QUANTUM MECHANICS

by

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$|t| > |x|$

$$x\mu + t\sqrt{k^2+\mu^2} = z$$

$$\left(x\mu + t\sqrt{k^2+\mu^2}\right)^2 - \left(t\mu + x\sqrt{k^2+\mu^2}\right)^2 = (x^2-t^2)\mu^2 + (t^2-x^2)(k^2+\mu^2) = (t^2-x^2)k^2 = \gamma^2$$

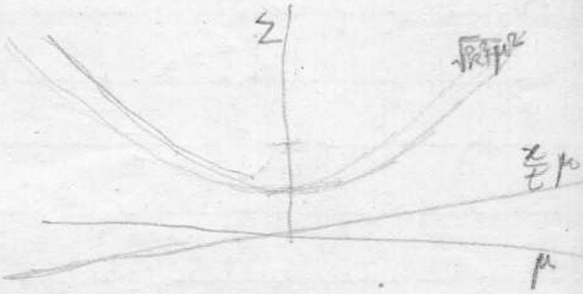
$$t\mu + x\sqrt{k^2+\mu^2} = \pm\sqrt{z^2-\gamma^2}$$

$$(t^2-x^2)\mu = \pm t\sqrt{z^2-\gamma^2} - xz \qquad (t^2-x^2)\sqrt{k^2+\mu^2} = tz \mp x\sqrt{z^2-\gamma^2}$$

$$\frac{\gamma^2}{k^2}\,d\mu = \left(\frac{tz}{\pm\sqrt{z^2-\gamma^2}} - x\right)dz = \frac{dz}{\pm\sqrt{z^2-\gamma^2}}\left(tz \mp x\sqrt{z^2-\gamma^2}\right)$$

$$\frac{\gamma^2}{k^2}\sqrt{k^2+\mu^2} = tz \mp x\sqrt{z^2-\gamma^2}$$

$$\frac{d\mu}{\sqrt{k^2+\mu^2}} = \pm\frac{dz}{\sqrt{z^2-\gamma^2}}$$



$$\int_{-\infty}^{\infty} e^{i(x\mu + t\sqrt{k^2+\mu^2})}\frac{d\mu}{\sqrt{k^2+\mu^2}} = 2\int_\gamma^\infty e^{iz}\frac{dz}{\sqrt{z^2-\gamma^2}} + \cancel{\int_{\infty}^{\gamma} e^{iz}\frac{dz}{-\sqrt{z^2-\gamma^2}}}$$

Inv. $K_1 = P(k^2(t^2-x^2)) = P(\gamma^2)$

$\alpha > 0$
$$P(\alpha) = \int_\alpha^\infty e^{iz}\frac{dz}{\sqrt{z^2-\alpha^2}} = \int_\alpha^\infty \frac{dz}{z}e^{iz}\left(1-\frac{\alpha^2}{z^2}\right)^{-\frac{1}{2}}$$

$z - \alpha = x$
$$\cancel{P(\alpha)} = e^{i\alpha}\int_0^\infty e^{ix}\frac{dx}{\sqrt{x(x+2\alpha)}}$$

$$\frac{d}{d\alpha}\left(P(\alpha)e^{-i\alpha}\right) = -\int_0^\infty e^{ix}\frac{dx}{x^{\frac{1}{2}}(x+2\alpha)^{3/2}} = -\int_\alpha^\infty e^{i(z-\alpha)}\frac{dz}{(z+\alpha)\sqrt{z^2-\alpha^2}}$$

$$P'(\alpha) = i\,P(\alpha) - \int_\alpha^\infty e^{iz}\frac{dz}{(z+\alpha)\sqrt{z^2-\alpha^2}} = \int_\alpha^\infty e^{iz}\frac{dz}{\sqrt{z^2-\alpha^2}}\left(i - \frac{1}{z+\alpha}\right)$$

$Q(\alpha) = P(\alpha)e^{-i\alpha}$
$$Q'(\alpha) = -\int_0^\infty e^{ix}\frac{dx}{(x+2\alpha)\sqrt{x(x+2\alpha)}}$$

There is no direct way of specifying the ~~constants of integration~~ state of a system by numbers. Such a specification must refer to a particular point of space time.

The code for specifying a state by numbers refers to a point of spacetime

Preface.

The modern theory of quantum mechanics was introduced by Heisenberg in a paper in the Zeits. f. Phys. vol. 33, p. 879 (1925). The following dissertation is a development of this theory from a slightly different point of view from that of Heisenberg's paper. The theory has been developed independently from Heisenberg's original point of view by Born and Jordan (Zeits. f. Phys. vol. 34, p. 858, 1925); Born, Heisenberg and Jordan (Zeits. f. Phys. vol. 35, p. 557, 1926) and Pauli (Zeits. f. Phys. vol. 36, p. 336, 1926). The general quantum conditions have been obtained independently by Born, Heisenberg and Jordan, by Kramers (Physica vol. 5, p. 369, 1925) and by the present author. The frequencies of the Balmer lines of hydrogen have been obtained independently by Pauli and the present author.

Dirac

① $\int_{-\infty}^{\infty} \frac{1}{\sqrt{k^2+\mu^2}} e^{i(x\mu + t\sqrt{k^2+\mu^2})} d\mu = \int_{-\infty}^{\infty} \frac{1}{\sqrt{k^2+\mu'^2}} e^{i(x'\mu' + t'\sqrt{k^2+\mu'^2})} d\mu'$

$$\frac{d\mu}{\sqrt{k^2+\mu^2}} = \frac{\left(l + \frac{m\mu'}{\sqrt{k^2+\mu'^2}}\right) d\mu'}{m\mu' + l\sqrt{k^2+\mu'^2}} = \frac{d\mu'}{\sqrt{k^2+\mu'^2}} \qquad \log(\mu+\sqrt{k^2+\mu^2}) = \log(\mu'+\sqrt{k^2+\mu'^2}) + \text{const.}$$

$$x\mu + t\sqrt{k^2+\mu^2} = x'\mu' + t'\sqrt{k^2+\mu'^2} \qquad (\mu+\sqrt{k^2+\mu^2}) = c(\mu'+\sqrt{k^2+\mu'^2})$$

$$d\mu\left(x + \frac{t\mu}{\sqrt{k^2+\mu^2}}\right) = d\mu'\left(x' + \frac{t'\mu'}{\sqrt{k^2+\mu'^2}}\right) \qquad = (l+m)(\mu'+\sqrt{k^2+\mu'^2})$$

$$\psi_2' = \psi_2$$

$$\psi_1 = -i\frac{\partial\psi_2}{\partial t} = -i\left(\frac{\partial\psi_2'}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial\psi_2'}{\partial x'}\frac{\partial x'}{\partial t}\right) = -i\left(l\frac{\partial\psi_2'}{\partial t'} + m\frac{\partial\psi_2'}{\partial x'}\right) = l\psi_1' - im\frac{\partial}{\partial x'}\psi_2'$$

$\psi_1' =$

General solution $\psi = a\psi_1 + b\psi_2$ $\qquad \psi_1 = -i\frac{\partial\psi_2}{\partial t}$

$\psi' = a'\psi_1' + b'\psi_2'$

$\psi', \frac{\partial\psi'}{\partial t}$ at any point depend only on $\psi, \frac{\partial\psi}{\partial t}$ in neighbourhood of that point.

$$\psi' = a'\left(l\psi_1 + im\frac{\partial}{\partial x}\psi_2\right) + b'\psi_2 = \alpha_1\psi + \alpha_2\frac{\partial\psi}{\partial t} + \alpha_3\frac{\partial\psi}{\partial x} + \alpha_4\frac{\partial^2\psi}{\partial x\partial t} + \cdots$$

$$= -ia'l\frac{\partial\psi_2}{\partial t} + ia'm\frac{\partial\psi_2}{\partial x} + b'\psi_2 = \alpha_1\left(-ia\frac{\partial\psi_2}{\partial t} + b\psi_2\right) + \alpha_2\frac{\partial}{\partial t}\left(-ia\frac{\partial\psi_2}{\partial t} + b\psi_2\right) + \alpha_3\frac{\partial}{\partial x}\left(-ia\frac{\partial\psi_2}{\partial t} + b\psi_2\right) + \cdots$$

If $a = 0$ and $a' = 0$ $\qquad b' = \alpha_1 b \quad \alpha_2 = \alpha_3 = \cdots = 0.$

Put $\mu + \sqrt{k^2+\mu^2} = \theta$ $\qquad d\mu\left(1 + \frac{\mu}{\sqrt{k^2+\mu^2}}\right) = d\theta \qquad \frac{d\theta}{\theta} = \frac{d\mu}{\sqrt{k^2+\mu^2}}$

$-\mu + \sqrt{k^2+\mu^2} = \frac{k^2}{\theta}$

$\mu = -\infty$

$\sqrt{k^2+\mu^2} = \mu\left(1+\frac{k^2}{\mu^2}\right)^{\frac{1}{2}}$

$= \mu + \frac{k^2}{2\mu} + \cdots$

$$x\mu + t\sqrt{k^2+\mu^2} = \frac{x+t}{2}(\mu+\sqrt{k^2+\mu^2}) + \frac{x-t}{2}(\mu - \sqrt{k^2+\mu^2}) = \frac{x+t}{2}\theta - \frac{x-t}{2}\frac{k^2}{\theta}$$

$$\int_{-\infty}^{\infty} \frac{d\mu}{\sqrt{k^2+\mu^2}} e^{i(x\mu + t\sqrt{k^2+\mu^2})} = \int_0^{\infty} \frac{d\theta}{\theta} e^{\frac{i}{2}\left[(x+t)\theta + (x-t)\frac{k^2}{\theta}\right]} = \int_0^{\infty} \frac{d\theta'}{\theta'} e^{\frac{i}{2}\left[\theta' + (x^2-t^2)\frac{k^2}{\theta'}\right]}$$

$$\theta' = (x+t)\theta$$

Contents.

$$\psi_1 = \int_{-\infty}^{\infty} \left[e^{i(x\mu + t\sqrt{k^2+\mu^2})} + e^{i(x\mu - t\sqrt{k^2+\mu^2})} \right] d\mu$$

$$\psi_2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{k^2+\mu^2}} \left[e^{i(x\mu + t\sqrt{k^2+\mu^2})} - e^{i(x\mu - t\sqrt{k^2+\mu^2})} \right] d\mu$$

$$-i\frac{\partial \psi_1}{\partial t} = \int_{-\infty}^{\infty} \sqrt{k^2+\mu^2} \left[e^{i(x\mu + t\sqrt{k^2+\mu^2})} - e^{i(x\mu - t\sqrt{k^2+\mu^2})} \right] d\mu$$

$$-i\frac{\partial \psi_2}{\partial t} = \int_{-\infty}^{\infty} \left[e^{i(x\mu + t\sqrt{k^2+\mu^2})} + e^{i(x\mu - t\sqrt{k^2+\mu^2})} \right] d\mu$$

$$x' = lx + mt \qquad l^2 - m^2 = 1 \qquad x'^2 - t'^2 = x^2 - t^2$$
$$t' = lt + mx$$

(1)

$$x'\mu' + t'\sqrt{k^2+\mu'^2} = x\mu + t\sqrt{k^2+\mu^2}$$

if $l\mu' + m\sqrt{k^2+\mu'^2} = \mu$

$$l\mu - m\sqrt{k^2+\mu^2} = \mu'$$
$$m\mu' + l\sqrt{k^2+\mu'^2} = \sqrt{k^2+\mu^2}$$
$$d\mu = \left(l + \frac{m\mu'}{\sqrt{k^2+\mu'^2}} \right) d\mu'$$

(2)

$$x'\mu' - t'\sqrt{k^2+\mu'^2} = x\mu - t\sqrt{k^2+\mu^2}$$

if $l\mu' - m\sqrt{k^2+\mu'^2} = \mu$

$$l\mu + m\sqrt{k^2+\mu^2} = \mu'$$
$$m\mu' - l\sqrt{k^2+\mu'^2} = -\sqrt{k^2+\mu^2}$$
$$d\mu = \left(l - \frac{m\mu'}{\sqrt{k^2+\mu'^2}} \right) d\mu'$$

(1)
$$\int_{-\infty}^{\infty} e^{i(x\mu + t\sqrt{k^2+\mu^2})} d\mu = \int_{-\infty}^{\infty} e^{i(x'\mu' + t'\sqrt{k^2+\mu'^2})} \left(l + \frac{m\mu'}{\sqrt{k^2+\mu'^2}} \right) d\mu'$$
$$= l \int_{-\infty}^{\infty} e^{i(x'\mu' + t'\sqrt{k^2+\mu'^2})} d\mu' - im\frac{\partial}{\partial x'} \int_{-\infty}^{\infty} \frac{1}{\sqrt{k^2+\mu'^2}} e^{i(x'\mu' + t'\sqrt{k^2+\mu'^2})} d\mu'$$

(2)
$$\int_{-\infty}^{\infty} e^{i(x\mu - t\sqrt{k^2+\mu^2})} d\mu = \int_{-\infty}^{\infty} e^{i(x'\mu' - t'\sqrt{k^2+\mu'^2})} \left(l - \frac{m\mu'}{\sqrt{k^2+\mu'^2}} \right) d\mu'$$
$$= l \int_{-\infty}^{\infty} e^{i(x'\mu' - t'\sqrt{k^2+\mu'^2})} d\mu' + im\frac{\partial}{\partial x'} \int_{-\infty}^{\infty} \frac{1}{\sqrt{k^2+\mu'^2}} e^{i(x'\mu' - t'\sqrt{k^2+\mu'^2})} d\mu'$$

$$\psi_1 = l\psi_1' - im\frac{\partial}{\partial x'}\psi_2'$$

$$\psi_1' = l\psi_1 + im\frac{\partial}{\partial x}\psi_2$$

gives
$$\psi_1 = l\left(l\psi_1 + im\frac{\partial \psi_2}{\partial x}\right) - im\left(\frac{\partial \psi_2}{\partial x}\overset{l}{\frac{\partial x}{\partial x'}} + \frac{\partial \psi_2}{\partial t}\overset{-m}{\frac{\partial t}{\partial x'}}\right)$$
$$= l^2\psi_1 - m^2\psi_1$$

§1 Systems involving t explicitly t a q-number

§2 $x\ y\ z\ t$ $p_x\ p_y\ p_z\ p_t$. $t' = t - \frac{x}{c}$ $W^{\prime c}$

§3 System in an electro-magnetic field.

Q.T. gives meaning to P.B. Emphasise fundamental nature of conj. variables. [interlinear: conj. with]

Any attempt to extend the applicability of the Q. mech. must be preceded by a formulation of the classical problem in terms of canonical variables & P.B's.

Relativity Quantum Mechanics of Moving Systems, with an Application to Compton Scattering.

§1. Introduction

The new quantum mechanics introduced by Heisenberg and since developed from different points of view by [various authors] — takes its simplest form if we assume ~~simply~~ [notably] that the dynamical variables are numbers of a special type (called q-numbers to distinguish them from ordinary or c-numbers) which ~~do not~~ [obey all the ordinary algebraic laws except] satisfy the commutative law of multiplication, and satisfy instead of this, the relations

$$qp - pq = - \qquad (1)$$

$$xy - yx = ih[x, y] \qquad (2)$$

where the p's and q's are a set of canonical variables, and where h is a c-number, equal to $(2\pi)^{-1}$ times the usual Planck's constant, and $[x, y]$ is a quantity which is closely analogous to the Poisson bracket expression of x and y on the classical theory, and ~~is~~, in particular, is equal to 0, or 1 when x and y are [two of a set of] canonical variables according to whether they are conjugate or not. Equations (1C) may be regarded as replacing the commutative law of the classical theory, and one can with ~~their~~ help, build up a complete algebraic theory of quantities that are analytic functions of a set of canonical variables. (1B) All the important equations of [classical] dynamics can be written in a form in which all differential coefficients have been replaced by P.B's, and these can then be taken over directly into the new mechanics.

$a e^{\frac{iw}{}} = e^{\frac{iw}{}} a'$

[It will be observed that the] The notion of canonical variables plays a very fundamental part in the ~~present~~ theory. It is absolutely necessary that all the dynamical variables of the system under consideration shall be functions of a set of variables which are assumed to be canonical. Any attempt to extend the domain of the present quantum mechanics must be preceded by the introduction of canonical variables into the corresponding classical theory, and a reformulation of this [classical] theory [with] ~~with~~ in terms of P.B's instead of diff. cos. *1A

§2 Quantum Theory

Consider a dynamical system [of n degrees of freedom] ~~for~~ which the Hamiltonian H involves the time explicitly. [It is known that on] On the classical theory one may solve the problem by considering the time t to be an extra co-ordinate of The principle of relativity demands that the time shall be [treated] on the same footing as the other variables, and it must therefore be a q-number

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+c\right)\psi=0$$

$y = xu$ $\quad u=\frac{y}{x}$ $\quad \left(\frac{\partial}{\partial x}\right)_y = \left(\frac{\partial}{\partial X}\right)_u \left(\frac{\partial X}{\partial x}\right)_y + \frac{\partial}{\partial u}\frac{\partial u}{\partial x}$

$X = x$

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial X}-\frac{y}{x^2}\frac{\partial}{\partial u}=\frac{\partial}{\partial X}-\frac{u}{X}\frac{\partial}{\partial u}$$

$$\frac{\partial}{\partial y}=\frac{1}{x}\frac{\partial}{\partial u}=\frac{1}{X}\frac{\partial}{\partial u}$$

$$\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial X^2}-\frac{\partial}{\partial X}\frac{u}{X}\frac{\partial}{\partial u}-\frac{u}{X}\frac{\partial}{\partial u}\frac{\partial}{\partial X}+\frac{u}{X}\frac{\partial}{\partial u}\frac{u}{X}\frac{\partial}{\partial u}-\frac{1}{X^2}\frac{\partial^2}{\partial u^2}$$

$$=\frac{\partial^2}{\partial X^2}-\frac{1}{X^2}\frac{\partial^2}{\partial u^2}+\frac{u}{X^2}\left(\frac{\partial}{\partial u}+u\frac{\partial^2}{\partial u^2}\right)-u\left(-\frac{1}{X^2}\frac{\partial}{\partial u}+\frac{1}{X}\frac{\partial^2}{\partial u\partial X}\right)$$

$$-\frac{u}{X}\frac{\partial^2}{\partial u\partial X}$$

$$=\frac{\partial^2}{\partial X^2}-\frac{1}{X^2}\frac{\partial^2}{\partial u^2}+\frac{2u}{X^2}\frac{\partial}{\partial u}+\frac{u^2}{X^2}\frac{\partial^2}{\partial u^2}-\frac{2u}{X}\frac{\partial^2}{\partial u\partial X}.$$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+c\right)X=0$$

$$X=\Sigma a_n r^n$$

$$\sum\left[n(n-1)a_n r^{n-2}+na_n r^{n-1}+ca_n r^n\right]=0$$

$$(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}+ca_n=0$$

$$\left(\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+rc\right)X=0$$

$$\phi=r^u X$$

$$u(u-1)r^{u-2}X+2ur^{u-1}\frac{\partial X}{\partial r}+r^u\frac{\partial^2 X}{\partial r^2}+\frac{1}{r}\left(ur^{u-1}X+r^u\frac{\partial X}{\partial r}\right)+cr^uX=0$$

$$u=-\frac{1}{2}$$

$$\frac{3}{4}\frac{1}{r^2}X+\frac{\partial^2 X}{\partial r^2}-\frac{1}{2}\frac{1}{r^2}X+cX=0 \qquad \frac{\partial^2 X}{\partial r^2}+\frac{1}{4r^2}X+cX=0$$

$$\frac{\partial^2\psi}{\partial x\partial y}=k\psi$$

$$\psi=f(x)\,g(y)$$

$$f'(x)\,g'(y)=k\,f(x)\,g(y)$$

$$\frac{f'(x)}{f(x)}=l \qquad \frac{g'(y)}{g(y)}=\frac{k}{l}$$

$$f(x)=e^{lx}$$

$$\psi=e^{\left(lx+\frac{k}{l}y\right)}$$

§2. Algebraic Axioms.

The branch of mathematics that is now to be developed deals with magnitudes of a kind that one cannot specify explicitly. These magnitudes can be ~~so~~ denoted by algebraic symbols and used analytically according to well-defined axioms, which are the same as those of ordinary algebra except for the commutative law of multiplication, which is not in general valid. We shall call these magnitudes q-numbers, to distinguish them from the numbers of ordinary mathematics, which will be called c-numbers, while the word number alone will be used to denote either a q-number or a c-number. Both q-numbers and c-numbers may appear together in the same piece of analysis, and even in the same equation. One cannot represent q-numbers in the decimal notation, and one cannot say that one q-number is greater or less than another.

If x and y are two q-numbers, ~~or as~~ they are assumed to have a sum $x+y$ and two products, xy and yx, which are not in general equal. They are assumed to satisfy all the laws of ordinary algebra except the commutative law of multiplication, i.e. if z is another q-number

$$x+y = y+x \tag{1.11}$$

$$(x+y)+z = x+(y+z) \tag{1.12}$$

$$(xy)z = x(yz) \tag{1.13}$$

$$x(y+z) = xy+xz, \quad (x+y)z = xz+yz. \tag{1.14}$$

$$\left.\begin{array}{ll} \text{and if} & xy=0 \\ \text{either} & x=0, \text{ or } y=0; \\ \text{but in general} & xy \neq yx. \end{array}\right\} \quad (1.15)$$

In the special case when xy does equal yx, we shall say that x commutes with y.

If x and y are one q-number and one c-number, they are also assumed to have a sum $x+y$ and a product xy or yx, these two quantities being now assumed to be equal, (i.e. a q-number commutes with a c-number). The algebraic axioms (1.11) – (1.15) are assumed to be still true when x, y and z are some of them q-numbers and the others c-numbers. One can of course use c-numbers for the counting of q-numbers, e.g. one can put

$\psi(\nu d) = \int (\nu d | x d)\, \psi(x d)\, dx$

$\psi(\nu) = \int (\nu | x)\, \psi(x)\, dx$

$\int_0^\infty \cos ax\, da = \delta(x)$

Phase variable $j = 1$ or 2

$\int_0^\infty \sin ax\, da = \frac{1}{x}$

$(\nu_1 | x) = \cos \nu x$

$\psi(x)$ real.

$\int_0^\infty a \cos ax\, da = -\frac{1}{x^2}$

$(\nu_2 | x) = \sin \nu x$

$\boxed{\psi(\nu) \text{ complex} \qquad (\nu | x) = e^{i\nu x}}$

$\nu(x' x'') = \int (x' | \nu')\, \nu' (\nu' | x'')\, d\nu' = \int \nu (\cos \nu x' \cos \nu x'' + \sin \nu x' \sin \nu x'')$

$= \int \nu \cos \nu (x' - x'')\, d\nu = \int \frac{\sin \nu (x' - x'')}{x' - x''}\, d\nu + \nu\, \delta(x' - x'')$

$= \nu\, \delta(x' - x'') + \frac{\delta'(x' - x'')}{x' - x''} \quad \frac{-1}{(x' - x'')^2}$

$(\nu x - x \nu)(x' x'') = \nu\, \delta(x' - x'')\, x'' + \frac{\delta'(x' - x'')}{x' - x''}\, x'' - x' \nu\, \delta(x' - x'')$

$= (x'' - x')\, \nu(x'; x'') - \frac{1}{x' - x''}$

$x(\nu' \nu'') = \int {}^{\cos}_{\sin} \nu' x \cdot x \cdot {}^{\cos}_{\sin} \nu'' x\, dx$

$\sum_n \left[\psi_n \frac{\partial \overline{\psi}_n}{\partial t} - \frac{\partial \psi_n}{\partial t} \overline{\psi}_n \right] \left[\psi_n' \frac{\partial \overline{\psi}_n'}{\partial t} - \frac{\partial \psi_n'}{\partial t} \overline{\psi}_n' \right]$ — same thing with dashed & undashed reversed

$= -\delta(x - x') \cancel{\frac{\partial \psi_n}{\partial t} \frac{\partial \overline{\psi}_n'}{\partial t}}$

$\sum_n \overline{\psi}_n \psi_n' = \delta(x - x')$

$+ \sum_n \frac{\partial \overline{\psi}_n}{\partial t} \psi_n' \left[\psi_n \frac{\partial \overline{\psi}_n'}{\partial t} - \frac{\partial \psi_n}{\partial t} \overline{\psi}_n' \right]$

$\sum_n \frac{\partial \overline{\psi}_n}{\partial t} \psi_n' + \sum_n \overline{\psi}_n \frac{\partial \psi_n'}{\partial t} = 0$

$+ \sum_n \overline{\psi}_n \frac{\partial \psi_n'}{\partial t} \frac{\partial \psi_n}{\partial t} \overline{\psi}_n'$ — reverse

$- \sum_n \frac{\partial \overline{\psi}_n}{\partial t} \frac{\partial \psi_n'}{\partial t} \psi_n \overline{\psi}_n'$ + reverse.

from which one can easily verify that x, y, z, p_x, p_y, p_z are canonical when it is assumed that the ξ's and η's are canonical. These ~~components~~ equations (35) are also the most convenient ones for evaluating the amplitudes of the various components of vibration, since they give at once

(This method shows that our previous θ + ϕ do commute)

and hence that the x, y, z, p_x, p_y, p_z are canonical. Equations (17) are equivalent to equations (15) and from eqns (16) and (17) eqns (14) can easily be deduced; so that this transformation of the ~~present~~ is the same as the ~~one of the~~ previous ~~preceding~~ one (and proves that the previous θ and ϕ commute).

The value of z in terms of the new variables is given by

$$z m_z = -(x m_x + y m_y) = -\tfrac{1}{2}(x+iy)(m_x - i m_y) - \tfrac{1}{2}(x-iy)(m_x + i m_y)$$

$$= +\tfrac{1}{4} r k^{-\frac{1}{2}} \{(\xi_1^2 - \eta_2^2)\xi_2\eta_1 + (\xi_2^2 - \eta_1^2)\xi_1\eta_2\} k^{-\frac{1}{2}}$$

$$= \tfrac{1}{4} r k^{-\frac{1}{2}} \{\xi_1\xi_2(\xi_1\eta_1 + \xi_2\eta_2) -$$

$$z = \tfrac{1}{2} r k^{-\frac{1}{2}} (\xi_1\xi_2 + \eta_1\eta_2) k^{-\frac{1}{2}} \qquad (18)$$

~~The formulae (16) and (18) may be written in terms of the k m_z w + ϕ.~~

$$\left.\begin{aligned} x+iy &= -\tfrac{1}{2} r \left\{ \frac{(k+m_z-\frac{1}{2}h)^{\frac{1}{2}}(k+m_z-\frac{3}{2}h)^{\frac{1}{2}}}{k^{\frac{1}{2}}(k-h)^{\frac{1}{2}}} e^{i(w+\phi)} + \frac{(k-m_z+\frac{1}{2}h)^{\frac{1}{2}}(k-m_z+\frac{3}{2}h)^{\frac{1}{2}}}{k^{\frac{1}{2}}(k+h)^{\frac{1}{2}}} e^{i(-w+\phi)} \right\} \\ x-iy &= \tfrac{1}{2} r \left\{ \frac{(k-m_z-\frac{1}{2}h)^{\frac{1}{2}}(k-m_z-\frac{3}{2}h)^{\frac{1}{2}}}{k^{\frac{1}{2}}(k-h)^{\frac{1}{2}}} e^{i(\theta-\phi)} + \frac{(k+m_z+\frac{1}{2}h)^{\frac{1}{2}}(k+m_z+\frac{3}{2}h)^{\frac{1}{2}}}{k^{\frac{1}{2}}(k+h)^{\frac{1}{2}}} e^{-i(\theta+\phi)} \right\} \\ z &= \tfrac{1}{2} r \left\{ \frac{(k+m_z-\frac{1}{2}h)^{\frac{1}{2}}(k-m_z-\frac{1}{2}h)^{\frac{1}{2}}}{k^{\frac{1}{2}}(k-h)^{\frac{1}{2}}} e^{i\theta} - \frac{(k+m_z+\frac{1}{2}h)^{\frac{1}{2}}(k-m_z+\frac{1}{2}h)^{\frac{1}{2}}}{k^{\frac{1}{2}}(k+h)^{\frac{1}{2}}} e^{-i\theta} \right\} \end{aligned}\right\} \quad (36)$$

~~From these relations the amplitudes of the different vibrations can be obtained by substituting for k and m_z in the coefts. their quantised values. (See §8)~~

~~For the case of the system of more than one electron there are no corresponding variables ξ and η which enable the transformation eqns to be put in a form corresponding to (16) & (17)~~

The simplicity of eqns (35) is due to the fact that one can associate each component of vibration with the product of two of the ξ, η variables that are not conjugate. For systems of more than one electron there are too many components of vibration for this to be done, so that there are no equations corresponding to (35) ~~for the~~ for such systems.

$$\psi = \int_{-\infty}^{\infty}\left[e^{i(x\mu+t\sqrt{k^2+\mu^2})}\left(a+\frac{b}{\sqrt{k^2+\mu^2}}\right)+e^{i(x\mu-t\sqrt{k^2+\mu^2})}\left(a-\frac{b}{\sqrt{k^2+\mu^2}}\right)\right]d\mu$$

$$-i\frac{\partial\psi}{\partial t} = \int_{-\infty}^{\infty}\left[e^{i(x\mu+t\sqrt{k^2+\mu^2})}\left(a\sqrt{k^2+\mu^2}+b\right)+e^{i(x\mu-t\sqrt{k^2+\mu^2})}\left(-a\sqrt{k^2+\mu^2}+b\right)\right]d\mu$$

$$x' = lx+mt \qquad x'^2-t'^2 = (l^2-m^2)(x^2-t^2) = x^2-t^2$$
$$t' = lt+mx \qquad l^2-m^2 = 1 \qquad l>0$$

$$\psi' = \int_{-\infty}^{\infty}\left[e^{i(x'\mu'+t'\sqrt{k^2+\mu'^2})}\left(a'+\frac{b'}{\sqrt{k^2+\mu'^2}}\right)+e^{i(x'\mu'} \quad - - \quad \right]d\mu'$$

$$x'\mu'+t'\sqrt{k^2+\mu'^2} = (lx+mt)\mu'+(lt+mx)\sqrt{k^2+\mu'^2}$$
$$= x\left(l\mu'+m\sqrt{k^2+\mu'^2}\right)+t\left(m\mu'+l\sqrt{k^2+\mu'^2}\right) = x\mu+t\sqrt{k^2+\mu^2}$$

$$l\mu'+m\sqrt{k^2+\mu'^2} = \mu \qquad \boxed{l\mu-m\sqrt{k^2+\mu^2} = \mu'}$$
$$m^2(k^2+\mu'^2) = (\mu-l\mu')^2 = \mu^2+l^2\mu'^2-2l\mu\mu'$$
$$\mu^2+\mu'^2-2l\mu\mu'+(l^2-1)k^2 = 0$$

$$\cancel{k^2+\mu^2 \quad \mu'^2}$$

$$m\mu'+l\sqrt{k^2+\mu'^2} = \pm\sqrt{k^2+\mu^2}$$
$$m^4\mu'^4+l^4(k^2+\mu'^2)^2+(k^2+\mu^2)^2-2m^2\mu'^2l^2(k^2+\mu'^2)-2m^2\mu'^2(k^2+\mu^2)$$
$$-2l^2(k^2+\mu'^2)(k^2+\mu^2) = 0$$

$$\left.\begin{aligned}&\mu'^4(m^4+l^4-2m^2l^2)+\mu^4-2\mu^2\mu'^2(m^2+l^2)\\&+2\mu'^2(l^4k^2-m^2l^2k^2-m^2k^2-l^2k^2)+2\mu^2(k^2-l^2k^2)\\&+l^4k^4+k^4-2l^2k^4\end{aligned}\right\} = 0$$

$$\mu'^4+\mu^4-2\mu^2\mu'^2(2l^2-1)-2\mu'^2k^2m^2-2\mu^2k^2m^2+k^4m^4 = 0$$
$$(\mu'^2+\mu^2-m^2k^2)^2-4l^2\mu^2\mu'^2 = 0$$

$$l^2\mu'+lm\sqrt{k^2+\mu'^2} = l\mu = \mu'+m\sqrt{k^2+\mu^2}$$
$$m^2\mu'+lm\sqrt{k^2+\mu'^2} = +m\sqrt{k^2+\mu^2}$$

$$d\mu = \left(l+\frac{m\mu'}{\sqrt{k^2+\mu'^2}}\right)d\mu'$$

$$\left(a\sqrt{k^2+\mu^2}+b\right)d\mu = \left[\frac{a}{l}\left(m\mu'+l\sqrt{k^2+\mu'^2}\right)+b\right]\left(l+\frac{m\mu'}{\sqrt{k^2+\mu'^2}}\right)d\mu'$$

$$= \left[lb+am\mu'+lam\mu'+al\sqrt{k^2+\mu'^2}+\frac{am^2\mu'^2}{l\sqrt{k^2+\mu'^2}}+\frac{bm\mu'}{\sqrt{k^2+\mu'^2}}\right]d\mu'$$

$$= \left[lb-\cancel{\frac{am^2}{l}k^2}+2am\mu'+al\sqrt{k^2+\mu'^2}+\frac{am^2}{l}\left(\sqrt{k^2+\mu'^2}-\frac{k^2}{\sqrt{k^2+\mu'^2}}\right)\right.$$
$$\left.+\frac{bm\mu'}{\sqrt{k^2+\mu'^2}}\right]d\mu'$$

$$\int_{-\infty}^{\infty}e^{i(x'\mu'+t'\sqrt{k^2+\mu'^2})}\left(x'+\frac{t'\mu'}{\sqrt{k^2+\mu'^2}}\right)d\mu' = 0$$

$$\int_{-\infty}^{\infty}e^{i(x\mu+t\sqrt{k^2+\mu^2})}\left(a\sqrt{k^2+\mu^2}+b\right)d\mu$$
$$= \int_{-\infty}^{\infty}e^{i(x'\mu'+t'\sqrt{k^2+\mu'^2})}\left[lb-\frac{mbx'}{t'}+\frac{mb}{t'}\left(x'+\frac{t'\mu'}{\sqrt{k^2+\mu'^2}}\right)\right.$$

$$\left.+2am\mu'+\frac{a}{l}(l^2+m^2)\sqrt{k^2+\mu'^2}-\frac{am^2k^2}{l\sqrt{k^2+\mu'^2}}\right]d\mu'$$

$$-i\frac{\partial\psi'}{\partial t'} =$$

§5. On q-numbers.

In the three preceeding ~~three~~ §§ two unsatisfactory features will have been observed which will here be discussed. The first of these is that it ~~is~~ was continually necessary to postulate that a q-number exists that satisfies certain conditions, and the second is that the expression "every q-number" has frequently be used, which implies that there is a definite domain of all q-numbers. In the mathematics of c-numbers these difficulties do not occur because one can rigorously define the domain of all c-numbers, and then the statement that a c-number exists that satisfies certain conditions can be proved (if it is true) and need not be assumed. The state of affairs for q-numbers must be essentially different, owing to the undefinable nature of q-numbers.

One can safely assume that a q-number exists that satisfies certain conditions whenever these conditions do not lead to an inconsistency, since by a q-number one means only a dummy symbol appearing in the analysis satisfying these conditions. ~~Further~~ Also, when one says that all q-numbers satisfy a certain condition, one needs this result to apply only to the q-numbers that one is actually dealing with in the problem. It would not do any harm if there was a q-number, entirely disconnected from all the q-numbers one is dealing with in the problem, that did not satisfy the condition, as the inconsistency, ~~would~~ which, of course, really exists, would not be "brought home" to one. Further, it might at a later stage in the problem actually be desirable to assume the existence of a q-number ~~that~~ that did not satisfy the condition. One would therefore have to consider the statement that all q-numbers satisfy the condition as applying only to those q-numbers with which one is dealing at the time. One is thus led to consider that the domain of all the q-numbers is elastic, and is liable at any time to be extended by fresh assumptions of the existence of q-numbers satisfying certain conditions, and that when one says that all q-numbers satisfy a certain condition, one means it to apply only to the then existing domain of q-numbers, and not to exclude the possibility of a later extension of the domain to q-numbers that do not satisfy the condition.

As an example of the necessity of this point of view, consider the definition of $x^{\frac{1}{2}}$. It is defined to commute with every number that commutes with x, and to satisfy the equation $x^{\frac{1}{2}}x^{\frac{1}{2}} = x$. Now from assumption (i) ~~of §2~~ at the end of §2, there must be a number b such that

$$\psi_m \frac{\partial \overline{\psi}_n}{\partial t} - \overline{\psi}_n \frac{\partial \psi_m}{\partial t}$$

conj imag. ψ_n

If $a_{11} = a_{22}$ $\quad b_{11} = b_{22}$

$$(ab)_{11} = a_{11} b_{11} + a_{12} b_{21} = a_{22} b_{22} + a_{21} b_{12} = (ba)_{22}$$

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = k^2 \psi \qquad \mu = \sqrt{k^2 + \alpha^2 + \beta^2 + \gamma^2}$$

$$\psi = \iiint_{-\infty}^{\infty} \left[e^{i(\alpha x + \beta y + \gamma z + \mu t)} + e^{i(\alpha x + \beta y + \gamma z - \mu t)} \right] d\alpha \, d\beta \, d\gamma$$

$\alpha = K \cos\theta$
$\beta = K \sin\theta \cos\phi$
$\gamma = K \sin\theta \sin\phi$

$$= \int_{-\infty}^{\infty} \left[e^{iK(x\cos\theta + y \sin\theta \cos\phi + z \sin\theta \sin\phi)} \, 2\cos\mu t \right] K^2 \, dK \, \sin\theta \, d\theta \, d\phi$$

$$\int_0^{2\pi} e^{ae^{i\phi} + be^{-i\phi}} \, d\phi = 2\pi \Big\{ \text{const term in} \Big[1 + (ae^{i\phi} + be^{-i\phi}) + \frac{1}{2!}(ae^{i\phi} + be^{-i\phi})^2 + \frac{1}{4!}(ae^{i\phi} + be^{-i\phi})^4$$
$$+ \frac{1}{6!}(ae^{i\phi} + be^{-i\phi})^6 + \frac{1}{8!}(ae^{i\phi} + be^{-i\phi})^8 + \cdots \Big]$$

$$= 2\pi \left[1 + \frac{1}{2!} 2ab + \frac{1}{4!} 6a^2b^2 + \frac{1}{6!} \frac{6.5.4}{1\,2\,3} a^3b^3 + \cdots \right.$$

$$= 2\pi \sum_{r=0}^{\infty} \frac{1}{(2r)!} (ab)^r \frac{(2r)!}{r!\, r!}$$

Take $y = 0$, $z = 0$

$$\psi = \int_{-\infty}^{\infty} e^{iKx\cos\theta} \, 2\cos\mu t \, K^2 \, dK \, . \, 4\pi \, .$$

analogues as closely as possible, it is desirable that we should be able to give a meaning to real and imaginary q-numbers. We can do this by defining the conjugate imaginary of an algebraic expression involving only real numbers and i, to be the number obtained when one writes $-i$ for i and reverses the order of the factors of all products. Thus if p and q are real, which they are assumed to be if they are the analogues of real classical quantities, the conjugate imaginary of $(qp - pq)$ is $pq - qp$, equal to $-(\cancel{qp} - pq)$, and hence if

$$qp - pq = ih,$$

h is real. It is to be doubted, though, whether the notion of real and imaginary q-numbers ~~can be~~ will survive in the ultimate theory, since, if one ~~can~~ could ~~give dosting~~ distinguish between real and imaginary q-numbers, one could define a positive q-number to be one whose square roots ~~were~~ are real and a negative q-number to be one whose square roots ~~were~~ are imaginary, and one could then give a meaning to one q-number being greater or less than another. This appears to be carrying the analogy with c-numbers too far. This difficulty is probably connected with the difficulties in the theory of square roots.

We have been able to develop the pure mathematics of q-numbers thus far without any reference to the notion of ~~a~~ the limit of a sequence of q-numbers. It might appear to be desirable to carry through the whole of the theory without the use of limits. To attempt to do so at the present time, however, would completely destroy the very close analogy that exists between the present theory and classical dynamics, owing to the fact that limits are ~~so~~ repeatedly used in classical dynamics, particularly through the sums of Fourier series. We shall therefore use limits freely in the subsequent work in analogy to the classical theory, although it does not appear to be possible to give a general definition of the limit of a sequence of q-numbers.

$\rho_{mn} = \chi(x1,m)\psi(x1,n) + \chi(x2,n)\psi(x2,n)$ at time t

Wave fn for state n is $\psi(xi,n)$ $i = 1$ or 2

$$\psi_{t+\tau}(xi) \doteq \int_{x-\tau}^{x+\tau}\left[B(xi,x'1)\psi_t(x'1) + B(xi,x'2)\psi_t(x'2)\right]dx'$$

where

$$B(xj,x'k) = e^{2\pi i H(xj,x'k)\tau/h}$$

$i\hbar\frac{\partial\psi}{\partial t} = H\psi$ $\quad\psi_1 = P(t^2 - \cancel{x^2})$ $\quad\psi_2 = i\hbar\frac{\partial\psi_1}{\partial t}$

$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi_1 = -k^2\psi_1$

$i\hbar\frac{\partial\psi_1}{\partial t} = H_{11}\psi_1 + H_{12}\psi_2$ $\quad H_{11} = 0 \quad H_{12} = 1$

$\hbar\frac{\partial\psi_2}{\partial t} = \hbar\frac{\partial^2\psi_1}{\partial t^2} = -\hbar^2\left(k^2+\frac{\partial^2}{\partial x^2}\right)\psi_1$ $\quad H_{21} = -\hbar^2\left(k^2+\frac{\partial^2}{\partial x^2}\right) \quad H_{22} = 0$

$= H_{21}\psi_1 + H_{22}\psi_2$

$\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}$

$\psi_2 = \lambda\psi_1$

$a\psi_1 = \lambda\psi_2$

$a = \lambda^2$

$\psi_1' = \psi_1$

$\psi_2' = \frac{\partial\psi_1}{\partial t'} = l\psi_2 + m\frac{\partial\psi_1}{\partial x}$

$$H = \begin{pmatrix} 0 & 1 \\ -\hbar^2\left(k^2+\frac{\partial^2}{\partial x^2}\right) & 0 \end{pmatrix}$$

Variable $j = (1,2)$ commutes with x but not with H.

Does it commute with $\frac{dx}{dt} = [x,H]$

$[j,[x,H]] = [x[j,H]] + \cancel{[H[x,j]]}$

Find relation between p_x and $\dot{x}$

To find $\sqrt{a+b\frac{\partial^2}{\partial x^2}} = A(x'x'')$

$\cancel{A\psi}$ $\left(a+b\frac{\partial^2}{\partial x^2}\right)\psi(x) = \lambda\psi(x)$

$\frac{\partial^2\psi}{\partial x^2} = \frac{\lambda-a}{b}\psi$ $\quad \frac{a-\lambda}{b} = n^2 \quad \lambda = a - bn^2$

$\psi = e^{\pm i\sqrt{\frac{a-\lambda}{b}}x}$

$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\quad 0 = \lambda\psi_1$

$\psi_1 = \lambda\psi_2$

$\lambda \doteq 0$ or $\psi_1 = 0$

$\lambda = 0$ or $\psi_2 = 0$

$\psi_2 = \frac{\partial\psi_1}{\partial t} + \Sigma_{xyz}\alpha_x\frac{\partial\psi_1}{\partial x}$

$\frac{\partial\psi_2}{\partial t} = (k^2+\nabla^2)\psi_1 + \Sigma_{xyz}\alpha_x\left(\frac{\partial\psi_2}{\partial x} - \Sigma_{x'y'z'}\alpha_{x'}\frac{\partial^2\psi_1}{\partial x\partial x'}\right)$

Try $\psi_2 = \frac{\partial\psi_1}{\partial t} + \alpha\frac{\partial\psi_1}{\partial x}$ $\quad\left(\frac{\partial\psi_2}{\partial x} - \alpha\frac{\partial^2\psi_1}{\partial x^2}\right)$

$G = \begin{pmatrix} \alpha\frac{\partial}{\partial x} & 1 \\ k^2+(1-\alpha^2)\frac{\partial^2}{\partial x^2} & \alpha \end{pmatrix}$ Hermitian if $\alpha = \pm 1$, $k = 1$

$\frac{\partial\psi_2}{\partial t} = \left(k^2+\frac{\partial^2}{\partial x^2}\right)\psi_1 + \alpha\frac{\partial^2\psi_1}{\partial x\partial t} = \alpha\frac{\partial\psi_2}{\partial x} + k^2\psi_1 + (1-\alpha^2)\frac{\partial^2\psi_1}{\partial x^2}$

Equation (8.3) gives

$$[k^2,[k,z]] = k[k,[k,z]] + [k,[k,z]]k = -(kz + zk)$$

Hence

$$[k^2,[k^2,z]] = k[k^2,[k,z]] + [k^2,[k,z]]k = -(k^2 z + 2kzk + zk^2)$$

$$= -2(k^2 z + zk^2) + (k^2 z - 2kzk + zk^2)$$

$$= -2(k^2 z + zk^2) - h^2[k,[k,z]]$$

$$= -2(k^2 z + zk^2) + h^2 z$$

or

$$\tfrac{1}{2}[k^2,[k^2,z]] = -(k^2 - \tfrac{1}{4}h^2)z - z(k^2 - \tfrac{1}{4}h^2)$$

$$= -k_1 k_2 z - z k_1 k_2 \qquad (8{\cdot}31)$$

where

$$k_1 = k + \tfrac{1}{2}h, \quad k_2 = k - \tfrac{1}{2}h.$$

(In general we shall take the suffix 1 attached to any action variable to denote the value of that variable increased by $\frac{1}{2}h$, and the suffix 2 to denote its value reduced by $\frac{1}{2}h$.)

From (8.23) and (8.26)

$$\tfrac{1}{2}[m^2,z] = \tfrac{1}{2}[m_x^2 + m_y^2, z] = \tfrac{1}{2}(-m_x y - y m_x + m_y x + x m_y)$$

$$= m_y x - m_x y + ih z = m_y x - y m_x = x m_y - m_x y \qquad (8{\cdot}32)$$

Similar relations hold for $[m^2,x]$ and $[m^2,y]$. Hence

$$\tfrac{1}{2}[m^2,[m^2,z]] = m_y[m^2,x] - m_x[m^2,y] + ih[m^2,z]$$

$$= 2m_y(y m_z - m_y z) - 2m_x(m_x z - x m_z) + ih[m^2,z]$$

$$= 2(m_x x + m_y y)m_z - 2(m_x^2 + m_y^2)z + ih[m^2,z]$$

$$= -2m^2 z + ih[m^2,z]$$

$$= -m^2 z - z m^2. \qquad (8{\cdot}33)$$

Comparing this with equation (8.31), we see that they agree if we take

$$m^2 = k_1 k_2 = k^2 - \tfrac{1}{4}h^2 \qquad (8{\cdot}34)$$

With k^2 defined by (8.4), equation (8.31) follows from equation (8.33). We shall assume that equation (8.3) then follows from equation (8.31), although the present theory of square roots does not enable one to demonstrate this rigorously. Corresponding to (8.3) we have the equations

$$[k,[k,x]] = -x \qquad [k,[k,y]] = -y \qquad (8{\cdot}5)$$

$$i\hbar \frac{\partial \psi_m}{\partial t} = G_{mn}\psi_n \qquad G = \begin{pmatrix} 0 & 1 \\ -\hbar^2(k^2 + \nabla^2) & 0 \end{pmatrix} \qquad \psi_n = \psi_n(x\, j)$$

$$x_{mn} = \Sigma_j \int \chi_m\, x\, \psi_n\, dx$$

$$\dot{x}_{mn} = \Sigma_j \int [\dot{\chi}_m x \psi_n + \chi_m x \dot{\psi}_n]\, dx$$

$$i\hbar \dot{x}_{mn} = \Sigma_k \Sigma_j \int dx [\chi_m x G_{nk} \psi_k - \chi_k G_{km} x \psi_n]$$

$$= (xG - Gx)_{mn}$$

$$i\hbar \dot{x}(x'j', x''j'') = x' G(x'j', x''j'') - G(x'j', x''j'')x'' =$$

$\cancel{G(x'x, 1x)}$

$G(x'1, x''1) = 0$

$G(x'1, x''2) = \delta(x'-x'')\delta(y'-y'')\delta(z'-z'')$

$\cancel{G(x'2, x''1) = -\hbar^2 k^2 \delta(x'-x'') \hbar^2 \delta''(x'-x'')}$

$G(x'2, x''2) = 0$

$G(x'2, x''1) = -\hbar^2 k^2 \delta(x'-x'')\delta(y'-y'')\delta(z'-z'')$
$- \hbar^2 \delta''(x'-x'')\delta(y'-y'')\delta(z'-z'')$
$- \hbar^2 \delta''(y'-y'')\delta(x'-x'')\delta(z'-z'')$
$- \hbar^2 \delta''(z'-z'')\delta(x'-x'')\delta(y'-y'')$

$$\begin{vmatrix} 0 & 0 \\ a & 0 \end{vmatrix} \begin{vmatrix} 0 & 0 \\ b & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$$

Reversed correspondence between rows & columns

Define $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1 b_3 + a_2 b_1 & a_1 b_4 + a_2 b_2 \\ a_3 b_3 + a_4 b_1 & a_3 b_4 + a_4 b_2 \end{pmatrix}$ Addition same as usual

second rotation works as for multiplication $\begin{pmatrix} b_3 & b_4 \\ b_1 & b_2 \end{pmatrix}$

$a(b+c) = ab + ac$

$a\mathbf{1} = a$ where $\mathbf{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\cancel{\mathbf{1}a = a \text{ where } \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$

$\mathbf{1}b = b$ where $\mathbf{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}^2 = \begin{pmatrix} a_1(a_2 + a_3) & a_2^2 + a_1 a_4 \\ a_3^2 + a_1 a_4 & a_4(a_2 + a_3) \end{pmatrix}$$

Define $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1 b_1 + a_2 b_3 & a_2 b_2 + a_1 b_4 \\ a_3 b_3 + a_4 b_1 & a_4 b_4 + a_3 b_2 \end{pmatrix}$ If a and b commute $(ab)_{mn} = a_{mn} b_{mn}$

$a_{1\,or\,4} b_1 \quad a_{1\,or\,4} b_2$

$a_{1\,or\,4} b_3 \quad a_{1\,or\,4} b_4$

a & b commute if

$a_3 = a_4 = b_3 = b_4 = 0$

Labels apply to the diagonals instead of rows & columns?

Work out canonical transformation

One can take as another action variable the quantity m_z ($= p$ say), since from (8.23)

$$\left.\begin{aligned} [p\,[p,x]] &= [p,y] = -x \\ [p\,[p,y]] &= -[p,x] = -y. \end{aligned}\right\} \quad \begin{matrix} 8.6 \\ (\cancel{8.46}) \end{matrix}$$

These equations show that x and y are periodic functions of ϕ, the variable conjugate of p, of period 2π, and that all the coefficients in their Fourier expansions vanish except those of $e^{i\phi}$ and $e^{-i\phi}$ terms, and also, since $[p,z]=0$, all the coefficients in the Fourier expansion of z vanish except those of terms independent of ϕ. The ordinary selection rules for p follow from this.

Begin a new page →

§9. ~~The~~ Motion of a Particle in a Central Field: the Angle Variables.

We must now consider how θ and ϕ are to be defined. On the classical theory an angle variable w of this type is defined by e^{iw} being equal to the square root of the ratio of two quantities that are conjugate imaginaries, i.e. by a relation of the type

$$e^{iw} = \left(\frac{a+ib}{a-ib}\right)^{\frac{1}{2}} \quad \begin{matrix} 9.11 \\ (\cancel{8.51}) \end{matrix}$$

where a and b are real. This, of course, makes w real, since if one writes $-i$ for i in (9.11 ~~8.51~~) it remains true.

On the quantum theory, however, there are two corresponding ways in which one might define e^{iw}, namely

$$e^{iw} = \left\{(a+ib)\frac{1}{(a-ib)}\right\}^{\frac{1}{2}}$$

and

$$e^{iw} = \left\{\frac{1}{(a-ib)}(a+ib)\right\}^{\frac{1}{2}}$$

but neither of these makes w real. The correct quantum generalisation of (9.11 ~~8.51~~) is the more symmetrical relation

$$e^{iw}(a-ib)e^{iw} = a+ib. \quad \begin{matrix} 9.12 \\ (\cancel{8.52}) \end{matrix}$$

This becomes, when one equates the conjugate imaginaries of both sides,

$$e^{-iw}(a+ib)e^{-iw} = a-ib,$$

which is equivalent to (9.12 ~~8.52~~), so that w defined in this way is real. We may solve (9.12 ~~8.52~~) for e^{iw} in either of two ways, i.e.

$$e^{iw}(a-ib)e^{iw}(a-ib) = (a+ib)(a-ib)$$

giving

$$e^{iw}(a-ib) = \{(a+ib)(a-ib)\}^{\frac{1}{2}} = \{(a+ib)(a-ib)\}^{-\frac{1}{2}}(a+ib)(a-ib)$$

so that

$$e^{iw} = \{(a+ib)(a-ib)\}^{-\frac{1}{2}}(a+ib); \quad \begin{matrix} 9.13 \\ (\cancel{8.53}) \end{matrix}$$

or alternatively

$$(a-ib)e^{iw}(a-ib)e^{iw} = (a-ib)(a+ib)$$

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = k^2\psi$$

$$\boxed{-p^2 + E^2 = m^2c^2}$$

$$\psi = e^{i(\nu ct + \mu x)}$$

$$\nu^2 - \mu^2 = k^2$$

Initially $\psi = \beta\delta(x) = \int_{-\infty}^{\infty} e^{i\mu x}\,d\mu$

$$\psi = \int e^{i(\mu x + \nu ct)}\,d\mu = \int e^{i\mu(x+ct)}\left(1 + i\alpha ct - \frac{\alpha^2c^2t^2}{2} + \dots\right)d\mu$$

$$\nu = \pm\sqrt{k^2+\mu^2} = \mu + \alpha$$

$$\alpha = \pm\sqrt{k^2+\mu^2} - \mu = \mu\left[1 + \frac{k^2}{2\mu^2} - 1\right] \quad \mu \text{ large}$$

α always small

if
$$az + \sqrt{1+z^2} = 0$$
$$a^2z^2 = 1 + z^2$$
$$1 = (a^2-1)z^2$$
$$z = \frac{1}{\sqrt{a^2-1}}$$

$$\psi = e^{-\mu|x| \pm \lambda ct}$$
$$\mu^2 = k^2 + \lambda^2$$

$$e^{i(kx + \sqrt{k^2+z^2}\,y)}$$

for $|t| < |x|$

$$\int_{-\infty}^{\infty}\left[e^{i(\mu x + \sqrt{k^2+\mu^2}\,t)} + e^{i(\mu x - \sqrt{k^2+\mu^2}\,t)}\right]d\mu = 0$$

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{k^2+\mu^2}}\left[e^{i(\mu x + \sqrt{k^2+\mu^2}\,t)} - e^{i(\mu x - \sqrt{k^2+\mu^2}\,t)}\right]d\mu = 0$$

$$\int_{-\infty}^{\infty}\left[e^{i(x\mu + t\sqrt{k^2+\mu^2})}\left(a + \frac{b}{\sqrt{k^2+\mu^2}}\right) + e^{i(x\mu - t\sqrt{k^2+\mu^2})}\left(a - \frac{b}{\sqrt{k^2+\mu^2}}\right)\right]d\mu$$

$$\int_0^{\infty} e^{i(\mu x + \sqrt{k^2+\mu^2}\,y)}\,d\mu = \left[\int_0^k + \int_k^{\infty}\right]e^{i(\mu x + \sqrt{k^2+\mu^2}\,y)}\,d\mu$$

$$= \int_k^{\infty} e^{i\left[x\mu + y\mu\left(1 + \frac{k^2}{2\mu^2} - \frac{k^4}{8\mu^4} \dots\right)\right]}d\mu + \int_0^{k} e^{i\left[\mu x + yk\left(1 + \frac{\mu^2}{2k^2} - \frac{\mu^4}{8k^4} \dots\right)\right]}d\mu$$

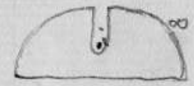
$$= k\int_1^{\infty} e^{ik\left[(x+y)\lambda + \frac{1}{2}y\frac{1}{\lambda} - \frac{1}{8}y\frac{1}{\lambda^3} \dots\right]}d\lambda + k\int_0^1 e^{ik\left[y + x\lambda + \frac{1}{2}y\lambda^2 - \frac{1}{8}y\lambda^4 \dots\right]}d\lambda$$

$\mu = k\lambda$ $\qquad$ $\mu = k\lambda$

$$k\int_0^{\infty} e^{ik(\lambda x + \sqrt{1+\lambda^2}\,y)}\,d\lambda \qquad x > y \text{ or } < -y$$

new $k = ky$, $\quad$ new $k.a = kx$
$y\lambda = z$

$$\int e^{ik(az + \sqrt{1+z^2})}\,dz \qquad z = re^{i\theta}$$



$$= \int_{\theta_1}^{\theta_2} e^{ik(are^{i\theta} \pm \sqrt{1+r^2e^{2i\theta}})}\,ire^{i\theta}\,d\theta = ir\int e^{i\left[k(a\pm1)re^{i\theta} + \theta \pm \frac{1}{2r}e^{-i\theta} + \dots\right]}d\theta$$

r very large

$$= ir\int e^{i\left[k(a\pm1)r\cos\theta + \theta \pm \frac{1}{2r}\cos\theta + \dots\right]}e^{-k(a\pm1)r\sin\theta \pm \frac{1}{2r}\sin\theta \dots}d\theta$$

Vanishes for $a > 1$, $\theta > 0$

Should vanish when $x > y$ or $x < -y$ $\quad a > 1$ or < -1

$$\int_0^{\alpha} f(\theta)e^{-R\theta}\,d\theta = \left[f(\theta)\frac{e^{-R\theta}}{-R}\right]_0^{\alpha} - \int f'(\theta)\frac{e^{-R\theta}}{-R}\,d\theta = \frac{f(0)}{R} + \frac{--}{R^2}\dots$$

§12. The Elimination of the Nodes.

In the present § the work of §§ 8, 9 will be extended to the problem of the system with two electrons moving in an approximately central field of force. or of the core-series electron atom On the classical theory an initial simplification can be made, known as the elimination of the nodes, which consists in obtaining a contact transformation to a set of canonical variables, all of which except three are independent of the orientation of the system as a whole. It can be shown that the new variables may be taken to be the r, p_r, k and θ of either electron, the θ's now being measured from the line of nodes instead of from the line of intersection of the orbital plane with the plane xy, together with the resultant angular momentum j, with the azimuth ψ of the line of nodes about the direction of this resultant angular momentum for conjugate variable, and the component M_z ~~p_z say~~ of total angular momentum in a given direction, z say, with the azimuth of the direction of resultant angular momentum about the z axis for conjugate variable. All the variables except the last three are independent of the orientation of the system as a whole. The condition for this on the quantum theory must of course be expressed analytically, and is then that the variables are invariant under the linear transformation

$$\left.\begin{aligned} \bar{x} &= l_1 x + l m_1 y + n_1 z & \qquad \bar{p}_x &= l_1 p_x + m_1 p_y + n_1 p_z \\ \bar{y} &= & & \\ \bar{z} &= l_3 x + m_3 y + n_3 z & \qquad \bar{p}_z &= \end{aligned}\right\} \qquad (12\cdot 1)$$

where the coefficients l, m, n are c-numbers satisfying the conditions that they satisfy on the classical theory, namely
$$l_1^2 + m_1^2 + n_1^2 = 1, \qquad l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \qquad \text{etc.}$$

On the quantum theory, we may still define the r, p_r, k of the first electron by (8·11) (8·12) and (8·4), as these variables are obviously invariant under the transformation (12·1). If we use dashed letters to refer to the second electron, we can define r', p_r', k' in a corresponding way, and then r', p_r' and k' will commute with r, p_r, k, so that ~~they~~ these six quantities may be taken to be six of the new variables.

The components of total angular momentum are
$$M_x = m_x + m_x' \qquad M_y = m_y + m_y' \qquad M_z = m_z + m_z'.$$
It is now easily verified that
$$[M_z, x] = y \qquad [M_z, y] = -x \qquad \cancel{[M_z, z] = 0} \qquad (12\cdot 23)$$
$$[M_z, z] = 0 \qquad (12\cdot 24)$$

$$(\xi,\eta) = \int (\xi|\lambda)\,\lambda\, d\lambda\,(\lambda|\eta)$$

(ξ,η) is degenerate on the right when a $(\eta|.)$ exists such that

$$\int (\xi,\eta)(\eta|.)\,d\eta = 0$$

$$e^{Ht}(x'x'') = \delta(x'-x''+ict'')$$

$$e^{H\delta t}\,\psi(x') = \psi(x'+ict) = \lambda\,\psi(x')$$

$$\psi(x) = e^{ipx} \qquad \lambda = e^{-pct}$$

$$e^{Ht}(p'p'') = \iint e^{i(p'x'-p''x'')}\delta(x'-x''+ict)\,dx'\,dx''$$

$$= \int e^{i(p'-p'')x'-p''ct}\,dx''$$

$$= e^{p'ct}\,\delta(p'-p'')$$

$$H(p'p'') = +p'c\,\delta(p'-p'')$$

$$H(x'x'') = \int p'c\,e^{ip'(x'-x'')}\,dp' = c\delta'(x'-x'')$$

$$a_{r\Lambda} = \Sigma_\lambda (r|\lambda) f(\lambda)(\lambda|\Lambda)$$

$$b_{r\Lambda} = \Sigma_\lambda (r|\lambda) g(\lambda)(\lambda|\Lambda)$$

$$a_{r\Lambda} = \Sigma_\mu (r|\mu) f(\mu)(\mu|\Lambda)$$

$$b_{r\Lambda} = \Sigma_{\mu\mu'} (r|\mu)\, b(\mu\mu')(\mu'|\Lambda)$$

$$|a_{r\Lambda} + \gamma b_{r\Lambda}| = 0 \text{ when } f(\lambda)+\gamma g(\lambda) = 0$$

$$\cancel{e^{H(x'x'')t}} \qquad e^{Ht}(x'x'') = \int (x'|\xi')\,e^{f(\xi')t}(\xi'|x'')\,d\xi'$$

H is diagonal in ξ scheme

$$= \int (x'|\xi,1)\,e^{f_1(\xi)t}(\xi_1|x'')\,d\xi + \int (x'|\xi_2)\,e^{f_2(\xi)t}(\xi_2|x'')\,d\xi_2$$

$f_1(\xi)$ $f_2(\xi)$

1 is propagated with vel. c

2 " $-c$

$$\psi_1(x, t+\delta t) = (1-\alpha\delta t)\,\psi_1(x-c\delta t, t) + \alpha\delta t\,\psi_2(x+c\delta t, t)$$

$$e^{H\delta t}(x_1'x_1'') = (1-\alpha\delta t)\,\delta(x'-x''-c\delta t) + \alpha\delta t$$

1 2	$\alpha\delta t$	$\delta(x'-x''+c\delta t)$
2 1	$\alpha\delta t$	$\delta(x'-x''-c\delta t)$
2 2	$(1-\alpha\delta t)$	$\delta(x'-x''+c\delta t)$

$$H(x_1'x_1'') = -\alpha\,\delta(x'-x''-c\delta t)$$

1	2	$= \alpha\,\delta(x'-x''+c\delta t)$
2	1	$\alpha\,\delta(x'-x''-c\delta t)$
2	2	$= -\alpha\,\delta\, x'-x''+0$

$c\delta t$

$$e^{-\alpha n\delta t}$$

$$(\alpha\delta t)^2(n-1)\,e^{-\alpha\delta t(n-2)}$$

$$(\alpha\delta t)^2(n-2)\,e^{-\alpha\delta t(n-2)} + (\alpha\delta t)^4 - \ldots$$

derived from (1). To avoid having two symbols i and j both denoting roots of -1 we shall take $j = i$, and must then modify the above rules to read :— from any equation one may obtain another equation by writing $-i$ for i wherever it occurs and at the same time writing $-h$ for h, or by reversing the orders of all factors and at the same time writing $-h$ for h, or by applying the two previous operations together, which ~~comes~~ reduces to reversing the orders of all factors and writing $-i$ for i. This third operation applied to any quantity gives what may be defined as the conjugate imaginary quantity. A quantity is defined as real if it is equal to its conjugate imaginary.

The remainder of this section will be devoted to some simple analytical rules which will be of use in the subsequent work. All the symbols denote q-numbers except h, i or except when otherwise stated.

When forming the reciprocal of a quantity composed of two or more factors one must reverse their order, i.e.

$$\frac{1}{(xy)} = \frac{1}{y} \cdot \frac{1}{x}. \qquad (4)$$

This equation may be verified by multiplying each side by xy either in front or behind.

To differentiate the reciprocal of a quantity x one must proceed as follows;—

$$0 = \frac{d}{dt}\left(\frac{1}{x} \cdot x\right) = \frac{d}{dt}\left(\frac{1}{x}\right) . x + \frac{1}{x}\dot{x}.$$

Hence, dividing by x behind, one gets

$$\frac{d}{dt}\left(\frac{1}{x}\right) = -\frac{1}{x}\dot{x}\frac{1}{x}.$$

The binomial theorem is the same as in ordinary algebra. Also one defines e^x by the same power series in x as in ordinary algebra. The ordinary exponential law, however, is not valid, i.e. e^{x+y} is not in general equal to $e^x e^y$ except when $xy = yx$, or, as we may say, when x commutes with y.

The Poisson bracket expression $[x, y]$ may be put equal to its classical theory value when there is no ambiguity concerning the orders of factors of products in this value.

Thus we have

$$\Big\{\sum_{xyzt}\Big(p_\nu - \frac{e}{c}A_\nu\Big)^2 + m^2c^2\Big\}\psi = 0$$

$-m^2c^2$

A_ν means $\int A_\nu(xyzt\; x'y'z't')\,\psi\, dx_1' dx_2' dx_3' dt'$

$$A_\nu = \frac{\dfrac{dx_\nu'}{ds}}{(x_\mu' - x_\mu)\dfrac{dx_\mu'}{ds'}}$$ for value of $x_1' x_2' x_3' x_4'$ which makes $(x_\mu - x_\mu')^2 = 0$

$$= \frac{p_\nu' - \frac{e}{c}A_\nu'}{(x_\mu' - x_\mu)(p_\mu' - \frac{e}{c}A_\mu')}$$

$$\Big(p_\nu - \frac{e}{c}A_\nu\Big)^2 (x_\mu' - x_\mu)\Big(p_\mu' - \frac{e}{c}A_\mu\Big)(x_\sigma' - x_\sigma)\Big(p_\sigma' - \frac{e}{c}A_\sigma'\Big) = \Big\{p_\nu\Big[(x_\mu' - x_\mu)\Big(p_\mu' - \frac{e}{c}A_\mu'\Big)\Big] - \frac{e}{c}\Big(p_\nu' - \frac{e}{c}A_\nu'\Big)\Big\}^2$$

$$A_\nu A_\nu = \frac{1}{\Big\{(x_\mu' - x_\mu)\dfrac{dx_\mu'}{ds'}\Big\}^2} = \frac{1}{\Big[\dfrac{d}{2ds'}(x_\mu' - x_\mu)^2\Big]^2}$$

$A_\nu\Big(\cancel{p_\nu - \frac{e}{c}A_\nu}\Big)$

$$\Big\{\Big\{p_\nu(x_\mu' - x_\mu)p_\mu' - \frac{e}{c}p_\nu'\Big\}^2 + m^2c^2\Big[(x_\mu' - x_\mu)p_\mu\Big]^2\Big\}\psi = 0$$

ψ very small except when $(x_\mu - x_\mu')^2$ is very small

$x_\nu\, y_\nu\, z_\nu\; \chi_n = \sum_n a_{mn} e^{i\omega_m(t_\nu - x_\nu/c)}\chi_n$

ψ contains factor $\exp\Big[(x_\mu - x_\mu')^2 \dfrac{m^2c^2}{h^2}\Big]^2$ $\qquad \psi \in e^{i\omega\phi}$

$p_\nu\; x_\nu$

$\psi = \psi_0 f(t-t')$

$f(x) = \dfrac{\varepsilon^2}{x^2+a^2}$

$f'(x) = \dfrac{-\varepsilon^2\, 2x}{(x^2+a^2)^2}$

$f''(x) = \dfrac{8x^2\varepsilon^2}{(x^2+a^2)^3} - \dfrac{2\varepsilon^2}{(x^2+a^2)^2}$

$\Big\{H - W + i\,V(t-t')\Big\}\psi = 0$

$\psi = \psi_1\psi_2$

$\psi_2(H-W)\psi_1 + i\,\psi_1(t-t')V\psi_2 = 0$

$\dfrac{(H-W)\psi_1}{\psi_1} = -i(t-t')\dfrac{V\psi_2}{\psi_2} = \text{const}$

$\Big\{(H-W)V + i(t-t')\Big\}\psi = 0$

$\psi = \psi_1\psi_2$

$(H-W)\psi_1 \cdot V\psi_2 + i(t-t')\psi_1\psi_2 = 0$

$\dfrac{(H-W)\psi_1}{\psi_1} = -\dfrac{i(t-t')\psi_2}{V\psi_2} = \text{const}$

$f(x) = e^{-x^2/a^2}$

$f'(x) = \dfrac{-2x}{a^2}e^{-x^2/a^2}$

$f''(x) = \Big(-\dfrac{2}{a^2} + \dfrac{4x^2}{a^4}\Big)e^{-x^2/a^2}$

$= \dfrac{4x^2 - 2a^2}{a^4}f(x)$

to the occurrence of the term $k_1 k_2/r^3 = (k^2 - \frac{1}{4}h^2)/r^3$ in (15) corresponding to the term k^2/r^3 in (12). There is thus no integral of (12) of the form (13).

We can, however, easily get an integral of (12) by making a small change in (13). We ~~may~~ transform from the variables r, θ, p_r, k to the variables r, θ', p_r, k', where

$$k' = \sqrt{k^2 + \tfrac{1}{4}h^2}\ , \qquad \theta' = \theta\, k'/k,$$

which are canonical since

$$[\theta', k'] = [\theta, k']\,\frac{k'}{k} = \frac{k}{\sqrt{k^2+\frac{1}{4}h^2}}\;\frac{k'}{k} = 1 .$$

Now try

$$1/r = a_0 + a_1 e^{i\theta'} + a_2 e^{-i\theta'} . \tag{13'}$$

Proceeding exactly as before, but using the Hamiltonian function

$$H = \frac{1}{2m}\left(p_r^{\,2} + \frac{k'^2 - \frac{1}{4}h^2}{r^2}\right) - \frac{e^2}{r},$$

we find that

$$\frac{d}{dt}\, e^{i\theta'} = \frac{i}{m}\, e^{i\theta'}\,\frac{k_1'}{r^2}, \qquad \frac{d}{dt}\, e^{-i\theta'} = -\frac{i}{m}\, e^{-i\theta'}\,\frac{k_2'}{r^2}$$

where

$$k_1' = k' + \tfrac{1}{2}h, \qquad k_2' = k' - \tfrac{1}{2}h,$$

and further that

$$m\dot{p}_r = \frac{k_1' k_2'}{r^3} - \frac{a_0 k_1' k_2'}{r^2} = \frac{k^2}{r^3} - \frac{a_0 k^2}{r^2},$$

which agrees with (12) if we take $a_0 = me^2/k^2$. Hence (13′) is an integral of the equation of motion (12).

This means that the orbit of the electron is an ellipse with a rotating apse line. If the Cartesian co-ordinates $x = r\cos\theta$ and $y = r\sin\theta$ are expanded in multiple Fourier series, two angle variables will be required, which will give two orbital frequencies. There must therefore necessarily be a two-fold infinity of energy levels, which disagrees with experiment (when one disregards the relativity fine-structure of the hydrogen spectrum). ~~One is therefore~~ Hence one is forced to modify the Hamiltonian in order to make the motion degenerate. The Hamiltonian which makes the quantum orbit most closely resemble the classical orbit is

$$H = \frac{1}{2m}\left(p_r^{\,2} + \frac{k_1 k_2}{r^2}\right) - \frac{e^2}{r}, \tag{16}$$

$$\bar{x} = x + \varepsilon T \qquad \bar{x}' = x' + \varepsilon T$$

$$\bar{t} = T + \frac{\varepsilon x}{c^2} \qquad \bar{t}' = T + \frac{\varepsilon x'}{c^2}$$

$$\bar{T} = T \qquad \bar{x} = x + \varepsilon \bar{T} + \dot{x}(\bar{T} - \bar{t}) = x + \varepsilon \bar{T} - \dot{x}\frac{\varepsilon x}{c^2}$$

$$\bar{x}' = x' + \varepsilon \bar{T} + \dot{x}'(\bar{T} - \bar{t}') = x' + \varepsilon \bar{T} - \dot{x}'\frac{\varepsilon x'}{c^2}$$

$$\bar{x} = x - \frac{\partial H}{\partial p_x}\frac{\varepsilon x}{c^2}$$

$$\bar{x}' = x' - \frac{\partial H}{\partial p_x'}\frac{\varepsilon x'}{c^2}$$

$$[\bar{x}, \bar{x}'] = \cancel{[x, x']} - \left[x, \frac{\partial H}{\partial p_x'}\right]\frac{\varepsilon x'}{c^2} + \left[x', \frac{\partial H}{\partial p_x}\right]\frac{\varepsilon x}{c^2}$$

$$[x, x'] = 0$$

$$H = p_1^2 + b p_2^2 b^{-1} + \cos(q_1 - b q_2 b^{-1})$$

$$[p_1 + b p_2 b^{-1}, q_1 - b q_2 b^{-1}] = [b p_2 b^{-1}, q_1] - [p_1, b q_2 b^{-1}]$$

$$= [b, q_1] p_2 b^{-1} + b p_2 b^{-1}[b, q_1] b^{-1} - [p_1, b] q_2 b^{-1} + b q_2 b^{-1}[p_1, b] b^{-1}$$

$$p_1 + b p_2 b^{-1} = p_1 + p_2 + i h [b, p_2] b^{-1}$$

$$q_1 - b q_2 b^{-1} = q_1 - q_2 - i h [b, q_2] b^{-1}$$

$$H = p_1^2 + p_2^2 + \dot{p}_2 \Delta t + \cos(q_1 - q_2 - \dot{q}_2 \Delta t)$$

$$= p_1 + p_2 + \cos(q_1 - q_2) + \{\dot{p}_2 + \sin(q_1 - q_2)\dot{q}_2\}\Delta t$$

$$\dot{p}_2 = -\frac{\partial H}{\partial q_2} = -\sin(q_1 - q_2)$$

$$\dot{q}_2 = 1$$

$$H = f(p, q) + \dot{p}_1 \Delta t \cdots$$

$$f(p_1 + \dot{p}_1\Delta t, p_2 - p_1, q_1 + \dot{q}_1\Delta t, q_2 - q_1)$$

$$= f(p, q) + \frac{\partial f}{\partial p_1}\dot{p}_1 \Delta t + \frac{\partial f}{\partial q_1}\dot{q}_1 \Delta t$$

$$H = f(b p_1 b^{-1}, p_2 - p_1, b q_1 b^{-1}, q_2 - q_1)$$

$$H = f(b p_1 b^{-1}, b q_1 b^{-1}) * f_0(p_2, q_2)$$

$$= b f(p_1, q_1) b^{-1} * f_0$$

$$\dot{q}_2 = \frac{\partial f}{\partial p_2}(p, q) \qquad \frac{\partial f}{\partial p_2}(p_1 + \dot{p}_1\Delta t, q_1 + \dot{q}_1\Delta t, p_2, q_2)$$

$$\partial f(p_1 + \dot{p}_1 \Delta t, \qquad = \frac{\partial f}{\partial p_2}(p_1, q_1, p_2, q_2) + \Delta t\left\{\frac{\partial^2 f}{\partial p_2 \partial p_1}\dot{p}_1 + \frac{\partial f}{\partial p_2 \partial q_1}\dot{q}_1\right\}$$

$\alpha(1-\ell_1)J' = B\, A^{\frac12} B^{-1} A^{\frac12} = A^{+\frac12} J A^{-\frac12}$ $\qquad [q_1, A^{\frac12}] = \dfrac{1-\ell_1}{2A^{\frac12}}$

$\dfrac{2H'}{c} = \dfrac{B}{\alpha(1-\ell_1)J'} = A^{-\frac12} B A^{-\frac12} = A^{\frac12} \dfrac{2H}{c} A^{-\frac12}$

$[\varpi, A] = \alpha(1-\ell_1) + \dfrac{2A p_2 (1-\ell_1)\alpha \cos\alpha q_1}{B} = \dfrac{\alpha(1-\ell_1)}{B} B \qquad [\varpi, B] = 0$

$A^{\frac12} \varpi A^{-\frac12} - \varpi$

$[\alpha q_1, J'] = B\left(\dfrac{1}{2A^{\frac12}} B^{-1} A^{\frac12} + A^{\frac12} B^{-1} \dfrac{1}{2A^{\frac12}}\right) = \dfrac{B}{2A^{\frac12}}\left(\dfrac{1}{B}A + A\dfrac{1}{B}\right)\dfrac{1}{2A^{\frac12}}$ $\qquad 2J'[\alpha q_1, J']\alpha(1-\ell_1) = B\{A^{\frac12} B^{-2} A^{\frac12} + A^{\frac12} B^{-1} A$

$= \frac12 \alpha(1-\ell_1)\left(\frac{1}{A} J + J \frac{1}{A}\right)$

$X = \ell_3 x - \dfrac{\ell_2\ell_3}{1-\ell_1} y + \dfrac{1-\ell_1-\ell_3^2}{1-\ell_1} z = \dfrac{1}{(1-\ell_1)}\{\ell_3(q_1 + c q_4 + \ell_2 q_2 + \ell_3 q_3) - \ell_2\ell_3 q_2 + (1-\ell_1-\ell_3^2) q_3\} = \dfrac{\ell_3}{1-\ell_1}(q_1 + c q_4) + q_3$

$Y = \ell_2 x + \dfrac{1-\ell_1-\ell_2^2}{1-\ell_1} y - \dfrac{\ell_2\ell_3}{1-\ell_1} z = \dfrac{\ell_2}{1-\ell_1}(q_1 + c q_4) + q_2$

$[X, J']\alpha(1-\ell_1) = A^{\frac12}[q_3, \frac{B}{B}] A^{\frac12} = 2A^{\frac12}\left(p_3 \frac{1}{B} + B \frac{1}{B} p_3 \frac{1}{B}\right) A^{\frac12} = 2p_3 A^{\frac12} \dfrac{2A p_2 \cos\alpha q_1}{B^2} A^{\frac12}$

$X = \dfrac{\ell_3}{1-\ell_1}(q_1 + c q_4) - q_1 \dfrac{p_3 + \ell_3 H/c}{(1-\ell_1)H/c} + \mathrm{const}(H) = A^{\frac12}\left\{\dfrac{\ell_3}{1-\ell_1}(q_1 + c q_4) - q_1 \dfrac{p_3 + \ell_3 H/c}{(1-\ell_1)H/c}\right\} A^{-\frac12} + \mathrm{const}(H')$

$Y = \dfrac{\ell_2}{1-\ell_1}(q_1 + c q_1) - q_1 \dfrac{p_2 + \ell_2 H/c}{(1-\ell_1)H/c} - \dfrac{A \sin\alpha q_1}{\alpha(1-\ell_1)H/c} + \mathrm{const}(H)$

$(1-\ell_1)X = \ell_3(A^{\frac12} q_1 A^{-\frac12} + c q_4) - A^{\frac12} q_1 A^{-\frac12}\left(\dfrac{p_3 c}{H'} + \ell_3\right) + \mathrm{const}(H') = \ell_3 c q_4 - A^{\frac12} q_1 A^{-\frac12} \dfrac{p_3 c}{H'} + \mathrm{const}(H')$

$X = \dfrac{\ell_3}{1-\ell_1} c q_4 - q_1 \dfrac{p_3}{(1-\ell_1)H/c} + \mathrm{const}(H)$ $\qquad 2A\dfrac{p_2 p_3}{B(1-\ell_1)} \quad \dfrac{2\alpha(1-\ell_1)J}{\alpha B}$

$Y = \dfrac{\ell_2}{1-\ell_1} c q_4 - q_1 \dfrac{p_2}{(1-\ell_1)H/c} - \dfrac{A \sin\alpha q_1}{\alpha(1-\ell_1)H/c}$

$\dfrac{2A p_2}{\alpha B} \sin\varpi \dfrac{p_2}{(1-\ell_1)} \dfrac{2\alpha(1-\ell_1)J}{B} - A \sin\varpi \dfrac{2\alpha(1-\ell_1)J}{\alpha B(1-\ell_1)}$

$= 2A \sin\varpi\left(\dfrac{2p_2^2 J}{B^2} - \dfrac{J}{B}\right)$

Intensity :: $4A^2\left\{p_2^2 p_3^2 \dfrac{J(J+h)}{B^4} + \dfrac{(p_2^2 - B)^2 J(J+h)}{B^4}\right\} = \dfrac{4A^2}{B^4}\left(p_2^2 p_3^2 + p_2^4 - p_2^2 B + \tfrac12 B^2\right)(J[J+h)$

$= \dfrac{A^2}{4m^4c^4}\{\ell_2^2\ell_3^2 + \ell_2^4 - 2\ell_2^2(1-\ell_1) + (1-\ell_1)^2\} J(J+h)$ $\qquad \{\ \} = (1-\ell_1)\{(1+\ell_1)\ell_2^2 - 2\ell_2^2 + 1 - \ell_1\}$

$\qquad = (1-\ell_1)^2(1-\ell_2^2)$

$= \dfrac{A^2(1-\ell_2^2)}{4m^2c^2\alpha^2}\left[1 + \gamma(1-\ell_1)\right]$

$\qquad \{\ \} = \frac{1}{\ell_3}\{(1-\ell_1)^2 - (1-\ell_1)(\ell_2^2\ell_3^2)\}$

$\qquad = \frac{1}{\ell_3}(1-\ell_1)\{1-\ell_1-\ell_2^2\ell_3^2\} = -\frac{\ell_1}{\ell_3}(1-\ell_1)^2$

$(1-\ell_1-\ell_2^2)X + \ell_2\ell_3 Y = (1-\ell_1-\ell_2^2)\dfrac{(1-\ell_1)}{\ell_3} X + \ell_2^2 \dfrac{1-\ell_1}{\ell_2} y = (1-\ell_1-\ell_2^2+\ell_2^2)(1-\ell_1)x + \left\{(1-\ell_1-\ell_2^2)\dfrac{(1-\ell_1-\ell_3^2)}{\ell_3} - \ell_2^2\ell_3\right\} z \quad : \ell_3 x - \ell_1 z$

$$m^2c^2 = \left(p_t + \frac{e}{c}\phi\right)^2 - \left(p_x + \frac{e}{c}A_x\right)^2 - \left(p_y + \frac{e}{c}A_y\right)^2 - \left(p_z + \frac{e}{c}A_z\right)^2 \qquad \text{for charge } -e.$$

Potl. produced at $(x'y'z't')$ by charge $-e$ at $(xyzt)$ is if $(x-x')^2+(y-y')^2+(z-z')^2-c^2(t-t')^2=0$

$$\phi = \frac{-ec}{c^2(t-t') - (x-x')\frac{dx}{ds} - (y-y')\frac{dy}{ds} - (z-z')\frac{dz}{ds}} = \frac{-ec\frac{dt}{ds}}{c^2(t-t')\frac{dt}{ds} - (x-x')\frac{dx}{ds} - \cdots}$$

$$A_x = \frac{-e\frac{dx}{ds}}{c^2(t-t')\frac{dt}{ds} - (x-x')\frac{dx}{ds} - \cdots} \qquad A_i = \frac{-e\frac{dx_i}{ds}}{g_{kl}(x_k - x_k')\frac{dx^l}{ds}}$$

$$p_i' + \frac{e}{c}A_i = m\frac{dx_i'}{ds} - \frac{\frac{e^2}{c}\frac{dx_i}{ds}}{g_{kl}(x_k-x_k')\frac{dx^l}{ds}} = \frac{m\frac{dx_i'}{ds}\frac{dx_l}{ds}g_{kl}(x_k-x_k') - \frac{e^2}{c}\frac{dx_i}{ds}}{g_{kl}(x_k-x_k')\frac{dx^l}{ds}}$$

$$g_{ij}\left(m\frac{dx_i'}{ds}\frac{dx_l}{ds}g_{kl}(x_k-x_k') - \frac{e^2}{c}\frac{dx_i}{ds}\right)\left(m\frac{dx_j'}{ds}\frac{dx_m}{ds}g_{nm}(x_n-x_n') - \frac{e^2}{c}\frac{dx_j}{ds}\right) = m^2c^2\left\{g_{kl}(x_k-x_k')\frac{dx_l}{ds}\right\}^2$$

$$\left\{g_{kl}(x_k-x_k')\frac{dx_l}{ds}\right\}^2\left(m^2c^2 - m^2 g_{ij}\frac{dx_i'}{ds}\frac{dx_j'}{ds}\right) = -\frac{2e^2m}{c}g_{ij}\frac{dx_i}{ds}\frac{dx_j'}{ds}\frac{dx_m}{ds}g_{mn}(x_n-x_n') + \frac{e^4}{c^2}g_{ij}\frac{dx_i}{ds}\frac{dx_j}{ds}$$

Can we find such a relation which commutes with $(x-x')^2+(y-y')^2+(z-z')^2-c^2(t-t')^2$?

$$A_\mu = \frac{+e\frac{dx_\mu}{ds}}{(X_\nu - x_\nu)\frac{dx_\nu}{ds}}$$

$$\frac{e}{A_\mu} = \frac{(X_\nu - x_\nu)\left(p_\nu - \frac{e}{c}A_\nu\right)}{p_\mu - \frac{e}{c}A_\mu}$$

$$\frac{e}{A_\mu} = (X_\nu - x_\nu)\frac{dx_\nu}{dx_\mu} = -\frac{1}{2}\frac{d}{dx_\mu}(X_\nu - x_\nu)^2$$

$A = A(x) \qquad B = B(x')$

$$AB(kl:mn) = \tfrac{1}{2}\iint(\psi_k\psi_l' - \psi_l\psi_k')AB(\psi_m\psi_n' - \psi_n\psi_m')\,d\tau\,d\tau'$$

$$= \tfrac{1}{2}\int\psi_k A\psi_m d\tau\int\psi_l' B\psi_n' d\tau$$

$$+ \tfrac{1}{2}\int\psi_l A\psi_n d\tau\int\psi_k' B\psi_m' d\tau$$

$$- \tfrac{1}{2}\int\psi_k A\psi_n d\tau\int\psi_l' B\psi_m' d\tau$$

$$- \tfrac{1}{2}\int\psi_l A\psi_m d\tau\int\psi_k' B\psi_n' d\tau$$

$$\tfrac{1}{2}(AB+BA)(kl:mn) = \tfrac{1}{2}\left\{A(km)B(ln) + A(ln)B(km) - A(kn)B(lm) - A(lm)B(kn)\right\}$$

By any quantity to imply the mean of that quantity and the symmetrically related one

$$i\hbar \dot N_r = \Sigma_s \mathrm{V}_{rs} N_r^{\frac12} N_s^{\frac12} e^{i(\theta_r-\theta_s)/\hbar} - \Sigma_s \mathrm{V}_{sr} N_r^{\frac12} N_s^{\frac12} e^{i(\theta_s-\theta_r)/\hbar}$$ (7)

The ar's must be normalised initially and will then always remain normalised.

The eigenfunctions for the perturbed system must satisfy the wave equation

$$i\hbar \frac{\partial \psi}{\partial t} = (H_0 + A)\psi .$$

where H_0+A is an operator. If the solution of this equation satisfying the preliminary conditions is $\psi = \Sigma_r a_r \psi_r$ where the ψ_r's are the eigenfunctions for the unperturbed system each associated with one stationary state labelled by the suffix r, ~~and are normalised~~, and the a_r's are functions of the time only, then $|a_r|^2$ is the probability of the system being in the rth state at any time. The theory will apply directly to an assembly of N similar independent systems if we multiply each of ~~these~~ a_r by $N^{\frac12}$ so as to make $\Sigma_r |a_r|^2 = N$. We shall now have that $|a_r|^2$ is the probable number of systems in the rth state. The equation that determines the rate of change of the a's is [see equation (25) of the author's previous paper]

$$i\hbar\, \dot a_r = \Sigma_s \mathrm{V}_{rs} a_s \qquad (3)$$

where the V_{rs}'s ~~being~~ are the elements of the matrix representing V.

(p2)

interchange θ & ϕ

~~It is convenient to~~ We can transform to the canonical variables N_r, ϕ_r by the contact transformation

$$a_r = N_r^{\frac12} e^{-i\phi_r/\hbar} \qquad a_r^* = N_r^{\frac12} e^{+i\phi_r/\hbar}$$

This transf. makes the ~~so that~~ N_r and ϕ_r ~~are~~ real, ~~N_r being equal to $a_r a_r^* = |a_r|^2$~~ $N_r = a_r a_r^*$ ~~is~~ the probable number of systems in state r and $\phi_r/\hbar$ being the phase of the eigenfunction that represents them. The Hamiltonian F_1 now becomes

$$F_1 = \Sigma_{rs} \mathrm{V}_{rs} N_r^{\frac12} N_s^{\frac12} e^{i(\phi_r-\phi_s)/\hbar}$$

and the equations of ~~motion~~ that determine the rate at which transitions occur ~~take place~~ have ~~take~~ the canonical form

$$\dot N_r = -\frac{\partial F_1}{\partial \phi_r} \qquad \dot \phi_r = \frac{\partial F_1}{\partial N_r}$$

~~The co-ordinates that are the canonical conjugates of the numbers of atoms in the different states are the phases of the eigenfunctions.~~

A ~~slightly~~ more convenient way of putting the ~~equations~~ ~~transition~~ equations in the Hamiltonian form may be obtained with the help of the quantities $b_r = a_r e^{-iW_r t/\hbar}$, $b_r^* = a_r^* e^{iW_r t/\hbar}$, W_r being the energy of the rth state. We now have $|b_r|^2$ equal to $|a_r|^2$ = the probable number of systems in the rth state. For $\dot b_r$ we find

$$i\hbar\, \dot b_r = W_r b_r + i\hbar\, \dot a_r e^{-iW_r t/\hbar} = W_r b_r + \Sigma_s \mathrm{V}_{rs} b_s e^{i(W_s-W_r)t/\hbar}$$

p iv

This differs from the previous equation (4) only in the notation, a single suffix r being there used to denote a stationary state instead of a set of numerical values ξ'_r for the variables ξ_r, and b_r being used instead of $\psi(\xi')$. Equation (6) or (4) (and therefore also eqn (3)) can ~~thus~~ still be used ~~written down~~ when the Hamiltonian is of the more general type which cannot be expressed as an algebraic function of a set of canonical variables, but can ~~still~~ be represented by a matrix $H(\alpha'\alpha'')$.

We ~~now~~ take b_r and $i\hbar b_r^*$ to be canonical conjugates instead of a_r and $i\hbar a_r^*$. ~~and~~ The equations (4) and its ~~canonical~~ conjugate will now ~~will~~ take the Hamiltonian form with the Hamiltonian function

$$F = \Sigma_{rs} b_r^* H_{rs} b_s . \qquad (5)$$ (3a)

$$r^2\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)() = \frac{\partial^2}{\partial r^2}r^2()$$

$r = \kappa r_1$

$r_2 r_1 \to \kappa_2 r_1$

$$\frac{\partial}{\partial r} = \frac{\partial \kappa}{\partial r}\frac{\partial}{\partial \kappa} + \frac{\partial r_1}{\partial r}\frac{\partial}{\partial r_1} = \frac{1}{r_1}\frac{\partial}{\partial \kappa}$$

$r_n = \kappa_n r_{n-1}$

$$p_n = \frac{1}{r_{n-1}} P_{n-1}$$

Initial variables $r_1\ r_2\ r_3 \ldots r_u$ $\quad r_1 \geq r_2 \geq r_3 \ldots \geq r_u \geq 0$

$p_1\ p_2\ p_3 \ldots p_u$

Final — $r_1\ \kappa_2\ \kappa_3 \ldots \kappa_u$ $\quad \kappa_n = \frac{r_n}{r_{n-1}}\ (n>1)$

$P_1\ P_2\ P_3 \ldots P_u$

$\cancel{P_n = r_n\, p_{n+1}}$

$s>t$, $r_s < r_t$: $$\frac{1}{r_{st}} = \frac{1}{r_t}\sum_n \left(\frac{r_s}{r_t}\right)^n P_n(\mu_{rs})$$

$$H\psi = \sum_k\left(\frac{\partial^2}{\partial r_k^2}r_k^2 + \frac{1}{\sin\theta_k}\frac{\partial}{\partial\theta_k}\sin\theta_k\frac{\partial}{\partial\theta_k} + \frac{1}{\sin^2\theta_k}\frac{\partial^2}{\partial\phi_k^2}\right)\psi$$

$$+\sum_{k>j}\sum \frac{1}{r_j}\sum_n\left(\frac{r_k}{r_j}\right)^n P_n(\mu_{kj})\,\psi$$

$\mu_{kj} = \cos\theta_k\cos\theta_j + \sin\theta_k\sin\theta_j\cos(\phi_j-\phi_k)$

$$\psi = \sum_{m_1 m_2 \ldots} c_{m_1 m_2 m_3, n_1 n_2} r_1^{m_1} r_2^{m_2} r_3^{m_3}\ldots S_{n_1}(\theta_1\phi_1) S_{n_2}(\theta_2\phi_2) S_{n_3}(\theta_3\phi_3)\ldots$$

$$P_n(\mu_{rs})\, P_{m_r k_r}(\theta_r\phi_r)\, P_{m_s k_s}(\theta_s\phi_s) =$$

$$g(\xi'\eta') = (\xi'|\eta')\, g_{\xi'\eta'} = \frac{1}{\sqrt{2\pi h}}\, e^{i\xi'\eta'/h}\, g_{\xi'\eta'}$$

$$g(\eta'\xi') = \frac{1}{\sqrt{2\pi h}}\, e^{-i\xi'\eta'/h}\, g_{\eta'\xi'}$$

$$g_{\eta'\xi'} = \sqrt{2\pi h}\, e^{i\xi'\eta'/h}\, g(\eta'\xi')$$

$$= \sqrt{2\pi h}\, e^{i\xi'\eta'/h} \iint (\eta'|\xi'')\, g(\xi''\eta'')\, (\eta''|\xi')\, d\xi''\, d\eta''$$

$$= \cancel{\sqrt{2\pi h}}\, \frac{e^{i\xi'\eta'/h}}{\sqrt{2\pi h}} \iint g(\xi''\eta'')\, e^{-i(\xi''\eta' + \xi'\eta'')/h}\, d\xi''\, d\eta''$$

$$= \frac{1}{2\pi h}\, e^{i\xi'\eta'/h} \iint e^{i\xi''\eta''/h}\, g_{\xi''\eta''}\, e^{-i(\xi''\eta' + \xi'\eta'')/h}\, d\xi''\, d\eta''$$

$$= \frac{1}{2\pi h} \iint g_{\xi''\eta''}\, e^{i(\xi''-\xi')(\eta''-\eta')/h}\, d\xi''\, d\eta''$$

$$(ab)_{\xi'\eta'} = \sqrt{2\pi h}\, e^{-i\xi'\eta'/h}\, (ab)(\xi'\eta')$$

$$= \sqrt{2\pi h}\, e^{-i\xi'\eta'/h} \iint a(\xi'\eta'')\, d\eta''\, (\eta''|\xi'')\, d\xi''\, b(\xi''\eta')$$

$$= (2\pi h)^{-1}\, e^{-i\xi'\eta'/h} \iint a_{\xi'\eta''}\, e^{i\xi'\eta''/h}\, e^{-i\xi''\eta''/h}\, e^{i\xi''\eta'/h}\, b_{\xi''\eta'}$$

$$= \frac{1}{2\pi h} \iint a_{\xi'\eta''}\, b_{\xi''\eta'}\, e^{-i(\xi'-\xi'')(\eta'-\eta'')/h}\, d\xi''\, d\eta''.$$

~~This~~ is probably not
true ~~when the~~ a's involve
the time explicitly

$$\ldots c_m c_m^* \, \eta_{mn} \, \eta_{nm} \int_0^T K(t)\, e^{i(W_m-W_n)t/h}\, dt \int_0^t K(s)\, e^{-i(W_m-W_n)s/h}\, ds$$

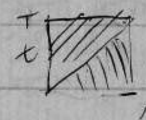
$$-1/h^2c^2 \sum_n c_m c_m^* \, \eta_{mn}^* \, \eta_{nm}^* \int_0^T K(t)\, e^{-i(W_m-W_n)t/h}\, dt \int_0^t K(s)\, e^{+i(W_m-W_n)s/h}\, ds$$

which reduces to

$$1/h^2c^2 \sum_n \{|c_n|^2-|c_m|^2\}\, |\eta_{mn}|^2 \left| \int_0^T K(t)\, e^{i(W_m-W_n)t/h}\, dt \right|^2 \qquad (24)$$

ΔN_m

This gives the increase in the number of atoms in the m-th state due to the perturbation from the time t=0 to the time t=T.



On the classical theory one of the variables to be introduced, namely k, is just equal to m. The quantum variable k may not be equal to m, but must be chosen such that x, y and z are ~~multiply~~ periodic fns of its canonically conjugate variable θ, of period 2π. On the classical theory, if a co-ordinate x, y or z, say z, is expanded as a Fourier series in θ_3 the angle variable, the coeffs. of the terms involving $e^{ni\theta}$ all vanish unless $n = \pm 1$. This fact is expressible analytically by the equation $\partial^2 z/\partial\theta_k^2 = -z$ or, in P.B.'s, by $[k,[k\ z]] = -z$. We try to choose our quantum variable k so as also to satisfy

5A*

$$[k,[k\ z]] = -z \tag{12}$$

Eqn (12) gives

$$[k^2,[k,z]] = k[k,[k,z]] + [k[k,z]]k = -(kz + zk)$$

Hence

$$[k^2,[k^2,z]] = k[k^2,[k,z]] + [k^2,[k,z]]k = -(k^2z + 2kzk + zk^2)$$

$$= -2(k^2z + zk^2) + (k^2z - 2kzk + zk^2)$$

$$= -2(k^2z + zk^2) - h^2[k,[k,z]]$$

$$= -2(k^2z + zk^2) + h^2z$$

or

$$\tfrac{1}{2}[k^2,[k^2,z]] = -(k^2 - \tfrac{1}{4}h^2)z - z(k^2 - \tfrac{1}{4}h^2) \tag{13}$$

(16)

Comparing this with (13) we see that they agree if we ~~we must~~ take $m^2 = k^2 - \tfrac{1}{4}h^2 = k_1 k_2$ where $k_1 = k + \tfrac{1}{2}h$, $k_2 = k - \tfrac{1}{2}h$

(In general we shall take the suffix 1 attached to any ~~angle~~ action variable to denote the value of that variable increased by $\tfrac{1}{2}h$ and the suffix 2 to denote its value reduced by $\tfrac{1}{2}h$)

$$\left[\left(p_x - \frac{e}{c}A_x\right)^2 \cdots \right]\psi$$

X is a fn of xyz

$$x\,\psi_k = \sum x_{nk}\,\psi_n$$

$$\Box\,\phi = 4\pi\rho$$

$$t_1' = t - \frac{l_1x + l_2y + l_3z}{c}$$

Choose the α's such that

$$[\alpha_1 t'] = [\alpha_2 t'] = [\alpha_3 t'] = 0$$

$$[\alpha_4, t'] = 1$$

Put $\alpha_4 = f(\alpha_1\alpha_2\alpha_3)$

X(~~$\alpha\alpha'$~~) ~~chosen such that~~ $[X, \alpha_4] = 0$

$$\int X(\alpha\alpha')\, e^{i\left[f(\alpha) - \alpha_4 - f(\alpha') + \alpha_4'\right]t'/h}\, d\alpha_4'$$

$X(\alpha\alpha')$ is a fn of α_4 and α_4' only through $(\alpha_4 - \alpha_4')$

X has matrix components $X(\alpha\alpha')$

t' — $\quad \delta'(\alpha_4 - \alpha_4')\,\delta(\alpha_1 - \alpha_1')\,\delta(\alpha_2 - \alpha_2')\,\delta(\alpha_3 - \alpha_3')$

$$(Xt' - t'X)(\alpha\alpha') = \int \left[X(\alpha\alpha'')\,\delta'(\alpha_4'' - \alpha_4') - \delta'(\alpha_4 - \alpha_4'')\,X(\alpha''\alpha')\right] d\alpha''$$

$$= \int \left[X(\alpha, \alpha_1''\alpha_2''\alpha_3''\alpha_4') - X(\alpha_1''\alpha_2''\alpha_3''\alpha_4, \alpha')\right] d\alpha_1''\,d\alpha_2''\,d\alpha_3''$$

Contact transf. can be made only from one q-number to another with same characteristic values. Perhaps there is no unique product of two q-numbers, which fits in with idea that a q-numb. is completely defined by its charac. values

One wants only just sufficient eigenfunctions

in same way in which the electrons in an ionised stellar atmosphere settle into their quantised orbits of lowest energy when the gas is cooled to ordinary temp

$a_{mm} b_{mn} = b_{mn} a_{nn}$
$b_{mn} = 0$ unless $a_{mm} = a_{nn}$

$\psi_{mn} = \overline{\Psi}_m \Psi_n \qquad \psi_m^* = \Sigma \psi_n a_{nm} \qquad \overline{\Psi}_m^* = \Sigma \overline{\Psi}_n \overline{a}_{nm}$

$x \psi_m^* = \Sigma x_{km}^* \psi_k^* = \Sigma x_{km}^* a_{nk} \psi_n$

$= x \Sigma a_{km} \psi_k = \Sigma a_{km} x_{nk} \psi_n$

$\Sigma_k x_{nk} a_{km} = \Sigma_k a_{nk} x_{km}^* \qquad x^* = a^{-1} x a$

$\Sigma_k a_{nk} \psi_{km}^* = \Sigma_k \psi_{nk} a_{km} = \Sigma_k \overline{\Psi}_n \psi_k a_{km} = \overline{\Psi}_n \psi_m^*$

$\Sigma_k a_{nk} \psi_{km}^* = \Sigma_k \psi_{nk} a_{km} = \overline{\Psi}_n \Sigma_k \psi_k a_{km} = \overline{\Psi}_n \psi_m^*$

$\Sigma_k a_{nk} \overline{a}_{nl} \psi_{km}^* = \Sigma_k \overline{a}_{nl} \overline{\Psi}_n \psi_m^* = \overline{\Psi}_l^* \psi_m^*$

$\Sigma_k \int a_{nk} \psi_{km}^* \, d\tau = \Sigma_k \delta_{nk} a_{km} = a_{nm}$

$= \int \psi_{mm'}(x_1' x_2'') \psi_{nn'}(x_1'' x_2'')$

$= \iint \psi_{mm'}(x_1' x_2') \psi_{nn'}(x_1'' x_2'') \, dx_2' \, dx_1'' \; \delta(x_1'-x_1'') \delta(x_2'-x_2'')$

$\Sigma_{kk'} \int \psi_{mm'}(x_1' x_2') . \psi_{kk'}(x_1' x_2') \, dx_2'$

$\int \psi_{kk'}(x_1'' x_2'') \psi_{nn'}(x_1'' x_2'') \, dx_1''$

$\int \rho_{mm',nn'} \, d\tau' = A_{mm',nn'} \qquad \Sigma_{kk'} A_{mm',kk'} A'_{kk',nn'} = \rho_{mm'nn'}$

$\int \rho_{mm',nn'} \, d\tau = A'_{mm',nn'} \qquad = \Sigma_{kk'} \int \rho_{mm'kk'} \, d\tau' . \int \rho_{kk'nn'} \, d\tau.$

If $\rho_{mm',nn'} = (\overline{\Psi}_m \psi_{m'} - \psi'_m \psi_{m'})(\Psi_n \Psi'_{n'} - \psi'_n \psi_{n'})$ $\qquad xyz \neq x'y'z'$

$A_{mm',kk'} = \overline{\Psi}_m \psi_k \delta_{m'k'} + \overline{\Psi}_{m'} \psi_{k'} \delta_{mk} - \overline{\Psi}_m \psi_{k'} \delta_{m'k} - \overline{\Psi}_{m'} \psi_k \delta_{mk'}$

$A'_{kk',nn'} = \overline{\Psi}'_{k'} \psi'_{n'} \delta_{kn} + \overline{\Psi}'_k \psi'_n \delta_{k'n'} - \overline{\Psi}'_k \psi'_{n'} \delta_{k'n} - \overline{\Psi}'_{k'} \psi'_n \delta_{kn'}$

$\Sigma_{kk'} A_{mm',kk'} A'_{kk'nn'} = 2\overline{\Psi}_m \psi_n \overline{\Psi}'_{m'} \psi'_{n'} + \cancel{\overline{\Psi}_m \Sigma_k \psi_k \overline{\Psi}_k \psi_n \delta_{m'n'}} - \cancel{\overline{\Psi}_m \Sigma_k \psi_k \overline{\Psi}_k \psi_n} - 2\overline{\Psi}_m \psi_{n'} \overline{\Psi}'_{m'} \psi'_n$

$+ 2\overline{\Psi}_{m'} \psi_{n'} \overline{\Psi}'_m \psi'_n - 2\overline{\Psi}_{m'} \psi_n \overline{\Psi}'_m \psi'_{n'} - \cancel{\overline{\Psi}_m \psi_{n'} \overline{\Psi}'_{m'} \psi'_n}$

$+ \cancel{\overline{\Psi}_m \psi_n \overline{\Psi}'_{m'} \psi'_{n'}} - \cancel{\overline{\Psi}_{m'} \psi_n \overline{\Psi}'_m \psi'_{n'}} - \cancel{\overline{\Psi}_{m'} \psi_{n'} \overline{\Psi}'_m \psi'_n}$

$= 2(\overline{\Psi}_m \overline{\Psi}'_{m'} - \overline{\Psi}_{m'} \overline{\Psi}'_m)(\psi_n \psi'_{n'} - \psi_{n'} \psi'_n)$

$= \psi_{mm',nn'}$

$\Sigma_k a_{mk} b_{kn} = 0$ $\qquad$ $\gamma_{mn} = \Sigma_k x_{mk} a_k a_n$

Matrix of rank p is multiplied by matrix of rank q

Product matrix has a rank not greater than p or q and not less than $n-p-q$

(n = no. of rows or columns)

$\Psi(\alpha_1\alpha_2, \beta_1\beta_2)$ orthogonal $\quad \Psi(\alpha_1 k_2\, \beta_1 k) = \theta_k(\alpha_1\beta_1) \quad \cancel{\Psi(\alpha\beta)}$

$$\Psi_{\alpha\alpha'} = \Sigma_\beta\, \theta_\alpha(x)\,\theta_\beta(x)\,\theta_\beta(x')\,\theta_{\alpha'}(x')$$

$$\int \theta_\alpha(x)\theta_\beta(x)\,dx = n\delta_{\alpha\beta}$$

$$\int \theta_\alpha(x)\,\theta_\alpha(x')\,d\alpha = n\,\delta_{xx'}$$

$$\iint \theta_\alpha(x')\,\theta_\alpha(x)\,\theta_\beta(x)\,d\alpha\,dx = \int \theta_\alpha(x')\,n\,\delta_{\alpha\beta}\,d\alpha = n\theta_\beta(x')$$

$$\cancel{\rho(\alpha_1\alpha_2, \beta_1\beta_2) = \Psi_1(\alpha_1\beta_1)\Psi_2(\alpha_2\beta_2)}$$

$$\cancel{\int a(\alpha_1\alpha_2)\,\rho(\alpha_1\alpha_2, \beta_1\beta_2)\,d\alpha_1 d\alpha_2 = a(\beta_1\beta_2)}$$

$$\rho(\alpha\alpha', \beta\beta') = \bar{\Psi}(\alpha\beta)\,\Psi(\alpha'\beta')$$

$\delta(x) = 0$ when $x = 0$

$\int \delta(x)\,dx = 1$

$$\int \rho(\alpha\alpha', \beta\beta)\,d\beta = \delta(\alpha-\alpha') \qquad 1(\alpha\alpha') = \delta(\alpha-\alpha')$$

$$\int \rho(\alpha\alpha, \beta\beta')\,d\alpha = \delta(\beta-\beta')$$

$$\int \rho(\alpha\alpha', \beta\beta')\,x(\beta\beta')\,d\beta\,d\beta' = x(\alpha\alpha')$$

$$\int x(\alpha\alpha')\,y(\alpha'\alpha'')\,d\alpha' = \int \rho(\alpha\alpha', \beta\beta')\,x(\beta\beta')\,\rho(\alpha'\alpha'', \beta''\beta''')\,y(\beta''\beta''')\,d\beta\,d\beta'\,d\beta''\,d\beta'''\,d\alpha'$$

$$= \int \rho(\alpha\alpha'', \beta\beta'')\,x(\beta\beta')\,y(\beta'\beta'')\,d\beta\,d\beta'\,d\beta''$$

$$\iiint \rho(\alpha\alpha', \beta\beta')\,\rho(\alpha'\alpha'', \beta''\beta''')\,y(\beta''\beta''')\,d\beta''\,d\beta'''\,d\alpha' = \int \rho(\alpha\alpha'', \beta\beta''')\,y(\beta'\beta''')\,d\beta'''$$

$$= \int \rho(\alpha\alpha'', \beta\beta''')\,\delta(\beta'-\beta'')\,y(\beta''\beta''')\,d\beta''\,d\beta'''$$

$$\int \rho(\alpha\alpha', \beta\beta')\,\rho(\alpha'\alpha'', \beta''\beta''')\,d\alpha' = \rho(\alpha\alpha'', \beta\beta''')\,\delta(\beta'-\beta'')$$

$$\cancel{\rho = \Sigma_k \phi}$$

$$\rho(\alpha\alpha', \beta\beta') = \Sigma_k\, \phi_k(\alpha\beta)\,\Psi_k(\alpha'\beta')$$

$$\Sigma_{kl}\int \phi_k(\alpha\beta)\,\Psi_k(\alpha'\beta')\,\phi_l(\alpha'\beta'')\,\Psi_l(\alpha''\beta''')\,d\alpha' = \Sigma_k\, \phi_k(\alpha\beta)\,\Psi_k(\alpha''\beta''')\,\delta(\beta'-\beta'')$$

$$= \Sigma_{kl}\, \phi_k(\alpha\beta)\,\Psi_l(\alpha''\beta''')\,\delta_{kl}\,\delta(\beta'-\beta'')$$

$$\int \Psi_k(\alpha'\beta')\,\phi_l(\alpha'\beta'')\,d\alpha' = \delta_{kl}\,\delta(\beta'-\beta'')$$

$$\int \phi_m(\alpha\beta')\,\Psi_k(\alpha'\beta')\,\phi_l(\alpha'\beta'')\,d\alpha'\,d\beta' = \delta_{kl}\int \phi_m(\alpha\beta')\,\delta(\beta'-\beta'')\,d\beta' = \delta_{kl}\,\phi_m(\alpha\beta'')$$

$$= \delta_{kl}\int \phi_m(\alpha'\beta'')\,\delta(\alpha-\alpha')\,d\alpha'$$

If $\phi(x)$ is any of the improper functions we have to deal with $\int_{-\infty}^{\infty} \phi(x)\, f(xy)\, dx$ is always a continuous fn of y regular fn of y for any regular $f(xy)$ and is one of our improper fns if $f(xy)$ is any improper fn.

$$\eta_1(\xi''\xi') = -i\hbar\, \delta'(\xi_1''-\xi_1')\, \delta(\xi_2''-\xi_2')$$

$$\eta_1(\alpha'\xi') = \int (\alpha'|\xi'')\, d\xi''\, .\, \eta_1(\xi''\xi')$$

$$\delta'(\xi'-\xi'')\frac{f(\xi')}{f(\xi'')} = \delta'(\xi'-\xi'')$$

$$= \delta'(\xi'-\xi'') + \delta'(\xi'-\xi'')\frac{\frac{\partial f(\xi'')}{\partial \xi''}(\xi'-\xi'')}{f(\xi'')}$$

$$\eta_r + \frac{\partial f}{\partial \xi_r}$$

$(\xi'|\alpha_1')$ and $(\alpha_2'|\xi')$ are conj imaginaries

$$\int (\alpha_2''|\xi')\, d\xi'\, (\xi'|\alpha_1'') = (\alpha_2'|\alpha_1'')$$

and $$\int (\alpha_2''|\xi')\, d\xi'\, (\xi'|\alpha_1') = (\alpha_2''|\alpha_1')$$

are conj. imaginaries

$$x\,\delta'(x) = -\delta(x)$$

$$\left(\frac{\xi''}{\xi''-\xi'''} - \frac{\xi'''}{\xi'-\xi'''}\right)\delta(\xi''-\xi''')\,\delta(\xi'-\xi''')\, d\xi'''$$

$$= \frac{\xi'^2 - \xi'\xi''' - \xi''\xi''' + \xi'''^2}{(\xi''-\xi''')(\xi'-\xi''')} = 1 + \frac{\xi'(\xi'-\xi'')}{(\xi''-\xi''')(\xi'-\xi''')}$$

$$\int \frac{1}{x} f(x)\, \delta(x)$$

$$f'(x) = -\frac{1}{x} f(x)$$

$$\frac{df}{dx} = -\frac{f}{x}$$

$$\log f = -\log x + \text{const}$$

$$f = \frac{c}{x}$$

$$g(\xi'\xi'') = \int a(\xi'\xi''\alpha'\alpha'')\, g(\alpha'\alpha'')\, d\alpha' d\alpha''$$

$\xi'\ \xi'' = m, n.$
$\alpha'\alpha'' = q$

$$fg(\xi'\xi'') = \int a(\xi'\xi''\alpha'\alpha'')\, d\alpha' d\alpha''\, f(\alpha'\alpha''')\, d\alpha'''\, g(\alpha'''\alpha'')$$

$$= \int f(\xi'\xi''')\, d\xi'''\, g(\xi'''\xi'') = \int a(\xi'\xi'''\alpha'\alpha'')\, f(\alpha'\alpha'')\, d\alpha' d\alpha''\, d\xi'''\, a(\xi'''\xi''\alpha'''\alpha^{IV})\, g(\alpha'''\alpha^{IV})\, d\alpha''' d\alpha^{IV}$$

$$\left[a(\xi'\xi''\alpha'\alpha^{IV})\, \delta(\alpha''-\alpha''') = \int a(\xi'\xi'''\alpha'\alpha'')\, d\xi'''\, a(\xi'''\xi''\alpha'''\alpha^{IV}). \right]$$

$$\left[a_{mn}(q'q^{IV})\, \delta(q''-q''') = \sum_k a_{mk}(q'q'')\, a_{kn}(q'''q^{IV}) \right.$$

Suppose $g(\alpha'\alpha'') = g(\alpha')\,\delta(\alpha'-\alpha'')$ $\quad f(\alpha'\alpha'') = f(\alpha')\,\delta(\alpha'-\alpha'')$.

$$\int a(\xi'\xi''\alpha'\alpha')\, f(\alpha')\, g(\alpha')\, d\alpha' = \int a(\xi'\xi'''\alpha')\, f(\alpha')\, d\alpha'\, d\xi'''\, a(\xi'''\xi''\alpha''')\, g(\alpha''')\, d\alpha'''.$$

$$a(\xi'\xi''\alpha')\,\delta(\alpha'-\alpha'') = \int a(\xi'\xi'''\alpha')\, d\xi'''\, a(\xi'''\xi''\alpha'')$$

$$a_{mn}(q')\,\delta(q'-q'') = \sum_k a_{mk}(q')\, a_{kn}(q'')$$

~~$a(\xi'\xi''\alpha')$~~

$$a_{mk}(q') = \frac{\partial\psi_m}{\partial t}\bar\psi_k - \frac{\partial\bar\psi_k}{\partial t}\psi_m + \gamma\bar\psi_k\psi_m$$

~~$a_{kn}(q'')$~~

$$\sum_k a_{mk}(q')\, a_{kn}(q'') = \sum_k\left(\frac{\partial\psi_m}{\partial t}\bar\psi_k - \frac{\partial\bar\psi_k}{\partial t}\psi_m + \gamma\bar\psi_k\psi_m\right)\left(\frac{\partial\psi_k'}{\partial t}\bar\psi_n' - \frac{\partial\bar\psi_n'}{\partial t}\psi_k' + \gamma\bar\psi_n'\psi_k'\right)$$

$$= \gamma^2\bar\psi_n\psi_m\,\delta(q'-q'') - \gamma\left(\psi_m\frac{\partial\bar\psi_n}{\partial t} - \frac{\partial\psi_m}{\partial t}\bar\psi_n\right)\delta(q'-q'') - \frac{\partial\psi_m}{\partial t}\frac{\partial\bar\psi_n}{\partial t}\delta(q'-q'')$$

$$+ \frac{\partial\psi_m}{\partial t}\bar\psi_n'\sum_k\bar\psi_k\frac{\partial\psi_k'}{\partial t} + \gamma\psi_m\bar\psi_n'\sum_k\bar\psi_k\frac{\partial\psi_k'}{\partial t} - \gamma\psi_m\bar\psi_n'\sum_k\frac{\partial\bar\psi_k}{\partial t}\psi_k' - \sum_k\frac{\partial\bar\psi_k}{\partial t}\frac{\partial\psi_k'}{\partial t}\psi_m\bar\psi_n'$$

$$+ \frac{\partial\bar\psi_n'}{\partial t}\psi_m\sum_k\psi_k'\frac{\partial\bar\psi_k}{\partial t}$$

$$a_{mn}(q'q^{IV})\,\delta(q''-q''') = \sum_k a_{mk}(q'q'')\, a_{kn}(q'''q^{IV})$$

Put $a_{mn}(q'q'') = \sum_\lambda f_{m\lambda}(q')\, g_{n\lambda}(q'')$

$$\sum_\lambda f_{m\lambda}(q')\, g_{n\lambda}(q'')\cdot\delta(q''-q''') = \sum_\lambda f_{m\lambda}(q')\sum_{\lambda'}\sum_k g_{k\lambda}(q'')\, f_{k\lambda'}(q''')\, g_{n\lambda'}(q^{IV}).$$

$$= \sum_{\lambda,\lambda'} f_{m\lambda}(q')\, g_{n\lambda'}(q^{IV})\cdot\sum_k g_{k\lambda}(q'')\, f_{k\lambda'}(q''').$$

$$\int a_{mn}(q'q'')\, dq' = b_{mn}(q'')$$

$$b_{mn}(q^{IV}) = \sum_k b_{mk}(q'')\, b_{kn}(q^{IV})$$

$$\cancel{a_m(\tau) = \delta_{m0} + \frac{1}{i\hbar c}\eta_{m0}\int_0^\tau \kappa(t)e^{i\omega_{m0}t}dt - \frac{1}{\hbar^2c^2}\sum_n \eta_{mn}\eta_{n0}\int_0^\tau \kappa(t)e^{i\omega_{mn}t}dt\int_0^t \kappa(s)e^{i\omega_{n0}s}ds}$$

$\cancel{\dot a_n(\tau) =}$ $a_m = c_m + c_m' + c_m'' + c_m''' + \cdots$

$$i\hbar c\,\dot c_m^{(r)} = \sum_n c_n^{(r-1)}\kappa\,\eta_{mn}e^{i\omega_{mn}t} \qquad c_m^{(r)}(T) = \frac{1}{i\hbar c}\sum_n \eta_{mn}\int_0^T \kappa(\tau)e^{i\omega_{mn}\tau}d\tau\; c_n^{(r-1)}(\tau)$$

$$c_m'''(T) = \frac{1}{(i\hbar c)^3}\sum_{nkj}\eta_{mn}\eta_{nk}\eta_{kj}c_j\int_0^T \kappa(\tau)e^{i\omega_{mn}\tau}d\tau\int_0^\tau \kappa(t)e^{i\omega_{nk}t}dt\int_0^t \kappa(s)e^{i\omega_{kj}s}ds.$$

If $\cancel{c_0=1}$ $c_m = 0$ $(m\neq 1)$

$(m\neq 1)$ at time T

$$c_m' = \frac{c_0}{i\hbar c}\eta_{m0}\int_0^T \kappa(\tau)e^{i\omega_{m0}\tau}d\tau \qquad c_m'' = \frac{c_0}{(i\hbar c)^2}\sum_n \eta_{mn}\eta_{n0}\int_0^T \kappa(\tau)e^{i\omega_{mn}\tau}d\tau\int_0^\tau \kappa(t)e^{i\omega_{n0}t}dt$$

$$c_m''' = \frac{c_0}{(i\hbar c)^3}\sum_{nk}\eta_{mn}\eta_{nk}\eta_{k0}\int_0^T \kappa(\tau)e^{i\omega_{mn}\tau}d\tau\int_0^\tau \kappa(t)e^{i\omega_{nk}t}dt\int_0^t \kappa(s)e^{i\omega_{k0}s}ds.$$

$$\bar c_m' c_m'' = \frac{|c_0|^2}{i\hbar^3c^3}\sum_n \eta_{mn}\eta_{n0}\eta_{0m}\int_0^T \kappa(\tau)e^{-i\omega_{m0}\tau}d\tau\cdot\int_0^T \kappa(\tau)e^{i\omega_{mn}\tau}d\tau\int_0^\tau \kappa(t)e^{i\omega_{n0}t}dt.$$

$$(X t' - t' X)(\alpha\alpha'') = \int \Big[X(\alpha\alpha')\, \delta(\alpha_1'-\alpha_1'')\,\delta(\alpha_2'-\alpha_2'')\,\delta(\alpha_3'-\alpha_3'')\,\delta'(\alpha_4'-\alpha_4'')$$

$$- \delta(\alpha_1-\alpha_1')\,\delta(\alpha_2-\alpha_2')\,\delta(\alpha_3-\alpha_3')\,\delta'(\alpha_4-\alpha_4')\, X(\alpha'\alpha'') \Big]\, d\alpha_1' d\alpha_2' d\alpha_3' d\alpha_4'$$

$$= \int \Big[X(\alpha, \alpha_1''\alpha_2''\alpha_3''\alpha_4')\, \delta'(\alpha_4'-\alpha_4'') - X(\alpha_1 \alpha_2 \alpha_3 \alpha_4', \alpha'')\, \delta'(\alpha_4-\alpha_4') \Big]\, d\alpha_4'$$

$$= \int \Big[-\frac{\partial}{\partial\alpha_4'} X(\alpha, \alpha_1''\alpha_2''\alpha_3''\alpha_4') . \delta(\alpha_4'-\alpha_4'') - \frac{\partial}{\partial\alpha_4'} X(\alpha_1 \alpha_2 \alpha_3 \alpha_4', \alpha'') . \delta(\alpha_4-\alpha_4'') \Big]\, d\alpha_4'$$

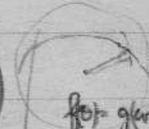
$$= -\Big[\frac{\partial}{\partial\alpha_4'} X(\alpha, \alpha_1''\alpha_2''\alpha_3''\alpha_4')\Big]_{\alpha_4'=\alpha_4} + \Big[\frac{\partial}{\partial\alpha_4'} X(\alpha_1''\alpha_2''\alpha_3''\alpha_4', \alpha'')\Big]_{\alpha_4'=\alpha_4''}$$

$$= -\frac{\partial}{\partial\alpha_4''} X(\alpha\alpha'') - \frac{\partial}{\partial\alpha_4} X(\alpha\alpha'')$$

If this $= 0$ then $X(\alpha\alpha'')$ involves α_4 and α_4'' only through their differences

$-\frac{\cos\theta}{\sin\theta} \mp \frac{1}{\sin\theta}$

dS . $r^2 dr\, d\theta\, d\phi$. $r\,d\theta$



$$\int\left(\frac{\partial V}{\partial n}\, dS\right) = \int\left(\frac{\partial V}{\partial r}\, d\theta\, d\phi\, r^2\sin\theta + \frac{\partial V}{r\,\partial\theta}\, dr\, r\sin\theta\, d\phi + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\, dr\, r\, d\theta\right)$$

$$= \int\left\{\frac{\partial}{\partial r}\left(\frac{\partial V}{\partial r} r^2\sin\theta\right) + \frac{\partial}{\partial\theta}\left(\frac{\partial V}{\partial\theta}\sin\theta\right) + \frac{\partial}{\partial\phi}\left(\frac{\partial V}{\partial\phi}\frac{1}{\sin\theta}\right)\right\} dr\, d\theta\, d\phi$$

$$\nabla^2 V = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}\left(\frac{\partial V}{\partial r} r^2\sin\theta\right) + \frac{\partial}{\partial\theta}\left(\frac{\partial V}{\partial\theta}\sin\theta\right) + \frac{\partial}{\partial\phi}\left(\frac{\partial V}{\partial\phi}\frac{1}{\sin\theta}\right)\right\}$$

$$= \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

$f(\theta) = g(\cos\theta)$

$\frac{\partial f}{\partial\theta} = -g' . \sin\theta$

$f'\sin\theta = -g'\sin^2\theta$

$\frac{\partial}{\partial\theta}(f'\sin\theta) = +g''\sin^3\theta - g' . 2\sin\theta\cos\theta$

$S = f(\theta)e^{im\phi}$

$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} = \cancel{k(k+1)}\,\frac{V}{r^2}$

$\frac{\partial^2}{\partial r^2}(rV) = \cancel{k(k+1)}\, k^2\, rV$

$rV = e^{ikr}$

$\left(\frac{\partial S}{\partial r}\right)^2 + \frac{\partial^2 S}{\partial r^2} + \frac{2}{r}$

$V = e^S$

$\frac{\partial V}{\partial r} = e^S\frac{\partial S}{\partial r}$

$\frac{\partial^2 V}{\partial r^2} = e^S\frac{\partial^2 S}{\partial r^2} + e^S\left(\frac{\partial S}{\partial r}\right)^2$

$V = r^m$

$m(m-1) + 2m = k(k+1)$

$m = k$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 S}{\partial\phi^2} + k(k+1)S = 0$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(f'\sin\theta) - \frac{m^2}{\sin^2\theta}f = k(k+1)f$$

$f(\theta) = g(\cos\theta) = g(x)$

$$g''(1-x^2) - 2g'.x - \frac{m^2 g}{1-x^2} = k(k+1)g$$

$$x_1(x_2yz + yzx_2 + zx_2y) - x_1(yx_2z + x_2zy + zyx_2)$$
$$(x_1yz + yzx_1 + zx_1y)x_2 - (yx_1z + x_1zy + zyx_1)x_2$$
$$(x_1x_2yz + yzx_1x_2 + zx_1x_2y) - (x_1x_2zy + zyx_1x_2 + yx_1x_2z) = \cancel{(d)^2[x_1x_2,[y,z]]}$$

$x_1 x_2$

theory (e.g. the question; - What is the fraction of the

$F_{x_1 y}$

$D \sin an = |a| \sin an$

$D \cos an = |a| \cos an$

$D e^{ian} = a e^{ian}$

$D e^{-ian} = -a e^{-ian}$

Discuss general transf. to variables that satisfy eqn. of motion for a particular Hamiltonian. This transf. theory will probably fit in better with the Hamiltonian equations

$F(p\,q\,W\,t) = 0 \qquad W = -p_0 \quad t = q_0$

$F(p\,q) = 0$

$XF = 0$

$$\frac{dq_r}{ds} = \frac{\partial (XF)}{\partial p_r} = \frac{\partial X}{\partial p_r} F + X \frac{\partial F}{\partial p_r}$$

$\bar p_r = p_r + \theta_r F$

$\bar q_r = q_r + \theta'_r F$

$F(p_r q_r) = \bar F(\bar p_r \bar q_r)$

$$\frac{\partial F}{\partial q_r} = \frac{\partial F}{\partial \bar q_s}\frac{\partial \bar q_s}{\partial q_r} + \frac{\partial F}{\partial \bar p_s}\frac{\partial \bar p_s}{\partial q_r} = \frac{\partial F}{\partial \bar q_s}\left(\delta_{rs} - \theta'_s \frac{\partial \bar F}{\partial q_r}\right) + \frac{\partial F}{\partial \bar p_s}\theta_s \frac{\partial F}{\partial q_r}$$

$$\frac{\partial F}{\partial q_r} = \frac{\partial F}{\partial \bar q_r} + \theta'_s \frac{\partial F}{\partial q_s}\frac{\partial \bar F}{\partial q_r} + \theta_s \frac{\partial F}{\partial p_s}\frac{\partial F}{\partial q_r} = \frac{\partial \bar F}{\partial q_r}\left(1 + \theta'_s \frac{\partial F}{\partial q_s} + \theta_s \frac{\partial F}{\partial p_s}\right)$$

$$\frac{\partial \bar p_r}{\partial p_t}\frac{\partial \bar q_s}{\partial q_t} - \frac{\partial \bar p_r}{\partial q_t}\frac{\partial \bar q_s}{\partial p_t} = \left(\delta_{rt} + \theta_r \frac{\partial F}{\partial p_t}\right)\left(\delta_{st} + \theta'_s \frac{\partial F}{\partial q_t}\right) - \theta_r \frac{\partial F}{\partial q_t}\cdot \theta'_s \frac{\partial F}{\partial p_t}$$

$$= \delta_{rs} + \theta'_s \frac{\partial F}{\partial q_r} + \theta_r \frac{\partial F}{\partial p_s}$$

$$\frac{\partial x}{\partial q_r}\frac{\partial y}{\partial p_r} = \left\{\frac{\partial x}{\partial \bar q_s}\left(\delta_{rs} + \theta'_s \frac{\partial F}{\partial q_r}\right) + \frac{\partial x}{\partial \bar p_s}\theta_s \frac{\partial F}{\partial q_r}\right\}\left\{\frac{\partial y}{\partial \bar p_t}\left(\delta_{rt} + \theta_t \frac{\partial F}{\partial p_r}\right) + \frac{\partial y}{\partial \bar q_t}\theta'_t \frac{\partial F}{\partial p_r}\right\}$$

$$\frac{\partial F}{\partial q_r}\frac{\partial F}{\partial p_r}\left(\frac{\partial x}{\partial \bar q_s}\theta'_s + \frac{\partial x}{\partial \bar p_s}\theta_s\right)\left(\frac{\partial y}{\partial \bar q_t}\theta'_t + \frac{\partial y}{\partial \bar p_t}\theta_t\right)$$

$$= \frac{\partial x}{\partial \bar q_r}\frac{\partial y}{\partial \bar p_r} + \frac{\partial x}{\partial \bar q_r}\frac{\partial y}{\partial \bar p_t}\theta_t \frac{\partial F}{\partial p_r} + \frac{\partial x}{\partial \bar q_r}\frac{\partial y}{\partial \bar q_t}\theta'_t \frac{\partial F}{\partial p_r} + \frac{\partial x}{\partial \bar q_s}\theta'_s \frac{\partial F}{\partial q_r}\frac{\partial y}{\partial \bar p_r} + \frac{\partial x}{\partial \bar p_s}\theta_s \frac{\partial F}{\partial q_r}\frac{\partial y}{\partial \bar p_r} + \left(\frac{\partial x}{\partial \bar q_s}\theta'_s \frac{\partial F}{\partial q_r} + \frac{\partial x}{\partial \bar p_s}\theta_s \frac{\partial F}{\partial q_r}\right)\left(\frac{\partial y}{\partial \bar q_t}\theta'_t \frac{\partial F}{\partial p_r} + \frac{\partial y}{\partial \bar p_t}\theta_t \frac{\partial F}{\partial p_r}\right)$$

$$= \frac{\partial x}{\partial \bar q_r}\frac{\partial y}{\partial \bar p_r} + \frac{\partial x}{\partial \bar q_r}\frac{\partial y}{\partial \bar p_s}\left(\frac{\partial F}{\partial p_r}\theta_s + \frac{\partial F}{\partial q_s}\theta'_r\right) + \frac{\partial x}{\partial \bar q_s}\frac{\partial y}{\partial \bar q_r}\theta'_r \frac{\partial F}{\partial p_s} + \frac{\partial x}{\partial \bar p_s}\frac{\partial y}{\partial \bar q_r}\theta_s \frac{\partial F}{\partial q_r} + \text{(ditto) interch. } x+y.$$

$F(\xi_r) = 0$

$\bar\xi_r = \xi_r + \theta_r F$

$$\frac{\partial F}{\partial \bar\xi_r} = \frac{\partial F}{\partial \xi_r}\left(1 - \frac{\partial F}{\partial \bar\xi_s}\theta_s\right)$$

$$\frac{\partial x}{\partial \xi_r} = \frac{\partial x}{\partial \bar\xi_r} + \frac{\partial x}{\partial \bar\xi_s}\theta_s \frac{\partial F}{\partial \xi_r} = \frac{\partial x}{\partial \bar\xi_r} + \frac{\partial x}{\partial \bar\xi_s}(\bar\xi_s - \xi_s)\frac{1}{F}\frac{\partial F}{\partial \xi_r}$$

$$\frac{\partial x}{\partial \xi_r} - \frac{\partial x}{\partial \bar\xi_r} = (\bar\xi_s - \xi_s)\frac{\partial x}{\partial \bar\xi_s}\frac{1}{F}\frac{\partial F}{\partial \xi_r}$$

$$\frac{\partial x}{\partial \bar\xi_r} - \frac{\partial x}{\partial \xi_r} = (\xi_s - \bar\xi_s)\frac{\partial x}{\partial \xi_s}\frac{1}{F}\frac{\partial F}{\partial \bar\xi_r}$$

$\dfrac{\partial F / \partial \bar\xi_r}{\partial F / \partial \xi_r}$ is ind. of r.

$$\frac{\bar\xi_s - \xi_s}{F}\left(\frac{\partial x}{\partial \bar\xi_s}\frac{\partial F}{\partial \xi_r} - \frac{\partial x}{\partial \xi_s}\frac{\partial F}{\partial \bar\xi_r}\right) = 0$$

$\delta F = \frac{\partial F}{\partial \xi_1}\delta\xi_1 = -\frac{\partial F}{\partial \xi_2}\delta\xi_2$

$\delta x = \frac{\partial x}{\partial \xi_1}\delta\xi_1 + \frac{\partial x}{\partial \xi_2}\delta\xi_2 \qquad \dfrac{\delta x}{\delta F} = \dfrac{\partial x/\partial \xi_1}{\partial F/\partial \xi_1} - \dfrac{\partial x/\partial \xi_2}{\partial F/\partial \xi_2}$

$$[x,F] = \overline{[x,F]} + \frac{\partial F}{\partial \bar p_s}\left\{\frac{\partial x}{\partial \bar q_r}\left(\frac{\partial F}{\partial p_r}\theta_s + \frac{\partial F}{\partial q_s}\theta'_r\right)\right\} + \frac{\partial x}{\partial \bar q_s}\frac{\partial F}{\partial q_r}\theta'_r + \frac{\partial x}{\partial \bar p_r}\theta_r \frac{\partial F}{\partial q_s} - \frac{\partial F}{\partial \bar q_s}\{\}$$

$$= \overline{[x,F]} + \frac{\partial F}{\partial \bar p_s}\frac{\partial F}{\partial p_r}\frac{\partial x}{\partial \bar q_r}\theta_s + \frac{\partial F}{\partial p_r}\frac{\partial F}{\partial \bar q_s}\frac{\partial x}{\partial \bar q_r}\theta'_s - \frac{\partial F}{\partial \bar q_s}\frac{\partial F}{\partial q_r}\frac{\partial x}{\partial \bar p_r}\theta'_s - \frac{\partial F}{\partial q_r}\frac{\partial F}{\partial \bar p_s}\frac{\partial x}{\partial \bar p_r}\theta_s$$

$$= \overline{[x,F]} + \left(\frac{\partial F}{\partial p_r}\frac{\partial x}{\partial \bar q_r} - \frac{\partial F}{\partial q_r}\frac{\partial x}{\partial \bar p_r}\right)\left(\frac{\partial F}{\partial \bar p_s}\theta_s + \frac{\partial F}{\partial \bar q_s}\theta'_s\right)$$

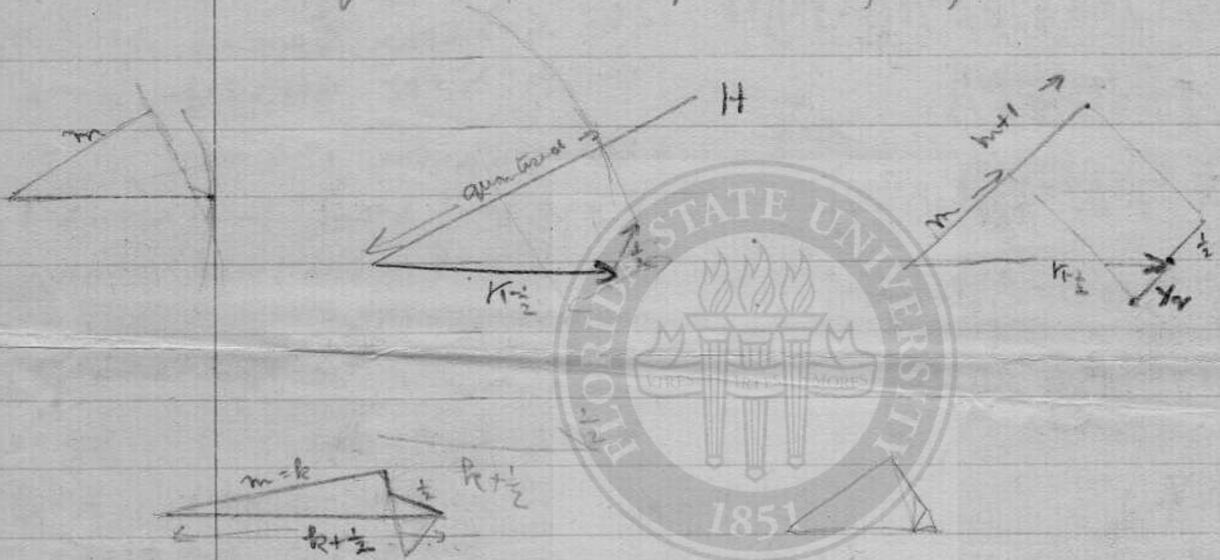
$$= \overline{[x,F]}\left\{1 + \frac{\partial F}{\partial \bar p_s}\theta_s + \frac{\partial F}{\partial \bar q_s}\theta'_s\right\}$$

is constant, but will be increased by $-eE_x$, and p_x will be increased by the same amount. Hence from (IV)

$$\dot{p}_x =$$

The Hamiltonian equations thus still hold with the new Hamiltonian H'.

Equation (III) can no longer be exactly true, since H can



$(q'|\xi') = a(q',\xi')\, e^{i\phi(q',\xi')/\hbar}$

$(\xi'|q') = a\, e^{-i\phi/\hbar}$

$a^2 dq'$ is the volume of η-space for which $q' < q < q'+dq'$ when $\xi = \xi'$

This volume is symmetrical about the point $\eta = \frac{\partial\phi}{\partial\xi'} = \eta_0$

$\delta(q-q') = (\xi'|q')(q'|\xi'')$

$[\eta\,\delta(q-q') + \delta(q-q')\eta](\xi'|\xi'')$

$= -i\hbar \int [\delta'(\xi'-\xi''')(\xi'''|q')(q'|\xi'') + (\xi'|q')(q'|\xi''')\delta'(\xi'''-\xi'')]\, d\xi'''$

$= -i\hbar\left\{\frac{\partial}{\partial\xi'}[(\xi'|q')(q'|\xi'')] - \frac{\partial}{\partial\xi''}[(\xi'|q')(q'|\xi'')]\right\}$

$= -i\hbar(\xi'|q')(q'|\xi'')\left[\frac{\partial}{\partial\xi'}\log(\xi'|q') - \frac{\partial}{\partial\xi''}\log(q'|\xi'')\right]$

$= -i\hbar(\xi'|q')(q'|\xi'')\left[\frac{\partial}{\partial\xi'}\log a(q'\xi') - \frac{\partial}{\partial\xi''}\log a(q'\xi'')\right]$

$+(\xi'|q')(q'|\xi'')\left[\frac{\partial}{\partial\xi'}\phi(q'|\xi') + \frac{\partial}{\partial\xi''}\phi(q'|\xi'')\right]$

$H_T = H_0 + \Sigma_n H(n) + \Sigma_{mn} V(mn)$

$V(r_1r_2\ldots; s_1s_2\ldots) = 0$ except when 2 and only 2 s's differ from the corr. r's.

$= V_{r_m r_n s_m s_n}$ when $s_m \neq r_m$, $s_n \neq r_n$ and every other $s_k = r_k$

$i\hbar\dot b(r_1r_2\ldots) = \ldots + \Sigma_{mn}\Sigma_{s_m\neq r_m}\Sigma_{s_n\neq r_n} V_{r_m r_n s_m s_n}\, b(r_1r_2..s_m..s_n..r_k)$

$= \ldots + \sum_{s_m s_n} \frac{N_{r_m}N_{r_n}}{2}\,[N_1!\,N_2!\ldots(N_{r_m}-1)!\ldots(N_{r_n}-1)!\ldots(N_{s_m}+1)!\ldots(N_{s_n}+1)!\ldots/N!]^{\frac12}\; b(N_1N_2\ldots N_{r_m}-1\ldots N_{r_n}-1\ldots N_{s_m}+1\ldots N_{s_n}+1\ldots)\,V_{r_m r_n s_m s_n}$

$(ab)_1 = a_1b_1 + a_2b_2 \qquad \bar a_1 = \lambda_1a_1+\lambda_2a_2$

$(ab)_2 = a_1b_2 + a_2b_1 \qquad \bar a_2 = \mu_1a_1+\mu_2a_2$

$\overline{(ab)}_1 = \lambda_1(ab)_1+\lambda_2(ab)_2 = \lambda_1(a_1b_1+a_2b_2)+\lambda_2(a_1b_2+a_2b_1)$

$=(\bar a\bar b)_1 = \bar a_1\bar b_1+\bar a_2\bar b_2 = (\lambda_1a_1+\lambda_2a_2)(\lambda_1b_1+\lambda_2b_2) + (\mu_1a_1+\mu_2a_2)(\mu_1b_1+\mu_2b_2)$

$\overline{(ab)}_2 = \mu_1(ab)_1+\mu_2(ab)_2 = \mu_1(a_1b_1+a_2b_2)+\mu_2(a_1b_2+a_2b_1)$

$=(\bar a\bar b)_2 = \bar a_1\bar b_2+\bar a_2\bar b_1 = (\lambda_1a_1+\lambda_2a_2)(\mu_1b_1+\mu_2b_2) + (\mu_1a_1+\mu_2a_2)(\lambda_1b_1+\lambda_2b_2)$

a_1b_1	$\lambda_1 = \lambda_1^2+\mu_1^2$	1)	$\mu_1 = 2\lambda_1\mu_1$	5) ✓
a_1b_2	$\lambda_2 = \lambda_1\lambda_2+\mu_1\mu_2$	2)	$\mu_2 = \lambda_1\mu_2+\lambda_2\mu_1$	6)
a_2b_1	$\lambda_2 = \lambda_1\lambda_2+\mu_1\mu_2$	3)	$\mu_2 = \lambda_2\mu_1+\lambda_1\mu_2$	7) ✓
a_2b_2	$\lambda_1 = \lambda_2^2+\mu_2^2$	4)	$\mu_1 = 2\lambda_2\mu_2$	8)

5) $\lambda_1 = \frac12$ or $\mu_1 = 0$

If $\mu_1 = 0$, from 1) $\lambda_1 = 0$ or 1 from 2) $\lambda_2 = 0$ or $\lambda_1 = 1$

from 6) $\mu_2 = 0$ or $\lambda_1 = 1$ from 8) $\lambda_2 = 0$ or $\mu_2 = 0$

$\lambda_1 = 0,\ \lambda_2 = 0,\ \mu_2 = 0$ | $\lambda_1 = 1 \quad \mu_2 = 0 \quad \lambda_2 = \pm1$ or $\lambda_2 = 0 \quad \mu_2 = \pm1$

If $\lambda_1 = \frac12$ from 1) $\mu_1 = \pm\frac12$ from 2) $\lambda_2 = 2\mu_1\mu_2 = \pm\mu_2$

from 6) $\mu_2 = 2\lambda_2\mu_1 = \pm\lambda_2 \qquad \lambda_2^2 = \mu_2^2$

from 4) $\frac12 = 2\lambda_2^2$ from 8) $\pm\frac12 = \pm 2\lambda_2^2$

$\lambda_2^2 = \mu_2^2 = \frac14 \qquad \mu_1 = \frac12,\ \lambda_2 = \mu_2 = \frac12;\quad \mu_1 = -\frac12,\ \lambda_2 = -\mu_2 = \frac12$

$\bar a_1 = \frac12(a_1+a_2) \qquad \bar a_2 = \frac12(a_1-a_2)$

$\bar b_1 = \frac12(b_1+b_2) \qquad \bar b_2 = \frac12(b_1-b_2)$

$\overline{(ab)}_1 = \frac12(a_1b_1+a_2b_2)+\frac12(a_1b_2+a_2b_1) = \frac12(a_1+a_2)(b_1+b_2)$

$\overline{(ab)}_2 = \frac12(a_1b_1+a_2b_2)-\frac12(a_1b_2+a_2b_1)$

$a_{n,n+1}\psi_{n+1} = \lambda\psi_n$

$\cancel{b_n}\quad \frac{\psi_{n+1}}{\psi_n} = \frac{\lambda}{b_n}$

Use only incomplete fn system satisfying

$F\psi_n = 0$.

$\Sigma_m W_n a_{nm}\psi_m = \lambda\psi_n \qquad a_{nm} = \delta_{nm} + \varepsilon_{nm}$

$\cancel{x_{11} f_{11} + x_{21} f_{21}}$

$(\lambda - W_n)\psi_n = \Sigma_m \varepsilon_{nm}\psi_m$

$f_{11}x_{11}g_{11} + f_{11}x_{12}g_{21} + f_{12}x_{21}g_{11} + f_{12}x_{22}g_{21}$

$\lambda = W_k \qquad \Sigma_m \varepsilon_{km}\psi_m = 0$

$f_{11}g_{11}f_{21}g_{22}$

All ψ's small except $\psi_k \qquad \varepsilon_{kk} = 0$

$= \alpha\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{vmatrix}$

$\Sigma_r b_r(\alpha\xi)\, c_r(\beta\eta)$

$$\begin{array}{cc} (11) & (21) \\ \end{array}\begin{vmatrix} a_{11}x_{11} + a_{12}x_{12} + a_{22}x_{22} + a_{21}x_{21} & b_{11}x_{11} + \cdots \\ c_{11}x_{11} + \cdots & d_{11}x_{11} + \cdots \end{vmatrix}\begin{array}{cc} (12) & (22) \end{array}$$

$x_{11}^2 \qquad x_{11}x_{12}$

✓ $a_{11}d_{11} = b_{11}c_{11} \qquad a_{11}d_{12} + a_{12}d_{11} - b_{11}c_{12} - b_{12}c_{11} = 0$

$a_{12}d_{12} = b_{12}c_{12} \qquad a_{22}d_{12} + a_{12}d_{22} - \cdots = 0$

$a_{rs}d_{rs} = b_{rs}c_{rs} \qquad a_{11}d_{22} + a_{22}d_{11} - b_{11}c_{22} - b_{22}c_{11} = C$

✓ $\qquad a_{12}d_{21} + a_{21}d_{12} - b_{12}c_{21} - b_{21}c_{12} = -C$

$a_{11} = f_{11}g_{11}$

$b_{11} = f_{21}g_{11}$

$c_{11} = f_{11}g_{12}$

$d_{11} = f_{21}g_{12}$

$\begin{vmatrix} b & 2 & - & - & 1 \\ rs & - & rs & - \end{vmatrix}$

$a_{12} = f_{11}g_{21}$

$a_{12}f_{21}g_{12} - b_{12}f_{11}g_{12} - c_{12}f_{21}g_{11} + d_{12}f_{11}g_{11} = 0 \qquad \frac{a_{12}}{b_{12}} = \frac{c_{12}}{d_{12}} = \gamma_{12}$

$(f_{21}a_{12} - f_{11}b_{12})g_{12} + (f_{11}d_{12} - f_{21}c_{12})g_{11} = 0$

$(f_{21}\gamma_{12} - f_{11})(b_{12}g_{12} - d_{12}g_{11}) = 0$

$(f_{22}\gamma_{12} - f_{12})(b_{12}g_{22} - d_{12}g_{21}) = 0$

$a_{22} = f_{12}g_{21}$

$b_{22} = f_{22}g_{21}$

$c_{22} = f_{12}g_{22}$

$d_{22} = f_{22}g_{22}$

$t' = t - \frac{x}{c}$

$J_1 \; J_2 \ldots \quad W$

$w_1 \; w_2 \ldots \quad t'$

$A = \frac{d^{k+m}}{dx^{k+m}}(1-x^2)^k \qquad B = \frac{d^{k+m+1}}{dx^{k+m+1}}(1-x^2)^{k+1}$

$C = \frac{d^{k+m-1}}{dx^{k+m-1}}(1-x^2)^{k-1}$

$t_1' = t' + a \cos \nu t'$

a, ν c-numbers

$W_1 = W \frac{1}{1 - a\nu \sin \nu t'}$; $[W_1, t_1] = [W, t_1] \frac{1}{1-a\nu \sin \nu t'} = (1 - a\nu \sin \nu t') \frac{1}{1 - a\nu \sin \nu t'}$

$J_{r_1} = J_r \frac{1}{1 - a\nu_r \sin \nu t'}$

$w_{r_1} = w_r (1 - a\nu_r \sin \nu t')$

$B = \frac{d^{k+m-1}}{dx^{k+m-1}} \frac{d^2}{dx^2}\left[(1-x^2)^{k-1}\{1-x^2\}^2\right]$

$W_1 = W_0 \frac{\partial J_0}{\partial J_1}$

$w_0 = \overline{w}_r \frac{\partial J_r}{\partial J_0}$

$J = q$

$w = -p$

$H_0(q_r) \frac{1}{1 - a\nu_r \sin \nu t'} + \frac{\partial S}{\partial t_1} = 0$

$S = H_0 f(t_1) + \Sigma \alpha_r q_r$

$Q_r =$ $\overline{H} = 1 \quad \overline{W} = 0$

$Q_r = q_r \quad \beta_r = q_r$

$P_r =$

$p_r = \frac{\partial H_0}{\partial q_r} f(t_1) + \alpha_r$

$\overline{J}_r = J_r$

$\overline{w}_r = -\overline{\alpha}_r = -p_r + \frac{\partial H_0}{\partial q_r} f(t_1) = w_r + \frac{\partial H_0}{\partial J_r} f(t_1)$

$f'(t_1) = \frac{1}{1 - a\nu_r \sin \nu t'}$

$\overline{W} = W + f(J_r)$

$\overline{J}_r = J_r$

$\cancel{\overline{w}_r = w_r + t}$

$w_0 = \overline{w}_0 + \overline{E} \frac{\partial f}{\partial J_0}$

$t = t$

$\overline{W} = W - H_0 f'(t_1)$

$[\overline{w}_r, \overline{W}] = -\frac{\partial H_0}{\partial J_r} f'(t_1) + \frac{\partial H_0}{\partial J_r}[f(t_1), W]$

In physical interpretation, instead of substituting c-number values, substitute c-equation i.e. equation disagreeing with quantum conditions.

$\underset{(w)}{W e^{i\nu t}} = e^{i\nu t} \underset{(J+1)}{W}$

If t occurs only through $e^{i\nu t}$ would Hamiltonian eqn be consistent with Q conditions?

$H \Sigma \frac{(\dot{w}t+\epsilon)^n}{n!}$

$ab + ba = 0 \qquad \underline{a}$ diagonal

$a_{mm} b_{mn} + b_{mn} a_{nn} = 0$

$b_{mn} = 0$ or $a_{mm} = -a_{nn}$

$K\alpha + \alpha K = 0$

$\alpha_{11} = -\alpha_{22}$

$0 = K_{11} = K_{22}$

$H(w) - wH = \dot{w}$

$H(\dot{w}t+\epsilon) - (\dot{w}t+\epsilon)H = \dot{w}$

$F = \left(ih \frac{\partial}{\partial x} + K_x\right)^2 + \ldots$

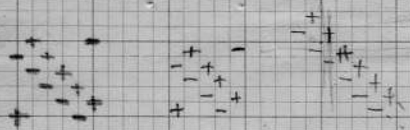
F

$\phi(f, g) = \Sigma_{mn} a_{mn} f_m \overline{g}_n$

$\phi(f, Tg) = \Sigma_{mnk} f_m a_{mn} \overline{T}_{nk} \overline{g}_k$

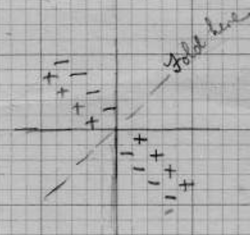
$\phi(T^x f, g) = \Sigma_{mnk} T^x{}_{mn} f_n a_{nk} \overline{g}_k$

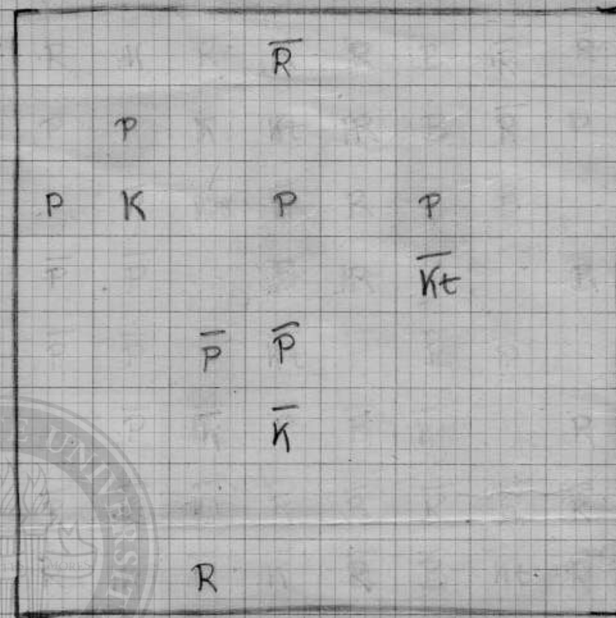
$a_{nm} \overline{T}_{mk} = T^{x}{}_{mn} a_{nk}$



$$\int \delta'(\alpha - \alpha'')\,\delta'(\alpha'' - \alpha')\,d\alpha''$$

$$= \int \delta'(\alpha'' - \alpha)\,\delta'(\alpha' - \alpha'')\,d\alpha''$$





$\mathfrak{H} = \text{fn}(X\,P)$

$H = \text{fn}(x\,\mu\,X\,P)$

$(\mathfrak{H} + \varepsilon H - W)\psi_n = 0$

$\psi_n = \psi_n' + \varepsilon\psi_n''$

$(\mathfrak{H} - W_n)\psi_n' = 0.$

$\psi = \psi_n' + \varepsilon\psi_n''$

$\cancel{(W_1 + H)\psi_n' = (\mathfrak{H} - W)\psi_n''}$

$W_n\psi_n' - W\psi_n' + \varepsilon H\psi_n' + \varepsilon(\mathfrak{H} - W)\psi_n'' = 0$

$(T_2 + \varepsilon H_1 + \varepsilon^2 V_2 - W)\psi = 0$

$H\psi_n' + (\mathfrak{H} - W)\psi_n'' = 0$

$\psi = \psi_n' + \varepsilon\psi_n''$

$T_2\psi_n'' + (H_1 - W)\psi_n' = 0$

ψ_n' is fn of x_i's only

$\psi = \psi_1\psi_2$ $\quad\psi_2$ nearly ind. of $x_1\,x_2\,x_3$

fn ψ_2 small

$B\psi_k = \sum_l b_{lk}\psi_l$

$\psi_k = \sum a_{kl}\bar{\psi}_l$

$B\sum_l a_{kl}\bar{\psi}_l = \sum_{lm} b_{lk}\,a_{lm}\bar{\psi}_m$

$\sum_l a_{kl}\,a_{lm} = \delta_{km}$

$B\bar{\psi}_l = \sum_m \bar{b}_{ml}\bar{\psi}_m$

$B\sum a_{..}\bar{\psi}$ ...

Boundary values are ... $(\lambda^2-\mu^2)=0$

$$\delta(g(\xi'\eta)-g') = \delta(\eta-\eta')\Big/\frac{\partial g(\xi'\eta')}{\partial\eta'} = \int e^{i(\eta-\eta')x}\,dx\Big/\frac{\partial g}{\partial\eta'}$$

Suppose $g(\xi'\eta') = g'$

$$\int\{a(\alpha')\,b(\alpha'')\,d\alpha''\,\delta'(\alpha''-\alpha''') - \delta'(\alpha'-\alpha'')\,d\alpha''\,a(\alpha'')\,b_{\alpha'''}\}$$

$$= -a(\alpha')\frac{\partial b(\alpha'')}{\partial\alpha'''} - \frac{\partial a(\alpha')}{\partial(\alpha')}\,b(\alpha''') = -\left(\frac{\partial}{\partial\alpha'}+\frac{\partial}{\partial\alpha'''}\right)\left[a(\alpha')\,b(\alpha''')\right]$$

$$\theta(\alpha'\alpha'') = \delta'(\alpha'-\alpha'')$$

$$\phi(\alpha'\alpha'') = a(\alpha')\,b(\alpha'')$$

$$(\theta\phi+\phi\theta)(\alpha'\alpha'') = \int\{\delta'(\alpha'-\alpha''')\,d\alpha'''\,a(\alpha''')\,b(\alpha'') + a(\alpha')\,b(\alpha''')\,d\alpha'''\,\delta'(\alpha'''-\alpha'')\}$$

$$= \frac{\partial a(\alpha')}{\partial\alpha'}\,b(\alpha'') - a(\alpha')\frac{\partial b(\alpha'')}{\partial(\alpha'')}$$

$$(\theta\phi+\phi\theta)(\alpha'\alpha') = \frac{\partial a(\alpha')}{\partial\alpha'}\,b(\alpha') - a(\alpha')\frac{\partial b(\alpha')}{\partial(\alpha')} = a(\alpha')\,b(\alpha')\frac{\partial}{\partial\alpha'}\log\frac{a(\alpha')}{b(\alpha')}$$

$$\delta'(\alpha'-\alpha'')\frac{\sqrt{a(\alpha')\,b(\alpha')}\,.\,\sqrt{a(\alpha'')\,b(\alpha'')}}{a(\alpha'')\,b(\alpha'')}$$

$$\delta'(\alpha'-\alpha'')\sqrt{\frac{a(\alpha')}{a(\alpha'')}}\sqrt{\frac{b(\alpha'')}{b(\alpha')}} = \Theta(\alpha'\alpha'')$$

$$a(\alpha')\,b(\alpha'') = \Phi(\alpha'\alpha'')$$

$$(\Theta\Phi+\Phi\Theta)(\alpha'\alpha'') = \pm\frac{\partial}{\partial\alpha'''}\left[\sqrt{\frac{a(\alpha')}{a(\alpha''')}}\sqrt{\frac{b(\alpha''')}{b(\alpha')}}\,a(\alpha''')\,b(\alpha'')\right]_{\alpha'''=\alpha'}$$

$$\mp a(\alpha')\frac{\partial}{\partial\alpha'''}\left[b(\alpha''')\sqrt{\frac{a(\alpha''')}{a(\alpha')}}\sqrt{\frac{b(\alpha'')}{b(\alpha''')}}\right]_{\alpha'''=\alpha''}$$

$$= b(\alpha'')\sqrt{\frac{a(\alpha')}{b(\alpha')}}\frac{\partial}{\partial\alpha'}\sqrt{a(\alpha')\,b(\alpha')}$$

$$-a(\alpha')\sqrt{\frac{b(\alpha'')}{a(\alpha'')}}\frac{\partial}{\partial\alpha''}\sqrt{a(\alpha'')\,b(\alpha'')}$$

$$(\Theta\Phi+\Phi\Theta)(\alpha'\alpha') = 0$$

matrix not an $\alpha'\alpha''$ $(\alpha'\lambda'')$

$\boxed{\frac{h}{i}K_x \text{ must be a matrix with central symmetry like } p_x}$

The x's and their time com. conjugates form a system for which an arb q-number is not a single-valued function

$\sin(xp_x + yp_y + zp_z - tW)/h$

$\cos(xp_x + yp_y + zp_z - tW)/h$

$\left.\begin{matrix}\cancel{W>0} \\ \cancel{W<0}\end{matrix}\right\} W>0$

$\int_{-\infty}^{\infty} e^{iax/h}dx = h\,\delta(a)$

$\int_0^{\infty} \cos ax\,dx$

$$\int_{-\infty}^{\infty}\{\sin(x_r p_r' - tW')/h\ \sin(x_r p_r'' - tW'')/h\ dx_r\,dt$$

$$= \frac{1}{2}\int\{\cos(x_r\,\overline{p_r'-p_r''} - t\,\overline{W'-W''})/h - \cos(x_r\,\overline{p_r'+p_r''} - t\,\overline{W'+W''})/h\}dx_r\,dt$$

$= 0$ unless $p_r' - p_r'' = 0$ and $W' - W'' = 0$

$$\int_{-\infty}^{\infty}\int_0^{\infty}\{\sin(x_r'p_r - t'W)/h\ \sin(x_r''p_r - t''W)/h + \cos(x_r'p_r - t'W)/h\ \cos(x_r''p_r - t''W)/h\}\,dp_r\,dW$$

$$= \int_{-\infty}^{\infty}\int_0^{\infty}\cos(\overline{x_r'-x_r''}\,p_r - \overline{t'-t''}\,W)/h\ dp_r\,dW = 0 \text{ unless } x_r' = x_r'',\ t' = t''$$

$$p_r(x'x'') = \iint (x'|p')\,p_r'\,dp'\,\delta(p'-p'')\,dp''(p''|x'') = \int (x'|p')\,dp'\,p_r'\,(p'|x'')$$

$$= \int p_r'(\sin x'p' \sin x''p' + \cos x'p' \cos x''p')\,dp'$$

$r \neq 4$ $$= \int p_r' \cos[(x'-x'')p'] dp' = -\int \frac{\sin(x'-x'')p'}{x_r'-x_r''}\,dp'$$

$r \neq 4$ $= 0$ except when $x_\rho' = x_\rho''$ $(\rho = 1 \ldots 4)$ or except when $r = 4$, $(\rho = 1 \ldots 3)$

$\int x\cos(ax + by)\,dxdy$ $y>0$

$y > 0$

$= -\frac{1}{a}\int \sin(ax+by)\,dxdy$

$$\int p_1' \exp i\{(x'-x'')p_1 + (y'-y'')p_2 + (z'-z'')p_3 - (t'-t'')W\}/h\ dp_1\,dp_2\,dp_3\,dW$$

$$= \delta(y'-y'')\,\delta(z'-z'')\,\delta'(x'-x'')\cdot\left\{\frac{h}{i(t'-t'')}\right)$$

$$\Sigma \left(p_r - \frac{e}{c} \kappa_r\right)^2 = 0$$

~~[illegible]~~

$$\Sigma_r \left(-i h \frac{\partial}{\partial q_r} - \frac{e}{c} \kappa_r\right)^2 \psi_\kappa = 0 \qquad \overline{\Psi}_{\kappa_n} = \psi_{-\kappa_n} \quad \text{one of}$$

$$\Sigma \left(-i h \frac{\partial}{\partial q_r} + \frac{e}{c} \kappa_r\right) \overline{\Psi}_\kappa = 0$$

Do the ψ_κ corresponding to + ve energy and the $\psi_{-\kappa}$ — — — — — ve satisfy orthogonal relations

$$\int_{-\infty}^{\infty} \psi'_{\kappa W} \overline{\Psi}''_{\kappa W} \, dW = 0$$

n positive

$e^{-iWt/h}$	$e^{iWt/h}$
$\psi_{\kappa n} = \overline{\Psi}_{-\kappa n}$	$\psi_{\kappa, -n}$
$\overline{\Psi}_{\kappa, -n}$	$\overline{\Psi}_{\kappa n}$
$= \psi_{-\kappa, -n}$	$= \psi_{-\kappa, n}$

$$= \int_0^\infty \psi'_{\kappa W} \overline{\Psi}''_{\kappa W} \, dW + \int_{-\infty}^{0} \psi''_{-\kappa, -W} \overline{\Psi}'_{-\kappa, -W} \, d(W)$$

we want

$$\int_0^\infty \left(\psi'_{\kappa W} \overline{\Psi}''_{\kappa W} + \psi''_{\kappa W} \overline{\Psi}'_{\kappa W}\right) dW' \text{ to vanish}$$

or $\int_0^\infty \{(x'|W')(W'|x'') + (x''|W')(W'|x')\} dW'$

$\psi_{-\kappa, -W}$ refers to same equations of motion as $\psi_{\kappa, W} = \overline{\Psi}_{-\kappa, -W}$

$$\int \{(q'|\alpha')(\alpha'|q'') + (q''|\alpha')(\alpha'|q')\} d\alpha' = 0 \ (?)$$

for +ve energy

$$\int (q''|\alpha')(\alpha'|q'') \, d\alpha' = \text{pure imaginary} (?)$$

$$= f[i(q' - q'')]$$

$$(\alpha|1)(1|\alpha') + (\alpha|2)(2|\alpha') = 0 \qquad \alpha \neq \alpha'$$

$$\frac{(\alpha|1)}{(\alpha|2)} = -\frac{(2|\alpha')}{(1|\alpha')}$$

$$\int (\alpha'|q') \, dq' (q'|\alpha'') = \delta(\alpha' - \alpha'')$$

$$\int \int (\alpha'|q') \, dq' \int_+ (q'|\alpha'') \, d\alpha'' (\alpha''|q'') = \int_+ \delta(\alpha' - \alpha'') \, d\alpha'' (\alpha''|q'') = (\alpha'|q'') \text{ when } \alpha' > 0$$
$$= 0 \quad \text{,, } \alpha' < 0$$

$$\int (\alpha'|q') \left(\int \{ d\alpha'' (q'|\alpha'')(\alpha''|q'') \} - \delta(q' - q'') \right) dq' = 0 \qquad \alpha' > 0$$
$$= -(\alpha'|q'') \quad \alpha' < 0$$

$$\int (\alpha'|q')(\alpha'|q'') \, dq' \, dq'' \int_+ (q'|\alpha'') \, d\alpha'' (\alpha''|q'') = \int (\alpha'|q'')^2 dq'' \quad \alpha' > 0$$
$$= 0 \qquad \alpha' < 0$$

should vanish (to 1st order in h) if $\int_+$ is to be antisym. in q' & q''.

$$(\alpha'|q'')^2 \doteq (\alpha'|q') \text{ with } h \text{ halved.}$$

$$\cancel{xy - yx = a + ib}$$

$$xy = a + ib$$

$$yx = a + ib'$$

$$(p + c)\psi = 0$$

$$\left(ih\frac{\partial}{\partial q} + c\right)\psi = 0$$

$$\Delta \dot{p}_q = -p_s\left(p_r \frac{\partial \kappa_s}{\partial x_r \partial x_q}\right) + \frac{\partial^2 \kappa_s}{\partial x_r \partial x_q} p_r$$

$$\Delta \dot{x}_q = p_r \frac{\partial \kappa_s}{\partial x_r}$$

$$\sum_{nk}\left[\int_0^T \kappa' e^{i(W_m - W_n)t/h}\, dt \int_0^t \kappa'(s)\, e^{i(W_n - W_k)s/h}\, ds\right]$$

$$\dot{Y}_{mn}\, a_k a_m^* \dot{Y}_{nk}$$

$$\sum_{nk}\left[\left(p_r - \frac{e}{c}\kappa_r\right)^2 + m^2c^2 = 0\right]^*$$

$$\dot{Y}_{mn}^*\, a_k^* a_m \dot{Y}_{nk}^*$$

$$\left(p_r^2 + \frac{e^2}{c^2}\kappa_r^2 + m^2c^2\right)^2 = 4\frac{e^2}{c^2}\left(p_r \kappa_r\right)^2$$

$$a^2 = b^2$$

$$a = p_r^2 + \frac{e^2}{c^2}\kappa_r^2 + m^2c^2 \qquad b = 2\frac{e}{c}p_r\kappa_r$$

$$[a, b]$$

$$\cancel{p_r^2 p_s^2 - 2\frac{e^2}{c^2} p_r^2 \kappa_s^2}$$

$$p_r^2 p_s^2 + 2\frac{e^2}{c^2} p_r^2 \kappa_s^2 + \frac{e^4}{c^4}\kappa_r^2\kappa_s^2 + 2m^2c^2\left(p_r^2 + \frac{e^2}{c^2}\kappa_r^2\right) + m^4c^4 - 4\frac{e^2}{c^2} p_r \kappa_r p_s \kappa_s = 0$$

$$[a, b] = 2\frac{e}{c}\left[p_r^2 + \frac{e^2}{c^2}\kappa_r^2,\ p_s\kappa_s\right] = 2\frac{e}{c}\left\{-p_s\left(p_r\frac{\partial \kappa_s}{\partial x_r} + \frac{\partial \kappa_s}{\partial x_r}p_r\right) + \frac{e^2}{c^2}\frac{\partial \kappa_r^2}{\partial x_s}\kappa_s\right\}$$

$$a^2 - b^2 = (a+b)(a-b) + ih[b, a] = (a+b)\left\{a - b + \frac{ih[b, a]}{a+b}\right\}$$

$$\int_0^\infty \cos(\xi'\eta')\cos(\xi''\eta')\,d\eta' = \tfrac{\pi}{2}\left[\delta(\xi'-\xi'')+\delta(\xi'+\xi'')\right]$$

$$(\xi'|\eta') = \begin{cases}\cos(\xi'\eta') & \xi'>0 \quad \eta'>0 \\ \sin(\xi'\eta') & \xi'<0 \quad \eta'<0\end{cases}$$

?

$$(\eta'|\xi') = \begin{cases}\cos(\xi'\eta') & \eta'>0 \\ \sin(\xi'\eta') & \eta'<0\end{cases}$$

$$\int(\xi'|\eta')(\eta'|\xi'')\,d\eta' = \int_{-\infty}^{\infty}\cos(\xi'\eta')\begin{pmatrix}\cos(\xi''\eta') & \eta'>0 \\ \sin(\xi''\eta') & \eta'<0\end{pmatrix}d\eta'$$

$$\tfrac{1}{2} J^{\frac{1}{2}} e^{i\psi} J^{\frac{1}{2}} \left(\lambda_2 + i[J, \lambda_2]\right) = \tfrac{1}{4}(J_2^2 - M_2^2)^{\frac{1}{2}} \Big\{ -(J_2 - k + k')^{\frac{1}{2}}(-J_2 + k + k')^{\frac{1}{2}} (J - k_2 + k')^{\frac{1}{2}}(-J + k_2 + k')^{\frac{1}{2}} \tfrac{1}{2} k^{-\frac{1}{2}} e^{i\theta} k^{-\frac{1}{2}}$$

$$+ (J_2 + k + k')^{\frac{1}{2}} (J_2 + k - k')^{\frac{1}{2}} (J + k_1 + k')^{\frac{1}{2}} (J + k_1 - k')^{\frac{1}{2}} \tfrac{1}{2} k^{-\frac{1}{2}} e^{-i\theta} k^{-\frac{1}{2}} \Big\}$$

$$(J+h)^{\frac{1}{2}} J^{\frac{1}{2}} \left(\lambda_2 - i[J, \lambda_2]\right) = \tfrac{1}{4}(J_1^2 - M_2^2)^{\frac{1}{2}} \Big\{ - F_{+1}' \frac{J_1}{k^{\frac{1}{2}}(k-h)^{\frac{1}{2}}} e^{i(\theta - \psi)} + F_{+1} \frac{J_1}{k^{\frac{1}{2}}(k+h)^{\frac{1}{2}}} e^{-i(\theta+\psi)}$$

which gives

where

$$F_{+1} = \tfrac{1}{4}(J + k + k' + \tfrac{1}{2}h)^{\frac{1}{2}} (J + k - k' + \tfrac{1}{2}h)^{\frac{1}{2}} (J + k + k' + \tfrac{3}{2}h)^{\frac{1}{2}} (J + k - k' + \tfrac{3}{2}h)^{\frac{1}{2}} / (J + \tfrac{1}{2}h)^{\frac{1}{2}} k^{\frac{1}{2}} (k+h)^{\frac{1}{2}}$$

$$F_{+1}' = \tfrac{1}{4}(J - k + k' + \tfrac{1}{2}h)^{\frac{1}{2}} (-J + k + k' - \tfrac{1}{2}h)^{\frac{1}{2}} (J - k + k' + \tfrac{3}{2}h)(-J + k + k' - \tfrac{3}{2}h) / (J + \tfrac{1}{2}h)^{\frac{1}{2}} k^{\frac{1}{2}} (k-h)^{\frac{1}{2}}$$

$$\lambda_2 - i[J, \lambda_2] = \tfrac{1}{4}(J_1^2 - M_2^2)^{\frac{1}{2}} \Big\{ \frac{J_1^{\frac{1}{2}}}{J^{\frac{1}{2}}(J+h)^{\frac{1}{2}} \cancel{k^{\frac{1}{2}}(k+h)^{\frac{1}{2}}}} F_{+1} e^{-i(\theta+\psi)} - \frac{J_1^{\frac{1}{2}}}{J^{\frac{1}{2}}(J+h)^{\frac{1}{2}} \cancel{k^{\frac{1}{2}}(k-h)^{\frac{1}{2}}}} F_{+1}' e^{i(\theta-\psi)} \Big\} \quad (55)$$

Similarly it may be shown that

$$\lambda_2 + i[J, \lambda_2] = \tfrac{1}{4}(J_2^2 - M_2^2)^{\frac{1}{2}} \Big\{ \frac{J_2^{\frac{1}{2}}}{J^{\frac{1}{2}}(J-h)^{\frac{1}{2}} \cancel{k^{\frac{1}{2}}(k-h)^{\frac{1}{2}}}} F_{-1}' e^{i(\theta+\psi)} - \frac{J_2^{\frac{1}{2}}}{J^{\frac{1}{2}}(J-h)^{\frac{1}{2}} \cancel{k^{\frac{1}{2}}(k+h)^{\frac{1}{2}}}} F_{-1} e^{-i(\theta-\psi)} \Big\} \quad (56)$$

where F_{-1}' is the quantity obtained by writing $-h$ for h in F_{+1} and F_{-1} the quantity obtained in the same way from F_{+1}'

$$F_{-1}' = \tfrac{1}{4}(J + k + k' - \tfrac{1}{2}h)(J + k - k' - \tfrac{1}{2}h)(J + k + k' - \tfrac{3}{2}h)(J + k - k' - \tfrac{3}{2}h) / (J - \tfrac{1}{2}h)^{\frac{1}{2}} k^{\frac{1}{2}} (k-h)^{\frac{1}{2}}$$

$$F_{-1} = \tfrac{1}{4}(J - k + k' - \tfrac{3}{2}h)^{\frac{1}{2}} (-J + k + k' + \tfrac{3}{2}h)^{\frac{1}{2}} (J - k + k' - \tfrac{1}{2}h)(-J + k + k' + \tfrac{1}{2}h) / (J - \tfrac{1}{2}h)^{\frac{1}{2}} k^{\frac{1}{2}} (k+h)^{\frac{1}{2}}$$

and also

$$q \cdot m' = \tfrac{1}{2}\Big\{ (J + k + k' - \tfrac{1}{2}h)^{\frac{1}{2}} (J + k - k' - \tfrac{1}{2}h)^{\frac{1}{2}} (J - k + k' + \tfrac{1}{2}h)^{\frac{1}{2}} (-J + k + k' - \tfrac{1}{2}h)^{\frac{1}{2}} k^{-\frac{1}{2}} e^{i\theta} k^{-\frac{1}{2}}$$

$$+ (J + k + k' + \tfrac{1}{2}h)^{\frac{1}{2}} (J + k - k' + \tfrac{1}{2}h)^{\frac{1}{2}} (J - k + k' - \tfrac{1}{2}h)^{\frac{1}{2}} (-J + k + k' + \tfrac{1}{2}h)^{\frac{1}{2}} k^{-\frac{1}{2}} e^{-i\theta} k^{-\frac{1}{2}} \Big\}$$

omit

$$= \left(\frac{J_1 J_2}{J}\right)^{\frac{1}{2}} F_0 e^{-i\theta} + \left(\frac{J_1 J_2}{J}\right)^{\frac{1}{2}} F_0' e^{i\theta} \quad (57)$$

where

$$F_0 = \tfrac{1}{2}(J + k + k' + \tfrac{1}{2}h)^{\frac{1}{2}} (J + k - k' + \tfrac{1}{2}h)^{\frac{1}{2}} (J - k + k' - \tfrac{1}{2}h)^{\frac{1}{2}} (-J + k + k' + \tfrac{1}{2}h)^{\frac{1}{2}} J^{\frac{1}{2}} / (J_1 J_2)^{\frac{1}{2}} k^{\frac{1}{2}} (k+h)^{\frac{1}{2}}$$

and F_0' is the quantity obtained by writing $-h$ for h in F_0

$$F_0' = \tfrac{1}{2}(J + k + k' - \tfrac{1}{2}h)^{\frac{1}{2}} (J + k - k' - \tfrac{1}{2}h)^{\frac{1}{2}} (J - k + k' + \tfrac{1}{2}h)^{\frac{1}{2}} (-J + k + k' - \tfrac{1}{2}h)^{\frac{1}{2}} J^{\frac{1}{2}} / (J_1 J_2)^{\frac{1}{2}} k^{\frac{1}{2}} (k-h)^{\frac{1}{2}}$$

$$z = \frac{M_2}{J_1 J_2} q \cdot m' + \tfrac{1}{2} i \frac{1}{J_1}\left(\lambda_2 - i[J, \lambda_2]\right) - \tfrac{1}{2} i \frac{1}{J_2}\left(\lambda_2 + i[J, \lambda_2]\right)$$

$$= \frac{M_2}{[illegible]} \left(F_0 e^{-i\theta} + F_0' e^{i\theta}\right) + \tfrac{1}{2} i \frac{(J + M_2 + \tfrac{1}{2}h)^{\frac{1}{2}} (J - M_2 + \tfrac{1}{2}h)^{\frac{1}{2}}}{J^{\frac{1}{2}} (J + \tfrac{1}{2}h)^{\frac{1}{2}} [illegible]}$$

$$\dot{j}(\dot{j}-\hbar)(\mu_z+i[\dot{j},\mu_z])(\lambda_z+i[\dot{j},\lambda_z]) = (\dot{j}_2^2-\hbar^2)(\mu_x\lambda_x+\mu_y\lambda_y+\mu_z\lambda_z) + i\hbar M_z(\mu_x\lambda_y - [illegible]) + i\dot{j}_2\{M_x(\mu_y\lambda_z-\mu_z\lambda_y) + M_y(\mu_z\lambda_x-\mu_x\lambda_z)\} \tag{55}$$

now $\mu_x\lambda_x+\mu_y\lambda_y+\mu_z\lambda_z = \Sigma_{xyz}(m_y m_z' - m_z m_y')(M_y z - M_z y - i\hbar x)$

Put dashed letters first

$$= \Sigma_{xyz}\{(m_y m_z' - m_z m_y')M_y - (m_z m_x' - m_x m_z')M_z - i\hbar(m_x m_y' - m_y m_x')\} z$$

$$= \Sigma_{xyz}\{m_z'(M_x M_x + M_y M_y + M_z M_z) - (m_x' M_x + m_y' M_y + m_z' M_z) m_z\} z$$

$$= m \cdot M\, m' \cdot q + \Sigma_{xyz}\{m_x(m_z' M_x - M_x m_z') + m_y(m_z' M_y - M_y m_z')\} z$$

using (5). Also $= m \cdot M\, m' \cdot q + i\hbar\mu\cdot q$

$$[\lambda_x, \mu_y] = [M_y z - y M_z, \mu_y] = M_y[z, \mu_y] - [y, \mu_y]M_z + y\mu_x$$

$$= M_y(-y m_z') - (x m_x' + z m_z')M_z + y(M_y m_z' - m_y' M_z) = -q\cdot m' M_z = -M_z q\cdot m'$$

$$\mu_x\lambda_y - \mu_y\lambda_x = \mu_x(x M_z - M_x z) - \mu_y(M_y z - y M_z)$$

$$= (\mu_x x + \mu_y y + \mu_z z)M_z - (\mu_x M_x + \mu_y M_y + \mu_z M_z) z \quad = \mu\cdot q\, M_z$$

and similarly $\mu_y\lambda_z - \mu_z\lambda_y = \mu\cdot q\, M_x \qquad \mu_z\lambda_x - \mu_x\lambda_z = \mu\cdot q\, M_y$

while

and $\mu\cdot q = \Sigma_{xyz}(m_y x - m_x y) m_z' = k[k, q\cdot m'] - \frac{1}{2}i\hbar\, q\cdot m'$

from (31). Using these results, and the fact that $\mu\cdot q$ commutes with M_x, M_y and M_z, eqn (55) becomes

$(M_x M_y M_z)$ commute with $\mu\cdot q$ (since they commute with k and $q\cdot m'$)

$$\dot{j}(\dot{j}-\hbar)(\mu_z+i[\dot{j},\mu_z])\dot{j}(\lambda_z-i[\dot{j},\lambda_z]) = (\dot{j}_2^2-\hbar^2)(M\cdot m' \overset{q\cdot m'}{} + i\hbar\mu\cdot q) + i\{\hbar M_z^2 - \dot{j}_2(M_x^2+M_y^2)\}\mu\cdot q$$

$$= (\dot{j}_2^2 - M_z^2)(M\cdot m' \overset{q\cdot m'}{} + i\hbar\mu\cdot q - i\dot{j}_1\,\mu\cdot q)$$

$$= \tfrac{1}{2}(J_2^2 - M_z^2)\{(\dot{j}_1\dot{j}_2 + k^2 - k'^2)q\cdot m' - 2i\dot{j}_2(k[k, q\cdot m'] - \tfrac{1}{2}i\hbar\, q\cdot m')\}$$

$$= \tfrac{1}{4}(J_2^2 - M_z^2)\{(-k+k'+\dot{j}_2)(k+k'-\dot{j}_2)(q\cdot m' + i[k, q\cdot m']) + (k+k'+\dot{j}_2)(k-k'+\dot{j}_2)(q\cdot m' - i[k, q\cdot m'])$$

Hence, substituting for $(\mu_z + i[\dot{j}, \mu_z])$, $(q\cdot m' + i[k, q\cdot m'])$ and $(q\cdot m' - i[k, q\cdot m'])$ from — we find

$$(\dot{j}+\hbar)\dot{j}(\lambda_z - i[J, \lambda_z]) = \tfrac{1}{4}(J_1^2 - M_z^2)^{(J_1-k+k')^{\frac{1}{2}}}$$

after slight rearrangement of the order of factors

$$H = \left(p_x^2 + p_y^2 + p_z^2\right) - \left(\frac{W}{c} + \frac{e^2}{cr}\right)^2 + m^2c^2 = 0$$

$$\frac{1}{2m}\left(p_r^2 + \frac{k_1k_2}{r^2}\right) - \frac{1}{2m}\left(\frac{W}{c} + \frac{e^2}{cr}\right)^2 + \frac{mc^2}{2} = 0$$

$$\frac{1}{2m}\left(p_r^2 + \frac{\bar{k}_1\bar{k}_2}{r^2}\right) - \frac{\bar{e}^2}{r} - \bar{W} = 0$$

$$\bar{k}^2 = k^2 - \frac{e^4}{c^2}$$

$$\bar{e}^2 = \frac{W}{mc^2}e^2$$

$$\bar{W} = -\tfrac{1}{2}mc^2 + \tfrac{1}{2}\frac{W^2}{mc^2}$$

We know the fns that transform from

$\bar{x}\ \bar{y}\ \bar{z}\ \bar{W}$ to $J_0^2, \bar{W}$

and hence we know those that transform from

$x\ y\ z\ W$ to J'_0, W

$$(x'W'|J'W'') = X(x'J'|W')\,\delta(W'-W'')$$

$$\iint (J''W'''|x'W')(x'W'|J'W'')\,dx'dW' = \int \bar{X}(x'J''|W'')\,dx'\,X(x'J'|W'')\,\delta(W''-W''') = \delta(J'-J'')\,\delta(W''-W''')$$

$$\iint (x'W'|J'W'')(J'W''|x''W''')\,dJ'dW''$$

$$= \int X(x'J'|W')\,dJ'\,\bar{X}(x''J'|W')\,\delta(W'-W''') = \delta(x'-x'')\,\delta(W'-W''')$$

$$(x'|J') = (x't'|J'W') = X(x'J'|W')\,e^{iW't'/h}$$

W is true canonical conjugate of t.

$$\int (J''x')\,dx'\,dt'\,(x'|J') = \int \bar{X}(x'J''|W'')\,dx'\,X(x'J'|W')\,\delta(W'-W'')$$

$$(a'\xi'|b'\xi'') = (a'|b')\,\delta(\xi'-\xi'')$$. For such a transf. the canonical conjugates of the ξ's are not changed.

$$\cancel{(a'\xi'|b'\xi'')\,f(\xi'')\,\delta(\xi''-\xi''')\,d\xi'''}$$

$$\iiiint (a'\xi'|b'\xi'')\,db'd\xi''\,f(\xi'')\,\delta(b'-b'')\,\delta(\xi''-\xi''')\,db''d\xi'''\,(b''\xi'''|a''\xi^{iv})$$

$$= \iint (a'\xi'|b'\xi'')\,db'd\xi''\,f(\xi'')\,(b'\xi''|a'\xi^{iv}) = f(\xi')\,\delta(\xi'-\xi^{iv})\,\delta(a'-a'')$$

$$x_0\psi_n(x) = \sum x_{mn}\psi_m(x)$$

$$x_{mn} = f(t)$$

$$x_0\psi_n(x_0) = \sum_m x_{mn}\psi_m(x_0)$$

$$P_{mn} = \psi_n(x_0 t)$$

$$\sum_k x_{mk}P_{kk} = x_0 P_{mk}$$

$$x_0 P_{kn} = \sum_m P_{km}x_{mn}$$

$$P_{kn}x_0 = \sum_m P_{km}x_{mn} = \{P\}\{x\}$$

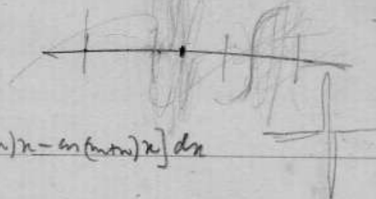
$$\{P\}x_0 = \{P\}\{x\}$$

$0 < x < \pi$

x

$p = \frac{\partial}{\partial x}$

Eigenfns $\frac{1}{\sqrt{\pi/2}} \sin nx$



$$x_{nm} = \frac{2}{\pi}\int_0^{\pi} x \sin nx \sin mx \, dx = \frac{1}{\pi}\int_0^{\pi} x\left[\cos(m-n)x - \cos(m+n)x\right] dx$$

$(m \neq n)$

$$= \frac{-1}{\pi}\int_0^{\pi}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right] dx = 0 \qquad m+n \text{ even}$$

$$= \frac{-2}{\pi}\left[\frac{1}{(m-n)^2} - \frac{1}{(m+n)^2}\right] \quad m+n \text{ odd}$$

$p(xx') = \delta'(x'-x)$ except when x and x' are separated by a multiple of π

$$\int_0^{\pi} \delta'(x-x')\,\phi(x')\,dx' = \lambda\phi(x)$$

$$= \int_0^{\pi} \delta(x-x')\,\phi'(x')\,dx'$$

$$+\left[\delta(x-x')\phi(x')\right]_0^{\pi} = \delta(x-\pi)\phi(\pi) - \delta(x)\phi(0)$$

$$\lambda\phi(x) = \phi'(x) + \delta(x-\pi)\phi(\pi) - \delta(x)\phi(0)$$

$$\phi(x) = e^{\lambda x} + \alpha\,\delta(x-\pi) + \beta\,\delta(x)$$

$$\int_{\pi-\varepsilon}^{\pi} \delta(x-\pi)\,dx + \int_{\pi}^{\pi+\varepsilon} \delta(x'-\pi)\,dx' = 1$$

$$+\int_0^{\varepsilon} \delta(x)\,dx = 1$$

$0 \leq x' \leq \pi$

$$\int_0^{\pi} \delta(x-x')f(x')\,dx' = f(x) - \int_{-\varepsilon}^{0} \delta(x-x')f(x')\,dx' - \int_{\pi}^{\pi+\varepsilon} \cdots$$

$$= f(x) - f(0)\int_{-\varepsilon}^{0} \delta(x-x')\,dx' - f(\pi)\int_0^{\varepsilon} \delta(x-\pi-x')\,dx'$$

$$\int_0^{\varepsilon} \delta(a-x')\,dx' = \int_{a-\varepsilon}^{a} \delta(y)\,dy$$

$f(x)$ a continuous fn.

$$\int_{-\pi/2}^{\pi/2} p(xx')\,f(x')\,dx' = f'(x) \qquad x \neq 0$$

$x \neq 0,$

$$\int_{-\infty}^{b} p(xx')\,dx' = -\delta(x-b)$$

$-\frac{\pi}{2} < x < \frac{\pi}{2}$

$$\int_{-\infty}^{+} p(0,x')\,dx' = \infty$$

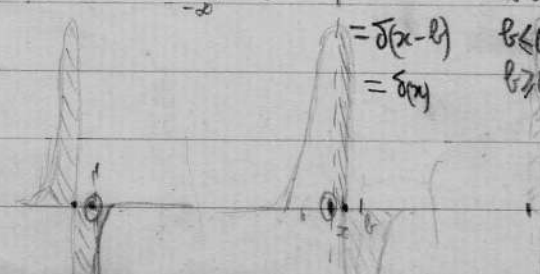
Suppose $p(xx')$ made periodic in x and x'

$$p(x,x') = p(x+\pi, x') = p(x, x'+\pi).$$

$$\int_{-\infty}^{b} p(xx')\,dx' = \delta(x-b) - \delta(x)_{b\geq 0} \quad \left.\begin{matrix} b \leq 0 \\ \end{matrix}\right\} x > 0$$

$$= \delta(x-b) \qquad b \leq 0$$

$$= \delta(x) \qquad b \geq 0 \quad \Big\} x < 0$$



$\phi(\tfrac{\pi}{2}) = \phi(-\tfrac{\pi}{2})$

$-\pi < x' < \pi$

$$\int_{-\infty}^{x'} p(x x')\,dx' = \delta(x-x') - \delta(x) \qquad x>0$$

$$\left.\begin{aligned} &= \delta(x-x') & x' \le 0 \\ &= \delta(x) & x' \ge 0 \end{aligned}\right\} x<0$$

$$= P(x x')$$

$$\int_{-\pi/2}^{\pi/2} p(x x')\phi(x')\,dx' = \Big[\phi(x')\,P(x x')\Big]_{-\pi/2}^{\pi/2}$$

$$-\int_{-\pi/2}^{\pi/2} \phi'(x')\,P(x x')\,dx' = \phi(\tfrac{\pi}{2})\begin{cases} x>0 & \delta(x-\frac{\pi}{2}) - \delta(x) - \delta(x+\frac{\pi}{2}) + \delta(x) \\ x<0 & \delta(x) - \delta(x+\frac{\pi}{2}) \end{cases}$$

correct

spurious

$$\text{1st term} = \phi(\tfrac{\pi}{2})\begin{Bmatrix} \delta(x-\frac{\pi}{2}) - \delta(x) \\ \delta(x) - \delta(x+\frac{\pi}{2}) \end{Bmatrix} \quad \begin{matrix} x \ge 0 \\ x \le 0 \end{matrix} \quad = \mp\phi(\tfrac{\pi}{2})\,\delta(x)$$

$$-(\text{2nd term}) = \int_0^{\pi/2} \phi'(x')\begin{Bmatrix} \delta(x-x') \\ \delta(x) \end{Bmatrix} dx' + \int_{-\pi/2}^{0} \phi'(x')\begin{Bmatrix} \delta(x) \\ \delta(x-x') \end{Bmatrix} dx'$$

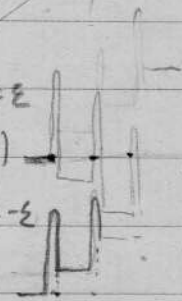
$$= \begin{cases} \phi'(x) - \int_{-\varepsilon}^{0} \phi'(x')\,\delta(x-x')\,dx' + \delta(x)\left[\phi(0) - \phi(\frac{\pi}{2})\right] \\ \delta(x)\left[\phi(\frac{\pi}{2}) - \phi(0)\right] + \phi'(x) - \int_{0}^{\varepsilon} \phi'(x')\,\delta(x-x')\,dx' \end{cases}$$

$x = \varepsilon$

$p(x x')$

x'

$x = -\varepsilon$

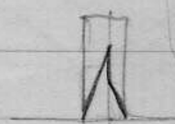


$$\left.\begin{aligned} P(x x') &= \delta(x) & x' \le 0 \\ &= \delta(x-x') & x' \ge 0 \end{aligned}\right\} x \ge 0$$

$$\left.\begin{aligned} &= \delta(x-x') & x' \le 0 \\ &= \delta(x) & x' \ge 0 \end{aligned}\right\} x \le 0$$

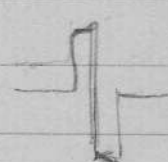
$$\text{Sum} + \phi'(x) = \begin{cases} -\delta(x)\,\phi(0) + \int_{-\varepsilon}^{0} \phi'(x')\,\delta(x-x')\,dx' \\ \delta(x)\,\phi(0) + \int_{0}^{\varepsilon} \phi'(x')\,\delta(x-x')\,dx' \end{cases}$$

If ϕ is continuous, this is $\quad \mp\,\delta(x)\,\phi(0) + \begin{Bmatrix} \int_{-\varepsilon}^{0} \\ \int_{0}^{\varepsilon} \end{Bmatrix} \phi'(x')\,\delta(x-x')\,dx'.$



$$= \mp\,\delta(x)\,\phi(0) + \begin{Bmatrix} \int_{x}^{x+\varepsilon} \delta(y)\,\phi'(x+y)\,dy \\ \int_{x-\varepsilon}^{x} \delta(y)\,\phi'(x+y)\,dy \end{Bmatrix} \qquad x - x' = y$$

$$\begin{aligned} \delta_1(x) &= \delta(x) & x<0 \\ &= -\delta(x) & x>0 \end{aligned}$$



$$d\sigma^2 = \gamma_{\mu\nu}\, dx^\mu dx^\nu \qquad (\mu\nu = 0 \cdot\cdot 4) \qquad (i,k = 1, \cdot\cdot 4)$$

$$\bar x^0 = x^0 + \psi_0(x_1 \cdots x_4) \qquad \bar\gamma_{\mu\nu} = \gamma_{\rho\sigma}\frac{\partial x^\rho}{\partial \bar x_\mu}\frac{\partial x^\sigma}{\partial \bar x^\nu}$$

$$\bar x^i = \psi_i(x_1 \cdots x_4) \qquad \gamma_{\rho\sigma} = \bar\gamma_{\mu\nu}\frac{\partial \bar x^\mu}{\partial x^\rho}\frac{\partial \bar x^\nu}{\partial x^\sigma} = \bar\gamma_{\mu\nu}\left\{\delta^\mu_0\delta^0_\rho + \frac{\partial\psi^\mu}{\partial x^\rho}\right\}\left\{\delta^\nu_0\delta^0_\sigma + \frac{\partial\psi^\nu}{\partial x^\sigma}\right\}$$

$$\bar x^\mu = \delta^\mu_0 x^0 + \psi^\mu(x_1 \cdots x_4) \qquad \gamma_{00} = \bar\gamma_{00} \qquad \gamma_{0k} = \bar\gamma_{0\mu}\frac{\partial\psi^\mu}{\partial x^k} \qquad \gamma_{k0} = \bar\gamma_{\mu 0}\frac{\partial\psi^\mu}{\partial x^k}$$

$$\gamma_{ik} = \bar\gamma_{\mu\nu}\frac{\partial\psi^\mu}{\partial x^i}\frac{\partial\psi^\nu}{\partial x^k}$$

$$d\bar x^\mu = \frac{\partial \bar x^\mu}{\partial x^\nu}dx^\nu = \left\{\delta^\mu_0\delta^0_\nu + \frac{\partial\psi^\mu}{\partial x^\nu}\right\}dx^\nu \qquad d\bar x^0 = dx^0 + \frac{\partial\psi^0}{\partial x^k}dx^k \qquad d\bar x^k = \frac{\partial\psi^k}{\partial x^i}dx^i$$

$$d\bar x_0 + \frac{\bar\gamma_{0i}}{\bar\gamma_{00}}d\bar x^i = dx_0 + \frac{\partial\psi^0}{\partial x^k}dx^k + \frac{\bar\gamma_{0i}}{\bar\gamma_{00}}\frac{\partial\psi^i}{\partial x^k}dx^k = dx_0 + \frac{\gamma_{0k}}{\gamma_{00}}dx^k = d\theta$$

$$\bar\gamma_{0i}\frac{\partial\psi^i}{\partial x^k} = \bar\gamma_{0\mu}\frac{\partial\psi^\mu}{\partial x^k} - \bar\gamma_{00}\frac{\partial\psi^0}{\partial x^k} = \gamma_{0k} - \bar\gamma_{00}\frac{\partial\psi^0}{\partial x^k}$$

$$\delta\int P\sqrt{-\gamma}\,dx^0 \cdots dx^4 = -\int\left(G^{\mu\nu} - \tfrac{1}{2}\gamma^{\mu\nu}G\right)\delta\gamma_{\mu\nu}\sqrt{-g}\,dx^0 \cdots dx^4$$

$$X(t) = \int x(\alpha)\, e^{\alpha t}\, d\alpha$$

$\overline{d\mathfrak{f}}$

$A(j_1, j_2)\, dj_1$ $A(j_2, j_1)\, dn^x$

$\overline{R}\, dn^x$

$\frac{2\pi i}{t}$

n^{x_2}

$A(\nu)\, d\nu\, e^{2\pi i \nu t}$

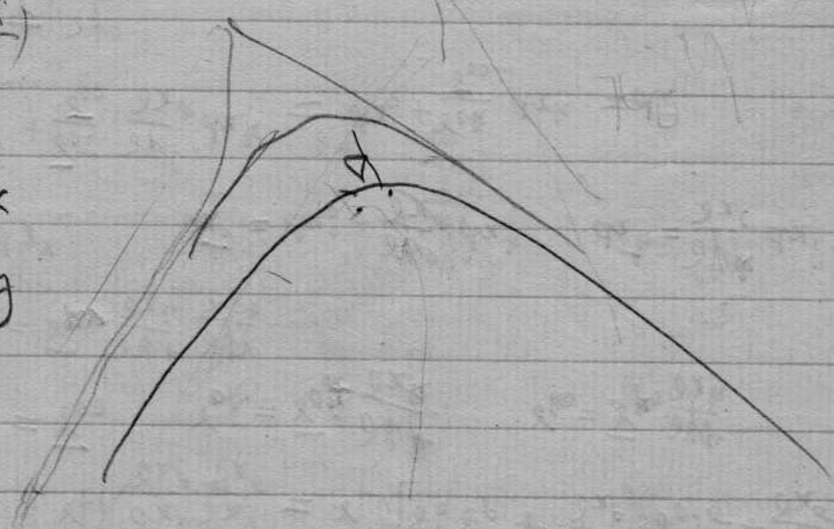
d

$\overline{n}$ Ek_m

dE

$\left(\frac{x}{y}\right)_{mn}$ $\left(\frac{x}{y}\right)$

$\left(\frac{x}{y}\right)_m$ $\frac{x}{y}$



$$L = L_0 - \frac{e}{c}\sum(A_x\dot x + A_y\dot y + A_z\dot z) \qquad A = \frac{1}{2}[H, r]$$

$$p_x = \frac{\partial L}{\partial \dot x} = m\dot x - \frac{e}{c}A_x \qquad \dot p_x = -\frac{\partial H^*}{\partial x} \quad \dot x = \frac{\partial H^*}{\partial p_x}$$

Ad. change

$$\dot p_x = -\frac{\partial H^*}{\partial x} - \frac{e}{c}\dot A_x$$

$$H^* = H_0^* + \frac{e}{c}(A v)$$
$$= H_0^* + \frac{e}{2mc}H p_\phi.$$

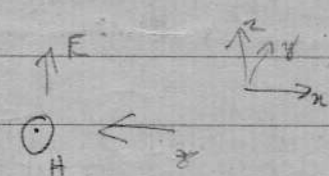
Electric field given by $\operatorname{curl} E = -\frac{1}{c}\dot H \qquad H = \operatorname{curl} A$

$$E = -\frac{1}{c}\dot A - \operatorname{grad}\phi. \qquad \phi \text{ of order } \dot A$$

This results in $m\ddot x$ being increased by $+\frac{e}{c}\dot A_x + e\frac{\partial\phi}{\partial x}$

$\dot p_x$ becomes $-\frac{\partial H^*}{\partial x} + e\frac{\partial \phi}{\partial x} = -\frac{\partial}{\partial x}(H^* - e\phi)$

$$\dot x = +\frac{\partial}{\partial p_x}(H^* - e\phi)$$



$$\frac{\partial H_y}{\partial x} = -\frac{1}{c}\frac{\partial E_z}{\partial t}$$

$K_1 = \alpha_1 \cos(l_i x_i - ct)\nu \qquad \alpha_i l_i - \alpha_4 c^2 = 0$

$K_2 = \alpha_2 \cos(l_i x_i - ct)\nu$

$K_\mu = \alpha_\mu \cos l_i x_i \nu t \; e^{i\nu ct}$

$K_4 = \alpha_4 \cos(l_i x_i \nu t)\, e^{i\nu ct}$

$\alpha_i l_i - \alpha_4 = 0$

$$E_2 = \frac{\partial K_2}{c\partial t} - \frac{\partial K_4}{\partial x_2} = \nu \sin(l_i x_i - ct)\nu \cdot \{\alpha_2 - l_2 \alpha_4\}$$

$$E_2 = \frac{\partial K_2}{c\partial t} - \frac{\partial K_4}{\partial x_2} = \alpha_2$$

$K_\mu = \alpha_\mu \sin l_1 x_1 \nu \sin l_2 x_2 \nu \sin l_3 x_3 \nu \sin ct\nu .$

$$E_2 = \frac{\partial K_2}{c\partial t} - \frac{\partial K_4}{\partial x_2} = \nu \sin l_1 x_1 \nu \sin l_3 x_3 \nu \{\alpha_2 \sin l_2 x_2 \nu \cos ct\nu - \alpha_4 l_2 \cos l_2 x_2 \nu \sin ct \nu^2\}$$

$e^{i\omega\theta} \qquad e^{-i\omega\theta} \qquad e^{-i\omega t}$

Light quantum eigenfunction may be reforms simply a constant. The quantum then goes out of existence

Different terms in expansion $\kappa\psi_k = \Sigma a_{kl}\psi_l$ refer to different states of atom + radiation field of same total energy

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} - n^2\psi = 0 \qquad rn = \bar r$$

$$\bar r^2 \frac{\partial^2\psi}{\partial \bar r^2} + \bar r \frac{\partial\psi}{\partial \bar r} - \bar r^2 \psi$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) - n^2 r\psi = 0$$

$$\frac{\partial^2}{\partial r^2}\left(r^{\frac12}\psi\right) = r^{\frac12}\frac{\partial^2\psi}{\partial r^2} + r^{-\frac12}\frac{\partial\psi}{\partial r} - \frac14 r^{-\frac32}\psi$$

$$= \left(n^2 - \frac14 r^{-2}\right) r^{\frac12}\psi$$

$$\psi = J_0(inr)$$

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-)^m \left(\frac12 z\right)^{n+2m}}{m!\,(n+m)!}$$

$$Y_0(z) = 2\left[\sum_{m=0}^{\infty} \frac{(-)^m\left(\frac12 z\right)^{2m}}{(m!)^2}\left\{\log \tfrac12 z - \frac{\partial}{\partial\nu}\Gamma(\nu+m+1)\right\}\right]_{\nu=0}$$

If x and the J's are real and if $\bar{x}_\alpha$ denotes the conjugate imaginary of x_α, then by equating the conjugate imaginaries of both sides of (8) we get

$$x = \Sigma_\alpha e^{-i(\alpha w)} \bar{x}_\alpha(J_r) = \Sigma_\alpha \bar{x}_\alpha(J_r + \alpha_r h) e^{-i(\alpha w)}$$

Comparing this with eq (8) we find that

$$\bar{x}_\alpha(J_r + \alpha_r h) = x_{-\alpha}(J_r)$$

This relation is brought out more clearly if we change the notation. For $x_\alpha(J)$ write $x(J, J-\alpha h)$.

Then $$\bar{x}(J+\alpha h, J) = x(J, J+\alpha h)$$

which shows that there is some kind of symmetry in the way in which ~~x~~ the amplitude $x(J, J+\alpha h)$ is related to the two sets of variables to which it explicitly refers. Our expansion for x is now

$$x = \Sigma_\alpha x(J, J-\alpha h) e^{i(\alpha w)} = \Sigma_\alpha e^{i(\alpha w)} x(J+\alpha h, J)$$

If y can also be expanded into form $$y = \Sigma_\beta y(J, J-\beta h) e^{i(\beta w)}$$

then $$xy = \Sigma_{\alpha\beta} x(J, J-\alpha h) e^{i(\alpha w)} y(J, J-\beta h) e^{i(\beta w)}$$

by again using (6), $$= \Sigma_{\alpha\beta} x(J, J-\alpha h) y(J-\alpha h, J-\alpha h-\beta h) e^{i(\alpha+\beta, w)}$$

or, the amplitude of xy are given by

$$xy(J, J-\gamma h) = \Sigma_\alpha x(J, J-\alpha h) y(J-\alpha h, J-\gamma h) \qquad (11)$$

These formulae provide a way of representing q-numbers by means of c-numbers. Suppose that in the expressions $x(J, J-\alpha h)$ and $\omega(J, J-\alpha h)$, considered merely as functions of the J's, we ~~write~~ substitute for each J_r the c-number $n_r h$, and denote the resulting c-numbers by $x(n, n-\alpha)$, $\omega(n, n-\alpha)$. We may consider the aggregate of all the c-numbers $x(n, n-\alpha) \exp. i\omega(n, n-\alpha)t$, where the n_r take a series of values all differing from one another by integers, as representing the values of the q-number x for all values of the q-numbers J_r.

Eq. (10) shows that $\dot{x}(n, n-\alpha) = i\omega(n, n-\alpha)\, x(n, n-\alpha)$

while eq. (8) gives $$xy(n, n-\gamma) = \Sigma_\alpha x(n, n-\alpha) y(n-\alpha, n-\gamma)$$

which is just Heisenberg's law of multiplication. By means of this law one can obtain the c-numbers which represent any function of the p's and q's that has the necessary periodic properties when one knows the c-numbers ~~that~~ which represent the p's and q's. Our representation of a q-number is thus complete in itself.

$W(J-\alpha h) < W(J)$

Suppose all the $x(J, J-\alpha h)$ $(\alpha > 0)$ vanish when some sets of c-numbers such as (the set of zero's say) is substituted for the J's. These c-numbers would then define a meta-stable state of the atom. If in addition each $x(J, J-\alpha h)$ vanishes when any $J-\alpha h$ is put equal to any negative integral multiple of h, the state would be the normal state. In this case every $x(n,m)$ would vanish if an n or m is negative, provided we take the n's and m's all integers. It appears that we should have to take the n's and m's all integers in our representation in order that the resulting c-numbers may be of physical importance.

Uses ξ's & η's

$$b = e^{-i(H'-W')t/h} e^{i(H-W)t'/h}.$$

$$e^{-iWt'/h} t\, e^{iWt'/h} = t+t'$$

$$b t b^{-1} = t'$$

$$b W b^{-1} = W + W' - H'$$

$$b = e^{i(H-W)t'/h} e^{-i(H'-W')t/h} e^{i(H-W)t'/h}$$

$$b t b^{-1} = t'$$

$$b(H-W)b^{-1} = H'-W'$$

$\frac{dx}{dt} = [x, H-W]$ if x is fn of undashed variables only

$\frac{dx'}{dt'} = [x', H'-W']$ if x' — dashed —

H is a fn only of undashed variables and dashed variables, that commute with $H'-W'$.

H' is a fn only of dashed variables and fn of dashed of undashed variables that commute with $H-W$.

$$b_1 b_2 b_1 = b_2 b_1 b_2$$

$$b_2^{-1} b_1 b_2 = b_1 b_2 b_1^{-1}$$

$$\cancel{b_1^{-1} b_2}$$

$$b_1 b_2 = b_3$$

$$b_3^{3} = b_2 b_3 b_2$$

$b x b^{-1}$ commutes with $H-W$

$b^{-1} x' b$ — $H'-W'$

$$b = e^{i(Ht' - H't)/h}$$

$$[t+t', (H'-W')t - (H-W)t'] = -t'+t$$

$$[t-t', \quad] =$$

$$[t, (H'-W')t - (H-W)t'] = -t'$$

$$[t', (H'-W')t \; (H-W)t'] = t$$

$$H - W = \text{fn.}(x, b^{-1}x' b)$$

$$b(H-W)b^{-1}$$

$$= H'-W' = \text{fn.}(b x b^{-1}, x')$$

$$b t b^{-1} = t'$$

$$b = e^{iW_0}$$

$$\left.\begin{aligned} \frac{dX}{dt} &= [X, H-W] \\ \frac{dX}{dt'} &= [X, H'-W'] \end{aligned}\right\} \text{Any } X$$

$X^2 = x$

$bXb^{-1} = -X$

Any no that commutes with x either commutes with X or is a fn of the angle variable b, or is a fn of b and numbers that commute with X.

Suppose ξ is a fn of b and of nos that commute with X.

$e^{\pi i J w}$

$X r X^{-1} = X' r X'^{-1}$

$X^2 = X'^2$

$$2mH = 4 c_2 c_1 - m^2 e^4 / k_2^2$$

Put

$$2mH = -m^2 e^4 / P^2$$

P, of course, commutes with k, c_1 and c_2. We have

$$c_1 c_2 = \tfrac{1}{4} m^2 e^4 \left(\frac{1}{k_1^2} - \frac{1}{P^2}\right) = \tfrac{1}{4} m^2 e^4 \frac{\varepsilon_1^2}{k_1^2}$$

$$c_2 c_1 = \tfrac{1}{4} m^2 e^4 \left(\frac{1}{k_2^2} - \frac{1}{P^2}\right) = \tfrac{1}{4} m^2 e^4 \frac{\varepsilon_2^2}{k_2^2} \qquad (27)\ (28)$$

$X^2 r X^{-2} = (XX') r (XX')^{-1}$

$= X'^2 r X'^{-2}$

where

$$\varepsilon_1 = \sqrt{1 - \frac{k_1^2}{P^2}}, \quad \varepsilon_2 = \sqrt{1 - \frac{k_2^2}{P^2}}$$

$\frac{X'}{X} X^{-1}$ c.w. r.

The eccentricities ε_1 and ε_2 are constants, and commute with P and k and with each other. Put

$$c_1 = \tfrac{1}{2} m e^2\, \varepsilon_1 / k_1 \,.\, e^{-i\chi}$$

χ is a constant and so commutes with P.

Since k commutes with $c_1 e^{i\theta}$ and with ε_1/k_1, it must commute with $e^{-i\chi} e^{i\theta}$, so that, i.e.

$$k e^{-i\chi} e^{i\theta} = e^{-i\chi} e^{i\theta} k = e^{-i\chi} (k - h) e^{i\theta}$$

Hence

$$k e^{-i\chi} = e^{-i\chi} (k - h)$$

This law for the interchange of χ and k shows that χ is canonically conjugate to k. We now have

χ corresponds in the classical theory to the angle between the major axis of the ellipse and the line $\theta = 0$.

$$c_1 = \tfrac{1}{2} m e^2\, \varepsilon_1/k_1\, e^{-i\chi} = \tfrac{1}{2} m e^2\, e^{-i\chi}\, \varepsilon_2/k_2$$

$e^{\pi i J w}$

and from (27) ~~or (28)~~

$$c_2 = \tfrac{1}{2} m e^2\, e^{i\chi}\, \varepsilon_1/k_1 = \tfrac{1}{2} m e^2\, \varepsilon_2/k_2\, e^{i\chi}.$$

$J^{\frac{1}{2}} = f(J)$

$f(n) = n^{\frac{1}{2}}$ n a +ve int

$f(n) = 0$ otherwise

The expression (17) for $1/r$ thus takes the form

$$\frac{1}{r} = \frac{m e^2}{k_1 k_2}\left(1 + \tfrac{1}{2} \frac{k_2}{k} \varepsilon_1 e^{-i\chi} e^{i\theta} + \tfrac{1}{2} \frac{k_1}{k} \varepsilon_2 e^{i\chi} e^{-i\theta}\right) \qquad (28)$$

$J^{\frac{1}{2}} e^{iw} = e^{iw} (J + h)^{\frac{1}{2}}$ is true only if $J^{\frac{1}{2}}$ is a fn of the single variable J.

$J = -n$

$$x = J^{\frac{1}{2}} e^{iw} + e^{-iw} J^{\frac{1}{2}}$$

$$[x, J] = i\{J^{\frac{1}{2}} e^{iw} - e^{-iw} J^{\frac{1}{2}}\}$$

fn of a fn is a fn.

Algebraic equations may be used for definition or part definition of a fn.

We do not define at the start all the properties of the q-numbers in use, ~~at the start~~ but add to them later as required.

$$f(x) = x^{\frac{1}{2}} e^{2\pi i n x^{\frac{1}{2}}} \qquad n \text{ an arb. integer}$$

is ind. of n only when $x^{\frac{1}{2}}$ is a positive integer

$n = J$

so that $f(x)$ ind. of n is nec. for $f(x)$ to be a fn of x only, i.e. to commute with w.

$$[S, J_r] = \tfrac{1}{2}\sum_k \{p_k[q_k\, J_r] + [q_k, J_r]p_k\}$$ $\frac{\partial S}{\partial w_r} = \sum_k p_k \frac{\partial q_k}{\partial w_r}$ $\binom{c}{j}$

$$[S, q_r] = \tfrac{1}{2}\sum_k \{w_k[J_k, q_r] + [J_k, q_r]w_k\}$$ $\frac{\partial S}{\partial p_r} = \sum w_k \frac{\partial J_k}{\partial p_r}$

$$[S[q_i\, J_j]] = \tfrac{1}{2}\sum_k \{p_k[q_k[q_i J_j]] + [q_k[q_i J_j]]p_k + w_k[J_k[q_i\, J_j]] + [J_k[q_i, J_j]]w_k\}$$

$$= -[q_i[J_j, S]] + [J_j, [q_i S]]$$

$$= \tfrac{1}{2}\left[q_i, \sum_k\{p_k[q_k J_j] + [q_k, J_j]p_k\}\right] - \tfrac{1}{2}\left[J_j, \sum_k\{w_k[J_k, q_i] + [J_k, q_i]w_k\}\right]$$

$$= \tfrac{1}{2}\sum_k\{p_k[q_i[q_k, J_j]] + 2[q_i, p_k][q_k, J_j] + [q_i[q_k J_j]]p_k - w_k[J_j[J_k, q_i]] - 2[J_j, w_k][J_k, q_i] - [J_j[J_k q_i]]w_k\}$$

$$= \tfrac{1}{2}\sum_k\{p_k[q_k[q_i J_j]] + [q_k[q_i\, J_j]]p_k - w_k[J_k[J_j\, q_i]] - [J_k[J_j, q_i]]w_k\} + \cancel{[q_i, J_j] - [J_j, q_i]}$$

$$a.f(b) = \sum_0^\infty \frac{(ih)^s}{s!} f^s(b)\, a_s$$ $a_s = [[[a, b]b]b] \ldots b]$

$a_0 = a$ $a_{bbb\ldots(s)}$

$$a e^{ib} = \sum_0^\infty \frac{(-h)^s}{s!} e^{ib} a_s$$ $p_k \Delta dq_k = \Delta(p_k dq_k) - \Delta p_k dq_k$

$\cancel{\Delta p_k}\; p_k d\Delta q_k = d(p_k \Delta q_k) - dp_k \Delta q_k$

$$e^{ib} a = \sum_0^\infty \frac{h^s}{s!} a_s e^{ib}$$

$\varepsilon dS = p_k dq_k + \cancel{q_k}\, d\bar{p}_k$

$$e^{ib} a e^{-ib} = \sum \frac{h^s}{s!} a_s$$ $= p_k dq_k - \bar{p}_k d\bar{q}_k + d(p_k q_k)$

$b = S/h$ $d(p_k q_k - \varepsilon S) = \Delta p_k dq_k + p_k \Delta dq_k$

$$e^{iS/h} a e^{-iS/h} = \sum \frac{1}{s!} a_{SSS\ldots(s)}$$ $d(p_k q_k - \varepsilon S - \cancel{p_k \Delta q_k}) = \Delta p_k dq_k$

$d(p_k q_k - \varepsilon S - p_k \Delta q_k) = + \Delta p_k dq_k - dp_k \Delta q_k$

$a = a(H, q)$ $e^{2\pi i J/h} q\, e^{-2\pi i J/h} = q = \sum \frac{1}{s!} q_{2\pi J, 2\pi J, \ldots (s)}$

$$e^{iS/h} p\, e^{-iS/h} = f(H) = J$$ $e^{iS^*/h} = c\, e^{2\pi i (nJ)/h} e^{iS/h}$

$H = e^{+iS/h} f^{-1}(J) e^{-iS/h}$ $q = e^{-iS/h} w\, e^{iS/h}$

$e^{2\pi i J/h} w\, e^{-2\pi i J/h} = w + 2\pi$

$[f(J_s), q_r] = \tfrac{1}{2}\sum_k\{\phi(J_s)[J_k + f(J), q_r] + [J_k + f(J), q_r]\phi(J)\}$

$S^* = S$ with $w_r + 2\pi$ substituted for w_r

$e^{2\pi i S/h} e^{-iS/h} e^{-2\pi i J/h}$

$p_r = e^{-iS/h} J_r e^{iS/h} = e^{-iS^*/h} J_r e^{iS^*/h}$

$q_r = e^{iS/h} w_r e^{+iS/h} = e^{iS^*/h}(w_r + 2\pi)e^{iS^*/h} = e^{-iS^*/h} e^{2\pi i J/h} w_r e^{-2\pi i J/h} e^{iS^*/h}$

$e^{iS^*/h} = K e^{iS/h}$ where K commutes with the J's

$e^{-2\pi i (nJ)/h}$ K commutes with the w's and J's, and is a c-number

$e^{-2\pi i n_r J/h}$ K commutes with w_r

10	7	3		4	7	3
4	1	3		2	1	3
2	1	1		0	1	1

~~algebra. It is a particular case of a more general axiom, which may be enunciated thus:— any number that commutes with ev~~

The papers consist ~~deal~~ entirely (except no ~~which is pure mathematics~~) of contributions to the new ~~quo~~ mechanics of the atom that was introduced by Heisenberg in 1925. The main ~~essential~~ point of H's theory is that the dynamical variables of the classical theory are to be replaced by matrices satisfying ~~formally~~ (as far as possible) the same equations as the classical variables. The classical equation of the ~~classical theory~~ $xy = yx$, where x & y are arbitrary variables, which expresses the commutative law of multiplication ~~is~~ are no longer in general true for matrices and the ~~equation that replace~~ in paper 1 the hypothesis is put forward that this equation is to be replaced in the Q.T. by $xy - yx = \frac{ih}{2\pi}[x, y]$ (x & y now being matrices) where h is Planck's constant and the expression $[x, y]$ is the analogue of the P.B. of the ~~analy~~ in the classical theory. The justification of this assumption is the similarity in the algebraic laws governing the expression $xy - yx$ and the P.B $[x, y]$ of the classical theory.

In paper 2 it is attempted to establish the theory from a new point of view in which pre-eminence is given to the fact that the dynamical variables in the quantum theory do not obey the comm. law of multiplication, and the fact that matrices can be found to represent the variables is considered of less importance. A method ~~theory~~ of information variables is given ~~obtained~~ which is applied to the Hydrogen atom, the Balmer formula being obtained, and also, in paper 3, to a number of problems relating to the orbital motion of electrons in an atom, ~~[illegible]~~ such as the anomalous Zeeman effect and the ratio of the intensities of the spectral lines in a multiplet. In paper 4 the methods ~~theory~~, after being modified so as to give results in agreement with the principle of relativity, is applied to the scattering of radiation by a free electron. Compton's formula for the frequency of the scattered radn. is obtained, and also a new formula for the intensity of the scattered radiation, which is in agreement ~~agrees with~~ experiment, and a law for the pol. of the scatt. radn., which has subsequently been conf. by expts. The same question is treated in paper 7 by a different method. Paper 5 is entirely pure mathematics, and the axioms there made are not suitable for the later form of the quantum theory (see paper 8)

The whole work is based on the idea of a quantum mechanics which was put forward by H in Z f P. --

The papers after the 5th have been greatly influenced by Schrödinger's works in Ann d. Phys and are based on his ~~methods~~ ideas

The individual results in the paper have in part been obtained indepently by other investigators at about the same time e.g. the quantum condition given in paper 1 were obtained independently by B+J, the Balmer formula deduced in paper 2 was obtained ind. by Pauli

The main original points in the papers are as follows:—

The relation introduced in paper 1 and used in 2, 3, 4 between $xy - yx$ on the Q.T and the $P.B\,[x, y]$ of the classical theory.

The method of uniformising variables given in paper 2 and its application in papers 2 & 3.

The theory of the Compton effect given in paper 4, which leads to a formula for the intensity of the scattered radiation in agreement with experiment.

In places use has been made of the work of other investigators in Q.M., to which reference is given in footnotes, ~~the most important~~ chief cases of this ~~being that~~ papers 6–10 ~~are largely based~~ depend very much on S's ideas, and that the theory of paper 8 is a development of Heisenberg's ~~theory~~ of fluctuations given in Z f Phys —
Statement of originality of the Papers

The papers contain original contributions to the theory of Q.M. introduced by Heisenberg in Z f Phys, which work of Heisenberg provided the fundamental idea on which the whole of the papers are based. The main original points of the papers are as follows:—

The quantum conditions of paper 1.

The theory of uniformising variables of paper 2 and its applications in papers 2, 3, 4.

The theory of the Compton effect in paper 4.

The ~~theory of~~ statistical theory of paper 7 §3, 4. (The statistics ~~which are~~ required by Pauli's exclusion principle had been previously worked out, without reference to Q.M., by Fermi, Z f Phys —)

The new formulation of quantum mechanics in paper 8.

The radiation theory of paper 9 and its application to dispersion in paper 10.

The perturbation theory of paper 9 §6.

~~Some~~ Two of these results have been obtained independently at about the same time by other investigators, namely, the quantum conditions by Born and Jordan, the statistical theory of paper 7 (in part) by Heisenberg, and a theory mathematically equivalent to that of paper 8 has been given by Jordan.

algebra. It is a particular case of a more general axiom which may be enunciated thus:- If a number commutes with every number that commutes with the q-numbers $q, q_1, q_2 \ldots q_n$ and with one number that does not commute with q but commutes with $q_1, q_2 \ldots q_n$, then it commutes with every number that commutes with $q_1, q_2 \ldots q_n$. (Note that a number x that commutes with every number must be a c-number, on account of the ~~axiom~~ assumption that if x is a q-number, there m

In paper 9 ~~it is shown that~~ the problem is considered, on the basis of paper 8, of the interaction of a number of light quanta with an atom. It is shown that, owing to the special statistics ~~satisfied~~ obeyed by light quanta, the Hamiltonian function that describes the interaction is of the same form as that which describes the interaction of the atom with quantised electromagnetic waves. This enables one to build up a self-consistent theory of radiation which contains the essential features of both the corpuscular and wave theories. This theory completes that of paper 7 & 5 by accounting in a natural manner for spontaneous emission. In paper 10 this theory is applied to dispersion.

The later papers (6-10) ~~have been greatly influenced by S's work~~ are based on S's ~~ideas~~ papers

which ~~may be~~ provide a powerful mathematical method of obtaining the matrices of H's theory although the physical ideas ~~stated in~~ of these papers, which involve practically a return to classical mechanics, has ~~not~~ proved to be untenable. An example of this method is given in paper 6, where it is applied to the Compton effect ~~is treated~~ and results are obtained in agreement with those of paper 14. In paper 7 ~~it is shown that two~~ the kinds of statistics are allowed by Q.M. ~~are~~ discussed (§ 3, 4) and also in § 5 a ~~general~~ perturbation theory is given which is shown to account correctly for the abs. and stimulated emission of radiation by an atom, although not the spontaneous emission.

In paper 8 a new and more general basis is provided for ~~Q.M.~~ ~~the whole theory~~ which supersedes the theory of paper 2. It is shown that the dynamical ~~quantum~~ variables of the quantum theory ~~are to~~ can be represented by matrices in a number of different matrix schemes, not only in one as in H's ~~first~~ theory. The ~~quantum~~ variables are to be regarded as completely defined when one is given the matrices that represent them in any one scheme, ~~and one can then obtain~~ it being then possible to obtain the matrices that represent them in any other scheme by ~~from~~ simple transformation laws. The theory provides a general method for the physical interpretation of Q.M. It appears that it is permissible to talk about dynamical variables having specified numerical values provided that one does not give numerical values simultaneously to two variables that do not commute, e.g. one can state that for a given atomic ~~system~~ system $x = a$ and $y = b$ where x and y are dynamical variables (co-ordinates or velocities ~~or distances~~ etc) and a and b are numbers, provided $xy = yx$. If xy have $\neq yx$, then for a system in which $x = a$ one can only say that there is a certain probability of y having any specified value. This probability is given by the theory, and is ~~[illegible]~~ (apart from certain factors) the diagonal element $(a\ a)$ of the matrix representing y in a matrix scheme in which x is a diagonal matrix. Thus if one is given, say the initial values of certain of the dynamical variables, one can calculate the prob. of any variable having any specified value at any subsequent time. From this theory all the special assumptions contained in previous forms of the Q.T. can be deduced as special cases.