Exercises and Problems in Calculus

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Preface

This is a set of *exercises* and *problems* for a (more or less) standard beginning calculus sequence. While a fair number of the *exercises* involve only routine computations, many of the *exercises* and most of the *problems* are meant to illuminate points that in my experience students have found confusing.

Virtually all of the *exercises* have fill-in-the-blank type answers. Often an *exercise* will end with something like, " ...so the answer is $a\sqrt{3} + \dfrac{\pi}{b}$ where $a =$ ______ and $b =$ ______ ." One advantage of this type of answer is that it makes it possible to provide students with feedback on a substantial number of homework exercises without a huge investment of time. More importantly, it gives students a way of checking their work without giving them the answers. When a student works through the *exercise* and comes up with an answer that doesn't look anything like $a\sqrt{3} + \dfrac{\pi}{b}$, he/she has been given an obvious invitation to check his/her work.

The major drawback of this type of answer is that it does nothing to promote good communication skills, a matter which in my opinion is of great importance even in beginning courses. That is what the *problems* are for. They require logically thought through, clearly organized, and clearly written up reports. In my own classes I usually assign *problems* for group work outside of class. This serves the dual purposes of reducing the burden of grading and getting students involved in the material through discussion and collaborative work.

This collection is divided into parts and chapters roughly by topic. Many chapters begin with a "background" section. This is most emphatically **not** intended to serve as an exposition of the relevant material. It is designed only to fix notation, definitions, and conventions (which vary widely from text to text) and to clarify what topics one should have studied before tackling the *exercises* and *problems* that follow.

The flood of elementary calculus texts published in the past half century shows, if nothing else, that the topics discussed in a beginning calculus course can be covered in virtually any order. The divisions into chapters in these notes, the order of the chapters, and the order of items within a chapter is in no way intended to reflect opinions I have about the way in which (or even if) calculus *should* be taught. For the convenience of those who might wish to make use of these notes I have simply chosen what seems to me one fairly common ordering of topics. Neither the *exercises* nor the *problems* are ordered by difficulty. Utterly trivial problems sit alongside ones requiring substantial thought.

Each chapter ends with a list of the solutions to all the odd-numbered *exercises*.

The great majority of the "applications" that appear here, as in most calculus texts, are best regarded as jests whose purpose is to demonstrate in the very simplest ways some connections between physical quantities (area of a field, volume of a silo, speed of a train, *etc.*) and the mathematics one is learning. It does *not* make these "real world" problems. No one seriously imagines that some Farmer Jones is really interested in maximizing the area of his necessarily rectangular stream-side pasture with a fixed amount of fencing, or that your friend Sally just happens to notice that the train passing her is moving at 54.6 mph. To my mind genuinely interesting "real world" problems require, in general, way too much background to fit comfortably into an already overstuffed calculus course. You will find in this collection just a very few serious applications, problem 15 in Chapter 29, for example, where the background is either minimal or largely irrelevant to the solution of the problem.

I make no claims of originality. While I have dreamed up many of the items included here, there are many others which are standard calculus exercises that can be traced back, in one form or another, through generations of calculus texts, making any serious attempt at proper attribution quite futile. If anyone feels slighted, please contact me.

There will surely be errors. I will be delighted to receive corrections, suggestions, or criticism at

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I have placed the the LaTeX source files on my web page so that anyone who wishes can download the material, edit it, add to it, and use it for any noncommercial purpose.

Part 1

PRELIMINARY MATERIAL

CHAPTER 1

INEQUALITIES AND ABSOLUTE VALUES

1.1. Background

Topics: inequalities, absolute values.

1.1.1. Definition. If x and a are two real numbers the DISTANCE between x and a is $|x-a|$. For most purposes in calculus it is better to think of an inequality like $|x-5|<2$ geometrically rather then algebraically. That is, think "The number x is within 2 units of 5," rather than "The absolute value of x minus 5 is strictly less than 2." The first formulation makes it clear that x is in the open interval $(3,7)$.

1.1.2. Definition. Let a be a real number. A NEIGHBORHOOD of a is an open interval (c,d) in $\mathbb{R}$ which contains a. An open interval $(a-\delta,a+\delta)$ which is centered at a is a SYMMETRIC NEIGHBORHOOD (or a δ-NEIGHBORHOOD) of a.

1.1.3. Definition. A DELETED (or PUNCTURED) NEIGHBORHOOD of a point $a\in\mathbb{R}$ is an open interval around a from which a has been deleted. Thus, for example, the deleted δ-neighborhood about 3 would be $(3-\delta,3+\delta)\setminus\{3\}$ or, using different notation, $(3-\delta,3)\cup(3,3+\delta)$.

1.1.4. Definition. A point a is an ACCUMULATION POINT of a set $B\subseteq\mathbb{R}$ if every deleted neighborhood of a contains at least one point of B.

1.1.5. Notation (For Set Operations)**.** Let A and B be subsets of a set S. Then

(1) $x\in A\cup B$ if $x\in A$ or $x\in B$ (*union*);
(2) $x\in A\cap B$ if $x\in A$ and $x\in B$ (*intersection*);
(3) $x\in A\setminus B$ if $x\in A$ and $x\notin B$ (*set difference*); and
(4) $x\in A^c$ if $x\in S\setminus A$ (*complement*).

If the set S is not specified, it is usually understood to be the set $\mathbb{R}$ of real numbers or, starting in Part 6, the set $\mathbb{R}^n$, Euclidean n-dimensional space.

1.2. Exercises

(1) The inequality $|x-2|<6$ can be expressed in the form $a<x<b$ where $a =$ _____ and $b =$ _____ .

(2) The inequality $-15 \le x \le 7$ can be expressed in the form $|x-a| \le b$ where $a =$ _____ and $b =$ _____ .

(3) Solve the equation $|4x+23| = |4x-9|$. Answer: $x =$ ________ .

(4) Find all numbers x which satisfy $|x^2+2| = |x^2-11|$.

Answer: $x =$ ________ and $x =$ ________ .

(5) Solve the inequality $\dfrac{3x}{x^2+2} \ge \dfrac{1}{x-1}$. Express your answer in interval notation.

Answer: [_____ , _____) $\cup$ [2 , _____) .

(6) Solve the equation $|x-2|^2 + 3|x-2| - 4 = 0$.

Answer: $x =$ _____ and $x =$ _____ .

(7) The inequality $-4 \le x \le 10$ can be expressed in the form $|x-a| \le b$ where $a =$ _____ and $b =$ _____ .

(8) Sketch the graph of the equation $x - 2 = |y-3|$.

(9) The inequality $|x+4| < 7$ can be expressed in the form $a<x<b$ where $a =$ _____ and $b =$ _____ .

(10) Solve the inequality $|3x+7| < 5$. Express your answer in interval notation.

Answer: (_____ , _____).

(11) Find all numbers x which satisfy $|x^2-9| = |x^2-5|$.

Answer: $x =$ _____ and $x =$ _____ .

(12) Solve the inequality $\left|\dfrac{2x^2-3}{14}\right| \le \dfrac{1}{2}$. Express your answer in interval notation.

Answer: [_____ , _____].

(13) Solve the inequality $|x-3| \ge 6$. Express your answer in interval notation.

Answer: (_____ , _____] $\cup$ [_____ , _____) .

(14) Solve the inequality $\dfrac{x}{x+2} \ge \dfrac{x+3}{x-4}$. Express your answer in interval notation.

Answer: (_____ , _____) $\cup$ [_____ , _____).

(15) In interval notation the solution set for the inequality $\dfrac{x+1}{x-2} \le \dfrac{x+2}{x+3}$ is $(-\infty,$ _____) $\cup$ [_____ , 2).

(16) Solve the inequality $\dfrac{4x^2-x+19}{x^3+x^2+4x+4} \ge 1$. Express your answer in interval notation.

Answer: (_____ , _____].

(17) Solve the equation $2|x+3|^2 - 15|x+3| + 7 = 0$.

Answer: $x =$ _____ , $x =$ _____ , $x =$ _____ , and $x =$ _____ .

(18) Solve the inequality $x \ge 1 + \dfrac{2}{x}$. Express your answer in interval notation.

Answer: [_____ , 0) $\cup$ [_____ , _____).

1.3. Problems

(1) Let a, $b \in \mathbb{R}$. Show that $\big|\,|a| - |b|\,\big| \leq |a - b|$.

(2) Let a, $b \in \mathbb{R}$. Show that $|ab| \leq \frac{1}{2}(a^2 + b^2)$.

1.4. Answers to Odd-Numbered Exercises

(1) -4, 8

(3) $-\dfrac{7}{4}$

(5) $[-\frac{1}{2}, 1) \cup [2, \infty)$

(7) 3, 7

(9) -11, 3

(11) $-\sqrt{7}$, $\sqrt{7}$

(13) $(-\infty, -3] \cup [9, \infty)$

(15) $(-\infty, -3) \cup [-\frac{7}{4}, 2)$

(17) -10, $-\dfrac{7}{2}$, $-\dfrac{5}{2}$, 4

CHAPTER 2

LINES IN THE PLANE

2.1. Background

Topics: equations of lines in the plane, slope, x- and y-intercepts, parallel and perpendicular lines.

2.1.1. Definition. Let (x_1, y_1) and (x_2, y_2) be points in the plane such that $x_1 \neq x_2$. The SLOPE of the (nonvertical straight) line L which passes through these points is

$$m_L := \frac{y_2 - y_1}{x_2 - x_1}.$$

The *equation* for L is

$$y - y_0 = m_L(x - x_0)$$

where (x_0, y_0) is any point lying on L. (If the line L is vertical (that is, parallel to the y-axis) it is common to say that it has *infinite slope* and write $m_L = \infty$. The equation for a vertical line is $x = x_0$ where (x_0, y_0) is any point lying on L.)

Two nonvertical lines L and L' are PARALLEL if their respective slopes m_L and $m_{L'}$ are equal. (Any two vertical lines are parallel.) They are PERPENDICULAR if their respective slopes are negative reciprocals; that is, if $m_{L'} = \frac{1}{m_L}$. (Vertical lines are always perpendicular to horizontal lines.)

2.2. Exercises

(1) The equation of the line passing through the points $(-7, -3)$ and $(8, 2)$ is $ay = x + b$ where $a =$ _____ and $b =$ _____

(2) The equation of the perpendicular bisector of the line segment joining the points $(2, -5)$ and $(4, 3)$ is $ax + by + 1 = 0$ where $a =$ _____ and $b =$ _____ .

(3) Let L be the line passing through the point $(4, 9)$ with slope $\frac{3}{4}$. The x-intercept of L is _____ and its y-intercept is _____ .

(4) The equation of the line which passes through the point $(4, 2)$ and is perpendicular to the line $x + 2y = 1$ is $ax + by + 1 = 0$ where $a =$ ________ and $b =$ ________ .

(5) The equation of the line which is parallel to the line $x + \frac{3}{2}y = \frac{5}{2}$ and passes through the point $(-1, -3)$ is $2x + ay + b = 0$ where $a =$ ________ and $b =$ ________ .

2.3. Problems

(1) The town of Plainfield is 4 miles east and 6 miles north of Burlington. Allentown is 8 miles west and 1 mile north of Plainfield. A straight road passes through Plainfield and Burlington. A second straight road passes through Allentown and intersects the first road at a point somewhere south and west of Burlington. The angle at which the roads intersect is $\pi/4$ radians. Explain how to find the location of the point of intersection and carry out the computation you describe.

(2) Prove that the line segment joining the midpoints of two sides of a triangle is half the length of the third side and is parallel to it. *Hint.* Try not to make things any more complicated than they need to be. A thoughtful choice of a coordinate system may be helpful. One possibility: orient the triangle so that one side runs along the x-axis and one vertex is at the origin.

2.4. Answers to Odd-Numbered Exercises

(1) 3, -2

(3) -8, 6

(1) 3, 11

CHAPTER 3

FUNCTIONS

3.1. Background

Topics: functions, domain, codomain, range, bounded above, bounded below, composition of functions.

3.1.1. Definition. If S and T are sets we say that f is a FUNCTION from S to T if for every x in S there corresponds one and only one element $f(x)$ in T. The set S is called the DOMAIN of f and is denoted by $\operatorname{dom} f$. The set T is called the CODOMAIN of f. The RANGE of f is the set of all $f(x)$ such that x belongs to S. It is denoted by $\operatorname{ran} f$. The words *function, map, mapping*, and *transformation* are synonymous.

A function $f\colon A \to B$ is said to be *real valued* if $B \subseteq \mathbb{R}$ and is called a *function of a real variable* if $A \subseteq \mathbb{R}$.

The notation $f\colon S \to T\colon x \mapsto f(x)$ indicates that f is a function whose domain is S, whose codomain is T, and whose value at x is $f(x)$. Thus, for example, $f\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto x^2$ defines the real valued function whose value at each real number x is given by $f(x) = x^2$. We use $\operatorname{dom} f$ to denote the domain of f and $\operatorname{ran} f$ to denote its range.

3.1.2. Definition. A function $f\colon S \to \mathbb{R}$ is BOUNDED ABOVE by a number M is $f(x) \leq M$ for every $x \in S$, It is BOUNDED BELOW by a number K if $K \leq f(x)$ for every $x \in S$. And it is BOUNDED if it is bounded both above and below; that is, if there exists $N > 0$ such that $|f(x)| \leq N$ for every $x \in S$.

3.1.3. Definition. Let f and g be real valued functions of a real variable. Define the COMPOSITE of g and f, denoted by $g \circ f$, by

$$(g \circ f)(x) := g(f(x))$$

for all $x \in \operatorname{dom} f$ such that $f(x) \in \operatorname{dom} g$. The operation $\circ$ is called *composition*.

For problem 2, the following fact may be useful.

3.1.4. Theorem. *Every nonempty open interval in* $\mathbb{R}$ *contains both rational and irrational numbers.*

3.2. Exercises

(1) Let $f(x) = \dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}$. Then:

(a) $f(\frac{1}{2}) =$ ________ .

(b) The domain of f is the set of all real numbers except _____ , _______ , and _____ .

(2) Let $f(x) = \dfrac{7-\sqrt{x^2-9}}{\sqrt{25-x^2}}$. Then $\operatorname{dom} f = (\ \_\_\_\_\_\ ,\ \_\_\_\_\_\ ] \cup [\ \_\_\_\_\_\ ,\ \_\_\_\_\_\ )$.

(3) Find the domain and range of the function $f(x) = 2\sqrt{4-x^2} - 3$.

Answer: $\operatorname{dom} f = [\ \_\_\_\_\_\ ,\ \_\_\_\_\_\ ]$ and $\operatorname{ran} f = [\ \_\_\_\_\_\ ,\ \_\_\_\_\_\ ]$.

(4) Let $f(x) = x^3 - 4x^2 - 11x - 190$. The set of all numbers x such that $|f(x) - 40| < 260$ is $(\ \_\_\_\_\_\ ,\ \_\_\_\_\_\ ) \cup (\ \_\_\_\_\_\ ,\ \_\_\_\_\_\ )$.

(5) Let $f(x) = x + 5$, $g(x) = \sqrt{x}$, and $h(x) = x^2$. Then $(g \circ (h - (g \circ f)))(4) =$ ________ .

(6) Let $f(x) = \dfrac{1}{1-\dfrac{2}{1+\dfrac{1}{1-x}}}$.

(a) Find $f(1/2)$. Answer. _____ .

(b) Find the domain of f. Answer. The domain of f is the set of all real numbers *except* _____ , _____ , and _____ .

(7) Let $f(x) = \dfrac{\sqrt{x^2-4}}{5-\sqrt{36-x^2}}$. Then, in interval notation, that part of the domain of f which is to the right of the origin is $[2, a) \cup (a, b]$ where $a =$ ______ and $b =$ _____ .

(8) Let $f(x) = (-x^2 - 7x - 10)^{-1/2}$.

(a) Then $f(-3) =$ ________ .

(b) The domain of f is (_____ , _____) .

(9) Let $f(x) = x^3 - 4$ for all real numbers x. Then for all $x \neq 0$ define a new function g by $g(x) = (2x)^{-1}(f(1+x) - f(1-x))$. Then $g(x)$ can be written in the form $ax^2 + bx + c$ where $a =$ _____ , $b =$ _____ , and $c =$ _____ .

(10) The cost of making a widget is 75 cents. If they are sold for \$1.95 each, 3000 widgets can be sold. For every cent the price is lowered, 60 more widgets can be sold.

(a) If x is the price of a widget in cents, then the net profit is $p(x) = ax^2 + bx + c$ where $a =$ ______ , $b =$ _____________ , and $c =$ ______________________ .

(b) The "best" price (that is, the price that maximizes profit) is $x =$ \$ ___ . ___ ___ .

(c) At this best price the profit is \$______________ .

(11) Let $f(x) = 3\sqrt{25-x^2} + 2$. Then $\operatorname{dom} f = [\ \_\_\_\_\_\ ,\ \_\_\_\_\_\ ]$ and $\operatorname{ran} f = [\ \_\_\_\_\_\ ,\ \_\_\_\_\_\ ]$.

(12) Find a formula exhibiting the area A of an equilateral triangle as a function of the length s of one of its sides.

Answer: $A(s) =$ _________ .

(13) Let $f(x) = 4x^3 - 18x^2 - 4x + 33$. Find the largest set S on which the function f is bounded above by 15 and below by -15.

Answer: $S = [\ \_\_\_\_\_\ ,\ \_\_\_\_\_\ ] \cup [\ \_\_\_\_\_\ ,\ \_\_\_\_\_\ ] \cup [\ \_\_\_\_\_\ ,\ \_\_\_\_\_\ ]$.

(14) Let $f(x) = \sqrt{x}$, $g(x) = \dfrac{4}{5-x}$, and $h(x) = x^2$. Find $(h \circ ((h \circ g \circ f) - f))(4)$.

Answer: ____________ .

(15) Let $f(x) = x + 7$, $g(x) = \sqrt{x+2}$, and $h(x) = x^2$. Find $(h \circ ((f \circ g) - (g \circ f)))(7)$.

Answer: ____________ .

(16) Let $f(x) = \sqrt{5-x}$, $g(x) = \sqrt{x+11}$, $h(x) = 2(x-1)^{-1}$, and $j(x) = 4x - 1$.

Then $(f \circ (g + (h \circ g)(h \circ j)))(5) =$ ____________ .

(17) Let $f(x) = x^2$, $g(x) = \sqrt{9+x}$, and $h(x) = \dfrac{1}{x-2}$. Then $(h \circ (f \circ g - g \circ f))(4) =$ ______ .

(18) Let $f(x) = x^2$, $g(x) = \sqrt{9+x}$, and $h(x) = (x-1)^{1/3}$.

Then $(h \circ ((f \circ g)(g \circ f)))(4) =$ _____ .

(19) Let $f(x) = \dfrac{5}{x}$, $g(x) = \sqrt{x}$, and $h(x) = x + 1$. Then $(g(f \circ g) + (g \circ f \circ h))(4) =$ _____ .

(20) Let $g(x) = 5 - x^2$, $h(x) = \sqrt{x+13}$, and $j(x) = \dfrac{1}{x}$. Then $(j \circ h \circ g)(3) =$ ______ .

(21) Let $h(x) = \dfrac{1}{\sqrt{x+6}}$, $j(x) = \dfrac{1}{x}$, and $g(x) = 5 - x^2$. Then $(g \circ j \circ h)(3) =$ ______ .

(22) Let $f(x) = x^2 + \dfrac{2}{x}$, $g(x) = \dfrac{2}{2x+3}$, and $h(x) = \sqrt{2x}$. Then $(h \circ g \circ f)(4) =$ ______ .

(23) Let $f(x) = 3(x+1)^3$, $g(x) = \dfrac{x^5 + x^4}{x+1}$, and $h(x) = \sqrt{x}$.

Then $(h \circ (g + (h \circ f)))(2) =$ ______ .

(24) Let $f(x) = x^2$, $g(x) = \sqrt{x+11}$, $h(x) = 2(x-1)^{-1}$, and $j(x) = 4x - 1$.

Then $(f \circ ((h \circ g) + (h \circ j)))(5) =$ ________ .

(25) Let $f(x) = x^3 - 5x^2 + x - 7$. Find a function g such that $(f \circ g)(x) = 27x^3 + 90x^2 + 78x - 2$.

Answer: $g(x) =$ ________________ .

(26) Let $f(x) = \cos x$ and $g(x) = x^2$ for all x. Write each of the following functions in terms of f and g. *Example.* If $h(x) = \cos^2 x^2$, then $h = g \circ f \circ g$.

(a) If $h(x) = \cos x^2$, then $h =$ __________ .

(b) If $h(x) = \cos x^4$, then h= ________________ .

(c) If $h(x) = \cos^4 x^2$, then $h =$ ______________________ .

(d) If $h(x) = \cos(\cos^2 x)$, then $h =$ ________________ .

(e) If $h(x) = \cos^2(x^4 + x^2)$, then $h =$ __________________________________ .

(27) Let $f(x) = x^3$, $g(x) = x - 2$, and $h(x) = \sin x$ for all x. Write each of the following functions in terms of f, g, and h. *Example.* If $k(x) = \sin^3(x-2)^3$, then $k = f \circ h \circ f \circ g$.

(a) If $k(x) = \sin^3 x$, then $k =$ ____________ .

(b) If $k(x) = \sin x^3$, then $k =$ ____________ .

(c) If $k(x) = \sin(x^3 - 2)$, then $k =$ ________________ .

(d) If $k(x) = \sin(\sin x - 2)$, then $k =$ ________________ .

(e) If $k(x) = \sin^3(\sin^3(x-2)$, then $k =$ _____________________________ .

(f) If $k(x) = \sin^9(x-2)$, then $k =$ ______________________ .

(g) If $k(x) = \sin(x^3 - 8)$, then $k =$ _________________________________ .

(h) If $k(x) = \sin(x^3 - 6x^2 + 12x - 8)$, then $k =$ ________________ .

(28) Let $g(x) = 3x - 2$. Find a function f such that $(f \circ g)(x) = 18x^2 - 36x + 19$.
Answer: $f(x) =$ ____________________ .

(29) Let $h(x) = \arctan x$ for $x \geq 0$, $g(x) = \cos x$, and $f(x) = (1 - x^2)^{-1}$. Find a number p such that $(f \circ g \circ h)(x) = 1 + x^p$. Answer: $p =$ ________ .

(30) Let $f(x) = 3x^2 + 5x + 1$. Find a function g such that $(f \circ g)(x) = 3x^4 + 6x^3 - 4x^2 - 7x + 3$.
Answer: $g(x) =$ ____________________ .

(31) Let $g(x) = 2x - 1$. Find a function f such that $(f \circ g)(x) = 8x^3 - 28x^2 + 28x - 14$.
Answer: $f(x) =$ __________________________ .

(32) Find two solutions to the equation

$$8\cos^3(\pi(x^2 + \tfrac{8}{3}x + 2)) + 16\cos^2(\pi(x^2 + \tfrac{8}{3}x + 2)) + 16\cos(\pi(x^2 + \tfrac{8}{3}x + 2)) = 13.$$

Answer: ________ and ________ .

(33) Let $f(x) = (x + 4)^{-1/2}$, $g(x) = x^2 + 1$, $h(x) = (x - 3)^{1/2}$, and $j(x) = x^{-1}$.
Then $(j \circ ((g \circ h) - (g \circ f)))(5) =$ ________ .

(34) Let $g(x) = 3x - 2$. Find a function f such that $(f \circ g)(x) = 18x^2 - 36x + 19$.
Answer: $f(x) =$ ____________________ .

(35) Let $f(x) = x^3 - 5x^2 + x - 7$. Find a function g such that $(f \circ g)(x) = 27x^3 + 90x^2 + 78x - 2$.
Answer: $g(x) =$ ________________ .

(36) Let $f(x) = x^2 + 1$. Find a function g such that $(f \circ g)(x) = 2 + \dfrac{2}{x} + \dfrac{1}{x^2}$.
Answer: $g(x) =$ _________________ .

(37) Let $f(x) = x^2 + 3x + 4$. Find two functions g such that $(f \circ g)(x) = 4x^2 - 6x + 4$.
Answer: $g(x) =$ _________ and $g(x) =$ _________ .

(38) Let $h(x) = x^{-1}$ and $g(x) = \sqrt{x} + 1$. Find a function f such that $(f \circ g \circ h)(x) = x^{-3/2} + 4x^{-1} + 2x^{-1/2} - 6$.
Answer: $f(x) =$ _____________________ .

(39) Let $g(x) = x^2 + x - 1$. Find a function f such that $(f \circ g)(x) = x^4 + 2x^3 - 3x^2 - 4x + 6$.
Answer: $f(x) =$ ___________________ .

(40) Let $S(x) = x^2$ and $P(x) = 2^x$.
Then $(S \circ S \circ S \circ S \circ P \circ P)(-1) =$ ___________ .

3.3. Problems

(1) Do there exist functions f and g defined on $\mathbb{R}$ such that

$$f(x) + g(y) = xy$$

for all real numbers x and y? Explain.

(2) Your friend Susan has become interested in functions $f\colon \mathbb{R} \to \mathbb{R}$ which preserve both the operation of addition and the operation of multiplication; that is, functions f which satisfy

$$f(x + y) = f(x) + f(y) \tag{3.1}$$

and

$$f(xy) = f(x)f(y) \tag{3.2}$$

for all $x, y \in \mathbb{R}$. Naturally she started her investigation by looking at some examples. The trouble is that she was able to find only two very simple examples: $f(x) = 0$ for all x and $f(x) = x$ for all x. After expending considerable effort she was unable to find additional examples. She now conjectures that there are no other functions satisfying (3.1) and (3.2). Write Susan a letter explaining why she is correct.

Hint. You may choose to pursue the following line of argument. Assume that f is a function (not identically zero) which satisfies (3.1) and (3.2) above.

(a) Show that $f(0) = 0$. [In (3.1) let $y = 0$.]
(b) Show that if $a \neq 0$ and $a = ab$, then $b = 1$.
(c) Show that $f(1) = 1$. [How do we know that there exists a number c such that $f(c) \neq 0$? Let $x = c$ and $y = 1$ in (3.2).]
(d) Show that $f(n) = n$ for every natural number n.
(e) Show that $f(-n) = -n$ for every natural number n. [Let $x = n$ and $y = -n$ in (3.1). Use (d).]
(f) Show that $f(1/n) = 1/n$ for every natural number n. [Let $x = n$ and $y = 1/n$ in (3.2).]
(g) Show that $f(r) = r$ for every rational number r. [If $r \geq 0$ write $r = m/n$ where m and n are natural numbers; then use (3.2), (d), and (e). Next consider the case $r < 0$.]
(h) Show that if $x \geq 0$, then $f(x) \geq 0$. [Write x as $\sqrt{x}\sqrt{x}$ and use (3.2).]
(i) Show that if $x \leq y$, then $f(x) \leq f(y)$. [Show that $f(-x) = -f(x)$ holds for all real numbers x. Use (h).]
(j) Now prove that f must be the identity function on $\mathbb{R}$. [Argue by contradiction: Assume $f(x) \neq x$ for some number x. Then there are two possibilities: either $f(x) > x$ or $f(x) < x$. Show that both of these lead to a contradiction. Apply theorem 3.1.4 to the two cases $f(x) > x$ and $f(x) < x$ to obtain the contradiction $f(x) < f(x)$.]

(3) Let $f(x) = 1 - x$ and $g(x) = 1/x$. Taking composites of these two functions in all possible ways ($f \circ f$, $g \circ f$, $f \circ g \circ f \circ f \circ f$, $g \circ g \circ f \circ g \circ f \circ f$, etc.), how many distinct functions can be produced? Write each of the resulting functions in terms of f and g. How do you know there are no more? Show that each function on your list has an inverse which is also on your list. What is the common domain for these functions? That is, what is the largest set of real numbers for which all these functions are defined?

(4) Prove or disprove: composition of functions is commutative; that is $g \circ f = f \circ g$ when both sides are defined.

(5) Let f, g, $h\colon \mathbb{R} \to \mathbb{R}$. Prove or disprove: $f \circ (g + h) = f \circ g + f \circ h$.

(6) Let f, g, $h\colon \mathbb{R} \to \mathbb{R}$. Prove or disprove: $(f + g) \circ h = (f \circ h) + (g \circ h)$.

(7) Let $a \in \mathbb{R}$ be a constant and let $f(x) = a - x$ for all $x \in \mathbb{R}$. Show that $f \circ f = I$ (where I is the identity function on $\mathbb{R}$: $I(x) = x$ for all x).

3.4. Answers to Odd-Numbered Exercises

(1) (a) $\dfrac{3}{4}$

(b) $-1,\ -\dfrac{1}{2},\ 0$

(3) $[-2,2],\ [-3,1]$

(5) $\sqrt{13}$

(7) $\sqrt{11},\ 6$

(9) $1,\ 0,\ 3$

(11) $[-5,5],\ [2,17]$

(13) $[-\frac{3}{2},-1]\cup[1,2]\cup[4,\frac{9}{2}]$

(15) 36

(17) $\dfrac{1}{6}$

(19) 6

(21) -4

(23) 5

(25) $3x+5$

(27) (a) $f\circ h$
(b) $h\circ f$
(c) $h\circ g\circ f$
(d) $h\circ g\circ h$
(e) $f\circ h\circ f\circ h\circ g$
(f) $f\circ f\circ h\circ g$
(g) $h\circ g\circ g\circ g\circ g\circ f$
(h) $h\circ f\circ g$

(29) -2

(31) x^3-4x^2+3x-6

(33) $\dfrac{9}{17}$

(35) $3x+5$

(37) $-2x,\ 2x-3$

(39) x^2-2x+3

Part 2

LIMITS AND CONTINUITY

CHAPTER 4

LIMITS

4.1. Background

Topics: limit of $f(x)$ as x approaches a, limit of $f(x)$ as x approaches infinity, left- and right-hand limits.

4.1.1. Definition. Suppose that f is a real valued function of a real variable, a is an accumulation point of the domain of f, and $\ell \in \mathbb{R}$. We say that ℓ is the *limit of* $f(x)$ *as* x *approaches* a if for every neighborhood V of ℓ there exists a corresponding deleted neighborhood U of a which satisfies the following condition:

for every point x in the domain of f which lies in U the point $f(x)$ lies in V.

Once we have convinced ourselves that in this definition it doesn't matter if we work only with symmetric neighborhoods of points, we can rephrase the definition in a more conventional algebraic fashion: ℓ is the *limit of* $f(x)$ *as* x *approaches* a provided that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ and $x \in \operatorname{dom} f$, then $|f(x) - \ell| < \epsilon$.

4.1.2. Notation. To indicate that a number ℓ is the limit of $f(x)$ as x approaches a, we may write either

$$\lim_{x\to a} f(x) = l \qquad \text{or} \qquad f(x) \to \ell \text{ as } x \to a.$$

(See problem 2.)

4.2. Exercises

(1) $\lim_{x\to 3} \dfrac{x^3 - 13x^2 + 51x - 63}{x^3 - 4x^2 - 3x + 18} = \dfrac{a}{5}$ where $a =$ ______ .

(2) $\lim_{x\to 0} \dfrac{\sqrt{x^2 + 9x + 9} - 3}{x} = \dfrac{a}{2}$ where $a =$ _____ .

(3) $\lim_{x\to 1} \dfrac{x^3 - x^2 + 2x - 2}{x^3 + 3x^2 - 4x} = \dfrac{3}{a}$ where $a =$ _____ .

(4) $\lim_{t\to 0} \dfrac{t}{\sqrt{4-t} - 2} =$ ______ .

(5) $\lim_{x\to 0} \dfrac{\sqrt{x+9} - 3}{x} = \dfrac{1}{a}$ where $a =$ _____ .

(6) $\lim_{x\to 2} \dfrac{x^3 - 3x^2 + x + 2}{x^3 - x - 6} = \dfrac{1}{a}$ where $a =$ ______ .

(7) $\lim_{x\to 2} \dfrac{x^3 - x^2 - 8x + 12}{x^3 - 10x^2 + 28x - 24} = -\dfrac{a}{4}$ where $a =$ _____ .

(8) $\lim_{x\to 0} \dfrac{\sqrt{x^2 - x + 4} - 2}{x^2 + 3x} = -\dfrac{1}{a}$ where $a =$ ________ .

(9) $\lim_{x\to 1} \dfrac{x^3 + x^2 - 5x + 3}{x^3 - 4x^2 + 5x - 2} =$ ______ .

(10) $\lim_{x\to 3} \dfrac{x^3 - 4x^2 - 3x + 18}{x^3 - 8x^2 + 21x - 18} =$ _____ .

(11) $\lim_{x\to -1} \dfrac{x^3 - x^2 - 5x - 3}{x^3 + 6x^2 + 9x + 4} = -\dfrac{4}{a}$ where $a =$ _____ .

(12) $\lim_{x\to 0} \dfrac{2x \sin x}{1 - \cos x} =$ _____ .

(13) $\lim_{x\to 0} \dfrac{1 - \cos x}{3x \sin x} = \dfrac{1}{a}$ where $a =$ _____ .

(14) $\lim_{x\to 0} \dfrac{\tan 3x - \sin 3x}{x^3} = \dfrac{a}{2}$ where $a =$ ______ .

(15) $\lim_{h\to 0} \dfrac{\sin 2h}{5h^2 + 7h} =$ __________ .

(16) $\lim_{h\to 0} \dfrac{\cot 7h}{\cot 5h} =$ __________ .

(17) $\lim_{x\to 0} \dfrac{\sec x - \cos x}{3x^2} = \dfrac{1}{a}$ where $a =$ _____ .

(18) $\lim_{x\to\infty} \dfrac{(9x^8 - 6x^5 + 4)^{1/2}}{(64x^{12} + 14x^7 - 7)^{1/3}} = \dfrac{a}{4}$ where $a =$ _____ .

(19) $\lim_{x\to\infty} \sqrt{x}(\sqrt{x+3} - \sqrt{x-2}) = \dfrac{a}{2}$ where $a =$ _____ .

(20) $\lim_{x\to\infty} \dfrac{7 - x + 2x^2 - 3x^3 - 5x^4}{4 + 3x - x^2 + x^3 + 2x^4} = \dfrac{a}{2}$ where $a =$ ______ .

(21) $\lim_{x\to\infty} \dfrac{(2x^4 - 137)^5}{(x^2 + 429)^{10}} =$ ________ .

(22) $\lim_{x\to\infty} \dfrac{(5x^{10}+32)^3}{(1-2x^6)^5} = -\dfrac{a}{32}$ where $a =$ __________ .

(23) $\lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right) = \dfrac{1}{a}$ where $a =$ _____ .

(24) $\lim_{x\to\infty} x(256x^4 + 81x^2 + 49)^{-1/4} = \dfrac{1}{a}$ where $a =$ _____ .

(25) $\lim_{x\to\infty} x\left(\sqrt{3x^2+22} - \sqrt{3x^2+4}\right) = a\sqrt{a}$ where $a =$ _____ .

(26) $\lim_{x\to\infty} x^{\frac{2}{3}}\left((x+1)^{\frac{1}{3}} - x^{\frac{1}{3}}\right) = \dfrac{1}{a}$ where $a =$ _____ .

(27) $\lim_{x\to\infty} \left(\sqrt{x+\sqrt{x}} - \sqrt{x-\sqrt{x}}\right) =$ _____ .

(28) Let $f(x) = \begin{cases} 2x-1, & \text{if } x < 2; \\ x^2+1, & \text{if } x > 2. \end{cases}$ Then $\lim_{x\to 2^-} f(x) =$ _____ and $\lim_{x\to 2^+} f(x) =$ _____ .

(29) Let $f(x) = \dfrac{|x-1|}{x-1}$. Then $\lim_{x\to 1^-} f(x) =$ ______ and $\lim_{x\to 1^+} f(x) =$ _____ .

(30) Let $f(x) = \begin{cases} 5x-3, & \text{if } x < 1; \\ x^2, & \text{if } x \geq 1. \end{cases}$ Then $\lim_{x\to 1^-} f(x) =$ _____ and $\lim_{x\to 1^+} f(x) =$ _____ .

(31) Let $f(x) = \begin{cases} 3x+2, & \text{if } x < -2; \\ x^2+3x-1, & \text{if } x \geq -2. \end{cases}$ Then $\lim_{x\to -2^-} f(x) =$ ______ and $\lim_{x\to -2^+} f(x) =$ ______ .

(32) Suppose $y = f(x)$ is the equation of a curve which always lies between the parabola $x^2 = y - 1$ and the hyperbola $yx + y - 1 = 0$. Then $\lim_{x\to 0} f(x) =$ _____ .

4.3. Problems

(1) Find $\lim_{x\to 0^+} \left(e^{-1/x}\sin(1/x) - (x+2)^3 \right)$ (if it exists) and give a careful argument showing that your answer is correct.

(2) The notation $\lim_{x\to a} f(x) = \ell$ that we use for limits is somewhat optimistic. It assumes the uniqueness of limits. Prove that limits, if they exist, are indeed unique. That is, suppose that f is a real valued function of a real variable, a is an accumulation point of the domain of f, and ℓ, $m \in \mathbb{R}$. Prove that if $f(x) \to \ell$ as $x \to a$ and $f(x) \to m$ as $x \to a$, then $l = m$. (Explain carefully why it was important that we require a to be an accumulation point of the domain of f.)

(3) Let $f(x) = \dfrac{\sin \pi x}{x+1}$ for all $x \neq -1$. The following information is known about a function g defined for all real numbers $x \neq 1$:
   (i) $g = \dfrac{p}{q}$ where $p(x) = ax^2 + bx + c$ and $q(x) = dx + e$ for some constants a, b, c, d, e;
   (ii) the only x-intercept of the curve $y = g(x)$ occurs at the origin;
   (iii) $g(x) \geq 0$ on the interval $[0, 1)$ and is negative elsewhere on its domain;
   (iv) g has a vertical asymptote at $x = 1$; and
   (v) $g(1/2) = 3$.

   Either find $\lim_{x\to 1} g(x)f(x)$ or else show that this limit does not exist.

   Hints. Write an explicit formula for g by determining the constants $a \dots e$. Use (ii) to find c; use (ii) and (iii) to find a; use (iv) to find a relationship between d and e; then use (v) to obtain an explicit form for g. Finally look at $f(x)g(x)$; replace $\sin \pi x$ by $\sin(\pi(x-1)+\pi)$ and use the formula for the sine of the sum of two numbers.

(4) Evaluate $\lim_{x\to 0} \dfrac{\sqrt{|x|}\cos\left(\pi^{1/x^2}\right)}{2+\sqrt{x^2+3}}$ (if it exists). Give a careful proof that your conclusion is correct.

4.4. Answers to Odd-Numbered Exercises

(1) -4

(3) 5

(5) 6

(7) 5

(9) -4

(11) 3

(13) 6

(15) $\dfrac{2}{7}$

(17) 3

(19) 5

(21) 32

(23) 2

(25) 3

(27) 1

(29) -1, 1

(31) -4, -3

CHAPTER 5

CONTINUITY

5.1. Background

Topics: continuous functions, *intermediate value theorem. extreme value theorem.*

There are many ways of stating the *intermediate value theorem.* The simplest says that *continuous functions take intervals to intervals.*

5.1.1. Definition. A subset J of the real line $\mathbb{R}$ is an INTERVAL if $z \in J$ whenever a, $b \in J$ and $a < z < b$.

5.1.2. Theorem (Intermediate Value Theorem)**.** *Let J be an interval in $\mathbb{R}$ and $f\colon J \to \mathbb{R}$ be continuous. Then the range of f is an interval.*

5.1.3. Definition. A real-valued function f on a set A is said to have a MAXIMUM at a point a in A if $f(a) \geq f(x)$ for every x in A; the number $f(a)$ is the MAXIMUM VALUE of f. The function has a MINIMUM at a if $f(a) \leq f(x)$ for every x in A; and in this case $f(a)$ is the MINIMUM VALUE of f. A number is an EXTREME VALUE of f if it is either a maximum or a minimum value. It is clear that a function may fail to have maximum or minimum values. For example, on the open interval $(0, 1)$ the function $f : x \mapsto x$ assumes neither a maximum nor a minimum.

The concepts we have just defined are frequently called GLOBAL (or ABSOLUTE) MAXIMUM and GLOBAL (or ABSOLUTE) MINIMUM.

5.1.4. Definition. Let $f\colon A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$. The function f has a LOCAL (or RELATIVE) MAXIMUM at a point $a \in A$ if there exists a neighborhood J of a such that $f(a) \geq f(x)$ whenever $x \in J$ and $x \in \operatorname{dom} f$. It has a LOCAL (or RELATIVE) MINIMUM at a point $a \in A$ if there exists a neighborhood J of a such that $f(a) \leq f(x)$ whenever $x \in J$ and $x \in \operatorname{dom} f$.

5.1.5. Theorem (Extreme Value Theorem)**.** *Every continuous real valued function on a closed and bounded interval in $\mathbb{R}$ achieves its (global) maximum and minimum value at some points in the interval.*

5.1.6. Definition. A number p is a FIXED POINT of a function $f\colon \mathbb{R} \to \mathbb{R}$ if $f(p) = p$.

5.1.7. Example. If $f(x) = x^2 - 6$ for all $x \in \mathbb{R}$, then 3 is a fixed point of f.

5.2. Exercises

(1) Let $f(x) = \dfrac{x^3 - 2x^2 - 2x - 3}{x^3 - 4x^2 + 4x - 3}$ for $x \neq 3$. How should f be defined at $x = 3$ so that it becomes a continuous function on all of $\mathbb{R}$?

Answer: $f(3) = \dfrac{a}{7}$ where $a =$ ________ .

(2) Let $f(x) = \begin{cases} 1 & \text{if } x < 0 \\ x & \text{if } 0 < x < 1 \\ 2 - x & \text{if } 1 < x < 3 \\ x - 4 & \text{if } x > 3 \end{cases}$.

(a) Is it possible to define f at $x = 0$ in such a way that f becomes continuous at $x = 0$? Answer: ______ . If so, then we should set $f(0) =$ ______ .

(b) Is it possible to define f at $x = 1$ in such a way that f becomes continuous at $x = 1$? Answer: ______ . If so, then we should set $f(1) =$ ______ .

(c) Is it possible to define f at $x = 3$ in such a way that f becomes continuous at $x = 3$? Answer: ______ . If so, then we should set $f(3) =$ ______ .

(3) Let $f(x) = \begin{cases} x + 4 & \text{if } x < -2 \\ -x & \text{if } -2 < x < 1 \\ x^2 - 2x + 1 & \text{if } 1 < x < 3 \\ 10 - 2x & \text{if } x > 3 \end{cases}$.

(a) Is it possible to define f at $x = -2$ in such a way that f becomes continuous at $x = -2$? Answer: ______ . If so, then we should set $f(-2) =$ ______ .

(b) Is it possible to define f at $x = 1$ in such a way that f becomes continuous at $x = 1$? Answer: ______ . If so, then we should set $f(1) =$ ______ .

(c) Is it possible to define f at $x = 3$ in such a way that f becomes continuous at $x = 3$? Answer: ______ . If so, then we should set $f(3) =$ ______ .

(4) The equation $x^5 + x^3 + 2x = 2x^4 + 3x^2 + 4$ has a solution in the open interval $(n, n+1)$ where n is the positive integer ______ .

(5) The equation $x^4 - 6x^2 - 53 = 22x - 2x^3$ has a solution in the open interval $(n, n+1)$ where n is the positive integer ______ .

(6) The equation $x^4 + x + 1 = 3x^3 + x^2$ has solutions in the open intervals $(m, m+1)$ and $(n, n+1)$ where m and n are the distinct positive integers ______ and ______ .

(7) The equation $x^5 + 8x = 2x^4 + 6x^2$ has solutions in the open intervals $(m, m+1)$ and $(n, n+1)$ where m and n are the distinct positive integers ______ and ______ .

5.3. Problems

(1) Prove that the equation

$$x^{180} + \frac{84}{1 + x^2 + \cos^2 x} = 119$$

has at least two solutions.

(2) (a) Find all the fixed points of the function f defined in example 5.1.7.

Theorem: Every continuous function $f\colon [0,1] \to [0,1]$ has a fixed point.

(b) Prove the preceding theorem. *Hint.* Let $g(x) = x - f(x)$ for $0 \leq x \leq 1$. Apply the *intermediate value theorem* 5.1.2 to g.

(c) Let $g(x) = 0.1\,x^3 + 0.2$ for all $x \in \mathbb{R}$, and h be the restriction of g to $[0,1]$. Show that h satisfies the hypotheses of the theorem.

(d) For the function h defined in (c) find an approximate value for at least one fixed point with an error of less than 10^{-6}. Give a careful justification of your answer.

(e) Let g be as in (c). Are there other fixed points (that is, points not in the unit square where the curve $y = g(x)$ crosses the line $y = x$)? If so, find an approximation to each such point with an error of less than 10^{-6}. Again provide careful justification.

(3) Define f on $[0,4]$ by $f(x) = x + 1$ for $0 \leq x < 2$ and $f(x) = 1$ for $2 \leq x \leq 4$. Use the *extreme value theorem* 5.1.5 to show that f is not continuous.

(4) Give an example of a function defined on $[0,1]$ which has no maximum and no minimum on the interval. Explain why the existence of such a function does not contradict the *extreme value theorem* 5.1.5.

(5) Give an example of a continuous function defined on the interval $(1,2]$ which does not achieve a maximum value on the interval. Explain why the existence of such a function does not contradict the *extreme value theorem* 5.1.5.

(6) Give an example of a continuous function on the closed interval $[3,\infty)$ which does not achieve a minimum value on the interval. Explain why the existence of such a function does not contradict the *extreme value theorem* 5.1.5.

(7) Define f on $[-2,0]$ by $f(x) = \dfrac{-1}{(x+1)^2}$ for $-2 \leq x < -1$ and $-1 < x \leq 0$, and $f(-1) = -3$. Use the *extreme value theorem* 5.1.5 to show that f is not continuous.

(8) Let $f(x) = \dfrac{1}{x}$ for $0 < x \leq 1$ and $f(0) = 0$. Use the *extreme value theorem* 5.1.5 to show that f is not continuous on $[0,1]$.

5.4. Answers to Odd-Numbered Exercises

(1) 13

(3) (a) yes, 2
(b) no, —
(c) yes, 4

(5) 3

(7) 1, 2

Part 3

DIFFERENTIATION OF FUNCTIONS OF A SINGLE VARIABLE

CHAPTER 6

DEFINITION OF THE DERIVATIVE

6.1. Background

Topics: definition of the derivative of a real valued function of a real variable at a point

6.1.1. Notation. Let f be a real valued function of a real variable which is differentiable at a point a in its domain. When thinking of a function in terms of its graph, we often write $y = f(x)$, call x the *independent variable*, and call y the *dependent variable*. There are many notations for the derivative of f at a. Among the most common are

$$Df(a), \quad f'(a), \quad \left.\frac{df}{dx}\right|_a, \quad y'(a), \quad \dot{y}(a), \quad \text{and} \quad \left.\frac{dy}{dx}\right|_a .$$

6.2. Exercises

(1) Suppose you know that the derivative of $\sqrt{x}$ is $\dfrac{1}{2\sqrt{x}}$ for every $x > 0$. Then

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \frac{1}{a} \text{ where } a = \underline{\qquad}.$$

(2) Suppose you know that the derivatives of $x^{\frac{1}{3}}$ is $\frac{1}{3}x^{-\frac{2}{3}}$ for every $x \neq 0$. Then

$$\lim_{x\to 8} \frac{\left(\frac{x}{8}\right)^{\frac{1}{3}}-1}{x-8} = \frac{1}{a} \text{ where } a = \underline{\qquad}.$$

(3) Suppose you know that the derivative of e^x is e^x for every x. Then

$$\lim_{x\to 2} \frac{e^x-e^2}{x-2} = \underline{\qquad}.$$

(4) Suppose you know that the derivative of $\ln x$ is $\dfrac{1}{x}$ for every $x > 0$. Then

$$\lim_{x\to e} \frac{\ln x^3 - 3}{x-e} = \underline{\qquad}.$$

(5) Suppose you know that the derivative of $\tan x$ is $\sec^2 x$ for every x. Then

$$\lim_{x\to \frac{\pi}{4}} \frac{\tan x - 1}{4x-\pi} = \underline{\qquad}.$$

(6) Suppose you know that the derivative of $\arctan x$ is $\dfrac{1}{1+x^2}$ for every x. Then

$$\lim_{x\to \sqrt{3}} \frac{3\arctan x - \pi}{x-\sqrt{3}} = \underline{\qquad}.$$

(7) Suppose you know that the derivative of $\cos x$ is $-\sin x$ for every x. Then

$$\lim_{x\to \frac{\pi}{3}} \frac{2\cos x - 1}{3x-\pi} = -\frac{1}{a} \text{ where } a = \underline{\qquad}.$$

(8) Suppose you know that the derivative of $\cos x$ is $-\sin x$ for every x. Then

$$\lim_{t\to 0} \frac{\cos(\frac{\pi}{6}+t) - \frac{\sqrt{3}}{2}}{t} = -\frac{1}{a} \text{ where } a = \underline{\qquad}.$$

(9) Suppose you know that the derivative of $\sin x$ is $\cos x$ for every x. Then

$$\lim_{x\to -\pi/4} \frac{\sqrt{2}\sin x + 1}{4x+\pi} = \frac{1}{a} \text{ where } a = \underline{\qquad}.$$

(10) Suppose you know that the derivative of $\sin x$ is $\cos x$ for every x. Then

$$\lim_{x\to \frac{7\pi}{12}} \frac{2\sqrt{2}\sin x - \sqrt{3} - 1}{12x-7\pi} = \frac{1-\sqrt{a}}{b} \text{ where } a = \underline{\qquad} \text{ and } b = \underline{\qquad}.$$

(11) Let $f(x) = \begin{cases} x^2, & \text{for } x \leq 1 \\ 1, & \text{for } 1 < x \leq 3 \\ 5-2x, & \text{for } x > 3 \end{cases}$. Then $f'(0) = \underline{\qquad}$, $f'(2) = \underline{\qquad}$, and $f'(6) = \underline{\qquad}$.

(12) Suppose that the tangent line to the graph of a function f at $x = 1$ passes through the point $(4, 9)$ and that $f(1) = 3$. Then $f'(1) =$ ______ .

(13) Suppose that g is a differentiable function and that $f(x) = g(x) + 5$ for all x. If $g'(1) = 3$, then $f'(1) =$ _____ .

(14) Suppose that g is a differentiable function and that $f(x) = g(x + 5)$ for all x. If $g'(1) = 3$, then $f'(a) = 3$ where $a =$ _____ .

(15) Suppose that f is a differentiable function, that $f'(x) = -2$ for all x, and that $f(-3) = 11$. Find an algebraic expression for $f(x)$. Answer: $f(x)$ $=$ ______________ .

(16) Suppose that f is a differentiable function, that $f'(x) = 3$ for all x, and that $f(3) = 3$. Find an algebraic expression for $f(x)$. Answer: $f(x)$ $=$ ________________ .

6.3. Problems

(1) Let $f(x) = \dfrac{1}{x^2 - 1}$ and $a = -3$. Show how to use the definition of *derivative* to find $Df(a)$.

(2) Let $f(x) = \dfrac{1}{\sqrt{x+7}}$. Show how to use the definition of *derivative* to find $f'(2)$.

(3) Let $f(x) = \dfrac{1}{\sqrt{x+3}}$. Show how to use the definition of *derivative* to find $f'(1)$.

(4) Let $f(x) = \sqrt{x^2 - 5}$. Show how to use the definition of *derivative* to find$f'(3)$.

(5) Let $f(x) = \sqrt{8 - x}$. Show how to use the definition of *derivative* to find $f'(-1)$.

(6) Let $f(x) = \sqrt{x - 2}$. Show how to use the definition of *derivative* to find $f'(6)$.

(7) Let $f(x) = \dfrac{x}{x^2 + 2}$. Show how to use the definition of *derivative* to find $Df(2)$.

(8) Let $f(x) = (2x^2 - 3)^{-1}$. Show how to use the definition of *derivative* to find $Df(-2)$.

(9) Let $f(x) = x + 2x^2 \sin \dfrac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. What is the derivative of f at 0 (if it exists)? Is the function f' continuous at 0?

6.4. Answers to Odd-Numbered Exercises

(1) 6

(3) e^2

(5) $\dfrac{1}{2}$

(7) $\sqrt{3}$

(9) 4

(11) 0, 0, -2

(13) 3

(15) $-2x+5$

CHAPTER 7

TECHNIQUES OF DIFFERENTIATION

7.1. Background

Topics: *rule for differentiating products*, *rule for differentiating quotients*, *chain rule*, tangent lines, implicit differentiation.

7.1.1. Notation. We use $f^{(n)}(a)$ to denote the n^{th} derivative of f at a.

7.1.2. Definition. A point a in the domain of a function f is a STATIONARY POINT of f is $f\,'(a) = 0$. It is a CRITICAL POINT of f if it is either a stationary point of f or if it is a point where the derivative of f does not exist.

Some authors use the terms *stationary point* and *critical point* interchangeably—especially in higher dimensions.

7.2. Exercises

(1) If $f(x) = 5x^{-1/2} + 6x^{3/2}$, then $f'(x) = ax^p + bx^q$ where $a =$ ________ , $p =$ ________ , $b =$ ______ , and $q =$ ______ .

(2) If $f(x) = 10\sqrt[5]{x^3} + \dfrac{12}{\sqrt[6]{x^5}}$, then $f'(x) = ax^p + bx^q$ where $a =$ _____ , $p =$ ________ , $b =$ ________ , and $q =$ __________ .

(3) If $f(x) = 9x^{4/3} + 25x^{2/5}$, then $f''(x) = ax^p + bx^q$ where $a =$ _____ , $p =$ __________ , $b =$ ______ , and $q =$ ___________ .

(4) If $f(x) = 18\sqrt[6]{x} + \dfrac{8}{\sqrt[4]{x^3}}$, then $f''(x) = ax^p + bx^q$ where $a =$ __________ , $p =$ ___________ , $b =$ ___________ , and $q =$ _____________ .

(5) Find a point a such that the tangent line to the graph of the curve $y = \sqrt{x}$ at $x = a$ has y-intercept 3. Answer: $a =$ ______ .

(6) Let $f(x) = ax^2 + bx + c$ for all x. We know that $f(2) = 26$, $f'(2) = 23$, and $f''(2) = 14$. Then $f(1) =$ ______ .

(7) Find a number k such that the line $y = 6x + 4$ is tangent to the parabola $y = x^2 + k$. Answer: $k =$ ________ .

(8) The equation for the tangent line to the curve $y = x^3$ which passes through the point $(0, 2)$ is $y = mx + b$ where $m =$ _____ and $b =$ _____ .

(9) Let $f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - 3x^2 + \frac{7}{4}$. Find all points x_0 such that the tangent line to the curve $y = f(x)$ at the point $(x_0, f(x_0))$ is horizontal. Answer: $x_0 =$ ______ , _____ , and _____ .

(10) In the land of Oz there is an enormous statue of the Good Witch Glinda. Its base is 20 feet high and, on a surveyor's chart, covers the region determined by the inequalities

$$-1 \leq y \leq 24 - x^2 .$$

(The chart coordinates are measured in feet.) Dorothy is looking for her little dog Toto. She walks along the curved side of the base of the statue in the direction of increasing x and Toto is, for a change, sitting quietly. He is at the point on the positive x-axis 7 feet from the origin. How far from Toto is Dorothy when she is first able to see him? Answer: $5\sqrt{a}$ ft. where $a =$ ______ .

(11) Let $f(x) = \dfrac{x - \frac{3}{2}}{x^2 + 2}$ and $g(x) = \dfrac{x^2 + 1}{x^2 + 2}$. At what values of x do the curves $y = f(x)$ and $y = g(x)$ have parallel tangent lines? Answer: at $x =$ ______ and $x =$ _____ .

(12) The tangent line to the graph of a function f at the point $x = 2$ has x-intercept $\dfrac{10}{3}$ and y-intercept -10. Then $f(2) =$ ______ and $f'(2) =$ _____ .

(13) The tangent line to the graph of a function f at $x = 2$ passes through the points $(0, -20)$ and $(5, 40)$. Then $f(2) =$ _____ and $f'(2) =$ ______ .

(14) Suppose that the tangent line to the graph of a function f at $x = 2$ passes through the point $(5, 19)$ and that $f(2) = -2$. Then $f'(2) =$ _____ .

(15) Let $f(x) = \begin{cases} x^2, & \text{for } x \leq 1 \\ 1, & \text{for } 1 < x \leq 3 \\ 5 - 2x, & \text{for } x > 3 \end{cases}$. Then $f'(0) =$ _____ , $f'(2) =$ _____ , and $f'(6) =$ ______ .

(16) Suppose that g is a differentiable function and that $f(x) = g(x + 5)$ for all x. If $g'(1) = 3$, then $f'(a) = 3$ where $a =$ ______ .

(17) Let $f(x) = |2 - |x-1|| - 1$ for every real number x. Then

$f'(-2) =$ ______ , $f'(0) =$ _____ , $f'(2) =$ ______ , and $f'(4) =$ _____ .

(18) Let $f(x) = \tan^3 x$. Then $Df(\pi/3) =$ ________ .

(19) Let $f(x) = \dfrac{1}{x}\csc^2\dfrac{1}{x}$. Then $Df(6/\pi) = \dfrac{\pi^2}{a}\left(\dfrac{\pi}{\sqrt{b}} - 1\right)$ where $a =$ _____ and $b =$ _____ .

(20) Let $f(x) = \sin^2(3x^5+7)$. Then $f'(x) = ax^4\sin(3x^5+7)f(x)$ where $a =$ ______ and $f(x) =$ ______________________ .

(21) Let $f(x) = (x^4+7x^2-5)\sin(x^2+3)$. Then $f'(x) = f(x)\cos(x^2+3) + g(x)\sin(x^2+3)$ where $f(x) =$ _________________________ and $g(x) =$ ________________ .

(22) Let $j(x) = \sin^5(\tan(x^2+6x-5)^{1/2})$. Then $Dj(x) = p(x)\sin^n(g(a(x)))f(g(a(x)))h(a(x))\big(a(x)\big)^r$ where

$f(x) =$ _________ ,
$g(x) =$ _________ ,
$h(x) =$ ___________ ,
$a(x) =$ ____________________ ,
$p(x) =$ ___________ ,
$n =$ _____ , and
$r =$ ______ .

(23) Let $j(x) = \sin^4(\tan(x^3-3x^2+6x-11)^{2/3})$. Then $j'(x) = 8p(x)f(g(a(x)))\cos(g(a(x)))h(a(x))\big(a(x)\big)^r$ where

$f(x) =$ ___________ ,
$g(x) =$ _________ ,
$h(x) =$ ___________ ,
$a(x) =$ ____________________________ ,
$p(x) =$ ______________ ,
$r =$ ________ .

(24) Let $j(x) = \sin^{11}(\sin^6(x^3-7x+9)^3)$. Then

$$Dj(x) = 198(3x^2+b)\sin^p(g(a(x)))h(g(a(x)))\sin^q(a(x))h(a(x))\big(a(x)\big)^r$$

where

$g(x) =$ _________ ,
$h(x) =$ ___________ ,
$a(x) =$ ____________________ ,
$p =$ ______ ,
$q =$ _____ ,
$r =$ ______ , and
$b =$ ______ ,

(25) Let $f(x) = (x^2 + \sin\pi x)^{100}$. Then $f'(1) =$ _________________ .

(26) Let $f(x) = (x^2-15)^9(x^2-17)^{10}$. Then the equation of the tangent line to the curve $y = f(x)$ at the point on the curve whose x-coordinate is 4 is $y = ax+b$ where $a =$ ______ and $b =$ ______ .

(27) Let $f(x) = (x^2-3)^{10}(x^3+9)^{20}$. Then the equation of the tangent line to the curve $y = f(x)$ at the point on the curve whose x-coordinate is -2 is $y = ax + b$ where $a =$ ________ and $b =$ ________ .

(28) Let $f(x) = (x^3-9)^8(x^3-7)^{10}$. Then the equation of the tangent line to the curve $y = f(x)$ at the point on the curve whose x-coordinate is 2 is $y = ax + b$ where $a =$ ______ and $b =$ ________ .

(29) Let $f(x) = (x^2 - 10)^{10}(x^2 - 8)^{12}$. Then the equation of the tangent line to the curve $y = f(x)$ at the point on the curve whose x-coordinate is 3 is $y = ax+b$ where $a =$ ______ and $b =$ ________ .

(30) Let $h = g \circ f$ and $j = f \circ g$ where f and g are differentiable functions on $\mathbb{R}$. Fill in the missing entries in the table below.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$h(x)$	$h'(x)$	$j(x)$	$j'(x)$
0		-3			1		1	$-\frac{3}{2}$
1	0			$\frac{3}{2}$	0	$\frac{1}{2}$		

(31) Let $f = g \circ h$ and $j = g \cdot h$ where g and h are differentiable functions on $\mathbb{R}$. Fill in the missing entries in the table below.

x	$g(x)$	$g'(x)$	$h(x)$	$h'(x)$	$f(x)$	$f'(x)$	$j(x)$	$j'(x)$
0			2			-4	-6	3
1	-2					4	-4	2
2		4	4		13	24	4	19

Also, $g(4) =$ ______ and $g'(4) =$ _____ .

(32) Let $h = g \circ f$, $j = g \cdot f$, and $k = g + f$ where f and g are differentiable functions on $\mathbb{R}$. Fill in the missing entries in the table below.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$h(x)$	$h'(x)$	$j(x)$	$j'(x)$	$k(x)$	$k'(x)$
-1			-2	4				4	-2	
0		0					0	-1		1
1			2			2	0			6

(33) Let $y = \log_3(x^2+1)^{1/3}$. Then $\dfrac{dy}{dx} = \dfrac{2x}{a(x^2+1)}$ where $a =$ ___________ .

(34) Let $f(x) = \ln \dfrac{(6+\sin^2 x)^{10}}{(7+\sin x)^3}$. Then $Df(\pi/6) = \dfrac{a}{5}$ where $a =$ ___________ .

(35) Let $f(x) = \ln(\ln x)$. What is the domain of f? Answer: (_____ , _____). What is the equation of the tangent line to the curve $y = f(x)$ at the point on the curve whose x-coordinate is e^2? Answer: $y - a = \dfrac{1}{b}(x - e^2)$ where $a =$ ________ and $b =$ ________ .

(36) Find when $y = (\tan x)^{\sin x}$ for $0 < x < \pi/2$. Then $\dfrac{dy}{dx} = (\tan x)^{\sin x}(f(x) + \cos x \ln \tan x)$ where $f(x) =$ ________ .

(37) Find when $y = (\sin x)^{\tan x}$ for $0 < x < \pi/2$. Then $\dfrac{dy}{dx} = (\sin x)^{\tan x}(a + f(x)\sec^2 x)$ where $a =$ _____ and $f(x) =$ _______________ .

(38) $\dfrac{d}{dx}\sqrt{x}\ln x = x^p(1 + g(x))$ where $p =$ ________ and $g(x) =$ __________ .

(39) If $f(x) = x^3e^x$, then $f'''(x) = (ax^3 + bx^2 + cx + d)e^x$ where $a =$ _____ , $b =$ _____ , $c =$ _______ , and $d =$ _____ .

(40) Let $f(x) = x^2\cos x$. Then $(ax^2 + bx + c)\sin x + (Ax^2 + Bx + C)\cos x$ is an antiderivative of $f(x)$ if $a =$ _____ , $b =$ _____ , $c =$ ______ , $A =$ _____ , $B =$ _____ , and $C =$ _____ .

(41) Let $f(x) = (x^4 - x^3 + x^2 - x + 1)(3x^3 - 2x^2 + x - 1)$. Use the rule for differentiating products to find $f'(1)$. Answer: _____ .

(42) Let $f(x) = \dfrac{x^{3/2} - x}{3x - x^{1/2}}$. Then $f'(4) = \dfrac{9}{a}$ where $a =$ __________ .

(43) Find a point on the curve $y = \dfrac{x^2}{x^3 - 2}$ where the tangent line is parallel to the line $4x + 6y - 5 = 0$. Answer: (_____ , _______).

(44) Let $f(x) = 5x\cos x - x^2\sin x$. Then $(ax^2 + bx + c)\sin x + (Ax^2 + Bx + C)\cos x$ is an antiderivative of $f(x)$ if $a =$ _____ , $b =$ _____ , $c =$ _____ , $A =$ _____ , $B =$ _____ , and $C =$ _____ .

(45) Let $f(x) = (2x - 3)\csc x + (2 + 3x - x^2)\cot x\csc x$. Then $\dfrac{ax^2 + bx + c}{\sin x}$ is an antiderivative of $f(x)$ if $a =$ _____ , $b =$ ______ , and $c =$ ______ .

(46) Let $f(x) = \dfrac{x^2 - 10}{x^2 - 8}$. Find the equation of the tangent line to the curve $y = f(x)$ at the point on the curve whose x-coordinate is 3. Answer: $y = ax + b$ where $a =$ ______ and $b =$ ________ .

(47) Let $f(x) = (x^4 + x^3 + x^2 + x + 1)(x^5 + x^3 + x + 2)$. Find the equation of the tangent line to the curve $y = f(x)$ at the point on the curve whose x-coordinate is -1. Answer: $y = ax + b$ where $a =$ ______ and $b =$ ______ .

(48) Let $y = \dfrac{x^2 - 2x + 1}{x^3 + 1}$. Then $\left.\dfrac{dy}{dx}\right|_{x=2} = \dfrac{2}{a}$ where $a =$ ________ .

(49) Let $f(x) = \dfrac{x - \frac{3}{2}}{x^2 + 2}$ and $g(x) = \dfrac{x^2 + 1}{x^2 + 2}$. At what values of x do the curves $y = f(x)$ and $y = g(x)$ have parallel tangent lines? Answer: at $x =$ ______ and _____ .

(50) Let $f(x) = x\sin x$. Find constants a, b, A, and B so that $(ax + b)\cos x + (Ax + B)\sin x$ is an antiderivative of $f(x)$. Answer: $a =$ ______ , $b =$ _____ , $A =$ _____ , and $B =$ _____ .

(51) Find $\dfrac{d}{dx}\left(\dfrac{1}{x}\dfrac{d^2}{dx^2}\left(\dfrac{1}{1 + x}\right)\right) = a\,\dfrac{bx + 1}{x^2(1 + x)^b}$ where $a =$ ______ and $b =$ _____ .

(52) $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\cdot\dfrac{d^2}{dx^2}\left(\dfrac{1}{x^2}\right)\right) = ax^p$ where $a =$ ________ and $p =$ ______ .

(53) Let $f(x) = \dfrac{x + 3}{4 - x}$. Find $f^{(15)}(x)$. Answer: $\dfrac{7n!}{(4 - x)^p}$ where $n =$ ______ and $p =$ ______ .

(54) Let $f(x) = \dfrac{x}{x + 1}$. Then $f^{(4)}(x) = a(x + 1)^p$ where $a =$ __________ and $p =$ ______ .

(55) Let $f(x) = \dfrac{x+1}{2-x}$. Then $f^{(4)}(x) = a(2-x)^p$ where $a =$ ______ and $p =$ ______ .

(56) Find the equation of the tangent line to the curve $2x^6 + y^4 = 9xy$ at the point $(1, 2)$. Answer: $23y = ax + b$ where $a =$ _____ and $b =$ ______ .

(57) For the curve $x^3 + 2xy + \frac{1}{3}y^3 = \frac{11}{3}$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point $(2, -1)$.

Answer: $y'(2) =$ ________ and $y''(2) = \dfrac{a}{5}$ where $a =$ _____ .

(58) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for the devil's curve $y^4 + 5y^2 = x^4 - 5x^2$ at the point $(3, 2)$.
Answer: $y'(3) =$ ______ and $y''(3) =$ ________ .

(59) Find $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, and $\dfrac{d^3y}{dx^3}$ at the point $(1, 8)$ on the astroid $x^{2/3} + y^{2/3} = 5$.

Answer: $y'(1) =$ ______ : $y''(1) = \dfrac{a}{6}$ where $a =$ _____ ; and $y'''(1) = \dfrac{b}{24}$ where $b =$ __________ .

(60) Find the point of intersection of the tangent lines to the curve $x^2 + y^3 - 3x + 3y - xy = 18$ at the points where the curve crosses the x-axis. Answer: (_______ , _______).

(61) Find the equation of the tangent line to the curve $x \sin y + x^3 = \arctan e^y + x - \frac{\pi}{4}$ at the point $(1, 0)$. Answer: $y = ax + b$ where $a =$ ______ and $b =$ _____ .

(62) The equation of the tangent line to the lemniscate $3(x^2 + y^2)^2 = 25(x^2 - y^2)$ at the point $(2, 1)$ is $y - 1 = m(x - 2)$ where $m =$ __________ .

(63) The points on the ovals of Cassini $(x^2+y^2)^2 - 4(x^2-y^2)+3 = 0$ where there is a horizontal tangent line are $\left(\pm\dfrac{\sqrt{a}}{b\sqrt{b}}, \pm\dfrac{1}{b\sqrt{b}}\right)$ where $a =$ ______ and $b =$ _____ .

(64) The points on the ovals of Cassini $(x^2 + y^2)^2 - 4(x^2 - y^2) + 3 = 0$ where there is a vertical tangent line are $(\pm\sqrt{a}, b)$ and $(\pm c, b)$ where $a =$ _____ , $b =$ _____ , and $c =$ _____ .

(65) At the point $(1, 2)$ on the curve $4x^2 + 2xy + y^2 = 12$, $\dfrac{dy}{dx} =$ ______and $\dfrac{d^2y}{dx^2} =$ ________ .

(66) Let f and g be differentiable real valued functions on $\mathbb{R}$. We know that the points $(-4, 1)$ and $(3, 4)$ lie on the graph of the curve $y = f(x)$ and the points $(-4, 3)$ and $(3, -2)$ lie on the graph of $y = g(x)$. We know also that $f'(-4) = 3$, $f'(3) = -4$, $g'(-4) = -2$, and $g'(3) = 6$.
 (a) If $h = f \cdot g$, then $h'(-4) =$ _____ .
 (b) If $j = (2f + 3g)^4$, then $j'(3) =$ ________ .
 (c) If $k = f \circ g$, then $k'(-4) =$ _____ .
 (d) If $\ell = \dfrac{f}{g}$, then $\ell'(3) =$ ______ .

(67) Let $f(x) = 5 \sin x + 3 \cos x$. Then $f^{(117)}(\pi) =$ ______ .

(68) Let $f(x) = 4 \cos x - 7 \sin x$. Then $f^{(87)}(0) =$ ______ .

7.3. Problems

(1) Let (x_0, y_0) be a point in $\mathbb{R}^2$. How many tangent lines to the curve $y = x^2$ pass through the point (x_0, y_0)? What are the equations of these lines? *Hint.* Consider the three cases: $y_0 > {x_0}^2$, $y_0 = {x_0}^2$, and $y_0 < {x_0}^2$.

(2) For the purposes of this problem you may assume that the differential equation

$$y'' + y = 0 \tag{$*$}$$

has at least one nontrivial solution on the real line. (That is, there exists at least one twice differentiable function y, not identically zero, such that $y''(x) + y(x) = 0$ for all $x \in \mathbb{R}$.)

(a) Show that if u and v are solutions of $(*)$ and a, $b \in \mathbb{R}$, then $w = au + bv$ and u' are also solutions of $(*)$.
(b) Show that if y is a solution of $(*)$ then $y^2 + (y')^2$ is constant.
(c) Show that if y is a nontrivial solution of $(*)$, then either $y(0) \neq 0$ or $y'(0) \neq 0$. *Hint.* Argue by contradiction. Show that if y is a solution of $(*)$ such that both $y(0) = 0$ and $y'(0) = 0$, then $y(x) = 0$ for all x.
(d) Show that there exists a solution s of $(*)$ such that $s(0) = 0$ and $s'(0) = 1$. *Hint.* Let y be a nontrivial solution of $(*)$. Look for a solution s of the form $a\,y + b\,y'$ (with a, $b \in \mathbb{R}$) satisfying the desired conditions.
(e) Show that if y is a solution of $(*)$ such that $y(0) = a$ and $y'(0) = b$, then $y = b\,s + a\,s'$. *Hint.* Let $u(x) = y(x) - b\,s(x) - a\,s'(x)$ and show that u is a solution of $(*)$ such that $u(0) = u'(0) = 0$. Use (c).
(f) Define $c(x) = s'(x)$ for all x. Show that $(s(x))^2 + (c(x))^2 = 1$ for all x.
(g) Show that s is an odd function and that c is even. *Hint.* To see that s is odd let $u(x) = s(-x)$ for all x. Show that u is a solution of $(*)$. Use (e). Once you know that s is odd, differentiate to see that c is even.
(h) Show that $s(a + b) = s(a)c(b) + c(a)s(b)$ for all real numbers a and b. *Hint.* Let $y(x) = s(x + b)$ for all x. Show that y is a solution of $(*)$. Use (e).
(i) Show that $c(a+b) = c(a)c(b)-s(a)s(b)$ for all real numbers a and b. *Hint.* Differentiate the formula for $s(x + b)$ that you derived in (h).
(j) Define $t(x) = \dfrac{s(x)}{c(x)}$ and $\sigma(x) = \dfrac{1}{c(x)}$ for all x such that $c(x) \neq 0$. Show that $t'(x) = (\sigma(x))^2$ and $\sigma'(x) = t(x)\sigma(x)$ wherever $c(x) \neq 0$.
(k) Show that $1 + (t(x))^2 = (\sigma(x))^2$ wherever $c(x) \neq 0$.
(l) Explain carefully what the (mathematical) point of this problem is.

(3) Suppose that f is a differentiable function such that $f'(x) \geq \dfrac{3}{2}$ for all x and that $f(1) = 2$. Prove that $f(5) \geq 8$.

(4) Suppose that f is a differentiable function such that $f'(x) \geq 3$ for all x and that $f(0) = -4$. Prove that $f(3) \geq 5$.

(5) Suppose that f is a differentiable function such that $f'(x) \leq -2$ for all $x \in [0, 4]$ and that $f(1) = 6$.
(a) Prove that $f(4) \leq 0$.
(b) Prove that $f(0) \geq 8$.

(6) Give a careful proof that $\sin x \leq x$ for all $x \geq 0$.

(7) Give a careful proof that $1 - \cos x \leq x$ for all $x \geq 0$.

(8) Prove that if $x^2 = \dfrac{1 - y^2}{1 + y^2}$, then $\left(\dfrac{dx}{dy}\right)^2 = \dfrac{1 - x^4}{1 - y^4}$ at points where $y \neq \pm 1$.

(9) For the circle $x^2 + y^2 - 1 = 0$ use implicit differentiation to show that $y'' = -\dfrac{1}{y^3}$ and $y''' = -\dfrac{3x}{y^5}$.

(10) Explain how to calculate $\dfrac{d^2y}{dx^2}$ at the point on the folium of Déscartes

$$x^3 + y^3 = 9xy$$

where the tangent line is parallel to the asymptote of the folium.

(11) Explain carefully how to find the curve passing through the point $(2, 3)$ which has the following property: the segment of any tangent line to the curve contained between the (positive) coordinate axes is bisected at the point of tangency. Carry out the computation you have described.

7.4. Answers to Odd-Numbered Exercises

(1) $-\frac{5}{2}$, $-\frac{3}{2}$, 9, $\frac{1}{2}$

(3) 4, $-\frac{2}{3}$, -6, $-\frac{8}{5}$

(5) 36

(7) 13

(9) -3, 0, 2

(11) -1, 2

(13) 4, 12

(15) 0, 0, -2

(17) -1, 1, -1, 1

(19) 9, 3

(21) $2x^5 + 14x^3 - 10x$, $4x^3 + 14x$

(23) $\sin^3 x$, $\tan x$, $\sec^2 x$, $\left(x^3 - 3x^2 + 6x - 11\right)^{\frac{2}{3}}$, $x^2 - 2x + 2$, $-\frac{1}{2}$

(25) $100(2 - \pi)$

(27) 200, 401

(29) 12, -35

(31) -3, 0, -1, 1 (first row)
2, 2, 1, 1 (second row)
1, 3 (third row)
13, 8

(33) $3 \ln 3$

(35) 1, ∞, $\ln 2$, $2e^2$

(37) 1, $\ln \sin x$

(39) 1, 9, 18, 6

(41) 8

(43) 2, $\frac{2}{3}$

(45) 1, -3, -2

(47) 11, 10

(49) -1, 2

(51) -2, 4

(53) 15, 16

(55) 72, -5

(57) -2, 4

(59) -2, 5, -25

(61) -4, 4

(63) 15, 2

(65) $-2, -\dfrac{4}{3}$

(67) -5

CHAPTER 8

THE MEAN VALUE THEOREM

8.1. Background

Topics: *Rolle's theorem*, the *mean value theorem*, the *intermediate value theorem*.

8.1.1. Definition. A real valued function f defined on an interval J is INCREASING on J if $f(a) \leq f(b)$ whenever a, $b \in J$ and $a \leq b$. It is STRICTLY INCREASING on J if $f(a) < f(b)$ whenever a, $b \in J$ and $a < b$. The function f is DECREASING on J if $f(a) \geq f(b)$ whenever a, $b \in J$ and $a \leq b$. It is STRICTLY DECREASING on J if $f(a) > f(b)$ whenever a, $b \in J$ and $a < b$.
NOTE: In many texts the word "nondecreasing" is used where "increasing" in these notes; and "increasing" is used for "strictly increasing".

8.2. Exercises

(1) Let $M > 0$ and $f(x) = x^3$ for $0 \le x \le M$. Find a value of c which satisfies the conclusion of the *mean value theorem* for the function f over the interval $[0, M]$. Answer: $c = \dfrac{M}{a}$ where $a =$ ______ .

(2) Let $f(x) = x^4 + x + 3$ for $0 \le x \le 2$. Find a point c whose existence is guaranteed by the *mean value theorem*. Answer: $c = 2^p$ where $p =$ ______ .

(3) Let $f(x) = \sqrt{x}$ for $4 \le x \le 16$. Find a point c whose existence is guaranteed by the *mean value theorem*. Answer: $c =$ _____ .

(4) Let $f(x) = \dfrac{x}{x+1}$ for $-\frac{1}{2} \le x \le \frac{1}{2}$. Find a point c whose existence is guaranteed by the *mean value theorem*. Answer: $c = \dfrac{a}{2} - 1$ where $a =$ ______ .

8.3. Problems

(1) Use *Rolle's theorem* to derive the *mean value theorem*.

(2) Use the *mean value theorem* to derive *Rolle's theorem*.

(3) Use the *mean value theorem* to prove that if a function f has a positive derivative at every point in an interval, then it is increasing on that interval.

(4) Let $a \in \mathbb{R}$. Prove that if f and g are differentiable functions with $f'(x) \leq g'(x)$ for every x in some interval containing a and if $f(a) = g(a)$, then $f(x) \leq g(x)$ for every x in the interval such that $x \geq a$.

(5) Suppose that f is a differentiable function such that $f'(x) \leq -2$ for all $x \in [0, 4]$ and that $f(1) = 6$.
  (a) Prove that $f(4) \leq 0$.
  (b) Prove that $f(0) \geq 8$.

(6) Your friend Fred is confused. The function $f\colon x \mapsto x^{\frac{2}{3}}$ takes on the same values at $x = -1$ and at $x = 1$. So, he concludes, according to *Rolle's theorem* there should be a point c in the open interval $(-1, 1)$ where $f\,'(c) = 0$. But he cannot find such a point. Help your friend out.

(7) Consider the equation $\cos x = 2x$.
  (a) Use the *intermediate value theorem* to show that the equation has at least one solution.
  (b) Use the *mean value theorem* to show that the equation has at most one solution.

(8) Let $m \in \mathbb{R}$. Use *Rolle's theorem* to show that the function f defined by $f(x) = x^3 - 3x + m$ can not have two zeros in the interval $[-1, 1]$.

(9) Use the *mean value theorem* to show that if $0 < x \leq \pi/3$, then $\frac{1}{2}x \leq \sin x \leq x$.

(10) Use the *mean value theorem* to show that on the interval $[0, \pi/4]$ the graph of the curve $y = \tan x$ lies between the lines $y = x$ and $y = 2x$.

(11) Let $x > 0$. Use the *mean value theorem* to show that $\dfrac{x}{x^2 + 1} < \arctan x < x$.

(12) Use the *mean value theorem* to show that

$$x + 1 < e^x < 2x + 1$$

whenever $0 < x \leq \ln 2$.

(13) Show that the equation $e^x + x = 0$ has exactly one solution. Locate this solution between consecutive integers.

(14) Prove that the equation $\sin x = 1 - 2x$ has exactly one solution. Explain how the *intermediate value theorem* can be used to produce an approximation to the solution which is correct to two decimal places.

(15) Give a careful proof that at one time your height (in inches) was exactly equal to your weight (in pounds). Be explicit about any physical assumptions you make.

8.4. Answers to Odd-Numbered Exercises

(1) $\sqrt{3}$

(3) 9

CHAPTER 9

L'HÔPITAL'S RULE

9.1. Background

Topics: l'Hôpital's rule.

9.2. Exercises

(1) $\lim\limits_{x\to 0} \dfrac{\frac{1}{3}x^3 + 2x - 2\sin x}{4x^3} = \dfrac{1}{a}$ where $a =$ ____ .

(2) $\lim\limits_{x\to 0} \left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right) =$ ____ .

(3) $\lim\limits_{t\to 1} \dfrac{nt^{n+1} - (n+1)t^n + 1}{(t-1)^2} =$ ________________ when $n \geq 2$.

(4) $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x - \sin x} =$ ____ .

(5) $\lim\limits_{x\to 2} \dfrac{x^4 - 4x^3 + 5x^2 - 4x + 4}{x^4 - 4x^3 + 6x^2 - 8x + 8} = \dfrac{a}{6}$ where $a =$ ____ .

(6) Let n be a fixed integer. Then the function f given by $f(x) = \dfrac{\sin(n + \frac{1}{2})x}{\sin \frac{1}{2}x}$ is not defined at points $x = 2m\pi$ where m is an integer. The function f can be extended to a function continuous on all of $\mathbb{R}$ by defining
$f(2m\pi) =$ __________ for every integer m.

(7) $\lim\limits_{x\to 1} \dfrac{x^5 - 1}{6x^5 - 4x^3 + x - 3} = \dfrac{5}{a}$ where $a =$ _____ .

(8) Suppose that g has derivatives of all orders, that $g(0) = g'(0) = g''(0) = 0$, that $g'''(0) = 27$, and that there is a deleted neighborhood U of 0 such that $g^{(n)}(x) \neq 0$ whenever $x \in U$ and $n \geq 0$. Define $f(x) = x^{-4}g(x)(1 - \cos x)$ for $x \neq 0$ and $f(0) = 0$. Then $f'(0) = \dfrac{a}{4}$ where $a =$ ____ .

(9) Suppose that g has derivatives of all orders, that $g(0) = g'(0) = 0$, that $g''(0) = 10$, that $g'''(0) = 12$, and that there is a deleted neighborhood of 0 in which $g(x)$, $g'(x)$, $xg'(x) - g(x) - 5x^2$, and $g''(x) - 10$ are never zero. Let $f(x) = \dfrac{g(x)}{x}$ for $x \neq 0$ and $f(0) = 0$. Then $f''(0) =$ ____ .

(10) Suppose that g has derivatives of all orders, that $g(0) = g'(0) = g''(0) = g'''(0) = 0$, and that $g^{(4)}(0) = 5$. Define $f(x) = \dfrac{x\,g(x)}{2\cos x + x^2 - 2}$ for $x \neq 0$ and $f(0) = 0$. Then $f'(0) = \dfrac{a}{2}$ where $a =$ ____ .

(11) $\lim\limits_{x\to 0} \dfrac{x^2 + 2\cos x - 2}{x^4} = \dfrac{1}{a}$ where $a =$ _____ .

(12) $\lim\limits_{x\to 0} \dfrac{\cos x + \frac{1}{2}x^2 - 1}{5x^4} = \dfrac{1}{a}$ where $a =$ _______ .

(13) $\lim\limits_{x\to 0} \dfrac{x^2 + 2\ln(\cos x)}{x^4} = -\dfrac{1}{a}$ where $a =$ ____ .

(14) $\lim\limits_{x\to\infty} \left(1 - \dfrac{5}{2x}\right)^{4x} =$ _______ .

(15) $\lim\limits_{x\to\infty} \left(\dfrac{\ln x}{x}\right)^{1/\ln x} =$ _______ .

(16) $\lim\limits_{x\to 0} \left(\dfrac{\sin x}{x^3} - \dfrac{1}{x^2}\right) = -\dfrac{1}{a}$ where $a =$ ____ .

(17) $\displaystyle\lim_{x\to 1}\left[\frac{1}{x-1}-\frac{\ln x}{(x-1)^2}\right]=\frac{1}{a}$ where $a=$ _____ .

(18) $\displaystyle\lim_{x\to 1}\left[\frac{1}{2(x-1)}-\frac{1}{(x-1)^2}+\frac{\ln x}{(x-1)^3}\right]=\frac{1}{a}$ where $a=$ _____ .

(19) $\displaystyle\lim_{x\to 0^+}\left(\sqrt{\frac{1}{x^2}+\frac{1}{x}}-\sqrt{\frac{1}{x^2}-\frac{1}{x}}\right)=$ _____ .

(20) $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{x-\arctan x}=$ ______ .

(21) Let $f(x)=x^2e^{1/x}$ for all $x\neq 0$. Then $\displaystyle\lim_{x\to 0^-}f(x)=$ _____ and $\displaystyle\lim_{x\to 0^+}f(x)=$ _____ .

(22) $\displaystyle\lim_{x\to\infty}(x\ln(5x))^{3/\ln x}=$ ______ .

(23) $\displaystyle\lim_{x\to 0}\left[\frac{1}{x^2}+\frac{2}{x^4}\ln\cos x\right]=$ ________ .

(24) $\lim_{x\to 0^+}(\sin x)^{\tan x}=$ _____ .

(25) $\displaystyle\lim_{x\to\frac{\pi}{2}^-}(\sec x-\tan x)=$ _____ .

(26) $\displaystyle\lim_{x\to\frac{\pi}{2}^-}(\sec^2 x-\tan^2 x)=$ _____ .

(27) $\displaystyle\lim_{x\to\frac{\pi}{2}^-}(\sec^3 x-\tan^3 x)=$ _____ .

(28) $\displaystyle\lim_{x\to\infty}\frac{(\ln x)^{25}}{x}=$ _____ .

(29) $\displaystyle\lim_{x\to\infty}\left(1-\frac{5}{7x}\right)^{2x}=e^{-a/7}$ where $a=$ ______ .

(30) $\displaystyle\lim_{x\to\infty}\left(\frac{3x}{e^{2x}+7x^2}\right)^{1/x}=e^a$ where $a=$ ______ .

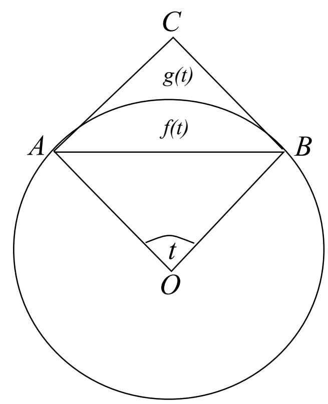
(31) $\displaystyle\lim_{x\to\infty}\left(\frac{\ln x}{x}\right)^{1/\ln x}=e^a$ where $a=$ ______ .

9.3. Problems

(1) Is the following a correct application of l'Hôpital's rule? Explain.

$$\lim_{x\to 1}\frac{2x^3-3x+1}{x^4-1}=\lim_{x\to 1}\frac{6x^2-3}{4x^3}=\lim_{x\to 1}\frac{12x}{12x^2}=\lim_{x\to 1}\frac{1}{x}=1.$$

(2) Let t be the measure of a central angle $\angle AOB$ of a circle. The segments AC and BC are tangent to the circle at points A and B, respectively. The triangular region $\triangle ABC$ is divided into the region outside the circle whose area is $g(t)$ and the region inside the circle with area $f(t)$. Find $\displaystyle\lim_{t\to 0}\frac{f(t)}{g(t)}$.



(3) Let $f(x)=x^{(x-1)^{-1}}$ for $x>0$, $x\neq 1$. How should $f(1)$ be defined so that f is continuous on $(0,\infty)$? Explain your reasoning carefully.

(4) Show that the curve $y=x(\ln x)^2$ does not have a vertical asymptote at $x=0$.

(5) Define $f(x)=(x^2)^x$ for all $x\neq 0$. Define $f(0)$ in such a way as to make f a continuous function on $\mathbb{R}$. Find all critical points of f. Determine the intervals on which f is increasing, decreasing, concave up, concave down. Take special care to describe what happens at $x=0$. Use Newton's method to find to 4 decimal place accuracy any points of inflection which may occur.

(6) Let $f(x)=\dfrac{x\ln x}{x-1}$ for $x>0$ and $x\neq 1$.
 (a) How should f be defined at $x=1$ so that f will be continuous on $(0,\infty)$? Explain how you know your answer is correct.
 (b) Suppose $f(1)$ has the value you found in (a). Then find $f'(1)$ (and explain what you are doing).
 (c) Suppose $f(1)$ has the value you found in (a). Find $f''(1)$ (and explain what you are doing).
 (d) Suppose $f(1)$ has the value you found in (a). Give a careful proof that f'' is continuous at $x=1$.

(7) Your good friend Fred is confused again. He is trying to find $\ell=\displaystyle\lim_{x\to 0^+}(1+x)^{1/x}$. It is clear him that for $x>0$ the quantity in parentheses, $1+x$, is always strictly greater than 1. Further more the power $\dfrac{1}{x}$ is going to infinity as x approaches 0 from the right. So ℓ is the result of taking a number strictly greater than 1 to higher and higher powers and, therefore, $l=\infty$. On the other hand he sees that $1+x$ is approaching 1 as x approaches 0, and 1 taken to any power whatever is 1. So $\ell=1$. Help Fred by pointing out to him the error of his ways.

9.4. Answers to Odd-Numbered Exercises

(1) 6

(3) $\frac{1}{2}n(n+1)$

(5) 5

(7) 19

(9) 4

(11) 12

(13) 6

(15) $\frac{1}{e}$

(17) 2

(19) 1

(21) 0, ∞

(23) $-\frac{1}{6}$

(25) 0

(27) ∞

(29) 10

(31) -1

CHAPTER 10

MONOTONICITY AND CONCAVITY

10.1. Background

Topics: increasing, decreasing, monotone, concave up, concave down.

10.1.1. Definition. A real valued function f defined on an interval J is INCREASING on J if $f(a) \leq f(b)$ whenever $a, b \in J$ and $a \leq b$. It is STRICTLY INCREASING on J if $f(a) < f(b)$ whenever $a, b \in J$ and $a < b$. The function f is DECREASING on J if $f(a) \geq f(b)$ whenever $a, b \in J$ and $a \leq b$. It is STRICTLY DECREASING on J if $f(a) > f(b)$ whenever $a, b \in J$ and $a < b$.

10.1.2. Definition. Let $f \colon A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$. The function f has a LOCAL (or RELATIVE) MAXIMUM at a point $a \in A$ if there exists $r > 0$ such that $f(a) \geq f(x)$ whenever $|x - a| < r$ and $x \in \operatorname{dom} f$. It has a LOCAL (or RELATIVE) MINIMUM at a point $a \in A$ if there exists $r > 0$ such that $f(a) \leq f(x)$ whenever $|x - a| < r$ and $x \in \operatorname{dom} f$. The point a is a RELATIVE EXTREMUM of f if it is either a relative maximum or a relatives minimum.

The function $f \colon A \to \mathbb{R}$ is said to attain a MAXIMUM at a if $f(a) \geq f(x)$ for all $x \in \operatorname{dom} f$. This is often called a GLOBAL (or ABSOLUTE) MAXIMUM to help distinguish it from the local version defined above. It is clear that every global maximum is also a local maximum but not *vice versa.* (Of course, similar definitions hold for *global* or *absolute minima* and *global* or *absolute extrema.*)

10.1.3. Definition. A real valued function f defined on an interval J is CONCAVE UP on J if the chord line connecting any two points $(a, f(a))$ and $(b, f(b))$ on the curve (where $a, b \in J$) always lies on or above the curve. It is CONCAVE DOWN if the chord line always lies on or below the curve. A point on the curve where the concavity changes is a POINT OF INFLECTION.

When f is twice differentiable it is concave up on J if and only if $f''(c) \geq 0$ for all $c \in J$ and is concave down on J if and only if $f''(c) \leq 0$ for all $c \in J$.

10.2. Exercises

(1) Suppose that the derivative of a function f is given by $f'(x) = \dfrac{x}{x+2} - \dfrac{x+3}{x-4}$. Then the intervals on which f is increasing are (______ , ______) and (________ , _____).

(2) Suppose that the derivative of a function f is given by $f'(x) = \dfrac{-x}{(x^2+1)^2}$.
- (a) The interval on which the function f is increasing is (______ , _____).
- (b) Estimate to two decimal places the location of a point $x > 0$ where f has a point of inflection. Answer: ___ . ___ ___ .

(3) A function f is defined on the interval $[0, \pi]$. Its derivative is given by $f'(x) = \cos x - \sin 2x$.
- (a) The intervals on which f is increasing are $\left(a\,,\dfrac{\pi}{b}\right)$ and $\left(\dfrac{\pi}{c}\,,\dfrac{d\pi}{b}\right)$ where $a =$ _____, $b =$ _____, $c =$ _____, and $d =$ _____.
- (b) Estimate to two decimal places the location of points of inflection. Answer: 1. ___ ___ and 2. ___ ___ .

(4) Suppose that the derivative of a function f is given by $f'(x) = (x-2)^2(x+4)$.
- (a) The interval on which f is increasing is (______ , _____) .
- (b) f has no local ________________ .
- (c) f has a local ________________ at $x =$ ______ .

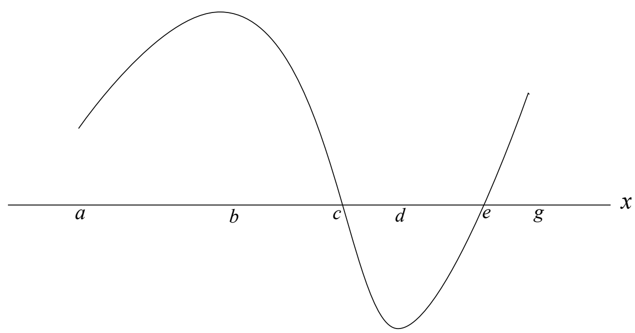
(5) Suppose that the derivative of a function f is given by $f'(x) = \dfrac{x+1}{\sqrt{x^2+1}}$.
- (a) The interval on which f is increasing is (______ , _____) .
- (b) f has no local ________________ .
- (c) f has a local ________________ at $x =$ ______ .

(6) Suppose that the derivative of a function f is given by $f'(x) = \ln(1 + x^2)$.
- (a) The interval on which f is increasing is (______ , _____) .
- (b) At how many points does f have a local maximum? Answer: _____ .
- (c) At how many points does f have a local minimum? Answer: _____ .
- (d) The interval on which f is concave up is (_____ , _____) .
- (e) f has a point of inflection at $x =$ _____ .

(7) Suppose that the derivative of a function f is given by $f'(x) = \dfrac{1}{x^2+1}$.
- (a) The interval on which f is increasing is (______ , _____) .
- (b) At how many points does f have a local maximum? Answer: _____ .
- (c) At how many points does f have a local minimum? Answer: _____ .
- (d) The interval on which f is concave up is (______ , _____) .
- (e) f has a point of inflection at $x =$ _____ .

(8) Suppose that the derivative of a function f is given by $f'(x) = \dfrac{x}{x^2+1}$.
- (a) The interval on which f is increasing is (_____ , _____) .
- (b) At how many points does f have a local maximum? Answer: _____ .
- (c) At how many points does f have a local minimum? Answer: _____ .
- (d) The interval on which f is concave up is (______ , _____) .
- (e) f has points of inflection at $x =$ ______ and $x =$ _____ .

(9) The domain of a function f is $[a, g]$. Below is a sketch of the graph of the *derivative* of f.



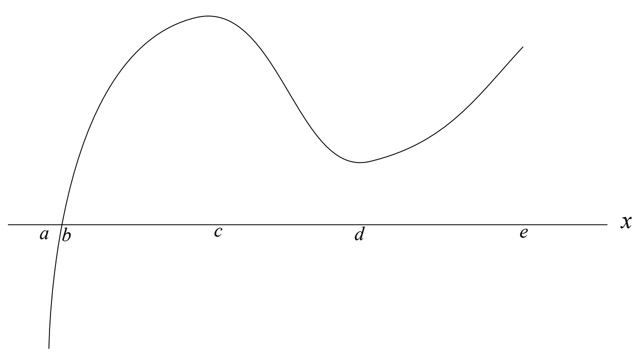
(a) The largest intervals on which f is increasing are (_____ , _____) and (_____ , _____) .

(b) f has local minima at $x =$ _____ and $x =$ _____ .

(c) The largest intervals on which f is concave up are (_____ , _____) and (_____ , _____) .

(d) f has points of inflection at: $x =$ _____ and $x =$ _____ .

(10) The domain of a function f is $[a, e]$. Below is a sketch of the graph of the *derivative* of f.



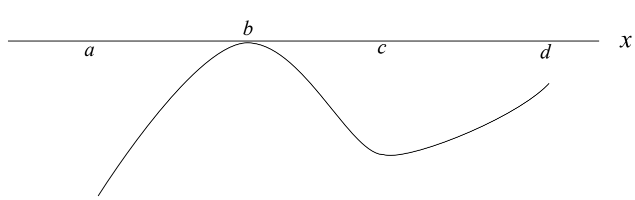
(a) The largest interval on which f is increasing is (_____ , _____) .

(b) f has local maxima at $x =$ _____ and $x =$ _____ .

(c) The largest intervals on which f is concave up are (_____ , _____) and (_____ , _____) .

(d) f has points of inflection at: $x =$ _____ and $x =$ _____ .

(11) The domain of a function f is $[a, d]$. Below is a sketch of the graph of the *derivative* of f.



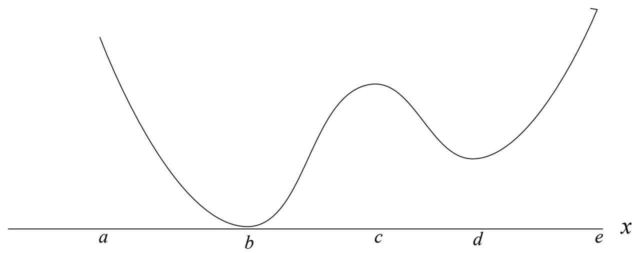
(a) The largest interval on which f is decreasing is (_____ , _____) .

(b) f has a local maximum at $x =$ _____ .

(c) The largest intervals on which f is concave up are (_____ , _____) and (_____ , _____) .

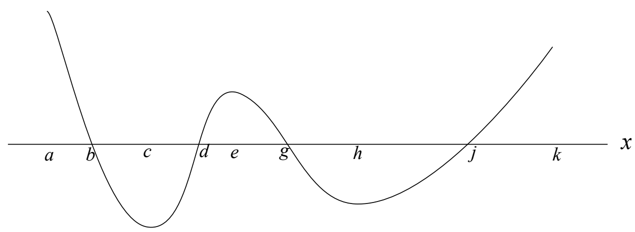
(d) f has points of inflection at: $x =$ _____ and $x =$ _____ .

(12) The domain of a function f is $[a, e]$. Below is a sketch of the graph of the *derivative* of f.



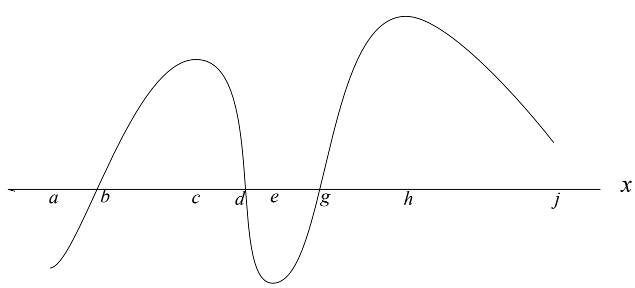
(a) The largest interval on which f is increasing is (_____ , _____) .
(b) f has local minimum at $x =$ _____ .
(c) The largest intervals on which f is concave up are (_____ , _____) and (_____ , _____) .
(d) f has points of inflection at: $x =$ _____ , $x =$ _____ , and $x =$ _____ .

(13) The domain of a function f is $[a, k]$. Below is a sketch of the graph of the *derivative* of f.



(a) The largest intervals on which f is decreasing are (_____ , _____) and (_____ , _____) .
(b) f has local minima at $x =$ _____ , $x =$ _____ , and $x =$ _____ .
(c) The largest intervals on which f is concave up are (_____ , _____) and (_____ , _____) .
(d) f has points of inflection at: $x =$ _____ , $x =$ _____ , and $x =$ _____ .

(14) The domain of a function f is $[a, j]$. Below is a sketch of the graph of the *derivative* of f.



(a) The largest intervals on which f is increasing are (_____ , _____) and (_____ , _____) .
(b) f has local maxima at $x =$ _____ , $x =$ _____ , and $x =$ _____ .
(c) The largest intervals on which f is concave up are (_____ , _____) and (_____ , _____) .
(d) f has points of inflection at: $x =$ _____ , $x =$ _____ , and $x =$ _____ .

(15) Consider the function $f\colon x \mapsto x^2 e^{-x}$.
(a) The interval on which f is increasing is (_____ , _____) .
(b) f has a local minimum at $x =$ _____ .
(c) The interval on which f is concave down is $(a - \sqrt{a}, a + \sqrt{a})$ where $a =$ _____ .

(16) The intervals on which the function $f(x) = x^2 + \dfrac{16}{x^2}$ is increasing are (_____ , _____) and (_____ , _____) .

(17) Consider the function $f\colon x \mapsto \dfrac{1}{x}e^x$.
  (a) The intervals on which f is decreasing are (______ , _____) and (_____ , _____) .
  (b) How many local maxima does f have? Answer: _____ .
  (c) The interval on which f is concave up is (_____ , _____) .

(18) Consider the function $f\colon x \mapsto x\exp\left(-\frac{1}{2}x^2\right)$.
  (a) The intervals on which f is decreasing are (______ , ______) and (_____ , _____) .
  (b) The intervals on which f is concave up are $(-a, 0)$ and $(a\,, \infty\,)$ where $a =$ ______ .
  (c) f has how many points of inflection? Answer: _____ .

(19) Consider the function $f\colon x \mapsto \ln(4 - x^2)$.
  (a) The domain of f is the interval (_____ , _____) .
  (b) The interval on which f is increasing is (_____ , _____) .
  (c) f is concave __________ .

(20) Consider the function $f\colon x \mapsto x\ln x$.
  (a) The domain of f is the interval (_____ , _____) .
  (b) $\lim\limits_{x\to 0^+} f(x) =$ _____ .
  (c) The interval on which f is positive is (_____ , _____) .
  (d) The interval on which f is increasing is (________ , _____) .
  (e) The function f attains its minimum value of $-\dfrac{1}{a}$ at $x = \dfrac{1}{b}$ where $a =$ ________ and $b =$ _______ .
  (f) f is concave __________ .

(21) Consider the function $f\colon x \mapsto x^2\ln x$.
  (a) The domain of f is the interval (_____ , _____) .
  (b) $\lim\limits_{x\to 0^+} f(x) =$ _____ .
  (c) The interval on which f is positive is (_____ , _____) .
  (d) The interval on which f is increasing is (________ , _____) .
  (e) The function f attains its minimum value of $-\dfrac{1}{a}$ at $x = \dfrac{1}{b}$ where $a =$ ________ and $b =$ _______ .
  (f) f has a point of inflection at $x = e^p$ where $p =$ ________ .

(22) Consider the function $f\colon x \mapsto x(\ln x)^2$.
  (a) The domain of f is the interval (_____ , _____) .
  (b) $\lim\limits_{x\to 0^+} f(x) =$ _____ .
  (c) The intervals on which f is increasing are $\left(0\,, \dfrac{1}{a}\right)$, where $a =$ ________ , and (_____ , _____) .
  (d) The function f attains its minimum value of _____ at $x =$ _____ .
  (e) f has a point of inflection at $x = e^p$ where $p =$ ______ .

(23) Consider the function $f\colon x \mapsto \dfrac{1}{x}\ln x$.

(a) The domain of f is the interval (_____ , _____) .

(b) $\lim_{x\to 0^+} f(x) =$ ______ .

(c) The interval on which f is increasing is (_____ , _____) .

(d) The function f attains its maximum value of ________ at $x =$ _____ .

(e) f has a point of inflection at $x = e^p$ where $p =$ ________ .

(24) Let $f(x) = e^x \sin x$ for $0 \le x \le \pi$. Then f has its global maximum at $x =$ __________ ; it has its global minimum at $x =$ _____; and it has a point of inflection at $x =$ ________ .

(25) Find real numbers a and b such that $x = 1$ is a critical point of the function f where $f(x) = ax + \dfrac{b}{x^2}$ for all $x \neq 0$ and $f(1) = 3$. Answer: $a =$ _____ and $b =$ _____ . Then the point $(1, 3)$ a local _____________.

(26) Let $f(x) = \dfrac{x^2-5}{x^2+3}$ for all $x \ge -1$. The function f has a global minimum at $x =$ _____ and a local maximum at $x =$ _____ .

(27) Consider the function $f\colon x \mapsto \dfrac{x^2-2x}{(x+1)^2}$.

(a) The intervals on which f is increasing are (______ , ______) and (______ , _____) .

(b) f has a global minimum at $x =$ _____ .

(c) The intervals on which f is concave up are (______ , ______) and (______ , ________) .

(d) f has a point of inflection at: $x =$ ________ .

(28) Consider the function $f\colon x \mapsto \dfrac{2x^2}{x^2+2}$.

(a) The interval on which f is increasing is (_____ , _____) .

(b) f has global minimum at $x =$ _____ .

(c) The interval on which f is concave up is $(-\sqrt{a}, \sqrt{a})$ where $a =$ ________ .

(29) Consider the function $f\colon x \mapsto \dfrac{6}{x^2} - \dfrac{6}{x}$.

(a) The intervals on which f is increasing are (______ , _____) and (_____ , _____) .

(b) f has a global minimum at $x =$ _____ .

(c) The intervals on which f is concave up are (______ , _____) and (_____ , _____) .

(30) Consider the function $f\colon x \mapsto \dfrac{|x-1|}{|x|-1}$.

(a) The intervals on which f is strictly increasing are (______ , ______) and (______ , _____) .

(b) f is constant on the intervals [_____ , _____) and (_____ , _____) .

(c) f has a vertical asymptote at $x =$ ______ and a horizontal asymptote at $y =$ _____ .

(d) The only point in the domain of f at which f is not differentiable is $x =$ _____ .

(e) f has how many points of inflection? Answer: _____ .

(31) Let $f(x) = \frac{1}{4}x^3 - 3x + 7$ for $-4 \le x \le 3$. Then f has (local) maxima at $x =$ _____ and $x =$ _____ . The global maximum of f occurs at $x =$ _____ . The maximum value of f is _____ .

(32) Let $f(x) = \sqrt{x} + \dfrac{4}{x}$ for $\frac{1}{4} \le x \le 100$. The maximum value attained by $f(x)$ is $\dfrac{a}{2}$ where $a =$ ______ .

10.3. Problems

(1) Suppose that a function f is increasing on the interval $(-\infty, -5)$ and is also increasing on the interval $(-5, \infty)$. Is it necessarily the case that f must be increasing on the set $(-\infty, -5) \cup (-5, \infty)$? Explain.

(2) An EQUATION OF STATE of a substance is an equation expressing a relationship between the pressure P, the volume V, and the temperature T of the substance. A VAN DER WAALS GAS is a gas for which there exist positive constants a and b (depending on the particular gas) such that the following equation of state holds:

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT. \tag{$*$}$$

(Here R is a universal constant, *not* depending on the particular gas.) For each fixed value of T the equation of state $(*)$ can be used to express P as a function of V, say $P = f(V)$. A CRITICAL TEMPERATURE, which we denote by T_c, is a value of T for which the corresponding function f possesses a critical point which is also a point of inflection. The V and P coordinates of this critical point are denoted by V_c and P_c and are called the CRITICAL VOLUME and the CRITICAL PRESSURE.

Show that every van der Waals gas has a critical temperature. Compute the critical values RT_c, V_c, and P_c (in terms of the gas constants a and b). Explain how you know that the point (V_c, P_c) is a point of inflection.

(3) A water storage tank consists of two parts: the bottom portion is a cylinder with radius 10 feet and height 50 feet; the top portion is a sphere of radius 25 feet. (A small bottom portion of the sphere is missing where it connects to the cylinder.) The tank is being filled from the bottom of the cylindrical portion with water flowing in at a constant rate of 100 cubic feet per minute. Let $h(t)$ be the height of the water in the tank at time t. Sketch a graph of the function h from the time the filling starts to the time the tank is full. Explain carefully the reasoning behind all properties of your graph—paying particular attention to its concavity properties.

(4) For what values of $k > 0$ does the function f defined by

$$f(x) = \frac{\ln x}{k} - \frac{kx}{x+1}$$

have local extrema? For each such k locate and classify the extrema. Explain the reasons for your conclusions carefully.

10.4. Answers to Odd-Numbered Exercises

(1) $-\infty$, -2, $-\frac{2}{3}$, 4

(3) (a) 0, 6, 2, 5
(b) 0, 0, 1, 4

(5) (a) -1, ∞
(b) maximum
(c) minimum. -1

(7) (a) $-\infty$, ∞
(b) 0
(c) 0
(d) $-\infty$, 0
(e) 0

(9) (a) a, c, e, g
(b) a, e
(c) a, b, d, g
(d) b, d

(11) (a) a, d
(b) a
(c) a, b, c, d
(d) b, c

(13) (a) b, d, g, j
(b) a, d, j
(c) c, e, h, k
(d) c, e, h

(15) (a) 0, 2
(b) 0
(c) 2

(17) (a) $-\infty$, 0, 0, 1
(b) 0
(c) 0, ∞

(19) (a) -2, 2
(b) -2, 0
(c) down

(21) (a) 0, ∞
(b) 0
(c) 1, ∞
(d) $\frac{1}{\sqrt{e}}$, ∞
(e) $2e$, $\sqrt{e}$
(f) $-\frac{3}{2}$

(23) (a) 0, ∞
(b) $-\infty$
(c) 0, e
(d) $\frac{1}{e}$, e
(e) $\frac{3}{2}$

(25) 2, 1, minimum

(27) (a) $-\infty$, -1, $\frac{1}{2}$, ∞
(b) $\frac{1}{2}$
(c) $-\infty$, -1, -1, $\frac{5}{4}$
(d) $\frac{5}{4}$

(29) (a) $-\infty$, 0, 2, ∞
(b) 2
(c) $-\infty$, 0, 0, 3

(31) -2, 3, -2, 11

CHAPTER 11

INVERSE FUNCTIONS

11.1. Background

Topics: inverse functions and their derivatives, logarithmic functions, the natural logarithm, exponential functions, trigonometric and inverse trigonometric functions, implicit differentiation.

The following two facts may be helpful in solving problem 1

11.1.1. Proposition. *Every real number is the limit of a sequence of rational numbers. That is, if a is a real number, then there are rational numbers x_1, x_2, x_3, ... such that* $\lim_{n\to\infty} x_n = a$.

11.1.2. Proposition. *If g is a continuous function and x_1, x_2, x_3, ... are real numbers such that* $\lim_{n\to\infty} x_n = a$, *then* $\lim_{n\to\infty} g(x_n) = g(a)$.

11.2. Exercises

(1) Let $f(x) = x^5 + 3x^3 + x - 10$. Then $Df^{-1}(48) = \dfrac{1}{a}$ where $a =$ ________ .

(2) Let $f(x) = \dfrac{3}{(x-1)^4}$ for $x \geq 1$. Then $f^{-1}(243) = -a\,3^p$ where $a =$ _____ and $p =$ _____ .

(3) Let $f(x) = \ln(x-2) + e^{x^2}$ for $x > 2$. Then $Df^{-1}(e^9) = (1 + ae^b)^{-1}$ where $a =$ _____ and $b =$ _____ .

(4) Let $f(x) = \ln \dfrac{1+x}{1-x}$ for $-1 < x < 1$. Then $Df^{-1}(\ln 5) = \dfrac{5}{a}$ where $a =$ ________ .

(5) Let $f(x) = \dfrac{2+x}{5-x}$. Then $f^{-1}(x) = \dfrac{ax+b}{x+1}$ where $a =$ _____ and $b =$ ______ .

(6) Let $f(x) = \exp\left(\dfrac{1}{1-x}\right)$. Then $f^{-1}(x) = 1 - \big(g(x)\big)^p$ where $g(x) =$ ________ and $p =$ _____ .

(7) Let $f(x) = \arctan(8x^3 + 2)$. Then $f^{-1}(x) = \dfrac{1}{a}(\tan x + b)^p$ where $a =$ _____ , $b =$ _____ , and $p =$ ________ .

(8) Let $f(x) = \sin^3 2x$ for $\frac{-\pi}{4} \leq x \leq \frac{\pi}{4}$. Then $Df^{-1}(\frac{1}{8}) = \dfrac{a}{b\sqrt{b}}$ where $a =$ _____ and where $b =$ _____ .

(9) Let $f(x) = \frac{4}{3}x^4 - 8x^3 + 18x^2 - 18x + \frac{27}{4}$ for $x < \frac{3}{2}$. Then $Df^{-1}(\frac{27}{4}) = -\dfrac{1}{a}$ where $a =$ ______ .

(10) Let $f(x) = x^3 + \ln(x-1)$ for $x > 1$. Then $Df^{-1}(8) = \dfrac{1}{a}$ where $a =$ ______ .

(11) Let $f(x) = \ln \dfrac{x^2+1}{x^2-1}$ for $x > 1$. Then $Df^{-1}(\ln 5 - \ln 3) = -\dfrac{a}{8}$ where $a =$ ______ .

(12) What is the area of the largest rectangle that has one corner at the origin, one corner on the negative y-axis, one corner on the positive x-axis, and one corner on the curve $y = \ln x$?

Answer: the area is ___________ .

(13) What is the area of the largest rectangle that has one corner at the origin, one corner on the negative x-axis, one corner on the positive y-axis, and one corner on the curve $y = e^x$?

Answer: the area is ___________ .

(14) Solve the equation: $1 + \log_{10}(x-4) = \log_{10}(x+5)$. Answer: $x =$ _____ .

(15) Let $f(x) = \log_3(\log_2 x)$. Then $Df(e) = \dfrac{1}{ae}$ where $a =$ ________ .

(16) Suppose p, $q > 0$ and $\log_9(p) = \log_{12}(q) = \log_{16}(p+q)$. Find $\dfrac{q}{p}$. Express your answer in a form that involves neither exponentials nor logarithms.

Answer: $\dfrac{q}{p} = \dfrac{1+a}{2}$ where $a =$ _____ .

(17) A triangle is bounded by the x-axis, the y-axis, and the tangent line to the curve $y = 2^x$ at $x = 0$. The area of this triangle is $\dfrac{1}{a \ln a}$ where $a =$ _____ .

(18) $\displaystyle\lim_{x\to 0} \frac{3^{4+x} - 3^4}{x} = a \ln b$ where $a =$ ______ and $b =$ _____ .

(19) $\lim_{t\to 0} \dfrac{\log_5(t+0.04)+2}{t} = \dfrac{a}{\ln b}$ where $a =$ ________ and $b =$ _____ .

(20) If $y = \arcsin\left(\dfrac{x^2}{3}\right)$, then $\dfrac{dy}{dx} = \dfrac{ax}{\sqrt{b-x^4}}$ where $a =$ _____ and $b =$ _____ .

(21) A tapestry 30 feet high is hung so that its lower edge is 24 feet above the eye of an observer. How far from the tapestry should the observer stand in order to maximize the visual angle subtended by the tapestry? Answer: ________ ft.

(22) Let $f(x) = \arctan\left(\dfrac{x^2}{1+x}\right)$. Then $Df(1) = \dfrac{a}{5}$ where $a =$ _____ .

(23) Let $f(x) = \arctan\left(\dfrac{1}{x}\right)$. Then $f\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{a}$ and $f'\left(\dfrac{1}{\sqrt{3}}\right) = -\dfrac{a}{b}$ where $a =$ _____ and $b =$ _____ .

(24) Let $f(x) = x^{\arcsin x}$. Then $f(1) =$ _____ and $f'(1) =$ __________ .

(25) A solution to the equation $\arcsin x - \arccos x = 0$ is $x = \dfrac{1}{a}$ where $a =$ ________ .

(26) Let $f(x) = \arctan 2x - \arctan x$ for $x \geq 0$.

  (a) The function f is increasing on the interval (_____ , ________).

  (b) The function f has a local maximum at $x =$ ________ .

  (c) The function f has a local minimum at $x =$ ________ .

(27) Let $f(x) = \ln(\arctan\sqrt{x^2-1})$. Then $f'(2) = \dfrac{\sqrt{a}}{b\pi}$ where $a =$ _____ and $b =$ _____ .

(28) The expression $e^{-\frac{3}{4}\ln 81}$ is a complicated way of writing the integer _____ .

(29) The expression $\dfrac{\ln 81}{(\ln 27)^2}\ln 3\sqrt{3}$ is a complicated way of writing the fraction $\dfrac{a}{3}$ where $a =$ _____ .

(30) The solution to the differential equation $y' = (2x-1)y$ which satisfies the initial condition $y(0) = 3$ is $y = ae^{f(x)}$ where $a =$ _____ and $f(x) =$ _____________ .

(31) The solution to the differential equation $y' = 4x^3y$ which satisfies the initial condition $y(0) = 7$ is $y = ae^{f(x)}$ where $a =$ _____ and $f(x) =$ ________ .

(32) Let $f(x) = e^{x^2+\ln x}$ for $x > 0$. Then $Df^{-1}(e) = \dfrac{1}{ae}$ where $a =$ _____ .

(33) The equation of the tangent line at the point $(1,0)$ to the curve whose equation is

$$x\sin y + x^3 = \arctan(e^y) + x - \frac{\pi}{4}$$

is $y = -ax + a$ where $a =$ _____.

11.3. Problems

(1) Show that the natural logarithm is the only continuous function f defined on the interval $(0, \infty)$ which satisfies

$$f(xy) = f(x) + f(y) \qquad \text{for all } x, y > 0$$

and

$$f(e) = 1.$$

Hint. Assume that you are given a function $f\colon (0, \infty) \to \mathbb{R}$ about which you know only three things:
(i) f is continuous;
(ii) $f(xy) = f(x) + f(y)$ for all $x, y > 0$; and
(iii) $f(e) = 1$.

What you must prove is that

$$f(x) = \ln x \qquad \text{for every } x > 0. \tag{11.1}$$

The crucial result that you will need to prove is that

$$f(u^r) = r\, f(u) \tag{11.2}$$

holds for every real number $u > 0$ and every rational number r. Once you have this, then you can use propositions 11.1.1 and 11.1.2 to conclude that

$$f(e^y) = y \qquad \text{for every real number } y.$$

Then substituting $\ln x$ for y will give you the desired result (11.1).

Prove (11.2) first for the case $r = n$ where n is a natural number. Then prove it for the case $r = 1/n$ where n is a natural number. Use these results to show that (11.2) holds for every positive rational number. Next deal with the case $r = 0$. Finally verify (11.2) for the case where r is a negative rational number. (To do this prove that $f(1/v) = -f(v)$ for all $v > 0$ by substituting v for x and $1/v$ for y in (ii).)

(2) Prove that

$$\arctan x + \arctan y = \arctan \frac{x + y}{1 - xy}$$

whenever $xy \neq 1$. *Hint.* Let y be an arbitrary, but fixed, real number. Define $f(x) = \arctan x + \arctan y$ and $g(x) = \arctan \dfrac{x + y}{1 - xy}$. Compare the derivatives of f and g.

(3) Prove that $\arctan x$ and $\arctan \dfrac{1 + x}{1 - x}$ differ by constants on the intervals $(-\infty, 1)$ and $(1, \infty)$. Find the appropriate constants. Show how to use this information to find $\lim\limits_{x \to 1^-} \arctan \dfrac{1 + x}{1 - x}$ and $\lim\limits_{x \to 1^+} \arctan \dfrac{1 + x}{1 - x}$.

(4) Give a careful proof that

$$\frac{x}{x^2 + 1} \leq \arctan x \leq x$$

for all $x \geq 0$.

(5) Define $f(x) = (x^2)^x$ for all $x \neq 0$. Define $f(0)$ in such a way as to make f a continuous function on $\mathbb{R}$. Sketch the function f. Locate all critical points and identify the intervals on which f is increasing, is decreasing, is concave up, and is concave down. Take special care to describe what happens at $x = 0$. Use Newton's method to find to 4 decimal place accuracy any points of inflection which may occur.

(6) Let $f(x) = 2x + \cos x + \sin^2 x$ for $-10 \le x \le 10$. Show that f has an inverse.

(7) Let $f(x) = \dfrac{4x+3}{x+2}$.
   (a) Show that f is one-to-one.
   (b) Find $f^{-1}(-2)$.
   (c) Find dom f^{-1}.

(8) Let $f(x) = e^{3x} + \ln x$ for $x > 0$. Prove that f has an inverse and calculate $Df^{-1}(e^3)$.

(9) Let $f(x) = \ln(1+x) - \ln(1-x)$ for $-1 < x < 1$. Prove that f has an inverse and find $f^{-1}(x)$.

(10) Show that there is exactly one number x such that $e^{-x} = x^3 - 9$. Locate the number between consecutive integers.

(11) Show that there is exactly one number x such that $e^{2x} = 10 - x^3$. Locate the number between consecutive integers.

(12) Show that there is exactly one number x such that $\ln x + x = 0$.

(13) Use the *mean value theorem* to show that $x + 1 < e^x < 2x + 1$ whenever $0 < x \le \ln 2$.

(14) (a) Find $\displaystyle\lim_{x\to 1} \frac{1}{x-1}$.
   (b) Find $\displaystyle\lim_{x\to 1} \frac{\ln x}{(x-1)^2}$.
   (c) Find $\displaystyle\lim_{x\to 1} \left(\frac{1}{x-1} - \frac{\ln x}{(x-1)^2}\right)$.

(15) Suppose that $f\colon (0,\infty) \to \mathbb{R}$ is a continuous function on $(0,\infty)$ such that $f(x) = \dfrac{x \ln x}{x-1}$ for every $x > 0$ except $x = 1$. Prove that $f''(x)$ (exists and) is continuous at $x = 1$.

(16) Let $0 < a < b$. Use the *mean value theorem* to show that

$$1 - \frac{a}{b} < \ln\frac{b}{a} < \frac{b}{a} - 1.$$

11.4. Answers to Odd-Numbered Exercises

(1) 117

(3) 6, 9

(5) 5, -2

(7) 2, -2, $\frac{1}{3}$

(9) 18

(11) 15

(13) $\frac{1}{e}$

(15) $\ln 3$

(17) 2

(19) 25, 5

(21) 36

(23) 3, 4

(25) $\sqrt{2}$

(27) 3, 2

(29) 2

(31) 7, x^4

(33) 4

CHAPTER 12

APPLICATIONS OF THE DERIVATIVE

12.1. Background

Topics: antiderivatives, related rates, optimization, Newton's method.

This chapter makes no pretense of presenting interesting "real-world" applications of the differential calculus. Its purpose is simply to make some elementary connections between the mathematical concept of *derivative* and various instances of *rates of change* of physical quantities.

Newton's Law of Cooling: the rate of cooling of a hot body is proportional to the difference between its temperature and that of the surrounding medium.

12.2. Exercises

(1) One leg of a right triangle decreases at 1 in./min. and the other leg increases at 2 in./min. At what rate is the area changing when the first leg is 8 inches and the second leg is 6 inches? Answer: _____ in^2/min.

(2) The volume of a sphere is increasing at the rate of 3 cubic feet per minute. At what rate is the radius increasing when the radius is 8 feet? Answer: $\dfrac{a}{b\pi}$ ft/minẇhere $a =$ _____ and $b =$ ___________ .

(3) A beacon on a lighthouse 1 mile from shore revolves at the rate of 10π radians per minute. Assuming that the shoreline is straight, calculate the speed at which the spotlight is sweeping across the shoreline as it lights up the sand 2 miles from the lighthouse. Answer: _________ miles/min.

(4) Two boats are moving with constant speed toward a marker, boat A sailing from the south at 8 mph and boat B approaching from the east. When equidistant from the marker the boats are $4\sqrt{2}$ miles apart and the distance between them is decreasing by $7\sqrt{2}$ mph. How fast is boat B going? Answer: _____ mph.

(5) A (right circular) cylinder is expanding in such a way that its height is increasing three times as rapidly as the radius of its base. At the moment when its height is 5 inches and the radius of its base is 3 inches its height is increasing at a rate of 12 inches per minute. At that moment its volume is increasing at a rate of ___________ cubic inches per minute.

(6) A cube is expanding in such a way that its edge is increasing at a rate of 4 inches per second. When its edge is 5 inches long, what is the rate of change of its volume? Answer: ________ in^3/sec.

(7) A kite 100 feet above the ground is being blown away from the person holding its string in a direction parallel to the ground and at a rate of 10 feet per second. At what rate must the string be let out when the length of string already let out is 200 feet? Answer: ________ ft/sec.

(8) A plane flying 4000 feet above the ground at a speed of 16,000 feet per minute is followed by a searchlight. It is flying in a straight line and passes directly over the light. When the angle between the beam and the ground is $\pi/3$ radians, what is the angular velocity of the beam? Answer: _____ radians/min.

(9) A lighthouse is 3 miles from (a straight) shore. The light makes 4 revolutions per minute. How fast does the light move along the shoreline when it makes an angle of $\pi/4$ radians with the shoreline? Answer: _____________ mi/min.

(10) Water leaking onto a floor creates a circular pool with an area that increases at the rate of 3 square inches per minute. How fast is the radius of the pool increasing when the radius is 10 inches? Answer: $\dfrac{a}{b\pi}$ in/minẇhere $a =$ _____ and $b =$ ________ .

(11) A cube is expanding in such a way that the length of its diagonal is increasing at a rate of 5 inches per second. When its edge is 4 inches long, the rate at which its volume is increasing is ___________ in^3/sec.

(12) You are standing on a road, which intersects a railroad track at right angles, one quarter of a mile from the intersection. You observe that the distance between you and the approaching train is decreasing at a constant rate of 25 miles per hour. How far from the intersection is the train when its speed is 40 miles per hour? Answer: $\dfrac{5}{4\sqrt{a}}$ mi where $a =$ _________.

(13) A light shines on top of a lamppost 30 feet above the ground. A woman 5 feet tall walks away from the light. Find the rate at which her shadow is increasing if she is walking at 3 ft./sec. Answer: ______________ ft/sec.

(14) A balloon is going up, starting at a point on the ground. An observer 300 feet away looks at the balloon. The angle θ which a line to the balloon makes with the horizontal is observed to increase at $\frac{1}{10}$ rad./sec. How rapidly is the balloon rising when $\theta = \pi/6$? Answer: _________ ft/sec.

(15) A man is walking along a sidewalk at 6 ft./sec. A searchlight on the ground 24 feet from the walk is kept trained on him. At what rate is the searchlight revolving when the man is 18 feet from the point on the walk nearest the light? Answer: ____________ rad/sec.

(16) A 20 foot long ramp has one end on the ground and the other end at a loading dock 5 feet off the ground. A person is pushing a box up the ramp at the rate of 3 feet per second. How fast is the box rising? Answer: _________ ft/sec.

(17) What is the area of the largest rectangle (with sides parallel to the coordinate axes) which lies above the x-axis and below the parabola $y = 48 - x^2$? Answer: Area is _________ .

(18) A piece of cardboard is to be made into an open box by cutting out the corners and folding up the sides. Given a piece of cardboard 12 in. $\times$ 12 in. what size should the corner notches be so that the resulting box has maximum volume? Answer: they should be squares _____ inches on each side.

(19) Express 20 as the sum of two positive numbers x and y such that $x^3 + y^2$ is as small as possible. Answer: $x =$ ________ and $y =$ ________.

(20) The combined resistance R of two resistors R_1 and R_2 is given by $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$ (where $R_1,\ R_2 > 0$). Suppose $R_1 + R_2$ is a constant. How does one obtain maximum combined resistance? Answer: ________________ .

(21) Find the point on the curve $y^2 = \frac{5}{2}(x+1)$ which is nearest the origin. Answer: (___ , ___).

(22) Find the point on the curve $y = x^2$ which is closest to the point $(3, 0)$. Answer: (___ , ___).

(23) Find the lengths of the sides of the rectangle of largest area which can be inscribed in a semicircle of radius 8. (The lower base of the rectangle lies along the diameter of the semicircle.) Answer: the sides should have lengths ________ and ________ .

(24) Consider triangles in the first quadrant bounded by the x-axis, the y-axis, and a tangent line to the curve $y = e^{-x}$. The largest possible area for such a triangle is _________ .

(25) An open cylindrical tank of volume 192π cubic feet is to be constructed. If the material for the sides costs \$3 per square foot, and the material for the bottom costs \$9 per square foot, find the radius and height of the tank which will be most economical.

Answer: radius = _____ ft; height = ______ ft.

(26) Find the dimensions of the cylinder with the greatest volume which can be inscribed in a sphere of radius 1.

Answer: radius = ________ ; height = ________ .

(27) A farmer has 100 pigs each weighing 300 pounds. It costs \$.50 a day to keep one pig. The pigs gain weight at 10 pounds a day. They sell today for \$.75 a pound, but the price is falling by \$.01 a day. How many days should the farmer wait to sell his pigs in order to maximize his profit? Answer: ________ days.

(28) Consider a parallelogram inscribed in a triangle ABC in such a way that one vertex coincides with A while the others fall one on each side of the triangle. The maximum

possible area for such a parallelogram is what fraction of the area of the original triangle? *Hint.* Orient the triangle so that its vertices are $A = (0, 0)$, $B = (a, 0)$, and $C = (b, c)$. Let $(t, 0)$ be the vertex of the parallelogram lying on AB and (x, y) be the vertex lying on BC. Use t as the independent variable. Find x and y in terms of t (and the constants a, b, and c). Answer: $\frac{r}{s}$ where $r =$ _____ and $s =$ _____ .

(29) Two men carry a $14\sqrt{7}$ ft. ladder down a $10\sqrt{5}$ ft. wide corridor. They turn into a second corridor, perpendicular to the first one, while keeping the ladder horizontal. Find the minimum possible width of the second corridor. Answer: ________ feet.

(30) At each point $a > 0$ the tangent line to the parabola $y = 1-x^2$ and the positive coordinate axes form a triangle. The minimum possible area of such a triangle is $\dfrac{a}{b\sqrt{b}}$ where $a =$ _____ and $b =$ _____ .

(31) A window is in the shape of a rectangle surmounted by a semicircle. If the perimeter is to be 18 feet, find the dimensions which maximize the area.
Answer: the radius of the semicircle should be $\dfrac{a}{4+\pi}$ ft and the height of the rectangle should be $\dfrac{b}{4+\pi}$ ft where $a =$ ________ and $b =$ ________.

(32) What is the distance from the point $(8, 4)$ to the tangent line to the curve $f(x) = 3x^2-4x+6$ at $x = 1$? Answer: ________ .

(33) What are the dimensions of a rectangular box—with no top—of greatest volume that can be constructed from 120 sq. in. of material if the base of the box is to be twice as long as it is wide? Answer: width of base $= 2\sqrt{a}$ and height of box $= \frac{4}{b}\sqrt{a}$ where $a =$ _____ in. and $b =$ _____ in.

(34) Consider all rectangles which have two sides on the positive coordinate axes and which lie under the curve $y = 2\cos x$. The one with the largest perimeter has width ______ and height ______ .

(35) Consider all rectangles which have one side on the positive x-axis and which lie under the curve $y = 4\sin x$ with $0 \leq x \leq \pi$. The one with the largest perimeter has width ______ and height ______ .

(36) Suppose that $f(-1) = -6$ and that $f'(x) = 6x^2 - 2x + 7$ for all real numbers x. Then $f(1) =$ ________ .

(37) Suppose that $f''(x) = 18x-14$, that $f'(-1) = 8$, and that $f(-1) = 9$. Then $f(1) =$ ________ .

(38) Suppose $f''(x) = 12x - 10$, $f(2) = -6$, and $f(-1) = -18$. Then $f(1) =$ ________ .

(39) Suppose that $f'''(x) = 6x + 6$, $f(0) = -7$, $f(1) = \frac{1}{4}$, and $f(2) = 19$. Then $f(x) = ax^4 + x^3 + bx^2 + cx + d$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $d =$ _____ .

(40) A pan of warm water (109°F) was put in a refrigerator. Fifteen minutes later, the water's temperature was 97° F; fifteen minutes after that, it was 87°F. Using *Newton's law of cooling* we can conclude that the temperature of the refrigerator was ________ ^{o} F.

(41) An object is heated to 838° and then allowed to cool in air that is 70°. Suppose that it takes 2 hours to cool the object to 313°. Then it takes ________ minutes to cool the object to 646°. *Hint.* Use *Newton's law of cooling.*

(42) A quantity y varies with time. The rate of increase of y is proportional to $\cos^2 y$. The initial value of y is $\pi/6$, while its value at $t = 1$ is $\pi/3$.
(a) For what value of t does $y = \pi/4$? Answer: $t =$ ____________ .

(b) What is the long-run value of y? Answer: $\lim_{t \longrightarrow \infty} y(t) =$ ______ .

(43) A point is moving along the x-axis in such a way that its acceleration at each time t is $\frac{3}{4}\pi^2 \sin \frac{\pi}{2}t$. Initially the point is located 4 units to the left of the origin. One second later it is at the origin. Where is it at time $t = 5$?

Answer: ________ units to the ___________ of the origin.

(44) A cylindrical water tank standing on end has diameter 9 ft and height 16 ft. The tank is emptied through a valve at the bottom of the tank. The rate at which the water level decreases when the valve is open is proportional to the square root of the depth of the water in the tank. Initially the tank is full of water. Three minutes after the valve is opened the tank is only 1/4 full. How long does it take from the time the valve is opened to empty the tank? Answer: _____ minutes.

(45) A function f satisfies the following conditions:
(i) $f''(x) = 6x - 12$ for all x, and
(ii) the graph of the curve $y = f(x)$ passes through the point $(2, 5)$ and has a horizontal tangent at that point.
Then $f(x) = x^3 + ax^2 + bx + c$ where $a =$ _____ , $b =$ _____ , and $c =$ _____ .

(46) A physical quantity y, which takes on only positive values, varies with time t. It is known that the rate of change of y is proportional to $y^3(t+1)^{-1/2}$, that initially $y = 1/3$, and that after 8 minutes $y = 1/5$.
(a) What is the value of y after 35 minutes? Answer: $y(35) =$ ________ .
(b) Approximately how many hours must one wait for y to become less than 1/15? Answer: ________ hours.

(47) The solution to the differential equation $\dfrac{dy}{dx} = 3x^{1/3}$ subject to the condition $y = 25$ when $x = 8$ is $y(x) = \frac{a}{4}x^{p/3} + b$ where $a =$ _____ , $p =$ _____ , and $b =$ _____ .

(48) The solution to the differential equation $\dfrac{d^2y}{dx^2} = \dfrac{6}{x^4}$ which satisfies the conditions $\dfrac{dy}{dx} = 3$ and $y = 2$ when $x = 1$ is $y(x) = ax^p + bx + c$ where $a =$ _____ , $p =$ _____ , $b =$ _____ , and $c =$ _____ .

(49) The solution to the differential equation $y'(x) = \sin x e^{\cos x}$ which satisfies the condition $y = 2$ when $x = \dfrac{\pi}{2}$ is $y(x) =$ ___________________ .

(50) The decay equation for (radioactive) radon gas is $y = y_0 e^{-0.18\,t}$ with t in days. About how long will it take the radon in a sealed sample of air to fall to 80% of its original value? (Give an approximate answer to two decimal places.)

Answer: __.2__ days.

(51) If the half-life of carbon 14 is approximately 5730 years, how old is a wooden axe handle that is found to contain only $\dfrac{1}{2\sqrt{2}}$ times the atmospheric proportion of carbon 14? Answer: ______________ years.

(52) The half-life of a radioactive substance is 10 years. If we start with 20 grams of this substance, then the amount remaining after 5 years is $a\sqrt{b}$ where $a =$ _____ and $b =$ _____ .

(53) If we assume exponential growth, what was the population of a city in 1930 if its population in 1940 was 750,000 and in 1970 was 1,296,000? Answer: _____________ .

(54) In 1920 the population of a city was 135,000 and in 1950 it was 320,000. Assuming exponential growth, the population in 1940 was approximately ________________ .

(55) An electric condenser discharges through a resistance, losing voltage at a rate proportional to the voltage remaining. If the initial voltage of 100 volts decreases to 50 volts in 3 seconds, then the function representing the voltage on the condenser at any time t is $100a^{f(t)}$ where $a =$ ______ and $f(t) =$ ________ .

(56) Find a function ϕ such that $y(x) = \sin\dfrac{1}{x}$ is a solution to the differential equation

$$\big(\phi(x)\,y'(x)\big)' + \frac{y(x)}{x^2} = 0\,.$$

Answer: $\phi(x) =$ _________ will work.

(57) Use *Newton's method* to find the first three estimates to $\sqrt{5}$ starting at $x = 2$.

Answer: $x_1 = \dfrac{a}{4}$ where $a =$ _____.

$x_2 = \dfrac{a}{72}$ where $a =$ __________.

$x_3 = \dfrac{a}{23184}$ where $a =$ ________________.

(58) Use *Newton's method* to find the first four estimates to $\sqrt{3}$ starting at $x = 1$.

Answer: $x_1 =$ _____.

$x_2 = \dfrac{a}{4}$ where $a =$ __________.

$x_3 = \dfrac{a}{56}$ where $a =$ ________________.

$x_4 = \dfrac{a}{10864}$ where $a =$ ________________.

(59) Find the first 6 approximations given by *Newton's method* to the root of the polynomial $x^3 - x - 1$ starting with $x_0 = 1$. Carry out your answers to 9 decimal places.

Answer: $x_0 = 1.000\ 000\ 000$; $x_1 = 1.$__ __0 000 000;

$x_2 = 1.347\ 8$__6 0__7; $x_3 = 1.325\ 2$_0 3__9;

$x_4 = 1.324\ 71$__ 1_4; $x_5 = 1.324\ 717\ 9$__7.

(60) Use *Newton's method* to find six successive approximations to each root of the polynomial $x^4 - 2x^3 - x^2 - 2x + 2$. Carry out your work to nine decimal places. In each case use the starting value of the form n or $n.5$ (where n is an integer) which is closest to the root your are trying to approximate.

Answer: For the first root: $x_0 = 0.$ __00 000 000; $x_1 = 0.640$ __25 __00;

$x_2 = 0.630\ 1$__1 5__1; $x_3 = 0.630\ 115\ 3$__8;

$x_4 = 0.630\ 115\ 39$__; $x_5 = 0.630\ 115\ 39$__.

For the second root: $x_0 =$ __ . __00 000 000; $x_1 = 2.57$__ 86__ 111;

$x_2 = 2.573$ __1__ 023; $x_3 = 2.573\ 271\ 9$__5;

$x_4 = 2.573\ 271\ 96$__; $x_5 = 2.573\ 271\ 96$__.

(61) A ball is thrown upward from the edge of the roof of a building 176 feet high with an initial velocity of 56 ft/sec. (Assume that the acceleration due to gravity is 32 ft/sec^2.)

(a) How high does the ball go? Answer: __________ ft.

(b) When does it reach the ground? Answer: after ________ sec.

(62) A ball is thrown upward from the edge of the roof of a building with a velocity of 40 ft./sec. The ball hits the ground at 120 ft./sec. (Assume that the acceleration due to gravity is 32 ft/sec^2.)

(a) How long does it take the ball to reach the ground? Answer: _____ sec.

(b) How tall is the building? Answer: ________ ft.

(c) What is the maximum height reached by the ball? Answer: ________ ft.

(63) A ball is thrown upward from the edge of the roof of a building at 72 ft./sec. It hits the ground 10 seconds later. (Use 32 ft./sec.^{2} as the magnitude of the acceleration due to gravity.)

(a) How tall is the building? Answer: 8_ _ ft.

(b) What is the maximum height reached by the ball? Answer: _ 6 _ ft.

(64) A ball is thrown upward from the edge of the roof of a building 160 feet tall at a velocity of 48 ft./sec. At what velocity does the ball hit the ground? (Use 32 ft./sec.^{2} as the magnitude of the acceleration due to gravity.) Answer: _________ ft/sec.

(65) A falling stone is observed to be at a height of 171 feet. Two seconds later it is observed to be at a height of 75 feet. From what height was it dropped? (Use 32 ft./sec.^{2} as the magnitude of the acceleration due to gravity.) Answer: _________ ft.

(66) A falling stone is observed to be at a height of 154 feet. Two seconds later it is observed to be at a height of 14 feet. If the stone was initially thrown upwards with a speed of 10 ft./sec., from what height was it thrown? (Use 32 ft./sec.^{2} as the magnitude of the acceleration due to gravity.) Answer: _________ ft.

(67) Two seconds after being thrown upward an object is rising at 176 ft./sec. How far does it travel before returning to the position from which it was thrown? Answer: ___________ ft.

(68) A predator-prey system is modeled by the equations

$$\frac{dx}{dt} = 4x - 5y\sqrt{x}$$
$$\frac{dy}{dt} = 7y\sqrt{x}$$

where the variable y represents the predator population while the variable x represents the prey population. Explain briefly how we know that the predator must have an alternate source of food.

Answer: ____ .

12.3. Problems

(1) A piston P moves within a cylinder. A connecting rod of length 7 inches connects the piston with a point Q on a crankshaft, which is constrained to move in a circle with center C and radius 2 inches. Assuming that the angular velocity of Q is 5π radians per second, find the speed of the piston at the moment when the line segment CQ makes an angle of $\pi/4$ radians with the horizontal.

(2) Part of the northern boundary of a body of water is a straight shoreline running east and west. A lighthouse with a beacon rotating at a constant angular velocity is situated 600 yards offshore. An observer in a boat 200 yards east of the lighthouse watches the light from the beacon move along the shore. At the moment t_1 when the observer is looking directly northeast the angular velocity of his line of sight is 2.5 radians per second.
   (a) How many revolutions per minute does the beacon make?
   (b) How fast (in miles per hour) is the light moving along the shore at time t_1?
   (c) Although the beacon rotates with constant angular velocity, the observer's line of sight does not. Locate the points on the shoreline where the angular velocity of the line of sight is greatest and where it is least. What is the limiting angular velocity of the line of sight as the light disappears down the shoreline?

(3) A wire 24 inches long is cut in two parts. One part is bent into the shape of a circle and the other into the shape of a square. How should it be cut if the sum of the areas of the circle and the square is to be (a) minimum, (b) maximum?

12.3.1. Theorem. *Let f be a function such that $f(x) \geq 0$ for every x in its domain. Then f has a local maximum at a point a if and only if the function f^2 has a local maximum there. Similarly, f has a local minimum at a if and only if f^2 does.*

(4) (a) Prove the preceding theorem.
   (b) Suppose that $0 < k < l$. Let $f(x) = |k \cos x - l \sin x|$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Without using the theorem above find all local maxima and minima of f.
   (c) Let f be as in (b). Use the theorem above to find all local maxima and minima of f.
   (d) Show (if you have not already done so) that the answers you got in parts (b) and (c) are in agreement.

(5) When a sector is removed from a thin circular disk of metal, the portion of the disk which remains can be formed into a cone. Explain how the sector should be chosen so that the resulting cone has the greatest capacity.

(6) Your good friend George, who is working for the Acme Widget Corporation, has a problem. He knows that you are studying calculus and writes a letter asking for your help. His problem concerns solutions to a system of two differential equations:

$$\begin{cases} \dfrac{dx}{dt} = x(t) \\ \dfrac{dy}{dt} = x(t) + y(t) \end{cases} \tag{1}$$

subject to the initial conditions

$$x(0) = a \qquad \text{and} \qquad y(0) = b, \tag{2}$$

where a and b are arbitrary constants. He has already found one set of solutions:

$$\begin{cases} x(t) = ae^t \\ y(t) = (b + at)e^t \end{cases}$$

What Fred is unable to discover is whether or not there are other solutions. Write a letter to Fred helping him out.

Hint. Suppose

$$\begin{cases} x(t) = u(t) \\ y(t) = v(t) \end{cases}$$

is a solution to the system (1) which satisfies the initial conditions (2). Consider the functions $p(t) = e^{-t}u(t)$ and $q(t) = e^{-t}v(t)$.

(7) Use *Newton's method* to approximate the solutions to the equation

$$\sin x = x^2 - x + 0.5$$

to eight decimal places. Use starting approximations of 0.3 and 1.3.

Explain carefully how we know that there are exactly two solutions. Explain how one might reasonably have chosen the numbers 0.3 and 1.3 as initial approximations. Discuss fully the problem of deciding when to stop.

(8) Explain carefully and fully how to use *Newton's method* to find the first point of intersection of the curves $y = \sin x$ and $y = e^{-x}$. Give your answer correct to 8 decimal places.

(9) Suppose we are given $a > 0$. Explain why it is that if x_1 is arbitrary and for each $n \in \mathbb{N}$ we let $x_{n+1} = \frac{1}{2}(x_n + a{x_n}^{-1})$, then (x_n) converges to the square root of a. *Hint.* Use *Newton's method*. Use this sequence to compute the square root of 10^7 to ten decimal places.

(10) Explain carefully and fully how to use *Newton's method* to find, correct to eight decimal places, an approximate value for the reciprocal of 2.74369.

(11) A chord subtends an arc of a circle. The length of the chord is 4 inches; the length of the arc is 5 inches. Find the central angle θ of the circle subtended by the chord (and the arc). The *law of cosines* yields an equation involving the angle θ. Explain carefully and fully how to use *Newton's method* to solve the equation (in radians) to four decimal places.

(12) Explain carefully and fully how to use *Newton's method* to find, correct to six decimal places, the slope of the tangent line to the curve $y = -\sin x$ $(\pi/2 \le x \le 3\pi/2)$ which passes through the origin.

12.4. Answers to Odd-Numbered Exercises

(1) 5

(3) 40π

(5) 228π

(7) $5\sqrt{3}$

(9) 48π

(11) $80\sqrt{3}$

(13) $\dfrac{3}{5}$

(15) $\dfrac{4}{25}$

(17) 256

(19) $\dfrac{10}{3}$, $\dfrac{50}{3}$

(21) -1, 0

(23) $4\sqrt{2}$, $8\sqrt{2}$

(25) 4, 12

(27) 20

(29) $4\sqrt{2}$

(31) 18, 18

(33) 5, 3

(35) $\dfrac{\pi}{3}$, $2\sqrt{3}$

(37) -15

(39) $\dfrac{1}{4}$, 1, 5, -7

(41) 30

(43) 28, right

(45) -6, 12, -3

(47) 9, 4, -11

(49) $3 - \exp(\cos x)$

(51) 8595

(53) $625,000$

(55) $\dfrac{1}{2}$, $\dfrac{t}{3}$

(57) 9, 161, 51841

(59) 5, 0, 2, 8, 0, 9, 8, 7, 5

(61)
   (a) 225
   (b) $\dfrac{11}{2}$

(63)
   (a) 8, 0

(b) 9, 1

(65) 175

(67) 1800

Part 4

INTEGRATION OF FUNCTIONS OF A SINGLE VARIABLE

CHAPTER 13

THE RIEMANN INTEGRAL

13.1. Background

Topics: summation notation, Riemann sums, Riemann integral, upper and lower Darboux sums, definite and indefinite integrals.

Here are two formulas which may prove helpful.

13.1.1. Proposition. *For every natural number n*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}\,.$$

13.1.2. Proposition. *For every natural number n*

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}\,.$$

13.1.3. Definition. Let $J = [a, b]$ be a fixed interval in the real line and $P = (x_0, x_1, \dots, x_n)$ be $n+1$ points of J. Then P is a PARTITION of the interval J if:

(1) $x_0 = a$,
(2) $x_n = b$, and
(3) $x_{k-1} < x_k$ for $k = 1, 2, \dots, n$.

We denote the length of the k^{th} subinterval by Δx_k; that is, $\Delta x_k = x_k - x_{k-1}$. A partition $P = (x_0, x_1, \dots, x_n)$ is REGULAR if all the subintervals $[x_{k-1}, x_k]$ have the same length. In this case

$$\Delta x_1 = \Delta x_2 = \cdots = \Delta x_n$$

and we write Δx for their common value.

13.1.4. Notation. Let f be a bounded function defined on the interval $[a, b]$ and $P = (x_0, x_1, \dots, x_n)$ be a partition of $[a, b]$. Then we define

$$\mathrm{R}(P) := \sum_{k=1}^{n} f(x_k)\,\Delta x_k \tag{13.1}$$

$$\mathrm{L}(P) := \sum_{k=1}^{n} f(x_{k-1})\,\Delta x_k \tag{13.2}$$

$$\mathrm{M}(P) := \sum_{k=1}^{n} f\big((\tfrac{1}{2}(x_{k-1} + x_k)\big)\,\Delta x_k. \tag{13.3}$$

These are, respectively, the RIGHT, LEFT, and MIDPOINT SUMS OF f ASSOCIATED WITH THE PARTITION P. If P is a *regular* partition of $[a, b]$ consisting of n subintervals, then we may write R_n for $R(P)$, L_n for $L(P)$, and M_n for $M(P)$.

13.1.5. Definition. The AVERAGE VALUE of a function f over an interval $[a, b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f$.

13.2. Exercises

(1) $\displaystyle\sum_{k=1}^{5} k^2 =$ __________ .

(2) $\displaystyle\sum_{k=3}^{10} 4 =$ ________ .

(3) $\displaystyle\sum_{k=1}^{100} k(k-3) =$ __________________ .

(4) $\displaystyle\sum_{m=1}^{200} m^3 - \sum_{m=1}^{199} m^3 =$ ________________ .

(5) $\displaystyle\sum_{k=1}^{4} (-1)^k k^k =$ _____________ .

(6) Let $a_k = 2^k$ for each k. Then $\displaystyle\sum_{k=3}^{8} (a_k - a_{k-1}) =$ _____________ .

(7) $\displaystyle\sum_{k=3}^{50} \frac{1}{k^2-k} = \frac{12}{a}$ where $a =$ ________ . *Hint.* Find numbers p and q such that $\dfrac{1}{k^2-k} = \dfrac{p}{k-1} - \dfrac{q}{k}$.

(8) Express $1 + \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \dfrac{1}{81} + \dfrac{1}{243}$ in summation notation. Answer: $\displaystyle\sum_{k=0}^{a} b^k$ where $a =$ ___________ and $b =$ ________ .

(9) $\displaystyle\sum_{k=1}^{4} (k-1)\,k\,(k+1) =$ ________ .

(10) $\displaystyle\sum_{k=0}^{5} 3^{k+4} = \sum_{j=a}^{b} 3^j$ where $a =$ ________ and $b =$ ________ .

(11) $\displaystyle\sum_{k=7}^{60} \frac{1}{3^{k-2}} = \sum_{j=-2}^{a} \frac{1}{3^{j+b}}$ where $a =$ ________ and $b =$ ________ .

(12) $\displaystyle\sum_{j=-3}^{70} \frac{1}{5^{j-7}} = \sum_{i=a}^{61} \frac{1}{5^{i+b}}$ where $a =$ ________ and $b =$ ________ .

(13) $\displaystyle\sum_{j=-4}^{18} \frac{1}{2^{j+3}} = \sum_{k=a}^{7} 2^{k-b}$ where $a =$ ________ and $b =$ ______ .

(14) $\displaystyle\sum_{k=4}^{60} \frac{1}{k^2-1} = \frac{a}{3660}$ where $a =$ ___________ . *Hint.* Write $\dfrac{1}{k^2-1}$ as the difference of two simpler fractions.

(15) $\displaystyle\sum_{k=2}^{34} \frac{1}{k^2+2k} = \frac{a}{2520}$ where $a =$ ________ . *Hint.* Write $\dfrac{1}{k^2+2k}$ as the difference of two simpler fractions.

(16) Let $f(x) = x^2$ on the interval $[0,4]$ and let $P = (0,1,2,4)$. Find the right,,left, and midpoint sums of f associated with the partition P.

Answer: $\mathrm{R}(P) =$ ________ ; $\mathrm{L}(P) =$ ______ ; and $\mathrm{M}(P) = \dfrac{a}{2}$ where $a =$ ________ .

(17) Let $f(x) = x^3 - x$ on the interval $[-2,3]$ and let $P = (-2,0,1,3)$. Find the right, left, and midpoint sums of f associated with the partition P.

Answer: $\mathrm{R}(P) =$ ________ ; $\mathrm{L}(P) =$ ________ ; and $\mathrm{M}(P) = \dfrac{a}{8}$ where $a =$ ________ .

(18) Let $f(x) = 3 - x$ on the interval $[0,2]$ and let P_n be the regular partition of $[0,2]$ into n subintervals. Then

(a) $\mathrm{R}_n = a + \dfrac{b}{n}$ where $a =$ _____ and $b =$ _____ .

(b) $\mathrm{L}_n = c + \dfrac{d}{n}$ where $c =$ _____ and $d =$ _____ .

(c) $\int_0^2 f =$ _____ .

(19) Let $f(x) = 2x - 3$ on the interval $[0,4]$ and let P_n be the regular partition of $[0,4]$ into n subintervals. Then

(a) $\mathrm{R}_n = a + \dfrac{b}{n}$ where $a =$ _____ and $b =$ _____ .

(b) $\mathrm{L}_n = c + \dfrac{d}{n}$ where $c =$ _____ and $d =$ _____ .

(c) $\int_0^4 f =$ _____ .

(20) Let $f(x) = x - 2$ on the interval $[1,7]$ and let P_n be the regular partition of $[1,7]$ into n subintervals. Then

(a) $\mathrm{R}_n = a + \dfrac{b}{n}$ where $a =$ _____ and $b =$ _____ .

(b) $\mathrm{L}_n = c + \dfrac{d}{n}$ where $c =$ _____ and $d =$ _____ .

(c) $\int_1^7 f =$ _____ .

(21) If $\displaystyle\int_1^e \ln x\,dx = 1$ and $\displaystyle\int_1^{e^2} \ln x\,dx = 1 + e^2$, then $\displaystyle\int_e^{e^2} \ln x\,dx =$ ________ .

(22) Suppose that $\displaystyle\int_{-10}^{17} f = 3$, $\displaystyle\int_{-7}^{8} f = 7$, $\displaystyle\int_{-3}^{1} f = -1$, $\displaystyle\int_{-3}^{8} f = 4$, $\displaystyle\int_{-1}^{2} f = 5$, $\displaystyle\int_{-1}^{17} f = 6$, and $\displaystyle\int_1^2 f = 1$. Then $\displaystyle\int_{-10}^{-7} f =$ ________ .

(23) For what value of x is $\displaystyle\int_4^{\sqrt{x}} f(t)\,dt$ sure to be 0? Answer: ________ .

(24) Suppose $\displaystyle\int_{-2}^{3} f(x)\,dx = 8$. Then $\displaystyle\int_{3}^{-2} f(\Xi)\,d\Xi =$ ________ .

(25) Find the value of the integral $\int_{-3}^{3} \sqrt{9-x^2}\,dx$ by regarding it as the area under the graph of an appropriately chosen function and using an area formula from plane geometry. Answer: ________ .

(26) Find the value of the integral $\int_{-2}^{2} (4-|x|)\,dx$ by regarding it as the area under the graph of an appropriately chosen function and using area formulas from plane geometry. Answer: ________ .

(27) Let $a>0$. Then $\int_0^a (\sqrt{a^2-x^2} - a + x)\,dx = \frac{1}{b}a^p(c-2)$ where $b =$ _____ , $p =$ _____ , and $c =$ _____ . *Hint.* Interpret the integral as an area.

(28) If the average value of a continuous function f over the interval $[0,2]$ is 3 and the average value of f over $[2,7]$ is 4, then the average value of f over $[0,7]$ is $\frac{a}{7}$ where $a =$ _____ .

(29) Let $f(x) = |2-|x-3||$. Then $\int_0^8 f(x)\,dx =$ _____ .

(30) Let $f(x) = \begin{cases} 2+\sqrt{2x-x^2}, & \text{for } 0 \le x \le 2 \\ 4-x, & \text{for } x > 2 \end{cases}$. Then

$\int_0^2 f(x)\,dx = a + \frac{\pi}{b}$ where $a =$ _____ and $b =$ _____ ;
$\int_0^4 f(x)\,dx = c + \frac{\pi}{d}$ where $c =$ _____ and $d =$ _____ ; and
$\int_1^6 f(x)\,dx = p + \frac{\pi}{q}$ where $p =$ _____ and $q =$ _____ .

(31) Suppose $\int_0^3 f(x)\,dx = 4$, $\int_2^5 f(x)\,dx = 5$, and $\int_2^3 f(x)\,dx = -1$. Then $\int_0^2 f(x)\,dx =$ ______ , $\int_0^1 f(x+2)\,dx =$ ______ , $\int_0^2 \big(f(x)+2\big)\,dx =$ ______ , $\int_2^5 f(x-2)\,dx =$ ______ , $\int_0^5 f(x)\,dx =$ ______ , $\int_5^7 5f(x-2)\,dx =$ ______ , and $\int_{-2}^3 f(x+2)\,dx =$ ______ .

(32) $\int_{-1}^4 (|x| + |x-2|)\,dx =$ ________ .

(33) $\int_0^3 (|x-1| + |x-2|)\,dx =$ _____ .

13.3. Problems

(1) Prove proposition 13.1.1.

(2) Prove proposition 13.1.2.

(3) Show that $\displaystyle\sum_{k=1}^{n} 2^{-k} = 1 - 2^{-n}$ for each n. *Hint.* Let $s_n = \displaystyle\sum_{k=1}^{n} 2^{-k}$ and consider the quantity $s_n - \frac{1}{2}s_n$.

(4) Let $f(x) = x^3 + x$ for $0 \le x \le 2$. Approximate $\int_0^2 f(x)\,dx$ using the *midpoint sum*. That is, compute, *and simplify*, the Riemann sum M_n for arbitrary n. Take the limit as $n \to \infty$ of M_n to find the value of $\int_0^2 f(x)\,dx$. Determine the smallest number of subintervals that must be used so that the error in the approximation M_n is less than 10^{-5}.

(5) Without evaluating the integral show that

$$\frac{7}{4} \le \int_{1/4}^{2} \left(\frac{4}{3}x^3 - 4x^2 + 3x + 1\right) dx \le 3.$$

(6) Let $f(x) = x^2 \sin\dfrac{1}{x}$ if $0 < x \le 1$ and $f(0) = 0$. Show that $\left|\displaystyle\int_0^1 f\right| \le \dfrac{1}{3}$.

(7) Suppose that $a < b$. Prove that $\displaystyle\int_a^b \big(f(x) - c\big)^2\,dx$ is smallest when c is the average value of f over the interval $[a, b]$.

(8) Show that if f is a continuous function on $[a, b]$, then

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Hint. Suppose that d is a positive number and we wish to prove that $|c| < d$. All we need to do is establish two things: that $c < d$ and that $-c < d$.

(9) Show that $1 \le \displaystyle\int_0^1 e^{x^2}\,dx \le \dfrac{e+1}{2}$. *Hint.* Examine the concavity properties of the curve $y = e^{x^2}$.

(10) Let $0 \le x \le 1$. Apply the *mean value theorem* to the function $f(x) = e^x$ over the interval $[0, x]$ to show that the curve $y = e^x$ lies between the lines $y = 1 + x$ and $y = 1 + 3x$ whenever x is between 0 and 1. Use this result to find useful upper and lower bounds for the value of $\int_0^1 e^x\,dx$ (that is, numbers m and M such that $m \le \int_0^1 e^x\,dx \le M$).

(11) Show that $\displaystyle\int_a^b \left(\int_c^d f(x)\,g(y)\,dy\right) dx = \left(\int_a^b f\right)\left(\int_c^d g\right)$.

(12) Without evaluating the integral show that

$$\frac{\pi}{3} \le \int_0^{\pi} \sin x\,dx \le \frac{5\pi}{6}.$$

(13) Consider the function $f(x) = x^2 + 1$ defined on the closed interval $[0, 2]$. For each natural number n let $P_n = (x_0, x_1, \dots, x_n)$ be a regular partition of the interval $[0, 2]$ into n subintervals. Denote the length of the k^{th} subinterval by Δx_k. (Thus for a regular partition $\Delta x_1 = \Delta x_2 = \dots = \Delta x_n$.)

Definition. Let P_n be a regular partition of $[0, 2]$ as above. For each k between 1 and n let a_k be the point in the k^{th} subinterval $[x_{k-1}, x_k]$ where f has its smallest value and b_k

be the point in $[x_{k-1}, x_k]$ where f has its largest value. Then let

$$L(n) = \sum_{k=1}^{n} f(a_k)\Delta x_k \qquad \text{and} \qquad U(n) = \sum_{k=1}^{n} f(b_k)\Delta x_k.$$

The number $L(n)$ is the LOWER SUM associated with the partition P and $U(n)$ is the UPPER SUM associated with P.

(a) Let $n = 1$. (That is, we do *not* subdivide $[0, 2]$.) Find P_1, Δx_1, a_1, b_1, $L(1)$, and $U(1)$. How good is $L(1)$ as an approximation to $\int_0^2 f$?
(b) Let $n = 2$. Find P_2. For $k = 1, 2$ find Δx_k, a_k, and b_k. Find $L(2)$ and $U(2)$. How good is $L(2)$ as an approximation to $\int_0^2 f$?
(c) Let $n = 3$. Find P_3. For $k = 1, 2, 3$ find Δx_k, a_k, and b_k. Find $L(3)$ and $U(3)$. How good is $L(3)$ as an approximation to $\int_0^2 f$?
(d) Let $n = 4$. Find P_4. For $k = 1, 2, 3, 4$ find Δx_k, a_k, and b_k. Find $L(4)$ and $U(4)$. How good is $L(4)$ as an approximation to $\int_0^2 f$?
(e) Let $n = 8$. Find P_8. For $k = 1, 2, \dots, 8$ find Δx_k, a_k, and b_k. Find $L(8)$ and $U(8)$. How good is $L(8)$ as an approximation to $\int_0^2 f$?
(f) Let $n = 20$. Find P_{20}. For $k = 1, 2, \dots, 20$ find Δx_k, a_k, and b_k. Find $L(20)$ and $U(20)$. How good is $L(20)$ as an approximation to $\int_0^2 f$?
(g) Now let n be an arbitrary natural number. (Note: "arbitrary" means "unspecified".) For $k = 1, 2, \dots, n$ find Δx_k, a_k, and b_k. Find $L(n)$ and $U(n)$. Explain carefully why $L(n) \le \int_0^2 f \le U(n)$. How good is $L(n)$ as an approximation to $\int_0^2 f$?
(h) Suppose we wish to approximate $\int_0^2 f$ by $L(n)$ for some n and have an error no greater than 10^{-5}. What is the smallest value of n that our previous calculations guarantee will do the job?
(i) Use the preceding to calculate $\int_0^2 f$ with an error of less than 10^{-5}.

13.4. Answers to Odd-Numbered Exercises

(1) 55

(3) $323,200$

(5) 232

(7) 25

(9) 90

(11) 51, 7

(13) -15, 6

(15) 979

(17) 48, -12, 93

(19) (a) 4, 16
(b) 4, -16
(c) 4

(21) e^2

(23) 16

(25) $\dfrac{9\pi}{2}$

(27) 4, 2, π

(29) 9

(31) 5, -1, 9, 4, 10, 30, 10

(33) 5

CHAPTER 14

THE FUNDAMENTAL THEOREM OF CALCULUS

14.1. Background

Topics: *Fundamental theorem of Calculus*, differentiation of indefinite integrals, evaluation of definite integrals using antiderivatives.

The next two results are versions of the most elementary form of the *fundamental theorem of calculus*. (For a much more sophisticated version see theorem 46.1.1.)

14.1.1. Theorem (Fundamental Theorem Of Calculus - Version I)**.** *Let a belong to an open interval J in the real line and $f\colon J \to \mathbb{R}$ be a continuous function. Define $F(x) = \int_a^x f$ for all $x \in J$. Then for each $x \in J$ the function F is differentiable at x and $DF(x) = f(x)$.*

14.1.2. Theorem (Fundamental Theorem of Calculus - Version II)**.** *Let a and b be points in an open interval $J \subseteq \mathbb{R}$ with $a < b$. If $f\colon J \to \mathbb{R}$ is continuous and g is an antiderivative of f on J, then*

$$\int_a^b f = g(b) - g(a)\,.$$

The next proposition is useful in problem 5. It says that the only circumstance in which a differentiable function F can fail to be continuously differentiable at a point a is when either the right- or left-hand limit of $F'(x)$ *fails to exist* at a.

14.1.3. Proposition. *Let F be a differentiable real valued function in some open interval containing the point a. If $l := \lim_{x\to a^-} F'(x)$ and $r := \lim_{x\to a^+} F'(x)$ both exist, then*

$$F'(a) = r = l = \lim_{x\to a} F'(x).$$

Proof. Suppose that F is differentiable on the interval $(a-\delta, a+\delta)$. For $x \in (a, a+\delta)$ the *mean value theorem* guarantees the existence of a point $c \in (a, x)$ such that

$$\frac{F(x) - F(a)}{x - a} = F'(c).$$

Taking the limit as x approaches a from the right we get $F'(a) = r$. A nearly identical argument yields $F'(a) = l$. This shows that F is continuously differentiable at a. $\square$

14.2. Exercises

(1) Evaluate $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(n+2k)^4}{n^5}$ by expressing it as an integral and then using the *fundamental theorem of calculus* to evaluate the integral. The integral is $\displaystyle\frac{1}{c}\int_a^b x^p\,dx$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $p =$ _____ . The value of the integral is $\dfrac{q}{r}$ where $q =$ ________ and $r =$ _____ .

(2) Evaluate $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+3k}$ by expressing it as an integral and then using the *fundamental theorem of calculus* to evaluate the integral. The integral is $\displaystyle\frac{1}{c}\int_a^b x^p\,dx$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $p =$ _____ . The value of the integral is $\dfrac{u}{v}\ln u$ where $u =$ _____ and $v =$ _____ .

(3) Evaluate $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2}$ by expressing it as an integral and then using the *fundamental theorem of calculus* to evaluate the integral. The integral is $\displaystyle\int_a^b f(x)\,dx$ where $a =$ _____ , $b =$ _____ , and $f(x) =$ ___________ . The value of the integral is _____________ .

(4) Evaluate $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{(2n+7k)^2}$ by expressing it as an integral and then using the *fundamental theorem of calculus* to evaluate the integral. The integral is $\displaystyle\frac{1}{c}\int_a^b x^p\,dx$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $p =$ ______ . The value of the integral is $\dfrac{1}{r}$ where $r =$ ________ .

(5) Evaluate $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(2n+5k)^2}{n^3}$ by expressing it as an integral and then using the *fundamental theorem of calculus* to evaluate the integral. The integral is $\displaystyle\frac{1}{c}\int_a^b x^p\,dx$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $p =$ _____ . The value of the integral is $\dfrac{r}{3}$ where $r =$ ________ .

(6) Evaluate $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(2n+4k)^2}{n^3}$ by expressing it as an integral and then using the *fundamental theorem of calculus* to evaluate the integral. The integral is $\displaystyle\frac{1}{c}\int_a^b x^p\,dx$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $p =$ _____ . The value of the integral is $\dfrac{r}{3}$ where $r =$ ________ .

(7) Let $J = \int_0^5 \sqrt{3x}\,dx$ and let P be the regular partition of $[0,5]$ into n subintervals. Find the left, right and midpoint approximations to J determined by P.

Answer: $\displaystyle L_n = \frac{5}{n}\sum_{k=p}^{q}\sqrt{\frac{15k}{n}}$ where $p =$ _____ and $q =$ ______ .

$\displaystyle R_n = \frac{5}{n}\sum_{k=r}^{s}\sqrt{\frac{15k}{n}}$ where $r =$ _____ and $s =$ _____ .

$\displaystyle M_n = \frac{5}{n}\sum_{k=1}^{n}\sqrt{\frac{tk-u}{vn}}$ where $t =$ ______ , $u =$ ______ , and $v =$ _____ .

(8) Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\ln(5k+n) - \ln n}{n}$ by expressing it as an integral and then using the *fundamental theorem of calculus* to evaluate the integral. The integral is $\frac{1}{c}\int_a^b f(x)\,dx$ where $a =$ ____ , $b =$ ____ , $c =$ ____ , and $f(x) =$ ____________ . The value of the integral is $\frac{r}{s}f(r) - 1$ where $r =$ ____ and $s =$ ____ .

(9) Let $g(x) = \int_3^{\frac{1}{2}x} \frac{t^3+4t+4}{1+t^2}\,dt$. Then $Dg(2) = \frac{a}{4}$ where $a =$ ____ .

(10) Let $g(x) = (5 + 7\cos^2(2\pi x) - \sin(4\pi x))^{-1}$ and $f(x) = \int_{x^3}^{2} g(t)\,dt$. Then $Df(\frac{1}{2}) = -\frac{1}{a}$ where $a =$ ______ .

(11) Let $g(x) = (1 + (x^4+7)^{1/3})^{-1/2}$ and $f(x) = \int_x^{x^3} g(t)\,dt$. Then $Df(1) = \frac{2}{\sqrt{a}}$ where $a =$ ____ .

(12) Let $f(x) = \int_{x^2}^{\sin \pi x} \frac{dt}{1+t^4}$. Then $Df(2) = a - \frac{4}{b}$ where $a =$ ____ and $b =$ _________ .

(13) Let $g(x) = \int_0^x \frac{u-1}{u-2}\,du$. Then
(a) the domain of g is (____ , ____);
(b) g is increasing on (____ , ____); and
(c) g is concave down on (____ , ____).

(14) Solve for x: $\int_0^x (2u-1)^2\,du = \frac{14}{3}$. Answer: $x =$ ____ .

(15) Solve for x: $\int_x^{x+2} u\,du = 0$. Answer: $x =$ ______ .

(16) Find a number $x > 0$ such that $\int_1^x (u-1)\,du = 4$. Answer: $x = 1 + a\sqrt{a}$ where $a =$ ____ .

(17) Find $\int_3^6 f'(x)\,dx$ given that the graph of f includes the points $(0,4)$, $(3,5)$, $(6,-2)$, and $(8,-9)$. Answer: ______ .

(18) Let $g(x) = \int_0^x xf(t)\,dt$ where f is a continuous function. Then

$$Dg(x) = \underline{\hspace{12em}} .$$

(19) Let $f(x) = \int_0^x \frac{1-t^2}{3+t^4}\,dt$. Then f is increasing on the interval (______ , ______) and is concave up on the intervals (______ , ______) and (______ , ______) .

(20) If $y = \int_0^s \sqrt{2+u^3}\,du$, then $\frac{dy}{ds} =$ ______________ .

(21) If $y = \int_2^{t^2} \cos\sqrt{x}\,dx$ and $t \geq 0$, then $\frac{dy}{dt} =$ ______________ .

(22) If $\int_{-2}^{x} f(t)\,dt = x^2\sin(\pi x)$ for every x, then $f(1/3) = \frac{\pi}{a} + \frac{1}{b}$ where $a =$ ____ and $b =$ ____ .

(23) Let $f(x) = \displaystyle\int_{x\ln x}^{x^3} \frac{dt}{3+\ln t}$ for $x \geq 1$. Then $Df(e) = \dfrac{1}{a}(b^2-1)$ where $a =$ ____ and $b =$ ____ .

(24) Let $f(x) = \displaystyle\int_{\ln(x+1)^4}^{\ln(x^2+1)} \frac{dt}{4+e^t}$ for $x > 0$. Then $Df(1) = \dfrac{1}{a}$ where $a =$ ______ .

(25) Let $f(x) = \displaystyle\int_{-2}^{\frac{1}{2}x^2e^{x-1}} \frac{t^2}{(4+\sin\pi t)^2}\,dt$. Then $Df(1) = \dfrac{3}{a}$ where $a =$ ________ .

(26) Let $f(x) = \displaystyle\int_0^{xe^{x^2}} \frac{dt}{5+(\ln t)^2}$. Then $Df(1) = \dfrac{a}{b}$ where $a =$ ____ and $b =$ ____ .

(27) Let $f(x) = \displaystyle\int_0^{e^{x^3}} \frac{dt}{6+(\ln t)^2}$. Then $Df(2) = \dfrac{6e^a}{b}$ where $a =$ ____ and $b =$ _____ .

(28) Let $f(x) = \displaystyle\int_{\ln x}^{\ln(x^2+3)} \frac{dt}{3+e^t}$ for $x \geq 1$. Then $Df(2) = -\dfrac{3}{a}$ where $a =$ ______ .

(29) Let $f(x) = (x^2+2x+2)^{-1}$ for all $x \in \mathbb{R}$. Then the interval on which the curve $y = \int_0^x f(t)\,dt$ is concave up is (_____ , _____).

(30) $\displaystyle\lim_{h\to 0} \frac{1}{h}\int_2^{2+h} \sqrt{1+x^2}\,dx =$ _____ .

(31) $\displaystyle\lim_{\lambda\to 0^+} \int_\lambda^{2\lambda} e^{-x}x^{-1}\,dx =$ ________ . *Hint:* $\dfrac{e^{-x}}{x} = \dfrac{e^{-x}-1}{x} + \dfrac{1}{x}$.

(32) $\displaystyle\lim_{x\to 0} \frac{1}{x}\int_1^{1+5x} (4-\cos 2\pi t)^3\,dt =$ ________ .

(33) $\displaystyle\lim_{r\to 0} \frac{1}{r}\int_1^{e^{4r}} \sqrt{3+\frac{1}{x}}\,dx =$ ____ .

(34) $\displaystyle\lim_{u\to 0} \frac{1}{u}\int_2^{\ln(e^2+3u)} \sqrt{1+2t+5t^2}\,dt = ae^b$ where $a =$ _____ and $b =$ _____ .

(35) $\displaystyle\int_1^9 \frac{1}{x^{3/2}} = \frac{a}{3}$ where $a =$ ____ .

(36) $\displaystyle\int 12e^{4x}\,dx = ae^{bx} + c$ where $a =$ ____ , $b =$ ____ , and c is an arbitrary constant.

(37) $\displaystyle\int 40\cos 5x\,dx = a\sin bx + c$ where $a =$ ____ , $b =$ ____ , and c is an arbitrary constant.

(38) $\displaystyle\int_0^1 \frac{4}{\sqrt{4-x^2}}\,dx = \frac{a}{3}$ where $a =$ ______ .

(39) $\displaystyle\int_0^{\sqrt{3}} \frac{6}{9+x^2}\,dx =$ __________ .

(40) $\displaystyle\int_0^{\pi/2} \cos x e^{\sin x}\,dx = a - b$ where $a =$ _____ and $b =$ ____ .

(41) $\displaystyle\int_0^{\pi} \sec^2 \tfrac{1}{4}x\,dx =$ ____ .

(42) If $a = 0$ and $b = \frac{1}{5}(e-1)$, then $\displaystyle\int_a^b \frac{15}{5x+1}\,dx =$ _____ .

(43) Let $f(x) = |x| + |\cos x|$ for all x. Then $\displaystyle\int_{-\pi/2}^{\pi} f = a + \frac{b}{8}\pi^p$ where $a =$ _____ , $b =$ _____ , and $p =$ _____ .

(44) $\displaystyle\int_{-1}^{2} \left|x^3 - x\right|\,dx = \frac{a}{4}$ where $a =$ ________ .

(45) $\displaystyle\int_0^{2\pi} (|\sin x| + \cos x)\,dx =$ _____ .

(46) $\displaystyle\int_0^{\pi/4} \sin^5 x\,\cos x\,dx = \frac{1}{a}$ where $a =$ ________ .

(47) $\displaystyle\int_1^3 (x^3 - 6x^2 + 2x - 7)\,dx =$ ___________ .

(48) $\displaystyle\int_0^4 (x^3 + 3\sqrt{x})\,dx =$ __________ .

(49) $\displaystyle\int_0^3 (5 - 2x^2)\,dx =$ ________ .

(50) $\displaystyle\int_1^5 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx = a + \ln b$ where $a =$ ________ and $b =$ _____ .

(51) $\displaystyle\int_{\pi/6}^{\pi/2} \csc^2 x\,dx =$ ________ .

(52) $\displaystyle\int_0^5 \frac{dx}{25 + x^2} = \frac{\pi}{a}$ where $a =$ ________ .

(53) $\displaystyle\int_0^1 (x^3 + x)e^{x^4 + 2x^2}\,dx = \frac{1}{a}(e^p - 1)$ where $a =$ _____ and $p =$ _____ .

(54) Let $f(x) = \displaystyle\int_{3\pi}^{x} (7 + \cos(\sin t))\,dt$. Then $Df^{-1}(0) = \dfrac{1}{a}$ where $a =$ _____ .

(55) Let $f(x) = \displaystyle\int_{\pi/3}^{x^{1/3}} \arctan(2 + 2\sin t)\,dt$ for $x \geq 0$. Then $Df^{-1}(0) = \dfrac{4\pi}{a}$ where $a =$ _____ . *Hint.* What is $\tan\frac{5\pi}{12}$?

(56) If $\log_2 x = \displaystyle\int_2^x \frac{1}{t}\,dt$, then $x = \exp\left(\frac{a^2}{a-1}\right)$ where a $=$ ________ .

14.3. Problems

(1) Estimate $\displaystyle\sum_{k=1}^{10^4} \sqrt{k}$ by interpreting it as a Riemann sum for an appropriate integral.

(2) Let $f(x) = x^3 + x$, let n be an arbitrary natural number, and let $P = (x_0, x_1, \dots x_n)$ be a regular partition of the interval $[0, 2]$ into n subintervals. (Note: "arbitrary" means "unspecified".) For each k between 1 and n let c_k be the midpoint of the k^{th} subinterval $[x_{k-1}, x_k]$.
  (a) Find the width Δx_k of each subinterval.
  (b) Find x_k for each $k = 0, \dots, n$.
  (c) Find c_k for each $k = 1, \dots, n$.
  (d) Find the corresponding Riemann midpoint sum $\sum_{k=1}^{n} f(c_k)\Delta x_k$. Simplify the expression and put it in the form $a + b/n + c/n^2 + \dots$.
  (e) Find the limit of the Riemann sums in part (d) as $n \to \infty$.
  (f) Compute $\int_0^2 f$ using the *fundamental theorem of calculus*
  (g) What is the smallest number of subintervals we can use so that the Riemann sum found in (d) approximates the true value of the integral found in (f) with an error of less than 10^{-5}?

(3) Let $f(x) = -\frac{1}{2}x + \frac{3}{2}$ for $-1 \le x \le 3$. Partition the interval $[-1, 3]$ into n subintervals of equal length. Write down the corresponding right approximating sum R_n. Show how this expression can be simplified to the form $a + \frac{b}{n}$ for appropriate numbers a and b. Take the limit of this expression as n gets large to find the value of $\int_{-1}^{3} f(x)\,dx$. Check your answer in two different ways: using a geometrical argument and using the *fundamental theorem of calculus.*

(4) Let $f(x) = x^2 + 1$ for $0 \le x \le 3$. Partition the interval $[0, 3]$ into n subintervals of equal length. Write down the corresponding right approximating sum R_n. Show how this expression can be simplified to the form $a + \frac{b}{cn} + \frac{d}{cn^2}$ for appropriate numbers a, b, c, and d. Take the limit of this expression as n gets large to find the value of $\int_0^3 f(x)\,dx$. Check your answer using the *fundamental theorem of calculus.*

(5) Define functions f, g, and h as follows:

$$h(x) = \begin{cases} 1, & \text{for } 0 \le x \le 2 \\ x, & \text{for } 2 < x \le 4. \end{cases}$$

$$g(x) = \int_x^{x^2} h(t)\,dt \qquad \text{for } 0 \le x \le 2$$

$$f(x) = \int_0^x g(t)\,dt \qquad \text{for } 0 \le x \le 2.$$

  (a) For each of the functions h, g, and f answer the following questions:
    (i) Where is the function continuous? differentiable? twice differentiable?
    (ii) Where is the function positive? negative? increasing? decreasing? concave up? concave down?
    (iii) Where are the x-intercepts? maxima? minima? points of inflection?
  (b) Make a careful sketch of the graph of each of the functions.
  (c) What is the moral of this problem? That is, what do these examples suggest about the process of integration in general?

Hints for solution. When working with the first function h it is possible to get the "right answers" to questions (i)—(iii) but at the same time fail to give coherent reasons for the assertions made. This part of the problem is meant to encourage paying attention to the precise definitions of some of the terms. Indeed, the correct answers will vary from text to text. Some texts, for instance, distinguish between functions that are *increasing* and those that are *strictly increasing*. Other texts replace these terms by *nondecreasing* and *increasing*, respectively. Some texts define *concavity* only for functions which are twice differentiable; others define it in terms of the first derivative; still others define it geometrically.

This first part of the problem also provides an opportunity to review a few basic facts: differentiability implies continuity; continuity can be characterized (or defined) in terms of limits; and so on.

Unraveling the properties of the second function g is rather harder. Try not to be put off by the odd looking definition of g. The crucial insight here is that by carrying out the indicated integration it is possible to express g, at least piecewise, as a polynomial. From a polynomial expression it is a simple matter to extract the required information. Impatience at this stage is not a reliable friend. It is not a good idea to try to carry out the integration before you have thought through the problem and discovered the necessity of dividing the interval into two pieces. It may be helpful to compute the values of g at $x = 1.0, 1.1, 1.2, \dots, 1.9, 2.0$. Notice that about midway in these computations something odd happens. What is the precise point p where things change? Eventually one sees that g too is expressible as one polynomial on $[0, p]$ and as another polynomial on $(p\,, 2]$. Once g has been expressed piecewise by polynomials it is possible to proceed with questions (i)–(iii). To determine whether g is continuous at p, compute the right- and left-hand limits of g there.

The question of the differentiability of g is subtle and deserves some serious thought. It may be tempting to carry over the format of continuity argument to decide about the differentiability of g at p. Suppose we compute the right- and left-hand limits of the derivative of g at p and find that they are not equal. Can we then conclude that g is not differentiable at p? At first one is inclined to say *no*, that all we have shown is that the derivative of g is not continuous at p, which does not address the issue of the *existence* of $g'(p)$. Interestingly enough, it turns out that the only way in which a differentiable function F can fail to be continuously differentiable at a point a is for either the right- or left-hand limit of $F'(x)$ *to fail to exist* at a. The crucial result, which is a bit hard to find in beginning calculus texts, is proposition 14.1.3. Thus when we discover that a function F is differentiable at all points other than a, and that the limits $\lim_{x\to a^-} F'(x)$ and $\lim_{x\to a^+} F'(x)$ both exist but fail to be equal, there is only one possible explanation: F fails to be differentiable at a.

After finding a piecewise polynomial expression for g, another difficulty arises in determining whether g is concave up. It is easy to see that g is concave up on the intervals $(0, p)$ and $(p\,, 2)$. But this isn't enough to establish the property for the entire interval $(0, 2)$. In fact, according to the definition of concavity given in many texts g is *not* concave up. Why? Because, according to Finney and Thomas (see [2], page 237), for example, concavity is defined only for differentiable functions. A function is concave up on an interval only if its derivative is increasing on the interval. So if our function g fails to be differentiable at some point it can not be concave up. On the other hand, under any reasonable geometric definition of concavity g certainly *is* concave up on $(0, 2)$—although it is a bit hard to show. The solution to this dilemma is straightforward: pick a definition and stick to it.

Analysis of the last function f proceeds pretty much as for g. One new wrinkle is the difficulty in determining where f is positive. The point at which f changes sign is a root of a fifth degree polynomial. An approximation based either on the *intermediate value theorem* or *Newton's method* goes smoothly.

As with g, conclusions concerning the concavity of f may differ depending on the definitions used. This time both a geometrical definition and one based on the first derivative lead to one conclusion while a definition based on the second derivative leads to another.

Finally, for part (c) does it make any sense to regard integration as a "smoothing" operation? In what way?

(6) Show that if f is continuous, then

$$\int_0^x f(u)(x-u)\,du = \int_0^x \int_0^u f(t)\,dt\,du.$$

Hint. What can you say about functions F and G if you know that $F'(x) = G'(x)$ for all x and that $F(x_0) = G(x_0)$ at some point x_0?

(7) Let f be a continuous function and $a < b$. Show that $\displaystyle\int_a^b f(-x)\,dx = \int_{-b}^{-a} f(x)\,dx$.
Hint. Show that if F is an antiderivative of f, then the function $G\colon x \mapsto -F(-x)$ is an antiderivative of the function $g\colon x \mapsto f(-x)$.

(8) Let $a < b$, f be a continuous function defined on the interval $[a, b]$, and g be the function defined by $g(t) = \displaystyle\int_a^b (f(x) - t)^2\,dx$ for t in $\mathbb{R}$. Find the value for t at which g assumes a minimum. How do you know that this point is the location of a minimum (rather than a maximum)?

(9) Let λ be a positive constant. Define $F(x) = \displaystyle\int_x^{\lambda x} \frac{1}{t}\,dt$ for all $x > 0$. *Without mentioning logarithms* show that F is a constant function.

(10) Without computing the integrals give a simple *geometric* argument that shows that the sum of $\int_0^1 \sqrt{x}\,dx$ and $\int_0^1 x^2\,dx$ is 1. Then carry out the integrations.

14.4. Answers to Odd-Numbered Exercises

(1) 1, 3, 2, 4, 121, 5

(3) 0, 1, $\dfrac{1}{1+x^2}$, $\dfrac{\pi}{4}$

(5) 2, 7, 5, 2, 67

(7) 0, $n-1$, 1, n, 30, 15, 2

(9) 9

(11) 3

(13) (a) $-\infty$, 2
(b) $-\infty$, 1
(c) $-\infty$, 2

(15) -1

(17) -7

(19) -1, 1, $-\sqrt{3}$, 0, $\sqrt{3}$. ∞

(21) $2t\cos t$

(23) 2, e

(25) 200

(27) 8, 35

(29) $-\infty$, -1

(31) $\ln 2$

(33) 8

(35) 4

(37) 8, 5

(39) $\dfrac{\pi}{3}$

(41) 4

(43) 3, 5, 2

(45) 4

(47) -38

(49) -3

(51) $\sqrt{3}$

(53) 4, 3

(55) 5

CHAPTER 15

TECHNIQUES OF INTEGRATION

15.1. Background

Topics: antiderivatives, change of variables, trigonometric integrals, trigonometric substitutions, integration by parts, partial fractions, improper Riemann integrals.

15.2. Exercises

(1) $\displaystyle\int_0^1 \frac{x^{\frac{1}{2}}}{1+x^{\frac{3}{4}}}\,dx = \frac{a}{3}(1-\ln b)$ where $a =$ ____ and $b =$ ____.

(2) $\displaystyle\int_0^9 \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = a + 4\ln b$ where $a =$ ____ and $b =$ ____.

(3) $\displaystyle\int_{\frac{1}{3\sqrt{3}}}^{3\sqrt{3}} \frac{1}{x^{\frac{4}{3}}+x^{\frac{2}{3}}}\,dx =$ ______.

(4) $\displaystyle\int_0^{1/2} \frac{\arctan 2x}{1+4x^2}\,dx = \frac{\pi^2}{a}$ where $a =$ ____.

(5) $\displaystyle\int_0^{\sqrt{2}} x\,10^{1+x^2}\,dx = \frac{a}{\ln 10}$ where $a =$ ________ .

(6) $\displaystyle\int_1^{16} \frac{x-1}{x+\sqrt{x}}\,dx =$ _____ .

(7) $\displaystyle\int_1^9 \frac{dx}{(x+1)\sqrt{x}+2x} = \frac{1}{a}$ where $a =$ ____ .

(8) $\displaystyle\int_{27/8}^8 \frac{2\,dx}{x^{5/3}-3x^{4/3}+3x-x^{2/3}} =$ ____ .

(9) $\displaystyle\int_{\pi/6}^{\pi/2} \frac{\cos^3 x}{\sqrt{\sin x}}\,dx = \frac{a}{5} - \frac{b}{10\sqrt{2}}$ where $a =$ ____ and $b =$ _____ .

(10) $\displaystyle\int_0^{\pi/8} \tan 2x \sec^2 2x\,dx = \frac{1}{a}$ where $a =$ ____ .

(11) $\displaystyle\int_1^{4/3} \frac{1}{x^2}\sqrt{1-\frac{1}{x}}\,dx = \frac{1}{a}$ where $a =$ ______ .

(12) $\displaystyle\int_1^4 \frac{4x-1}{2x+\sqrt{x}}\,dx =$ ____ .

(13) $\displaystyle\int_0^{1/\sqrt{2}} x\sin^3(\pi x^2)\cos(\pi x^2)\,dx = \frac{1}{a}$ where $a =$ ________ .

(14) $\displaystyle\int_0^{\pi/4} \frac{\sec^2 x}{(5+\tan x)^2}\,dx = \frac{1}{a}$ where $a =$ ______ .

(15) $\displaystyle\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,dx = \frac{a}{3}$ where $a =$ ____ .

(16) $\displaystyle\int_0^2 \frac{x\,dx}{\sqrt{4x^2+9}} = \frac{1}{a}$ where $a =$ ____ .

(17) $\displaystyle\int_1^2 \frac{2x^2\,dx}{(x^3+1)^2} = \frac{a}{27}$ where $a =$ ____ .

(18) $\displaystyle\int_1^{\sqrt{10}} x\sqrt{x^2-1}\,dx =$ ____ .

(19) $\displaystyle\int_4^9 \frac{x-9}{3\sqrt{x}+x}\,dx =$ ______ .

(20) $\displaystyle\int \frac{r^5\,dr}{\sqrt{4-r^6}} = -\frac{1}{a}\sqrt{4-r^6} + c$ where $a =$ _____ and c is an arbitrary constant.

(21) $\displaystyle\int_0^{\pi/4} \frac{\tan^3 x \sec^2 x}{\left(1+\tan^4 x\right)^3}\,dx = \frac{3}{a}$ where $a =$ ________ .

(22) $\displaystyle\int_0^{\sqrt{3}} \frac{x\,dx}{x^2-4} = -\ln a$ where $a =$ _____ .

(23) $\displaystyle\int_1^3 \frac{dx}{x^{1/2}+x^{3/2}} = \frac{\pi}{a}$ where $a =$ _____ .

(24) $\displaystyle\int_e^{e^e} \frac{1}{x\ln x(1+(\ln\ln x)^2)}\,dx = \frac{\pi}{a}$ where $a =$ _____ .

(25) $\displaystyle\int_{1/8}^{\frac{1}{2\sqrt{2}}} \frac{dx}{\sqrt{x^{4/3}-x^2}} = \frac{a}{4}$ where $a =$ _____ .

(26) $\displaystyle\int_{-5}^{\sqrt{3}-5} \frac{dx}{\sqrt{-x^2-10x-21}} = \frac{\pi}{a}$ where $a =$ _____ .

(27) $\displaystyle\int_{-3/2}^{0} \frac{dx}{4x^2+12x+18} = \frac{\pi}{a}$ where $a =$ ________ .

(28) $\displaystyle\int_{\arctan e}^{\arctan e^3} \frac{\csc 2x}{\ln(\tan x)}\,dx = \frac{1}{2}\ln a$ where $a =$ _____ .

(29) $\displaystyle\int_0^1 \frac{(\arctan x)^2}{1+x^2}\,dx = \frac{\pi^p}{a}$ where $p =$ _____ and $a =$ ___________ .

(30) $\displaystyle\int_0^{\ln 3} \frac{e^{x/2}}{1+e^x}\,dx = \frac{\pi}{a}$ where $a =$ _____ .

(31) $\displaystyle\int_0^{1/2} \frac{3\arcsin x}{\sqrt{1-x^2}}\,dx = \frac{\pi^p}{a}$ where $p =$ _____ and $a =$ ________ .

(32) $\displaystyle\int_{1/4}^{1/2} \frac{dx}{\sqrt{x-x^2}} = \frac{\pi}{a}$ where $a =$ _____ .

(33) $\displaystyle\int_0^1 (x+2)e^{x^2+4x}\,dx = \frac{1}{a}\left(e^p-1\right)$ where $a =$ _____ and $p =$ _____ .

(34) $\displaystyle\int_{\sqrt{\ln\frac{\pi}{2}}}^{\sqrt{\ln\pi}} xe^{x^2}\cos(3e^{x^2})\,dx = \frac{1}{a}$ where $a =$ _____ .

(35) $\displaystyle\int_{\ln 2}^{\ln 8} \frac{1-e^x}{1+e^x}\,dx = a\ln\frac{a}{b}$ where $a =$ _____ and $b =$ _____ .

(36) $\displaystyle\int_0^1 x\,5^{-x^2}\,dx = \frac{a}{b\ln b}$ where $a =$ _____ and $b =$ _____ .

(37) $\displaystyle\int_e^{e^4} \frac{\log_7 x}{x}\,dx = \frac{a}{2\ln 7}$ where $a =$ ________ .

(38) $\displaystyle\int_{e^2}^{e^3} x^{-1}(\log_3 x)^2\,dx = \frac{a}{b(\ln b)^2}$ where $a =$ ________ and $b =$ _____ .

(39) $\displaystyle\int_0^{\pi/2} \sin^7 u\,du = \frac{16}{a}$ where $a =$ ________ .

(40) $\displaystyle\int_0^{\pi/3} \sec^5 x \tan x\,dx = \frac{a}{5}$ where $a =$ ________ .

(41) $\displaystyle\int_0^{\pi/4} \sec^3 x\,dx = a + \tfrac{1}{2}\ln b$ where $a =$ ________ and $b =$ ___________ .

(42) $\displaystyle\int_0^{\pi/3} \tan^3 x \sec^3 x\,dx = \frac{a}{15}$ where $a =$ ________ .

(43) $\displaystyle\int_0^{\pi/2} \sin^4 x\,dx = \frac{3\pi}{a}$ where $a =$ ________ .

(44) $\displaystyle\int_0^{\pi/2} \cos^3 x \sin^5 x\,dx = \frac{1}{a}$ where $a =$ ________ .

(45) $\displaystyle\int_0^{\pi/2} \sin^4 x\, \cos^5 x\,dx = \frac{8}{a}$ where $a =$ _________ .

(46) $\displaystyle\int_0^{\pi/3} \tan^3 x\,dx = a - \ln b$ where $a =$ ________ and $b =$ _____ .

(47) $\displaystyle\int_0^{\pi} \sin^6 x\,dx = \frac{a\pi}{16}$ where $a =$ _____ .

(48) $\displaystyle\int_0^{\pi/3} \sec^6 x\,dx = \frac{a}{5}\sqrt{3}$ where $a =$ ________ .

(49) $\displaystyle\int_0^{3/4} \frac{dx}{\sqrt{9-4x^2}} =$ _____________ .

(50) $\displaystyle\int_{1/\sqrt{2}}^{1} \frac{dx}{x\sqrt{4x^2-1}} = \frac{\pi}{a}$ where $a =$ ________ .

(51) $\displaystyle\int \frac{dx}{\sqrt{8-4x-4x^2}} = a\ \arcsin\bigl(\tfrac{1}{3}f(x)\bigr) + c$ where $a =$ ______ , $f(x) =$ __________ , and c is an arbitrary constant.

(52) $\displaystyle\int \frac{dx}{x^2+2x+5} = a\ \arctan(a\,f(x)) + c$ where $a =$ ______ , $f(x) =$ __________ , and c is an arbitrary constant.

(53) $\displaystyle\int_1^{2} \frac{dx}{x(1+x^4)} = \frac{1}{4}\ln\frac{32}{a}$ where $a =$ ________ .

(54) $\displaystyle\int_1^{\sqrt{13}} \frac{dx}{x^2\sqrt{3+x^2}} = a\left(1 - \frac{b}{\sqrt{13}}\right)$ where $a =$ ________ and $b =$ _____ .

(55) $\displaystyle\int_3^{6} \frac{\sqrt{x^2-9}}{x}\,dx = a\sqrt{3} - b$ where $a =$ _____ and $b =$ _____ .

(56) $\displaystyle\int_{3\sqrt{2}}^{6} \frac{dx}{\sqrt{x^2-9}} = \ln(a+\sqrt{b}) - \ln(1+\sqrt{2})$ where $a =$ _____ and $b =$ _____ .

(57) $\displaystyle\int_0^{3} \frac{dw}{\sqrt{9+w^2}} = \ln(a+\sqrt{b})$ where $a =$ _____ and $b =$ _____ .

(58) $\displaystyle\int \frac{x^2}{\left(4-x^2\right)^{3/2}}\,dx = \frac{x}{f(x)} - g(x/2) + c$ where $f(x) =$ ____________ , $g(x) =$ ____________ , and c is an arbitrary constant.

(59) $\displaystyle\int_0^{\sqrt{3}} \arctan x\,dx = \frac{a}{\sqrt{3}} - \ln b$ where $a =$ _____ and $b =$ _____ .

(60) $\displaystyle\int_0^{\sqrt{3}} x\arctan x\,dx = \frac{a\pi}{b} - \frac{1}{a}\sqrt{b}$ where $a =$ _____ and $b =$ _____ .

(61) $\displaystyle\int_1^{2} x^3\ln x\,dx = a\ln 2 - \frac{b}{16}$ where $a =$ _____ and $b =$ ________ .

(62) $\displaystyle\int_1^{e} x^2\ln x\,dx = \frac{1}{a}(be^p + 1)$ where $a =$ _____ , $b =$ _____ , and $p =$ _____ .

(63) $\displaystyle\int x^2e^x\,dx = p(x)e^x + c$ where $p(x) =$ _______________ and c is an arbitrary constant.

(64) $\displaystyle\int_0^{1} \arctan x\,dx = \frac{\pi}{a} - \frac{1}{b}\ln b$ where $a =$ _____ and $b =$ _____ .

(65) $\displaystyle\int_0^{1} \operatorname{arccot} x\,dx = \frac{a}{4} + \frac{1}{b}\ln b$ where $a =$ _____ and $b =$ _____ .

(66) $\displaystyle\int_0^{\pi/6} x\sin x\,dx = \frac{a-\sqrt{b}\,\pi}{12}$ where $a =$ _____ and $b =$ _____ .

(67) $\displaystyle\int x^2\cos x\,dx = f(x)\sin x + g(x)\cos x + c$ where $f(x) =$ ___________ , $g(x) =$ ________ , and c is an arbitrary constant.

(68) Expand $\dfrac{x^2+2x-2}{x^3(x-1)}$ by partial fractions.

Answer: $\dfrac{a}{x} + \dfrac{b}{x^2} + \dfrac{c}{x^3} + \dfrac{d}{x-1}$ where $a =$ ______ , $b =$ _____ , $c =$ _____ , and $d =$ _____ .

(69) Expand $\dfrac{x^3+x^2+7}{x^2+x-2}$ by partial fractions.

Answer: $f(x) + \dfrac{a}{x-1} + \dfrac{b}{x+2}$ where $f(x) =$ _____ , $a =$ _____ and $b =$ ______ .

(70) $\displaystyle\int_1^{2} \frac{2}{w^3+2w}\,dw = \frac{1}{a}\ln a$ where $a =$ _____ .

(71) $\displaystyle\int_{-3}^{0} \frac{-2w^3+w^2+2w+13}{w^2+2w+3}\,dw = a \;+ \ln b$ where $a =$ ________ and $b =$ _____ .

(72) $\displaystyle\int \frac{3x^2+x+6}{x^4+3x^2+2}\,dx = -a\ln(x^2+2) + a\ln(g(x)) + 3\arctan x + c$ where $a =$ ________ , $g(x) =$ ____________ , and c is an arbitrary constant.

(73) $\displaystyle\int \frac{1-4x-3x^2-3x^3}{x^4+x^3+x^2}\,dx = \frac{a}{x} - 5\ln x + \ln(g(x)) + c$ where $a =$ ______ , $g(x) =$ __________ , and c is an arbitrary constant.

(74) $\displaystyle\int \frac{4x^3-2x^2+x}{x^4-x^3-x+1}\,dx = f(x) + 2g(x) + \ln(x^2+x+1) + c$ where $f(x) =$ ____________ , $g(x) =$ ____________ , and c is an arbitrary constant.

(75) $\displaystyle\int \frac{x^2+3}{x^3+x}\,dx = a\ln x - \ln(g(x)) + c$ where $a =$ ____ , $g(x) =$ ________ , and c is an arbitrary constant.

(76) $\displaystyle\int \frac{2x^2-3x+9}{x^3-3x^2+7x-5}\,dx = a\ln(x-1) + \frac{1}{a}\arctan\left(\frac{1}{a}\,g(x)\right) + c$ where $a =$ _____ , $g(x) =$ _________ , and c is an arbitrary constant.

(77) $\displaystyle\int \frac{2x^3+x^2+2x-1}{x^4-1}\,dx = \ln(x-1) + \ln(x+1) + f(x) + c$ where $f(x) =$ __________ and c is an arbitrary constant.

(78) $\displaystyle\int \frac{x^5-2x^4+x^3-3x^2+2x-5}{x^3-2x^2+x-2}\,dx = g(x) + a\ln(x-2) + b\arctan x + c$ where $a =$ _____ , $b =$ ____ , $g(x) =$ ___________ , and c is an arbitrary constant.

(79) $\displaystyle\int \frac{x^7+9x^5+2x^3+4x^2+9}{x^4+9x^2}\,dx = f(x) - \frac{1}{x} + \ln(g(x)) + \arctan(h(x)) + c$ where $f(x) =$ ________ , $g(x) =$ ________ , $h(x) =$ _______ , and c is an arbitrary constant.

(80) $\displaystyle\int_0^{\pi/2} \frac{dx}{8+4\sin x+7\cos x} = \ln\left(\frac{a}{9}\right)$ where $a =$ _____ . *Hint.* Try substituting $u = \tan\frac{x}{2}$.

(81) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x}\,dx = \frac{\pi}{a}$ where $a =$ ____ . *Hint.* Try substituting $u = \tan\frac{x}{2}$.

(82) $\displaystyle\int \frac{2x^2+9x+9}{(x-1)(x^2+4x+5)}\,dx = a\ln(x-1) + f(x) + c$ where $a =$ ____ , $f(x) =$ ___________ , and c is an arbitrary constant.

(83) $\displaystyle\int \frac{2x^4+x^3+4x^2+2}{x^5+2x^3+x}\,dx = a\,f(x) + \frac{1}{a}\arctan x - \frac{1}{a}g(x) + c$ where $a =$ _____ , $f(x) =$ ________ , $g(x) =$ ___________ , and c is an arbitrary constant.

(84) $\displaystyle\int \frac{5u^2+11u-4}{u^3+u^2-2u}\,du = a\ln|u| + b\ln|u-1| - \ln|u+2| + c$ where $a =$ ____ , $b =$ ____ , and c is an arbitrary constant.

(85) $\displaystyle\int_0^{\pi/2} (\cot x - x\csc^2 x)\,dx =$ _____ .

(86) $\displaystyle\int_0^{e} x^2\ln x\,dx = ae^p$ where $a =$ _____ and $p =$ ____ .

(87) $\displaystyle\int_2^{4} \frac{x\,dx}{\sqrt{|9-x^2|}} = \sqrt{5} + \sqrt{a}$ where $a =$ ____ .

(88) $\displaystyle\int_0^{\infty} e^{-x}\sin x\,dx = \frac{1}{a}$ where $a =$ ____ .

(89) If we choose $k =$ ____ , then the improper integral $\displaystyle\int_0^{\infty}\left(\frac{k}{3x+1} - \frac{2x}{x^2+1}\right)dx$ converges. In this case the value of the integral is $2\ln a$ where $a =$ ____ .

(90) $\displaystyle\int_2^{4} \frac{1}{\sqrt{2x-4}}\,dx =$ ____ .

(91) $\displaystyle\int_{e^2}^{\infty} \frac{dx}{x(\ln x)^2} = \frac{1}{a}$ where $a =$ ____ .

(92) Does the improper integral $\int_0^8 x^{-1/3}\,dx$ converge? Answer: __________ . If it converges, its value is _____ .

(93) Does the improper integral $\int_{\frac{1}{2}}^1 \frac{1}{\sqrt{2x-1}}\,dx$ converge? Answer: __________ . If it converges, its value is __________ .

(94) Does the improper integral $\int_{-1}^1 \frac{1}{x^2}\,dx$ converge? Answer: __________ . If it converges, its value is _____ .

(95) Does the improper integral $\int_{\frac{2}{3}}^1 \frac{1}{3x-2}\,dx$ converges? Answer: __________ . If it converges, its value is __________ .

(96) Does the improper integral $\int_0^\infty \frac{1}{1+9x^2}\,dx$ converge? Answer: __________ . If it converges, its value is __________ .

(97) Does the improper integral $\int_0^\infty x^4 e^{-x^5}\,dx$ converge? Answer: __________ . If it converges, its value is __________ .

(98) $\int_{-1}^1 \frac{dx}{\sqrt{|x|}} =$ _____ .

(99) $\int_{3^{1/4}}^\infty \frac{x\,dx}{1+x^4} = \frac{\pi}{a}$ where $a =$ ______ .

(100) $\int_0^\infty \frac{x\,dx}{(1+x^2)^4} = \frac{1}{a}$ where $a =$ _____ .

(101) $\int_0^{\sqrt{3}} \frac{x}{\sqrt{9-x^4}}\,dx = \frac{\pi}{a}$ where $a =$ _____ .

(102) $\int_{\sqrt{3}}^\infty \frac{1}{1+x^2}\,dx = \frac{a}{6}$ where $a =$ _____ .

(103) $\int_1^\infty xe^{-x}\,dx = \frac{2}{a}$ where $a =$ _____ .

(104) $\int_0^\infty x^{12}e^{-x}\,dx = n!$ where $n =$ ______ .

(105) $\int_{1/2}^1 \frac{x}{\sqrt{1-x^2}}\,dx = \frac{a}{2}$ where $a =$ ______ .

(106) $\lim_{\lambda\to 0^+} \frac{1}{\ln\lambda}\int_\lambda^a \frac{\cos x}{x}\,dx =$ ______ . *Hint.* Problem 10 may help.

(107) Let $f(x) = \int_{3\pi}^x (7+\cos(\sin t))\,dt$. Then $Df^{-1}(0) = \frac{1}{a}$ where $a =$ _____ .

(108) Let $f(x) = \int_{\pi/3}^{x^{1/3}} \arctan(2+2\sin t)\,dt$ for $x \geq 0$. Then $Df^{-1}(0) = \frac{4\pi}{a}$ where $a =$ _____ . *Hint.* What is $\tan\frac{5\pi}{12}$?

(109) Let $f(x) = \int_1^x \frac{3t^2+t+1}{5t^4+t^2+2}\,dt$. Then $Df^{-1}(0) = \frac{a}{5}$ where $a =$ _____ .

(110) Let $f(x) = \displaystyle\int_0^x t^3\sqrt{t^4+9}\,dt$ for $x \geq 0$. Then $Df^{-1}\left(\frac{49}{3}\right) = \dfrac{1}{a}$ where $a =$ ________ .

(111) $\displaystyle\lim_{\lambda\to 0^+}\int_{\lambda}^{2\lambda}\frac{e^{-x}}{x}\,dx = \ln a$ where $a =$ _____ . *Hint:* $\dfrac{e^{-x}}{x} = \dfrac{e^{-x}-1}{x} + \dfrac{1}{x}$.

15.3. Problems

(1) Your brother Al, who works for the Acme Widget Corporation, has a problem. He knows that you are studying calculus and writes a letter asking for your help. His problem concerns solutions to a system of two differential equations:

$$\begin{cases} \dfrac{dx}{dt} = x(t) \\ \dfrac{dy}{dt} = x(t) + y(t) \end{cases} \tag{15.1}$$

subject to the initial conditions

$$x(0) = a \qquad \text{and} \qquad y(0) = b, \tag{15.2}$$

where a and b are arbitrary constants. He has already found one set of solutions:

$$\begin{cases} x(t) = ae^t \\ y(t) = (b + at)e^t \end{cases}$$

What Fred is unable to discover is whether or not there are other solutions. Write a letter to Fred helping him out.

Hint. Suppose

$$\begin{cases} x(t) = u(t) \\ y(t) = v(t) \end{cases}$$

is a solution to the system (15.1) which satisfies the initial conditions (15.2). Consider the functions $p(t) = e^{-t}u(t)$ and $q(t) = e^{-t}v(t)$.

(2) Your friend Fred is trying to find the value of $I = \displaystyle\int_{-1}^{1} \frac{1}{1+x^2}\,dx$, but he has forgotten that $\arctan x$ is an antiderivative of $\dfrac{1}{1+x^2}$. He reasons as follows:

If I multiply both numerator and denominator of the integrand by x^{-2} I will get

$$I = \int_{-1}^{1} \frac{x^{-2}}{x^{-2}+1}\,dx.$$

Then I make the substitution $u = \dfrac{1}{x}$ to obtain

$$I = -\int_{-1}^{1} \frac{1}{1+u^2}\,du.$$

Since the last expression is the negative of the original integral, I have shown that $I = -I$. The only way this can be true is if the integral I is zero.

Explain to Fred what he did wrong.

(3) Explain carefully how to use *integration by parts* to evaluate $\int e^x \cos 2x\,dx$.

(4) Show that the improper integral $I = \displaystyle\int_{2}^{\infty} (1+x^3)^{-1/2}\,dx$ converges and that $0 \le I \le \sqrt{2}$.

(5) Show that the improper integral $I = \displaystyle\int_{1}^{\infty} \frac{2x+1}{5x^3+7x^2-2x-1}\,dx$ converges and that $0 \le I \le 3/5$.

(6) Show that the improper integral $\int_1^\infty \sin x^2\,dx$ converges and that its value lies between -1 and $+1$. *Hint.* $\sin x^2 = \dfrac{1}{2x} \cdot 2x \sin x^2$.

(7) Show that the improper integral $\int_0^2 \dfrac{\cos x}{x^2}\,dx$ does not converge. *Hint.* Consider dividing the interval [0,2] into two pieces, say $[0, \pi/3]$ and $[\pi/3, 2]$.

(8) Show that the improper integral $\int_2^\infty \dfrac{dx}{(1+x^5)^{1/6}}$ does not converge. *Hint.* Show that $1 + x^5 \le x^6$ whenever $x \ge 2$.

(9) Discuss the convergence of the improper integral $\int_e^\infty \dfrac{1}{(\ln x)^2}\,dx$. *Hint.* First prove that $\ln x$ is smaller than $\sqrt{x}$ for all positive x.

(10) Let $a > 0$. Show that the improper integral $\int_0^a \dfrac{\cos x}{x}\,dx$ diverges.

(11) Show that the improper integral $\int_1^\infty \dfrac{\cos x^2}{x^2}\,dx$ converges and that the value of the integral lies between -1 and $+1$.

(12) Show that the improper integral $\int_e^\infty \dfrac{1}{(\ln x)^2}\,dx$ diverges. *Hint.* Show that $\ln x \le \sqrt{x}$ for every positive number x.

(13) For this problem assume no prior knowledge of the natural logarithm, the exponential function, or the number e. For every $x > 0$ define

$$\ln x := \int_0^x \frac{1}{t}\,dt.$$

(a) Show that ln is strictly increasing and concave down.
(b) Show that $\ln 1 = 0$ and that $\ln 4 > 1$.
(c) Conclude from part (b) that there exists a number, call it e, such that $1 < e < 4$ and $\ln e = 1$.
(d) Show that if $a, b > 0$, then $\ln(ab) = \ln a + \ln b$.
(e) Show that if $a > 0$ and n is a natural number, then $\ln(a^n) = n \ln a$.
(f) Show that if $a, b > 0$, then $\ln\left(\dfrac{a}{b}\right) = \ln a - \ln b$.
(g) Prove that $\lim_{x\to\infty} \ln x = \infty$ and that $\lim_{x\to 0^+} \ln x = -\infty$.
(h) Conclude from part (g) that the range of ln is all of $\mathbb{R}$.
(i) Conclude from parts (a) and (h) that the function ln has an inverse. Let $\exp = \ln^{-1}$. Then the domain of exp is all of $\mathbb{R}$ and its range is the interval $(0, \infty)$.
(j) Prove that $D\exp(x) = \exp(x)$.
(k) Show that exp is strictly increasing and concave up.
(l) Prove that if $x, y \in \mathbb{R}$, then $\exp(x + y) = \exp x \cdot \exp y$.
(m) For every $a > 0$ and $x \in \mathbb{R}$ define $a^x := \exp(x \ln a)$. Prove that $e^x = \exp x$ for every x.
(n) Prove that if $a > 0$ and $x, y \in \mathbb{R}$, then $a^{x+y} = a^x \cdot a^y$.

(14) For this problem assume no prior knowledge of the trigonometric functions, the inverse trigonometric functions, or the number π. For every $x \in \mathbb{R}$ define

$$\arctan x := \int_0^x \frac{1}{1+t^2}\,dt.$$

(a) Show that arctan is strictly increasing. Where is it concave up? down?

(b) Show that the improper integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+t^2}\,dt$ converges. *Hint.* Why is it sufficient to show that the integral from 1 to ∞ converges?

(c) Define $\pi := \displaystyle\int_{-\infty}^{\infty} \frac{1}{1+t^2}\,dt$. Show that $2 < \pi < 4$. *Hint.* Why does it suffice to show that $1 < \displaystyle\int_0^{\infty} \frac{1}{1+t^2}\,dt < 2$?

(d) Show that $\arctan 1 = \dfrac{\pi}{4}$. *Hint.* Show that $\arctan 1 = \displaystyle\int_1^{\infty} \frac{1}{1+t^2}\,dt$ by making the change of variables $u = 1/t$.

(e) Show that arctan has an inverse. Identify the domain and range of $\arctan^{-1}$. (From now on let $\tan := \arctan^{-1}$.)

For all x in the domain of the function tan define $\sec x := \sqrt{1+\tan^2 x}$, $\cos x := \dfrac{1}{\sec x}$, and $\sin x := -D\cos x$. (*Notation:* $\tan^2 x$ means $(\tan x)^2$, *etc.*)

(f) Show that $D\tan x = \sec^2 x$.

(g) Show that $D\sec x = \tan x \sec x$.

(h) Show that $\tan x = \dfrac{\sin x}{\cos x}$.

(i) Show that $\sin^2 x + \cos^2 x = 1$.

(j) Show that $D\sin x = \cos x$. *Hint.* Try writing $\sin x$ as $\tan x \cos x$.

(k) How can the function sin be extended to a differentiable function on all of $\mathbb{R}$?

(15) Let $a > 0$.

(a) Show that $\displaystyle\lim_{\lambda\to 0^+} \int_\lambda^a \frac{\cos x}{x}\,dx = \infty$.

(b) Find $\displaystyle\lim_{\lambda\to 0^+} \frac{1}{\ln\lambda} \int_\lambda^a \frac{\cos x}{x}\,dx$.

15.4. Answers to Odd-Numbered Exercises

(1) 4, 2

(3) $\dfrac{\pi}{2}$

(5) 495

(7) 2

(9) 8, 19

(11) 12

(13) 8π

(15) 4

(17) 7

(19) -1

(21) 32

(23) 6

(25) π

(27) 24

(29) 3, 192

(31) 2, 24

(33) 2, 5

(35) 2, 3

(37) 15

(39) 35

(41) $\dfrac{1}{\sqrt{2}}$, $1+\sqrt{2}$

(43) 16

(45) 315

(47) 5

(49) $\dfrac{\pi}{12}$

(51) $\dfrac{1}{2}$, $1+2x$

(53) 17

(55) 3, π

(57) 1, 2

(59) π, 2

(61) 4, 15

(63) x^2-2x+2

(65) π, 2

(67) x^2-2, $2x$

(69) x, 3, -1

(71) 24, 2

(73) -1, x^2+x+1

(75) 3, x^2+1

(77) $\arctan x$

(79) $\frac{1}{4}x^4$, x^2+9, $\frac{1}{3}x$

(81) 4

(83) 2, $\ln x$, $\dfrac{x}{x^2+1}$

(85) -1

(87) 7

(89) 6, 3

(91) 2

(93) yes, 1

(95) no, -

(97) yes, $\dfrac{1}{5}$

(99) 12

(101) 4

(103) e

(105) $\sqrt{3}$

(107) 8

(109) 8

(111) 2

CHAPTER 16

APPLICATIONS OF THE INTEGRAL

16.1. Background

Topics: area under a curve, long-run value of a physical quantity, half-life of a radioactive substance, area of a region in the plane, arc length of a curve, volume of a solid using the method of cross-sections, volume of a solid using the shell method, work, centroid of a plane region, moment of a plane region about a line.

As with the chapter 12 on *applications of the derivative* there are few interesting "real-world" problems here. The purpose of the material is to establish a few elementary links between some mathematical concepts and corresponding physical quantities.

Newton's law of cooling says that the rate of cooling of a hot body is proportional to the difference between its temperature and that of the surrounding medium.

If the position of an object is given as a twice differentiable function x of time t, the VELOCITY v of the object is the derivative of position; that is, $v(t) = Dx(t)$. The ACCELERATION a of the object is the second derivative of position; that is $a(t) = D^2x(t) = Dv(t)$.

16.1.1. Definition. Let R be a region in the plane of area A with nonempty interior, $M_x(R)$ be the moment of R about the x-axis, and $M_y(R)$ be the moment of R about the y-axis. Then the CENTROID of R is the point $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{M_y(R)}{A} \qquad \text{and} \qquad \overline{y} = \frac{M_x(R)}{A}.$$

16.1.2. Theorem (Pappus's Theorem)**.** *Let R be a region in the plane and L be a line which passes through no point in the interior of R. When R is revolved once about L the volume of the resulting solid is the product of the area of R and the distance traveled by its centroid.*

16.2. Exercises

(1) The area of the (bounded) region between the curves $y = x^3 - x + 3$ and $y = x^2 + x + 3$ is $\dfrac{a}{12}$ where $a =$ ________ .

(2) The area of the region between the curves $y = \dfrac{1}{x}$ and $y = \dfrac{x}{4}$ for $1 \le x \le 3$ is $\dfrac{1}{a} + \ln \dfrac{a}{b}$ where $a =$ _____ and $b =$ _____ .

(3) The total area of the region between the curve $y = x^3 - x$ and the x-axis for $0 \le x \le 2$ is $\dfrac{a}{2}$ where $a =$ _____ .

(4) The area of the region enclosed by the curves $x = y^2$ and $3x = -y^2 + 4$ is $\dfrac{a}{9}$ where $a =$ _______ .

(5) The area between the curves $y = x^3 + x^2 - 3x + 4$ and $y = x^2 - 2x + 4$ from $x = -1$ to $x = 2$ is $\dfrac{a}{4}$ where $a =$ _______ .

(6) The area of the region between the curves $y = x^2$ and $y = \sqrt{x}$ over the interval $0 \le x \le 4$ is $\dfrac{a}{3}$ where $a =$ ________ .

(7) The length of the curve $y = \dfrac{1}{2}(e^x + e^{-x})$ from $x = 0$ to $x = \ln 3$ is _____ .

(8) Find the equation for a curve through the point $(0, 3)$ whose length from $x = 0$ to $x = 2$ is $\displaystyle\int_0^2 \sqrt{1 + \frac{1}{(2x+1)^2}}\, dx$. Answer: $y = a + \frac{1}{2} f(x)$ where $a =$ _____ and $f(x) =$ ______________ .

(9) The length of the curve $y = 8 \ln \dfrac{2 + \sqrt{x}}{2 - \sqrt{x}} - 8\sqrt{x}$ for $0 \le x \le 2$ is $a \ln b - b$ where $a =$ _____ and $b =$ _____ .

(10) The length of the curve $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$ from $x = 1$ to $x = 2$ is $\dfrac{a}{24}$ where $a =$ ________ .

(11) The length of the curve $y = \frac{1}{3}(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$ is __________ .

(12) The arclength of the curve $y = \ln \sec x$ for $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is $\ln(a + b\sqrt{b})$ where $a =$ _____ and $b =$ _____ .

(13) The length of the curve $y = \cosh x$ between $x = -1$ and $x = 1$ is $a - \dfrac{1}{a}$ where $a =$ _____ .

(14) The length of the curve $y = 2 \ln \dfrac{4}{4 - x^2}$ between $x = 0$ and $x = 1$ is $a \ln b - 1$ where $a =$ _____ and $b =$ _____ .

(15) The length of the curve $y = x^{3/2}$ from $x = 0$ to $x = \frac{5}{9}$ is $\dfrac{a}{27}$ where $a =$ ________ .

(16) The length of the curve $y = \frac{1}{2}x^2 - \frac{1}{4}\ln x$ from $x = 1$ to $x = 2$ is $\dfrac{a}{b} + \dfrac{1}{4}\ln b$ where $a =$ _____ and $b =$ _____ .

(17) The length of the curve $y = x^2$ between the origin and the point $(1, 1)$ is $\dfrac{1}{a}\sqrt{b} + \dfrac{1}{4}\ln(a + \sqrt{b})$ where $a =$ _____ and $b =$ _____ .

(18) A train is moving along a straight track in Africa at a steady rate of 60 mph. The engineer suddenly sees on the tracks ahead a large rhinoceros. He slams on the brakes and the train

decelerates as a constant rate stopping in exactly two minutes just an inch from the angry rhinoceros's horn. While stopping the train traveled _____ mi.

(19) If the work done in stretching a spring from a length of 3 ft to a length of 4 ft is one-half the work done in stretching it from 4 ft to 5 ft, then the natural length of the spring is _____ ft.

(20) A spring exerts a force of 1/2 pound when stretched 4 inches beyond its natural length. The work done in stretching the spring from its unstretched position to 8 inches beyond its natural length is _____ inch-pounds.

(21) The left end of a spring is attached to the origin of the x-axis. Its natural length is l. The work required to stretch the right end of the spring from $x = 6$ to $x = 8$ is 11/7 times the work required to stretch it from $x = 4$ to $x = 6$. Then the spring's natural length l is __________ .

(22) It takes a 20 pound force to stretch a particular spring 5 feet beyond its natural length. A 12 foot chain of uniform linear density weighs 48 pounds. One end of the spring is attached to the ceiling. When the chain is hung from the other end of the spring, the chain just touches the floor. The amount of work done in pulling the chain down three feet is ________ foot-pounds.

(23) A horizontal cylindrical tank of radius 3 ft and length 8 ft is half full of oil weighing 60 lb/ft^3. The work done in pumping out the oil to the top of the tank is $a\pi + b$ where $a =$ _______________ and $b =$ _______________ .

(24) A horizontal cylindrical tank of radius 4 feet and length 10 feet is half full of a liquid whose density is 60 pounds per cubic foot. What is the work done in pumping the liquid to the top of the tank? Answer: $a + b\pi$ where $a =$ _____________ and $b =$ _____________ .

(25) A conical tank is 20 ft. tall and the diameter of the tank at the top is 20 ft. It is filled with a liquid whose density is 12 lb./ft.^{3}. What is the work done in lifting the liquid to a trough 5 ft above the top of the tank? Answer: ___________ π ft-lbs.

(26) A hemispherical tank of water of radius 10 feet is being pumped out. The pump is placed 3 feet above the top of the tank. The work done in lowering the water level from 2 feet below the top to 4 feet below the top of the tank is (to five decimal places) _____________________ foot-pounds.

(27) An oil tank in the shape of a horizontal elliptic cylinder is 25 feet long. The major axis of the elliptical cross-section is horizontal and 12 feet long. The (vertical) minor axis is 6 feet long. The work done in emptying the contents of the tank through an outlet at the top of the tank when it is half full of oil weighing 60 pounds per cubic foot is ___ ___ 000 + ___ 0___ 00 π foot-pounds.

(28) A horizontal tank of length 25 feet has parabolic cross-sections (vertex down) 8 feet across at the top and 4 feet deep. The work done in pumping out the tank from an outlet 2 feet above the top of the tank if it is filled with oil weighing 60 pounds per cubic foot is 11 ___ ___ 00 foot-pounds.

(29) A bag of sand originally weighing 100 lb. is lifted at a constant rate. The sand leaks out uniformly at such a rate that the bag is just empty when it reaches a height of 35 ft. How much work is done in lifting the bag of sand that distance? Answer: ___________ ft-lbs.

(30) One electron is fixed on the x-axis at $x = -2$ and a second at $x = -1$. The force exerted by one electron on another is inversely proportional to the square of the distance between them. Use k as the constant of proportionality. Then the work done in moving

a third electron along the x-axis from $x = 3$ to $x = 0$ is (in appropriate units) $\dfrac{a}{20}k$ where $a =$ ________ .

(31) Assume that two particles repel each other with a force inversely proportional to the cube of the distance between them. Suppose that one particle is fixed and you know that the work done in moving the second particle from a distance of 10 inches to a distance of 5 inches from the first is 48 inch-pounds. Then the work done in moving the second particle from a distance of 10 inches to a distance of 1 inch from the first is ______________ inch-pounds.

(32) Let R be the region between the curve $y = \sin x$ and the x-axis for $0 \leq x \leq \pi$. Use *Pappus's theorem* to find the volume of the solid generated by revolving R about the y-axis. Answer: the volume is $a\pi^a$ where $a =$ _____ .

(33) Let R be the region between the curve $y = \cos x$ and the x-axis for $\frac{3\pi}{2} \leq x \leq \frac{5\pi}{2}$. Use *Pappus's theorem* to find the volume of the solid generated by revolving R about the y-axis. Answer: the volume is $a\pi^p$ where $a =$ _____ and $p =$ _____ .

(34) Let R be the region bounded by the parabola $y = -4x^2 + 12x$ and the x-axis. Use *Pappus's theorem* to find the volume of the solid which results when R is revolved about the y-axis. Answer: the volume is $a\pi^p$ where $a =$ ________ and $p =$ _____ .

(35) The centroid of the triangle whose vertices are $(0, 0)$, $(1, 0)$, and $(0, 1)$ is the point (________ , ________).

(36) The centroid of the region bounded by the parabola $y = 1 - x^2$ and the x-axis is the point (___________ , __________).

(37) The coordinates of the centroid of the region lying between the curves $y = \sqrt{1 - x^2}$ and $y = -1 - x$ with $0 \leq x \leq 1$ are $\overline{x} = \dfrac{a}{b(\pi + c)}$ and $\overline{y} = \dfrac{d}{b(\pi + c)}$ where $a =$ ________ , $b =$ _____ , $c =$ _____ , and $d =$ ________ .

(38) The centroid of the region $\{(x, y) : x^2 + y^2 \leq r^2 \text{ and } x \geq 0\}$ is $\left(\dfrac{a}{3\pi}, b\right)$ where $a =$ ______ and $b =$ ___ .

(39) The base of a solid is the region bounded by the parabolas $x = y^2$ and $x = 3 - 2y^2$. Find the volume of the solid if the cross-sections perpendicular to the x-axis are squares. Answer: _____ .

(40) The volume of the solid obtained by revolving the curve $y = \tan x$ $(0 \leq x \leq \frac{\pi}{4})$ about the x-axis is $a\left(1 - \dfrac{a}{b}\right)$ where $a =$ _____ and $b =$ _____ .

(41) Let R be the region between the curve $y = \cos x$ and the x-axis for $\frac{3\pi}{2} \leq x \leq \frac{5\pi}{2}$. The volume of the solid generated by revolving R about the x-axis is ___________ .

(42) Let R be the region which lies above the x-axis and below the curve $y = 2x - x^2$. The volume of the solid obtained when R is revolved about the x-axis is $\dfrac{a}{b}\pi$ where $a =$ ________ and $b =$ ________ .

(43) The base of a solid is the region bounded by the parabola $y = 1 - x^2$ and the x-axis. Suppose the cross-sections perpendicular to the x-axis are squares. Then the volume of the solid is $\dfrac{a}{b}$ where $a =$ ________ and $b =$ ________ .

(44) Let $f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } 1 < x \le 2 \end{cases}$ and A be the region which lies above the x-axis and under the curve $y = f(x)$. Then the volume of the solid generated by revolving A about the x-axis is $\dfrac{a\pi}{b}$ where $a =$ ____ and $b =$ ____ .

(45) A hemispherical basin of radius 10 feet is being used to store water. To what percent of capacity is the basin filled when the water is 5 feet deep?

Answer: ___ ___ . ___ ___ %.

(46) The volume of the solid generated by revolving the region bounded by $y^2 = 8x$ and $x = 2$ about the line $x = 2$ is $\dfrac{a}{b}\pi$ where $a =$ ________ and $b =$ ______ .

(47) The base of a solid is the region bounded by the parabolas $x = y^2$ and $x = 3 - 2y^2$. Find the volume of the solid if the cross-sections perpendicular to the x-axis are equilateral triangles. Answer: $\dfrac{a}{b}\sqrt{a}$ where $a =$ ____ and $b =$ ____ .

(48) If R is the region bounded by $x = \dfrac{3}{\sqrt{y+1}}$, $x = 0$, $y = 0$, and $y = 8$, then the volume of the solid generated by revolving R about the y-axis is $a\pi \ln 3$ where $a =$ _____ .

(49) If R is the region bounded by $y = x^2 + 1$ and $y = -x + 3$, then the volume of the solid generated by revolving R about the x-axis is $\dfrac{a\pi}{5}$ where $a =$ ________ .

(50) The arch $y = 2x - x^2$, $0 \le x \le 2$, is revolved about the line $y = c$ to generate a solid. The value of c that minimizes this volume is $\dfrac{a}{b}$ where $a =$ ____ and $b =$ ____ .

(51) Let R be the region between the curve $y = \cos x$ and the x-axis for $\frac{3\pi}{2} \le x \le \frac{5\pi}{2}$. The volume of the solid generated by revolving R about the y-axis is ______ .

(52) Let R be the region bounded by the parabola $y = -4x^2 + 12x$ and the x-axis. Use the shell method to find the volume of the solid which results when R is revolved about the y-axis. Answer: $a\,\pi$ where $a =$ _____ .

(53) Let R be the region under the curve $y = x^{3/2}$ for $0 \le x \le 4$. The volume of the solid generated by revolving R about the y-axis is $\dfrac{a}{7}\pi$ where $a =$ _________ .

(54) Let R be the region bounded by the curves $\sqrt{x} + \sqrt{y} = 1$ and $x + y = 1$. Then the volume of the solid generated by revolving R about the y-axis is $\dfrac{a\pi}{b}$ where $a =$ ____ and $b =$ _____ .

(55) Let R be the region above the x-axis and below $y = 2x - x^2$. The volume generated when R is revolved about the y-axis is $\dfrac{a\pi}{b}$ where $a =$ ____ and $b =$ ____ .

(56) Use the shell method to find the volume generated by revolving the curve $y^2 = 8x$ about the line $x = 2$. Answer: $\dfrac{a}{15}\pi$ where $a =$ _________ .

(57) Let R be the region bounded by the parabola $y = -x^2 + 6x - 8$ and the x-axis. The volume of the solid generated by revolving R about the y-axis is $a\pi$ where $a =$ ____ .

(58) Let R be the region bounded by the curves $x = \dfrac{y^4}{4} - \dfrac{y^2}{2}$ and $x = \dfrac{y^2}{2}$. The volume of the solid obtained by revolving R about the x-axis is $\dfrac{a}{3}\pi$ where $a =$ ____ .

(59) Let R be the region bounded by the curves $y = \dfrac{1}{2x}$, $y = 0$, $x = 1$, and $x = 5$. The volume of the solid generated by revolving R about the y-axis is $a\pi$ where $a =$ _____ .

(60) Let R be the region bounded by the curves $y = |x|$ and $y = 2$. Use the shell method to find the volume of the solid generated by revolving R about the x-axis. Answer: $\dfrac{a}{3}\pi$ where $a =$ ________ .

(61) Let R be the region in the first quadrant bounded by the curves $y = x^2$ and $y = 2x$. Use the shell method to find the volume of the solid generated by revolving R about the line $x = 3$. Answer: ____________________ .

16.3. Problems

The first problem below is a slight variant of the famous "snowplow" problem, which I first came across in Ralph Palmer Agnew's classic text [1] on *Differential Equations*. This wonderful problem is much more like a "real-world" problem than most of the others included here in that it requires one to make a number of rather strong assumptions (simplifications) even to get started.

(1) One morning it started snowing in Portland. A snowplow was dispatched at noon to clear a highway. It had plowed 2 miles by 1 pm and 3 miles by 2 pm. At what time did it begin snowing? *Hint.* Suppose w is the width of the plow blade, $y(t)$ is the depth of the snow at time t, and $x(t)$ is the distance the plow has traveled by time t. What reasonable assumptions might you make that would justify taking $wy(t)x'(t)$ to be constant?

(2) You are considering the problem of calculating the surface area of a solid of revolution. The curve $y = f(x)$ (for $a \leq x \leq b$) has been rotated about the x-axis. You set up a partition $(x_0, x_1, \dots, x_n)$ of $[a, b]$ and approximate the surface area of the resulting solid by connecting consecutive points $\big(x_{k-1}, f(x_{k-1})\big)$ and $\big(x_k, f(x_k)\big)$ by a straight line segment, which, when rotated about the x-axis, produces a portion of a cone. Taking the limit of these conical approximations you end up with the integral

$$\int_a^b 2\pi f(x)\sqrt{1 + \big(f'(x)\big)^2}\,dx$$

which you claim represents the surface area of the solid.

Your friend Fred has chosen to make cylindrical approximations rather than conical ones and produces the integral

$$\int_a^b 2\pi f(x)\,dx$$

which he claims represents the surface area. So you and Fred argue.

He says your formula is too complicated. You say his is too simple to work. Unable to convince him, you look up the formula in a well known calculus text and find that the author agrees with you. Fred is unimpressed; he says the author probably made the same mistake you did. You find several more references to support your work. Fred thinks they all probably copied from the person who made the original mistake.

Find a *completely convincing* argument that even skeptical Fred will accept that his formula can't be correct and that yours is better.

(3) The sides of a square are initially 4 inches in length and increase at a constant rate of 3 inches per second. One corner of the square moves along a line L at a speed of 2 inches per second for 5 seconds. During this time the square makes one complete revolution about L, revolving at a constant rate and remaining always perpendicular to L. Explain carefully how to compute the volume of the solid generated by the square. Carry out the computation you describe.

(4) Consider a cable of uniform linear density hanging between two smooth pegs (of negligible diameter) at the same height. The two free ends of the cable hang straight down and are 12 feet in length. Place the origin of the coordinate system at the point where the sag of the cable is greatest. Between the two pegs the cable hangs in a curve $y = f(x)$ for $-M \leq x \leq M$ (where $2M$ is the distance between the pegs). Let $\theta(x)$ be the angle the (tangent to the) cable makes with the horizontal at each x between $-M$ and M. The value of θ just to the left of M is $\pi/3$. What is the total length of the cable? How far does it sag? How far apart are the pegs?

Hint. Suppose the cable's linear density is δ pounds per foot. Let $T(x)$ be the tension in the cable at the point $(x, f(x))$. Try the following steps.

(a) Start by considering the horizontal and vertical components of the forces acting on a small piece of cable lying between x and $x + \Delta x$. Show that $T(x)\cos(\theta(x))$ is a constant, say K.
(b) Find an expression for the derivative of $T(x)\sin(\theta(x))$. (It should involve the derivative of the arclength of f.)
(c) Use the results of (a) and (b) to calculate the derivative of $\tan(\theta(x))$.
(d) Express K in terms of δ.
(e) Substitute this in the result of (c).
(f) Use (e) to calculate the length of that portion of the cable lying between the pegs. Write down the total length of the cable.
(g) In (e) make the substitution $u = \tan(\theta(x))$. Treat the derivative $\dfrac{du}{dx}$ as a fraction. Move all terms containing x to one side of the equation and all terms containing u to the other. Integrate both sides. Solve for u. Then evaluate the constant of integration by examining the relationship between $u(-M)$ and $u(M)$.
(h) Since u is $\dfrac{dy}{dx}$ a second integration will produce a formula for y and a second constant of integration. Evaluate this second constant by looking at what happens at the origin.
(i) Evaluate M by setting the integral representing the arclength of the function you found in (h) equal to the value you computed in (f). Give the exact answer in terms of natural logarithms and a decimal approximation.
(j) Find the sag in the cable and the distance between the pegs.

(5) A rope of length ℓ feet that weighs σ lbs/ft is lying on the ground. What is the work done in lifting the rope so that it hangs from a beam 2ℓ feet high? Explain clearly how to solve this problem in two different ways.

(6) Let R be a bounded region in the plane and L a line which does not pass through any interior point of R. For the purposes of this problem assume the following facts about $M_L(R)$, the moment of R about L, to be known.
(1) If R is a rectangular region, then $M_L(R)$ is the product of the area of R and the distance between the center of R and L. (Of course, the *center* of a rectangle is the point at which the diagonals intersect.)
(2) If R is divided into nonoverlapping subregions $R_1, \dots, R_n$, then the moment of R about L is the sum of the moments of the subregions R_k about L. That is,

$$M_L(R) = \sum_{k=1}^{n} M_L(R_k).$$

Now suppose that f and g are continuous functions on the interval $[a, b]$ and that $0 \leq g(x) \leq f(x)$ for $a \leq x \leq b$. Let R be the region between the curves $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$.

Set up appropriate Riemann sums to approximate the moment $M_x(R)$ of R about the x-axis and the moment $M_y(R)$ of R about the y-axis. Take limits of these sums to find integral formulas for $M_x(R)$ and $M_y(R)$.

(7) A right circular cylinder of radius 3 is standing on end. It is cut by two planes; one of these is horizontal and the other makes an angle of 60° with the first and passes through a diameter of the cross section of the cylinder made by the first plane. This creates two wedges. Find the volume of (either) one of them.

(8) **[This problem is suitable only for those with access to a decent Computer Algebra System (CAS)]** We are given two ellipses

$$\frac{(x-5)^2}{3^2} + \frac{(y-5)^2}{2^2} = 1 \qquad \text{and} \qquad \frac{(x-5)^2}{1^2} + \frac{(y-5)^2}{4^2} = 1.$$

Write a program in a CAS designed to approximate the area of the region R common to the two ellipses. The program should do the following.

(i) Choose points at random in the square $[0, 10] \times [0, 10]$.
(ii) Test each point that is chosen to see whether or not it falls inside the region R.
(iii) Count the number of points which pass this test and the number that fail.
(iv) Use the results of (iii) and the area of the original square to estimate the area of R.

Document your program carefully. Try running the program using different numbers of points. What do the results indicate to you about the accuracy of your best answer? Suggest alternative methods (that is, other than using more points) for improving the accuracy.

(9) Show how to use the method of cross-sections to derive the formula for the volume of a right circular cone with base radius r and height h.

(10) Show how to use the shell method to derive the formula for the volume of a right circular cone with base radius r and height h.

(11) Show how to use the method of cross-sections to derive the formula for the volume of a sphere with radius r.

(12) Show how to use the shell method to derive the formula for the volume of a sphere with radius r.

(13) Find the centroid of the region in the plane lying between the x-axis and the curve $y = \dfrac{1}{x^2}$ on the interval $[1, \infty)$. Discuss as thoroughly as you can the problem of applying this result to actual physical laminae of uniform density.

(14) For all $x > 0$ define

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt.$$

(a) Show that this definition makes sense. That is, show that the improper integral converges. *Hint.* Prove first that $\lim_{t \longrightarrow \infty} t^p e^{-\frac{1}{2}t} = 0$ for all real numbers p. Use this to show that there exists a number $M > 0$ (depending on p) such that $0 \le t^p e^{-\frac{1}{2}t} \le M$ whenever $t \ge 1$. And then conclude that $\int_1^\infty t^{x-1} e^{-t}\, dt$ exists for all $x > 0$.
(b) Show that $\Gamma(x + 1) = x\Gamma(x)$ for all $x > 0$.
(c) Show that $\Gamma(n + 1) = n!$ for each natural number n.
(d) Using double integrals in polar coordinates one can show that $\int_0^\infty e^{-u^2}\, du = \frac{1}{2}\sqrt{\pi}$ (see problem 7 in chapter 33). Use this fact to calculate $\Gamma(\frac{1}{2})$ and $\Gamma(\frac{3}{2})$.

16.4. Answers to Odd-Numbered Exercises

(1) 37

(3) 5

(5) 11

(7) $\dfrac{4}{3}$

(9) 8, 2

(11) 12

(13) e

(15) 19

(17) 2, 5

(19) $\dfrac{5}{2}$

(21) $\dfrac{3}{2}$

(23) 6480, 8640

(25) $80,000$

(27) 5, 4, 4, 5

(29) 1750

(31) 1584

(33) 8, 2

(35) $\dfrac{1}{3}$, $\dfrac{1}{3}$

(37) 14, 3, 6, -10

(39) 6

(41) $\frac{1}{2}\pi^2$

(43) 16, 15

(45) 3, 1, 2, 5

(47) 3, 2

(49) 117

(51) $8\pi^2$

(53) 512

(55) 8, 3

(57) 8

(59) 4

(61) $\dfrac{16\pi}{3}$

Part 5

SEQUENCES AND SERIES

CHAPTER 17

APPROXIMATION BY POLYNOMIALS

17.1. Background

Topics: Taylor polynomials, Maclaurin polynomials, Lagrange form for the remainder (error term).

17.2. Exercises

(1) The Maclaurin polynomial of degree three for $f(x) = \arcsin x$ is $a + bx + cx^2 + dx^3$ where $a =$ ____ , $b =$ ____ , $c =$ ____ , and $d =$ ______ .

(2) The Taylor polynomial of degree 4 for $\sec x$ is $a + bx^2 + cx^4$ where $a =$ ____ , $b =$ _____ , and $c =$ _______ .

(3) The Taylor polynomial of degree 4 for $f(x) = 2(x-1)^3 + 2(x+1)^{1/2}$ is $7x - ax^2 + \frac{17}{8}x^3 - bx^4$ where $a =$ _______ and $b =$ _______ .

(4) The Taylor polynomial of degree 3 for $f(x) = (x+4)^{3/2} - (x+1)^{3/2}$ is $7 + \frac{a}{2}x + \frac{b}{16}x^2 + \frac{c}{128}x^3$ where $a =$ ____ , $b =$ _____ , and $c =$ ____ .

(5) The Taylor polynomial of degree 4 for $f(x) = \dfrac{1 - \cos 2x}{x^2}$ is $a + bx^2 + cx^4$ where $a =$ ____ , $b =$ _____ , and $c =$ ______ .

(6) Express the polynomial $p(x) = x^3 - x^2 + 3x - 5$ as a polynomial in powers of $x - 2$. Answer: $a + b(x-2) + c(x-2)^2 + (x-2)^3$ where $a =$ ____ , $b =$ ______ , and $c =$ ____ .

(7) Find an approximate value of $\displaystyle\int_0^1 \cos x^2\, dx$ by replacing the integrand with an appropriate polynomial of degree 8. Answer: $1 - \dfrac{1}{a} + \dfrac{1}{b}$ where $a =$ _____ and $b =$ ______ .

(8) The Taylor polynomial of degree 4 for $f(x) = e^{2x}\sin x$ is $a + x + bx^2 + cx^3 + x^4$ where $a =$ ____ , $b =$ ____ , and $c =$ ____ .

(9) The Taylor polynomial of degree 5 for $f(x) = \sin 3x$ is $ax + \frac{b}{2}x^3 + \frac{c}{40}x^5$ where $a =$ ____ , $b =$ _____ , and $c =$ ______ .

(10) Write $p(x) = x^4 - 2x^3 + 4x^2 - 7x + 6$ as a polynomial in powers of $x - 1$.
Answer: $a + b(x-1) + c(x-1)^2 + d(x-1)^3 + (x-1)^4$ where
$a =$ ______ , $b =$ ______ , $c =$ ______ , and $d =$ ______ .

(11) Write $x^4 + x^3 + x^2 + 3x + 5$ as a polynomial in powers of $x + 2$.
Answer: $a + b(x+2) + c(x+2)^2 + d(x+2)^3 + (x+2)^4$ where
$a =$ ______ , $b =$ ______ , $c =$ ______ , and $d =$ ______ .

(12) Write $p(x) = x^4 + 3x^3 + 4x^2 + 5x + 7$ as a polynomial in powers of $x + 1$. Answer: $a + b(x+1) + c(x+1)^2 + d(x+1)^3 + (x+1)^4$ where $a =$ ____ , $b =$ ____ , $c =$ ____ , and $d =$ _____ .

(13) Write $p(x) = x^4 + x^3 - x^2 - x + 4$ as a polynomial in powers of $x - 1$.
Answer: $a + b(x-1) + c(x-1)^2 + d(x-1)^3 + (x-1)^4$ where
$a =$ ______ , $b =$ ______ , and $c =$ ______ , and d ______ .

(14) Write $x^4 - x^3 + 3x^2 - 5x - 6$ as a polynomial in powers of $x - 2$.
Answer: $a + b(x-2) + c(x-2)^2 + d(x-2)^3 + (x-2)^4$ where
$a =$ ______ , $b =$ ______ , and $c =$ ______ , and d ______

(15) Find the Maclaurin polynomial of degree 4 for the function $f(x) = (x+1)^{3/2} + (x+1)^{1/2}$.
Answer: $A + Bx + Cx^2 + Dx^3 + Ex^4$ where $A =$ ____ , $B =$ ____ , $C =$ ____ , $D =$ ____ , and $E =$ _______ .

(16) Find the Maclaurin polynomial of degree 4 for the function $f(x) = \dfrac{x-1}{x+1}$.

Answer: $A + Bx + Cx^2 + Dx^3 + Ex^4$ where $A =$ ____ , $B =$ ____ , $C =$ ____ , $D =$ ____ , and $E =$ ____ .

(17) Find the Maclaurin polynomial of degree 4 for $f(x) = \ln\cos x$.

Answer: $A + Bx + Cx^2 + Dx^3 + Ex^4$ where $A =$ ____ , $B =$ ____ , $C =$ ____ , $D =$ ____ , and $E =$ ______ .

(18) Approximate $\cos 1$ making use of an appropriate polynomial of degree 4. Express your answer as a single fraction in lowest terms. Answer: $\dfrac{a}{24}$ where $a =$ ________ .

(19) Find an approximate value for $\displaystyle\int_0^1 \sin x^3\,dx$ by replacing the integrand with an appropriate polynomial of degree 15. Answer: $\dfrac{a}{1920}$ where $a =$ _________ .

(20) Using the Lagrange form for the remainder, determine the smallest number of non-zero terms a Taylor polynomial for $\sin x$ must have to guarantee an approximation of $\sin 0.1$ which is accurate to within 10^{-10}. Answer: ____ .

(21) Find $e^{1/5}$ with an error of less than 10^{-5}. Express your answer as a sum of fractions in lowest terms. Answer: $1+\dfrac{1}{5}+\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$ where $a =$ ____ , $b =$ _____ , and $c =$ ________ .

(22) Let $f(x) = \dfrac{1-\cos 2x}{x^2}$. Then $f^{(10)}(0) = -\dfrac{2^p}{a}$ where $p =$ ____ and $a =$ ____ .

(23) Let $f(x) = x^3 \sin x$. Then $f^{(8)}(0) =$ ________ .

17.3. Problems

(1) Let $f(x) = \sin x$. Use an appropriate polynomial of degree 3 to approximate $\sin(0.3)$. Show that the error in this approximation is less than $2.1 \cdot 10^{-5}$.

(2) Use an appropriate polynomial to approximate $\ln 1.5$ with an error of $< 10^{-2}$. (Express your answer as the quotient of two natural numbers.) Give a careful proof that the error in this approximation is less than 10^{-2}.

(3) Find an approximate value for $\displaystyle\int_0^{1/2} e^{x^2}\,dx$ by replacing the integrand with an appropriate polynomial of degree 4. (Express your answer as the quotient in lowest form of two natural numbers.) Give a careful argument to show that the error in this approximation is less than $4 \cdot 10^{-4}$.

(4) Find an approximate value of $\displaystyle\int_0^{1/2} \exp(x^3)\,dx$ by replacing the integrand with an appropriate polynomial of degree 6. (Express your answer as the quotient of two natural numbers.) Give a careful argument to show that the error in this approximation is less than $\frac{1}{15}2^{-11}$.

17.4. Answers to Odd-Numbered Exercises

(1) $0,\ 1,\ 0,\ \dfrac{1}{6}$

(3) $\dfrac{25}{4},\ \dfrac{5}{64}$

(5) $2,\ -\dfrac{2}{3},\ \dfrac{4}{45}$

(7) $10,\ 216$

(9) $3,\ -\dfrac{9}{2},\ \dfrac{81}{40}$

(11) $11,\ -21,\ 19,\ -7$

(13) $4,\ 4,\ 8,\ 5$

(15) $2,\ 2,\ \dfrac{1}{4},\ 0,\ -\dfrac{1}{64}$

(17) $0,\ 0,\ -\dfrac{1}{2},\ 0,\ -\dfrac{1}{12}$

(19) 449

(21) $50,\ 750,\ 15000$

(23) 336

CHAPTER 18

SEQUENCES OF REAL NUMBERS

18.1. Background

Topics: sequences, monotone (increasing and decreasing) sequences, bounded sequences, convergence of sequences.

18.1.1. Notation. Technically a SEQUENCE of real numbers is a function from the set $\mathbb{N}$ of natural numbers (or, occasionally, from the set $\mathbb{Z}^+$ of positive integers) into $\mathbb{R}$. In the following material we will denote sequences by whichever of the following four notations

$$a = \big(a_n\big)_{n=1}^{\infty} = (a_n) = (a_1, a_2, a_3, \dots)$$

seems most convenient at the moment.

In elementary algebra texts authors go to a great deal of trouble to distinguish between the notation for finite sets and for n-tuples. For example, the set containing the numbers 3 and 5 is denoted by $\{3, 5\}$ while the ordered pair whose first entry is 3 and whose second entry is 5 is denoted by $(3, 5)$. The principal difference is that $\{3, 5\} = \{5, 3\}$ while $(3, 5) \neq (5, 3)$. Sequences have an order just as ordered pairs, ordered triples, and so on have. So it is somewhat surprising that many authors of calculus texts choose the notation $\{a_1, a_2, a_3, \dots\}$ for sequences.

18.1.2. Definition. A sequence (a_n) of real numbers is INCREASING if $a_{n+1} \geq a_n$ for every $n \in \mathbb{N}$; it is STRICTLY INCREASING if $a_{n+1} > a_n$ for every n. A sequence is DECREASING if $a_{n+1} \leq a_n$ for every n, and is STRICTLY DECREASING if $a_{n+1} < a_n$ for every n. A sequence is MONOTONE if it is either increasing or decreasing.

Here are three useful facts about convergence of sequences of real numbers.

18.1.3. Theorem (Monotone sequence theorem)**.** *Every bounded monotone sequence of real numbers converges.*

18.1.4. Theorem. *If (a_n) is a convergent sequence with $a_n \to l$ as $n \to \infty$ and f is a function which is continuous at l, then the sequence $\big(f(a_n)\big)$ converges and*

$$\lim_{n\to\infty} f(a_n) = f(l).$$

18.1.5. Theorem. *If $\big(a_n\big)$ is a sequence of real numbers, then*

$$\lim_{n\to\infty} {a_n}^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

if the limit on the right exists.

18.2. Exercises

(1) Let $\big(a_k\big)_{k=1}^{\infty}$ be the sequence whose first few terms are $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \frac{1}{14}, \frac{1}{17}, \dots$. Then an explicit formula for a_k which works for all values of k is $a_k = (b(k))^{-1}$ where $b(k) =$ __________ .

(2) Find the n^{th} term of the sequence $(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots)$. Answer: $a_n =$ __________ .

(3) Find the n^{th} term of the sequence $(\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots)$. Answer: $a_n =$ ______________ .

(4) Let $\big(a_k\big)_{k=1}^{\infty}$ be the sequence whose first few terms are $\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \frac{1}{26}, \frac{1}{37}, \dots$. Then an explicit formula for a_k which works for all values of k is $a_k = (b(k))^{-1}$ where $b(k) =$ __________ .

(5) Find the n^{th} term of the sequence $(1, 5, 1, 5, 1, \dots)$. Answer: $a_{2n} =$ _____ and $a_{2n-1} =$ _____.

(6) Let $\big(a_k\big)_{k=1}^{\infty}$ be the sequence whose first few terms are $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}, \frac{1}{42}, \frac{1}{56}, \frac{1}{72}, \frac{1}{90}, \dots$. Then an explicit formula for a_k which works for all values of k is $a_k = (b(k))^{-1}$ where $b(k) =$ _____________ .

(7) Let $a_n = \dfrac{2}{3n+1}$ for every $n \in \mathbb{N}$. Then the sequence (a_n)

is ________________________ (increasing/decreasing/not monotone)

and _____________ (is/is not) bounded.

(8) Let $a_n = \dfrac{2n}{n+1}$ for every $n \in \mathbb{N}$. Then the sequence (a_n)

is ________________________ (increasing/decreasing/not monotone)

and _____________ (is/is not) bounded.

(9) Let $a_n = \dfrac{n^2+2}{n^2+1}$ for every $n \in \mathbb{N}$. Then the sequence (a_n)

is ________________________ (increasing/decreasing/not monotone)

and _____________ (is/is not) bounded.

(10) Let $a_n = \ln \dfrac{n^2+2}{n+1}$ for every $n \in \mathbb{N}$. Then the sequence (a_n)

is ________________________ (increasing/decreasing/not monotone)

and _____________ (is/is not) bounded.

(11) Let $a_n = \dfrac{\sqrt{n+1}}{5n+3}$ for every $n \in \mathbb{N}$. Then the sequence (a_n)

is ________________________ (increasing/decreasing/not monotone)

and _____________ (is/is not) bounded.

(12) $\lim\limits_{n\to\infty} (\sqrt{n^2 - 7n} - n) = -\dfrac{a}{2}$ where $a =$ _____ .

(13) $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{5n}\right)^{10n} = e^p$ where $p =$ _____ .

(14) $\lim\limits_{n\to\infty} \displaystyle\int_{n-3}^{n+4} \frac{x^2+5}{x^2+1}\,dx =$ _____ .

(15) $\lim\limits_{n\to\infty} \displaystyle\int_{2n}^{3n} \frac{x+3}{x^2+1}\,dx = \ln\frac{a}{2}$ where $a =$ _____ .

(16) $\lim_{n\to\infty} n(\sqrt{n^2+8}-\sqrt{n^2+3}) = \dfrac{a}{2}$ where $a =$ _____ .

(17) $\lim_{n\to\infty} n^{2/3}((n+2)^{1/3}-n^{1/3}) = \dfrac{a}{3}$ where $a =$ _____ .

(18) $\lim_{n\to\infty} \left(\dfrac{(n!)^2}{(2n)!}\right)^{1/n} = \dfrac{1}{a}$ where $a =$ _____ .

(19) $\lim_{n\to\infty} \left(1+\dfrac{2}{3n}\right)^{4n} = e^{a/3}$ where $a =$ _____ .

(20) $\lim_{n\to\infty} n^2\left(\sqrt{n^4+11}-\sqrt{n^4+1}\right) =$ _____ .

(21) $\lim_{n\to\infty} \left(\dfrac{(3n)!}{(n!)^3}\right)^{1/n} =$ ________ .

(22) $\lim_{n\to\infty} n^{3(n-1)^{-1}} =$ _____ .

(23) Find $\lim_{n\to\infty} \displaystyle\int_{n+3}^{n+5} \dfrac{x^{5/2}-1}{x^3+4}\,dx =$ _____ .

(24) $\lim_{n\to\infty} \dfrac{2n}{\sqrt[4]{256n^4+81n^2+49}} = \dfrac{1}{a}$ where $a =$ _____ .

(25) $\lim_{n\to\infty} \left(\dfrac{1+2+\cdots+n}{n+2}-\dfrac{n}{2}\right) = -\dfrac{1}{a}$ where $a =$ _____ .

(26) $\lim_{n\to\infty} \displaystyle\int_{2/n}^{1/n} \dfrac{x+1}{x^2+1}\,dx =$ _____ .

(27) $\lim_{n\to\infty} \displaystyle\int_{n}^{n+1} \dfrac{x+1}{x^2+1}\,dx =$ _____ .

(28) $\lim_{n\to\infty} \displaystyle\int_{n}^{2n} \dfrac{x+1}{x^2+1}\,dx =$ ________ .

(29) $\lim_{n\to\infty} \displaystyle\int_{n}^{n^2} \dfrac{x+1}{x^2+1}\,dx =$ _____ .

(30) $\lim_{n\to\infty} \dfrac{2^n+3n}{3^n-2n} =$ _____ .

(31) $\lim_{n\to\infty} \dfrac{n}{(n!)^{1/n}} =$ _____ .

(32) $\lim_{n\to\infty} \left[\dfrac{(3n)!}{(n!)^3}\right]^{1/n} =$ ________ .

(33) $\lim_{n\to\infty} \left[\dfrac{1}{2}\left(\dfrac{2}{3}\right)^2\left(\dfrac{3}{4}\right)^3\cdots\left(\dfrac{n}{n+1}\right)^n\right]^{1/n} = \dfrac{1}{a}$ where $a =$ _____ .

(34) Let $a_n = \dfrac{3n^2-4}{5-6n^2}$ for all $n \in \mathbb{N}$. Then the sequence (a_n) converges to $-\dfrac{1}{a}$ where $a =$ _____ .

(35) Let $a_n = \dfrac{3n^2-4}{5-6n^3}$ for all $n \in \mathbb{N}$. Then the sequence (a_n) converges to _____ .

(36) Let $a_n = \arctan 3n$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} a_n = \dfrac{\pi}{a}$ where $a =$ _____ .

(37) Let $a_n = \ln(4n+5) - \ln 2n$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} a_n =$ ________ .

(38) Let $a_n = \dfrac{(-4)^n}{n!}$ for each $n \in \mathbb{N}$. Then the sequence (a_n) converges to _____ .

(39) Let $a_n = \dfrac{\sin^3(3n^2-4)}{5n-2}$ for all $n \in \mathbb{N}$. Then the sequence (a_n) converges to _____ .

(40) $\displaystyle\lim_{n\to\infty} \int_{n-3}^{n+5} \frac{x^2+4}{x^2+1}\,dx =$ _____ .

(41) For each natural number k let $a_k = (\ln x)^k$. Then the sequence (a_k) converges if and only if $a < x \le b$ where $a =$ ______ and $b =$ _____ .

(42) For each natural number k let $a_k = \dfrac{\ln(1+k^2)}{\ln(4+3k)}$. Then, to four decimal places, $a_{100} = 1.$ _ _ 11, $a_{100{,}000} = 1.$ _ _ 58, and $\lim_{k\to\infty} a_k =$ _____ .

(43) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{(n+2k)^4}{n^5} = \frac{a}{5}$ where $a =$ ________ .

(44) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+3k} = \frac{1}{a}\ln b$ where $a =$ _____ and $b =$ _____ .

(45) Let $a_n = \dfrac{1^2}{n^3} + \dfrac{2^2}{n^3} + \cdots + \dfrac{n^2}{n^3}$ for each n. Then $\lim_{n\to\infty} a_n = \dfrac{1}{a}$ where $a =$ _____ .

(46) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2+n^2}{kn^2+n^3} = a\ln a - \frac{1}{a}$ where $a =$ _____ .

(47) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{15}{60k+4n} = \ln a$ where $a =$ _____ .

(48) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{(2n+7k)^2} = \frac{1}{a}$ where $a =$ ______ .

(49) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\ln(5k+n)-\ln n}{n} = \frac{a}{b}\ln a - 1$ where $a =$ _____ and $b =$ _____ .

(50) Let $a_n = \displaystyle\sum_{k=1}^{n} \frac{k^2}{n^3}$ for all n. Then $\lim_{n\to\infty} a_n = \dfrac{1}{a}$ where $a =$ _____ .

18.3. Problems

(1) Let $x_1 = -7$ and $x_{n+1} = \frac{1}{5}(3x_n + 4)$ for all integers $n \geq 1$.
  (a) Show that $x_n \leq 2$ for all n.
  (b) Show that (x_n) is increasing.
  (c) Show that (x_n) converges.
  (d) Find $\lim\limits_{n\to\infty} x_n$.

(2) Let

$$\begin{aligned} a_1 &= \sqrt{2} \\ a_2 &= \sqrt{2+\sqrt{2}} \\ a_3 &= \sqrt{2+\sqrt{2+\sqrt{2}}} \\ &\vdots \end{aligned}$$

  (a) Express a_{n+1} in terms of a_n.
  (b) Prove that (a_n) has a limit, say r. *Hint.* Show first that $\sqrt{2} \leq a_n < 2$ for all n.
  (c) Find r.

(3) Let $a_n = \left(\displaystyle\sum_{k=1}^{n} \frac{1}{k}\right) - \ln n$ for $n \geq 1$.
  (a) Show that $\ln(n+1) - \ln n \geq \dfrac{1}{n+1}$ for $n \geq 1$.
  (b) Show that (a_n) is a decreasing sequence.
  (c) Show that $\ln n \leq \displaystyle\sum_{k=2}^{n} \frac{1}{k-1}$. *Hint.* For any integrable function

$$\int_1^n f = \sum_{k=2}^{n} \int_{k-1}^{k} f.$$

  (d) Show that (a_n) converges.

(4) Let $x_1 = 1$ and $x_{n+1} = \frac{1}{3}(x_n^3 - 2)$ for every $n \geq 1$.
  (a) Show that (x_n) is bounded. *Hint.* Show $-1 < x_n < 2$ for every n.)
  (b) Show that (x_n) is decreasing.
  (c) Show that (x_n) converges.
  (d) Find $\lim\limits_{n\to\infty} x_n$.

(5) Let $x_1 = -10$ and $x_{n+1} = 1 - \sqrt{1 - x_n}$ for every $n \geq 1$.
  (a) Show that $x_n < 0$ for every n.
  (b) Show that (x_n) is increasing.
  (c) Show that (x_n) converges.
  (d) Find $\lim\limits_{n\to\infty} x_n$.
  (e) Find $\lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n}$ (if this limit exists.)

(6) Suppose that a sequence (x_n) satisfies $(7 + x_n)x_{n+1} = 7(1 + x_n)$ for all $n \geq 1$, and suppose that $x_1 = 10^9$.
  (a) Show that $x_n > \sqrt{7}$ for every n.
  (b) Show that (x_n) is decreasing.
  (c) Show that (x_n) converges.
  (d) Find $\lim\limits_{n\to\infty} x_n$.

18.4. Answers to Odd-Numbered Exercises

(1) $3k - 1$

(3) $\dfrac{1}{2n+1}$

(5) 5, 1

(7) decreasing, is

(9) decreasing, is

(11) decreasing, is

(13) 2

(15) 3

(17) 2

(19) 8

(21) 27

(23) 0

(25) 2

(27) 0

(29) ∞

(31) e

(33) e

(35) 0

(37) $\ln 2$

(39) 0

(41) $\dfrac{1}{e}$, e

(43) 121

(45) 3

(47) 2

(49) 6, 5

CHAPTER 19

INFINITE SERIES

19.1. Background

Topics: infinite series, sum of an infinite series, geometric series, Taylor series.

19.1.1. Remark. *Are sequences and series the same thing?* In ordinary discourse the words "sequence" and "series" are ordinarily used interchangeably. There seems to be no distinction between saying that *a sequence of events led to an outcome* and saying that *a series of events led to an outcome.* So naturally the first thing a conscientious calculus textbook writer has to do is make sure that students understand the difference. Sequences are ordered; they are lists; they are functions whose domain is the set of natural numbers. Series are sums, albeit infinite sums. [*Digression*: it is unfortunate that in many texts the distinction is immediately muddied by using the notation $\{a_k\}$ for sequences. Beginners expend much effort in learning that $\{a, b\}$ is a *set* (an unordered pair) whereas (a, b) is an *ordered pair.* This leads rational students to expect $\{a_k\}$ to denote a set and (a_k) a sequence. It is a cruel trick to switch and use $\{a_k\}$ for a sequence.]

Once it is agreed that sequences are lists, what exactly are series? One approach is to think of them as being (infinite) sums of numbers. The (finite) series $2+4+8$ is just another way of writing the number 14. So when we write $\sum_{k=1}^{\infty} 2^{-k} = 1$, isn't it clear that what we are saying is that the infinite series $\sum_{k=1}^{\infty} 2^{-k}$ is just the number 1? But this can't be right. If an infinite series is just a number, how could we possibly make sense of the assertion that the harmonic series $\sum_{k=1}^{\infty} k^{-1}$ diverges? A single number can't diverge.

The usual way around this is to distinguish between an infinite series and its sum. Given a sequence $(a_1, a_2, \dots)$ define a new sequence $(s_1, s_2, \dots)$ by setting $s_n = \sum_{k=1}^{n} a_k$ for each natural number n. Then define the infinite series $\sum_{k=1}^{\infty} a_k$ to be this sequence (s_n) of *partial sums.* So saying that the sequence (s_n) converges is the same thing as saying that the series $\sum_{k=1}^{\infty} a_k$ converges. If $s_n \to L$ as $n \to \infty$ we say that L is the *sum* of the series. Now what would be a reasonable notation for the sum of a series? It's hard to think of a worse choice than the one that history has conferred on us: if a series $\sum_{k=1}^{\infty}$ converges, we use the name of the series $\sum_{k=1}^{\infty}$ to denote its sum. And so $\sum_{k=1}^{\infty} 2^{-k}$ equals the number 1 while $\sum_{k=1}^{\infty}(-1^k)$ doesn't equal any number at all.

Thus it happens that every series is a sequence. It is the sequence of its partial sums. Is every sequence a series? In other words, is every sequence the sequence of partial sums of some other sequence? Certainly. Suppose we start with a sequence $(s_1, s_2, \dots)$. Construct a new sequence by letting $a_1 = s_1$ and for $n > 1$ let $a_n = s_n - s_{n-1}$. Then it is clear that our original sequence $(s_1, s_2, \dots)$ is the sequence of partial sums of the sequence $(a_1, a_2, \dots)$, that is to say, an infinite series. So every sequence is an infinite series. This leads to a nice puzzle. Since every sequence is a series and every series is a sequence, can the two concepts really be distinguished?

19.2. Exercises

(1) Let $a_n = (5 + 8^{-n})^{-1}$ for each $n \geq 3$. We know that the series $\displaystyle\sum_{3}^{\infty} a_n$ does not converge since $a_n \to$ _____ $\neq 0$ as $n \to \infty$.

(2) Let $a_n = n^{-1/n}$ for each $n \geq 1$. We know that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ does not converge since $a_n \to$ _____ $\neq 0$ as $n \to \infty$.

(3) Let $a_n = \left(1 + \dfrac{1}{n}\right)^n$ for each natural number n. Then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges because $a_n \to$ _____ $\neq 0$ as $n \to \infty$.

(4) The series $\displaystyle\sum_{1}^{\infty} \frac{\ln k}{\ln(3 + k^2)}$ diverges because the terms of the series approach ________ , which is not zero.

(5) $\displaystyle\sum_{k=5}^{\infty} \frac{1}{k(k-1)} = \frac{1}{a}$ where $a =$ _____ .

(6) $\displaystyle\sum_{1}^{\infty} \frac{1}{25k^2 + 15k - 4} = \frac{1}{a}$ where $a =$ ______ .

(7) $\displaystyle\sum_{n=2}^{\infty} \frac{4n}{(n^2 - 2n + 1)(n^2 + 2n + 1)} = \frac{a}{4}$ where $a =$ _____ .

(8) $\displaystyle\sum_{0}^{\infty} \frac{k!}{(k+5)!} = \frac{1}{a}$ where $a =$ ______ . *Hint.* Suppose that $a_k := \dfrac{k!}{(k+4)!}$. Then what can you say about $\displaystyle\sum_{0}^{n}(a_k - a_{k+1})$?

(9) $\displaystyle\sum_{0}^{\infty} \frac{k!}{(k+6)!} = \frac{1}{a}$ where $a =$ _________ . *Hint.* Suppose that $a_k := \dfrac{k!}{(k+5)!}$ Then what can you say about $\displaystyle\sum_{0}^{n}(a_k - a_{k+1})$?

(10) $\displaystyle\sum_{n=0}^{\infty} \arctan(n^2 + n + 1)^{-1} = \frac{\pi}{a}$ where $a =$ _____ . *Hint.* $\dfrac{1}{n^2 + n + 1} = \dfrac{(n+1) - n}{1 + (n+1)n}$.

(11) Let $S_n = \displaystyle\sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+2}}\right)$ for each natural number n. Then $\displaystyle\lim_{n\to\infty} S_n = a + b\sqrt{2}$ where $a =$ _____ and $b =$ ______ .

(12) $\displaystyle\sum_{n=0}^{\infty}(\arctan(n+1) - \arctan n) = \frac{\pi}{a}$ where $a =$ _____ .

(13) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 4n + 1}{(n^2 + n)^2} =$ _____ . *Hint.* Partial fractions.

(14) Suppose the n^{th} partial sum of the series $\sum_{k=1}^{\infty} a_k$ is $4 - n3^{-n}$. Then $a_k = \dfrac{b}{c}$ where $b =$ _________ and $c =$ _____ ; and $\sum_{k=1}^{\infty} a_k =$ _____ .

(15) Suppose the n^{th} partial sum of the series $\sum_{k=1}^{\infty} a_k$ is $5 - 7n2^{-n}$. Then $a_k = \dfrac{b}{c}$ where $b =$ ___________ and $c =$ ______ ; and $\sum_{k=1}^{\infty} a_k =$ _____ .

(16) $\displaystyle\sum_{3}^{\infty} \frac{(-1)^k 3^{k+1}}{2^{2k-4}} = -\frac{a}{7}$ where $a =$ ______ .

(17) $\displaystyle\sum_{6}^{\infty} \frac{(-1)^n 2^{n-3}}{3^{n-2}} = \frac{8}{a}$ where $a =$ _________ .

(18) $\displaystyle\sum_{2}^{\infty} \frac{(-1)^{n-1} 3^{3n-5}}{7^{2n-2}} = -\frac{3}{a}$ where $a =$ ________ .

(19) $\displaystyle\sum_{7}^{\infty} \frac{5^{n-4}}{3^{3n-17}} = \frac{125}{a}$ where $a =$ ________ .

(20) The repeating decimal $0.36\overline{36} = \dfrac{4}{a}$ where $a =$ ________ .

(21) The repeating decimal $0.108\overline{108} = \dfrac{4}{a}$ where $a =$ ________ .

(22) Let $S_n = \displaystyle\sum_{k=0}^{n} \frac{1}{2^{k+4}}$ for each $n \in \mathbb{N}$. Then $\displaystyle\lim_{n\to\infty} S_n = \frac{1}{a}$ where $a =$ _____ .

(23) Let $S_n = \displaystyle\sum_{k=5}^{n} \left(-\frac{1}{2}\right)^k$ for each natural number n. Then $\displaystyle\lim_{n\to\infty} S_n = -\frac{1}{a}$ where $a =$ ________ .

(24) The series $\displaystyle\sum_{k=1}^{\infty} \left(\frac{3x+1}{2}\right)^k$ converges for those x which satisfy _____ $< x <$ ________ and for no others.

(25) The series $\displaystyle\sum_{k=5}^{\infty} (2x-3)^k$ converges for those x which satisfy _____ $< x <$ _____ and for no others.

(26) $\displaystyle\sum_{k=0}^{\infty} e^{-k} = \frac{e}{a}$ where $a =$ ________ .

(27) A ball is dropped from a height of 15 feet. Each time it bounces it rises four-fifths the distance it previously fell. The total distance traveled by the ball is _________ feet.

19.3. Problems

(1) Criticize the following argument.

Claim: $0 = 1$.

Proof:

$$\begin{aligned} 0 &= 0 + 0 + 0 + \cdots \\ &= (1-1) + (1-1) + (1-1) + \cdots \\ &= 1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots \\ &= 1 + ((-1)+1) + ((-1)+1) + ((-1)+1) + \cdots \\ &= 1 + 0 + 0 + 0 + \cdots \\ &= 1\,. \end{aligned}$$

(2) Criticize the following argument.

Claim: $1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots = \dfrac{1}{2}$.

Proof: Let $S = 1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots$. Then it follows that

$$\begin{aligned} 1 - S &= 1 - (1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots) \\ &= 1 + (-1) + 1 + (-1) + 1 + (-1) + 1 + \cdots \\ &= S\,. \end{aligned}$$

Thus $2S = 1$ and $S = \dfrac{1}{2}$.

(3) Explain carefully what point (if any) is being made in the two preceding problems.

(4) On the interval $[0, 1]$, use the graphs of the functions $f_n(x) = x^n$ to give a *geometric* argument that the sum of the series $\sum_{n=1}^{\infty} n^{-1}(n+1)^{-1}$ is 1. *Hint.* What is the area under the curve $y = f^n(x)$?

(5) Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$ converges and find its sum.

(6) Let P be the set of all the natural numbers which have no prime factors other than 2 and 3. (In particular,1 belongs to P.) Find the sum of the reciprocals of the members of P.

19.4. Answers to Odd-Numbered Exercises

(1) $\frac{1}{5}$

(3) e

(5) 4

(7) 5

(9) 600

(11) 1, $\frac{1}{2}$

(13) 3

(15) $7k - 14$, 2^k, 5

(17) 135

(19) 66

(21) 37

(23) 48

(25) 1, 2

(27) 135

CHAPTER 20

CONVERGENCE TESTS FOR SERIES

20.1. Background

Topics: *integral test*, *p-test*, *comparison test*, *limit comparison test*, *ratio test*, *root test*.

The *integral test* for the convergence of a series has multiple formulations. Here is the one most useful for some of the following exercises.

20.1.1. Theorem (Integral Test)**.** *If a is a continuous, positive, decreasing function on the interval $[1, \infty)$ and either $\sum_{k=1}^{\infty} a_k$ or $\int_1^{\infty} a(x)\, dx$ converges, then so does the other and*

$$\int_1^{\infty} a(x)\, dx \le \sum_{k=1}^{\infty} a_k \le a_1 + \int_1^{\infty} a(x)\, dx\,.$$

(In the preceding statement a_k is just another way of writing $a(k)$.) A very readable discussion, and proof, of this result can be found on pages 577–580 of [3].

20.1.2. Theorem (p-test)**.** *For $p > 0$ the p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ and diverges if $p \le 1$.*

20.2. Exercises

(1) The infinite series $\sum_{k=1}^{\infty} ke^{-k^2}$ ______________ (converges/diverges) because the integral $\int_1^{\infty} xe^{-x^2}\,dx =$ _________ .

(2) The infinite series $\sum_3^{\infty} \dfrac{1}{n(\ln n)(\ln\ln n)^2}$ ______________ (converges/diverges) because the integral $\int_3^{\infty} \dfrac{1}{x(\ln x)(\ln\ln x)^2}\,dx =$ _________ .

(3) The infinite series $\sum_2^{\infty} \dfrac{1}{n\sqrt{\ln n}}$ ______________ (converges/diverges) because the integral $\int_2^{\infty} \dfrac{1}{x\sqrt{\ln x}}\,dx =$ _________ .

(4) Let $n \geq 2$ and f be a continuous, positive, decreasing function defined on the interval $[1, n]$. For $k = 1, 2, \dots, n$ let $a_k = f(k)$. Put the numbers x, y, and z in increasing order, where $x = \int_1^n f(t)\,dt$, $y = \sum_{k=1}^{n-1} a_k$, and $z = \sum_{k=2}^{n} a_k$. Answer: ____ $\leq$ ____ $\leq$ ____ .

(5) By the *integral test* the sum S of the series $\sum_{n=0}^{\infty} \dfrac{1}{n^2+1}$ satisfies $\dfrac{a}{2} < S < \dfrac{b}{2}$ where $a =$ ____ and $b =$ _________ .

(6) By the *integral test* the sum S of the series $\sum_{n=1}^{\infty} ne^{-n}$ satisfies $\dfrac{a}{e} < S < \dfrac{b}{e}$ where $a =$ ____ and $b =$ ____ .

(7) The series $\sum_{n=2}^{\infty} \dfrac{1}{n(\ln n)^p}$ converges for $a < p < b$ where $a =$ ____ and $b =$ ____, and nowhere else.

(8) By the *integral test* the sum S of the series $\sum_{n=1}^{\infty} \dfrac{\arctan n}{n^2+1}$ satisfies $S \leq a\pi + b\pi^2$ where $a =$ _______ and $b =$ _________ .

(9) By the integral test the sum S of the series $\sum_{n=2}^{\infty} \dfrac{\ln n}{n^2}$ satisfies $S \leq \dfrac{a + b\ln 2}{c}$ where $a =$ ____ , $b =$ ____ , and $c =$ ____ .

(10) Let $a_k = (k^2 - k - 1)^{-1/2}$ for $k \geq 5$. To show that the series $\sum_{k=5}^{\infty} a_k$ diverges we choose $b_k =$ _____ for each $k \geq 5$ and notice that $a_k \geq b_k$ for each $k \geq 5$ and that $\sum_{k=5}^{\infty} b_k$ diverges.

(11) Let $a_k = \dfrac{3k^2 - 2k - 1}{k^4 + 3k^2 + 5}$ for $k \geq 2$. To show that the series $\sum_{k=2}^{\infty} a_k$ converges we choose $b_k =$ _________ for each $k \geq 2$ and notice that $a_k \leq b_k$ for each $k \geq 2$ and that $\sum_{k=2}^{\infty} b_k$ converges.

(12) Let $a_k = \dfrac{1}{\sqrt{k^4 + 7k^2 + 20}}$ for $k \geq 2$. To show that the series $\sum_{k=2}^{\infty} a_k$ converges we choose $b_k =$ ______ for each $k \geq 2$ and notice that $a_k \leq b_k$ for each $k \geq 2$ and that $\displaystyle\sum_{k=2}^{\infty} b_k$ converges.

(13) Let $a_k = \dfrac{1}{\sqrt{5k^3 - 20k^2 + 7k - 3}}$ for $k \geq 4$. To show that the series $\sum_{k=4}^{\infty} a_k$ converges we choose $b_k = k^{-3/2}$ for $k \geq 4$ and use the __________________________ test. The series $\sum_{k=4}^{\infty} b_k$ converges and $\displaystyle\lim_{k\to\infty} \frac{a_k}{b_k} = \frac{1}{a}$ where $a =$ ______ .

(14) Let $\displaystyle\sum_{1}^{\infty} \frac{2k^{3/2} + 5k^{1/2} - 1}{7k^{5/2} - 6k^{3/2} + 3k - 2}$ for $k \geq 1$. To show that the series $\sum_{k=1}^{\infty} a_k$ diverges we choose $b_k = 1/k$ for $k \geq 1$ and use the __________________________ test. The series $\sum_{k=1}^{\infty} b_k$ diverges and $\displaystyle\lim_{k\to\infty} \frac{a_k}{b_k} = \frac{2}{a}$ where $a =$ _____ .

(15) Let $a_k = \dfrac{1}{5k^2 - 3k + 5}$ for $k \geq 1$. To show that the series $\sum_{k=1}^{\infty} a_k$ converges we choose $b_k = k^{-2}$ for $k \geq 1$ and use the _____________________ test. The series $\sum_{k=1}^{\infty} b_k$ converges and $\displaystyle\lim_{k\to\infty} \frac{a_k}{b_k} = \frac{1}{a}$ where $a =$ ______ .

(16) Let $a_n = \dfrac{n!\,3^n}{n^n}$ for $n \geq 1$. Then the series $\sum_{n=1}^{\infty} a_n$ _______________ (converges/diverges) because $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} =$ ________ .

(17) Let $a_n = \dfrac{3^{100n}}{3 \cdot 6 \cdots (3n)}$ for $n \geq 1$. Then the series $\sum_{n=1}^{\infty} a_n$ _______________ (converges/diverges) because $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} =$ ________ .

(18) Let $a_k = \dfrac{(k!)^2}{(2k)!}$ for $k \geq 1$. Then the series $\sum_{k=1}^{\infty} a_k$ _______________ (converges/diverges) because $\displaystyle\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{1}{a}$ where $a =$ ______ .

(19) Let $a_n = \dfrac{(2n-1)!}{4^{2n-1}}$ for $n \geq 1$. Then the series $\sum_{n=1}^{\infty} a_n$ _______________ (converges/diverges) because $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} =$ ______ .

(20) Let $a_k = \left(\dfrac{2k}{3k+1}\right)^k$ for $k \geq 4$. Then the series $\sum_{k=4}^{\infty} a_k$ _______________ (converges/diverges) because $\lim_{k\to\infty}(a_k)^{1/k} =$ ________ .

(21) Let $a_n = \left(1 + \dfrac{1}{n}\right)^{-n^2}$ for $n \geq 1$. Then the series $\sum_{n=1}^{\infty} a_n$ _______________ (converges/diverges) because $\lim_{n\to\infty}(a_n)^{1/n} = \dfrac{1}{a}$ where $a =$ ________ .

(22) Let $a_n = \dfrac{(\ln n)^n}{n!}$ for $n \geq 2$. Then the series $\sum_{n=2}^{\infty} a_n$ _______________ (converges/diverges) because $\lim_{n\to\infty}(a_n)^{1/n} =$ ________ .

(23) The series $\sum_{2}^{\infty} \frac{(-1)^k}{k \ln k}$ converges ____________________ . By computing the partial sums S_6 and S_7 we know that the sum S of the series satisfies

$$0.4\_\ \_60 < S < 0.5\_\ \_03.$$

(24) The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ converges ____________________ . By computing the partial sums S_{24} and S_{25} we know that the sum S of the series satisfies

$$0.\_\ \_38 < S < 0.\_\ \_39.$$

If we want the error in approximating the sum S by a partial sum S_n to be less than 10^{-2} the smallest n that the *alternating series test* assures us will work is $n =$ _________ .

(25) The series $\sum_{k=0}^{\infty} \frac{(-1)^k (k!)^2}{(2k)!}$ converges _______________ . By computing the partial sums S_5 and S_6 we know that the sum S of the series satisfies $\frac{a}{126} < S < \frac{b}{2772}$ where $a =$ ________ and $b =$ ____________ .

(26) The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 + 2^k}$ converges _______________ . To achieve an accuracy of 0.005 the *alternating series test* tells us to take at least $n =$ _____ . For that value of n the partial sum S_n is approximately $-0.2\_\ \_10$.

20.3. Problems

(1) Use the *integral test* to find upper and lower bounds for the series $\displaystyle\sum_{k=1}^{\infty} \frac{(\arctan k)^3}{1+k^2}$. Explain your reasoning carefully.

(2) Use the *comparison test* and the *p-test* to discuss the convergence of the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{n}}{n^2+3}$.

(3) Discuss the convergence of the series $\displaystyle\sum_{n=37}^{\infty} \frac{n^4 - 105n^2 + 70n - 5}{n^5 + 317n^3 - 150n - 3}$.

(4) Discuss the convergence of the series $\displaystyle\sum \frac{(3n)^n n!}{(2n)! 2^{2n}}$.

(5) Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{n^{4n}}(3n)!$. *Hint.* $\ln 54 \approx 3.988984$.

(6) Discuss the convergence of the series $\displaystyle\sum_{1}^{\infty} \frac{k^k}{2^k k!}$.

(7) Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty} a_n$ where $a_n = \begin{cases} \left(\frac{3}{4}\right)^n & \text{if } n \text{ is odd} \\ 2\left(\frac{3}{4}\right)^n & \text{if } n \text{ is even.} \end{cases}$

(8) Discuss the convergence of the series $\displaystyle\sum_{1}^{\infty} \frac{(-1)^k}{1+3^{-k}}$.

(9) Discuss the convergence of the series $\displaystyle\sum_{2}^{\infty} (-1)^k \frac{\ln k}{k}$.

(10) Discuss the convergence of the series $\displaystyle\sum_{2}^{\infty} \frac{(-1)^k}{k^2 \ln k}$.

(11) Discuss the convergence of the series $\displaystyle\sum_{1}^{\infty} (-1)^n \frac{n+2}{n^2+3n-1}$.

(12) Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{n!}$.

(13) Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$.

20.4. Answers to Odd-Numbered Exercises

(1) converges, $\dfrac{1}{2e}$

(3) diverges, ∞

(5) π, $\pi + 2$

(7) 1, ∞

(9) 2, 3, 4

(11) $\dfrac{3}{k^2}$

(13) limit comparison, $\sqrt{5}$

(15) limit comparison, 5

(17) converges, 0

(19) diverges, ∞

(21) converges, e

(23) conditionally, 9, 3, 6, 7

(25) absolutely, 79, 1741

CHAPTER 21

POWER SERIES

21.1. Background

Topics: power series, Maclaurin series, Taylor series, Lagrange form of the remainder, binomial series, radius of convergence, interval of convergence, use of power series to solve differential equations.

Despite the impression given my many beginning calculus texts the natural habitat of power series is the field of complex numbers—*not* the real number line. Here is an example to illustrate this point.

Consider two functions: the *sine* and the *arctangent*. They are both bounded and infinitely differentiable. The *arctangent* is extraordinarily well behaved: it is strictly increasing and changes concavity only once. The *sine* function, on the other hand, not only changes from increasing to decreasing infinitely often, but also changes concavity infinitely often. Why then should it turn out that the *sine* function has an *infinite* radius of convergence while the apparently much nicer *arctangent* function has radius of convergence of only *one*? When we think of these functions as being defined only on the real line this behavior seems totally inexplicable. But if we regard them as functions on the complex plane, everything seems quite natural. The *arctangent* function and its derivative, the function $z \mapsto \left(1 + z^2\right)^{-1}$ must have the same radius of convergence, and the latter function clearly blows up at $z = i$.

One calculus text that discusses complex power series is Spivak's beautiful classic [4].

21.2. Exercises

(1) The radius of convergence of the power series $\sum_{1}^{\infty} \frac{n^n}{n!}(x-1)^n$ is $\frac{1}{a}$ where $a =$ _____ .

(2) The radius of convergence of the power series $\sum_{n=1}^{\infty} \left(\frac{x^n}{n^2}\right) \frac{1}{2^n}$ is _____ .

(3) The radius of convergence of the power series $\sum \left[\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right]^4 \left(\frac{x}{3}\right)^n$ is _____ .

(4) The radius of convergence of the power series $\sum_{1}^{\infty} \frac{3^k (2k)!}{k^3 (k!)^2}(x+1)^k$ is $\frac{1}{a}$ where $a =$ ________ .

(5) The radius of convergence of the power series $\sum \frac{n^{2n}}{(2n)!}(x-1)^n$ is $\frac{a}{e^p}$ where $a =$ _____ and $p =$ _____ .

(6) The interval of convergence of the power series $\sum_{n=1}^{\infty}(-1)^n n x^n$ is (_____ , _____) .

(7) The interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is (_____ , _____) .

(8) The interval of convergence of the power series $\sum_{2}^{\infty} \frac{2^k}{\ln k}(x+3)^k$ is $\left[\frac{a}{2}, \frac{b}{2} \right)$ where $a =$ _______ and $b =$ _______ .

(9) The interval of convergence of the power series $\sum \frac{(-1)^{n+1}}{(n+1)^2} \frac{(x+1)^{2n}}{5^n}$ is $[a - \sqrt{b}\ , a + \sqrt{b}\ ]$ where $a =$ _______ and $b =$ _____ .

(10) The interval of convergence of the power series $1 + x + x^2 + 2x^3 + x^4 + 3x^5 + x^6 + 4x^7 + \cdots$ is (_______ , _____) .

(11) The interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ is _ -1 , 1 _ .

(12) The interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^2 2^n}$ is _ -2 , 2 _ .

(13) The interval of convergence of the power series $1 + \frac{1}{3}x + x^2 + \frac{1}{3}x^3 + x^4 + \frac{1}{3}x^5 + x^6 + \cdots$ is _ -1 , 1 _ .

(14) The interval of convergence of the power series $\sum_{n=27}^{\infty} \frac{(-1)^n (x+3)^n}{\sqrt{n}}$ is _ -4 , -2 _ .

(15) The interval of convergence of the power series $\sum \frac{1}{k 2^k} x^k$ is _ _____ , _____ _ .

(16) The interval of convergence of the power series $\sum_{1}^{\infty}(-1)^n \frac{(x+2)^n}{n 2^n}$ is _ _____ , _____ _ .

(17) The interval of convergence of the power series $\sum_{1}^{\infty} \frac{(x-5)^n}{\sqrt{n}\, 3^n}$ is _ _____ , _____ _ .

(18) The sum of the series $\displaystyle\sum_{n=3}^{\infty} \frac{n-1}{3^n}$ is $\dfrac{5}{a}$ where $a =$ ________ .

(19) Express $\displaystyle\sum_{1}^{\infty}(n+1)x^n$ as the value at x of an elementary function f.

Answer: $f(x) = \dfrac{g(x)}{(1-x)^2}$ where $g(x) =$ ________________ .

(20) Express $\displaystyle\sum_{1}^{\infty} nx^n$ as the value at x of an elementary function f.

Answer: $f(x) = \dfrac{g(x)}{(1-x)^2}$ where $g(x) =$ ______ .

(21) Express $\displaystyle\sum_{1}^{\infty} n^2x^n$ as the value at x of an elementary function f.

Answer: $f(x) = \dfrac{g(x)}{(1-x)^p}$ where $g(x) =$ ______________ and $p =$ _____ .

(22) The sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5^n}$ is $\dfrac{15}{a}$ where $a =$ ________ .

(23) The sum of the series $\displaystyle\sum_{k=1}^{\infty} \frac{2k-1}{4^k}$ is $\dfrac{a}{9}$ where $a =$ _____ . *Hint.* $4^k = 2^{2k}$.

(24) Express $\displaystyle\sum_{1}^{\infty} n(n+1)x^n$ as the value at x of an elementary function f.

Answer: $f(x) = \dfrac{g(x)}{(1-x)^p}$ where $g(x) =$ ________ and $p =$ _____ .

(25) The sum of the series $\dfrac{1\cdot 2}{2} + \dfrac{2\cdot 3}{4} + \dfrac{3\cdot 4}{8} + \dfrac{4\cdot 5}{16} + \cdots$ is _____ .

(26) The sum of the series $1 - \dfrac{1}{3\cdot 3} + \dfrac{1}{5\cdot 3^2} - \dfrac{1}{7\cdot 3^3} + \dfrac{1}{9\cdot 3^4} - \dots$ is $\dfrac{a}{2\sqrt{b}}$ where $a =$ _____ and $b =$ _____ .

(27) Express $1+x+x^2+3x^3+x^4+5x^5+x^6+7x^7+x^8+\cdots$ as the value at x of an elementary function f.

Answer: $f(x) = \dfrac{a}{1-x^2} + \dfrac{b(x)}{(1-x^2)^2}$ where $a =$ _____ and $b(x) =$ _____________ .

(28) A power series expansion of $\displaystyle\int \frac{x - \arctan x}{x^2}\,dx$ is $\displaystyle\sum_{1}^{\infty} \frac{(-1)^{k+1}}{a(k)} x^{2k} + c$ where $a(k) =$ ______________ and c is an arbitrary constant.

(29) $x\ln(1+x) = \displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{a_k} x^k$ where (for each k) $a_k =$ __________ . The radius of convergence of the series is $R =$ _____ .

(30) $\dfrac{1}{1+4x^2} = \displaystyle\sum_{k=0}^{\infty} a_k x^{2k}$ where (for each k) $a_k =$ _______________ . The radius of convergence of the series is $R = \dfrac{1}{a}$ where $a =$ _____ .

(31) $\dfrac{1}{(1+x)^2} = \displaystyle\sum_{k=0}^{\infty}(-1)^k a_k x^k$ where (for each k) $a_k =$ _________ . The radius of convergence of the series is $R =$ _____ .

(32) Let $f(x) = \dfrac{\arctan x}{x}$. A power series expansion for $f''(x)$ is $\displaystyle\sum_{0}^{\infty} \frac{(-1)^{n+1}a(n)}{2n+3} x^{2n}$ where $a(n) =$ _____________________ .

(33) A power series expansion for $\displaystyle\int \frac{\frac{1}{2}x^2 - x + \ln(1+x)}{x^3}\,dx$ is $\displaystyle\sum_{1}^{\infty} \frac{(-1)^{n-1}}{a(n)} x^n + c$ where $a(n) =$ __________________ and c is an arbitrary constant.

(34) $\dfrac{x^2}{(1-2x)^2} = \displaystyle\sum_{k=2}^{\infty} a_k x^k$ where (for each k) $a_k =$ _______________ . The radius of convergence of the series is $R =$ _______ .

(35) $\ln\left(\dfrac{1+x}{1-x}\right) = \displaystyle\sum_{k=0}^{\infty} \frac{2}{a_k} x^{2k+1}$ where (for each k) $a_k =$ _________ . The interval of convergence of the series is $(-b, b)$ where $b =$ _____ .

(36) $\displaystyle\int \frac{x}{1+x^5}\,dx = c + \sum_{k=0}^{\infty} \frac{(-1)^k}{a_k} x^{a_k}$ where (for each k) $a_k =$ __________ and c is an arbitrary constant. The radius of convergence of the series is _____ .

(37) A power series expansion of $\displaystyle\int_0^x \frac{\ln(1+t)}{t}\,dt$ is $c + \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{a(n)} x^n$ where $a(n) =$ _______ and c is an arbitrary constant.

(38) To six decimal places $\displaystyle\int_0^{1/2} \arctan x^2\,dx = 0.\_\,\_\,1\,\_\,\_\,3.$

(39) To six decimal places $\displaystyle\int_0^{1/2} \frac{1}{1+x^6}\,dx = 0.\_\,\_\,8\,\_\,\_\,3$.

(40) A power series expansion of $\displaystyle\int \frac{(1-t)\ln(1-t)}{t}\,dt$ is $-t + \displaystyle\sum_{k=2}^{\infty} \frac{1}{a(k)} t^k + c$ where c is an arbitrary constant and (for each k) $a(k) =$ ____________ .

(41) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k} =$ _____ .

(42) $\displaystyle\sum_{k=2}^{\infty} \frac{k^2-k}{2^k} =$ _____ .

(43) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{2^k} =$ _____ .

(44) Define $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k^2}$. Then

the interval of convergence for $f(x)$ is _ -1 , 1_ ;
the interval of convergence for $f'(x)$ is _ -1 , 1_ ; and
the interval of convergence for $f''(x)$ is _ -1 , 1_ .

(45) Express $\sum_{k=0}^{\infty} \frac{1}{2}(n+1)(n+2)x^n$ as an elementary function.
Answer: $f(x) = (1-x)^p$ where $p =$ ______ .

(46) The Taylor polynomial of degree 4 for $\sec x$ is $a + bx^2 + cx^4$ where $a =$ _____ , $b =$ ______ , and $c =$ _________ .

(47) The Taylor polynomial of degree 4 for $f(x) = 2(x-1)^3 + 2(x+1)^{1/2}$ is $7x - \frac{a}{4}x^2 + \frac{17}{8}x^3 - \frac{5}{b}x^4$ where $a =$ ________ and $b =$ ________ .

(48) The Taylor polynomial of degree 5 for $f(x) = \sin 3x$ is $ax + \frac{b}{2}x^3 + \frac{c}{40}x^5$ where $a =$ _____ , $b =$ ______ , and $c =$ ________ .

(49) The Taylor polynomial of degree 3 for $f(x) = (x+4)^{3/2} - (x+1)^{3/2}$ is $7 + \frac{a}{2}x + \frac{b}{16}x^2 + \frac{c}{128}x^3$ where $a =$ _____ , $b =$ ______ , and $c =$ _____ .

(50) Write $x^4 + x^3 + x^2 + 3x + 5$ as a polynomial in powers of $x+2$. Answer: $a + b(x+2) + c(x+2)^2 + d(x+2)^3 + (x+2)^4$ where $a =$ ______ , $b =$ _________ , $c =$ ________ , and $d =$ ______ .

(51) Write $p(x) = x^4 - 2x^3 + 4x^2 - 7x + 6$ as a polynomial in powers of $x - 1$. Answer: $a + b(x-1) + c(x-1)^2 + d(x-1)^3 + (x-1)^4$ where $a =$ _____ , $b =$ ______ , $c =$ _____ , and $d =$ _____ .

(52) Write $p(x) = x^4 + 3x^3 + 4x^2 + 5x + 7$ as a polynomial in powers of $x + 1$. Answer: $a + b(x+1) + c(x+1)^2 + d(x+1)^3 + (x+1)^4$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $d =$ ______ .

(53) Express the polynomial $p(x) = x^3 - x^2 + 3x - 5$ as a polynomial in powers of $x - 2$. Answer: $a + b(x-2) + c(x-2)^2 + (x-2)^3$ where $a =$ _____ , $b =$ ________ , and $c =$ _____ .

(54) Approximate $\cos 1$ making use of an appropriate polynomial of degree 4. Express your answer as a single fraction in lowest terms. Answer: $\dfrac{a}{b}$ where $a =$ _________ and $b =$ _________ .

(55) Find an approximate value for $\int_0^1 \sin x^3\, dx$ by replacing the integrand with an appropriate polynomial of degree 15. Answer: $\dfrac{a}{1920}$ where $a =$ ___________ .

(56) Using the Lagrange form for the remainder, determine the smallest number of <u>non-zero</u> terms a Taylor polynomial for $\sin x$ must have to guarantee an approximation of $\sin 0.1$ which is accurate to within 10^{-10}. Answer: _____ .

(57) Let $f(x) = \dfrac{1}{x}e^x - \dfrac{1}{x^2}e^x + \dfrac{1}{x^2}$ for $x \neq 0$.
   (a) If we define $f(0) =$ ________ , then f becomes a continuous function on the whole real line.
   (b) The function f, extended as in (a), has derivatives of all orders at $x = 0$ and $f^{(n)}(0) = \dfrac{1}{a(n)}$ where $a(n) =$ ___________ .

(58) Find an approximate value of $\int_0^1 \cos x^2\, dx$ by replacing the integrand with an appropriate polynomial of degree 8. Answer: $1 - \dfrac{1}{a} + \dfrac{1}{b}$ where $a =$ ______ and $b =$ ________ .

(59) Find $e^{1/5}$ with an error of less than 10^{-5}. Express your answer as a sum of fractions in lowest terms. Answer: $1 + \dfrac{1}{5} + \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$ where $a =$ ______ , $b =$ ________ , and $c =$ __________ .

(60) Let $f(x) = \dfrac{1 - \cos 2x}{x^2}$. Then $f^{(10)}(0) = -\dfrac{2^p}{a}$ where $p =$ _______ and $a =$ _______ .

(61) Let $f(x) = x^3 \sin x$. Then $f^{(8)}(0) =$ __________ .

(62) Let $f(x) = x^3 \cos x$. Then $f^{(11)}(0) =$ _________ .

(63) Express $\displaystyle\sum_1^\infty \frac{n}{(n+1)!} x^{n+1}$ as the value at x of an elementary function f.
Answer: $f(x) =$ ___________________ .

(64) Express $\displaystyle\sum_0^\infty \frac{(n+1)x^n}{(n+2)!}$ as the value at x of an elementary function f.

Answer: $f(x) = \dfrac{g(x)}{x^2}$ where $g(x) =$ __________________ .

(65) Express $\displaystyle\sum_0^\infty (-1)^n \frac{(2n+2)x^{2n+1}}{(2n)!}$ as the value at x of an elementary function f.
Answer: $f(x) = a(x)\sin x + b(x)\cos x$ where $a(x) =$ _______ and $b(x) =$ _______ .

(66) The sum of the series $\displaystyle\sum_1^\infty (-1)^{n+1} \frac{2n+1}{2^n n!}$ is ____ .

(67) Express $\displaystyle\sum_0^\infty \frac{(-1)^{n+1}(2n+2)x^{2n+1}}{(2n+3)!}$ as the value at x of an elementary function f.

Answer: $f(x) = \dfrac{g(x)}{x^2}$ where $g(x) =$ ______________________ .

(68) Express $\displaystyle\sum_0^\infty (-1)^n \frac{(2n+2)^2}{(2n+1)!} x^{2n+1}$ as the value at x of an elementary function f.
Answer: $f(x) = a(x)\sin x + b(x)\cos x$ where $a(x) =$ ________ and $b(x) =$ _______ .

(69) Let $f(x) = \displaystyle\sum_0^\infty \frac{k+1}{k!} x^k$. Then the domain of f is the interval (____ , ____) and $f(\ln 3) = a\ln 3 + b$ where $a =$ ____ and $b =$ ____ .

(70) The sum of the series $\displaystyle\sum_0^\infty (-1)^{k+1} \frac{(2k+1)^2}{(2k)!}$ is $a \sin b$ where $a =$ ____ and $b =$ ____ .

(71) The sum of the series $\dfrac{1}{3!} + \dfrac{2}{5!} + \dfrac{3}{7!} + \dfrac{4}{9!} + \dots$ is $\dfrac{a}{be}$ where $a =$ ____ and $b =$ ____ .

(72) Let $f(x) = \dfrac{\sin x}{x}$ for all $x \neq 0$. We can extend f to a function continuous on the whole real line by defining $f(0) = 1$. Find $f^{(n)}(0)$ for all $n \in \mathbb{N}$. Answer: If n is odd, say $n = 2k+1$, then $f^{(2k+1)}(0) =$ ____ ; and if n is even, say $n = 2k$, then $f^{(2k)}(0) = \dfrac{(-1)^k}{ak+b}$ where $a =$ ____ and $b =$ ____ .

(73) The sum of the series $\displaystyle\sum_1^\infty \frac{k(k+1)}{k!} 3^k$ is ae^p where $a =$ _______ and $p =$ ____ .

(74) The sum of the series $\displaystyle\sum_{k=2}^{\infty} \frac{k^4}{k!}$ is $ae + b$ where $a =$ ________ and $b =$ _______ .

(75) The sum of the series $\dfrac{\pi^2}{4^2 2!} - \dfrac{\pi^4}{4^4 4!} + \dfrac{\pi^6}{4^6 6!} - \dfrac{\pi^8}{4^8 8!} + \cdots$ is $1 - \dfrac{1}{a}$ where $a =$ _______ .

(76) The coefficient of the term containing x^{12} in the power series expansion of $(1 + x^3)^{3/5}$ is $-\dfrac{a}{625}$ where $a =$ ________ .

(77) The coefficient of the term containing x^9 in the power series expansion of $(1 - x^3)^{2/3}$ is $-\dfrac{a}{81}$ where $a =$ _____ .

(78) The sum of the series $\displaystyle\sum_{0}^{\infty} \binom{1/2}{n} \frac{(-1)^n}{2n+1}$ is $\dfrac{a}{b}$ where $a =$ _____ and $b =$ _____ . *Hint.* Integrate the power series expansion of $(1 - x^2)^{\frac{1}{2}}$ over the interval $[0, 1]$.

(79) The sum of the series

$$1 - \frac{1}{2^2} - \frac{1}{2!\,2^4} - \frac{1 \cdot 3}{3!\,2^6} - \frac{1 \cdot 3 \cdot 5}{4!\,2^8} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{5!\,2^{10}} - \dots$$

is $\dfrac{1}{\sqrt{a}}$ where $a =$ _____ .

(80) Use power series to solve the differential equation $f'(x) = xf(x)$ subject to the initial condition $f(0) = 1$.
Answer: $f(x) =$ ____________________ .

(81) Use power series to solve the differential equation $f'(x) = x^2 f(x)$ subject to the initial condition $f(0) = 5$.
Answer: $f(x) =$ _______________________ .

(82) Use power series to solve the differential equation $f''(x) = f(x)$ subject to the initial conditions $f(0) = 3$ and $f'(0) = 1$.
Answer: $f(x) = 2f(x) + e^{-x}$ where $f(x) =$ __________ .

(83) Use power series to solve the differential equation $y'' - xy' = y$ subject to the initial conditions $y(0) = 1$ and $y'(0) = 0$.
Answer: $y(x) = \exp(f(x))$ where $f(x) =$ ___________ .

(84) Use power series to solve the differential equation $y'' + x^2 y' + xy = 0$ subject to the initial conditions $y(0) = 0$ and $y'(0) = 1$.
Answer: $y(x) = x + \displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{a(k)}{(3k+1)!} x^{3k+1}$ where $a(k) =$ _________________________________ .

(85) Use series to solve the differential equation $(1 + x^2)y'' + 2xy' - 2y = 0$ subject to the initial conditions $y(0) = 1$ and $y'(0) = 3$.
Answer: $y(x) = 1 + 3x + f(x)$ where $f(x) =$ _________________________ .

21.3. Problems

(1) Find the power series expansion of $\dfrac{1+2x}{1-x-x^2}$.

(2) Suppose you have never heard of exponential or logarithmic functions. *Define* exp, the EXPONENTIAL FUNCTION, by

$$\exp(x) := \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

On the basis of this definition develop the properties of the *exponential function*. Include at least the following:

(a) The series $\sum_{k=0}^{\infty} \frac{1}{k!} x^k$ converges absolutely for all x in $\mathbb{R}$; so $\exp(x)$ is defined for every real number x.

(b) The exponential function is differentiable and satisfies the differential equation

$$y' - y = 0.$$

(c) The exponential function is positive, increasing, and concave up on $\mathbb{R}$.

(d) If x, $y \in \mathbb{R}$, then

$$\exp(x) \cdot \exp(y) = \exp(x+y).$$

(e) If $x \in \mathbb{R}$, then

$$\big(\exp(x)\big)^{-1} = \exp(-x).$$

In this problem you may use without proof the following result: If the series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ are both absolutely convergent then their product is given by

$$\sum_{k=0}^{\infty} a_k \cdot \sum_{k=0}^{\infty} b_k = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k}.$$

(3) Let $f(x) = \sin x$. Use an appropriate polynomial of degree 3 to approximate $\sin(0.3)$. Show that the error in this approximation is less than $2.1 \cdot 10^{-5}$.

(4) Use an appropriate polynomial to approximate $\ln 1.5$ with an error of less than 10^{-2}. (Express your answer as the quotient in lowest terms of two natural numbers.) Give a careful proof that the error in this approximation is less than 10^{-2}.

(5) Find an approximate value for $\displaystyle\int_0^{1/2} e^{x^2}\,dx$ by replacing the integrand with an appropriate polynomial of degree 4. (Express your answer as the quotient in lowest form of two natural numbers.) Give a careful argument to show that the error in this approximation is less than $4 \cdot 10^{-4}$.

(6) Use the *alternating series test* to show that $1 - \frac{x^2}{6} < \frac{\sin x}{x} < 1$ whenever $0 < |x| < 1$.

(7) Use the *alternating series test* to show that $\dfrac{1}{2} - \dfrac{x^2}{24} < \dfrac{1-\cos x}{x^2} < \dfrac{1}{2}$ whenever $0 < |x| < 1$.

(8) Prove that the number e is irrational. *Hint.* Argue by contradiction. Suppose that e is a rational number; that is, suppose $e = \dfrac{p}{q}$ where p and q are natural numbers. Fix a natural number $n > q$ and define

$$\alpha = n!\left(e - \sum_{k=0}^{n} \frac{1}{k!}\right).$$

Prove that α is a natural number and that $\alpha < \dfrac{1}{n}$.

(9) Make use of the relation $\sec x \cos x = 1$ to find the first four nonzero terms of the Maclaurin expansion of $\sec x$.

(10) Suppose one uses the first two terms of the binomial series to approximate $(627)^{1/4}$ as follows:

$$(627)^{1/4} = 5\left(1 + \frac{2}{625}\right)^{1/4} \approx 5\left(1 + \frac{1}{4} \cdot \frac{2}{625}\right) = 5.004.$$

Show that the error in this approximation is less than 10^{-5}.

(11) Explain in detail how to use power series to solve the differential equation

$$y'' + y = \sin x$$

subject to the initial conditions

$$y'(0) = y(0) = 0.$$

21.4. Answers to Odd-Numbered Exercises

(1) e

(3) 3

(5) 4, 2

(7) $-\infty$, ∞

(9) -1, 5

(11) [,)

(13) (,)

(15) [, -2, 2,)

(17) [, 2, 8,)

(19) $2x - x^2$

(21) $x + x^2$, 3

(23) 5

(25) 8

(27) 1, $x + x^3$

(29) $k - 1$, 1

(31) $k + 1$, 1

(33) $n(n + 2)$

(35) $2k + 1$, 1

(37) n^2

(39) 4, 9, 8, 9

(41) 2

(43) 6

(45) -3

(47) 25, 64

(49) 3, -3, 7

(51) 2, -1, 4, 2

(53) 5, 11, 5

(55) 449

(57) (a) $\frac{1}{2}$
(b) $n + 2$

(59) 50, 750, 15,000

(61) 336

(63) $xe^x - e^x + 1$

(65) $-x^2$, $2x$

(67) $x\cos x - \sin x$

(69) $-\infty$, ∞, 3, 3

(71) 1, 2

(73) 15, 3

(75) $\sqrt{2}$

(77) 4

(79) 2

(81) $5\exp\left(\frac{1}{3}x^3\right)$

(83) $\frac{1}{2}x^2$

(85) $x \arctan x$

Part 6

SCALAR FIELDS AND VECTOR FIELDS

CHAPTER 22

VECTOR AND METRIC PROPERTIES of $\mathbb{R}^n$

22.1. Background

Topics: $\mathbb{R}^n$, vectors in $\mathbb{R}^n$, addition of vectors, scalars, scalar multiplication, inner product, Schwarz inequality, perpendicularity (orthogonality), norm of a vector, unit vector, cross products, neighborhood of a point, deleted neighborhood, distance, open set, closed set, scalar fields, vector fields, standard basis vectors, equations of a line in $\mathbb{R}^3$, equation of a plane in $\mathbb{R}^3$.

22.1.1. Definition. The set $\mathbb{R}^n$ of all n-tuples of real numbers is *Euclidean n-dimensional space*—or more briefly, just *n-space*. We make a standard notational convention. If $\mathbf{x}$ belongs to $\mathbb{R}^n$, then $\mathbf{x}$ is the n-tuple whose coordinates are $x_1, x_2, \dots, x_n$; that is,

$$\mathbf{x} = (x_1, x_2, \dots, x_n)\,.$$

It must be confessed that we do not *always* use this convention. For example, the temptation to denote a member of $\mathbb{R}^3$ by (x, y, z), rather than by (x_1, x_2, x_3), is usually just too strong to resist.

We will often refer to the elements of $\mathbb{R}^n$ as *vectors* and real numbers as *scalars*. We give $\mathbb{R}^n$ the structure of a vector space by defining operations of *addition* and *scalar multiplication*. For n-tuples $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ in $\mathbb{R}^n$ define

$$\mathbf{x} + \mathbf{y} := (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)\,.$$

Thus we say that addition in $\mathbb{R}^n$ is defined *coordinatewise*. Scalar multiplication is also defined in a coordinatewise fashion. That is, if $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, then we define

$$\alpha\mathbf{x} := (\alpha x_1, \alpha x_2, \dots, \alpha x_n)\,.$$

It is convenient to define the *distance* between two points in $\mathbb{R}^n$ in terms of a scalar valued function defined on pairs of vectors called an *inner product*. If $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ are vectors in $\mathbb{R}^n$, then the INNER PRODUCT (or DOT PRODUCT) of $\mathbf{x}$ and $\mathbf{y}$, denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$, is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{k=1}^{n} x_k y_k\,.$$

(For the fundamental facts about the inner product see problem 1.)

We *define* the ANGLE between two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ by

$$\measuredangle(\mathbf{x}, \mathbf{y}) := \arccos\left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|\,\|\mathbf{y}\|}\right);$$

and we *define* two vectors $\mathbf{x}$ and $\mathbf{y}$ to be PERPENDICULAR (or ORTHOGONAL) if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

The *norm* of a vector is defined using the inner product. If $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ define

$$\|\mathbf{x}\| = \left(\sum_{k=1}^{n} {x_k}^2\right)^{1/2}.$$

This is the (EUCLIDEAN) NORM on $\mathbb{R}^n$. The expression $\|\mathbf{x}\|$ may be read as "the *norm* of $\mathbf{x}$" or "the *length* of $\mathbf{x}$". A vector in $\mathbb{R}^n$ which has norm 1 is a UNIT VECTOR.

22.1.2. Theorem (Schwarz inequality)**.** *If* $\mathbf{x}$*,* $\mathbf{y} \in \mathbb{R}^n$*, then*

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \le \|\,\mathbf{x}\|\,\|\,\mathbf{y}\|.$$

The *distance* between two vectors in $\mathbb{R}^n$ is the length of their difference. If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$ define

$$d(\mathbf{x}, \mathbf{y}) = \|\,\mathbf{x} - \mathbf{y}\|.$$

This is the DISTANCE between $\mathbf{x}$ and $\mathbf{y}$. (Here, as in real arithmetic, $\mathbf{x} - \mathbf{y}$ means $\mathbf{x} + (-1)\mathbf{y}$.)

We generalize the notion of δ-neighborhood (see definition 1.1.2) in $\mathbb{R}$ to *open balls of radius* δ in $\mathbb{R}^n$. Let $\mathbf{a}$ be a vector in $\mathbb{R}^n$ and $\delta > 0$. The OPEN BALL ABOUT $\mathbf{a}$ OF RADIUS δ is defined by

$$B_\delta(\mathbf{a}) := \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{a}) < \delta\}.$$

We will also refer to this set as the

δ-NEIGHBORHOOD OF $\mathbf{a}$. The DELETED δ-NEIGHBORHOOD of the point $\mathbf{a} \in \mathbb{R}^n$ is the open ball around $\mathbf{a}$ of radius δ from which the point $\mathbf{a}$ has been deleted. More generally, we will refer to any open set containing the point $\mathbf{a}$ as a NEIGHBORHOOD of $\mathbf{a}$.

A subset of $\mathbb{R}^n$ is OPEN if it is a union of open balls in $\mathbb{R}^n$. A set is CLOSED if its complement is open.

22.1.3. Definition. If $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ are vectors in $\mathbb{R}^3$, then their CROSS PRODUCT, denoted by $\mathbf{x} \times \mathbf{y}$, is the vector $(x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$.

22.1.4. Notation. Define vectors $\mathbf{e}^1, \dots, \mathbf{e}^n$ in $\mathbb{R}^n$ by

$$\begin{aligned}
\mathbf{e}^1 &:= (1, 0, 0, \dots, 0)\\
\mathbf{e}^2 &:= (0, 1, 0, \dots, 0)\\
&\ \vdots\\
\mathbf{e}^n &:= (0, 0, \dots, 0, 1).
\end{aligned}$$

In other words, for $1 \le j \le n$ and $1 \le k \le n$, the k^{th} coordinate of the vector $\mathbf{e}^j$ (denote it by $(\mathbf{e}^j)_k$ or e^j_k) is 1 if $j = k$ and 0 if $j \ne k$. The vectors $\mathbf{e}^1, \dots, \mathbf{e}^n$ are the STANDARD BASIS VECTORS in $\mathbb{R}^n$. (Note that the superscripts here have nothing to do with powers.) In $\mathbb{R}^3$ the three standard basis vectors are often denoted by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ rather than $\mathbf{e}^1$, $\mathbf{e}^2$, and $\mathbf{e}^3$, respectively.

Every vector in $\mathbb{R}^n$ is a linear combination of the standard basis vectors in that space: that is, if $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, then

$$\mathbf{x} = \sum_{k=1}^{n} x_k \mathbf{e}^k\,.$$

22.1.5. Definition. In these exercise/problem sets we restrict our attention to functions F which map one Euclidean space, say $\mathbb{R}^n$, to another, say $\mathbb{R}^m$. In the first twenty chapters the emphasis has been on the case where $n = m = 1$, real valued functions of a real variable. *Curves* are mappings $f : \mathbb{R} \to \mathbb{R}^m$ (where $m \ge 2$). Thus curves are vector valued functions of a real variable. Much of the subsequent material in calculus is devoted to the study of *scalar fields*. These are real valued (that is, scalar valued) functions of a vector variable; that is, mappings $f : \mathbb{R}^n \to \mathbb{R}$ (where $n \ge 2$). As we will see shortly, scalar fields are also called 0*-forms*.

Also of importance are *vector fields* These are mappings of $\mathbb{R}^n$ into $\mathbb{R}^m$ (where n, $m \ge 2$). Thus a vector field is a vector valued function of a vector variable. Such a mapping $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ comprises m *coordinate functions* $\mathbf{F} = (F^1, F^2, \dots, F^m)$, where each F^k is itself a scalar field. Thus for each vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ in the domain of $\mathbf{F}$ we have

$$\mathbf{F}(\mathbf{x}) = \big(F^1(\mathbf{x}), F^2(\mathbf{x}), \dots, F^m(\mathbf{x})\big)\,.$$

As an example consider the function $\mathbf{F}$ which gives rectangular coordinates in terms of spherical coordinates. Then $\mathbf{F}\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ and

$$\mathbf{F}(\rho, \phi, \theta) = \big(F^1(\rho, \phi, \theta), F^2(\rho, \phi, \theta), F^3(\rho, \phi, \theta)\big)$$

where

$$\begin{aligned} F^1(\rho, \phi, \theta) &= \rho \sin\phi \cos\theta, \\ F^2(\rho, \phi, \theta) &= \rho \sin\phi \sin\theta, \text{ and} \\ F^3(\rho, \phi, \theta) &= \rho \cos\phi. \end{aligned}$$

22.1.6. Definition. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in the plane. We say that z is a LINEAR COMBINATION of x and y if there exist scalars α and β such that

$$\mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}.$$

22.2. Exercises

(1) Three vertices of a parallelogram are $P = (1, 3, 2)$, $Q = (4, 5, 3)$, and $R = (2, -1, 0)$. What are the possible locations of the fourth vertex?
Answer: (-1 , _____ , _____), or (3 , _____ , _____), or (5 , _____ , _____).

(2) Given the points $A = (1, -2)$ and $B = (4, -6)$, the unit vector in the direction of $\overline{AB}$ is $a\mathbf{i} + b\mathbf{j}$ where $a =$ ______ and $b =$ ______ .

(3) Find a vector parallel to the line $8x + 6y = 7$. Answer: $9\mathbf{i} + b\mathbf{j}$ where $b =$ ______ .

(4) Find a vector of length 26 which is parallel to the line $24x - 10y = 13$.
Answer: $a\mathbf{i} + b\mathbf{j}$ where $a =$ _____ and $b =$ ______ .

(5) In $\mathbb{R}^2$, $x + y = 4$ is the equation of a _____________ . In $\mathbb{R}^3$, $x + y = 4$ is the equation of a _____________ .

(6) In $\mathbb{R}^2$, $y = 3x^2$ is the equation of a ___________________________ . In $\mathbb{R}^3$, $y = 3z^2$ is the equation of a ___________________________ .

(7) In $\mathbb{R}^2$, $x^2+y^2 = 25$ is the equation of a ___________________________ . In $\mathbb{R}^3$, $x^2+y^2 = 25$ is the equation of a ___________________________ .

(8) The orthogonal projection of the point $(2, 3, 5)$ on the xy-plane is (__ , __ , __); on the yz-plane is (__ , __ , __); and on the xz-plane is (__ , __ , __).

(9) Let $\mathbf{x} = (3, 2)$, $\mathbf{y} = (2, -1)$ and $\mathbf{z} = (7, 1)$ be vectors in the plane. We know that $\mathbf{z}$ is a linear combination of $\mathbf{x}$ and $\mathbf{y}$ because $\mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}$ where $\alpha =$ _____ and $\beta =$ _____ .

(10) Let $\mathbf{a} = (5, 0, 2)$ and $\mathbf{b} = (1, -3, -2)$. Then
$\|\mathbf{a}\| =$ ______________ .
$\mathbf{a} + \mathbf{b} =$ _____ $\mathbf{i} +$ _____ $\mathbf{j} +$ _____ $\mathbf{k}$.
$\mathbf{a} - \mathbf{b} =$ _____ $\mathbf{i} +$ _____ $\mathbf{j} +$ _____ $\mathbf{k}$.
$3\mathbf{a} =$ _____ $\mathbf{i} +$ _____ $\mathbf{j} +$ _____ $\mathbf{k}$.
$3\mathbf{a} - 2\mathbf{b} =$ _____ $\mathbf{i} +$ _____ $\mathbf{j} +$ _____ $\mathbf{k}$.

(11) Let $\mathbf{x} = (1, 0, 1)$, $\mathbf{y} = (0, 1, 1)$, and $\mathbf{z} = (1, 2, 3)$. The only number α such that $\alpha\mathbf{x} + 2\mathbf{y}$ is perpendicular to $\mathbf{z}$ is $\alpha =$ ________ .

(12) Find all numbers α such that the angle between the vectors $2\mathbf{i} + 2\mathbf{j} + (\alpha - 2)\mathbf{k}$ and $2\mathbf{i} + (\alpha - 2)\mathbf{j} + 2\mathbf{k}$ is $\frac{\pi}{3}$. Answer: $\alpha =$ _____ and ______ .

(13) Find all numbers α such that the vectors $2\alpha\,\mathbf{i} - 2\,\mathbf{j} - \mathbf{k}$ and $2\alpha\mathbf{i} + 3\alpha\mathbf{j} - 2\mathbf{k}$ are perpendicular.
Answer: $\alpha =$ ______ and _____ .

(14) Find all numbers α such that the vectors $2\alpha\,\mathbf{i} - \mathbf{j} + 12\,\mathbf{k}$ and $\alpha\,\mathbf{i} + 2\alpha\,\mathbf{j} - \mathbf{k}$ are perpendicular.
Answer: $\alpha =$ _____ and _____ .

(15) The angle in $\mathbb{R}^3$ between the vectors $(-3, 1, 2)$ and $(1, 2, -3)$ is $a\pi$ where $a =$ ________ .

(16) The angle in $\mathbb{R}^4$ between the vectors $(1, 0, -1, 3)$ and $(1, \sqrt{3}, 3, -3)$ is $a\pi$ where $a =$ ________ .

(17) Let $\mathbf{x} = (1, 1, -1)$ and $\mathbf{y} = (2, 0, 3)$. Find a scalar α such that $\mathbf{x} + \alpha\mathbf{y} \perp \mathbf{x}$.
Answer: $\alpha =$ _____ .

(18) In $\mathbb{R}^3$ which of the angles of triangle ABC, with vertices $A = (1, -2, 0)$, $B = (2, 1, -2)$, and $C = (6, -1, -3)$, is a right angle? Answer: the right angle is at vertex _____ .

(19) Suppose that the hydrogen atoms of a methane molecule CH_4 are located at $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 1)$, and $(1, 0, 1)$ while the carbon atom is at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Find the cosine of

the angle θ between two rays starting at the carbon atom and going to different hydrogen atoms. Answer: $\cos\theta =$ _____ .

(20) The length of the vector $(1, 2, -1, -3, 1)$ in $\mathbb{R}^5$ is ________ .

(21) The length of the vector $(2, -2, 0, 3, -2, 2)$ in $\mathbb{R}^6$ is ________ .

(22) Find the angle θ between the vectors $\mathbf{x} = (3, -1, 1, 0, 2, 1)$ and $\mathbf{y} = (2, -1, 0, \sqrt{2}, 2, 1)$ in $\mathbb{R}^6$. Answer: $\theta =$ _____ .

(23) If a, b, c, d, e, $f \in \mathbb{R}$, then

$$|ad + be + cf| \le \sqrt{a^2 + b^2 + c^2}\sqrt{d^2 + e^2 + f^2}.$$

The proof of this inequality is obvious since this is just the *Schwarz inequality* where $\mathbf{x} =$ (___ , ___ , ___) and $\mathbf{y} =$ (___ , ___ , ___).

(24) If $a_1, \dots, a_n > 0$, then

$$\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \ge n^2.$$

The proof of this is obvious from the *Schwarz inequality* 22.1.2 when we choose

$\mathbf{x} =$ ________________________ and $\mathbf{y} =$ ________________________ .

(25) The volume of the parallelepiped generated by the three vectors $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{j} + \mathbf{k}$, and $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ is ____ .

(26) The equations of the line containing the points $(3, -1, 4)$ and $(7, 9, 10)$ are

$$\frac{x-3}{2} = \frac{y-j}{b} = \frac{z-k}{c}$$

where $b =$ _____ , $c =$ _____ , $j =$ _____ , and $k =$ _____ .

(27) The equations of the line containing the points $(5, 2, -1)$ and $(9, -4, 1)$ are

$$\frac{x-h}{a} = \frac{y-2}{-3} = \frac{z-k}{c}$$

where $a =$ _____ , $c =$ _____ , $h =$ _____ , and $k =$ _____ .

(28) Find the equations of the line containing the point $(1, 0, -1)$ which is parallel to the line $\dfrac{x-4}{2} = \dfrac{2y-3}{5} = \dfrac{3z-7}{6}$.
Answer: $\dfrac{x-h}{a} = \dfrac{y-j}{b} = \dfrac{z+1}{4}$ where $a =$ _____ , $b =$ _____ , $h =$ _____ , and $j =$ _____ .

(29) The equation of the plane containing the points $(0, -1, 1)$, $(1, 0, 2)$, and $(3, 0, 1)$ is $x + by + cz = d$ where $b =$ _____ , $c =$ _____ , and $d =$ _____ .

(30) The equation of the plane which passes through the points $(0, -1, -1)$, $(5, 0, 1)$, and $(4, -1, 0)$ is $ax + by + cz = 1$ where $a =$ _____ , $b =$ _____ , and $c =$ _____ .

(31) The angle between the planes $4x + 4z - 16 = 0$ and $-2x + 2y - 13 = 0$ is $\dfrac{a}{b}\pi$ where $a =$ _____ and $b =$ _____ .

(32) Suppose that $\mathbf{u} \in \mathbb{R}^3$ is a vector which lies in the first quadrant of the xy-plane and has length 3 and that $\mathbf{v} \in \mathbb{R}^3$ is a vector that lies along the positive z-axis and has length 5. Then
 (a) $\|\,\mathbf{u} \times \mathbf{v}\| =$ ____ ;
 (b) the x-coordinate of $\mathbf{u} \times \mathbf{v}$ is _____ 0 (choose $<$, $>$, or $=$);
 (c) the y-coordinate of $\mathbf{u} \times \mathbf{v}$ is _____ 0 (choose $<$, $>$, or $=$); and
 (d) the z-coordinate of $\mathbf{u} \times \mathbf{v}$ is _____ 0 (choose $<$, $>$, or $=$).

(33) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^7$ both of length $2\sqrt{2}$ and that the length of $\mathbf{u}-\mathbf{v}$ is also $2\sqrt{2}$. Then $\|\,\mathbf{u}+\mathbf{v}\| =$ _______ and the angle between $\mathbf{u}$ and $\mathbf{v}$ is _______ .

22.3. Problems

(1) Verify the following properties of the inner product on $\mathbb{R}^n$: If $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are vectors in $\mathbb{R}^n$ and α is a scalar. Then

(a) $\langle \mathbf{x}+\mathbf{y}, \mathbf{z}\rangle = \langle \mathbf{x}, \mathbf{z}\rangle + \langle \mathbf{y}, \mathbf{z}\rangle$;
(b) $\langle \alpha\mathbf{x}, \mathbf{y}\rangle = \alpha\langle \mathbf{x}, \mathbf{y}\rangle$;
(c) $\langle \mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{y}, \mathbf{x}\rangle$;
(d) $\langle \mathbf{x}, \mathbf{x}\rangle \geq 0$;
(e) $\langle \mathbf{x}, \mathbf{x}\rangle = 0$ only if $\mathbf{x} = \mathbf{0}$; and
(f) $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle}$.

Items (a) and (b) say that the inner product is *linear* in its first variable; (c) says it is *symmetric*; and (d) and (e) say that it is *positive definite*. It is virtually obvious that the inner product on $\mathbb{R}^n$ is also linear in its second variable. Thus an inner product may be characterized as a *positive definite, symmetric, bilinear functional on* $\mathbb{R}^n$.

(2) Verify the fundamental facts about the norm on $\mathbb{R}^n$: if $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$ and α is a scalar, then

(a) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$;
(b) $\|\alpha\mathbf{x}\| = |\alpha|\,\|\mathbf{x}\|$; and
(c) if $\|\mathbf{x}\| = 0$, then $\mathbf{x} = \mathbf{0}$.

(3) Prove that if $(a_1, a_2, \dots)$ is a sequence of real numbers such that the series $\displaystyle\sum_{k=1}^{\infty} {a_k}^2$ converges, then the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}\, a_k$ converges absolutely.

Hint for proof. You may find the following steps helpful in organizing your solution.

(i) The key to this problem is *Monotonic Sequence Theorem* (MST)—see 18.1.3.
(ii) The hypothesis of the result we are trying to prove is that the series $\displaystyle\sum_{k=1}^{\infty} {a_k}^2$ converges. What, exactly, does this mean?
(iii) For each natural number n let $b_n = \displaystyle\sum_{k=1}^{n} {a_k}^2$. Rephrase (ii) in terms of the sequence (b_n).
(iv) Is the sequence (b_n) increasing?
(v) What, then, does the MST say about the sequence (b_n)?
(vi) For each natural number n let $c_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k^2}$. What do we know about the sequence (c_n)? (If in doubt consult the *p-test* 20.1.2). What does the MST say about the sequence (c_n)?
(vii) The conclusion we are trying to prove is that the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} a_k$ converges absolutely. What does this mean?
(viii) For each natural number n let $s_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k}\,|a_k|$. Rephrase (vii) in terms of the sequence (s_n).
(ix) Explain how for each n we may regard the number s_n as the dot product of two vectors in $\mathbb{R}^n$.
(x) Apply the *Schwarz inequality* 22.1.2 to the dot product in (ix). Use (v) and (vi) to establish that the sequence (s_n) is bounded above.
(xi) Use the MST one last time—keeping in mind what you said in (viii).

(4) Explain how to use the *Schwarz inequality* to show that if a, b, $c > 0$, then

$$\left(\frac{1}{2}\,a + \frac{1}{3}\,b + \frac{1}{6}\,c\right)^2 \le \frac{1}{2}\,a^2 + \frac{1}{3}\,b^2 + \frac{1}{6}\,c^2.$$

(5) Show that for all real numbers a, b, and θ

$$|a\cos\theta + b\sin\theta| \le \sqrt{a^2+b^2}\,.$$

(6) Explain carefully how to use the *Schwarz inequality* to prove that

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \ge 16$$

for all numbers a, b, c, $d > 0$.

(7) Suppose that $a_1, \dots, a_n > 0$. Use the *Schwarz inequality* to show that

$$\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \ge n^2.$$

(8) Show that the *parallelogram law* holds in $\mathbb{R}^n$. That is, prove that if $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$, then

$$\|\,\mathbf{x}+\mathbf{y}\|^2 + \|\,\mathbf{x}-\mathbf{y}\|^2 = 2\|\,\mathbf{x}\|^2 + 2\|\,\mathbf{y}\|^2.$$

(9) Prove that if $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^3$, then

$$\|\,\mathbf{a}\times\mathbf{b}\|^2 = \|\,\mathbf{a}\|^2\,\|\,\mathbf{b}\|^2 - \langle \mathbf{a},\mathbf{b}\rangle^2\,.$$

22.4. Answers to Odd-Numbered Exercises

(1) -3, -1, 9, 5, 1, 1

(3) -12

(5) line, plane

(7) circle, right circular cylinder

(9) $\frac{9}{7}$, $\frac{11}{7}$

(11) $-\frac{5}{2}$

(13) $\frac{1}{2}$, 1

(15) $\frac{3}{4}$

(17) -2, 3

(19) $-\frac{1}{3}$

(21) 5

(23) a, b, c, d, e, f

(25) 12

(27) 2, 1, 5, -1

(29) -3, 2, 5

(31) 1 (or 2), 3

(33) $2\sqrt{6}$, $\frac{\pi}{3}$

CHAPTER 23

LIMITS OF SCALAR FIELDS

23.1. Background

Topics: scalar fields, limits of scalar fields.

An important, if rather obvious, theorem says that a scalar field has zero limit at a point in $\mathbb{R}^n$ if and only if its absolute value does.

23.1.1. Theorem. *Let f be a scalar field defined in a neighborhood of a point $\mathbf{a}$ in $\mathbb{R}^n$ but not necessarily at a. Then*

$$\lim_{\mathbf{x} \longrightarrow \mathbf{a}} f(\mathbf{x}) = 0 \qquad \textit{if and only if} \qquad \lim_{\mathbf{x} \longrightarrow \mathbf{a}} |f(\mathbf{x})| = 0\,.$$

The next theorem, also very useful, is sometimes referred to as the *sandwich theorem* for scalar fields.

23.1.2. Theorem. *Let f, g, and h be scalar fields defined in a deleted neighborhood U of a point $\mathbf{a}$ in $\mathbb{R}^n$. If $f(\mathbf{x}) \le g(\mathbf{x}) \le h(\mathbf{x})$ for all $\mathbf{x} \in U$, $\lim\limits_{\mathbf{x} \longrightarrow \mathbf{a}} f(\mathbf{x}) = \ell$, and $\lim\limits_{\mathbf{x} \longrightarrow \mathbf{a}} h(\mathbf{x}) = \ell$, then $\lim\limits_{\mathbf{x} \longrightarrow \mathbf{a}} g(\mathbf{x}) = \ell$.*

23.2. Exercises

(1) Let $f(x,y) = \dfrac{x^2y}{x^3+y^3}$ for all $(x,y) \neq (0,0)$.

(a) $\lim\limits_{x \longrightarrow 0} f(x,0) =$ ______ .

(b) $\lim\limits_{x \longrightarrow 0} f(x,x) =$ ______ .

(c) Together what do (a) and (b) tell us about $\lim\limits_{(x,y) \longrightarrow (0,0)} f(x,y)$?

Answer: ____________________________ .

(2) Let f be the scalar field defined by $f(x,y) = x^3y^3(2x^{12}+3y^4)^{-1}$ for all $(x,y) \neq (0,0)$.

(a) $\lim\limits_{x \longrightarrow 0} f(x,0) =$ ______ .

(b) $\lim\limits_{x \longrightarrow 0} f(x,x^3) =$ ______ .

(c) Together what do (a) and (b) tell us about $\lim\limits_{(x,y) \longrightarrow (0,0)} f(x,y)$?

Answer: ____________________________ .

(3) Let f be the scalar field defined by $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$. Then f is defined at every point of $\mathbb{R}^2$ except the origin. If we further define $f(0,0) =$ _____ , then f is continuous on all of $\mathbb{R}^2$.

(4) The limit of $e^{\sqrt{x+3y}}$ as (x,y) approaches $(4,4)$ is ______________ .

(5) The limit of $\dfrac{2xy}{x^2+y^2}$ as (x,y) approaches $(0,0)$ is ______________ .

(6) The limit of $\dfrac{2x^2y}{x^4+3y^2}$ as (x,y) approaches $(0,0)$ is ______________ .

(7) The limit of $\dfrac{x^3+xy^2}{x^2+y^2}$ as (x,y) approaches $(0,0)$ is ______________ .

(8) The limit of $\dfrac{x^2+y^2}{\sqrt{x^2+y^2+4}-2}$ as (x,y) approaches $(0,0)$ is ______________ .

(9) The limit of $\dfrac{3xy+(\cos y)z^2+xy^2z^3}{\sqrt{x^4+4y^4+7z^8}}$ as (x,y,z) approaches $(0,0,0)$ is ______________ .

(10) The limit of $\dfrac{3xy+4yz+xz}{\sqrt{x^2+4y^2+7z^4}}$ as (x,y,z) approaches $(0,0,0)$ is ______________ .

(11) The limit of $\dfrac{xy+yz+zx}{x^2+y^2+z^2}$ as (x,y,z) approaches $(0,0,0)$ is ______________ .

(12) The limit of $\dfrac{xy+yz+zx}{\sqrt{x^2+y^2+z^2}}$ as (x,y,z) approaches $(0,0,0)$ is ______________ .

(13) The limit of $\dfrac{xyz}{(x^2+y^2+z^2)^{1/2}}$ as (x,y,z) approaches $(0,0,0)$ is ______________ .

Hint. Spherical coordinates.

(14) The limit of $\dfrac{xyz}{(x^2+y^2+z^2)^{3/2}}$ as (x,y,z) approaches $(0,0,0)$ is ______________ .

(15) Let $g(t) = e^t + \sqrt{t}$, $f(x,y) = x - 2y - 8$, and h be the composite function $g \circ f$. (That is, $h(x,y) = g(f(x,y))$ whenever this is defined.) Then the domain of h is the set of all

points (x, y) in the plane $\mathbb{R}^2$ which lie ___________ (above/below) the line $y = ax + b$ where $a =$ _____ and $b =$ _____ .

23.3. Problems

(1) Prove or disprove: $\displaystyle\lim_{(x,y)\longrightarrow(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$ exists.

Hint. Use problem 2 of chapter 1.

(2) Prove or disprove: $\displaystyle\lim_{(x,y)\longrightarrow(0,0)} \frac{x^2y^2}{x^2+y^2}$ exists.

(3) Prove or disprove: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ exists.

(4) Prove or disprove: $\displaystyle\lim_{(x,y)\longrightarrow(0,0)} \frac{x^2y}{x^3+y^3}$ exists.

(5) Prove or disprove: $\displaystyle\lim_{(x,y)\longrightarrow(0,0)} \frac{3x^2y^2}{7x^6+4y^3}$ exists.

(6) Prove or disprove: $\displaystyle\lim_{(x,y)\longrightarrow(0,0)} \frac{x^2y^3}{x^5+y^5}$ exists.

(7) Let $f(x,y) = \dfrac{\sin xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 1$. Prove or disprove: f is continuous.

(8) Your friend Fred R. Dimm conjectures that whenever the double limit $\displaystyle\lim_{(x,y)\longrightarrow(a,b)} f(x,y)$ exists then the iterated limits $\displaystyle\lim_{x\longrightarrow a}\big(\lim_{y\longrightarrow b} f(x,y)\big)$ and $\displaystyle\lim_{y\longrightarrow b}\big(\lim_{x\longrightarrow a} f(x,y)\big)$ exist. Write a note to him explaining why his conjecture is not correct.

Hint. Consider the function f defined by $f(x,y) = y\sin\frac{1}{x}$ whenever $x \neq 0$ and $f(0,y) = 0$ for all y.

(9) Prove or disprove: $\displaystyle\lim_{(x,y)\longrightarrow(0,0)} \frac{x^2}{x^2+y^2}$ exists.

(10) Prove or disprove: $\displaystyle\lim_{(x,y)\longrightarrow(0,0)} \frac{x^4+y^4}{xy}$ exists.

(11) Prove or disprove: $\displaystyle\lim_{(x,y)\longrightarrow(0,0)} \frac{x^4+y^8}{x^2y^2}$ exists.

(12) Use the *Schwarz inequality* to show that if $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$, then

$$\lim_{\mathbf{x}\longrightarrow\mathbf{a}} \langle \mathbf{x}, \mathbf{b}\rangle = \langle \mathbf{a}, \mathbf{b}\rangle.$$

(13) Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Prove that $\displaystyle\lim_{\mathbf{x}\longrightarrow\mathbf{0}} \frac{x_j x_k}{\|\mathbf{x}\|} = 0$ for j, $k = 1, \dots, n$ with $j \neq k$.

Show also that $\displaystyle\lim_{\mathbf{x}\longrightarrow\mathbf{0}} \frac{{x_k}^2}{\|\mathbf{x}\|} = 0$ for $k = 1, \dots, n$.

(14) Let f be a real valued function defined in a deleted neighborhood of the origin in $\mathbb{R}^2$. Prove or disprove: if $f(x,y)$ approaches 0 as (x,y) approaches the origin along any straight line, then $\displaystyle\lim_{(x,y)\longrightarrow(0,0)} f(x,y) = 0$.

Hint. Consider the function f defined by $f(x,y) = \dfrac{xy^3}{x^2+y^6}$.

23.4. Answers to Odd-Numbered Exercises

(1) (a) 0
    (b) $\dfrac{1}{2}$
    (c) It does not exist.

(3) 1

(5) not defined

(7) 0

(9) not defined

(11) not defined

(13) 0

(15) below, $\dfrac{1}{2}$, -4

Part 7

DIFFERENTIATION OF FUNCTIONS OF SEVERAL VARIABLES

CHAPTER 24

PARTIAL DERIVATIVES

24.1. Background

Topics: partial derivatives, tangent planes.

24.1.1. Definition. If U is an open subset of $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}$ is a scalar field on U, its k^{th} PARTIAL DERIVATIVE f_k (for $1 \le k \le n$) is defined by

$$f_k(\mathbf{a}) = \lim_{t \to 0} \frac{f(\mathbf{a} + t\,\mathbf{e}^k) - f(\mathbf{a})}{t}$$

for every $\mathbf{a} \in U$ for which this limit exists.

A vector field $\mathbb{F} = (F^1, F^2, \dots, F^n)$ defined on an open subset of $\mathbb{R}^m$ may also have partial derivatives. Let $\mathbf{F}_k = (F^1_k, F^2_k, \dots, F^n_k)$ whenever the partial derivatives of all the coordinate functions F^k exist.

24.1.2. Notation. Among the notations for the k^{th} partial derivative of a scalar field f are f_k, $\dfrac{\partial f}{\partial x_k}$, and $D_k f$. When we work in $\mathbb{R}^3$ and use coordinates (x, y, z)—instead of (x_1, x_2, x_3)—alternative notations for f_1 are f_x, $\dfrac{\partial f}{\partial x}$, and $D_x f$. Similar notations are used for f_2 and f_3.

If the k^{th} partial derivative f_k of f has a j^{th} partial derivative, this *second order* partial derivative $\bigl(f_k\bigr)_j$ is denoted by f_{kj} or $D_{kj} f$ or $\dfrac{\partial^2 f}{\partial x_j \partial x_k}$. If $j = k$ the last of these becomes $\dfrac{\partial^2 f}{\partial {x_k}^2}$. Higher order partial derivatives are treated in a similar fashion.

For a vector field $\mathbf{F}$ the notation F^j_k is unambiguous since it can be proved, at least for smooth vector fields (see definition 28.1.6), that $\bigl(F^j\bigr)_k = \bigl(\mathbf{F}_k\bigr)^j$ for all appropriate indices j and k.

24.2. Exercises

(1) Let $\phi(x,y) = e^x \ln y$. Then $\phi_x(0,e) =$ ____ and $\phi_y(0,e) =$ ________ .

(2) Let $f(x,y,z) = 3xy^3 + z^2e^x$ and $\mathbf{a} = (\ln 2, 1, -2)$. Then $\dfrac{\partial f}{\partial x}(\mathbf{a}) =$ ______ , $\dfrac{\partial f}{\partial y}(\mathbf{a}) =$ _________ , and $\dfrac{\partial f}{\partial z}(\mathbf{a}) =$ ______ .

(3) Let $f(x,y) = \displaystyle\int_{10}^{x^3 \sin y} \frac{dt}{1+\cos^2(\frac{\pi}{16}t)}$. Then $\dfrac{\partial f}{\partial x}(2, \frac{\pi}{6}) =$ ____ .

(4) Let $f(x,y,z) = x^y \arctan z$. Then $\dfrac{\partial^2 f}{\partial z \partial y}(2,3,1) = a \ln 2$ where $a =$ ____ .

(5) Let $\phi(u,v) = \dfrac{u}{u+v}$. Then $\phi_u(1,2) =$ _________ and $\phi_v(1,2) =$ ________ .

(6) Let $\psi(x,y) = e^{-x}\sin(x+2y)$. Then $\dfrac{\partial \psi}{\partial x}(0, \frac{\pi}{4}) =$ _____ and $\dfrac{\partial \psi}{\partial y}(0, \frac{\pi}{4}) =$ ____ .

(7) Let $g(x,y) = e^{1-6x}\sin(4x+2y)$. Then $\dfrac{\partial g}{\partial x}(0, \frac{\pi}{6}) = (a - b\sqrt{b})e$ where $a =$ ____ and $b =$ ____ .

(8) Let $f(x,y,z) = \arctan(xyz)$. Then $\dfrac{\partial^2 f}{\partial x \partial z}(1,2,-1) = -\dfrac{a}{b}$ where $a =$ ____ and $b =$ ______ .

(9) Let $f(u,v,w) = \ln(u^2 + 2uv) + \sin^3(u+w)$. Then $\dfrac{\partial f}{\partial u}(1, \frac{7}{2}, \frac{\pi}{3} - 1) = \dfrac{a}{4}$ where $a =$ ____ .

(10) Let $z = x^c e^{-y/x}$. If the constant $c =$ ______ , then z satisfies the partial differential equation

$$\frac{\partial z}{\partial x} = y\,\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y}\,.$$

(11) Let $\phi(w,x,y,z) = wxy + x^2 \tan z$ and $\mathbf{a} = (1, -2, 3, \frac{\pi}{4})$. Then $\phi_w(\mathbf{a}) =$ _____ , $\phi_x(\mathbf{a}) =$ _____ , $\phi_y(\mathbf{a}) =$ _____ , and $\phi_z(\mathbf{a}) =$ ____ .

(12) Let $f(x,y) = \displaystyle\int_{-7}^{e^{x^2+5x}\sin y} \frac{dt}{8+\cos^4(2\pi t)}$. Then $\dfrac{\partial f}{\partial x}(0, \frac{\pi}{6}) = \dfrac{5}{a}$ where $a =$ ____ .

(13) Let $f(x,y) = \displaystyle\int_3^{\sqrt{x}y^2} \frac{2t}{8+t^2+t^4}\,dt$. Then $\dfrac{\partial f}{\partial x}(4, \sqrt{2}) = \dfrac{1}{a}$ where $a =$ ______ and $\dfrac{\partial f}{\partial y}(4, \sqrt{2}) = \dfrac{b\sqrt{2}}{35}$ where $b =$ ____ .

(14) Let $f(x,y,z) = x^{y^z}$. Then $\dfrac{\partial f}{\partial x} = y^z x^{k(y,z)}$ where $k(y,z) =$ ______________ ,

$\dfrac{\partial f}{\partial y} = x^{y^z} g(x,y,z)$ where $g(x,y,z) =$ __________________ , and

$\dfrac{\partial f}{\partial z} = x^{y^z} y^z h(x,y)$ where $h(x,y) =$ __________________ .

(15) Let $f(x,y,z) = \displaystyle\int_{x^2 \sin y}^{z^3 \tan y} \frac{dt}{1+t^2+t^4}$. Then $\dfrac{\partial f}{\partial y}(-1, \frac{\pi}{4}, 1) = \dfrac{a}{b} - \dfrac{a}{c}\sqrt{a}$ where $a =$ ____ , $b =$ ____ , and $c =$ ____ .

(16) Let $f(x,y) = xy(x^2 - y^2)(x^2+y^2)^{-1}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Then $f_{xy}(0,0) =$ _____ and $f_{yx}(0,0) =$ ____ .

(17) Let $f(x,y) = 5xe^{2y} + 2x^2(y+3)\sin\frac{1}{x}\cos y$ for $x \neq 0$ and $f(0,y) = 0$. Then $\dfrac{\partial f}{\partial x}(0,0) =$ ________ .

(18) Let $g(x,y) = \displaystyle\int_1^{x^3y^2} \frac{1}{1+\sqrt{t}+t^2}\,dt$. Then $\dfrac{\partial g}{\partial x}(1,2) = \dfrac{a}{19}$ where $a =$ ________ .

24.3. Problems

(1) Show that $u = \dfrac{x^2y^2}{x+y}$ satisfies the partial differential equation

$$x\,u_x + y\,u_y = 3u.$$

(2) Show that for any constants a, b, and c the function $u = ax^4 + 2bx^2y^2 + cy^4$ is a solution to the partial differential equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 4u.$$

(3) Show that $u = x^2y + y^2z + z^2x$ is a solution to the partial differential equation

$$u_x + u_y + u_z = (x+y+z)^2.$$

24.4. Answers to Odd-Numbered Exercises

(1) $1,\ \dfrac{1}{e}$

(3) 4

(5) $\dfrac{2}{9},\ -\dfrac{1}{9}$

(7) 2, 3

(9) 9

(11) $-6,\ -1,\ -2,\ 8$

(13) 70, 4

(15) 2, 3, 7

(17) 5

CHAPTER 25

GRADIENTS OF SCALAR FIELDS AND TANGENT PLANES

25.1. Background

Topics: gradient, tangent plane, directional derivative, path of steepest descent

For the following definition and theorem suppose that f is a real valued function whose domain is an open subset V of $\mathbb{R}^n$. Suppose also that $\mathbf{a}$ is a vector in the domain of f and that $\mathbf{u}$ is a unit vector in $\mathbb{R}^n$.

25.1.1. Definition. The DIRECTIONAL DERIVATIVE of f at $\mathbf{a}$ in the direction $\mathbf{u}$, denoted by $D_{\mathbf{u}}f(\mathbf{a})$, is defined to be

$$\lim_{\lambda \longrightarrow 0} \frac{f(\mathbf{a} + \lambda\mathbf{u}) - f(\mathbf{a})}{\lambda}$$

if this limit exists.

25.1.2. Theorem. *If f is differentiable at $\mathbf{a}$, then the directional derivative $D_{\mathbf{u}}f(\mathbf{a})$ exists and*

$$D_{\mathbf{u}}f(\mathbf{a}) = \langle \nabla f(\mathbf{a}), \mathbf{u} \rangle.$$

25.2. Exercises

(1) Find a function (if one exists) whose gradient is

$$\left(\frac{y^2}{x} + 2xy^3 - \frac{1}{1+x^2}, 2y\ln x + 3x^2y^2 - \sin y\right).$$

Answer: ______________________ .

(2) Find a function (if one exists) whose gradient is

$$\left(4x^3y - \frac{1}{1+x^2} + e^y\right)\mathbf{i} + (x^4 + xe^y + x)\,\mathbf{j}.$$

Answer: ______________________ .

(3) Find a function (if one exists) whose gradient is

$$(y^3 + 2xy + 3x^2 + 2xy^2)\,\mathbf{i} + (4y^3 + x^2 + 2x^2y + 3xy^2)\,\mathbf{j}.$$

Answer: ______________________ .

(4) Find a function (if one exists) whose gradient is

$$\left(x^2 \arcsin y, \frac{x^3}{3\sqrt{1-y^2}} - \ln y\right).$$

Answer: ______________________ .

(5) Find a function (if one exists) whose gradient is

$$\left(\ln x + 2xye^y + \frac{x}{\sqrt{1-x^2}} + 1\,,\; x^2e^y + \frac{1}{\sqrt{1-y^2}} + x^2ye^y\right).$$

Answer: ______________________ .

(6) Find a function (if one exists) whose gradient is

$$\left(\frac{\arctan y}{\sqrt{1-x^2}} + \frac{x}{y}\,, \frac{\arcsin x}{1+y^2} - \frac{x^2}{2y^2} + 1\right).$$

Answer: ______________________ .

(7) Find a function (if one exists) whose gradient is

$$(2xy + z^2)\,\mathbf{i} + (x^2 + e^y + 2yz^3)\,\mathbf{j} + (2xz + \pi\cos\pi z + 3y^2z^2)\,\mathbf{k}.$$

Answer: ______________________ .

(8) At what angle θ do the sphere $x^2 + y^2 + z^2 = 8$ and the plane $y = 2$ intersect?

Answer: $\theta =$ ________ .

(9) The equation of the tangent plane to the surface $x^3 + y^3 = 3xyz$ at the point $(1, 2, \frac{3}{2})$ is $ax + by + 4z = 0$ where $a =$ _____ and $b =$ _____ .

(10) Let $f(x, y) = x^2 - y^2 - 2x + 3y - 4$ and $\mathbf{p} = (2, 1)$. The equation of the tangent plane to the surface $z = f(x, y)$ at $\mathbf{p}$ is $z = ax + by + c$ where $a =$ _____ , $b =$ _____ , and $c =$ _____ . The value of $\Delta f_{\mathbf{p}}$ at (x, y) is $x^2 + ay^2 + bx + cy + d$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $d =$ _____ . The equation of the tangent plane to the surface $z = \Delta f_{\mathbf{p}}(x, y)$ at $(0, 0)$ is $z = ax + by + c$ where $a =$ _____ , $b =$ _____ , and $c =$ _____ .

(11) Let $f(x,y) = x^2 + 2y^2 - 4x - 4y + 6$ and $\mathbf{p} = (1,-1)$. The equation of the tangent plane to the surface $z = f(x,y)$ at $\mathbf{p}$ is $z = ax + by + c$ where $a =$ ____ , $b =$ ____ , and $c =$ ____ . The value of $\Delta f_{\mathbf{p}}$ at (x,y) is $x^2 + ay^2 + bx + cy + d$ where $a =$ ____ , $b =$ ____ , $c =$ ____ , and $d =$ ____ . The equation of the tangent plane to the surface $z = \Delta f_{\mathbf{p}}(x,y)$ at $(0,0)$ is $z = ax + by + c$ where $a =$ ____ , $b =$ ____ , and $c =$ ____ .

(12) The equation of the tangent plane to the surface $x^3 + y^3 + z^3 = 8$ at the point $(1,2,-1)$ is $x + ay + z = b$ where $a =$ ____ and $b =$ ____ .

(13) The sum of the intercepts of a plane tangent to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ is ________ .

(14) The sum of the squares of the intercepts of a plane tangent to the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$ is ________ .

(15) The equation of the tangent plane to the surface $x^4 + y^4 + z^4 = 18$ at the point $(1,-2,1)$ is $ax + by + cz = 18$ where $a =$ ____ , $b =$ ____ , and $c =$ ____ .

(16) The equation of the tangent plane to the surface $x^3z + xyz + y^2z^2 = 1$ at the point $(1,2,-1)$ is $5x + ay + bz = c$ where $a =$ ____ , $b =$ ____ , and $c =$ ____ .

(17) The equation of the tangent plane to the surface $ye^{xy} + z^2 = 0$ at the point $(0,-1,1)$ is $ax + by + cz = 1$ where $a =$ ____ , $b =$ ____ , and $c =$ ____ .

(18) The equation of the tangent plane to the cylinder $xz = 4$ at the point $(1,0,4)$ is $ax + by + cz = 8$ where $a =$ ____ , $b =$ ____ , and $c =$ ____ .

(19) The equation of the tangent plane to the ellipsoid $2x^2 + y^2 + z^2 = 12$ at the point $(1,-3,1)$ is $ax + by + cz = 12$ where $a =$ ____ , $b =$ ____ , and $c =$ ____ .

(20) The equation of the tangent plane to the surface $z = x^2y - xy^3 + 7$ at the point $(1,2,1)$ is $ax + by + cz = 27$ where $a =$ ____ , $b =$ ____ , and $c =$ ____ .

(21) Let $f(x,y,z) = x^2y - y^2z$. The directional derivative of f at the point $(2,1,1)$ in the direction of the curve $\mathbf{r}(t) = (2\cos(t-1), t, \exp(t^4 - t))$ is $-\dfrac{1}{\sqrt{a}}$ where $a =$ ____ .

(22) Let $f(x,y,z) = xy + 2xz - y^2 + z^2$ and $P = (1,-2,1)$.
  (a) The directional derivative of f at P in the direction of the curve $\mathbf{r}(t) = (t, t-3, t^2)$ is $\dfrac{a}{\sqrt{6}}$ where $a =$ ____ .
  (b) A vector pointing in the direction of the greatest increase of f at P is ___$\mathbf{i}$+___$\mathbf{j}$+4$\mathbf{k}$.
  (c) A vector normal to the surface $f(x,y,z) = -3$ at P is ___$\mathbf{i}$ + ___$\mathbf{j}$ + 4$\mathbf{k}$.

(23) Let $f(x,y) = \dfrac{x^2}{16} + \dfrac{y^2}{9}$ and $\mathbf{P} = (-2, \frac{3}{2}\sqrt{3})$.
  (a) A vector pointing in the direction of greatest increase of f at P is ____ $\mathbf{i} + 4\mathbf{j}$.
  (b) A vector normal to the level curve $f(x,y) = 1$ at the point P is ____ $\mathbf{i} + 4\mathbf{j}$.
  (c) A vector tangent to the level curve $f(x,y) = 1$ at the point P is $4\mathbf{i} +$ ____ $\mathbf{j}$.

(24) Let $f(x,y,z) = xy + yz + xz$. The directional derivative of f at $(1,-1,1)$ in the direction of $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ is $\dfrac{a}{3}\sqrt{6}$ where $a =$ ____ .

(25) Let $f(x,y,z) = xy + yz + xz$. The directional derivative of f at $(3,4,-1)$ in the direction of $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ is $\dfrac{a}{3}\sqrt{6}$ where $a =$ ____ .

(26) Let $f(x,y) = x^2y$. Then the directional derivative of f at $\mathbf{a} = (2,4)$ in the direction of the curve $\mathbf{r}(t) = (t^3 + 1)\,\mathbf{i} + (t^2 + 2t + 1)\,\mathbf{j}$ is $\dfrac{a}{5}$ where $a =$ ______ .

(27) The directional derivative of $\phi(x, y) = x^2 \ln y$ at the point $(2, 4)$ in the direction of the curve $\mathbf{r}(t) = (t^3 + 1, 2t + 2)$ is $\dfrac{1}{\sqrt{a}}(b + 24 \ln b)$ where $a =$ _____ and $b =$ _____ .

(28) Find the derivatives of $v = xy^2 \sin z$ at the point $(1, 1, \pi/6)$ in the (two) directions of the line $x = y$, $z = 0$.

Answer: ___________ and ___________ .

(29) If $f(x, y, z) = xy + y^2 z$ and $\mathbf{r}(t) = (\frac{1}{2}t^2, t - 4, \frac{1}{2}e^{t^2-4})$, then $(f \circ \mathbf{r})'(2) =$ _____ .

(30) The directional derivative of the function $f\colon (x, y, z) \mapsto x \exp(y^2 - z^2)$ at $(1, 2, -2)$ in the direction of the curve $\mathbf{r}\colon t \mapsto (t, 2\cos(t - 1), -2\exp(t - 1))$ is $\dfrac{a}{\sqrt{5}}$ where $a =$ ______ .

25.3. Problems

(1) Let $\phi(x,y) = 2x^2 + 6y^2$ and $\mathbf{a} = (2,-1)$. Find the steepest downhill path on the surface $z = \phi(x,y)$ starting at the point $\mathbf{a}$ and ending at the minimum point on the surface.

Hints.

(a) It is enough to find the equation of the projection of the curve onto the xy-plane; every curve $t \mapsto (x(t), y(t))$ in the xy-plane is the projection along the z-axis of a unique curve $t \mapsto \big(x(t), y(t), \phi(x(t), y(t))\big)$ on the surface $z = \phi(x,y)$.

(b) If $\mathbf{c}\colon t \mapsto (x(t), y(t))$ is the desired curve and we set $\mathbf{c}(0) = \mathbf{a}$, then the unit vector $\mathbf{u}$ which minimizes the directional derivative $D_{\mathbf{u}}(\mathbf{b})$ at a point $\mathbf{b}$ in $\mathbb{R}^2$ is the one obtained by choosing $\mathbf{u}$ to point in the direction of $-\nabla\phi(\mathbf{b})$. Thus in order for the curve to point in the direction of the most rapid decrease of ϕ at each point $\mathbf{c}(t)$, the tangent vector to the curve at $\mathbf{c}(t)$ must be some positive multiple $p(t)$ of $-\nabla\phi(\mathbf{c}(t))$. The function p will govern the speed of descent; since this is irrelevant in the present problem, set $p(t) = 1$ for all t.

(c) Recall that on an interval the only nonzero solution to an equation of the form $Dx(t) = kx(t)$ has the form $x(t) = x(0)\, e^{kt}$.

(d) The parameter t which we have introduced is artificial. Eliminate it to obtain an equation of the form $y = f(x)$.

(2) This, like the preceding problem, is a steepest descent problem. However, we now suppose (probably contrary to fact) that for some reason we are unable to solve explicitly the resulting differential equations. Instead we invoke an approximation technique. Let

$$\phi(x) = 13{x_1}^2 - 42x_1 + 13{x_2}^2 + 6x_2 + 10x_1x_2 + 9$$

for all $\mathbf{x}$ in $\mathbb{R}^2$. The goal is to approximate the path of steepest descent. Start at an arbitrary point $\mathbf{x}^0$ in $\mathbb{R}^2$ and choose a number $h > 0$. At $\mathbf{x}^0$ compute the gradient of ϕ, take $\mathbf{u}^0$ to be the unit vector pointing in the direction of $-\nabla\phi(\mathbf{x}^0)$, and then move h units in the direction of $\mathbf{u}^0$ arriving at a point $\mathbf{x}^1$. Repeat the procedure: find the unit vector $\mathbf{u}^1$ in the direction of $-\nabla\phi(\mathbf{x}^1)$, then from $\mathbf{x}^1$ move h units along $\mathbf{u}^1$ to a point $\mathbf{x}^2$. Continue in this fashion. In other words, $\mathbf{x}^0 \in \mathbb{R}^2$ and $h > 0$ are arbitrary, and for $n \geq 0$

$$\mathbf{x}^{n+1} = \mathbf{x}^n + h\mathbf{u}^n$$

where $\mathbf{u}^n = -\|\,\nabla\phi(\mathbf{x}^n)\|^{-1}\nabla\phi(\mathbf{x}^n)\,$.

(a) Start at the origin $\mathbf{x}^0 = (0,0)$ and choose $h = 1$. Compute 25 or 30 values of $\mathbf{x}^n$. Explain geometrically what is happening here. Why is h "too large"? *Hint.* Don't attempt to do this by hand. Write a program for a computer or a programmable calculator. In writing your program don't ignore the possibility that $\nabla\phi(\mathbf{x}^n)$ may be zero for some n. Also don't forget when you write up your report that the reader probably has no idea how to read the language in which you write your program. It must be well enough documented so that the reader can easily understand what you are doing at each step.

(b) Describe what happens when h is "too small". Again start at the origin, take $h = 0.001$ and compute 25 or 30 values of $\mathbf{x}^n$.

(c) By altering the values of h at appropriate times, find a succession of points $\mathbf{x}^0, \dots, \mathbf{x}^n$ (starting with $\mathbf{x}^0 = (0,0)$) such that the distance between $\mathbf{x}^n$ and the point where ϕ assumes its minimum value is less than 0.001. (By examining the points $\mathbf{x}^0, \dots, \mathbf{x}^n$ you should be able to guess, for this particular function, the exact location of the minimum.)

(d) Alter the program in part (a) to eliminate division by $\|\,\nabla\phi(\mathbf{x}^n)\|$. (That is, let $\mathbf{x}^{n+1} = \mathbf{x}^n - h\nabla\phi(\mathbf{x}^n)$.) Explain what happens in this case when h is "too large" (say $h = 1$). Explain why the altered program works better (provided that h is chosen appropriately) than the program in (a) for the present function ϕ.

(3) Let $P(x,y) = 3x^2 \arctan y + ye^{xy} + y^2$ and $Q(x,y) = \dfrac{x^3}{1+y^2} + xe^{xy} + x^2y$. Explain clearly how you know that the vector field $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ is not the gradient of any scalar field ϕ defined on $\mathbb{R}^2$.

(4) Use the *definition* of directional derivative to find $D_{\mathbf{u}}f(\mathbf{a})$ when $f(x,y) = \ln\sqrt{xy}$, $\mathbf{u} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and $\mathbf{a} = (2,3)$.

(5) Use the *definition* of directional derivative to find $D_{\mathbf{u}}f(\mathbf{a})$ when $f(x,y) = \ln\sqrt{x^2+y^2}$, $\mathbf{u} = \left(\frac{3}{5}, \frac{4}{5}\right)$, and $\mathbf{a} = (1,1)$.

(6) Let $f(x,y) = \exp(x+y)$, $\mathbf{u}$ be the unit vector in the direction of $(1,1)$, and $\mathbf{a} = (5,-3)$. Use the *definition* of directional derivative to find $D_{\mathbf{u}}f(\mathbf{a})$.

(7) Let $f(x,y,z) = xy + \cos z$, $\mathbf{u}$ be the unit vector in the direction of $(4,-2,2)$ and $\mathbf{a} = (1,1,-\frac{\pi}{3})$. Use the *definition* of directional derivative to find $D_{\mathbf{u}}f(\mathbf{a})$.

(8) Use the *definition* of directional derivative to find $D_{\mathbf{u}}f(\mathbf{a})$ when $f(x,y) = \ln(x+y^2)^5$, $\mathbf{u} = (\frac{3}{5}, \frac{4}{5})$, and $\mathbf{a} = (2,1)$.

(9) A force field $\mathbf{F}\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is CONSERVATIVE if there exists a scalar field $V\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that $\mathbf{F} = -\nabla V$; such a scalar field is a POTENTIAL FUNCTION for $\mathbf{F}$. Let the position of a particle P at time t be denoted by $\mathbf{r}(t)$. Suppose that P is acted upon by a conservative force field $\mathbf{F}$. Assume *Newton's second law*:

$$\mathbf{F} \circ \mathbf{r} = m\mathbf{a}$$

where $\mathbf{a}$ is the acceleration of P and m is its mass. The KINETIC ENERGY of P is defined by

$$KE := \frac{1}{2}m\|\mathbf{v}\|^2$$

where $\mathbf{v}$ is the velocity of P; its POTENTIAL ENERGY is defined by

$$PE := V \circ \mathbf{r}$$

where V is a potential function for $\mathbf{F}$. The TOTAL ENERGY of P is the sum of its kinetic and potential energies. Prove for this situation the *law of conservation of energy*; that is, show that the total energy of P is constant.

(10) Suppose that the temperature ϕ at a point (x,y) on a flat surface is given by the formula $\phi(x,y) = x^2 - y^2$. A heat-seeking bug is placed at a point (a,b) on the surface. What path should the bug follow to get warm as quickly as possible?

(11) Let $P(x,y) = 3x^2 \arctan y + ye^{xy} + y^2$ and $Q(x,y) = \dfrac{x^3}{1+y^2} + xe^{xy} + x^2y$. Explain clearly how you know that the vector field $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ is not the gradient of any scalar field ϕ defined on $\mathbb{R}^2$.

(12) Determine in non-parametric form the path of steepest descent (projected onto the xy–plane) along the surface $z = 9x^2 + 3y^2$ starting from the point $(2,2)$.

25.4. Answers to Odd-Numbered Exercises

(1) $y^2 \ln x + x^2y^3 - \arctan x + \cos y$

(3) $xy^3 + x^2y + x^3 + x^2y^2 + y^4$

(5) $x^2ye^y + \arcsin y + x \ln x - \sqrt{1 - x^2}$

(7) $x^2y + e^y + xz^2 + \sin \pi z + y^2z^3$

(9) 4, −5

(11) −2, −8, 3, 2, −2, −8, 0, −2, −8, 0

(13) a

(15) 1, −8, 1

(17) 1, 1, 2

(19) 2, −3, 1

(21) 10

(23) (a) $-\sqrt{3}$
(b) $-\sqrt{3}$
(c) $\sqrt{3}$

(25) 7

(27) 13, 2

(29) 4

CHAPTER 26

MATRICES AND DETERMINANTS

26.1. Background

Topics: matrix, transpose, symmetric, determinant, minor, cofactor.

The Arithmetic of Matrices. Let m and n be natural numbers. An $m \times n$ (read "m by n") MATRIX is a rectangular array of numbers with m rows and n columns. If A is a matrix, the entry in the i^{th} row and j^{th} column is denoted by A^i_j. (Occasionally we use the notation A_{ij} instead.) The matrix A itself may be denoted by $\left[A^i_j\right]_{i=1}^{m}\,{}_{j=1}^{n}$, by $[A^i_j]$, or by a rectangular array

$$\begin{bmatrix} A^1_1 & A^1_2 & \dots & A^1_n \\ A^2_1 & A^2_2 & \dots & A^2_n \\ \vdots & \vdots & \ddots & \vdots \\ A^m_1 & A^m_2 & \dots & A^m_n \end{bmatrix}.$$

In light of this notation it is reasonable to refer to the index i in the expression A^i_j as the ROW INDEX and to call j the COLUMN INDEX. When we speak of the "value of a matrix A at (i,j)," we mean the entry in the i^{th} row and j^{th} column of A. Thus, for example,

$$A = \begin{bmatrix} 1 & 4 \\ 3 & -2 \\ 7 & 0 \\ 5 & -1 \end{bmatrix}$$

is a 4×2 matrix and $A^3_1 = 7$.

Two matrices of the same size can be added. Addition of matrices is done pointwise. The sum $A + B$ of two $m \times n$ matrices is the $m \times n$ matrix whose value at (i,j) is $A^i_j + B^i_j$, That is,

$$(A + B)^i_j = A^i_j + B^i_j$$

for $1 \le i \le m$ and $1 \le j \le n$. So, for example,

$$\begin{bmatrix} 1 & 2 & 7 \\ 3 & 0 & -4 \end{bmatrix} + \begin{bmatrix} 1 & 3 & -1 \\ 2 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 1+1 & 2+3 & 7+(-1) \\ 3+2 & 0+4 & (-4)+(-1) \end{bmatrix} = \begin{bmatrix} 2 & 5 & 6 \\ 5 & 4 & -5 \end{bmatrix}.$$

Scalar multiplication is also defined pointwise. If A is an $m \times n$ matrix and $\alpha \in \mathbb{R}$, then αA is the $m \times n$ matrix whose value at (i,j) is αA^i_j. That is,

$$(\alpha A)^i_j = \alpha A^i_j$$

for $1 \le i \le m$ and $1 \le j \le n$. For example,

$$(-3) \cdot \begin{bmatrix} 1 & 3 & -2 \\ -1 & 4 & -5 \end{bmatrix} = \begin{bmatrix} -3 & -9 & 6 \\ 3 & -12 & 15 \end{bmatrix}.$$

We may also subtract two matrices of the same size. By $-B$ we mean $(-1)B$, and by $A - B$ we mean $A + (-B)$. So,

$$\begin{bmatrix} 6 & 4 \\ 5 & -2 \\ 3 & 7 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 3 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 2 & -1 \\ 7 & 5 \end{bmatrix}.$$

If A is an $m \times n$ matrix and B is an $n \times p$ matrix, the product of A and B is the $m \times p$ matrix whose value at (i, j) is $\sum_{k=1}^{n} A^i_k B^k_j$. That is,

$$(AB)^i_j = \sum_{k=1}^{n} A^i_k B^k_j$$

for $1 \le i \le m$ and $1 \le j \le p$. Notice that in order for the product AB to be defined the number of columns of A must be the same as the number of rows of B. Here is a slightly different way of thinking of the product of A and B. Define (as usual) the INNER PRODUCT (or DOT PRODUCT) of two n-tuples $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ to be $\sum_{k=1}^{n} x_k y_k$. Regard the rows of the matrix A as n-tuples (read from left to right) and the columns of B as n-tuples (read from top to bottom). Then the entry in the i^{th} row and j^{th} column of the product AB is the dot product of the i^{th} row of A and the j^{th} column of B.

26.1.1. Example. Let A be the 3×4 matrix $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 3 & -4 & 5 \\ 0 & 1 & 8 & 2 \end{bmatrix}$ and B be the 4×2 matrix $\begin{bmatrix} 2 & 3 \\ -4 & 7 \\ 1 & -1 \\ 5 & 6 \end{bmatrix}$.

Then the product AB is the 3×2 matrix $\begin{bmatrix} -1 & 23 \\ 11 & 58 \\ 14 & 11 \end{bmatrix}$.

Matrix multiplication is not commutative. If A is a 2×3 matrix and B is a 3×4 matrix, then AB is defined but BA is not.

Even in situations where both products AB and BA are defined, they need not be equal. For example, if $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$, then $AB = \begin{bmatrix} 3 & 7 \\ -1 & 1 \end{bmatrix}$ whereas $BA = \begin{bmatrix} 0 & -2 \\ 5 & 4 \end{bmatrix}$.

We now define the ACTION of a matrix on a vector. If A is an $m \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$, then $A\mathbf{x}$, the RESULT OF A ACTING ON $\mathbf{x}$, is defined to be the vector in $\mathbb{R}^m$ whose j^{th} coordinate is $\sum_{k=1}^{n} A^j_k x_k$ (this is just the dot product of the j^{th} row of A with $\mathbf{x}$). That is,

$$(A\mathbf{x})_j := \sum_{k=1}^{n} A^j_k x_k$$

for $1 \leq j \leq m$. Here is another way of saying the same thing: Regard $\mathbf{x}$ as an $n \times 1$ matrix

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

(sometimes called a COLUMN VECTOR). Now multiply the $m \times n$ matrix A by the $n \times 1$ matrix $\mathbf{x}$. The result will be an $m \times 1$ matrix (another column vector), say

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}.$$

Then $A\mathbf{x}$ is the m-tuple $(y_1, \dots, y_m)$. Thus an $m \times n$ matrix A may be thought of as a mapping from $\mathbb{R}^n$ into $\mathbb{R}^m$.

26.1.2. Example. Let $A = \begin{bmatrix} 3 & 0 & -1 & -4 \\ 2 & 1 & -1 & -2 \\ 1 & -3 & 0 & 2 \end{bmatrix}$ and $\mathbf{x} = (2, 1, -1, 1)$. Then

$$A\mathbf{x} = \begin{bmatrix} 3 & 0 & -1 & -4 \\ 2 & 1 & -1 & -2 \\ 1 & -3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = (3, 4, 1).$$

One matrix for which we have a special name is I_n, the $n \times n$ IDENTITY MATRIX. It has 1's on the main diagonal (from upper left to lower right) and 0's everywhere else. Thus, for example,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that it acts as the identity function on any vector (that is, $I_n\mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^n$) and that $I_nA = AI_n = A$ for every $n \times n$ matrix A. (When it will cause no confusion, we often write I for I_n.)

26.1.3. Definition. Let A be an $m \times n$ matrix. The TRANSPOSE of A, denoted by A^t, is the $n \times m$ matrix obtained by interchanging the rows and columns of A. That is, if $B = A^t$, then $B^i_j = A^j_i$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. For example, if $A = \begin{bmatrix} 2 & 3 \\ 4 & -5 \\ 6 & 11 \end{bmatrix}$, then $A^t = \begin{bmatrix} 2 & 4 & 6 \\ 3 & -5 & 11 \end{bmatrix}$.

A square matrix A is SYMMETRIC if $A^t = A$.

Some Facts about Determinants. Determinants are a useful tool for dealing with matrices (and the linear transformations they represent).

Fact 1. Let $n \in \mathbb{N}$ and $\mathbf{M}_{n\times n}$ be the collection of all $n \times n$ matrices. There is exactly one function

$$\det\colon \mathbf{M}_{n\times n} \longrightarrow \mathbb{R}\colon A \mapsto \det A$$

which satisfies

(a) $\det I_n = 1$.
(b) If $A \in \mathbf{M}_{n\times n}$ and A' is the matrix obtained by interchanging two rows of A, then $\det A' = -\det A$.
(c) If $A \in \mathbf{M}_{n\times n}$, $c \in \mathbb{R}$, and A' is the matrix obtained by multiplying each element in one row of A by the number c, then $\det A' = c \det A$.
(d) If $A \in \mathbf{M}_{n\times n}$, $c \in \mathbb{R}$, and A' is the matrix obtained from A by multiplying one row of A by c and adding it to another row of A (that is, choose i and j between 1 and n with $i \neq j$ and replace A^j_k by $A^j_k + cA^i_k$ for $1 \leq k \leq n$), then $\det A' = \det A$.

26.1.4. Definition. The unique function $\det\colon \mathbf{M}_{n\times n} \longrightarrow \mathbb{R}$ described above is the $n \times n$ DETERMINANT FUNCTION.

Fact 2. If $A \in \mathbb{R}$ ($= \mathbf{M}_{1\times 1}$), then $\det A = A$; if $A \in \mathbf{M}_{2\times 2}$, then $\det A = A^1_1 A^2_2 - A^1_2 A^2_1$.

Fact 3. If $A, B \in \mathbf{M}_{n\times n}$, then $\det(AB) = (\det A)(\det B)$.

Fact 4. If $A \in \mathbf{M}_{n\times n}$, then $\det A^t = \det A$. (An obvious corollary of this: in conditions (b), (c), and (d) of fact 1 the word "columns" may be substituted for the word "rows".)

26.1.5. Definition. Let A be an $n \times n$ matrix. The MINOR of the element A^j_k, denoted by M^j_k, is the determinant of the $(n-1) \times (n-1)$ matrix which results from the deletion of the j$^{\text{th}}$ row and k$^{\text{th}}$ column of A. The COFACTOR of the element A^j_k, denoted by C^j_k is defined by

$$C^j_k := (-1)^{j+k} M^j_k.$$

Fact 5. If $A \in \mathbf{M}_{n\times n}$ and $1 \leq j \leq n$, then

$$\det A = \sum_{k=1}^{n} A^j_k C^j_k.$$

This is the (LAPLACE) EXPANSION of the determinant along the j$^{\text{th}}$ row.

In light of fact 4, it is clear that expansion along columns works as well as expansion along rows. That is,

$$\det A = \sum_{j=1}^{n} A^j_k C^j_k$$

for any k between 1 and n. This is the (LAPLACE) EXPANSION of the determinant along the k$^{\text{th}}$ column.

26.2. Exercises

(1) Let $A = \begin{bmatrix} 4 & 2 & 0 & -1 \\ -1 & -3 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -5 & 3 & -1 \\ 3 & 0 & 1 & -1 \end{bmatrix}$. Then

$$A + B = \begin{bmatrix} & & & \\ & & & \end{bmatrix}, \quad 3A = \begin{bmatrix} & & & \\ & & & \end{bmatrix}, \text{ and}$$

$$A - 2B = \begin{bmatrix} & & & \\ & & & \end{bmatrix}.$$

(2) Let $A = \begin{bmatrix} 4 & 3 & 1 & 2 \\ 0 & -1 & -1 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 1 & 0 \\ -3 & 2 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} & \\ & \\ & \end{bmatrix} \text{ and } BA = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}.$$

(3) Let $A = \begin{bmatrix} 2 & 0 \\ 1 & -3 \\ 5 & 1 \end{bmatrix}$ and $\mathbf{x} = \mathbf{i} - 2\,\mathbf{j}$. Then $A\mathbf{x} =$ _____ $\mathbf{i} +$ _____ $\mathbf{j} +$ _____ $\mathbf{k}$.

(4) Let $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $C = AB$. Evaluate the following.

(a) $A^{37} = \begin{bmatrix} & \\ & \end{bmatrix}$ (b) $B^{63} = \begin{bmatrix} & \\ & \end{bmatrix}$

(c) $B^{138} = \begin{bmatrix} & \\ & \end{bmatrix}$ (d) $C^{42} = \begin{bmatrix} & \\ & \end{bmatrix}$

Note: If M is a matrix M^p is the product of p copies of M.

(5) Let $A = \begin{bmatrix} 1 & 1/3 \\ c & d \end{bmatrix}$. Find numbers c and d such that $A^2 = 0$.

Answer: $c =$ ______ and $d =$ ______ .

(6) Let $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 1 & -1 \\ 2 & 4 & 0 & 3 \\ -3 & 1 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 0 & -2 \\ 4 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & -2 & 0 & 5 \\ 1 & 0 & -3 & 4 \end{bmatrix}$.

(a) Does the matrix $D = ABC$ exist? ________ If so, then $d_{34} =$ ____________ .
(b) Does the matrix $E = BAC$ exist? ________ If so, then $e_{22} =$ ____________ .
(c) Does the matrix $F = BCA$ exist? ________ If so, then $f_{43} =$ ____________ .
(d) Does the matrix $G = ACB$ exist? ________ If so, then $g_{31} =$ ____________ .
(e) Does the matrix $H = CAB$ exist? ________ If so, then $h_{21} =$ ____________ .
(f) Does the matrix $J = CBA$ exist? ________ If so, then $j_{13} =$ ____________ .

(7) Let $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 1 & -1 \\ 2 & 4 & 0 & 3 \\ -3 & 1 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 0 & -2 \\ 4 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -2 & 0 & 5 \\ 1 & 0 & -3 & 4 \end{bmatrix}$, and

$M = 3A^3 - 5(BC)^2$. Then $M_{14} =$ ____________ and $M_{41} =$ ____________ .

(8) Evaluate each of the following determinants.

(a) $\det \begin{bmatrix} 6 & 9 & 39 & 49 \\ 5 & 7 & 32 & 37 \\ 3 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} =$ ________ .

(b) $\det \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 2 & 0 \\ 2 & -1 & 3 & 1 \\ 4 & 17 & 0 & -5 \end{bmatrix} =$ ______ .

(c) $\det \begin{bmatrix} 13 & 3 & -8 & 6 \\ 0 & 0 & -4 & 0 \\ 1 & 0 & 7 & -2 \\ 3 & 0 & 2 & 0 \end{bmatrix} =$ ________ .

(9) Find the determinants of the following matrices.

$$A = \begin{bmatrix} -73 & 78 & 24 \\ 92 & 66 & 25 \\ -80 & 37 & 10 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -73 & 78 & 24 \\ 92 & 66 & 25 \\ -80 & 37 & 10.01 \end{bmatrix}.$$

Hint. Use a calculator. Answer: $\det A =$ ______ and $\det B =$ ____________ .

(10) Find the determinant of the following matrix.

$$\begin{bmatrix} 283 & 5 & \pi & 347.86 \times 10^{15^{83}} \\ 3136 & 56 & 5 & \cos(2.7402) \\ 6776 & 121 & 11 & 5 \\ 2464 & 44 & 4 & 2 \end{bmatrix}.$$

Hint. Do not use a calculator. Answer: ________ .

(11) Find the determinant of the matrix $\begin{bmatrix} 4 & 5 & 16 & 30 \\ 4 & 8 & 29 & 55 \\ 4 & 5 & 18 & 48 \\ 4 & 5 & 16 & 36 \end{bmatrix}$. Answer: ___________ .

(12) $\det \begin{bmatrix} 1 & t & t^2 & t^3 \\ t & 1 & t & t^2 \\ t^2 & t & 1 & t \\ t^3 & t^2 & t & 1 \end{bmatrix} = (1 - a(t))^p$ where $a(t) =$ ________ and $p =$ _____ .

(13) Find the determinant of the following matrix.

$$\begin{bmatrix} 3 & 3 & 3.95 \times 10^{43^{119}} & P^{53} \\ 3 & 5 & 5 & 4.73 \times 273^{681} \\ 3 & 5 & 6 & 6 \\ 3 & 5 & 6 & 9 \end{bmatrix}$$

where P is the smallest prime number greater than $10^{10^{10}}$. Answer: ________ .

(14) Solve the following equation for x:

$$\det \begin{bmatrix} 3 & 4 & \sqrt{2} & 3 & -6 & 8 \\ 2 & -6 & 4 & -1 & 9 & 17 \\ 17 & x & \pi & 11 & -15 & 4 \\ 2 & 0 & 37 & -4 & 0 & -7 \\ -4 & 8 & -6 & 8 & -12 & 6 \\ 6 & -2 & 2 & -1 & 3 & 1 \end{bmatrix} = 0.$$

Answer: $x =$ ______ .

26.3. Problems

(1) Show that the matrix $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 1 \\ 1 & -2 & 1 \end{bmatrix}$ satisfies the equation

$$A^3 - 4A^2 + 8A - 9I_3 = 0.$$

(2) Let $A = \begin{bmatrix} 0 & a & a^2 & a^3 \\ 0 & 0 & a & a^2 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $B = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} A^k$.

(a) Explain why, for this matrix A, there are no convergence difficulties in the definition of B. Express B as a single matrix.

(b) Compute $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k!} B^k$.

(3) (a) Give an example of two symmetric matrices whose product is not symmetric. *Hint.* Matrices containing only 0's and 1's will suffice.

(b) Now suppose that A and B are symmetric $n \times n$ matrices. Prove that AB is symmetric if and only if A commutes with B.

Hint. To prove that A statement P holds if and only if a statement Q holds you must first show that P implies Q and then show that Q implies P. In the current problem, there are 4 conditions to be considered:

(i) $A^t = A$ (A is symmetric),
(ii) $B^t = B$ (B is symmetric),
(iii) $(AB)^t = AB$ (AB is symmetric), and
(iv) $AB = BA$ (A commutes with B).

One important additional fact about transposes will be helpful here. If C and D are any two matrices whose product is defined, then

(v) $(CD)^t = D^t C^t$.

The first task is to derive (iv) from (i), (ii), (iii), and (v). Then try to derive (iii) from (i), (ii), (iv), and (v).

(4) Let M be the matrix $\begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & 5 & 5 & 5 \\ 3 & 5 & 6 & 6 \\ 3 & 5 & 6 & 9 \end{bmatrix}$.

(a) Explain how to express the determinant of M as a constant times the determinant of a single 3×3 matrix. (What is the constant and what is the resulting matrix?)

(b) Explain how to express the determinant of the 3×3 matrix you found in (a) as a constant times the determinant of a single 2×2 matrix. (What is the constant and what is the resulting matrix?)

(c) Explain how to use (a) and (b) to find the determinant of M. (What is $\det M$?)

(5) Let A and B be $n \times n$-matrices. Your good friend Fred R. Dimm believes that

$$\det \begin{bmatrix} A & B \\ B & A \end{bmatrix} = \det(A+B)\det(A-B).$$

He offers the following argument to support this claim:

$$\begin{aligned} \det \begin{bmatrix} A & B \\ B & A \end{bmatrix} &= \det(A^2 - B^2) \\ &= \det[(A+B)(A-B)] \\ &= \det(A+B)\det(A-B)\,. \end{aligned}$$

(a) Comment (helpfully) on his "proof". In particular, explain carefully why each of the three steps in his "proof" is correct or incorrect. (That is, provide a proof or a counterexample to each step.)

(b) Is the result he is trying to prove actually true?

Hint: Consider the product $\begin{bmatrix} I & B \\ 0 & A-B \end{bmatrix} \begin{bmatrix} A+B & 0 \\ 0 & I \end{bmatrix}$.

26.4. Answers to Odd-Numbered Exercises

(1) $A+B=\begin{bmatrix} 5 & -3 & 3 & -2 \\ 2 & -3 & 2 & 4 \end{bmatrix}$, $3A=\begin{bmatrix} 12 & 6 & 0 & -3 \\ -3 & -9 & 3 & 15 \end{bmatrix}$, $A-2B=\begin{bmatrix} 2 & 12 & -6 & 1 \\ -7 & -3 & -1 & 7 \end{bmatrix}$.

(3) 2, 7, 3

(5) -3, -1

(7) -2060, -1562

(9) 1, -118.94

(11) 144

(13) 18

CHAPTER 27

LINEAR MAPS

27.1. Background

Topics: vector spaces, linear maps (transformations),

27.1.1. Definition. Let V be a set. Suppose there is an operation (called ADDITION) which associates with each pair $\mathbf{x}$ and $\mathbf{y}$ of elements of V an element $\mathbf{x}+\mathbf{y}$ of V. And suppose there is a second operation (called SCALAR MULTIPLICATION) which associates with each real number α and each member $\mathbf{x}$ of V an element $\alpha \cdot \mathbf{x}$ (or just $\alpha\mathbf{x}$) in V. Then V is said to be a VECTOR SPACE if the following conditions are satisfied:

(I) Addition is associative. That is,

$$\mathbf{x}+(\mathbf{y}+\mathbf{z}) = (\mathbf{x}+\mathbf{y})+\mathbf{z} \qquad \text{for all } \mathbf{x}, \mathbf{y}, \mathbf{z} \in V.$$

(II) In V there is an element $\mathbf{0}$ (called the ZERO VECTOR) such that

$$\mathbf{x}+\mathbf{0} = \mathbf{x} \qquad \text{for all } \mathbf{x} \in V.$$

(III) For each $\mathbf{x}$ in V there is a corresponding element $-\mathbf{x}$ (the ADDITIVE INVERSE of $\mathbf{x}$) such that

$$\mathbf{x}+(-\mathbf{x}) = \mathbf{0}.$$

(IV) Addition is commutative. That is,

$$\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x} \qquad \text{for all } \mathbf{x}, \mathbf{y} \in V.$$

(V) If $\alpha \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in V$, then

$$\alpha(\mathbf{x}+\mathbf{y}) = (\alpha\mathbf{x}) + (\alpha\mathbf{y}).$$

(VI) If $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x} \in V$, then

$$(\alpha+\beta)\mathbf{x} = (\alpha\mathbf{x}) + (\beta\mathbf{x}).$$

(VII) If $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x} \in V$, then

$$\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}.$$

(VIII) If $\mathbf{x} \in V$, then

$$1 \cdot \mathbf{x} = \mathbf{x}.$$

An element of V is a VECTOR; an element of $\mathbb{R}$ is, in this context, often called a SCALAR. Concerning the order of performing operations, we agree that scalar multiplication takes precedence over addition. Thus, for example, condition (V) above may be unambiguously written as

$$\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}.$$

(Notice that the parentheses on the left may not be omitted.) If $\mathbf{x}$ and $\mathbf{y}$ are vectors, we define $\mathbf{x}-\mathbf{y}$ to be $\mathbf{x}+(-\mathbf{y})$.

27.1.2. Definition. A function $T\colon V \longrightarrow W$ between vector spaces is LINEAR if

$$T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$$

for all $\mathbf{x}$, $\mathbf{y} \in V$ and

$$T(\alpha\mathbf{x}) = \alpha T(\mathbf{x})$$

for all $\mathbf{x} \in V$ and $\alpha \in \mathbb{R}$. A linear function is often called a LINEAR TRANSFORMATION, a LINEAR MAP, or a LINEAR OPERATOR. There are two notational oddities associated with linear maps:

(i) The value of a linear map T at a vector $\mathbf{x}$ in V, is usually written $T\mathbf{x}$ rather than $T(\mathbf{x})$. Of course, sometimes parentheses are necessary for clarity as in the expression $T(\mathbf{x}+\mathbf{y})$.

(ii) The composite of two linear maps $S\colon U \longrightarrow V$ and $T\colon V \longrightarrow W$ is usually written TS rather than $T \circ S$.

27.1.3. Definition. If $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear map, we define $[T]$ to be the $m \times n$ matrix whose entry in the j^{th} row and k^{th} column is $(T\mathbf{e}^k)_j$, the j^{th} component of the vector $T\mathbf{e}^k$ in $\mathbb{R}^m$. That is, $[T]^j_k = (T\mathbf{e}^k)_j$. The matrix $[T]$ is the MATRIX REPRESENTATION of T

27.1.4. Example. Let $T\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^3\colon (w,x,y,z) \mapsto (w+2x+3y, 5w+6x+7y+8z, -2x-3y-4z)$. Then T is linear and

$$\begin{aligned} T\mathbf{e}^1 &= T(1,0,0,0) = (1,5,0) \\ T\mathbf{e}^2 &= T(0,1,0,0) = (2,6,-2) \\ T\mathbf{e}^3 &= T(0,0,1,0) = (3,7,-3) \\ T\mathbf{e}^4 &= T(0,0,0,1) = (0,8,-4). \end{aligned}$$

Having computed $T\mathbf{e}^1, \dots, T\mathbf{e}^4$, we use these as the successive columns of $[T]$. Thus

$$T = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 5 & 6 & 7 & 8 \\ 0 & -2 & -3 & -4 \end{bmatrix}.$$

27.1.5. Example. If $I\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the identity map on $\mathbb{R}^n$, then its matrix representation $[I]$ is just the $n \times n$ identity matrix I_n.

27.1.6. Theorem. *Let T, $U\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be linear maps and $\alpha \in \mathbb{R}$. Then*

(a) $[T+U] = [T] + [U]$, *and*

(b) $[\alpha T] = \alpha[T]$.

If, in addition, $S\colon \mathbb{R}^p \longrightarrow \mathbb{R}^n$, then $[TS] = [T][S]$.

27.1.7. Definition. Let $T\colon V \longrightarrow W$ be a linear map between vector spaces. The KERNEL of T, denoted by $\ker T$, is the set of all $\mathbf{x} \in V$ such that $T\mathbf{x} = \mathbf{0}$.

27.2. Exercises

(1) If $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear map satisfying

$$\begin{aligned} T\mathbf{i} &= 3\mathbf{i} - 5\mathbf{j} \\ T\mathbf{j} &= 2\mathbf{i} - \mathbf{j} \end{aligned}$$

then,

$$T(4\mathbf{i} - 5\mathbf{j}) = \underline{\qquad}\ \mathbf{i} + \underline{\qquad}\ \mathbf{j}.$$

(2) Define $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ by

$$T\mathbf{x} = (x_1 - x_3, x_1 + x_2, x_3 - x_2, x_1 - 2x_2)$$

for all $\mathbf{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$.
 (a) Then $T(1, -2, 3) = (\ \underline{\qquad}\ ,\ \underline{\qquad}\ ,\ \underline{\qquad}\ ,\ \underline{\qquad})$.
 (b) Find a vector $\mathbf{x} \in \mathbb{R}^3$ such that $T\mathbf{x} = (8, 9, -5, 0)$.
 Answer: $\mathbf{x} = (\ \underline{\qquad}\ ,\ \underline{\qquad}\ ,\ \underline{\qquad}\ )$.

(3) If $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a linear map satisfying

$$\begin{aligned} T\mathbf{i} &= 2\mathbf{i} + 4\mathbf{j} \\ T\mathbf{j} &= \mathbf{i} + \mathbf{j} + \mathbf{k} \\ T\mathbf{k} &= 3\mathbf{i} - \mathbf{j} - 2\mathbf{k} \end{aligned}$$

then,

$$T(2\mathbf{i} - \mathbf{j} - \mathbf{k}) = \underline{\qquad}\ \mathbf{i} + \underline{\qquad}\ \mathbf{j} + \underline{\qquad}\ \mathbf{k}.$$

(4) Your friend Fred R. Dimm took an exam in which he made the following **incorrect** calculations. Each error can be explained by supposing that Fred believes (wrongly) that some particular function f is linear. Specify the function.

$$\begin{aligned} \sqrt{3^2 + 4^2} = \sqrt{3^2} + \sqrt{4^2} = 7 &\qquad \text{Ans.} f(x) = \underline{\qquad\qquad}\ . \\ \sqrt{3^2 + 4^2} = \sqrt{(3+4)^2} = 7 &\qquad \text{Ans.} f(x) = \underline{\qquad\qquad}\ . \\ \sin 75^\circ = \frac{1}{2} + \frac{1}{\sqrt{2}} &\qquad \text{Ans.} f(x) = \underline{\qquad\qquad}\ . \\ \int_1^2 \frac{1}{x^2 + x}\, dx = \frac{1}{2} + \ln 2 &\qquad \text{Ans.} f(x) = \underline{\qquad\qquad}\ . \\ e^{\ln 4 + \ln 7} = e^{\ln 4} + e^{\ln 7} = 11 &\qquad \text{Ans.} f(x) = \underline{\qquad\qquad}\ . \\ e^{\ln 4 + \ln 7} = e^{\ln(11)} = 11 &\qquad \text{Ans.} f(x) = \underline{\qquad\qquad}\ . \\ \arctan \frac{1}{2} = \frac{\pi}{8} &\qquad \text{Ans.} f(x) = \underline{\qquad\qquad}\ . \end{aligned}$$

(5) Let $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be defined by

$$T\mathbf{x} = (x_1 - 3x_3\ ,\ x_1 + x_2 - 6x_3\ ,\ x_2 - 3x_3\ ,\ x_1 - 3x_3)$$

for every $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. (The map T is linear, but you need not prove this.) Then

(a) $[T] = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$.

(b) $T(3,-2,4) =$ ________________________ .

(6) Let $T\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be defined by

$$T\mathbf{x} = (x_1 - 3x_3 + x_4\,,\ 2x_1 + x_2 + x_3 + x_4\,,\ 3x_2 - 4x_3 + 7x_4)$$

for every $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. (The map T is linear, but you need not prove this.)

(a) Find $[T]$. Answer: $\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$.

(b) Find $T(1,-2,1,3)$. Answer: ________________________ .

27.3. Problems

(1) Consider an interval $[a, b]$ in $\mathbb{R}$. Let $\mathcal{C}([a, b])$ be the family of all continuous real valued functions defined on $[a, b]$. For functions $\mathbf{f}, \mathbf{g} \in \mathcal{C}([a, b])$ define the function $\mathbf{f} + \mathbf{g}$ by

$$(\mathbf{f} + \mathbf{g})(x) := \mathbf{f}(x) + \mathbf{g}(x) \qquad \text{for all } x \in [a, b].$$

(It should be clear that the two "+" signs in the preceding equation denote operations in different spaces. The one on the left (which is being defined) represents addition in the space $\mathcal{C}([a, b])$; the one on the right is ordinary addition in $\mathbb{R}$.) Because we specify the value of $\mathbf{f} + \mathbf{g}$ at each point x by adding the values of $\mathbf{f}$ and $\mathbf{g}$ at that point, we say that we add $\mathbf{f}$ and $\mathbf{g}$ POINTWISE.

We also define scalar multiplication to be a pointwise operation. That is, if $\mathbf{f} \in \mathcal{C}([a, b])$ and $\alpha \in \mathbb{R}$, then we define the function $\alpha\mathbf{f}$ by

$$(\alpha\mathbf{f})(x) := \alpha(\mathbf{f}(x)) \qquad \text{for every } x \in [a, b].$$

We know from first term calculus that according to the definitions above, both $\mathbf{f}+\mathbf{g}$ and $\alpha\mathbf{f}$ belong to $\mathcal{C}([a, b])$. (Sums of continuous functions are continuous, and constant multiples of continuous functions are continuous.)

Prove that under these pointwise operations $\mathcal{C}([a, b])$ is a vector space.

(2) Show that a vector space has at most one zero vector. That is, if $\mathbf{0}$ and $\mathbf{0}'$ are members of a vector space V which satisfy $\mathbf{x} + \mathbf{0} = \mathbf{x}$ and $\mathbf{x} + \mathbf{0}' = \mathbf{x}$ for all $\mathbf{x} \in V$, then $\mathbf{0} = \mathbf{0}'$.

(3) Show that for every vector $\mathbf{x}$ in a vector space V there exists only one vector $-\mathbf{x}$ such that

$$\mathbf{x} + (-\mathbf{x}) = \mathbf{0}\,.$$

(4) Show that if $\mathbf{x}$ is a vector (in some vector space) and $\mathbf{x} + \mathbf{x} = \mathbf{x}$, then $\mathbf{x} = \mathbf{0}$. *Hint.* Add $\mathbf{0}$ to $\mathbf{x}$; then write $\mathbf{0}$ as $\mathbf{x} + (-\mathbf{x})$.

(5) Let $\mathbf{x}$ be a vector in a vector space V and let α be a real number. Prove that $\alpha\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$ or $\alpha = 0$.

Hint. You need to show three things:
(a) $\alpha\mathbf{0} = \mathbf{0}$,
(b) $0\,\mathbf{x} = \mathbf{0}$, and
(c) If $\alpha \neq 0$ and $\alpha\mathbf{x} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$.
To prove (a) write $\mathbf{0} + \mathbf{0} = \mathbf{0}$, multiply both sides by α, and use a previous problem. For (c) use the fact that if $\alpha \in \mathbb{R}$ is not zero, it has a reciprocal. What happens if we multiply the vector αx by the scalar $1/\alpha$?

(6) Suppose that $S \colon U \longrightarrow V$ and $T \colon V \longrightarrow W$ are linear maps between vector spaces. Show that their composite (usually denoted by TS rather than $T \circ S$) is also linear.

(7) Prove that the map $T \colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by

$$T(x, y, z) = (x + y - z, x - 2y + 3z)$$

is linear.

(8) Let A be a 2×2 matrix. Show that the action of A on vectors in $\mathbb{R}^2$ is linear. Conversely, show that every linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ can be represented by the action of a 2×2 matrix.

Note: A more general statement is also true. (You need not prove this.) Every $m \times n$ matrix acts as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$; and every linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be represented by an $m \times n$ matrix.

(9) If $T \colon V \longrightarrow W$ is a linear transformation between vector spaces, then it takes the zero vector of V to the zero vector of W.

(10) Denote by $\mathcal{C}$ the family of all continuous functions on the real line $\mathbb{R}$ and by $\mathcal{C}^1$ the family of all continuously differentiable functions on $\mathbb{R}$. (A function f is CONTINUOUSLY DIFFERENTIABLE if it is differentiable and its derivative f' is continuous.) Show that the differentiation operator $D\colon \mathcal{C}^1 \longrightarrow \mathcal{C}$ defined by $D(f) = f'$ is linear.

(11) Denote by $\mathcal{C}([a,b])$ the family of all continuous functions on the interval $[a,b]$. Show that integration is linear: that is, show that the function $\phi\colon \mathcal{C}([a,b]) \longrightarrow \mathbb{R}$ defined by $\phi(f) = \int_a^b f(x)\,dx$ is linear.

(12) Let a be a fixed point in $\mathbb{R}$. As in problem 10, let $\mathcal{C}$ be the family of all continuous functions on $\mathbb{R}$ and $\mathcal{C}^1$ be the family of all continuously differentiable functions on $\mathbb{R}$. Define an operator $J\colon \mathcal{C} \longrightarrow \mathcal{C}^1$ as follows: for every $f \in \mathcal{C}$ let Jf be the function whose value at each x is the integral of f from a to x. That is,

$$(Jf)(x) = \int_a^x f(t)\,dt$$

for each $x \in \mathbb{R}$. Show that the operator J is linear. Also show that DJ is the identity operator on $\mathcal{C}$. Is JD the identity operator on $\mathcal{C}^1$? Why or why not?

(13) Identify all the linear transformations mapping $\mathbb{R}$ into $\mathbb{R}$. *Hint.* What does the graph of such a function look like?

(14) Suppose that $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear map. Prove that if we compute the action of its matrix representation $[T]$ on a vector $\mathbf{x}$ in $\mathbb{R}^n$, what we get is the value of the function T at $\mathbf{x}$. That is, show that for every $\mathbf{x} \in \mathbb{R}^n$

$$T\mathbf{x} = [T]\mathbf{x}.$$

Show, moreover, the representation is unique; that is, two distinct matrices cannot represent the same linear map.

Hint. For the last assertion show that if A is any $m \times n$ matrix which satisfies

$$T\mathbf{x} = A\mathbf{x} \qquad \text{for all } \mathbf{x} \in \mathbb{R}^n,$$

then $A = [T]$.

(15) Let $T\colon V \longrightarrow W$ be a linear map between vector spaces. Show that T is one-to-one if and only if the kernel of T is $\{\mathbf{0}\}$.

27.4. Answers to Odd-Numbered Exercises

(1) 2, -15

(3) 0, 8, 1

(5) (a) $\begin{bmatrix} 1 & 0 & -3 \\ 1 & 1 & -6 \\ 0 & 1 & -3 \\ 1 & 0 & -3 \end{bmatrix}$

(b) $(-9, -23, -14, -9)$

CHAPTER 28

DEFINITION OF DERIVATIVE

28.1. Background

Topics: differential = derivative = total derivative, differentiable at a point, differentiable function, tangent plane to a surface, Jacobian matrix, smooth function.

28.1.1. Definition. Let $\mathbf{F}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a vector field and $\mathbf{p}$ be a point in $\mathbb{R}^n$. Define a function $\Delta\mathbf{F}_{\mathbf{p}}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by

$$\Delta\mathbf{F}_{\mathbf{p}}(\mathbf{h}) = \mathbf{F}(\mathbf{p}+\mathbf{h}) - \mathbf{F}(\mathbf{p})$$

for every $\mathbf{h} \in \mathbb{R}^n$.

28.1.2. Definition. A function $\mathbf{F}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is DIFFERENTIABLE AT a point $\mathbf{p} \in \mathbb{R}^n$ if there exists a linear transformation (equivalently, an $m \times n$ matrix) $d\mathbf{F}_{\mathbf{p}}$ taking $\mathbb{R}^n$ to $\mathbb{R}^m$ such that

$$\lim_{\mathbf{h} \longrightarrow 0} \frac{\Delta\mathbf{F}_{\mathbf{p}}(\mathbf{h}) - d\mathbf{F}_{\mathbf{p}}(\mathbf{h})}{\|\mathbf{h}\|} = \mathbf{0}\,.$$

When this happens we say that the linear transformation (or matrix) $d\mathbf{F}_{\mathbf{p}}$ is TANGENT TO the function $\Delta\mathbf{F}_{\mathbf{p}}$. We call $d\mathbf{F}_{\mathbf{p}}$ the DIFFERENTIAL (or DERIVATIVE, or TOTAL DERIVATIVE) of $\mathbf{F}$ at $\mathbf{p}$.

A function $\mathbf{F}$ is DIFFERENTIABLE if it is differentiable at each point in its domain.

28.1.3. Theorem. *If a function $\mathbf{F}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is* DIFFERENTIABLE AT A POINT $\mathbf{p} \in \mathbb{R}^n$, *then*

$$d\mathbf{F}_{\mathbf{p}} = \left[F^i_j(\mathbf{p})\right] = \left[\frac{\partial F^i}{\partial x_j}(\mathbf{p})\right]_{i=1}^{m}\,{}_{j=1}^{n}\,.$$

28.1.4. Definition. The matrix in the preceding theorem is the JACOBIAN MATRIX OF $\mathbf{F}$ AT $\mathbf{p}$.

28.1.5. Definition. A function $\mathbf{F}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is TWICE DIFFERENTIABLE AT a point $\mathbf{p} \in \mathbb{R}^n$ if it is differentiable there and the function $d\mathbf{F}$ is also differentiable at $\mathbf{p}$. Higher order differentiability is defined similarly.

28.1.6. Definition. A function $\mathbf{F}\colon U \to \mathbb{R}^n$, where U is an open subset of $\mathbb{R}^m$, is SMOOTH (or INFINITELY DIFFERENTIABLE) if it has derivatives of all orders.

28.1.7. Convention. When we say that a function f is *differentiable* (or *smooth*) on some region in $\mathbb{R}^n$, we will mean that it is differentiable (or smooth) on some open set in $\mathbb{R}^n$ which contains the region.

28.2. Exercises

(1) Let $f(x) = \sin x$ and $p = \pi/6$. The slope of the tangent line to the curve $y = f(x)$ at $x = p$ is ________ . The value of Δf_p at x is $\frac{1}{2}(a \sin x + b \cos x - b)$ where $a =$ _____ and $b =$ _____ . The slope of the tangent line to the curve $y = \Delta f_p(x)$ at $x = 0$ is ________ .

(2) Let $f(x) = -x^4 + 5x^3 - 7x^2 + 3x + 1$ and $p = 2$. The slope of the tangent line to the curve $y = f(x)$ at $x = p$ is _____ . The value of Δf_p at x is $-x^4 + ax^3 - x^2 + bx + c$ where $a =$ _____ , $b =$ _____ , and $c =$ _____ . The slope of the tangent line to the curve $y = \Delta f_p(x)$ at $x = 0$ is _____ .

(3) Let $f(x) = \tan x$ and $p = \pi/4$. The slope of the tangent line to the curve $y = f(x)$ at $x = p$ is _____ . The value of Δf_p at x is $\dfrac{g(x)}{1 - \tan x}$ where $g(x) =$ _____________ . The slope of the tangent line to the curve $y = \Delta f_p(x)$ at $x = 0$ is ________ .

(4) Let $f(x) = e^{2x}$ and $p = \ln 3$. The slope of the tangent line to the curve $y = f(x)$ at $x = p$ is _____ . The value of Δf_p at x is $ae^{2x} - b$ where $a =$ _____ and $b =$ _____ . The slope of the tangent line to the curve $y = \Delta f_p(x)$ at $x = 0$ is ________ .

(5) Let $f(x, y) = xy^2$ and $\mathbf{p} = (1, -1)$. Then $\dfrac{\partial f}{\partial x}(\mathbf{p}) =$ _____ and $\dfrac{\partial f}{\partial y}(\mathbf{p}) =$ _____ . The function $\Delta f_{\mathbf{p}}(x, y) = a(x)b(y) - c$ where $a(x) =$ ________ , $b(y) =$ ________ , and $c =$ ___ . The partial derivatives of this function at $\mathbf{0} = (0, 0)$ are given by $\dfrac{\partial(\Delta f_{\mathbf{p}})}{\partial x}(\mathbf{0}) =$ _____ and $\dfrac{\partial(\Delta f_{\mathbf{p}})}{\partial y}(\mathbf{0}) =$ _____ .

(6) Let $f(x, y, z) = xy + yz$ and $\mathbf{p} = (1, -1, 2)$. Then $\dfrac{\partial f}{\partial x}(\mathbf{p}) =$ _____ , $\dfrac{\partial f}{\partial y}(\mathbf{p}) =$ _____ , and $\dfrac{\partial f}{\partial z}(\mathbf{p}) =$ _____ . The function $\Delta f_{\mathbf{p}}(x, y, z) = xy + yz + ax + by + cz$ where $a =$ _____ , $b =$ _____ , and $c =$ _____ . The partial derivatives of this function at $\mathbf{0} = (0, 0, 0)$ are given by $\dfrac{\partial(\Delta f_{\mathbf{p}})}{\partial x}(\mathbf{0}) =$ _____ , $\dfrac{\partial(\Delta f_{\mathbf{p}})}{\partial y}(\mathbf{0}) =$ _____ , and $\dfrac{\partial(\Delta f_{\mathbf{p}})}{\partial z}(\mathbf{0}) =$ _____ .

(7) Let $f(x, y) = x^2 + y^2$ and $\mathbf{p} = (1, 1)$. If we choose $df_{\mathbf{p}}$ to be the 1×2-matrix $\begin{bmatrix} a & b \end{bmatrix}$, where $a =$ _____ and $b =$ _____, then $\Delta f_{\mathbf{p}}$ is tangent to $df_{\mathbf{p}}$.

(8) Let $\mathbf{F}(x, y) = (2x, 3y^2)$ and $\mathbf{p} = (1, 1)$. If we choose $d\mathbf{F}_{\mathbf{p}}$ to be the 2×2-matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $d =$ _____ , then $\Delta\mathbf{F}_{\mathbf{p}}$ is tangent to $d\mathbf{F}_{\mathbf{p}}$.

(9) Let $\mathbf{F} \colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined by $\mathbf{F}(x, y, z) = x^2 y e^z\, \mathbf{i} + (x^3 + y^3 + z^3)\, \mathbf{j}$ and $\mathbf{p} = (1, 2, 0)$. Then the derivative of $\mathbf{F}$ at $\mathbf{p}$ is given by

$$d\mathbf{F}_{\mathbf{p}} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}.$$

(10) Let $\mathbf{F}(x,y) = (x^2 - \frac{1}{2}xy^3, 4xy)$ and $\mathbf{p} = (5,2)$. Then the differential of $\mathbf{F}$ at $\mathbf{p}$ is given by

$$d\mathbf{F}_{\mathbf{p}} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}.$$

(11) Let $\mathbf{F}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be defined by $\mathbf{F}(x,y) = (x^2, 2x - 3y, 5xy)$ and $\mathbf{p} = (1,-1)$. Then the Jacobian matrix of $\mathbf{F}$ at $\mathbf{p}$ is given by

$$d\mathbf{F}_{\mathbf{p}} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{bmatrix}.$$

(12) Let $\mathbf{F}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be defined by $\mathbf{F}(x,y) = xy\,\mathbf{i} + (x^2y + xy^2)\,\mathbf{j} + y^2 \sin \pi x\,\mathbf{k}$ and $\mathbf{p} = (3,-1)$. Then the Jacobian matrix of $\mathbf{F}$ at $\mathbf{p}$ is given by

$$d\mathbf{F}_{\mathbf{p}} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{bmatrix}.$$

(13) Let $\mathbf{T}$ be a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $\mathbf{p} \in \mathbb{R}^n$. Then $d\mathbf{T}_{\mathbf{p}} =$ _____ .

$+$

28.3. Problems

(1) Let $f(x) = x^2 - 4x + 6$ and $p = 3$. On the same set of axes make a careful sketch of the curves $y = f(x)$ and $y = \Delta f_p(x)$. Draw the tangent line to the first curve at $x = p$ and the tangent line to the second curve at $x = 0$.

(2) Let $f(x) = \frac{1}{6}x^2 - \frac{5}{6}x + \frac{11}{6}$ and $p = -1$. On the same set of axes make a careful sketch of the curves $y = f(x)$ and $y = \Delta f_p(x)$. Draw the tangent line to the first curve at $x = p$ and the tangent line to the second curve at $x = 0$.

(3) Let $\mathbf{F}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $\mathbf{F}(x, y) = (3xy, x^2 + y^2)$, and let $\mathbf{p} = (1, 2)$.
   (a) Compute the derivative $d\mathbf{F}_{\mathbf{p}}$.
   (b) Use the *definition* of “differentiable” to show that $\mathbf{F}$ is differentiable at $\mathbf{p}$.

(4) Let $\mathbf{F}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $\mathbf{F}(x, y) = (3x - y + 7, x + 4y)$, and let $\mathbf{p} = (1, 2)$. Use the *definition* of “differentiable” to show that $\mathbf{F}$ is differentiable at $\mathbf{p}$.

(5) Let $\mathbf{F}\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined by $\mathbf{F}(x, y, z) = (3x + z^2)\,\mathbf{i} + 5yz\,\mathbf{j}$, and let $\mathbf{p} = (3, -2, 1)$. Use the *definition* of “differentiable” to show that $\mathbf{F}$ is differentiable at $\mathbf{p}$.

(6) Let $\mathbf{F}\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined by $\mathbf{F}(x, y, z) = (xy - 3)\,\mathbf{i} + (y + 2z^2)\,\mathbf{j}$, and let $\mathbf{p} = (1, -1, 2)$. Use the *definition* of “differentiable” to show that $\mathbf{F}$ is differentiable at $\mathbf{p}$.

(7) Let $\mathbf{F}\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be defined by $\mathbf{F}(x, y, z) = (x + 2yz, y^3 - z, 3y^2, 2xy - 5z)$, and let $\mathbf{p} = (1, 2, -5)$. Use the *definition* of “differentiable” to show that $\mathbf{F}$ is differentiable at $\mathbf{p}$.

(8) Let T be a symmetric $n \times n$ matrix and let $\mathbf{p} \in \mathbb{R}^n$. Define a function $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ by $f(\mathbf{x}) = \langle T\mathbf{x}, \mathbf{x}\rangle$. Show that
$$df_{\mathbf{p}}(\mathbf{h}) = \langle T\mathbf{p}, \mathbf{h}\rangle$$
for every $\mathbf{h} \in \mathbb{R}^n$.

28.4. Answers to Odd-Numbered Exercises

(1) $\dfrac{\sqrt{3}}{2}$, $\sqrt{3}$, 1, $\dfrac{\sqrt{3}}{2}$

(3) 2, $2\tan x$, 2

(5) 1, -2, $x+1$, $(y-1)^2$, 1, 1, -2

(7) 2, 2

(9) $\begin{bmatrix} 4 & 1 & 2 \\ 3 & 12 & 0 \end{bmatrix}$

(11) $\begin{bmatrix} 2 & 0 \\ 2 & -3 \\ -5 & 5 \end{bmatrix}$

(13) T

CHAPTER 29

DIFFERENTIATION OF FUNCTIONS OF SEVERAL VARIABLLES

29.1. Background

Topics: chain rule

Differentiation is about approximating (a suitable translation of) a smooth function by a linear one. What the *chain rule* says (although it may be difficult to see this from what you are told in many texts) is that the best linear approximation to the composite of two smooth functions is the composite of their best linear approximations. That is, the differential of the composite is the composite of the differentials. Here is the formal statement.

29.1.1. Theorem (The Chain Rule)**.** *If* $F\colon \mathbb{R}^m \longrightarrow \mathbb{R}^n$ *is differentiable at* $\mathbf{a} \in \mathbb{R}^m$ *and* $G\colon \mathbb{R}^n \longrightarrow \mathbb{R}^p$ *is differentiable at* $F(\mathbf{a}) \in \mathbb{R}^n$*, then* $G \circ F$ *is differentiable at* $\mathbf{a}$ *and*

$$d(G \circ F)_{\mathbf{a}} = (dG)_{F(\mathbf{a})}\, dF_{\mathbf{a}}.$$

For computational purposes it is convenient to rephrase this theorem in terms of partial derivatives.

29.1.2. Theorem (The Chain Rule)**.** *If* $F\colon \mathbb{R}^m \longrightarrow \mathbb{R}^n$ *is differentiable at* $\mathbf{a} \in \mathbb{R}^m$ *and* $G\colon \mathbb{R}^n \longrightarrow \mathbb{R}^p$ *is differentiable at* $F(\mathbf{a}) \in \mathbb{R}^n$*, then* $G \circ F$ *is differentiable at* $\mathbf{a}$ *and*

$$(G \circ F)^k_i(\mathbf{a}) = \sum_{j=1}^{n} G^k_j(F(\mathbf{a}))F^j_i(\mathbf{a})$$

for $i = 1, \dots, m$ *and* $k = 1, \dots, p$.

29.1.3. Notation. Scientists like to work with variables. And frequently they use the name of a function for its corresponding dependent variable. As a consequence of this highly dubious notation one variable may depend explicitly and/or implicitly on another variable. For example, suppose that $w = w(x, y, t)$ where $x = x(s, t)$ and $y = y(s, t)$ and that all the functions mentioned are differentiable. In this case we say that w depends *explicitly* on the variables x and y; it depends *implicitly* on s; and it depends *both explicitly and implicitly* on t. This can lead to notational confusion. For example, it is perhaps tempting to write

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial t} \\ &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial t} \end{aligned} \tag{29.1}$$

(since $\frac{\partial t}{\partial t} = 1$). The trouble with this is that the $\frac{\partial w}{\partial t}$ on the left is not the same as the one on the right. The $\frac{\partial w}{\partial t}$ on the right refers only to the rate of change of w with respect to t insofar as t appears *explicitly* in the formula for w; the one on the left takes into account the fact that in addition w depends *implicitly* on t *via* the variables x and y.

One way of dealing with the resulting ambiguity is to give the functions involved names of their own. Relate the variables by functions as follows.

$$\begin{matrix} s \\ t \end{matrix} \xrightarrow{\ f\ } \begin{matrix} x \\ y \\ t \end{matrix} \xrightarrow{\ g\ } w \tag{29.2}$$

Also let $h = g \circ f$. Notice that $f^3(s,t) = t$. Then according to the *chain rule* 29.1.2

$$h_2 = \sum_{k=1}^{3} (g_k \circ f)\, f_2^k\,.$$

But $f_2^3 = 1$ (that is, $\frac{\partial t}{\partial t} = 1$). So

$$h_2 = (g_1 \circ f) f_2^1 + (g_2 \circ f) f_2^2 + g_3 \circ f. \tag{29.3}$$

The ambiguity of (29.1) has been eliminated in (29.3). The $\frac{\partial w}{\partial t}$ on the left is seen to be the derivative with respect to t of the composite $h = g \circ f$, whereas the $\frac{\partial w}{\partial t}$ on the right is just the derivative with respect to t of the function g.

But what does one do if, as is often the case in scientific work, one refuses to give names to functions? Here is one standard procedure. Look back at diagram (29.2) and remove the names of the functions.

$$\begin{matrix} s \\ t \end{matrix} \longrightarrow \begin{matrix} x \\ y \\ t \end{matrix} \longrightarrow w \tag{29.4}$$

The problem now is that the symbol "t" occurs twice. To specify differentiation of the composite function (our h) with respect to t, indicate that the "t" you are interested in is the one in the left column of (29.4). This may be done by listing everything else that appears in that column. That is, specify which variables are held constant. This specification conventionally appears as a subscript outside parentheses. Thus the $\frac{\partial w}{\partial t}$ on the left of (29.1) (our h_2) is written as $\left(\frac{\partial w}{\partial t}\right)_s$ (and is read, "$\frac{\partial w}{\partial t}$ with s held constant"). Similarly, the $\frac{\partial w}{\partial t}$ on the right of (29.1)) (our g_3) involves differentiation with respect to t while x and y are fixed. So it is written $\left(\frac{\partial w}{\partial t}\right)_{x,y}$ (and is read, "$\frac{\partial w}{\partial t}$ with x and y held constant"). Thus (29.1) becomes

$$\left(\frac{\partial w}{\partial t}\right)_s = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \left(\frac{\partial w}{\partial t}\right)_{x,y} \tag{29.5}$$

It is not necessary to write, for example, an expression such as $\left(\frac{\partial w}{\partial x}\right)_{t,y}$ because there is no ambiguity; the symbol "x" occurs only once in (29.4). If you choose to use the convention just presented, it is best to use it only to avoid confusion; use it because you *must*, not because you *can*.

29.1.4. CAUTION. Be careful not to mix the notational convention using subscripts described above with the one introduced in 24.1.2.

29.1.5. Definition. A smooth function $\mathbf{F}$ mapping from a subset of $\mathbb{R}^n$ into $\mathbb{R}^n$ is said to be LOCALLY INVERTIBLE at a point $\mathbf{a}$ in its domain if there exists a neighborhood U of $\mathbf{a}$ such that the restriction of $\mathbf{F}$ to U is a bijection between U and $\mathbf{F}(U)$ and such that the inverse function $\mathbf{F}^{-1}\colon \mathbf{F}(U) \longrightarrow U$ is smooth.

29.1.6. Example. The function $x \mapsto x^2$ is locally invertible at every point $a \in \mathbb{R}$ except for $a = 0$.

Here is a special case of the *inverse function theorem*. .

29.1.7. Theorem. *Let* $\mathbf{F}$ *be a smooth function from an open subset of* $\mathbb{R}^n$ *into* $\mathbb{R}^n$*. If its differential* $d\mathbf{F}_{\mathbf{a}}$ *is an invertible linear map at a point* $\mathbf{a}$*, then* $\mathbf{F}$ *is locally invertible at* $\mathbf{a}$ *and*

$$d\big(\mathbf{F}^{-1}\big)_{\mathbf{F}(\mathbf{x})} = \big[d\mathbf{F}_{\mathbf{x}}\big]^{-1}$$

for all $\mathbf{x}$ *in some neighborhood of* $\mathbf{a}$*.*

29.2. Exercises

(1) Suppose that $f(u,v) = g(x,y)$ where $u = x^2 - y^2$ and $v = 2xy$. Then

$$y\frac{\partial g}{\partial x} - x\frac{\partial g}{\partial y} = \underline{\qquad\qquad}\ \frac{\partial f}{\partial u} - \underline{\qquad\qquad}\ \frac{\partial f}{\partial v}\ .$$

(2) Suppose $w = \frac{1}{4}ux^2y$ where $x = u^2 - v^2$ and $y = 2uv$. At the point where $u = -1$ and $v = -2$

$$\left(\frac{\partial w}{\partial u}\right)_{x,y} = \underline{\qquad} \qquad \text{and} \qquad \left(\frac{\partial w}{\partial u}\right)_{v} = \underline{\qquad}\ .$$

(3) Suppose $w = \frac{1}{12}t^2x^3 - 5y^2\ln t$ where $x = t^2 - u^2$ and $y = \dfrac{t}{u}$. At the point where $t = 2$ and $u = 1$

$$\left(\frac{\partial w}{\partial t}\right)_{x,y} = \underline{\qquad\quad} \qquad \text{and} \qquad \left(\frac{\partial w}{\partial t}\right)_{u} = a - b\ln c$$

where $a = \underline{\qquad\qquad}$, $b = \underline{\qquad\qquad}$, and $c = \underline{\qquad}$.

(4) Suppose $w = \frac{1}{16}x^4y + y^2\arctan u$ where $x = t^2 + u^3$ and $y = t^3 - 5u$. At the point where $t = u = 1$

$$\left(\frac{\partial w}{\partial u}\right)_{x,y} = \underline{\qquad} \qquad \text{and} \qquad \left(\frac{\partial w}{\partial u}\right)_{t} = a\pi - b$$

where $a = \underline{\qquad\qquad}$ and $b = \underline{\qquad\qquad}$.

(5) Suppose $w = \frac{1}{2}x^2y + \arctan(tx)$ where $x = t^2 - 3u^2$ and $y = 2tu$. At the point where $t = 2$ and $u = -1$

$$\left(\frac{\partial w}{\partial t}\right)_{x,y} = \frac{a}{5} \qquad \text{and} \qquad \left(\frac{\partial w}{\partial t}\right)_{u} = -\frac{b}{5}$$

where $a = \underline{\qquad}$ and $b = \underline{\qquad\qquad}$.

(6) Let $w = ux^2 + \arctan yz$, where $x = u + v$, $y = u^2 - v$, and $z = uv - 3$. At the point where $u = 2$ and $v = 3$

$$\left(\frac{\partial w}{\partial u}\right)_{x,y,z} = \underline{\qquad\quad} \qquad \text{and} \qquad \left(\frac{\partial w}{\partial u}\right)_{v} = \frac{a}{2}$$

where $a = \underline{\qquad\qquad}$.

(7) Suppose that $w = f(x,y) = g(u,v)$ where $x = u^2 - v^2$ and $y = 2uv$, and that f and g are twice continuously differentiable functions. Then

$$\frac{\partial^2 g}{\partial u^2} - \frac{\partial^2 g}{\partial v^2} = \underline{\qquad\quad}\ \frac{\partial^2 f}{\partial x^2} + \underline{\qquad\quad}\ \frac{\partial^2 f}{\partial x \partial y} - \underline{\qquad\quad}\ \frac{\partial^2 f}{\partial y^2} + \underline{\qquad}\ \frac{\partial f}{\partial x} + \underline{\qquad}\ \frac{\partial f}{\partial y}.$$

(8) Suppose that $z = f(x,y) = g(r,\theta)$ where $x = r\cos\theta$, $y = r\sin\theta$, and f and g are twice continuously differentiable functions. Then

$$r\frac{\partial^2 g}{\partial r \partial \theta} = \underline{\qquad\quad}\ \left(\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x^2}\right) + \underline{\qquad\qquad\qquad}\ \frac{\partial^2 f}{\partial x \partial y} + \underline{\qquad}\ \frac{\partial f}{\partial y} - \underline{\qquad}\ \frac{\partial f}{\partial x}.$$

(9) Let $f(x,y) = \dfrac{x^2 - y^2}{\sqrt{x^2 + y^2}}$. Then at the point where $x = 1$ and $y = 1$

$$xy\left(\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x^2}\right) + (x^2 - y^2)\frac{\partial^2 f}{\partial x \partial y} + x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} = -a\sqrt{a}$$

where $a = \underline{\qquad}$. *Hint.* Use the preceding exercise.

(10) (Conversion of the Laplacian to polar coordinates) If $f(x,y) = g(r,\theta)$, then

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} = \underline{\qquad}\ \frac{\partial^2 f}{\partial x^2} + \underline{\qquad}\ \frac{\partial^2 f}{\partial x \partial y} + \underline{\qquad}\ \frac{\partial^2 f}{\partial y^2} + \underline{\qquad}\ \frac{\partial f}{\partial x} + \underline{\qquad}\ \frac{\partial f}{\partial y}.$$

(11) Let $f(x,y) = \dfrac{x^2 - y^2}{\sqrt{x^2+y^2}}$. Then $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = \underline{\qquad\qquad\qquad\qquad}$.
Hint. Use the preceding exercise.

(12) Let $w = f(x,y) = g(u,v)$ where $x = u^2 + v^2$ and $y = 2uv$. Then

$$\frac{\partial^2 g}{\partial u \partial v} = \underline{\qquad}\ \frac{\partial^2 f}{\partial x^2} + \underline{\qquad}\ \frac{\partial^2 f}{\partial x \partial y} + \underline{\qquad}\ \frac{\partial^2 f}{\partial y^2} + \underline{\qquad}\ \frac{\partial f}{\partial x} + \underline{\qquad}\ \frac{\partial f}{\partial y}.$$

(13) Let $f(x,y) = \dfrac{\sqrt{x+y} + \sqrt{x-y}}{\sqrt{x+y} - \sqrt{x-y}} + \sqrt{\sqrt{x+y} - \sqrt{x-y}}$. Then the value of the expression $2y\dfrac{\partial^2 f}{\partial x^2} + 4x\dfrac{\partial^2 f}{\partial x \partial y} + 2y\dfrac{\partial^2 f}{\partial y^2} + 2\dfrac{\partial f}{\partial y}$ at the point where $x = 5$ and $y = 4$ is $\underline{\qquad}$.
Hint. Use the preceding exercise.

(14) Suppose that $w = G(x,y,z) = F(x, xz, xy)$ where F and G are differentiable functions. Find $x\dfrac{\partial G}{\partial x} - y\dfrac{\partial G}{\partial y} - z\dfrac{\partial G}{\partial z}$ in terms of the partial derivatives of F. Answer: $\underline{\qquad\qquad\qquad}$.

(15) At the point $(1,\pi)$ the vector field $\mathbf{F}$ defined by $\mathbf{F}(x,y) = (x^2y^3, x\tan y)$ is locally invertible. Then $d\big(\mathbf{F}^{-1}\big)_{(\pi^3,0)} = \begin{bmatrix} \qquad & \qquad \\ \qquad & \qquad \end{bmatrix}$.

(16) The vector field $\mathbf{G}$ defined by $\mathbf{G}(x,y) = (\ln y, xy^2)$ for all $y > 0$ is locally invertible at each point in its domain. Then $d\big(\mathbf{G}^{-1}\big)_{(-\ln 2,1)} = \begin{bmatrix} \qquad & \qquad \\ \qquad & \qquad \end{bmatrix}$.

29.3. Problems

(1) Derive theorem 29.1.2 from theorem 29.1.1.

(2) Your good friend Fred R. Dimm is hopelessly confused. He knows the formulas for changing polar to rectangular coordinates ($x = r\cos\theta$ and $y = r\sin\theta$), and wants to find the partial derivative of the variable r with respect to the variable x. Fred likes to check his answers by doing problems two different ways. So he calculates

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} = \cos\theta.$$

But when he rewrites the formula for x as $r = \dfrac{x}{\cos\theta}$ and differentiates he gets

$$\frac{\partial r}{\partial x} = \frac{1}{\cos\theta} = \sec\theta.$$

Even Fred knows that $\cos\theta$ and $\sec\theta$ are not always equal, but he just can't seem to find the error in his work. Write a note to your friend helping him out. Make sure that you explain with exemplary clarity exactly what he did that was right and exactly what was wrong.

(3) Show that if $z = f\left(\dfrac{x-y}{y}\right)$, then $xz_x + yz_y = 0$.

(4) Show that if $z = xy + xf\left(\dfrac{y}{x}\right)$, then $xz_x + yz_y = xy + z$.

(5) Show that if $w = f(x-y, y-z, z-x)$ where f is a differentiable function, then $w_x + w_y + w_z = 0$.

(6) Let $z = f(x+ct) + g(x-ct)$, $u = x+ct$, and $v = x-ct$ (where c is a constant). Show that $\dfrac{\partial^2 z}{\partial t^2} = c^2\dfrac{\partial^2 z}{\partial x^2} = c^2(f''(u) + g''(v))$.

(7) Let $w = f\left(\dfrac{y-x}{xy}, \dfrac{z-x}{xz}\right)$. Show that $x^2\dfrac{\partial w}{\partial x} + y^2\dfrac{\partial w}{\partial y} + z^2\dfrac{\partial w}{\partial z} = 0$.

(8) Let $z = f(u,v)$ where $u = x+y$ and $v = x-y$. Show that $\dfrac{\partial^2 z}{\partial x\partial y} = \dfrac{\partial^2 z}{\partial u^2} - \dfrac{\partial^2 z}{\partial v^2}$.

(9) Let $w = f(u^2-t^2, t^2-u^2)$. Show that $tw_u + uw_t = 0$.

(10) Let n be a fixed positive integer. Prove that if a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ satisfies $f(tx,ty) = t^n f(x,y)$ for all t, x, $y \in \mathbb{R}$, then $xf_x + yf_y = nf$.

(11) Let n be a fixed positive integer. Give a coherent proof that if a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ satisfies the relation $f(tx,ty) = t^n f(x,y)$ for all t, x, $y \in \mathbb{R}$, then $x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy} = n(n-1)f$.

(12) Suppose that a variable y is defined implicitly in terms of x by an equation of the form $G(x, y(x)) = 0$, where G is a smooth real valued function of two variables. Derive a formula for $\dfrac{dy}{dx}$ in terms of the partial derivatives $G_1(x,y)$ and $G_2(x,y)$. *Hint.* Let $h(x) = (x, y(x))$ and consider $D(G\circ h)(x)$.

Illustrate your result by showing how it can be used to find the slope of the tangent line to the curve

$$x^2y\,\ln y + xy^2e^{xy} + x^3y^2 = 1$$

at the point $(e, \frac{1}{e})$.

(13) Here is a problem from a thermodynamics text:

Show that if $f(x,y,z)=0$, then

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1\,.$$

Your old pal, Fred Dimm, is in trouble again. He is taking a class in thermodynamics and is baffled by the statement of the problem. Among other things he notices that there is a mysterious f in the hypothesis that does not appear in the conclusion. He wonders, not unreasonably, how assuming something about f is going to help him prove something that makes no reference whatever to that quantity. He is also convinced that the answer is wrong: he thinks that the product of the partial derivatives should be $+1$. He claims it should be just like the single variable case: the chain rule says $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$ because we can cancel the dx's.

Help Fred by explaining the exercise to him. And comment on his "correction" of the problem.

Unfortunately, once you have explained all this to him, he still can't do the problem. So also show him how to solve it.

Hint. Let $w = f(x,y,z)$. By considering the mappings

$$\begin{matrix} y \\ \\ z \end{matrix} \;\longrightarrow\; \begin{matrix} x \\ y \\ z \end{matrix} \;\xrightarrow{\;f\;}\; w$$

see if you can find a simple expression for $-\dfrac{\left(\dfrac{\partial f}{\partial y}\right)_{x,z}}{\left(\dfrac{\partial f}{\partial x}\right)_{y,z}}$.

(14) What, precisely, do people mean when they write the formula

$$\left(\frac{\partial y}{\partial x}\right)_z = -\frac{\left(\dfrac{\partial y}{\partial z}\right)_x}{\left(\dfrac{\partial x}{\partial z}\right)_y} \quad ?$$

Give a careful proof that (under appropriate conditions) this formula is correct.

(15) Regard thermodynamics as having 5 fundamental variables: T (temperature), V (volume), U (internal energy), P (pressure), and S (entropy). Any two of these may be regarded as "independent". If we take T and V to be our independent variables, then two basic laws of thermodynamics may be stated as follows:

$$T\left(\frac{\partial S}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V \tag{I}$$

$$T\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial U}{\partial V}\right)_T + P \tag{II}$$

Using (I) and (II) derive the following "Maxwell relations":

$$\left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T \tag{III}$$

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V \tag{IV}$$

$$\left(\frac{\partial T}{\partial P}\right)_S = \left(\frac{\partial V}{\partial S}\right)_P. \tag{V}$$

Hints: You may wish to develop some tools. For example, you may be able to derive (and use) the following formulas. (Keep in mind that "formulas" are typically *conclusions* of theorems: and theorems have *hypotheses* as well as conclusions.)

$$\left(\frac{\partial y}{\partial t}\right)_z \left(\frac{\partial t}{\partial x}\right)_z = \left(\frac{\partial y}{\partial x}\right)_z \tag{A}$$

$$\left(\frac{\partial y}{\partial x}\right)_z = \frac{1}{\left(\frac{\partial x}{\partial y}\right)_z} \tag{B}$$

$$\left(\frac{\partial y}{\partial x}\right)_z = -\frac{\left(\frac{\partial z}{\partial x}\right)_y}{\left(\frac{\partial z}{\partial y}\right)_x} \tag{C}$$

Once you have succeeded in deriving (A)–(C), you may choose to proceed as follows. For (III) differentiate (I) with respect to V and (II) with respect to T. For (IV) use (C) to write the left side of (IV) in terms of the variables T and V; use (A) to write the left side of (III) as a product of two terms one of which is $\left(\frac{\partial P}{\partial S}\right)_V$ then apply (III). For (V) write $\left(\frac{\partial P}{\partial T}\right)_S$ as a product of two terms one of which is $\left(\frac{\partial P}{\partial V}\right)_S$, using (A). Apply (C) to both terms of the product. In the denominator of the resulting expression there will be a term with subscript V; apply (B) to it. Use (A) once more, and then (III).

(16) Define a function $\mathbf{F}$ by $\mathbf{F}(x, y, z) = \dfrac{x}{y^2z^2}\,\mathbf{i} + yz\,\mathbf{j} + \ln y\,\mathbf{k}$ for x, y, $z > 0$. For this vector field verify the equation in the conclusion of the *inverse function theorem* 29.1.7 by computing each side separately.

29.4. Answers to Odd-Numbered Exercises

(1) $2v$, $2u$

(3) -1, 35, 20, 2

(5) 1, 76

(7) $4x$, $8y$, $4x$, 4, 0

(9) 2

(11) $-3\,\dfrac{x^2-y^2}{\left(x^2+y^2\right)^{\frac{3}{2}}}$

(13) -1

(15) $\begin{bmatrix} \dfrac{1}{2\pi^3} & -\dfrac{3}{2\pi} \\ 0 & 1 \end{bmatrix}$

CHAPTER 30

MORE APPLICATIONS OF THE DERIVATIVE

30.1. Background

Topics: optimization of functions of several variables, global extrema, local extrema, saddle points, Lagrange multipliers.

Classification of Critical Points. The *second derivative test* as stated in many texts works only for functions of two (or three) variables. Here is a procedure for classifying critical points of functions of any (finite) number of variables.

30.1.1. Definition. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth scalar field and $\mathbf{p} \in \mathbb{R}^n$. The HESSIAN MATRIX (or SECOND DERIVATIVE MATRIX) of f at $\mathbf{p}$, denoted by $H_f(\mathbf{p})$, is the $n \times n$ matrix

$$H_f(\mathbf{p}) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{p})\right]_{i=1}^{n}\,{}_{j=1}^{n} = \big[f_{ij}(\mathbf{p})\big].$$

30.1.2. Definition. An $n \times n$ matrix M is POSITIVE DEFINITE if $\langle M\mathbf{x}, \mathbf{x}\rangle > 0$ for every $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$. It is NEGATIVE DEFINITE if $\langle M\mathbf{x}, \mathbf{x}\rangle < 0$ for every $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$. It is INDEFINITE if there are vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ such that $\langle M\mathbf{x}, \mathbf{x}\rangle > 0$ and $\langle M\mathbf{y}, \mathbf{y}\rangle < 0$.

30.1.3. Theorem (Second Derivative Test)**.** *Let* $\mathbf{p}$ *be a critical point of a smooth scalar field* f*. If the Hessian matrix* H_f *is positive definite at* $\mathbf{p}$*, then* f *has a local minimum there. If* H_f *is negative definite at* $\mathbf{p}$*, then* f *has a local maximum there. If* H_f *is indefinite at* $\mathbf{p}$*, then* f *has a saddle point there.*

Is there some simple way of telling whether a matrix is positive definite?

> Yes. It is positive definite if all its eigenvalues are strictly positive. It is negative definite if all its eigenvalues are strictly negative. It is indefinite if it has at least one strictly positive and at least one strictly negative eigenvalue.

What's an eigenvalue?

> An EIGENVALUE of a square matrix M is a root of its characteristic polynomial.

OK. What's a characteristic polynomial then?

> Notice that if M is an $n\times n$ matrix, then $\det(M - \lambda I_n)$ is a polynomial in λ of degree n. This is the CHARACTERISTIC POLYNOMIAL of M. Incidentally, it turns out that if the matrix is symmetric (the Hessian matrix of a smooth scalar field, for example, is symmetric), then all the roots of the characteristic polynomial are real.

Summary: To classify the critical points of a scalar field $f : \mathbb{R}^n \longrightarrow \mathbb{R}$

(A) Compute ∇f.
(B) Solve $\nabla f = 0$ to find the critical points.
(C) Compute the Hessian matrix H_f.
(D) For each critical point $\mathbf{p}$ find the roots of the polynomial $\det(H_f - \lambda I_n)$.
(E) If all these roots are strictly positive, f has a local minimum at $\mathbf{p}$. If they are all strictly negative, f has a local maximum at $\mathbf{p}$. And if there is at least one strictly positive and one strictly negative root, then f has a saddle point at p.

Some definitions relevant to Problem 1.

30.1.4. Definition. A RAY from the origin in $\mathbb{R}^2$ is a parametrized constant speed curve $\mathbf{r}\colon [0, \infty) \longrightarrow \mathbb{R}^2$ such that $\mathbf{r}(0) = (0, 0)$ and the range of $\mathbf{r}$ lies in some straight line.

30.1.5. Definition. A function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is LOCALLY INCREASING at the origin on a ray $\mathbf{r}$ if the real valued function $f \circ \mathbf{r}$ is increasing on some interval of the form $[0, t_0]$ $(t_0 > 0)$. The term *locally decreasing* is defined similarly.

30.1.6. Definition. Two collections of distinct rays $(\mathbf{r}_1, \dots \mathbf{r}_n)$ and $(\mathbf{R}_1, \dots \mathbf{R}_n)$ are SEPARATING if it is not possible to move between any two members of the first collection (in either the clockwise or counterclockwise direction) without encountering a member of the second.

30.2. Exercises

In the first 7 exercises below you are asked to classify critical points as *local minima*, *local maxima*, or *saddle points*.

(1) Let $f(x,y,z) = x^2y - ye^z + 2x + z$. The only critical point of the function f is located at (_____ , ____ , ____) and it is a ______________________________ .

(2) Let $f(x,y) = x^3 - 3xy + y^3 - 2$. Then f has two critical points. One is located at (0 , ____) and is a __________________________. The other is located at (____ , ____) and is a __________________________ .

(3) Let $f(x,y,z) = x^2y + 2xy + y - ye^{z-1} + 2x + z + 7$. The only critical point of f is located at (_____ , ____ , ____) and it is a __________________________ .

(4) Let $f(x,y) = -\frac{1}{2}xy + \dfrac{2}{x} - \dfrac{1}{y}$. The only critical point of f is located at (____ , _____) and it is a ____________________________ .

(5) Let $f(x,y) = \sin x + \sin y + \sin(x+y)$ for $0 < x < \pi$, $0 < y < \pi$. The only critical point of f is located at (_______ , _______) and it is a ________________________________ .

(6) Let $f(x,y,z) = x^3y + z^2 - 3x - y + 4z + 5$. The only critical point of the function f is located at (____ , ____ , _____) and it is a __________________________ .

(7) Let $f(x,y,z) = x^2y - 4x - y\sin z$ for $0 < z < \pi$. Then f has two critical points located at (a,b,c) and at (-a,-b,c) where $a =$ _____ , $b =$ _____ , and $c =$ ________ . Both of these critical points are ______________________________ .

(8) Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(2,1) = -3$, $\nabla f(2,1) = (2,0)$ and the Hessian matrix of f at (a,b) is $\begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}$ for all (a,b). Then $f(x,y,z) =$ ______________________________________ .

(9) Let $f(x,y) = x^2 - xy + 2y^2 - x - 3y + 1$ be defined over the region bounded by the triangle whose vertices are (0,0), (2,0), and (0,2). The global minimum of f is ______ and occurs at the point (____ , ____). The global maximum of f is _____ and occurs at two points (0 , ____) and (____ , ____).

(10) Let $f(x,y) = xy - x^2 + 10$ be defined over the rectangular region $[0,5] \times [0,4]$. The global minimum of f is ________ and occurs at the point (____ , ____). The global maximum of f is ______ and occurs at the point (____ , ____).

(11) Let $f(x,y) = x^3 - 3xy + y^3 - 2$ be defined over the rectangular region $[0,3] \times [0,2]$. The global minimum of f is ________ and occurs at the point (____ , ____). The global maximum of f is ________ and occurs at the point (____ , ____).

(12) Let $f(x,y) = 3x^2 - 2xy + 3y^2 - 10x + 6y + 8$ be defined on the square region $[0,2] \times [-1,1]$. The global minimum of f is ______ and occurs at the point (________ , ________). The global maximum of f is ______ and occurs at the point (____ , ____).

(13) Let $f(x,y) = x^4 + y^2 - 2x^2 - 4y$ be defined on the rectangular region $[0,3] \times [0,5]$. The global minimum of f is ______ and occurs at the point (____ , ____). The global maximum of f is ________ and occurs at the point (____ , ____).

(14) Let $f(x,y) = x^2 + 2xy + 4x - y^2 - 8y - 6$ be defined on the triangular region whose vertices are $(0,0)$, $(0,-8)$, and $(4,-4)$. The global minimum of f is ______ and occurs at the point (_____ , _____). The global maximum of f is ______ and occurs at the point (_____ , _____).

(15) Let $f(x,y) = \dfrac{6+2xy-y}{3+x^2}$ be defined over the rectangular region $[0,3]\times[0,6]$. The global minimum of f is ____ and occurs at the point (___ , ___). The global maximum of f is _______ and occurs at the point (____ , ___).

(16) Let $D = \{(x,y) : x^2+y^2 \le 1\}$ be the closed unit disk in $\mathbb{R}^2$. At each point (x,y) in D the temperature $\mathcal{T}$ is given by $\mathcal{T}(x,y) = x^2+2y^2-y+3$. The coldest point on the disk is (____ , ______); at that point the temperature is $\dfrac{a}{8}$ where $a =$ ______ . The hottest point on the disk is (____ , ______); at that point the temperature is ____ .

(17) The maximum value achieved by xy^2z^3 on the unit sphere $x^2+y^2+z^2=1$ is $\dfrac{1}{a\sqrt{3}}$ where $a =$ _______ .

(18) The maximum value achieved by $x+3y+4z$ on the sphere $x^2+y^2+z^2=13$ is $a\sqrt{2}$ where $a =$ _______ .

(19) The maximum value achieved by $x+2y+4z$ on the sphere $x^2+y^2+z^2=7$ is $a\sqrt{3}$ where $a =$ ____ .

(20) A rectangular box without a top is to be made from 18 ft^2 of a given material. The largest possible volume of such a box is $a\sqrt{2a}$ ft^3 where $a =$ ____ .

(21) Locate the maximum value of xy^2z^3 on the plane $x+y+z=1$.

Answer: ____________________________ .

(22) The largest value achieved by xyz on the curve of intersection of the circular cylinder $x^2+y^2=3$ and the plane $y=2z$ is ____ . One point where this maximum occurs is (a, b, b^{-1}) where $a =$ ____ and $b =$ ______ .

(23) The largest value achieved by $x+y+z$ on the curve of intersection of the circular cylinder $x^2+y^2=2$ and the plane $x+z=1$ is $a+\sqrt{b}$ where $a =$ ____ and $b =$ ____ . One point where this maximum occurs is $(c, \sqrt{b}, a)$ where $c =$ ____ .

(24) The (right circular) cylinder whose axis is the z-axis and whose radius is 2 intersects the plane $y+3z=9$ in an ellipse C. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be the scalar field defined by $f(x,y,z) = -x+2y+6z$. Then the maximum value attained by f on C is ______ and the minimum value of f on C is ______ .

30.3. Problems

(1) Following are three plausible conjectures. In each case if the conjecture is true, prove it; and if it is false, give an example to show that it can fail. (In the latter case, don't forget to *prove* that your example really does what you claim.)

Conjecture 1. If on every parametrized ray from the origin a function f of two variables is locally increasing at the origin, then f has a local minimum at the origin.

Conjecture 2. If a function f of two variables has a saddle point at the origin, then there are at least two separating pairs of rays from the origin such that f is locally increasing at the origin on one pair and locally decreasing at the origin on the other.

Conjecture 3. It is possible to find a function f of two variables for which there exist two separating triples of rays from the origin such that on one of the triples f is locally increasing at the origin and on the other f is locally decreasing at the origin.

Hint. A careful analysis of the behavior of the following surfaces may provide some insight into a couple of the conjectures.

$$f(x, y) = x^3 - 3xy^2,$$
$$g(x, y) = y^2 - 3x^2y + 2x^4.$$

(2) Let $f(w, x, y, z) = \left(w + \dfrac{1}{w}\right)^2 + \left(x + \dfrac{1}{x}\right)^2 + \left(y + \dfrac{1}{y}\right)^2 + \left(z + \dfrac{1}{z}\right)^2$ for $w,\ x,\ y,\ z > 0$. Prove that on the hyperplane $w + x + y + z = 16$ in $\mathbb{R}^4$ the function f is bounded below by $\dfrac{289}{4}$.

(3) Explain carefully how to use Lagrange multipliers to find the distance from $(0, 1)$ to the parabola $x^2 = 4y$.

(4) Use Lagrange multipliers to show that of all triangles inscribed in a circle, the equilateral triangle has the largest product of the lengths of the sides. *Hint.* As the constraint use the fact that the sum of the central angles of the circle determined by the sides of the triangle is 2π. Use the *law of cosines*.

(5) Use the theorem concerning Lagrange multipliers to argue that the function $f(x, y, z) = x + y + z$ does *not* achieve either a maximum or a minimum on the curve of intersection of the hyperbolic cylinder $x^2 - y^2 = 1$ and the plane $2x + z = 1$.

(6) Find the point on the plane $2x - 3y - 4z = 25$ which is nearest the point $(3, 2, 1)$ using
   (a) a geometric method;
   (b) the *second derivative test*; and
   (c) *Lagrange multipliers.*

30.4. Answers to Odd-Numbered Exercises

(1) -1, 1, 0, saddle point

(3) -2, 1, 1, saddle point

(5) $\frac{\pi}{3}$, $\frac{\pi}{3}$, local maximum

(7) 1, 2, $\frac{\pi}{2}$, saddle points

(9) -1, 1, 1, 3, 2, 2, 0

(11) -3, 1, 1, 25, 3, 0

(13) -5, 1, 2, 68, 3, 5

(15) 0, 0, 6, $2\sqrt{3}$, $\sqrt{3}$, 6

(17) 12

(19) 7

(21) no maximum exists

(23) 1, 2, 0

Part 8

PARAMETRIZED CURVES

CHAPTER 31

PARAMETRIZED CURVES

31.1. Background

Topics: parametrized curves, arclength of parametrized curves and of curves specified in polar coordinates, areas of regions bounded by closed curves, tangent lines to curves.

31.2. Exercises

(1) Express the curve $\begin{cases} x = \sin t \\ y = 1 - \cos^2 t \end{cases}$ $(t \in \mathbb{R})$ in nonparametric form.

Answer: $y = f(x)$ where $f(x) =$ ________ and _____ $\leq x \leq$ _____ .

(2) Express the curve $\begin{cases} x = 3\cos t \\ y = 4\sin t \end{cases}$ in nonparametric form.

Answer: $\dfrac{x^2}{a} + \dfrac{y^2}{b} = 1$ where $a =$ _____ and $b =$ _____ .

(3) Express the curve $\begin{cases} x = \sin t \\ y = \cos 2t \end{cases}$ in nonparametric form.

Answer: $y = f(x)$ where $f(x) =$ ______________ .

(4) Express the curve $\begin{cases} x = e^t \\ y = t\, e^{2t} \end{cases}$ in nonparametric form.

Answer: $y = f(x)$ where $f(x) =$ ______________ .

(5) Express the curve $\begin{cases} x = e^t \\ y = e^{t^2} \end{cases}$ in nonparametric form.

Answer: $y = x^{f(x)}$ where $f(x) =$ ____________ .

(6) Express the curve $\begin{cases} x = \dfrac{t}{1+t} \\ y = \dfrac{1-t}{1+t} \end{cases}$ in nonparametric form.

Answer: $y = f(x)$ where $f(x) =$ ______________ .

(7) Let L be the line segment connecting the points $(2, -2)$ and $(10, 2)$. Find a parametrization of L starting at $(10, 2)$ and ending at $(2, -2)$ with parameter interval $[0, 1]$.

Answer: $x = a + bt$ and $y = c + dt$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , $d =$ _____ , and $t \in [0, 1]$.

(8) Let T be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Find a counterclockwise parametrization of T with parameter interval $[0, 1]$.

Answer: Let $f(t) = ($ _____ , $0)$ for $0 \leq t \leq \frac{1}{3}$, $f(t) = ($ ________ , ________ $)$ for $\frac{1}{3} < t \leq \frac{2}{3}$, and $f(t) = ($ 0 , ________ $)$ for $\frac{2}{3} < t \leq 1$.

(9) Let C be the circle of radius 1 whose center is at $(0, 1)$. Find a parametrization of C with parameter interval $[0, 2\pi]$ which traverses the curve once in a counterclockwise direction starting at $(1, 1)$.

Answer: Let $x(t) = a + b\sin t + c\cos t$ and $y(t) = A + B\sin t + C\cos t$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , $A =$ _____ , $B =$ _____ , and $C =$ _____ .

(10) Express the curve $\begin{cases} x = \dfrac{\sqrt{t}}{1+t} \\ y = \dfrac{1-t}{1+t} \end{cases}$ in nonparametric form.

Answer: The curve lies on the ellipse $ax^2 + by^2 = 1$ where $a =$ _____ and $b =$ _____ ; but it contains only those points (x, y) such that $p \leq x \leq q$ and $r < y \leq s$, where $p =$ _____ , $q =$ _____ , $r =$ _____ , and $s =$ _____ .

(11) Express the curve $\begin{cases} x = \dfrac{2t+1}{(t+1)^2} \\ y = \dfrac{t}{t+1} \end{cases}$ in nonparametric form.

Answer: $x = f(y)$ where $f(y) =$ ________________ .

(12) Let C be the circle of radius 1 whose center is at $(0, 1)$. Find a parametrization of C which traverses the curve once in a clockwise direction starting at $(1, 1)$ and has parameter interval $[0, 1]$.

Answer: Let $x(t) = a + b\sin kt + c\cos kt$ and $y(t) = A + B\sin kt + C\cos kt$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , $A =$ _____ , $B =$ _____ , $C =$ _____ , and $k =$ _____ .

(13) Let S be the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$. Find a counterclockwise parametrization of S with parameter interval $[0, 1]$.

Answer: Let $f(t) = ($ _____ $, 0)$ for $0 \le t \le \frac{1}{4}$, $f(t) = (1$, ________ $)$ for $\frac{1}{4} < t \le \frac{1}{2}$, $f(t) = ($________ $, 1)$ for $\frac{1}{2} < t \le \frac{3}{4}$, and $f(t) = (0,$ ________ $)$ for $\frac{3}{4} < t \le 1$.

(14) Let D be the region in the plane satisfying

$$x^2 + y^2 \le 4, \qquad x \ge 0, \qquad \text{and } y \ge 0.$$

Find a counterclockwise parametrization of the curve which bounds D, has parameter interval $[0, 1]$, and starts at $(0, 0)$.

Answer: Let

$$f(t) = \begin{cases} (\ \_\_\_\_\_\_\ , 0) \text{ for } 0 \le t \le \frac{1}{3}, \\ (\ \_\_\_\_\_\_\_\_\_\_\_\_\ ,\ \_\_\_\_\_\_\_\_\_\_\_\_\ ) \text{ for } \frac{1}{3} < t \le \frac{2}{3}, \text{ and} \\ (\ 0\ ,\ \_\_\_\_\_\_\_\_\_\ ) \text{ for } \frac{2}{3} < t \le 1. \end{cases}$$

(15) The positions of two particles A and B at time t are given by

$$\begin{cases} \mathbf{r}(t) = (2\sin t, -2\cos t) \\ \mathbf{R}(t) = (2t^3, 2(t^3 - 1)) \end{cases} \quad \text{for } -2\pi \le t \le 2\pi.$$

(a) The paths intersect at (_____ , _____) and (_____ , _____) .

(b) The particles collide at (_____ , _____) at time $t =$ _____ .

(16) The positions of two particles A and B at time t are given by

$$\begin{cases} \mathbf{r}(t) = (t^2 - 5, t^2 - 3) \\ \mathbf{R}(t) = (2t - 5, 4t^2 - 20t + 25) \end{cases} \quad \text{for } t \ge 0\ .$$

(a) The paths intersect at (_____ , _____) and (_____ , _____) .

(b) The particles collide at (_____ , _____) at time $t =$ _____ .

(17) The positions of two particles A and B at time t are given by

$$\begin{cases} \mathbf{r}(t) = (16\cos^2 t - 12, 4\sin t) \\ \mathbf{R}(t) = (4 - 2t^3, t^3). \end{cases}$$

(a) The paths intersect at (_____ , _____) and (_____ , _____) .

(b) The particles collide at (_____ , _____) at time $t =$ _____ .

(18) The positions of two particles A and B at time t are given by

$$\begin{cases} \mathbf{r}(t) = (\sqrt{1 + t^2}, t) \\ \mathbf{R}(t) = \left(4 - \frac{1}{8}t^2, \sqrt{\frac{1}{4}t^2 + 6}\right) \end{cases}$$

(a) The only point at which the paths intersect is (_____ , _______) .
(b) The particles collide at this point when $t =$ _______ .

(19) The positions of two particles A and B at time t are given by

$$\begin{cases} \mathbf{r}(t) = \left(1 - 2\cos\frac{t}{2}, 4\cos^2\frac{t}{2} - 4\cos\frac{t}{2}\right) \\ \mathbf{R}(t) = \left(\frac{9}{4}t - 1, \frac{9}{4}t\right) \end{cases}$$

(a) The paths intersect at (_____ , _____) and (_____ , _____) .
(b) The particles collide at (_____ , _____) at time $t =$ _____ .

(20) The positions of two particles A and B at time t are given by

$$\begin{cases} \mathbf{r}(t) = \left(2\sin\left[\dfrac{\pi t}{2(t+1)}\right], 2\sin\left[\dfrac{\pi}{2(t+1)}\right]\right) \\ \mathbf{R}(t) = \left(\frac{1}{2}\sqrt{8+t^2}, \frac{1}{12}(8+t^2)\right) \end{cases} \quad \text{for } 0 \le t \le 4.$$

(a) The only point at which the paths intersect is (_____ , _____) .
(b) The particles collide at this point when $t =$ _____ .

(21) The positions of two particles A and B at time t are given by

$$\begin{cases} \mathbf{r}(t) = (\sin^3 t, \cos^3 t) \\ \mathbf{R}(t) = \left(\tan^2\frac{t}{4}, 2 - \sec^2\frac{t}{4}\right) \end{cases} \quad \text{for } -\pi < t < \pi.$$

(a) The paths intersect at (_____ , _____) and (_____ , _____) .
(b) The particles collide at (_____ , _____) at time $t =$ _____ .

(22) The positions of two particles A and B at time t are given by

$$\begin{cases} \mathbf{r}(t) = (\sin^3 t, \cos^3 t) \\ \mathbf{R}(t) = \left(\tan^2\frac{t}{2}, 2 - \sec^2\frac{t}{2}\right) \end{cases} \quad \text{for } -\frac{\pi}{2} \le t \le \frac{\pi}{2}.$$

The particles collide at the point (_____ , _____) when $t =$ _____ and at the point (_____ , _____) when $t =$ _____ .

(23) Let L be the line segment connecting the points $(-2, 4)$ and $(1, 0)$.

(a) Find a parametrization of L starting at $(-2, 4)$ and ending at $(1, 0)$ with parameter interval $[-2, 1]$.

Answer: $x = r$ and $y = c + dr$ where $c =$ ________ , $d =$ ________ , and $r \in [-2, 1]$.

(b) Find a parametrization of L starting at $(1, 0)$ and ending at $(-2, 4)$ with parameter interval $[-1, 2]$. *Hint.* Let $s = -r$ in (a).

Answer: $x = -s$ and $y = c + ds$ where $c =$ ________ , $d =$ ________ , and $s \in [-1, 2]$.

(c) Find a parametrization of L starting at $(1, 0)$ and ending at $(-2, 4)$ with parameter interval $[0, 3]$. *Hint.* Let $t = s + 1$ in (b).

Answer: $x = a + bt$ and $y = c + dt$ where $a =$ ________ , $b =$ ________ , $c =$ ________ , $d =$ ________ , and $t \in [0, 1]$.

(d) Find a parametrization of L starting at $(1, 0)$ and ending at $(-2, 4)$ with parameter interval $[0, 1]$. *Hint.* Let $u = \frac{1}{3}t$ in (c).

Answer: $x = a + bu$ and $y = c + du$ where $a =$ ________ , $b =$ ________ , $c =$ ________ , $d =$ ________ , and $u \in [0, 1]$.

(24) The equation of a cycloid is given by $\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}$. At the point $t = \frac{\pi}{6}$, $\dfrac{dy}{dx} = a + \sqrt{b}$ and $\dfrac{d^2y}{dx^2} = -4(c + d\sqrt{b})$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $d =$ _____ .

(25) Find the tangent lines to the curve whose parametric equations are $\begin{cases} x = t^2 - 2t + 1 \\ y = t^4 - 4t^2 + 4 \end{cases}$ at the point $(1, 4)$. Answer: $y = ax + b$ and $y = c$ where $a =$ _____ , $b =$ _____ , and $c =$ _____ .

(26) Consider the curve $r = 3\theta$. At the point where $\theta = \pi$, $\dfrac{dy}{dx} = a\pi + b$ and $\dfrac{d^2y}{dx^2} = -\frac{1}{3}(c + \pi^d)$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $d =$ _____ .

(27) The slope of the tangent line to the cardioid $r = 2 - 2\cos\theta$ at the point on the curve where $\theta = \frac{\pi}{4}$ is $a + \sqrt{b}$ where $a =$ _____ and $b =$ _____ .

(28) At the point on the limaçon $r = 1 + 2\sin\theta$ where $\theta = \frac{\pi}{4}$, $\dfrac{dy}{dx} = a - 2\sqrt{b}$ and $\dfrac{d^2y}{dx^2} = -c(d\sqrt{b} + 2)$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $d =$ _____ .

(29) Consider the curve given by the parametric equations $\begin{cases} x = \cos t + t\sin t \\ y = \sin t - t\cos t \end{cases}$. At the point where $t = \frac{\pi}{3}$, $\dfrac{dy}{dx} = \sqrt{a}$ and $\dfrac{d^2y}{dx^2} = \dfrac{b}{c}$ where $a =$ _____ , $b =$ _______ , and $c =$ _____ .

(30) Consider the curve given by the parametric equations $\begin{cases} x = \dfrac{1 - t^2}{1 + t^2} \\ y = \dfrac{2t}{1 + t^2} \end{cases}$. At the point where $t = 3$, the slope of the tangent line to the curve is $\dfrac{a}{3}$ where $a =$ _____ .

(31) Let C be the curve whose parametric equations are $\begin{cases} x = \tan t \\ y = \sec t \end{cases}$. At the point $\left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$, $\dfrac{dy}{dx} = \dfrac{1}{a}$ where $a =$ _____ and $\dfrac{d^2y}{dx^2} = \dfrac{b}{c}\sqrt{b}$ where $b =$ _____ and $c =$ _____ .

(32) The length of that portion of the curve $\begin{cases} x = 6t^2 \\ y = \frac{4}{3}t^3 - 9t \end{cases}$ which lies between the origin and the point (54,9) is ________ .

(33) Let $a > 0$. The arc length of the cycloid $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$ for $0 \le t \le 4\pi$ is ________ .

(34) The length of the curve $\begin{cases} x = \sin^3 t \\ y = \cos^3 t \end{cases}$ for $0 \le t \le \pi$ is _____ .

(35) The length of the curve $\begin{cases} x = 3 + \arctan t \\ y = 2 - \ln\sqrt{1 + t^2} \end{cases}$ from $t = 0$ to $t = 1$ is $\ln(a + \sqrt{b})$ where $a =$ _____ and $b =$ _____ .

(36) The length of the curve $\begin{cases} x = e^t \cos t \\ y = e^t \sin t \end{cases}$ from $t = 0$ to $t = \ln 8$ is $a\sqrt{b}$ where $a =$ _____ and $b =$ _____ .

(37) Estimate the length of the curve $\begin{cases} x = t \\ y = \frac{1}{3}t^3 \end{cases}$ from $t = 0$ to $t = \frac{1}{2}$.

Answer: with an error of less than 10^{-4} the length is $\dfrac{161}{a}$ where $a =$ _____. *Hint:* Use the *binomial theorem* to approximate an intractable integrand.

(38) The area of the region that lies outside the circle $r = a$ but inside the lemniscate $r^2 = 2a^2 \cos 2\theta$ is $a^2(\sqrt{b} + c\pi)$ where $b =$ _____ and $c =$ ______ .

(39) The area enclosed by the curve $r = 6 \sin 3\theta$ is ________ .

(40) The area of the region common to the circles $(x-1)^2 + y^2 = 1$ and $x^2 + (y-1)^2 = 1$ is $a + b\pi$ where $a =$ _____ and $b =$ ______ .

(41) The area enclosed by the curve $r = 1 + \cos 2\theta$ is ________ .

(42) Use integration in polar coordinates to find the area of the region which lies inside the circle $x^2 + y^2 = 2$ and to the right of the line $x = 1$.

Answer: $a\pi + b$ where $a =$ ______ and $b =$ ______ .

(43) The area between the two loops of the limaçon $r = 1 + \sqrt{2}\cos\theta$ is $a + b\pi$ where $a =$ _____ and $b =$ _____ .

(44) The area of the region bounded by the astroid $x^{2/3} + y^{2/3} = 1$ is $\dfrac{a}{8}$ where $a =$ ______ .

(45) The length of the curve $r = 1 - \cos\theta$ for $0 \le \theta \le \frac{2\pi}{3}$ is _____ .

(46) The length of the curve $r = \sin^2 \dfrac{\theta}{2}$ for $0 \le \theta \le \frac{\pi}{2}$ is $a - \sqrt{a}$ where $a =$ _____ .

(47) The arclength of the spiral of Archimedes, $r = 3\theta$, where $0 \le \theta \le \sqrt{3}$ is $a\sqrt{a} + \dfrac{a}{b}\ln(b + \sqrt{a})$ where $a =$ _____ and $b =$ _____ .

(48) The length of the hyperbolic spiral $r = \dfrac{1}{\theta}$ for $1 \le \theta \le \sqrt{3}$ is $\sqrt{a} - \dfrac{a}{\sqrt{b}} + \ln \dfrac{a + \sqrt{b}}{1 + \sqrt{a}}$ where $a =$ _____ and $b =$ _____ .

(49) The length of the curve $\mathbf{r}(t) = (2t, t^2, \frac{4}{3}\sqrt{2}\,t^{3/2})$ between the points where $t = 0$ and $t = 3$ is ________ .

(50) The length of the curve $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j} + \sqrt{2}\,t\,\mathbf{k}$ between the points where $t = 0$ and $t = 1$ is $a - \dfrac{1}{a}$ where $a =$ _____ .

(51) The length of the curve $\mathbf{r}(t) = (\arctan t, \frac{1}{2}\ln(1 + t^2), -5)$ between the points where $t = 0$ and $t = 1$ is $\ln(a + \sqrt{b})$ where $a =$ _____ and $b =$ _____ .

(52) The length of the logarithmic spiral $\mathbf{r}(t) = e^\theta$ between $\theta = 0$ and $\theta = 4\pi$ is $a(e^b - 1)$ where $a =$ ______ and $b =$ ________ .

(53) A particle follows the path $\mathbf{r}(t) = (t, \frac{2}{3}\sqrt{2}\,t^{\frac{3}{2}}, \frac{1}{2}t^2)$. The distance the particle travels between $t = 0$ and $t = 4$ is ________ .

(54) The length of the curve forming the intersection of the surfaces $2x^3 = 3y$ and $x^2 = z$ between the origin and the point $(3, 18, 9)$ is ________ .

(55) The length of the curve $\mathbf{r}(t) = \frac{1}{3}(1+t)^{\frac{3}{2}}\,\mathbf{i} + \frac{1}{3}(1-t)^{\frac{3}{2}}\,\mathbf{j} + \frac{1}{2}t\,\mathbf{k}$ between the points where $t = -1$ and $t = 1$ is ______ .

(56) The length of the curve $\mathbf{r}(t) = (8t, 6t^2, 3t^3)$ between the points where $t = 0$ and $t = 2$ is ________ .

(57) The length of the curve forming the intersection of the surfaces $y = 2\sqrt{2x}$ and $z = \ln x$ between the points $(1, 2\sqrt{2}, 0)$ and $(2, 4, \ln 2)$ is $a + \ln b$ where $a =$ _____ and $b =$ _____ .

(58) The curves $\mathbf{r}(t) = (3t^2 - 1, t - 1, \ln t - 3t)$ and $\mathbf{R}(u) = (u^2 + 1, -u - 1, e^{u+1} - u^2 + u - 2)$ intersect at the point (_____ , _____ , _____) . The cosine of their angle of intersection is $-\sqrt{\dfrac{a}{41}}$ where $a =$ ________ .

(59) The curves $\mathbf{r}(t) = (e^t - 1, 2\sin t, \ln(t+1))$ and $\mathbf{R}(u) = (u + 1, u^2 - 1, u^3 + 1)$ intersect at the origin. The angle (in radians) at which the curves intersect is __________ .

(60) The curves $\mathbf{r}(t) = (t^2 - 1, t + 3, -(t + 4))$ and $\mathbf{R}(u) = (u - 3, 2u - 4, u^3 - 4u^2 - 4u + 18)$ intersect at the point (_____ , _____ , _____). The cosine of their angle of intersection is $\dfrac{1}{a}$ where $a =$ _____ .

(61) The curves $\mathbf{r}(t) = (t^2-2, t^2-2t-1, 2\ln(t-1))$ and $\mathbf{R}(u) = (u^2+3u+4, u^2+2u, e^{u+1}-2u-3)$ intersect at the point (_____ , _______ , _____). The cosine of their angle of intersection is $\dfrac{1}{2\sqrt{a}}$ where $a =$ _____ .

(62) The curves $\mathbf{r}(t) = t\,\mathbf{i} + (t^2 + t - 4)\,\mathbf{j} + (3 + \ln t)\,\mathbf{k}$ and $\mathbf{R}(u) = (u^2 - 8)\,\mathbf{i} + (u^2 - 2u - 5)\,\mathbf{j} + (u^3 - 3u^2 - 3u + 12)\,\mathbf{k}$ intersect at the point (_____ , _______ , _____). The cosine of their angle of intersection is $\dfrac{6}{a}\sqrt{2}$ where $a =$ _____ .

(63) The curves $\mathbf{r}(t) = e^t\,\mathbf{i} + 2\sin(t + \frac{\pi}{2})\,\mathbf{j} + (t^2 - 2)\,\mathbf{k}$ and $\mathbf{R}(u) = u\,\mathbf{i} + 2\,\mathbf{j} + (u^2 - 3)\,\mathbf{k}$ intersect at the point (_____ , _____ , _______). The cosine of their angle of intersection is $\dfrac{1}{\sqrt{a}}$ where $a =$ _____ .

(64) Let $\mathbf{f}(t) = (t^3 + t + 1, t^2 - 4, t + \cos \pi t)$ and $\mathbf{g}(t) = (e^{t^2} + 2t + 5, 3, t^3 + t - 4)$. Then the derivative of $\mathbf{f} \times \mathbf{g}$ at $t = 0$ is (_______ , _______ , _______) .

(65) Let $\mathbf{f}(t) = (t^4 - 2t^3 + t^2 + t + 1, t^4 - t^3 + t^2 - t + 1, 2t^3 - t^2 - t + 2)$ and $g(t) = \|\,\mathbf{f}(t)\|^2$. Then $Dg(1) =$ ________ .

(66) Let $\mathbf{f}(t) = 2t^2\,\mathbf{i} + -3\,\mathbf{j}$ and $\mathbf{g}(t) = \mathbf{i} + t\,\mathbf{j} + t^2\,\mathbf{k})$. Then

$$D(\mathbf{f} \times \mathbf{g})(-1) = (\_\_\_\_\_ , \_\_\_\_\_ , \_\_\_\_\_) .$$

(67) Let $\mathbf{f}(t) = (f^1(t), f^2(t), f^3(t))$, where $f^1(t) = t^2 + t - 1 + t^{-1}$, $f^2(t) = t^3 - t^2 + 3 - t^{-2}$, and $f^3(t) = t^4 - t^3 + t^2 - t + 3$. Also let $\mathbf{g}(t) = (g^1(t), g^2(t), g^3(t))$, where $g^1(t) = 2t(1+t)^{-1}$, $g^2(t) = (1-t)(1+t^2)^{-1}$, and $g^3(t) = t^{-1} + t^{-2} + \ln t$.

(a) If $p = \langle \mathbf{f}, \mathbf{g} \rangle$, then $Dp(1) =$ _____ .

(b) If $\mathbf{u} = \mathbf{f} \times \mathbf{g}$, then $D\mathbf{u}(1) =$ (________ , ________ , ______) .

(c) If $n = \|\,\mathbf{f}\|^2$, then $Dn(1) =$ ________ .

(68) Let $\mathbf{f}(t) = (e^t, \ln(t + 1), \arctan t)$. Then $D\mathbf{f}(2) =$ (_____ , _____ , _____) and $\displaystyle\int_0^1 \mathbf{f}(t)\,dt =$ (__________ , ___________________ , ______________________) .

(69) Let $\mathbf{f}(t) = (t^2+t+1)\,\mathbf{i} + (t^3-4)\,\mathbf{j} + \cos\pi t\,\mathbf{k}$ and $\mathbf{g}(t) = (e^{t^2}+5)\,\mathbf{i} + 3\,\mathbf{j} + (t^3+t+5)\,\mathbf{k}$. Then $D(\mathbf{f}\times\mathbf{g})(0) =$ (______ , ______ , ______) .

(70) Let $\mathbf{f}(t) = (t^3+2t^2-4t+1)\,\mathbf{i} + (t^4-2t^3+t^2+3)\,\mathbf{j} + (t^3-t^2+t-2)\,\mathbf{k}$ and $g(t) = \|\,\mathbf{f}(t)\|^2$. Then $Dg(1) =$ ______ .

(71) Let $\mathbf{f}(t) = (t^2\sin t + \cos t, e^{t^2}+e^t-1, t^3-t^2)$ and $\mathbf{g}(t) = (e^t\sin t + e^t\cos t, \arctan t^2, t^4-t^2+3t-2)$. Then $D(\mathbf{f}\times\mathbf{g})(0) =$ (_____ , ______ , ______) .

(72) Let $\mathbf{G}(t) = \displaystyle\int_0^{t^2} (\cos u, e^{-u^2}, \tan u)\,du$. Then

$\mathbf{G}'(t) =$ (________________ , _______________ , __________________) .

(73) Let $\mathbf{f}(t) = f^1(t)\,\mathbf{i} + f^2(t)\,\mathbf{j} + f^3(t)\,\mathbf{k}$ where $f^1(t) = \sin t + \cos t + t^2 - 3t$, $f^2(t) = t + 2 + \tan t$, and $f^3(t) = t^3 - t^2 + \arctan t$. Also let $\mathbf{g}(t) = g^1(t)\,\mathbf{i} + g^2(t)\,\mathbf{j} + g^3(t)\,\mathbf{k}$ where $g^1(t) = t + e^t + e^{-t}$, $g^2(t) = t^2 + e^t - e^{-t}$, and $g^3(t) = \arctan t^2 + e^{t^2} - 3$.

(a) If $\mathbf{h} = \mathbf{f} + \mathbf{g}$, then $D\mathbf{h}(0) =$ (_____ , _____ , _____) .

(b) If $p = \langle \mathbf{f}, \mathbf{g}\rangle$, then $Dp(0) =$ ______ .

(c) If $\mathbf{r} = \mathbf{f}\times\mathbf{g}$, then $D\mathbf{r}(0) =$ (_____ , _____ , _____) .

(d) If $\mathbf{u} = \mathbf{g}\times\mathbf{f}$, then $D\mathbf{u}(0) =$ (_____ , _____ , _____) .

(74) Let $\mathbf{f}(t) = f^1(t)\,\mathbf{i} + f^2(t)\,\mathbf{j} + f^3(t)\,\mathbf{k}$ where $f^1(t) = t^3 + 2t^2 - 4t + 1$, $f^2(t) = t^4 - 2t^3 + t^2 + 3$, and $f^3(t) = t^3 - t^2 + t - 2$. Let $g(t) = \|\,\mathbf{f}(t)\|^2$. Then $Dg(1) =$ ______ .

(75) $\displaystyle\lim_{t\longrightarrow 0}\left(\frac{te^t}{1-e^t}\,\mathbf{i} + \frac{e^{t-1}}{\cos t}\,\mathbf{j}\right) =$ (______ , _______) .

(76) $\displaystyle\lim_{t\longrightarrow 0}\left(\frac{\sin 3t}{\sin 2t}\,\mathbf{i} + \frac{\ln(\sin t)}{\ln(\tan t)}\,\mathbf{j} + t^t\,\mathbf{k}\right) =$ (_______ , _____ , _____) .

(77) Let $\mathbf{G}(t) = \dfrac{t^2+3t-10}{t^2-t-2}\mathbf{i} + \sin\dfrac{\pi t}{2}\mathbf{j} + \dfrac{\sin(t-2)}{t-2}\mathbf{k}$. Then $\displaystyle\lim_{t\longrightarrow 2}\mathbf{G}(t) =$ (_______ , _____ , _____) .

(78) $\displaystyle\lim_{t\longrightarrow 0}\left(\frac{1-e^{3t^2}}{t^2}, \frac{4\arctan(1+t^2)-\pi}{t^2}, (1+t)^{3/t}\right) =$ (_____ , _____ , _____).

(79) Let $\mathbf{f}(t) = \begin{cases} (t^2+t+1)\mathbf{i} + \dfrac{\sin t}{t}\mathbf{j} + e^{t^2}\mathbf{k} & \text{if } t < 0 \\ \cos(\pi t)\mathbf{i} + (t+1)\mathbf{j} + e^t\mathbf{k} & \text{if } 0 \le t < 1 \\ (2t-3)\mathbf{i} + (t^2-2)\mathbf{j} + e^{\sqrt{t}}\mathbf{k} & \text{if } t \ge 1 \end{cases}$. The function $\mathbf{f}$ is continuous at every real number except $t =$ _____ .

(80) If $\mathbf{r}(t) = \left(\dfrac{1}{t-1}, \ln(t+1), \sqrt{5-t}\right)$, then the domain of $\mathbf{r}$ is (_____ , _____) $\cup$ (_____ , _____] .

(81) Let $\mathbf{F}(t) = 3t\,\mathbf{j} + t^{-1}\,\mathbf{k}$ and $\mathbf{G}(t) = 5t\,\mathbf{i} + \sqrt{10-t}\,\mathbf{j}$. Then the domain of $\mathbf{F}+\mathbf{G}$ is (_____ , _____) $\cup$ (_____ , _____] .

(82) Let $\mathbf{F}(t) = t^2\,\mathbf{i} + t\,\mathbf{j} + 2t\,\mathbf{k}$ and $\mathbf{G}(t) = \dfrac{1}{t+2}\,\mathbf{i} + (t+4)\,\mathbf{j} - \sqrt{-t}\,\mathbf{k}$. Then the domain of $\mathbf{F}\times\mathbf{G}$ is (_____ , _____) $\cup$ (_____ , _____] .

31.3. Problems

(1) Show that the equations

$$\begin{cases} x = \dfrac{1-t^2}{1+t^2} \\[2ex] y = \dfrac{2t}{1+t^2} \end{cases}$$

parametrize the entire unit circle except for the point $(-1, 0)$. (It is quite easy to show that for any number t these equations produce a point lying on the unit circle. What is not so easy is to show that if (a, b) is a point on the unit circle other than $(-1, 0)$, then there exists a number t such that $x(t) = a$ and $y(t) = b$.)

(2) Show that if an object moves with constant velocity, then its angular momentum (with respect to the origin) is constant. (The ANGULAR MOMENTUM $\mathbf{L}$ about the origin is defined by $\mathbf{L}(t) = \mathbf{r}(t) \times m\mathbf{v}(t)$, where m is the mass of the object, $\mathbf{r}$ is its position, and $\mathbf{v}$ its velocity.)

(3) Explain why the curve $\mathbf{r}(t) = (2\cos 2t)\,\mathbf{i} + (3\cos t)\,\mathbf{j}$ for $0 \leq t \leq 4\pi$ is not a smooth curve. Include a sketch of the curve, and on your sketch identify the points at which the tangent vector vanishes.

(4) Fred is riding a ferris wheel at an amusement park. The radius of the wheel is 20 feet. His friends Sally and Leslie are on the ground. Leslie sees that Sally is standing in the plane of the ferris wheel 75 feet to the right of the bottom of the wheel. Leslie also notices that the wheel is making 5 counterclockwise revolutions each minute. Sally throws a baseball to Fred as he moves upwards. At the instant Fred is halfway up Sally throws the ball with a speed of 60 ft/sec at an angle of 60° above the horizontal. The point at which the ball leaves Sally's hand is the same distance from the ground as the bottom of the wheel. How close to Fred does the ball get? Give your answer to the nearest inch.

(5) Let $\mathbf{f}$ be a differentiable curve. Show that $\|\,\mathbf{f}(t)\|$ is constant if and only if $\mathbf{f}(t) \perp D\mathbf{f}(t)$ for all t.

(6) The position of a particle at time t is given by $\mathbf{r}(t) = \dfrac{1-2t^2}{1+4t^2}\,\mathbf{i} + \dfrac{4t}{1+4t^2}\,\mathbf{j} + \dfrac{2t\sqrt{3t^2-1}}{1+4t^2}\,\mathbf{k}$ for all $t > 1$. Without actually calculating the velocity vector $\mathbf{v}(t)$ explain how you know that it must be perpendicular to the position vector $\mathbf{r}(t)$ for every $t > 1$.

31.4. Answers to Odd-Numbered Exercises

(1) x^2, -1, 1

(3) $1 - 2x^2$

(5) $\ln x$

(7) 10, -8, 2, -4

(9) 0, 0, 1, 1, 1, 0

(11) $1 - y^2$

(13) $4t$, $4t - 1$, $3 - 4t$, $4 - 4t$

(15) (a) 0, -2, 2, 0
(b) 0, -2, 0

(17) (a) 4, 0, 0, 2
(b) 4, 0, 0

(19) (a) -1, 0, 2, 3
(b) -1, 0, 0

(21) (a) 0, 1, 1, 0
(b) 0, 1, 0

(23) (a) $\frac{4}{3}$, $-\frac{4}{3}$
(b) $\frac{4}{3}$, $\frac{4}{3}$
(c) 1, -1, 0, $\frac{4}{3}$
(d) 1, -3, 0, 4

(25) 8, -4, 4

(27) 1, 2

(29) 3, 24, π

(31) 2, 3, 8

(33) $16a$

(35) 1, 2

(37) 320

(39) 9π

(41) $\frac{3\pi}{2}$

(43) 3, 1

(45) 2

(47) 3, 2

(49) 15

(51) 1, 2

(53) 12

(55) $\sqrt{3}$

(57) 1, 2

(59) $\dfrac{\pi}{2}$

(61) 2, −1, 0, 3

(63) 1, 2, −2, 5

(65) 20

(67) (a) 0
(b) $\dfrac{7}{2}$, $\dfrac{7}{2}$, −5
(c) 32

(69) −4, −6, 3

(71) 1, −3, −3

(73) (a) −1, 4, 1
(b) −1
(c) −4, −2, −4
(d) 4, 2, 4

(75) −1, $\dfrac{1}{e}$

(77) $\dfrac{7}{3}$, 0, 1

(79) 1

(81) $-\infty$, 0, 0, 10

CHAPTER 32

ACCELERATION AND CURVATURE

32.1. Background

Topics: velocity, speed, acceleration, tangential component of the acceleration (denoted by a_T), normal component of the acceleration of a curve (denoted by a_N), arclength, parametrization by arclength, curvature.

32.1.1. Definition. Let C be a curve parametrized by arclength. The CURVATURE κ of C is the rate of change of the unit tangent vector with respect to arclength; that is, $\kappa(s) = \left\| \dfrac{d\mathbf{T}}{ds} \right\|$ where $\mathbf{T}$ is the unit tangent vector and s is arclength.

32.2. Exercises

(1) A particle P moves along a curve C. The position of P at time t is given by $\mathbf{r}(t) = 6t\,\mathbf{i} + \sqrt{2}\,t^2\,\mathbf{j} + 5\,\mathbf{k}$.

(a) When $t = 6$ the tangential component of the acceleration of P is $\dfrac{a}{3}$ where $a =$ _____ .

(b) When $t = 6$ the normal component of the acceleration of P is $\dfrac{a\sqrt{a}}{b}$ where $a =$ _____ and $b =$ _____ .

(c) At the point $\mathbf{r}(6)$ the curvature of C is $\dfrac{\sqrt{2}}{a}$ where $a =$ ___________ .

(2) A particle P moves along a curve C. The position of P at time t is given by $\mathbf{r}(t) = 3t\,\mathbf{i} + \frac{1}{2}t^2\,\mathbf{j} - 7\,\mathbf{k}$.

(a) At the point $\mathbf{r}(3)$ the equation of the tangent line to the curve C is $\mathbf{R}(u) = (a + 3u)\,\mathbf{i} + (\frac{9}{2} + bu)\,\mathbf{j} + (c + du)\,\mathbf{k}$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $d =$ _____ .

(b) When $t = 3$ the tangential component of acceleration of P is $\dfrac{1}{\sqrt{a}}$ where $a =$ _____.

(c) When $t = 3$ the normal component of acceleration of P is $\dfrac{1}{\sqrt{a}}$ where $a =$ _____.

(d) At the point $\mathbf{r}(3)$ the curvature of C is $\dfrac{1}{a\sqrt{2}}$ where $a =$ ________ .

(3) The motion of a particle is given by $\mathbf{r}(t) = 6t^2\,\mathbf{i} + (\frac{4}{3}t^3 - 9t)\,\mathbf{j}$. Then at time t its

(a) velocity is $a(t)\,\mathbf{i} + b(t)\,\mathbf{j}$ where $a(t) =$ ________ and $b(t) =$ ___________ .

(b) speed is ____________________ .

(c) acceleration is $a(t)\,\mathbf{i} + b(t)\,\mathbf{j}$ where $a(t) =$ _____ and $b(t) =$ _____ .

(d) tangential component of acceleration is ________ .

(e) normal component of acceleration is ________ .

(f) curvature is $\dfrac{12}{(a(t))^2}$ where $a(t) =$ ___________________ .

(4) A particle P moves along a curve C. Its position at time t is given by $\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k}$. Then

(a) the velocity of P at time t is _____ $\mathbf{i}$ + ________ $\mathbf{j}$ + ______ $\mathbf{k}$;

(b) the speed of P at time t is $a + bt^p$ where $a =$ _____ , $b =$ _____ , and $p =$ _____ ;

(c) the tangential component of the acceleration of P at time t is ________ ;

(d) the normal component of the acceleration of P at time t is _____ ;

(e) the unit tangent vector to C at time t is $\dfrac{2}{a(t)}\,\mathbf{i} + \dfrac{b(t)}{a(t)}\,\mathbf{j} + \dfrac{c(t)}{a(t)}\,\mathbf{k}$ where $a(t) =$ ___________ , $b(t) =$ _________ , and $c(t) =$ _________ ;

(f) the unit normal vector to C at time t is $\dfrac{-2t}{a(t)}\,\mathbf{i} + \dfrac{b(t)}{a(t)}\,\mathbf{j} + \dfrac{c(t)}{a(t)}\,\mathbf{k}$ where $a(t) =$ ___________ , $b(t) =$ ______________ , and $c(t) =$ _________ ; and

(g) the curvature of C at time t is $\dfrac{2}{a(t)}$ where $a(t) =$ ________________ .

(5) A particle P moves along a curve C. Its position at time t is given by

$$\mathbf{r}(t) = \frac{4}{5}\cos t\,\mathbf{i} + (1 - \sin t)\,\mathbf{j} - \frac{3}{5}\cos t\,\mathbf{k}.$$

Then

(a) the velocity of P at time t is $-\dfrac{4}{5}a(t)\,\mathbf{i} + b(t)\,\mathbf{j} + c\,a(t)\,\mathbf{k}$ where $a(t) =$ ___________ , $b(t) =$ _________ , and $c =$ ___________ ;

(b) the speed of P at time t is _____ ;

(c) the tangential component of the acceleration of P at time t is _____ ;
(d) the normal component of the acceleration of P at time t is _____ ;
(e) the unit tangent vector to C at time t is $-\frac{4}{5}a(t)\,\mathbf{i}+b(t)\,\mathbf{j}+c\,a(t)\,\mathbf{k}$ where $a(t) =$ ________ , $b(t) =$ _________ , and $c =$ ___________ ; and
(f) the unit normal vector to C at time t is $-\frac{4}{5}a(t)\,\mathbf{i}+b(t)\,\mathbf{j}+c\,a(t)\,\mathbf{k}$ where $a(t) =$ ________ , $b(t) =$ _________ , and $c =$ ___________ .

(6) The motion of a particle is given by $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j} + \sqrt{2}t\,\mathbf{k}$. Then at time t its
(a) velocity is $a(t)\,\mathbf{i} + b(t)\,\mathbf{j} + c(t)\,\mathbf{k}$ where $a(t) =$ ________ , $b(t) =$ ___________ , and $c(t) =$ ______ .
(b) speed is ____________________ .
(c) acceleration is $a(t)\,\mathbf{i}+b(t)\,\mathbf{j}+c(t)\,\mathbf{k}$ where $a(t) =$ _____ , $b(t) =$ _____ , and $c(t) =$ ______ .
(d) tangential component of acceleration is ____________ .
(e) normal component of acceleration is ______ .

(7) A particle P moves along a curve C in a plane. Its position at time t is given by

$$\mathbf{r}(t) = (\cos t + t\sin t)\,\mathbf{i} + (\sin t - t\cos t)\,\mathbf{j}$$

for all $t \geq 0$.
(a) The unit tangent to C at time t is $\mathbf{T}(t) = a(t)\,\mathbf{i} + b(t)\,\mathbf{j}$ where $a(t) =$ ________ and $b(t) =$ ________ .
(b) The unit normal to C at time t is $\mathbf{N}(t) = a(t)\,\mathbf{i} + b(t)\,\mathbf{j}$ where $a(t) =$ ________ and $b(t) =$ ________ .
(c) The tangential component of the acceleration of P at time t is $a_{\mathbf{T}}(t) =$ _____ .
(d) The normal component of the acceleration of P at time t is $a_{\mathbf{N}}(t) =$ _____ .
(e) The curvature of C at time t is $\kappa(t) = \dfrac{1}{a(t)}$ where $a(t) =$ _____ .

(8) The curvature of the curve $x^2 + 4xy - 2y^2 = 10$ at the point $(2, 1)$ is $\dfrac{a}{2\sqrt{b}}$ where $a =$ _____ and $b =$ _____ .

(9) The curvature of the ellipse $x^2 + 4y^2 = 8$ at the point $(2, 1)$ is $\dfrac{a}{b\sqrt{b}}$ where $a =$ _____ and $b =$ _____ .

(10) The curvature of the spiral of Archimedes $r = \theta$ at the point where $\theta = 2\sqrt{2}$ is $\dfrac{a}{b}$ where $a =$ ________ and $b =$ ________ .

(11) The curvature of the curve $x^3 - xy + y^2 = 7$ at the point $(1, 3)$ is $\dfrac{a}{b}$ where $a =$ _____ and $b =$ _____ .

(12) The curvature of the curve $\mathbf{r}(t) = (t - \sin t)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j}$ at the point $\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\mathbf{i} + \frac{1}{2}\,\mathbf{j}$ is $\dfrac{1}{a}$ where $a =$ _____ .

(13) The curvature of the logarithmic spiral $r = e^{a\theta}$ at an arbitrary point is $\left(e^b\sqrt{c}\right)^{-1}$ where $b =$ ________ and $c =$ ____________ .

(14) The curvature of the curve $\mathbf{r}(t) = t^2\,\mathbf{i} + (t^4 - t^5)\,\mathbf{j}$ at the point $\mathbf{r}(0)$ is ______________ .

(15) The curvature of the curve $2x^2 + 2xy - y^2 = 2$ at the point $(1, 0)$ is $\dfrac{a}{b\sqrt{b}}$ where $a =$ _____ and $b =$ _____ .

(16) The curvature of the curve $y = \ln x$ is greatest at the point $x = \dfrac{a}{\sqrt{b}}$ where $a =$ _____ and $b =$ _____ . At that point the curvature is $\dfrac{b}{c\sqrt{c}}$ where $c =$ _____ .

(17) At the point where the curve $y = 3x - x^3$ attains a local maximum its curvature is _____ .

(18) The curvature of the cardioid $r = 2(1 - \cos\theta)$ is $\dfrac{a}{b}r^p$ where $a =$ _____ , $b =$ _____ , and $p =$ ________ .

32.3. Problems

(1) Explain carefully why the principal unit normal vector $\mathbf{N}$ to a curve is perpendicular to the unit tangent vector $\mathbf{T}$.

(2) Consider a parametrized curve in $\mathbb{R}^3$ with velocity $\mathbf{v}$ and acceleration $\mathbf{a}$. Give a careful and informative derivation of the formula for its curvature
$$\kappa = \frac{\|\,\mathbf{v}\times\mathbf{a}\|}{\|\,\mathbf{v}\|^3}$$
from the *definition* of curvature (see 32.1.1). *Hint.* Derive the formulas
$$\mathbf{v} = \frac{ds}{dt}\,\mathbf{T} \qquad \text{and} \qquad \mathbf{a} = \frac{d^2s}{dt^2}\,\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}.$$
From these conclude that
$$\|\,\mathbf{v}\times\mathbf{a}\| = \kappa\left(\frac{ds}{dt}\right)^3.$$

(3) Let $\mathbf{r}(t) = t\,\mathbf{i} + \frac{2}{3}(t^2+1)\,\mathbf{j}$ and $P = \mathbf{r}(1)$.
   (a) Make a careful sketch of the graph of the range of $\mathbf{r}$ for $0 \le t \le 2$.
   (b) Add to your sketch the vector $\mathbf{a}(1)$, with its initial point at $\mathbf{r}(1)$. (Include an explanation of how you found this vector.)
   (c) Add to your sketch the vectors $a_T(1)\mathbf{T}(1)$ and $a_N(1)\mathbf{N}(1)$ with initial point $\mathbf{r}(1)$. (Include an explanation of how you found these vectors.)
   (d) Compute the arc length along the curve from 0 to an arbitrary point t. (Include an explanation *but not the details* of the calculation that produced your answer.)
   (e) Compute the curvature of the curve at $t = 1$. (Include an explanation *but not the details* of the calculation that produced your answer.)

(4) A parametrized curve $\mathbf{r}$ in $\mathbb{R}^3$ is a UNIT SPEED CURVE if its speed $\|\,\mathbf{v}(t)\|$ is 1 for each t. Recall that the length of arc between the points $\mathbf{r}(0)$ and $\mathbf{r}(t)$ on such a curve is $s = \int_0^t \|\,\mathbf{v}(\tau)\|\,d\tau = t$; thus we may say that a unit speed curve has been PARAMETRIZED BY ARCLENGTH.

In this problem we wish to attach a moving orthonormal frame of coordinates to a curve in such a way that one of the axes always points in the direction of the unit tangent vector $\mathbf{T}$ and a second axis points in the direction of the unit normal $\mathbf{N}$. (The third axis is called $\mathbf{B}$, for *unit binormal.*) When a geometric curve (as opposed to a parametrized one) has been parametrized by arclength (or when a parametrized curve has been reparametrized by arclength), the business of deriving properties of the moving $\mathbf{T}-\mathbf{N}-\mathbf{B}$ frame is fairly straightforward. That is the task here. The idea is to write up a coherent introduction to moving frames. Include the material in items (a)–(i) below, but do *not* break up your exposition into chunks called (a), (b), (c), *etc.*

   (a) Let $a, b > 0$. Find a constant c with the property that the helix $\mathbf{r}$ defined by
   $$\mathbf{r}(t) = \big(a\cos(ct), a\sin(ct), bct\big)$$
   is a unit speed curve.
   (b) For an arbitrary (unspecified) unit speed curve define for each t
   $$\mathbf{T}(t) = \mathbf{v}(t),$$
   $$\mathbf{N}(t) = \frac{\mathbf{a}(t)}{\|\mathbf{a}(t)\|}, \quad \text{and}$$
   $$\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t).$$

Show that for each t the vectors $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ are mutually perpendicular unit vectors.

(c) Make a careful sketch of the unit speed helix in part (a) showing the vectors $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ at several different points.

(d) If $\mathbf{r}$ is a unit speed curve define its CURVATURE $\kappa(t)$ at t by $\kappa(t) = \|\mathbf{a}(t)\|$. Show that $\mathbf{T}'(t) = \kappa(t)\mathbf{N}(t)$ for each t.

(e) Let $\mathbf{r}$ be a unit speed curve. Show that for each t there exists a number $\tau(t)$ (which we call the TORSION of $\mathbf{r}$ at t) such that $\mathbf{B}'(t) = -\tau(t)\mathbf{N}(t)$. (Don't worry about the sign; it's just a convention. Many people use the opposite one.)

(f) Let $\mathbf{r}$ be a unit speed curve. Show that

$$\mathbf{N}'(t) = -\kappa(t)\mathbf{T}(t) + \tau(t)\mathbf{B}(t)$$

for every t.

(g) Let $A(t)$ be the matrix $\begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix}$. What vector do we obtain by letting the matrix $A(t)$ act on the vector $\big(\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\big)$?

(h) For a unit speed curve derive the formula

$$\tau(t) = \frac{[\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)]}{(\kappa(t))^2}$$

where $[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ is the scalar triple (or box) product of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.

(i) Compute the curvature and the torsion of the unit speed helix in part (a). Use these computations to illustrate the following two assertions. (i) Curvature measures how far a curve is from being a straight line.
(ii) Torsion measures how far a curve is from being a plane curve.

(5) For the path traversed by a particle discuss the appropriate units for curvature (in either the English or metric system).

32.4. Answers to Odd-Numbered Exercises

(1) (a) 8
  (b) 2, 3
  (c) 486

(3) (a) $12t$, $4t^2 - 9$
  (b) $4t^2 + 9$
  (c) 12, $8t$
  (d) $8t$
  (e) 12
  (f) $4t^2 + 9$

(5) (a) $\sin t$, $-\cos t$, $\dfrac{3}{5}$
  (b) 1
  (c) 0
  (d) 1
  (e) $\sin t$, $-\cos t$, $\dfrac{3}{5}$
  (f) $\cos t$, $\sin t$, $\dfrac{3}{5}$

(7) (a) $\cos t$, $\sin t$
  (b) $-\sin t$, $\cos t$
  (c) 1
  (d) t
  (e) t

(9) 4, 5

(11) 6, 5

(13) $a\theta$, $1 + a^2$

(15) 6, 5

(17) 6

Part 9

MULTIPLE INTEGRALS

CHAPTER 33

DOUBLE INTEGRALS

33.1. Background

Topics: Riemann sums, double integrals, iterated integrals, double integrals in polar coordinates, improper double integrals, moment of a plane region about a line, density and mass of a plane lamina, center of gravity (or mass).

33.1.1. Theorem (Fubini's theorem for double integrals)**.** *If a real valued function f is continuous on the rectangle $R = [a, b] \times [c, d]$, then*

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx\,.$$

33.2. Exercises

(1) Let $f(x,y) = 2y - x^2$ and R be the region in the xy-plane bounded by the coordinate axes and the line $x + y = 1$. Then $\int_R f\,dA = \dfrac{1}{a}$ where $a =$ _____ .

(2) Let $f(x,y) = x^2 + y^2$. Then the integral of f over the region bounded by the straight line $y = x$ and the parabola $y = x^2$ is $\dfrac{3}{a}$ where $a =$ __________ .

(3) Let f be a continuous function of two variables. Then

$$\int_0^2 \int_y^{\sqrt{8-y^2}} f(x,y)\,dx\,dy = \int_0^a \int_b^{g(x)} f(x,y)\,dy\,dx + \int_a^c \int_b^{h(x)} f(x,y)\,dy\,dx$$

where $a =$ _____ , $b =$ _____ , $c =$ ______ , $g(x) =$ _____ , and $h(x) =$ _____________ .

(4) The volume of the region in the first octant bounded by the surfaces $z = x^2 y$, $y^2 = x$, and $y = x^2$ is given by the double integral

$$\int_a^b \int_{g(x)}^{h(x)} j(x,y)\,dy\,dx$$

where $a =$ _____ , $b =$ _____ , $g(x) =$ ______ , $h(x) =$ ______ , and $j(x,y) =$ __________ .

(5) The volume of the region satisfying $x^2 + y^2 \le 4$ and $0 \le z \le 2x$ can be expressed as the iterated integral

$$\int_a^b \int_{f(y)}^{g(y)} 2x\,dx\,dy$$

where $a =$ _____ , $b =$ _____ , $f(y) =$ _____ , and $g(y) =$ _____________ or as the iterated integral

$$\int_c^d \int_{k(x)}^{l(x)} 2x\,dy\,dx$$

where $c =$ _____ , $d =$ _____ , $k(x) =$ ___________ , and $l(x) =$ ___________ .

(6) Let f be a continuous function of two variables. Then

$$\int_0^4 \int_{\frac{1}{3}x}^{2x} f(x,y)\,dy\,dx + \int_4^6 \int_{\frac{1}{3}x}^{-3x+20} f(x,y)\,dy\,dx$$
$$= \int_0^a \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy + \int_a^b \int_{g(y)}^{j(y)} f(x,y)\,dx\,dy$$

where $a =$ _____ , $b =$ _____ , $g(y) =$ ________ , $h(y) =$ ________ , and $j(y) =$ __________________ .

(7) Let f be a continuous function of two variables. Then

$$\int_0^2 \int_0^{\sqrt{3}\,y} f(y,z)\,dz\,dy + \int_2^4 \int_0^{\sqrt{16-y^2}} f(y,z)\,dz\,dy = \int_0^a \int_{g(z)}^{h(z)} f(y,z)\,dy\,dz$$

where $a =$ ______ , $g(z) =$ __________ , and $h(z) =$ ________________ .

(8) Let f be a continuous function of two variables. Then

$$\int_0^3 \int_{\frac{1}{6}x^2}^{\frac{1}{2}x} f(x,y)\,dy\,dx = \int_a^b \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy$$

where $a =$ _____ , $b =$ ________ , $g(y) =$ ________ , and $h(y) =$ __________ .

(9) $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \cos x^3 \, dx \, dy = \frac{a}{b} \sin a$ where $a =$ _____ and $b =$ _____ .

(10) $\displaystyle\frac{1}{100} \int_0^{10} \int_0^{10} \min\{x, y\} \, dx \, dy = \frac{a}{b}$ where $a =$ _____ and $b =$ _____ . [Note: $\min\{x, y\}$ denotes the smaller of the numbers x and y.]

(11) $\displaystyle\int_0^2 \int_0^1 e^{\max\{x^2, 4y^2\}} \, dy \, dx = \frac{a}{b}(e^p - a)$ where $a =$ _____ , $b =$ _____ , and $p =$ _____ . [Note: $\max\{u, v\}$ denotes the larger of the numbers u and v.]

(12) $\displaystyle\int_0^1 \int_{\arctan y}^{\pi/4} \sec^5 x \, dx \, dy = \frac{1}{a}(b\sqrt{2} - 1)$ where $a =$ _____ and $b =$ _____ .

(13) $\displaystyle\int_0^1 \int_{2y}^2 \frac{x}{1 + x^3} \, dx \, dy = \frac{a}{b} \ln b$ where $a =$ _____ and $b =$ _____ .

(14) $\displaystyle\int_{-1}^1 \int_{|y|}^1 (x + y)^2 \, dx \, dy = \frac{a}{b}$ where $a =$ _____ and $b =$ _____ .

(15) $\displaystyle\int_0^1 \int_{-1/2}^2 (x + y) e^{x^2 + 2xy} \, dy \, dx = \frac{1}{a} e^p + \frac{b}{2}$ where $a =$ _____ , $p =$ _____ , and $b =$ _____ .

(16) Let R be the rectangle $[0, 3] \times [0, 2]$ in the xy-plane and let P be the partition of this rectangle into squares induced by partitioning both $[0, 3]$ and $[0, 2]$ into subintervals of length one. Estimate the volume of the solid lying above R and below the surface $z = x^2y + 2x$ by calculating the Riemann sum associated with the partition P and choosing the upper right corner of each square as a sample point.

Answer: ______ .

(17) $\displaystyle\int_0^1 \int_0^\pi x^2 \cos xy \, dx \, dy =$ _____ .

(18) If $R = [0, 3] \times [4, 5]$, then $\displaystyle\iint_R \frac{xy}{\sqrt{x^2 + y^2 - 16}} \, dx \, dy = a(\sqrt{b} - 1)$ where $a =$ ______ and $b =$ _____ .

(19) Express $\displaystyle\int_0^2 \int_2^{\sqrt{8 - y^2}} xy \, dx \, dy$ as an integral in polar coordinates.

Answer: $\displaystyle\int_a^b \int_{f(\theta)}^c r^p \sin\theta \cos\theta \, dr \, d\theta$ where $a =$ _____ , $b =$ ________ , $c =$ _____ , $f(\theta) =$ _____________ , and $p =$ _____ .

(20) The integral of $g(x, y) = \cos(x^2 + y^2)$ over the unit disk is $a \sin b$ where $a =$ _____ and $b =$ _____ . *Hint.* Polar coordinates.

(21) Let R be the region defined by $1 \leq x^2 + y^2 \leq 2$, and $x \geq 0$. The integral of $f(x, y) = x^2$ over this region is $\dfrac{a}{8}\pi$ where $a =$ _____ .

(22) Let D be the closed unit disk in the plane. Then $\displaystyle\iint_D e^{-(x^2 + y^2)} \, dx \, dy = \left(1 - \frac{1}{a}\right) b$ where $a =$ _____ and $b =$ _____ .

(23) In polar coordinates $\displaystyle\int_0^3 \int_{2y}^{\sqrt{45-y^2}} \left(x^4 + x^2y^2\right) dx\, dy$ can be written as $\displaystyle\int_0^a \int_0^b r^m \sin^n \theta \cos^p \theta \, dr\, d\theta$ where $a =$ ______________ , $b =$ _________ , $m =$ _____ , $n =$ _____ , and $p =$ _____ .

(24) The volume of the solid bounded below by the xy-plane, above by the cone $x^2 + y^2 = z^2$, and on the sides by the cylinder $x^2 + y^2 = 2y$ is $\dfrac{a}{9}$ where $a =$ ________ .

(25) The area of the region bounded by the lemniscate $\left(x^2+y^2\right)^2 = 2a^2(x^2-y^2)$ is _________ .

(26) $\displaystyle\int_0^1 \int_y^1 \frac{1}{(1+x^2+y^2)^{3/2}}\, dx\, dy = \frac{\pi}{a}$ where $a =$ ________ .

(27) The volume of the solid bounded below by the $xy-$plane, above by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} + \dfrac{z^2}{b^2} = 1$, and on the sides by the cylinder $x^2 + y^2 - ay = 0$ is $\frac{2}{9}a^2 b(m\pi - n)$ where $m =$ _____ and $n =$ _____ .

(28) Let R be the region bounded by the limaçon $r = 2 + \cos\theta$. Then $\displaystyle\iint_R (x^2+y^2)^{\frac{1}{2}}\, dA = \frac{a}{3}\pi$ where $a =$ ________ .

(29) Let $\alpha > 0$ and $0 < \beta < \frac{\pi}{2}$. Then $\displaystyle\int_0^{\alpha \sin\beta} \int_{y\cot\beta}^{\sqrt{\alpha^2-y^2}} \ln(x^2+y^2)\, dxdy = \alpha^p \beta^q (\ln r - s)$ where $p =$ _____ , $q =$ _____ , $r =$ _____ , and $s =$ _____ .

(30) Rotate that portion of the curve $y = \dfrac{1}{x}$ where $x \geq 1$ about the x-axis. Then the volume of the resulting solid is _____ and its surface area is _____ .

(31) $\displaystyle\int_0^\infty \frac{\sin t}{t}\, dt = a$ where $a =$ ________.

Hint. Consider the iterated integral $\displaystyle\int_0^\infty \int_0^\infty e^{-st} \sin t \, ds\, dt$. You may use the fact that in this problem the order of integration of the improper integrals may be reversed.

(32) Let $f(x,y) = \dfrac{x^2-y^2}{(x^2+y^2)^2}$. Then $\displaystyle\int_0^1 \int_0^1 f(x,y)\, dx\, dy =$ _________ and $\displaystyle\int_0^1 \int_0^1 f(x,y)\, dy\, dx =$ _________ . This result does not violate Fubini's theorem because ____ . *Hint.* Although these integrals can be evaluated by hand, they take some effort. Use a CAS.

(33) Let D be the region in the xy-plane lying above the line $y = x$ and below the parabola $y = -x^2 + 2$. Then the moments of D about the x-axis and the y-axis are given by $M_x = \dfrac{a}{5}$ where $a =$ _____ and $M_y = \dfrac{b}{4}$ where $b =$ ______ . The centroid is at $\left(-\dfrac{1}{c}, \dfrac{2}{d}\right)$ where $c =$ _____ and $d =$ _____.

(34) A plane lamina L is bounded by the lines $y = x$, $y = 2 - x$, and the x-axis; its density function ρ is given by $\rho(x,y) = 2x + y + 1$ grams per square centimeter. Then the mass of L is _________ grams and its center of mass is located at the point $\left(\dfrac{a}{10}, \dfrac{b}{20}\right)$ where $a =$ ______ and $b =$ _____ .

(35) Let L be a plane lamina in the shape of the upper half of a circular disk of radius 3 centered at the origin. The density of the lamina at any point is proportional to the square of the

distance of the point from the center of the circle. Then the center of mass of L is located at the point $\left(a, \dfrac{b}{5\pi}\right)$ where $a =$ ____ and $b =$ ________ .

(36) Let E be the solid region determined by the surfaces $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 2 - x^2 - y^2$. The base of this region is the square $S = [0,1] \times [0,1]$ in the xy-plane. Estimate the volume of E by using the *midpoint rule for double integrals* to approximate the integral $\iint_S f(x,y)\,dxdy$ where $f(x,y) = 2 - x^2 - y^2$. For this approximation divide S into nine squares each having side $\frac{1}{3}$. Calculate the corresponding Riemann sum evaluating f at the center of each square. According to this estimate the volume of E is approximately $\dfrac{a}{54}$ where $a =$ ____ . The error made in this approximation (that is, the absolute value of the difference between the approximate volume and the true volume) is $\dfrac{b}{54}$ where $b =$ ____ .

(37) Let E be the solid region determined by the inequalities $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 2 - x - y$. The base of this region is the square $S = [0,1] \times [0,1]$ in the xy-plane. Estimate the volume of E by using the *midpoint rule for double integrals* to approximate the integral $\iint_S f(x,y)\,dxdy$ where $f(x,y) = 2 - x - y$. For this approximation divide S into nine squares each having side $\frac{1}{3}$. Calculate the corresponding Riemann sum evaluating f at the center of each square.According to this estimate the volume of E is approximately ______ . The error made in this approximation (that is, the absolute value of the difference between the approximate volume and the true volume) is ______ .

(38) If $R = [0,4] \times [0, \frac{\pi}{6}]$, then $\displaystyle\iint_R x\cos y\,dx\,dy =$ ____ .

(39) If $R = [0,4] \times [0, \frac{\pi}{6}]$, then $\displaystyle\iint_R x\cos(xy)\,dx\,dy = \frac{a}{\pi}$ where $a =$ ____ .

33.3. Problems

(1) Let $f(x,y) = \dfrac{1}{2x-y}$ over the region R where $0 \le y \le 2x$ and $x \le 3$. Determine whether the function f is integrable over R. Find $\iint\limits_R f(x,y)\,dA$ if it exists.

(2) Let $f(x,y) = \dfrac{1}{\sqrt{xy}}$ over the region $R = [0,4] \times [0,4]$. Show that f is integrable over R and find $\iint\limits_R f(x,y)\,dA$.

(3) Let $f(x,y) = \dfrac{1}{(2x-y)^2}$ over the region R where $0 \le y \le 2x$ and $x \le 3$. Determine whether the function f is integrable over R. Find $\iint\limits_R f(x,y)\,dA$ if it exists.

(4) Let $f(x,y) = \dfrac{1}{\sqrt{2x-y}}$ over the region R where $0 \le y \le 2x$ and $x \le 3$. Determine whether the function f is integrable over R. Find $\iint\limits_R f(x,y)\,dA$ if it exists.

(5) Let $f(x,y) = \dfrac{1}{\sqrt{|x| \cdot |y|}}$ over the region $R = [-1,4] \times [-1,4]$. Explain carefully why $\iint\limits_R f(x,y)\,dA$ is improper. Show that f is integrable over R and find $\iint\limits_R f(x,y)\,dA$.

(6) Let $f(x,y) = \dfrac{1}{\sqrt{|x^2-y|}}$ over the region $R = [0,1] \times [0,1]$. Explain carefully why $\iint\limits_R f(x,y)\,dA$ is improper. Show that f *is* integrable over R and find $\iint\limits_R f(x,y)\,dA$.

(7) Derive the formula $\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$. *Hint.* Do exercise 22.

33.4. Answers to Odd-Numbered Exercises

(1) 4

(3) $2,\ 0,\ 2\sqrt{2},\ x,\ \sqrt{8-x^2}$

(5) $-2,\ 2,\ 0,\ \sqrt{4-y^2},\ 0,\ 2,\ -\sqrt{4-x^2},\ \sqrt{4-x^2}$

(7) $2\sqrt{3},\ \dfrac{z}{\sqrt{3}},\ \sqrt{16-z^2}$

(9) $1,\ 3$

(11) $1,\ 2,\ 4$

(13) $1,\ 3$

(15) $4,\ 5,\ -3$

(17) π

(19) $0,\ \dfrac{\pi}{4},\ 2\sqrt{2},\ 2\sec\theta,\ 3$

(21) 3

(23) $\arctan\frac{1}{2},\ 3\sqrt{5},\ 5,\ 0,\ 2$

(25) $2a^2$

(27) $3,\ 4$

(29) $2,\ 1,\ \alpha,\ \dfrac{1}{2}$

(31) $\dfrac{\pi}{2}$

(33) $9,\ -9,\ 2,\ 5$

(35) $0,\ 24$

(37) $1,\ 0$

(39) 9

CHAPTER 34

SURFACES

34.1. Background

Topics: quadric surfaces, spheres, ellipsoids, elliptic paraboloids, hyperboloids of 1 and 2 sheets, parametrized surface, level surface.

34.2. Exercises

(1) The surface $z = \frac{1}{3}\sqrt{36(x^2-1)+4y^2}$ is the upper half of which quadric surface?

Answer. ____

The curve obtained when the surface is cut by the plane $z = 2\sqrt{3}$ is an ellipse whose major axis is of length ______ and whose minor axis is of length _____ .

(2) The surface $z = 2\sqrt{\frac{1}{2}(x^2+2)+\frac{1}{3}y^2}$ is the upper half of which quadric surface?

Answer. ________________________________.

The curve obtained when the surface is cut by the plane $z = 2\sqrt{3}$ is an ellipse whose major axis is of length ________ and whose minor axis is of length _____ .

(3) Let $w = f(x,y,z) = \left[\left(\dfrac{y}{\sqrt{x}}\right)^2 + \left(\dfrac{2z}{\sqrt{x}}\right)^2\right]^{1/2}$. The level surface which results when the value of w is set to 4 is a(n)

____

which opens along the positive _____ -axis.

(4) Let $w = f(x,y,z) = 3 - 6z + \sqrt{9x^2+4y^2}$. The level surface at $w = 3$ is the upper half of a(n) _________________________ which opens along the positive _____ -axis.

(5) Consider the surface in $\mathbb{R}^3$ defined by $f(x,y) = \dfrac{2xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Then the plane $z = c$ intersects the surface for all values of c in the interval $[a,b]$ where $a =$ ______ and $b =$ ______ .

(6) Let $f(x,y) = \dfrac{1+\sqrt{y^2-4}}{3+\sqrt{25-x^2}}$. The domain of f is the set of all (x,y) in $\mathbb{R}^2$ such that $x \in [$ ___ , ___ $]$ and $y \in ($ ___ , ___ $] \cup [$ ___ , ___ $)$. The range of f is $[$ _____ , _____ $)$.

(7) The surface whose parametrization is given by $\mathbf{r}(u,v) = (u, v, u-3)$ with $u,\ v \in \mathbb{R}$ is a plane whose equation is $ax + by - z + c = 0$ where $a =$ _____ , $b =$ _____ , and $c =$ _____ .

(8) The surface whose parametrization is given by

$$\mathbf{r}(u,v) = (u, 2u\cos v, 3u\sin v)$$

with $u \geq 0$ and $0 \leq v \leq 2\pi$ is the upper half of an elliptic ______________ opening along the ___ -axis.

(9) The surface whose parametrization is given by

$$\mathbf{r}(u,v) = (u, u\cos v, u\sin v)$$

with $u \geq 0$ and $0 \leq v \leq 2\pi$ is the upper half of a(n) ______________ cone opening along the ___ -axis.

(10) The surface whose parametrization is given by

$$\mathbf{r}(u,v) = (\cos u, v, \sin u)$$

with $0 \leq u \leq 2\pi$ and $v \in \mathbb{R}$ is a circular ___________ whose axis lies along the ___ -axis.

(11) The surface whose parametrization is given by

$$\mathbf{r}(u,v) = (\cos u, \sin u, v)$$

with $0 \leq u \leq 2\pi$ and $v \in \mathbb{R}$ is a ___________ cylinder whose axis lies along the ___ -axis.

(12) Let S be the portion of the cylinder $x^2 + z^2 = 1$ in $\mathbb{R}^3$ that lies between the planes $y = 0$ and $x + y = 2$. A parametric representation of S is

$$\mathbf{r}(u, v) = (\cos u\,,\ \underline{\qquad}\,,\ \underline{\qquad\qquad})$$

where $0 \le u \le 2\pi$ and $\underline{\quad} \le v \le \underline{\qquad\qquad}$.

(13) Let S be that portion of the hyperboloid $x^2 + y^2 - z^2 = -1$ that lies above the rectangle $[-1, 1] \times [-2, 2]$ in the xy-plane. A parametric representation of S is

$$\mathbf{r}(u, v) = (\,u\,,\ v\,,\ \underline{\qquad\qquad\qquad})$$

where $-1 \le u \le 1$ and $-2 \le v \le 2$.

(14) Let S be that portion of the elliptic paraboloid $x + y^2 + 2z^2 = 4$ for which $x \ge 0$. A parametric representation of S is

$$\mathbf{r}(u, v) = (\underline{\qquad\qquad}\,,\ u\,,\ \underline{\quad})$$

where $-2 \le u \le 2$ and $-\sqrt{a(u)} \le v \le \sqrt{a(u)}$ with $a(u) = \underline{\qquad\qquad}$.

34.3. Problems

(1) Suppose $0 < a < b$. In $\mathbb{R}^3$ let C_0 be the circle in the xz-plane of radius a centered at the point $(b, 0, 0)$. Let $\mathbb{T}$ be the surface of revolution formed by revolving C_0 about the z-axis. Find a (reasonably simple) parametrization of the torus $\mathbb{T}$.

Hint. Let P be a point on $\mathbb{T}$. Then there exists a circular cross-section C of $\mathbb{T}$ which contains the point P and which is coplanar with the z-axis. This circle can be specified by a single parameter θ: the angle between the positive x-axis and the vector from the origin to the center of C. The location of P on C can also be specified by a single parameter ϕ: the angle between the vector in the xy-plane from the center of C to the outer portion of the surface $\mathbb{T}$ and the vector from the center of C to the point P.

34.4. Answers to Odd-Numbered Exercises

(1) hyperboloid of one sheet, 12, 4

(3) elliptic paraboloid, x

(5) -1, 1

(7) 1, 0, -3

(9) circular, x

(11) circular, z

(13) $\sqrt{1 + u^2 + v^2}$

(15) 0, 2π, 0, $\sin u$, $2 - \cos u$

CHAPTER 35

SURFACE AREA

35.1. Background

Topics: surfaces of revolution, surface area.

35.2. Exercises

(1) The area of the surface which is formed when the curve

$$2x = y\sqrt{y^2-1} + \ln\big(y - \sqrt{y^2-1}\big)$$

for $2 \le y \le 5$ is revolved about the x-axis is ________ .

(2) The area of the surface generated when the curve

$$y^2 - 2\ln y = 4x$$

from $y = 1$ to $y = 2$ is revolved about the x-axis is $\dfrac{a}{3}\pi$ where $a =$ ____ .

(3) The area of the surface obtained by revolving that portion of the astroid

$$\begin{cases} x = \sin^3 t \\ y = \cos^3 t \end{cases}$$

for which $0 \le t \le \dfrac{\pi}{2}$ about the x-axis is $\dfrac{a\pi}{5}$ where $a =$ ____ .

(4) A *zone* of a sphere is the portion of its surface which lies between two parallel planes which intersect the sphere. The *altitude* of the zone is the distance between the planes. The surface area of a zone of height h of a sphere of radius a is ___________ .

(5) The surface area of the solid which results from revolving the curve $3x^2 + 4y^2 = 12$ about the x-axis is $\dfrac{a}{\sqrt{3}}\pi^p + b\pi$ where $a =$ ____ , $p =$ ____ , and $b =$ ____ .

(6) A surface is defined parametrically by

$$\mathbf{R}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\,\mathbf{k}$$

where $0 \le u \le 1$ and $0 \le v \le 2\pi$.

(a) The equation of the tangent plane to the surface at the point $(-\frac{1}{2}, 0, \frac{1}{2})$ is $ax+by+z = c$ where $a =$ ____ , $b =$ ____ , and $c =$ ____ .

(b) The area of the surface is _____________ .

(7) A surface is defined parametrically by

$$\mathbf{R}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$$

where $0 \le u \le 1$ and $0 \le v \le 2\pi$.

(a) The equation of the tangent plane to the surface at the point $(-\frac{1}{2}, 0, \pi)$ is $ax+by+z = c$ where $a =$ ____ , $b =$ ____ , and $c =$ ____ .

(b) The area of the surface is $a(b + \ln(1+b))$ where $a =$ ____ and $b =$ _____ .

(8) The surface area in the first octant cut from the cylindrical surface $x^2 + y^2 = a^2$ by the plane $z = x$ is _______ .

(9) Rotate the portion of the parabola $y = x^2$ where $0 \le x \le 1$ about the y-axis. The area of the surface thus obtained is $\dfrac{\pi}{a}(b\sqrt{b} - 1)$ where $a =$ ____ and $b =$ ____ .

(10) Let $\mathbf{R}(u,v) = (u-v, u+v, uv)$ and D be the unit disk in the plane. Then the area of $\mathbf{R}(D)$ is $\dfrac{\pi}{a}(b\sqrt{b} - 8)$ where $a =$ ____ and $b =$ ____ .

(11) The surface area of that portion of the sphere $x^2 + y^2 + z^2 = 14z$ that is inside the paraboloid $x^2 + y^2 = 5z$ is $a\pi$ where $a =$ _______ .

(12) The surface area of that portion of the parabolic cylinder $x^2 = 1 - z$ bounded by the planes $y = 0$, $z = 0$, and $y = x$ with $x, y, z \geq 0$ is $\frac{1}{a}(b\sqrt{b} - 1)$ where $a =$ ________ and $b =$ _____ .

(13) The circular cylinder $x^2 + y^2 = x$ divides the unit sphere into 2 pieces. Let U be the piece inside the cylinder and V be the one outside. Then the ratio of the surface area of V to the surface area of U is $\frac{a+b}{a-b}$ where $a =$ _____ and $b =$ _____ .

(14) The area of that portion of the plane $x + y + z = 1$ lying inside the elliptic cylinder $x^2 + 2y^2 \leq 1$ is $\sqrt{a}\pi$ where $a =$ __________ .

(15) The surface area of that portion of the cone $z^2 = x^2 + y^2$ lying between the planes $z = 0$ and $z = 1$ is _____________ .

(16) Represent the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (where a, b, and c are positive constants) parametrically using the spherical coordinates ϕ and θ. Then the surface are of the ellipse is $\int_0^{2\pi} \int_0^{\pi} \sqrt{f(\phi,\theta)}\, d\phi d\theta$ where

$f(\phi, \theta) =$ ____ .

35.3. Problems

(1) You are considering the problem of calculating the surface area of a solid of revolution. The curve $y = f(x)$ (for $a \le x \le b$) has been rotated about the x-axis. You set up a partition $(x_0, x_1, \dots, x_n)$ of $[a, b]$ and approximate the surface area of the resulting solid by connecting consecutive points $\bigl(x_{k-1}, f(x_{k-1})\bigr)$ and $\bigl(x_k, f(x_k)\bigr)$ by a straight line segment, which, when rotated about the x-axis, produces a portion of a cone. Taking the limit of these conical approximations you end up with the integral

$$2\pi \int_a^b f(x)\sqrt{1 + \bigl(f'(x)\bigr)^2}\,dx$$

which you claim represents the surface area of the solid.

Your friend Fred has chosen to make cylindrical approximations rather than conical ones and produces the integral

$$2\pi \int_a^b f(x)\,dx$$

which he claims represents the surface area. So you and Fred argue.

He says your formula is too complicated to compute. You say his is too simple to work. Unable to convince him, you look up the formula in a well known calculus text and find that the author agrees with you. Fred is unimpressed; he says the author probably made the same mistake you did. You find several more references to support your work. Fred thinks they all probably copied from the person who made the original mistake.

Find a *completely convincing* argument that even skeptical Fred will accept that his formula can't be correct and that yours is better.

(2) In many calculus texts the authors derive the formula for the arc length of a smooth curve by taking the limit of sums of lengths of polygonal paths inscribed in the curve. The purpose of this problem is to inquire into the possibility of doing the "same thing" to find the surface area of a smooth surface. In particular, consider a (vertical) right circular cylinder with height h and base radius r. Partition the surface of this cylinder by $m + 1$ equally spaced horizontal circles, thus creating m sub-cylinders of height h/m (and radius r). Then partition each circle into n circular arcs by means of n equally spaced points. (The points on one circle need not necessarily be directly above those on the circle below.) Now approximate the surface area of each sub-cylinder by the area of triangles inscribed in the sub-cylinder using these points as vertices. In each sub-cylinder there will be $2n$ such triangles—n having two vertices on the upper circle and n having two vertices on the lower circle. Thus, altogether, there will be $2mn$ triangles the sum of whose areas approximates the surface area of the original cylinder.

Now, of course, the question is: what can we say about the limit of the sum of the areas of these triangles as m and n get large? Does it give us the correct expression for the area of a cylinder? (How, incidentally, can we be sure what the surface area of a right circular cylinder really *ought* to be?) Thinking about the problem in more precise terms, we need to ask whether this double limit even exists. If not, can we make the corresponding iterated limits exist by restricting m and n in some way? Does the answer depend on the relative rates of growth of m and n? What happens if we require m and n to be in some fixed ratio? That is, will the limit depend on the shape of the triangles?

After considering these matters, what do you think of the prospects of generalizing the derivation of the formula for arc length to one for surface area?

(3) Let the curve $y = f(x)$ (where $a \le x \le b$) be rotated about the y-axis. Show that the surface area swept out is $2\pi \displaystyle\int_a^b |x|\sqrt{1 + (f'(x))^2}\,dx$. *Hint.* Parametrize the surface.

35.4. Answers to Odd-Numbered Exercises

(1) 78π

(3) 6

(5) 4, 2, 6

(7) (a) 0, 2, π
(b) π, $\sqrt{2}$

(9) 6, 5

(11) 70

(13) π, 2

(15) $\sqrt{2}\pi$

CHAPTER 36

TRIPLE INTEGRALS

36.1. Background

Topics: triple integrals, iterated integrals, cylindrical coordinates, spherical coordinates, density, mass, center of gravity, moment of inertia of a solid about a line.

36.2. Exercises

(1) $\displaystyle\int_0^3 \int_0^y \int_0^y z\,dx\,dz\,dy = \frac{a}{8}$ where $a =$ ______ .

(2) Sketch the region over which the function f is being integrated in the integral

$$\int_0^1 \int_0^{2-2z} \int_0^{1-z} f(x,y,z)\,dx\,dy\,dz.$$

(3) Choose functions g and h so that the expression

$$\int_0^1 \int_0^{h(y)} \int_0^{g(x,y)} dz\,dx\,dy$$

gives the volume of the solid bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x+y+3z = 1$. (Do not integrate.)

Answer: $g(x,y) =$ __________________ and $h(y) =$ __________ .

(4) Sketch the region over which the function f is being integrated in the integral

$$\int_0^2 \int_0^{4-y^2} \int_0^{2-y} f(x,y,z)\,dz\,dx\,dy.$$

(5) Use a triple integral to find the volume of the tetrahedron whose vertices are the points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

Answer: the volume is $\dfrac{1}{a}$ where $a =$ _________ .

(6) The volume of the region bounded by the planes $z = y$, $z = 0$, $x = 1$, and the surface $y = x^2$ can be computed using any one of the following iterated triple integrals:

$$\int_0^1 \int_c^d \int_a^b dx\,dy\,dz$$

where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $d =$ _____ ;

$$\text{or} \qquad \int_0^1 \int_c^d \int_a^b dy\,dx\,dz$$

where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $d =$ _____ ;

$$\text{or} \qquad \int_0^1 \int_c^d \int_a^b dx\,dz\,dy$$

where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $d =$ _____ ;

$$\text{or} \qquad \int_0^1 \int_c^d \int_a^b dz\,dx\,dy$$

where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $d =$ _____ ;

$$\text{or} \qquad \int_0^1 \int_c^d \int_a^b dy\,dz\,dx$$

where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $d =$ _____ ;

$$\text{or} \qquad \int_0^1 \int_c^d \int_a^b dz\,dy\,dx$$

where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $d =$ _____ .

(7) The volume of the solid in the first octant bounded by the planes $x = y$, $x = 1$, $z = 0$, and $z = y$ is $\dfrac{1}{a}$ where $a =$ ____ .

(8) The volume of the solid bounded by $y = 2x$, $y = 2$, $z = 0$, $x = 0$, and $z = 5 - x^2 - y^2$ is $\dfrac{a}{6}$ where $a =$ ______ .

(9) The integral of the function $f(x, y, z) = x^3 \cos xz$ over the solid bounded by the planes $x = y$, $z = y$, $x = 1$, and $z = 0$ is $\frac{1}{2}(a - \sin a)$ where $a =$ ____ .

(10) Let E be the solid bounded above by the cylinder $y^2 + z = 4$, below by the plane $y + z = 2$, and on the sides by the planes $x = 0$ and $x = 2$. If its density is y^2 at each point (x, y, z) in E, then its volume is ____ and its mass is $\dfrac{a}{10}$ where $a =$ _____ .

(11) Consider a block in the shape of a cube with edges of length a. Its density at each point P is proportional to the distance between P and one fixed face of the cube. (Let k be the constant of proportionality.) Then its mass is __________ .

(12) Consider a block in the shape of a cube with edges of length a. Suppose that the block's density at each point is proportional to the square of the distance between the point and one fixed vertex. (Let k be the constant of proportionality.) Then the mass of the block is ________ .

(13) The volume of the region in the first octant lying under the plane $3x+2y+6z = 6$ is ____ .

(14) Sketch the region over which the function f is being integrated in the integral

$$\int_0^1 \int_0^{1-x} \int_0^{\cos(\pi x/2)} f(x, y, z)\, dz\, dy\, dx.$$

(15) The volume of the region you sketched in the preceding exercise is $a\pi^p$ where $a =$ ____ and $p =$ ____ .

(16) The volume of the solid bounded by the torus $\rho = 3 \sin \phi$ is $\dfrac{a}{4}\pi^p$ where $a =$ _______ and $p =$ ____ .

(17) A homogeneous solid sphere of radius a is centered at the origin. The center of gravity of the hemisphere lying above the xy-plane is at the point $(0\,,0\,,$ ________ $)$.

(18) Let $a > 0$. A solid is bounded below by the xy-plane, above by the sphere $x^2+y^2+z^2 = 4a^2$, and on the sides by the cylinder $r = 2a \cos \theta$. Then the moment of inertia of the solid about the z-axis is $\dfrac{b}{15}a^p\left(\pi - \dfrac{c}{15}\right)$ where $b =$ _______ , $p =$ ____ , and $c =$ _______ .

(19) Let R be the solid region which lies above the cone $x^2 + y^2 = 3z^2$ and inside the sphere $x^2 + y^2 + z^2 = 1$. The moment of inertia of R about the z-axis is $\dfrac{\pi}{a}$ where $a =$ ____ .

(20) Suppose the density of a spherical object of radius a at a point p is proportional to $\exp(-d^3/a^3)$ where d is the distance between p and the center of the sphere. Then the mass of the object is ___________________ .

(21) The solid R is bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$. The density at each point of R is the distance from the point to the z-axis (in kilograms per cubic meter). Then the mass of R is $\dfrac{a}{15}\pi$ kilograms where $a =$ _________ .

(22) The volume of the solid bounded above by the plane $z = y$ and below by the paraboloid $z = x^2 + y^2$ is $\dfrac{\pi}{a}$ where $a =$ _______ .

(23) The volume of the solid bounded by $z = 2r$ $(r \geq 0)$, $r = 1 - \cos\theta$, and $z = 0$ is $\dfrac{a\pi}{3}$ where $a =$ ________ .

(24) Let S be the solid bounded above by the plane $z = y$ and below by the paraboloid $z = x^2 + y^2$. If the density at each point of S is 20 times its distance from the z-axis, then the mass of S is $\dfrac{a}{15}$ where $a =$ ________ .

(25) The triple integral in cylindrical coordinates giving the volume of the solid bounded above by the (lower of the two) surfaces $x^2 + y^2 + z^2 = 4$ $(z \geq 0)$ and $3z = x^2 + y^2$ and bounded below by the plane $z = 0$ can be written either as

$$\int_0^{2\pi}\int_0^b\int_0^a r\,dz\,dr\,d\theta + \int_0^{2\pi}\int_b^d\int_0^c r\,dz\,dr\,d\theta$$

where $a =$ _________ , $b =$ ______ , $c =$ ___________ , and $d =$ _____ ; or as

$$\int_0^{2\pi}\int_0^c\int_a^b r\,dr\,dz\,d\theta$$

where $a =$ ________ , $b =$ _____________ , and $c =$ _____ .

(26) The volume of the solid which lies above the plane $z = \sqrt{3}$ and within the sphere $x^2 + y^2 + z^2 = 4$ is $\left(\dfrac{a}{b} - b\sqrt{b}\right)\pi$ where $a =$ ________ and $b =$ _____ .

(27) The equation of the sphere centered at $(0, 0, b)$ with radius b in spherical coordinates is $\rho =$ ___________________ .

(28) Let C be a circular cone of base radius 1 and height 1. If the density at each point of C is 5 times its distance from the base, then the mass of C is $\dfrac{5\pi}{a}$ where $a =$ ________ .

(29) Let C be a circular cone of base radius 1 and height 1. If the density at each point of C is 5 times the square of its distance from the vertex then the mass of C is __________ .

(30) A hole of radius 1 millimeter is drilled through the center of a ball bearing of radius 2 millimeters. The volume of the solid which remains is $a\pi$ cubic millimeters where $a =$ __________ .

(31) The value of $\iiint\limits_R f$ where $f(x, y, z) = \sqrt{z}$ and R is the solid region determined by the inequalities $(1/\sqrt{3})x \leq y \leq x$, $z \geq 0$, and $x^2 + y^2 + z^2 \leq 16$ is $\dfrac{a}{63}\pi$ where $a =$ ________ .

(32) Find numbers a, b, c, and d and a function f such that the integral (in spherical coordinates)

$$\int_0^d\int_{\arctan\sqrt{b}}^c\int_{f(\phi)}^{\sqrt{a}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

gives the volume of the solid bounded above by the (lower of the two) surfaces $x^2+y^2+z^2 = 15$ $(z \geq 0)$ and $2z = x^2 + y^2$ and below by the plane $z = 0$. *Answer:* $a =$ ______ , $b =$ ________ , $c =$ ________ , $d =$ ______ , and $f(\phi) =$ ______________________ .

36.3. Answers to Odd-Numbered Exercises

(1) 81

(3) $\frac{1}{3}(1-2x-y)$, $\frac{1}{2}(1-y)$

(5) 6

(7) 6

(9) 1

(11) $\frac{1}{2}ka^4$

(13) 1

(15) 4, -2

(17) $\dfrac{3a}{8}$

(19) 12

(21) 128

(23) 10

(25) $\dfrac{r^2}{3}$, $\sqrt{3}$, $\sqrt{4-r^2}$, 2, $\sqrt{3z}$, $\sqrt{4-z^2}$, 1

(27) $2b\cos\phi$

(29) $\dfrac{3\pi}{2}$

(31) 128

CHAPTER 37

CHANGE OF VARIABLES IN AN INTEGRAL

37.1. Background

Topics: change of coordinates, one-to-one (= injective), onto (= surjective), Jacobian matrix, Jacobian

37.1.1. Theorem (Abel's theorem). *Let* $\sum_{k=0}^{\infty} c_k x^k$ *be a power series with radius of convergence* $R > 0$. *If the series converges at* $x = R$, *then it converges uniformly on the closed interval* $[0, R]$ *and the function* f *defined by*

$$f(x) = \sum_{k=0}^{\infty} c_k x^k$$

is continuous on $[0, R]$.

(A proof of this theorem may be found on page 644 of *Advanced Calculus*, 3^{rd} ed; by A. E. Taylor and W. R. Mann.)

37.2. Exercises

(1) Let C be the circle of radius 4 in the uv-plane. A transformation $\mathbf{T}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by $\mathbf{T}(u,v) = (u^2 - v^2, 2uv)$. Then the image of C under $\mathbf{T}$ is a ______________ of ________________ .

(2) Consider the transformation $\mathbf{T}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $\mathbf{T}(u,v) = (4u+13v, 2u+7v)$. Then the image under $\mathbf{T}$ of the square $[0,1]\times[0,1]$ is a parallelogram whose vertices are (0, ___), (4, ___), (_____ , 7), and (______ , _____). The area of the image is _____ .
Hint. Recall properties of the cross product.

(3) Let $\mathbf{T}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the transformation defined by $\mathbf{T}(u,v) = (u+v, v-u^2)$. Let D be the region bounded by the curves $y = x$ $(0 \le x \le 1)$, $y = -x^2 + 2x$ $(1 \le x \le 2)$, $y = x - 2$ $(1 \le x \le 2)$, and $y = -x^2$ $(0 \le x \le 1)$. There are two regions D_1^* and D_2^* in the uv-plane which $\mathbf{T}$ maps onto D. The region D_1^* is a parallelogram whose vertices are (-1 , _____), (_____ , 4), (-2 , _____), and (_____ , 2); and D_2^* is a ______________ whose vertices are (0 , 0), (_____ , 0), (_____ , _____), and (0 , _____).

(4) Let T be the triangular region whose vertices are $(1,0)$, $(0,1)$, and $(0,0)$. Then $\displaystyle\iint_T \sin(x+y)\cos(x-y)\,dx\,dy = \frac{1}{a} - \frac{1}{b}\sin a$ where $a =$ _____ and $b =$ _____ .

(5) The area of that portion of the plane $x+y+z=1$ which lies inside the elliptic cylinder $x^2 + 2y^2 = 1$ is $\frac{1}{2}\sqrt{a}\pi$ where $a =$ _____ .

(6) The volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \le 1$ is __________________ .

(7) Let D be the region bounded by the lines $x+y=0$, $x+y=5$, $3x-2y=0$, and $3x-2y=5$. Let $f(x,y) = xy - y$. Then $\displaystyle\iint_D f(x,y)\,dx\,dy = \frac{a}{12}$ where $a =$ ________ .
Hint. Try $x = 2r + s$, $y = 3r - s$.

(8) The value of $\displaystyle\iint_D \exp\left(\frac{2x-y}{2x+y}\right)dx\,dy$ where D is the triangular region $(0,0)$, $(1,0)$, and $(\frac{1}{2},1)$ is $\frac{1}{a}(b-1)$ where $a =$ _____ and $b =$ _____ .

(9) The area of that portion of the plane $4x+6y-z=1$ which lies inside the elliptic paraboloid $z = 4x^2 + 9y^2$ is $\dfrac{\sqrt{a}\,\pi}{6}$ where $a =$ ________ .

(10) Let E be the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \le 1$. Then

$$\iiint_E \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)dx\,dy\,dz = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}.$$

(11) Let D be the square with vertices $(\pm1,0)$ and $(0,\pm1)$. Then

$$\iint_D e^{-(x+y)}(x-y)^2\,dx\,dy = \frac{1}{a}\left(b - \frac{1}{b}\right)$$

where $a =$ _____ and $b =$ _____ .

(12) $\displaystyle\int_0^2 \int_{2x-3}^{\frac{1}{2}x} \exp\left(\frac{x+y}{2x-y}\right)dy\,dx = \frac{a}{2}\left(b - \frac{1}{b}\right)$ where $a =$ _____ and $b =$ _____ .

(13) The integral of $f(x,y) = \cos\left(\dfrac{\pi(2x-y)}{3x+y}\right)$ over the region bounded by the lines $y = 2x$, $y = -3x + 20$, and $y = \frac{1}{3}x$ is $\dfrac{a}{\pi}$ where $a =$ ______ .

(14) Let D be the triangular region in the xy-plane with vertices $(0,1)$, $(1,0)$, and $\left(\frac{3}{2}, \frac{1}{2}\right)$. Then $\displaystyle\iint_D (x+y-1)\cos\left(\frac{\pi}{16}(x-y+1)^3\right)dx\,dy = \frac{1}{a}$ where $a =$ ______ . *Hint.* Try the change of variables $u = x + y - 1$ and $v = x - y + 1$.

37.3. Problems

(1) Consider the transformation $\mathbf{T}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $\mathbf{T}(u,v) = (3u - 4v, u + 5v)$. Prove or disprove:
  (a) $\mathbf{T}$ is one-to-one.
  (b) $\mathbf{T}$ is onto.

(2) Prove or disprove: the mapping $\mathbf{T}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $\mathbf{T}(u,v) = (u^2 - v^2, 2uv)$ is one-to-one.

(3) Consider the transformation $\mathbf{T}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $\mathbf{T}(u,v) = (4u + 13v, 2u + 7v)$. Prove or disprove:
  (a) $\mathbf{T}$ is one-to-one.
  (b) $\mathbf{T}$ is onto.

(4) Consider the transformation $\mathbf{T}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $\mathbf{T}(u,v) = (8u + 12v, 6u + 9v)$. Prove or disprove:
  (a) $\mathbf{T}$ is one-to-one.
  (b) $\mathbf{T}$ is onto.

(5) Let D be the region bounded by the lines $x + y = 0$, $x + y = 6$, $x = 2y$, and $x = 2y - 3$, and let $F(x,y) = x + 2y$. Explain in detail how to find $\displaystyle\iint_D F(x,y)\,dx\,dy$ by making use of the change of variables $T \begin{cases} x &= 2r - s \\ y &= r + s \end{cases}$. Carry out the computation.

(6) Let D be the region bounded by $y = x$ $(0 \le x \le 1)$, $y = -x^2 + 2x$ $(1 \le x \le 2)$, $y = x - 2$ $(1 \le x \le 2)$, and $y = -x^2$ $(0 \le x \le 1)$. Explain in detail how to compute $\displaystyle\iint_D (x - y)\,dx\,dy$ in three different ways. (Carry out the computations to check that you get the same answer each time.) *Hint.* For two of the ways try the change of variables $x = u + v$ and $y = v - u^2$.

(7) Give an argument showing that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Hint. Start by showing that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \int_0^1 \int_0^1 \frac{dx\,dy}{1 - xy}.$$

To evaluate the integral change variables: obtain u and v axes by rotating the x and y axes counterclockwise by $\pi/4$ radians. Then make use of *Abel's theorem* 37.1.1.

37.4. Answers to Odd-Numbered Exercises

(1) circle, radius 16

(3) 1, -2, 3, -1, square, 1, 1, 1, 1

(5) 6

(7) 55

(9) 53

(11) 3, e

(13) 40

CHAPTER 38

VECTOR FIELDS

38.1. Background

Topics: vector fields, flow lines, divergence, curl, conservative fields, potential functions, open sets, connected sets, simply connected sets.

38.1.1. Definition. A vector field is INCOMPRESSIBLE (or SOLENOIDAL) if its divergence is zero. It is IRROTATIONAL if its curl is zero.

38.2. Exercises

(1) The (nonparametric) equation of the flow line of the vector field $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$ which passes through the point $(4, 3)$ is ________________ .

(2) The (nonparametric) equation of the flow line of the vector field $\mathbf{F}(x, y) = y\,\mathbf{i} - x\,\mathbf{j}$ which passes through the point $(4, 3)$ is ________________ .

(3) The (nonparametric) equation of the flow line of the vector field $\mathbf{F}(x, y) = x\,\mathbf{i} + x^3\,\mathbf{j}$ which passes through the point $(1, 1)$ is $3y = f(x)$ where $f(x) =$ ________________ .

(4) Let $f(x, y) = \sqrt{x^2 + y^2}$ for all $(x, y) \neq (0, 0)$. Then $\nabla f(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ where:

$$P(x, y) \;=\; \underline{\hspace{3cm}} \text{ and } Q(x, y) \;=\; \underline{\hspace{3cm}} .$$

Sketch the gradient field of f.

(5) The (parametric) equation of the flow line of the vector field $\mathbf{F}(x, y, z) = 2x\,\mathbf{i} + z\,\mathbf{j} - z^2\,\mathbf{k}$ which passes through the point $(e^2, 0, 1)$ at time $t = 1$ is $\mathbf{r}(t) = a(t)\,\mathbf{i} + b(t)\,\mathbf{j} + c(t)\,\mathbf{k}$ where $a(t) =$ ________ , $b(t) =$ __________ , and $c(t) =$ ______ .

(6) The (parametric) equation of the flow line of the vector field $\mathbf{F}(x, y, z) = (y + 1)\,\mathbf{i} + 2\,\mathbf{j} + (2z)^{-1}\,\mathbf{k}$ which passes through the point $(1, 1, 1)$ at time $t = 1$ is $\mathbf{r}(t) = a(t)\,\mathbf{i} + b(t)\,\mathbf{j} + c(t)\,\mathbf{k}$ where $a(t) =$ _____ , $b(t) =$ __________ , and $c(t) =$ ______ .

(7) A (parametric) equation of the flow line of the vector field $\mathbf{F}(x, y) = 3t^2x\,\mathbf{i} + y\,\mathbf{j}$ which passes through the point $(2, 1)$ is $\mathbf{r}(t) = a(t)\,\mathbf{i} + b(t)\,\mathbf{j}$ where $a(t) =$ ____________ and $b(t) =$ ______________ .

(8) An example of a potential function f for the vector field $\mathbf{F}(x, y) = (y \cos x - \cos y)\,\mathbf{i} + (\sin x + x \sin y)\,\mathbf{j}$ is $f(x, y) =$ ________________ .

(9) Consider the set $S = \{(x, y) \colon x \neq 0\}$. The set S _________ (is/is not) open; it _________ (is/is not) connected; it _________ (is/is not) simply connected.

(10) Consider the set $S = \{(x, y) \colon x^2 + y^2 \le 1 \text{ or } 4 < x^2 + y^2 < 9\}$. The set S _________ (is/is not) open; it _________ (is/is not) connected; it _________ (is/is not) simply connected.

(11) Consider the set $S = \{(x, y) \colon 1 < x^2 + y^2 < 25\}$. The set S _________ (is/is not) open; it _________ (is/is not) connected; it _________ (is/is not) simply connected.

(12) Let $f(x, y) = (x^2 + y^2 + 4x - 6y + 13)^{-1}$. Then the domain of f _________ (is/is not) open; it _________ (is/is not) connected; it _________ (is/is not) simply connected.

(13) Let $f(x, y, z) = x^2y^3e^z$. Find each of the following (if they make sense):
   (a) $\operatorname{grad} f =$ __________________________ ;
   (b) $\operatorname{div} f =$ __________________________ ;
   (c) $\operatorname{curl} f =$ __________________________ .

(14) Let $\mathbf{F}(x, y, z) = yz^2\,\mathbf{i} + x^2y^3\,\mathbf{j} + (x^2 + y^2 + z^2)\,\mathbf{k}$. Find each of the following (if they make sense):
   (a) $\operatorname{grad} \mathbf{F} =$ __________________________ ;
   (b) $\operatorname{div} \mathbf{F} =$ __________________________ ;
   (c) $\operatorname{curl} \mathbf{F} =$ __________________________ .

(15) Let $\mathbf{F}(x, y, z) = 2xye^z\,\mathbf{i} + x^2e^z\,\mathbf{j} + (x^2ye^z + z^2)\,\mathbf{k}$. Then
   (a) $\nabla \cdot \mathbf{F} =$ __________________________ ;
   (b) $\nabla \times \mathbf{F} =$ __________________________ ; and
   (c) a function f such that $\mathbf{F} = \nabla f$ is given by $f(x, y, z) =$ __________________________ .

(16) Let $\mathbf{F}(x,y,z) = x\,\mathbf{i} + x^2y^2\,\mathbf{j} + x^3y^3z^3\,\mathbf{k}$. Then

(a) the divergence of $\mathbf{F}$ is ______________________________

(b) and the curl of $\mathbf{F}$ is __________ $\mathbf{i}$ + __________ $\mathbf{j}$ + __________ $\mathbf{k}$.

(17) Let $\mathbf{F}(x,y,z) = \dfrac{x}{x^2+y^2}\,\mathbf{i} + \dfrac{y}{x^2+y^2}\,\mathbf{j}$. Then the divergence of $\mathbf{F}$ is _____ and the curl of F is _____ $\mathbf{i}$ + _____ $\mathbf{j}$ + _____ $\mathbf{k}$.

(18) Let $\mathbf{F}(x,y,z) = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$. Then

(a) the divergence of $\mathbf{F}$ is ______________________________

(b) and the curl of $\mathbf{F}$ is __________ $\mathbf{i}$ + __________ $\mathbf{j}$ + __________ $\mathbf{k}$.

(19) Consider the following vector fields:

(a) $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$;

(b) $(x^2+y^2)\,\mathbf{i} + (y^2+z^2)\,\mathbf{j} + (z^2+x^2)\,\mathbf{k}$; and

(c) $yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$.

The ones that are incompressible (solenoidal) are __________________ .

The ones that are irrotational are __________________ .

(20) Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $\mathbf{F} = \dfrac{\mathbf{r}}{\|\,\mathbf{r}\|^p}$. Then the divergence of $\mathbf{F}$ is zero when $p =$ _____ .

(21) Let $\mathbf{F}(x,y) = xy^2\,\mathbf{i} + (x+y)\,\mathbf{j}$ and let D be the region bounded by the curves $y = x^2$ and $y = x$. Then

$$\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA = \underline{\hspace{3cm}}\ .$$

38.3. Answers to Odd-Numbered Exercises

(1) $xy = 12$

(3) $x^3 + 2$

(5) e^{2t}, $\ln t$, $\dfrac{1}{t}$

(7) $2e^{t^3}$, e^t

(9) is, is not, is not

(11) is, is, is not

(13) (a) $(2xy^3e^z, 3x^2y^2e^z, x^2y^3e^z)$
(b) does not make sense
(c) does not make sense

(15) (a) $2ye^z + x^2ye^z + 2z$
(b) $(0, 0, 0)$
(c) $x^2ye^z + \frac{1}{3}z^3$

(17) 0, 0, 0, 0

(19) c, a and c

(21) $\dfrac{1}{12}$

Part 10

THE CALCULUS OF DIFFERENTIAL FORMS

CHAPTER 39

DIFFERENTIAL FORMS

39.1. Background

Topics: differential forms, wedge product.

Although differential forms can be defined in spaces of any dimension, in this course we will work only with forms defined on subsets of $\mathbb{R}^3$. In the following take U to be a subset of $\mathbb{R}^3$.

39.1.1. Definition. A 0-FORM on U is a smooth scalar field defined on U.

Example: $\omega = x^2y + \sin(xz)$ is a 0-form on $\mathbb{R}^3$.

39.1.2. Definition. A 1-FORM on U is an expression of the form

$$\omega = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz \tag{1}$$

where P, Q, and R are smooth scalar fields (0-forms) defined on U. We usually write just

$$\omega = P\,dx + Q\,dy + R\,dz.$$

Example: $\omega = x^2y\,dx + xyz\,dy + e^yz^3\,dz$ is a 1-form on $\mathbb{R}^3$.

Example: $\omega = x\sin x\,dx + y\ln x\,dy$ is a 1-form on the right half plane.

Example: $\omega = y^2 \arctan y\,dy$ is a 1-form on the real line.

39.1.3. Definition. A 2-FORM on U is an expression of the form

$$\omega = P(x,y,z)\,dy\wedge dz + Q(x,y,z)\,dz\wedge dx + R(x,y,z)\,dx\wedge dy \tag{2}$$

where P, Q, and R are smooth scalar fields (0-forms) defined on U. We usually write

$$\omega = P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy.$$

Example: $\omega = xyz\,dy\wedge dz + yz\,dz\wedge dx + z\,dx\wedge dy$ is a 2-form on $\mathbb{R}^3$.

Example: $\omega = \dfrac{xy}{x^2+y^2}\,dx\wedge dy$ is a 2-form on $\mathbb{R}^2$ with the origin excluded.

39.1.4. Definition. A 3-FORM on U is an expression of the form

$$\omega = P(x,y,z)\,dx\wedge dy\wedge dz \tag{3}$$

where P is a smooth scalar field (0-form) defined on U. We usually write

$$\omega = P\,dx\wedge dy\wedge dz.$$

Example: $\omega = (x^2+y^2+z^2)\,dx\wedge dy\wedge dz$ is a 3-form on $\mathbb{R}^3$.

The n-forms above (for $n = 0,1,2,3$) are examples of DIFFERENTIAL FORMS. It will be convenient (for the purposes of this course) to regard a p-form for $p > 3$ as identically 0. We denote by $G^p(U)$ the family of p-forms on the region U. Thus (in this course), $G^4(U) = G^5(U) = \cdots = 0$.

Terms of differential forms which are multiplied by 0 are omitted. For example, the 1-form $P\,dx$ is understood to be $P\,dx + 0\,dy + 0\,dz$ and the 2-form $Q\,dz\wedge dx$ is understood to be $0\,dy\wedge dz + Q\,dz\wedge dx + 0\,dx\wedge dy$.

Let P, Q, and R be scalar fields on U. We say that $\omega = P\,dx + Q\,dy + R\,dz$ is the 1-form ASSOCIATED WITH the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. (And, of course, $\mathbf{F}$ is the vector field associated with ω.) Notice that a vector field is identically zero if and only if its associated 1-form is.

Example: If $\mathbf{F}(x,y,z) = x^2y\,\mathbf{i} + y\sin z\,\mathbf{j} + e^{xz}\,\mathbf{k}$, then its associated 1-form is $x^2y\,dx + y\sin z\,dy + e^{xz}\,dz$.

Differential forms can be ADDED. Addition of 0-forms (scalar fields) is familiar. We add 1-forms by adding "coefficients". If $\omega = P\,dx + Q\,dy + R\,dz$ and $\mu = S\,dx + T\,dy + U\,dz$, then

$$\omega + \mu = (P+S)\,dx + (Q+T)\,dy + (R+U)\,dz.$$

Addition of 2-forms and 3-forms is similar.

Differential forms can be multiplied. The (WEDGE) PRODUCT of a p-form ω and a q-form μ is a $(p+q)$-form denoted by $\omega \wedge \mu$. The wedge product has the following properties:

(1) The operation $\wedge$ is associative. That is, if ω, μ, and ν are differential forms, then

$$(\omega \wedge \mu) \wedge \nu = \omega \wedge (\mu \wedge \nu);$$

(2) The distributive laws hold: if $\omega \in G^p$ and $\mu,\nu \in G^q$, then

$$\omega \wedge (\mu + \nu) = \omega \wedge \mu + \omega \wedge \nu$$

and

$$(\mu + \nu) \wedge \omega = \mu \wedge \omega + \nu \wedge \omega;$$

(3) If ω is a 1-form, then $\omega \wedge \omega = \mathbf{0}$.

Note: The wedge symbol is usually not written when taking the product of a 0-form with another differential form. For example, if we multiply the 0-form $3x^2z$ by the 2-form $dx \wedge dy$ we would write $3x^2z\,dx \wedge dy$ instead of $3x^2z \wedge dx \wedge dy$. Furthermore, 0-forms are usually written on the left. Thus, for example, if $\omega = 3x\,dx$ and $\mu = 2xz\,dy$, then $\omega \wedge \mu = 6x^2z\,dx \wedge dy$. Note also that the number 0 acts as an annihilator on differential forms; that is, if ω is a k-form, then $0 \cdot \omega$ is the zero k-form.

39.1.5. Theorem. *If ω and μ are 1-forms, then*

$$\omega \wedge \mu = -\mu \wedge \omega.$$

PROOF. You are asked to prove this fact as problem 1 below.

39.1.6. Example. If $\omega = x\,dx + y\,dy$ and $\mu = y\,dx + x\,dy$, then

$$\begin{aligned}\omega \wedge \mu &= (x\,dx + y\,dy) \wedge (y\,dx + x\,dy) \\ &= xy\,dx \wedge dx + x^2\,dx \wedge dy + y^2\,dy \wedge dx + xy\,dy \wedge dy \qquad \text{[by (2) above]} \\ &= x^2\,dx \wedge dy + y^2\,dy \wedge dx \qquad \text{[by (3) above]} \\ &= (x^2 - y^2)\,dx \wedge dy \qquad \text{[by the preceding theorem]}\end{aligned}$$

39.1.7. Example. If $\omega = 2xy\,dx + x^2\,dy$ and $\mu = x\,dx + y\,dy + z\,dz$, then

$$\begin{aligned}\omega \wedge \mu &= (2xy\,dx + x^2\,dy) \wedge (x\,dx + y\,dy + z\,dz) \\ &= 2x^2y\,dx \wedge dx + 2xy^2\,dx \wedge dy + 2xyz\,dx \wedge dz + x^3\,dy \wedge dx + x^2y\,dy \wedge dy + x^2z\,dy \wedge dz \\ &= (2xy^2 - x^3)\,dx \wedge dy - 2xyz\,dz \wedge dx + x^2z\,dy \wedge dz.\end{aligned}$$

39.2. Exercises

(1) Let $\omega = (3x + yz)\,dx + \cos(xz)\,dy + (x^2 + y)\,dz$ and $\mu = (x + x^2)\,dx + e^y\,dy + (y + z)\,dz$.
Then $\omega + \mu =$ ______________ $dx +$ ______________ $dy +$ ______________ dz.

(2) Let $\omega = e^{xy}\,dz \wedge dx$ and $\mu = e^{-xy}\,dy$. Then $\omega \wedge \mu =$ ________________ .

(3) Let $\omega = 3x\,dx + y\,dy$ and $\mu = x^3\,dx + z^5\,dz$.
Then $\omega \wedge \mu =$ __________ $dy \wedge dz +$ __________ $dz \wedge dx +$ __________ $dx \wedge dy$.

(4) Let $\omega = 3xy\,dx + y^2z\,dy + xy\,dz$ and $\mu = xz\,dy \wedge dz + 2y\,dz \wedge dx - dx \wedge dy$.
Then $\omega \wedge \mu =$ ____ .

(5) Let $\omega = P\,dx + Q\,dy + R\,dz$ and $\mu = S\,dx + T\,dy + U\,dz$ be 1-forms.
Then $\omega \wedge \mu =$ ___________ $dy \wedge dz +$ ___________ $dz \wedge dx +$ ___________ $dx \wedge dy$.

(6) Let $\omega = 3yz\,dy \wedge dz - 5xy\,dx \wedge dy$ and $\mu = x^2\,dy \wedge dz + xyz\,dz \wedge dx + z^2\,dx \wedge dy$.
Then $\omega \wedge \mu =$ _______ .

(7) Let $\omega = x^2y\,dy \wedge dz + xyz\,dz \wedge dx$ and $\mu = dx + dy$. Then

$$\omega \wedge \mu = \underline{\hspace{6cm}}\ .$$

(8) Let $\mathbf{F} = x\,\mathbf{i} - y\,\mathbf{j} + z\,\mathbf{k}$ and $\mathbf{G} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$. Then the 1-form associated with the vector field $\mathbf{F} \times \mathbf{G}$ is ____________ $dx +$ ____________ $dy +$ ____________ dz.

39.3. Problems

(1) Prove that if ω and μ are 1-forms, then

$$\omega \wedge \mu = -\mu \wedge \omega.$$

Hint. Consider $(\omega + \mu)^2$.

(2) Let f be a scalar field (that is, a 0-form) on a region in $\mathbb{R}^3$. Show that df is the 1-form associated with ∇f.

39.4. Answers to Odd-Numbered Exercises

(1) $4x + yz + x^2$, $\cos(xz) + e^y$, $x^2 + 2y + z$

(3) yz^5, $-3xz^5$, $-x^3y$

(5) $QU - RT$, $RS - PU$, $PT - QS$

(7) $(x^2y + xyz)\, dx \wedge dy \wedge dz$

CHAPTER 40

THE EXTERIOR DIFFERENTIAL OPERATOR

40.1. Background

Topics: exterior differential operator.

40.1.1. Definition. The EXTERIOR DIFFERENTIATION OPERATOR d is a mapping which takes k-forms to $(k+1)$-forms. That is $d\colon G^k \longrightarrow G^{k+1}$. It has the following properties:

(i) If f is a 0-form on $\mathbb{R}^3$, then $d(f)$ is just the differential (or total derivative) df of f. That is,
$$d(f) = df = f_x\,dx + f_y\,dy + f_z\,dz;$$
(ii) d is linear (that is, if ω and μ are k-forms and c is a constant, then $d(\omega+\mu) = d\omega + d\mu$ and $d(c\omega) = c\,d\omega$);
(iii) $d^2 = \mathbf{0}$ (that is, $d(d\omega) = \mathbf{0}$ for every k-form ω);
(iv) If ω is a k-form and μ is any differential form
$$d(\omega \wedge \mu) = (d\omega)\wedge\mu + (-1)^k\omega\wedge d\mu.$$

Example: If $\omega = xy^2\sin z$, then
$$d\omega = y^2\sin z\,dx + 2xy\sin z\,dy + xy^2\cos z\,dz \qquad \text{[by (i)]}$$

Example: If $\omega = x^2e^z$, then
$$d\omega = 2xe^z\,dx + x^2e^z\,dz \qquad \text{[by (i)]}$$

Example: If $\omega = xy^2z^3\,dy$, then
$$\begin{aligned}
d\omega &= d(xy^2z^3)\wedge dy + xy^2z^3\,d(dy) \qquad \text{[by (iv)]}\\
&= d(xy^2z^3)\wedge dy \qquad \text{[by (iii)]}\\
&= (y^2z^3\,dx + 2xyz^3\,dy + 3xy^2z^2\,dz)\wedge dy \qquad \text{[by (i)]}\\
&= y^2z^3\,dx\wedge dy + 2xyz^3\,dy\wedge dy + 3xy^2z^2\,dz\wedge dy\\
&= y^2z^3\,dx\wedge dy - 3xy^2z^2\,dy\wedge dz
\end{aligned}$$

Example: If $\omega = xy\,dx + x^2yz\,dy + z^3\,dz$, then
$$\begin{aligned}
d\omega &= d(xy\,dx) + d(x^2yz\,dy) + d(z^3\,dz) \qquad \text{[by (ii)]}\\
&= d(xy)\wedge dx + xy\,d(dx) + d(x^2yz)\wedge dy + x^2yz\,d(dy)\\
&\qquad\qquad + d(z^3)\wedge dz + z^3\,d(dz) \qquad \text{[by (iv)]}\\
&= d(xy)\wedge dx + d(x^2yz)\wedge dy + d(z^3)\wedge dz \qquad \text{[by (iii)]}\\
&= (y\,dx + x\,dy)\wedge dx + (2xyz\,dx + x^2z\,dy + x^2y\,dz)\wedge dy\\
&\qquad\qquad + (3z^2\,dz)\wedge dz \qquad \text{[by (i)]}\\
&= x\,dy\wedge dx + 2xyz\,dx\wedge dy + x^2y\,dz\wedge dy\\
&= x(2yz-1)\,dx\wedge dy - x^2y\,dy\wedge dz.
\end{aligned}$$

Remark: It simplifies computations to notice that $d(dx \wedge dy) = d(dx) \wedge dy - dx \wedge d(dy) = \mathbf{0} \wedge dy - dx \wedge \mathbf{0} = \mathbf{0}$.

Example: If $\omega = xz^2\, dx \wedge dy + xyz\, dz \wedge dx$, then

$$\begin{aligned}
d\omega &= d(xz^2\, dx \wedge dy) + d(xyz\, dz \wedge dx) \qquad \text{[by (ii)]} \\
&= d(xz^2) \wedge (dx \wedge dy) + xz^2\, d(dx \wedge dy) + d(xyz) \wedge (dz \wedge dx) \\
&\qquad\qquad + xyz\, d(dz \wedge dx) \qquad \text{[by (iv)]} \\
&= d(xz^2) \wedge (dx \wedge dy) \\
&\qquad\qquad + d(xyz) \wedge (dz \wedge dx) \qquad \text{[by the Remark above]} \\
&= (z^2\, dx + 2xz\, dz) \wedge dx \wedge dy \\
&\qquad\qquad + (yz\, dx + xz\, dy + xy\, dz) \wedge dz \wedge dx \qquad \text{[by (i)]} \\
&= 2xz\, dz \wedge dx \wedge dy + xz\, dy \wedge dz \wedge dx \\
&= -2xz\, dx \wedge dz \wedge dy - xz\, dy \wedge dx \wedge dz \\
&= 2xz\, dx \wedge dy \wedge dz + xz\, dx \wedge dy \wedge dz \\
&= 3xz\, dx \wedge dy \wedge dz
\end{aligned}$$

Example: If $\omega = xy^2z^3\, dx \wedge dy \wedge dz$, then $d\omega = \mathbf{0}$. [**Proof:** The differentiation operator takes 3-forms to 4-forms and (in this course) all 4-forms are zero. Or, you can give essentially the same argument as in the Remark above.]

40.2. Exercises

(1) Let $f(x,y,z) = x^3 + y^2 + 2x\sin z$. Then

$$df = \underline{\hspace{6em}}\,dx + \underline{\hspace{3em}}\,dy + \underline{\hspace{5em}}\,dz\,.$$

(2) If $\omega = x^3y^2z^5\,dy$, then

$$d\omega = \underline{\hspace{7em}}\,dy\wedge dz + \underline{\hspace{3em}}\,dz\wedge dx + \underline{\hspace{6em}}\,dx\wedge dy.$$

(3) Let $\omega = \cos(xy^2)\,dx\wedge dz$. Then (in simplified form)

$$d\omega = \underline{\hspace{18em}}\ .$$

(4) If $\omega = xy^2z^3\,dx + \sin(xy)\,dy + e^{yz}\,dz$, then

$$d\omega = \underline{\hspace{4em}}\,dy\wedge dz + \underline{\hspace{4em}}\,dz\wedge dx + \underline{\hspace{8em}}\,dx\wedge dy.$$

(5) Let $\omega = x^2y\,dy - xy^2\,dx$. Then (in simplified form)

$$d\omega = \underline{\hspace{12em}}\ .$$

(6) Let $\omega = x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$. Then (in simplified form)

$$d\omega = \underline{\hspace{12em}}\ .$$

(7) Let f be a 0-form. Then (in simplified form)

$$d(f\,dx) = \underline{\hspace{2em}}\,dy\wedge dz + \underline{\hspace{2.5em}}\,dz\wedge dx + \underline{\hspace{2.5em}}\,dx\wedge dy.$$

(8) Let $\omega = 3xz\,dx + xy^2\,dy$ and $\mu = x^2y\,dx - 6xy\,dz$. Then (in simplified form)

$$d(\omega\wedge\mu) = \underline{\hspace{16em}}\ .$$

(9) Let $\omega = yz\,dx + xz\,dy + xy\,dz$. Then

$$d\omega = \underline{\hspace{2.5em}}\,dy\wedge dz + \underline{\hspace{2.5em}}\,dz\wedge dx + \underline{\hspace{2.5em}}\,dx\wedge dy.$$

(10) Let $\omega = 2x^5e^z\,dx + y^3\sin z\,dy + (x^2+y)\,dz$. Then

$$d\omega = \underline{\hspace{5em}}\,dy\wedge dz + \underline{\hspace{6em}}\,dz\wedge dx + \underline{\hspace{4em}}\,dx\wedge dy.$$

(11) Let $\omega = x\,dx + xy\,dy + xyz\,dz$. Then

$$d\omega = \underline{\hspace{4em}}\,dy\wedge dz + \underline{\hspace{4em}}\,dz\wedge dx + \underline{\hspace{4em}}\,dx\wedge dy.$$

40.3. Problems

(1) Let U be a region in $\mathbb{R}^2$ and $\mathbf{F} : U \longrightarrow \mathbb{R}^2$ be a smooth vector field. Consider the change of coordinates in the plane given by

$$(x, y) = \mathbf{F}(u, v).$$

(a) Show that

$$dx \wedge dy = \det[d\mathbf{F}]\, du \wedge dv.$$

(b) Apply (a) to the transformation taking polar to rectangular coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

(2) Let U be a region in $\mathbb{R}^3$ and $\mathbf{F} \colon U \longrightarrow \mathbb{R}^3$ be a smooth vector field. Consider the change of coordinates in $\mathbb{R}^3$ given by

$$(x, y, z) = \mathbf{F}(u, v, w).$$

(a) Show that

$$dx \wedge dy \wedge dz = \det[d\mathbf{F}]\, du \wedge dv \wedge dw.$$

(b) Apply (a) to the transformation taking cylindrical to rectangular coordinates.
(c) Apply (a) to the transformation taking spherical to rectangular coordinates.

40.4. Answers to Odd-Numbered Exercises

(1) $3x^2 + 2\sin z,\ 2y,\ 2x\cos z$

(3) $2xy\sin(xy^2)\,dx\wedge dy\wedge dz$

(5) $4xy\,dx\wedge dy$

(7) $0,\ f_3,\ -f_2$

(9) $0,\ 0,\ 0$

(11) $xz,\ -yz,\ y$

CHAPTER 41

THE HODGE STAR OPERATOR

41.1. Background

Topics: Hodge star operator.

41.1.1. Definition. In $\mathbb{R}^3$ the HODGE STAR OPERATOR $*$ maps k-forms to $(3-k)$-forms for $0 \leq k \leq 3$. That is, if ω is a k-form then $*\,\omega$ is a $(3-k)$-form. The operator has the following properties:

(i) $*\,(\omega + \mu) = *\,\omega + *\,\mu$ if ω and μ are k-forms;
(ii) $*\,(f\omega) = f(*\,\omega)$ if f is a 0-form and ω is a k-form;
(iii) $*\,1 = dx \wedge dy \wedge dz$;
(iv) $*\,dx = dy \wedge dz$; $*\,dy = dz \wedge dx$; $*\,dz = dx \wedge dy$; and
(v) $*\,*\,\omega = \omega$ for every k-form ω.

Note: In even dimensional spaces $\mathbb{R}^n$ (n an *even* integer) condition (v) must be replaced by $*\,*\,\omega = (-1)^{k(n-k)}\omega$ when ω is a k-form.

41.1.2. Example. Let $\omega = x^2yz\,dy \wedge dz + y\sin z\,dx \wedge dy + e^{xy}\,dx \wedge dz$. Then

$$*\,\omega = x^2yz\,dx - e^{xy}\,dy + y\sin z\,dz.$$

41.2. Exercises

(1) If ω is the 3-form $(x^2y + 3y^2z^5)\,dx \wedge dy \wedge dz$ then $*\,\omega =$ ________________ .

(2) If ω is the 2-form $z\,dx\wedge dy + yz\,dx\wedge dz + xy^2\,dy\wedge dz$, then $*\,\omega =$ ________________________ .

(3) Let $\omega = 3xz\,dx + y^2z\,dy + x^2y^3\,dz$. Then $*\,d*\,\omega =$ ________________ .

(4) Let $\omega = 3xz\,dx + y^2z\,dy + x^2y^3\,dz$. Then $*\,d\omega = P\,dx + Q\,dy + R\,dz$ where $P =$ ________________ , $Q =$ ________________ , and $R =$ ________ .

(5) Suppose we wish to define the INNER PRODUCT (or DOT PRODUCT) of two 1-forms $\omega = P\,dx + Q\,dy + R\,dz$ and $\mu = S\,dx + T\,dy + U\,dz$ to be the 0-form $PS + QT + RU$. Rephrase this definition without mentioning the components of ω and μ.

Answer: $\langle\omega,\mu\rangle = \omega\cdot\mu =$ ________________ .

(6) Let $\omega = \sin(yz)\,dy \wedge dz + xyz\,dz \wedge dx + e^{xz}\,dx \wedge dy$. Then $*\,d*\,\omega = P\,dx + Q\,dy + R\,dz$ where $P =$ ______ , $Q =$ ________________ , and $R =$ ________________ .

(7) Let $\omega = \sin(yz)\,dy \wedge dz + xyz\,dz \wedge dx + e^{xz}\,dx \wedge dy$. Then $d*\,d\omega = P\,dx + Q\,dy + R\,dz$ where $P =$ ________________ , $Q =$ ______ , and $R =$ ________________ .

(8) Let f be a scalar field. Write *Laplace's equation* $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$ in terms of d, $*$, and f only.

Answer: ________________ $= 0$.

(9) Suppose we wish to define the SCALAR TRIPLE PRODUCT $[\omega,\mu,\eta]$ of three 1-forms $\omega = P\,dx + Q\,dy + R\,dz$, $\mu = S\,dx + T\,dy + U\,dz$, and $\eta = V\,dx + W\,dy + X\,dz$ to be the 0-form $\det\begin{bmatrix} P & Q & R \\ S & T & U \\ V & W & X \end{bmatrix}$. Rephrase this definition without mentioning the components of ω, μ, and η.

Answer: $[\omega,\mu,\eta] =$ ________________ .

41.3. Problems

(1) Let $\mathbf{F}$ and $\mathbf{G}$ be vector fields on a region in $\mathbb{R}^3$. Show that if ω and μ are, respectively, their associated 1-forms, then $*(\omega \wedge \mu)$ is the 1-form associated with $\mathbf{F} \times \mathbf{G}$.

41.4. Answers to Odd-Numbered Exercises

(1) $x^2y + 3y^2z^5$

(3) $3z + 2yz$

(5) $*(\omega \wedge (*\mu)\,)$ (**or** $*(\,(*\omega) \wedge \mu)$)

(7) $(xz+1)e^{xz} + z,\ 0,\ x(1 + xe^{xz})$

(9) $*(\omega \wedge \mu \wedge \eta)$

CHAPTER 42

CLOSED AND EXACT DIFFERENTIAL FORMS

42.1. Background

Topics: closed differential forms, exact differential forms.

42.1.1. Definition. A k-form ω is CLOSED if $d\omega = 0$. It is EXACT if there exists a $(k-1)$-form η such that $\omega = d\eta$.

42.2. Exercises

(1) The 1-form $\omega = ye^{xy}\,dx + xe^{xy}\,dy$ ____________ (is/is not) exact. If it is then $\omega = df$ where $f(x, y, z) =$ ________________ .

(2) The 1-form $\omega = x\sin y\,dx + y\cos x\,dy$ ____________ (is/is not) exact. If it is then $\omega = df$ where $f(x, y, z) =$ ________________ .

(3) The 1-form $\omega = \left(\dfrac{\arctan y}{\sqrt{1-x^2}} + \dfrac{x}{y} + 3x^2\right) dx + \left(\dfrac{\arcsin x}{1+y^2} - \dfrac{x^2}{2y^2} + e^y\right) dy$ ________ (is/is not) exact. If it is then $\omega = df$ where

$f(x, y, z) =$ ____ .

(4) The 1-form $\omega = \left(\dfrac{2x^3y + 2xy + 1}{1+x^2}\right) dx + (x^2 + e^z)\,dy + (ye^z + 2z)\,dz$ ________ (is/is not) exact. If it is then $\omega = df$ where

$f(x, y, z) =$ ____ .

(5) The 1-form $\omega = (yze^{xyz} + 2xy^3)\,dx + (xze^{xyz} + 3x^2y^2 + \sin z)\,dy + (xye^{xyz} + y\cos z + 4z^3)\,dz$ _________ (is/is not) exact. If it is, then $\omega = df$ where

$f(x, y, z) =$ ____ .

(6) Solve the initial value problem

$$e^x \cos y + 2x - e^x(\sin y)y' = 0, \qquad y(0) = \pi/3.$$

Hint. Why is the 1-form $(e^x \cos y + 2x)\,dx - e^x(\sin y)\,dy$ exact?

Answer: $y(x) =$ ____________________________ .

(7) Solve the differential equation $2x^3y^2 + x^4y\,y' = 0$ on the interval $1 \le x \le 10$ subject to the condition $y(2) = \frac{1}{2}$. *Hint.* Is $2x^3y^2\,dx + x^4y\,dy$ exact?

Answer: $y(x) =$ ____________ .

42.3. Problems

(1) Show that every exact k-form is closed.

(2) Show that if ω and μ are closed differential forms, then so is $\omega \wedge \mu$.

(3) Show that if ω is exact and μ is closed, then $\omega \wedge \mu$ is exact.

(4) Show that if the 1-form $\omega = P\,dx + Q\,dy + R\,dz$ is exact, then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$.

(5) Suppose that $\mathbf{F}$ is a smooth vector field in $\mathbb{R}^3$ and that ω is its associated 1-form. Show that $*\,d\omega$ is the 1-form associated with $\operatorname{curl}\mathbf{F}$.

(6) Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ and ω be its associated 1-form. Show that $*\,d*\omega = \operatorname{div}\mathbf{F}$.

(7) Let f be a smooth scalar field (that is, a 0-form) in $\mathbb{R}^3$. Use differential forms (but not partial derivatives) to show that $\operatorname{curl}\operatorname{grad} f = \mathbf{0}$.

(8) Let $\mathbf{F}$ be a vector field on an open subset of $\mathbb{R}^3$. Use differential forms (but not partial derivatives) to show that $\operatorname{div}\operatorname{curl}\mathbf{F} = 0$.

(9) Use differential forms to show that the cross product of two irrotational vector fields is incompressible (solenoidal). *Hint.* Show (without using partial derivatives) that $\operatorname{div}(\omega \times \mu) = \operatorname{curl}\omega \cdot \mu - \omega \cdot \operatorname{curl}\mu$.

(10) Explain how you know that there does not exist a vector field defined on $\mathbb{R}^3$ whose curl is $yz^2\,\mathbf{i} + x^4yz\,\mathbf{j} + y^2z\,\mathbf{k}$.

42.4. Answers to Odd-Numbered Exercises

(1) is, $\exp(xy)$

(3) is, $\arcsin x \, \arctan y + \dfrac{x^2}{2y} + x^3 + e^y$

(5) is, $\exp(xyz) + x^2y^3 + y\sin z + z^4$

(7) $\dfrac{2}{x^2}$

Part 11

THE FUNDAMENTAL THEOREM OF CALCULUS

CHAPTER 43

MANIFOLDS AND ORIENTATION

43.1. Background—The Language of Manifolds

Topics: manifolds, parametrization, orientation, unit normal vector

In these notes, for simplicity, we consider only manifolds contained in $\mathbb{R}^3$. For more serious applications manifolds in Euclidean spaces $\mathbb{R}^n$ of arbitrary dimensions must be considered.

A 0-MANIFOLD is a point (or finite collection of points).

A function is SMOOTH if it is infinitely differentiable (that is, if it has derivatives of all orders).

A CURVE is a continuous image of a closed line segment in $\mathbb{R}$.

Let C be a curve. The choice of an interval $[a, b]$ and a continuous function f such that $C = f([a, b])$ is a PARAMETRIZATION of C. If the function f is smooth, we say that C is a SMOOTH CURVE.

A 1-MANIFOLD is a curve (or finite collection of curves). A 1-manifold is FLAT if it is contained in some line in $\mathbb{R}^3$. For example, the line segment connecting two points in $\mathbb{R}^3$ is a flat 1-manifold.

A SURFACE is a continuous image of a closed rectangular region in $\mathbb{R}^2$.

Let S be a surface. The choice of an interval $R = [a_1, b_1] \times [a_2, b_2]$ and a continuous function f such that $S = f(R)$ is a PARAMETRIZATION of S. If the function f is smooth, we say that S is a SMOOTH SURFACE.

A 2-MANIFOLD is a surface (or finite collection of surfaces). A 2-manifold is FLAT if it is contained in some plane in $\mathbb{R}^3$. For example, the triangular region connecting the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ is a flat 2-manifold.

A SOLID is a continuous image of the 3-dimensional region determined by a closed rectangular parallelepiped (to avoid a ten-syllable name many people say *rectangular solid* or even just *box*) in $\mathbb{R}^3$.

Let E be a solid. The choice of a rectangular parallelepiped $P = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ and a continuous function f such that $E = f(P)$ is a PARAMETRIZATION of E. If the function f is smooth, we say that E is a SMOOTH SOLID.

A 3-MANIFOLD is a solid (or finite collection of solids).

43.1.1. Notation. Let P, Q, and R be points in $\mathbb{R}^3$. We denote by $\langle P, Q\rangle$ the oriented line segment starting at P and ending at Q. (It should be clear from context when this notation indicates a line segment and when it indicates an inner product.) Also we denote by $\langle P, Q, R\rangle$ the oriented triangular region whose vertices are P, Q, and R and whose orientation is P first, then Q, then R.

Notice that $\partial\langle P, Q\rangle = Q - P$ and that $\partial\langle P, Q, R\rangle = \langle P, Q\rangle + \langle Q, R\rangle + \langle R, P\rangle$.

In the following paragraphs we will lay out some conventions for assigning orientations to the manifolds we are considering. It should be noted that not every manifold admits an orientation. When, for example, there is no way of assigning in a continuous fashion a unit tangent vector to each point of a smooth 2-manifold, as is the case for a Möbius strip, we say that the manifold is *nonorientable*.

Each of the manifolds we consider can be given two orientations, which we call, somewhat arbitrarily, *positive* and *negative*. If M is an *oriented manifold* (that is, a manifold together with an orientation), we denote by $-M$ the same manifold with the opposite orientation. Furthermore, if M_1 and M_2 are oriented k-manifolds, we denote by $M_1 + M_2$ the union of M_1 and M_2 where each part keeps its original orientation.

Is there any point to defining $-M$ and $M_1 + M_2$? For our purposes there is exactly one reason: it has to do with integration. Shortly we will discuss (for $k = 0$, 1, 2, and 3) what it means to *integrate* a differential form *over* an oriented manifold. We will adopt the following very important conventions:

(1) Whenever M is an oriented k-manifold and ω is a k-form defined on some open set containing M, we *define*
$$\int_{-M} \omega := -\int_M \omega\,.$$
(2) Whenever M_1 and M_2 are oriented k-manifolds and ω is a k-form defined on some open set containing $M_1 \cup M_2$, we *define*
$$\int_{M_1+M_2} \omega := \int_{M_1} \omega \;+ \int_{M_2} \omega,.$$

Oriented points

A point P can be assigned one of two orientations: $+1$ or -1. There is no geometrical significance in this assignment.

Oriented curves

An example of a *flat* 1-manifold is an interval in $\mathbb{R}$. Its *positive* orientation is taken to be in the direction of increasing values. That is, if you think of the interval as a subset of the x-axis; its positive orientation is from left to right.

The orientations of an orientable 1-manifold are induced by its parametrization. If $\mathbf{r}(t) = (r^1(t), r^2(t), r^3(t))$ (where $a \le t \le b$) is the parametrization of a smooth curve C in $\mathbb{R}^3$, the *positive* orientation of C is in the direction from its “starting point” $\mathbf{r}(a)$ to its “ending point” $\mathbf{r}(b)$.

The orientation of a 1-manifold induces an orientation on its boundary. If $\mathbf{r}(t) = (r^1(t), r^2(t), r^3(t))$ $(a \le t \le b)$ is the parametrization of the curve C, then the boundary of C consists of its positively oriented ending point together with its negatively oriented starting point; that is,
$$\partial C = \mathbf{r}(b) + (-\mathbf{r}(a))\,.$$

43.1.2. CAUTION. If P and Q are points most people write $Q - P$ for $Q + (-P)$. Do *not* interpret this or the right side of the preceding formula in terms of the ordinary vector arithmetic on coordinates of points in $\mathbb{R}^3$.

Oriented surfaces

An example of a *flat* 2-manifold is a rectangle in $\mathbb{R}^2$. Let R be the rectangle $[a_1, b_1] \times [a_2, b_2]$ in the xy-plane. Its *positive* orientation is the one determined by the unit vector $\mathbf{k} = \mathbf{i} \times \mathbf{j}$. This is often described as the “counterclockwise” orientation. Of course, the connection between these two descriptions is given by the familiar “right-hand rule”.

An orientation of a 2-manifold is induced by its parametrization. Suppose that $\mathbf{r}(u,v) = (r^1(u,v), r^2(u,v), r^3(u,v))$ (where $a_1 \le u \le b_1$ and $a_2 \le v \le b_2$) is the parametrization of a surface S. Then we take the *positive* orientation of S to be in the direction of its normal vector $\mathbf{r}_u \times \mathbf{r}_v$.

An orientation of a 2-manifold induces an orientation on its boundary. Suppose that $\mathbf{r}(u,v) = (r^1(u,v), r^2(u,v), r^3(u,v))$ ($a_1 \le u \le b_1$ and $a_2 \le v \le b_2$) is the parametrization of a surface S and that S is bounded by a finite collection of simple closed piecewise smooth curves C_1, ..., C_p. We write

$$\partial S = \sum_{k=1}^{p} C_k$$

where we have chosen the orientation on each C_k by selecting a parametrization of C_k in such a way that at each point p where the parametrization is differentiable the binormal vector $\mathbf{B}(p) = \mathbf{T}(p) \times \mathbf{N}(p)$ points *away* from the surface. (Here, $T(p)$ is the unit tangent vector to the curve C_k at p and $\mathbf{N}(p)$ is the usual unit normal which results from normalizing $\mathbf{r}_u \times \mathbf{r}_v$.) A somewhat more informal way of saying the same thing is, "if you walk around each of the curves C_k in a positive direction with your head pointing in the direction of $\mathbf{N}$, then the surface will always be on your left."

Oriented solids

An example of a *flat* 3-manifold is a rectangular parallelepiped in $\mathbb{R}^3$. Let $R = [a,b]\times[c,d]\times[e,f]$ in $\mathbb{R}^3$. Just as we think of the positive (counterclockwise) orientation in $\mathbb{R}^2$ as "going from the x-axis to the y-axis" we specify the *positive* orientation in $\mathbb{R}^3$ by selecting first the x-axis, then the y-axis, and finally the z-axis. Any cyclic permutation of this choice (first y then z then x—or—first z then x then y) is also regarded as positive. Other permutations (first y then x then z, for example) produce *negative* orientations.

An orientation of a 3-manifold is induced by its parametrization. If $\mathbf{r}(u,v,w) = (r^1(u,v,w), r^2(u,v,w), r^3(u,v,w))$ (where $a_1 \le u \le b_1$ and $a_2 \le v \le b_2$ and $a_3 \le w \le b_3$) is the parametrization of a solid E, the orientation of E is *positive* if the Jacobian of $\mathbf{r}$ is everywhere positive.

An orientation of a 3-manifold induces an orientation on its boundary. Suppose that $\mathbf{r}(u,v,w) = (r^1(u,v,w), r^2(u,v,w), r^3(u,v,w))$ (where $a_1 \le u \le b_1$ and $a_2 \le v \le b_2$ and $a_3 \le w \le b_3$) is a positive parametrization of a solid E and that E is bounded by a finite collection of simple closed piecewise smooth surfaces S_1, ..., S_p. We write

$$\partial E = \sum_{k=1}^{p} S_k$$

where we have chosen the orientation on each S_k by selecting a parametrization of S_k in such a way that at each point p where the parametrization is differentiable the normal vector $\mathbf{N}(p)$ points *away* from the solid. That is, the positive orientation induced by a solid on its boundary is the "outward directed normal".

43.2. Exercises

(1) Let R be the surface parametrized by

$$\mathbf{r}\colon [0,2]\times[0,1] \longrightarrow \mathbb{R}^3\colon (u,v)\mapsto (u,v,0)\,.$$

Then the (positive) orientation induced by $\mathbf{r}$ is determined by the unit normal vector, which is given by

$$\mathbf{N}(u,v) = (\_\_\_\_\_\,,\,\_\_\_\_\_\,,\,\_\_\_\_\_\,)\,.$$

We can write the boundary of R as the sum of four simple smooth curves

$$\partial R = C_1 + C_2 + C_3 + C_4$$

whose induced orientations are given by

(a) $r_1(t) = (\,\_\_\_\_\_\,,\,0\,,\,0\,)$ where $0\le t\le 1$,
(b) $r_2(t) = (\,2\,,\,\_\_\_\_\_\,,\,0\,)$ where $0\le t\le 1$,
(c) $r_3(t) = (\,\_\_\_\_\_\_\_\_\,,\,1\,,\,0\,)$ where $0\le t\le 1$, and
(d) $r_4(t) = (\,0\,,\,\_\_\_\_\_\_\_\,,\,0\,)$ where $0\le t\le 1$.

(2) Let R be the surface parametrized by

$$\mathbf{r}\colon [0,2]\times[0,1] \longrightarrow \mathbb{R}^3\colon (u,v)\mapsto (v,u,0)\,.$$

Then the (positive) orientation induced by $\mathbf{r}$ is determined by the unit normal vector, which is given by

$$\mathbf{N}(u,v) = (\_\_\_\_\_\,,\,\_\_\_\_\_\,,\,\_\_\_\_\_\,)\,.$$

We can write the boundary of R as the sum of four simple smooth curves

$$\partial R = C_1 + C_2 + C_3 + C_4$$

whose induced orientations are given by

(a) $r_1(t) = (\,0\,,\,\_\_\_\_\_\,,\,0\,)$ where $0\le t\le 1$
(b) $r_2(t) = (\,\_\_\_\_\_\,,\,1\,,\,0\,)$ where $0\le t\le 1$,
(c) $r_3(t) = (\,2\,,\,\_\_\_\_\_\_\_\,,\,0\,)$ where $0\le t\le 1$, and
(d) $r_4(t) = (\,\_\_\_\_\_\_\_\_\,,\,0\,,\,0\,)$ where $0\le t\le 1$.

(3) Let C be the curve parametrized by

$$\mathbf{r}(t) = (\cos t, \sin t, 2t) \quad \text{where } -\pi \le t \le \tfrac{9}{2}\pi.$$

Then $\partial C = Q - P$ where P is the point whose coordinates are $(\,\_\_\_\_\_\_\,,\,\_\_\_\_\_\_\,,\,\_\_\_\_\_\_\,)$ and Q is the point whose coordinates are $(\,\_\_\_\_\_\_\,,\,\_\_\_\_\_\_\,,\,\_\_\_\_\_\_\,)$.

(4) Let $D = \{(u,v)\colon 0\le v\le 20 \text{ and } 0\le u\le 2v\}$ and T be the region in the xy-plane given by the parametrization

$$\mathbf{r}(u,v)\colon D \longrightarrow \mathbb{R}^2\colon (u,v)\mapsto \left(\tfrac{1}{5}u+\tfrac{1}{5}v, -\tfrac{3}{5}u+\tfrac{2}{5}v\right).$$

Then $\partial T = \langle O,P\rangle + \langle P,Q\rangle + \langle Q,O\rangle$ where O is the origin, P is the point with coordinates $(\,\_\_\_\,,\,\_\_\_\,)$, and Q is the point with coordinates $(\,\_\_\_\,,\,\_\_\_\,)$.

(5) Let A be the rectangular solid $[0,1]\times[0,\pi]\times[0,2\pi]$ and E be the solid parametrized by

$$\mathbf{R}(u,v,w) = (u\sin v\cos w, u\sin v\sin w, u\cos v)$$

where $(u,v,w)\in A$. List four geometric/topological properties which characterize the solid E:

(a) ____________________________;
(b) ____________________________;
(c) ____________________________; and
(d) ____________________________.

We know that E is positively parametrized by $\mathbf{R}$ because the Jacobian of $\mathbf{R}$ is given by

$$\mathcal{J}\mathbf{R}(u,v,w) = \det[d\mathbf{R}_{(u,v,w)}] = \underline{\hspace{3cm}}\ ,$$

which is positive for all $(u,v,w) \in A$.

Parametrize the boundary of E by setting

$$\mathbf{r}\colon [0,\pi] \times [0,2\pi] \longrightarrow \mathbb{R}^3 \colon (v,w) \mapsto \mathbf{R}(1,v,w)\,.$$

The unit normal vector $\mathbf{N}$ to ∂E is given by

$$\mathbf{N}(v,w) = (\underline{\hspace{3cm}}\,,\ \underline{\hspace{3cm}}\,,\ \underline{\hspace{3cm}}).$$

We know that $\mathbf{r}$ is the parametrization of ∂E induced by the parametrization $\mathbf{R}$ of E because

$$\underline{\hspace{12cm}}\ .$$

(6) Let E be the unit cube $[0,1] \times [0,1] \times [0,1]$. Also let
S_1 be the face of the cube lying in the xy-plane,
S_2 be the face of the cube lying in the xz-plane,
S_3 be the face of the cube lying in the plane $x = 1$,
S_4 be the face of the cube lying in the plane $y = 1$,
S_5 be the face of the cube lying in the yz-plane, and
S_6 be the face of the cube lying in the plane $z = 1$.
In the following D denotes the unit square $[0,1] \times [0,1]$. By an "appropriate" parametrization we mean one that produces the usual positive (outwards directed) orientation.

(a) An appropriate parametrization for S_1 is
$$\mathbf{r}\colon D \longrightarrow \mathbb{R}^3 \colon (u,v) \mapsto (\underline{\hspace{1cm}}\,,\ \underline{\hspace{1cm}}\,,\ \underline{\hspace{1cm}}).$$
(b) An appropriate parametrization for S_2 is
$$\mathbf{r}\colon D \longrightarrow \mathbb{R}^3 \colon (u,v) \mapsto (\underline{\hspace{1cm}}\,,\ \underline{\hspace{1cm}}\,,\ \underline{\hspace{1cm}}).$$
(c) An appropriate parametrization for S_3 is
$$\mathbf{r}\colon D \longrightarrow \mathbb{R}^3 \colon (u,v) \mapsto (\underline{\hspace{1cm}}\,,\ \underline{\hspace{1cm}}\,,\ \underline{\hspace{1cm}}).$$
(d) An appropriate parametrization for S_4 is
$$\mathbf{r}\colon D \longrightarrow \mathbb{R}^3 \colon (u,v) \mapsto (\underline{\hspace{1cm}}\,,\ \underline{\hspace{1cm}}\,,\ \underline{\hspace{1cm}}).$$
(e) An appropriate parametrization for S_5 is
$$\mathbf{r}\colon D \longrightarrow \mathbb{R}^3 \colon (u,v) \mapsto (\underline{\hspace{1cm}}\,,\ \underline{\hspace{1cm}}\,,\ \underline{\hspace{1cm}}).$$
(f) An appropriate parametrization for S_6 is
$$\mathbf{r}\colon D \longrightarrow \mathbb{R}^3 \colon (u,v) \mapsto (\underline{\hspace{1cm}}\,,\ \underline{\hspace{1cm}}\,,\ \underline{\hspace{1cm}}).$$

43.3. Problems

(1) Let T be the triangular region consisting of all those points (x, y) in $\mathbb{R}^2$ such that $x,\, y \geq 0$ and $x + y \leq 1$. Show by direct calculation that $\partial^2 T = 0$.

(2) Let T be the tetrahedron consisting of all those points (x, y, z) in $\mathbb{R}^3$ such that $x,\, y,\, z \geq 0$ and $x + y + z \leq 1$. Show by direct calculation that $\partial^2 T = 0$.

(3) Sketch the boundaries of the following parametrized surfaces and use arrows to indicate the direction of orientation.
   (a) $\mathbf{r}(u, v) = \sin u \cos v\, \mathbf{i} + \sin u \sin v\, \mathbf{j} + \cos u\, \mathbf{k}$ with $0 \leq u \leq \frac{\pi}{2}$ and $0 \leq v \leq 2\pi$.
   (b) $\mathbf{R}(u, v) = \sin u \sin v\, \mathbf{i} + \sin u \cos v\, \mathbf{j} + \cos u\, \mathbf{k}$ with $0 \leq u \leq \frac{\pi}{2}$ and $0 \leq v \leq 2\pi$.

(4) Sketch the boundaries of the following parametrized surfaces and use arrows to indicate the direction of orientation.
   (a) $\mathbf{r}(u, v) = u\, \mathbf{i} + v\, \mathbf{j} + (1 - u - v)\, \mathbf{k}$ with $u,\, v \geq 0$ and $u + v \leq 1$.
   (b) $\mathbf{R}(u, v) = v\, \mathbf{i} + u\, \mathbf{j} + (1 - u - v)\, \mathbf{k}$ with $u,\, v \geq 0$ and $u + v \leq 1$.

(5) Sketch the boundaries of the following parametrized surfaces and use arrows to indicate the direction of orientation.
   (a) $\mathbf{r}(u, v) = u\, \mathbf{i} + v\, \mathbf{j} + (1 - v^2)\, \mathbf{k}$ with $0 \leq u \leq 1$ and $-1 \leq v \leq 1$.
   (b) $\mathbf{R}(u, v) = v\, \mathbf{i} + u\, \mathbf{j} + (1 - u^2)\, \mathbf{k}$ with $-1 \leq u \leq 1$ and $0 \leq v \leq 1$.

43.4. Answers to Odd-Numbered Exercises

(1) 0, 0, 1, $2t$, t, $2 - 2t$, $1 - t$

(3) -1, 0, -2π, 0, 1, 9π

(5) closed, ball, centered at the origin, has radius 1 (the preceding in any order), $u \sin v$, $\sin v \cos w$, $\sin v \sin w$, $\cos v$, the normal $\mathbf{N}$ is directed outwards

CHAPTER 44

LINE INTEGRALS

44.1. Background

Topics: line integrals, *Green's theorem*.

44.1.1. Definition. Let $\mathbf{r}\colon [a,b] \longrightarrow \mathbb{R}^3$ be a parametrization of a curve C:

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

If f is a scalar field in $\mathbb{R}^3$ whose domain is an open set containing C, we define

$$\int_C f\,ds := \int_a^b f(\mathbf{r}(t))\,\|\,\mathbf{r}'(t)\|\,dt.$$

It would seem appropriate to call this integral a *curve integral*; it is usually called a LINE INTEGRAL.

Suppose next that $\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$ is a vector field in $\mathbb{R}^3$ whose domain is an open set containing C. Then we define the

INTEGRAL OF (THE TANGENTIAL COMPONENT OF) $\mathbf{F}$ OVER C by:

$$\int_C \mathbf{F}\cdot d\mathbf{r} := \int_a^b \langle\,\mathbf{F}(\mathbf{r}(t)), \mathbf{r}'(t)\,\rangle\,dt.$$

(Some people remember this formula by pretending that the equation $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ actually makes sense, and then substituting $\dfrac{dx}{dt}\,dt$ for dx, and so on.) Another notation for this integral is $\displaystyle\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ (where $\mathbf{T}$ is the unit tangent vector $d\mathbf{r}/ds$).

Suppose finally that ω is the 1-form

$$P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz.$$

Then we define

$$\int_C \omega := \int_C \mathbf{F}\cdot d\mathbf{r}$$

where $\mathbf{F}$ is the vector field associated with the 1-form ω.

44.2. Exercises

(1) Let C be the curve given by $\mathbf{r}(t) = \mathbf{i} + 2\,\mathbf{j} + t^2\,\mathbf{k}$ for $0 \le t \le 1$. Then $\displaystyle\int_C e^{\sqrt{z}}\,ds =$ ____ .

(2) Let C be the curve $x = y^2$ starting at $(1,1)$ and ending at $(9,3)$. Then

$$\int_C (-y\,dx + 5x\,dy) = \underline{\qquad\qquad}\,.$$

(3) Let $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and C be the curve given by $\mathbf{r}(t) = t\,\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k}$ for $0 \le t \le 1$. Then $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} =$ ________ .

(4) Let ω be the 1-form associated with the vector field $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and let C be the curve given by $\mathbf{r}(t) = t\,\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k}$ for $0 \le t \le 1$. Then $\displaystyle\int_C \omega =$ ________ .

(5) Let C_1 be the shortest path along the unit circle $x^2 + y^2 = 1$ from $(1,0)$ to $(0,1)$ and C_2 be the line segment from $(0,1)$ to $(4,3)$. Also let $\omega = y\,dx - x\,dy$. Then $\displaystyle\int_{C_1+C_2} \omega = a - \tfrac{1}{2}b$ where $a =$ ____ and $b =$________ .

(6) Let C be the curve given by $\mathbf{r}(t) = t\,\mathbf{i} + 2t\,\mathbf{j} + 3t\,\mathbf{k}$ for $0 \le t \le 1$. Then $\displaystyle\int_C yz\,ds = a\sqrt{b}$ where $a =$ ____ and $b =$ ______ .

(7) Let C be the curve given by $\mathbf{r}(t) = t\,\mathbf{i} + \frac{2}{3}t^{3/2}\,\mathbf{j} + t\,\mathbf{k}$ for $2 \le t \le 7$. Then $\displaystyle\int_C \frac{x+y}{y+z}\,ds = \frac{a}{3}$ where $a =$ ________ .

(8) The value of the line integral of the vector field $x\,\mathbf{i} + y\,\mathbf{j}$ around the astroid $x^{2/3} + y^{2/3} = 1$ is ______ .

(9) A thin wire has the shape of a quarter of the circle $x^2 + y^2 = 64$ with $x \ge 0$ and $y \ge 0$. Suppose that its density function is $\rho(x,y) = x+y$. Then the mass of the wire is __________ and its center of mass is (__________ , __________) .

(10) A wire lies along the intersection of the sphere $x^2+y^2+z^2 = 1$ and the plane $x+y+z = 0$. Its linear density at each point (x,y,z) is x^2 grams per unit length. The mass of the wire is __________ grams.

(11) Let C be the curve $y^2 = x^3$ starting at $(1,-1)$ and ending at $(1,1)$ and ω be the differential form $(5y - x)\,dx + x^2y\,dy$. Then $\displaystyle\int_C \omega =$ ________ .

(12) Let ω be the 1-form associated with the vector field $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and let C be the curve given by $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j}$ for $0 \le t \le 2\pi$. Then $\displaystyle\int_C \omega =$ ________ .

(13) Let $\omega = x\,dx + y\,dy + z\,dz$ and C be the curve given by $\mathbf{r}(t) = t^2\,\mathbf{i} + 3t\,\mathbf{j} + 2t^3\,\mathbf{k}$ for $-1 \le t \le 2$. Then $\displaystyle\int_C \omega =$ _________ .

(14) A particle moves in a force field given by $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. The work done in moving the particle along the parabola $y = x^2$ in the xy-plane from $x = -1$ to $x = 2$ is _____ .

(15) Let C be the curve given by $\mathbf{r}(t) = (20\cos^3 t, 20\sin^3 t)$ for $0 \le t \le \pi/2$ and let $f(x,y,z) = 1 + \frac{1}{2}y$. Then $\int_C f(x,y)\,ds =$ ________ .

(16) If the curve C is given by $\mathbf{r}(t) = t\,\mathbf{i} + t^n\,\mathbf{j}$ where $0 \le t \le 1$ and n is a natural number, then $\displaystyle\int_C y\,dx + (3y^3 - x)\,dy + z\,dz = \frac{a-n}{b+bn}$ where $a =$ _____ and $b =$ _____ .

(17) If C is the portion of the parabola $y = x^2$ lying between $(-2,4)$ and $(1,1)$ and $\omega = (x - 2y^2)\,dy$, then $\displaystyle\int_C \omega =$ ________ .

(18) Let C be the path consisting of the straight lines segments connecting (in order) the points $(0,0,0)$, $(2,0,0)$, $(1,3,-1)$, and $(1,3,0)$. Let ω be the 1-form $yz\,dx + xz\,dy + xy\,dz$. Then $\displaystyle\int_C \omega =$ _____ .

(19) A thin wire has the shape of a portion of the helix $x = t$, $y = \cos t$, $z = \sin t$ where $0 \le t \le 2\pi$. Suppose that its density at any point is the square of its distance from the origin. Then the mass of the wire is $a\sqrt{a}b\left(\dfrac{a^2b^2}{c} + 1\right)$ and its center of mass is $\left(bc\dfrac{ab^2+1}{a^2b^2+c}, 0, 0\right)$ where $a =$ _____ , $b =$ _____ , and $c =$ _____ .

(20) Let $\mathbf{F}(x,y,z) = 2xy^3e^z\,\mathbf{i} + 3x^2y^2e^z\,\mathbf{j} + x^2y^3e^z\,\mathbf{k}$ and C be the curve given by $\mathbf{r}(t) = 4te^t\,\mathbf{i} + t^2\,\mathbf{j} + \ln(1+2t)\,\mathbf{k}$ for $0 \le t \le 1$. Then $\int_C \mathbf{F}\cdot d\mathbf{r} = a\,e^b$ where $a =$ ________ and $b =$ _____ .

(21) Let $\mathbf{F}(x,y) = e^x\cos y\,\mathbf{i} - e^x\sin y\,\mathbf{j}$ and C be the curve given by $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j}$ for $0 \le t \le \frac{\pi}{2}$. Then $\int_C \mathbf{F}\cdot d\mathbf{r} = (\cos a) - b$ where $a =$ _____ and $b =$ _____ .

(22) Let $\mathbf{F}(x,y,z) = \dfrac{y + yz + x^2y^3z}{1+x^2y^2}\,\mathbf{i} + \dfrac{x + xz + x^3y^2z}{1+x^2y^2}\,\mathbf{j} + xy\,\mathbf{k}$ and C be the curve given by $\mathbf{r}(t) = (\sin\frac{\pi}{2}t, 1 - \cos\frac{\pi}{2}t, t^3)$ for $0 \le t \le 1$. Then $\int_C \mathbf{F}\cdot d\mathbf{r} = \dfrac{\pi + a}{a}$ where $a =$ _____ .

(23) If C is the path given parametrically by $\mathbf{r}(t) = (t^8\sin^5\frac{\pi}{2}t)\,\mathbf{i} + t^{10}(1 - \cos^9\frac{\pi}{2}t)\,\mathbf{j}$ for $0 \le t \le 1$, then $\int_C (x^5 - 2xy^3)\,dx - 3x^2y^2\,dy = -\dfrac{a}{6}$ where $a =$ _____ .

(24) Let C be the curve given by $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$ for $0 \le t \le 2$ and let $\mathbf{F}(x,y,z) = 2xy^3z^4\,\mathbf{i} + 3x^2y^2z^4\,\mathbf{j} + 4x^2y^3z^3\,\mathbf{k}$. The vector field $\mathbf{F}$ is conservative because $\mathbf{F} = \nabla f$ where f is the scalar field $f(x,y,z) =$ ____________ . Thus $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = 2^p$ where $p =$ ______ .

(25) Let C be the curve given by $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$ for $0 \le t \le 2$ and let $\omega = 2xy^3z^4\,dx + 3x^2y^2z^4\,dy + 4x^2y^3z^3\,dz$. The 1-form ω is exact because $\omega = df$ where f is the 0-form $f(x,y,z) =$ ____________ . Thus $\displaystyle\int_C \omega = 2^p$ where $p =$ ______ .

(26) If C is the path given parametrically by $\mathbf{r}(t) = t^{7/3}\,\mathbf{i} + \frac{1}{3}\arcsin\frac{1}{2}(t+1)\,\mathbf{j}$ for $0 \le t \le 1$, then $\int_C (xy\cos xy + \sin xy)\,dx + (x^2\cos xy)\,dy = \dfrac{1}{a}$ where $a =$ _____ .

(27) Let C be the path given parametrically by $\mathbf{r}(t) = (t-1)e^{t^2-t}\,\mathbf{i} + (4t^2-1)\,\mathbf{j}$ for $0 \le t \le 1$, then $\int_C (y - x^2)\,dx + (x + y^2)\,dy =$ _____ .

(28) Let $\mathbf{F}(x,y,z) = (x^2+y^2+z^2)(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})$ and C be the curve given by $\mathbf{r}(t) = t^2\,\mathbf{i}+t^5\,\mathbf{j}+t^8\,\mathbf{k}$ for $0 \le t \le 1$. Then $\int_C \mathbf{F}\cdot d\mathbf{r} = \dfrac{9}{a}$ where $a =$ _____ .

(29) A force field in the plane is given by $\mathbf{F}(x,y) = \dfrac{y^2}{x^2}\,\mathbf{i} - \dfrac{2y}{x}\,\mathbf{j}$. The work done in moving a particle from the point $(2,4)$ to the point $(1,2)$ is _____ .

(30) Let $\mathbf{F}(x,y) = (x^2\sin x - y, 2x + y^3e^y)$, R be the rectangular region $[0,2]\times[0,1]$, and C be the counterclockwise path around the boundary of R. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \underline{\qquad}\,.$$

(31) Let $\mathbf{F}(x,y) = (2x - x^2y, 3y + xy^2)$ and let C_a and C_b be the circles centered at the origin with radii a and b, respectively, where $a < b$. Suppose that C_a is oriented clockwise and C_b is oriented counterclockwise. Then

$$\int_{C_a} \mathbf{F}\cdot d\mathbf{r} + \int_{C_b} \mathbf{F}\cdot d\mathbf{r} = \underline{\qquad\qquad\qquad\qquad}\,.$$

(32) Let C be the positively oriented path bounding the rectangle $[0,2]\times[0,3]$. Then

$$\int_C (x^3 - xy^2)\,dx + 2xy\,dy = \underline{\qquad\quad}\,.$$

(33) The integral of the differential form $\omega = (y^2 - \arctan x)\,dx + (3x + \sin y)\,dy$ along the curve which forms the boundary of the region $\{(x,y)\colon y \ge x^2 \text{ and } y \le 4\}$ is $-\dfrac{a}{5}$ where $a =$ ________ .

(34) The area of the region bounded by the curve $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin^3 t\,\mathbf{j}$ (where $0 \le t \le 2\pi$) is $\dfrac{a}{4}$ where $a =$ ________ .

(35) Let $\mathbf{F}(x,y) = (2xy - x\ln x)\,\mathbf{i} + (x^2y + y^2e^y)\,\mathbf{j}$, let R be the rectangular region $[1,7]\times[2,5]$, and let C be the counterclockwise path bordering R. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \underline{\qquad\qquad}\,.$$

(36) Let C be the (positively oriented) triangle whose vertices are $(0,0)$, $(1,1)$, and $(0,1)$. Then

$$\int_C (x^2 + y^2)\,dx + (x^2 - y^2)\,dy = -\frac{1}{a}$$

where $a =$ _____ .

(37) Let $\mathbf{F}(x,y) = (-y, x)$ and let C_a and C_b be the circles centered at the origin with radii a and b, respectively, where $a < b$. Suppose that C_a is oriented clockwise and C_b is oriented

counterclockwise. Then

$$\int_{C_a} \mathbf{F} \cdot d\mathbf{r} + \int_{C_b} \mathbf{F} \cdot d\mathbf{r} = \underline{\hspace{4cm}} .$$

(38) The area inside the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $b\pi a^2$ where $b =$ _________ .
Hint. $\mathbf{r}(t) = (a\cos^3 t)\,\mathbf{i} + (a\sin^3 t)\,\mathbf{j}$ (where $0 \le t \le 2\pi$) is a parametrization of the astroid.

(39) Evaluate $\displaystyle\oint_{\partial S} x^2y\,dx + xy^5\,dy$ where S is the square $\{(x, y) : |x| \le 1 \text{ and } |y| \le 1\}$.
Answer: $-\dfrac{a}{3}$ where $a =$ _____ .

(40) Let D be the unit disk $\{(x, y) : x^2 + y^2 \le 1\}$. Then $\displaystyle\oint_{\partial D} x^2y\,dx - 3y^2\,dy = -\frac{a}{4}$ where $a =$ _____ .

(41) Evaluate $\displaystyle\oint_{\partial R} (x^3 - y^3)\,dx + (x^3 + y^3)\,dy$ where R is the annular region $\{(x, y) : 1 \le x^2 + y^2 \le 9\}$.
Answer: $a\pi$ where $a =$ ___________ .

(42) Let D be the upper half of the disk of radius 2 centered at the origin of the plane and $\mathbf{F}$ be a force field given by $\mathbf{F}(x, y) = x\,\mathbf{i} + (x^3 + 3xy^2)\,\mathbf{j}$. Then the work done by the field in moving a particle counterclockwise around the boundary of D starting at the point $(-2, 0)$ is $a\pi$ where $a =$ ________ .

44.3. Problems

(1) Suppose that the smoothly parametrized curve C has length ℓ and that the vector field $\mathbf{F}$ is bounded (that is, there is a constant $M > 0$ such that $\|\mathbf{F}(x,y,z)\| \le M$ for all $(x,y,z) \in \mathbb{R}^3$). Show that

$$\left| \int_C \mathbf{F} \cdot d\mathbf{r} \right| \le \ell M .$$

(2) Let $\mathbf{F}(x,y) = F^1(x,y)\,\mathbf{i} + F^2(x,y)\,\mathbf{j} = \dfrac{-y}{x^2+y^2}\,\mathbf{i} + \dfrac{x}{x^2+y^2}\,\mathbf{j}$ and C be the unit circle given its usual parametrization $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j}$ for $0 \le t \le 2\pi$.
  (a) Check that $F^1_2(x,y) = F^2_1(x,y)$ for all (x,y) in the domain of f.
  (b) Calculate $\int\limits_C \mathbf{F} \cdot d\mathbf{r}$.
  (c) Either prove that $\mathbf{F}$ is conservative or prove that it is not.

(3) Let $\omega = \dfrac{-y}{x^2+y^2}\,dx + \dfrac{x}{x^2+y^2}\,dy$. Show that ω is a closed 1-form but that it is not exact.

(4) Let R be a region in the xy-plane bounded by a simple closed curve C. Use *Green's theorem* to show that the area A of R is given by the following expressions

$$A = \oint_C x\,dy = -\oint_C y\,dx = \tfrac{1}{2}\oint_C x\,dy - y\,dx .$$

(5) Explain how to use problem 4 above to compute the area under one loop of the cycloid $\mathbf{r}(t) = a(t - \sin t)\,\mathbf{i} + a(1 - \cos t)\,\mathbf{j}$ (where $0 \le t \le 2\pi$).

(6) Explain how to use problem 4 above to compute the area inside the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(7) Use *Green's theorem* to show that the coordinates of the centroid of a plane region bounded by a simple closed curve C are given by

$$\overline{x} = \frac{1}{2A}\oint_C x^2\,dy \quad \text{and} \quad \overline{y} = -\frac{1}{2A}\oint_C y^2 dx$$

where A is the area of the region.

(8) Explain how to use problem 7 to find the centroid of the upper half of the disk of radius a centered at the origin in $\mathbb{R}^2$. Carry out the computation you describe.

44.4. Answers to Odd-Numbered Exercises

(1) 2

(3) $\dfrac{3}{2}$

(5) 4, π

(7) 38

(9) 128, $\pi + 2$, $\pi + 2$

(11) 4

(13) 147

(15) 150

(17) 48

(19) 2, π, 3

(21) 1, e

(23) 5

(25) $x^2y^3z^4$, 20

(27) 8

(29) 0

(31) $\dfrac{\pi}{2}(b^4 - a^4)$

(33) 96

(35) 360

(37) $2\pi(b^2 - a^2)$

(39) 4

(41) 120

CHAPTER 45

SURFACE INTEGRALS

45.1. Background

Topics: surface integrals.

45.1.1. Definition. Let $\mathbf{r}\colon D \longrightarrow \mathbb{R}^3$ be a parametrization of a surface S:

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$$

If f is a scalar field in $\mathbb{R}^3$ whose domain is an open set containing S, we define

$$\iint\limits_S f\,dS := \iint\limits_D f(\mathbf{r}(u,v))\,\|\,\mathbf{r}_u \times \mathbf{r}_v\|\,du\,dv.$$

Suppose next that $\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$ is a vector field in $\mathbb{R}^3$ whose domain is an open set containing S. Then we define

$$\iint\limits_S \mathbf{F}\cdot d\mathbf{S} := \iint\limits_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA$$

Thus

$$\iint\limits_S \mathbf{F}\cdot d\mathbf{S} = \iint\limits_D \Big[P\big(x(u,v), y(u,v), z(u,v)\big)\frac{\partial(y,z)}{\partial(u,v)} + Q\big(x(u,v), y(u,v), z(u,v)\big)\frac{\partial(z,x)}{\partial(u,v)} + R\big(x(u,v), y(u,v), z(u,v)\big)\frac{\partial(x,y)}{\partial(u,v)}\Big]\,du\,dv.$$

Some people remember this formula by pretending that the equation

$$d\mathbf{S} = (dy \wedge dz)\,\mathbf{i} + (dz \wedge dx)\,\mathbf{j} + (dx \wedge dy)\,\mathbf{k}$$

actually makes sense, and then substituting $\dfrac{\partial(y,z)}{\partial(u,v)}\,du \wedge dv$ for $dy \wedge dz$ (see problem 2), and so on. Another notation for this integral is $\displaystyle\iint\limits_S \mathbf{F}\cdot\mathbf{N}\,dS$ where $\mathbf{N}(u,v) = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{\|\,\mathbf{r}_u \times \mathbf{r}_v\|}$.

Suppose finally that ω is the 2-form

$$P(x,y,z)\,dy \wedge dz + Q(x,y,z)\,dz \wedge dx + R(x,y,z)\,dx \wedge dy.$$

Then we define

$$\int\limits_S \omega := \iint\limits_S \mathbf{F}\cdot d\mathbf{S}$$

where $\mathbf{F}$ is the vector field associated with the 1-form $*\,\omega$.

45.2. Exercises

(1) Let $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}\colon (x,y,z) \mapsto xz$. Then the surface integral of the scalar field f over the triangle whose vertices are $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ is $\dfrac{\sqrt{3}}{a}$ where $a =$ ________ .

(2) Let $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and S be the triangular region whose vertices are $(2,0,0)$, $(0,-4,0)$, and $(0,0,3)$. Then $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S} =$ ________ .

(3) Let $\mathbf{F}(x,y,z) = -x\,\mathbf{i} - y\,\mathbf{j} + z^2\,\mathbf{k}$ and S be that portion of the cone $z = \sqrt{x^2+y^2}$ which lies between the planes $z = 1$ and $z = 2$. Then the surface integral of (the normal component of) the vector field $\mathbf{F}$ over the surface S is $\dfrac{a}{6}\pi$ where $a =$ ________ .

(4) Let $\omega = x\,dy \wedge dz + y\,dz \wedge dx + 2z\,dx \wedge dy$ and S be the portion of the paraboloid $z = 5 - x^2 - y^2$ that lies above the xy-plane. Then the surface integral of ω over S is $a\pi$ where $a =$ ________ .

(5) If S is that portion of the unit sphere which lies in the first octant (that is, where x, y, $z \geq 0$), then $\displaystyle\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy =$ ________ .

(6) Let $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}\colon (x,y,z) \mapsto \sqrt{1+x^2+y^2}$. Then the surface integral of the scalar field f over the helicoid given parametrically by $\mathbf{r}\colon [0,1] \times [0,\pi]\colon (u,v) \mapsto u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$ is $\dfrac{a}{3}\pi$ where $a =$ _____ .

(7) Find the surface integral of the scalar field $f(x,y,z) = xyz$ over that portion of the unit sphere centered at the origin which lies inside the cone $z = \sqrt{x^2+y^2}$. Answer: _____ .

(8) Let $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a smooth scalar field. Suppose we know that the the value $f(\mathbf{p})$ of f at each point $\mathbf{p}$ depends only on the distance of $\mathbf{p}$ from the origin and that $f(1,0,\sqrt{3}) = 7$. Then the surface integral of f over the sphere of radius 2 centered at the origin is $a\pi$ where $a =$ ________ .

(9) Let V be the solid region bounded by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$, and let $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}\colon (x,y,z) \mapsto xy$. Then the surface integral of the scalar field f over the boundary of V is $-\left(a + \dfrac{1}{a\sqrt{a}}\right)\pi$ where $a =$ _____ .

(10) Let C be the solid cylinder $\{(x,y,z)\colon x^2+y^2 \leq 9 \text{ and } 0 \leq z \leq 2\}$. Then

$$\iint_{\partial C} (x^2+y^2+z^2)\,dS = a\pi \text{ where } a = \text{\_\_\_\_\_\_\_\_\_\_} .$$

(11) Let $\mathbf{F}(x,y,z) = x^2y\,\mathbf{i} - 3xy^2\,\mathbf{j} + 4y^3\,\mathbf{k}$ and S be that portion of the elliptic paraboloid $z = x^2 + y^2 - 9$ which lies below the rectangle $[0,2] \times [0,1]$ in the xy-plane. Then the surface integral of (the normal component of) the vector field $\mathbf{F}$ over the surface S is ______ .

(12) Let $\mathbf{F}(x,y,z) = -y\,\mathbf{i} + x\,\mathbf{j} + 3z\,\mathbf{k}$ and S be the upper half of the sphere of radius 4 centered at the origin. Then the surface integral of (the normal component of) the vector field $\mathbf{F}$ over the surface S is $a\pi$ where $a =$ ________ .

(13) Let $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$ and V be that portion of the solid cylinder $x^2 + z^2 = 1$ which lies between the xz-plane and the plane $x + y = 2$. Then the surface integral of (the normal component of) the vector field $\mathbf{F}$ over the surface ∂V is $a\pi$ where $a =$ _____ .

(14) Let $\omega = x\,dy \wedge dz + xy\,dz \wedge dx + xz\,dx \wedge dy$ and S be the (upward oriented) triangular region whose vertices are $(2,0,0)$, $(0,3,0)$, and $(0,0,6)$. Then the integral of the 2-form ω over S is ________ .

(15) Let $\omega = xz\,dx + yz\,dy$ and H be the upper half of the (solid) ball of radius 2 centered at the origin. Then $\displaystyle\int_H d*\omega =$ ________ and $\displaystyle\int_{\partial H} *\omega =$ ________ .

(16) If S is the surface $z = x^2 + y^2$ (where $x^2 + y^2 \leq 1$), then $\displaystyle\iint_S z\,dS = \frac{\pi}{a}(b\sqrt{5} + 1)$ where $a =$ ________ and $b =$ ________ .

45.3. Problems

(1) Let $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$ where $u, v \in D$ be a parametrization of a surface S. Define $\mathbf{n}(u,v) := \mathbf{r}_u \times \mathbf{r}_v$. Show that

$$\mathbf{n} = \frac{\partial(y,z)}{\partial(u,v)}\,\mathbf{i} + \frac{\partial(z,x)}{\partial(u,v)}\,\mathbf{j} + \frac{\partial(x,y)}{\partial(u,v)}\,\mathbf{k}$$

and that consequently

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot \mathbf{n}\,du\,dv.$$

(2) Let $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$ where $u, v \in D$ be a parametrization of a surface S. Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ be a vector field on some open set containing S and μ be its associated 1-form.

(a) Show that $dx \wedge dy = \dfrac{\partial(x,y)}{\partial(u,v)}\,du \wedge dv$.

(b) Show that if $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$ and $F_N = \mathbf{F} \cdot \mathbf{n}$, then $F_N\,du \wedge dv = *\,\mu$, so that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_S *\,\mu.$$

45.4. Answers to Odd-Numbered Exercises

(1) 24

(3) 73

(5) $\dfrac{\pi}{2}$

(7) 0

(9) 2

(11) -1

(13) 4

(15) 8π, 8π

CHAPTER 46

STOKES' THEOREM

46.1. Background

Topics: *Stokes' theorem*, *divergence theorem*, *Green's theorem*, *generalized Stokes' theorem.*

Here is perhaps the most important theorem in calculus. This is certainly the result that *deserves* to be called the *fundamental theorem of calculus.*

46.1.1. Theorem (generalized Stokes' theorem)**.** *Let M be a bounded smooth oriented k-manifold and ω be a smooth $(k-1)$-form on an open set containing M. If M has a nonempty piecewise smooth boundary ∂M given the induced orientation , then*

$$\int_M d\omega = \int_{\partial M} \omega\,.$$

If ∂M is empty, then

$$\int_M d\omega = 0\,.$$

46.2. Exercises

(1) Let $\mathbf{F}(x,y,z) = y^2z\,\mathbf{i} + xz\,\mathbf{j} + x^2y^2\,\mathbf{k}$ and S be that portion of the paraboloid $z = x^2 + y^2$ which lies inside the cylinder $x^2 + y^2 = 1$. Take S to be oriented inward. Then the surface integral of (the normal component of) the vector field $\nabla \times \mathbf{F}$ over the surface S is ____ .

(2) Let $\mathbf{F}(x,y,z) = x^2\,\mathbf{i} + e^y\,\mathbf{j} + (x^2+y^2)\,\mathbf{k}$ and let P be the solid in the first quadrant bounded by the paraboloid $z = 1 - x^2 - y^2$ and the coordinate planes. Use *Stokes' theorem* to find the integral of (the tangential component of) the vector field $\mathbf{F}$ over the boundary of P. Answer: $\displaystyle\int_{\partial P} \mathbf{F}\cdot d\mathbf{r} =$ ____ .

(3) Let $\mathbf{F}(x,y,z) = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ and H be the upper half of the unit sphere centered at the origin in $\mathbb{R}^3$.
  (a) Without using any form of *Stokes' theorem* evaluate the integral of the normal component of the curl of $\mathbf{F}$ over H. Answer: ____ .
  (b) Repeat part (a), this time making use of *Stokes' theorem*. Answer: ____ .

(4) (a) Let $\mathbf{F}(x,y,z) = (x+y)\,\mathbf{i} + (y-x)\,\mathbf{j} + z^3\,\mathbf{k}$. Then the surface integral of the normal component of the curl of $\mathbf{F}$ over the upper half of the unit sphere centered at the origin is ________ .
  (b) Let $\omega = (x+y)\,dx + (y-x)\,dy + z^3\,dz$ and S be the upper half of the unit sphere centered at the origin. Then $\int_S d\omega =$ ________ .

(5) Let $\omega = x^2\,dx + (2xy + x)\,dy + z\,dz$ and S be the closed unit disk centered at the origin in the plane $z = 0$. Then, using the *generalized Stokes' theorem*, we get $\displaystyle\int_S d\omega =$ ____ .

(6) Let $\mathbf{F}(x,y,z) = (e^x + \arctan(y^2z^3))\,\mathbf{i} + y^2z\,\mathbf{j} + z\,\mathbf{k}$ and S be that portion of the hemisphere $x = \sqrt{9 - y^2 - z^2}$ which lies inside the cylinder $y^2 + z^2 = 4$. Use *Stokes' theorem* to find the surface integral of (the normal component of) the curl of $\mathbf{F}$ over the surface S. Answer: $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = -a\pi$ where $a =$ ____ .

(7) Let $\mathbf{F}(x,y,z) = 4x^3z\,\mathbf{i} + \arctan(xyz)\,\mathbf{j} - 3xe^z\,\mathbf{k}$ and P be the pyramid whose base is the square $S_0 = [0,1]\times[0,1]$ in the xy-plane and whose vertex is $(0,0,1)$. The pyramid has four slanting sides S_1, S_2, S_3, and S_4 (whose orientation in each case is induced by the outward unit normal). Then $\displaystyle\iint_{S_1+S_2+S_3+S_4} (\nabla\times\mathbf{F})\cdot d\mathbf{S} =$ ____ .

(8) Let $\mathbf{F}(x,y,z) = z^2\,\mathbf{i} + y^2\,\mathbf{j} + xy\,\mathbf{k}$ and T be the triangular region whose vertices are $(1,0,0)$, $(0,1,0)$, and $(0,0,2)$. Then $\displaystyle\int_{\partial T} \mathbf{F}\cdot d\mathbf{r} = \frac{a}{3}$ where $a =$ ____ .

(9) Let $\omega = x^2\,dx + xy^2\,dy + z^2\,dz$ and S be the surface parametrized by

$$\mathbf{r}(u,v) = \big(2(u+v), 3(u-v), 4uv\big)$$

for $(u,v) \in [-1,1]\times[-1,1]$. Then $\displaystyle\int_{\partial S} \omega =$ __________ .

(10) Let $\omega = (x^2 + y - 4)\,dx + 3xy\,dy + (2xz + z^2)\,dz$ and S be the surface $x^2 + y^2 + z^2 = 16$, $z \geq 0$. Then, using the *generalized Stokes' theorem*, we get $\displaystyle\int_S d\omega =$ __________ .

(11) Let ω be the 2-form $(ax^3 - 9xz^2)\,dy \wedge dz + (3x^2y + by^3)\,dz \wedge dx + cz^3\,dx \wedge dy$ and C be the curve parametrized by $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos 2t\,\mathbf{k}$ for $0 \le t \le 2\pi$. Find numbers a, b, and c so that $\int_S \omega$ has the same value for every surface S whose boundary is C. Answers: $a =$ ________ , $b =$ _____ , and $c =$ _____ .

(12) Let $\omega = 3y\,dx - xz\,dy + yz^2\,dz$ and S be that portion of the surface $2z = x^2 + y^2$ which lies below the plane $z = 2$.

(a) Using the *generalized Stokes' theorem*, we find that $\int_S d\omega =$ _____________ .

(b) *Without* using any form of *Stokes' theorem*, we find that $\int_S d\omega =$ _____________ .

(13) Let ω be the 1-form $(y + \sin x)\,dx + (z^2 + \cos y)\,dy + x^3\,dz$ and C be the curve whose parametrization is $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin 2t\,\mathbf{k}$ for $0 \le t \le 2\pi$. Then $\int_C \omega =$ ________ . *Hint.* $\sin 2t = 2\sin t \cos t$.

(14) Let $\omega = \dfrac{x^2 + y^2}{y}\,dx + \dfrac{y}{x^2 + y^2}\,dy + xy^2\,dz$ and S be the surface $x^2 + y^2 + z^2 = 1$, $z \ge 0$. Then $\int_S d\omega =$ __________ .

(15) Let $\mathbf{F}(x, y, z) = 3x^2y^2e^{z^2}\,\mathbf{i} + 2x^3ye^{z^2}\,\mathbf{j} + \arctan(1 + xy^3z^4)\,\mathbf{k}$. The following argument explains how we know that the integral of the tangential component of $\mathbf{F}$ around the unit circle C in the xy-plane is zero.

Since $F^1_2 =$ _____ , the __________ _______________ of the _______ of $\mathbf{F}$ over the _______ _______ , call it D, in the _____________ is zero. Since C is the _____________ of D, we know by __________ ____________ that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

(16) Let $\mathbf{F}(x, y, z) = x^3\,\mathbf{i} + 2xz^2\,\mathbf{j} + 3y^2z\,\mathbf{k}$ and V be the solid region bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane. Use the *divergence theorem* to find the surface integral of (the normal component of) the vector field $\mathbf{F}$ over ∂V. Answer: $\iint_{\partial V} \mathbf{F} \cdot d\mathbf{S} = a\pi$ where $a =$ ________ .

(17) Let $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j}$ and H be the upper hemisphere of radius 2 centered at the origin. Use the *divergence theorem* to calculate $\iint_H \mathbf{F} \cdot d\mathbf{S}$. (What do we do about the fact that by itself H isn't the boundary of anything?) Answer: $a\pi$ where $a =$ _____ .

(18) Let $\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + xz\,\mathbf{k}$ and V be the solid region consisting of the set of all points (x, y, z) such that $x^2 + y^2 \le z \le 1$ and $x \ge 0$. Then $\iint_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \dfrac{a}{15}$ where $a =$ _____ .

(19) Let $\omega = 3xy^2\,dy \wedge dz + 3x^2y\,dz \wedge dx + z^3\,dx \wedge dy$ and S be the unit sphere in $\mathbb{R}^3$. Then $\int_S \omega = \dfrac{a}{5}\pi$ where $a =$ ________ .

(20) Let $\mathbf{F}(x,y,z) = x^2y\,\mathbf{i} - x^2z\,\mathbf{j} + yz^2\,\mathbf{k}$ and R be the rectangular box $[0,3]\times[0,2]\times[0,1]$. Use the *divergence theorem* to find the surface integral of (the normal component of) the vector field $\mathbf{F}$ over the boundary of the box. Answer: $\displaystyle\iint_{\partial R} \mathbf{F}\cdot d\mathbf{S} =$ ________ .

(21) Let $\mathbf{F}$ be the vector field on $\mathbb{R}^3$ defined by $\mathbf{F}(x,y,z) = x^2\mathbf{i} + z\mathbf{j} + yz\mathbf{k}$ and E be the unit cube $[0,1]\times[0,1]\times[0,1]$. Also let

S_1 be the face of the cube lying in the xy-plane,
S_2 be the face of the cube lying in the xz-plane,
S_3 be the face of the cube lying in the plane $x = 1$,
S_4 be the face of the cube lying in the plane $y = 1$,
S_5 be the face of the cube lying in the yz-plane, and
S_6 be the face of the cube lying in the plane $z = 1$.

Then $\partial E = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$. Giving each S_k its positive (outwards directed) orientation compute the following integrals.

(a) $\displaystyle\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} =$ ______ .

(b) $\displaystyle\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} =$ ______ .

(c) $\displaystyle\iint_{S_3} \mathbf{F}\cdot d\mathbf{S} =$ ______ .

(d) $\displaystyle\iint_{S_4} \mathbf{F}\cdot d\mathbf{S} =$ ______ .

(e) $\displaystyle\iint_{S_5} \mathbf{F}\cdot d\mathbf{S} =$ ______ .

(f) $\displaystyle\iint_{S_6} \mathbf{F}\cdot d\mathbf{S} =$ ______ .

(g) $\displaystyle\iint_{\partial E} \mathbf{F}\cdot d\mathbf{S} =$ ______ .

(h) $\iiint_E \operatorname{div}\mathbf{F}\,dV =$ ______ .

(22) Let $\mathbf{F}(x,y,z) = 3xy\,\mathbf{i} + y^2\,\mathbf{j} - x^6y^5\,\mathbf{k}$ and T be the tetrahedron whose vertices are $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and the origin. Use the *divergence theorem* to find the surface integral of (the normal component of) the vector field $\mathbf{F}$ over the boundary of the tetrahedron. Answer: $\displaystyle\iint_{\partial T} \mathbf{F}\cdot d\mathbf{S} = \frac{a}{24}$ where $a =$ _____ .

(23) Let $\mathbf{F}(x,y,z) = e^z\arctan y^4\,\mathbf{i} + z^6\ln(x^4+5)\,\mathbf{j} + z\,\mathbf{k}$. Find the (outward) flux of $\mathbf{F}$ across that portion S of the paraboloid $z = 2 - x^2 - y^2$ that lies above the plane $z = 1$. Answer: the flux of $\mathbf{F}$ across S is $\dfrac{a}{2}\pi$ where $a =$ _____ .

(24) Let S be the unit sphere in $\mathbb{R}^3$ centered at the origin. Use the *divergence theorem* to find $\iint_S (3x + 7y + 5z^2)\,dS$. Answer: $\dfrac{a}{3}\pi$ where $a =$ ________ .

(25) Let $\mathbf{F}(x,y,z) = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ and S be the unit sphere in $\mathbb{R}^3$. Then $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = \frac{a}{5}\pi$ where $a =$ ________ .

(26) Let $\mathbf{F}(x,y,z) = y\,\mathbf{i} + z\,\mathbf{j} + xz\,\mathbf{k}$ and S be the set of points (x,y,z) such that $x^2+y^2 \le z \le 1$. Then $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} =$ _____ .

(27) Let E be the cylinder given by $x^2+y^2 \le 1$ and $-1 \le z \le 1$. Then

$$\iint_{\partial E} xy^2\,dy\wedge dz + x^2y\,dz\wedge dx + y\,dx\wedge dy = \underline{\qquad}\ .$$

(28) Let $\mathbf{F}(x,y,z) = 2x\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$ and S be the unit sphere in $\mathbb{R}^3$. Then $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = \frac{a}{3}\pi$ where $a =$ _____ .

(29) Let E be the solid region bounded above by the paraboloid $z = 5 - x^2 - y^2$ and below by the plane $z = -7$ and $\mathbf{F}(x,y,z) = (x^2yz^3\,,\ \sin(yz)\,,\ x^3e^z)$. Then $\displaystyle\iint_{\partial E} \nabla\times\mathbf{F}\cdot d\mathbf{S} =$ _____ . *Hint.* Think. Don't write anything down.

(30) Let $\omega = x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$ and E be the rectangular box $[0,2]\times[0,3]\times[0,4]$ from which the cube $[0,1]\times[0,1]\times[0,1]$ has been removed. Then $\displaystyle\int_{\partial E}\omega =$ ________ . *Hint.* You should not have to write anything down to solve this.

46.3. Problems

(1) Let M be a closed bounded interval in $\mathbb{R}$ and let f be a 0-form on M (that is, a smooth real valued function on M). What classical theorem do we get when we apply the *generalized Stokes' theorem* to the 1-manifold M and the differential 0-form f? Explain.

(2) Let M be a smoothly parametrized curve in $\mathbb{R}^3$ and let f be a 0-form on some open subset U of $\mathbb{R}^3$ containing M (that is, a smooth real valued function on U). What classical theorem do we get when we apply the *generalized Stokes' theorem* to the 1-manifold M and the differential 0-form f? Explain.

(3) Let R be a simple region in the plane $\mathbb{R}^2$ whose boundary is a positively oriented simple closed curve, let $\mathbf{F}$ be a vector field on a region in $\mathbb{R}^2$ containing R, and let ω be the 1-form associated with $\mathbf{F}$. What classical theorem do we get when we apply the *generalized Stokes' theorem* to the 2-manifold R and the differential 1-form ω? Explain.

(4) Let S be a smoothly parametrized surface in $\mathbb{R}^3$ whose boundary is a positively oriented simple closed curve, let $\mathbf{F}$ be a vector field on a region in $\mathbb{R}^3$ containing S, and let ω be the 1-form associated with $\mathbf{F}$. What classical theorem do we get when we apply the *generalized Stokes' theorem* to the 2-manifold S and the differential 1-form ω? Explain.

(5) Let E be a simple solid region whose boundary has positive (outward) orientation, let $\mathbf{F}$ be a vector field on a region in $\mathbb{R}^3$ containing E, and let ω be the 1-form associated with $\mathbf{F}$. What classical theorem do we get when we apply the *generalized Stokes' theorem* to the 3-manifold E and the differential 2-form $*\omega$? Explain.

(6) By direct computation verify *Stokes' theorem* for the case of the vector field $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + x^2\,\mathbf{k}$ and the surface S which is the boundary of the unit cube $[0, 1] \times [0, 1] \times [0, 1]$ with the face in the xy-plane missing.

(7) By direct computation verify *Stokes' theorem* for the case of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ and the helicoid S whose parametrization is $\mathbf{r}(u, v) = (u \cos v, u \sin v, v)$ for $(u, v) \in [0, 1] \times [0, \frac{1}{2}\pi]$.

(8) Suppose that C is a closed curve which is the boundary of a surface S and that f and g are smooth scalar fields. Derive the following formulas:

(a) $\displaystyle\int_C f\,\nabla g \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot d\mathbf{S}$; and

(b) $\displaystyle\int_C (f\,\nabla g + g\,\nabla f) \cdot d\mathbf{r} = 0.$

(9) Let $\omega = (z - y)\,dx + (x - z)\,dy + (y - x)\,dz$ and E be the tetrahedron whose vertices are $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Also let T be the face of E lying in the plane $x + y + z = 1$, C be the boundary of T, and T_1, T_2, and T_3 be the faces of E lying, respectively, in the yz-, xz-, and xy-planes. Compute separately and without using any version of *Stokes' theorem* $\displaystyle\int_T d\omega$, $\displaystyle\int_{T_1+T_2+T_3} d\omega$, and $\displaystyle\int_C \omega$. Explain any variations in sign which occur.

(10) Verify the *divergence theorem* directly for the special case where the simple solid region is the tetrahedron consisting of all those points (x, y, z) in $\mathbb{R}^3$ such that $x, y, z \geq 0$ and $x + y + z \leq 1$ and the vector field is $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

(11) Verify the *divergence theorem* directly for the special case where the simple solid region is the unit cube $[0, 1] \times [0, 1] \times [0, 1]$ and the vector field is $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

(12) Explain how to use the *divergence theorem* (or the *generalized Stokes' theorem*) to evaluate $\displaystyle\iint_S (x^2+y+z)\,dS$ where S is the unit sphere in $\mathbb{R}^3$. *Hint.* To make use of the *divergence theorem* the integral $\displaystyle\iint_S (x^2+y+z)\,dS$ must be expressed in the form $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S}$ (equivalently, $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{N}\,dS$) where $\mathbf{F}$ is an appropriate vector field and $\mathbf{N}$ is the unit normal to S.

(13) Verify the *divergence theorem* directly for the special case where the simple solid region is the upper half of the solid ball of radius a centered at the origin and the vector field is $\mathbf{F}(x,y,z) = a(x+y)\,\mathbf{i} + a(y-x)\,\mathbf{j} + z^2\,\mathbf{k}$.

46.4. Answers to Odd-Numbered Exercises

(1) π

(3) (a) π
(b) π

(5) π

(7) 4

(9) -288

(11) -1, 0, 3

(13) $-\pi$

(15) F_1^2, normal component, curl, unit disk, xy-plane, boundary, Stokes' theorem

(17) 8

(19) 12

(21) (a) 0
(b) $-\frac{1}{2}$
(c) 1
(d) $\frac{1}{2}$
(e) 0
(f) $\frac{1}{2}$
(g) $\frac{3}{2}$
(h) $\frac{3}{2}$

(23) 3

(25) 12

(27) π

(29) 0

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