr{I}$-measurable. Now similar reasoning, together with the martingale convergence theorem, gives

$$\begin{aligned} h(X_0, X_1, \ldots) &= \lim_{n\to\infty} \mathbf{E}[h(X_0, X_1, \ldots) \mid X_0, \ldots, X_n] \\ &= \lim_{n\to\infty} \mathbf{E}[h(X_n, X_{n+1}, \ldots) \mid X_0, \ldots, X_n] = \lim_{n\to\infty} f(X_n) = f(X_0) \end{aligned}$$

by the lemma. ◀

As an immediate corollary, we get a criterion for ergodicity:

Corollary 16.6. $\mathbf{p} \times \mu$ *is ergodic for the shift iff* $\mathscr{I}$ *is trivial.*

In our setting, this says that $\Theta \times \mu$ is ergodic (for the shift map) iff every Θ-invariant set of trees has μ-measure 0 or 1, where a Borel set A of trees is called Θ***-invariant*** if

$$\mathbf{p}_\Theta(T, A) = \mathbf{1}_A(T) \quad \mu\text{-a.s.}$$

Moreover, even without ergodicity, shift-invariant functions on $\mathsf{RaysInTrees}$ correspond to Θ-invariant functions; in particular, they depend (a.s.) only on their second coordinate.

We call two measures ***equivalent*** if they are mutually absolutely continuous.

Proposition 16.7. *Let* Θ *be a flow rule such that for* $\mathbf{GW}$*-a.e. tree* T *and for all* $|x| = 1$, $\Theta_T(x) > 0$. *Then the Markov chain with transition probabilities* $\mathbf{p}_\Theta$ *and initial distribution* $\mathbf{GW}$ *is ergodic, though not necessarily stationary. Hence, if a (possibly infinite)* Θ*-stationary measure* μ *exists that is absolutely continuous with respect to* $\mathbf{GW}$, *then* μ *is equivalent to* $\mathbf{GW}$ *and the associated stationary Markov chain is ergodic.*

Proof. Let A be a Borel set of trees that is Θ-invariant. It follows from our assumption that for $\mathbf{GW}$-a.e. T,

$$T \in A \iff T^x \in A \quad \text{for every } |x| = 1\,.$$

Thus, the result follows from Proposition 5.6. ◀

Question 16.8. If a flow rule has a stationary measure equivalent to $\mathbf{GW}$, must the associated Markov chain be ergodic?

Given a Θ-stationary probability measure μ on the space of trees, we follow Furstenberg (1970) and define the ***entropy*** of the associated stationary Markov chain as

$$\begin{aligned}\mathsf{Ent}_\Theta(\mu) :&= \int \sum_{|x|=1} \mathbf{p}_\Theta(T, T^x) \log \frac{1}{\mathbf{p}_\Theta(T, T^x)}\, d\mu(T) \\ &= \int \sum_{|x|=1} \Theta_T(x) \log \frac{1}{\Theta_T(x)}\, d\mu(T) \\ &= \int\!\!\int \log \frac{1}{\Theta_T(\xi_1)}\, d\Theta_T(\xi)\, d\mu(T) \\ &= \int \log \frac{1}{\Theta_T(\xi_1)}\, d(\Theta \times \mu)(\xi, T)\,.\end{aligned}$$

Define

$$g_\Theta(\xi, T) := \log \frac{1}{\Theta_T(\xi_1)}$$

and let S be the shift on $\mathsf{RaysInTrees}$. The ergodic theorem tells us that the Hölder exponent (Section 14.4) is actually a limit a.s.:

$$\begin{aligned}\mathsf{H\ddot{o}}(\Theta_T)(\xi) &= \lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\Theta_T(\xi_n)} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \frac{\Theta_T(\xi_k)}{\Theta_T(\xi_{k+1})} \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \frac{1}{\Theta(T)^{\xi_k}(\xi_{k+1})} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} S^k g_\Theta(\xi, T)\end{aligned}$$

exists $\Theta \times \mu$-a.s.; and it satisfies

$$\int \mathsf{H\ddot{o}}(\Theta_T)(\xi)\, d(\Theta \times \mu)(\xi, T) = \mathsf{Ent}_\Theta(\mu)\,.$$

If the Markov chain is ergodic, then

$$\mathsf{H\ddot{o}}(\Theta_T)(\xi) = \mathsf{Ent}_\Theta(\mu) \quad \Theta \times \mu\text{-a.s.} \tag{16.1}$$

This is our principal tool for calculating Hausdorff dimension. Note that even if the Markov chain is not ergodic, the Hölder exponent $\mathsf{H\ddot{o}}(\Theta_T)(\xi)$ is constant Θ_T-a.s. for μ-a.e. T: since $(\xi, T) \mapsto \mathsf{H\ddot{o}}(\Theta_T)(\xi)$ is a shift-invariant function, it depends only on T (a.s.) (Proposition 16.5).

§16.3. The Hölder Exponent of Limit Uniform Measure.

By the Seneta-Heyde theorem, if Z_n and Z'_n are two i.i.d. Galton-Watson processes without extinction, then $\lim_{n\to\infty} Z_n/Z'_n$ exists a.s. This allows us to define (a.s.) a probability measure UNIF_T on the boundary of a Galton-Watson tree by

$$\mathsf{UNIF}_T(x) := \lim_{n\to\infty} \frac{|T^x \cap T_n|}{Z_n},$$

where T_n denotes the vertices of the nth generation of T. We call this measure ***limit uniform*** since, before the limit is taken, it is uniform on T_n. Figure 5.1 was drawn by considering the uniform measure on generation 9 and inducing the masses on the preceding generations. Figures 16.1 and 16.2 show this same tree drawn using the uniform measure on generations 14 and 19, respectively.

We may write limit uniform measure another way: Let c_n be constants such that $c_{n+1}/c_n \to m$ and

$$\tilde{W}(T) := \lim_{n\to\infty} Z_n/c_n$$

exists and is finite and non-zero a.s. Then we have

$$\mathsf{UNIF}_T(x) = \frac{\tilde{W}(T^x)}{m^{|x|}\tilde{W}(T)}\,. \tag{16.2}$$

Note that

$$\tilde{W}(T) = \frac{1}{m}\sum_{|x|=1} \tilde{W}(T^x). \tag{16.3}$$

According to the Kesten-Stigum theorem, when $\mathbf{E}[Z_1 \log Z_1] < \infty$, we may take c_n to be m^n and so W may be used in place of $\tilde{W}$ in (16.2) and (16.3). A theorem of Athreya (1971) gives that

$$\int \tilde{W}(T)\, d\mathbf{GW}(T) < \infty \iff \mathbf{E}[Z_1 \log Z_1] < \infty\,. \tag{16.4}$$

As we mentioned in the introduction, $\dim(\partial T) = \log m$ a.s. Now Hawkes (1981) showed that the Hölder exponent of UNIF_T is $\log m$ a.s. provided $\mathbf{E}[Z_1(\log Z_1)^2] < \infty$. One could anticipate Hawkes's result that UNIF_T has full Hausdorff dimension, $\log m$, since limit uniform measure "spreads out" the most possible (at least under some moment condition on Z_1). Furthermore, one might guess that no other measure that comes from a consistent flow rule can have full dimension. We will show that this is indeed true provided that the flow rule has a finite stationary measure for the associated Markov chain that is absolutely continuous with respect to $\mathbf{GW}$. We will then show that such is the case for

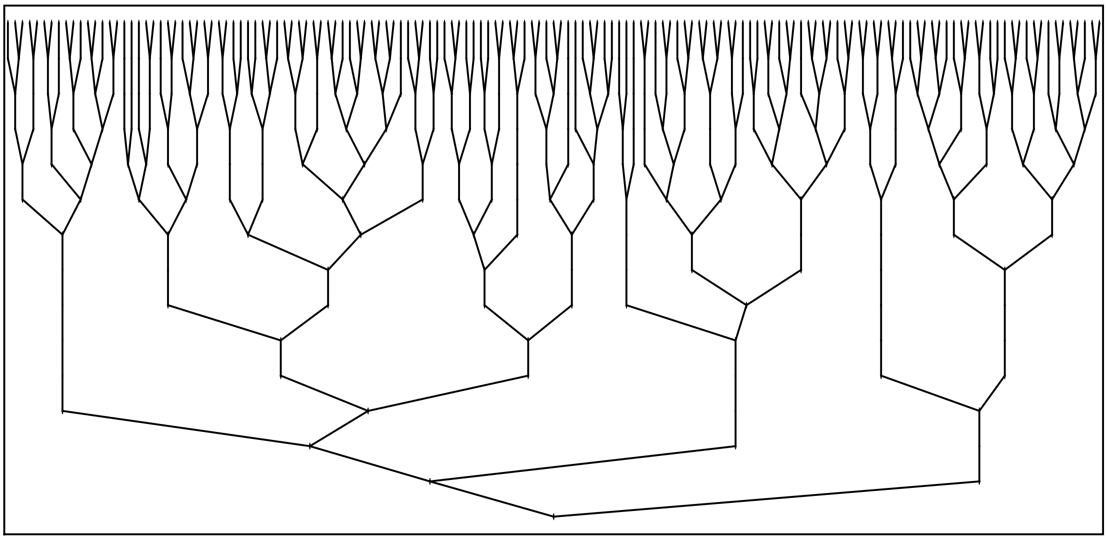


Figure 16.1. Generations 0 to 14 of a typical Galton-Watson tree for $f(s) = (s + s^2)/2$.

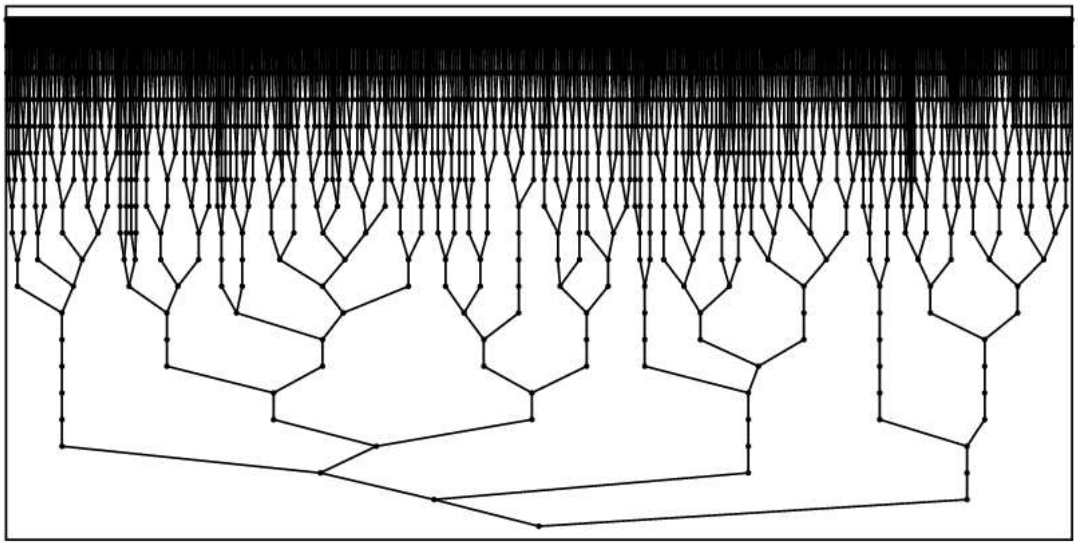


Figure 16.2. Generations 0 to 19 of a typical Galton-Watson tree for $f(s) = (s + s^2)/2$.

harmonic measure of simple random walk. Incidentally, this method will allow us to give a simpler proof of Hawkes's theorem, as well as to extend its validity to the case where $\mathbf{E}[Z_1 \log Z_1] < \infty$. The value of $\dim \mathsf{UNIF}_T$ is unknown when $\mathbf{E}[Z_1 \log Z_1] = \infty$.

In this section, we prove and extend the theorem of Hawkes (1981) on the Hölder exponent of limit uniform measure and study further the associated Markov chain. We begin by showing that a (possibly infinite) UNIF-stationary measure on trees is $\tilde{W}(T)\, d\mathbf{GW}(T)$. This was also observed by Hawkes (1981), p. 378. It can be seen intuitively as follows: start a Markov chain with transition probabilities $\mathbf{p}_{\mathsf{UNIF}}$ from the initial distribution $\mathbf{GW}$. If the path followed through T is ξ, then the nth state of the Markov chain is T^{ξ_n}. It is clear that $\tilde{W}(T^{\xi_n})$ is sampled from Z_n independent copies of $\tilde{W}$ biased according to size. Since $Z_n \to \infty$, the law of $\tilde{W}(T^{\xi_n})$ tends weakly to that of size-biased $\tilde{W}$. Actually, this only makes sense when $\tilde{W}$ has finite mean, i.e., when $\mathbf{E}[Z_1 \log Z_1] < \infty$ by Athreya's theorem (16.4). In this case, this intuitive argument is easily turned into a rigorous proof. Related ideas occur in Joffe and Waugh (1982). However, we present the following direct verification of stationarity that is valid even in the case when $\tilde{W}$ has infinite mean.

Proposition 16.9. *The Markov chain with transition probabilities $\mathbf{p}_{\mathsf{UNIF}}$ and initial distribution $\tilde{W} \cdot \mathbf{GW}$ is stationary and ergodic.*

Proof. Apply the definition of stationarity with $\Theta_T(x) = \tilde{W}(T^x)/\big(m\tilde{W}(T)\big)$: for any Borel set A of trees, we have

$$
\begin{aligned}
\big((\tilde{W} \cdot \mathbf{GW})\mathbf{p}_{\mathsf{UNIF}}\big)(A) &= \int \sum_{\substack{|x|=1 \\ T^x \in A}} \frac{\tilde{W}(T^x)}{m\tilde{W}(T)} \cdot \tilde{W}(T)\, d\mathbf{GW}(T) \\
&= \sum_{k=1}^{\infty} p_k \frac{1}{m} \int_{T^{(1)}} \cdots \int_{T^{(k)}} \sum_{i=1}^{k} \mathbf{1}_{\{T^{(i)} \in A\}} \tilde{W}(T^{(i)}) \prod_{j=1}^{k} d\mathbf{GW}(T^{(j)}) \\
&= \sum_{k=1}^{\infty} p_k \frac{1}{m} \sum_{i=1}^{k} \int_{T^{(1)}} \cdots \int_{T^{(k)}} \mathbf{1}_{\{T^{(i)} \in A\}} \tilde{W}(T^{(i)}) \prod_{j=1}^{k} d\mathbf{GW}(T^{(j)}) \\
&= \sum_{k=1}^{\infty} p_k \frac{1}{m} \sum_{i=1}^{k} \int_A \tilde{W}\, d\mathbf{GW} = \int_A \tilde{W}\, d\mathbf{GW} \\
&= (\tilde{W} \cdot \mathbf{GW})(A)\,,
\end{aligned}
$$

as desired. Since $\tilde{W} > 0$ $\mathbf{GW}$-a.s., ergodicity is guaranteed by Proposition 16.7. ◀

▷ **Exercise 16.5.**

This chain is closely connected to the size-biased Galton-Watson trees of Section 12.1. Show that in case $\mathbf{E}[Z_1 \log Z_1] < \infty$, the distribution of a UNIF_T-path is $\widehat{\mathbf{GW}}_*$.

In order to calculate the Hölder exponent of limit uniform measure, we will use the following well-known lemma of ergodic theory:

Lemma 16.10. *If S is a measure-preserving transformation on a probability space, g is finite and measurable, and $g - Sg$ is bounded below by an integrable function, then $g - Sg$ is integrable with integral zero.*

Proof. The ergodic theorem implies the a.s. convergence of

$$\lim_{k\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} S^k(g - Sg)(x) = \lim_{k\to\infty} \frac{1}{n}(g - S^n g)(x)\,.$$

Since the distribution of $S^n g$ is the same as that of g, taking this limit in probability gives 0. Hence we have a.s. convergence to 0 and $g - Sg$ is integrable with integral 0. ◀

Theorem 16.11. *If $\mathbf{E}[Z_1 \log Z_1] < \infty$, then the Hölder exponent at ξ of limit uniform measure UNIF_T is equal to $\log m$ for UNIF_T-a.e. ray $\xi \in \partial T$ and $\mathbf{GW}$-a.e. tree T. In particular, $\dim \mathsf{UNIF}_T = \log m$ for $\mathbf{GW}$-a.e. T.*

Proof. The hypothesis and Proposition 16.9 ensure that $W\cdot\mathbf{GW}$ is a stationary probability distribution. Let S be the shift on the ray bundle $\mathsf{RaysInTrees}$ with the invariant measure $\mathsf{UNIF} \times W \cdot \mathbf{GW}$. Define $g(\xi, T) := \log W(T)$ for a Galton-Watson tree T and $\xi \in \partial T$. Then

$$\begin{aligned}(g - Sg)(\xi, T) &= \log W(T) - \log W(T^{\xi_1}) = \log \frac{mW(T)}{W(T^{\xi_1})} - \log m \\ &= \log \frac{1}{\mathsf{UNIF}_T(\xi_1)} - \log m\,.\end{aligned}$$

In particular, $g - Sg \geq -\log m$, whence the lemma implies that $g - Sg$ has integral zero.

Now, for $\mathsf{UNIF} \times W \cdot \mathbf{GW}$-a.e. (ξ, T) (hence for $\mathsf{UNIF} \times \mathbf{GW}$-a.e. (ξ, T)), we have that

$$\mathsf{H\ddot{o}}(\mathsf{UNIF}_T)(\xi) = \mathsf{Ent}_{\mathsf{UNIF}}(W \cdot \mathbf{GW})$$

by ergodicity. By definition and the preceding calculation, this in turn is

$$\begin{aligned}\mathsf{Ent}_{\mathsf{UNIF}}(W \cdot \mathbf{GW}) &= \iint \log \frac{1}{\mathsf{UNIF}_T(\xi_1)}\, d\mathsf{UNIF}_T(\xi)\, W(T)\, d\mathbf{GW}(T) \\ &= \log m + \iint (g - Sg)\, d\mathsf{UNIF}_T(\xi)\, W(T)\, d\mathbf{GW}(T) \\ &= \log m.\end{aligned}$$

◀

Aïdékon (2010) showed that $\dim \mathsf{UNIF}_T = 0$ when $\mathbf{E}[Z_1 \log Z_1] = \infty$.

§16.4. Dimension Drop for Other Flow Rules.

It was conjectured in Lyons, Pemantle, and Peres (1995b) that any flow rule other than limit uniform gives measures of dimension less than $\log m$ **GW**-a.s. In this section, we prove (following Lyons, Pemantle, and Peres (1995b)) that this is the case when the flow rule has a finite stationary measure equivalent to **GW**. Our theorem is valid even when $\mathbf{E}[Z_1 \log Z_1] = \infty$. Shannon's inequality will be the tool we use to compare the dimension of measures arising from flow rules to the dimension of the whole boundary: the inequality states that

$$a_i, b_i \in [0, 1], \quad \sum a_i = \sum b_i = 1 \quad \Longrightarrow \quad \sum a_i \log \frac{1}{a_i} \le \sum a_i \log \frac{1}{b_i}\,,$$

with equality iff $a_i \equiv b_i$.

▷ **Exercise 16.6.**

Prove Shannon's inequality.

Theorem 16.12. *If Θ is a flow rule such that $\Theta_T \ne \mathsf{UNIF}_T$ for **GW**-a.e. T and there is a finite Θ-stationary measure μ absolutely continuous with respect to **GW**, then for μ-a.e. T, we have $\mathsf{H\ddot{o}}(\Theta_T) < \log m$ Θ_T-a.s. and $\dim(\Theta_T) < \log m$.*

Proof. Recall that the Hölder exponent of Θ_T is constant Θ_T-a.s. for μ-a.e. T and equal to the Hausdorff dimension of Θ_T. Thus, it suffices to show that the set of trees

$$A := \{T\,;\ \dim \Theta_T = \log m\} = \{T\,;\ \mathsf{H\ddot{o}}(\Theta_T) = \log m \quad \Theta_T\text{-a.s.}\}$$

has μ-measure 0. Suppose that $\mu(A) > 0$. Recall that μ_A denotes μ conditioned on A. Now since $\mu \ll \mathbf{GW}$, the limit uniform measure UNIF_T is defined and satisfies (16.2) for μ_A-a.e. T. Let $g(\xi, T) := \log \widetilde{W}(T)$. As in the proof of Theorem 16.11, $g - Sg \ge -\log m$. Since the entropy is the mean Hölder exponent, we have by Shannon's inequality and Lemma 16.10,

$$\begin{aligned}
\log m = \mathsf{Ent}_\Theta(\mu_A) &= \int \sum_{|x|=1} \Theta_T(x) \log \frac{1}{\Theta_T(x)}\, d\mu_A(T) \\
&< \int \sum_{|x|=1} \Theta_T(x) \log \frac{1}{\mathsf{UNIF}_T(x)}\, d\mu_A(T) \\
&= \int \log \frac{1}{\mathsf{UNIF}_T(\xi_1)}\, d\Theta_T(\xi)\, d\mu_A(T) \\
&= \log m + \int (g - Sg)\, d\Theta_T(\xi)\, d\mu_A(T) \\
&= \log m\,.
\end{aligned}$$

This contradiction shows that $\mu(A) = 0$, as desired. ◀

In order to apply this theorem to harmonic measure, we need to find a stationary measure for the harmonic flow rule with the above properties. We do not know a general condition for a flow rule to have such a stationary measure.

§16.5. Harmonic-Stationary Measure.

Consider the set of "last exit points"

$$\mathsf{Exit} := \left\{(\overleftrightarrow{x}, T) \in \mathsf{PathsInTrees}\,;\; x_{-1} \in x_{-\infty}\,,\; \forall n > 0 \; x_n \neq x_{-1}\right\}.$$

This is precisely the event that the path has just exited, for the last time, a horoball centered at $x_{-\infty}$. Since regeneration points are exit points, it follows from Proposition 13.16 that the set Exit has positive measure and for a.e. $(\overleftrightarrow{x}, T)$, there is an $n > 0$ such that $S^n(\overleftrightarrow{x}, T) \in \mathsf{Exit}$. (This also follows directly merely from the almost sure transience of simple random walk.) Inducing on this set will yield the measure we need to apply Theorem 16.12.

We recall some more terminology from ergodic theory for this. Fix $A \in \mathscr{F}$ with $0 < \mu(A) < \infty$. Define the ***return time*** to A by $n_A(x) := \inf\{n \geq 1\,;\; S^n x \in A\}$ for $x \in A$ and, if $n_A(x) < \infty$, the ***return map*** $S_A(x) := S^{n_A(x)}(x)$. The Poincaré recurrence theorem (Petersen (1983), p. 34) states that if $\mu(X) < \infty$, then $n_A(x) < \infty$ for a.e. $x \in A$. In this case, $(A, \mathscr{F} \cap A, \mu_A, S_A)$ is a measure-preserving system (Petersen (1983), p. 39, or Exercise 6.45), called the ***induced system***. Given two measure-preserving systems, $(X_1, \mathscr{F}_1, \mu_1, S_1)$ and $(X_2, \mathscr{F}_2, \mu_2, S_2)$, the second is called a ***factor*** of the first if there is a measurable map $f : (X_1, \mathscr{F}_1) \to (X_2, \mathscr{F}_2)$ such that $\mu_2 = \mu_1 \circ f^{-1}$ and $f \circ S_1 = S_2 \circ f$ μ_1-a.e.

Theorem 16.13. *There is a unique ergodic* HARM*-stationary measure* μ_{HARM} *equivalent to* $\mathbf{GW}$.

Proof. The key point is that for $(\overleftrightarrow{x}, T) \in \mathsf{Exit}$, the path of vertices $\langle x'_k \rangle$ in the tree T given by the roots of the second components of the sequence $\langle S^k_{\mathsf{Exit}}(\overleftrightarrow{x}, T)\rangle_{k \geq 0}$ is a sample from the ray generated by the Markov chain associated to $\mathsf{HARM}_{T \setminus T^{x_{-1}}}$. (Recall that T is rooted at x_0.) Note that the Markov property of this factor of the system induced on Exit is a consequence of the fact that HARM is a consistent flow rule.

Now since $\mathbf{AGW}_{\mathsf{Exit}} \ll \mathbf{AGW}$, we have that the $(\mathsf{SRW} \times \mathbf{AGW})_{\mathsf{Exit}}$-law of $T \setminus T^{x_{-1}}$ is absolutely continuous with respect to $\mathbf{GW}$. From Proposition 16.7, it follows that the $(\mathsf{SRW} \times \mathbf{AGW})_{\mathsf{Exit}}$-law of $T \setminus T^{x_{-1}}$ is equivalent to $\mathbf{GW}$. [This can also be seen directly: for $\mathbf{AGW}$-a.e. T, the SRW_T-probability that $(\overleftrightarrow{x}, T) \in \mathsf{Exit}$ is positive, whence the

$(\mathsf{SRW} \times \mathbf{AGW})_{\mathsf{Exit}}$-law of T is equivalent to $\mathbf{AGW}$. This gives that the $(\mathsf{SRW} \times \mathbf{AGW})_{\mathsf{Exit}}$-law of $T \setminus T^{x_{-1}}$ is equivalent to $\mathbf{GW}$.]

Therefore, the above natural factor of the induced measure-preserving system

$$(\mathsf{Exit}, (\mathsf{SRW} \times \mathbf{AGW})_{\mathsf{Exit}}, S_{\mathsf{Exit}})$$

obtained by mapping $(\vec{x}, T) \mapsto \langle \Gamma(k) \setminus \Gamma(k)^{x'_{k-1}} \rangle$, where $\Gamma(k) := (T, x'_k)$, is a HARM-stationary Markov chain on trees with a stationary measure μ_{HARM} equivalent to $\mathbf{GW}$.

The fact that $\mathsf{HARM} \times \mu_{\mathsf{HARM}}$ is ergodic follows from our general result on ergodicity, Proposition 16.7. Ergodicity implies that μ_{HARM} is the unique HARM-stationary measure absolutely continuous with respect to $\mathbf{GW}$. ◀

Since increases in distance from the root can be considered to come only at exit points, it is natural that the speed is also the probability of being at an exit point:

Proposition 16.14. *The measure of the exit set is the speed:* $(\mathsf{SRW} \times \mathbf{AGW})(\mathsf{Exit}) = \mathbf{E}[(Z_1 - 1)/(Z_1 + 1)]$.

▷ **Exercise 16.7.**

Prove Proposition 16.14.

Define $T_\Delta := [\Delta \bullet\!\!-T]$, where Δ is a single vertex not in T, to be thought of as representing the past. Let $\gamma(T)$ be the probability that simple random walk started at Δ never returns to Δ:

$$\gamma(T) := \mathsf{SRW}_{T_\Delta}(\forall n > 0 \quad x_n \neq \Delta)\,.$$

This is also equal to $\mathsf{SRW}_{[T \bullet\!\!-\Delta]}(\forall n > 0\ x_n \neq \Delta)$. Let $\mathscr{C}(T)$ denote the effective conductance of T from its root to infinity when each edge has unit conductance. Clearly,

$$\gamma(T) = \frac{\mathscr{C}(T)}{1 + \mathscr{C}(T)} = \mathscr{C}(T_\Delta)\,.$$

The notation γ is intended to remind us of the word "conductance".

The next proposition is intuitively obvious, but crucial.

Proposition 16.15. *For* $\mathbf{GW}$*-a.e.* T*,* $\mathsf{HARM}_T \neq \mathsf{UNIF}_T$*.*

Proof. In view of the zero-one law, Proposition 16.3, we need merely show that we do not have $\mathsf{HARM}_T = \mathsf{UNIF}_T$ a.s. Now, for any tree T and any $x \in T$ with $|x| = 1$, we have

$$\mathsf{HARM}_T(x) = \frac{\gamma(T^x)}{\sum_{|y|=1} \gamma(T^y)}$$

while

$$\mathsf{UNIF}_T(x) = \frac{\tilde{W}(T^x)}{\sum_{y=1} \tilde{W}(T^y)} .$$

Therefore, if $\mathsf{HARM}_T = \mathsf{UNIF}_T$, the vector

$$\left\langle \frac{\gamma(T^x)}{\tilde{W}(T^x)} \right\rangle_{|x|=1} \tag{16.5}$$

is a multiple of the constant vector $\mathbf{1}$. For Galton-Watson trees, the components of this vector are i.i.d. with the same law as that of $\gamma(T)/\tilde{W}(T)$. The only way (16.5) can be a (random) multiple of $\mathbf{1}$, then, is for $\gamma(T)/\tilde{W}(T)$ to be a constant $\mathbf{GW}$-a.s. But $\gamma < 1$ and, since Z_1 is not constant, $\tilde{W}$ is obviously unbounded, so this is impossible. ◀

Taking stock of our preceding results, we obtain our main theorem:

Theorem 16.16. *The dimension of harmonic measure is* $\mathbf{GW}$*-a.s. less than* $\log m$*. The Hölder exponent exists a.s. and is constant.*

Proof. The hypotheses of Theorem 16.12 are verified in Theorem 16.13 and Proposition 16.15. The constancy of the Hölder exponent follows from (16.1). ◀

Note that no moment assumptions (other than $m < \infty$) were used.

Question 16.17. We saw in Section 14.4 that the dimension of visibility measure is a.s. $\mathbf{E}[\log Z_1]$. In the direction of comparison opposite to that of Theorem 16.1, is this a lower bound for $\dim \mathsf{HARM}_T$? This question, due to Ledrappier (personal communication, 1994) was posed in Lyons, Pemantle, and Peres (1995b, 1997).

Question 16.18. For $0 \le \lambda < m$, is the dimension of harmonic measure for RW_λ on a Galton-Watson tree T monotonic nondecreasing in the parameter λ? Is it strictly increasing? This was asked in Lyons, Pemantle, and Peres (1997).

▷ **Exercise 16.8.**

Suppose that $p_0 > 0$. Show that given nonextinction, the dimension of harmonic measure is a.s. less than $\log m$.

§16.6. Confinement of Simple Random Walk.

We now demonstrate how the drop in dimension of harmonic measure, proved in the preceding section, implies the confinement of simple random walk to a much smaller subtree. A visual comparison of harmonic measure, uniform measure (these two calculated based on generation 19), and visibility measure appears in Figure 16.3.

Given a tree T and positive integer n, recall that T_n denotes the set of the vertices of T at distance n from the root and $|T_n|$ is the cardinality of T_n.

Corollary 16.19. *For* **GW**-*almost all trees T and for every $\epsilon > 0$, there is a subtree $T(\epsilon) \subseteq T$ such that*

$$\mathsf{SRW} \times \mathbf{GW}\Big\{x_n \in T(\epsilon) \textit{ for all } n \geq 0\Big\} \geq 1 - \epsilon \tag{16.6}$$

and

$$\frac{1}{n} \log |T(\epsilon)_n| \to d\,, \tag{16.7}$$

where $d < \log m$ is the dimension of HARM_T. Furthermore, any subtree $T(\epsilon)$ satisfying (16.6) must have growth

$$\liminf \frac{1}{n} \log |T(\epsilon)_n| \geq d\,. \tag{16.8}$$

Proof. Let $n_k := 1 + \max\{n\,;\ |x_n| = k\}$ be the k-th exit epoch and $D(x,k)$ be the set of descendants y of x with $|y| \leq |x| + k$. We will use three sample path properties of simple random walk on a fixed tree:

$$\textit{Speed:} \quad l := \lim_{n\to\infty} \frac{|x_n|}{n} > 0 \quad \text{a.s.} \tag{16.9}$$

$$\textit{Hölder exponent:} \quad \lim_{n\to\infty} \frac{1}{k} \log \frac{1}{\mathsf{HARM}_T(x_{n_k})} = d \quad \text{a.s.} \tag{16.10}$$

$$\textit{Neighborhood size:} \quad \forall \delta > 0 \qquad \limsup_{n\to\infty} \frac{\log |D(x_n, \delta |x_n|)|}{|x_n|} \leq \delta \log m \quad \text{a.s.} \tag{16.11}$$

We have already shown (16.9) and (16.10). In fact, the limit in (16.11) exists and equals the right-hand side, but this is not needed. In order to see that (16.11) holds for **GW**-a.e. tree, denote by y_k the k-th fresh point visited by simple random walk. Then (16.11) can be written as

$$\forall \delta > 0 \qquad \limsup_{k\to\infty} |y_k|^{-1} \log |D(y_k, \delta |y_k|)| \leq \delta \log m$$

and since $|y_k|/k$ has a positive a.s. limit, this is equivalent to

$$\forall \delta^* > 0 \qquad \limsup_{k} k^{-1} \log |D(y_k, \delta^* k)| \leq \delta^* \log m\,. \tag{16.12}$$

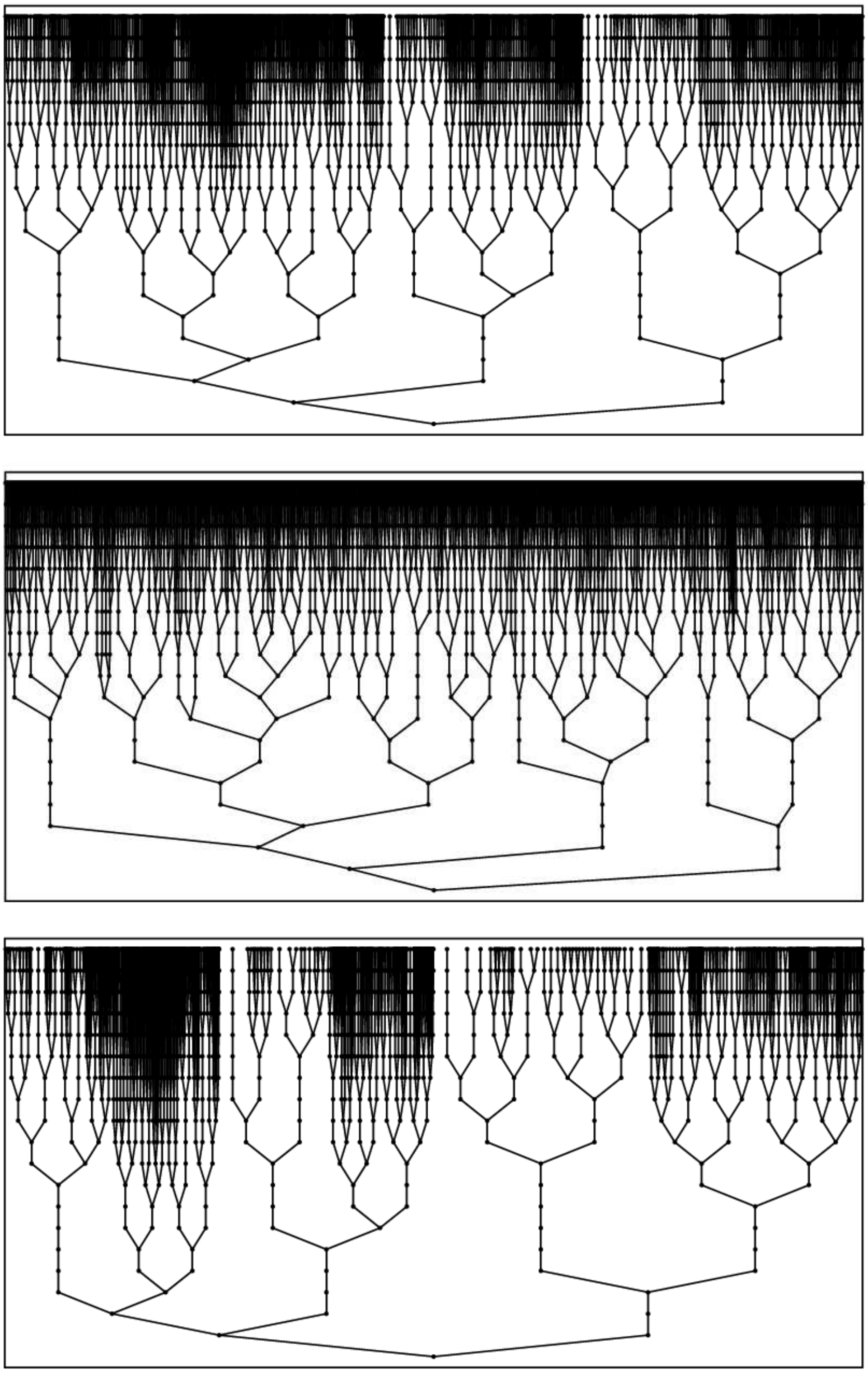


Figure 16.3. Generations 0 to 19 of a typical Galton-Watson tree for $f(s) = (s + s^2)/2$ displayed according to harmonic, uniform, and visibility measure, respectively. This means that if Θ is the flow rule, then the vertex x is centered in an interval of length $\Theta_T(x)$, where the total width of the figure is 1.

Now the random variables $|D(y_k, \delta^* k)|$ are identically distributed, though not independent. Indeed, the descendant subtree of y_k has the law of $\mathbf{GW}$. Since the expected number of descendants of y_k at generation $|y_k| + j$ is m^j for every j, we have by Markov's inequality that for every $\delta' > 0$,

$$(\mathsf{SRW} \times \mathbf{GW})\big\{|D(y_k, \delta^* k)| \ge m^{\delta' k}\big\} \le m^{-\delta' k} \sum_{j=0}^{\delta^* k} m^j .$$

If $\delta' > \delta^*$, then the right-hand side decays exponentially in k, so by the Borel-Cantelli lemma, we get (16.12), hence (16.11).

Now (16.10) alone implies (16.8) (Exercise 16.9).

Applying Egorov's theorem to the two almost sure asymptotics (16.9) and (16.10), we see that for each $\epsilon > 0$, there is a set of paths A_ϵ with $\mathsf{SRW} \times \mathbf{GW}(A_\epsilon) > 1 - \epsilon$ and such that the convergence in (16.9) and (16.10) is uniform on A_ϵ. Thus, we can choose $\langle \delta_n \rangle$ decreasing to 0 such that on A_ϵ, for all k and all n,

$$\mathsf{HARM}_T(x_{n_k}) > e^{-k(d+\delta_k)} \quad \text{and} \quad \left| \frac{|x_n|}{nl} - 1 \right| < \delta_n . \tag{16.13}$$

Now since δ_n is eventually less than any fixed δ, (16.11) implies that

$$\limsup_{n\to\infty} |x_n|^{-1} \log |D(x_n, 3\delta_{|x_n|}|x_n|)| = 0 \text{ a.s.},$$

so applying Egorov's theorem once more and replacing A_ϵ by a subset thereof (which we continue to denote A_ϵ), we may assume that there exists a sequence $\langle \eta_n \rangle$ decreasing to 0 such that

$$|D(x_n, 3\delta_{|x_n|}|x_n|)| \le e^{|x_n|\eta_n} \quad \text{for all } n \tag{16.14}$$

on A_ϵ.

Define $F_0^{(\epsilon)}$ to consist of all vertices $v \in T$ such that either $\delta_{|v|} \ge 1/3$ or both

$$\mathsf{HARM}_T(v) \ge e^{-|v|(d+\delta_{|v|})} \quad \text{and} \quad |D(v, 3\delta_{|v|}|v|)| \le e^{|v|\eta_{|v|}} . \tag{16.15}$$

Finally, let

$$F^{(\epsilon)} = \bigcup_{v \in F_0^{(\epsilon)}} D(v, 3\delta_{|v|}|v|)$$

and denote by $T(\epsilon)$ the component of the root in $F^{(\epsilon)}$. Since the number of vertices $v \in T_n$ satisfying $\mathsf{HARM}_T(v) \ge e^{-|v|(d+\delta_{|v|})}$ is at most $e^{n(d+\delta_n)}$, the bounds (16.15) yield for sufficiently large n

$$|T(\epsilon)_n| \le \sum_{\substack{v \in F_0^{(\epsilon)} \\ n - 3\delta_{|v|}|v| \le |v| \le n}} |D(v, 3\delta_{|v|}|v|)| \le \sum_{k=1}^{n} e^{k(d+\delta_k)} e^{k\eta_k} = e^{n(d+\alpha_n)} ,$$

where $\lim_{n\to\infty} \alpha_n = 0$. Hence

$$\limsup \frac{1}{n} \log |T(\epsilon)_n| \le d \,.$$

In combination with the lower bound (16.8), this gives (16.7).

It remains to establish that the walk stays inside $F^{(\epsilon)}$ forever on the event A_ϵ, since that will imply that the walk is confined to $T(\epsilon)$ on this event. The points visited at exit epochs n_k are in $F_0^{(\epsilon)}$ by the first part of (16.13) and (16.14). Fix a path $\{x_j\}$ in A_ϵ and a time n, and suppose that the last exit epoch before n is n_k, so that $n_k \le n < n_{k+1}$. Denote by $N := n_{k+1} - 1$ the time preceding the next exit epoch, and observe that $x_N = x_{n_k}$. If $\delta_n \ge 1/3$, then x_n is in $F_0^{(\epsilon)}$ since $\delta_{|x_n|} \ge \delta_n$, so consider the case that $\delta_n < 1/3$. By the second part of (16.13), we have

$$\frac{|x_n|}{nl} < 1 + \delta_n \qquad \text{and} \qquad \frac{|x_N|}{nl} \ge \frac{|x_N|}{Nl} > 1 - \delta_N \ge 1 - \delta_n \,.$$

Dividing these inequalities, we find that

$$|x_n| \le \frac{1+\delta_n}{1-\delta_n} |x_N| \le (1+3\delta_n)|x_N| \,.$$

It follows that x_n is in $D(x_{n_k}, 3\delta_{|x_{n_k}|}|x_{n_k}|)$. Since $x_{n_k} \in F_0^{(\epsilon)}$, we arrive at our desired conclusion that $x_n \in F^{(\epsilon)}$. ◀

▷ **Exercise 16.9.**
Prove (16.8).

§16.7. Calculations.

Our primary aim in this section is to compute the dimension of harmonic measure, d, numerically. Recall the notation $\gamma(T)$ and $\mathscr{C}(T)$ of Section 16.5. Note that

$$\sum_{|x|=1} \gamma(T^x) = \mathscr{C}(T) = \frac{\gamma(T)}{1-\gamma(T)} \,,$$

whence for $|x| = 1$,

$$\mathsf{HARM}_T(x) = \frac{\gamma(T^x)}{\sum_{|y|=1} \gamma(T^y)} = \gamma(T^x)\big(1-\gamma(T)\big)/\gamma(T) \,. \tag{16.16}$$

Thus, we have

$$\begin{aligned} d = \mathsf{Ent}_{\mathsf{HARM}}(\mu_{\mathsf{HARM}}) &= \int \log \frac{1}{\mathsf{HARM}_T(\xi_1)}\, d\mathsf{HARM} \times \mu_{\mathsf{HARM}}(\xi, T) \\ &= \int \log \frac{\gamma(T)}{\gamma(T^{\xi_1})\big(1-\gamma(T)\big)}\, d\mathsf{HARM} \times \mu_{\mathsf{HARM}}(\xi, T) \\ &= \int \log \frac{1}{1-\gamma(T)}\, d\mu_{\mathsf{HARM}}(T) = \int \log\big(1 + \mathscr{C}(T)\big)\, d\mu_{\mathsf{HARM}}(T) \end{aligned}$$

by stationarity.

In order to compute such an integral, the following expression for the Radon-Nikodým derivative of the HARM-stationary Galton-Watson measure μ_{HARM} with respect to $\mathbf{GW}$ is useful; it has not appeared in print before, though it was alluded to in Lyons, Pemantle, and Peres (1995b). Denote by $\mathscr{R}(T)$ the effective resistance of T from its root to infinity.

Proposition 16.20. *The Radon-Nikodým derivative of μ_{HARM} with respect to $\mathbf{GW}$ is*

$$\frac{d\mu_{\mathsf{HARM}}}{d\mathbf{GW}}(T) = \frac{1}{l} \int_{T'} \frac{1}{1 + \mathscr{R}(T) + \mathscr{R}(T')}\, d\mathbf{GW}(T')\,. \tag{16.17}$$

Proof. Since the $\mathsf{SRW} \times \mathbf{AGW}$-law of $T \setminus T^{x_{-1}}$ is $\mathbf{GW}$, we have for every event A,

$$\mathbf{GW}(A) = (\mathsf{SRW} \times \mathbf{AGW})[T \setminus T^{x_{-1}} \in A]$$

and

$$\mu_{\mathsf{HARM}}(A) = (\mathsf{SRW} \times \mathbf{AGW})[T \setminus T^{x_{-1}} \in A \mid \mathsf{Exit}]\,.$$

Thus, using Proposition 16.14, we have

$$\begin{aligned} \frac{d\mu_{\mathsf{HARM}}}{d\mathbf{GW}}(\Gamma) &= \frac{(\mathsf{SRW} \times \mathbf{AGW})[T \setminus T^{x_{-1}} = \Gamma \mid \mathsf{Exit}]}{(\mathsf{SRW} \times \mathbf{AGW})[T \setminus T^{x_{-1}} = \Gamma]} \\ &= \frac{1}{l} (\mathsf{SRW} \times \mathbf{AGW})[\mathsf{Exit} \mid T \setminus T^{x_{-1}} = \Gamma]\,. \end{aligned}$$

Note that on Exit, we have $x_{-1} = (x_{-\infty})_1$. Thus,

$$\frac{d\mu_{\mathsf{HARM}}}{d\mathbf{GW}}(\Gamma) = \frac{1}{l} (\mathsf{SRW} \times \mathbf{AGW})[\vec{x} \subset \Gamma \text{ and } x_{-\infty} \notin \partial\Gamma \bigm| T \setminus T^{x_{-1}} = \Gamma]\,. \tag{16.18}$$

Recall that under SRW_T, $\overleftarrow{x}$ and $\vec{x}$ are independent simple random walks starting at $\mathrm{root}(T)$. Given disjoint trees T_1, T_2, define $[T_1 \bullet\!\!- T_2]$ to be the tree rooted at $\mathrm{root}(T_1)$ formed by joining $\mathrm{root}(T_1)$ and $\mathrm{root}(T_2)$ by an edge. If μ is a measure on trees, then $[T_1 \bullet\!\!- \mu]$ denotes the law of $[T_1 \bullet\!\!- T_2]$ when T_2 has the law of μ; and similarly for other notation. Also, the $(\mathsf{SRW} \times \mathbf{AGW} \mid T \setminus T^{x_{-1}} = \Gamma)$-law of $T^{x_{-1}}$ is $\mathbf{GW}$, whence the $(\mathsf{SRW} \times \mathbf{AGW} \mid T \setminus T^{x_{-1}} = \Gamma)$-law of T is $[\Gamma \bullet\!\!-\mathbf{GW}]$. Since the conditioning in (16.18) forces $x_{-1} \notin \Gamma$, we have

$$\begin{aligned}
\frac{d\mu_{\mathsf{HARM}}}{d\mathbf{GW}}(\Gamma) &= \frac{1}{l}\gamma(\Gamma)\int \mathsf{HARM}_{[T'\bullet\!-\Gamma]}(\partial T')\, d\mathbf{GW}(T') \\
&= \frac{\gamma(\Gamma)}{l}\int \frac{\mathscr{C}(T')}{\gamma(\Gamma)+\mathscr{C}(T')}\, d\mathbf{GW}(T') \\
&= \frac{1}{l}\int \frac{1}{\gamma(\Gamma)^{-1}+\mathscr{C}(T')^{-1}}\, d\mathbf{GW}(T') \\
&= \frac{1}{l}\int \frac{1}{1+\mathscr{R}(\Gamma)+\mathscr{R}(T')}\, d\mathbf{GW}(T')\,. \qquad \blacktriangleleft
\end{aligned}$$

Of course, it follows that

$$l = \iint \frac{1}{1+\mathscr{R}(T)+\mathscr{R}(T')}\, d\mathbf{GW}(T)\, d\mathbf{GW}(T')\,;$$

the right-hand side may be thought of as the $[\mathbf{GW} \bullet\!\!-\mathbf{GW}]$-expected effective conductance from $-\infty$ to $+\infty$, where the boundary of one of the $\mathbf{GW}$ trees is $-\infty$ and the boundary of the other is $+\infty$.

The next computational step is to find the $\mathbf{GW}$-law of $\mathscr{R}(T)$, or, equivalently, of $\gamma(T)$. Since

$$\gamma(T) = \frac{\mathscr{C}(T)}{1+\mathscr{C}(T)} = \frac{\sum_{|x|=1}\gamma(T^x)}{1+\sum_{|x|=1}\gamma(T^x)}\,, \tag{16.19}$$

we have for $s \in (0,1)$,

$$\gamma(T) \le s \iff \sum_{|x|=1}\gamma(T^x) \le \frac{s}{s-1}\,.$$

Since $\gamma(T^x)$ are i.i.d. under $\mathbf{GW}$ with the same law as γ, the $\mathbf{GW}$-c.d.f. F_γ of γ satisfies

$$F(s) = \begin{cases} \sum_k p_k F^{*k}\left(\dfrac{s}{1-s}\right) & \text{if } s \in (0,1); \\ 0 & \text{if } s \le 0; \\ 1 & \text{if } s \ge 1. \end{cases} \tag{16.20}$$

Theorem 16.21. *The functional equation (16.20) has exactly two solutions, F_γ and the Heaviside function $\mathbf{1}_{[0,\infty)}$. Define the operator on c.d.f.'s*

$$\mathscr{K} : F \mapsto \sum_k p_k F^{*k}\left(\frac{s}{1-s}\right) \qquad \big(s \in (0,1)\big)\,.$$

For any initial c.d.f. F with $F(0) = 0$ and $F(1) = 1$ other than the Heaviside function, we have weak convergence under iteration to F_γ:

$$\lim_{n\to\infty} \mathscr{K}^n(F) = F_\gamma\,.$$

For a proof, see Lyons, Pemantle, and Peres (1997).

Theorem 16.21 provides a method of calculating F_γ. It is not known that F_γ has a density, but calculations support a conjecture that it does. In the case that the offspring distribution is bounded and always at least 2, Perlin (2001) proved that this holds and, in fact, that the effective conductance has a bounded density. Some graphs of the **GW**-density of $\gamma(T)$ for certain progeny distributions appear in Figures 16.4–16.6. These graphs reflect the stochastic self-similarity of the Galton-Watson trees. Consider, for example, Figure 16.4. Roughly speaking, the peaks represent the number of generations with no branching. For example, note that the full binary tree has conductance 1, whence its γ value is 1/2. Thus, the tree with one child of the root followed by the full binary tree has conductance 1/2 and γ value 1/3. The wide peak at the right of Figure 16.4 is thus due entirely to those trees that begin with two children of the root. The peak to its left, roughly lying over 0.29, is due, at first approximation, to an unspecified number of generations without branching, while the nth peak to the left of it is due to n generations without branching (and an unspecified continuation). Of course, the next level of approximation deals with further resolution of the peaks; for example, the central peak over 0.29 is actually the sum of two nearby peaks.

Numerical calculations (still for the case $p_1 = p_2 = 1/2$) give the mean of $\gamma(T)$ to be about 0.297, the mean of $\mathscr{C}(T)$ to be about 0.44, and the mean of $\mathscr{R}(T)$ to be about 2.76. This last can be compared with the mean energy of the equally-splitting flow VIS_T, which is exactly 3:

▷ **Exercise 16.10.**

Show that the mean energy of VIS_T is $(1 - \mathbf{E}[1/Z_1])^{-1} - 1$.

In terms of F_γ, we have

$$d = -\int \log\big(1-\gamma(T)\big)\,d\mu_{\mathsf{HARM}}(T) = -\frac{1}{l}\int_{s=0}^{1}\int_{t=0}^{1} \frac{\log(1-s)}{s^{-1}+t^{-1}-1}\,dF_\gamma(t)\,dF_\gamma(s)\,.$$

Using this equation for $p_1 = p_2 = 1/2$, we have calculated that the dimension of harmonic measure is about 0.38, i.e., about $\log 1.47$, which should be compared with the dimensions of visibility measure, $\log\sqrt{2}$, and of limit uniform measure, $\log 1.5$. One can also calculate that the mean number of children of the vertices visited by a HARM_T path, which is the same as the μ_{HARM}-mean degree of the root, is about 1.58. This is about halfway between the average seen by the entire walk (and by simple forward walk), namely, exactly 1.5, and the average seen by a UNIF_T path, 5/3. This last calculation comes from Exercise 16.5 that a UNIF_T-path has the law of $\widehat{\mathbf{GW}}_*$; from Section 12.1, we know that this implies that the number of children of a vertex on a UNIF_T-path has the law of the size-biased variable $\widehat{Z}_1$.

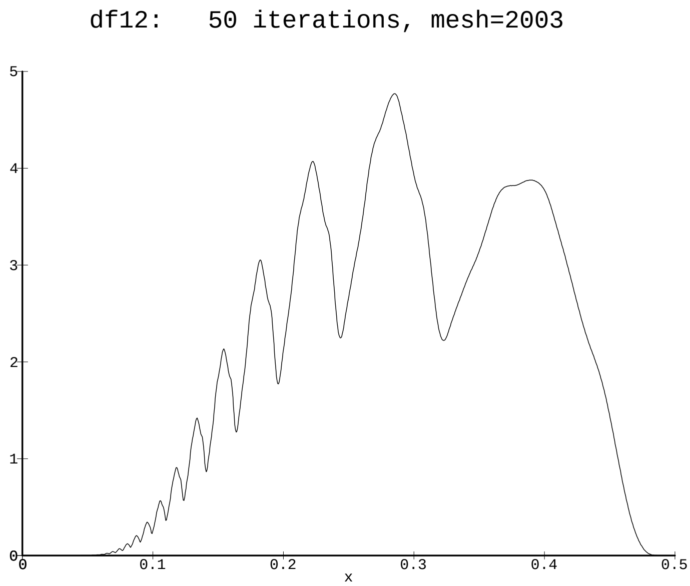


Figure 16.4. The **GW**-density of $\gamma(T)$ for $f(s) = (s + s^2)/2$.

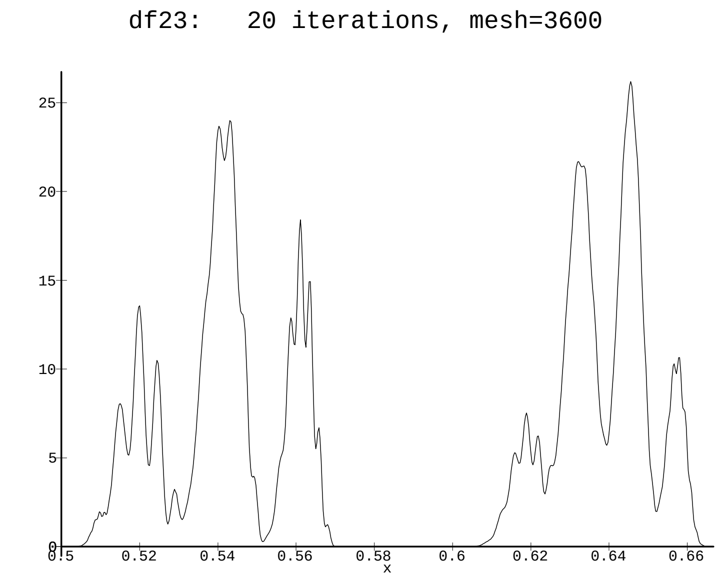


Figure 16.5. The **GW**-density of $\gamma(T)$ for $f(s) = (s^2 + s^3)/2$.

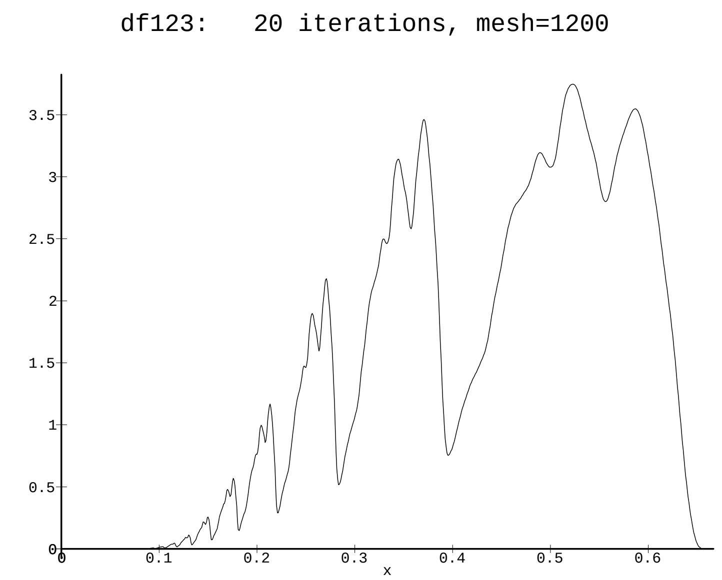


Figure 16.6. The **GW**-density of $\gamma(T)$ for $f(s) = (s + s^2 + s^3)/3$.

§16.8. Notes.

Other uses of some of the ideas in Section 16.2 appear in Furstenberg and Weiss (2003), who show "tree-analogues" of theorems of van der Waerden and Szemerédi on arithmetic progressions.

It was shown in Theorem 13.17 that the speed of simple random walk on a Galton-Watson tree with mean m is strictly smaller than the speed of simple random walk on a deterministic tree where each vertex has m children ($m \in \mathbb{N}$). Since we have also shown that simple random walk is essentially confined to a smaller subtree of growth e^d, it is natural to ask whether its speed is, in fact, smaller than $(e^d - 1)/(e^d + 1)$. This is true and was shown by Virág (2000b).

§16.9. Collected In-Text Exercises.

16.1. Show that the Hausdorff dimension of harmonic measure is a.s. constant.

16.2. Let $\mathscr{T}$ be the space of trees in Exercise 5.2. Call two trees ***(rooted) isomorphic*** if there is a bijection of their vertex sets preserving adjacency and mapping one root to the other. For $T \in \mathscr{T}$, let $[T]$ denote the set of trees that are isomorphic to T. Let $\mathscr{T}_0$ denote the set of trees whose only automorphism is the identity (it is not required that the root be fixed) and let $[\mathscr{T}_0] := \{[T]\,;\ T \in \mathscr{T}_0\}$. Show that the metric on $\mathscr{T}$ induces an incomplete separable metric on $[\mathscr{T}_0]$. Show that $\mathbf{GW}$ is concentrated on $\mathscr{T}_0$ provided $p_k \neq 1$ for all k.

16.3. Define formally the space of flows on trees with a natural metric. Define "consistent flow rule" on a space of flows on $[\mathscr{T}_0]$.

16.4. Show that Lemma 16.4 may not be true if μ is an infinite stationary measure.

16.5. The Markov chain of Proposition 16.9 is closely connected to the size-biased Galton-Watson trees of Section 12.1. Show that in case $\mathbf{E}[Z_1 \log Z_1] < \infty$, the distribution of a UNIF_T-path is $\widehat{\mathbf{GW}}_*$.

16.6. Prove Shannon's inequality.

16.7. Prove Proposition 16.14.

16.8. Suppose that $p_0 > 0$. Show that given nonextinction, the dimension of harmonic measure is a.s. less than $\log m$.

16.9. Prove (16.8).

16.10. Show that the mean energy of VIS_T is $(1 - \mathbf{E}[1/Z_1])^{-1} - 1$.

§16.10. Additional Exercises.

16.11. Show that the hypothesis of Proposition 16.3 is needed. You may want to use two flow rules that both follow a 2-ray when it exists (see Exercise 16.12) but do different things otherwise.

16.12. Give an example of a flow rule Θ with a Θ-stationary measure that is absolutely continuous with respect to **GW** but whose associated Markov chain is *not* ergodic as follows. Call a ray $\xi \in \partial T$ an ***n-ray*** if every vertex in the ray has exactly n children and write $T \in A_n$ if ∂T contains an n-ray. Note that A_n are pairwise disjoint. Consider the Galton-Watson process with $p_3 := p_4 := 1/2$. Show that $\mathbf{GW}(A_n) > 0$ for $n = 3, 4$. Define Θ_T to choose equally among all children of the root on $(A_3 \cup A_4)^c$ and to choose equally among all children of the root belonging to an n-ray when $T \in A_n$. Show that $\mathbf{GW}_{A_n}$ is Θ-stationary for both $n = 3, 4$, whence the Θ-stationary measure $(\mathbf{GW}_{A_3} + \mathbf{GW}_{A_4})/2$ gives a non-ergodic Markov chain.

16.13. Let T be the Fibonacci tree of Exercise 13.23. Show that the dimension of harmonic measure for RW_λ on T is

$$\frac{1+\sqrt{\lambda+1}}{2+\sqrt{\lambda+1}} \log(1+\sqrt{\lambda+1}) - \frac{\sqrt{\lambda+1}}{(2+\sqrt{\lambda+1})} \log\sqrt{\lambda+1}\,.$$

The following sequence of exercises, 16.14–16.22, treats the ideas of Furstenberg (1970) that inspired those of Section 16.2. They will not be used in the sequel. We adopt the setting and notation of Section 14.5. In particular, fix an integer r and let $\mathbf{T}$ be the r-ary tree.

16.14. Let $\mathscr{U}$ be the space of unit flows on $\mathbf{T}$. Give $\mathscr{U}$ a natural compact topology.

16.15. A Markov chain on $\mathscr{U}$ is called ***canonical*** if it has transition probabilities $p(\theta, \theta^v) = \theta(v)$ for $|v| = 1$. Any Borel probability measure μ on $\mathscr{U}$ can be used as an initial distribution to define a canonical Markov chain on $\mathscr{U}$, denoted $\mathsf{Markov}(\mu)$. Regard $\mathsf{Markov}(\mu)$ as a Borel probability measure on path space $\mathscr{U}^\infty$ (which has the product topology). Show that the set of canonical Markov chains on $\mathscr{U}$ is weak*-compact and convex.

16.16. Let S be the left shift on $\mathscr{U}^\infty$. For a probability measure μ on $\mathscr{U}$, let $\mathsf{Stat}(\mu)$ be the set of weak*-limit points of

$$\frac{1}{N}\sum_{n=0}^{N-1} S^n \mathsf{Markov}(\mu)\,.$$

Show that $\mathsf{Stat}(\mu)$ is nonempty and consists of stationary canonical Markov chains.

16.17. Let f be a continuous function on $\mathscr{U}^\infty$, μ be a probability measure on $\mathscr{U}$, and $\nu \in \mathsf{Stat}(\mu)$. Show that $\int f\,d\nu$ is a limit point of the numbers

$$\frac{1}{N}\sum_{n=0}^{N-1} \int \sum_{|v|=n} \theta(v) \int f(\theta^v, \theta^{v_1}, \theta^{v_2}, \ldots)\, d\theta^v(v_1, v_2, \ldots)\, d\mu(\theta)\,,$$

where we identify $\mathscr{U}$ with the set of Borel probability measures on $\partial\mathbf{T}$.

16.18. For any probability measure μ on $\mathscr{U}$, define its ***entropy*** as $\mathsf{Ent}(\mu) := \int F(\theta)\, d\mu(\theta)$, where

$$F(\theta) := \sum_{|v|=1} \theta(v) \log \frac{1}{\theta(v)}\,.$$

Define this also to be the entropy of the associated Markov chain $\mathsf{Markov}(\mu)$. Show that $\mathsf{Ent}(\mu) = \int H\, d\mathsf{Markov}(\mu)$, where

$$H(\theta, \theta^{v_1}, \theta^{v_2}, \ldots) := \log \frac{1}{\theta(v_1)}\,.$$

Show that if $\nu \in \mathsf{Stat}(\mu)$, then $\int H\, d\nu$ is a limit point of the numbers

$$\frac{1}{N} \sum_{n=0}^{N-1} \int \sum_{|v|=n} \theta(v) \log \frac{1}{\theta(v)}\, d\mu(\theta)\,.$$

16.19. Suppose that the initial distribution is concentrated at a single unit flow θ_0, so that $\mu = \delta_{\theta_0}$. Show that if for all v with $\theta(v) \neq 0$, we have

$$\alpha \le \frac{1}{|v|} \log \frac{1}{\theta_0(v)} \le \beta\,,$$

then for all $\nu \in \mathsf{Stat}(\mu)$, we have $\alpha \le \mathsf{Ent}(\nu) \le \beta$.

16.20. Show that if ν is a stationary canonical Markov chain with initial distribution μ, then its entropy is

$$\mathsf{Ent}(\nu) = \lim_{N \to \infty} \int \frac{1}{N} \sum_{|v|=N} \theta(v) \log \frac{1}{\theta(v)}\, d\mu(\theta)\,.$$

16.21. Show that if ν is a stationary canonical Markov chain, then

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{1}{\theta(v_n)}$$

exists for almost every trajectory $\langle \theta^{v_1}, \theta^{v_2}, \ldots \rangle$ and has expectation $\mathsf{Ent}(\nu)$. If the Markov chain is ergodic, then this limit is $\mathsf{Ent}(\nu)$ a.s.

16.22. Show that if T is a tree of uniformly bounded degree, then there is a stationary canonical Markov chain ν such that for almost every trajectory $\langle \theta, \theta^{v_1}, \theta^{v_2}, \ldots \rangle$, the flow θ is carried by a derived tree of T and

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{1}{\theta(v_n)} = \mathsf{Ent}(\nu) = \dim\sup \partial T\,.$$

Show similarly that there is a stationary canonical Markov chain ρ such that for almost every trajectory $\langle \theta, \theta^{v_1}, \theta^{v_2}, \ldots \rangle$, the flow θ is carried by a derived tree of T and

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{1}{\theta(v_n)} = \mathsf{Ent}(\rho) \le \dim\inf \partial T\,.$$

16.23. Let T be a Galton-Watson tree without extinction. Suppose that $\mathbf{E}[L^2] < \infty$. Consider the flow θ on T of strength $W(T)$ given by $\theta(e(x)) := W(T^x)/m^{|x|}$, i.e., the flow corresponding to the measure $W(T)\mathsf{UNIF}_T$. Show that $\mathbf{E}[W^2] = 1 + \mathrm{Var}(L)/(m-1)$. Show that if $\lambda < m$, then for the conductances $c(e) := \lambda^{-|e|}$, we have $\mathbf{E}[\mathscr{E}_c(\theta)] = \lambda \mathbf{E}[W^2]/(m-\lambda)$. Show that for these same conductances, the expected effective conductance from the root to infinity is at least $(m-\lambda)/(\lambda \mathbf{E}[W^2])$. Use this to give another proof that for every Galton-Watson tree of mean $m > 1$ (without restriction on $\mathbf{E}[L^2]$), RW_λ is transient a.s. given non-extinction for all $\lambda < m$.

16.24. Show that under the assumptions of Theorem 13.8, harmonic measure of simple random walk has positive Hausdorff dimension. This gives another proof that μ-a.e. tree has branching number > 1.

16.25. Define loop-erased simple random walk as the limit x_∞ of a simple random walk path $\vec{x}$. Show that the (expected) probability that the path of a loop-erased simple random walk from the root of T does not intersect the path of an independent (not loop-erased) simple random walk from the root of T is the speed $\mathbf{E}[(Z_1 - 1)/(Z_1 + 1)]$. Hence by Proposition 10.19, the chance that the root of T does not belong to the same tree as a uniformly chosen neighbor in the wired uniform spanning forest on T is the speed $\mathbf{E}[(Z_1 - 1)/(Z_1 + 1)]$.

Comments on Exercises

Chapter 1

1.1. Let $\lambda > \underline{\mathrm{gr}}\, T$. Then $\lim_{n\to\infty} |T_n|\lambda^{-n} = 0$. The total amount of water that can flow from o to infinity is bounded by $|T_n|\lambda^{-n}$.

Chapter 2

2.1. **(c)** Use part (a).

2.8. By symmetry, all vertices at a given distance from o have the same voltage. Therefore, they can be identified without changing any voltages (or currents). This yields the graph $\mathbb{N}$ with multiple edges and loops. We may remove the loops. We see that the parallel edges between $n-1$ and n are equivalent to a single edge whose conductance is C_n. These new edges are in series.

2.10. The complete bipartite graph $K_{4,4}$.

2.13. Use that the minimum occurs iff F is harmonic at each $x \notin A \cup Z$. Or, let i be the unit current flow from A to Z and use the Cauchy-Schwarz inequality:

$$\begin{aligned}\sum_{e\in\mathsf{E}_{1/2}} dF(e)^2 c(e)\mathscr{R}(A\leftrightarrow Z) &= \sum_{e\in\mathsf{E}_{1/2}} dF(e)^2 c(e) \sum_{e\in\mathsf{E}_{1/2}} i(e)^2 r(e)\\ &\ge \left(\sum_{e\in\mathsf{E}_{1/2}} i(e)dF(e)\right)^2 = \left(\sum_{x\in\mathsf{V}} d^*i(x)F(x)\right)^2\\ &= \left(\sum_{x\in A} d^*i(x)F(x)\right)^2 = 1\,.\end{aligned}$$

See Griffeath and Liggett (1982), Theorem 2.1 for essentially the same statement. The minimum is the same even if we allow all F with $F \ge 1$ on A and 0 on Z.

2.14. Since $\{\chi^e\,;\ e\in\mathsf{E}_{1/2}\}$ form a basis of $\ell^2_-(\mathsf{E}, r)$ and the norms of θ_n are bounded, it follows that θ_n tend weakly to θ. Hence the norm of θ is at most $\liminf_n \mathscr{E}(\theta_n)$. Furthermore, as $d^*\theta_n(x)$ is the inner product of θ_n with the star at x, it converges to $d^*\theta$.

2.15. (This is due to T. Lyons (1983).) Let U_1, U_2 be independent uniform $[0,1]$ random variables. Take a path in W_f that stays fairly close to the points $(n, U_1 n, U_2 f(n))$ $(n \ge 1)$.

2.20. **(a)** Consider a path from x to y and sum $d\widehat{\alpha}$ along this path.

(c) The random variables X have such a density.

(d) The orthogonal projection that gives $\nabla\widehat{\alpha}$ produces $|\mathsf{V}| - 1$ linearly independent random variables.

2.21. Use Exercise 2.1.

2.24. This is called the ***Riesz decomposition***. The decomposition also exists and is unique if instead of $f \ge 0$, we require $\mathcal{G}|f| < \infty$; we still have $f \ge 0$.

2.26. Express the equations for i_c by Exercise 2.2. Cramer's rule gives that i_c is a rational function of c.

2.27. The corresponding conductances are $c^h(x, y) := c(x, y)h(x)h(y)$.

2.30. Decompose the random walk run for infinite time and starting at x into the excursions between visits to x. On each excursion, use (2.4) to calculate the probability that A is visited given that $A \cup Z$ is visited. See Berger, Gantert, and Peres (2003) for details.

2.31. *Solution 1.* Let u be the first vertex among $\{x, y\}$ that is visited by a random walk starting from a. Before being absorbed on Z, the walk is as likely to make cycles at u in one direction as in the other by reversibility. This leaves at most one net traversal of the edge between x and y.

Solution 2. Let, say, $v(x) \ge v(y)$. Let $\Pi := \{[u, w]\,;\ v(u) \ge v(x),\ v(w) \le v(x)\}$. Then Π is a cutset separating a from Z (see Section 3.1), whence $\sum_{[u,w]\in\Pi} i(u, w) = 1$ since $\Pi \setminus \{e\,;\ i(e) = 0\}$ is a minimal cutset. Since $i(u, w) \ge 0$ for all $[u, w] \in \Pi$ and since $[x, y] \in \Pi$, it follows that $i(x, y) \le 1$.

2.32. Use Proposition 2.2.

2.33. Use the stationary distribution from Exercise 2.1.

2.34. Verify that $\mathcal{G}_\tau(a, \bullet)$ is a stationary measure. This is due to Aldous and Fill (2002).

2.35. Assume that $\mathbf{E}_a[\tau_a^+] < \infty$ and use the ideas of Exercise 2.34 to show that there is a stationary probability measure.

2.37. This is well known. One proof of (a) and (b) is to apply the martingale convergence theorem to $\langle f(X_n)\rangle$, where f is harmonic. For (c), apply (b) to the reversed chain.

2.39. Show that f is harmonic by using Exercise 2.34. This is due to Aldous and Fill (2002).

2.40. Use Fubini's theorem. It suffices that $\sum_{e\in\xi} r(e) < \infty$ for every ray ξ.

2.41. **(b)** From part (a), we have

$$\mathbf{P}_z[\tau_A < \tau_z^+]\mu(x) = \pi(x) \sum_{\mathcal{P}} \prod_{e\in\mathcal{P}} c(e) \Big/ \prod_{w\in\mathcal{P}} \pi(w) = \pi(x)\nu(x)/\pi(z)\,,$$

where the sum is over paths $\mathcal{P}$ from z to x that visit z and x just once and do not visit $A \setminus \{x\}$.

2.45. This is known as Parrondo's paradox, as it combines games that are not winning into a winning game. See Parrondo (1996) and Harmer and Abbott (1999). For additional analysis, see Pyke (2003) and Ethier and Lee (2009). For other aspects of turning fair games into unfair ones, see Durrett, Kesten, and Lawler (1991).

2.46. **(a)** 29/63, 29/35, 17/20.

2.47. **(d)** Let G'' be G' with a loop at each x of conductance $\pi(x)p_W(x,x)$. Then the escape probability from each x to each y is the same in G as in G'', as is the sum of the conductances around x, whence we deduce equality of effective resistances. But the loops do not affect the effective resistances, whence the result. We remark that the network walk on G'' is the walk on G after inducing on W: see Exercise 6.45 and Exercise 6.46.

(e) A remarkable use of this transformation is in Caputo, Liggett, and Richthammer (2009).

2.50. Fix $o \in \mathsf{V}$ and let i_x be the unit current flow from x to o. Use $\mathscr{R}(u \leftrightarrow x) = \|i_u - i_x\|_r^2$ or use (2.20). This is equivalent to saying that the effective resistance metric (see Exercise 2.58) has 1-negative type, i.e., $\sum_{x,y\in\mathsf{V}} \mathscr{R}(x \leftrightarrow y)\alpha_x\alpha_y \le 0$ whenever $\sum_{x\in\mathsf{V}} \alpha_x = 0$. In this form, the result is due to Jorgensen and Pearse (2009) (see Theorem 5.1).

2.51. Use superposition and $\mathscr{E}(i) = (i, dv) = (d^*i, v) = (f, v)$.

2.53. **(a)** Use superposition of currents.

It is worth noting also that $v_a(x,y) = \mathcal{G}(x,y)/\pi(y)$ for the random walk killed at a (by Proposition 2.1).

2.54. This is noted by Coppersmith, Doyle, Raghavan, and Snir (1993). See also Ponzio (1998). Use Exercise 2.53.

2.56. Use Exercise 2.13 and the fact that the minimum of linear functions is concave.

2.57. Use Exercise 2.13.

2.58. There are several solutions. One involves superposition of unit currents, one from u to x and one from x to w. One can also use Exercise 2.59 (which explains the left-hand side minus the right-hand side) or Corollary 2.20.

2.59. There are many interesting proofs. For one, use Exercise 2.53. For another solution, see Tetali (1991), who discovered this formula.

2.60. Use that $\partial i/\partial r(e)$ is a flow with strength 0.

2.61. Use Exercise 2.60 and Exercise 2.57.

2.63. **(a)** The probability that the random walk leaves G_n before visiting a or z tends to 0 as $n \to \infty$. We may couple random walks X_k on G_n and Y_k on G as follows. Start at $X_0 := Y_0 := x$. Define Y_k as the usual network random walk on G, stopped when it reaches $\{a, z\}$. Define $\tau := \inf\{k\,;\ Y_k \notin G_n\}$. For $k < \tau$, let $X_k := Y_k$, while for $k \ge \tau$, continue the random walk X_k independently on G_n, stopped at $\{a, z\}$. Thus, $v_n(x)$ is the probability that X_k reaches a, while $v(x)$ is the probability that Y_k reaches a. For large n, it is likely that $\tau = \infty$, on which event either both random walks reach a or neither does.

2.65. Let $\langle G_n \rangle$ be an exhaustion by finite subnetworks. Note that the star spaces of G_n^{W} increase to $\bigstar$ and the cycle spaces of G_n increase to $\diamondsuit$. It follows from Exercises 2.64 and 2.63 that the projections of χ^e onto $\bigstar$ and the orthocomplement of $\diamondsuit$ agree for each e.

2.66. The expression given by Thomson's Principle can also be regarded as an extremal width. This identity is due to Duffin (1962). Given ℓ, find F with $|dF| \le \ell$ and use Dirichlet's principle. The minimum is the same even if we allow all ℓ with the distance between any point of A and any point of Z to be at least 1.

2.70. **(a)** Cycles at a are equally likely to be traversed in either direction. Thus, cycles contribute nothing to $\mathbf{E}[S_e - S_{-e}]$.

2.72. By Proposition 2.11, $v(X_n) = P_{X_n}[\exists k \ge n \ X_k = a]$. Now the intersection of the events $\{\exists k \ge n \ X_k = a\}$ is the event that a is visited infinitely often, which has probability 0. Since these events are also decreasing, their limiting probability is 0. That is, $E[v(X_n)] \to 0$. On the other hand, $\langle v(X_n) \rangle$ is a nonnegative supermartingale, whence it converges a.s. and $E[\lim v(X_n)] = \lim E[v(X_n)] = 0$.

2.73. Use Exercise 2.13.

2.74. Use Exercise 2.56.

2.75. Use Exercise 2.57.

2.76. Consider the random walk conditioned to return to o. (This is due to Benjamini, Gurel-Gurevich, and Lyons (2007).)

2.77. See the beginning of the proof of Theorem 3.1.

2.78. The tree of Example 1.2 will do. Order the cutsets according to the (first) edge on the "main" ray that they contain. Let A_n be the size of the nth cutset. Let $T(k)$ be the tree coming off the vertex at level k of the main ray; thus, $T(k)$ begins with two children of its root for $k \ge 1$. Show that $T(k)$ contains k levels of a binary tree. Let $B_{n,k} := \Pi_n \cap T(k)$. Thus, $A_n \ge \sum_{k=1}^{n-1} |B_{n,k}|$. Reduce the problem to the case where all vertices in $B_{n,k}$ are at level at most $3k$ for $n \le 2k$. Use (2.15) to show that $\sum_{n=k+1}^{2k} |B_{n,k}|^{-1} < 2$ for each k. Deduce that $\sum_n A_n^{-1} < \infty$ by using the Arithmetic-Harmonic Mean Inequality.

Alternatively, use Exercise 2.79.

2.79. This is due to Yoram Gat (1997, personal communication). Write $A := \{\Pi_n \,;\ n \ge 1\}$ and $S(A) := \sum_n |\Pi_n|^{-1}$. We may clearly assume that each Π_n is minimal. Suppose that there is some edge $e \notin \bigcup_n \Pi_n$. Now there is some Π_k that does not separate the root from e. Let Π'_k be the cutset obtained from Π_k by replacing all the descendants of e in Π_k by e. Then $|\Pi'_k| \le |\Pi_k|$. Let $A' := \{\Pi_n \,;\ n \ne k\} \cup \{\Pi'_k\}$. Then A' is also a sequence of disjoint cutsets and $S(A') \ge S(A)$.

If we order the edges of T in any fashion that makes $|e|$ nondecreasing and apply the above procedure recursively to all edges not in the current collection of cutsets, we obtain a sequence A_i of collections of disjoint cutsets with $S(A_i)$ nondecreasing in i. Let $A_\infty := \liminf A_i$. Then A_∞ is also a sequence of disjoint cutsets, and $S(A_\infty) \ge S(A)$. Furthermore, each edge of T appears in some element of A_∞.

Call two cutsets ***comparable*** if one separates the root from the other. Suppose that there are two cutsets of A_∞ that are not comparable, Π_n and Π_m. Then we may create two new cutsets Π'_n and Π'_m, that are comparable and whose union contains the same edges as $\Pi_n \cup \Pi_m$. Since $|\Pi'_n| + |\Pi'_m| = |\Pi_n| + |\Pi_m|$ and $\min\{|\Pi'_n|, |\Pi'_m|\} \le \min\{|\Pi_n|, |\Pi_m|\}$, it follows that $|\Pi'_n|^{-1} + |\Pi'_m|^{-1} \ge |\Pi_n|^{-1} + |\Pi_m|^{-1}$. By replacing Π_n and Π_m in A_∞ by Π'_n and Π'_m and repeating this procedure as long as there are at least two cutsets in the collection that are not comparable, we obtain in the limit a collection A'_∞ of disjoint pairwise comparable cutsets containing each edge of T and such that $S(A'_\infty) \ge S(A)$. But the only collection of disjoint pairwise comparable cutsets containing each edge of T is $A'_\infty = \{T_n\,;\ n \ge 1\}$.

2.80. This is due to I. Benjamini (personal communication, 1996).

(a) Given cutsets, consider the new network where the edges of the cutsets are divided in two by an extra vertex for each edge, each half getting half the resistance.

(b) Reverse time.

(c) Note that A_n are independent.

2.85. Use Proposition 2.19 and Exercise 2.59. This is due to Tetali (1991).

2.86. Decompose a path of length k that starts at x and visits y into the part to the last visit of y and the rest (which may be empty). Reverse the first part and append the result of moving the second part to x via a fixed automorphism. The 1-skeleton of the truncated tetrahedron is an example of a transitive graph for which there are two vertices x and y such that no automorphism interchanges x and y.

2.87. The first identity (2.21) follows from a path-reversal argument. For the second, (2.22), use Exercise 2.85. This proof was given by Coppersmith, Tetali, and Winkler (1993), who discovered the result. The first equality of (2.22) does not follow from an easy path-reversal argument: consider, e,g,, the path $\langle x, y, x, y, z, x\rangle$, where $\tau_{y,z,x} = 6$, but for the reversed path, $\tau_{z,y,x} = 4$.

2.88. A tricky bijective proof was given by Tanushev and Arratia (1997). A simpler proof proceeds by showing the equality with "$\le k$" in place of "$= k$". Decompose a path of length k into a cycle at x that completes the tour plus a path that does not return to x; reverse the cycle.

2.89. Run the chain from x until it first visits a and then z. This will also be the first visit to z from x, unless $\tau_z < \tau_a$. In the latter case, the path from x to a to z involves an extra commute from z to a beyond time τ_z. Taking expectations yields

$$\mathbf{E}_x[\tau_a] + \mathbf{E}_a[\tau_z] = \mathbf{E}_x[\tau_z] + \mathbf{P}_x[\tau_z < \tau_a]\big(\mathbf{E}_z[\tau_a] + \mathbf{E}_a[\tau_z]\big)$$

This yields the formula. In the reversible case, the cycle identity (2.22) yields

$$\mathbf{E}_x[\tau_a] + \mathbf{E}_a[\tau_z] - \mathbf{E}_x[\tau_z] = \mathbf{E}_a[\tau_x] + \mathbf{E}_z[\tau_a] - \mathbf{E}_z[\tau_x]\,.$$

Adding these two quantities gives a sum of two commute times minus a third. Let γ denote the sum of all edge conductances. Then by the commute time formula Corollary 2.20, the denominator in (2.23) is $2\gamma\mathscr{R}(a \leftrightarrow z)$ and the numerator is $\gamma[\mathscr{R}(x \leftrightarrow a) + \mathscr{R}(a \leftrightarrow z) - \mathscr{R}(z \leftrightarrow x)]$.

2.90. Use Proposition 2.19. As shown by Peres, in the case that G is a tree, we also have that $\mathbf{E}_x\left[\binom{\tau_y}{k}\right] \in \mathbb{N}$ for all $k \in \mathbb{N}$. Here is a sketch: This is the same as saying that $\mathbf{E}_x[(1+s)^{\tau_y}]$ has integer coefficients in s. Writing τ_y as a sum of hitting times over the path from x to y shows that it suffices to consider x and y neighbors. Of course, we may also assume y is a leaf. In fact, we may consider the return time $\mathbf{E}_y[(1+s)^{\tau_y^+}] = \mathbf{E}_x[(1+s)^{1+\tau_y}]$. The return time to y is $1 +$ a sum of a random number G of excursion lengths from x in the tree minus y, the random number G having a geometric distribution with parameter $1/d$, where d is the degree of x. Note that since $\mathbf{P}[G = k] = (1-1/d)^k/d$ for $k \ge 0$, we have $\mathbf{E}[(1+s)^G] = \sum (d-1)^k s^k$. In the tree where y was deleted, each excursion is to one of the $d-1$ neighbors of x other than y, each with probability $1/(d-1)$. Furthermore, the excursion lengths above are return times to x in a smaller tree. Putting all this together gives the result.

2.91. Use the fact that the expected number of visits to x by time n is at most $\mathbf{P}_o[\tau_x \le n]\mathcal{G}(x,x)$.

2.92. Use Propositions 2.1 and 2.2.

2.94. This is due to Thomas Sauerwald (personal communication, 2007). Let the distance be n and let Π_k be the edges with one endpoint at distance k from a and the other at distance $k+1$ from a, where $0 \le k < n$. Write $A_k := \sum_{e \in \Pi_k} c(e)$. Corollary 2.20 and the finitistic analogue of (2.15) give that the commute time is

$$2\mathscr{R}(a \leftrightarrow z) \sum_{e \in \mathsf{E}_{1/2}} c(e) \ge 2 \sum_{k=0}^{n-1} A_k^{-1} \sum_{k=0}^{n-1} A_k \ge 2n^2$$

by the Cauchy-Schwarz inequality.

2.95. The voltage is constant on the vertices with fixed coordinate sum.

2.96. **(a)** This is a deterministic result.

(b) Use Exercise 2.20 with $\alpha := v$. This also follows from Exercise 2.53 and the standard result relating the covariance of Gaussians to their density (note that $\|dU\|_c^2 = (U, \Delta_G U)$).

2.97. Add an edge from every vertex to a new vertex and apply Exercise 2.96.

Chapter 3

3.1. The random path $\langle Y_n \rangle$ visits x at most once.

3.3. Let $Z := \{e^+ ;\ e \in \Pi\}$ and let G be the subtree of T induced by those vertices that Π separates from ∞. Then the restriction of θ to G is a flow from o to Z, whence the result is a consequence of Lemma 2.8.

3.4. Consider spherically symmetric trees.

3.10. **(a)** Let $\beta > \inf a_n/n$ and let m be such that $a_m/m < \beta$. Write $a_0 := 0$. For any n, write $n = qm + r$ with $0 \le r \le m-1$; then we have

$$a_n = a_{qm+r} \le a_m + \cdots + a_m + a_r = qa_m + a_r\,,$$

whence

$$\frac{a_n}{n} \le \frac{qa_m + a_r}{qm + r} < \frac{a_m}{m} + \frac{a_r}{n} < \beta + \frac{a_r}{n}\,,$$

whence

$$\limsup_{n\to\infty} \frac{a_n}{n} \le \beta\,.$$

(b) Modify the proof so that if an infinite a_r appears, then it is replaced by a_{m+r} instead.
(c) Observe that $\langle \log |T_n| \rangle$ is subadditive.

3.14. Suppose that w_x is broken arbitrarily as the concatenation of two words w_1 and w_2. Let x_i be the product of the generators in w_i ($i = 1, 2$). Then $x = x_1 x_2$. If for some i, we had $w_{x_i} \ne w_i$, then we could substitute the word w_{x_i} for w_i and find another word w whose product was x yet would be either shorter than w_x or come earlier lexicographically than w_x. Either of these circumstances would contradict the definition of w_x as minimal, whence $w_i = w_{x_i}$ for both i.

This fact for $i = 1$ shows that T is a tree; for $i = 2$, it shows that T is subperiodic.

3.15. Replace each vertex x by two vertices x' and x'' with an edge from x' to x'' of capacity $c(x)$. All edges that led into x now lead into x', while all edges that led out of x now lead out of x''. Apply the Max-Flow Min-Cut Theorem for directed networks (with edge capacities).

3.17. Use Exercise 3.15(a).

3.19. A minimal cutset is the same as a cut, as defined in Exercise 2.52.

3.21. (This is Proposition 13 of Hoffman, Holroyd, and Peres (2006).) Find a flow $\theta \le q$. Consider θ' of Proposition 3.2. Show that $\mathscr{E}(\theta') < \infty$.

3.22. (This is due to Peres.)

3.23. Let G be the subtree of T induced by those vertices that Π separates from ∞. If G were infinite, then we could find a path $o = x_0, x_1, x_2, \ldots$ of vertices in G such that each x_{i+1} is a child of x_i by choosing x_{i+1} to be a child of x_i with an infinite number of descendants in G. This would produce a path from o to ∞ that did not intersect Π, which contradicts our assumption. Hence G is finite. Let Π' be the set of edges joining a vertex of G to a vertex of $T \setminus G$ that has an infinite number of descendants.

3.24. If T is transient, then the voltage function, normalized to take the value 1 at the root o and vanish at infinity, works as F. Conversely, given F, use the Max-Flow Min-Cut Theorem to find a nonzero flow θ from the root, with $\theta(e) \le dF(e)c(e)$ for all e. Thus $\sum_{e\in\xi} \theta(e)/c(e) \le F(o)$ for every path ξ from o. For each path ξ connecting o to x, multiply this inequality by $\theta\big(e(x)\big)$ and sum over $x \in T_n$. This implies that θ has finite energy.

3.27. This an immediate consequence of the Max-Flow Min-Cut Theorem. In the form of Exercise 14.14, it is due to Frostman (1935).

3.28. Let $x \in T$ with $|x| > k$. Let u be the ancestor of x which has $|u| = |x| - k$. Then the embedding of T^u into T embeds T^x into T^w for some $w \in T_k$.

3.30. Consider the directed graph on $0, 1, \ldots, k$ with edges $\langle i, j \rangle$ for $i \le j$ and also $\langle k, 0 \rangle$.

3.31. This is part of Theorem 5.1 of Lyons (1990).

3.32. This seems to be new. Let β be an irrational number. For real x, y and $k \in \mathbb{N}$, set $f_k(x, y) := \big(\mathbf{1}_{[0,1/2]}(\lfloor x + k\beta \rfloor), \mathbf{1}_{[0,1/2]}(\lfloor y + k\beta \rfloor)\big)$ and let $F_n(x, y)$ be the sequence $\langle f_k(x, y) ;\ 0 \le k < n\rangle$. Let T be the tree of all finite sequences of the form $F_n(x, y)$ for $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$, together will the null sequence, which is the root of T. Join $F_n(x, y)$ to $F_{n+1}(x, y)$ by an edge. Then $|T_n| = 4n^2$. Let λ_2 be Lebesgue measure on $[0, 1]^2$ and for vertices $w \in T$, set $\theta(w) := \lambda_2\{(x, y) ;\ F_{|w|}(x, y) = w\}$. Now choose β not well-approximable by rationals, e.g., $\sqrt{5}$. Then θ is approximately uniform on T_n, whence has finite energy for unit conductances. That is, simple random walk is transient.

3.33. Given $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ that correspond to vertices in T, the sequence obtained by concatenating them with N zeros in between also corresponds to a vertex in T. Thus T is N-superperiodic and $\operatorname{br} T = \operatorname{gr} T$. To rule out $(N-1)$-superperiodicity, use rational approximations to α. If $\alpha > 1/2$ or $\alpha = 1/2$, then $\operatorname{gr} T = 2$ by the SLLN or the Ballot Theorem respectively. If $\alpha < 1/2$, then Cramer's theorem on large deviations, or Stirling's formula, imply that $\operatorname{gr} T$ is the binary entropy of α.

CHAPTER 4

4.1. For the random walk version, this is proved in Sections 6.5 and 7.3 of Lawler (1991). Use the "craps principle" (Pitman (1993), p. 210). First prove equality for the distribution of the first step.

4.3. Use also Rayleigh's monotonicity law.

4.6. (This is also due to Feder and Mihail (1992).) We follow the proof of Theorem 4.5. We induct on the number of edges of G. Given $\mathscr{A}$ and $\mathscr{B}$ as specified, there is an edge e on which $\mathscr{A}$ depends (and so $\mathscr{B}$ ignores) such that $\mathscr{A}$ is positively correlated with the event $e \in T$ (since $\mathscr{A}$ is negatively correlated with those edges that $\mathscr{A}$ ignores). Thus, $\mathbf{P}[\mathscr{A} \mid e \in T] \ge \mathbf{P}[\mathscr{A} \mid e \notin T]$. Now

$$\mathbf{P}[\mathscr{A} \mid \mathscr{B}] = \mathbf{P}[e \in T \mid \mathscr{B}]\mathbf{P}[\mathscr{A} \mid \mathscr{B}, e \in T] + \mathbf{P}[e \notin T \mid \mathscr{B}]\mathbf{P}[\mathscr{A} \mid \mathscr{B}, e \notin T]\,. \tag{17.1}$$

The induction hypothesis implies that (17.1) is at most

$$\mathbf{P}[e \in T \mid \mathscr{B}]\mathbf{P}[\mathscr{A} \mid e \in T] + \mathbf{P}[e \notin T \mid \mathscr{B}]\mathbf{P}[\mathscr{A} \mid e \notin T]\,. \tag{17.2}$$

By Theorem 4.5, we have that $\mathbf{P}[e \in T \mid \mathscr{B}] \le \mathbf{P}[e \in T]$ and we have chosen e so that $\mathbf{P}[\mathscr{A} \mid e \in T] \ge \mathbf{P}[\mathscr{A} \mid e \notin T]$. Therefore, (17.2) is at most

$$\mathbf{P}[e \in T]\mathbf{P}[\mathscr{A} \mid e \in T] + \mathbf{P}[e \notin T]\mathbf{P}[\mathscr{A} \mid e \notin T] = \mathbf{P}[\mathscr{A}]\,.$$

4.7. (Compare Theorem 3.2 in Thomassen (1990).) The number of components of T when restricted to the subgraph induced by V_n is at most $|\partial_{\mathsf{E}} \mathsf{V}_n|$. A tree with k vertices has $k-1$ edges. The statement on expectation follows from the bounded convergence theorem.

4.10. You need to use $H(1, 1)$ to find $Y(e, f) = 1/2 - 1/\pi$ for e the edge from $(0, 0)$ to $(1, 0)$ and f the edge from $(0, 0)$ to $(0, 1)$. From this, the values of $Y(e, g)$ for the other edges g incident to the origin follow. Use the transfer-current theorem directly to find $\mathbf{P}[\deg_T(0,0) = 4]$. Other probabilities can be computed by Exercise 4.37 or by computing $\mathbf{P}[\deg_T(0,0) \ge 3]$ and using the fact that the expected degree is 2 (Exercise 4.7). See Burton and Pemantle (1993), p. 1346, for some of the details.

It is not needed for the solution to this problem, but here are some values of H.

4	$80 - \frac{736}{3\pi}$	$-49 + \frac{160}{\pi}$	$12 - \frac{472}{15\pi}$	$-1 + \frac{48}{5\pi}$	$\frac{704}{105\pi}$
3	$17 - \frac{48}{\pi}$	$-8 + \frac{92}{3\pi}$	$1 + \frac{8}{3\pi}$	$\frac{92}{15\pi}$	$-1 + \frac{48}{5\pi}$
2	$4 - \frac{8}{\pi}$	$-1 + \frac{8}{\pi}$	$\frac{16}{3\pi}$	$1 + \frac{8}{3\pi}$	$12 - \frac{472}{15\pi}$
1	1	$\frac{4}{\pi}$	$-1 + \frac{8}{\pi}$	$-8 + \frac{92}{3\pi}$	$-49 + \frac{160}{\pi}$
0	0	1	$4 - \frac{8}{\pi}$	$17 - \frac{48}{\pi}$	$80 - \frac{736}{3\pi}$
(x_1, x_2)	0	1	2	3	4

Such a table was first constructed by McCrea and Whipple (1940). It is used for studying harmonic measure there and in Spitzer (1976), Section 15. See these references for proofs that

$$\lim_{|x_1|+|x_2|\to\infty} \left[H(x_1, x_2) - \frac{2}{\pi} \log \sqrt{|x_1|^2 + |x_2|^2} \right] = \frac{2\gamma + \log 8}{\pi},$$

where γ is Euler's constant. In fact, the convergence is quite rapid. See Kozma and Schreiber (2004) for more precise estimates. Using the preceding table, we can calculate the transfer currents, which we give in two tables. We show $Y(e, f)$ for e the edge from $(0,0)$ to $(1,0)$ and f varying. The first table is for the horizontal edges f, labelled with the left-hand endpoint of f:

4	$-\frac{129}{2}+\frac{608}{3\pi}$	$\frac{95}{2}-\frac{746}{5\pi}$	$-\frac{37}{2}+\frac{872}{15\pi}$	$\frac{7}{2}-\frac{1154}{105\pi}$	0
3	$-\frac{25}{2}+\frac{118}{3\pi}$	$\frac{17}{2}-\frac{80}{3\pi}$	$-\frac{5}{2}+\frac{118}{15\pi}$	0	$-\frac{7}{2}+\frac{1154}{105\pi}$
2	$-\frac{5}{2}+\frac{8}{\pi}$	$\frac{3}{2}-\frac{14}{3\pi}$	0	$\frac{5}{2}-\frac{118}{15\pi}$	$\frac{37}{2}-\frac{872}{15\pi}$
1	$-\frac{1}{2}+\frac{2}{\pi}$	0	$-\frac{3}{2}+\frac{14}{3\pi}$	$-\frac{17}{2}+\frac{80}{3\pi}$	$-\frac{95}{2}+\frac{746}{5\pi}$
0	$\frac{1}{2}$	$\frac{1}{2}-\frac{2}{\pi}$	$\frac{5}{2}-\frac{8}{\pi}$	$\frac{25}{2}-\frac{118}{3\pi}$	$\frac{129}{2}-\frac{608}{3\pi}$
(x_1,x_2)	0	1	2	3	4

The second table is for the vertical edges f, labelled with the lower endpoint of f:

4	$-138+\frac{6503}{15\pi}$	$79-\frac{1241}{5\pi}$	$-25+\frac{1649}{21\pi}$	$4-\frac{1321}{105\pi}$
3	$-26+\frac{245}{3\pi}$	$13-\frac{613}{15\pi}$	$-3+\frac{47}{5\pi}$	$\frac{1}{2}-\frac{167}{105\pi}$
2	$-5+\frac{47}{3\pi}$	$2-\frac{19}{3\pi}$	$-\frac{1}{2}+\frac{23}{15\pi}$	$-3+\frac{47}{5\pi}$
1	$-1+\frac{3}{\pi}$	$\frac{1}{2}-\frac{5}{3\pi}$	$2-\frac{19}{3\pi}$	$13-\frac{613}{15\pi}$
0	$-\frac{1}{2}+\frac{1}{\pi}$	$-1+\frac{3}{\pi}$	$-5+\frac{47}{3\pi}$	$-26+\frac{245}{3\pi}$
(x_1,x_2)	1	2	3	4

Symmetries of the plane give some other values from these.

4.11. (From Propp and Wilson (1998).) Let T be a tree rooted at some vertex x. Choose any directed path $x = u_0, u_1, \ldots, u_l = x$ from x back to x that visits every vertex. For $1 \le i < l$, let $\mathcal{P}_i$ be the path $u_0, u_1, \ldots, u_i$ followed by the path in T from u_i to x. In the trajectory $\mathcal{P}_{l-1}, \mathcal{P}_{l-2}, \ldots, \mathcal{P}_1$, the last time any vertex $u \neq x$ is visited, it is followed by its parent in T. Therefore, if the chain on spanning trees begins at any spanning tree, following this trajectory (which has positive probability of happening) will lead to T. (This also shows aperiodicity.)

4.13. We have

$$\tilde{p}(T, B(T,e)) = \frac{\pi(B(T,e))}{\pi(T)} p(B(T,e),T) = \frac{\alpha(B(T,e))}{\alpha(T)} p(g) = p(e)\,.$$

4.14. A solution using the Aldous/Broder algorithm was noted by Broder (1989).

4.15. This is due to Aldous (1990).

4.18. This is due to Aldous (1990). Use Exercise 4.15 and torus grids.

4.19. There are too many spanning trees.

4.20. This is due to Edmonds (1971).

4.21. These results are due to Asadpour, Goemans, Mądry, Oveis Gharan, and Saberi (2010). Use Lagrange multipliers for (a). Deduce (b) from (a).

4.24. (From Propp and Wilson (1998).) We sum over the number of transitions out of x that are needed. We may find this by popping cycles at x. The number of such is the number of visits to x starting at x before visiting r. This is $\pi(x)(\mathbf{E}_x[\tau_r] + \mathbf{E}_r[\tau_x])$, as can be seen by considering the frequency of appropriate events in a bi-infinite path of the Markov chain. This also follows from Exercise 2.34 by using the stopping time equal to the first visit to x after the first visit to r. In the reversible case, we can also use the formulas in Section 2.2 to get

$$\sum_{x \text{ a state}} \sum_{e^- = x} c(e)\mathscr{R}(x \leftrightarrow r)\,,$$

which is the same as what is to be shown.

4.25. Use the craps principle as in the solution to Exercise 4.1.

4.26. This is originally due to Meir and Moon (1970).

4.28. This is due to Foster (1948).

4.29. Use Exercises 2.92 and 4.28. This formula is due to Coppersmith, Doyle, Raghavan, and Snir (1993).

4.30. **(a)** Add a (new) edge e from a to z of unit conductance. Then spanning trees of $G \cup \{e\}$ containing e are in 1-1 correspondence preserving $\beta(\bullet)$ with spanning trees of $G/\{a,z\}$. Thus, the right-hand side of (4.16) is $(1 - \mathbf{P}[e \in T])/\mathbf{P}[e \in T]$ in $G \cup \{e\}$. Now apply Corollary 4.4.

(b) Note that any cycles in a random walk from a to z have equal likelihood of traversing e as $-e$.

4.32. Use Exercise 4.21.

4.33. (Due to R. Lyons.) Let X be the set of subsets A of vertices of G such that both A and $\mathsf{V} \setminus A$ induce connected subgraphs. Identify each $A \in X$ with the subnetwork it induces. On X, put the measure $\mu(A) := \beta(A)\beta(A^c)/(2\beta(G))$. Map a vertex x in G to the function on X given by $A \mapsto \mathbf{1}_A(x)$. Use Exercise 4.30(a) to verify that this is an isometry into $\ell^1(X,\mu)$.

4.34. Use Wilson's algorithm with a given rung in place and a random walk started far away. Alternatively, use the Transfer-Current Theorem. This problem was originally analyzed by Häggström (1994), who used the theory of subshifts of finite type.

4.36. Use the fact that $i - i' - [i(f) - i'(f)]\chi^f$ is a flow between the endpoints of f.

4.37. For the first part, compare coefficients of monomials in x_i. The second part is Cor. 4.4 in Burton and Pemantle (1993).

4.38. Calculate the chance that all the edges incident to a given vertex are present.

4.41. This is a special case of what is proved in Feder and Mihail (1992). An earlier and more straightforward proof of a different generalization is given by Joag-Dev and Proschan (1983), paragraph 3.1(c).

4.42. Use the result of Exercise 4.41.

4.43. Compare this exact result to Proposition 2.15.

Chapter 5

5.1. Various theorems will work. The monotone convergence theorem works even when $m = \infty$.

5.2. Compare Neveu (1986).

5.3. (Compare Neveu (1986).) Let Ω be the probability space on which the random variables $L_i^{(n)}$ are defined. Then $\mathbf{GW}$ is the law of the random variable $T : \Omega \to \mathscr{T}$ defined by

$$T := \{\langle i_1, \ldots, i_n \rangle \,;\; n \geq 0,\; i_j \in \mathbb{Z}^+ \;(1 \leq j \leq n),\; i_{j+1} \leq L^{j+1}_{I(i_1,\ldots,i_j)} \;(1 \leq j \leq n-1)\}\,,$$

where $I(i_1, \ldots, i_j)$ is the index appropriate to the individual $\langle i_1, \ldots, i_j \rangle$. Actually, any injection I from the set of finite sequences to $\mathbb{Z}^+$ will do, rather than the one implicit in the definition of a Galton-Watson process, i.e., (5.1). For example, if $\langle P_n \rangle$ denotes the sequence of prime numbers, then we may use $I(i_1, \ldots, i_j) := \prod_{l=1}^{j} P_{i_l}$.

5.4. We will soon see (Corollary 5.10) that also $\operatorname{br} T = m$ a.s. given nonextinction.

5.5. Show that for each n, the event that the diameter of $K(x)$ is at least n is measurable.

5.6. One way is to let $\langle G_n \rangle$ be an exhaustion of G by finite subgraphs containing x. If

$$\mathbf{P}_p[\exists \text{ infinite-diameter cluster }] > 0\,,$$

then for some n, we have $\mathbf{P}_p[\exists u \in G_n\; u \leftrightarrow \infty] > 0$. This latter event is independent of the event that all the edges of G_n are present. The intersection of these two events is contained in the event that $x \leftrightarrow \infty$.

5.7. Let $p \geq p_c(G)$ and $p' \in [0, 1]$. Given ω_p, the law of $\omega_{pp'}$ is precisely that of percolation on ω_p with survival parameter p'. Therefore, we want to show that if $p' < p_c(G)/p$, i.e., if $pp' < p_c(G)$, then $\omega_{pp'}$ a.s. has no infinite components, while if $pp' > p_c(G)$, then $\omega_{pp'}$ a.s. has an infinite component. But this follows from the definition of $p_c(G)$.

5.13. Use Exercise 5.49 with $n := 1$ and Proposition 5.23(ii).

5.14. We have

$$G_d(s) = 1 - (1-s)^{d+1}p^{d+1} + (d+1)(1-s)^d p^d (1-p+ps).$$

Since $G_d(0) > 0$ and $G_d(1) = 1$, there is a fixed point of G_d in $(0,1)$ if $G_d(s) < s$ for some $s \in (0,1)$. Consider

$$g(s) := 1 - (1-s)^{d+1} + (d+1)(1-s)^d s$$

obtained by taking $p \to 1$. Since $g'(0) = 0$, there is certainly some $s \in (0,1)$ with $g(s) < s$. Hence the same is true for G_d when p is sufficiently close to 1.

5.15. Generate the labels at the same time as you generate the tree. That is, the root is labeled i with probability $1/k$; then there are j children of the root with probability $e^{-c}c^j/j!$ and they are labeled by a subset of $[1,k] \setminus \{i\}$ with probability $\binom{k-1}{j}^{-1}$ each, etc.

5.16. This type of percolation is known as the ***Erdős-Rényi random graph***. For (b) and (c), consider a random total ordering of the vertices of K_n. It induces a relative ordering of the vertices of $C(o)$, which in turn induces a labeling. For (c), calculate the exact value of the left-hand side before the limit is taken.

5.17. Let f_n and f be the corresponding p.g.f.'s. Then $f_n(s) \to f(s)$ for each $s \in [0,1]$.

5.22. **(a)** By symmetry, we have on the event A that

$$\begin{aligned}\mathbf{E}\left[\frac{L_i^{(n+1)}}{\sum_{j=1}^{Z_n} L_j^{(n+1)} + \sum_{j=1}^{Z'_n} {L'}_j^{(n+1)}} \;\middle|\; \mathcal{F}_n\right] &= \mathbf{E}\left[\frac{{L'}_i^{(n+1)}}{\sum_{j=1}^{Z_n} {L'}_j^{(n+1)} + \sum_{j=1}^{Z'_n} L_j^{(n+1)}} \;\middle|\; \mathcal{F}_n\right] \\ &= \frac{1}{Z_n + Z'_n}.\end{aligned}$$

(b) By part (a), we have

$$\begin{aligned}\mathbf{E}[Y_{n+1} \mid \mathcal{F}_n] &= \frac{Z_n}{Z_n + Z'_n}\mathbf{P}[A \mid \mathcal{F}_n] + Y_n \mathbf{P}[\neg A \mid \mathcal{F}_n] \\ &= \mathbf{E}\left[\frac{Z_n}{Z_n + Z'_n}\mathbf{1}_A \;\middle|\; \mathcal{F}_n\right] + \mathbf{E}[Y_n \mathbf{1}_{\neg A} \mid \mathcal{F}_n] \\ &= \mathbf{E}[Y_n \mathbf{1}_A + Y_n \mathbf{1}_{\neg A} \mid \mathcal{F}_n] = Y_n.\end{aligned}$$

Since $0 \le Y_n \le 1$, the martingale converges to $Y \in [0,1]$.

(c) Assume that $1 < m < \infty$. We have

$$\mathbf{E}[Y \mid Z_0,\ Z'_0] = Y_0 = Z_0/(Z_0 + Z'_0). \tag{17.3}$$

Now $\langle Z'_{k+n} \rangle_{n \ge 0}$ is also a Galton-Watson process, so

$$Y^{(k)} := \lim_n \frac{Z_n}{Z_n + Z'_{k+n}}$$

exists a.s. too with
$$\mathbf{E}[Y^{(k)} \mid Z_0,\ Z'_k] = Z_0/(Z_0 + Z'_k)$$
by (17.3). We have
$$\begin{aligned}
\mathbf{P}[Y = 1,\ Z_n \not\to 0,\ Z'_n \not\to 0] &= \mathbf{P}\left[\frac{Z'_n}{Z_n} \to 0,\ Z_n \not\to 0,\ Z'_n \not\to 0\right] \\
&= \mathbf{P}\left[\frac{Z'_{n+k}}{Z_n} \to 0,\ Z_n \not\to 0,\ Z'_n \not\to 0\right] \\
&= \mathbf{P}[Y^{(k)} = 1,\ Z_n \not\to 0,\ Z'_n \not\to 0] \\
&\le \mathbf{E}[Y^{(k)} \mathbf{1}_{\{Z'_k > 0\}}] = \mathbf{E}\left[\frac{Z_0}{Z_0 + Z'_k} \mathbf{1}_{\{Z'_k > 0\}}\right] \\
&\to 0 \text{ as } k \to \infty,
\end{aligned}$$
where the second equality is due to the fact that, by the weak law of large numbers and Proposition 5.1, $Z'_{n+k}/Z'_n \xrightarrow{\mathbf{P}} m^k$. Hence $\mathbf{P}[Y = 1,\ Z_n \not\to 0,\ Z'_n \not\to 0] = 0$. By symmetry, $\mathbf{P}[Y = 0,\ Z_n \not\to 0,\ Z'_n \not\to 0] = 0$ too.

5.24. Take $c_n := Z'_n$ for almost any particular realization of $\langle Z'_n \rangle$ with $Z'_n \not\to 0$. Part (iii) of the Seneta-Heyde theorem, the fact that $c_{n+1}/c_n \to m$, follows from $Z_{n+1}/Z_n \xrightarrow{\mathbf{P}} m$. That is, if $Z_n/c_n \to V$ a.s., $0 < V < \infty$ a.s. on nonextinction, then
$$\frac{Z_{n+1}}{Z_n} \cdot \frac{c_n}{c_{n+1}} \to \frac{V}{V} = 1 \tag{17.4}$$
a.s. on nonextinction. Since $Z_{n+1}/Z_n \xrightarrow{\mathbf{P}} m$, it follows that $c_n/c_{n+1} \to 1/m$.

5.25. This follows either from (17.4) in the solution to Exercise 5.24 or from the Seneta-Heyde theorem.

5.27. There are various ways to prove these; see Pitman (1998) for some of them and the history. For (a), one way to proceed is to replace the tree in stages as follows: replace the initial individuals by their progeny; then replace each of these in turn by their progeny; etc. Each replacement decreases the total by a copy of $L - 1$. For (b), given L_j with $S_n = -k$, show that among the n cyclic permutations of $\langle L_j\ ;\ j \le n \rangle$, there are exactly k for which the first time the sum is $-k$ is n (i.e., for which the event in (a) occurs). You might want to do this first for $k = 1$.

5.28. The distribution in part (a) is known as the ***Borel distribution***.

5.29. This is due to Mathieu and Wilson (2009). Write $g := f_\infty$ in the notation of Exercise 5.26. Then $g(s) = se^{g(s)-1}$, whence $\log s = \log g(s) + 1 - g(s)$ and $ds/s = (1/g - 1)dg$. Therefore, $\int_0^1 [g(s)/s]ds = \int_0^1 g[1/g - 1]dg = 1/2$.

5.32. We have $1 - \theta_n(p) = \big(1 - p\theta_n(p)\big)^n$.

5.33. The bilinear form $(\mu_1, \mu_2) \mapsto \mathbf{E}[X(\mu_1)X(\mu_2)]$ gives the seminorm $\mathscr{E}(\bullet)^{1/2}$, whence the seminorm satisfies the parallelogram law, which is the desired identity.

5.34. Use Exercise 5.33.

5.35. Use Theorem 5.15. This result is due originally to Grimmett and Kesten (1983), Lemma 2, whose proof was very long. See also Exercises 16.10 and 16.23. For another proof that is direct and short, see Collevecchio (2006).

5.36. **(a)** Use Exercise 2.57. See Exercise 16.23 for another upper bound on the expected effective conductance.

(b) This is due to Chen (1997).

5.40. Part (a) is used by Bateman and Katz (2008).

5.41. $\mathscr{C}(o \leftrightarrow \partial T(n))/(1+\mathscr{C}(o \leftrightarrow \partial T(n))) = 2/(n+2)$ while $\mathbf{P}_{1/2}[o \leftrightarrow \partial T(n)] \sim 4/n$, so the inequalities with "2" in them are better for large n. To see this, let $p_n := \mathbf{P}_{1/2}[o \leftrightarrow \partial T(n)]$. We have $p_{n+1} = p_n - p_n^2/4$. For $\epsilon > 0$ and t_1, t_2, N chosen appropriately, show that $a_n := (4-\epsilon)/(n-t_1)$ and $b_n := (4+\epsilon)/(n-t_2)$ satisfy $a_N = b_N = p_N$ and $a_{n+1} < a_n - a_n^2/4$, $b_{n+1} > b_n - b_n^2/4$ for $n > N$. Deduce that $a_n < p_n < b_n$ for $n > N$. Theorem 12.7 determines this kind of asymptotic more generally for critical Galton-Watson processes.

5.42. $\mathscr{C}(o \leftrightarrow \partial T)/(1+\mathscr{C}(o \leftrightarrow \partial T)) = (2p-1)/p$ and $\mathbf{P}_p[o \leftrightarrow \partial T] = (2p-1)/p^2$.

5.43. For related results, see Adams and Lyons (1991).

5.45. See Pemantle and Peres (1996).

5.46. This is due to Lyons (1992).

5.48. Parts (a) and (b) are due to Lyons (1992), while parts (c) and (d) are due to Marchal (1998).

5.50. This follows from the facts that $B_{n,d,1}(s) < s$ for $s > 0$ small enough, that the functions $B_{n,d,p}(s)$ converge uniformly to $B_{n,d,1}(s)$ as $p \to 1$, and that $B_{n,d,p}(0) > 0$ for $p > 0$.

5.51. This is due to Balogh, Peres, and Pete (2006).

5.52. Set $h_a(t) := (t-a)^+$. For one direction, just note that h_a is a nonnegative increasing convex function. For the other, given h, let g give a line of support of h at the point $\big(\mathbf{E}Y, \mathbf{E}h(Y)\big)$. Then $\mathbf{E}h(Y) = g(\mathbf{E}Y) = \mathbf{E}g(Y) \le \mathbf{E}g^+(Y) \le \mathbf{E}g^+(X) \le \mathbf{E}h(X)$.

5.53. **(a)** Condition on X_i and Y_i for $i < n$.

(b) Show that $n \mapsto \mathbf{E}[h(\sum_{i=1}^n X_i)]$ is an increasing convex function when h is. See Ross (1996), p. 444 for details.

5.54. **(a)** Use Exercise 5.53.

(b) Use (a) and Exercise 5.52. See Ross (1996), p. 446.

CHAPTER 6

6.1. In fact, for every finite connected subset K of $\mathbb{T}_{b+1}$, we have $|\partial_{\mathsf{E}} K| = (b-1)|K| + 2$.

6.5. Let the transition probabilities be $p(x, y)$. Then the Cauchy-Schwarz inequality gives us

$$\begin{aligned}\|Pf\|_\pi^2 &= \sum_{x \in \mathsf{V}} \pi(x)(Pf)(x)^2 = \sum_{x \in \mathsf{V}} \pi(x) \left[\sum_{y \in \mathsf{V}} p(x, y) f(y)\right]^2 \\ &\le \sum_{x \in \mathsf{V}} \pi(x) \sum_{y \in \mathsf{V}} p(x, y) f(y)^2 = \sum_{y \in \mathsf{V}} f(y)^2 \sum_{x \in \mathsf{V}} \pi(x) p(x, y) \\ &= \sum_{y \in \mathsf{V}} f(y)^2 \pi(y) = \|f\|_\pi^2 .\end{aligned}$$

6.6. The first equality depends only on the fact that P is self-adjoint. Suppose that $|(Pf, f)| \le C(f, f)$ for all f. Then for all f, g, we have

$$\begin{aligned}|(Pf, g)| &= \left|\frac{(P(f+g), f+g) - (P(f-g), f-g)}{4}\right| \\ &\le C[(f+g, f+g) + (f-g, f-g)]/4 = C[(f, f) + (g, g)]/2 .\end{aligned}$$

Put $g = Pf\|f\|/\|Pf\|$ to get $\|Pf\| \le C\|f\|$. This shows that $\|P\| = \sup\{|(Pf, f)|/(f, f)\,;\ f \in \mathbf{D}_{00} \setminus \{0\}\}$.

6.7. The fact that the Markov chain is irreducible implies that the right-most quantity does not depend on the choice of x, y. Furthermore, it is clear that the middle term is at least $\rho(G)$. To see that the middle term is at most $\|P\|_\pi$, use $p_n(x, z) = \left(\mathbf{1}_{\{x\}},\, P^n \mathbf{1}_{\{z\}}\right)_\pi / \pi(x)$ to deduce that $p_n(x, z) \le \sqrt{\pi(z)/\pi(x)}\|P^n\|_\pi$. Finally, to show that $\|P\|_\pi \le \rho(G)$, suppose that $f \in \mathbf{D}_{00}$ with $\|f\|_\pi = 1$. Write $f = \sum_x f(x) \mathbf{1}_{\{x\}}$. Then

$$\limsup_{n \to \infty} \|P^n f\|_\pi^{1/n} = \limsup_{n \to \infty} \left[\sum_{x, y} f(x) f(y) p_{2n}(x, y)\right]^{1/2n} \le \rho(G) .$$

On the other hand, $\|Pf\|_\pi^n \le \|P^n f\|_\pi$ by the spectral theorem and the power-mean inequality (i.e., convexity of $t \mapsto t^n$ for $t > 0$). [In order to use only the spectral theorem from linear algebra, one may use the same reduction as in the solution to Exercise 13.3.] Hence $\|P\|_\pi \le \rho(G)$.

6.9. Use the fact that $\Phi_{\mathsf{E}}(\mathbb{T}_{b+1}) = (b-1)/(b+1)$ to get an upper bound on $\rho(\mathbb{T}_{b+1})$. To get a lower bound, calculate $\|Pf\|$ for $f := \sum_{n=1}^N b^{-n/2} \mathbf{1}_{S_n}$, where N is arbitrary and S_n is the sphere of radius n. For another proof, see Proposition 7.32.

6.13. To prove this, do not condition that a vertex belongs to an infinite cluster. Instead, grow the cluster from x and continue to explore even if the cluster turns out finite. Disjoint large balls will be encountered infinitely often and will have a certain probability of containing only a long path. These events are independent and so one will occur a.s. For more details of this kind of proof, see the proof of Lemma 7.22.

6.19. For finite networks and the corresponding definition of expansion constant (see Exercise 6.38), the situation is quite different; see Chung and Tetali (1998).

6.20. (This was the original proof of Theorem 6.2 in BLPS (1999b).) Let θ be an antisymmetric function on E with $|\theta(e)| \le c(e)$ for all edges e and $d^*\theta(x) = \Phi_{\mathsf{E}}(G)D(x)$ for all vertices x. We may assume that θ is acyclic, i.e., that there is no cycle of oriented edges on each of which $\theta > 0$. (Otherwise, we modify θ by subtracting appropriate cycles.) Let $K \subset \mathsf{V}$ be finite and nonempty. Suppose for a contradiction that $|\partial_{\mathsf{E}} K|_c / |K|_D = \Phi_{\mathsf{E}}(G)$. The proof of Theorem 6.1 shows that for all $e \in \partial_{\mathsf{E}} K$, we have $\theta(e) = c(e)$ if e is oriented to point out of K; in particular, $\theta(e) > 0$. Let $(x_1, x_0) \in \partial_{\mathsf{E}} K$ with $x_1 \in K$ and $x_0 \notin K$ and let g be an automorphism of G that carries x_0 to x_1. Write x_2 for the image of x_1 and gK for the image of K. Since we also have $|\partial_{\mathsf{E}} gK|_c / |gK|_D = \Phi_{\mathsf{E}}(G)$ and $(x_2, x_1) \in \partial_{\mathsf{E}} gK$, it follows that $\theta(x_2, x_1) > 0$. We may similarly find x_3 such that $\theta(x_3, x_2) > 0$ and so on, until we arrive at some x_k that equals some previous x_j or is outside K. Both lead to a contradiction, the former contradicting the acyclicity of θ and the latter the fact that on all edges leading out of K, we have $\theta > 0$.

6.21. (Due to R. Lyons.) Use the method of solution of Exercise 6.20.

6.23. (Due to R. Lyons.)

6.26. Consider cosets.

6.27. (This was the original proof of Theorem 6.4 in BLPS (1999b).) Let θ be an antisymmetric function on E with $\mathsf{flow}_+(\theta, x) \le 1$ and $d^*\theta(x) = \Phi_{\mathsf{V}}(G)$ for all vertices x. Let $K \subset \mathsf{V}$ be finite and nonempty. Suppose for a contradiction that $|\partial_{\mathsf{V}} K|/|K| = \Phi_{\mathsf{V}}(G)$. The proof of Theorem 6.3 shows that for all $x \in \partial_{\mathsf{V}} K$, we have $\mathsf{flow}_+(\theta, x) = 1$ and $\mathsf{flow}_+(-\theta, v) = \Phi_{\mathsf{V}}(G) + 1$; in particular, there is some e leading to x with $\theta(e) \ge 1/d$ and some e leading away from x with $\theta(e) \ge (\Phi_{\mathsf{V}}(G) + 1)/d \ge 1/d$, where d is the degree in G. Since G is transitive, the same is true for all $x \in \mathsf{V}$. Therefore, we may find either a cycle or a bi-infinite path with all edges e having the property that $\theta(e) \ge 1/d$. We may then subtract $1/d$ from these edges, yielding another function θ' that satisfies $\mathsf{flow}_+(\theta', v) \le 1$ and $d^*\theta'(x) = \Phi_{\mathsf{V}}(G)$ for all vertices x. But then $\mathsf{flow}_+(\theta', x) < 1$ for some x, a contradiction.

6.28. (Due to R. Lyons.)

6.30. This is Exercise $6.17\frac{1}{2}$ in Gromov (1999). Consider the graph G_k formed by adding all edges $[x, y]$ with $\mathrm{dist}_G(x, y) \le k$.

6.35. Only part (b).

6.36. (Prasad Tetali suggested that Theorem 6.1 might be used for this purpose, by analogy with the proof in Alon (1986).) Let θ be an antisymmetric function on E such that $|\theta| \le c$ and $d^*\theta = \Phi_{\mathsf{E}}(G)\pi$. Then for all $f \in \mathbf{D}_{00}$, we have $(f, f)_\pi = (f^2, \pi) = \Phi_{\mathsf{E}}(G)^{-1}(f^2, d^*\theta) = \Phi_{\mathsf{E}}(G)^{-1}(d(f^2), \theta) \le \Phi_{\mathsf{E}}(G)^{-1} \sum_{e \in \mathsf{E}_{1/2}} c(e)|f(e^+)^2 - f(e^-)^2|$.

6.38. This is due to Chung and Tetali (1998).

6.39. We use the Cauchy-Schwarz inequality and Exercise 6.7:

$$\begin{aligned} \mathbf{P}_x[X_n \in A] &= (\mathbf{1}_{\{x\}}, P^n \mathbf{1}_A)_\pi / \pi(x) \le \sqrt{|x|_\pi} \|P^n \mathbf{1}_A\|_\pi / \pi(x) \\ &\le \|P\|_\pi^n \sqrt{|A|_\pi / \pi(x)} = \rho(G)^n \sqrt{|A|_\pi / \pi(x)}\,. \end{aligned}$$

6.40. We may apply Exercise 6.39 to bound $\mathbf{P}[|X_n| \le 2\alpha n]$ for $\alpha < -\log \rho(G)/\log b$. (This appears as Lemma 4.2 in Virág (2000a).)

6.41. This confirms a conjecture of Jan Swart (personal communication, 2008), who used it in Swart (2009). Consider $(P^n \mathbf{1}_A, \mathbf{1}_A)_\pi$.

6.42. Recall from Proposition 2.1 and Exercise 2.1 that $v(x) = \mathcal{G}(o,x)/\pi(x) = \mathcal{G}(x,o)/\pi(o)$. Therefore

$$\sum_{x \in \mathsf{V}} \pi(x) v(x)^2 = \sum_{x \in \mathsf{V}} \mathcal{G}(o,x)\mathcal{G}(x,o)/\pi(o)\,.$$

Now

$$\mathcal{G}(o,x)\mathcal{G}(x,o) = \sum_{m,n} p_m(o,x) p_n(x,o)\,,$$

whence

$$\sum_{x \in \mathsf{V}} \mathcal{G}(o,x)\mathcal{G}(x,o) = \sum_k (k+1) p_k(o,o)\,.$$

Since $p_k(o,o) \le \rho^k$, this sum is finite.

6.43. (Due to Benjamini, Nachmias, and Peres (2009).) Define

$$\tau := \inf\{n \ge 0\,;\; X_n \in A\} \qquad \text{and} \qquad \tau^+ := \inf\{n > 0\,;\; X_n \in A\}\,.$$

Let

$$f(x) := \mathbf{P}_x[\tau < \infty]\,.$$

This is the voltage function from A to ∞. By Exercise 6.42, we have $f \in \ell^2(\mathsf{V}, \pi)$. Observe that $f \equiv 1$ on A. For all $x \in \mathsf{V}$,

$$(Pf)(x) = \mathbf{P}_x[\tau^+ < \infty]\,.$$

In particular, $Pf = f$ on $\mathsf{V} \setminus A$. Thus $((I-P)f)(x) = \mathbf{P}_x[\tau^+ = \infty]$ for $x \in A$ and $((I-P)f)(x) = 0$ for $x \in \mathsf{V} \setminus A$. Therefore

$$(f, (I-P)f)_\pi = \sum_{x \in A} \pi(x)\, \mathbf{P}_x[\tau^+ = \infty] = \pi(A)\, \mathbf{P}_{\pi_A}[\tau^+ = \infty]\,.$$

On the other hand, clearly,

$$(f,f)_\pi \ge \sum_{x \in A} \pi(x) f(x)^2 = \pi(A)\,.$$

The claim follows by combining the last two displays with Exercise 6.6.

6.44. Use the fact that $\|A\| = \sqrt{\|A^* A\|}$.

6.45. **(a)** We may assume that our stationary sequence is bi-infinite. Shifting the sequence to the left preserves the probability measure on sequences. Write $Y := X_{\tau_A^+}$. We want to show that for measurable $B \subseteq A$, we have $\mathbf{P}[Y \in B \mid X_0 \in A] = \mu_A(B)$. Write $\sigma_A^- := \sup\{n \le -1\,;\; X_n \in A\}$. Now $\mathbf{P}[Y \in B, X_0 \in A] = \sum_{n \ge 1} \mathbf{P}[X_0 \in A, \tau_A^+ = n, X_n \in B]$. By shifting the nth set here by n to the left, we obtain $\mathbf{P}[Y \in B, X_0 \in A] = \sum_{n \ge 1} \mathbf{P}[\sigma_A^- = -n, X_0 \in B]$, which equals $\mathbf{P}[X_0 \in B]$. This gives the first desired result. The same method shows that shifting the entire sequence $\langle X_n \rangle$ to the left by τ_A^+ preserves the measure given that $X_0 \in A$.

We give two proofs of the Kac lemma.

First, write $\sigma_A := \sup\{n \le 0 ;\ X_n \in A\}$. We have $\mathbf{E}[\tau_A^+; X_0 \in A] = \sum_{k \ge 0} \mathbf{P}[X_0 \in A, \tau_A^+ > k]$. If we shift the kth set in this sum to the left by k, then we obtain the set where $\sigma_A = -k$, whence $\mathbf{E}[\tau_A^+; X_0 \in A] = \sum_{k \ge 0} \mathbf{P}[\sigma_A = -k] = 1$.

Second, consider the asymptotic frequency that $X_n \in A$. By the ergodic theorem, this equals $\mathbf{P}[X_0 \in A] = \mu(A)$ a.s. Let τ_A^k be the time between the kth visit to A and the succeeding visit to A. By decomposing into visits to A, the asymptotic frequency of visits to A is also the reciprocal of the asymptotic average of τ_A^k. Since the random variables $\langle \tau_A^k \rangle$ are stationary when conditioned on $X_0 \in A$ by the first part of our proof, the ergodic theorem again yields that the asymptotic average of τ_A^k equals $\mathbf{E}[\tau_A^+ \mid X_0 \in A]$ a.s. Equating these asymptotics gives the Kac lemma.

6.46. (Due to Aldous, unpublished, 1999.) Let g_A be the spectral gap of the induced chain and g be the original spectral gap. Write π_A for the induced stationary probability measure on A and P_A for the induced transition operator. Choose $\phi : A \to \mathbb{R}$ with $(\phi, \mathbf{1})_{\pi_A} = 0$ and $g_A = \big((I - P_A)\phi, \phi\big)_{\pi_A} / \|\phi\|_{\pi_A}^2$. Let $\psi : \mathsf{V} \to \mathbb{R}$ be the harmonic extension of ϕ. Show that $P_A\phi$ is the restriction of $P\psi$ to A, that $\big((I - P_A)\phi, \phi\big)_{\pi_A} = \big((I - P)\psi, \psi\big)_\pi / \pi(A)$, and that $\|\phi\|_{\pi_A}^2 \le \|\psi - (\psi, \mathbf{1})_\pi\|_\pi^2 / \pi(A)$. We remark that a similar proof shows the same inequality for the gaps of the network Laplacians (Exercise 2.53), which are the gaps for continuous-time random walk. A special case of this (in view of Exercise 2.47) is Proposition 2.1 of Caputo, Liggett, and Richthammer (2009).

6.48. This is from Häggström, Jonasson, and Lyons (2002).

6.49. (Due to Y. Peres and published in Häggström, Jonasson, and Lyons (2002).) The amenable case is trivial, so assume that G is non-amenable. According to the reasoning of the first paragraph of the proof of Theorem 6.15 and (6.17), we have

$$|(\widehat{K})'|/|E((\widehat{K})')| + |K|/|E^*(K)| \le |(\widehat{K})'|/|E((\widehat{K})')| + |\widehat{K}|/|E^*(\widehat{K})| \le 1 + 1/|E((\widehat{K})')| \,. \tag{17.5}$$

Write

$$\kappa_n := |\partial_{\mathsf{E}} K_n|/|K_n| - \Phi'_{\mathsf{E}}(G)$$

and

$$\lambda_n := |\partial_{\mathsf{E}} L_n|/|L_n| - \Phi'_{\mathsf{E}}(G^\dagger) \,.$$

Also write $d := d_G$, $d^\dagger := d_{G^\dagger}$, $\iota := \Phi'_{\mathsf{E}}(G)$, and $\iota^\dagger := \Phi'_{\mathsf{E}}(G^\dagger)$. We may rewrite (17.5) as

$$\frac{2}{d^\dagger - |\partial_{\mathsf{E}} L_n|/|L_n|} + \frac{2}{d + |\partial_{\mathsf{E}} K_n|/|K_n|} \le 1 + \frac{1}{|E(L_n)|} \,,$$

or, again, as

$$\frac{2}{d^\dagger - \iota^\dagger - \lambda_n} + \frac{2}{d + \iota + \kappa_n} \le 1 + \frac{1}{|E(L_n)|} = \frac{2}{d^\dagger - \iota^\dagger} + \frac{2}{d + \iota} + \frac{1}{|E(L_n)|} \,,$$

whence

$$\frac{2\lambda_n}{(d^\dagger - \iota^\dagger)(d^\dagger - \iota^\dagger - \lambda_n)} + \frac{2\kappa_n}{(d+\iota)(d+\iota+\kappa_n)} \le \frac{1}{|E(L_n)|}\,.$$

Therefore

$$\begin{aligned} 2\lambda_n &\le \frac{(d^\dagger - \iota^\dagger)(d^\dagger - \iota^\dagger - \lambda_n)}{(d+\iota)(d+\iota+\kappa_n)}(2\kappa_n) + \frac{(d^\dagger - \iota^\dagger)(d^\dagger - \iota^\dagger - \lambda_n)}{|E(L_n)|} \\ &\le \left(\frac{d^\dagger - \iota^\dagger}{d+\iota}\right)^2 2\kappa_n + \frac{(d^\dagger - \iota^\dagger)^2}{|E(L_n)|}\,. \end{aligned}$$

Similarly, we have

$$2\kappa_{n+1} \le \left(\frac{d-\iota}{d^\dagger + \iota^\dagger}\right)^2 2\lambda_n + \frac{(d-\iota)^2}{|E(K_{n+1})|}\,.$$

Putting these together, we obtain

$$2\kappa_{n+1} \le a(2\kappa_n) + b_n\,,$$

where

$$a := \left(\frac{(d-\iota)(d^\dagger - \iota^\dagger)}{(d+\iota)(d^\dagger + \iota^\dagger)}\right)^2$$

and

$$b_n := \left(\frac{(d-\iota)(d^\dagger - \iota^\dagger)}{d^\dagger + \iota^\dagger}\right)^2 \frac{1}{|E(L_n)|} + \frac{(d-\iota)^2}{|E(K_{n+1})|}\,.$$

Therefore

$$2\kappa_n \le 2\kappa_0 a^{n-1} + \sum_{j=0}^{n-2} a^j b_{n-j}\,.$$

Since $a < 1$ and $b_n \to 0$, we obtain $\kappa_n \to 0$. Hence $\lambda_n \to 0$ too.

6.50. **(a)** Subdivide edges the original network where the voltage would equal $v(z)/2$, identify the vertices where the voltage is $v(z)/2$ to a new vertex b, and apply Lemma 6.19 either to a, b or to b, z. This result is very similar to Benjamini and Kozma (2005).

(b) Note that $\psi \ge 1$.

6.51. These results are due to Benjamini and Schramm (2004). There, they refer to Benjamini and Schramm (2001b) for an example of a tree with balls having cardinality in $[r^d/c, cr^d]$, yet containing arbitrarily large finite subsets with only one boundary vertex. That reference has a minor error; to fix it, Δ on p. 10 there should be assumed to equal the diameter. The hypothesis for the principal result can be weakened to $|B(x,r)| \le ca^r$ and $\lim_{R\to\infty} |B(x,R)|/|B(x,R-r)| \ge a^r/c$.

6.52. This is due to Benjamini, Lyons, and Schramm (1999). Use the method of proof of Theorem 3.10.

6.54. Use Proposition 6.25.

6.55. Conjecture 7.12 says that a proper two-dimensional isoperimetric inequality implies $p_c(G) < 1$. Itai Benjamini (personal communication) has conjectured that the stronger conclusion

$|\mathcal{A}_n| \le e^{Cn}$ for some $C < \infty$ also holds under this assumption. This exercise shows that these conjectures are not true with the weakened assumption of anchored isoperimetry.

6.57. Suppose that the distribution ν of L does not have an exponential tail. Then for every $c > 0$ and every $\epsilon > 0$, we have $\mathbf{P}[\sum_{i=1}^n L_i \ge cn] \ge \mathbf{P}[L_1 \ge cn] \ge e^{-\epsilon n}$ for infinitely many n's, where $\{L_i\}$ are i.i.d. with law ν. Let G be a binary tree with the root o as the basepoint. Pick a collection of 2^n pairwise disjoint paths from level n to level $2n$. Then

$$\mathbf{P}\left[\text{along at least one of these } 2^n \text{ paths } \sum_{i=1}^n L_i \ge cn\right] \ge 1 - (1 - e^{-\epsilon n})^{2^n}$$
$$\ge 1 - \exp(-e^{-\epsilon n} 2^n) \to 1\,.$$

With probability very close to 1 (depending on n), there is a path from level n to $2n$ along which $\sum_{i=1}^n L_i \ge cn$. Take such a path and extend it to the root, o. Let S be the set of vertices in the extended path from the root o to level $2n$. Then

$$\frac{|\partial_{\mathsf{E}} S|}{\sum_{e \in E(S)} L_e} \le \frac{2n+1}{cn} \approx \frac{2}{c}\,.$$

Since c can be arbitrarily large, $\Phi^*_{\mathsf{E}}(G^\nu) = 0$ a.s.

6.60. Consider a random permutation of X.

6.61. Let $A \subset \{0, \ldots, n-1\}^d$ with $|A| < \frac{n^d}{2}$. Let m be such that $|\mathcal{P}_m(A)|$ is maximal over all projections, and let

$$F = \{a \in \mathcal{P}_m(A) : |\mathcal{P}_m^{-1}(a)| = n\}\,.$$

Notice that for any $a \notin F$ there is at least one edge in $\partial_{\mathsf{E}} A$, and for different a's we get disjoint edges, i.e., $|\partial_{\mathsf{E}} A| \ge |\mathcal{P}_m(A) \setminus F|$. By Theorem 6.30 we get

$$|A|^{d-1} \le \prod_{j=1}^d |\mathcal{P}_j(A)| \le |\mathcal{P}_m(A)|^d\,,$$

and so $|\mathcal{P}_m(A)| \ge |A|/|A|^{1/d} \ge 2^{1/d}|A|/n \ge 2^{1/d}|F|$, which together yield

$$|\partial_{\mathsf{E}} A| \ge |\mathcal{P}_m(A) \setminus F| \ge (1 - 2^{-1/d})|\mathcal{P}_m(A)| \ge (1 - 2^{-1/d})|A|^{\frac{d-1}{d}}\,.$$

Chapter 7

7.1. Prove this by induction, removing one leaf of $K \cap T$ at a time.

7.2. Divide A into two pieces depending on the value of $\omega(e)$.

7.6. Let $A := \{y \leftrightarrow \infty\}$ and $B := \{x \leftrightarrow y\}$.

7.11. The transitive case is due to Lyons (1995).

7.12. **(a)** Show that for every finite F, there is some infinite component K of $G \setminus F$ such that $|K \cap \omega| = \infty$ a.s. by considering large balls that are likely to intersect ω.

7.13. Let us first suppose that $\mathbf{P}_o[X_k = o]$ is strictly positive for all large k. Note that for any k and n, we have $\mathbf{P}_o[X_n = o] \ge \mathbf{P}_o[X_k = o]\mathbf{P}_o[X_{n-k} = o]$. Thus by Fekete's Lemma (Exercise 3.10), we have $\rho(G) = \lim_{n\to\infty} \mathbf{P}_o[X_n = o]^{1/n}$ and $\mathbf{P}_o[X_n = o] \le \rho(G)^n$ for all n.

On the other hand, if $\mathbf{P}_o[X_k = o] = 0$ for odd k, it is still true that $\mathbf{P}_o[X_k = o]$ is strictly positive for all even k, whence $\lim_{n\to\infty} \mathbf{P}_o[X_{2n} = o]^{1/2n}$ exists and equals $\rho(G)$.

7.14. Suppose that a large square is initially occupied. Then the chance is close to 1 that it will grow to occupy everything.

7.17. Use Exercise 6.4.

7.19. This is due to Angel and Szegedy (2010).

7.20. (These generators were introduced by Revelle (2001).) It is the same as the graph of Example 7.2 would be if the trees in that example were chosen both to be 3-regular.

7.23. (This is noted in BLPS (1999b).) Use Lemma 7.7.

7.24. This is due to Lyons, Pichot, and Vassout (2008). It is not hard to see that it suffices to establish the case where every tree of $\mathfrak{F}$ is infinite a.s. Let $K \subset \mathsf{V}$ be finite and write $\overline{K} := K \cup \partial_{\mathsf{V}} K$. Let Y be the subgraph of G spanned by those edges of $\mathfrak{F}$ that are incident to some vertex of K. This is a forest with no isolated vertices, whence

$$\begin{aligned}\sum_{x\in K} \deg_{\mathfrak{F}} x \le \sum_{x\in \overline{K}} \deg_Y x - |\mathsf{V}(Y)\setminus K| &= 2|\mathsf{E}(Y)| - |\mathsf{V}(Y)\setminus K| \\ &< 2|\mathsf{V}(Y)| - |\mathsf{V}(Y)\setminus K| = 2|K| + |\mathsf{V}(Y)\setminus K| \le 2|K| + |\partial_{\mathsf{V}} K|\,.\end{aligned}$$

Take the expectation and divide by $|K|$ to get the result.

7.29. Use Theorem 6.24.

7.30. See Grimmett (1999), pp. 18–19.

7.31. It is not known whether the hypothesis holds for every Cayley graph of at least quadratic growth.

7.34. (This fact is folklore, but was published for the first time by Peres and Steif (1998).) If there is an infinite cluster with positive probability, then by Kolmogorov's 0-1 law, there is an infinite cluster a.s. Let A be the set of vertices x of T for which $\omega \cap T^x$ contains an infinite cluster a.s. Then A is a subtree of T and clearly cannot have a finite boundary. Furthermore, since A is countable, for a.e. ω, each $x \in A$ has the property that $\omega \cap T^x$ contains an infinite cluster. Also, a.s. for all $x \in A$, $\omega \cap T^x \neq T^x \cap A$. Therefore, a.e. ω has the property

$$\forall x \in A \;\; \omega \cap T^x \text{ contains an infinite cluster different from } T^x \cap A\,. \tag{17.6}$$

For any ω with this property, for all $x \in A$, there is some $y \in T^x \cap A \setminus \omega$. Therefore, such an ω contains infinitely many infinite clusters. Since a.e. ω does have property (17.6), it follows that ω contains infinitely many infinite clusters a.s.

7.36. (This is from Kesten (1982).) Let $a_{n,\ell}$ denote the number of connected subgraphs $(\mathsf{V}', \mathsf{E}')$ of G such that $o \in \mathsf{V}'$, $|\mathsf{V}'| = n$, and $|\partial_{\mathsf{V}} \mathsf{V}'| = \ell$. Note that for such a subgraph, $\ell \le (d-1)|\mathsf{V}'| = (d-1)n$ provided $n \ge 2$. Let $p := 1/d$ and consider Bernoulli(p) site percolation on G. Writing the fact that 1 is at least the probability that the cluster of o is finite, we obtain

$$1 \ge \sum_{n,\ell} a_{n,\ell} p^n (1-p)^\ell \ge \sum_{n \ge 2} a_n p^n (1-p)^{(d-1)n}\,.$$

Therefore $\limsup_{n\to\infty} a_n^{1/n} \le 1/[p(1-p)^{d-1}]$. Putting in the chosen value of p gives the result.

7.37. (From Häggström, Jonasson, and Lyons (2002).) Let $b_{n,\ell}$ denote the number of connected subgraphs $(\mathsf{V}', \mathsf{E}')$ of G such that $o \in \mathsf{V}'$, $|\mathsf{E}'| = n$, and $|\{[x, y]\,;\; x \in \mathsf{V}',\; y \notin \mathsf{V}'\}| = \ell$. Note that for such a subgraph,

$$\ell \le d|\mathsf{V}'| - 2n \le d(n+1) - 2n = (d-2)n + d\,. \tag{17.7}$$

Let $p := 1/(d-1)$ and consider Bernoulli(p) bond percolation on G. Writing the fact that 1 is at least the probability that the cluster of o is finite and using (17.7), we obtain

$$1 \ge \sum_{n,\ell} b_{n,\ell} p^n (1-p)^\ell \ge \sum_{n} b_n p^n (1-p)^{(d-2)n+d}\,.$$

Therefore $\limsup_{n\to\infty} b_n^{1/n} \le 1/[p(1-p)^{d-2}]$. Putting in the chosen value of p gives the result.

7.38. Since the graph has bounded degree, it contains a bi-infinite geodesic passing through o iff it contains geodesics of length $2k$ for each k with o in the middle. By transitivity, it suffices to find a geodesic of length $2k$ anywhere. But this is trivial.

7.39. (This is from Babson and Benjamini (1999).) We give the solution for bond percolation. Let d be the degree of vertices and $2t$ be an upper bound for the length of cycles in a set spanning all cycles. If $K(o)$ is finite, then $\partial_{\mathsf{E}} K(o)$ is a cutset separating o from ∞. Let $\Pi \subseteq \partial_{\mathsf{E}} K(o)$ be a minimal cutset. All edges in Π are closed, which is an event of probability $(1-p)^{|\Pi|}$. We claim that the number of minimal cutsets separating o from ∞ and with n edges is at most CnD^n for some constants C and D that do not depend on n, which implies that $p_c(G) < 1$ as in the proof of Theorem 7.15. Fix any bi-infinite geodesic $\langle x_k \rangle_{k\in\mathbb{Z}}$ with $x_0 = o$ (see Exercise 7.38). Let Π be any minimal cutset separating o from ∞ with n edges. Since Π separates o from ∞, there are some $j, l \ge 0$ such that $[x_{-j-1}, x_{-j}], [x_l, x_{l+1}] \in \Pi$.

Since Π is connected in G_E^t and $\langle x_k \rangle$ is a geodesic, it follows that $j+l < nt$. By Exercise 7.36, the number of connected subgraphs of G_E^t that have n vertices and that include the edge $[x_l, x_{l+1}]$ is at most cD^n, where c is some constant and D is the degree of G_E^t. Since there are no more than nt choices for l, the bound we want follows with $C := ct$.

7.40. (This is due to Häggström, Peres, and Schonmann (1999), where a direct proof can be found.) Follow the method of proof of Lemma 7.30.

7.42. Take alternating edge-levels of $\mathbb{T}_{b+1}$ cross $\mathbb{Z}^+$ or $\mathbb{Z}^-$ to preserve independence among (though not within) levels and make a Galton-Watson process, except that the root has 8 children. The probability in Bernoulli(p) percolation that the endpoints of the bottom rung of an infinite ladder are in the same cluster is $p/[1-p^2(1-p)]$; to see this, consider the first open rung.

7.43. This is due to Benjamini and Schramm (1996b).

7.45. See Theorem 1.4 of Balogh, Peres, and Pete (2006) for details.

7.46. The first formula of (7.23) translates to the statement $\pi(d,1) = 1/d$, while the second formula follows from Exercise 5.51. The formulae of (7.23) were first given in Chalupa, Reich, and Leath (1979), which was the first paper to introduce bootstrap percolation into the statistical physics literature.

7.47. The first inequality follows immediately from viewing T as a subgraph of $\mathbb{T}_{d+1}$. To prove the positivity of the critical probability $b(\mathbb{T}_{d+1}, k)$, consider the probability that a simple path of length n starting from a fixed vertex x does not intersect any vacant $(k-1)$-fort of the initial Bernoulli(p) configuration of occupied vertices. Using Exercise 5.50, show that this probability is bounded above by some $O(z(p)^n)$, where $z(p) \to 0$ as $p \to 0$. But there is a fixed exponential bound on the number of simple paths of length n, so we can deduce that for p small enough, any infinite simple path started at x eventually intersects a vacant $(k-1)$-fort a.s., hence infinite occupied clusters are impossible.

The main idea of this proof came from Howard (2000). That paper, together with Fontes, Schonmann, and Sidoravicius (2002), used bootstrap percolation to understand the zero-temperature Glauber dynamics of the Ising model.

7.49. Let $R(x, T_\xi)$ be the event {the vertex x of T_ξ is in an infinite vacant 1-fort}, and set $r(T_\xi) = \mathbf{P}_p[R(o, T_\xi)]$. This is not an almost sure constant, so let us take expectation over all Galton-Watson trees: $r = \mathbf{E}[r(T_\xi)]$. With a recursion as in Theorem 5.24, one can write the equation $r = \frac{1}{2}(1-p)(2r - r^2 + 4r^3 - 3r^4)$. So we need to determine the infimum of p's for which there is no solution $r \in (0,1]$; that infimum will be $p(T_\xi, 2)$. Setting $f(r) = 2 - r + 4r^2 - 3r^3$, an examination of $f'(r)$ gives that $\max\{f(r) : r \in [0,1]\} = f((4+\sqrt{7})/9) = 2.2347\ldots$. So there is no solution $r > 0$ iff $2/(1-p) > 2.2347\ldots$, which gives $p(T_\xi, 2) = 0.10504\ldots < 1/9$.

CHAPTER 8

8.3. Fix $u, w \in \mathsf{V}$. Let $f(x, y)$ be the indicator that $y \in \Gamma_{u,x} w$.

8.4. Use Exercise 8.3.

8.8. A quantitative strengthening is that the weights μ'_j for Γ' are sums of the weights μ_i for Γ as follows: $\mu'_j = \sum_{o_i \in \Gamma' o'_j} \mu_i$.

8.9. Use the same proof as of Theorem 8.14, but only put mass on vertices in ω.

8.10. (This is from BLPS (1999b).) Note that if K is a finite tree, then $\alpha_K < 2$.

8.11. (This is from BLPS (1999b).) Use the method of proof of the first part of Theorem 8.19.

8.13. If there were two faces with an infinite number of sides, then a ball that intersects both of them would contain a finite number of vertices whose removal would leave more than one infinite component. If there were only one face with an infinite number of sides, then let x_n ($n \in \mathbb{Z}$) be the vertices on that face. Quasi-transitivity would imply that there is a maximum distance M of any vertex to $A := \{x_n\}$. Since the graph has one end, for all large n, there is a path from x_{-n} to x_n that avoids the M-neighborhood of x_0. Since this path stays within distance M of A, there are two vertices $x_{-r(n)}$, $x_{s(n)}$ $(r(n), s(n) > 0)$ within distance M of the path and within distance $2M + 1$ of each other. If $\sup_n r(n)s(n) = \infty$, then there would be points (either $x_{-r(n)}$ or $x_{s(n)}$) the removal of whose $(2M+1)$-neighborhood would leave an arbitrarily large finite component (one containing x_0). This would contradict the quasi-transitivity. Hence all paths from x_{-n} to x_n intersect some fixed neighborhood of x_0. But this contradicts having just one end.

8.14. By the Mass-Transport Principle, we have

$$\mathbf{E}|\{x \in \mathsf{V};\ o \in \gamma_x L\}| = \sum_{x \in \mathsf{V}} \mathbf{P}[o \in \gamma_x L] = \sum_{x \in \mathsf{V}} \mathbf{P}[x \in \gamma_o L] = \mathbf{E}|\gamma_o L| = |L|\,.$$

8.16. For any x, consider the elements of $S(x)$ that fix $x_1, \ldots, x_n$.

8.18. It is enough to prove (8.4) for bounded f.

8.19. It is unknown whether $\inf \Phi_{\mathsf{V}}(G) = 0$ without the degree constraint.

8.21. (This is adapted from BLPS (1999b).)

8.22. (This is due to Salvatori (1992).)

8.23. (From BLPS (1999b).) Because of Proposition 8.12 and Exercise 8.21, we know that Γ is unimodular. Fix i and let r be the distance from o_1 to o_i. Let n_i be the number of vertices in Γo_i at distance r from o_1 and let n_1 be the number of vertices in Γo_1 at distance r from o_i. Begin with mass n_i at each vertex $x \in \Gamma o_1$ and redistribute it equally among those vertices in Γo_i at distance r from x. When n is large, $|K_n|$ dominates the number of vertices at distance r from $\partial_{\mathsf{V}} K_n$. Hence, the total mass transported from vertices in $\Gamma o_1 \cap K_n$ is asymptotically proportional to the total mass transported into vertices in $\Gamma o_i \cap K_n$; that is,

$$\lim_{n \to \infty} \frac{n_1 |\Gamma o_i \cap K_n|}{n_i |\Gamma o_1 \cap K_n|} = 1\,. \tag{17.8}$$

By the Mass-Transport Principle, we also have

$$n_i \mu_{o_i} = n_1 \mu_{o_1} . \tag{17.9}$$

Since (17.8) and (17.9) hold for all i, their conjunction imply the desired result.

8.24. Use Exercise 8.20.

8.25. This was noted by Häggström (private communication, 1997). The same sharpening was shown another way in BLPS (1999b) by using Theorem 6.2.

8.28. This is from BLPS (1999b).

8.29. These are due to R. Lyons.

8.30. Fixing x and y, define $f_n(p)$ to be the $\mathbf{P}_p$-probability that there is an open path between x and y that has length at most n. Then f_n is a polynomial, hence continuous, and increases to τ as $n \to \infty$. Furthermore, f_n is an increasing function. This implies the claim.

8.32. Verify Corollary 8.11 for probabilities equal to the reciprocal of the degrees. This is a special case of Example 9.6 of Aldous and Lyons (2007).

8.33. Verify Corollary 8.11 for the following probability measure: Let o be a vertex of G. Let $Z := 1 + \deg o/2 + \sum_{f \sim o} 1/\deg f$, where f denotes a face of G and $\deg f$ denotes its degree in $G^\dagger$. Choose $\hat{o}$ equal to o with probability $1/Z$, equal to v_e with probability $1/(2Z)$ for each $e \sim o$, and equal to f with probability $1/(Z \deg f)$ for each $f \sim o$. This is a special case of Example 9.6 of Aldous and Lyons (2007).

8.36. This is from BLPS (1999b). An equivalent version of it appeared earlier in Levitt (1995).

8.37. (This is from BLPS (1999b).) Combine Lemma 8.32 with Theorem 8.17. See also Corollary 8.18.

8.38. This is from BLPS (1999b).

8.39. (This is from BLPS (1999b).) Let $\mathbb{Z}_3 := \mathbb{Z}/3\mathbb{Z}$ be the group of order 3 and let T be the 3-regular tree with a distinguished end, ξ. On T, let every vertex be connected to precisely one of its offspring (as measured from ξ), each with probability 1/2. Then every component is a ray. Let ω_1 be the preimage of this configuration under the coordinate projection $T \times \mathbb{Z}_3 \to T$. For every vertex x in T that has distance 5 from the root of the ray containing x, add to ω_1 two edges at random in the $\mathbb{Z}_3$ direction, in order to connect the 3 preimages of x in $T \times \mathbb{Z}_3$. The resulting configuration is a stationary spanning forest with 3 ends per tree and expected degree $(1/2)1 + (1/2)2 + 2^{-(5+1)}((2/3)1 + (1/3)2)$.

8.40. (This is from BLPS (1999b).) Use the notation of the solution to Exercise 8.39. For every vertex x that is a root of a ray in T, add to ω_1 two edges at random in the $\mathbb{Z}_3$ direction to connect the 3 preimages of x, and for every edge e in a ray in T that has distance 5 from the root of the ray, delete two of its preimages at random.

CHAPTER 9

9.1. Find $\theta \perp \bigstar$ such that $\chi^e = i^e_{\mathrm{W}} + \theta$.

9.2. It suffices to do the case $H = \bigcup H_n$. Let $K_1 := H_1$ and define K_n for $n > 1$ by $H_n = H_{n-1} \oplus K_n$. Then

$$H = \bigoplus_{n=1}^{\infty} K_n := \left\{ \sum x_n \,;\; x_n \in K_n,\; \sum \|x_n\|^2 < \infty \right\}.$$

This makes the result obvious.

9.3. Follow the proof of Proposition 9.1, but use the stars more than the cycles.

9.5. The free effective resistance in G' between a and z equals $r(e)i^e_{\mathrm{F}}(e)/[1 - i^e_{\mathrm{F}}(e)]$ when i^e_{F} is the free current in G.

9.6. **(a)** If $x \in \mathsf{V}$, then $\nabla \mathbf{1}_{\{x\}}$ is the star at x.

(d) Use Exercise 2.73.

(g) Show that $(\nabla f, i_x)_r = (\nabla g, i_x)_r$ and then show that $g(x) = (\nabla g, i_x)_r$ when g has finite support. [This is an example where (df, i_x) might not equal $(f, d^* i_x)$.]

(h) Use part (g) or the open mapping theorem.

9.7. (Thomassen, 1989) Fix vertices $a_i \in \mathsf{V}_i$. Use Theorem 9.7 with

$$W_n := \{(x_1, x_2) \,;\; |x_1 - a_1| \vee |x_2 - a_2| = n\}.$$

We remark that the product with $\mathsf{V} := \mathsf{V}_1 \times \mathsf{V}_2$ and $\mathsf{E} := \{((x_1, x_2), (y_1, y_2)) \,;\; (x_1, y_1) \in \mathsf{E}_1$ and $(x_2, y_2) \in \mathsf{E}_2\}$ is called the ***categorical product***. These terms are not universally used; other terms include "sum" for what we called the cartesian product and "product" for the categorical product.

9.8. The set of $g \geq |f|$ in $\mathbf{D}_0$ is closed and convex. Use Exercise 9.6(h).

9.10. Let θ be a unit flow of finite energy from a vertex o to ∞. Since θ has finite energy, there is some $K \subset \mathsf{V}(G)$ such that the energy of θ on the edges with some endpoint not in K is less than $1/m$. That is, the effective resistance from K to infinity is less than $1/m$.

9.13. If f is bounded and harmonic, let $\langle X_n \rangle$ be the random walk whose increments are i.i.d. with distribution μ. Then $\lim f(X_n)$ exists and belongs to the exchangeable σ-field.

9.14. **(a)** $P_{\bigstar_n}(\theta \restriction \mathsf{E}(G^{\mathrm{W}}_n)) = i_n$.

(b) Take θ to be a limit point of $\langle i_n \rangle$.

9.16. $\operatorname{div} \nabla = I - P$; use Exercises 9.6(g–h) and 9.15.

9.17. Use Exercise 9.16.

9.18. Let $f_1 := (-f) \vee \mathbf{0}$ and $f_2 := f \vee \mathbf{0}$. Apply Exercise 9.6(f) to f_1 and f_2 and use Exercise 9.17.

9.20. Rather than Exercise 9.6 for the implication that **HD** $= \mathbb{R}$, one can easily deduce this directly: For any a, if u is harmonic, then $u \vee a$ is superharmonic. But we just proved that this means it is harmonic. Since this holds for all a, u is constant.

9.21. Let the function values be certain probabilities.

9.22. The wired effective resistance is $2(b^n - 1)/[b^{n-1}(b^2 - 1)]$ if the branching number is b and the distance between the vertices is n.

9.27. Use Theorem 2.16.

9.29. The space **BD** is called the ***Dirichlet algebra***. The maximal ideal space of **BD** is called the ***Royden compactification*** of the network.

9.30. **(a)** Use Exercise 2.55.

(b) Use Exercise 2.55 and Proposition 2.11. Alternatively, use the fact that the subspace onto which we are projecting is 1-dimensional.

9.32. Use Exercise 9.6(e).

9.33. By Exercise 9.6(e),(f), it suffices to prove this for $f \in \mathbf{D}_0$. Therefore, it suffices to prove it for $f \in \mathbf{D}_{00}$. Let $g := (I - P)f$. By (6.9), we have $\|df\|_c^2 - \|dPf\|_c^2 = 2\|g\|_\pi^2 - (g, (I-P)g)_\pi \ge 0$ since $\|I - P\|_\pi \le 2$.

9.35. This result shows that restricted to $\bigstar$, the map $\theta \mapsto F$ is the inverse of the map of Exercise 9.6(h).

9.37. Write the current as the appropriate orthogonal projection of a path from a to z and move the projection to the other side of the inner product.

9.38. Imitate the proof of Exercise 2.13 and use Exercise 9.37.

9.40. Since $\{\chi^e\, ;\ e \in \mathsf{E}_{1/2}\}$ is a basis for $\ell^2_-(\mathsf{E}, r)$ and $P_{\nabla\mathbf{HD}} = P_\diamondsuit^\perp - P_\bigstar$, the linear span of $\{i^e_{\mathrm{F}} - i^e_{\mathrm{W}}\}$ is dense in $\nabla\mathbf{HD}$. (This also shows that the bounded Dirichlet functions are dense in $\mathbf{D}$. Furthermore, in combination with Exercise 2.37, it gives another proof of Corollary 9.6.)

9.42. Define the random walk

$$Y_n := \begin{cases} X_n & \text{if } n \le \tau_{\mathsf{V}\setminus W}, \\ X_{\tau_{\mathsf{V}\setminus W}} & \text{otherwise.} \end{cases}$$

It follows from a slight extension of the ideas leading to (9.8) that for $f \in \mathbf{D}$ harmonic at all vertices in W, the sequence $\langle f(Y_n) \rangle$ is an L^2 martingale, whence $f(X_0) = \mathbf{E}[\lim_{n\to\infty} f(Y_n)]$. This is 0 if f is supported on W.

9.44. Consider linearly independent elements of $\mathbf{D}/\widetilde{\mathbf{D}_0}$.

9.48. If $e \in \mathsf{E}$ with $v(e^-) < \alpha < v(e^+)$ and we subdivide e by a vertex x, giving the two resulting edges resistances

$$r(e^-, x) = r(e)\frac{\alpha - v(e^-)}{v(e^+) - v(e^-)}$$

and

$$r(x, e^+) = r(e)\frac{v(e^+) - \alpha}{v(e^+) - v(e^-)}\,,$$

and if the corresponding random walk on the graph with e subdivided is observed only when vertices of G are visited and, further, consecutive visits to the same vertex are replaced by a single visit, then we see the original random walk on G and $v(x)$, the probability of never visiting o, is α.

For each $k = 2, 3, \ldots$ in turn, subdivide each edge e where $v(e^-) < 1 - 1/k < v(e^+)$ as just described with a new vertex x having $v(x) = 1 - 1/k$. Let G' be the network that includes all these vertices. Let Π_k be the set of all vertices $x \in G'$ with $v(x) = 1 - 1/k$ and let τ_k be the first time the random walk on G' visits Π_k. This stopping time is finite since $v(X_n) \to 1$ by Exercise 2.72. The limit distribution of $J(X_n)$ on the circle is the same for G as for G' and is the limit of $J(X_{\tau_k})$ since $J(X_n)$ converges a.s.

Let G'_k be the subnetwork of G' determined by all vertices x with $v(x) \le 1 - 1/k$. Because all vertices of Π_k are at the same potential in G', identifying them to a single vertex will not change the current flow in G'_k. Thus, the current flow along an edge e in G'_k incident to a vertex in Π_k is proportional to the chance that e is the edge taken when Π_k is first visited. This means that the chance that $J(X_{\tau_k})$ is an arc $J(x)$ is exactly the length of $J(x)$ if $x \in \Pi_k$. Hence the limit distribution of $J(X_{\tau_k})$ is Lebesgue measure.

CHAPTER 10

10.4. Let $A \subset \mathsf{E}$ be a minimal set whose removal leaves at least 2 transient components. Show that there is a finite subset B of endpoints of the edges of A such that $\mathsf{FSF}[\exists x, y \in B\ x \leftrightarrow y] = 1 > \mathsf{WSF}[\exists x, y \in B\ x \leftrightarrow y]$. Here, $x \leftrightarrow y$ means that x and y are in the same component (tree). One can also use Exercise 9.21, or, alternatively, one can derive a new proof of Exercise 9.21 by using this exercise and Proposition 10.13.

10.5. This is due to Häggström (1998). The same holds if we assume merely that $\sum_n r(e_n) = \infty$ for any path $\langle e_n \rangle$ of edges in G.

10.6. **(b)** This holds for any probability measure on spanning forests with infinite trees by the bounded convergence theorem.

10.8. Use Corollary 10.5.

10.9. Use Exercise 10.4 or 10.5.

10.10. (This is due to Medolla and Soardi (1995).) Use Corollary 10.9.

10.12. The free uniform spanning forest has one tree a.s. since this joining edge is present in every finite approximation. But the wired uniform spanning forest has two a.s.; use Proposition 10.1.

10.13. Orthogonality of $\{G_x\}$ is obvious. Completeness of $\{G_x\}$ follows from the density of trigonometric polynomials in $L^2(\mathbb{T}^d)$. This proves the identity for $F \in L^2(\mathbb{T}^d)$. Furthermore, density of trigonometric polynomials in $C(\mathbb{T}^d)$ shows that f determines F uniquely (given $F \in L^1(\mathbb{T}^d)$, if $f = \mathbf{0}$, then $\int_{\mathbb{T}^d} F(\alpha)p(\alpha)\,d\alpha = 0$ for all trigonometric polynomials p, whence for all continuous p, so $F = \mathbf{0}$). Therefore, if $f \in \ell^2(\mathbb{Z}^d)$, then $F \in L^2(\mathbb{T}^d)$, so the identity also holds when $F \notin L^2(\mathbb{T}^d)$.

10.14. In fact, we need to assume only that $\sum_n r(e_n) = \infty$ for any path $\langle e_n \rangle$ of edges in T.

10.15. Join $\mathbb{Z}$ and $\mathbb{Z}^3$ by an edge.

10.18. This is from BLPS (2001), Proposition 14.1.

10.20. Show that p_c is a solution of $3 = 4p^2 + 2p^3$.

10.23. This extends to planar transitive graphs.

10.24. The covariances can be expressed via the free and wired transfer current matrices. Note that if jointly normal random variables Y with mean 0 have covariance matrix M, then there are independent standard normal random variables Z such that $Y = \sqrt{M}Z$. Using the notions of Gaussian Hilbert spaces (see Janson (1997)), one can also define the free and wired Gaussian networks directly without limits.

10.27. Use Exercise 8.34 or (10.3) on the dual graph, together with symmetry.

10.30. This analogue of Theorem 4.8 is due to R. Pemantle after seeing Theorem 4.8.

10.31. Before Theorem 10.17 was proved, Pemantle (1991) had proved this. This property is called ***strong Følner independence*** and is stronger than tail triviality. To prove it, let $\langle G_n \rangle$ be an exhaustion of G with edge set E_n. Given n, let $K \subset \mathsf{E} \setminus \mathsf{E}_n$. Let $\mathscr{A}$ be an elementary cylinder event of the form $\{D \subseteq T\}$ for some set of edges $D \subseteq \mathsf{E}_n$ and let $\mathscr{B}$ be a cylinder event in $\mathscr{F}(K)$ with positive probability. By Rayleigh monotonicity, we have for all sufficiently large $m \ge n$,

$$\mu_n^{\mathrm{W}}(\mathscr{A}) \le \mu_m^{\mathrm{F}}(\mathscr{A} \mid \mathscr{B}) \le \mu_n^{\mathrm{F}}(\mathscr{A}) \,.$$

Therefore $\mu_n^{\mathrm{W}}(\mathscr{A}) \le \mathsf{FSF}(\mathscr{A} \mid \mathscr{B}) \le \mu_n^{\mathrm{F}}(\mathscr{A})$ and so the same is true of any $\mathscr{B} \in \mathscr{F}(K)$ of positive probability (not just cylinders $\mathscr{B}$). The hypothesis that $\mathsf{FSF} = \mathsf{WSF}$ now gives the result.

10.33. Use Theorem 10.25 and Exercise 6.42.

10.34. This is due to Le Gall and Rosen (1991).

10.36. This is due to BLPS (2001), Remark 9.5.

10.37. We do not know whether it is true if one of the graphs is assumed to be transitive, but this seems likely.

10.40. A theorem of Gromov (1981) and Trofimov (1985) says that all quasi-transitive graphs of at most polynomial growth satisfy the hypothesis.

10.41. This is due to BLPS (2001), Remark 9.8.

10.42. This is due to BLPS (2001), Remark 9.9.

CHAPTER 11

11.2. The left-hand side divided by the right-hand side turns out to be 109872/109561. An outline of the calculation is given by Lyons, Peres, and Schramm (2006).

11.5. If the endpoints of e are x and y, then W is the vertex set of the component of x or the component of y in the set of edges lower than e.

11.7. Use Exercise 11.5.

11.12. Each spanning tree has all but two edges of G. Condition on the values of these two missing edges to calculate the chance that they are the missing edges. It turns out that there are 3 trees with probability 4/45, 6 with probability 7/72, and 2 with probability 3/40. There are 3 edge conductances equal to $9\sqrt{3}/(2\sqrt{7})$, two equal to $16/\sqrt{21}$, and one equal to $5\sqrt{7}/(2\sqrt{3})$.

11.13. There are 4 trees of probability 1/15 and 12 of probability 11/180.

11.14. Show that p_c is a solution of $3 = 4p^2 + 2p^3$.

11.15. This question was asked by Lyons, Peres, and Schramm (2006). The answer is no; use Exercise 7.24, Proposition 11.6, Corollary 7.37, and the fact that there are non-amenable groups that are not of uniformly exponential growth (see Section 3.5).

11.17. The chance is

$$\int_0^1 \frac{1-x^2}{1-x^2+x^3}\,dx = 0.72301^+ .$$

This is just slightly less than the chance for the uniform spanning tree.

11.18. Use Exercise 10.6.

11.19. Use Exercise 11.7. In contrast to the WSF, the number of trees in the WMSF is not always an a.s. constant: see Example 6.2 of Lyons, Peres, and Schramm (2006).

11.20. It is a tail random variable.

11.22. This is due to Lyons, Peres, and Schramm (2006).

11.23. This is due to Lyons, Peres, and Schramm (2006).

11.25. Use Exercise 8.34.

CHAPTER 12

12.4. For any $\lambda > 0$, we have $\mathbf{P}[\widehat{X}_n < \lambda] = \mathbf{E}[X_n\,;\ X_n < \lambda]/\mathbf{E}[X_n] \le \lambda \mathbf{P}[X_n > 0]/\mathbf{E}[X_n] \to 0$.

12.7. Use Laplace transforms.

12.8. It is not necessary to assume that A has finite mean.

12.9. We have $\mathbf{E}[A_i] \to 0$, whence $\mathbf{E}\big[\sum_{i=1}^n A_i/n\big] \to 0$, which implies the first result. The second result follows similarly since $\mathbf{E}\big[\sum_{j=1}^{A_i} |C_{i,j}|\big] \le \mathbf{E}[A_i]$.

12.10. (This is due independently to J. Geiger and G. Alsmeyer (private communications, 2000).) We need to show that for any x, we have $\mathbf{P}[A > x \mid A \ge B] \le \mathbf{P}[A > x \mid A \ge C]$. Let F be the c.d.f. of A. For any fixed x, we have

$$\begin{aligned}
\mathbf{P}[A > x \mid A \ge B] &= \frac{\mathbf{P}[A > x, A \ge B]}{\mathbf{P}[A \ge B]} = \frac{\int_{y>x} \mathbf{P}[B \le y]\, dF(y)}{\int_{y\in\mathbb{R}} \mathbf{P}[B \le y]\, dF(y)} \\
&= \left(1 + \frac{\int_{y\le x} \mathbf{P}[B \le y]\, dF(y)}{\int_{y>x} \mathbf{P}[B \le y]\, dF(y)}\right)^{-1} \\
&\le \left(1 + \frac{\int_{y\le x} \mathbf{P}[C \le y] \frac{\mathbf{P}[B\le x]}{\mathbf{P}[C\le x]}\, dF(y)}{\int_{y>x} \mathbf{P}[C \le y] \frac{\mathbf{P}[B\le x]}{\mathbf{P}[C\le x]}\, dF(y)}\right)^{-1} \\
&= \left(1 + \frac{\int_{y\le x} \mathbf{P}[C \le y]\, dF(y)}{\int_{y>x} \mathbf{P}[C \le y]\, dF(y)}\right)^{-1} = \mathbf{P}[A > x \mid A \ge C]\,.
\end{aligned}$$

To show that the hypothesis holds for geometric random variables, we must show that for all $k \ge 1$, the function $p \mapsto (1-p^{k+1})/(1-p^k)$ is decreasing in p. Taking the logarithmic derivative of this function and rearranging terms, we obtain that this is equivalent to $p \le (k+p^{k+1})/(k+1)$, which is a consequence of the arithmetic mean-geometric mean inequality.

12.14. To deduce the Cauchy-Schwarz inequality, apply the arithmetic mean-quadratic mean inequality to the probability measure $(Y^2/\mathbf{E}[Y^2])\mathbf{P}$ and the random variable X/Y.

12.17. This is due to Zubkov (1975). In the notation of the proof of Theorem 12.7, show that $i\mathbf{P}[X_i' > 0] \to 1$ as $i \to \infty$. See Geiger (1999) for details.

12.19. **(a)** By l'Hôpital's rule, Exercise 5.1, and Exercise 12.18, we have

$$\begin{aligned}
\lim_{s\uparrow 1} \delta(s) = \lim_{s\uparrow 1} \frac{f(s) - s}{(1-s)[1-f(s)]} &= \lim_{s\uparrow 1} \frac{f'(s) - 1}{f(s) - 1 - f'(s)(1-s)} \\
&= \lim_{s\uparrow 1} \frac{f''(s)}{2f'(s) - f''(s)(1-s)} = \sigma^2/2\,.
\end{aligned}$$

(b) We have

$$[1 - f^{(n)}(s)]^{-1} - (1-s)^{-1} = \sum_{k=0}^{n-1} \delta\left(f^{(k)}(s)\right)\,.$$

Since $f^{(k)}(s_n) \to 1$ uniformly in n as $k \to \infty$, it follows that

$$\frac{[1 - f^{(n)}(s_n)]^{-1} - (1-s_n)^{-1}}{n} \to \sigma^2/2$$

as $n \to \infty$. Applying the hypothesis that $n(1-s_n) \to \alpha$ gives the result.

12.20. **(i)** Take $s_n := 0$ in Exercise 12.19 to get $n[1 - f^{(n)}(0)] \to 2/\sigma^2$.

(ii) By the continuity theorem for Laplace transform (Feller (1971), p. 431), it suffices to show that the Laplace transform of the law of Z_n/n conditional on $Z_n > 0$ converges to the

Laplace transform of the exponential law with mean $\sigma^2/2$, i.e., to $x \mapsto (1 + x\sigma^2/2)^{-1}$. Now the Laplace transform at x of the law of Z_n/n conditional on $Z_n > 0$ is

$$\begin{aligned}\mathbf{E}[e^{-xZ_n/n} \mid Z_n > 0] = \mathbf{E}[e^{-xZ_n/n}\mathbf{1}_{\{Z_n>0\}}]/\mathbf{P}[Z_n > 0] &= \frac{\mathbf{E}[e^{-xZ_n/n}] - \mathbf{E}[e^{-xZ_n/n}\mathbf{1}_{\{Z_n=0\}}]}{1 - f^{(n)}(0)} \\ = \frac{f^{(n)}(e^{-x/n}) - f^{(n)}(0)}{1 - f^{(n)}(0)} &= 1 - \frac{n[1 - f^{(n)}(e^{-x/n})]}{n[1 - f^{(n)}(0)]}\,.\end{aligned}$$

Application of Exercise 12.19 with $s_n := e^{-x/n}$, together with part (i), gives the result.

Chapter 13

13.1. Let d_u denote the number of children of a vertex u in a tree. Use the SLLN for bounded martingale differences (Theorem 13.1) to compare the speed to

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k\le n} \frac{d_{X_k} - 1}{d_{X_k} + 1}\,.$$

Show that the frequency of visits to vertices with at least 2 children is at least $1/N$ by using the SLLN for L^2-martingale differences applied to the times between successive visits to vertices with at least 2 children. Note that if x has only 1 child, then x has a descendant at distance less than N with at least 2 children and also an ancestor with the same property (unless x is too close to the root).

An alternative solution goes as follows. Let $v(u)$ be the ancestor of u closest to u in the set $\{v : d_v > 2\} \cup \{o\}$ (we allow $v(u) = u$) and let $w(u)$ be the descendant of u of degree > 2 chosen such that $L(u) = |w(u)| - |u| > 0$ is minimal. Write $k(u) = |u| - |v(u)| \ge 0$ so that $k(u) + L(u) < N$ always. Check that

$$Y_t := |X_t| - [t + k(X_t)L(X_t)]/(3N)$$

is a submartingale with bounded increments by considering separately the cases where $d(X_t) > 2$ or X_t is the root, and the remaining cases. Apply Exercise 13.16 to finish.

13.2. The expected time to exit a unary tree, once entered, is infinite; see Exercise 2.35.

13.3. This is an immediate consequence of the spectral theorem. If the reader is unfamiliar with the spectral theorem for bounded self-adjoint operators on Hilbert space, then he may reduce to the spectral theorem from linear algebra on finite-dimensional spaces as follows. Let H_n be finite-dimensional subspaces increasing to the entire Hilbert space. Let A_n be the orthogonal projection onto H_n. Then A_n converges to the identity operator in the strong operator topology, i.e., $A_n f$ converges to f in norm for every f. Since A_n has norm 1, it follows easily that A_nTA_n converges to T in the strong operator topology. Furthermore, A_nTA_n is really a self-adjoint operator of norm at most $\|T\|$ on H_n. Since $Q(A_nTA_n)$ converges to $Q(T)$ in the strong operator topology and thus $\|Q(A_nTA_n)\| \to \|Q(T)\|$, the result can be applied to A_nTA_n and then deduced for T. (Even the spectral theorem on finite-dimensional spaces can be avoided by more work. For example, one can use the numerical range and the spectral mapping theorem for polynomials.)

13.4. We may take $k \ge 1$. Given any complex number z, there is a number w such that $z = (w + w^{-1})/2$. Since $(w^k + w^{-k})/2 - (2z)^k/2$ is a Laurent polynomial of degree $k-1$ and is unchanged when w is mapped to w^{-1}, induction on k shows that there is a unique polynomial Q_k such that

$$Q_k(z) = Q_k((w + w^{-1})/2) = (w^k + w^{-k})/2\,. \tag{17.10}$$

In case $z = e^{i\theta}$, we get

$$Q_k(\cos\theta) = \cos k\theta\,. \tag{17.11}$$

Furthermore, any polynomial satisfying (17.11) satisfies (17.10) for infinitely many z, whence for all z. Thus, Q_k are unique. Finally, any $s \in [-1, 1]$ is of the form $\cos\theta$, so the inequality is immediate.

13.6. Distances in G' are no smaller than in G.

13.7. Use the Nash-Williams criterion.

13.8. Think about simulating large conductances via multiple edges.

13.11. (This generalizes Häggström (1997). It was proved in a more elementary fashion in Proposition 8.30.) If there are ≥ 3 isolated boundary points in some tree, then replace this tree by the tree spanned by the isolated points. The new tree has countable boundary and still gives a translation-invariant random forest, so contradicts Corollary 13.12. If there is exactly one isolated boundary point, go from it to the first vertex encountered of degree ≥ 3 and choose 2 rays by visibility measure from there; if there are exactly 2 isolated boundary points, choose one of them at random and then do the same as when there is only one isolated boundary point. In either case, we obtain a translation-invariant random forest with a tree containing exactly 3 boundary points, again contradicting Corollary 13.12.

13.12. Use (local) reversibility of simple random walk.

13.13. To prove this intuitively clear fact, note that the **AGW**-law of $T \setminus T^{x_{-1}}$ is **GW** since x_{-1} is uniformly chosen from the neighbors of the root of T. Let A be the event that the walk remains in $T \setminus T^{x_{-1}}$:

$$\begin{aligned} A :&= \big\{(\overleftrightarrow{x}, T) \in \mathsf{PathsInTrees}\,;\ \forall n > 0\ x_n \in T \setminus T^{x_{-1}}\big\} \\ &= \big\{(\overleftrightarrow{x}, T) \in \mathsf{PathsInTrees}\,;\ \overrightarrow{x} \subset T \setminus T^{x_{-1}}\big\} \end{aligned}$$

and B_k be the event that the walk returns to the root of T exactly k times:

$$B_k := \big\{(\overleftrightarrow{x}, T) \in \mathsf{PathsInTrees}\,;\ \big|\{i \ge 1\,;\ x_i = x_0\}\big| = k\big\}\,.$$

Then the $(\mathsf{SRW} \times \mathbf{AGW} \mid A, B_k)$-law of $(\overrightarrow{x}, T \setminus T^{x_{-1}})$ is equal to the $(\mathsf{SRW} \times \mathbf{GW} \mid B_k)$-law of $(\overrightarrow{x}, T)$, whence the $(\mathsf{SRW} \times \mathbf{AGW} \mid A)$-law of $(\overrightarrow{x}, T \setminus T^{x_{-1}})$ is equivalent to the $(\mathsf{SRW} \times \mathbf{GW})$-law of $(\overrightarrow{x}, T)$. By Theorem 13.17, this implies that the speed of the latter is almost surely $\mathbf{E}[(Z_1 - 1)/(Z_1 + 1)]$.

13.14. For the numerator, calculate the probability of extinction by calculating the probability that each child of the root has only finitely many descendants; while for the denominator, calculate the probability of extinction by regarding **AGW** as the result of joining two **GW** trees by an edge, so that extinction occurs when each of the two **GW** trees is finite.

13.15. We show that a simple random walk on the hypercube satisfies

$$\mathbf{E}d(Y_t, Y_0) \ge \frac{t}{2} \qquad \forall t < \frac{n}{4}\,.$$

It follows from Jensen's inequality that $\mathbf{E}d^2(Y_t, Y_0) > t^2/4$ for $t < k/4$, implying that the hypercubes do not have uniform Markov type 2. Indeed, at each step at time $t < \frac{k}{4}$ we have probability at least $\frac{3}{4}$ to increase the distance by 1 and probability at most $\frac{1}{4}$ of decreasing it by 1, so

$$\mathbf{E}d(Y_t, Y_0) \ge \frac{3}{4}t - \frac{1}{4}t = \frac{t}{2}\,.$$

13.16. This follows from Theorem 13.1 by subtracting the conditional expectations given the past at every stage (elementary Doob decomposition).

13.17. **(a)** This is due to Lyons (1988). Define $n_1 := 1$ and then recursively n_{k+1} as the smallest $n > n_k$ so that $a_n/n < (n - n_k)/n_k^2$.

(b) This is due to Lyons (1988). Prove it first along the subsequence n_k given by (a), then for all n by imitating the proof of Theorem 13.1.

13.18. **(a)** This refinement of the principle of Cauchy condensation is due to Dvoretzky (1949). Choose b_n to be increasing and tending to infinity such that $\sum_n b_n a_n/n < \infty$. Define $m_1 := 1$ and then recursively $m_{k+1} := m_k + \lceil m_k/b_{m_k} \rceil$. Define $n_k \in [m_k, m_{k+1})$ so that $a_{n_k} = \min\{a_n\,;\ n \in [m_k, m_{k+1})\}$.

(b) A special case appeared in Dvoretzky (1949). The general result is essentially due to Davenport, Erdős, and LeVeque (1963); see also Lyons (1988). Prove it first along the subsequence n_k given by (a), then for all n by imitating the proof of Theorem 13.1.

13.19. For positive integers L, let $N(L)$ be the number of paths in G of length L. Then $\log b(G) = \lim_{L\to\infty} \log N(L)/L$. Let A denote the directed adjacency matrix of G. Consider the Markov chain with transition probabilities $p(x, y) := A(x, y)/d(x)$. Our hypothesis on G guarantees that a stationary measure is $\sigma(x) := d(x)/D$, where $D := \sum_x d(x)$. Write $n := |\mathsf{V}|$. The entropy of this chain is, by convexity of the function $t \mapsto t \log t$,

$$\sum_x [d(x)/D] \log d(x) = \sum_x [d(x)/D] \log[d(x)/D] + \log D \ge nz \log z + \log D = \log(D/n)\,,$$

where $z := (1/n)\sum_x d(x)/D = 1/n$. The method of proof of (13.5) now shows the result.

13.20. This is called the ***Shannon-Parry measure***. Let λ be the Perron eigenvalue of the adjacency matrix A of G with left eigenvector L and right eigenvector R. Define $\pi(x) := L(x)R(x)$, where we assume R is normalized so that this is a probability vector. Define the transition probabilities $p(x, y) := [\lambda R(x)]^{-1} A(x, y) R(y)$.

13.21. This is due to Lyons, Pemantle, and Peres (1997), Example 2.1.

13.22. See Example 2.2 of Lyons, Pemantle, and Peres (1997).

13.23. See Example 2.3 of Lyons, Pemantle, and Peres (1997). For more general calculations of speed on directed covers, see Takacs (1997, 1998).

13.24. (Cf. Barlow and Perkins (1989).) Use Theorem 13.4 and summation by parts in estimating $\mathbf{P}[|X_n| \ge \sqrt{(d+\epsilon)}\sqrt{n \log n}]$.

13.25. Cf. Pittet and Saloff-Coste (2001).

13.26. Consider spherically symmetric trees.

13.27. Consider the proof of Theorem 13.8 and the number of returns to $T^\diamond$ until the walk moves along $T^\diamond$.

13.28. These observations are due to Lyons and Peres.

(a) This is an immediate consequence of the ergodic theorem.

(b) In each direction of time, the number of visits to the root is a geometric random variable. To establish that Γ_k decreases with k, first prove the elementary inequality

$$a_i > 0 \quad (1 \le i \le k+1) \quad \Longrightarrow \quad \frac{1}{k+1} \sum_{i=1}^{k+1} \frac{k}{\sum_{j \ne i} a_j} \ge \frac{k+1}{\sum_{i=1}^{k+1} a_i}\,.$$

We have $\lim_{k\to\infty} \Gamma_k = 1/\int \gamma \, d\mathbf{GW}$.

(c) Let

$$N_k(\overleftrightarrow{x}, T) := \lim_{n\to\infty} \frac{\sum_{j \in D_k(\overleftrightarrow{x}, T),\ |j| \le n} N(\overleftrightarrow{x}, T)}{|D_k(\overleftrightarrow{x}, T) \cap [-n, n]|}$$

when the limit exists; this is the average number of visits to vertices of degree $k+1$. By the ergodic theorem and part (b), $N_k(\overleftrightarrow{x}, T) = \Gamma_k$ **AGW**-a.s. Let $D'_k(\overleftrightarrow{x}, T) := \{j \in \mathbb{N};\ \deg x_j = k+1,\ S^j(\overleftrightarrow{x}, T) \in \mathsf{Fresh}\}$. Then

$$N_k(\overleftrightarrow{x}, T) = \lim_{n\to\infty} \frac{\sum_{j \in D'_k(\overleftrightarrow{x}, T)\ , |j| \le n} N(\overleftrightarrow{x}, T)^2}{|D'_k(\overleftrightarrow{x}, T) \cap [-n, n]|}$$

when the limit exists. Thus, N_k measures the second moment of the number of visits to fresh vertices, not the first, which indeed does not depend on k (see part (d)). The fact that this decreases with k is consistent with the idea that the variance of the number of visits to a fresh vertex decreases in k since a larger degree gives behavior closer to the mean.

(d) It is $1/\int \gamma \, d\mathbf{GW}$. We give two proofs.

First proof. It cannot depend on k since the chance of being at a vertex of degree $k+1$ given that the walk is at a fresh vertex is p_k, which is the same as the proportion of time spent at vertices of degree k. Since it doesn't depend on k, we can simply calculate the expected number of visits to a fresh vertex. Since the number of visits is geometric, its mean is the reciprocal of the non-return probability.

Second proof. A fresh epoch is an epoch of last visit for the reversed process. For $\mathsf{SRW} \times$ **AGW**, if $x_n \ne x_0$ for $n > 0$, then the descendant tree T^{x_1} has an escape probability $\widehat{\gamma}$ with the size-biased distribution of the **GW**-law of $\gamma(T)$. By reversibility, then, so does the tree $T^{x_{-1}}$ when $(\overleftrightarrow{x}, T) \in \mathsf{Fresh}$. Assume now that $(\overleftrightarrow{x}, T) \in \mathsf{Fresh}$ and $\deg x_0 = k+1$. Let $y_1, \ldots, y_k$ be the neighbors of x_0 other than x_{-1}. Then $\gamma(T^{y_i})$ are i.i.d. with the distribution of the **GW**-law of $\gamma(T)$ and $\widehat{\gamma}, \gamma(T^{y_1}), \ldots, \gamma(T^{y_k})$ are independent. Hence the expected number of visits to x_0 is $\mathbf{E}[(k+1)/(\widehat{\gamma} + \gamma(T^{y_1}) + \cdots + \gamma(T^{y_k})]$. Now use Exercise 12.13.

13.31. (Enflo, 1969) As an example, take f to be the identity embedding $\Omega_d \to \mathbb{R}^d$. For the lower bound, a simple computation gives that for any $x_1, x_2, x_3, x_4 \in \ell^2(\mathbb{N})$, we have that the sum of squares of the diagonals is smaller than the sum of squared edge lengths:

$$\|x_1 - x_4\|^2 + \|x_2 - x_3\|^2 \le \|x_1 - x_2\|^2 + \|x_2 - x_4\|^2 + \|x_3 - x_4\|^2 + \|x_1 - x_3\|^2 .$$

Indeed, the right-hand side minus the left-hand side equals $\|x_1 + x_3 - x_2 - x_4\|^2$. This can be easily extended by induction on d to

$$\sum_{x \in \Omega_d} \|f(x) - f(-x)\|^2 \le \sum_{x \sim y} \|f(x) - f(y)\|^2 .$$

Assume now that C satisfies

$$d(x, y) \le \|f(x) - f(y)\|_2 \le C d(x, y) .$$

Then

$$\sum_{x \sim y} \|f(x) - f(y)\|^2 \le 4C^2 d 2^d ,$$

and also

$$\sum_{x \in \Omega_d} \|f(x) - f(-x)\|^2 \ge 4d^2 2^d ,$$

so that $C \ge \sqrt{d}$, as required.

13.32. This result is due to Linial, London, and Rabinovich (1995), but the following proof is due to Linial, Magen, and Naor (2002). Let $\{X_j\}$ be the simple random walk on the expander with X_0 uniform on the vertices, V_n. Since the family is d-regular, the random walk has uniform stationary distribution. By Theorem 6.9, it follows that for any $x, y \in \mathsf{V}_n$

$$p_t(x, y) \le \pi(y) + e^{-gt} ,$$

where $g := 1 - \lambda$. Take $t := \lceil \log n \rceil$. Then

$$p_t(x, y) \le \frac{1}{n} + e^{-g \log n} \le 2e^{-g \log n} .$$

Fix $\gamma > 0$ small enough that $d^\gamma e^{-g} < 1$. We wish to show that up to time t, the random walk on the expander has positive speed. More specifically, for any $x \in \mathsf{V}_n$, since the ball $B(x, \gamma \log n)$ of radius $\gamma \log n$ around x has at most $d^{\gamma \log n}$ vertices, it follows that

$$\mathbf{P}_x[X_t \in B(x, \gamma \log n)] \le d^{\gamma \log n} 2e^{-g \log n} \to 0 .$$

This in turn implies that for large enough n,

$$\mathbf{E} d^2(X_0, X_t) > \frac{\gamma^2 \log^2 n}{2} .$$

Let $f_n : \mathsf{V}_n \to \ell^2(\mathbb{N})$ satisfy (13.17) with $r = 1$. In the proof of Theorem 13.19, we actually proved that

$$\mathbf{E} d^2(f_n(X_0), f_n(X_t)) \le (1 + \lambda + \lambda^2 + \cdots + \lambda^{t-1}) \mathbf{E} d^2(f_n(X_0), f_n(X_1)) .$$

This immediately implies that

$$\mathbf{E} d^2(f_n(X_0), f_n(X_t)) \le \frac{C^2}{g} .$$

Now, similarly to the proof of Proposition 13.21, we conclude that $C \ge \sqrt{g}\gamma \log n / \sqrt{2}$.

13.33. This is due to Lyons, Pemantle, and Peres (1996a), who also show that the speed is a positive constant a.s. when $f'(q) < \lambda < m$. Use Proposition 5.23 and show that the expected time spent in finite descendant trees between moves on the reduced tree is infinite.

CHAPTER 14

14.1. Show that if $\alpha_1 < \alpha_2$ and $\mathcal{H}_{\alpha_2}(E) > 0$, then $\mathcal{H}_{\alpha_1}(E) = +\infty$. (In fact, $\alpha_0 \le d$.)

14.5. Since $\bigcap E_n \subseteq E_m$, we have $\dim \bigcap E_n \le \dim E_m$ for each m.

14.8. It is usually, but not always, $e^{-|v|}$.

14.12. Take $E_n \subseteq \partial T$ with $\dim E_n$ decreasing to the infimum. Set $E := \bigcap E_n$.

14.14. This is due to Frostman (1935).

(a) Take the largest b-adic cube containing the center of the ball. A bounded number of b-adic cubes of the same size cover the ball.

(b) Use the result of Exercise 3.27.

14.16. Use Exercise 3.10 on subadditivity. We get that $\dim\sup \partial T = \inf_n \max_v \frac{1}{n}\log M_n(v)$ and $\dim\inf \partial T = \sup_n \min_v \frac{1}{n}\log M_n(v)$.

14.17. Use Furstenberg's theorem (Theorem 3.8) and Theorem 3.9.

14.19. Use Theorem 5.26. (The statement of this exercise was proved in increasing generality by Hawkes (1981), Graf (1987), and Falconer (1987), Lemma 4.4(b).)

14.20. Let α be the minimum appearing in Theorem 14.10. If $\gamma < 1$, then $\alpha < 1$ and so two applications of Hölder's inequality yield

$$1 = \mathbf{E}\left[\sum_1^L A_i^\alpha\right] = \mathbf{E}\left[\sum_1^L A_i^\alpha 1^{1-\alpha}\right] \le \mathbf{E}\left[\left(\sum_1^L A_i\right)^\alpha \left(\sum_1^L 1\right)^{1-\alpha}\right]$$
$$\le \left(\mathbf{E}\left[\sum A_i\right]\right)^\alpha \mathbf{E}[L]^{1-\alpha} = \gamma^\alpha m^{1-\alpha}\,.$$

If $\gamma > 1$, then $\alpha > 1$ and the inequalities are reversed. Of course, if $\gamma = 1$, then $\alpha = 1$ always.

14.22. This is due to Mauldin and Williams (1986).

14.23. Use the law of large numbers.

14.24. Use the law of large numbers for Markov chains.

CHAPTER 15

15.1. Use convexity of the energy (from, say, (2.14)) to deduce that energy is minimized by the spherically symmetric flow.

15.4. The only significant change is the replacement of (15.3) and (15.4). For the former, note that diagonals of cubes are longer than sides when $d > 1$. For the latter, notice that if $|x - y| \le b^{-n}$ and x is in a certain b-adic cube, then y must be in either the same b-adic cube or a neighboring one. See Pemantle and Peres (1995b), Theorem 3.1, for the details; the appearance differs slightly due to a different convention in the relation of f to the resistances.

15.8. The intersection of m independent Brownian traces, stopped at independent exponential times, is intersection equivalent in the cube to the random set $Q_{2,2}\,(p_n^m)$, where $p_n = n/(n+1)$ for $n \ge 1$. It is easy to see that for any m, percolation on a binary tree with edge probabilities $p_e = p_{|e|}^m$ survives with positive probability. Hence the m-wise intersection is non-empty with positive probability. We get the almost sure result the same way as in part (ii).

15.12. One can also deduce Theorem 5.21 from this result. It suffices to show (5.16). Consider the Markov chain on $\partial_{\mathrm{L}} T$ that moves from left to right in planar embedding of T by simply hopping from one leaf to the next that is connected to the root. This turns out to give a kernel that differs slightly along the diagonal from the one in (5.16). Details are in Benjamini, Pemantle, and Peres (1995).

15.13. Bound the potential of μ at each point x by integrating over $B_{2^{-n}}(x) \setminus B_{2^{-n-1}}(x)$, $n \ge 1$.

15.14. For the lower bound, just use the probability that $\tau \ge 1$. For the upper bound, we follow the hint. Clearly $\int_0^a h(r) f_2(r)\,dr \le \psi(a) \int_0^a h(r) f_1(r)\,dr$. Write $T_a = \int_a^\infty f_1(r)\,dr$. We have

$$\begin{aligned}\int_a^\infty h(r) f_2(r)\,dr &= T_a \int_a^\infty h(r)\psi(r) \frac{f_1(r)}{T_a}\,dr \\ &\le T_a \int_a^\infty h(r) \frac{f_1(r)}{T_a}\,dr \int_a^\infty \psi(r) \frac{f_1(r)}{T_a}\,dr \\ &= \frac{1}{T_a} \int_a^\infty h(r) f_1(r)\,dr \int_a^\infty f_2(r)\,dr\end{aligned}$$

by Chebyshev's inequality. Combining these two inequalities proves the lemma. Now apply the lemma with f_j the density of B_{t_j} and

$$h(r) := \int_{|y|=r} \mathbf{P}_y[B(0,s) \cap A \ne \varnothing]\,d\sigma_r(y)\,,$$

where σ_r is the normalized surface area measure on the sphere $\{|y| = r\}$ in $\mathbb{R}^d$. This gives an upper bound of

$$\frac{\mathbf{P}_0[B(t_2, t_2+s) \cap A \ne \varnothing]}{\mathbf{P}_0[B(t_1, t_1+s) \cap A \ne \varnothing]} \le \frac{f_2(a)}{f_1(a)} + (\mathbf{P}_0[|B_{t_1}| > a])^{-1} \le e^{\frac{|a|^2}{2t_1}} + (\mathbf{P}_0[|B_{t_1}| > a])^{-1}\,.$$

Finally, let $H(I) := \mathbf{P}_0[B(I) \cap A \ne \varnothing]$, where I is an interval. Then H satisfies

$$H(t, t + \frac{1}{2}) \le C_{a,d} H(\frac{1}{2}, 1) \text{ for } t \ge \frac{1}{2}\,,$$

where $C_{a,d} = e^{|a|^2} + (\mathbf{P}_0[|B_{1/2}| > a])^{-1}$. Hence,

$$\begin{aligned}\mathbf{P}_0[B(0,\tau) \cap A \ne \varnothing] = \mathbf{E} H(0,\tau) &\le H(0,1) + \sum_{j=2}^\infty e^{-j/2} H(\frac{j}{2}, \frac{j+1}{2}) \le C_{a,d} \sum_{j=0}^\infty e^{-j/2} H(0,1) \\ &= \frac{C_{a,d}}{1 - e^{-1/2}} \mathbf{P}_0[B(0,1) \cap A \ne \varnothing]\,.\end{aligned}$$

15.18. Let $X := \{0, 1\}$ and $K(x, y) := \infty \mathbf{1}_{\{x \ne y\}}$.

CHAPTER 16

16.1. Use Proposition 5.6.

16.6. Use concavity of log.

16.7. Use the Kac lemma, Exercise 6.45. Recall that the system $(\mathsf{PathsInTrees}, \mathsf{SRW} \times \mathbf{AGW}, S)$ was proved to be ergodic in Section 13.5.

16.8. Given nonextinction, the subtree of a Galton-Watson tree consisting of those individuals with an infinite line of descent has the law of another Galton-Watson process still with mean m (Section 5.5). Theorem 16.16 applies to this subtree, while harmonic measure on the whole tree is equal to harmonic measure on the subtree.

16.10. (This is Lyons, Pemantle, and Peres (1995b), Lemma 9.1). For a flow θ on T, define

$$\mathscr{E}_n(\theta) := \sum_{1 \le |x| \le n} \theta(x)^2 ,$$

so that its energy for unit conductances is $\mathscr{E}(\theta) = \lim_{n\to\infty} \mathscr{E}_n(\theta)$. Set $a_n := \int \mathscr{E}_n(\mathsf{VIS}_T)\, d\mathbf{GW}(T)$. We have $a_0 = 0$ and

$$a_{n+1} = \int \left\{ \sum_{|x|=1} \frac{1}{Z_1^2} (1 + \mathscr{E}_n(\mathsf{VIS}_{T^x})) \right\} d\mathbf{GW}(T) .$$

Conditioning on Z_1 gives

$$\begin{aligned} a_{n+1} &= \sum_{k\ge1} p_k \frac{1}{k^2} \sum_{i=1}^{k} \int \left(1 + \mathscr{E}_n(\mathsf{VIS}_{T^{(i)}})\right) d\mathbf{GW}(T^{(i)}) \\ &= \sum_{k\ge1} p_k \frac{1}{k^2} k(1+a_n) = \mathbf{E}[1/Z_1](1+a_n) . \end{aligned}$$

Therefore, by the monotone convergence theorem,

$$\int \mathscr{E}(\mathsf{VIS}_T)\, d\mathbf{GW}(T) = \lim_{n\to\infty} a_n = \sum_{k=1}^{\infty} \mathbf{E}[1/Z_1]^k = \frac{\mathbf{E}[1/Z_1]}{1 - \mathbf{E}[1/Z_1]} .$$

16.13. See Lyons, Pemantle, and Peres (1997).

16.21. This exercise is relevant to the proof of Lemma 6 in Furstenberg (1970), which is incorrect. If the definition of dimension of a measure as given in Section 14.4 is used instead of Furstenberg's definition, thus implicitly revising his Lemma 6, then the present exercise together with Billingsley's Theorem 14.15 give a proof of this revision.

16.23. A similar calculation for VIS_T was done in Lyons, Pemantle, and Peres (1995b), Lemma 9.1, but it does not work for all $\lambda < m$; see Exercise 16.10. See also Pemantle and Peres (1995b), Lemma 2.2, for a related statement. See Exercise 5.36 for a lower bound on the expected effective conductance.

16.25. $S^{-1}(\mathsf{Exit})$ has the same measure as Exit and for $(\overleftrightarrow{x}, T) \in S^{-1}(\mathsf{Exit})$, the ray $x_{-\infty}$ is a path of a loop-erased simple random walk while $\vec{x}$ is a disjoint path of simple random walk.

Bibliography

ACHLIOPTAS, D. and MCSHERRY, F.

(2001) Fast computation of low rank matrix approximations. In *Proceedings of the Thirty-Third Annual ACM Symposium on Theory of Computing*, pages 611–618 (electronic), New York. ACM. Held in Hersonissos, 2001, Available electronically at `http://portal.acm.org/toc.cfm?id=380752`.

(2007) Fast computation of low-rank matrix approximations. *J. ACM* **54**, Art. 9, 19 pp. (electronic).

ADAMS, S.

(1988) Indecomposability of treed equivalence relations. *Israel J. Math.* **64**, 362–380 (1989).

(1990) Trees and amenable equivalence relations. *Ergodic Theory Dynamical Systems* **10**, 1–14.

ADAMS, S. and LYONS, R.

(1991) Amenability, Kazhdan's property and percolation for trees, groups and equivalence relations. *Israel J. Math.* **75**, 341–370.

ADAMS, S.R. and SPATZIER, R.J.

(1990) Kazhdan groups, cocycles and trees. *Amer. J. Math.* **112**, 271–287.

ADLER, J. and LEV, U.

(2003) Bootstrap percolation: visualizations and applications. *Brazilian J. of Phys.* **33**, 641–644.

AÏDÉKON, E.

(2008a) Large deviations for random walks in random environment on a Galton-Watson tree. Preprint, `http://www.arxiv.org/abs/0811.0438`.

(2008b) Transient random walks in random environment on a Galton-Watson tree. *Probab. Theory Related Fields* **142**, 525–559.

(2010) Uniform measure on a Galton-Watson tree without the $X \log X$ condition. Preprint.

AIZENMAN, M.

(1985) The intersection of Brownian paths as a case study of a renormalization group method for quantum field theory. *Comm. Math. Phys.* **97**, 91–110.

AIZENMAN, M. and BARSKY, D.J.

(1987) Sharpness of the phase transition in percolation models. *Comm. Math. Phys.* **108**, 489–526.

AIZENMAN, M., BURCHARD, A., NEWMAN, C.M., and WILSON, D.B.

(1999) Scaling limits for minimal and random spanning trees in two dimensions. *Random Structures Algorithms* **15**, 319–367. Statistical physics methods in discrete probability, combinatorics, and theoretical computer science (Princeton, NJ, 1997).

AIZENMAN, M., KESTEN, H., and NEWMAN, C.M.
(1987) Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation. *Comm. Math. Phys.* **111**, 505–531.

AIZENMAN, M. and LEBOWITZ, J.L.
(1988) Metastability effects in bootstrap percolation. *J. Phys. A* **21**, 3801–3813.

ALDOUS, D.J.
(1990) The random walk construction of uniform spanning trees and uniform labelled trees. *SIAM J. Discrete Math.* **3**, 450–465.

(1991) Random walk covering of some special trees. *J. Math. Anal. Appl.* **157**, 271–283.

ALDOUS, D.J. and FILL, J.A.
(2002) *Reversible Markov Chains and Random Walks on Graphs*. In preparation. Current version available at `http://www.stat.berkeley.edu/users/aldous/RWG/book.html`.

ALDOUS, D.J. and LYONS, R.
(2007) Processes on unimodular random networks. *Electron. J. Probab.* **12**, no. 54, 1454–1508 (electronic).

ALEXANDER, K.S.
(1995) Percolation and minimal spanning forests in infinite graphs. *Ann. Probab.* **23**, 87–104.

ALEXANDER, K.S. and MOLCHANOV, S.A.
(1994) Percolation of level sets for two-dimensional random fields with lattice symmetry. *J. Statist. Phys.* **77**, 627–643.

ALON, N.
(1986) Eigenvalues and expanders. *Combinatorica* **6**, 83–96. Theory of computing (Singer Island, Fla., 1984).

(2003) Problems and results in extremal combinatorics. I. *Discrete Math.* **273**, 31–53. EuroComb'01 (Barcelona).

ALON, N., HOORY, S., and LINIAL, N.
(2002) The Moore bound for irregular graphs. *Graphs Combin.* **18**, 53–57.

ALON, N. and MILMAN, V.D.
(1985) λ_1, isoperimetric inequalities for graphs, and superconcentrators. *J. Combin. Theory Ser. B* **38**, 73–88.

ANANTHARAM, V. and TSOUCAS, P.
(1989) A proof of the Markov chain tree theorem. *Statist. Probab. Lett.* **8**, 189–192.

ANCONA, A.
(1988) Positive harmonic functions and hyperbolicity. In Král, J., Lukeš, J., Netuka, I., and Veselý, J., editors, *Potential Theory—Surveys and Problems (Prague, 1987)*, pages 1–23, Berlin. Springer.

ANCONA, A., LYONS, R., and PERES, Y.
(1999) Crossing estimates and convergence of Dirichlet functions along random walk and diffusion paths. *Ann. Probab.* **27**, 970–989.

ANGEL, O. and SZEGEDY, B.
(2010) Personal communication.

ASADPOUR, A., GOEMANS, M.X., MĄDRY, A., OVEIS GHARAN, S., and SABERI, A.
(2010) An $O(\log n/\log\log n)$-approximation algorithm for the asymmetric traveling salesman problem. In *Proceedings of the Twenty-First Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 379–389, New York. Association for Computing Machinery (ACM). Held in Austin, TX, January 17–19, 2010.

ASMUSSEN, S. and HERING, H.
(1983) *Branching Processes.* Birkhäuser Boston Inc., Boston, Mass.

ATHREYA, K.B.
(1971) A note on a functional equation arising in Galton-Watson branching processes. *J. Appl. Probability* **8**, 589–598.

ATHREYA, K.B. and NEY, P.E.
(1972) *Branching Processes.* Springer-Verlag, New York. Die Grundlehren der mathematischen Wissenschaften, Band 196.

ATIYAH, M.F.
(1976) Elliptic operators, discrete groups and von Neumann algebras. In *Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan*, pages 43–72. Astérisque, No. 32–33. Soc. Math. France, Paris. Tenu le 17–20 juin 1974 à Orsay.

AVEZ, A.
(1974) Théorème de Choquet-Deny pour les groupes à croissance non exponentielle. *C. R. Acad. Sci. Paris Sér. A* **279**, 25–28.

AZUMA, K.
(1967) Weighted sums of certain dependent random variables. *Tôhoku Math. J. (2)* **19**, 357–367.

BABAI, L.
(1997) The growth rate of vertex-transitive planar graphs. In *Proceedings of the Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 564–573, New York. ACM. Held in New Orleans, LA, January 5–7, 1997.

BABSON, E. and BENJAMINI, I.
(1999) Cut sets and normed cohomology with applications to percolation. *Proc. Amer. Math. Soc.* **127**, 589–597.

BAHAR, I., ATILGAN, A.R., and ERMAN, B.
(1997) Direct evaluation of thermal fluctuations in protein using a single parameter harmonic potential. *Folding & Design* **2**, 173–181.

BALL, K.
(1992) Markov chains, Riesz transforms and Lipschitz maps. *Geom. Funct. Anal.* **2**, 137–172.

BALOGH, J., PERES, Y., and PETE, G.
(2006) Bootstrap percolation on infinite trees and non-amenable groups. *Combin. Probab. Comput.* **15**, 715–730.

BARLOW, M.T. and PERKINS, E.A.
(1989) Symmetric Markov chains in $\mathbf{Z}^d$: how fast can they move? *Probab. Theory Related Fields* **82**, 95–108.

BARSKY, D.J., GRIMMETT, G.R., and NEWMAN, C.M.
(1991) Percolation in half-spaces: equality of critical densities and continuity of the percolation probability. *Probab. Theory Related Fields* **90**, 111–148.

BARTHOLDI, L.

(2003) A Wilson group of non-uniformly exponential growth. *C. R. Math. Acad. Sci. Paris* **336**, 549–554.

BASS, R.F.

(1995) *Probabilistic Techniques in Analysis*. Probability and its Applications (New York). Springer-Verlag, New York.

BATEMAN, M. and KATZ, N.H.

(2008) Kakeya sets in Cantor directions. *Math. Res. Lett.* **15**, 73–81.

BEARDON, A.F. and STEPHENSON, K.

(1990) The uniformization theorem for circle packings. *Indiana Univ. Math. J.* **39**, 1383–1425.

BEKKA, M.E.B. and VALETTE, A.

(1997) Group cohomology, harmonic functions and the first L^2-Betti number. *Potential Anal.* **6**, 313–326.

BEN AROUS, G., FRIBERGH, A., GANTERT, N., and HAMMOND, A.

(2007) Biased random walks on a Galton-Watson tree with leaves. Preprint, `http://www.arxiv.org/abs/0711.3686`.

BENEDETTI, R. and PETRONIO, C.

(1992) *Lectures on Hyperbolic Geometry*. Universitext. Springer-Verlag, Berlin.

BENJAMINI, I., GUREL-GUREVICH, O., and LYONS, R.

(2007) Recurrence of random walk traces. *Ann. Probab.* **35**, 732–738.

BENJAMINI, I., KESTEN, H., PERES, Y., and SCHRAMM, O.

(2004) Geometry of the uniform spanning forest: transitions in dimensions $4, 8, 12, \ldots$. *Ann. of Math. (2)* **160**, 465–491.

BENJAMINI, I. and KOZMA, G.

(2005) A resistance bound via an isoperimetric inequality. *Combinatorica* **25**, 645–650.

BENJAMINI, I., LYONS, R., PERES, Y., and SCHRAMM, O.

(1999a) Critical percolation on any nonamenable group has no infinite clusters. *Ann. Probab.* **27**, 1347–1356.

(1999b) Group-invariant percolation on graphs. *Geom. Funct. Anal.* **9**, 29–66.

(2001) Uniform spanning forests. *Ann. Probab.* **29**, 1–65.

BENJAMINI, I., LYONS, R., and SCHRAMM, O.

(1999) Percolation perturbations in potential theory and random walks. In Picardello, M. and Woess, W., editors, *Random Walks and Discrete Potential Theory*, Sympos. Math., pages 56–84, Cambridge. Cambridge Univ. Press. Papers from the workshop held in Cortona, 1997.

BENJAMINI, I., NACHMIAS, A., and PERES, Y.

(2009) Is the critical percolation probability local? *Probab. Theory Related Fields*. To appear.

BENJAMINI, I., PEMANTLE, R., and PERES, Y.

(1995) Martin capacity for Markov chains. *Ann. Probab.* **23**, 1332–1346.

(1998) Unpredictable paths and percolation. *Ann. Probab.* **26**, 1198–1211.

BENJAMINI, I. and PERES, Y.

(1992) Random walks on a tree and capacity in the interval. *Ann. Inst. H. Poincaré Probab. Statist.* **28**, 557–592.

(1994) Tree-indexed random walks on groups and first passage percolation. *Probab. Theory Related Fields* **98**, 91–112.

BENJAMINI, I. and SCHRAMM, O.

(1996a) Harmonic functions on planar and almost planar graphs and manifolds, via circle packings. *Invent. Math.* **126**, 565–587.

(1996b) Percolation beyond $\mathbf{Z}^d$, many questions and a few answers. *Electron. Comm. Probab.* **1**, no. 8, 71–82 (electronic).

(1996c) Random walks and harmonic functions on infinite planar graphs using square tilings. *Ann. Probab.* **24**, 1219–1238.

(1997) Every graph with a positive Cheeger constant contains a tree with a positive Cheeger constant. *Geom. Funct. Anal.* **7**, 403–419.

(2001a) Percolation in the hyperbolic plane. *Jour. Amer. Math. Soc.* **14**, 487–507.

(2001b) Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.* **6**, no. 23, 13 pp. (electronic).

(2004) Pinched exponential volume growth implies an infinite dimensional isoperimetric inequality. In Milman, V.D. and Schechtman, G., editors, *Geometric Aspects of Functional Analysis*, volume 1850 of *Lecture Notes in Math.*, pages 73–76. Springer, Berlin. Papers from the Israel Seminar (GAFA) held 2002–2003.

BENNIES, J. and KERSTING, G.

(2000) A random walk approach to Galton-Watson trees. *J. Theoret. Probab.* **13**, 777–803.

BERGER, N., GANTERT, N., and PERES, Y.

(2003) The speed of biased random walk on percolation clusters. *Probab. Theory Related Fields* **126**, 221–242.

VAN DEN BERG, J. and KEANE, M.

(1984) On the continuity of the percolation probability function. In Beals, R., Beck, A., Bellow, A., and Hajian, A., editors, *Conference in Modern Analysis and Probability (New Haven, Conn., 1982)*, pages 61–65. Amer. Math. Soc., Providence, RI.

VAN DEN BERG, J. and KESTEN, H.

(1985) Inequalities with applications to percolation and reliability. *J. Appl. Probab.* **22**, 556–569.

VAN DEN BERG, J. and MEESTER, R.W.J.

(1991) Stability properties of a flow process in graphs. *Random Structures Algorithms* **2**, 335–341.

BIGGS, N.L., MOHAR, B., and SHAWE-TAYLOR, J.

(1988) The spectral radius of infinite graphs. *Bull. London Math. Soc.* **20**, 116–120.

BILLINGSLEY, P.

(1965) *Ergodic Theory and Information.* John Wiley & Sons Inc., New York.

(1995) *Probability and Measure.* Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, third edition. A Wiley-Interscience Publication.

BLACKWELL, D.

(1955) On transient Markov processes with a countable number of states and stationary transition probabilities. *Ann. Math. Statist.* **26**, 654–658.

Bollobás, B.
(1998) *Modern Graph Theory*, volume 184 of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

Borre, K. and Meissl, P.
(1974) Strength analysis of leveling-type networks. An application of random walk theory. *Geodaet. Inst. Medd.*, 80.

Bourgain, J.
(1985) On Lipschitz embedding of finite metric spaces in Hilbert space. *Israel J. Math.* **52**, 46–52.

Bowen, L.
(2004) Couplings of uniform spanning forests. *Proc. Amer. Math. Soc.* **132**, 2151–2158 (electronic).

Broadbent, S.R. and Hammersley, J.M.
(1957) Percolation processes. I. Crystals and mazes. *Proc. Cambridge Philos. Soc.* **53**, 629–641.

Broder, A.
(1989) Generating random spanning trees. In *30th Annual Symposium on Foundations of Computer Science (Research Triangle Park, North Carolina)*, pages 442–447, New York. IEEE.

Brooks, R.L., Smith, C.A.B., Stone, A.H., and Tutte, W.T.
(1940) The dissection of rectangles into squares. *Duke Math. J.* **7**, 312–340.

Burton, R.M. and Keane, M.
(1989) Density and uniqueness in percolation. *Comm. Math. Phys.* **121**, 501–505.

Burton, R.M. and Pemantle, R.
(1993) Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer-impedances. *Ann. Probab.* **21**, 1329–1371.

Buser, P.
(1982) A note on the isoperimetric constant. *Ann. Sci. École Norm. Sup. (4)* **15**, 213–230.

Campanino, M.
(1985) Inequalities for critical probabilities in percolation. In Durrett, R., editor, *Particle Systems, Random Media and Large Deviations*, volume 41 of *Contemp. Math.*, pages 1–9. Amer. Math. Soc., Providence, RI. Proceedings of the AMS-IMS-SIAM joint summer research conference in the mathematical sciences on mathematics of phase transitions held at Bowdoin College, Brunswick, Maine, June 24–30, 1984.

Cannon, J.W., Floyd, W.J., and Parry, W.R.
(1994) Squaring rectangles: the finite Riemann mapping theorem. In Abikoff, W., Birman, J.S., and Kuiken, K., editors, *The Mathematical Legacy of Wilhelm Magnus: Groups, Geometry and Special Functions*, volume 169 of *Contemp. Math.*, pages 133–212. Amer. Math. Soc., Providence, RI. Proceedings of the conference held at Polytechnic University, Brooklyn, New York, May 1–3, 1992.

Caputo, P., Liggett, T.M., and Richthammer, T.
(2009) A recursive proof of Aldous' spectral gap conjecture. Preprint, `http://www.arxiv.org/abs/0906.1238`.

CARLESON, L.
(1967) *Selected Problems on Exceptional Sets.* Van Nostrand Mathematical Studies, No. 13. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London.

CARNE, T.K.
(1985) A transmutation formula for Markov chains. *Bull. Sci. Math. (2)* **109**, 399–405.

CAYLEY, A.
(1889) A theorem on trees. *Quart. J. Math.* **23**, 376–378.

CHABOUD, T. and KENYON, C.
(1996) Planar Cayley graphs with regular dual. *Internat. J. Algebra Comput.* **6**, 553–561.

CHALUPA, J., REICH, G.R., and LEATH, P.L.
(1979) Bootstrap percolation on a Bethe lattice. *J. Phys. C* **12**, L31–L35.

CHANDRA, A.K., RAGHAVAN, P., RUZZO, W.L., SMOLENSKY, R., and TIWARI, P.
(1996/97) The electrical resistance of a graph captures its commute and cover times. *Comput. Complexity* **6**, 312–340.

CHAUVIN, B., ROUAULT, A., and WAKOLBINGER, A.
(1991) Growing conditioned trees. *Stochastic Process. Appl.* **39**, 117–130.

CHAYES, J.T., CHAYES, L., and DURRETT, R.
(1988) Connectivity properties of Mandelbrot's percolation process. *Probab. Theory Related Fields* **77**, 307–324.

CHAYES, J.T., CHAYES, L., and NEWMAN, C.M.
(1985) The stochastic geometry of invasion percolation. *Comm. Math. Phys.* **101**, 383–407.

CHEEGER, J.
(1970) A lower bound for the smallest eigenvalue of the Laplacian. In Gunning, R.C., editor, *Problems in Analysis*, pages 195–199, Princeton, N. J. Princeton Univ. Press. A symposium in honor of Salomon Bochner, Princeton University, Princeton, N.J., 1–3 April 1969.

CHEEGER, J. and GROMOV, M.
(1986) L_2-cohomology and group cohomology. *Topology* **25**, 189–215.

CHEN, D.
(1997) Average properties of random walks on Galton-Watson trees. *Ann. Inst. H. Poincaré Probab. Statist.* **33**, 359–369.

CHEN, D. and PERES, Y.
(2004) Anchored expansion, percolation and speed. *Ann. Probab.* **32**, 2978–2995. With an appendix by Gábor Pete.

CHEN, D.Y. and ZHANG, F.X.
(2007) On the monotonicity of the speed of random walks on a percolation cluster of trees. *Acta Math. Sin. (Engl. Ser.)* **23**, 1949–1954.

CHOQUET, G. and DENY, J.
(1960) Sur l'équation de convolution $\mu = \mu * \sigma$. *C. R. Acad. Sci. Paris* **250**, 799–801.

CHUNG, F.R.K., GRAHAM, R.L., FRANKL, P., and SHEARER, J.B.
(1986) Some intersection theorems for ordered sets and graphs. *J. Combin. Theory Ser. A* **43**, 23–37.

CHUNG, F.R.K. and TETALI, P.

(1998) Isoperimetric inequalities for Cartesian products of graphs. *Combin. Probab. Comput.* **7**, 141–148.

COLBOURN, C.J., PROVAN, J.S., and VERTIGAN, D.

(1995) A new approach to solving three combinatorial enumeration problems on planar graphs. *Discrete Appl. Math.* **60**, 119–129. ARIDAM VI and VII (New Brunswick, NJ, 1991/1992).

COLLEVECCHIO, A.

(2006) On the transience of processes defined on Galton-Watson trees. *Ann. Probab.* **34**, 870–878.

CONSTANTINE, G.M.

(2003) Graphs, networks, and linear unbiased estimates. *Discrete Appl. Math.* **130**, 381–393.

COPPERSMITH, D., DOYLE, P., RAGHAVAN, P., and SNIR, M.

(1993) Random walks on weighted graphs and applications to on-line algorithms. *J. Assoc. Comput. Mach.* **40**, 421–453.

COPPERSMITH, D., TETALI, P., and WINKLER, P.

(1993) Collisions among random walks on a graph. *SIAM J. Discrete Math.* **6**, 363–374.

COULHON, T. and SALOFF-COSTE, L.

(1993) Isopérimétrie pour les groupes et les variétés. *Rev. Mat. Iberoamericana* **9**, 293–314.

COURNOT, A.

(1847) *De l'Origine et des Limites de la Correspondance entre l'Algèbre et la Géométrie.* Hachette, Paris.

CROYDON, D.

(2008) Convergence of simple random walks on random discrete trees to Brownian motion on the continuum random tree. *Ann. Inst. Henri Poincaré Probab. Stat.* **44**, 987–1019.

CROYDON, D. and KUMAGAI, T.

(2008) Random walks on Galton-Watson trees with infinite variance offspring distribution conditioned to survive. *Electron. J. Probab.* **13**, no. 51, 1419–1441.

DAI, J.J.

(2005) A once edge-reinforced random walk on a Galton-Watson tree is transient. *Statist. Probab. Lett.* **73**, 115–124.

DAVENPORT, H., ERDŐS, P., and LEVEQUE, W.J.

(1963) On Weyl's criterion for uniform distribution. *Michigan Math. J.* **10**, 311–314.

DEKKING, F.M. and MEESTER, R.W.J.

(1990) On the structure of Mandelbrot's percolation process and other random Cantor sets. *J. Statist. Phys.* **58**, 1109–1126.

DELLACHERIE, C. and MEYER, P.A.

(1978) *Probabilities and Potential*, volume 29 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam.

DEMBO, A., GANTERT, N., PERES, Y., and ZEITOUNI, O.

(2002) Large deviations for random walks on Galton-Watson trees: averaging and uncertainty. *Probab. Theory Related Fields* **122**, 241–288.

DEMBO, A., GANTERT, N., and ZEITOUNI, O.
(2004) Large deviations for random walk in random environment with holding times. *Ann. Probab.* **32**, 996–1029.

DEMBO, A., PERES, Y., ROSEN, J., and ZEITOUNI, O.
(2001) Thick points for planar Brownian motion and the Erdős-Taylor conjecture on random walk. *Acta Math.* **186**, 239–270.

DEMBO, A. and ZEITOUNI, O.
(1998) *Large Deviations Techniques and Applications*, volume 38 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition.

DERRIENNIC, Y.
(1980) Quelques applications du théorème ergodique sous-additif. In *Journées sur les Marches Aléatoires*, volume 74 of *Astérisque*, pages 183–201, 4. Soc. Math. France, Paris. Held at Kleebach, March 5–10, 1979.

DIACONIS, P. and EVANS, S.N.
(2002) A different construction of Gaussian fields from Markov chains: Dirichlet covariances. *Ann. Inst. H. Poincaré Probab. Statist.* **38**, 863–878. En l'honneur de J. Bretagnolle, D. Dacunha-Castelle, I. Ibragimov.

DIESTEL, R. and LEADER, I.
(2001) A conjecture concerning a limit of non-Cayley graphs. *J. Algebraic Combin.* **14**, 17–25.

DODZIUK, J.
(1979) L^2 harmonic forms on rotationally symmetric Riemannian manifolds. *Proc. Amer. Math. Soc.* **77**, 395–400.

(1984) Difference equations, isoperimetric inequality and transience of certain random walks. *Trans. Amer. Math. Soc.* **284**, 787–794.

DODZIUK, J. and KENDALL, W.S.
(1986) Combinatorial Laplacians and isoperimetric inequality. In Elworthy, K.D., editor, *From Local Times to Global Geometry, Control and Physics*, pages 68–74. Longman Sci. Tech., Harlow. Papers from the Warwick symposium on stochastic differential equations and applications, held at the University of Warwick, Coventry, 1984/85.

DOOB, J.L.
(1984) *Classical Potential Theory and Its Probabilistic Counterpart*. Springer-Verlag, New York.

DOYLE, P.
(1988) Electric currents in infinite networks (preliminary version). Unpublished manuscript, available at `http://arxiv.org/abs/math/0703899`.

DOYLE, P.G. and SNELL, J.L.
(1984) *Random Walks and Electric Networks*. Mathematical Association of America, Washington, DC. Also available at `http://arxiv.org/abs/math/0001057`.

DUFFIN, R.J.
(1962) The extremal length of a network. *J. Math. Anal. Appl.* **5**, 200–215.

DUPLANTIER, B.
(1992) Loop-erased self-avoiding walks in $2D$. *Physica A* **191**, 516–522.

DUQUESNE, T. and LE GALL, J.F.
(2002) Random trees, Lévy processes and spatial branching processes. *Astérisque*, vi+147.

Durrett, R.

(1986) Reversible diffusion processes. In Chao, J.A. and Woyczyński, W.A., editors, *Probability Theory and Harmonic Analysis*, pages 67–89. Dekker, New York. Papers from the conference held in Cleveland, Ohio, May 10–12, 1983.

(2005) *Probability: Theory and Examples.* Duxbury Press, Belmont, CA, third edition.

Durrett, R., Kesten, H., and Lawler, G.

(1991) Making money from fair games. In Durrett, R. and Kesten, H., editors, *Random Walks, Brownian Motion, and Interacting Particle Systems*, volume 28 of *Progr. Probab.*, pages 255–267. Birkhäuser Boston, Boston, MA. A Festschrift in honor of Frank Spitzer.

Dvoretzky, A.

(1949) On the strong stability of a sequence of events. *Ann. Math. Statistics* **20**, 296–299.

Dvoretzky, A., Erdös, P., and Kakutani, S.

(1950) Double points of paths of Brownian motion in n-space. *Acta Sci. Math. Szeged* **12**, 75–81.

Dvoretzky, A., Erdős, P., Kakutani, S., and Taylor, S.J.

(1957) Triple points of Brownian paths in 3-space. *Proc. Cambridge Philos. Soc.* **53**, 856–862.

Dwass, M.

(1969) The total progeny in a branching process and a related random walk. *J. Appl. Probability* **6**, 682–686.

Dymarz, T.

(2009) Bilipschitz equivalence is not equivalent to quasi-isometric equivalence for finitely generated groups. Preprint, `http://www.arxiv.org/abs/0904.3764`.

Dynkin, E.B.

(1980) Markov processes and random fields. *Bull. Amer. Math. Soc. (N.S.)* **3**, 975–999.

Dynkin, E.B. and Maljutov, M.B.

(1961) Random walk on groups with a finite number of generators. *Dokl. Akad. Nauk SSSR* **137**, 1042–1045. English translation: *Soviet Math. Dokl.* (1961) **2**, 399–402.

Edmonds, J.

(1971) Matroids and the greedy algorithm. *Math. Programming* **1**, 127–136.

Enflo, P.

(1969) On the nonexistence of uniform homeomorphisms between L_p-spaces. *Ark. Mat.* **8**, 103–105 (1969).

Eskin, A., Fisher, D., and Whyte, K.

(2009) Coarse differentiation of quasi-isometries II: Rigidity for Sol and Lamplighter groups. Preprint, `http://www.arxiv.org/abs/0706.0940`.

Ethier, S. and Lee, J.

(2009) Limit theorems for Parrondo's paradox. Preprint, `http://www.arxiv.org/abs/0902.2368`.

Evans, S.N.

(1987) Multiple points in the sample paths of a Lévy process. *Probab. Theory Related Fields* **76**, 359–367.

Falconer, K.J.

(1986) Random fractals. *Math. Proc. Cambridge Philos. Soc.* **100**, 559–582.

(1987) Cut-set sums and tree processes. *Proc. Amer. Math. Soc.* **101**, 337–346.

FAN, A.H.

(1989) *Décompositions de Mesures et Recouvrements Aléatoires*. Ph.D. thesis, Université de Paris-Sud, Département de Mathématique, Orsay.

(1990) Sur quelques processus de naissance et de mort. *C. R. Acad. Sci. Paris Sér. I Math.* **310**, 441–444.

FARAUD, G.

(2008) A central limit theorem for random walk in random environment on marked Galton-Watson trees. Preprint, `http://www.arxiv.org/abs/0812.1948`.

FEDER, T. and MIHAIL, M.

(1992) Balanced matroids. In *Proceedings of the Twenty-Fourth Annual ACM Symposium on Theory of Computing*, pages 26–38, New York. Association for Computing Machinery (ACM). Held in Victoria, BC, Canada.

FELLER, W.

(1971) *An Introduction to Probability Theory and Its Applications. Vol. II.* John Wiley & Sons Inc., New York, second edition.

FISHER, M.E.

(1961) Critical probabilities for cluster size and percolation problems. *J. Mathematical Phys.* **2**, 620–627.

FITZSIMMONS, P.J. and SALISBURY, T.S.

(1989) Capacity and energy for multiparameter Markov processes. *Ann. Inst. H. Poincaré Probab. Statist.* **25**, 325–350.

FLANDERS, H.

(1971) Infinite networks. I: Resistive networks. *IEEE Trans. Circuit Theory* **CT-18**, 326–331.

FØLNER, E.

(1955) On groups with full Banach mean value. *Math. Scand.* **3**, 243–254.

FONTES, L.R., SCHONMANN, R.H., and SIDORAVICIUS, V.

(2002) Stretched exponential fixation in stochastic Ising models at zero temperature. *Comm. Math. Phys.* **228**, 495–518.

FORD, JR., L.R. and FULKERSON, D.R.

(1962) *Flows in Networks.* Princeton University Press, Princeton, N.J.

FORTUIN, C.M., KASTELEYN, P.W., and GINIBRE, J.

(1971) Correlation inequalities on some partially ordered sets. *Comm. Math. Phys.* **22**, 89–103.

FOSTER, R.M.

(1948) The average impedance of an electrical network. In *Reissner Anniversary Volume, Contributions to Applied Mechanics*, pages 333–340. J. W. Edwards, Ann Arbor, Michigan. Edited by the Staff of the Department of Aeronautical Engineering and Applied Mechanics of the Polytechnic Institute of Brooklyn.

FROSTMAN, O.

(1935) Potentiel d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions. *Meddel. Lunds Univ. Mat. Sem.* **3**, 1–118.

FUKUSHIMA, M.

(1980) *Dirichlet Forms and Markov Processes.* North-Holland Publishing Co., Amsterdam.

(1985) Energy forms and diffusion processes. In Streit, L., editor, *Mathematics + Physics. Vol. 1*, pages 65–97. World Sci. Publishing, Singapore. Lectures on recent results.

FURSTENBERG, H.

(1967) Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation. *Math. Systems Theory* **1**, 1–49.

(1970) Intersections of Cantor sets and transversality of semigroups. In Gunning, R.C., editor, *Problems in Analysis*, pages 41–59. Princeton Univ. Press, Princeton, N.J. A symposium in honor of Salomon Bochner, Princeton University, Princeton, N.J., 1–3 April 1969.

FURSTENBERG, H. and WEISS, B.

(2003) Markov processes and Ramsey theory for trees. *Combin. Probab. Comput.* **12**, 547–563. Special issue on Ramsey theory.

GABORIAU, D.

(1998) Mercuriale de groupes et de relations. *C. R. Acad. Sci. Paris Sér. I Math.* **326**, 219–222.

(2000) Coût des relations d'équivalence et des groupes. *Invent. Math.* **139**, 41–98.

(2002) Invariants l^2 de relations d'équivalence et de groupes. *Publ. Math. Inst. Hautes Études Sci.*, 93–150.

(2005) Invariant percolation and harmonic Dirichlet functions. *Geom. Funct. Anal.* **15**, 1004–1051.

GANDOLFI, A., KEANE, M.S., and NEWMAN, C.M.

(1992) Uniqueness of the infinite component in a random graph with applications to percolation and spin glasses. *Probab. Theory Related Fields* **92**, 511–527.

GEIGER, J.

(1995) Contour processes of random trees. In Etheridge, A., editor, *Stochastic Partial Differential Equations*, volume 216 of *London Math. Soc. Lecture Note Ser.*, pages 72–96. Cambridge Univ. Press, Cambridge. Papers from the workshop held at the University of Edinburgh, Edinburgh, March 1994.

(1999) Elementary new proofs of classical limit theorems for Galton-Watson processes. *J. Appl. Probab.* **36**, 301–309.

GERL, P.

(1988) Random walks on graphs with a strong isoperimetric property. *J. Theoret. Probab.* **1**, 171–187.

GLASNER, E. and WEISS, B.

(1997) Kazhdan's property T and the geometry of the collection of invariant measures. *Geom. Funct. Anal.* **7**, 917–935.

GOLDMAN, J.R. and KAUFFMAN, L.H.

(1993) Knots, tangles, and electrical networks. *Adv. in Appl. Math.* **14**, 267–306.

GRAF, S.

(1987) Statistically self-similar fractals. *Probab. Theory Related Fields* **74**, 357–392.

GRAF, S., MAULDIN, R.D., and WILLIAMS, S.C.

(1988) The exact Hausdorff dimension in random recursive constructions. *Mem. Amer. Math. Soc.* **71**, x+121.

GREY, D.R.

(1980) A new look at convergence of branching processes. *Ann. Probab.* **8**, 377–380.

GRIFFEATH, D. and LIGGETT, T.M.

(1982) Critical phenomena for Spitzer's reversible nearest particle systems. *Ann. Probab.* **10**, 881–895.

GRIGORCHUK, R.I.

(1983) On the Milnor problem of group growth. *Dokl. Akad. Nauk SSSR* **271**, 30–33.

GRIGOR'YAN, A.A.

(1985) The existence of positive fundamental solutions of the Laplace equation on Riemannian manifolds. *Mat. Sb. (N.S.)* **128(170)**, 354–363, 446. English translation: *Math. USSR-Sb.* **56** (1987), no. 2, 349–358.

GRIMMETT, G.

(1999) *Percolation*. Springer-Verlag, Berlin, second edition.

GRIMMETT, G. and KESTEN, H.

(1983) Random electrical networks on complete graphs II: Proofs. Unpublished manuscript, available at `http://www.arxiv.org/abs/math.PR/0107068`.

GRIMMETT, G.R., KESTEN, H., and ZHANG, Y.

(1993) Random walk on the infinite cluster of the percolation model. *Probab. Theory Related Fields* **96**, 33–44.

GRIMMETT, G.R. and MARSTRAND, J.M.

(1990) The supercritical phase of percolation is well behaved. *Proc. Roy. Soc. London Ser. A* **430**, 439–457.

GRIMMETT, G.R. and NEWMAN, C.M.

(1990) Percolation in $\infty + 1$ dimensions. In Grimmett, G.R. and Welsh, D.J.A., editors, *Disorder in Physical Systems*, pages 167–190. Oxford Univ. Press, New York. A volume in honour of John M. Hammersley on the occasion of his 70th birthday.

GRIMMETT, G.R. and STACEY, A.M.

(1998) Critical probabilities for site and bond percolation models. *Ann. Probab.* **26**, 1788–1812.

GROMOV, M.

(1981) Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, 53–73.

(1999) *Metric Structures for Riemannian and non-Riemannian Spaces*. Birkhäuser Boston Inc., Boston, MA. Based on the 1981 French original [MR 85e:53051], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.

HÄGGSTRÖM, O.

(1994) *Aspects of Spatial Random Processes*. Ph.D. thesis, Göteborg, Sweden.

(1995) Random-cluster measures and uniform spanning trees. *Stochastic Process. Appl.* **59**, 267–275.

(1997) Infinite clusters in dependent automorphism invariant percolation on trees. *Ann. Probab.* **25**, 1423–1436.

(1998) Uniform and minimal essential spanning forests on trees. *Random Structures Algorithms* **12**, 27–50.

HÄGGSTRÖM, O., JONASSON, J., and LYONS, R.
(2002) Explicit isoperimetric constants and phase transitions in the random-cluster model. *Ann. Probab.* **30**, 443–473.

HÄGGSTRÖM, O. and PERES, Y.
(1999) Monotonicity of uniqueness for percolation on Cayley graphs: all infinite clusters are born simultaneously. *Probab. Theory Related Fields* **113**, 273–285.

HÄGGSTRÖM, O., PERES, Y., and SCHONMANN, R.H.
(1999) Percolation on transitive graphs as a coalescent process: Relentless merging followed by simultaneous uniqueness. In Bramson, M. and Durrett, R., editors, *Perplexing Problems in Probability*, pages 69–90, Boston. Birkhäuser. Festschrift in honor of Harry Kesten.

HÄGGSTRÖM, O., SCHONMANN, R.H., and STEIF, J.E.
(2000) The Ising model on diluted graphs and strong amenability. *Ann. Probab.* **28**, 1111–1137.

HAMMERSLEY, J.M.
(1959) Bornes supérieures de la probabilité critique dans un processus de filtration. In *Le Calcul des Probabilités et ses Applications. Paris, 15-20 juillet 1958*, Colloques Internationaux du Centre National de la Recherche Scientifique, LXXXVII, pages 17–37. Centre National de la Recherche Scientifique, Paris.
(1961a) Comparison of atom and bond percolation processes. *J. Mathematical Phys.* **2**, 728–733.
(1961b) The number of polygons on a lattice. *Proc. Cambridge Philos. Soc.* **57**, 516–523.

HAN, T.S.
(1978) Nonnegative entropy measures of multivariate symmetric correlations. *Information and Control* **36**, 133–156.

HARA, T. and SLADE, G.
(1990) Mean-field critical behaviour for percolation in high dimensions. *Comm. Math. Phys.* **128**, 333–391.
(1994) Mean-field behaviour and the lace expansion. In Grimmett, G., editor, *Probability and Phase Transition*, pages 87–122. Kluwer Acad. Publ., Dordrecht. Proceedings of the NATO Advanced Study Institute on Probability Theory of Spatial Disorder and Phase Transition held at the University of Cambridge, Cambridge, July 4–16, 1993.

HARMER, G.P. and ABBOTT, D.
(1999) Parrondo's paradox. *Statist. Sci.* **14**, 206–213.

DE LA HARPE, P. and VALETTE, A.
(1989) La propriété (T) de Kazhdan pour les groupes localement compacts (avec un appendice de Marc Burger). *Astérisque*, 158. With an appendix by M. Burger.

HARRIS, T.E.
(1952) First passage and recurrence distributions. *Trans. Amer. Math. Soc.* **73**, 471–486.
(1960) A lower bound for the critical probability in a certain percolation process. *Proc. Cambridge Philos. Soc.* **56**, 13–20.

HAWKES, J.
(1970/71) Some dimension theorems for the sample functions of stable processes. *Indiana Univ. Math. J.* **20**, 733–738.

(1981) Trees generated by a simple branching process. *J. London Math. Soc. (2)* **24**, 373–384.

HE, Z.X. and SCHRAMM, O.
(1995) Hyperbolic and parabolic packings. *Discrete Comput. Geom.* **14**, 123–149.

HEATHCOTE, C.R.
(1966) Corrections and comments on the paper "A branching process allowing immigration". *J. Roy. Statist. Soc. Ser. B* **28**, 213–217.

HEATHCOTE, C.R., SENETA, E., and VERE-JONES, D.
(1967) A refinement of two theorems in the theory of branching processes. *Teor. Verojatnost. i Primenen.* **12**, 341–346.

HEYDE, C.C.
(1970) Extension of a result of Seneta for the super-critical Galton-Watson process. *Ann. Math. Statist.* **41**, 739–742.

HEYDE, C.C. and SENETA, E.
(1977) *I. J. Bienaymé. Statistical Theory Anticipated.* Springer-Verlag, New York. Studies in the History of Mathematics and Physical Sciences, No. 3.

HIGUCHI, Y. and SHIRAI, T.
(2003) Isoperimetric constants of (d, f)-regular planar graphs. *Interdiscip. Inform. Sci.* **9**, 221–228.

HOEFFDING, W.
(1963) Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.* **58**, 13–30.

HOFFMAN, C., HOLROYD, A.E., and PERES, Y.
(2006) A stable marriage of Poisson and Lebesgue. *Ann. Probab.* **34**, 1241–1272.

HOLOPAINEN, I. and SOARDI, P.M.
(1997) p-harmonic functions on graphs and manifolds. *Manuscripta Math.* **94**, 95–110.

HOLROYD, A.E.
(2003) Sharp metastability threshold for two-dimensional bootstrap percolation. *Probab. Theory Related Fields* **125**, 195–224.

HOWARD, C.D.
(2000) Zero-temperature Ising spin dynamics on the homogeneous tree of degree three. *J. Appl. Probab.* **37**, 736–747.

ICHIHARA, K.
(1978) Some global properties of symmetric diffusion processes. *Publ. Res. Inst. Math. Sci.* **14**, 441–486.

IMRICH, W.
(1975) On Whitney's theorem on the unique embeddability of 3-connected planar graphs. In Fiedler, M., editor, *Recent Advances in Graph Theory (Proc. Second Czechoslovak Sympos., Prague, 1974)*, pages 303–306. (Loose errata). Academia, Prague.

INDYK, P. and MOTWANI, R.
(1999) Approximate nearest neighbors: towards removing the curse of dimensionality. In *Proceedings of the 30th Annual ACM Symposium on Theory of Computing held in Dallas, TX, May 23–26, 1998*, pages 604–613. ACM, New York.

JACKSON, T.S. and READ, N.
(2009a) Theory of minimum spanning trees I: Mean-field theory and strongly disordered spin-glass model. Preprint, `http://www.arxiv.org/abs/0902.3651`.
(2009b) Theory of minimum spanning trees II: Exact graphical methods and perturbation expansion at the percolation threshold. Preprint, `http://www.arxiv.org/abs/0909.5343`.

JANSON, S.
(1997) *Gaussian Hilbert Spaces*, volume 129 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge.

JERRUM, M. and SINCLAIR, A.
(1989) Approximating the permanent. *SIAM J. Comput.* **18**, 1149–1178.

JOAG-DEV, K., PERLMAN, M.D., and PITT, L.D.
(1983) Association of normal random variables and Slepian's inequality. *Ann. Probab.* **11**, 451–455.

JOAG-DEV, K. and PROSCHAN, F.
(1983) Negative association of random variables, with applications. *Ann. Statist.* **11**, 286–295.

JOFFE, A. and WAUGH, W.A.O'N.
(1982) Exact distributions of kin numbers in a Galton-Watson process. *J. Appl. Probab.* **19**, 767–775.

JOHNSON, W.B. and LINDENSTRAUSS, J.
(1984) Extensions of Lipschitz mappings into a Hilbert space. In Beals, R., Beck, A., Bellow, A., and Hajian, A., editors, *Conference in Modern Analysis and Probability*, volume 26 of *Contemp. Math.*, pages 189–206. Amer. Math. Soc., Providence, RI. Held at Yale University, New Haven, Conn., June 8–11, 1982, Held in honor of Professor Shizuo Kakutani.

JORGENSEN, P.E.T. and PEARSE, E.P.J.
(2009) *Operator Theory of Electrical Resistance Networks*. Preprint, `http://www.arxiv.org/abs/0806.3881`.

JOYAL, A.
(1981) Une théorie combinatoire des séries formelles. *Adv. in Math.* **42**, 1–82.

KAHN, J.
(2003) Inequality of two critical probabilities for percolation. *Electron. Comm. Probab.* **8**, 184–187 (electronic).

KAIMANOVICH, V.A.
(1992) Dirichlet norms, capacities and generalized isoperimetric inequalities for Markov operators. *Potential Anal.* **1**, 61–82.
(2007) Poisson boundary of discrete groups. In preparation.

KAĬMANOVICH, V.A. and VERSHIK, A.M.
(1983) Random walks on discrete groups: boundary and entropy. *Ann. Probab.* **11**, 457–490.

KAKUTANI, S.
(1944) Two-dimensional Brownian motion and harmonic functions. *Proc. Imp. Acad. Tokyo* **20**, 706–714.

Kanai, M.

(1985) Rough isometries, and combinatorial approximations of geometries of noncompact Riemannian manifolds. *J. Math. Soc. Japan* **37**, 391–413.

(1986) Rough isometries and the parabolicity of Riemannian manifolds. *J. Math. Soc. Japan* **38**, 227–238.

Kargapolov, M.I. and Merzljakov, J.I.

(1979) *Fundamentals of the Theory of Groups*. Springer-Verlag, New York. Translated from the second Russian edition by Robert G. Burns.

Kasteleyn, P.

(1961) The statisics of dimers on a lattice I. The number of dimer arrangements on a quadratic lattice. *Physica* **27**, 1209–1225.

Kendall, D.G.

(1951) Some problems in the theory of queues. *J. Roy. Statist. Soc. Ser. B.* **13**, 151–173; discussion: 173–185.

Kenyon, R.

(1997) Local statistics of lattice dimers. *Ann. Inst. H. Poincaré Probab. Statist.* **33**, 591–618.

(1998) Tilings and discrete Dirichlet problems. *Israel J. Math.* **105**, 61–84.

(2000) The asymptotic determinant of the discrete Laplacian. *Acta Math.* **185**, 239–286.

(2001) Dominos and the Gaussian free field. *Ann. Probab.* **29**, 1128–1137.

(2008) Height fluctuations in the honeycomb dimer model. *Comm. Math. Phys.* **281**, 675–709.

Kenyon, R.W., Propp, J.G., and Wilson, D.B.

(2000) Trees and matchings. *Electron. J. Combin.* **7**, Research Paper 25, 34 pp. (electronic).

Kesten, H.

(1959a) Full Banach mean values on countable groups. *Math. Scand.* **7**, 146–156.

(1959b) Symmetric random walks on groups. *Trans. Amer. Math. Soc.* **92**, 336–354.

(1980) The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. *Comm. Math. Phys.* **74**, 41–59.

(1982) *Percolation Theory for Mathematicians*. Birkhäuser Boston, Mass.

(1986) Subdiffusive behavior of random walk on a random cluster. *Ann. Inst. H. Poincaré Probab. Statist.* **22**, 425–487.

Kesten, H., Ney, P., and Spitzer, F.

(1966) The Galton-Watson process with mean one and finite variance. *Teor. Verojatnost. i Primenen.* **11**, 579–611.

Kesten, H. and Stigum, B.P.

(1966) A limit theorem for multidimensional Galton-Watson processes. *Ann. Math. Statist.* **37**, 1211–1223.

Kifer, Y.

(1986) *Ergodic Theory of Random Transformations*. Birkhäuser Boston Inc., Boston, MA.

Kirchhoff, G.

(1847) Ueber die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird. *Ann. Phys. und Chem.* **72**, 497–508.

KOLMOGOROV, A.
(1938) On the solution of a problem in biology. *Izv. NII Matem. Mekh. Tomskogo Univ.* **2**, 7–12.

KOZMA, G. and SCHREIBER, E.
(2004) An asymptotic expansion for the discrete harmonic potential. *Electron. J. Probab.* **9**, no. 1, 1–17 (electronic).

KURATOWSKI, K.
(1966) *Topology. Vol. I.* Academic Press, New York. New edition, revised and augmented. Translated from the French by J. Jaworowski.

LALLEY, S.P.
(1998) Percolation on Fuchsian groups. *Ann. Inst. H. Poincaré Probab. Statist.* **34**, 151–177.

LAMPERTI, J.
(1967) The limit of a sequence of branching processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **7**, 271–288.

LAWLER, G.F.
(1980) A self-avoiding random walk. *Duke Math. J.* **47**, 655–693.

(1986) Gaussian behavior of loop-erased self-avoiding random walk in four dimensions. *Duke Math. J.* **53**, 249–269.

(1988) Loop-erased self-avoiding random walk in two and three dimensions. *J. Statist. Phys.* **50**, 91–108.

(1991) *Intersections of Random Walks.* Birkhäuser Boston Inc., Boston, MA.

LAWLER, G.F., SCHRAMM, O., and WERNER, W.
(2004) Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.* **32**, 939–995.

LAWLER, G.F. and SOKAL, A.D.
(1988) Bounds on the L^2 spectrum for Markov chains and Markov processes: a generalization of Cheeger's inequality. *Trans. Amer. Math. Soc.* **309**, 557–580.

LAWRENCENKO, S., PLUMMER, M.D., and ZHA, X.
(2002) Isoperimetric constants of infinite plane graphs. *Discrete Comput. Geom.* **28**, 313–330.

LE GALL, J.F. and LE JAN, Y.
(1998) Branching processes in Lévy processes: the exploration process. *Ann. Probab.* **26**, 213–252.

LE GALL, J.F. and ROSEN, J.
(1991) The range of stable random walks. *Ann. Probab.* **19**, 650–705.

LEVITT, G.
(1995) On the cost of generating an equivalence relation. *Ergodic Theory Dynam. Systems* **15**, 1173–1181.

LIGGETT, T.M.
(1987) Reversible growth models on symmetric sets. In Itō, K. and Ikeda, N., editors, *Probabilistic Methods in Mathematical Physics*, pages 275–301. Academic Press, Boston, MA. Proceedings of the Taniguchi International Symposium held in Katata, June 20–26, 1985, and at Kyoto University, Kyoto, June 27–29, 1985.

LIGGETT, T.M., SCHONMANN, R.H., and STACEY, A.M.
(1997) Domination by product measures. *Ann. Probab.* **25**, 71–95.

Linial, N., London, E., and Rabinovich, Y.

(1995) The geometry of graphs and some of its algorithmic applications. *Combinatorica* **15**, 215–245.

Linial, N., Magen, A., and Naor, A.

(2002) Girth and Euclidean distortion. *Geom. Funct. Anal.* **12**, 380–394.

Loomis, L.H. and Whitney, H.

(1949) An inequality related to the isoperimetric inequality. *Bull. Amer. Math. Soc* **55**, 961–962.

Lück, W.

(2009) L^2-invariants from the algebraic point of view. In Bridson, M.R., Kropholler, P.H., and Leary, I.J., editors, *Geometric and Cohomological Methods in Group Theory*, volume 358 of *London Math. Soc. Lecture Note Ser.*, pages 63–161. Cambridge Univ. Press, Cambridge. Papers from the London Mathematical Society Symposium on Geometry and Cohomology in Group Theory held in Durham, July 2003.

Lyons, R.

(1988) Strong laws of large numbers for weakly correlated random variables. *Michigan Math. J.* **35**, 353–359.

(1989) The Ising model and percolation on trees and tree-like graphs. *Comm. Math. Phys.* **125**, 337–353.

(1990) Random walks and percolation on trees. *Ann. Probab.* **18**, 931–958.

(1992) Random walks, capacity and percolation on trees. *Ann. Probab.* **20**, 2043–2088.

(1995) Random walks and the growth of groups. *C. R. Acad. Sci. Paris Sér. I Math.* **320**, 1361–1366.

(1996) Diffusions and random shadows in negatively curved manifolds. *J. Funct. Anal.* **138**, 426–448.

(2000) Phase transitions on nonamenable graphs. *J. Math. Phys.* **41**, 1099–1126. Probabilistic techniques in equilibrium and nonequilibrium statistical physics.

(2005) Asymptotic enumeration of spanning trees. *Combin. Probab. Comput.* **14**, 491–522.

(2009) Random complexes and ℓ^2-Betti numbers. *J. Top. Analysis* **1**, 153–175.

(2010) Identities and inequalities for tree entropy. *Combin. Probab. Comput.* **19**, 303–313.

Lyons, R., Morris, B.J., and Schramm, O.

(2008) Ends in uniform spanning forests. *Electron. J. Probab.* **13**, no. 58, 1702–1725.

Lyons, R., Pemantle, R., and Peres, Y.

(1995a) Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. *Ann. Probab.* **23**, 1125–1138.

(1995b) Ergodic theory on Galton-Watson trees: speed of random walk and dimension of harmonic measure. *Ergodic Theory Dynam. Systems* **15**, 593–619.

(1996a) Biased random walks on Galton-Watson trees. *Probab. Theory Related Fields* **106**, 249–264.

(1996b) Random walks on the lamplighter group. *Ann. Probab.* **24**, 1993–2006.

(1997) Unsolved problems concerning random walks on trees. In Athreya, K.B. and Jagers, P., editors, *Classical and Modern Branching Processes*, pages 223–237. Springer, New York. Papers from the IMA Workshop held at the University of Minnesota, Minneapolis, MN, June 13–17, 1994.

LYONS, R., PERES, Y., and SCHRAMM, O.

(2003) Markov chain intersections and the loop-erased walk. *Ann. Inst. H. Poincaré Probab. Statist.* **39**, 779–791.

(2006) Minimal spanning forests. *Ann. Probab.* **34**, 1665–1692.

LYONS, R., PICHOT, M., and VASSOUT, S.

(2008) Uniform non-amenability, cost, and the first l^2-Betti number. *Groups Geom. Dyn.* **2**, 595–617.

LYONS, R. and SCHRAMM, O.

(1999) Indistinguishability of percolation clusters. *Ann. Probab.* **27**, 1809–1836.

LYONS, R. and STEIF, J.E.

(2003) Stationary determinantal processes: Phase multiplicity, Bernoullicity, entropy, and domination. *Duke Math. J.* **120**, 515–575.

LYONS, T.

(1983) A simple criterion for transience of a reversible Markov chain. *Ann. Probab.* **11**, 393–402.

MADER, W.

(1970) Über den Zusammenhang symmetrischer Graphen. *Arch. Math. (Basel)* **21**, 331–336.

MADRAS, N. and SLADE, G.

(1993) *The Self-Avoiding Walk.* Birkhäuser Boston Inc., Boston, MA.

MAJUMDAR, S.N.

(1992) Exact fractal dimension of the loop-erased self-avoiding random walk in two dimensions. *Phys. Rev. Letters* **68**, 2329–2331.

MARCHAL, P.

(1998) The best bounds in a theorem of Russell Lyons. *Electron. Comm. Probab.* **3**, 91–94 (electronic).

(2000) Loop-erased random walks, spanning trees and Hamiltonian cycles. *Electron. Comm. Probab.* **5**, 39–50 (electronic).

MARCKERT, J.F.

(2008) The lineage process in Galton-Watson trees and globally centered discrete snakes. *Ann. Appl. Probab.* **18**, 209–244.

MARCKERT, J.F. and MOKKADEM, A.

(2003) Ladder variables, internal structure of Galton-Watson trees and finite branching random walks. *J. Appl. Probab.* **40**, 671–689.

MASSEY, W.S.

(1991) *A Basic Course in Algebraic Topology.* Springer-Verlag, New York.

MASSON, R.

(2009) The growth exponent for planar loop-erased random walk. *Electron. J. Probab.* **14**, no. 36, 1012–1073.

MATHIEU, C. and WILSON, D.B.

(2009) To appear.

MATOUŠEK, J.
(2008) On variants of the Johnson-Lindenstrauss lemma. *Random Structures Algorithms* **33**, 142–156.

MATSUZAKI, K. and TANIGUCHI, M.
(1998) *Hyperbolic Manifolds and Kleinian Groups*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York. Oxford Science Publications.

MATTHEWS, P.
(1988) Covering problems for Brownian motion on spheres. *Ann. Probab.* **16**, 189–199.

MAULDIN, R.D. and WILLIAMS, S.C.
(1986) Random recursive constructions: asymptotic geometric and topological properties. *Trans. Amer. Math. Soc.* **295**, 325–346.

MCCREA, W.H. and WHIPPLE, F.J.W.
(1940) Random paths in two and three dimensions. *Proc. Roy. Soc. Edinburgh* **60**, 281–298.

MEDOLLA, G. and SOARDI, P.M.
(1995) Extension of Foster's averaging formula to infinite networks with moderate growth. *Math. Z.* **219**, 171–185.

MEIR, A. and MOON, J.W.
(1970) The distance between points in random trees. *J. Combinatorial Theory* **8**, 99–103.

MINC, H.
(1988) *Nonnegative Matrices*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York. A Wiley-Interscience Publication.

MOHAR, B.
(1988) Isoperimetric inequalities, growth, and the spectrum of graphs. *Linear Algebra Appl.* **103**, 119–131.

MONTROLL, E.
(1964) Lattice statistics. In Beckenbach, E., editor, *Applied Combinatorial Mathematics*, pages 96–143. John Wiley and Sons, Inc., New York-London-Sydney. University of California Engineering and Physical Sciences Extension Series.

MOON, J.W.
(1967) Various proofs of Cayley's formula for counting trees. In Harary, F. and Beineke, L., editors, *A seminar on Graph Theory*, pages 70–78. Holt, Rinehart and Winston, New York.

MORRIS, B.
(2003) The components of the wired spanning forest are recurrent. *Probab. Theory Related Fields* **125**, 259–265.

MÖRTERS, P. and PERES, Y.
(2010) *Brownian Motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge. With an appendix by Oded Schramm and Wendelin Werner.

MUCHNIK, R. and PAK, I.
(2001) Percolation on Grigorchuk groups. *Comm. Algebra* **29**, 661–671.

NASH-WILLIAMS, C.ST.J.A.
(1959) Random walk and electric currents in networks. *Proc. Cambridge Philos. Soc.* **55**, 181–194.

NEVEU, J.

(1986) Arbres et processus de Galton-Watson. *Ann. Inst. H. Poincaré Probab. Statist.* **22**, 199–207.

NEWMAN, C.M.

(1997) *Topics in Disordered Systems.* Birkhäuser Verlag, Basel.

NEWMAN, C.M. and SCHULMAN, L.S.

(1981) Infinite clusters in percolation models. *J. Statist. Phys.* **26**, 613–628.

NEWMAN, C.M. and STEIN, D.L.

(1996) Ground-state structure in a highly disordered spin-glass model. *J. Statist. Phys.* **82**, 1113–1132.

ORNSTEIN, D.S. and WEISS, B.

(1987) Entropy and isomorphism theorems for actions of amenable groups. *J. Analyse Math.* **48**, 1–141.

PAK, I. and SMIRNOVA-NAGNIBEDA, T.

(2000) On non-uniqueness of percolation on nonamenable Cayley graphs. *C. R. Acad. Sci. Paris Sér. I Math.* **330**, 495–500.

PAKES, A.G. and DEKKING, F.M.

(1991) On family trees and subtrees of simple branching processes. *J. Theoret. Probab.* **4**, 353–369.

PAKES, A.G. and KHATTREE, R.

(1992) Length-biasing, characterizations of laws and the moment problem. *Austral. J. Statist.* **34**, 307–322.

PARRONDO, J.

(1996) How to cheat a bad mathematician. *EEC HC&M Network on Complexity and Chaos.* #ERBCHRX-CT940546. ISI, Torino, Italy. Unpublished.

PATERSON, A.L.T.

(1988) *Amenability.* American Mathematical Society, Providence, RI.

PEKÖZ, E. and RÖLLIN, A.

(2009) New rates for exponential approximation and the theorems of Rényi and Yaglom. Preprint, `http://people.bu.edu/pekoz/exponential2.pdf`.

PEMANTLE, R.

(1988) Phase transition in reinforced random walk and RWRE on trees. *Ann. Probab.* **16**, 1229–1241.

(1991) Choosing a spanning tree for the integer lattice uniformly. *Ann. Probab.* **19**, 1559–1574.

PEMANTLE, R. and PERES, Y.

(1995a) Critical random walk in random environment on trees. *Ann. Probab.* **23**, 105–140.

(1995b) Galton-Watson trees with the same mean have the same polar sets. *Ann. Probab.* **23**, 1102–1124.

(1996) On which graphs are all random walks in random environments transient? In *Random Discrete Structures (Minneapolis, MN, 1993)*, pages 207–211. Springer, New York.

(2000) Nonamenable products are not treeable. *Israel J. Math.* **118**, 147–155.

PEMANTLE, R. and STACEY, A.M.
(2001) The branching random walk and contact process on Galton-Watson and nonhomogeneous trees. *Ann. Probab.* **29**, 1563–1590.

PERES, Y.
(1996) Intersection-equivalence of Brownian paths and certain branching processes. *Comm. Math. Phys.* **177**, 417–434.
(1999) Probability on trees: an introductory climb. In Bernard, P., editor, *Lectures on Probability Theory and Statistics*, volume 1717 of *Lecture Notes in Math.*, pages 193–280. Springer, Berlin. Lectures from the 27th Summer School on Probability Theory held in Saint-Flour, July 7–23, 1997.
(2000) Percolation on nonamenable products at the uniqueness threshold. *Ann. Inst. H. Poincaré Probab. Statist.* **36**, 395–406.

PERES, Y. and STEIF, J.E.
(1998) The number of infinite clusters in dynamical percolation. *Probab. Theory Related Fields* **111**, 141–165.

PERES, Y. and ZEITOUNI, O.
(2008) A central limit theorem for biased random walks on Galton-Watson trees. *Probab. Theory Related Fields* **140**, 595–629.

PERLIN, A.
(2001) *Probability Theory on Galton-Watson Trees*. Ph.D. thesis, Massachusetts Institute of Technology. Available at `http://hdl.handle.net/1721.1/8673`.

PETERSEN, K.
(1983) *Ergodic Theory*. Cambridge University Press, Cambridge.

PIAU, D.
(1996) Functional limit theorems for the simple random walk on a supercritical Galton-Watson tree. In Chauvin, B., Cohen, S., and Rouault, A., editors, *Trees*, volume 40 of *Progr. Probab.*, pages 95–106. Birkhäuser, Basel. Proceedings of the Workshop held in Versailles, June 14–16, 1995.
(1998) Théorème central limite fonctionnel pour une marche au hasard en environnement aléatoire. *Ann. Probab.* **26**, 1016–1040.
(2002) Scaling exponents of random walks in random sceneries. *Stochastic Process. Appl.* **100**, 3–25.

PINSKER, M.S.
(1973) On the complexity of a concentrator. *Proceedings of the Seventh International Teletraffic Congress (Stockholm, 1973)*, 318/1–318/4. Paper No. 318; unpublished.

PITMAN, J.
(1993) *Probability*. Springer, New York.
(1998) Enumerations of trees and forests related to branching processes and random walks. In Aldous, D. and Propp, J., editors, *Microsurveys in Discrete Probability*, volume 41 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 163–180. Amer. Math. Soc., Providence, RI. Papers from the workshop held as part of the Dimacs Special Year on Discrete Probability in Princeton, NJ, June 2–6, 1997.

PITT, L.D.
(1982) Positively correlated normal variables are associated. *Ann. Probab.* **10**, 496–499.

Pittet, C. and Saloff-Coste, L.
(2001) A survey on the relationships between volume growth, isoperimetry, and the behavior of simple random walk on Cayley graphs, with examples. In preparation.

Ponzio, S.
(1998) The combinatorics of effective resistances and resistive inverses. *Inform. and Comput.* **147**, 209–223.

Port, S.C. and Stone, C.J.
(1978) *Brownian Motion and Classical Potential Theory*. Academic Press [Harcourt Brace Jovanovich Publishers], New York. Probability and Mathematical Statistics.

Propp, J.G. and Wilson, D.B.
(1998) How to get a perfectly random sample from a generic Markov chain and generate a random spanning tree of a directed graph. *J. Algorithms* **27**, 170–217. 7th Annual ACM-SIAM Symposium on Discrete Algorithms (Atlanta, GA, 1996).

Pyke, R.
(2003) On random walks and diffusions related to Parrondo's games. In Moore, M., Froda, S., and Léger, C., editors, *Mathematical Statistics and Applications: Festschrift for Constance van Eeden*, volume 42 of *IMS Lecture Notes Monogr. Ser.*, pages 185–216. Inst. Math. Statist., Beachwood, OH.

Ratcliffe, J.G.
(2006) *Foundations of Hyperbolic Manifolds*, volume 149 of *Graduate Texts in Mathematics*. Springer, New York, second edition.

Reimer, D.
(2000) Proof of the van den Berg-Kesten conjecture. *Combin. Probab. Comput.* **9**, 27–32.

Revelle, D.
(2001) Biased random walk on lamplighter groups and graphs. *J. Theoret. Probab.* **14**, 379–391.

Rider, B. and Virág, B.
(2007) The noise in the circular law and the Gaussian free field. *Int. Math. Res. Not. IMRN*, Art. ID rnm006, 33.

Rosenblatt, M.
(1971) *Markov Processes. Structure and Asymptotic Behavior*. Springer-Verlag, New York. Die Grundlehren der mathematischen Wissenschaften, Band 184.

Ross, S.M.
(1996) *Stochastic Processes*. John Wiley & Sons Inc., New York, second edition.

Saloff-Coste, L.
(1995) Isoperimetric inequalities and decay of iterated kernels for almost-transitive Markov chains. *Combin. Probab. Comput.* **4**, 419–442.

Salvatori, M.
(1992) On the norms of group-invariant transition operators on graphs. *J. Theoret. Probab.* **5**, 563–576.

Schonmann, R.H.
(1992) On the behavior of some cellular automata related to bootstrap percolation. *Ann. Probab.* **20**, 174–193.

(1999a) Percolation in $\infty+1$ dimensions at the uniqueness threshold. In Bramson, M. and Durrett, R., editors, *Perplexing Problems in Probability*, pages 53–67. Birkhäuser Boston, Boston, MA. Festschrift in honor of Harry Kesten.

(1999b) Stability of infinite clusters in supercritical percolation. *Probab. Theory Related Fields* **113**, 287–300.

(2001) Multiplicity of phase transitions and mean-field criticality on highly non-amenable graphs. *Comm. Math. Phys.* **219**, 271–322.

SCHRAMM, O.

(2000) Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.* **118**, 221–288.

SENETA, E.

(1968) On recent theorems concerning the supercritical Galton-Watson process. *Ann. Math. Statist.* **39**, 2098–2102.

(1970) On the supercritical Galton-Watson process with immigration. *Math. Biosci.* **7**, 9–14.

SHEFFIELD, S.

(2007) Gaussian free fields for mathematicians. *Probab. Theory Related Fields* **139**, 521–541.

SHEPP, L.A.

(1972) Covering the circle with random arcs. *Israel J. Math.* **11**, 328–345.

SHROCK, R. and WU, F.Y.

(2000) Spanning trees on graphs and lattices in d dimensions. *J. Phys. A* **33**, 3881–3902.

SOARDI, P.M.

(1993) Rough isometries and Dirichlet finite harmonic functions on graphs. *Proc. Amer. Math. Soc.* **119**, 1239–1248.

SOARDI, P.M. and WOESS, W.

(1990) Amenability, unimodularity, and the spectral radius of random walks on infinite graphs. *Math. Z.* **205**, 471–486.

SOLOMON, F.

(1975) Random walks in a random environment. *Ann. Probability* **3**, 1–31.

SPITZER, F.

(1976) *Principles of Random Walk*. Springer-Verlag, New York, second edition. Graduate Texts in Mathematics, Vol. 34.

STACEY, A.M.

(1996) The existence of an intermediate phase for the contact process on trees. *Ann. Probab.* **24**, 1711–1726.

STEELE, J.M.

(1997) *Probability Theory and Combinatorial Optimization*, volume 69 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA.

STRASSEN, V.

(1965) The existence of probability measures with given marginals. *Ann. Math. Statist.* **36**, 423–439.

SWART, J.M.

(2009) The contact process seen from a typical infected site. *J. Theoret. Probab.* **22**, 711–740.

SZNITMAN, A.S.

(1998) *Brownian Motion, Obstacles and Random Media.* Springer Monographs in Mathematics. Springer-Verlag, Berlin.

TAKACS, C.

(1997) Random walk on periodic trees. *Electron. J. Probab.* **2**, no. 1, 1–16 (electronic).

(1998) Biased random walks on directed trees. *Probab. Theory Related Fields* **111**, 123–139.

TANNY, D.

(1988) A necessary and sufficient condition for a branching process in a random environment to grow like the product of its means. *Stochastic Process. Appl.* **28**, 123–139.

TANUSHEV, M. and ARRATIA, R.

(1997) A note on distributional equality in the cyclic tour property for Markov chains. *Combin. Probab. Comput.* **6**, 493–496.

TAYLOR, S.J.

(1955) The α-dimensional measure of the graph and set of zeros of a Brownian path. *Proc. Cambridge Philos. Soc.* **51**, 265–274.

TEMPERLEY, H.N.V.

(1974) Enumeration of graphs on a large periodic lattice. In McDonough, T.P. and Mavron, V.C., editors, *Combinatorics (Proc. British Combinatorial Conf., Univ. Coll. Wales, Aberystwyth, 1973)*, pages 155–159, London. Cambridge Univ. Press. London Math. Soc. Lecture Note Ser., No. 13.

TETALI, P.

(1991) Random walks and the effective resistance of networks. *J. Theoret. Probab.* **4**, 101–109.

THOMASSEN, C.

(1989) Transient random walks, harmonic functions, and electrical currents in infinite electrical networks. Technical Report Mat-Report n. 1989-07, The Technical Univ. of Denmark.

(1990) Resistances and currents in infinite electrical networks. *J. Combin. Theory Ser. B* **49**, 87–102.

(1992) Isoperimetric inequalities and transient random walks on graphs. *Ann. Probab.* **20**, 1592–1600.

TIMÁR, Á.

(2006) Ends in free minimal spanning forests. *Ann. Probab.* **34**, 865–869.

(2007) Cutsets in infinite graphs. *Combin. Probab. Comput.* **16**, 159–166.

TJUR, T.

(1991) Block designs and electrical networks. *Ann. Statist.* **19**, 1010–1027.

TONGRING, N.

(1988) Which sets contain multiple points of Brownian motion? *Math. Proc. Cambridge Philos. Soc.* **103**, 181–187.

TROFIMOV, V.I.

(1985) Groups of automorphisms of graphs as topological groups. *Mat. Zametki* **38**, 378–385, 476. English translation: *Math. Notes* **38** (1985), no. 3-4, 717–720.

TRUEMPER, K.

(1989) On the delta-wye reduction for planar graphs. *J. Graph Theory* **13**, 141–148.

VAROPOULOS, N.TH.
(1985a) Isoperimetric inequalities and Markov chains. *J. Funct. Anal.* **63**, 215–239.
(1985b) Long range estimates for Markov chains. *Bull. Sci. Math. (2)* **109**, 225–252.

VATUTIN, V.A. and ZUBKOV, A.M.
(1985) Branching processes. I. In *Teoriya Veroyatnostei. Matematicheskaya Statistika. Teoreticheskaya Kibernetika. Tom 23*, Itogi Nauki i Tekhniki, pages 3–67, 154. Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow. English translation in *J. Soviet Math.* **39**, no. 1, pp. 2431–2475.
(1993) Branching processes. II. *J. Soviet Math.* **67**, 3407–3485. Probability theory and mathematical statistics, 1.

VIRÁG, B.
(2000a) Anchored expansion and random walk. *Geom. Funct. Anal.* **10**, 1588–1605.
(2000b) On the speed of random walks on graphs. *Ann. Probab.* **28**, 379–394.
(2002) Fast graphs for the random walker. *Probab. Theory Related Fields* **124**, 50–72.

WATKINS, M.E.
(1970) Connectivity of transitive graphs. *J. Combinatorial Theory* **8**, 23–29.

WAYMIRE, E.C. and WILLIAMS, S.C.
(1994) A general decomposition theory for random cascades. *Bull. Amer. Math. Soc. (N.S.)* **31**, 216–222.

WHITNEY, H.
(1932) Congruent graphs and the connectivity of graphs. *Amer. J. Math.* **54**, 150–168.

WHYTE, K.
(1999) Amenability, bi-Lipschitz equivalence, and the von Neumann conjecture. *Duke Math. J.* **99**, 93–112.

WIELAND, B. and WILSON, D.B.
(2003) Winding angle variance of Fortuin-Kasteleyn contours. *Physical Review E* **68**, 056101. doi:10.1103/PhysRevE.68.056101.

WILSON, D.B.
(1996) Generating random spanning trees more quickly than the cover time. In *Proceedings of the Twenty-eighth Annual ACM Symposium on the Theory of Computing*, pages 296–303, New York. ACM. Held in Philadelphia, PA, May 22–24, 1996.
(1997) Determinant algorithms for random planar structures. In *Proceedings of the Eighth Annual ACM-SIAM Symposium on Discrete Algorithms (New Orleans, LA, 1997)*, pages 258–267, New York. ACM. Held in New Orleans, LA, January 5–7, 1997.
(2004a) Red-green-blue model. *Phys. Rev. E (3)* **69**, 037105.

WILSON, J.S.
(2004b) On exponential growth and uniformly exponential growth for groups. *Invent. Math.* **155**, 287–303.
(2004c) Further groups that do not have uniformly exponential growth. *J. Algebra* **279**, 292–301.

WOESS, W.
(1991) Topological groups and infinite graphs. *Discrete Math.* **95**, 373–384. Directions in infinite graph theory and combinatorics (Cambridge, 1989).

(2000) *Random Walks on Infinite Graphs and Groups*, volume 138 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge.

WOLF, J.A.
(1968) Growth of finitely generated solvable groups and curvature of Riemanniann manifolds. *J. Differential Geometry* **2**, 421–446.

YAGLOM, A.M.
(1947) Certain limit theorems of the theory of branching random processes. *Doklady Akad. Nauk SSSR (N.S.)* **56**, 795–798.

ZEMANIAN, A.H.
(1976) Infinite electrical networks. *Proc. IEEE* **64**, 6–17. Recent trends in system theory.

ZUBKOV, A.M.
(1975) Limit distributions of the distance to the nearest common ancestor. *Teor. Verojatnost. i Primenen.* **20**, 614–623. English translation: *Theor. Probab. Appl.* **20**, 602–612.

ŻUK, A.
(1996) La propriété (T) de Kazhdan pour les groupes agissant sur les polyèdres. *C. R. Acad. Sci. Paris Sér. I Math.* **323**, 453–458.

Index

References to definitions have page numbers in roman font, while others (such as to theorems, examples, exercises) are in *slant* font. Of course, some results appear on the same page as the definition.

Glossary of Notation

$\langle \cdots \rangle$	a sequence, vii
$\asymp$	equal up to bounded factors, 159, 452
$\restriction$	restriction, vii
$\lvert \bullet \rvert$	cardinality, viii
G/e	contraction of e in G, 100
$G\backslash e$	deletion of e in G, 101
$\deg x$	degree of x, 21
$\mathsf{E}(G)$	edge set of G, 1
$G_1 \times G_2$	cartesian product graph, 196
$\mathsf{V}(G)$	vertex set of G, 1
G^{W}	G with its boundary wired, 39
e^+	the head of e, 2
e^-	the tail of e, 2
$\langle x, y \rangle$	the oriented edge with endpoints x and y, 1
$-e$	the reverse of e, 2
$[x, y]$	the unoriented edge with endpoints x and y, 1
$x \sim y$	x and y are adjacent, 1
G/K	G with K contracted, 38
$\partial_{\mathsf{E}} K$	edge boundary, 156
$\partial_{\mathsf{E}} K$	edge boundary of K, 106
$\partial_{\mathsf{V}}^{\mathrm{int}} K$	internal vertex boundary, 180
$G \restriction K$	the network G induces on K, 2
$G^\dagger$	dual graph, 259
$e^\dagger$	dual edge, 259
$\omega^\times$	dual configuration, 259
$\mathbb{T}_{b+1}$	regular tree of degree $b+1$, 83
∂T	boundary at infinity (rays) of T, 13
$\mathrm{br}\, T$	branching number of T, 3
$\mathrm{gr}\, T$	exponential growth rate of T, 3
$\underline{\mathrm{gr}}\, T$	lower exponential growth rate of T, 3
$\overline{\mathrm{gr}}\, T$	upper exponential growth rate of T, 3
T_n	level n of the tree T, 2
T^x	the descendant subtree of x, 76
$x < y$	x is between the root and y and $x \neq y$, 76
$x \le y$	x is between the root and y, 76
$x \to y$	y is a child of x, 76

$\lvert x-y\rvert$	the distance between x and y, 75
$\lvert x\rvert$	distance between x and the root, 4
$T_{[b]}(E)$	b-adic coding tree of E, 16

(G, c, D)	graph G with edge weights c and vertex weights D, 156
$(f, g)_h$	inner product with respect to h, 35
$C_{\rm eff}$	effective conductance, 25
$P_\bigstar$	orthogonal projection onto $\bigstar$, 36
$R_{\rm eff}$	effective resistance, 25
$Y(e, e')$	transfer current along e' from e, 37
χ^e	unit flow along e, 35
$\diamondsuit$	cycle space, 36
$\mathbf{D}$	space of Dirichlet functions, 280
$\mathbf{D}_0$	Dirichlet closure of functions with finite support, 282
$\mathbf{D}_{00}$	functions with finite support, 162
Δ_G	network Laplacian, 62
$\mathscr{D}(f)$	Dirichlet energy of f, 281
$\mathscr{C}(a \leftrightarrow Z; G)$	effective conductance in G between a and Z, 25
$\mathscr{R}(a \leftrightarrow Z; G)$	effective resistance in G between a and Z, 25
$\ell^2_-(\mathsf{E})$	antisymmetric functions on E, 33
$\mathbf{HD}$	space of harmonic Dirichlet functions, 281
$\bigstar$	star space, 36
$\Phi^*_{\mathsf{E}}(G)$, $\Phi^*_{\mathsf{V}}(G)$	anchored expansion constants, 181
$\mathscr{E}(\theta)$	energy of θ, 35
$\Phi_{\mathsf{E}}(G)$	expansion (isoperimetric) constant of G, 156
∇f	gradient of f, 280
$\psi(G, t)$	expansion profile, 178
$\pi(x)$	sum of conductances at x, stationary measure, 19
$\lVert f\rVert_h$	norm with respect to h, 35
$c(e)$	conductance of e, 19
d	coboundary operator, 33
d^*	boundary operator, 33
$i(e)$	current through e, 24
$v(x)$	voltage at x, 24
$\lvert K\rvert_D$	D-weight of K, 156

$\Gamma * \Gamma'$	free product of groups, 82
$\langle S \mid R\rangle$	the group with generators S and relators R, 82

$\mathbf{E}[X\,;\ A]$	expectation of X on A, viii
$\mathsf{H\ddot{o}}(\mu)$	Hölder exponent of μ, 434
FMSF	free minimal spanning forest, 354
$\mathfrak{F}$	random spanning forest, 307
FSF	free spanning forest, 307
MST	minimal spanning tree, 352
WMSF	wired minimal spanning forest, 355
WSF	wired spanning forest, 308

$\mathsf{LE}(\mathcal{P})$ .... loop-erasure of $\mathcal{P}$, 91
$\mathbf{P}[a \to Z]$ .... probability of hitting Z before returning to a, 25
$\mathbf{P}_x$ .... law of random walk started at x, 20, 55
$\mathcal{G}(x, y)$ .... expected number of visits to x from x, 20, 55
$\rho(G)$ .... spectral radius, 164, 195
τ_x .... first hitting time of x, 20, 55
τ_x^+ .... first hitting time of x after 0, 20, 55
RW_λ .... homesick random walk biased by λ, 74

L .... offspring random variable, 124
W .... limit of martingale Z_n/m^n, 9
Z_n .... size of nth generation, 9
$\widehat{L}$ .... size-biased random variable, 375
$\overline{q}$ .... $1 - q$, 142
$f(s)$ .... offspring p.g.f., 124
m .... mean number of offspring, 9
p_k .... the probability of k children in a branching process, 9
q .... probability of extinction, 124

$\mathbf{P}_p$ .... Bernoulli(p) percolation, 130
$\hat{o}$ .... a normalized random root in a quasi-transitive unimodular graph, 251
ω_p .... open subgraph at level p, 130
$G[p]$ .... open subgraph at level p, 351
$p_c(G)$ .... critical probability of G, 130
$p_u(G)$ .... critical probability for uniqueness, 221
$\theta(p)$ .... probability that a vertex belongs to an infinite cluster, 202

Ich bin dein Baum
by Friedrich Rückert

Ich bin dein Baum: o Gärtner, dessen Treue
Mich hält in Liebespfleg' und süßer Zucht,
Komm, daß ich in den Schoß dir dankbar streue
Die reife dir allein gewachsne Frucht.

Ich bin dein Gärtner, o du Baum der Treue!
Auf andres Glück fühl' ich nicht Eifersucht:
Die holden Äste find' ich stets aufs neue
Geschmückt mit Frucht, wo ich gepflückt die Frucht.

I am your tree, O gardener, whose care
Holds me in sweet restraint and loving ban.
Come, let me find your lap and scatter there
The ripened fruit, grown for no other man.

I am your gardener, then, O faithful tree!
I covet no one else's bliss instead;
Your lovely limbs in perpetuity
Yield fruit, however lately harvested.

Transl. by Walter Arndt
Set to music by Robert Schumann