

Black-hole electrodynamics: an absolute-space/ universal-time formulation*

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Summary. This paper reformulates and extends the Blandford–Znajek theory of a stationary, axisymmetric magnetosphere anchored in a black hole and in its accretion disc. Such a magnetosphere should transfer much of the rotational energy of the hole and orbital energy of the disc into an intense flux of electromagnetic energy – which in turn might be the energizer for quasars and active galactic nuclei.

Our reformulation of the theory attempts to make it accessible to plasma astrophysicists who have little experience with general relativity. This is done by replacing the relativist's 'unified spacetime' viewpoint with an equivalent Galilean-type 'absolute-space-plus-universal-time' viewpoint, and by replacing the electromagnetic field tensor $F_{\mu\nu}$ with electric and magnetic fields \mathbf{E} and \mathbf{B} that reside in the absolute space outside the black hole. The resulting formalism resembles the theory of axisymmetric pulsar magnetospheres; it will, we hope, permit a fairly easy transfer of physical intuition and results from the pulsar problem to the black-hole problem.

The Blandford–Znajek theory focused primarily on force-free regions of the magnetosphere. This paper, in addition to recasting the force-free theory in new language, extends it to encompass regions that are degenerate ($\mathbf{E} \cdot \mathbf{B} = 0$) but not force-free, and regions that are neither degenerate nor force-free. Blandford & Znajek showed that the magnetospheric structure in the force-free region is determined by a general relativistic 'stream differential equation'. This paper presents an action principle for the stream equation, it elucidates the boundary conditions that one must pose on the stream function ψ , and it shows that ψ and the poloidal magnetic field distribute themselves over the hole's horizon in such a manner as to extremize the horizon's electromagnetic surface energy.

This paper also constructs a general relativistic version of DC electronic circuit theory and uses it to elucidate the flows of electric current and of electromagnetic power in the magnetosphere. The circuit-theory analysis, and independently a torque-balance analysis, suggests that those magnetic field

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lines which thread the hole will be dragged into rotation with roughly half the angular velocity of the hole — and, consequently, that the hole will deliver to the magnetosphere the maximum electromagnetic power permitted by the horizon strengths of the magnetic fields.

1 Introduction

Within one year after the discovery of pulsars (Hewish *et al.* 1968) it became evident that they are rotating neutron stars, and that the rotational energy is transmitted to radiating particles by a strong magnetic field embedded in the star (Gold 1968; Pacini 1968). During the subsequent year Goldreich & Julian (1969) laid the foundations for the theory of this ‘pulsar electrodynamics’; during the decade since then scores of outstanding researchers have explored many different variants of the theory (see, e.g. the review by Arons 1979).

Quasars were discovered four years before pulsars (Schmidt 1963), and almost immediately a number of astrophysicists proposed that they might be energized by black holes (Robinson, Schild & Schücking 1965). However, a fully viable and compelling mechanism by which black holes can energize quasars was not found until 1976, 13 years after the quasar discovery (Blandford 1976; Lovelace 1976; Harrison 1976). The long delay in finding this mechanism is surprising, since the mechanism is essentially the same as in the pulsar case: Magnetic fields, embedded in a rotating black hole and a surrounding accretion disc, transmit rotational and orbital energy to distant, radiating particles. Equally surprising is the fact that scores of theorists have *not*, since 1976, explored many different variants of the theory of ‘black-hole and accretion-disc electrodynamics’ (for the modest amount of work that has been done, see Blandford & Znajek 1977; Znajek 1977, 1978b; Lovelace, MacAulian & Burns 1979).

Why has the theory of black-hole and accretion-disc electrodynamics developed so slowly? We think a significant factor is the arena of curved four-dimensional spacetime in which crucial parts of the theory must reside. Many astrophysicists feel uncomfortable in curved spacetime, even when the subject they are exploring, electrodynamics, is totally familiar.

Relativity theorists are largely responsible for the astrophysicists’ discomfort. The relativist prefers to think about physics in a geometric, frame-independent way, representing the electromagnetic field by the tensor \mathbf{F} , a four-dimensional geometric object. The astrophysicist, on the other hand, would prefer to split this tensor into a three-dimensional electric field \mathbf{E} and magnetic field \mathbf{B} , sacrificing the general covariance of the theory for the insight to be gained from a comparison with the well-developed flat-spacetime theory of pulsar electrodynamics.

Moreover, the relativist likes to regard spacetime as a global, four-dimensional manifold, free of any global reference frames. In calculating, he may introduce a local reference frame here, a different local frame there, a third one elsewhere, and flit back and forth from frame to frame as suits his convenience. The astrophysicist is apt to be uncomfortable with this slippery mode of study. His intuition is based largely on the Galilean–Newtonian viewpoint, in which physics occurs in a fixed, absolute three-dimensional space with an associated universal reference frame, and events are determined by the passage of a universal time. This absolute-space/universal-time viewpoint has given the astrophysicist a firm foundation on which to develop a vast lore of insights into pulsar electrodynamics and other astrophysical problems.

The relativist's slippery viewpoint is essential when spacetime is highly dynamical; and it has been enormously powerful in studies of cosmological singularities and of dynamically evolving black holes. However, its track record in treating black-hole electrodynamics has not been good.

Fortunately, in stationary, curved spacetimes such as those outside most astrophysical black holes, one can reformulate electrodynamics in terms of an absolute but curved three-dimensional space and a universal time. The variables in this reformulation are those familiar from flat-space electrodynamics: electric and magnetic fields \mathbf{E} and \mathbf{B} , charge density ρ_e and current density \mathbf{j} . We present and utilize that reformulation in this paper, with the hope that it may catalyse pulsar-experienced astrophysicists to begin research on black-hole electrodynamics and to bring to bear on this topic their lore about the 'axisymmetric pulsar problem' (e.g. Mestel, Phillips & Wang 1979).

Our absolute-space/universal-time formulation of stationary general relativity has deep roots in the '3+1 hypersurface formulations' of dynamical relativity, which are used nowadays by numerical relativists; and those 3+1 formulations in turn have deep roots in the Hamiltonian formulation of geometrodynamics, which was introduced in the 1950s as a tool for quantizing general relativity. We describe those roots in an accompanying paper (Thorne & Macdonald 1982, cited henceforth as Paper I); and, more important, we there use the 3+1 formulations of general relativity to derive the absolute-space/universal-time formalism of this paper.

In this paper we present without derivation the absolute-space/universal-time formalism, restricted to the space outside a rotating, axially symmetric black hole. We then use that formalism to derive the basic equations governing black-hole accretion discs and their magnetospheres.

More specifically, the remainder of this paper does the following: Section 2 presents without derivation the absolute-space/universal-time formulation of general relativity outside stationary black holes. Section 2.1 focuses on the mathematical structure of the formalism and its physical interpretation. Section 2.2 focuses (i) on the equations of electrodynamics, (ii) on the local laws of energy balance and force balance (first law of thermodynamics, and Euler or Navier–Stokes equations for an arbitrary continuous medium and for the electromagnetic field), and (iii) on the global laws of conservation of angular momentum and 'redshifted energy'. Section 2.3 presents the Znajek (1978b)–Damour (1978) theory of boundary conditions at the black hole's horizon, reformulated in our language of absolute space. Derivations of these results are to be found in Paper I.

Section 3 uses the absolute-space formalism of Section 2 to give an overview of black-hole and accretion-disc electrodynamics. More specifically, Faraday's law is used to describe the manner in which an accretion disc dynamically squeezes magnetic field lines on to a black hole and holds them there (Fig. 1). The discussion is fully dynamical; it does not require stationarity or axial symmetry of the accretion disc or its magnetic field.

The remainder of the paper, Sections 4–7, develops the theory of a stationary, axially symmetric magnetosphere. Sections 4, 5 and 6 develop the fundamental magnetosphere equations with successively increasing degrees of specialization; Section 7 shows how to put together a coherent magnetosphere model based on those equations.

The general equations for a stationary, axisymmetric magnetosphere are developed in Section 4. Section 5 imposes the constraint that the electromagnetic field be degenerate, $\mathbf{E} \cdot \mathbf{B} = 0$, and derives the consequences of degeneracy for the magnetosphere equations. Section 6 imposes, in addition to degeneracy, the constraint that the fields and currents be force-free, $\rho_e \mathbf{E} + (\mathbf{j}/c) \times \mathbf{B} = 0$, and derives the consequences of force-freeness. Sections 4.1, 5.1 and 6.1 focus on regions of the magnetosphere outside the black hole's horizon;

sections 4.2, 5.2 and 6.2 present the boundary conditions at the horizon for the unspecialized case, the degenerate case, and the degenerate force-free case, respectively. Section 6.3 presents action principles for the magnetosphere structure in force-free regions and for the distribution of the magnetic field on the horizon.

The magnetosphere theory for force-free regions has been developed previously by Blandford & Znajek (1977) using the usual four-dimensional spacetime formulation of general relativity; the general four-dimensional theory of boundary conditions has been developed by Znajek (1977, 1978b) and Damour (1978). Our versions of these theories are isomorphic to theirs; we merely rewrite them in our absolute-space language, and extend them to include non-force-free regions of the magnetosphere.

Our general magnetosphere equations can act as a foundation for a plethora of different magnetosphere models. The details of a model will depend on the detailed assumptions made about the behaviour of the plasma and charged particles in the non-force-free regions; and those details are left unspecified in our equations. Section 7 sketches various aspects of global magnetosphere models which are independent of the detailed assumptions about the non-force-free regions. Section 7 makes only the mild constraining assumptions that the fields in the disc are degenerate, that just outside the disc and hole there is a force-free region, and that beyond the force-free region is a non-degenerate, non-force-free region, the acceleration region, where the magnetosphere's rotation-induced power output is deposited into charged particles (*cf.* Fig. 2). Section 7.1 presents an overview of this assumed global magnetosphere structure.

Section 7.2 analyses the balance that must exist between the torques exerted at one end of a magnetic flux tube by the hole or disc, and at the other end by charged particles in the acceleration region. This torque balance determines the angular velocity Ω^F of the flux tube; and, in the case of tubes threading the disc (but not those threading the hole), it also determines the current flowing in the force-free region of the magnetosphere. Section 7.2 also describes the rotation-induced power flow from disc and hole to acceleration region, and presents an argument (generalized from pulsar theory) which suggests that torque balance of flux tubes threading the hole will lead to a flux-tube angular velocity roughly half that of the hole $\Omega^F \approx \Omega^H/2$ – and will thereby lead to optimal power output.

Section 7.3 develops a quantitative version of a DC circuit analysis of power flow through the force-free region, which was proposed qualitatively by Znajek (1978b) and Blandford (1979). This analysis shows that the angular velocity Ω^F of a flux tube threading the horizon is determined by the ratio of impedances across the tube in the horizon, ΔZ^H , and in the acceleration region, ΔZ^A : $\Omega^F/(\Omega^H - \Omega^F) = \Delta Z^A/\Delta Z^H$, where Ω^H is the hole's angular velocity. The standard circuit-theory condition for optimal power flow, (load impedance) $\equiv \Delta Z^A =$ (source impedance) $\equiv \Delta Z^H$, agrees with the torque-balance condition for optimal power flow, $\Omega^F = \Omega^H/2$. Estimates of the impedance of the acceleration region (e.g. Lovelace *et al.* 1979) suggest that this impedance matching may be roughly achieved in Nature.

Section 7.4 discusses the mathematical structure of the problem of constructing a precise model for the force-free region. Roughly speaking, one must specify at the interface with the disc and acceleration regions, the normal components of the magnetic field and electric current and the angular velocity of the field lines. One does not specify any boundary conditions at all on the black-hole horizon. One then solves a single non-linear partial differential equation – the relativistic stream equation of Blandford & Znajek (1977) – subject to these boundary conditions; and from the resulting stream function ψ , one computes the electric and magnetic fields and the current and charge densities throughout the force-free region and on the horizon. The resulting fields and currents will automatically satisfy the Znajek–Damour boundary conditions at the horizon.

Section 7.5 proves that a magnetic field loop cannot exist in the stationary, force-free region with both its feet anchored in the horizon. Presumably such a loop, if formed, will annihilate itself on a time-scale Δt of the order of the light travel time across the loop.

Because the mathematical details of Sections 2, 4, 5 and 6 may seem formidable at first sight (actually they are not because they closely mirror flat-space axisymmetric pulsar theory), readers may find it helpful to peruse the astrophysical sections of the paper (Sections 3 and 7) before going on to a more detailed reading.

Throughout this paper we use cgs and Gaussian units (\mathbf{B} measured in Gauss, \mathbf{E} in statvolts per centimetre); and we use the terminology ‘equation (I,2.4)’ to denote ‘equation (2.4) of Paper I (Thorne & Macdonald 1982)’.

2 The absolute-space formulation of black-hole physics

2.1 THE MATHEMATICS OF ABSOLUTE SPACE

In the standard four-dimensional spacetime formalism of general relativity, a stationary axisymmetric black hole is characterized by the spacetime geometry

$$ds^2 = -\alpha^2 c^2 dt^2 + \varpi^2 (d\phi - \omega dt)^2 + \exp(2\mu_1) dr^2 + \exp(2\mu_2) d\theta^2 \quad (2.1)$$

(see, e.g. Carter 1979; Bardeen 1973). Here the metric coefficients α , ϖ , ω , μ_1 and μ_2 are functions of r and θ , and c is the speed of light.

In our absolute-space formalism, the hole is characterized instead by an absolute three-dimensional space with curved geometry

$$\begin{aligned} ds^2 &= \gamma_{jk} dx^j dx^k \\ &= \varpi^2 d\phi^2 + \exp(2\mu_1) dr^2 + \exp(2\mu_2) d\theta^2 \quad \text{in the above coordinates.} \end{aligned} \quad (2.2)$$

We denote by $\nabla \mathbf{A}$ (components $A_{j|k}$) the gradient (covariant derivative) of a vector \mathbf{A} in this absolute space. The axial symmetry of the space is embodied in the fact that ϖ , μ_1 and μ_2 are independent of ϕ – or, more abstractly, in the fact that $\mathbf{m} = \varpi^2 \nabla \phi$ is a Killing vector field

$$\begin{aligned} \mathbf{m} &= \varpi^2 \nabla \phi, \\ m_{j|k} + m_{k|j} &= 0. \end{aligned} \quad (2.3)$$

Note that ϖ is the length of this Killing vector

$$\varpi \equiv |\mathbf{m}|; \quad (2.4)$$

i.e. $2\pi\varpi$ is the circumference of the circle of constant radius and latitude, $(r, \theta) = \text{constant}$, to which \mathbf{m} is tangent. We shall call such circles ‘ \mathbf{m} -loops’, and we shall call ϖ the ‘cylindrical radius’ of an \mathbf{m} -loop.

Going hand in hand with our absolute space is an absolute global time t (equal to the time coordinate t of the four-dimensional spacetime metric (2.1)). A vector field \mathbf{A} in absolute space can evolve with time: $\mathbf{A} = \mathbf{A}(\mathbf{x}, t)$. Its time derivative at fixed absolute-space location will be denoted

$$\begin{aligned} \dot{\mathbf{A}} &= \partial \mathbf{A} / \partial t, \\ \dot{A}^j &= \partial A^j / \partial t. \end{aligned} \quad (2.5)$$

(In the ‘3+1’ formalism of Paper I this derivative is denoted $\mathcal{L}_t \mathbf{A}$ (equations I,2.15 and I,5.5); it is the ‘Lie derivative’ of \mathbf{A} along the spacetime Killing vector field $\mathbf{k} = \partial/\partial t$.)

Living in our absolute space is a family of fiducial observers called ‘zero-angular-momentum observers’, or ZAMOs (Bardeen, Press & Teukolsky 1973). In our absolute-space formalism all the laws of physics are formulated in terms of physical quantities (observables) measured by the ZAMOs. For example, a particle with charge q and velocity \mathbf{v} as measured by the ZAMOs, moving through electric and magnetic fields \mathbf{E} and \mathbf{B} as measured by the ZAMOs, experiences a Lorentz force $q[\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B}]$ (see Section 2.2).

If the black hole is non-rotating, then the ZAMOs are at rest relative to absolute space. However, rotation of the hole drags the ZAMOs into toroidal motion along \mathbf{m} -loops; their angular velocity relative to absolute space is

$$(d\phi/dt)_{\text{of ZAMO rest frame}} = \omega. \quad (2.6)$$

This ZAMO angular velocity ω is the same quantity that appears in the spacetime metric (2.1). When one adopts the four-dimensional spacetime viewpoint (which we do not), one discovers that, despite the toroidal ZAMO motion $d\phi/dt = \omega$, the ZAMO world lines are orthogonal to the three-dimensional hypersurfaces of constant time t — i.e. orthogonal to our absolute space (*cf.* equation 2.1). Thus, our absolute space at time t is regarded by the ZAMOs as a space of constant time in their local Lorentz frames.

If the black hole did not gravitate, the clock carried by a ZAMO would read absolute global time t . However, the gravity of the hole produces a gravitational redshift of ZAMO clocks; their lapse of proper time $d\tau$ is related to the lapse of global time dt by:

$$(d\tau/dt)_{\text{of ZAMO clock}} = \alpha. \quad (2.7)$$

Here α , the ‘lapse function’, is the same quantity that appears in the spacetime metric (2.1). The gravitational acceleration \mathbf{g} measured by a gravimeter carried by a ZAMO (which is the negative of the acceleration \mathbf{a} measured by his accelerometers) is (equation I, 2.4)

$$\mathbf{g} = -\mathbf{a} = -c^2 \nabla \ln \alpha. \quad (2.8)$$

Because absolute space is axially symmetric, the ZAMO angular velocity ω , the lapse function α , and the cylindrical radius function ϖ are all constant on \mathbf{m} -loops

$$\mathbf{m} \cdot \nabla \omega = d\omega/d\phi = 0, \quad \mathbf{m} \cdot \nabla \alpha = d\alpha/d\phi = 0, \quad \mathbf{m} \cdot \nabla \varpi = d\varpi/d\phi = 0. \quad (2.9)$$

The horizon \mathcal{H} of the black hole is the two-dimensional surface of infinite gravitational redshift, $\alpha = 0$. Everywhere on \mathcal{H} the ZAMOs are dragged into motion with the uniform angular velocity Ω^H of the black hole (equations I, 5.22a)

$$\alpha \rightarrow 0, \quad \omega \rightarrow \Omega^H \quad \text{at horizon } \mathcal{H}. \quad (2.10)$$

ZAMOs at the horizon feel an infinitely strong gravitational acceleration \mathbf{g} . However, if one multiplies \mathbf{g} by $\alpha = d\tau/dt$ to convert the acceleration from a ‘per unit ZAMO proper time’ basis to a ‘per unit global time’ basis, one obtains a finite result (equation I, 5.22b or page 252 of Bardeen 1973):

$$\alpha \mathbf{g} \rightarrow -\kappa \mathbf{n} \quad \text{at } \mathcal{H}. \quad (2.11)$$

Here \mathbf{n} is a unit vector pointing orthogonally out of \mathcal{H} , and κ is the ‘surface gravity’ of the hole. (κ and Ω^H are both constant over \mathcal{H} , see e.g. Carter 1979.) In calculations very near \mathcal{H} it is useful to introduce a coordinate system (α, λ, ϕ) , where $\lambda \equiv$ (proper distance along \mathcal{H} from the north pole toward the equator). In this coordinate system the metric of absolute space reads (equation I, 5.30)

$$ds^2 = (c^2/\kappa)^2 d\alpha^2 + d\lambda^2 + \varpi^2 d\phi^2 \quad \text{near } \mathcal{H}, \quad (2.12a)$$

and unit vectors along the ‘toroidal’ (i.e. ϕ) direction, the ‘poloidal’ (i.e. λ) direction, and the ‘normal’ (i.e. α) direction are – in the notation of modern differential geometers –

$$\mathbf{e}_{\hat{\phi}} = \frac{1}{\varpi} \frac{\partial}{\partial \phi} = \frac{1}{\varpi} \mathbf{m}, \quad \mathbf{e}_{\hat{\lambda}} = \frac{\partial}{\partial \lambda}, \quad \mathbf{n} = \frac{\kappa}{c^2} \frac{\partial}{\partial \alpha}. \quad (2.12b)$$

Observers freely falling into the hole move at the speed of light c relative to the infinitely accelerated ZAMOs at the horizon:

$$\frac{ds}{d\tau} = - \frac{(c^2/\kappa) d\alpha}{\alpha dt} = c;$$

consequently, they move along trajectories

$$\alpha = \text{const} \times \exp(-\kappa t/c) \quad \text{near } \mathcal{H}. \quad (2.13)$$

Because the infalling observers are physically non-pathological, all physical quantities which they measure must approach well-behaved limits along their trajectories. These limits are defined mathematically by

$$\begin{aligned} \text{‘}\rightarrow\text{’ means ‘approaches, as one approaches the horizon along a time-evolving trajectory} \\ \alpha = \text{const} \times \exp(-\kappa t/c) \end{aligned} \quad (2.14)$$

(for a further discussion see Section 5.3 of Paper I).

For the special case of a non-rotating Schwarzschild black hole of mass M and a Schwarzschild spatial coordinate system, the quantities appearing in the above discussion are:

$$\begin{aligned} ds^2 &= (1 - 2GM/c^2 r)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2; \\ \alpha &= (1 - 2GM/c^2 r)^{1/2}, \quad \omega = 0, \quad \varpi = r \sin \theta; \\ \mathbf{g} &= - \frac{GM/r^2}{(1 - 2GM/c^2 r)^{1/2}} \mathbf{e}_{\hat{r}} \quad \text{with } \mathbf{e}_{\hat{r}} \equiv \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} \frac{\partial}{\partial r} = (\text{unit radial vector}); \end{aligned} \quad (2.15)$$

$$\mathcal{H} \text{ is at } r = 2GM/c^2; \text{ there } \kappa = c^4/4GM, \quad \Omega^H = 0, \quad \mathbf{n} = \mathbf{e}_{\hat{r}};$$

$$\lambda = (2GM/c^2)\theta \quad \text{near } \mathcal{H};$$

$$\alpha = \text{const} \times \exp(-\kappa t/c) \iff r - 2GM/c^2 = \text{const} \times \exp(-c^3 t/2GM) \text{ for freely falling observers near } \mathcal{H}.$$

Here G is Newton’s gravitational constant.

For the special case of a rotating Kerr black hole of mass M and angular momentum per unit mass $a \equiv L^H/Mc$, and a Boyer–Lindquist spatial coordinate system:

$$\begin{aligned} ds^2 &= (\rho^2/\Delta) dr^2 + \rho^2 d\theta^2 + (A \sin^2 \theta/\rho^2) d\phi^2, \\ \rho^2 &\equiv r^2 + a^2 \cos^2 \theta, \quad \Delta \equiv r^2 - 2GMr/c^2 + a^2, \quad A \equiv (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta; \\ \alpha &= (\rho^2 \Delta/A)^{1/2}, \quad \omega = 2aGMr/cA, \quad \varpi = (A/\rho^2)^{1/2} \sin \theta; \\ \mathbf{g} &= - \frac{c^2 A}{2\rho^3 \Delta} \left[\Delta^{1/2} \frac{\partial}{\partial r} \left(\frac{\rho^2 \Delta}{A} \right) \mathbf{e}_{\hat{r}} + \frac{\partial}{\partial \theta} \left(\frac{\rho^2 \Delta}{A} \right) \mathbf{e}_{\hat{\theta}} \right] \end{aligned} \quad (2.16)$$

with $\mathbf{e}_{\hat{r}} \equiv (\Delta^{1/2}/\rho) \partial/\partial r$ and $\mathbf{e}_{\hat{\theta}} \equiv (1/\rho) \partial/\partial \theta$ unit vectors;

\mathcal{H} is at $r = r_+ \equiv GM/c^2 + [(GM/c^2)^2 - a^2]^{1/2}$;

$$\text{At } \mathcal{H}, \quad \kappa = c^4(r_+ - GM/c^2)/2GMr_+, \quad \Omega^H = c^3 a/2GMr_+, \quad \mathbf{n} = \mathbf{e}_{\hat{r}}; \quad (2.16)$$

$$\lambda = \int_0^\theta (r_+^2 + a^2 \cos^2 \theta)^{1/2} d\theta = (r_+^2 + a^2)^{1/2} E[\theta, a/(r_+^2 + a^2)^{1/2}] \quad \text{near } \mathcal{H},$$

with $E \equiv$ (elliptic integral of the second kind);

$$\alpha = \text{const} \times \exp(-\kappa t/c) \iff r - r_+ = \text{const} \times \exp[-c^3 t(r_+ - GM/c^2)/GMr_+] \\ \text{for freely falling observers near } \mathcal{H}.$$

Our absolute-space formalism is also applicable to stationary, axially symmetric black holes whose spatial metric γ_{jk} , lapse function α , and ZAMO angular velocity ω are modified away from Kerr by the gravitational effects of surrounding matter.

2.2 ELECTRODYNAMICS IN ABSOLUTE SPACE

When studying electrodynamic phenomena around black holes we deal with the electric field \mathbf{E} , the magnetic field \mathbf{B} , the electric charge density ρ_e , and the current density \mathbf{j} , all as measured by the ZAMOs. In terms of these quantities, Maxwell's equations are (equations I,5.8)

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e, \quad (2.17a)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (2.17b)$$

$$\nabla \times (\alpha \mathbf{B}) = 4\pi\alpha \mathbf{j}/c + (1/c) [\dot{\mathbf{E}} + \omega \mathfrak{L}_{\mathbf{m}} \mathbf{E} - (\mathbf{E} \cdot \nabla \omega) \mathbf{m}], \quad (2.17c)$$

$$\nabla \times (\alpha \mathbf{E}) = -(1/c) [\dot{\mathbf{B}} + \omega \mathfrak{L}_{\mathbf{m}} \mathbf{B} - (\mathbf{B} \cdot \nabla \omega) \mathbf{m}]. \quad (2.17d)$$

Here $\mathfrak{L}_{\mathbf{m}} \mathbf{E}$ is the 'Lie derivative' of \mathbf{E} along the toroidal Killing vector \mathbf{m}

$$\mathfrak{L}_{\mathbf{m}} \mathbf{E} \equiv (\mathbf{m} \cdot \nabla) \mathbf{E} - (\mathbf{E} \cdot \nabla) \mathbf{m}. \quad (2.18)$$

Notice that the unfamiliar expression in square brackets

$$\dot{\mathbf{E}} + \omega \mathfrak{L}_{\mathbf{m}} \mathbf{E} - (\mathbf{E} \cdot \nabla \omega) \mathbf{m} = \partial \mathbf{E} / \partial t + \mathfrak{L}_{\omega \mathbf{m}} \mathbf{E} \quad (2.19)$$

is just the 'Lie-type' time derivative, moving with the ZAMOs ($dx/dt = \omega \mathbf{m}$), of the electric field. When the electric and magnetic fields are stationary and axisymmetric, the terms $\dot{\mathbf{E}}$, $\mathfrak{L}_{\mathbf{m}} \mathbf{E}$, $\dot{\mathbf{B}}$ and $\mathfrak{L}_{\mathbf{m}} \mathbf{B}$ will vanish. Notice also that the lapse function α is introduced to convert the curl and current terms in (2.17) over to the same 'per unit global time t ' basis as the time derivative terms. The fields $\alpha \mathbf{E}$ and $\alpha \mathbf{B}$, which we shall meet extensively below, are the electric and magnetic fields measured by ZAMOs, if the ZAMOs use global time t rather than ZAMO proper time τ in computing the rate of change of momenta:

$$(dp/dt)_{\text{Lorentz force}} = \alpha(dp/d\tau) = q(\alpha \mathbf{E} + \mathbf{v} \times \alpha \mathbf{B}).$$

The four Maxwell equations (2.17) can be re-expressed in integral form (equations I,4.1–I,4.4)

$$\int_{\partial \mathcal{V}} \mathbf{E} \cdot d\boldsymbol{\Sigma} = 4\pi \int_{\mathcal{V}} \rho_e dV \quad (\text{Gauss's law for } \mathbf{E}), \quad (2.20a)$$

$$\int_{\partial \mathcal{V}} \mathbf{B} \cdot d\boldsymbol{\Sigma} = 0 \quad (\text{Gauss's law for } \mathbf{B}), \quad (2.20b)$$

$$\int_{\partial \mathcal{V}(t)} \alpha \left(\mathbf{B} - \frac{1}{c} \mathbf{v} \times \mathbf{E} \right) \cdot d\mathbf{l} = \frac{1}{c} \frac{d}{dt} \int_{\mathcal{A}(t)} \mathbf{E} \cdot d\mathbf{\Sigma} + \frac{4\pi}{c} \int_{\mathcal{A}(t)} \alpha (\mathbf{j} - \rho_e \mathbf{v}) \cdot d\mathbf{\Sigma} \quad (\text{Ampere's law}), \quad (2.20c)$$

$$\int_{\partial \mathcal{V}(t)} \alpha \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \cdot d\mathbf{l} = - \frac{1}{c} \frac{d}{dt} \int_{\mathcal{A}(t)} \mathbf{B} \cdot d\mathbf{\Sigma} \quad (\text{Faraday's law}). \quad (2.20d)$$

Here \mathcal{V} is any three-dimensional volume entirely outside the horizon and $\partial \mathcal{V}$ is its two-dimensional boundary; $dV = d(\text{proper volume})$ and $d\mathbf{\Sigma} = (\text{outward pointing unit normal to } \partial \mathcal{V}) d(\text{proper area})$. Also $\mathcal{A}(t)$ is any two-dimensional surface entirely outside the horizon and $\partial \mathcal{A}(t)$ is its boundary curve; $d\mathbf{l} = d(\text{proper distance along boundary curve})$; and the orientations of $d\mathbf{\Sigma}$ and $d\mathbf{l}$ must be chosen in accordance with the right-hand rule. Finally, \mathbf{v} is the physical velocity

$$\mathbf{v} = \left[\frac{d(\text{proper distance})}{d\tau} \right]_{\text{relative to ZAMOs}} \quad (2.21a)$$

of a point on $\mathcal{A}(t)$ or $\partial \mathcal{A}(t)$, relative to and as measured by the ZAMOs. (Thus

$$\alpha \mathbf{v} + \omega \mathbf{m} = \left[\frac{d(\text{proper distance})}{dt} \right]_{\text{relative to absolute space}} \quad (2.21b)$$

is the velocity, per unit global time, relative to absolute space.)

The Maxwell equations (2.17) imply the differential law of charge conservation (equation I,5.11)

$$\partial \rho_e / \partial t + \omega \mathbf{m} \cdot \nabla \rho_e + \nabla \cdot (\alpha \mathbf{j}) = 0. \quad (2.22)$$

Note that $\partial / \partial t + \omega \mathbf{m} \cdot \nabla$ is the global time derivative along the ZAMO trajectories (where ρ_e and \mathbf{j} are measured), and the factor α converts \mathbf{j} over from 'charge per unit area per unit proper time τ ' to 'charge per unit area per unit global time t '. The integral formulation of this law of charge conservation is (equation I,4.5)

$$\frac{d}{dt} \int_{\mathcal{V}(t)} \rho_e dV = - \int_{\partial \mathcal{V}(t)} \alpha (\mathbf{j} - \rho_e \mathbf{v}) \cdot d\mathbf{\Sigma}. \quad (2.23)$$

The electric and magnetic fields can be derived from a scalar potential A_0 and a vector potential \mathbf{A} (equations I,5.9)

$$\mathbf{E} = \alpha^{-1} [\nabla A_0 + (\omega/c) \nabla A_\phi] - (\alpha c)^{-1} (\dot{\mathbf{A}} + \omega \mathbf{f}_m \mathbf{A}) \quad (2.24a)$$

where $A_\phi \equiv \mathbf{A} \cdot \mathbf{m}$,

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (2.24b)$$

When these expressions are used, the source-free Maxwell equations (2.17b,d) are automatically satisfied.

A test particle of charge q and rest mass μ , moving through absolute space, obeys the equation of motion (equation I,5.12)

$$\alpha^{-1} [\partial / \partial t + (\alpha \mathbf{v} + \omega \mathbf{m}) \cdot \nabla] \mathbf{p} = \mu \Gamma \mathbf{g} + \alpha^{-1} [\omega (\mathbf{p} \cdot \nabla) \mathbf{m} - (\mathbf{p} \cdot \mathbf{m}) \nabla \omega] + q [\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B}]. \quad (2.25)$$

Here \mathbf{v} is the particle's physical velocity relative to the ZAMOs (equation 2.21), $\mu\Gamma$ is its mass-energy and \mathbf{p} its momentum as measured by the ZAMOs

$$\Gamma \equiv \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-1/2},$$

$$\mathbf{p} = \mu\Gamma\mathbf{v}. \quad (2.26)$$

$\alpha^{-1}[\partial/\partial t + (\alpha\mathbf{v} + \omega\mathbf{m}) \cdot \nabla]$ is the ZAMO proper time derivative moving with the particle, $\mu\Gamma\mathbf{g}$ is the 'gravitational acceleration' of the hole on the particle, $\alpha^{-1}[\omega(\mathbf{p} \cdot \nabla)\mathbf{m} - (\mathbf{p} \cdot \mathbf{m})\nabla\omega]$ is the 'frame-dragging' force of the hole's rotation on the particle and $q[\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B}]$ is the Lorentz force.

The ZAMOs will characterize the electromagnetic field and/or any continuous medium present by

$$\epsilon \equiv (\text{total mass-energy density, erg cm}^{-3}, \text{ as measured by ZAMOs}),$$

$$\mathbf{S} \equiv (\text{total flux of energy, erg cm}^{-2} \text{ s}^{-1}, \text{ as measured by ZAMOs}),$$

$$\mathbf{W} \equiv (\text{stress tensor, dyn cm}^{-2}, \text{ as measured by ZAMOs}). \quad (2.27)$$

In terms of these we can also define densities and fluxes of 'redshifted energy' (also often called 'energy-at-infinity'), and of 'angular momentum about the hole's symmetry axis' (equations I, 5.15 and I, 5.16)

$$\epsilon_E \equiv \alpha\epsilon + \omega\mathbf{S} \cdot \mathbf{m}/c^2, \quad \mathbf{S}_E \equiv \alpha\mathbf{S} + \omega\mathbf{W} \cdot \mathbf{m}, \quad (2.28a)$$

$$\epsilon_L \equiv \mathbf{S} \cdot \mathbf{m}/c^2, \quad \mathbf{S}_L \equiv \mathbf{W} \cdot \mathbf{m}. \quad (2.28b)$$

For the electromagnetic field

$$\epsilon = (1/8\pi)(\mathbf{E}^2 + \mathbf{B}^2), \quad \mathbf{S} = (c/4\pi)(\mathbf{E} \times \mathbf{B}), \quad (2.29a)$$

$$\mathbf{W} = (1/4\pi) \left[-(\mathbf{E} \otimes \mathbf{E} + \mathbf{B} \otimes \mathbf{B}) + \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\mathbf{Y} \right], \quad (2.29b)$$

$$\epsilon_E = (\alpha/8\pi)(\mathbf{E}^2 + \mathbf{B}^2) + (\omega/4\pi c)(\mathbf{E} \times \mathbf{B}) \cdot \mathbf{m}, \quad (2.30a)$$

$$\mathbf{S}_E = (1/4\pi) \left[\alpha c \mathbf{E} \times \mathbf{B} - \omega(\mathbf{E} \cdot \mathbf{m})\mathbf{E} - \omega(\mathbf{B} \cdot \mathbf{m})\mathbf{B} + \frac{1}{2}\omega(\mathbf{E}^2 + \mathbf{B}^2)\mathbf{m} \right], \quad (2.30b)$$

$$\epsilon_L = (1/4\pi c)(\mathbf{E} \times \mathbf{B}) \cdot \mathbf{m}, \quad (2.31a)$$

$$\mathbf{S}_L = (1/4\pi) \left[-(\mathbf{E} \cdot \mathbf{m})\mathbf{E} - (\mathbf{B} \cdot \mathbf{m})\mathbf{B} + \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\mathbf{m} \right]. \quad (2.31b)$$

For a perfect fluid with density of mass-energy ρ (erg cm⁻³) and pressure p (dyn cm⁻²) as measured in its own rest frame, and with velocity \mathbf{v} as measured by the ZAMOs,

$$\epsilon = \Gamma^2(\rho + p\mathbf{v}^2/c^2), \quad \mathbf{S} = (\rho + p)\Gamma^2\mathbf{v}, \quad (2.32a)$$

$$\mathbf{W} = (1/c^2)(\rho + p)\Gamma^2\mathbf{v} \otimes \mathbf{v} + p\mathbf{Y}, \quad \Gamma \equiv (1 - \mathbf{v}^2/c^2)^{-1/2}. \quad (2.32b)$$

The law of local energy balance as formulated by the ZAMOs is (equation I, 5.13)

$$\alpha^{-1}[\partial/\partial t + \omega\mathbf{m} \cdot \nabla]\epsilon + \alpha^{-2}\nabla \cdot (\alpha^2\mathbf{S}) + \alpha^{-1}\mathbf{m} \cdot \mathbf{W} \cdot \nabla\omega$$

$$= 0 \quad \text{if all forms of energy and stress are included in } \epsilon, \mathbf{S}, \mathbf{W} \quad (2.33a)$$

$$= -\mathbf{j} \cdot \mathbf{E} \quad \text{if only electromagnetic contributions are included in } \epsilon, \mathbf{S}, \mathbf{W}.$$

(The third term on the left side of the equation is caused by the shear of the ZAMO trajectories relative to each other.) The law of local momentum balance (force balance) as formulated by the ZAMOs is (equation I,5.14)

$$\begin{aligned} \alpha^{-1} [\partial/\partial t + \omega \mathbf{f}_m] \mathbf{S} + \alpha^{-1} (\mathbf{S} \cdot \mathbf{m}) \nabla \omega - \epsilon \mathbf{g} + c^2 \alpha^{-1} \nabla \cdot (\alpha \mathbf{W}) \\ = 0 \quad \text{if all stress and energy are included in } \mathbf{S} \text{ and } \mathbf{W} \\ = -c^2 [\rho_e \mathbf{E} + (\mathbf{j}/c) \times \mathbf{B}] \quad \text{if only electromagnetism is included.} \end{aligned} \quad (2.33b)$$

(The test-particle equation of motion (2.25) can be derived in the usual way from this general force-balance equation.) From the laws of energy balance and force balance one can derive differential and integral conservation laws for redshifted energy and for angular momentum (equations I,5.17 and I,5.18)

$$\begin{aligned} \alpha^{-1} (\partial/\partial t + \omega \mathbf{m} \cdot \nabla) \epsilon_E + \alpha^{-1} \nabla \cdot (\alpha \mathbf{S}_E) = 0 \quad \text{if all included} \\ = -\alpha \mathbf{j} \cdot \mathbf{E} - \omega [\rho_e \mathbf{E} + (\mathbf{j}/c) \times \mathbf{B}] \cdot \mathbf{m} \quad \text{if only electromagnetism,} \end{aligned} \quad (2.34a)$$

$$\begin{aligned} \alpha^{-1} (\partial/\partial t + \omega \mathbf{m} \cdot \nabla) \epsilon_L + \alpha^{-1} \nabla \cdot (\alpha \mathbf{S}_L) = 0 \quad \text{if all included} \\ = -[\rho_e \mathbf{E} + (\mathbf{j}/c) \times \mathbf{B}] \cdot \mathbf{m} \quad \text{if only electromagnetism,} \end{aligned} \quad (2.34b)$$

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{V}(t)} \epsilon_E dV + \int_{\partial \mathcal{V}(t)} \alpha (\mathbf{S}_E - \epsilon_E \mathbf{v}) \cdot d\mathbf{\Sigma} = 0 \quad \text{if all included} \\ = - \int_{\mathcal{V}(t)} \{ \alpha^2 \mathbf{j} \cdot \mathbf{E} + \alpha \omega [\rho_e \mathbf{E} + (\mathbf{j}/c) \times \mathbf{B}] \cdot \mathbf{m} \} dV \quad \text{if only electromagnetism,} \end{aligned} \quad (2.35a)$$

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{V}(t)} \epsilon_L dV + \int_{\partial \mathcal{V}(t)} \alpha (\mathbf{S}_L - \epsilon_L \mathbf{v}) \cdot d\mathbf{\Sigma} = 0 \quad \text{if all included} \\ = - \int_{\mathcal{V}(t)} \alpha [\rho_e \mathbf{E} + (\mathbf{j}/c) \times \mathbf{B}] \cdot \mathbf{m} dV \quad \text{if only electromagnetism.} \end{aligned} \quad (2.35b)$$

Here \mathbf{v} is the velocity of the boundary $\partial \mathcal{V}(t)$ as measured by the ZAMOs (equation 2.21).

Far from the black hole space becomes flat, the lapse function α becomes unity, the ZAMO angular velocity ω goes to zero; and, consequently, our formalism reduces to standard flat-space physics in a global Lorentz frame — the rest frame of the hole. In the asymptotically flat region redshifted energy reduces to ordinary, every-day energy.

2.3 BOUNDARY CONDITIONS AT THE HORIZON

Znajek (1978b) and Damour (1978) have developed an elegant formalism for studying the boundary conditions at the horizon by defining surface charge and current densities lying in the horizon (for a beautiful review of this work see Carter 1979). This formalism can be expressed naturally in the language of absolute space and universal time, where the horizon is just the two-dimensional surface $\alpha = 0$ (see Section 5.4 of Paper I). The horizon's charge density

$$\sigma^H = (\text{charge per unit area on } \mathcal{H}) \quad (2.36)$$

and current density

$$\mathcal{J}^H = (\text{charge crossing a unit length perpendicular to } \mathcal{J}^H \text{ on } \mathcal{H}, \text{ per unit global time } t) \quad (2.37)$$

have these properties: (i) σ^H terminates all electric flux that intersects the horizon (equation I,5.42)

$$\mathbf{E} \cdot \mathbf{n} \equiv E_{\perp} \rightarrow 4\pi\sigma^H \quad (\text{Gauss's law at } \mathcal{H}), \quad (2.38)$$

(where \mathbf{n} is the unit outward normal to the horizon and ' \rightarrow ' means 'approaches as one approaches the horizon in the manner of equation 2.14'); (ii) \mathcal{J}^H completes the circuit of all electric currents that intersect the horizon (equation I,5.44)

$$\alpha \mathcal{J} \cdot \mathbf{n} \rightarrow -\partial \sigma^H / \partial t - {}^{(2)}\nabla \cdot \mathcal{J}^H \quad (\text{charge conservation at } \mathcal{H}), \quad (2.39)$$

(where ${}^{(2)}\nabla \cdot \mathcal{J}^H$ is the two-dimensional divergence of the surface current in the two-dimensional geometry of the horizon); (iii) \mathcal{J}^H terminates all tangential magnetic fields at the horizon (equation I,5.43)

$$\alpha \mathbf{B}_{\parallel} \rightarrow \mathbf{B}^H = (4\pi/c) \mathcal{J}^H \times \mathbf{n} \quad (\text{Ampere's law at } \mathcal{H}) \quad (2.40)$$

(where \mathbf{B}_{\parallel} is the component of the magnetic field tangential to the horizon and $\alpha \mathbf{B}_{\parallel}$ is this tangential field converted over to a 'per unit global time' basis); (iv) the horizon has a surface resistivity

$$R^H = 4\pi/c = 377 \text{ ohm} \quad (2.41)$$

in the sense that (equations I,5.41 and I,5.37a)

$$\alpha \mathbf{E}_{\parallel} \rightarrow \mathbf{E}^H = R^H \mathcal{J}^H \quad (\text{Ohm's law in } \mathcal{H}) \quad (2.42)$$

(where \mathbf{E}_{\parallel} is the component of the electric field tangential to the horizon and $\alpha \mathbf{E}_{\parallel}$ is this tangential field converted to a 'per unit global time' basis). (Note that the 'horizon fields' \mathbf{E}^H and \mathbf{B}^H of this paper are the same as the \mathbf{E}_{\parallel}^H and \mathbf{B}_{\parallel}^H of Paper I — i.e. in passing from Paper I to this paper we have deleted the null components E_{\perp}^H and B_{\perp}^H from \mathbf{E}^H and \mathbf{B}^H (equation I,5.36).)

Because \mathbf{E}^H and \mathbf{B}^H must be finite and because $\alpha \rightarrow 0$ at \mathcal{H} , the tangential fields \mathbf{E}_{\parallel} and \mathbf{B}_{\parallel} can diverge at \mathcal{H} as $1/\alpha$. By contrast, the normal fields $B_{\perp} \equiv \mathbf{B} \cdot \mathbf{n}$ and $E_{\perp} \equiv \mathbf{E} \cdot \mathbf{n}$ must be finite (Section 5.4 of Paper I). The fact that R^H is an impedance equal to that of free space at the end of an open waveguide, together with Ohm's law (2.42) and Ampere's law (2.40), guarantees that the electromagnetic field looks to ZAMOs near the horizon like an infinitely blueshifted, ingoing electromagnetic plane wave

$$E_{\perp} \text{ and } B_{\perp} \text{ finite at } \mathcal{H}, \quad (2.43a)$$

$$\mathbf{E}_{\parallel} \text{ and } \mathbf{B}_{\parallel} \text{ diverge as } 1/\alpha \text{ at } \mathcal{H} \quad (2.43b)$$

$$|\mathbf{E}_{\parallel} - \mathbf{n} \times \mathbf{B}_{\parallel}| \text{ goes to zero as } \alpha \text{ at } \mathcal{H}, \quad (2.43c)$$

(A derivation of the rate at which $|\mathbf{E}_{\parallel} - \mathbf{n} \times \mathbf{B}_{\parallel}|$ goes to zero is sketched in the paragraph preceding equation (I,5.38).)

The electromagnetic field produces a torque per unit area on the hole's horizon given by (cf. equations 2.31b, 2.38, 2.40, 2.42; also I,5.46)

$$\begin{aligned} -\alpha \mathbf{S}_L \cdot \mathbf{n} &\rightarrow \frac{d(\text{angular momentum of hole})}{d(\text{area of horizon}) dt} \equiv \frac{dL^H}{d\Sigma^H dt} \\ &= [\sigma^H \mathbf{E}^H + (\mathcal{J}^H/c) \times \mathbf{B}_{\perp}] \cdot \mathbf{m}. \end{aligned} \quad (2.44)$$

Here $\mathbf{B}_\perp \equiv \mathbf{B} \cdot \mathbf{n}$ is the component of \mathbf{B} perpendicular to \mathcal{H} . The fields also increase the hole's entropy by Joule heating (equation I, 5.48)

$$\Theta^H \frac{dS^H}{d\Sigma^H dt} = \mathcal{J}^H \cdot \mathbf{E}^H, \quad (2.45)$$

where $\Theta^H = (\hbar/2\pi kc)\kappa$ is the hole's temperature and $S^H = (c^3 k/4\hbar G) \times (\text{area of horizon})$ is its entropy (Hawking 1976). Here k is Boltzmann's constant and \hbar is Planck's constant. Finally, the fields increase the hole's mass at a rate given by the thermodynamic law $dM^H c^2 = \Omega^H dL^H + \Theta^H dS^H$ – and given equally well by the relation $dM^H c^2 = d(\text{redshifted energy entering the hole})$

$$\begin{aligned} -\alpha \mathbf{S}_E \cdot \mathbf{n} \rightarrow \frac{dM^H c^2}{d\Sigma^H dt} &\equiv -\frac{dP^H}{d\Sigma^H} = \frac{\Omega^H dL^H + \Theta^H dS^H}{d\Sigma^H dt} \\ &= \Omega^H [\sigma^H \mathbf{E}^H + (\mathcal{J}^H/c) \times \mathbf{B}_\perp] \cdot \mathbf{m} + \mathbf{E}^H \cdot \mathcal{J}^H. \end{aligned} \quad (2.46)$$

(cf. equations 2.10, 2.30b, 2.38, 2.40, 2.42; also 2.44, 2.45; also I, 5.47). Here $dP/d\Sigma^H$ is the redshifted energy per unit time t per unit area (i.e. the redshifted power per unit area) flowing out of the hole.

3 Overview of black-hole and accretion-disc electrodynamics

Consider a black hole surrounded by an accretion disc in which a magnetic field is embedded (Fig. 1). Initially make no simplifying assumptions about the disc and its magnetosphere: Let them evolve dynamically; let them have no axial symmetry; admit the possibility that the fields near the hole may be so strong that classical electromagnetic theory must be replaced by quantum electrodynamics, and so tangled that the field lines slip through the disc's plasma and reconnect. On the other hand, insist that far from the hole the field be weak enough to be described classically, and that in the equatorial plane far from the hole the field be frozen into the disc's plasma (perfect magnetohydrodynamic approximation).

We can get insight into the structure and evolution of the magnetic field by applying our curved-space Faraday law (2.20d) to the two-dimensional surface \mathcal{A} shown in Fig. 1. The boundary $\partial\mathcal{A}$ of this surface is an \mathbf{m} -loop (circle of constant r and θ) in the equatorial plane, at a sufficiently large radius for the field to be frozen in and classical. The surface \mathcal{A} stretches upward from this anchoring curve and over the hole's north pole like a circus

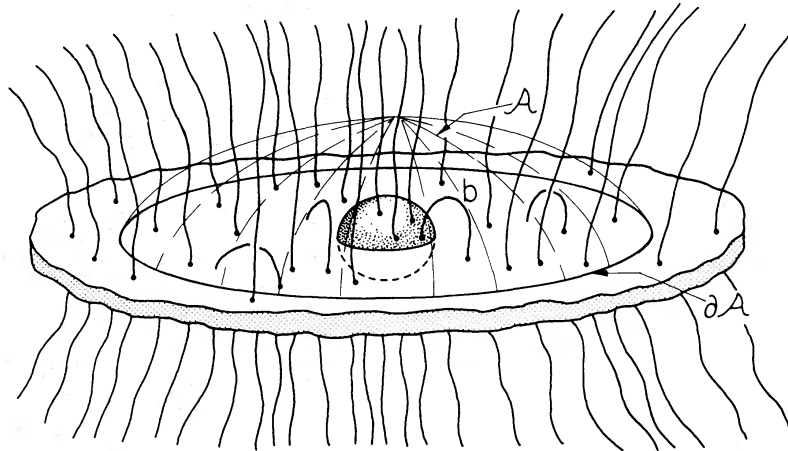


Figure 1. Accretion disc around a black hole, with magnetic field lines threading it. Although the disc is shown thin, nothing anywhere in our analysis constrains it to be so. The surface \mathcal{A} and its boundary $\partial\mathcal{A}$ are used in the mathematical discussion of magnetic flux conservation in Section 3.

tent — remaining always at a sufficiently large radius for classical theory to apply, and remaining everywhere axially symmetric. Require that \mathcal{A} be fixed in space — or, equivalently since \mathcal{A} is axially symmetric, that it be attached to and move toroidally with the ZAMOs.

Faraday's law (2.20d), applied to \mathcal{A} and $\partial\mathcal{A}$, says

$$d\Phi_{\mathcal{A}}/dt = -c \int_{\partial\mathcal{A}} \alpha \mathbf{E} \cdot d\mathbf{l}, \quad (3.1)$$

where $\Phi_{\mathcal{A}}$ is the magnetic flux crossing \mathcal{A} . The emf on the right-hand side can be re-expressed in terms of the magnetic field \mathbf{B} by invoking the freezing-in condition

$$\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B} = 0 \quad \text{in the disc at } \partial\mathcal{A}, \quad (3.2)$$

where \mathbf{v} is the velocity of the disc's plasma as measured by the ZAMOs:

$$d\Phi_{\mathcal{A}}/dt = \int_{\partial\mathcal{A}} \alpha \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}. \quad (3.3)$$

The right-hand side has an obvious physical interpretation as the rate (per unit global time t) at which the plasma carries magnetic flux inward across $\partial\mathcal{A}$. Thus, magnetic flux is conserved in time; the flux across \mathcal{A} can increase or decrease only as a result of field lines being physically transported inward or outward across $\partial\mathcal{A}$.

Consider, now, the fate of the magnetic field lines being transported in toward the hole by the accretion disc's plasma: As the plasma reaches the inner edge of the disc and then spirals down the hole, it becomes causally disconnected from the field lines it was transporting. This does not liberate the transported field, however. Our flux conservation law guarantees that the field lines, though disconnected from their sources, cannot escape. They are squeezed inward and are forced to thread the hole by the Maxwell pressure of surrounding field lines, which in turn are anchored in the disc. (An exception is a field line such as *b* in Fig. 1, which will annihilate itself once the disc has deposited both its feet on to the hole — see Section 7.5.) If the disc's conductivity were suddenly turned off, the field anchored in it would suddenly fly away, releasing its Maxwell pressure from the field lines threading the hole. The hole's gravity has little power to hold the threading field; with the Maxwell pressures gone it would quickly disperse as well ('Price's theorem', Price 1972).

In the remainder of this paper we shall assume that an accretion-disc-plus-magnetic-field structure like that in Fig. 1 has been set up, and that the electric and magnetic fields are weak enough for classical electrodynamics to be a good approximation. We shall assume, further, that the disc and its fields have settled down into a stationary, axisymmetric state. In the next three sections, we shall consider the electrodynamic equations for successively more specialized field configurations. Section 4 considers situations where the electromagnetic field is stationary and axisymmetric, but not otherwise special, Section 5 specializes to the case where the field is degenerate and Section 6 specializes still further to the case where the field is force-free. In Section 7 these three cases are put together into a coherent model of the fields in the disc and its magnetosphere.

4 Stationary, axisymmetric electrodynamics

4.1 OUTSIDE THE HORIZON

We now specialize the electrodynamic equations of Section 2 to fields and charge-current distributions that are stationary and axisymmetric:

$$\begin{aligned} \partial\mathbf{F}/\partial t \equiv \dot{\mathbf{F}} &= 0, & \mathcal{L}_{\mathbf{m}}\mathbf{F} &= 0 & \text{for all vector fields, } \mathbf{F}, \\ \partial f/\partial t \equiv \dot{f} &= 0, & \mathbf{m} \cdot \nabla f &= 0 & \text{for all scalar fields, } f. \end{aligned} \quad (4.1)$$

This specialization simplifies the theory so much that the electric field \mathbf{E} , the magnetic field \mathbf{B} , the charge density ρ_e , and the current density \mathbf{j} can all be derived from three freely-specifiable scalar potentials. (Put differently: given stationary, axisymmetric distributions of ρ_e and \mathbf{j} satisfying charge conservation $\nabla \cdot (\alpha \mathbf{j}) = 0$, the three independent functions in ρ_e and \mathbf{j} determine via Maxwell's equations three independent potentials, from which \mathbf{E} and \mathbf{B} can be computed by differentiation.)

We shall choose as our potentials A_0 , the electrical potential of equation (2.24a), I , the total current passing downward through an \mathbf{m} -loop and ψ , the total magnetic flux passing upward through an \mathbf{m} -loop. To define I and ψ more precisely, let \mathbf{x} be a location in absolute space at which they are to be evaluated, let $\partial\mathcal{A}$ be the \mathbf{m} -loop passing through \mathbf{x} , and let \mathcal{A} be any surface bounded by $\partial\mathcal{A}$ and not intersecting the horizon. Then

$$I(\mathbf{x}) \equiv - \int_{\mathcal{A}} \alpha \mathbf{j} \cdot d\mathbf{\Sigma} = - (\text{current through } \partial\mathcal{A}), \quad (4.2a)$$

$$\psi(\mathbf{x}) \equiv \int_{\mathcal{A}} \mathbf{B} \cdot d\mathbf{\Sigma} = (\text{magnetic flux through } \partial\mathcal{A}), \quad (4.2b)$$

where the orientation of $d\mathbf{\Sigma}$ is chosen 'upward' rather than 'downward' (i.e. along the direction of the hole's angular momentum vector rather than opposed to it). The integrals in equations (4.2) do not depend on the choice made for \mathcal{A} , as long as it is outside the horizon. Charge conservation (2.23), together with stationarity (including stationarity of the total charge on the horizon \mathcal{H}) guarantees that $I(\mathbf{x})$ is independent of our choice of the integration area \mathcal{A} (including whether it passes over the hole or under the hole). The magnetic Gauss law (2.20b) similarly guarantees that $\psi(\mathbf{x})$ is independent of our choice of \mathcal{A} . (If the total magnetic flux entering the hole were non-zero, $\psi(\mathbf{x})$ would be different depending on whether \mathcal{A} passed over the hole or under it. However, the only way that any hole's total flux can be non-zero is by the hole having been so created in the big bang (see equation I, 5.45). We exclude this by fiat, and thereby deal with a $\psi(\mathbf{x})$ which is uniquely defined everywhere outside the horizon.)

The magnetic flux $\psi(\mathbf{x})$ is simply related to the component $A_\phi \equiv \mathbf{A} \cdot \mathbf{m}$ of the vector potential \mathbf{A} : By integrating $\mathbf{B} = \nabla \times \mathbf{A}$ over the surface \mathcal{A} and then using Stokes's theorem (I, 2.23) to convert to an integral around $\partial\mathcal{A}$ we find

$$\begin{aligned} \psi(\mathbf{x}) &= \int_{\mathcal{A}} \mathbf{B} \cdot d\mathbf{\Sigma} = \int_{\mathcal{A}} (\nabla \times \mathbf{A}) \cdot d\mathbf{\Sigma} = \int_{\partial\mathcal{A}} \mathbf{A} \cdot d\mathbf{l} \\ &= 2\pi A_\phi. \end{aligned} \quad (4.3)$$

In contrast to other authors (Scharlemann & Wagoner 1973; Blandford & Znajek 1977), we prefer to use ψ as our potential rather than A_ϕ because of its simple physical interpretation.

When expressing \mathbf{E} and \mathbf{B} in terms of A_0 , I , ψ , we shall break them into their 'toroidal' parts (parts along the toroidal direction \mathbf{m}) and their 'poloidal' parts (parts orthogonal to \mathbf{m}):

$$\mathbf{E} \equiv \mathbf{E}^T + \mathbf{E}^P, \quad \mathbf{E}^T \equiv \varpi^{-2}(\mathbf{E} \cdot \mathbf{m})\mathbf{m}, \quad (4.4a)$$

$$\mathbf{B} \equiv \mathbf{B}^T + \mathbf{B}^P, \quad \mathbf{B}^T \equiv \varpi^{-2}(\mathbf{B} \cdot \mathbf{m})\mathbf{m}. \quad (4.4b)$$

Because the magnetic flux ψ through an \mathbf{m} -loop is constant in time (stationarity), the emf around the loop must be zero (Faraday's law 2.20d), and consequently the toroidal electric field must vanish

$$\mathbf{E}^T = 0, \quad \mathbf{E} \text{ is pure poloidal.} \quad (4.5)$$

Axisymmetry of \mathbf{B} ($\mathbf{e}_m \cdot \mathbf{B} = 0$, equation 2.18) guarantees that $\nabla \cdot \mathbf{B}^T = 0$; this, together with $\nabla \cdot \mathbf{B} = 0$, guarantees that

$$\nabla \cdot \mathbf{B}^T = 0 = \nabla \cdot \mathbf{B}^P, \quad (4.6)$$

i.e. the toroidal magnetic field and the poloidal magnetic field can be characterized separately by field lines that never end.

The general expression (2.24a) for \mathbf{E} in terms of A_0 and \mathbf{A} , together with stationarity, axisymmetry, and $A_\phi = \psi/2\pi$, implies the following expression for the electric field in terms of A_0 and ψ

$$\mathbf{E} = -\frac{1}{\alpha} \left(\nabla A_0 + \frac{\omega}{2\pi c} \nabla \psi \right). \quad (4.7)$$

Ampere's law (2.20c) applied to an \mathbf{m} -loop $\partial\mathcal{A}$ gives us the following expression for the toroidal magnetic field in terms of the current I through $\partial\mathcal{A}$

$$\mathbf{B}^T = -\frac{2I}{\alpha\omega^2 c} \mathbf{m}. \quad (4.8)$$

If we move our field point \mathbf{x} by $d\mathbf{x}$ and thereby also move the \mathbf{m} -loop passing through it, we change the flux $\psi(\mathbf{x})$ by $d\psi = \nabla\psi \cdot d\mathbf{x} = (d\mathbf{x} \times 2\pi\mathbf{m}) \cdot \mathbf{B} = (2\pi\mathbf{m} \times \mathbf{B}) \cdot d\mathbf{x}$. This fact permits us to write \mathbf{B}^P (the part of \mathbf{B} orthogonal to \mathbf{m}) as

$$\begin{aligned} \nabla\psi &= 2\pi\mathbf{m} \times \mathbf{B}^P, \\ \mathbf{B}^P &= -\frac{\mathbf{m} \times \nabla\psi}{2\pi\omega^2}. \end{aligned} \quad (4.9)$$

This expression can also be derived from $\mathbf{B} = \nabla \times \mathbf{A}$, together with $\psi/2\pi = A_\phi = \mathbf{A} \cdot \mathbf{m}$, and $\mathbf{m} \times \mathbf{B} = \mathbf{m} \times \mathbf{B}^P$. (Note that the same argument enables us to write similar expressions

$$\begin{aligned} \nabla I &= -2\pi\mathbf{m} \times (\alpha\mathbf{j}^P), \\ \alpha\mathbf{j}^P &= \frac{\mathbf{m} \times \nabla I}{2\pi\omega^2}, \end{aligned} \quad (4.10a)$$

for the poloidal part of the current density \mathbf{j}^P in terms of its flux, the current I .) Using equations (2.17a, c), (4.7), (4.8), (4.9) and (2.3) we can express the toroidal part of \mathbf{j} and the charge density ρ_e in the forms

$$4\pi\mathbf{j}^T \equiv 4\pi\mathbf{j}^T \cdot \frac{\mathbf{m}}{\omega} = -\frac{c\omega}{\alpha} \nabla \cdot \left(\frac{\alpha\nabla\psi}{2\pi\omega^2} \right) + \frac{\omega}{\alpha^2} \nabla\omega \cdot \left(\nabla A_0 + \frac{\omega\nabla\psi}{2\pi c} \right), \quad (4.10b)$$

$$4\pi\rho_e = \nabla \cdot \left[\frac{1}{\alpha} \left(\nabla A_0 + \frac{\omega\nabla\psi}{2\pi c} \right) \right]. \quad (4.10c)$$

One can regard equations (4.10) either as formulae for computing ρ_e and \mathbf{j} from specified potentials ψ , I , A_0 , or as differential equations for ψ , I , A_0 in terms of their sources, ρ_e , \mathbf{j}^T , and a divergence-free $\alpha\mathbf{j}^P$.

(Petterson (1975), as corrected and extended by Znajek (1978a), gives a general multipole expansion for vacuum, stationary, axisymmetric solutions of Maxwell's equations in Kerr spacetime. The absence of sources means that there are only two independent scalar potentials in his formalism, which he calls A_t and A_ϕ (equal to our A_0 and $\psi/2\pi$, respectively). The vacuum solutions found by Petterson and Znajek may be used to construct the Green's function of any problem containing no poloidal sources, e.g. a toroidal

current loop, but they may not be used in situations where \mathbf{j}^P and I are non-zero. Linet (1979) presents a procedure by which the field of any stationary, axisymmetric charge and current distribution in Kerr spacetime can be constructed using Debye potentials. His procedure is consistent with ours, although expressed in very different language.)

The flow of electromagnetic angular momentum and redshifted energy in the magnetosphere is described by the poloidal parts of the flux vectors \mathbf{S}_L and \mathbf{S}_E . Equation (2.31b) for the flux of angular momentum, together with $\mathbf{E}^T = 0$, implies

$$\mathbf{S}_L^P = -(\omega/4\pi) |\mathbf{B}^T| \mathbf{B}^P = (I/2\pi\alpha c) \mathbf{B}^P. \quad (4.11)$$

Thus, angular momentum flows poloidally along magnetic field lines; there is no angular momentum flow unless currents also flow; and the angular momentum will flow outward (away from disc and hole) only if the current I through an \mathbf{m} -loop flows in a direction opposite to that of the poloidal field (i.e. $I > 0$ if \mathbf{B}^P points upward; $I < 0$ if \mathbf{B}^P points downward). The torque per unit volume of the electric and magnetic fields on the matter (equation 2.34b) is

$$(\text{torque per unit volume}) = -\frac{1}{\alpha} \nabla \cdot (\alpha \mathbf{S}_L^P) = (\mathbf{j}/c) \times \mathbf{B} \cdot \mathbf{m}. \quad (4.12)$$

Equation (2.30b) for the flux of redshifted energy, together with the facts $\mathbf{E}^T = 0$ and $\mathbf{E}^P \times \mathbf{B}^P = (\text{a purely toroidal vector})$, implies

$$\begin{aligned} \mathbf{S}_E^P &= (\alpha c/4\pi) (\mathbf{E} \times \mathbf{B}^T) + \omega \mathbf{S}_L^P \\ &= \frac{I}{2\pi} \left(\frac{\omega}{\alpha c} \mathbf{B}^P - \frac{\mathbf{E} \times \mathbf{m}}{\omega^2} \right). \end{aligned} \quad (4.13)$$

Thus, the poloidal flow of redshifted energy is in part orthogonal to \mathbf{E} and in part along \mathbf{B}^P ; and there is no flow at all unless poloidal currents are present ($I \neq 0$). The rate at which electromagnetic fields transfer redshifted energy to matter (equation 2.34a) is

$$\begin{aligned} \left(\begin{array}{l} \text{rate, per unit } \tau \text{ time, that redshifted energy} \\ \text{is fed into a unit volume of matter} \end{array} \right) &= -\frac{1}{\alpha} \nabla \cdot (\alpha \mathbf{S}_E^P) \\ &= \alpha \mathbf{j} \cdot \mathbf{E} + (\omega/c) \mathbf{j} \times \mathbf{B} \cdot \mathbf{m}. \end{aligned} \quad (4.14)$$

4.2 BOUNDARY CONDITIONS AT THE HORIZON

In the black hole's horizon the vanishing of the toroidal electric field, together with Ohm's law (2.42), Ampere's law (2.40), and expression (4.8) for \mathbf{B}^T , implies

$$\mathcal{J}^H = \frac{I}{2\pi\omega} \mathbf{e}_{\hat{\lambda}}, \quad \mathbf{E}^H = \frac{2I}{\omega c} \mathbf{e}_{\hat{\lambda}}, \quad \mathbf{B}^H = -\frac{2I}{\omega c} \mathbf{e}_{\hat{\phi}}. \quad (4.15)$$

Here we have used the horizon basis vectors of equation (2.12b). Note that the horizon current and electric field are purely poloidal, while the horizon magnetic field is purely toroidal. Note also that $\mathcal{J}^H = (I/2\pi\omega) \mathbf{e}_{\hat{\lambda}}$, at some 'observation point' on the horizon, is precisely the surface current required to sink the total current I flowing into \mathcal{H} north of the observation point.

The precise vanishing of the toroidal electric field, together with the horizon boundary condition $|\mathbf{E}_{\parallel} - \mathbf{n} \times \mathbf{B}_{\parallel}| = O(\alpha)$ at \mathcal{H} , implies that the poloidal magnetic field as measured by the ZAMOs intersects the horizon orthogonally

$$\mathbf{B}^P \rightarrow \mathbf{B}_{\perp} + (\text{a parallel component that dies out as } \alpha). \quad (4.16a)$$

The toroidal magnetic field, of course, diverges as $1/\alpha$ (equations 2.40 and 4.15)

$$\alpha \mathbf{B}^T \rightarrow \mathbf{B}^H = -(2I/\omega c) \mathbf{e}_{\hat{\phi}}. \quad (4.16b)$$

Boundary condition (4.16b) is automatically satisfied by our expression (4.8) for \mathbf{B}^T in terms of the potential I , but condition (4.16a) is satisfied only if the potential ψ of equation (4.9) has the limiting form, in the coordinates of equation (2.12a),

$$\psi = \psi_0(\lambda) + O(\alpha^2) \quad \text{near } \mathcal{H}. \quad (4.17)$$

The tangential component of the (purely poloidal) electric field measured by the ZAMOs diverges at the horizon as $1/\alpha$ (equations 2.42 and 4.15)

$$\alpha \mathbf{E}_{\parallel} \rightarrow \mathbf{E}^H = (2I/\omega c) \mathbf{e}_{\hat{\lambda}}, \quad (4.18a)$$

while the perpendicular component remains finite (Gauss's law 2.38)

$$\mathbf{E}_{\perp} \rightarrow 4\pi\sigma^H \mathbf{n}. \quad (4.18b)$$

These boundary conditions are compatible with expression (4.7) for \mathbf{E} in terms of the potentials A_0 and ψ only if A_0 has the limiting form near the horizon,

$$A_0 = -\frac{\Omega^H}{2\pi c} \psi + \left[\int_0^{\lambda} \frac{2I}{\omega c} d\lambda \right]_{\alpha=0} + O(\alpha^2). \quad (4.19)$$

The hole's surface charge σ^H and $E_{\perp} = 4\pi\sigma^H$ are determined by the $O(\alpha^2)$ part of A_0 and ψ , whereas B_{\perp} is determined by $\psi_0(\lambda)$.

The torque of stationary, axisymmetric electromagnetic fields on a unit area of the horizon (equation 2.44) is

$$-\alpha \mathbf{S}_L \cdot \mathbf{n} \rightarrow \frac{dL^H}{d\Sigma^H dt} = \frac{1}{c} (\mathcal{J}^H \times \mathbf{B}_{\perp}) \cdot \mathbf{m} = -\frac{IB_{\perp}}{2\pi c}, \quad (4.20)$$

the Joule heating rate per unit area (equation 2.45) is

$$\Theta^H \frac{dS^H}{d\Sigma^H dt} = \mathcal{J}^H \cdot \mathbf{E}^H = \frac{1}{\pi c} \left(\frac{I}{\omega} \right)^2 \quad (4.21)$$

and the rate per unit area that redshifted energy flows into the hole, increasing its mass (equation 2.46) is

$$\begin{aligned} -\alpha \mathbf{S}_E \cdot \mathbf{n} \rightarrow \frac{dM^H c^2}{d\Sigma^H dt} &= -\frac{dP^H}{d\Sigma^H} = \frac{\Omega^H}{c} \mathbf{m} \cdot (\mathcal{J}^H \times \mathbf{B}_{\perp}) + \mathcal{J}^H \cdot \mathbf{E}^H \\ &= -\frac{\Omega^H IB_{\perp}}{2\pi c} + \frac{1}{\pi c} \left(\frac{I}{\omega} \right)^2. \end{aligned} \quad (4.22)$$

In equation (4.20) we see that the hole loses angular momentum when I and B_{\perp} have the same sign, which is consistent with equation (4.11).

5 Degenerate, stationary, axisymmetric electrodynamics

5.1 OUTSIDE THE HORIZON

In regions of space where

$$|\mathbf{E} \cdot \mathbf{B}| \ll |\mathbf{B}^2 - \mathbf{E}^2| \quad (5.1a)$$

it is reasonable to make the simplifying approximation that the fields are 'degenerate'

$$\mathbf{E} \cdot \mathbf{B} = 0. \quad (5.1b)$$

The degeneracy approximation is justified in a wide range of physical situations, most notably in the presence of a plasma with electrical conductivity so high that the electric field vanishes in the plasma rest frame and the magnetic field is thereby frozen into the plasma (see, e.g. Carter 1979 for a thorough relativistic discussion). For our stationary axisymmetric magnetosphere, degeneracy, together with the fact that \mathbf{E} is purely poloidal, guarantees the existence of a toroidal vector

$$\mathbf{v}^F \equiv \alpha^{-1}(\Omega^F - \omega)\mathbf{m} \quad (5.2)$$

such that

$$\mathbf{E} = -(\mathbf{v}^F/c) \times \mathbf{B}^P = -\frac{(\Omega^F - \omega)}{2\pi\alpha c} \nabla\psi \quad (5.3)$$

(cf. equation 4.9). One can interpret \mathbf{v}^F as the physical velocity of the magnetic field lines relative to the ZAMOs; note that nothing constrains \mathbf{v}^F to be less than the speed of light. Any observer who moves with the magnetic field lines, i.e. with velocity \mathbf{v}^F , sees a vanishing electric field $\mathbf{E}' = [1 - (\mathbf{v}^F)^2/c^2]^{-1/2} [\mathbf{E} + (\mathbf{v}^F/c) \times \mathbf{B}] = 0$. The non-zero electric field \mathbf{E} seen by the ZAMOs is entirely induced by the motion of the magnetic field. Note that just as ω is the angular velocity $d\phi/dt$ of the ZAMOs relative to absolute space, so the Ω^F in equation (5.2) is the angular velocity $d\phi/dt$ of the magnetic field lines relative to absolute space.

Each magnetic field line must rotate with constant angular velocity — i.e. Ω^F must be constant along field lines

$$\mathbf{B}^P \cdot \nabla \Omega^F = 0 \quad (5.4)$$

otherwise the magnetic field would ‘wind itself up’ in violation of stationarity. The mathematical proof of this ‘isorotation law’ is carried out by setting $\dot{\mathbf{B}} = 0$ in Maxwell’s evolution law (2.17d) for \mathbf{B} , imposing in addition axisymmetry and expression (5.3) for \mathbf{E} in terms of ψ , invoking $\nabla \times \nabla\psi = 0$, and then re-expressing $\nabla\psi$ in terms of \mathbf{B}^P by equation (4.9). The original, non-relativistic version of the isorotation law (5.4) is due to Ferraro (1937); and the relativistic version is due to Blandford & Znajek (1977) for force-free magnetospheres, and due to Carter (1979) for degenerate magnetospheres.

In studying the electromagnetic field structure, it is useful to think not only in terms of poloidal magnetic field lines, but also in terms of the axially symmetric field surfaces obtained by moving the poloidal field lines around the axis of symmetry along \mathbf{m} -loops. These axially symmetric field surfaces can be labelled by the flux ψ , since ψ is obviously constant on them. Since $\mathbf{B}^P \cdot \nabla \Omega^F = 0$ (isorotation) and $\mathbf{m} \cdot \nabla \Omega^F = 0$ (axial symmetry), Ω^F is also constant on the magnetic surfaces — which means that we can regard Ω^F as a function of ψ , $\Omega^F(\psi)$. From equations (5.3) and (4.7) we see that in the degenerate region A_0 is also a function of ψ , as is A_ϕ (equation 4.3):

$$\begin{aligned} dA_0/d\psi &= -\Omega^F/2\pi c, \\ dA_\phi/d\psi &= 1/2\pi. \end{aligned} \quad (5.5)$$

In the general stationary axisymmetric case a full solution of Maxwell’s equations (\mathbf{E} , \mathbf{B} , ρ_e , \mathbf{j}) was generated by three independent and freely specifiable scalar fields, I , ψ and A_0 . When degeneracy is also imposed, a full solution is generated by two freely specifiable scalar fields I and ψ , plus one function, $\Omega^F(\psi)$. The electric field \mathbf{E} is computed from expression (5.3), the toroidal and poloidal magnetic fields \mathbf{B}^T and \mathbf{B}^P from (4.8) and (4.9), and the

charge and current densities ρ_e and \mathbf{j} from the Maxwell equations (2.17a, c). The resulting expressions for ρ_e and \mathbf{j} are equations (4.10), specialized to the case $\nabla A_0 = -(\Omega^F/2\pi c)\nabla\psi$

$$\alpha \mathbf{j}^P = \frac{\mathbf{m} \times \nabla I}{2\pi\omega^2}, \quad (5.6a)$$

$$8\pi^2 j^T = -\frac{c\omega}{\alpha} \nabla \cdot \left(\frac{\alpha}{\omega^2} \nabla \psi \right) + \frac{\omega}{\alpha^2 c} (\Omega^F - \omega) \nabla \psi \cdot \nabla (\Omega^F - \omega) - \frac{\omega}{\alpha^2 c} (\Omega^F - \omega) \frac{d\Omega^F}{d\psi} (\nabla \psi)^2, \quad (5.6b)$$

$$8\pi^2 \rho_e = -\nabla \cdot \left[\left(\frac{\Omega^F - \omega}{\alpha c} \right) \nabla \psi \right]. \quad (5.6c)$$

Notice that the divergence-free $\alpha \mathbf{j}^P$ is still determined by I alone, while j^T and ρ_e are both determined by ψ and $\Omega^F(\psi)$. Thus ρ_e and j^T are not independent. The charge density and toroidal current must carefully adjust themselves in degenerate regions (e.g. in regions where there is a highly conducting plasma) so as to keep $\mathbf{E} \cdot \mathbf{B} = 0$.

Degeneracy simplifies expression (4.13) for the poloidal flux of redshifted energy. Using equations (5.2), (5.3) and (4.11), we make it read

$$\begin{aligned} \mathbf{S}_E^P &= \Omega^F \mathbf{S}_L^P = -\Omega^F (\omega/4\pi) |\mathbf{B}^T| \mathbf{B}^P \\ &= \Omega^F (I/2\pi\alpha c) \mathbf{B}^P. \end{aligned} \quad (5.7)$$

Similarly, the rate at which the fields deposit redshifted energy in matter (equation 4.14) simplifies to

$$\begin{aligned} \left(\begin{array}{l} \text{rate, per unit } \tau \text{ time, that redshifted} \\ \text{energy is fed into a unit volume of matter} \end{array} \right) &= -\frac{1}{\alpha} \nabla \cdot (\alpha \mathbf{S}_E^P) \\ &= \Omega^F \left(\begin{array}{l} \text{torque per unit volume of electro-} \\ \text{magnetic fields on matter} \end{array} \right) = -\frac{\Omega^F}{\alpha} \nabla \cdot (\alpha \mathbf{S}_L^P) \\ &= (\Omega^F/c) (\mathbf{j} \times \mathbf{B}) \cdot \mathbf{m}. \end{aligned} \quad (5.8)$$

Thus, angular momentum and redshifted energy both flow along poloidal magnetic field lines, and their ratio everywhere satisfies the 'energy-angular momentum relation' $dE = \Omega^F dL$.

5.2 BOUNDARY CONDITIONS AT THE HORIZON

Degeneracy tightens up and simplifies the boundary conditions at the horizon. All the boundary conditions of Section 4.2 remain valid, but they are now augmented by the new constraint (equations 5.2, 5.3, 2.10, 2.42 and 4.16a)

$$\mathbf{E}^H = -\frac{1}{c} (\Omega^F - \Omega^H) \mathbf{m} \times \mathbf{B}_\perp. \quad (5.9)$$

This constraint, together with our old expression (4.18a) for \mathbf{E}^H , implies that

$$I = \frac{1}{2} (\Omega^H - \Omega^F) \omega^2 B_\perp \quad \text{at } \mathcal{H}, \quad (5.10)$$

and combined with expression (4.9) for $\mathbf{B}_\perp = \mathbf{B}^P$, it says that the potentials ψ and I cannot be specified freely at the horizon; they must in fact be related by

$$I = (\Omega^H - \Omega^F)(\omega/4\pi) d\psi_0/d\lambda \quad \text{on } \mathcal{H}, \quad (5.11)$$

where $\psi_0(\lambda)$ is the horizon value of ψ (equation 4.17).

Condition (5.10) simplifies expressions (4.20)–(4.22) for the torque, dissipation, and mass increase on the horizon

$$-\alpha \mathbf{S}_L \cdot \mathbf{n} \rightarrow \frac{dL^H}{d\Sigma^H dt} = \frac{(\Omega^F - \Omega^H)}{4\pi c} (\omega B_\perp)^2, \quad (5.12)$$

$$\frac{\Theta^H dS^H}{d\Sigma^H dt} = \frac{(\Omega^F - \Omega^H)^2}{4\pi c} (\omega B_\perp)^2, \quad (5.13)$$

$$\begin{aligned} -\alpha \mathbf{S}_E \cdot \mathbf{n} \rightarrow \frac{dM^H c^2}{d\Sigma^H dt} &= -\frac{dP^H}{d\Sigma^H} = \frac{\Omega^F(\Omega^F - \Omega^H)}{4\pi c} (\omega B_\perp)^2 \\ &= \Omega^F \frac{dL^H}{d\Sigma^H dt}. \end{aligned} \quad (5.14)$$

Notice that in the horizon, as outside it, degeneracy enforces the energy–angular momentum relation $dM^H c^2 = \Omega^F dL^H$.

The above boundary conditions at the horizon for a degenerate magnetosphere are identical to those for a force-free magnetosphere (see Section 6.2 below). They were first derived for the force-free case by Znajek (1977) and Blandford & Znajek (1977).

6 Force-free, stationary, axisymmetric electrodynamics

6.1 OUTSIDE THE HORIZON

Consider a region of the magnetosphere where large amounts of plasma are freely available, and where the plasma has adjusted its charge and current densities to make $|\mathbf{E} \cdot \mathbf{B}| \ll |\mathbf{B}^2 - \mathbf{E}^2|$. If the inertial and gravitational forces on the plasma are small enough compared to the inertia of the electromagnetic field, the plasma will be unable to exert significant force on the field, i.e. it will further adjust itself so that

$$|\mathbf{E} \cdot \mathbf{B}| \ll |\mathbf{B}^2 - \mathbf{E}^2|, \quad |\rho_e \mathbf{E} + (\mathbf{j}/c) \times \mathbf{B}| \ll |\mathbf{j}/c| |\mathbf{B}|. \quad (6.1a)$$

In such regions we shall approximate the fields as precisely degenerate and force-free

$$\mathbf{E} \cdot \mathbf{B} = 0, \quad \rho_e \mathbf{E} + (\mathbf{j}/c) \times \mathbf{B} = 0. \quad (6.1b)$$

Note that precise force-freeness implies precise degeneracy. The relativistic theory of degenerate, force-free, stationary, axisymmetric electrodynamics has been developed previously by Blandford & Znajek (1977). The formulae which follow are simply a rewrite of their formalism in our ‘absolute space/universal time’ language.

When the constraint of force-freeness is added to the constraint of degeneracy, the formalism of Section 5 is thereby tightened: No longer are the ‘potentials’ I and ψ independent scalar fields; now I , like Ω^F , must be constant on magnetic field surfaces and therefore must be a function of ψ . To see this, note that because \mathbf{E} is pure poloidal, force-freeness (6.1b) implies that $(\mathbf{j} \times \mathbf{B})^T \equiv \mathbf{j}^P \times \mathbf{B}^P$ must vanish, which means that the poloidal part of the current \mathbf{j}^P is everywhere parallel to the poloidal part of the magnetic field \mathbf{B}^P , which in turn

means that $\nabla\psi = 2\pi\mathbf{m} \times \mathbf{B}^P$ (equation 4.9) and $\nabla I = -2\pi\mathbf{m} \times (\alpha\mathbf{j}^P)$ (equation 4.10a) are parallel, which in turn means that surfaces of constant I and ψ coincide; I is a function of ψ . Moreover, from $\nabla I = -2\pi\mathbf{m} \times (\alpha\mathbf{j}^P)$ and $\nabla\psi = 2\pi\mathbf{m} \times \mathbf{B}^P$ we can read off the proportionality constant relating \mathbf{j}^P to \mathbf{B}^P

$$\mathbf{j}^P = (-\alpha^{-1} dI/d\psi)\mathbf{B}^P. \quad (6.2)$$

Combining this with force-freeness $\rho_e \mathbf{E} + (\mathbf{j}/c) \times \mathbf{B} = 0$ and the expression $\mathbf{E} = -(\mathbf{v}^F/c) \times \mathbf{B}$ we see that the full current density (poloidal plus toroidal) is

$$\mathbf{j} = \rho_e \mathbf{v}^F - (\alpha^{-1} dI/d\psi)\mathbf{B}. \quad (6.3)$$

In general, stationary, axisymmetric regions there were three freely specifiable scalar fields in the most general solution of Maxwell's equations: ψ , I and A_0 — or, alternatively, ρ_e , j^T and the divergence-free $\alpha\mathbf{j}^P$. When degeneracy was imposed, the charges were forced to distribute themselves in a special manner so as to keep $\mathbf{E} \cdot \mathbf{B} = 0$ — and this reduced the number of freely specifiable scalar fields from three to two: ψ and I , or j^T and the divergence-free $\alpha\mathbf{j}^P$. Now, as we impose force-freeness, we suddenly reduce the number of freely specifiable scalar fields from two to zero: The degeneracy relationship (5.6b, c) between ρ_e and j^T is compatible with their force-free relationship $j^T = \rho_e(\Omega^F - \omega)\varpi/\alpha + (\alpha^{-1} dI/d\psi)(2I/\alpha\varpi c)$ (equation 6.3 with B^T replaced by equation 4.8) if and only if the potential ψ satisfies the partial differential equation

$$\nabla \cdot \left\{ \frac{\alpha}{\varpi^2} \left[1 - \frac{(\Omega^F - \omega)^2 \varpi^2}{\alpha^2 c^2} \right] \nabla \psi \right\} + \frac{(\Omega^F - \omega)}{\alpha c^2} \frac{d\Omega^F}{d\psi} (\nabla \psi)^2 + \frac{16\pi^2}{\alpha \varpi^2 c^2} I \frac{dI}{d\psi} = 0. \quad (6.4)$$

The general force-free solution of Maxwell's equations is obtained by selecting a solution ψ , $\Omega^F(\psi)$, $I(\psi)$ of this equation, then computing \mathbf{E} from (5.3), \mathbf{B} from (4.8) and (4.9) and ρ_e and \mathbf{j} from (5.6).

Following Newtonian practice, we call ψ the 'stream function' and equation (6.4) the 'stream equation', because the poloidal current and the poloidal magnetic field both point along 'stream lines' of ψ , i.e. along poloidal lines of constant ψ . Our general relativistic stream equation (6.4) agrees with that derived, in very different notation, in the pioneering paper of Blandford & Znajek (1977).

In the force-free regions the fluxes of angular momentum and redshifted energy carried by the electromagnetic field are conserved

$$\nabla \cdot (\alpha \mathbf{S}_E^P) = \nabla \cdot (\alpha \mathbf{S}_L^P) = 0 \quad (6.5)$$

(equations 5.8). The angular momentum and redshifted energy flow without loss along the poloidal magnetic field lines.

6.2 BOUNDARY CONDITIONS AT THE HORIZON

Because the horizon fields are degenerate, $\mathbf{E}^H \cdot \mathbf{B}^H = 0$, but not force-free, $\sigma^H \mathbf{E}^H + (\mathcal{J}^H/c) \times \mathbf{B}_\perp \neq 0$, the boundary conditions at the horizon are unchanged when we constrain the degenerate exterior fields ($\mathbf{E} \cdot \mathbf{B} = 0$) to be force-free [$\rho_e \mathbf{E} + (\mathbf{j}/c) \times \mathbf{B} = 0$]. The boundary conditions remain as described in Section 5.2.

However, it is important to examine the relationship between the stream equation (6.4) and the unchanged boundary conditions on ψ , $I(\psi)$, and $\Omega^F(\psi)$

$$\psi = \psi_0(\lambda) + O(\alpha^2) \quad \text{near } \mathcal{H}, \quad (6.6a)$$

$$4\pi I = (\Omega^H - \Omega^F)\varpi d\psi_0/d\lambda \quad \text{at } \mathcal{H} \quad (6.6b)$$

(equations 4.17 and 5.11). In the neighbourhood of the horizon, and in the coordinate system of equation (2.12a), the stream equation (6.4) can be put in the form

$$\begin{aligned} \frac{\kappa^2}{c^4} (\Omega^F - \Omega^H)^2 \alpha \frac{\partial}{\partial \alpha} \left(\frac{1}{\alpha} \frac{\partial \psi}{\partial \alpha} \right) + \left(\frac{1}{2\omega^2 \partial \psi / \partial \lambda} \right) \frac{\partial}{\partial \lambda} \left\{ \left[(\Omega^F - \Omega^H) \omega \frac{\partial \psi}{\partial \lambda} \right]^2 - (4\pi I)^2 \right\} \\ + [\text{coefficients of } O(\alpha)] \times [\text{derivatives of } \psi] \\ + [\text{coefficients of order unity}] \times [\partial \psi / \partial \alpha] = 0. \end{aligned} \quad (6.7)$$

Note that if ψ is finite at the horizon, then this stream equation requires that $\psi = \psi_0(\lambda) + O(\alpha^2 \ln \alpha) + O(\alpha^2)$ at \mathcal{H} . If, moreover, ψ and I are well behaved on the axis of symmetry ($\psi \propto \omega^2$ and $I \propto \omega^2$), then this stream equation says that the absence of $O(\alpha^2 \ln \alpha)$ terms from ψ is equivalent to the demand that $4\pi I = \pm (\Omega^H - \Omega^F) \omega d\psi_0/d\lambda$. Roman Znajek points out to us that the choice of sign (+ versus -) corresponds to a choice of the direction of the Poynting flux at the horizon (in versus out) and thence to a choice of whether the horizon is a future horizon (black hole) or a past horizon (white hole).

Thus, physically well-behaved solutions of the stream equation (solutions with ψ finite and no $O(\alpha^2 \ln \alpha)$ terms at \mathcal{H}) automatically satisfy the boundary conditions (6.6) — except for a possible sign error in (6.6b). In Section 7.4 we shall use this fact to elucidate the global structure of the force-free region of a magnetosphere, and to formulate the problem of constructing solutions for the force-free region.

6.3 ACTION PRINCIPLES FOR STREAM FUNCTION

In the force-free region we can regard the stream function ψ as governed either by the stream equation (6.4) or (equivalently) by an action principle. The action to be extremized is

$$\mathcal{J} = \int_{\mathcal{V}} (1/8\pi) [(\mathbf{B}^P)^2 - (\mathbf{B}^T)^2 - (\mathbf{E})^2] \alpha dV. \quad (6.8a)$$

Here \mathcal{V} is the region of integration (which must be force-free), and \mathbf{B}^P , \mathbf{B}^T , \mathbf{E} are to be expressed in terms of ψ , I and Ω^F via equations (4.9), (4.8) and (5.3), yielding

$$\mathcal{J} = \int_{\mathcal{V}} \frac{1}{8\pi} \left\{ \left[1 - \frac{(\Omega^F - \omega)^2 \omega^2}{\alpha^2 c^2} \right] \left(\frac{\nabla \psi}{2\pi \omega} \right)^2 - \left(\frac{2I}{\alpha \omega c} \right)^2 \right\} \alpha dV. \quad (6.8b)$$

In this action $dV \equiv (\det \|\gamma_{jk}\|)^{1/2} dx^1 dx^2 dx^3$ is the proper volume element of absolute space, α , ω and ω are known functions of x^1 , x^2 and x^3 describing the black hole and I and Ω^F are to be specified as explicit functions of ψ before the variation. The variation of ψ in this action leads to

$$\begin{aligned} \delta \mathcal{J} = - \frac{1}{16\pi^3} \int_{\mathcal{V}} \delta \psi (\text{left-hand side of stream equation 6.4}) dV \\ + \frac{1}{16\pi^3} \int_{\partial \mathcal{V}} \delta \psi \frac{\alpha}{\omega^2} \left[1 - \frac{(\Omega^F - \omega)^2 \omega^2}{\alpha^2 c^2} \right] \nabla \psi \cdot d\Sigma. \end{aligned} \quad (6.8c)$$

Thus, so long as the region of integration is bounded away from the horizon ($\alpha > 0$ throughout \mathcal{V} and $\partial \mathcal{V}$), the appropriate boundary conditions on ψ are:

$$\left. \begin{aligned} \text{At each point on } \partial \mathcal{V} \\ \text{either specify an arbitrary but smoothly changing value of } \psi (\delta \psi = 0) \\ \text{or specify that } \nabla \psi \text{ be parallel to } \partial \mathcal{V} (\nabla \psi \cdot d\Sigma = 0). \end{aligned} \right\} \quad (6.8d)$$

If ψ and $\nabla\psi$ are thus specified on $\partial\mathcal{V}$, the functions ψ which extremize \mathcal{J} ($\delta\mathcal{J} = 0$) will be precisely the solution of the stream equation.

If the region of integration \mathcal{V} is bounded in part by the black hole's horizon \mathcal{H} ($\alpha = 0$), then the action (6.8a, b) is infinite [$(\mathbf{B}^T)^2$ and $(\mathbf{E})^2$ both diverge as $1/\alpha^2$, so \mathcal{J} diverges as $\ln \alpha$]. This defect can be repaired by a renormalization of the action

$$\mathcal{J}' = \lim \left\{ \int_{\mathcal{V}} (1/8\pi) [(\mathbf{B}^P)^2 - (\mathbf{B}^T)^2 - (\mathbf{E})^2] \alpha dV - (c^2/\kappa) \ln \alpha \int_{\mathcal{H} \cap \partial\mathcal{V}} (1/8\pi) [(\mathbf{B}^H)^2 + (\mathbf{E}^H)^2] d\Sigma \right\}. \quad (6.9a)$$

Here and below 'lim' means 'take the limit as the boundary $\partial\mathcal{V}$ approaches the horizon \mathcal{H} , i.e. as $\alpha \rightarrow 0$ on $\mathcal{H} \cap \partial\mathcal{V}$ '; also, $\mathbf{B}^H = \lim(\alpha \mathbf{B}^T)$ and $\mathbf{E}^H = \lim(\alpha \mathbf{E}_{\parallel})$ are the horizon fields (cf. equations 4.16b and 4.18a). When \mathbf{B}^P , \mathbf{B}^T and \mathbf{E} are expressed in terms of ψ , I and Ω^F via equations (4.9), (4.8) and (5.3), this renormalized action becomes

$$\mathcal{J}' = \lim \left\{ \frac{1}{8\pi} \int_{\mathcal{V}} \left[1 - \frac{(\Omega^F - \omega)^2 \varpi^2}{\alpha^2 c^2} \right] \left(\frac{\nabla\psi}{2\pi\varpi} \right)^2 - \left(\frac{2I}{\alpha\varpi c} \right)^2 \right\} \alpha dV - \frac{\ln \alpha}{8\pi\kappa} \int_{\mathcal{H} \cap \partial\mathcal{V}} \left\{ \frac{(\Omega^F - \Omega^H)^2}{4\pi^2} \left(\frac{d\psi}{d\lambda} \right)^2 + \left(\frac{2I}{\varpi} \right)^2 \right\} \varpi d\phi d\lambda \right\}. \quad (6.9b)$$

Here we have used the coordinates of equation (2.12a) for the horizon surface integral. The variation of ψ in this action, with I and Ω^F taken to be fixed functions of ψ as before, yields

$$\delta\mathcal{J}' = \lim \left\{ -\frac{1}{16\pi^3} \int_{\mathcal{V}} \delta\psi \text{ (left-hand side of stream equation 6.4)} dV + \frac{1}{16\pi^3} \int_{\partial\mathcal{V}} \delta\psi \frac{\alpha}{\varpi^2} \left[1 - \frac{(\Omega^F - \omega)^2 \varpi^2}{\alpha^2 c^2} \right] \nabla\psi \cdot d\Sigma - \frac{\ln \alpha}{32\pi^3\kappa} \int_{\mathcal{H} \cap \partial\mathcal{V}} \left(\frac{\delta\psi}{\varpi d\psi/d\lambda} \right) \frac{d}{d\lambda} \left\{ (4\pi I)^2 - \left[\varpi(\Omega^F - \Omega^H) \frac{d\psi}{d\lambda} \right]^2 \right\} d\phi d\lambda - \frac{\ln \alpha}{16\pi^3\kappa} \int_{\partial(\mathcal{H} \cap \partial\mathcal{V})} \delta\psi \left[\varpi(\Omega^F - \Omega^H)^2 \frac{d\psi}{d\lambda} \right] d\phi \right\}. \quad (6.9c)$$

Thus, the appropriate boundary conditions on ψ are:

$$\left. \begin{array}{l} \text{At each point on } \mathcal{H} \cap \partial\mathcal{V} \\ \quad \text{specify } \psi \text{ as a fixed solution of the differential equation} \\ \quad 4\pi I = (\Omega^H - \Omega^F) \varpi d\psi/d\lambda \text{ (equation 6.6b) (so } \delta\psi = 0). \\ \text{At each point of } \partial\mathcal{V} \text{ that is not part of } \mathcal{H} \\ \quad \text{either specify an arbitrary but smoothly changing value of} \\ \quad \psi (\delta\psi = 0), \text{ or specify that } \nabla\psi \text{ be parallel to } \partial\mathcal{V} (\nabla\psi \cdot d\Sigma = 0). \end{array} \right\} \quad (6.9d)$$

The functions ψ which extremize \mathcal{J}' subject to these constraints are precisely the solutions of the stream equation.

The two action principles (6.8) and (6.9) are generalizations to black holes of the flat-space, pulsar-magnetosphere action principle of Scharlemann & Wagoner (1973). A third action principle, one without a pulsar analogue, governs the distribution of magnetic field on the horizon:

Suppose that just outside the horizon the magnetosphere is force-free. Let the total magnetic flux through the horizon, between the poles and equator, be specified (i.e. let $\psi_0 = 0$ at the north and south poles, and $\psi_0 = \psi_E$ at the equator be fixed). Specify, moreover, for each poloidal field line, its angular velocity Ω^F and the current I inside it (i.e. let the functions $\Omega^F(\psi_0)$ and $I(\psi_0)$ be fixed). Then the poloidal magnetic field lines will distribute themselves over the horizon in such a manner as to extremize the horizon's total surface energy of tangential electromagnetic field

$$\begin{aligned} \mathcal{E} &= \int_{\mathcal{H}} (1/8\pi) [(\mathbf{B}^H)^2 + (\mathbf{E}^H)^2] d\Sigma \\ &= \int_{\mathcal{H}} \frac{1}{8\pi c^2} \left[\frac{(\Omega^F - \Omega^H)^2}{4\pi^2} \left(\frac{d\psi_0}{d\lambda} \right)^2 + \left(\frac{2I}{\varpi} \right)^2 \right] \varpi d\phi d\lambda. \end{aligned} \quad (6.10)$$

In fact, the Euler–Lagrange equation for the action principle $\delta\mathcal{E} = 0$ is identical to the near-horizon form (6.7) of the stream equation

$$\frac{1}{\varpi d\psi_0/d\lambda} \frac{d}{d\lambda} \left\{ \left[(\Omega^F - \Omega^H) \varpi \frac{d\psi_0}{d\lambda} \right]^2 - (4\pi I)^2 \right\} = 0, \quad (6.11)$$

and this equation is satisfied by the ‘true’ magnetic field distribution ($4\pi I = (\Omega^H - \Omega^F) \varpi d\psi_0/d\lambda$ – equation (6.6b)). In Section 7.4 we shall discuss further the manner in which this action principle and equation (6.6b) govern the horizon's field.

R. Znajek (private communication) points out to us that the action \mathcal{E} is not merely extremized by the horizon's poloidal field configuration; it is actually minimized: the second variation of expression (6.10), evaluated at the extremal configuration (6.11), is

$$\delta^2 \mathcal{E} = \int_{\mathcal{H}} \frac{(\Omega^F - \Omega^H)^2}{32\pi^3 c^2} \left[\frac{d}{d\lambda} \delta\psi_0 - \delta\psi_0 \frac{d}{d\lambda} \ln \left(\varpi \frac{d\psi_0}{d\lambda} \right) \right]^2 \varpi d\phi d\lambda, \quad (6.12)$$

which is positive semi-definite. Znajek goes on to point out that, after the action \mathcal{E} has been extremized and one has found that $|\mathbf{B}^H| = |\mathbf{E}^H|$, then $\mathcal{E} = (1/c) \times$ (rate of Joule heating of the horizon). In an alternative description of the horizon (Section 5.4 of Paper I), Znajek (1978b) has attributed the horizon's Joule heating to a combination of electric current and magnetic current. In this description \mathcal{E} is $(1/c) \times$ (rate of Joule heating) even before minimization – which means that *the horizon's rate of entropy production is minimized by its equilibrium distribution of poloidal flux*, a result reminiscent of ‘Prigogine's Principle’ (cf. Kittel 1958).

7 Global model of stationary magnetosphere

7.1 OVERVIEW

We now use the results of Sections 4, 5 and 6 to elucidate features of the Blandford–Znajek (1977) model for the power sources of quasars and active galactic nuclei.

We assume that the black hole, the accretion disc and their magnetosphere have settled down into a stationary state that is axisymmetric about the hole's rotation axis and reflection symmetric in the hole's equatorial plane, and that has the qualitative character of Fig. 1. We assume, further, that the plasma in the disc is a sufficiently good conductor that, although the field lines might slip through it a bit, the fields are, nevertheless, degenerate ($|\mathbf{E} \cdot \mathbf{B}| \ll |\mathbf{B}^2 - \mathbf{E}^2|$). We shall idealize them as perfectly degenerate, $\mathbf{E} \cdot \mathbf{B} = 0$, but not as force-free: The inertia of the disc's plasma is absolutely crucial for containing the magnetic field. Without it the field would fly away. This degenerate, non-force-free region (which also includes the horizon) is marked ‘D’ in Fig. 2.

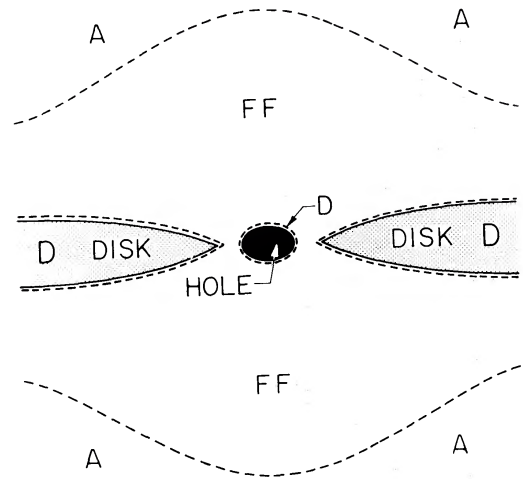


Figure 2. Cross-section through space at constant azimuth ϕ showing three types of regions where we make three different assumptions about the structure of the electromagnetic field. In region FF the field is force-free; in region D (which includes the disc and the horizon) it is degenerate but not force-free; in region A ('acceleration region') it is neither degenerate nor force-free. The boundaries between the FF, D and A regions are shown as dashed lines. Although the disc is drawn fairly thin, it might well be so thick as to reach up and intersect the acceleration region A.

Following Blandford (1976) and Blandford & Znajek (1977), we assume further that just outside the disc and hole the plasma becomes sufficiently rarefied that it no longer exerts significant force on the magnetic field – but it is still sufficiently dense to provide the charged particles needed for degeneracy and force-freeness

$$|\mathbf{E} \cdot \mathbf{B}| \ll B^2 - E^2,$$

$$|\rho_e \mathbf{E} + (\mathbf{j}/c) \times \mathbf{B}| \ll |\mathbf{j}/c| |\mathbf{B}|.$$

As discussed by Blandford (1976), on field lines threading the disc the necessary charged particles are likely extracted from the disc by the Goldreich–Julian (1969) mechanism: a component of \mathbf{E} along \mathbf{B} , which is so weak as to constitute a negligible violation of our force-free, degenerate assumption, pulls the charges out of the disc. However, magnetic field lines that thread the hole must get their charges and currents in some other manner. Blandford & Znajek (1977) argue that they come from the Ruderman–Sutherland (1975) 'spark-gap' process, a cascade production of electron–positron pairs in the force-free region – a production induced indirectly by a component of \mathbf{E} along \mathbf{B} , which again is so weak as to constitute a negligible violation of force-freeness and degeneracy. We shall assume that by these processes, or some others, the necessary charged particles are supplied and the region marked 'FF' in Fig. 2 becomes force-free. This region might extend through the annulus between the disc and the horizon, if the plasma there is tenuous enough for force-freeness to be established. We shall idealize the FF region as being precisely degenerate and force-free

$$[\mathbf{E} \cdot \mathbf{B} = 0, \rho_e \mathbf{E} + (\mathbf{j}/c) \times \mathbf{B} = 0].$$

Outside the FF region, where the magnetic field becomes weak and where it may be dragging charges into rotation at nearly the speed of light, the inertia of the charged particles begins to make itself felt. In this region, marked 'A' in Fig. 2, the field loses both its degeneracy and force-freeness, and it presumably uses its energy to accelerate charged particles and (hopefully) to form charged-particle beams. Blandford (1976), Lovelace

(1976) and Lovelace *et al.* (1979) have speculated extensively about the physics which occurs in this acceleration region. For our purposes the only important point (assumption) is that all the power being transported outward through the FF region by DC electromagnetic fields somehow gets transferred to charged particles – and subsequently perhaps into radiation – in the A region.

7.2 GLOBAL ENERGY AND ANGULAR MOMENTUM BALANCE

Each magnetic field line in the D and FF regions rotates rigidly, dragged around by the hole's rotation and the disc's orbital motion. In the A region, where particle inertia is strong, degeneracy breaks down and the concept of field-line angular velocity Ω^F ceases to be a useful one. (If one defines Ω^F so as to account, by \mathbf{B} -field motion, for the component of \mathbf{E} orthogonal to \mathbf{B} (analogue of equation 5.3), one finds that in the A region Ω^F is no longer constant along field lines.) Nevertheless, the fields in the A region continue to transport energy and angular momentum.

Consider an annular tube of magnetic flux $\Delta\psi$ intersecting the hole (flux tube 'a' in Fig. 3). The horizon exerts a net torque

$$-\left(\frac{d\Delta L^H}{dt}\right) = -\frac{1}{c}(\mathcal{J}^H \times \mathbf{n}\Delta\psi) \cdot \mathbf{m} = \frac{(\Omega^H - \Omega^F)}{4\pi c} \omega^2 B_\perp \Delta\psi \quad (7.1)$$

on this flux tube (equations 4.20 and 5.12) and with this torque it transmits a redshifted power (equation 5.14)

$$\Delta P = -\Omega^F \frac{d\Delta L^H}{dt} = \frac{\Omega^F(\Omega^H - \Omega^F)}{4\pi c} \omega^2 B_\perp \Delta\psi \quad (7.2)$$

up the flux tube and into the FF region. The torque (7.1) and power (7.2) are transmitted loss-free along the flux tube (equations 5.7 and 6.5) through the entire thickness of the FF

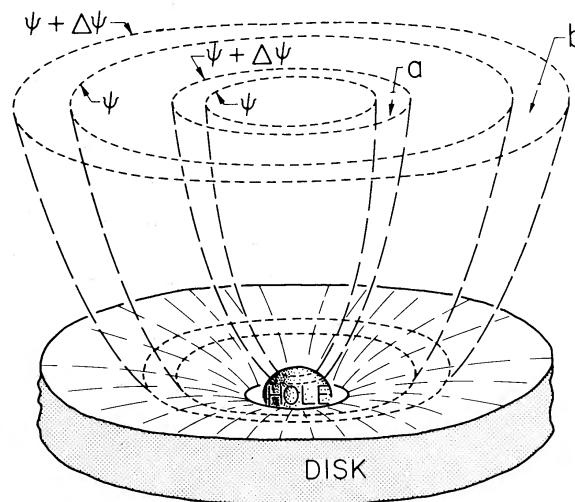


Figure 3. Annular magnetic flux tubes (dashed lines) used in the text's discussion of torque balance and power flow. Two annular flux tubes are shown. The inner tube (labelled 'a') intersects the hole. The outer tube (labelled 'b') intersects the disc. Magnetic field lines on the inner face of each tube are characterized by some value ψ of the flux parameter; those on the outer face are characterized by $\psi + \Delta\psi$, where $\Delta\psi$ is the total magnetic flux in the annular tube.

region and into the A region. In the A region the angular momentum continues to flow along the flux tube (equation 4.11), where it gradually gets deposited into charged particles (equation 4.12)

$$\begin{aligned} \frac{d\Delta L^A}{dt} &= \frac{1}{c} \int_{\text{flux tube in A}} \alpha \mathbf{j} \times \mathbf{B} \cdot \mathbf{m} dV \\ &= - \frac{d\Delta L^H}{dt}. \end{aligned} \quad (7.3)$$

The power, by contrast, flows away from the flux tube into adjoining regions of space (equation 4.13), where it presumably also gets deposited in charged particles (equation 4.14).

The total power output from the flux tube, expression (7.2) (which was derived by Blandford & Znajek), depends critically on the tube's angular velocity Ω^F . That angular velocity is determined by torque balance along the flux tube between the hole and the A region (equations 7.1 and 7.3). If the particles in the A region of the tube have enormous inertia, they will drag the tube down to $\Omega^F \ll \Omega^H$, and the power transmitted will be very small (equation 7.2). If the particles have very little inertia, the tube will be dragged up to $\Omega^F \simeq \Omega^H$, and once again the power transmitted will be very small. From this viewpoint it might appear to be a miracle if the charged particles conspired with the hole so as to have just the right amount of inertia to make $\Omega^F \sim \Omega^H/2$ and thereby produce large power output. Nevertheless, such a miracle may well occur, according to the following variant of a classic Goldreich–Julian (1969) argument for pulsars.

Near the black hole, magnetic field lines rotate backward relative to ZAMOs with velocities $\mathbf{v}^F = -(\Omega^H - \Omega^F)\mathbf{m}/\alpha$ which greatly exceed the speed of light ($\alpha \rightarrow 0$ at \mathcal{H}). Far from the hole and away from the symmetry axis ($\alpha \rightarrow 1$, $\omega \rightarrow 0$, ϖ large), field lines rotate forward relative to ZAMOs with velocities $\mathbf{v}^F = \Omega^F \mathbf{m}$ which also greatly exceed the speed of light. A charged particle that is constrained to move along a magnetic field line can move, relative to ZAMOs, more slowly than the field line. Near the hole it does this by sliding down the field line toward the horizon; far from the hole it does it by sliding out the field line. However, there is a minimum possible speed with which the particle can slide if it is to stay on the field line

$$v_{\min} = \frac{v^F}{[1 + (\mathbf{B}^T)^2/(\mathbf{B}^P)^2]^{1/2}}. \quad (7.4)$$

The boundary condition $I = \frac{1}{2}(\Omega^H - \Omega^F)\varpi^2 B_\perp$ (equation 5.10) at the horizon is perfectly designed to make this minimum sliding velocity equal to the speed of light (*cf.* equations 4.8, 4.16a, 5.2): charged particles slide along field lines into the horizon at precisely the speed of light relative to ZAMOs. Far from the hole v_{\min} will be of order c if and only if $\Omega^F \simeq \Omega^H/2$, since

$$\begin{aligned} v^F &\simeq \Omega^F \varpi, \\ B^T &\simeq 2I/\varpi c \simeq (1/\varpi c) [(\Omega^H - \Omega^F)\varpi^2 B_\perp]_{\text{at } \mathcal{H}} \simeq (\Omega^H - \Omega^F)\psi/\pi c \varpi, \\ B^P &= |\nabla\psi|/2\pi\varpi \simeq \psi/\pi\varpi^2, \\ v_{\min} &\simeq c\Omega^F/(\Omega^H - \Omega^F). \end{aligned}$$

Thus, if $\Omega^F \ll \Omega^H/2$, particles can easily slide along the field lines more slowly than light near the boundary of the FF and A regions and perhaps this will lead to small particle inertia in the A region and thereby to a spinning up of the field toward $\Omega^F \simeq \Omega^H/2$. On the other

hand, if $(\Omega^H - \Omega^F) \ll \Omega^H/2$, charged particles will be unable to achieve $v_{\min} \gg c$ at large radii; they will be thrown off the field lines, and they may well exert a back-reaction torque on the field lines sufficient to drive Ω^F back down near $\Omega^H/2$.

Of course, this argument is speculative. The analogous argument in the case of pulsars (where Ω^F is fixed but I is adjustable and determines v_{\min}) is highly controversial even today, a dozen years after the Goldreich–Julian work (see, e.g. Mestel *et al.* 1979; Arons 1979).

Turn attention now to field lines which thread the disc. For simplicity, assume that the disc is reflection symmetric, and restrict attention to the region above the equatorial plane. As in the case of the hole, consider an annular tube of magnetic flux which threads the disc (flux tube ‘b’ in Fig. 3). Let Ω^D be the angular velocity of the disc’s plasma inside this tube (appropriately averaged vertically if necessary). The field lines in the tube will rotate slightly more slowly than Ω^D , due to drag on the tube in the A region; this velocity difference will induce in the rest frame of the disc plasma a radial electric field proportional to $(\Omega^D - \Omega^F)B^P$, which in turn will drive a radial current $j \propto (\Omega^D - \Omega^F)B^P/(\text{resistivity})$ that will interact with \mathbf{B}^P to produce a torque on the flux tube

$$-\frac{d\Delta L^D}{dt} = \left(\frac{\Omega^D - \Omega^F}{4\pi^2 c^2} \right) \frac{\Delta\psi}{\Delta Z^D} \Delta\psi. \quad (7.5)$$

Here ΔZ^D is the total electrical resistance (impedance) in the disc, north of the equator, between the inner surface of the flux tube and the outer surface (for further details about ΔZ^D see the next section). This net torque will be transmitted, loss-free, through the FF region where it is described by equation (5.7)

$$-\frac{d\Delta L^D}{dt} = (I/2\pi c)\Delta\psi, \quad (7.6)$$

and into the A region where it will act on charged particles in the flux tube (equation 4.12)

$$-\frac{d\Delta L^D}{dt} = \frac{d\Delta L^A}{dt} = \frac{1}{c} \int_{\text{flux tube in A}} \alpha \mathbf{j} \times \mathbf{B} \cdot \mathbf{m} dV. \quad (7.7)$$

Associated with this torque is the redshifted power (equation 5.8)

$$\Delta P = \Omega^F \frac{d\Delta L^A}{dt} = \Omega^F \left(\frac{I}{2\pi c} \right) \Delta\psi, \quad (7.8)$$

which the magnetic flux tube extracts from the disc and transfers to charged particles in the A region.

Torque balance, i.e. equality of expressions (7.5), (7.6) and (7.7), determines both Ω^F (the tube’s angular velocity) and I (the current flowing in the disc across the tube) in terms of $\Delta\psi$ (the magnetic flux in the tube), Ω^D (the disc’s angular velocity at the foot of the tube), ΔZ^D (the disc’s impedance across the tube) and $-d\Delta L^A/dt$ (the torque of the acceleration-region plasma on the tube). Typically, the disc impedance ΔZ^D will be very small (high conductivity), and the field lines will therefore be locked into the disc, $\Omega^F \simeq \Omega^D$.

Our variant of the Goldreich–Julian argument suggests that the torque in the A region might regulate itself so as to make $v_{\min} \simeq c$, and thereby (equations 4.8 and 5.2)

$$I \simeq I_{\text{crit}} = \Omega^F \psi / 2\pi. \quad (7.9)$$

These conditions of self-regulation ($\Omega^F \simeq \Omega^H/2$ and thence $I \simeq \Omega^F \psi / 2\pi$ for lines threading the hole; $I \simeq \Omega^F \psi / 2\pi$ for lines threading the disc) lead to astrophysically interesting power outputs from both disc and hole (Blandford 1976, 1979; Blandford &

Znajek 1977): For a black hole of mass $M \approx 10^8 M_\odot$ rotating at $\Omega^H \approx 1 \text{ rad}/1000 \text{ s}$ (a/M near unity), on to which an accretion disc has deposited a 10^4 G magnetic field, the power outputs from the hole and the disc will both be roughly $10^{44} \text{ erg s}^{-1}$.

7.3 CIRCUIT ANALYSIS OF POWER FLOW

Znajek (1978b) and Blandford (1979) have described the above model, semi-quantitatively, in terms of a circuit analogy. Blandford says: ‘The massive black hole behaves like a battery with an emf of up to 10^{21} V and an internal resistance of about 30Ω . When a current flows, the power dissipated within the horizon, manifest as an increase in the irreducible mass (i.e. entropy), is comparable with that dissipated in particle acceleration etc. in the far field.’ In this section we shall give a mathematically precise version of this description.

We begin by defining the potential drop $\Delta V^{\mathcal{C}}$ along any curve segment \mathcal{C} in absolute space:

$$\Delta V^{\mathcal{C}} \equiv \int_{\mathcal{C}} \alpha \mathbf{E} \cdot d\mathbf{l}. \quad (7.10)$$

The factor α converts the forces produced by the electric field \mathbf{E} from a ‘per unit proper time $d/d\tau$ ’ basis to a ‘per unit global time d/dt ’ basis. Without the α , we would be unable to use Faraday’s law (2.20d) to deduce the potential drop around a circuit; and without it the total current $I' = \int \alpha \mathbf{j} \cdot d\mathbf{\Sigma}$ = (charge per unit global time t), flowing in a thin magnetic-free wire at rest relative to the ZAMOs, could not be expressed simply as $I' = \Delta V/R$ with R the resistance of the wire as measured in flat space. The α in expression (7.10) is required to balance the α in I' and to thereby make standard circuit theory valid in the presence of gravity. (We use a prime on I' merely to distinguish it from the potential I of our magnetosphere theory.)

We can compute the total potential drop around any closed curve \mathcal{C} from Faraday’s law (2.20d). Obviously, the total drop must be independent of the state of motion of the curve – and, in fact, the two magnetic terms in Faraday’s law conspire to make this so (see Section 2.5 of Paper I). For purposes of computation, assume that the curve is at rest in invariant space, so its shape and size are unchanging; for application to our magnetosphere assume that the magnetic field is time independent, $\dot{\mathbf{B}} = 0$. Then the time derivative of the flux through the curve vanishes, the velocity of the curve relative to the ZAMOs is $\mathbf{v} = -\omega \mathbf{m}/\alpha$, and Faraday’s law becomes

$$\Delta V^{\mathcal{C}} = (1/c) \oint_{\mathcal{C}} (\omega \mathbf{m} \times \mathbf{B}) \cdot d\mathbf{l}. \quad (7.11)$$

Thus, the interaction of a stationary magnetic field, $\dot{\mathbf{B}} = 0$, with the hole’s dragging of inertial frames (i.e. with its ‘gravitomagnetic field’) produces an emf around closed curves. Using our stationary, axisymmetric magnetosphere equation (4.9), we can rewrite this emf in terms of the flux potential

$$\Delta V^{\mathcal{C}} = (1/c) \oint_{\mathcal{C}} (\omega/2\pi) \nabla \psi \cdot d\mathbf{l}. \quad (7.12)$$

(Note: This could have been derived more quickly, but perhaps with less physical clarity, from equations (7.10) and (4.7).)

Fig. 4 shows several electric equipotential surfaces (surfaces orthogonal to $\alpha \mathbf{E}$, i.e. surfaces made up of curves along which $\Delta V = 0$). Consider the neighbouring equipotentials labelled 1 and 2, which intersect the horizon. In the D and FF regions these equipotentials coincide with the walls of the inner magnetic flux tube of Fig. 3 (\mathbf{E} is orthogonal to the flux

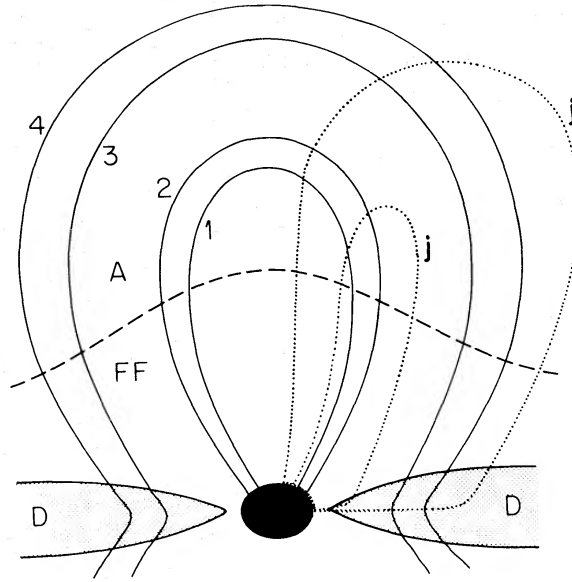


Figure 4. Cross-section showing several electric equipotential surfaces (solid lines, labelled 1, 2, 3 and 4) and several curves along which poloidal current flows (dotted curves labelled *j*). The degenerate, force free, and acceleration regions are marked D, FF and A as in Fig. 2. In the D and FF regions the equipotential surfaces coincide with the walls of the magnetic flux tubes of Fig. 3, but in the A region they deviate from the flux tubes and close up on themselves. In the FF region the current flows along equipotential surfaces, but in the D and A regions it can flow across them.

tube walls due to degeneracy, so there is no potential drop ΔV along any curve lying entirely in a flux tube wall). However, the equipotentials separate from the flux tube walls in region A, where degeneracy breaks down. In fact, the equipotentials must eventually close upon themselves, as shown, in order for spatial infinity to remain at zero potential. Everywhere in the FF region currents are constrained to flow in the equipotential surfaces ($\mathbf{j} \cdot \mathbf{E} = 0$); no current can cross them. However, in the horizon (a D region) current flows from the north polar region, across equipotentials, toward the equator. The total current crossing our two equipotential surfaces 1 and 2 in the horizon is I , where I is the current ‘potential’ of previous sections evaluated at the feet of the equipotential surfaces. By stationarity, this same total current I must flow back across surfaces 2 and 1 in the A region. (We assume that 2 and 1 are close enough together that the poloidal current between them in the FF region is negligible.)

In the horizon, with its surface resistivity $R^H = 4\pi/c = 377 \Omega$, the current crossing from surface 1 to surface 2 encounters a total resistance

$$\begin{aligned} \Delta Z^H &= R^H \frac{(\text{distance from 1 to 2})}{(\text{circumference across which } I \text{ flows})} = R^H \frac{\Delta \lambda}{2\pi \varpi} \\ &= R^H \frac{\Delta \psi}{4\pi^2 \varpi^2 B_\perp}. \end{aligned} \quad (7.13)$$

Here $\Delta \psi$, as in the last section, is the total magnetic flux in the tube between 1 and 2. The potential drop in the horizon between 1 and 2 is, of course,

$$\Delta V^H = I \Delta Z^H, \quad (7.14a)$$

and it can be expressed equally well from equation (5.3) and $\alpha \mathbf{E} \rightarrow \mathbf{E}^H$, as

$$\Delta V^H = \int_1^2 \mathbf{E}^H \cdot d\mathbf{l} = (\Omega^H - \Omega^F) \Delta \psi / 2\pi c. \quad (7.14b)$$

We assume for pedagogical purposes (and because it is likely true) that the entire A region is far enough from the hole that $\omega \ll \Omega^H$ there; we therefore approximate ω as zero in A. Then equation (7.11) guarantees that in the A region the potential drop ΔV^A between surfaces 2 and 1 is independent of where one computes it — near the symmetry axis, or at 15° latitude, or This unique potential drop can be thought of as produced by a resistance ΔZ^A to the flow of the current I :

$$\Delta V^A = I \Delta Z^A. \quad (7.15a)$$

An alternative expression for the potential drop, derivable by integration at the interface between the A and FF regions where expression (5.3) for \mathbf{E} is valid, is

$$\Delta V^A = \int_2^1 \alpha \mathbf{E} \cdot d\mathbf{l} = \Omega^F \Delta \psi / 2\pi c. \quad (7.15b)$$

The sum of the potential drops in the horizon \mathcal{H} (equations 7.14) and in region A (equations 7.15) is equal to the total emf around a closed curve that passes along \mathcal{H} from 1 to 2, then up 2 poloidally into region A, then from 2 to 1, then poloidally down 1 to its starting point at \mathcal{H} . This total closed-loop emf can be evaluated from equation (7.12), where the only non-zero contribution comes from \mathcal{H} (because $\omega = 0$ in A and $\nabla\psi \cdot d\mathbf{l} = 0$ in the FF portions of 1 and 2). The result is

$$\Delta V^H + \Delta V^A = \text{emf} = \Omega^H \Delta \psi / 2\pi c. \quad (7.16)$$

Equations (7.14)–(7.16) for the potential drops and equation (7.13) for the horizon resistance are the foundations for our circuit-theory analysis of power flow.

The ratio of the potential drops in the acceleration region and horizon, as computed from equations (7.14) and (7.15), is

$$\frac{\Delta V^A}{\Delta V^H} = \frac{\Delta Z^A}{\Delta Z^H} = \frac{\Omega^F}{\Omega^H - \Omega^F}; \quad (7.17)$$

the total current, as computed from (7.16), (7.17) and (7.13), is

$$I = \frac{\text{emf}}{\Delta Z^A + \Delta Z^H} = \frac{1}{2} (\Omega^H - \Omega^F) \varpi^2 B_\perp; \quad (7.18)$$

the total power transmitted to the A region, as computed from (7.18), (7.17) and (7.13), is

$$\Delta P = \Delta Z^A I^2 = \frac{\Omega^F (\Omega^H - \Omega^F)}{4\pi c} \varpi^2 B_\perp \Delta \psi; \quad (7.19)$$

and the power dissipated in the horizon is

$$\Theta^H \frac{d\Delta S^H}{dt} = \Delta Z^H I^2 = \frac{(\Omega^H - \Omega^F)^2}{4\pi c} \varpi^2 B_\perp \Delta \psi. \quad (7.20)$$

(The right-hand sides of equations (7.18)–(7.20) are all evaluated at the feet of the equipotentials in the horizon.)

Note that our circuit analysis produces the same result for the power output as was obtained from the torque-balance analysis of the last section (*cf.* equations 7.2 and 7.19). It also reproduces the standard horizon boundary condition on I (*cf.* equations 7.18 and 5.10). In addition, it gives new insight into the determination of the field line angular velocity: Ω^F is determined by the ratio of the resistance of the acceleration region to the resistance of the horizon (equation 7.17). The condition of maximum power output, $\Omega^F = \Omega^H/2$, corresponds precisely to the standard circuit-theory condition: the impedance ΔZ^A of the

load should equal the impedance ΔZ^H of the power source. Moreover, as in our torque-balance discussion so also here, there is reason to suspect that the optimal power output will be approximately achieved in nature: Lovelace *et al.* (1979) argue that the complex processes occurring in region A are likely to produce a total impedance between widely separated equipotentials of $\sim 25 \Omega$. Between our neighbouring surfaces the impedance will be smaller than this by $\sim (\text{thickness of flux tube})/(\text{distance to symmetry axis}) \sim \Delta\lambda/\varpi$. Comparison with (7.13) shows

$$\begin{aligned}\Delta Z^H &= (R^H/2\pi)(\Delta\lambda/\varpi) \sim (60 \text{ ohm})(\Delta\lambda/\varpi) \\ \Delta Z^A &\sim (25 \text{ ohm})(\Delta\lambda/\varpi),\end{aligned}\tag{7.21}$$

i.e. rough impedance matching. This conclusion, that the impedances will roughly match and therefore the power output will be roughly optimal, is due to Blandford (1979).

A circuit analysis can also be developed for a neighbouring pair of equipotentials threading the upper half disc. A current I flows, in the upper half disc, between the equipotential surfaces 3 and 4 of Fig. 4. This current produces a potential drop ΔV^D between surfaces 3 and 4 given by

$$I\Delta Z^D = \int_3^4 \alpha [\mathbf{E} + (\mathbf{v}^D/c) \times \mathbf{B}] \cdot d\mathbf{l} = \Delta V^D + (\Omega^D - \omega^D)\Delta\psi/2\pi c.\tag{7.22}$$

(One can regard this as defining the disc impedance ΔZ^D between surfaces 3 and 4.) Here $\mathbf{v}^D = (\Omega^D - \omega^D)\mathbf{m}/\alpha$ is the disc's orbital velocity relative to ZAMOs, and the integral is performed in the equatorial plane where the ZAMO angular velocity is $\omega \equiv \omega^D$. The current I flows back from equipotential 4 to equipotential 3 in region A, where it produces a potential drop

$$\Delta V^A = I\Delta Z^A = \Omega^F \Delta\psi/2\pi c\tag{7.23}$$

(equation 7.15). The total emf around the loop is

$$\Delta V^D + \Delta V^A = \text{emf} = \omega^D \Delta\psi/2\pi c\tag{7.24}$$

(equation 7.12). Combining equations (7.22)–(7.24) we obtain

$$\frac{\Delta Z^A}{\Delta Z^D} = \frac{\Omega^F}{\Omega^D - \Omega^F},\tag{7.25}$$

$$I = \Omega^F \frac{\Delta\psi}{2\pi c \Delta Z^A} = (\Omega^D - \Omega^F) \frac{\Delta\psi}{2\pi c \Delta Z^D},\tag{7.26}$$

$$\Delta P = I^2 \Delta Z^A = \Omega^F \left(\frac{I}{2\pi c} \right) \Delta\psi.\tag{7.27}$$

As for field lines threading the hole, so also here, the field line angular velocity Ω^F is determined by the ratio of acceleration-region impedance to disc impedance. If the disc impedance is very low, the disc will lock the field lines to itself, $\Omega^F \simeq \Omega^D$. Once Ω^F is fixed, the current I is determined by the A-region impedance.

7.4 CONSTRUCTION OF A MODEL FOR THE FORCE-FREE REGION

The details of the disc and of the acceleration region are highly dependent on ill-understood plasma physics. This is not the case for the force-free region and the horizon; they can be modelled with considerable confidence (as Blandford & Znajek emphasize) – except for

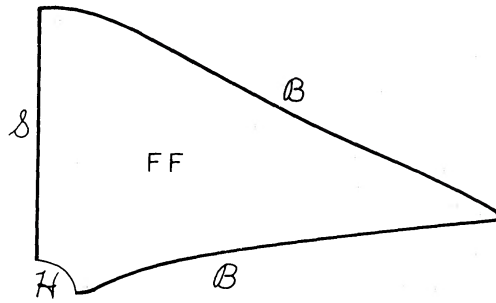


Figure 5. Poloidal diagram of the force-free region FF and its boundaries \mathcal{S} , \mathcal{B} and \mathcal{H} as used in Section 7.4 in formulating initial value data for solutions of the stream equation.

uncertain boundary conditions at the interface with the disc and the A-region. In this section we summarize the mathematical structure of the problem of modelling the force-free (FF) region.

A poloidal slice through the FF region is shown in Fig. 5. It is bounded by the horizon (labelled \mathcal{H}), the symmetry axis (labelled \mathcal{S}), and the boundaries with the disc and acceleration region (both labelled \mathcal{B}). Except at the end of this section, we shall regard the disc as extending all the way in to the horizon's equator. A solution for all details of the FF region and the horizon is generated by the stream function ψ and the two subsidiary functions $I(\psi)$ and $\Omega^F(\psi)$. Once $I(\psi)$, $\Omega^F(\psi)$, and suitable boundary conditions are specified, ψ is computed by solving the partial differential stream equation (6.4) or, equivalently, by extremizing the action (6.9).

The boundary conditions on ψ are determined by the poloidal magnetic field distribution at the boundary. (Recall: ψ is the flux through \mathbf{m} -loops; $\mathbf{B}^P = -\mathbf{m} \times \nabla \psi / 2\pi \varpi^2$.) To avoid unphysical singularities on the symmetry axis \mathcal{S} , one must set $\psi = 0$ there and $\psi \propto \varpi^2$ on \mathcal{B} near \mathcal{S} ; but otherwise there are no constraints of principle on ψ :

$$\begin{aligned} \psi &= 0 && \text{on } \mathcal{S}, \\ \psi &\propto \varpi^2 && \text{as } \varpi \rightarrow 0 \text{ on } \mathcal{B}, \\ \psi &\text{ otherwise arbitrary on } \mathcal{B}. \end{aligned} \tag{7.28a}$$

As one moves away from the symmetry axis along \mathcal{B} , one is free to make ψ increase for a while, then decrease and even go negative, then increase again and oscillate. Such complicated behaviour will lead, when solving the stream equation, to magnetic field loops going out through \mathcal{B} and then returning in various places, and to neutral points of field reversal. But, of course, one would prefer to choose ψ on \mathcal{B} to match as closely as possible boundary fields from realistic models of the disc and acceleration regions.

One must also specify $I(\psi)$ and $\Omega^F(\psi)$ on the boundary, subject to one non-obvious constraint

$$dI/d\psi = (\Omega^H - \Omega^F)/2\pi \quad \text{at the intersection of } \mathcal{B} \text{ and } \mathcal{S} \tag{7.28b}$$

(see below), and several obvious ones

$$\begin{aligned} I &= 0 && \text{on } \mathcal{S}, \\ I &\propto \varpi^2 && \text{as } \varpi \rightarrow 0 \text{ on } \mathcal{B}, \\ \Omega^F &= \text{constant on } \mathcal{S}, \end{aligned} \tag{7.28c}$$

I and Ω^F otherwise arbitrary on \mathcal{B} , except that they must be functions of ψ in the sense not that every point with the same ψ has the same I and Ω^F , but rather that points with the same (ψ, I, Ω^F) come in pairs so distributed that one can topologically draw non-crossing lines of force connecting them.

Of course, in practice one will try to choose I and Ω^F in accord with the physical conditions of the last two sections: Ω^F determined by Ω^H or Ω^D and by the relative A-region and D-region impedances; I determined by Ω^F and the A-region impedance, $I = \Omega^F \Delta\psi / (2\pi c \Delta Z^A)$ (equation 7.26).

One need not be concerned about any boundary conditions on the horizon. The stream differential equation, when it is solved, will automatically enforce the two horizon boundary conditions

$$\psi = \psi_0(\lambda) + O(\alpha^2) \quad \text{near } \mathcal{H}, \quad (7.29a)$$

$$4\pi I = (\Omega^H - \Omega^F) \varpi d\psi_0/d\lambda \quad \text{on } \mathcal{H}, \quad (7.29b)$$

except for a possible sign error in (7.29b) (*cf.* Section 6.2). The boundary condition (7.28b) is designed in part to ensure that, at least near the symmetry axis, the sign starts out correct. If a solution of the stream equation produces an incorrect sign reversal at a null point of the magnetic field ($d\psi_0/d\lambda = 0$) further along the horizon — an unlikely occurrence — then one must discard the solution as unphysical.

It is instructive to see how the stream equation distributes the poloidal magnetic field $B_\perp = (2\pi\varpi)^{-1} d\psi_0/d\lambda$ over the horizon. It distributes the field in such a manner as to extremize the horizon's electromagnetic surface energy (6.10), or equivalently in accordance with the ordinary differential equation (7.29b). Rewrite this equation as

$$d\psi_0/G(\psi_0) = d\lambda/\varpi(\lambda), \quad (7.30)$$

where $\varpi(\lambda)$ is a known function determined by the surface geometry of the horizon, and

$$G(\psi_0) \equiv \frac{4\pi I(\psi_0)}{\Omega^H - \Omega^F(\psi_0)} \quad (7.31)$$

is a known function determined by the choice of boundary conditions moving outward from the symmetry axis \mathcal{S} along \mathcal{B} . In the limit as one approaches the symmetry axis, smoothness of the horizon requires $\varpi = \lambda$, and our (previously unexplained) constraint (7.28b) on the boundary data guarantees $G = 2\psi$, which in turn guarantees that the solution of the differential equation (7.30) has the well-behaved form

$$\psi_0 = \pi\lambda^2 B_{\perp 0} \quad \text{as } \lambda \rightarrow 0. \quad (7.32)$$

The integration constant $B_{\perp 0}$ is the poloidal magnetic field strength on the symmetry axis of the horizon. Imagine integrating the differential equation (7.29b) along the horizon from pole toward equator, beginning with some trial value of $B_{\perp 0}$. Unless that trial value is chosen very carefully, upon reaching the intersection of \mathcal{H} with \mathcal{B} , one will have a value of ψ which does not match the chosen boundary value there. Obtaining the right match can be regarded as an eigenvalue problem for $B_{\perp 0}$. Presumably, by solving the stream differential equation (6.4) subject to the boundary conditions (7.28) on \mathcal{S} and \mathcal{B} , one automatically also solves the eigenvalue problem for $B_{\perp 0}$ and obtains a unique physically well-behaved solution on all of the horizon and throughout the force-free region.

However, we do not claim to have proved *rigorously* that there will always exist a solution to the stream equation with boundary conditions of the specified type, nor that when such a solution exists it will be unique.

The above formulation of the boundary-value problem is appropriate to situations where the magnetic field is firmly anchored in the disc all the way in to the horizon, so that it is appropriate to specify ψ on \mathcal{B} all the way into \mathcal{B} 's intersection with \mathcal{H} . When, instead, there is a force-free gap between the disc and the horizon, one must modify the boundary-value problem. The boundaries then have the form of Fig. 6, which is the same as Fig. 5 except for the presence of a force-free equatorial boundary segment \mathcal{F} . Since field lines passing through \mathcal{F} are force-free everywhere except in the distant acceleration region A, they presumably will be anchored in A and will thus have zero angular velocity $\Omega^F = 0$ and will carry zero torque, $\mathbf{S}_L^P = (I/2\pi\alpha c)\mathbf{B}^P = 0$ (equation 5.7). This means that the boundary values ψ , $\Omega^F(\psi)$, and $I(\psi)$ must be chosen such that

$$\Omega^F = I = 0 \quad \text{everywhere on } \mathcal{F}. \quad (7.33a)$$

By contrast with the boundaries \mathcal{S} and \mathcal{B} , one should not explicitly specify the distribution of ψ on \mathcal{F} ; rather, reflection symmetry dictates that one specify

$$\nabla\psi \quad \text{parallel to } \mathcal{F} \text{ everywhere on } \mathcal{F}. \quad (7.33b)$$

However, the values of ψ will be fixed at the inner and outer edges of \mathcal{F} , i.e. at the intersections with \mathcal{H} and \mathcal{B} :

$$\psi \quad \text{fixed at } \mathcal{H} \cap \mathcal{F} \text{ and } \mathcal{B} \cap \mathcal{F} \text{ as those values for which } \Omega^F(\psi) \text{ and } I(\psi) \text{ start departing from zero.} \quad (7.33c)$$

Boundary values ψ , $I(\psi)$, $\Omega^F(\psi)$ chosen in accord with equations (7.28) and (7.33) lead to a well-posed action principle (6.9) for the stream function ψ in the FF region – an action principle whose Euler–Lagrange equation is the stream differential equation (6.4).

7.5 ABSENCE OF MAGNETIC LOOPS THREADING THE HORIZON

In Section 3 we mentioned that a looped field line such as b in Fig. 1, with one foot on the horizon \mathcal{H} and the other anchored in the disc, will annihilate itself once the disc has deposited the loop's second foot onto \mathcal{H} . Presumably the annihilation will occur on a time-scale Δt roughly equal to the light travel time across the loop. (This is what one would compute from the flat-space theory of the diffusion of magnetic field lines through a medium with surface resistivity $R^H = 377 \Omega$.) However, we shall not attempt here a relativistic derivation of this time-scale. Rather, we shall content ourselves with a formal proof that the annihilation must occur – i.e. that a poloidal magnetic field loop \mathcal{L}_0 of the form shown in Fig. 7 cannot exist in the force-free region of a stationary axisymmetric magnetosphere. For simplicity of proof we presume that the topology of the poloidal field

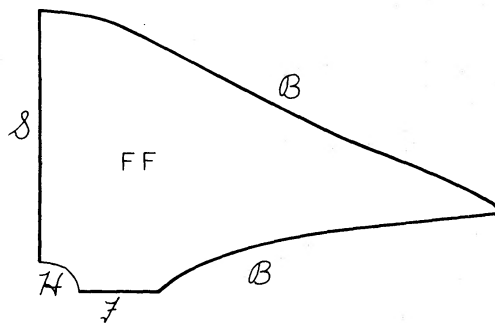


Figure 6. Poloidal diagram of the force-free region FF and its boundaries for situations where there is a force-free equatorial gap \mathcal{F} between disc and horizon.

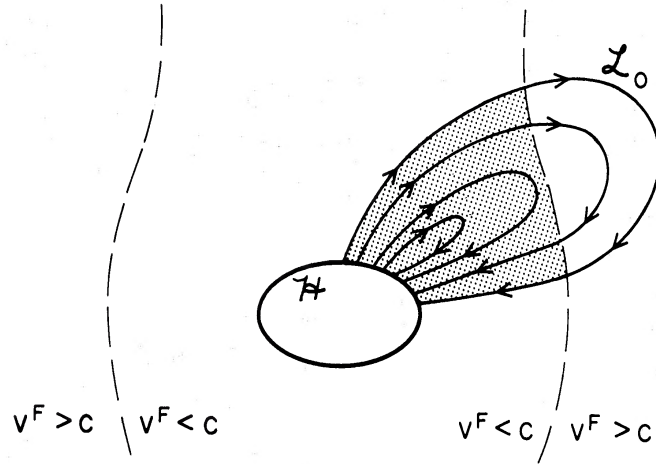


Figure 7. Diagram used in the proof that poloidal magnetic field loops such as \mathcal{L}_0 cannot exist in the force-free region of a stationary, axisymmetric magnetosphere.

inside \mathcal{L}_0 is as shown in Fig. 7: a series of simple nested loops with the magnetic field in the same direction on each loop (no neutral points). The reader can generalize the proof.

The first step in our proof is to show that on each loop inside \mathcal{L}_0 $\Omega^F = \Omega^H$ and $I = 0$. This follows from torque balance. Conserved angular momentum travels along each loop, outward from \mathcal{H} at one foot and back into \mathcal{H} at the other. But because Ω^F is constant along a loop, the direction of the angular momentum flow must be the same at both feet (outward if $\Omega^F < \Omega^H$; inward if $\Omega^F > \Omega^H$; equation 7.1). This is possible only if there is no flow of angular momentum at all, i.e. only if $\Omega^F = \Omega^H$ (equation 7.1), which means in turn that $I = 0$ (equation 5.11).

Because $\Omega^F = \Omega^H$ and $I = 0$ throughout the interior of \mathcal{L}_0 , the stream equation (6.4) is there a perfect divergence. Integrating it over the interior of \mathcal{L}_0 and converting to a surface integral by Gauss's theorem, and noting that the contribution from \mathcal{H} vanishes because $\Omega^F - \omega = \Omega^H - \omega = O(\alpha^2)$ and $\alpha \rightarrow 0$ at \mathcal{H} (see pp. 251–252 of Bardeen 1973), we obtain

$$\int_{\mathcal{L}_0} (\alpha/\omega^2) [1 - (v^F/c)^2] \nabla\psi \cdot d\Sigma = 0, \quad (7.34)$$

$v^F \equiv (\Omega^H - \omega)\omega/\alpha =$ (linear velocity of field line relative to ZAMOs).

Note that because \mathcal{L}_0 is a surface of constant flux, $\nabla\psi$ is parallel to $d\Sigma$; and because of our assumed simple loop topology (Fig. 7), $\nabla\psi \cdot d\Sigma$ has a constant sign (positive or negative) everywhere on \mathcal{L}_0 . Shown dashed in Fig. 7 is the 'velocity of light' surface on which $v^F = c$. If \mathcal{L}_0 lies entirely inside that surface, then $v^F < c$ everywhere on \mathcal{L}_0 , and (7.34) then demands that $\nabla\psi \cdot d\Sigma = 0$ everywhere on \mathcal{L}_0 , which in turn means that $\nabla\psi = 0$ and hence $\mathbf{B}^P = 0$ everywhere on \mathcal{L}_0 . Thus, there is no field at all, much less any field line, on \mathcal{L}_0 – and the same holds true for all loops inside \mathcal{L}_0 .

If \mathcal{L}_0 pierces through the velocity-of-light surface (the case shown in Fig. 7), then integrate the stream equation separately over the shaded and unshaded parts of the interior of \mathcal{L}_0 , convert to surface integrals, note that the integrals over \mathcal{H} and over the velocity-of-light surface vanish, and thereby conclude that (7.34) holds separately for the shaded ($v^F < c$) and unshaded ($v^F > c$) parts of \mathcal{L}_0 . This means, again, that $\nabla\psi \cdot d\Sigma = 0$ everywhere on \mathcal{L}_0 , which means that there is no poloidal magnetic field at all, much less any field line, on \mathcal{L}_0 – and the same holds true for all loops inside \mathcal{L}_0 .

Thus, a loop of the form \mathcal{L}_0 cannot exist in the force-free region – though it can exist if part of the loop exits from the force-free region, e.g. into the disc.

8 Conclusion

In this paper we have tried to focus exclusively on those aspects of black-hole electrodynamics which are independent of the complexities and uncertainties of realistic plasma physics. As a result, we have ignored the most important features of the theory: the processes by which the flowing electromagnetic power gets deposited into charged particles in the acceleration region, and the details of the resulting particle motions. However, we think and hope that our formalism can serve as a foundation for detailed studies of these phenomena and of other aspects of black-hole magnetospheres.

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