MATHEMATICS FOR ECONOMIC ANALYSIS

BA ECONOMICS

II Semester

(2011 Admission)

COMPLEMENTARY COURSE

UNIVERSITY OF CALICUT

SCHOOL OF DISTANCE EDUCATION

CALICUT UNIVERSITY.P.O., MALAPPURAM, KERALA, INDIA – 673 635

385

UNIVERSITY OF CALICUT

SCHOOL OF DISTANCE EDUCATION

STUDY MATERIAL

BA ECONOMICS

II Semester (2011 Admission)

COMPLEMENTARY COURSE

MATHEMATICS FOR ECONOMIC ANALYSIS-I

Prepared by:

Module I & IV : *Ibrahim.Y.C,*
Associate Professor in Economics,
Govt.College, Kodenchery,
Kozhikode – 673 580.

Module II & III: *Shabeer.K.P,*
Assistant Professor in Economics,
Govt.College, Kodenchery,
Kozhikode – 673 580.

Module V : *Rahul.K,*
Assistant Professorin Economics,
N.M.S.M. Government College,
Kalpetta, Wayanad – 673 121.

Edited and Scrutinised by

Dr. C.Krishnan,
Associate Professor in Economics,
Govt.College, Kodenchery,
Kozhikode – 673 580.

Layout & Settings

Computer Section, SDE

CONTENTS

CHAPTER I

THEORY OF SETS

Set theory plays a vital role in all branches of modern Mathematics and is increasingly using in Economics also. The set theory was developed by George Cantor (1845 -1918).

A Set is a collection of definite and well distinguished objects. There exist, a rule with the help of which we are able to say whether a particular object 'belong to' the set or 'does not belong to' the set.

Eg:-

The Students in a class

Days in a month

A team of foot ball players

The set of positive integers

Elements

The Objects in a set or the constituent members of a set are called elements of that set.

A set is known by its elements. A set is well defined if and only if it can be decided that a given object is an element of the set.

Ex: A = {3,5,7,9} is a set of odd numbers 3, 5, 7 and 9, we can say that A is a set of all odd numbers between 1 and 10 or 3, 5, 7, 9 are the elements of A.

Notations

It is convention to denote sets by capital letters A, B, C ………., X, Y, Z …………….. and Elements by small letters a, b, c, ………. x, y, z………..

If 'a' is an element of a set 'S', then we denote it as

'a' is an element of S $\quad a \in s$

'a' is not an element of S $\quad a \notin s$

The symbol ∈ (Epsilon) between 'a' and 's' indicates

'a' is an element of set S

'a' 'belong to' S

'a' 'is contained' in S

'a' 'is a member of' S

Similarly ∉ symbol shows 'a' is not an element of set S.

Ex. If A = {2, 4, 6, 8} is a set of even number between 1 and 9, then for the element 4, we write

$4 \in \{2, 4, 6, 8\}$

$3 \notin \{2, 4, 6, 8\}$

The element of the sets are separated by commas and written in Braces of Flowered Brackets i.e. {}. An element is never listed twice or more in a set. i.e. repeated elements being deleted.

Ex. Set of letters in the word 'good' is {g, o, d} though 'good' has two 'o's, only one 'o' appears in set notation.

Method of Representing Set

A set can be represented symbolically by three different methods.

A. Roster Method or Tabulation Method or Enumeration Method

If all the elements of a set are known and they are few in numbers we can use Roster Method for denoting a set. In this method, the elements of the set are written in flowered brackets.

Examples:

1. $A = \{a, e, i, o, u\}$
2. $B = \{2, 4, 8, 16\}$
3. $C = \{ 1, 2, 3, 4, 5\}$

B. Rule Method or Set Builder Form

This method is used when elements of a set have a specific property 'P' and if any object satisfying the property, x is the element of the set.

If 'x' is any element of a set 'A' having the property 'p', then A is denoted as

A = {x/x has the property 'p'}

or A = {x:x has the property 'p'}

Hence the vertical line '/' or ':' is read as "such that"

Examples:

(1) Let 'A' be the set of all odd numbers, then it can be expressed as
A = {x/x is odd}

(2) Set of add numbers less than '10' can be expressed as

A = {x:x is an odd number $<$10}

If we express the same thing in Tabular Form we get $A = \{1, 3, 5, 7, 9\}$

C. Descriptive Phrase Method

This is a method of expressing a set of elements by stating in words what its elements are

Examples:

(1) A is a set of first five even numbers
(2) The set of first five natural numbers

Examples of different methods of representing a set

$A = \{2, 4, 6, 8, 10\}$ Tabular form

A = { x:x is an even numbers less than 11} – Set Builder Form.

A is a set of first five even numbers – Descriptive Phrase Method

1.1 KINDS OF SETS

A. Finite Sets

A set is said to be finite if it consists of a specific numbers or finite number of elements. The elements of such a set can be counted by a finite number

Ex. (1) A = { 1, 2, 3, 4, 5}

(2) Set of even numbers between 10 and 20.

B. Infinite Sets

If a set contain infinite number of elements, it is a infinite set. The elements of such a set cannot be counted by a finite number.

Ex. (1) Set of odd numbers

(2) A = { x:x is a number between 0 and 1}

C. Null set of Empty set or Zero set

A set which does not contain any element is called a null set or empty set.

Null set is denoted by ϕ of {}

Examples (1) A = {x/x is real number whose square is negative}

(2) B = {x/x is a negative number greater than 0}

(3) C = {x/x is a child having an age of 70}

D. Singleton Set or Unit Set

A set containing only one element is called Unit set or singleton set.

Ex. (1) Set of Positive integers less than 2

(2) A = {5}

(3) A = {0}

E. Equal Sets or Identical Sets or Equality of Sets

Two sets A and B are equal sets or identical sets if and only if they contain the same elements i.e. A = B.

Ex. (1) If A = {2, 3, 4, 5} B = {4, 2, 5, 3} then A = B

(2) A = set of even numbers between 1 and 9

B = { 2, 4, 6, 8} the A = B

F. Subsets

Subset of a set B is a set which consists of some or all of the elements of B. A set ‘A’ is a ‘sub set’ of set ‘B’ if and only if each element of set ‘A’ is also an element of the set B.

We denote it as $A \subseteq B$ which is read as “A is a subset of B”. $a \in A$ implies $a \in B$ or ‘a’ contained in B.

If A is not a subset of B, we denote this by $A \not\subseteq B$.

G. Super Set

If A is a subset of B, then we say that B is a 'Super Set' of A. We denote this relation as $B \supseteq A$.

Ex. A = { 2, 3, 4, 5}, B = {1, 2, 3, 4, 5, 6, 7} the $A \subseteq B$ and $B \subseteq A$.

H. Proper Subset

Set A is a 'Proper Subset' of a set 'B', if an only if each element of the set A is an element of the set B and at least one element of the set B is not an element of the set A. We denote this as $A \subset B$. If A is not a proper subset of B, we denote this by $A \not\subset B$.

Ex. Let A = {1, 3, 5, 7, 9} B = {1, 2, 3, 4, 5, 6, 7 8, 9}

Hence all the elements of set A one also the elements of set B and the elements of the set B i.e. 2, 4, 6, 8, are not the elements of set A, so $A \subset B$.

1. Improper Subset

A set 'A' is called an 'Improper Subset of set B' if and only if A=B i.e. every set is "Improper subset of itself"

Ex. A = {1, 2, 3, 4} B = {4, 3, 2, 1}. Hence A=B, therefore A is improper subset of B.

J. Universal Set

Any set under discussion can be treated as a subset of a big set. This big set is the universe of the set under consideration. That is when we are considering a set 'A' we assume the existence of another set 'X' or U and form the set 'A' by selecting either all elements or some of the elements of 'X'. Such a set 'X' or 'U' is called 'Universal set' or "Master set". It is denoted by 'U' or 'X'.

Example: While considering the set of all girls students of a college, the universal set consults of all students of the college.

K. Power Set

If A be a given set then the family of sets each of whose elements is a subset of the given set A is called the power set of the set A. We denote this power set of a set A by P(A).

$P(A) = \{x : x \text{ is a subset of } A\}$

If 'n' is the number of elements in a set, then the total number of subsets will be 2^n. So it is called power set.

Ex. (1) If A = {1, 2} then

$P(A) = [\phi, \{1\}, \{2\}, \{1, 2\}]$ i.e. $2^2 = 4$ subsets.

(2) If B = {a, b, c}

$P(B) = [\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}]$ i.e. $2^3 = 8$ subsets.

L. Disjoint sets

If two sets A and B have no common elements then we say that A and B are 'Disjoint Sets'

Ex. If A = {1, 2, 3} and B = {4, 5, 6}

Hence A and B are disjoint sets because there is no common elements in A and B. The intersection of disjoint matrix is a null matrix.

1.2 OPERATIONS OF SETS

Basic set operations are

1. Union of Sets
2. Intersection of Sets
3. Difference of Sets
4. Complement of a Set

A. Union of Sets or Set Union or Sum

Union or sum of two sets A and B is a set of all those elements which belongs to A or B or both; we denote this by A∪B and is read as 'A Union B'

$A \cup B = \{x : x \in A, x \in B\}$

Examples:

1. If A = {1,2,3,4,5}, B = {2, 4,6,8,10}, then
   A∪B = {1, 2, 3, 4, 5, 6, 8, 10}
2. If A = {a, b, c, d, e}, B = {a, e, i, o, u} then
   A∪B = {a, b, c, d, e, i, o, u}

B. Intersection of Sets

The intersection of two sets A and B is the set of the elements which are common to A and B or which belongs to both A and B. We denote 'intersection by A∩B and read as "A Intersection B".

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

Intersection of A and B is also called the "Product of A and B" so it is also denote by AB.

Examples

1. If A = {1, 2, 3, 4, 5} and B = {2, 4, 6, 8 10}, then
   A∩B = {2, 4}
2. If A = {a, b, c, d, e} and B = {a, e, i, o, u} then
   A∩B = {a, e}

C. Difference of Sets

Difference of two sets A and B is the set of all elements which belongs to A but do not belongs to B. Set difference is denoted by A-B and it is read as "A difference B".

$A - B = \{ x : x \in A \; x \notin B\}$

Examples:

1. If A = {1, 2, 3, 4, 5} And B= {2, 3, 4, 5, 6}, then
   A-B = {1} (from A remove elements common to A & B)
   B-A = {6} (from B remove elements common to A & B)

2. If A = {m, a, t, h, c, s} B= {s, t, a, i, c}
   A-B = {m, h}

D. Complement of A set

Let 'U' be the Universal set and 'A' be the subset of the Universal set 'U'. Then the complement of the set A is the set of all those element, belonging to the Universal set U but not belonging to A.

The complement of A is denoted by A^1(A-prime) or A^c.

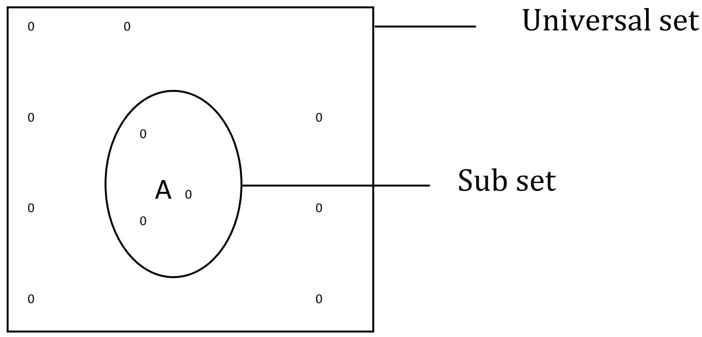
A^c=U-A or {x/x x∈U and x∉A}

Examples:

1. Let U = {1, 2, 3, 4, 5, 6, 7, 8} and A={2, 4, 6, 8}
   A^c or A^1 = {1, 3, 5, 7}
2. Let U be the set of all integers from 1 and 25 and A be the set of all odd integer from 1 and 25. Therefore A^1 will contain all even numbers between 1 and 25.

1.3 VENN DIAGRAMS

Most of the relationships between the sets can be represented by diagrams. These diagrams are known as 'Venn Diagrams' or "Venn-Euler Diagrams". They represent the set in pictorial way using rectangles and circles. Rectangle represents the universal set and circle represents any set. All the elements of the Universal set will be represented by points in it. We can also represents the set operations in the Venn Diagram.



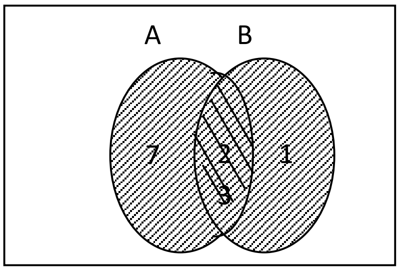
Universal set and subset

Set operation by means of Venn Diagram.

(a) AUB

If A = {2, 3, 7} and B = {1, 2, 3}

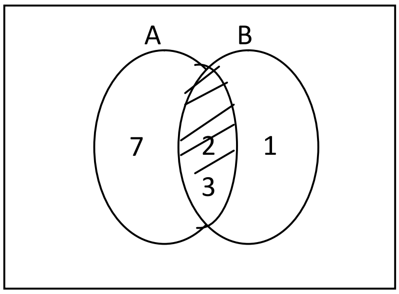
AUB {1, 2, 3, 7} or the shaded area in the figure



(b) A⋂B

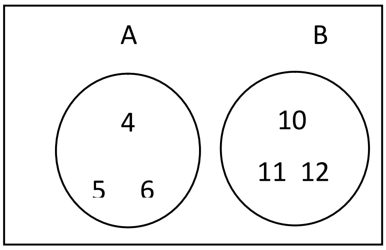
If $A = \{2, 3, 7\}$ and $B = \{1, 2, 3\}$ then

$A \cap B = \{2, 3\}$ or shaded area in the figure



(c) Disjoint Sets

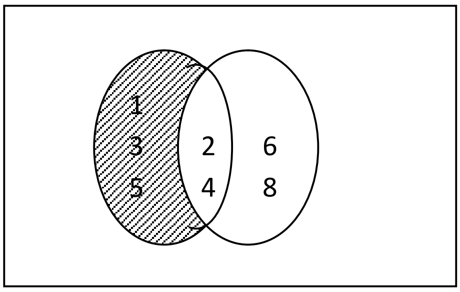
If $A = \{4, 5, 6\}$ and $B = \{10, 11, 12\}$ then A and B are 'Disjoint sets" (no common element) They are shown as



(d) Difference of Sets (A-B)

If $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8\}$ then

$A\text{-}B = \{1, 3, 5\}$ or the shaded area in the figure

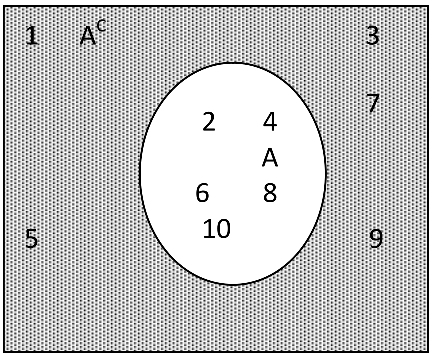


(e) Complement of the set (A^c or A^1)

If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and

$A = \{2, 4, 6, 8, 10\}$ then complement of A is

$A^c = \{1, 3, 5, 7, 9\}$ or shaded area in the figure.



Important Laws of Set Operations

1. Commutative Law
   (a) $A \cup B = B \cup A$
   (b) $A \cap B = B \cap A$
2. Associate Law
   (a) $A \cup (B \cup C) = (A \cup B) \cup C$
   (b) $A \cap (B \cap C) = (A \cap B) \cap C$
3. Distributive Law
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4. De Morgan's Law
   (a) $(A \cup B)^1 = A^1 \cap B^1$
   (b) $(A \cap B)^1 = A^1 \cup B^1$

Examples:

1. Using the following sets, verify that $A \cup (B \cup C) = (A \cup B) \cup C$
   $A = \{1, 2, 3\}$ $B = \{2, 4, 6\}$ $C = \{3, 4, 5\}$
   $B \cup C = \{2, 3, 4, 5, 6\}$
   $A \cup (B \cup C) = \{1, 2, 3, 4, 5, 6\}$
   $A \cup B = \{1, 2, 3, 4, 5, 6\}$
   $(A \cup B) \cup C = \{1, 2, 3, 4, 5, 6\}$
   $\therefore$ $A \cup (B \cup C) = (A \cup B) \cup C$
2. Using the following sets, verify that $A \cap (B \cap C) = (A \cap B) \cap C$ is $A = \{1, 2, 3, 4, 5, 6\}$ if
   $A = \{1, 2, 3, 4, 5, 6\}$
   $B = \{2, 4, 6\}$ $C = \{3, 6, 9\}$
   $B \cap C = \{6\}$ $A \cap (B \cup C) = \{6\}$
   $(A \cap B) = \{2, 4, 6\}$, $(A \cap B) \cap C = \{6\}$
   $\therefore$ $A \cap (B \cap C) = (A \cap B) \cap C$.

3. Using the following sets verify that
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ if $A = \{1, 2, 3\}$
   $B = \{2, 3, 4\}$ $C = \{3, 4, 5\}$

(a) $B \cap C = \{3, 4\}$, $A \cup (B \cap C) = \{1, 2, 3, 4\}$
$A \cup B = \{1, 2, 3, 4\}$, $A \cup C = (1, 2, 3, 4, 5\}$
$(A \cup B) \cap (A \cup C) = \{1, 2, 3, 4\}$

(b) $B \cup C = \{2, 3, 4, 5\}$
$A \cap (B \cup C) = \{2, 3\}$
$A \cap B = \{2, 3\}$, $A \cap C = \{3\}$
$(A \cap B) \cup (A \cap C) = \{2, 3\}$
$\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4. If $U = \{0, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{0, 2, 3\}$ $B = \{2, 4, 5\}$ verify that (a) $(A \cup B)^1 = A^1 \cap B^1$ (b) $(A \cap B)^1 = A^1 \cup B^1$
   (a) $A \cup B = \{0, 2, 3, 4, 5\}$
   $(A \cup B)^1 = \{6, 7, 8\}$
   $A^1 = \{4, 5, 6, 7, 8\}$
   $B^1 = \{0, 3, 6, 7, 8\}$
   $A^1 \cap B^1 = \{6, 7, 8\}$
   $\therefore (A \cup B)^1 = A^1 \cap B^1$
   (b) $A \cap B = \{2\}$
   $(A \cap B)^1 = \{0, 3, 4, 5, 6, 7, 8\}$
   $A^1 = \{4, 5, 6, 7, 8\}$
   $B^1 = \{0, 3, 6, 7, 8\}$
   $A^1 \cup B^1 = \{0, 3, 4, 5, 6, 7, 8\}$
   $\therefore (A \cap B)^1 = A^1 \cup B^1$

ORDERED PAIRS

An ordered pair consists of two elements say ‘a’ and ‘b’ such that one of them say ‘a’ is designated as the first element and the other ‘b’ is designated as the second element.

An ordered pair is represented by (a, b)

If (a, b) and (c, d) are two ordered pairs then, (a, b) = (c, d) if and only if a=c and b=d. Thus, the ordered pairs $(a, b) \neq (b, a)$ because $a \neq b$ and $b \neq a$.

1.4 CARTESIAN PRODUCTS

Let A and B be any two non-empty sets, then the Cartesian product of these two non-empty set A and B, is the set of all possible ordered pairs (a, b) where $a \in A$ and $b \in B$. Cartesian product is denoted by A X B (read as A cross B).

$A \times B = \{(a, b) / a \in A \text{ and } b \in B\}$

In the Cartesian Product the first element belongs to the first set and the second element belongs to the second set.

Examples:

1. If A = {1, 2, 3} and B = {4, 5} find A X B and B X A. Are they equal?
   A X B = {(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)}
   B X B = {(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)}
   ∴A X B ≠ B X A
2. If A = {4, 5}, B = {1, 2} C= {3, 4} find (a) A x (B∪C) (b) (A X B) ∩ (A X C) (c) A X A

(a) B∪C = {1, 2, 3, 4}
A X (B∪C) = {(4, 1) (4, 2), (4, 3), (4, 4), (5, 1), (5, 2), (5, 3), (5, 4)}

(b) A X B = {(4, 1), (4, 2), (5, 1), (5, 2)}
A X C = {(4, 3), (4, 4), (5, 3), (5, 4)}
(A X B) ∩ (A X C) = { }

(c) A X A = {(4, 4), (4, 5), (5, 4), (5, 5)}

1.5 RELATIONS

A relation is an association between two or more things. It may or may not be true. They are classified according to the number of elements associated

(a) Binary or Dyadic Relation

When a relation suggests a correspondence or an association between the elements of two sets it is called Binary or Dyadic Relation.

(b) Ternary or Triadic Relation

When a relation suggests a correspondence or an association between the elements of three sets it is called Ternary or Triadic relation.

Most of the relations in mathematics are binary for example, Equality, Identity, equivalence greater than or less than etc.

Example.1 $x \longrightarrow y$
$1 \longrightarrow 4$
$2 \longrightarrow 8$
$3 \longrightarrow 12$
$4 \longrightarrow 16$
…………..
…………..
$x \longrightarrow 4x$

In this the relation is y = 4x.

Example: 2 $x \longrightarrow y$

$2 \longrightarrow 4$

$3 \longrightarrow 9$

$4 \longrightarrow 16$

$5 \longrightarrow 25$

……………..

……………

$x \longrightarrow x^2$

In this the relation between 'x' and 'y' is given by 'y' = x^2

If A and B are two sets, then a subset of A X B is a relation from A to B and that relation is denoted by R.

Example: If A = {1, 4}, B = {1, 2}

Then A X B = {(1, 1), (1, 2), (4, 1), (4, 2)}

In this A X B the subsets {(1, 1), (4, 2)} shows some relation from A to B. The relation is first elements (1 and 4) are the squares of the second elements (1 and 2) i.e. $a = b^2$.

Relation 'R' consists of Domain and range.

<u>Domain (D)</u>

The domain (D) of the relation 'R' is the set of all first elements of the ordered pairs which belongs to R.

In the above relation R = (1, 1), (4, 2) domain is (1, 4)

<u>Range (E)</u>

The range (E) of the relation R is the set of all second elements of the ordered pairs which belongs to R.

In the above example R = (1, 1), (4, 2), range is (1, 2)

Example: If A = {3, 1} B= {9, 1} find A X B.

Form the relation R, "$a^2 = b$" specify the domain and range of the relation.

A X B = {(3, 9), (3, 1), (1, 9), (1, 1)}

The Relation R '$a^2 = b$' = {(3, 9), (1, 1)}

Domain 'D' = (3, 1)

Range of the relation (E) = (9, 1).

<u>1.6 FUNCTIONS</u>

If there are two sets A and B and for each element of A, a unique element of B is assigned in some manner or other we have a function or a functional relation from the set A to the set B. Such an assignment, which is a special kind of relation, is usually written as

$f : A \longrightarrow B$

or 'f' is a function of A into B

It implies that if $(x, y) \in f$ and $(x, z) \in f$,

then $y = z$

i.e. for each $x \in A$ there is almost one $y \in B$ with $(x, y) \in f$.

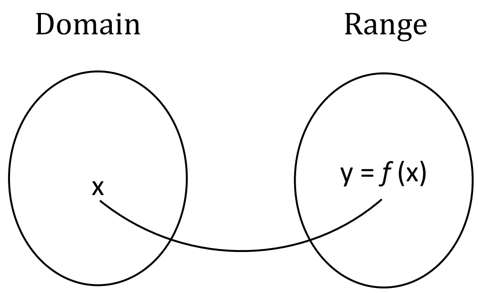
We define a function from the set X into the set y as a set of ordered pairs (x, y) where 'x' is an element of X and y is an element of Y such that for each x in X there is only one ordered pair (x, y) in the function P.

The notation used is

$f : X \longrightarrow Y$ or $y = f(x)$ or $x \longrightarrow f(x)$

or $y = f(x)$

A function is a mapping or transformation of 'x' into 'y' or $f(x)$. The variable 'x' presents elements of the 'Domain' and is called the independent variable. The variable 'y' represent elements of the 'Range' is called the dependent variable.



The function $y = f(x)$ is often called a 'single valued function' because there is a unique 'y' in the range for each specified x.

A function whose domain and range are sets of real number is called a real valued function of a real variable.

If the range consists of a single element it is called constant function. It may be written as $y = k$ or $f(x) = k$. Where 'k' is constant.

CHAPTER 2

FUNDAMENTAL OF LINEAR ALGEBRA-MATRICES

2.1 The Role of Linear Algebra

In the function, Y = f(x), if both variables appear to the first power, then it can be represented by a straight line and is known as linear function. In other words, a function of the form Y = $a_0 + a_1x$ is a linear function and represented graphically by a straight line.

The following are the role of linear algebra

(a) It permits the expression of a complicated system of equations in a simplified form.
(b) It provides a shorthand method to determine whether a solution exists before it is attempted.
(c) It furnishes the means of solving the equation system.

However, linear algebra can be applied only to systems of linear equations. Since many economic relationships can be approximated by linear equations and other can be converted to linear relationships, this limitation generally presents no serious problem.

2.2 The Matrices : Definitions and Terms

A matrix is a rectangular array of numbers, parameters or variables, each of which has a carefully ordered place within the matrix. The numbers (parameters or variables) are referred to as elements of the matrix and are usually enclosed in brackets. A matrix can be written in the following form.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Matrices, like sets, are denoted by capital letters and the elements of the matrix are usually represented by small letters. The members in the horizondal line are called rows and members in the vertical line are called columns. The number of rows and the number of columns in a matrix together define the dimensions or order of matrix. If a matrix contains 'm' rows and 'n' columns, it is said to be diamension of m × n (read as 'm by n'). The row number always precedes the column number. In that sence, the above matrix A is of dimension 3×3. Similarly,

$$B = \begin{bmatrix} 5 & 7 \\ 4 & 3 \end{bmatrix}_{2\times 2}$$

$$C = \begin{bmatrix} 9 & 5 \end{bmatrix}_{1\times 2}$$

$$D = \begin{bmatrix} 6 \\ 0 \\ 8 \end{bmatrix}_{3\times 1}$$

$$E = \begin{bmatrix} 0 & -2 & 3 \\ 4 & 0 & 5 \\ 8 & 4 & 1 \\ 3 & 2 & 7 \end{bmatrix}_{4\times 3}$$

The following are important types of matrices

(a) Square Matrix

A matrix with equal number of rows and columns is called square matrix. Thus, it is a special case where m = n. For example

$$\begin{bmatrix} 4 & 7 \\ 5 & 2 \end{bmatrix} \text{ is a square matrix of order 2}$$

$$\begin{bmatrix} 5 & 2 & 7 \\ 4 & 0 & 1 \\ 2 & 8 & 9 \end{bmatrix} \text{ is a square matrix of order 3}$$

(b)Row vector or Row Matrix

A matrix having only one row is called row vector or row matrix. A row vector will have a diamension of $1\times n$. For example,

$$A = [4 \quad 8]_{1\times2}$$

$$B = [3 \quad 0 \quad 2]_{1\times3}$$

$$C = [0 \quad 4 \quad 8 \quad 1]_{1\times4}$$

(c) Column Vector or Column Matrix

A matrix having only one column is called column vector or column matrix. A column vector will have a diamension of $m \times 1$. For example,

$$A = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_{2\times1}$$

$$B = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}_{3\times1}$$

$$C = \begin{bmatrix} 4 \\ 7 \\ 9 \\ 6 \end{bmatrix}_{4\times1}$$

are column vectors

(d) Diagonal Matrix

In a square matrix, the elements lie on the leading diagonal from left top to the right bottom are called diagonal elements. For example in the following square matrix,

$$A = \begin{bmatrix} 5 & 8 \\ 3 & 6 \end{bmatrix}$$

The elements 5 and 6 are diagonal elements. A square matrix in which all the elements except those in the diagonal are zero is called diagonal matrix. The following are examples of diagonal matrix.

$$A = \begin{bmatrix} 8 & 0 \\ 0 & 5 \end{bmatrix}_{2\times2}$$

$$B = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}_{3\times3}$$

(e)Triangular Matrix

If every element above or below the leading diagonal is zero, the matrix is called triangular matrix. A triangular matrix may be upper triangular or lower triangular. In the upper triangular matrix, all elements below the leading diagonal are zero. The following matrix A is a upper triangular matrix.

$$A = \begin{bmatrix} 4 & 7 & 3 \\ 0 & 8 & 6 \\ 0 & 0 & 5 \end{bmatrix}_{3\times3}$$

In the lower triangular matrix, all elements above the leading diagonal are zero. The following matrix B is an example of lower triangular matrix.

$$B = \begin{bmatrix} 5 & 0 & 0 \\ 9 & 3 & 0 \\ 4 & 8 & 7 \end{bmatrix}_{3\times3}$$

(f) Symmetric Matrix

The matrix obtained from any given matrix A, by interchanging its rows and columns is called its transpose and is denoted as A^t or A'. If matrix A is m × n matrix, then A' will be of n×m matrix. Any square matrix A is said to be symmetric if it is equal to its transpose. That is A is symmetric if A $=A^t$. For example,

$$A = \begin{bmatrix} 5 & 7 \\ 7 & 3 \end{bmatrix} \qquad A^t = \begin{bmatrix} 5 & 7 \\ 7 & 3 \end{bmatrix}$$

Since A $= A^t$, A is symmetric matrix.

$$B = \begin{bmatrix} 2 & 3 & 9 \\ 3 & 6 & 4 \\ 9 & 4 & 8 \end{bmatrix}$$

$$B^{\,t} = \begin{bmatrix} 2 & 3 & 9 \\ 3 & 6 & 4 \\ 9 & 4 & 8 \end{bmatrix}$$

Since B $= B^{\,t}$, B is symmetric

(g) Skew symmetric Matrix

Any square matrix A is said to be skew symmetric if it is equal to its negative transpose. That is, if A $=-A^t$, then matrix A is skew symmetric. For example,

$$A = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$$

$$-A^t = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$$

Since A $= -A^t$, A is skew symmetric

$$B = \begin{bmatrix} 0 & 4 & -9 \\ -4 & 0 & 5 \\ 9 & -5 & 0 \end{bmatrix}$$

$$B^t = \begin{bmatrix} 0 & -4 & 9 \\ 4 & 0 & -5 \\ -9 & 5 & 0 \end{bmatrix}$$

$$-B^t = \begin{bmatrix} 0 & 4 & -9 \\ -4 & 0 & 5 \\ 9 & -5 & 0 \end{bmatrix}$$

Since B $= -B^t$, B is skew symmetric

2.3 Addition and subtraction of Matrices

Two matrices can be added or subtracted if and only if they have the same dimension. That is, given two matrices A and B, their addition and subtraction, that is A+B and A-B requires that A and B have the same dimension. When this dimensional requirement is met, the matrices are said to be conformable for addition and subtraction. Then each element of one matrix is added to (or subtracted from) the corresponding element of the other matrix.

Example 1

$$\mathrm{A} = \begin{bmatrix} 8 & 9 \\ 12 & 7 \end{bmatrix}_{2\times 2} \qquad \mathrm{B} = \begin{bmatrix} 13 & 4 \\ 2 & 6 \end{bmatrix}_{2\times 2}$$

$$\mathrm{A} + \mathrm{B} = \begin{bmatrix} 8+13 & 9+4 \\ 12+2 & 7+6 \end{bmatrix}$$

$$\mathrm{A} + \mathrm{B} = \begin{bmatrix} 21 & 13 \\ 14 & 13 \end{bmatrix}_{2\times 2}$$

Example 2

$$\mathrm{A} = \begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}_{2\times 2} \quad \mathrm{B} = \begin{bmatrix} 9 & 6 \\ 2 & 4 \end{bmatrix}_{2\times 2}$$

$$\mathrm{A} - \mathrm{B} = \begin{bmatrix} 5-9 & 7-6 \\ 3-2 & 4-4 \end{bmatrix}$$

$$\mathrm{A} - \mathrm{B} = \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix}_{2\times 2}$$

Example 3

$$\mathrm{C} = \begin{bmatrix} 9 & 4 \\ 2 & 7 \\ 3 & 5 \\ 8 & 6 \end{bmatrix}_{4\times 2} \qquad \mathrm{D} = \begin{bmatrix} 1 & 3 \\ 6 & 5 \\ 2 & 8 \\ 9 & 2 \end{bmatrix}_{4\times 3}$$

$$\mathrm{C} + \mathrm{D} = \begin{bmatrix} 10 & 7 \\ 8 & 12 \\ 5 & 13 \\ 17 & 8 \end{bmatrix}_{4\times 2}$$

Example 4

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & -3 \\ 1 & 0 & 4 \end{bmatrix}_{3\times3} \quad B = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 0 & 5 \\ 6 & 9 & -1 \end{bmatrix}_{3\times3} \quad C = \begin{bmatrix} 4 & 4 & 4 \\ 5 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix}_{3\times3}$$

$$A + B - C = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 2 \\ 5 & 6 & 2 \end{bmatrix}_{3\times3}$$

2.4 Scalar Multiplication

In the matrix algebra, a simple number such as 1,2, -2, -1 or 0.5 is called a scalar. Multiplication of a matrix by a scalar or a number involves multiplication of every element of the matrix by the number. The process is called scalar multiplication.

Let 'A' by any matrix and 'K' any scalar, then the matrix obtained by multiplying every element of A by K and is said to be scalar multiple of A by K because it scales the matrix up or down according to the size of the scalar.

Example 1

$$K = 8 \text{ and } A = \begin{bmatrix} 6 & 9 \\ 2 & 7 \\ 8 & 4 \end{bmatrix}$$

$$KA = 8\begin{bmatrix} 6 & 9 \\ 2 & 7 \\ 8 & 4 \end{bmatrix}$$

$$KA = \begin{bmatrix} 8\times6 & 8\times9 \\ 8\times2 & 8\times7 \\ 8\times8 & 8\times4 \end{bmatrix}$$

$$KA = \begin{bmatrix} 48 & 72 \\ 16 & 56 \\ 64 & 32 \end{bmatrix}$$

Example 2

$$K = 7 \text{ and } A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & -3 \\ 1 & 0 & 4 \end{bmatrix}$$

$$KA = \begin{bmatrix} 14 & 14 & 14 \\ 14 & 7 & -21 \\ 7 & 0 & 28 \end{bmatrix}$$

Example 3

$$A = \begin{bmatrix} 6 & 7 & 4 \\ 3 & -1 & 0 \\ 2 & 4 & 5 \end{bmatrix} \text{ and } K = 5$$

$$KA = 5\begin{bmatrix} 6 & 7 & 4 \\ 3 & -1 & 0 \\ 2 & 4 & 5 \end{bmatrix}$$

$$KA = \begin{bmatrix} 30 & 35 & 20 \\ 15 & -5 & 0 \\ 10 & 20 & 25 \end{bmatrix}$$

Example 4

K = -2 and $A = \begin{bmatrix} 3 & -4 \\ 2 & -9 \end{bmatrix}$

$$KA = -2\begin{bmatrix} 3 & -4 \\ 2 & -9 \end{bmatrix}$$

$$KA = \begin{bmatrix} -6 & 8 \\ -4 & 18 \end{bmatrix}$$

2.5 Vector Multiplication

Multiplication of a row vector 'A' by a column vector B requires that each vector have precisely the same number of elements. The product is then found by multiplying the individual elements of the row vector by their corresponding elements in the column vector and summing the products. For example, if

$$A = [a \quad b \quad c]_{1\times 3}$$

and

$$B = \begin{bmatrix} d \\ e \\ f \end{bmatrix}_{3\times 1}$$

AB = (ad) + (bc) + (cf)

Thus, the product of row-column multiplication will be a single number or scalar. Row-column vector multiplication is very important because it serves the basis for all matrix multiplications.

Example 1

$$A = [4 \quad 2]_{1\times 2} \quad B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}_{2\times 1}$$

$$AB = [(4 \times 3) + (2 \times 5)]$$

$$AB = 22$$

Example 2

$$A = [5 \quad 1 \quad 3]_{1\times 3} \quad B = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}_{3\times 1}$$

$$AB = [(5 \times 2) + (1 \times 6) + (3 \times 3)]$$

$$AB = 25$$

Example 3

$$A = [12 \quad -5 \quad 6 \quad 11]_{1\times 4} \qquad B = \begin{bmatrix} 3 \\ 2 \\ -8 \\ 6 \end{bmatrix}$$

$$AB = [(12 \times 3) + (-5 \times 2) + (6 \times -8) + (11 \times 6)]$$

$$AB = 44$$

Example 4

$$A = [9 \quad 6 \quad 2 \quad 0 \quad -5]_{1\times 5} \qquad B = \begin{bmatrix} 2 \\ 13 \\ 5 \\ 8 \\ 1 \end{bmatrix}$$

$$AB = [(9 \times 3) + (6 \times 13) + (2 \times 5) + (0 \times 8) + (-5 \times 1)]$$

$$AB = 101$$

2.6 Multiplication of Matrices

The matrices A and B are said to be conformable for multiplication if and only if the number of columns in the matrix A is equal to the number of rows in matrix B. That is, to find the product AB, conformity condition for multiplication requires that the column dimension of A (the lead matrix in the expression AB) must be equal to the row dimension of B (the lag matrix).

In general, if A is of order $m \times n$, then B should be of the order $n\times p$ and dimension of AB will be $m\times p$. That is, if dimension of A is 1×2 and dimension of B is 2×3, then AB will be of 1×3 dimension. The procedure of multiplication is that take each row and multiply it with eac column. The row-column products, called 'inner products' are then used as elements in the formation of the product matrix.

Example 1

$$A = \begin{bmatrix} 6 & 12 \\ 5 & 10 \\ 13 & 2 \end{bmatrix}_{3\times 2} \qquad B = \begin{bmatrix} 1 & 7 & 8 \\ 2 & 4 & 3 \end{bmatrix}_{2\times 3}$$

Since matrix A is of 3×2 dimension and B is of 2×3 diamension, the matrices are conformable for multiplication and the product AB will be of 3×3 dimension.

$$AB = \begin{bmatrix} 6\times 1 + 12\times 2 & 6\times 7 + 12\times 4 & 6\times 8 + 12\times 3 \\ 5\times 1 + 10\times 2 & 5\times 7 + 10\times 4 & 5\times 8 + 10\times 3 \\ 13\times 1 + 2\times 2 & 13\times 7 + 2\times 4 & 13\times 8 + 2\times 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 30 & 90 & 84 \\ 25 & 75 & 70 \\ 17 & 99 & 110 \end{bmatrix}_{3\times 3}$$

Example 2

$$A = \begin{bmatrix} 12 & 14 \\ 20 & 5 \end{bmatrix}_{2\times 2} \; B = \begin{bmatrix} 3 & 9 \\ 0 & 2 \end{bmatrix}_{2\times 2}$$

$$AB = \begin{bmatrix} 12 \times 3 + 14 \times 0 & 12 \times 9 + 14 \times 2 \\ 20 \times 3 + 5 \times 0 & 20 \times 9 + 5 \times 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 36 & 136 \\ 60 & 190 \end{bmatrix}_{2\times2}$$

Example 3

$$A = \begin{bmatrix} 4 & 7 \\ 9 & 1 \end{bmatrix}_{2\times2} \qquad B = \begin{bmatrix} 3 & 8 & 5 \\ 2 & 6 & 7 \end{bmatrix}_{2\times3}$$

$$AB = \begin{bmatrix} 4 \times 3 + 7 \times 2 & 4 \times 8 + 7 \times 6 & 4 \times 5 + 7 \times 7 \\ 9 \times 3 + 1 \times 2 & 9 \times 8 + 1 \times 6 & 9 \times 5 + 1 \times 7 \end{bmatrix}$$

$$AB = \begin{bmatrix} 26 & 74 & 69 \\ 29 & 78 & 52 \end{bmatrix}_{2\times3}$$

Example 4

If $A = \begin{bmatrix} 6 & 1 \\ 9 & 4 \end{bmatrix}$ compute A^2

$$A^2 = A \times A = \begin{bmatrix} 6 & 1 \\ 9 & 4 \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 9 & 4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 6 \times 6 + 1 \times 9 & 6 \times 1 + 1 \times 4 \\ 9 \times 6 + 4 \times 9 & 9 \times 1 + 4 \times 4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 45 & 10 \\ 90 & 25 \end{bmatrix}$$

2.7 Commutative, Associative and Distributive Laws in Algebra

In ordinary algebra, the additive, subtractive and multiplicative operations obey commutative, associative and distributive laws. Most, but not all, of these laws also apply to matrix operations.

(a) Matrix Addition and Subtraction

Matrix addition and subtraction is commutative.

Commutative law of addition states that A + B = B+A

Commutative law of subtraction states that A –B = -B + A

Associative law of addition states that (A+B)+C = A +(B+C)

Associative law of subtraction states that (A +B)-C = A +(B-C)

(b)Matrix Multiplication

Matrix multiplication, with few exception is not commutative. That is AB≠BA. One exception is a case when A is a square matrix and B is on identity matrix. At the sametime, scalar multiplication is commutative. That is KA =AK, where K is a scalar. But matrix multiplication is a associative. That (AB)C = A(BC). If matrices are conformable, the matrix multiplication is also distributive. Distributive law states that A(B+C) = AB +AC.

Example 1

Given $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 2 \\ 3 & 4 \end{bmatrix}$ prove that matrix addition and subtraction is commutative.

Commutative law of addition states that A+B = B + A

$$A + B = \begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}$$

$$B + A = \begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}$$

A + B = B + A ∴ matrix subtraction is commutative

Example 2

Given $A = \begin{bmatrix} 6 & 2 & 7 \\ 9 & 5 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 9 & 1 & 3 \\ 4 & 2 & 6 \end{bmatrix}$ $C = \begin{bmatrix} 7 & 5 & 1 \\ 10 & 3 & 8 \end{bmatrix}$ prove that matrix addition satisfy associative law.

Associative law of addition (A+B)+C = A + (B+C)

$$A+B = \begin{bmatrix} 15 & 3 & 10 \\ 13 & 7 & 9 \end{bmatrix}$$

$$(A+B)+C = \begin{bmatrix} 22 & 8 & 11 \\ 23 & 10 & 17 \end{bmatrix}$$

$$B + C = \begin{bmatrix} 16 & 6 & 4 \\ 14 & 5 & 14 \end{bmatrix}$$

$$A + (B + C) = \begin{bmatrix} 22 & 8 & 11 \\ 23 & 10 & 17 \end{bmatrix}$$

(A+B) +C = A + (B+C) ∴ matrix addition satisfy associative law

Example 3

Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$ prove that $AB \neq BA$

$$AB = \begin{bmatrix} 12 & 13 \\ 24 & 25 \end{bmatrix}$$

$$BA = \begin{bmatrix} -3 & -4 \\ 27 & 40 \end{bmatrix}$$

$\therefore AB \neq BA$

Example 4

Given $A = \begin{bmatrix} 7 & 1 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 6 & 5 \\ 2 & 4 \\ 3 & 8 \end{bmatrix}$ $C = \begin{bmatrix} 9 & 4 \\ 3 & 10 \end{bmatrix}$

Prove that matrix multiplication is associative.

Associative law of multiplication (AB)C = A(BC)

$$AB = \begin{bmatrix} 59 & 79 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 768 & 1026 \end{bmatrix}$$

$$BC = \begin{bmatrix} 69 & 74 \\ 30 & 48 \\ 51 & 92 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 768 & 1026 \end{bmatrix}$$

(AB)C = A(BC) ∴ matrix multiplication is associative.

2.8 Identity and Null Matrices

A diagonal matrix in which each of the diagonal element is unity is said to be a identity or unit matrix and is denoted by I. when a subscript is used, as I_n, n denotes the dimension of the matrix (n × n). The identity matrix is similar to the number 1 in algebra since multiplication of a matrix by an identity matrix leaves the original matrix unchanged. That is, AI = IA=A. Multiplication of an identity matrix by itself leaves the identity matrix unchanged. That is , I × I=I^2= I. The identity matrix is symmetric and idempotent. Following are examples of identity matrix.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

On the other hand, a matrix in which every element is zero is called null matrix or zero matrix. Null matrix can be of any dimension. It is not necessarily square. Addition or subtraction of the null matrix leaves the original matrix unchanged. Multiplication by a null matrix produces a null matrix. Following are examples of null matrix.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3\times 2} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$$

2.9 Matrix Expression of a system of Linear Equations

Matrix algebra permits concise expression of a system of linear equation. Given the following system of linear equations.

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

can be expressed in the matrix form as

$$Ax=B$$

where,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}_{2\times 1}$$

Here 'A' is the coefficient matrix, 'x' is the solution vector and B is the vector of constant terms. X and B will always be column vectors. Since A is 2 × 2 matrix and X is 2 × 1 vector, they are conformable for multiplication and the product matrix will be 2 × 1.

Example 1

Given,

$$7x_1 + 3x_2 = 45$$

$$4x_1 + 5x_2 = 29$$

In matrix form AX = B

$$\begin{bmatrix} 7 & 3 \\ 4 & 5 \end{bmatrix}_{2\times2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times1} = \begin{bmatrix} 45 \\ 29 \end{bmatrix}_{2\times1}$$

Example 2

Given,

$$7x_1 + 8x_2 = 120$$

$$6x_1 + 9x_2 = 92$$

In matrix form AX = B

$$\begin{bmatrix} 7 & 8 \\ 6 & 9 \end{bmatrix}_{2\times2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times1} = \begin{bmatrix} 120 \\ 92 \end{bmatrix}_{2\times1}$$

Example 3

Given,

$$2x_1 + 4x_2 + 9x_3 = 143$$

$$2x_1 + 8x_2 + 7x_3 = 204$$

$$5x_1 + 6x_2 + 3x_3 = 168$$

In matrix form AX = B

$$\begin{bmatrix} 2 & 4 & 9 \\ 2 & 8 & 7 \\ 5 & 6 & 3 \end{bmatrix}_{3\times3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times1} = \begin{bmatrix} 143 \\ 204 \\ 168 \end{bmatrix}_{3\times1}$$

Example 4

Given,

8w + 12x – 7y + 2z = 139

3w – 13x + 4y + 9z = 242

In matrix form AX = B

$$\begin{bmatrix} 8 & 12 & -7 & 2 \\ 3 & -13 & 4 & 9 \end{bmatrix}_{2\times4} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}_{4\times1} = \begin{bmatrix} 139 \\ 242 \end{bmatrix}_{2\times1}$$

2.10 Row Operations

Row operations involve the application of simple algebraic operations to the rows of a matrix. With no change in the linear relationship, we have three basic row operations. They are

1. Any two rows of a matrix to be interchanged.
2. Any row or rows to be multiplied by a constant provided the constant does not equal to zero.
3. Any multiple of a row to be added to or subtracted from any other row.

For example, given $5x + 2y = 16$

$8x + 4y = 28$

Without any change in the linear relationship, we may

(1) Interchange the two rows

$8x + 4y = 28$

$5x + 2y = 16$

(2) Multiply any row by a constant, say first row by

$2x + y = 7$

$5x + 2y = 16$

(3) Subtract 2 times of first row from the second row, that is

$2[(2x + y) = 7] \Rightarrow 4x + 2y = 14$

$5x + 2y = 16$ -

$14x + 2y = 14$

$x = 2$

REFERENCE

(1) Edward .T. Dowling : " Introduction to Mathematical Economics"
Second Edition . Schaum's outline series

(2) Alpha C. chiang: Fundamental Methods of Mathematical Economics,
MC Graw Hill

(3) Manmohan Gupta : Mathematics for Business and Economics

CHAPTER 3

MATRIX INVERSION

3.1 Determinants and Nonsingularity

The determinant is a single number or scalar associated with a square matrix. Determinants are defined only for square matrix. In other words, the determinant of a square matrix A denoted as |A| is a uniquely defined number or scalar associated with that matrix.

It the determinant of a square matrix is equal to zero, the determinant is said to vanish and the matrix is termed as "singular". That is, a singular matrix is the one in which there exists a linear relationship or dependence between atleast two rows or columns. If |A| ≠ 0, matrix A is called "nonsingular" and all its rows and columns are linearly independent.

3.2 Determinants

If A = $[a_{11}]$ is a 1 × 1 matrix, then the determinant of A, that is |A| is a_{11} itself. If A is 2 × 2 matrix, like

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then |A| is derived by taking the product of the two elements on the principal diagonal and subtracting form it the product of the elements off principal diagonal. That is,

$$|A| = (a_{11}a_{22}) - (a_{21}\,a_{12})$$

Thus, |A| is obtained by cross multiplication of the elements. The determinant of a 2 × 2 square matrix is called second order determinant.

The determinant of order three is associated w ith a 3 × 3 matrix. Given.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then $$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|A| = \text{a Scalar}$$

Thus, the third order determinant is the summation of three products. To derive the three products;

1. Take the first element of the first row, that is, a_{11} by the determinant of the remaining element.

2. Take the second element of the first row, ie, a_{12} and mentally delete the row and column in which it appears. Then multiply a_{12} by -1 times the determinant of the remaining element.
3. Take the third element of first row, ie, a_{13} and mentally delete the row and column in which it appears. Then multiply a_{13} by the determinant of the remaining element.

In the like manner, determinant of a 4 x 4 matrix is the sum of four products. The determinant of a 5 x 5 matrix is the sum five products and so on.

Example 1

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}$$

$$|A| = (2 \times 6) - (3 \times 3)$$

$$|A| = \underline{\underline{3}}$$

Example 2

$$B = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

$$|B| = (2 \times 2) - (-3 \times 1)$$

$$|B| = \underline{\underline{7}}$$

Example 3

$$C = \begin{bmatrix} -3 & 6 & 2 \\ 1 & 5 & 4 \\ 4 & -8 & 2 \end{bmatrix}$$

$$|C| = -3 \begin{vmatrix} 5 & 4 \\ -8 & 2 \end{vmatrix} - 6 \begin{vmatrix} 1 & 4 \\ 4 & 2 \end{vmatrix} + 2 \begin{vmatrix} 1 & 5 \\ 4 & -8 \end{vmatrix}$$

$$|C| = \underline{\underline{-98}}$$

Example 4

$$D = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & -1 \\ 4 & 1 & 5 \end{bmatrix}$$

$$|D| = 1 \begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} - 3 \begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} + 2 \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix}$$

$$|C| = \underline{\underline{-17}}$$

3.3 Properties of a Determinant

The following are important proportion of a determinant.

(1) The value of the determinant does not change, if the rows and columns of it are interchanged. That is, the determinant of a matrix equals the determinant of its transpose. Thus, $|A| = |A^T|$

$$\text{If } A = \begin{bmatrix} 6 & 4 \\ 7 & 9 \end{bmatrix}$$

$$|A| = (6 \times 9) - (4 \times 7) = \underline{\underline{26}}$$

$$|A^T| = \begin{bmatrix} 6 & 4 \\ 7 & 9 \end{bmatrix}$$

$$|A^T| = (6 \times 9) - (4 \times 7) = \underline{\underline{26}}$$

$\therefore \quad |A| = |A^T|$

=======

(2) The inter change of any two rows or any two columns will alter the sign, but not the numerical value of the determinant.

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 7 & 5 & 2 \\ 1 & 0 & 3 \end{bmatrix} = |A| = \underline{26}$$

Now interchanging first and third column, forming B

$$B = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 5 & 7 \\ 3 & 0 & 1 \end{bmatrix} = |B| = -\underline{\underline{26}} \quad = - |A|$$

(3) If any two rows or columns of a matrix are identical or proportional, that is, lineally dependent, the determinant is zero.

$$\text{If } A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} = |A| = \underline{\underline{0}}, \text{ Since first and third row are identical.}$$

(4) The multiplication of any one row or one column by a scalar or constant, say 'K' will change the value of the determinant K fold. That is, multiplying a single row or column of a matrix by a scalar will cause the value of the determinant to be multiplied by the scalar.

$$\text{If } A = \begin{bmatrix} 3 & 5 & 7 \\ 2 & 1 & 4 \\ 4 & 2 & 3 \end{bmatrix} = |A| = \underline{\underline{35}}$$

Now, forming a new matrix B, by multiplying the first row of a by scalar 2, we have

$$B = \begin{bmatrix} 6 & 10 & 14 \\ 2 & 1 & 4 \\ 4 & 2 & 3 \end{bmatrix} \quad |B| = \underline{\underline{70}}, \text{ that is } 2 \times |A|$$

(5) The determinant of a triangular matrix is equal to the product of the elements on principal diagonal. For the following lower triangular matrix, A, where

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 2 & -5 & 0 \\ 6 & 2 & 3 \end{bmatrix} \quad |A| = \underline{\underline{60}}, \text{ that is } -3 \times -5 \times 4$$

(6) If all the elements of any row or column are zero then determinant is also zero.

$$\text{If } A = \begin{bmatrix} 3 & 8 & 4 \\ 9 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad |A| = \underline{\underline{0}}, \text{ Since all the elements of third row is zero.}$$

(7) Addition or subtraction of a non-zero multiple of any one row or column from another row or column does not change the value of determinant.

$$\text{If } A = \begin{bmatrix} 20 & 3 \\ 10 & 2 \end{bmatrix} = |A| = \underline{\underline{10}}$$

Subtracting two times of column two from column one and forming a new matrix B,

$$B = \begin{bmatrix} 14 & 3 \\ 6 & 2 \end{bmatrix} \quad |B| = \underline{\underline{10}}$$

(8) If every element in a row or column of a matrix is the sum of two numbers, then the given determinant can be expressed as sum of two determinant.

$$A = \begin{bmatrix} 2+3 & 1 \\ 4+1 & 5 \end{bmatrix} \quad |A| = \underline{\underline{20}}$$

That is,

$$A_1 = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} + A_2 = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix} = \ |A_1| = \underline{\underline{6}}$$

$$|A_2| = \underline{\underline{14}}$$

$$|A_1| + |A_2| = \underline{\underline{20}} \text{ which is } |A| \text{ itself}$$

3.4 Minors and cofactors

Every element of a square matrix has a minor. It is the value of determinant formed with elements obtained when the row and the column in which the element lies are deleted. Thus, a minor Mij is the determinant of the sub matrix formed by deleting the ith row and jth column of the matrix.

For example, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Minor of $a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ Minor of $a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ Minor of $a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Similarly

$$a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \quad a_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \quad a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$a_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \quad a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \quad a_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Since all the minors have two rows and columns, it is also called minor of order K = 2 or second order minor of determinant of A.

A cofactor (Cij) is a minor with a prescribed sign. Cofactor of an element is obtained by multiplying the minor of the element with $(-1)^{i+j}$, where 'i' is the number of row and j is the number of column.

That is $Cij = (-1)^{i+j} Mij$

In the previous matrix A

Cofactor of $a_{11=(-1)^{1+1}} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \ a_{12=(-1)^{1+2}} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and so on.

Cofactor matrix is a matrix in which every element of Cij is replaced with its cofactors Cij.

Example 1

If $A = \begin{bmatrix} 6 & 7 \\ 12 & 9 \end{bmatrix}$

Cofactor of 6 = $(-1)^{1+1}\ 9$

" $7 = (-1)^{1+2}\ 12$

" $12 = (-1)^{2+1}\ 7$

" $9 = (-1)^{2+2}\ 6$

$\therefore\ \text{Cij} = \begin{bmatrix} 9 & -12 \\ -7 & 6 \end{bmatrix}$

Example2

$B = \begin{bmatrix} -2 & -1 \\ 4 & 6 \end{bmatrix}$

Cofactor of -2 = $(-1)^{1+1}\ 6$

" -1= $(-1)^{1+2}\ 4$

" 4= $(-1)^{2+1}$ -1

" 6= $(-1)^{2+2}$ -2

$\therefore\ \text{Cij} = \begin{bmatrix} 6 & -4 \\ 1 & -2 \end{bmatrix}$

Example 3

$A = \begin{bmatrix} 6 & 2 & 7 \\ 5 & 4 & 9 \\ 3 & 3 & 1 \end{bmatrix}$

Matrix of cofactors Cij = $\begin{bmatrix} -23 & 22 & 3 \\ 19 & -15 & -12 \\ -10 & -19 & 14 \end{bmatrix}$

Example 4

$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 5 & 3 & 4 \end{bmatrix}$

Matrix of cofactors Cij = $\begin{bmatrix} -2 & -6 & 7 \\ -9 & 3 & 9 \\ 5 & 0 & -10 \end{bmatrix}$

3.5 Cofactor and Adjoint Matrix

An adjoint matrix is the transpose of cofactor matrix. That is adjoint of a given square matrix is the transpose of the matrix formed by cofactors of the elements of a given square matrix, taken in order. In other words, adjoint of

$A = [\text{Cij}]^T$

Example 1

$$A = \begin{bmatrix} 7 & 12 \\ 4 & 3 \end{bmatrix}$$

Matrix of cofactors $Cij = \begin{bmatrix} 3 & -4 \\ -12 & 7 \end{bmatrix}$

Adj A = $[Cij]^T = \begin{bmatrix} 3 & -12 \\ -4 & 7 \end{bmatrix}$

Example 2

$$A = \begin{bmatrix} 9 & -16 \\ -20 & 7 \end{bmatrix}$$

Matrix of cofactors $Cij = \begin{bmatrix} 7 & 20 \\ 16 & 9 \end{bmatrix}$

Adj A = $[Cij]^T = \begin{bmatrix} 7 & 16 \\ 20 & 9 \end{bmatrix}$

Example 3

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

Matrix of cofactors $Cij = \begin{bmatrix} 3 & -9 & -5 \\ -4 & 1 & 3 \\ -5 & 4 & 1 \end{bmatrix}$

Adj A = $[Cij]^T = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$

Example 4

$$A = \begin{bmatrix} 13 & -2 & 8 \\ -9 & 6 & -4 \\ -3 & 2 & -1 \end{bmatrix}$$

Matrix of cofactors $Cij = \begin{bmatrix} 2 & 3 & 0 \\ 14 & 11 & -20 \\ -40 & -20 & 60 \end{bmatrix}$

Adj A = $[Cij]^T = \begin{bmatrix} 2 & 14 & -40 \\ 3 & 11 & -20 \\ 0 & -20 & 60 \end{bmatrix}$

3.6 Inverse Matrices

For a square matrix A, if there exist another square matrix B such that AB = BA = I, then B is said to be inverse matrix of A and is denoted as A^{-1}.

That is, $AA^{-1} = A^{-1}A = I$

Multiplying a matrix by its inverse reduces it to an identity matrix. For a given matrix A, there exist inverse only if (1) A is a square matrix.

(2) A is non singular $|A| \neq 0$

Again, if B is the inverse of A, then A is inverse of B. The formula for deriving the inverse is

$$A^{-1} = \frac{\text{Adj.A}}{|A|}$$

Example 1

$$A = \begin{bmatrix} 24 & 15 \\ 8 & 7 \end{bmatrix}$$

$$|A| = (24 \times 7) - (8 \times 15) = 48$$

Matrix of cofactors Cij $= \begin{bmatrix} 7 & -8 \\ -15 & 24 \end{bmatrix}$

$$\text{Adj . A} = [\text{Cij}]^T = \begin{bmatrix} 7 & -15 \\ -8 & 24 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj.A}}{|A|} = \frac{\begin{matrix} 7 & -15 \\ -8 & 24 \end{matrix}}{48}$$

$$A^{-1} = \begin{bmatrix} \frac{7}{48} & \frac{-5}{16} \\ \frac{-1}{6} & \frac{1}{2} \end{bmatrix}$$

Example 2

$$A = \begin{bmatrix} -7 & 16 \\ -9 & 13 \end{bmatrix}$$

$$|A| = 53$$

Matrix of cofactors Cij $= \begin{bmatrix} 13 & 9 \\ -16 & -7 \end{bmatrix}$

$$\text{Adj. A} = \text{Cij}^T = \begin{bmatrix} 13 & -16 \\ 9 & -7 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj.A}}{|A|} = \frac{\begin{matrix} 13 & -16 \\ 9 & -7 \end{matrix}}{53}$$

$$A^{-1} = \begin{bmatrix} \frac{13}{53} & \frac{-16}{53} \\ \frac{9}{53} & \frac{-7}{53} \end{bmatrix}$$

Example 3

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

$$|A| = -11$$

Matrix of cofactors Cij $= \begin{bmatrix} 8 & -4 & -7 \\ -5 & -3 & 3 \\ -2 & 1 & -1 \end{bmatrix}$

$$\text{Adj. A} = \text{Cij}^{\text{T}} = \begin{bmatrix} 8 & -5 & -2 \\ -4 & -3 & 1 \\ -7 & 3 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj.A}}{|A|} = \begin{bmatrix} \frac{-8}{11} & \frac{5}{11} & \frac{2}{11} \\ \frac{4}{11} & \frac{3}{11} & \frac{-1}{11} \\ \frac{7}{11} & \frac{-3}{11} & \frac{1}{11} \end{bmatrix}$$

Example 4

$$A = \begin{bmatrix} 14 & 0 & 6 \\ 9 & 5 & 0 \\ 0 & 11 & 8 \end{bmatrix}$$

$$|A| = 1154$$

$$\text{Matrix of cofactors Cij} = \begin{bmatrix} 40 & -72 & 99 \\ 66 & 112 & -154 \\ -30 & 54 & 70 \end{bmatrix}$$

$$\text{Adj. A} = \text{Cij}^{\text{T}} = \begin{bmatrix} 40 & 66 & -30 \\ -72 & 112 & 54 \\ -99 & -154 & 70 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj.A}}{|A|} = \begin{bmatrix} \frac{20}{577} & \frac{33}{577} & \frac{-15}{577} \\ \frac{-36}{577} & \frac{56}{577} & \frac{27}{577} \\ \frac{99}{577} & \frac{-77}{577} & \frac{35}{577} \end{bmatrix}$$

3.7 Solving linear Equations with the Inverse

An inverse matrix can be used to solve linear equations. Consider the following set of equations

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

Now setting

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

In matrix form, the system of equation is

$$AX = B$$

In matrix A is nonsingular so that A^{-1} exist, then multiplication of both sides by A^{-1}

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

This gives the solution of above equation. Thus, the solving of equation is given by the product of inverse matrix of the coefficient matrix (A^{-1}) and the column vector of constants B.

Example 1

$$4x_1 + 3x_2 = 28$$

$$2x_1 + 5x_2 = 42$$

In matrix form AX = B

$$\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 28 \\ 42 \end{bmatrix}$$

$$X = A^{-1}B$$

$$X = A^{-1}B$$

$$A^{-1} = \frac{\text{Adj.A}}{|A|}$$

$$|A| = (4 \times 5) - (2 \times 3) = 14$$

Matrix of cofactors $Cij = \begin{bmatrix} 5 & -2 \\ -3 & 4 \end{bmatrix}$

$$\text{Adj. A} = Cij^T = \begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj.A}}{|A|} = \begin{bmatrix} \frac{5}{14} & \frac{-3}{14} \\ \frac{-2}{14} & \frac{4}{14} \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} \frac{5}{14} & \frac{-3}{14} \\ \frac{-2}{14} & \frac{4}{14} \end{bmatrix} \begin{bmatrix} 28 \\ 42 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$\therefore x_1 = 1 \qquad x_2 = 8$

Example 2

$$6x_1 + 7x_2 = 56$$

$$2x_1 + 3x_2 = 44$$

$$AX = B$$

$$\begin{bmatrix} 6 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 56 \\ 44 \end{bmatrix}$$

$$X = A^{-1}B$$

$$A^{-1} = \frac{\text{Adj.A}}{|A|}$$

$|A| = 4$

Adj. $A = \begin{bmatrix} 3 & -7 \\ -2 & 6 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{-7}{4} \\ \frac{-1}{2} & \frac{3}{2} \end{bmatrix}$

$X = A^{-1} B = \begin{bmatrix} \frac{3}{4} & \frac{-7}{4} \\ \frac{-1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 56 \\ 44 \end{bmatrix}$

$X = \begin{bmatrix} -35 \\ 38 \end{bmatrix}$

$\therefore x_1 = -35 \qquad x_2 = 38$

Example 3

$5x + 3y + 14z = 4$

$y + 2z = 1$

$x + y + 2z = 0$

$AX = B$

$\begin{bmatrix} 5 & 3 & 14 \\ 0 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$

$X = A^{-1}B$

$A^{-1} = \frac{\text{Adj.A}}{|A|}$

$|A| = 12$

Adj. $A = \begin{bmatrix} 4 & -20 & -8 \\ 2 & -4 & -10 \\ -1 & 8 & 5 \end{bmatrix}$

$X = A^{-1}B = \left[\frac{\begin{matrix} 4 & -20 & -8 \\ 2 & -4 & -10 \\ -1 & 8 & 5 \end{matrix}}{12} \right] \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$

$X = \begin{bmatrix} -1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$

$\therefore x = -1/3 \qquad y = 1/3 \qquad z = 1/3$

Example 4

$$2x_1 + 4x_2 - 3x_3 = 12$$

$$3x_1 - 5x_2 + 2x_3 = 13$$

$$-x_1 + 3x_2 + 2x_3 = 17$$

$$AX = B$$

$$\begin{bmatrix} 2 & 4 & -3 \\ 3 & -5 & 2 \\ -1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 17 \end{bmatrix}$$

$$X = A^{-1}B$$

$$A^{-1} = \frac{\text{Adj.A}}{|A|}$$

$$|A| = -76$$

$$\text{Adj. A} = \begin{bmatrix} -16 & -17 & -7 \\ -8 & 1 & -13 \\ 4 & -10 & -22 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} \dfrac{\begin{matrix} -16 & -17 & -7 \\ -8 & 1 & -13 \\ 4 & -10 & -22 \end{matrix}}{-76} \end{bmatrix} \begin{bmatrix} 12 \\ 13 \\ 17 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 6 \end{bmatrix}$$

$\therefore x_1 = 7 \qquad x_2 = 4 \qquad x_3 = 6$

3.8 Cramer's Rule for Matrix solutions

Cramer's rule provide a simplified method of solving a system of linear equations through the use of determinants. Cramer's rule states that

$$x_i = \frac{|A_i|}{|A|}$$

Where x_i is unknown variable and $|A|$ is the determinant of the coefficient matrix and $|A_i|$ is the determinant of special matrix formed from the original coefficient matrix by replacing the column of coefficient of x_i with column vector of constants.

Example 1

$$7x_1 + 2x_2 = 60$$

$$x_1 + 8x_2 = 78$$

In matrix form $AX = B$

$$\begin{bmatrix} 7 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 60 \\ 78 \end{bmatrix}$$

$$|A| = 54$$

To solve for x_1 forming new matrix A_1

$A_1 = \begin{bmatrix} 60 & 2 \\ 78 & 8 \end{bmatrix}$

$|A_i| = 324$

$x_1 = \frac{|A_1|}{|A|} = \frac{324}{54} = 6$

To solve for x_2 forming new matrix A_2

$A_2 = \begin{bmatrix} 7 & 60 \\ 1 & 78 \end{bmatrix}$

$|A_2| = 486$

$x_2 = \frac{|A_2|}{|A|} = \frac{486}{54} = 9$

$\therefore$ Solution is $x_1 = 6$ and $x_2 = 9$

Example 2

$$18x + y = 87$$

$$-2x + 36y = 98$$

In matrix form AX = B

$\begin{bmatrix} 18 & -1 \\ -2 & 36 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 87 \\ 98 \end{bmatrix}$

$|A| = 642$

$A_1 = \begin{bmatrix} 87 & -1 \\ 98 & 36 \end{bmatrix}$

$|A_1| = 3230$

$x_1 = \frac{|A_1|}{|A|} = \frac{3230}{642} = 5$

$A_2 = \begin{bmatrix} 18 & 87 \\ -2 & 98 \end{bmatrix}$

$|A_2| = 1938$

$y = \frac{|A_2|}{|A|} = \frac{1938}{642} = 3$

$\therefore$ Solution is $x = 5$ and $y = 3$

Example 3

$5x_1 - 2x_2 + 3x_3 = 16$

$2x_1 + 3x_2 - 5x_3 = 2$

$4x_1 - 5x_2 + 6x_3 = 7$

AX = B

$$\begin{bmatrix} 5 & -2 & 3 \\ 2 & 3 & -5 \\ 4 & -5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16 \\ 2 \\ 7 \end{bmatrix}$$

$|A| = -37$

$$A_1 = \begin{bmatrix} 16 & -2 & 3 \\ 2 & 3 & -5 \\ 7 & -5 & 6 \end{bmatrix}$$

$|A_1| = -111$

$$x_1 = \frac{|A_1|}{|A|} = \frac{-111}{-37} = 3$$

$$A_2 = \begin{bmatrix} 5 & 16 & 3 \\ 2 & 2 & -5 \\ 4 & 7 & 6 \end{bmatrix}$$

$|A_2| = -259$

$$x_2 = \frac{|A_2|}{|A|} = \frac{-259}{-37} = 3$$

$$A_3 = \begin{bmatrix} 5 & -2 & 16 \\ 2 & 3 & 2 \\ 4 & -5 & 7 \end{bmatrix}$$

$|A_3| = -185$

$$X_3 = \frac{|A_3|}{|A|} = \frac{-185}{-37} = 5$$

$\therefore$ Solution is $x_1 = 3$, $x_2 = 7$ and $x_3 = 5$

Example 4

$$11p_1 - p_2 - p_3 = 31$$

$$-p_1 + 6p_2 - 2p_3 = 26$$

$$-p_1 - 2p_2 + 7p_3 = 24$$

$AX = B$

$$\begin{bmatrix} 11 & -1 & -1 \\ -1 & 6 & -2 \\ -1 & -2 & 7 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 31 \\ 26 \\ 24 \end{bmatrix}$$

$|A| = 401$

$$A_1 = \begin{bmatrix} 31 & -1 & -1 \\ 26 & 6 & -2 \\ 24 & -2 & 7 \end{bmatrix}$$

$|A_1| = 1604$

$$p_1 = \frac{|A_1|}{|A|} = \frac{1604}{401} = 4$$

$$A_2 = \begin{bmatrix} 11 & 31 & -1 \\ -1 & 26 & -2 \\ -1 & 24 & 7 \end{bmatrix}$$

$$|A_2| = 2807$$

$$p_2 = \frac{|A_2|}{|A|} = \frac{2807}{401} = 7$$

$$A_3 = \begin{bmatrix} 11 & -1 & 31 \\ -1 & 6 & 26 \\ -1 & -2 & 24 \end{bmatrix}$$

$$|A_3| = 2406$$

$$p_3 = \frac{|A_3|}{|A|} = \frac{2406}{401} = 6$$

$\therefore$ Solution is $p_1 = 4$, $p_2 = 7$ and $p_3 = 6$

3.9 Rank of a matrix

The rank of matrix of A, denoted as P(A) is defined as the maximum number of linearly independent rows or columns. If square matrix A is non singular, such that $|A| \neq 0$, then rank of A will be the order of the matrix itself and rows and columns are linearly independent.

Example 1

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}$$

$$|A| = 3$$

Since A is non singular $P(A) = 2$

Example 2

$$B = \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix}$$

$$|B| = 0$$

B is singular and hence $P(B) = 1$

Example 3

$$C = \begin{bmatrix} -3 & 6 & 2 \\ 1 & 5 & 4 \\ 4 & -8 & 2 \end{bmatrix}$$

$$|C| = -98$$

C is non singular $P(C) = 3$

Example 4

$$D = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 2 & 4 & 6 \end{bmatrix}$$

$$|D| = 0$$

Since D = 0, D is singular and $P(D) \neq 3$. Now trying various 2×2 sub matrices, starting from upper left corner.

$$\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 6 - 6 = 0$$

$$\begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = 18 - 18 = 0$$

$$\begin{vmatrix} 3 & 6 \\ 2 & 4 \end{vmatrix} = 12 - 12 = 0$$

$$\begin{vmatrix} 6 & 9 \\ 4 & 6 \end{vmatrix} = 36 - 36 = 0$$

With all the determinants of different 2×2 submatrices are equal to zero, non of two or columns are independent. Hence $P(D) \neq 2$ and $P(D) = 1$.

Reference:

(1) Edward R Dowling : Introduction to Mathematical Economics Second Edition, Schaum's outline series.

(2) Alpha C Chiang : Fundamental Methods of Mathematical Economics, MC Graw Hill.

(3) G.S. Monga : Mathematics and Statics for Economics Vikad Publishing House Pvt. Ltd.

CHAPTER IV

BASIC MATHEMATICAL CONCEPTS

4. 1: EXPONENTS

In the expression a^n the base is 'a' and 'n' is the power or the exponent or the index. The value is obtained by multiplying 'a' by itself 'n' times. The expression a^n is called exponential expression. The theory that deals with the expressions of the type a^n is called theory of indices.

Some Basic Concepts

1. Positive Integral Power

   If n is a positive integer, a^n is defined as the product of n factors each of which is 'a'

   i.e., $a^n = a \times a \times a$ ………………….. (n factors)

   Eg: $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$

   $5^3 = 5 \times 5 \times 5 = 125$

2. Zero Power (Zero Exponent)

   Any number other than zero raised to zero = 1

   i.e. $a^0 = 1$ if $a \neq 0$

   Eg: (1) $5^0 = 1$

   (2) $(\frac{1}{2})^0 = 1$

3. Negative Integral Power

   If n is a positive integer, then $a^{-n} = \frac{1}{a^n}$, where $a \neq 0$

   Eg: (1) $2^{-4} = \frac{1}{2^4} = \frac{1}{16}$

   (2) $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

   Here a^{-n} is the reciprocal of a^n.

4. Root of the Number

   (a) Square root

   Square root of a number 'a' is another number 'b' whose square is 'a'.

   If $a^2 = b$, then 'a' is the square root of 'b'

   i.e. $a = \sqrt{b}$ or $a = b^{1/2}$

   Eg: If $10^2 = 100$ then $10 = \sqrt{100}$ or $100^{1/2}$

   (b) n^{th} Root of

   If $a^n = b$, then 'a' is the n^{th} root of 'b' and it is written as $\sqrt[n]{b}$ or $b^{1/n}$

   Eg: $3^4 = 81$ then $3 = \left(\sqrt[4]{81}\right) = 81^{1/4}$

5. **Positive Fractional Power**

If 'm' and 'n' are positive integers, then $a^{\frac{m}{n}}$ is defined as the n^{th} root of m^{th} power of a.

$a^{\frac{m}{n}} = (\sqrt[n]{a})^m$

Eg: (1) $(32)^{\frac{3}{5}} = (\sqrt[5]{32})^3 = 2^3 = 8$

(2) $(-64)^{\frac{4}{3}} = (\sqrt[3]{-64})^4 = -4^4 = 256$

6. **Negative Factional Power**

If 'm' and 'n' are positive integers, then $a^{\frac{-m}{n}}$ is defined as $\frac{1}{a^{\frac{m}{n}}}$

Eg: (1) $(32)^{\frac{-3}{5}} = \frac{1}{(32)^{\frac{3}{5}}} = \frac{1}{(\sqrt[5]{32})^3} = \frac{1}{2^3} = \frac{1}{8}$

(2) $(-64)^{\frac{-4}{3}} = \frac{1}{(-64)^{\frac{4}{3}}} = \frac{1}{(\sqrt[3]{-64})^4} = \frac{1}{-4^4} = \frac{1}{256}$

Laws of Indices (Rules of Exponent)

1. $a^m \cdot a^n = a^{m+n}$
2. $\frac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $(xy)^n = x^n \cdot y^n$
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
6. $\frac{1}{x^n} = x^{-n}$

Examples

1. Find the values of (a) 10^0 (b) 5^{-3} (c) $125^{1/3}$ (d) $(-8)^{2/3}$

   (a) $10^0 = 1$

   (b) $5^{-3} = \frac{1}{5^3} = \frac{1}{125}$

   (c) $125^{1/3} = \sqrt[3]{125} = 5$

   (d) $(-8)^{2/3} = (\sqrt[3]{-8})^2 = (-2)^2 = 4$

2. Simplify the following

   (a) $(0.0001)^{1/4}$ (b) $9^{3/2} \cdot 243^{4/5} \cdot 729^{5/6}$

   (c) $[(-8)^2]^2 \div (-4)^4$ (d) $x^{a-b} \cdot x^{b-c} \cdot x^{c-a}$

(a) $(0.0001)^{1/4} = 0.1 \times 0.1 \times 0.1 \times 0.1 = (0.1)^4$

$(0.0001)^{1/4} = [(0.1)^4]^{1/4} = 0.1^{4/4} = 0.1$

=====

(b) $9^{3/2} . 243^{4/5} . 729^{5/6} = (\sqrt{9})^3 \times (\sqrt[5]{243})^4 \times (\sqrt[6]{729})^5$

$= 3^3 \times 3^4 \times 3^5 = 3^{3+4+5} = 3^{12}$

$= 531441$

=======

(c) $[(-8)^2]^2 \div (-4)^4$

$(-8)^2 = [(-2^3)^2]^2 = (-2^3)^4 = (-2)^{12} = (-2^2)^6 = (2^2)^6 = 4^6$

$[(-8)^2]^2 \div (-4)^4 = \frac{4^6}{4^4} = 4^{6-4} = 4^2 = 16$

=======

(d) $x^{a-b}. x^{b-c} . x^{c-a} = x^{a-b+b-c+c-a} = x^0 = 1$

======

4.2 POLYNOMIALS

In the expression such as $10x^2$, x is called the variable because it can assume any number of different values, and 10 is reffered to as the coefficient of 'x'.

The expressions consisting simply of a real number or a coefficient times one or more variables raised to the power of a positive integer one called monomials. A monomial can also be a variable, like 'x' or 'y'. It can also be a combination of these.

Ex: (1) $4x^5$ (2) $3y^2$ (3) $10xyz$

A polynomial is an algebraic expression with a finite number of terms. Monomials can be added or subtracted to form polynomials. Each of the monomials comprising a polynomial is called a term. Terms that have the same variables and exponents are called like terms. Like terms in polynomials can be added or subtracted by adding their coefficients. Unlike items cannot be so added or subtracted.

Ex: (1) $10x^2 + 4x^2 = 14x^2$

(2) $25xy + 5xy – 20xy = 10xy$

(3) $(6x^5 + 5x^4 – 10x) + (4x^5 + 3x^4 + 8x) = 10x^5 + 8x^4 – 2x$

(4) $(2x^2 + 4y) + (4x^2 + 3z) = 6x^2 + 4y + 3z$

Like and unlike terms can be multiplied or divided by multiplying or dividing both the coefficient and variables.

Ex: (1) $(5x)(10y^2) = 50xy^2$

(2) $(5x^2y^3)(6x^4y^2) = 30x^6y^5$

(3) $\frac{25x^4y^3z^2}{5x^2y^2z} = 5x^2yz$

Multiplication of two polynomials

In multiplying two polynomials, each term in the first polynomial must be multiplied by each term in the second and their product is added.

Ex: (1) $(5x + 6y)(4x + 2y)$

$$= (5x)(4x) + (5x)(2y) + (6y)(4x) + (6y)(2y)$$

$$= 20x^2 + 10xy + 24xy + 12y^2$$

$$= 20x^2 + 34xy + 12y^2$$

====================

(2) $(6x – 4y)(3x – 5y)$

$$= (6x)(3x) + (6x)(-5y) – (4y)(3x) – (4y)(-5y)$$

$$= 18x^2 – 30xy – 12xy + 20y^2$$

$$= 18x^2 – 42xy + 20y^2$$

===================

Problems

(1) $(2x + 3y)(8x – 5y -7z)$

[Ans: $16x^2 + 14xy – 14xz – 21yz – 15y^2$]

(2) $(x + y)(x – y)$ [Ans: $x^2 – y^2$]

4.3 FACTORING

Factoring polynomial expressions is not quite the same as factoring numbers. But the concept of factoring is very similar in both cases. When factoring numbers or polynomials, we are finding numbers or polynomials that divide out evently from the original numbers or polynomials. But in the case of polynomials, we are dividing numbers and variables out of expressions, not just dividing number out of numbers.

Ex: $2x + 6 = 2(x) + 2(3) = 2(x + 3)$

Factoring means "dividing out and putting in front of the parentheses". (Here nothing "disappears" when we factor, but the things are merely rearranged.)

Ex: Factor $4x + 12$

In the above example the only thing common between the two terms (i.e. the only thing that can be divided out of each term and then moved up front) is '4'.

Therefore $4x + 12 = 4(\quad)$

When we divide the '4' out of the 4x, we get 'x'. So we have to put that 'x' as the first term inside the parentheses.

i.e. $4x + 12 = 3(x \quad)$

When we divide the '4' out of the +12 we get '4'. So we have to put it as the second term inside the parentheses.

$$4x + 12 = 3(x + 4)$$

Ex: (2) Factor $5x - 35$

Ans: $5(x - 7)$

(3) Factor $8x^2 - 5x$

Ans: $x(8x - 5)$

(4) Factor $x^2y^3 + xy$

Ans: $xy(xy^2 + 1)$

(5) Factor $3x^3 + 6x^2 - 15x$

Ans: Here '3' and 'x' are common. So we can factor a '3' and 'x' out of each term.

$3x^3 = 3x(x^2)$

$6x^2 = 3x(2x)$

$-15x = 3x\ (-5)$

$3x^3 + 6x^2 - 15x = 3x(x^2 + 2x -5)$

(6) $2(x-y) - b(x - y)$

Ans: $(x - y)(2 - b)$

(7) Factor $x(x-2) + 3(2 - x)$

Ans: $(x - 2)(x - 3)$

(8) Factor $xy - 5y - 2x + 10$

Here there is no common factor in all four terms. So, we have to factor "in pairs". For this split this expression into two pairs of terms and then factor the pairs separately.

$xy - 5y - 2x + 10 = y(x - 5) - 2x + 10 = y(x - 5) - 2(x - 5)$

Here $(x - 5)$ is common.

$(x - 5)(y - 2)$

(9) Factor $x^2 + 4x - x - 4$

Ans: $x(x + 4) - 1(x + 4)$

Here $(x + 4)$ is common

$(x + 4)(x - 1)$

===========

(10) $x^2 - 5x - 6$

Ans: Here to get two pairs we have to split up the expression '$-5x$' into $-6x+1x$ then the function will become.

$x^2 - 6x + 1x - 6 = x\ (x - 6) + 1\ (x - 6)]$

$= (x - 6)\ (x + 1)$

===========

(11) Factor $6x^2 - 13x + 6$

Ans: $6x^2 - 9x - 4x + 6 = 3x(2x - 3) - 2(2x - 3)$

$= (2x - 3)(3x - 2)$

============

4.4 EQUATIONS : LINEAR AND QUADRATIC

A mathematical statement setting two algebraic expressions equal to each other is called an equation.

(a) Linear Equations

An equation in which all variables are raised to the first power is known as linear equation. It can be solved by moving the unknown variable to the left handside of the equal sign and all the other terms to the right-hand side.

Ex. (1) $8x + 4 = 10x - 4$

Solution : $8x - 10x = -4 - 4$

$-2x = -8$

$x = {}^{-8}/_{-2} = 4$

=======

Ex. (2) $9(3x + 4) - 2x = 11 + 5(4x - 1)$

Ans: $27x + 36 - 2x = 11 + 20x - 5$

$27x - 2x - 20x = 11 - 5 - 36$

$5x = -30$

$x = {}^{-30}/_{-5} = -6$

========

Ex. (3) $\frac{x}{10} - 4 = \frac{x}{8} + 3$

Ans: $\frac{x}{10} - \frac{x}{8} = 3 + 4 = 7$

$\frac{x}{10} - \frac{x}{8} = 7$

Multiplying with '40' on both side to avoid fractional expression we get

$40(\frac{x}{10} - \frac{x}{8}) = 7 \times 40$

$\frac{40x}{10} - \frac{40x}{8} = 280$

$4x - 5x = 280$

$-x = 280$

$x = -280$

======

Quadratic Equation

An equation in the form of $ax^2 + bx + c = 0$ where a, b and c are constants and $a \neq 0$ is called quadratic equation. It can be solved by factoring method, completing the square method or by using the quadratic formula.

(i) **Factoring Method**

Factoring is the easiest way to solve a quadratic equation, provided the factors are easily recognized integers.

Ex 1: $x^2 + 3x + 30 = 0$

$(x + 3)x + (x + 3)10 = 0$

$(x + 3)(x + 10) = 0$

The equation is true only if either $(x + 3) = 0$ or $(x + 10) = 0$

If $(x + 3) = 0$ $x = -3$

If $(x + 10) = 0$ $x = -10$

Ex: $2\ x^2 - 6x + 6 = 0$

$(x - 2)(x - 4) = 0$

$x = 2$ or $x = 4$

(ii) **Quadratic Formula**

One of the commonly used methods for finding the root values of quadratic equations is quadratic formula. In the quadratic formula the unknown value of x is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Ex 1: $x^2 + 13x + 30 = 0$

Ans: Here a = 1, b = 13, c = 30

Substituting the values of a, b and c in the equation we get

$$= \frac{-13 \pm \sqrt{13^2 - 4 \times 1 \times 30}}{2 \times 1}$$

$$= \frac{-13 \pm \sqrt{169 - 120}}{2}$$

$$= \frac{-13 \pm \sqrt{49}}{2} = \frac{-13 \pm 7}{2}$$

$$= \frac{-13 + 7}{2} \text{ or } \frac{-13 - 7}{2}$$

$$= \frac{-6}{2} = -3 \text{ or } \frac{-20}{2} = -10$$

$x = -3$ or -10

==============

Ex 2: Solve the following quadratic equations using the quadratic formula.

$5x^2 + 23x + 12 = 0$

(Ans: x = -0.6 or -4)

4.4 COMPLETING THE SQUARE

(Shridhar's Method)

This is a method for obtaining the root values of the quadratic equations. Here use convert the equation into a complete square form, then into a simple linear equation and solve it.

Ex. 1 Solve $4x^2 - 12x + 9 = 0$

Ans: $(2x - 3)^2 = 0$

Since $(2x - 3)^2 = 0$, $(2x - 3) = 0$

$2x = 3$, $x = {}^3/_2 = 1.5$

====

Ex. 1 Solve $4x^2 - 16x + 16 = 0$

Ans: $(2x - 4)^2 = 0$

$2x - 4 = 0$

$2x = 4$, $x = {}^4/_2 = 2$

====

4.6 SIMULATANEOUS EQUATIONS

Simultaneous equations are a set of equations containing multiple variables. This set is often referred to as a system of equations.

Among the system of equations, the systems of linear equations are specially important.

(a) Simultaneous Equations of first degree in two unknown

Ex. Solve $4x + 2y = 6$

$5x + y = 6$

Here we can use diminations method. We can eliminate either 'x' or 'y'. For eliminating 'x' make the coefficient of 'x' in both the equations same or coefficient of 'y' in both equations same.

Ans: $4x + 2y = 6$ ---------- (1)

$5x + y = 6$ ---------- (2)

For eliminating 'x' multiplying equation (1) by 5 and the (2) by 4

$20x + 10y = 30$ ----------- (3)

$20x + 4y = 24$ ----------- (4)

Substracting (4) from (3) we get

$6y = 6$

$y = {}^{6}/_{6} = 1$

Substituting y = 1 in the first equation 4x + 2y = 6 we get

$4x + 2(1) = 6$

$4x = 6 - 2 = 4$

$x = {}^{4}/_{4} = 1$

Solutions x = 1 and y = 1

================

Ex. 2: Solve $2x - 5 = 5$

$3x - 4y = 10$ (Ans: x = -2, y = -1)

Ex. 3: Solve $-10x + 8y = 52$

$4x - 6y = -32$ (Ans: x = -2, y = 4)

(b) <u>Simultaneous Equations containing three unknown</u>

In the three unknown case there will be three equation. In order to solve such simultaneous equations using elimination method, at first take one pair of equations then eliminate one unknown. From another pair eliminate the same unknown. The two equations thus obtained contain only two unknown. So they can be solved. The values of two unknowns obtained may be substituted in one of the given equations so as to get the value of the third unknown.

Ex.1: Solve $9x + 3y - 4z = 35$

$x + y - z = 4$

$2x - 5y - 4z = -48$

Ans: $9x + 3y - 4z = 35$ --------- (1)

$x + y - z = 4$ ------------- (2)

$2x - 5y - 4z = -48$ ---------- (3)

(2) x 9 = $9x + 9y - 9z = 36$ -------- (4)

Subtracting (4) from (1) we get

$-6y + 5z = -1$ ---------- (5)

Equation (2) x 2 = $2x + 2y - 2z = 8$ ---------- (6)

(6) – (3) = $7y + 2z = 56$ ---------- (7)

==================

Equation (5) x 7 = $-42y + 35z = -7$ -------- (8)

(7) x 6 = $42y + 12z = 336$ ------- (9)

----

(8) + (9) = $42z = 329$

$z = {}^{329}/_{42} = 7$

=====

Substituting in (7) we get

$$7y + 2z = 56$$
$$7y + 2 \times 7 = 56$$
$$7y + 14 = 56$$
$$7y = 56 - 14 = 42$$
$$y = {}^{42}/_{7} = 6$$

Substitute in (2) $\quad x + y - z = 4$

i.e. $x + 6 - 7 = 4$

$x = 4 - 6 + 7 = -2 + 7 = 5$

$x = 5$

=====

Solution is $x = 5, y = 6, z = 7$

Ex.2 : Solve the following simultaneous equations

$$2x - y + z = 3$$
$$x + 3y - 2z = 11$$
$$3x - 2y + z = 4$$

(Ans: $x = 3, y = 2, z = -1$)

Note: Simultaneous Equations can also be solved by matrix inversion method and Crammer's Rule.

4.7 FUNCTIONS

The term function describes the manner in which one variable changes in relation to the changes in the other related variables. Therefore a function is a relation which associates any given variable with other variables. It is the rule or a formula that given the value of one variable if the value of the other variable is specified.

Ex: If 'x' and 'y' are the two variables such that $y = 2x^2$ shows every value of 'y' is obtained by squaring the corresponding value of x and multiplying with '2'. So we can say that 'y' is a function of x. i.e. $y = f(x)$.

If $x = 2 \qquad y = 2 \times 2^2 = 8$

If $x = 3 \qquad y = 2 \times 3^2 = 18$

Ex. 1: Given $f(x) = 3x^2 + 5x - 10$ find $f(4)$ and $f(-4)$

Substituting (4) for each occurrence of 'x' in the function we get.

$$= 3(4)^2 + 5(4) - 10 = 3 \times 16 + 20 - 10 = 48 + 20 - 10 = 58$$

Substituting (-4) for each occurrence of 'x' in the function we get.

$$= 3(-4)^2 + 5(-4) - 10$$
$$= 3 \times 16 - 20 - 10$$
$$= 48 - 30 = 18$$

=======

Ex.2 : Given $y = 2x^3 - 15x^2 + 2x - 10$ find f(2) and f(-1)

[Ans: f(2) = -58 f(-1) = -29]

Ex.3 : Check whether the following equations are functions or not and why?

(a) $y = 5x + 10$

This is a function because for each value of 'x' there is one and only one value of 'y'.

(b) $y^2 = x$

Here $y = \pm\sqrt{x}$. This is not a function because for each value of 'x' there are two values for 'y'.

(c) $x^2 + y^2 = 64$

This is not a function because if x = 0, y^2 became 64. Therefore $y = \sqrt{64} = \pm 8$. Two values.

(d) x = 4

This is not a function because at x = 4, y has many values.

(e) $y = 5x^2$

This is a function because there is a unique value for 'y' at each value of x.

Important functions used in Economics are:-

(1) Linear Functions

$fx = mx + c$

(2) Quadratic Functions

$f(x) = ax^2 + bx + c \quad (a \neq 0)$

(3) Polynomial Functions of degree n

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots\ldots\ldots + a_0$

(n = non-negative, $a_n \neq 0$)

(4) Rational Functions

$f(x) = \frac{g(x)}{n(x)}$

Where g(x) and f(x) are both polynomials and $h(x) \neq 0$

(5) Power Functions

$f(x) = ax^n \quad$ (n = any real number)

4.8 GRAPHS, SLOPES AND INTERCEPTS

For graphing a function such as y = f(x), the independent variable 'x' is placed on the horizontal axis and dependent variable 'y' is placed on the vertical axis. The graph of a linear function is a straight line. The slope of a line measures the change of 'y' divided by a change in x i.e. ${\Delta y}/{\Delta x}$. Slope indicates the steepness and directions of a line. A line will be more steeper if the absolute value of the slope is more and flatter if the absolute value of the slope is low.

Slope of a horizontal line for which $\Delta y = 0$ is zero became slope if $\frac{\Delta y}{\Delta x}$ ie. ${0}/{\Delta x} = 0$.

For a vertical line for which $\Delta x = 0$ the slope is undefined. i.e. slope is $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{0} = \infty$.

Intercepts

There are two intercepts for a straight line i.e. x intercept and y intercepts.

The ‘x’ intercept is the point where the line intersects the ‘x’ axis; it occurs when y = 0.

The ‘y’ intercept is the point where the line intersects the ‘y’ axis; it occurs when x = 0.

Ex.1: Graph the following linear equations and obtain the slope and x and y intercepts for $y = -{}^{1}/_{3}x+5$

Ans: First of all we have to put some convenient values of x and find the corresponding value of ‘y’.

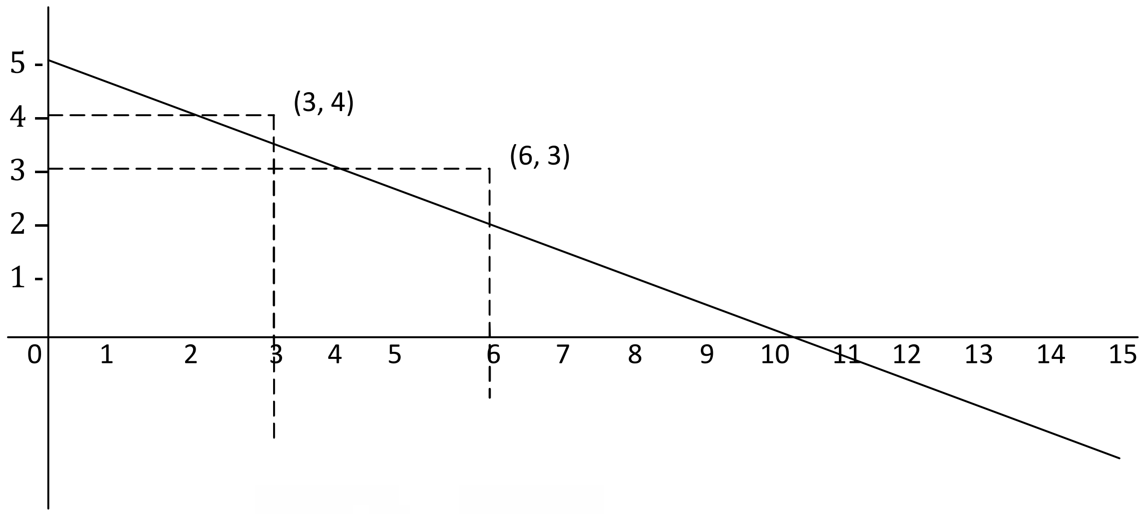
When x = 0, we have y = 2

When x = 15, y will be zero.

Similarly we have to put some more values for x and get corresponding values of y.

x	0	3	-3	6	-6	15
y	5	4	8	3	7	0

Since the equation $y = -{}^{1}/_{3}x+5$ is a linear equation we need only two points which satisfy the equation and correct them by a straight line. Since the graph of a linear function is a straight line, all the points satisfying the equation must be on the line.



Here slope is $-{}^{1}/_{3}$ i.e. $\frac{3-4}{6-3} = -{}^{1}/_{3}$

For a line passing through (x_1, y_1) and (x_2, y_2) the slope ‘m’ is calculated as

$$m = \frac{\Delta y}{\Delta x} = \frac{y2-y1}{x2-x1}, \quad x_1 \neq x_2$$

The slope is also the coefficient of ‘x’ is the y on x equation.

Since the equation is $y = -{}^{1}/_{3}x + 5$

Slope is $-1/3$

The vertical intercept (y intercept) is obtained by setting x = 0

Hence 'y' intercept is 5.

The horizontal intercept (x intercept) is obtained by setting y = 0

Here 'x' intercept is 15.

Ex.2: Graphs the following equations and indicate their respective slopes and intercepts.

(a) 3y + 15x = 30

(Ans: Slope m = -5, 'y' intercept = 5, x intercept = 2)

(b) 8y – 2x + 16 = 0

(Ans: Slop m = ¼ , y intercept = -2, x intercept = 8)

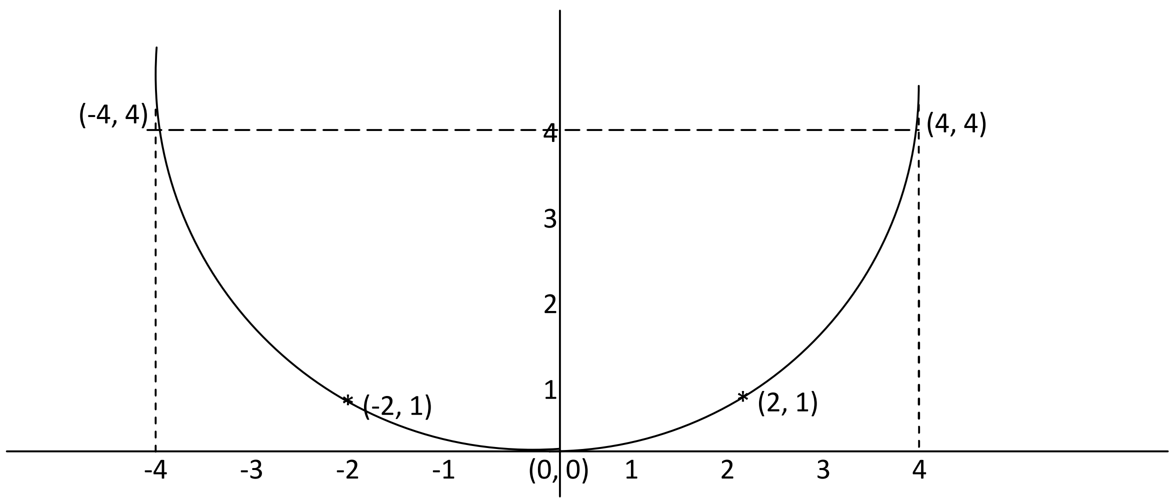
4.9 GRAPHS OF NON-LINEAR FUNCTIONS

We can draw the graphs of non-linear function such as quadratic functions, rational functions by preparing the schedule showing the values of 'x' and 'y' from the function and plotting the points and joining them.

Ex.1: Draw the graph of the function $y = \frac{x^2}{4}$

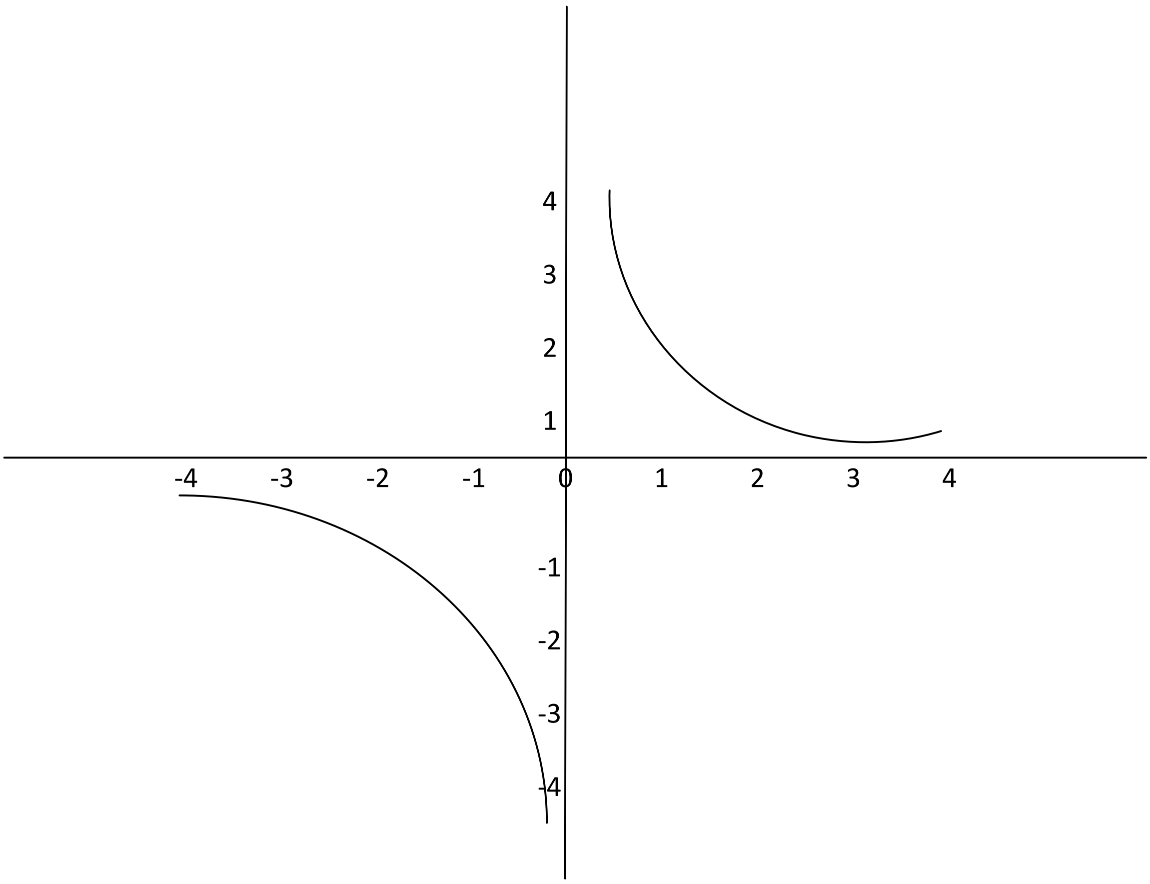
Ans: Schedule showing values of x and y

x	0	-1	1	-2	2	-4	4
y	0	0.25	0.25	1	1	4	4



Ex.2: Graph the rational function $y = 2/x$

x	$y = 2/x$	Points
-4	- ½	(-4, -½)
-2	-1	(-2, -1)
-1	-2	(-1, -2)
-½	-4	(- ½, -4)
½	4	(½, 4)
1	2	(1, 2)
2	1	(2, 1)
4	½	(4, ½)



CHAPTER V

ECONOMIC APPLICATIONS OF GRAPHS & EQUATIONS

5.1 ISOCOST LINES

An isocost line represents the different combinations of two inputs or factors of production that can be purchased with a given sum of money. If we assume that there are only two inputs, say capital (K) and labour(L), then the general formula for isocost line can be written as

$$P_L L + P_k K = E$$

Where K and L are capital and labour, P_k and P_L their respective prices, and E the amount allotted to expenditures. In isocost analysis the individual prices and the expenditure are initially held constant, only the combinations of inputs are allowed to change. The function can then be graphed by expressing one variable in terms of the other That is,

$$\text{Given} \quad P_L L + P_k K = E$$

$$P_k K = E - P_L L$$

$$K = \frac{E - P_L L}{P_k}$$

$$K = \frac{E}{P_k} - \frac{P_L L}{P_k}$$

This is the familiar linear function, where $\frac{E}{P_k}$ is the vertifical intercept and $(-\frac{P_L}{P_k})$ is the slop. If graphically expressed, we get a straight line or shown in figure 5.1 .

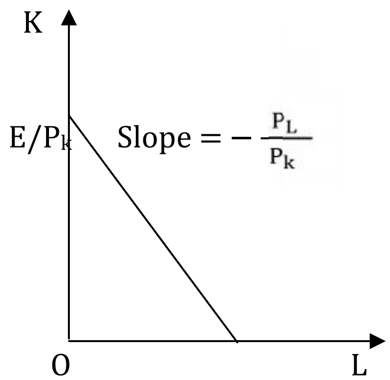


Figure 5.1

From the equation and graph, the effects of a change in any one of the parameters are easily discernible. An increase in the expenditure from E to 'E' will increase the vertical intercept and cause the isocost line to shift out to the right parallel to the old line as shown in figure 5.2(a)

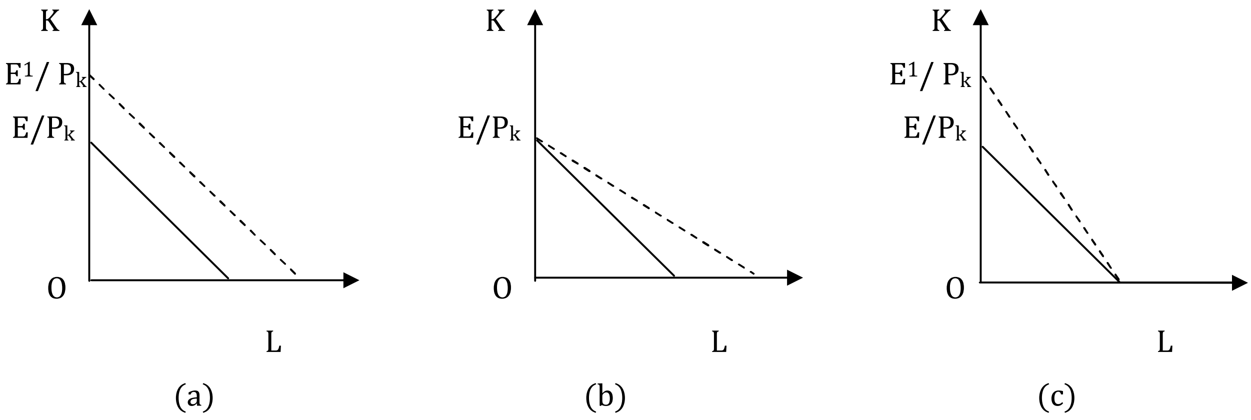


Figure 5.2

The slope is unaffected because the slope depends on the relative prices $(-P_L/P_k)$ and prices are not affected by expenditure changes. A charge in P_L will alter the slope of the line but leave the vertical intercept unchanged (figure 5.2 (b)). A change in P_k will alter the slope and the vertical intercept (figure 5.2 (C1))

5.2 SUPPLY AND DEMAND ANALYSIS

(a) Demand functions and curves.

Consider the amount of a definite group of consumers. In order to obtain a simple representation of the demand for X, we must assume that

(i) the number of consumers in the group.
(ii) the taster or preferences of each individual consumers for all goods on the market.
(iii) the income of each individual consumer, and
(iv) the prices of all goods other than X itself.

are fixed and known. The amount of X each consumer will take can then be considered as uniquely dependent on the price of X ruling on the market. By addition, it follows that the total amount of X demanded by the market depends uniquely on the market price of X The demand for X can only change if its market price varies.

This expression of market conditions can be translated at once into symbolic form. Let P denote the market price of X, and let x denote the amount of X demanded by the market. The x is a single valued function of P, which can be written as

$$X = f(P)$$

the demand function for X.

The variables x and p take only positive values.

For normal goods, demand is a negative function of price. The generally used demand functions are Downward sloping straight line.

(i) $P = a - b\,x$

(ii) $\left.\begin{array}{l} P = (a - bx)^2 \\ \text{or} \\ P = a - bx^2 \end{array}\right\}$ Parabola (to be taken in the positive quadrant only)

(iii) $(p + c)(x + b) = a$ Rectangular hyperbola with centre at $x = -b$, $p = -c$

The demand function can be represented as a demand curve in a plane referred to axes Op and Ox a long which prices and demands are respectively measured. It is clearly convenient, for theoretical purposes, to assume that the demand function and curve are continuous, that demand varies continuously with price figure 5.3 illustrates a hypothetical fitting of a continuous demand curve.

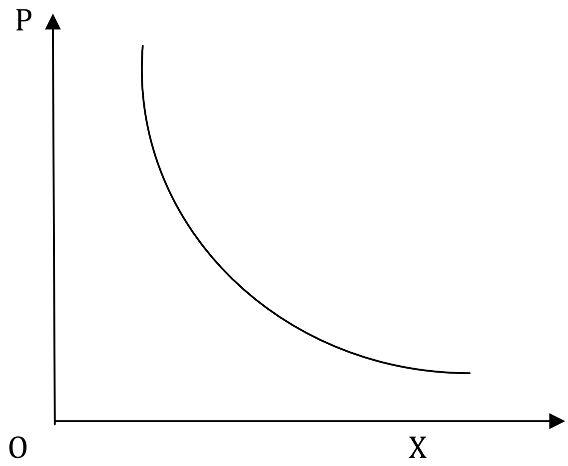


Figure 5.3

The continuity assumption is made for purpose of mathematical convenience, it can be given up, if necessary, at the cost of considerable complication in the theory.

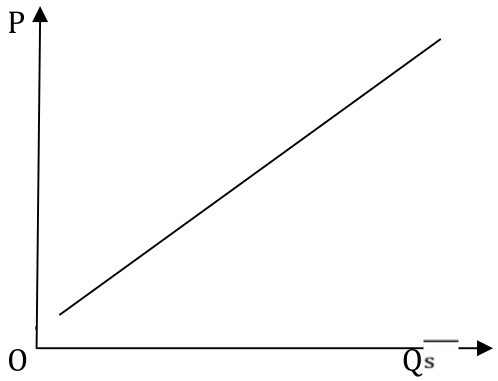
(b) Supply functions and curves.

The demand and supply functions are exactly similar. Supply means the quantities of a commodity that is available for sale in the market, at different prices during a given period of time. If all factors other than the price of the good concerned are fixed, the total supply of commodity in the market depends directly on the market price. Thus there is a functional relationship between the price of a commodity and its supply, which can expressed as

$$Qs = g(p)$$

Where Qs is the quantity supplied and P is the price.

Since the supply and price are directly related, supply curve is positively sloping. Figure 5.4 illustrates a hypothetical fitting of a continuous supply curve.



The positive sloping supply curve shows that as the price rises the quantity supplied of the commodity will also increase.

(C) Equilibrium

Equilibrium in supply and demand analysis occurs when quantity supplied equals quantity demanded. Tha is

$$Q_s = Q$$

By equating the supply and demand functions, the equilibrium price and quantity can be determined.

Example.

Given Qs = -5 + 3 p Q = 10 - 2p

In equilibrium

$$Q_s = Q$$

Solving for P,

$$-5 + 3P = 10 - 2P$$

$$5P = 15$$

$$P = 3$$

Substituting $P = 3$ in either of the equations,

$$Q_s = -5 + 3P = -5 + 3(3) = 4 = Q$$

Thus equilibrium price = 3 and

equilibrium quantity = 4

5.3 PRODUCTION POSSIBILITY FRONTIERS

Assume that a firm produces two good X and Y under given technical conditions and making use of fixed supplies of certain factors of production. The total cost of production is now given and the interest lies entirely in the varying amounts of the two good that can be produced. If a given amount x of the good X is produced, then the fixed resources of the firm can be adjusted so as to produce the largest amount y of the other good Y compatible with the given production of X. Here y is a single – valued function of x which can be taken as continuous and monotonic decreasing, the larger the amount of x required, the smaller is the amount of y obtainable. Conversely, if x is the largest amount of X that can be produced jointly with a given amount y of Y, then x is a single-valued, continuous and decreasing function of y. The two function must be inverse to each other, that is, two aspects of a single relation between the amounts of X and Y produced, a relation imposed by the condition of given resources. In symbols, we can write the relation.

$$F(x, y) = 0,$$

giving $y = f(x)$ and $x = g(y)$,

where f and g are single – valued and decreasing functions to be interpreted in the way described. The relation, $F(x,y) = 0$, can be called the transformation function of the firm and it serves to show the alternative productive possibilities of the given supplies of the factors. The corresponding transformation curve in the plane O_{xy} is cut by parallels to either axis in only single points and is downward sloping to both O_x and O_y. The curve is also called production possibility factors. Further, it can be taken, in the "normal" case, that the production of Y decreases at an increasing rate as the production of X is increased, and conversely. The transformation curve is thus concave to the origin. Figure 5.5 shows a normal transformation curve in the hypothetical case of a firm producing two good, given resources and technology.

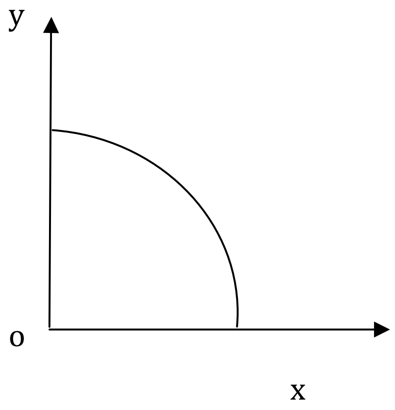


Figure 5.5

The transformation curve forms the outer boundary of the productive possibilities. Any point within the curve corresponds to productions of X and Y possible with the given resources, while productions represented by points outside the curve cannot be obtained no matter what adjustments of the given resources are made.

@@@@