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The quasi-normal modes of the Schwarzschild black hole

BY S. CHANDRASEKHAR, F.R.S. AND S. DETWEILER
University of Chicago, Chicago, Illinois, 60637

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The quasi-normal modes of a black hole represent solutions of the relevant perturbation equations which satisfy the boundary conditions appropriate for purely outgoing (gravitational) waves at infinity and purely ingoing waves at the horizon. For the Schwarzschild black hole the problem reduces to one of finding such solutions for a one-dimensional wave equation (Zerilli's equation) for a potential which is positive everywhere and is of short-range. The notion of quasi-normal modes of such one-dimensional potential barriers is examined with two illustrative examples; and numerical solutions for Zerilli's potential are obtained by integrating the associated Riccati equation.

1. INTRODUCTION

It is known that the evolution of an arbitrary perturbation of the metric coefficients of the Schwarzschild black hole can be fully described in terms of the reflexion (R) and the transmission (T) coefficients of the one-dimensional barrier represented by Zerilli's potential (Zerilli 1970; see also Chandrasekhar 1975). Nevertheless, the notion of *quasi-normal modes* of a black hole has been introduced in the literature in analogy with the normal modes of oscillation of a star. In the context of a black hole these quasi-normal modes are defined as proper solutions of the perturbation equations belonging to certain complex characteristic frequencies which satisfy the boundary conditions appropriate for purely ingoing waves at the horizon and purely outgoing waves at infinity.

It does not appear that the quasi-normal modes of a black hole serve the same purposes as the normal modes of oscillation of a star. Consider, for example, the spherically symmetric perturbations of an initially static configuration either in the Newtonian (Eddington 1918, 1919) or in the relativistic (Chandrasekhar 1964) framework. In either framework, the determination of the characteristic frequencies leads to a two-point boundary-value problem of the classical Sturmian type for a self-adjoint second order differential equation. Consequently, the associated proper solutions (i.e. the normal modes) form a *complete set* in the sense that any arbitrary spherically symmetric perturbation of the star (compatible with the boundary conditions of the problem) can be expressed as a linear superposition of the normal modes. And, therefore, the evolution of any such perturbation can be followed in terms of the normal modes and the characteristic frequencies to which they belong. Also, it follows that if any of the modes should belong to a (purely) imaginary characteristic frequency – this is the only possibility for the spherically symmetric systems under consideration – then the system can be considered as necessarily unstable. It is not at all clear that the enumeration of the quasi-normal modes of a black hole serves the same purposes as the normal modes of radial oscillation of a star. Thus, in the particular case of the Schwarzschild black hole, the evolution of an arbitrary perturbation of its metric coefficients can be followed, as we have already stated, in terms of the reflexion and the transmission coefficients of Zerilli's potential for various frequencies; and a knowledge of its quasi-normal modes is of no relevance for that purpose.

In spite of the adverse comments we have made concerning the usefulness of the notion of quasi-normal modes, their determination in the case of the Schwarzschild black hole has some interest, at least to the extent that it may illuminate their relevance (or, otherwise) for our understanding of the physics of black holes. This paper, then, is devoted to an examination of the general problem of the quasi-normal modes of one-dimensional potential barriers and of Zerilli's potential in particular.

2. The general theory

Consider the simple one-dimensional wave equation,

$$\frac{d^2\psi}{dx^2}+[\sigma^2-V(x)]\psi = 0 \quad (-\infty < x < +\infty), \tag{1}$$

where $V(x)$ is positive everywhere and is of 'short range' in the sense that

$$\int_{-\infty}^{+\infty} V(x)\,dx \quad \text{is finite.} \tag{2}$$

It is evident that if we have a plane-wave of unit amplitude $e^{+i\sigma x}$ incident on the barrier from the right,† then a part of it will be reflected and a part of it will be transmitted; i.e. there will be an admixture of the incident wave with a reflected wave $A\,e^{-i\sigma x}$ of amplitude A (say) at $+\infty$ and there will also be a transmitted wave $B\,e^{+i\sigma x}$ of amplitude B (say) at $-\infty$. The reflexion and the transmission coefficients are then given by

$$R = |A|^2 \quad \text{and} \quad T = |B|^2; \tag{3}$$

and it must always be true that

$$R+T = 1, \tag{4}$$

so long as σ is real. On the other hand, a quasi-normal mode belonging to a complex frequency σ is so defined that there is no wave incident on the barrier from the right and we only have a reflected wave at $+\infty$ and a transmitted wave at $-\infty$.

† The convention that $e^{+i\sigma x}$ represents an ingoing wave is the opposite of the one which is normally adopted in the quantum theory; it is a consequence of the assumption, normal in this theory, that the time-dependence of the normal modes is $e^{i\sigma t}$.

The problem of determining the reflexion and the transmission coefficients of a rectangular barrier is a standard exercise in elementary quantum mechanics; and from the solution of this problem given in textbooks (cf. Flugge & Marschall 1952, p. 40) one can readily obtain the equations which determine the quasi-normal modes. But for general potential barriers for which explicit solutions cannot be found (as is the case with Zerilli's potential) it is convenient to reformulate the problem explicitly as a standard characteristic-value problem by transforming equation (1) to the form of a Riccati equation by the substitution,

$$\psi = \exp\left(\mathrm{i}\int^{x} \phi \, \mathrm{d}x\right), \tag{5}$$

and obtaining

$$\mathrm{i}\,\mathrm{d}\phi/\mathrm{d}x + \sigma^2 - \phi^2 - V(x) = 0. \tag{6}$$

A quasi-normal mode corresponds to a solution of equation (6) which satisfies the boundary conditions

$$\phi \to -\sigma \quad \text{as} \quad x \to +\infty \quad \text{and} \quad \phi \to +\sigma \quad \text{as} \quad x \to -\infty, \tag{7}$$

with the real part of σ assumed to be positive. Solutions having these properties (generally) exist when σ assumes one of a discrete set of complex values; but the set need not be an enumerable infinity: it can be sometimes, but often it is not.

An identity, which follows from integrating equation (6) over the entire range of x and making use of the boundary conditions (7), is

$$-2\mathrm{i}\sigma + \int_{-\infty}^{+\infty} (\sigma^2 - \phi^2)\, \mathrm{d}x = \int_{-\infty}^{+\infty} V(x)\, \mathrm{d}x. \tag{8}$$

By virtue of the boundary conditions (7) and the assumed short-range character of $V(x)$, both the integrals which appear in equation (8) are finite.

In practice it is useful to separate the real and the imaginary parts of equation (6) by writing

$$\sigma = \sigma_1 + \mathrm{i}\sigma_2 \quad \text{and} \quad \phi = \phi_1 + \mathrm{i}\phi_2 \quad (\sigma_1 \geqslant 0). \tag{9}$$

We then obtain the pair of equations,

$$\mathrm{d}\phi_1/\mathrm{d}x = -2\sigma_1\sigma_2 + 2\phi_1\phi_2 \tag{10}$$

and

$$\mathrm{d}\phi_2/\mathrm{d}x = \sigma_1^2 - \sigma_2^2 - \phi_1^2 + \phi_2^2 - V, \tag{11}$$

together with the boundary conditions

$$\phi_1 \to -\sigma_1 \quad \text{as} \quad x \to +\infty \quad \text{and} \quad \phi_1 \to +\sigma_1 \quad \text{as} \quad x \to -\infty, \tag{12}$$

and

$$\phi_2 \to -\sigma_2 \quad \text{as} \quad x \to +\infty \quad \text{and} \quad \phi_2 \to +\sigma_2 \quad \text{as} \quad x \to -\infty. \tag{13}$$

It can sometimes happen that σ is purely imaginary so that $\sigma_1 = 0$. Since, in this case, ϕ must vanish at $\pm\infty$, it follows from equation (10) that $\phi_1 \equiv 0$; and we are left with asking whether there exist non-trivial solutions of the equation,

$$\mathrm{d}\phi_2/\mathrm{d}x = -\sigma_2^2 + \phi_2^2 - V, \tag{14}$$

which satisfy the boundary conditions (13).

The problem of the quasi-normal modes as formulated in terms of the phase ϕ appears explicitly as a problem in characteristic values though of a somewhat unconventional kind.

3. Two illustrative examples

Since the notion of quasi-normal modes of a one-dimensional potential barrier is not considered in the standard literature on quantum mechanics, it may be useful to consider them in the context of two elementary situations which admit of explicit solutions.

(a) *A rectangular barrier*

Let the potential barrier be defined by

$$\begin{aligned} V(x) &= U > 0 \quad \text{for} \quad 0 < x < a \\ &= 0 \quad \text{for} \quad x < 0 \quad \text{and} \quad x > a. \end{aligned} \tag{15}$$

Further, letting
$$\kappa^2 = U - \sigma^2, \tag{16}$$

we can write the required solution of equation (6) in the form

$$\left.\begin{aligned} \phi &= +\sigma && (x \leqslant 0) \\ &= -\mathrm{i}\kappa \tanh(\kappa x + \kappa c) && (0 \leqslant x \leqslant a) \\ &= -\sigma && (x \geqslant a), \end{aligned}\right\} \tag{17}$$

where c is a constant. The continuity of ϕ, at $x = 0$ and $x = a$, gives

$$\sigma = -\mathrm{i}\kappa \tanh \kappa c$$

and
$$\sigma = +\mathrm{i}\kappa \tanh(\kappa a + \kappa c). \tag{18}$$

The characteristic equation for σ will follow from the elimination of κc from the foregoing equations. In carrying out the elimination, it will be convenient to write

$$\sigma = Q \sin\alpha \quad \text{and} \quad \kappa = -Q\cos\alpha \quad \text{where} \quad Q^2 = U; \tag{19}$$

and we shall adopt the convention $\quad Q \geqslant 0. \quad (20)$

With the foregoing definitions, the elimination of κc from the pair of equations (18) gives

$$\cosh \kappa a + \mathrm{i} \cot 2\alpha \sinh \kappa a = 0. \tag{21}$$

In considering equation (21), we shall measure σ in the unit a^{-1} and let

$$\alpha = \alpha_1 + \mathrm{i}\alpha_2 \quad \text{and} \quad \kappa = \kappa_1 + \mathrm{i}\kappa_2. \tag{22}$$

Equation (21) then leads to the pair of equations,

$$\frac{\sin 4\alpha_1}{\cosh 4\alpha_2 - \cos 4\alpha_1} = \frac{\sin 2\kappa_2}{\cosh 2\kappa_1 - \cos 2\kappa_2} \tag{23}$$

and
$$-\frac{\sinh 4\alpha_2}{\cosh 4\alpha_2 - \cos 4\alpha_1} = \frac{\sinh 2\kappa_1}{\cosh 2\kappa_1 - \cos 2\kappa_2}. \tag{24}$$

From these equations it follows that

$$\kappa_1 = -2\alpha_2 \quad \text{and} \quad \kappa_2 = 2\alpha_1 - n\pi, \tag{25}$$

where n is an integer positive, negative, or zero. On the other hand (cf. equation (18))

$$\kappa_1 = -Q\cos\alpha_1 \cosh\alpha_2 \quad \text{and} \quad \kappa_2 = +Q\sin\alpha_1 \sinh\alpha_2. \tag{26}$$

By combining equations (25) and (26), we obtain the 'characteristic equation'

$$\tan\alpha_1 \tanh\alpha_2 = (\alpha_1 - \tfrac{1}{2}n\pi)/\alpha_2. \tag{27}$$

If (α_1, α_2) is a pair of values which satisfies equation (27), then the corresponding values of Q, σ_1, and σ_2 follow from the equations

$$Q = 2[(\alpha_1 - \tfrac{1}{2}n\pi)/\sin\alpha_1)]\operatorname{cosech}\alpha_2, \tag{28}$$

$$\sigma_1 = Q\sin\alpha_1\cosh\alpha_2 = 2\alpha_2\tan\alpha_1 \tag{29}$$

and

$$\sigma_2 = Q\cos\alpha_1\sinh\alpha_2 = 2\alpha_2\tanh\alpha_2. \tag{30}$$

The convention with respect to the outgoing and the ingoing waves requires that

$$\sigma_1 \geqslant 0. \tag{31}$$

From the requirements (20) and (31), it follows from equations (28)–(30) that

$$\text{if} \quad \alpha_2 > 0 \quad \text{then} \quad n \leqslant 0 \quad \text{and} \quad 0 \leqslant \alpha_1 \leqslant \tfrac{1}{2}\pi \tag{32}$$

or

$$\text{if} \quad \alpha_2 < 0 \quad \text{then} \quad n > 0 \quad \text{and} \quad \pi \geqslant \alpha_1 \geqslant \tfrac{1}{2}\pi. \tag{33}$$

We shall adopt the former conditions (32).

We observe that $\sigma_2 > 0$; and this is consistent with the fact that these waves are damped, though the proper solutions themselves diverge exponentially both for $x \to +\infty$ and $x \to -\infty$. These divergences are admissible in the present context.

For the case $n = 0$, corresponding to the lowest mode, the relevant equations are

$$\frac{\tan\alpha_1}{\alpha_1} = \frac{\coth\alpha_2}{\alpha_2}; \quad Q = 2\frac{\alpha_1}{\sin\alpha_1}\operatorname{cosech}\alpha_2, \tag{34}$$

$$\sigma_1 = 2\alpha_2\tan\alpha_1 \quad \text{and} \quad \sigma_2 = 2\alpha_2\tanh\alpha_2.$$

From these equations it follows that $Q \geqslant Q_{\min}$, where $Q_{\min}$ occurs for $\alpha_1 \to 0$ and α_2 is the root of the equation

$$\alpha_2\tanh\alpha_2 = 1. \tag{35}$$

Denoting the root of this equation by α_2^* ($\simeq 1.1997$), we find that

$$Q = 2\operatorname{cosech}\alpha_2^* \,(= 1.3255), \quad \sigma_1 = 0, \quad \text{and} \quad \sigma_2 = 2. \tag{36}$$

For $Q \leqslant Q_{\min}$, the characteristic values of σ are purely imaginary; and the appropriate solution, found in accordance with equation (14), is given by

$$Q = 2\alpha_2\operatorname{sech}\alpha_2 \quad \text{and} \quad \sigma_2 = 2\alpha_2\tanh\alpha_2. \tag{37}$$

Accordingly, along this branch of the solution, $Q = 0$ both when $\alpha_2 \to 0$ and when $\alpha_2 \to \infty$; and Q attains its maximum for precisely the same value of α_2^* (as the solution of equation (35)) when σ_2 has again the value 2.

The higher modes are obtained for $n = -1, -2, -3$, etc., in accordance with equations (28)–(30). It can be readily shown that along these higher modes

$$\alpha_1 \to 0, \quad \alpha_2 \to \infty \quad \text{while} \quad \alpha_1 \alpha_2 \to \tfrac{1}{2}|n|\pi; \tag{38}$$

and simultaneously

$$Q \to 4\alpha_2 e^{-\alpha_2}, \quad \sigma_1 \to |n|\pi, \quad \text{and} \quad \sigma_2 \to 2\alpha_2. \tag{39}$$

The behaviour of the characteristic values belonging to the various modes is illustrated in figure 1.

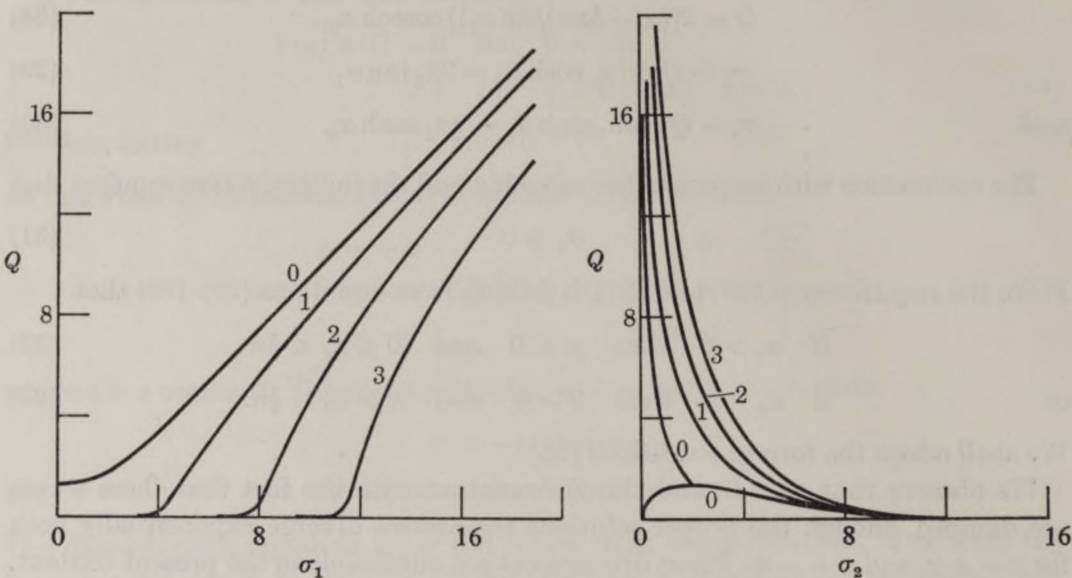

FIGURE 1. The complex frequencies $(\sigma_1 + i\sigma_2)$ belonging to the quasi-normal modes of a rectangular barrier. The frequencies are measured in units of the width of the barrier (a); and Q^2 $(= a^2 U)$ is a measure of the height of the barrier. The curves are labelled by the values of $-n$ to which they belong.

(b) *Price's model potential for the Schwarzschild black hole*

Since we do not expect to solve Zerilli's equation in terms of known functions, Price (1972*a*, *b*) has considered the following potential as having the same general behaviour as Zerilli's potential and for which analytic solutions can be found:

$$\begin{aligned} V(x) &= [l(l+1)]/x^2 \quad \text{for} \quad x \geqslant m \\ &= 0 \qquad\qquad\quad\ \ \text{for} \quad x < m. \end{aligned} \tag{40}$$

In the actual physical problem the variable x is identified with r_* (see §4 below).

The problem of the quasi-normal modes for Price's potential has been solved by Press (1973); but since his solution has not been published, we may briefly outline the solution.

For Price's potential (with the convention regarding outgoing and ingoing waves which we have adopted)

$$\psi(x) = \mathrm{e}^{\mathrm{i}\sigma x} \quad \text{for} \quad x \leqslant m \tag{41}$$

with $\mathrm{Re}(\sigma) \geqslant 0$. For $x \geqslant m$, the general solution of the wave equation is clearly a linear combination of $x^{\frac{1}{2}} J_{l+\frac{1}{2}}(\sigma x)$ and $x^{\frac{1}{2}} J_{-l-\frac{1}{2}}(\sigma x)$; and from the known series expansions for $J_{\pm(l+\frac{1}{2})}(Z)$ (cf. Watson 1922, pp. 53 and 54, equations (1) and (4)) we conclude that

$$\psi(x) = A\,\mathrm{e}^{-\mathrm{i}\sigma x} \sum_{s=0}^{l} \frac{(-\mathrm{i})^s (l+s)!\,(\sigma x)^{-s}}{2^s s!\,(l-s)!} \quad (x \geqslant m), \tag{42}$$

where A is a constant. The continuity of the logarithmic derivative of ψ (i.e. of the phase function ϕ) at $x = m$ leads to the following algebraic equation:

$$2\mathrm{i} \sum_{s=0}^{l} \frac{(-\mathrm{i})^s (l+s)!\,(\sigma m)^{-s}}{2^s s!\,(l-s)!} + \sum_{s=0}^{l} \frac{(-\mathrm{i})^s s(l+s)!\,(\sigma m)^{-s}}{2^s s!\,(l-s)!} = 0. \tag{43}$$

From the character and the order of this equation we infer that if l is odd there are exactly $\frac{1}{2}(l+1)$ complex roots for σ with both its real and imaginary parts positive; while if l is even there is one purely imaginary root and $\frac{1}{2}l$ complex roots with both its real and imaginary parts positive. All these roots correspond to damped waves though they exponentially diverge both for $x \to +\infty$ and for $x \to -\infty$.

Press's numerical solutions of equation (43) are listed along with those of Zerilli's equation in table 1 (see §4 below).

We observe that in contrast to the case of the rectangular barrier, there are only a finite number of quasi-normal modes for each l.

4. The quasi-normal modes of the Schwarzschild black hole

We now turn to the determination of the quasi-normal modes of Zerilli's equation

$$\mathrm{d}^2 Z/\mathrm{d}r_*^2 + (\sigma^2 - V_Z)\,Z = 0, \tag{44}$$

where

$$V_Z(r) = \frac{2n^2(n+1)\,r^3 + 6n^2 m r^2 + 18 n m^2 r + 18 m^3}{r^3 (nr+3m)^2}\left(1 - \frac{2m}{r}\right), \tag{45}$$

$$r_* = r + 2m \lg\left(\frac{r}{2m} - 1\right), \quad \text{and} \quad n = \tfrac{1}{2}(l-1)(l+2). \tag{46}$$

Zerilli's potential is clearly of short range; indeed, we have (cf. Chandrasekhar 1975, equation (38))

$$2m \int_{-\infty}^{+\infty} V_Z\,\mathrm{d}r_* = 2n + \tfrac{1}{2} = (l-1)(l+2) + \tfrac{1}{2}. \tag{47}$$

As $r_* \to \pm\infty$, equation (44) allows two independent solutions with the asymptotic behaviours

$$Z_\pm \to \mathrm{e}^{\pm \mathrm{i}\sigma r_*}. \tag{48}$$

As we are assuming a time dependence of the form $e^{i\sigma t}$, Z_- represents an outgoing wave and Z_+ represents an ingoing wave. A quasi-normal mode is one which belongs to a complex σ with $\mathrm{Re}(\sigma) \geqslant 0$, such that it represents a purely outgoing wave at $+\infty$ and a purely ingoing wave at $-\infty$.

A straightforward procedure for finding a quasi-normal mode would appear to be the following. For some chosen complex σ, start from a large positive value of r_*, where $|V(r_*)| \ll |\sigma|^2$, with $Z = Z_- = e^{-i\sigma r_*}$ and integrate backwards to some intermediate value of r_*; and similarly, start from a large negative value of r_*, where again $|V(r_*)| \ll |\sigma|^2$, with $Z = Z_+ = e^{+i\sigma r_*}$ and integrate forwards to the same intermediate value of r_*. The condition that the chosen value of σ belongs to a quasi-normal mode is, clearly, that the Wronskian of the two solutions, at the common point to which we have integrated them, vanishes. The vanishing of the Wronskian provides, then, with a criterion for searching in the positive half of the complex σ-plane for the characteristic values belonging to the quasi-normal modes. But this procedure, simple as it appears, is beset with grave numerical instabilities. They arise from the finite numerical accuracy of all methods of numerical integrations. Thus, for a large positive r_*, the solution for Z, with the asymptotic behaviour $e^{-i\sigma r_*}$, will very soon be contaminated by an admixture with the solution $e^{+i\sigma r_*}$ (which is exponentially small for large r_* when the imaginary part of σ is positive); and by the time we have integrated backwards to some finite r_*, the admixture with the unwanted solution will become appreciable and the solution, we shall be integrating, will no longer be of the kind that we have stipulated. The same thing will happen with the solution integrated forwards from a large negative r_*: the solution started with the asymptotic behaviour $e^{+i\sigma r_*}$ will be contaminated with the solution $e^{-i\sigma r_*}$ $(r_* \rightarrow -\infty)$; and, again, the solution will not be of the kind that we have stipulated by the time we have integrated it to some finite r_*.

The numerical instabiities we have described in the foregoing paragraph have prevented, so far, the determination of the quasi-normal modes of the Schwarzschild black hole (see, however, Detweiler 1975). But it appears that the integration of the first-order Riccati equation (6) is not beset with numerical instabilities, at least, to the same extent. The reason for the relative stability of integrating the Riccati equation for the phase function ϕ (as compared with the direct integration of the wave equation) appears to be that we can start the integrations, backwards from a positive r_* and forwards from a negative r_*, without regard to the requirement $|V(r_*)| \ll \sigma^2$, so long as we can ensure that we have convergent series expansions at both ends which are adequate enough to determine ϕ with sufficient accuracy to values of r_* where it differs substantially from its limiting values at $\pm\infty$.

For the particular case of Zerilli's equation the required series expansions are of the forms,

$$Z = e^{-i\sigma r_*} \sum_{j=0}^{\infty} \alpha_j r^{-j} \quad (r_* \rightarrow +\infty) \tag{49}$$

and

$$Z = e^{+i\sigma r_*} \sum_{j=0}^{\infty} \beta_j (r-2m)^j \quad (r_* \rightarrow -\infty), \tag{50}$$

where the coefficients α_j and β_j can be determined with the aid of the recurrence relations

$$\begin{aligned}2\mathrm{i}\sigma n^2(j+1)\,\alpha_{j+1}+[n^2j(j+1)-2n^2(n+1)+12\mathrm{i}\sigma mjn]\,\alpha_j\\ +m[6nj(j-1)-2n^2(j^2-1)-6n^2+18\mathrm{i}\sigma m(j-1)]\,\alpha_{j-1}\\ +m^2[9(j-1)(j-2)-12nj(j-2)-18n]\,\alpha_{j-2}\\ -18m^3[(j-1)(j-3)+1]\,\alpha_{j-3}=0,\end{aligned} \tag{51}$$

and

$$\begin{aligned}&2\mathrm{i}\sigma n^2(j-1)\,\beta_{j-1}+j[n^2(j-1+8\mathrm{i}\sigma m)+12\mathrm{i}\sigma mn(n+1)-(A/j)]\,\beta_j\\ &+m(j+1)\,[n^2(2j+2+8\mathrm{i}\sigma m)+6n(n+1)\,(j+8\mathrm{i}\sigma m)+6\mathrm{i}\sigma m(2n+1)\,(2n+3)\\ &\qquad -(B/m(j+1))]\,\beta_{j+1}\\ &+m^2(j+2)\,[6n(n+1)\,(2j+4+8\mathrm{i}\sigma m)+3(2n+1)\,(2n+3)\,(j+1+8\mathrm{i}\sigma m)\\ &\qquad +4\mathrm{i}\sigma m(2n+3)^2-(C/m^2(j+2))]\,\beta_{j+2}\\ &+m^3(j+3)\,[3(2n+1)\,(2n+3)\,(2j+6+8\mathrm{i}\sigma m)+2(2n+3)^2(j+2+8\mathrm{i}\sigma m)\\ &\qquad -(D/m^3(j+3))]\,\beta_{j+3}\\ &+4m^4(j+4)\,(2n+3)^2\,(j+4+4\mathrm{i}\sigma m)\,\beta_{j+4}=0,\end{aligned} \tag{52}$$

where

$$A=2n^2(n+1),\quad B=6n^2(2n+3)\,m,$$

$$C=6n(4n^2+8n+3)\,m^2\quad\text{and}\quad D=(16n^3+40n^2+36n+18)\,m^3. \tag{53}$$

With Z determined at both ends by the expansions (49) and (50), the phase function ϕ follows from the equation

$$\phi=\frac{1}{Z}\frac{\mathrm{d}Z}{\mathrm{d}r_*}. \tag{54}$$

The search for the characteristic values of σ belonging to the quasi-normal modes proceeds, then, as follows. We choose a complex value of σ in the positive-half plane ($\mathrm{Re}\,(\sigma)>0$) and determine the expansion coefficients α_j and β_j in accordance with equations (51)–(53) and evaluate ϕ for values of r_* (both positive and negative) for which the series expansions (49) and (50) suffice to determine it accurately enough.† We then continue by numerical integration, backwards from $+\infty$ and forwards from $-\infty$, to a common intermediate value of r_* (generally $3m$ where V_Z is approximately at its maximum). At this common point we find the difference

$$M(\sigma)=\phi_-(r_*)-\phi_+(r_*). \tag{55}$$

The condition that the chosen value of σ belongs to a quasi-normal mode is that $M(\sigma)$ vanishes (in view of the Riccati equation being of the first order).

It was found that the foregoing procedure enables the determination of the quasi-normal modes so long as

$$|\mathrm{Im}\,(\sigma)|\leqslant|\mathrm{Re}\,(\sigma)|. \tag{56}$$

† It is necessary to retain as many terms in the expansions as are necessary to determine ϕ until it is substantially different from its limiting values at $\pm\infty$.

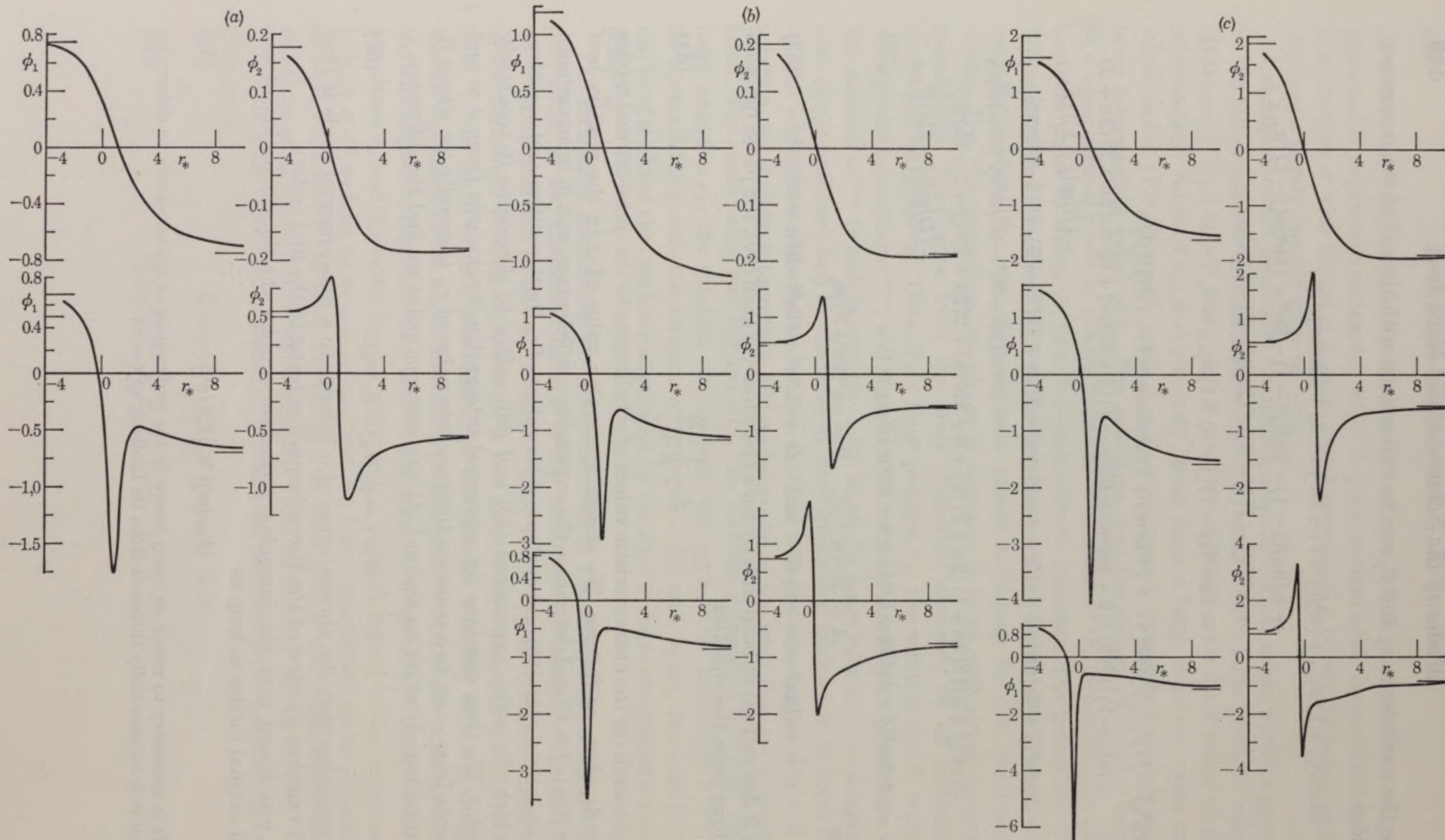

FIGURE 2. The complex phase $(\phi_1 + i\phi_2)$ belonging to the quasi-normal modes of Zerilli's potential for $l = 2$, 3, and 4 (figures (a), (b), and (c), respectively). The asymptotes $(\pm\sigma_1$ and $\pm\sigma_2)$ to which the phases ϕ_1 and ϕ_2 tend for $r_* \to +\infty$ and $-\infty$ are indicated. It will be observed that there are two modes for $l = 2$ and three each for $l = 3$ and 4.

When this condition is violated, the numerical integration appears to suffer from instabilities. The underlying cause is probably the same as in the direct integration of the wave equation though it is obscured (and made less unstable!) by the non-linearity of the Riccati equation.

In our search for the characteristic values of σ (for $l \leqslant 4$) in the positive half of the complex σ-plane, we limited ourselves to only those regions in which we were satisfied that the numerical procedure adopted was free from instabilities.

In figure 2 we illustrate the quasi-normal modes determined, in the fashion described, for $l = 2$, 3, and 4; and in table 1 we list the characteristic values of σ to which they belong. In table 1, we also list the characteristic values for Price's potential (as determined by Press). It will be observed that the two sets of values differ very markedly. This difference is perhaps not very surprising in view of the identity (8) and the fact that the integral,

$$2m\int_{m}^{\infty}\frac{l(l+1)}{r_*^2}\,\mathrm{d}r_* = 2l(l+1), \tag{57}$$

for Price's potential differs from the corresponding integral for Zerilli's potential (cf. equation (47)) by a factor exceeding 2.

TABLE 1. THE COMPLEX FREQUENCIES BELONGING TO THE QUASI-NORMAL MODES OF ZERILLI'S AND PRICE'S POTENTIALS: σ IS EXPRESSED IN THE UNIT $(2m)^{-1}$

	$2m\sigma$	
l	Zerilli's potential	Price's potential
2	0.74734 + 0.17792i	2.5087 + 1.6871i
	0.69687 + 0.54938i	2.6258i
3	1.19889 + 0.18541i	4.1604 + 2.2210i
	1.16402 + 0.56231i	1.3802 + 3.7790i
	0.85257 + 0.74546i	
4	1.61835 + 0.18832i	5.8840 + 2.6647i
	1.59313 + 0.56877i	2.9090 + 4.6967i
	1.12019 + 0.84658i	5.2771i

5. THE ODD-PARITY MODES

In determining the quasi-normal modes of the Schwarzschild black hole in §4, we restricted ourselves to the even-parity modes described by the Zerilli equation. But it is known that the odd-parity perturbations are described by the Regge–Wheeler equation (Regge & Wheeler 1957; see also Chandrasekhar 1975, appendix):

$$\mathrm{d}^2X/\mathrm{d}r_*^2 + (\sigma^2 - V_0)\,X = 0, \tag{58}$$

where
$$V_0 = 2\left(1-\frac{2m}{r}\right)\frac{(n+1)\,r-3m}{r^3}. \tag{59}$$

It is, however, not necessary to consider this equation separately, since, as we shall now show, it must yield the same complex frequencies (and, indeed, the same reflexion and transmission coefficients) as Zerilli's equation. (It may also be noticed here that the integral of V_0 over the range of r_* has the same value (47), as for V_Z.)

In the equation (Chandrasekhar 1975, equation (A 10)),

$$[\tfrac{2}{3}n(n+1)-2mi\sigma]X = \frac{r^2(nr+3m)}{3(r-2m)}Y - \frac{r^6}{6(r-2m)^2}(W_0+2i\sigma)\Lambda_- Y, \tag{60}$$

substitute for Y and $\Lambda_- Y$ (in terms of Z) in accordance with equations (51) and (52) of the same paper. On simplifying the resulting equation, we are left with

$$[\tfrac{2}{3}n(n+1)-2mi\sigma]X = \left[\tfrac{2}{3}n(n+1)+\frac{6m^2(r-2m)}{r^2(nr+3m)}\right]Z - 2m\frac{dZ}{dr_*}. \tag{61}$$

Therefore, a solution of Zerilli's equation with the asymptotic behaviour,

$$\begin{aligned} Z &\to e^{+i\sigma r_*} + A\,e^{-i\sigma r_*} \quad (r_* \to +\infty) \\ &\to B e^{+i\sigma r_*} \quad (r_* \to -\infty), \end{aligned} \tag{62}$$

will yield a solution of equation (58) with the behaviour

$$\begin{aligned} X &\to e^{+i\sigma r_*} + \frac{\frac{2}{3}n(n+1)+2mi\sigma}{\frac{2}{3}n(n+1)-2mi\sigma}A\,e^{-i\sigma r_*} \quad (r_* \to +\infty) \\ &\to B e^{+i\sigma r_*} \quad (r_* \to -\infty). \end{aligned} \tag{63}$$

The equality of the reflexion and transmission coefficients, that are determined by the two equations, is now manifest.† It is also clear that the complex frequencies belonging to the quasi-normal modes of the two parities must be the same; and the modes themselves must be related by equation (61).

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† This equality had been suspected by C. V. Vishveshwara (private communication) from the numerical agreement he had found from a direct evaluation of the coefficients. But the reason for the agreement does not appear to have been asked.