Relativistic theory of tidal Love numbers

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(Dated: September 16, 2009)

In Newtonian gravitational theory, a tidal Love number relates the mass multipole moment created by tidal forces on a spherical body to the applied tidal field. The Love number is dimensionless, and it encodes information about the body's internal structure. We present a relativistic theory of Love numbers, which applies to compact bodies with strong internal gravities; the theory extends and completes a recent work by Flanagan and Hinderer, which revealed that the tidal Love number of a neutron star can be measured by Earth-based gravitational-wave detectors. We consider a spherical body deformed by an external tidal field, and provide precise and meaningful definitions for electric-type and magnetic-type Love numbers; and these are computed for polytropic equations of state. The theory applies to black holes as well, and we find that the relativistic Love numbers of a nonrotating black hole are all zero.

I. INTRODUCTION AND SUMMARY

Context of this work

The exciting prospect of using gravitational-wave detectors to measure the tidal coupling of two neutron stars during the inspiral phase of their orbital evolution was recently articulated by Flanagan and Hinderer [1, 2]. The idea is as follows. The orbital motion of a binary system of neutron stars produces the emission of gravitational waves, which remove energy and angular momentum from the system. This causes the orbits to decrease in radius and increase in frequency, and leads to the inspiraling motion of the compact bodies. Late in the inspiral the gravitational waves enter the frequency band of the detector, and detailed features of the orbital motion are revealed in the shape and phasing of the wave. At the large orbital separations that correspond to the low-frequency threshold of the instrument, the tidal interaction between the bodies is negligible, and the bodies behave as point masses. As the frequency increases, however, the orbital separation decreases sufficiently that the influence of the tidal interaction becomes important. The bodies acquire a tidal deformation, and this affects their gravitational field and orbital motion; the effect is revealed in the shape and phasing of the gravitational waves.

Flanagan and Hinderer have provided a quantitative analysis of this story, and they have shown that the tidal coupling between neutron stars is accessible to measurement by the current generation of Earth-based gravitational-wave detectors (such as Enhanced LIGO). This prospect is exciting, because the details of the tidal interaction depend on the internal structure of each body, and the measurement can thus reveal important information regarding the compactness of each body, as well as its equation of state; and this information is released cleanly, during the inspiral phase of the orbital evolution,

well before the messy merger of the two companions.

Newtonian theory of tidal Love numbers

The effect of the tidal interaction on the orbital motion and gravitational-wave signal is measured by a quantity known as the *tidal Love number* of each companion [3]. In Newtonian gravity (see, for example, Ref. [4]), the tidal Love number is a constant of proportionality between the tidal field applied to the body and the resulting multipole moment of its mass distribution. In the quadrupolar case, the tidal field is characterized by the *tidal moment* $\mathcal{E}_{ab}(t) := -\partial_{ab}U_{\text{ext}}$, in which the external Newtonian potential U_{ext} is sourced by the companion body and evaluated (after differentiation with respect to the spatial coordinates) at the body's center-of-mass. Because the external potential satisfies Laplace's equation in the body's neighborhood, the tidal-moment tensor is not only symmetric but also tracefree; it is a symmetric-tracefree (STF) tensor.

The quadrupole moment is $Q^{ab} := \int \rho(x^a x^b - \frac{1}{3}\delta^{ab}r^2)\, d^3x$, where ρ is the mass density inside the body, x^a is a Cartesian coordinate system whose origin is at the center-of-mass, and $r := (\delta_{ab}x^a x^b)^{1/2}$ is the distance to the center-of-mass; the quadrupole moment is another STF tensor. In the absence of a tidal field the body would be spherical, and its quadrupole moment would vanish. In the presence of a (weak) tidal field, the quadrupole moment is proportional to the tidal field, and dimensional analysis requires an expression of the form $Q_{ab} = -\frac{2}{3}k_2 R^5 \mathcal{E}_{ab}$. (We use relativistic units and set $G = c = 1$.) Here R is the body's radius, and the factor of $\frac{2}{3}$ is conventional; the dimensionless constant k_2 is the tidal Love number for a quadrupolar deformation. Using these expressions, the Newtonian potential outside the body can be written as a sum of body and external

potentials, and we have

$$U = \frac{M}{r} - \frac{1}{2}\big[1 + 2k_2(R/r)^5\big]\mathcal{E}_{ab}(t)x^a x^b. \qquad (1.1)$$

The first term is evidently the monopole piece of the potential, which depends on the body's mass M. Within the square brackets, the first term represents the applied tidal field, and the second term is the body's response, measured in terms of the Love number k_2.

In Eq. (1.1) the total potential was truncated to the leading, quadrupole order in a Taylor expansion of the external potential; additional terms would involve tidal moments of higher multipole orders, and higher powers of the coordinates x^a. When the tidal field is a pure multipole of order l, Eq. (1.1) generalizes to

$$U = \frac{M}{r} - \frac{1}{(l-1)l}\big[1 + 2k_l(R/r)^{2l+1}\big]\mathcal{E}_L(t)x^L. \quad (1.2)$$

Here k_l is the Love number for this multipolar configuration, and $L := a_1 a_2 \cdots a_l$ is a multi-index that contains a number l of individual indices. The tidal moment is now defined by $\mathcal{E}_L(t) := -\partial_L U_{\text{ext}}/(l-2)!$, and it is symmetric and tracefree in all pairs of indices. We also introduced $x^L := x^{a_1}x^{a_2}\cdots x^{a_l}$. In this generalized case the l-pole moment of the mass distribution is the STF tensor $Q^L := \int \rho x^{\langle L \rangle}\, d^3x$, where the angular brackets indicate that all traces must be removed from the tensor x^L; it is related to the tidal moment by $Q_L = -[2(l-2)!/(2l-1)!!]k_l R^{2l+1}\mathcal{E}_L$.

Purpose of this work

Our purpose in this paper is to introduce a precise notion of tidal Love numbers in general relativity, something that was not pursued in the original work by Flanagan and Hinderer [1, 2]. In fact, we provide precise definitions for two types of tidal Love numbers: an electric-type Love number k_{el} that has a direct analogy with the Newtonian Love number introduced previously, and a magnetic-type Love number k_{mag} that has no analogue in Newtonian gravity. Magnetic-type Love numbers were introduced in post-Newtonian theory in the works of Damour, Soffel, and Xu [5] and Favata [6]. Our definitions apply to gravitational fields that are arbitrarily strong, and to (weak) tidal deformations of any multipolar order.

Our relativistic Love numbers are defined within the context of linear perturbation theory, in which an initially-spherical body is perturbed slightly by an applied tidal field. Our definitions are restricted to slowly-changing tidal fields; this means that while a tidal moment such as $\mathcal{E}_L(t)$ does depend on time, to reflect the changes in the external distribution of matter, the dependence is sufficiently slow that the body's response presents only a *parametric dependence* upon time. This

allows us to ignore time-derivative terms in the field equations, because they are much smaller than the spatial-derivative terms. For all practical purposes the perturbation is stationary, and t appears as an adiabatic parameter.

Gravitational perturbations of spherically-symmetric bodies are described by a metric perturbation $p_{\alpha\beta}$ that can be decomposed into tensorial spherical harmonics; each multipole can be considered separately. The complete spacetime metric is $g_{\alpha\beta} = g^0_{\alpha\beta} + p_{\alpha\beta}$, with $g^0_{\alpha\beta}$ denoting the (spherically-symmetric) metric of the unperturbed body. We work in the body's immediate neighborhood, and the external bodies that create the (multipolar) tidal field are assumed to live outside this neighborhood. To *define* the relativistic Love numbers it is sufficient to consider the vacuum region external to the body, and to construct $g_{\alpha\beta}$ in this region only; this metric will be a solution to the vacuum field equations, and will represent the relativistic generalization of Eq. (1.2). To *compute* the Love numbers it is necessary to construct $g_{\alpha\beta}$ in the body's interior also, and this requires the formulation of a stellar model. The external problem therefore applies to any type of body, while the internal problem refers to a specific choice of equation of state.

External problem

We review the external problem first. We erect a coordinate system (v, r, θ, ϕ) that is intimately tied to the behavior of light cones: The advanced-time coordinate v is constant on past light cones that converge toward the center at $r = 0$, r is both an areal radius and an affine-parameter distance along the null generators of each light cone, and the angular coordinates $\theta^A = (\theta, \phi)$ are constant on each generator. This choice of coordinates is inherited from previous work on the tidal deformation of black holes [7].

In these coordinates the external metric of the *unperturbed body* is given by $ds_0^2 = -f\, dv^2 + 2\, dv dr + r^2 d\Omega^2$, in which $f := 1 - 2M/r$ and $d\Omega^2 := d\theta^2 + \sin^2\theta\, d\phi^2$; this is the Schwarzschild metric presented in Eddington-Finkelstein coordinates. To construct the perturbation we impose the *light-cone gauge conditions* $p_{vr} = p_{rr} = p_{r\theta} = p_{r\phi} = 0$ to ensure that the coordinates keep their geometrical meaning in the perturbed spacetime [8]. (This property makes the light-cone gauge superior to the popular Regge-Wheeler gauge, which does not provide the coordinates with any geometrical meaning.) A perturbation of multipole order l can be decomposed into even-parity and odd-parity sectors, and each sector must be a solution to the Einstein field equations linearized about the Schwarzschild metric.

The even-parity sector is generated by the electric-type tidal moment $\mathcal{E}_L(v)$, an STF tensor defined in a quasi-Cartesian system x^a related in the usual way to the spherical coordinates (r, θ^A). The $(2l + 1)$ independent components of this tensor can be encoded in the

functions $\mathcal{E}_m^{(l)}(v)$, in which the azimuthal index m is an integer within the interval $-l \leq m \leq l$; the encoding is described by $\mathcal{E}_L x^L = r^l \sum_m \mathcal{E}_m^{(l)} Y^{lm}(\theta^A)$, in which Y^{lm} are the usual spherical-harmonic functions. We define the tidal potentials

$$\mathcal{E}^{(l)}(v, \theta^A) = \sum_m \mathcal{E}_m^{(l)}(v) Y^{lm}(\theta^A), \qquad (1.3a)$$

$$\mathcal{E}_A^{(l)}(v, \theta^A) = \frac{1}{l} \sum_m \mathcal{E}_m^{(l)}(v) Y_A^{lm}(\theta^A), \qquad (1.3b)$$

$$\mathcal{E}_{AB}^{(l)}(v, \theta^A) = \frac{2}{l(l-1)} \sum_m \mathcal{E}_m^{(l)}(v) Y_{AB}^{lm}(\theta^A), \qquad (1.3c)$$

in which Y_A^{lm} and Y_{AB}^{lm} are vector and tensor spherical harmonics of even parity; these are defined in Sec. II.

The odd-parity sector is generated by the magnetic-type tidal moment $\mathcal{B}_L(v)$, another STF tensor whose independent components can be encoded (as previously) in the functions $\mathcal{B}_m^{(l)}(v)$. The odd-parity tidal potentials are

$$\mathcal{B}_A^{(l)}(v, \theta^A) = \frac{1}{l} \sum_m \mathcal{B}_m^{(l)}(v) X_A^{lm}(\theta^A), \qquad (1.4a)$$

$$\mathcal{B}_{AB}^{(l)}(v, \theta^A) = \frac{2}{l(l-1)} \sum_m \mathcal{B}_m^{(l)}(v) X_{AB}^{lm}(\theta^A), \qquad (1.4b)$$

in which X_A^{lm} and X_{AB}^{lm} are vector and tensor spherical harmonics of odd parity; these also are defined in Sec. II. There is no scalar potential $\mathcal{B}^{(l)}$ in the odd-parity sector.

The metric outside any spherical body deformed by a tidal environment characterized by the tidal moments $\mathcal{E}_L$ and $\mathcal{B}_L$ is calculated in Sec. III. It is given by

$$g_{vv} = -f - \frac{2}{(l-1)l} r^l e_1(r) \mathcal{E}^{(l)}, \qquad (1.5a)$$

$$g_{vr} = 1, \qquad (1.5b)$$

$$g_{vA} = -\frac{2}{(l-1)(l+1)} r^{l+1} e_4(r) \mathcal{E}_A^{(l)}$$
$$+ \frac{2}{3(l-1)} r^{l+1} b_4(r) \mathcal{B}_A^{(l)}, \qquad (1.5c)$$

$$g_{AB} = r^2 \Omega_{AB} - \frac{2}{l(l+1)} r^{l+2} e_7(r) \mathcal{E}_{AB}^{(l)}$$
$$+ \frac{2}{3l} r^{l+2} b_7(r) \mathcal{B}_{AB}^{(l)}. \qquad (1.5d)$$

The radial functions are

$$e_1 = A_1 + 2k_{\rm el}(R/r)^{2l+1} B_1, \qquad (1.6a)$$

$$e_4 = A_4 - 2\frac{l+1}{l} k_{\rm el}(R/r)^{2l+1} B_4, \qquad (1.6b)$$

$$e_7 = A_7 + 2k_{\rm el}(R/r)^{2l+1} B_7, \qquad (1.6c)$$

$$b_4 = A_4 - 2\frac{l+1}{l} k_{\rm mag}(R/r)^{2l+1} B_4, \qquad (1.6d)$$

$$b_7 = A_7 + 2k_{\rm mag}(R/r)^{2l+1} B_7, \qquad (1.6e)$$

with

$$A_1 := f^2 F(-l+2, -l; -2l; 2M/r), \qquad (1.7a)$$

$$B_1 := f^2 F(l+1, l+3; 2l+2; 2M/r), \qquad (1.7b)$$

$$A_4 := F(-l+1, -l-2; -2l; 2M/r), \qquad (1.7c)$$

$$B_4 := F(l-1, l+2; 2l+2; 2M/r), \qquad (1.7d)$$

$$A_7 := \frac{l+1}{l-1} F(-l, -l; -2l; 2M/r)$$
$$- \frac{2}{l-1} F(-l, -l-1; -2l; 2M/r), \qquad (1.7e)$$

$$B_7 := \frac{l}{l+2} F(l+1, l+1; 2l+2; 2M/r)$$
$$+ \frac{2}{l+2} F(l, l+1; 2l+2; 2M/r). \qquad (1.7f)$$

Here R is the body's radius, and $F(a, b; c; z)$ is the hypergeometric function. The functions A_n are finite polynomials in $2M/r$, while the functions B_n have nonterminating expansions in powers of $2M/r$; for selected values of l they can be expressed in terms of elementary functions such as $\ln(1 - 2M/r)$ and finite polynomials (see Table I in Sec. III). Each one of these functions goes to one as r goes to infinity. And while A_n is finite at $r = 2M$, we observe that B_n diverges logarithmically when $r \to 2M$.

The metric of Eqs. (1.5) is valid in a neighborhood of the deformed body, and it provides a definition for the electric-type Love numbers $k_{\rm el}$ and the magnetic-type Love numbers $k_{\rm mag}$; these refer to the multipole order l, but we suppress the use of this label to keep the notation clean. While the definitions seem to rely on a specific choice of gauge for the metric perturbation, we prove in Sec. III that our Love numbers are gauge-invariant.

When the tidal moments are switched off the metric reduces to the Schwarzschild metric expressed in the light-cone coordinates (v, r, θ^A). When the mass parameter M is set equal to zero the metric describes the neighborhood of a geodesic world line in a Ricci-flat spacetime. In this limit the tidal moments can be related to the derivatives of the Weyl tensor evaluated at $r = 0$. According to Eqn. (1.3) of Ref. [9], we have that $\mathcal{E}_L = [(l-2)!]^{-1}(C_{ta_1 ta_2; a_3 \cdots a_l})^{\rm STF}$ and $\mathcal{B}_L = [\frac{2}{3}(l+1)(l-2)!]^{-1}(\epsilon_{a_1 bc} C^{bc}{}_{a_2 t; a_3 \cdots a_l})^{\rm STF}$, where ϵ_{abc} is the permutation symbol and the tensor components are listed in the quasi-Lorentzian coordinates $(t := v - r, x^a)$; the STF superscript indicates that the a_n indices are symmetrized and all traces are removed. In the spacetime of Eq. (1.5) the tidal moments $\mathcal{E}_L$ and $\mathcal{B}_L$ retain a similar relationship with the Weyl tensor, with the understanding that the relations are now approximate and refer to the asymptotic behavior of the Weyl tensor for $r \gg M$.

The perturbed metric of Eq. (1.5) can be compared with the Newtonian potential of Eq. (1.2). We define an effective Newtonian potential $U_{\rm eff}$ by $g_{vv} =: -(1 - 2U_{\rm eff})$, and our expression for g_{vv} implies that in general

relativity,

$$U_{\text{eff}} = -\frac{M}{r} - \frac{1}{(l-1)l}\Big[A_1 + 2k_{\text{el}}(R/r)^{2l+1}B_1\Big]\mathcal{E}_L(v)x^L.$$

$$(1.8)$$

In the nonrelativistic limit, A_1 and B_1 are both approximately equal to unity, and we recover Eq. (1.2); the electric-type Love number k_{el} reduces to the Newtonian number k_l. In the strong-field regime we still recognize the A_1 term as coming from the applied tidal field, while the B_1 term is clearly associated with the body's response. There is no confusion between these terms, because the structure of A_1 is that of the finite polynomial $1 + \cdots + \lambda(2M/r)^l$, which does not contain a term of order $(2M/r)^{2l+1}$; λ is a numerical factor that can be determined by expanding the hypergeometric function. Because r is geometrically well defined, we can always distinguish the tidal terms from the body terms in the metric.

The light-cone coordinates (v, r, θ^A) are well-behaved across an eventual event horizon of the perturbed spacetime, and our formalism is capable of handling black holes as well as material bodies. In general, however, the metric of Eqs. (1.5) is not regular at the event horizon, because of the presence of the B_n functions, which diverge logarithmically in the limit $r \to 2M$. To represent a perturbed black hole the metric must be devoid of these terms, and this can be accomplished by assigning $k_{\text{el}} = k_{\text{mag}} = 0$ to a black hole. This is one of the major conclusions of this work: *The relativistic Love numbers of a nonrotating black hole are all zero.* This result is contained implicitly in Ref. [7], but the formalism of this paper permits a much clearer articulation of this property.

Internal problem

To *compute* the relativistic Love numbers for a selected stellar model requires the construction of the internal metric (also expressed as a sum of unperturbed solution and linear perturbation) and its matching with the external metric at the perturbed boundary of the matter distribution. We carry out this exercise in Secs. IV and V, adapting the formalism of Thorne and Campolattaro [10] to our light-cone coordinates. We take the body to consist of a perfect fluid with a polytropic equation of state

$$p = K\rho^{1+1/n}.\qquad (1.9)$$

Here p is the fluid's pressure, ρ its proper energy density, K is a constant, and n is the polytropic index (another constant).

Our results are presented in Figs. 1–8, and tables of values are provided in the Appendix. In each figure we plot the Love number for a selected multipole order (from $l = 2$ to $l = 5$), and for selected values of the polytropic index n (from $n = 0.5$ to $n = 2.0$), as a function of the
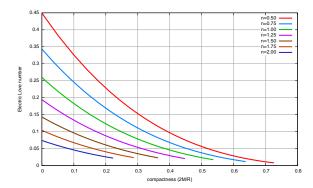

FIG. 1: Electric-type Love numbers for $l = 2$, plotted as functions of the compactness parameter $2M/R$. The uppermost curve corresponds to $n = 0.5$ and the stiffest equation of state. The lowermost curve corresponds to $n = 2.0$ and the softest equation of state. The curves in between are ordered by the value of n. The arrangement is the same in all other figures.

stellar compactness parameter $C := 2M/R$; this ranges from $C = 0$ — a weak-field, Newtonian configuration — to $C = C_{\text{max}}$, with C_{max} representing the compactness of the maximum-mass configuration for the selected equation of state.

For the electric-type Love numbers we observe the following features. (i) At $C = 0$ we recover the Newtonian values for polytropes, as tabulated by Brooker and Olle [11]. (ii) For a constant C, k_{el} decreases as the polytropic index increases; this reflects the fact that as n increases, the matter distribution becomes increasingly concentrated near the center, which inhibits the development of large multipole moments. (iii) For a constant n, k_{el} decreases as the compactness parameter increases; this reflects the fact that as C increases, the strength of the internal gravity increases, which produces an increased resistance to tidal deformations.

For the magnetic-type Love numbers we observe the following features. (i) At $C = 0$ the Love numbers are all zero; this reflects the fact that the magnetic-type tidal coupling is a purely relativistic effect that has a vanishing Newtonian limit. (ii) For a constant C, k_{mag} decreases as the polytropic index increases; this is explained as in the preceding paragraph. (iii) For a constant n, k_{mag} first increases as C increases, but then it decreases after reaching a maximum; this reflects the fact that the magnetic-type tidal coupling is the result of an internal competition: a strong field is required to produce an effect in the first place, but it eventually causes a large resistance to tidal deformation.

Damour and Nagar

After this work was completed we witnessed the appearance of an article by Damour and Nagar [12] in which almost identical work is presented. Their paper, like ours, is concerned with the tidal deformation of compact bod-
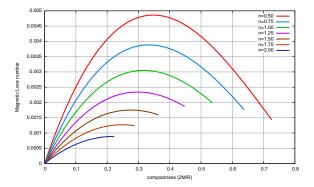
FIG. 2: Magnetic-type Love numbers for $l = 2$, plotted as functions of the compactness parameter $2M/R$.
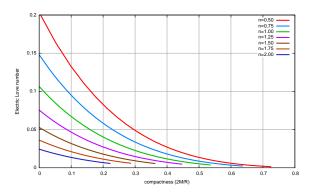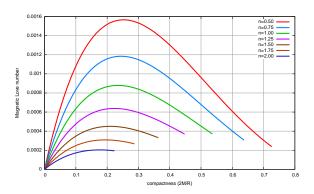

FIG. 3: Electric-type Love numbers for $l = 3$.
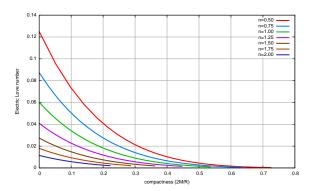

FIG. 4: Magnetic-type Love numbers for $l = 3$.

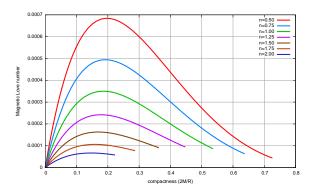FIG. 5: Electric-type Love numbers for $l = 4$.

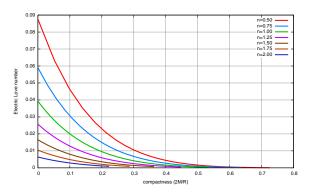
FIG. 6: Magnetic-type Love numbers for $l = 4$.

FIG. 7: Electric-type Love numbers for $l = 5$.

ies in full general relativity, and presents precise definitions for electric-type and magnetic-type Love numbers. And their paper, like ours, presents computations of Love numbers for selected matter models. Their coverage of the parameter space is wider: Damour and Nagar examine two types of polytropic equations of state, and two tabulated equations of state for realistic nuclear matter. In addition, Damour and Nagar define and compute "shape Love numbers," something that we did not pursue in this work.

There are superficial differences between our treatments. One concerns the choice of coordinates: Damour
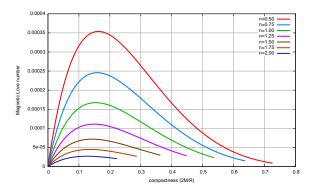

FIG. 8: Magnetic-type Love numbers for $l = 5$.

and Nagar work in Schwarzschild coordinates and adopt the Regge-Wheeler gauge for the metric perturbation; we work in Eddington-Finkelstein coordinates and the light-cone gauge. Another concerns notation: we adopt different normalization conditions for the Love numbers and the tidal moments. These differences are not important.

A more significant difference concerns the conclusion that the tidal Love numbers of a black hole must be zero. In this paper we boldly proclaim this conclusion, which we firmly believe to be a correct interpretation of our results. Damour and Nagar, however, shy away from the conclusion, although they agree with us on the basic results. We do not understand the reasons behind this reluctance. Damour and Nagar comment on the need to understand "diverging diagrams that enter the computation of interacting black holes at the 5-loop (or 5PN) level" before reaching a conclusion. But since the results presented here do not rely at all on a post-Newtonian expansion of the field equations, the fate of 5PN terms in a post-Newtonian representation of interacting black holes seems to us to be irrelevant. We point out, also, that the Damour-Nagar work does not provide a very clean foundation for the tidal deformation of black holes, because their coordinate system is ill-behaved on the event horizon. Our light-cone coordinates were selected precisely because they permit a unified treatment of material bodies and black holes.

Aside from this issue of interpretation, and as far as we can judge, the results presented here are in complete agreement with the Damour-Nagar results. The Damour-Nagar work was carried out in complete independence from us, and our work was carried out in complete independence from them. The near-simultaneous completion of our works provides evidence that the problem is interesting and timely, and the agreement is a reassuring confirmation that each team performed their calculations without error.

Fang and Lovelace

The deformation of a black hole produced by an applied tidal field was previously examined by Fang and Lovelace [13], who concluded that $Q_{ab} = 0$ when the perturbation is expressed in Regge-Wheeler gauge. Fang and Lovelace therefore anticipated our result that the quadrupole, electric-type Love number of a black hole is zero. These authors, however, qualified their conclusion by raising doubts about the gauge invariance of the result, and claiming that the induced quadrupole moment of a tidally deformed black hole is inherently ambiguous. We do not share these reservations.

We first discuss the issue of gauge invariance. The argument advanced by Fang and Lovelace in favor of a gauge dependence of the tidal Love number goes as follows. In Newtonian theory, the coordinate transformation $r = \bar{r}[1 + 2\chi(R/\bar{r})^5]^{1/2}$, where χ is an arbitrary constant, turns a pure tidal potential $\mathcal{E}_{ab}x^a x^b$ into $[1 + 2\chi(R/\bar{r})^5]\mathcal{E}_{ab}\bar{x}^a \bar{x}^b$, which appears to describe a sum of tidal and body potentials; the transformation shifts the Love number by χ. Fang and Lovelace correctly dismiss this coordinate dependence as irrelevant in Newtonian theory, because r has a well-defined meaning, but they point out that in a relativistic context, the coordinate transformation could be viewed as a change of gauge. The implication, then, is that the relativistic Love number can be altered by a gauge transformation. Notice that the argument applies to all types of compact bodies: material bodies and black holes.

We do not accept the validity of this argument. The coordinate transformation considered by Fang and Lovelace is not of a type that can be associated with a gauge transformation of the perturbation theory. A gauge transformation necessarily involves coordinate displacements that are of the same order of magnitude as the perturbation field. But the transformation from r to $\bar{r}$ does not involve the perturbation at all, and represents a large change of the background coordinates. The new coordinate $\bar{r}$ does not share the geometrical properties of the original r, and one would easily be able to distinguish the two coordinate systems. The argument, therefore, does not make a case for the gauge dependence of the Love numbers. And in fact, the *gauge invariance* of k_{el} and k_{mag} for all types of compact bodies (material bodies and black holes) is established in Sec. III.

We next discuss the issue of ambiguity. Unlike Fang and Lovelace, we believe that the relativistic Love numbers of compact bodies, as defined in this paper, are well defined and completely devoid of ambiguity. The reason is that the metric of Eqs. (1.5), which is presented in coordinates that have clear geometrical properties, defines a perfectly well-defined spacetime geometry. Given this spacetime, one could in principle monitor the motion of test masses and light rays and thereby measure its detailed features, including the mass M, the tidal moments $\mathcal{E}_L$ and $\mathcal{B}_L$, and the Love numbers. These measurements would contain no ambiguities.

The ambiguity identified by Fang and Lovelace concerns the coupling of Q_{ab}, the induced quadrupole moment, to $\mathcal{E}_{abc}$, the octupole moment of the applied tidal field. According to Newtonian ideas, this coupling should lead to a force $F^a = -\frac{1}{2}\mathcal{E}^a_{\ bc}Q^{bc}$ acting on the compact body. (Once more the argument applies to all types of compact bodies.) Fang and Lovelace associate F^a with $\dot{P}^a(r)$, the rate of change of three-momentum contained within a world tube of radius r that surrounds the compact body; this is calculated by integrating the flux of Landau-Lifshitz energy-momentum pseudotensor across the world tube. They observe that the result is indeed proportional to $\mathcal{E}^a_{\ bc}Q^{bc}$, but that the coefficient in front depends on r. They interpret this as a statement that the force is ambiguous, assign the ambiguity to Q_{ab}, and conclude that the induced quadrupole moment of a tidally-deformed compact body is inherently ambiguous.

We believe that the ambiguity in $\dot{P}^a(r)$ is genuine — the result does depend on the world tube's radius. It is

hasty, however, to conclude from this that F^a itself is ambiguous, because force calculations that rely on techniques of matched asymptotic expansions [14, 15] must involve a limiting procedure in which both M and r are taken to approach zero. Although ambiguities remain in this procedure, they are much smaller than those claimed by Fang and Lovelace. At the accuracy level of our calculations, the induced quadrupole moment of a tidally-deformed compact body is not ambiguous.

Suen

An earlier determination of the induced quadrupole moment of a tidally deformed black hole was made by Suen [16], who examined the specific case of a black hole perturbed by an axisymmetric ring of matter. Suen found that the black-hole quadrupole moment is $Q_{ab} = +\frac{4}{21}M^5\mathcal{E}_{ab}$, so that it gives rise to a negative Love number, $k_{el} = -\frac{1}{122}$. This result contradicts our own results.

Suen's result is wrong. The starting points of Suen's analysis is the perturbed metric presented in Eq. (2.6) of his paper. It is easy to show that while the metric does indeed satisfy the Einstein field equations (up to terms that are quadratic in the small parameter A), it fails to be regular at the event horizon. The metric does not, therefore, represent a perturbed black hole, and the nonzero result for k_{el} is a consequence of this fact. The regularity of the metric perturbation $p_{\alpha\beta}$ at $r = 2M$ can be judged by examining its components in the light-cone coordinates (v, r, θ, ϕ), which are regular on the event horizon. A simple calculation reveals that in Suen's notation, $p_{rr} = -2(2U-V)/f$, where $f = 1-2M/r$. This is singular at $r = 2M$ unless $2U-V$ vanishes there, but Eqs. (2.7) of Suen's paper show instead that $2U-V \to AM^2$ in the limit. The perturbation is singular.

Organization of the paper

In the remaining sections of this paper we present the details of our analysis, and describe how the results reviewed previously were obtained. We begin in Sec. II with a discussion of tidal moments and tidal potentials, and motivate the definitions presented in Eqs. (1.3) and (1.4). In Sec. III we solve the external problem, and show that the metric of Eqs. (1.5) is a solution to the vacuum field equations linearized about the Schwarzschild metric. In Sec. IV we formulate the internal problem for general stellar models, and we specialize this to polytropes in Sec. V. In Sec. VI we review the numerical techniques that were employed to generate the figures and the tables displayed in the Appendix.

II. TIDAL MOMENTS AND POTENTIALS

A spherical stellar model is perturbed by an external tidal field characterized by the electric-type tidal moments $\mathcal{E}_L(v)$ and the magnetic-type tidal moments $\mathcal{B}_L(v)$. These are symmetric-tracefree (STF) tensors, and L is a multi-index that contains a number l of individual indices. The tidal moments depend on v (and not on the spatial coordinates), but this time dependence is taken to be so slow that all v-derivatives will be ignored in the Einstein field equations.

We begin our discussion of tidal potentials by adopting quasi-Cartesian coordinates x^a related in the usual way to our spherical coordinates (r, θ^A). We write the transformation as $x^a = r\Omega^a(\theta^A)$, with $\Omega^a = [\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta]$ denoting the unit radial vector. We introduce

$$\gamma_{ab} := \delta_{ab} - \Omega_a\Omega_b \qquad (2.1)$$

as the projector to the transverse space orthogonal to Ω^a, and we let $\Omega_A^a := \partial\Omega^a/\partial\theta^A$. We note the helpful identities

$$\Omega_a\Omega_A^a = 0, \qquad (2.2a)$$

$$\Omega_{AB} = \gamma_{ab}\Omega_A^a\Omega_B^b = \delta_{ab}\Omega_A^a\Omega_B^b, \qquad (2.2b)$$

$$\Omega^{AB}\Omega_A^a\Omega_B^b = \gamma^{ab}. \qquad (2.2c)$$

Here $\Omega_{AB} = \text{diag}[1, \sin^2\theta]$ is the metric on the unit two-sphere, and Ω^{AB} is its inverse. We introduce D_A as the covariant-derivative operator compatible with Ω_{AB}, and ϵ_{AB} as the Levi-Civita tensor on the unit two-sphere (with nonvanishing components $\epsilon_{\theta\phi} = -\epsilon_{\phi\theta} = \sin\theta$). In addition to Eqs. (2.2) we also have

$$\epsilon_{AB} = \epsilon_{abc}\Omega_A^a\Omega_B^b\Omega^c, \qquad (2.3a)$$

$$\epsilon_A{}^B\Omega_B^b = -\Omega_A^a\epsilon_{ap}{}^b\Omega^p, \qquad (2.3b)$$

$$D_A D_B\Omega^a = D_B D_A\Omega^a = -\Omega^a\Omega_{AB}. \qquad (2.3c)$$

Here and below, upper-case latin indices are raised and lowered with Ω^{AB} and Ω_{AB}, respectively. Finally, we note that $D_C\Omega_{AB} = D_C\epsilon_{AB} = 0$.

For an electric-type tidal moment $\mathcal{E}_L$ of degree $l \geq 2$, the Cartesian version of the tidal potentials are defined by

$$\mathcal{E}^{(l)} := \mathcal{E}_L\Omega^L, \qquad (2.4a)$$

$$\mathcal{E}_a^{(l)} := \gamma_a{}^c\mathcal{E}_{c\,L-1}\Omega^{L-1}, \qquad (2.4b)$$

$$\mathcal{E}_{ab}^{(l)} := 2\gamma_a{}^c\gamma_b{}^d\mathcal{E}_{cd\,L-2}\Omega^{L-2} + \gamma_{ab}\mathcal{E}^{(l)}. \qquad (2.4c)$$

Here $\mathcal{E}^{(l)}$ is a scalar potential, $\mathcal{E}_a^{(l)}$ is a transverse vector potential, and $\mathcal{E}_{ab}^{(l)}$ is a transverse-tracefree tensor potential. The angular version of the tidal potentials are

$$\mathcal{E}^{(l)} = \mathcal{E}_L\Omega^L, \qquad (2.5a)$$

$$\mathcal{E}_A^{(l)} := \mathcal{E}_a^{(l)}\Omega_A^a = \Omega_A^a\mathcal{E}_{a\,L-1}\Omega^{L-1}, \qquad (2.5b)$$

$$\mathcal{E}_{AB}^{(l)} := \mathcal{E}_{ab}^{(l)}\Omega_A^a\Omega_B^b = 2\Omega_A^a\Omega_B^b\mathcal{E}_{ab\,L-2}\Omega^{L-2} + \Omega_{AB}\mathcal{E}^{(l)}. \tag{2.5c}$$

For a magnetic-type tidal moment $\mathcal{B}_L$ of degree $l \geq 2$, the Cartesian version of the tidal potentials are defined by

$$\mathcal{B}_a^{(l)} := \epsilon_{apq}\Omega^p\mathcal{B}_{\,L-1}^q\Omega^{L-1}, \tag{2.6a}$$

$$\mathcal{B}_{ab}^{(l)} := \left(\epsilon_{apq}\Omega^p\mathcal{B}_{d\,L-2}^q\gamma^d_{\ b} + \epsilon_{bpq}\Omega^p\mathcal{B}_{c\,L-2}^q\gamma^c_{\ a}\right)\Omega^{L-2}. \tag{2.6b}$$

Here $\mathcal{B}_a^{(l)}$ is a transverse vector potential, and $\mathcal{B}_{ab}^{(l)}$ is a transverse-tracefree tensor potential; there is no scalar potential in the magnetic case. The angular version of the tidal potentials are

$$\mathcal{B}_A^{(l)} := \mathcal{B}_a^{(l)}\Omega_A^a = \Omega_A^a\epsilon_{apq}\Omega^p\mathcal{B}_{\,L-1}^q\Omega^{L-1}, \tag{2.7a}$$

$$\mathcal{B}_{AB}^{(l)} := \mathcal{B}_{ab}^{(l)}\Omega_A^a\Omega_B^b = \left(\Omega_A^a\epsilon_{apq}\Omega^p\mathcal{B}_{b\,L-2}^q\Omega_B^b \right.$$
$$\left. + \Omega_B^b\epsilon_{bpq}\Omega^p\mathcal{B}_{a\,L-2}^q\Omega_A^a\right)\Omega^{L-2}. \tag{2.7b}$$

The tidal potentials can all be expressed in terms of (scalar, vector, and tensor) spherical harmonics. Let Y^{lm} be the standard (scalar) spherical-harmonic functions. The vector and tensor harmonics of even parity are $Y_A^{lm} := D_A Y^{lm}$, $\Omega_{AB}Y^{lm}$, and $Y_{AB}^{lm} := [D_A D_B + \frac{1}{2}l(l+1)\Omega_{AB}]Y^{lm}$; notice that $\Omega^{AB}Y_{AB}^{lm} = 0$ by virtue of the eigenvalue equation satisfied by the spherical harmonics. The vector and tensor harmonics of odd parity are $X_A^{lm} := -\epsilon_A^{\ B}D_B Y^{lm}$ and $X_{AB}^{lm} := -\frac{1}{2}(\epsilon_A^{\ C}D_B + \epsilon_B^{\ C}D_A)D_C Y^{lm}$; X_{AB}^{lm} also is tracefree: $\Omega^{AB}X_{AB}^{lm} = 0$.

We first express the electric-type tidal potentials in terms of the even-parity spherical harmonics. We begin with $\mathcal{E}^{(l)}$, which we decompose as

$$\mathcal{E}^{(l)}(v,\theta^A) = \sum_m \mathcal{E}_m^{(l)}(v)Y^{lm}(\theta^A), \tag{2.8}$$

in terms of harmonic components $\mathcal{E}_m^{(l)}(v)$. There are $2l+1$ terms in the sum, and the $2l+1$ independent components of $\mathcal{E}_L$ are in a one-to-one correspondence with the $2l+1$ coefficients $\mathcal{E}_m^{(l)}$. Returning to the original representation of Eq. (2.4), we find after differentiation that $D_A\mathcal{E}^{(l)} = l\Omega_A^a\mathcal{E}_{a\,L-1}\Omega^{L-1}$, and we conclude that

$$\mathcal{E}_A^{(l)} = \frac{1}{l}D_A\mathcal{E}^{(l)} = \frac{1}{l}\sum_m \mathcal{E}_m^{(l)}Y_A^{lm}. \tag{2.9}$$

An additional differentiation using the last of Eqs. (2.3) reveals that $D_A D_B \mathcal{E}^{(l)} = -l\Omega_{AB}\mathcal{E}^{(l)} + l(l-1)\Omega_A^a\Omega_B^b\mathcal{E}_{ab\,L-2}\Omega^{L-2}$. From this we conclude that

$$\mathcal{E}_{AB}^{(l)} = \frac{2}{l(l-1)}\left[D_A D_B + \frac{1}{2}l(l+1)\Omega_{AB}\right]\mathcal{E}^{(l)}$$
$$= \frac{2}{l(l-1)}\sum_m \mathcal{E}_m^{(l)}Y_{AB}^{lm}. \tag{2.10}$$

We next express the magnetic-type potentials in terms of the odd-parity spherical harmonics. We begin with $\mathcal{B}^{(l)} := \mathcal{B}_L\Omega^L$ and its decomposition $\mathcal{B}^{(l)} = $

$\sum_m \mathcal{B}_m^{(l)}Y^{lm}(\theta^A)$. Differentiating the first expression, multiplying this by the Levi-Civita tensor, and involving the second of Eqs. (2.3) returns $\epsilon_A^{\ B}D_B\mathcal{B}^{(l)} = -l\Omega_A^a\epsilon_{apq}\Omega^p\mathcal{B}_{\,L-1}^q\Omega^{L-1}$. From this we conclude that

$$\mathcal{B}_A^{(l)} = \frac{1}{l}\left(-\epsilon_A^{\ B}D_B\right)\mathcal{B}^{(l)} = \frac{1}{l}\sum_m \mathcal{B}_m^{(l)}X_A^{lm}. \tag{2.11}$$

A second differentiation yields $-\epsilon_A^{\ C}D_B D_C \mathcal{B}^{(l)} = l\epsilon_{AB}\mathcal{B}^{(l)} + l(l-1)\Omega_A^a\epsilon_{apq}\Omega^p\mathcal{B}_{b\,L-2}^q\Omega_B^b\Omega^{L-2}$, and after symmetrization we obtain

$$\mathcal{B}_{AB}^{(l)} = -\frac{1}{l(l-1)}\left(\epsilon_A^{\ C}D_B + \epsilon_B^{\ C}D_A\right)D_C\mathcal{B}^{(l)}$$
$$= \frac{2}{l(l-1)}\sum_m \mathcal{B}_m^{(l)}X_{AB}^{lm}. \tag{2.12}$$

III. EXTERNAL PROBLEM

A. Even-parity sector

In this subsection we determine the tidal deformation of the metric outside the matter distribution, in the even-parity sector. The unperturbed external solution is the Schwarzschild metric

$$ds_0^2 = -f\,dv^2 + 2dvdr + r^2\,d\Omega^2, \tag{3.1}$$

with $f := 1 - 2M/r$ and M denoting the body's mass; the metric is valid for $r > R$, where R is the body's radius. We employ the perturbation formalism of Martel and Poisson [17], and implement the light-cone gauge of Preston and Poisson [8].

In the light-cone gauge the even-parity metric perturbation is given by

$$p_{vv} = \sum_m h_{vv}^{lm}(r)Y^{lm}(\theta^A), \tag{3.2a}$$

$$p_{vA} = \sum_m j_v^{lm}(r)Y_A^{lm}(\theta^A), \tag{3.2b}$$

$$p_{AB} = r^2\sum_m K^{lm}(r)\Omega_{AB}Y^{lm}(\theta^A)$$
$$+ r^2\sum_m G^{lm}(r)Y_{AB}^{lm}(\theta^A). \tag{3.2c}$$

We consider each l-mode separately, and we henceforth omit the label lm on the perturbation variables h_{vv}, j_v, K, and G, which depend on r only. As discussed by Preston and Poisson, K^{lm} can always be set equal to zero when the perturbation satisfies the vacuum field equations; this represents a refinement of the light-cone gauge, and we shall make this choice here.

To simplify the task of solving the field equations we set

$$h_{vv} = -\frac{2}{(l-1)l}r^l e_1(r)\mathcal{E}_m^{(l)}, \tag{3.3a}$$

$$j_v = -\frac{2}{(l-1)l(l+1)}r^{l+1}e_4(r)\mathcal{E}_m^{(l)}, \qquad (3.3b)$$

$$G = -\frac{4}{(l-1)l^2(l+1)}r^l e_7(r)\mathcal{E}_m^{(l)}, \qquad (3.3c)$$

where the functions $e_1(r)$, $e_4(r)$, and $e_7(r)$ are to be determined. Substitution of Eqs. (3.3) into Eq. (3.2) produces

$$p_{vv} = -\frac{2}{(l-1)l}r^l e_1(r)\mathcal{E}^{(l)}, \qquad (3.4a)$$

$$p_{vA} = -\frac{2}{(l-1)(l+1)}r^{l+1}e_4(r)\mathcal{E}_A^{(l)}, \qquad (3.4b)$$

$$p_{AB} = -\frac{2}{l(l+1)}r^{l+2}e_7(r)\mathcal{E}_{AB}^{(l)}, \qquad (3.4c)$$

where $\mathcal{E}^{(l)}$, $\mathcal{E}_A^{(l)}$, and $\mathcal{E}_{AB}^{(l)}$ are the tidal potentials introduced in Eq. (2.5).

The motivation behind the introduction of the functions e_1, e_4, and e_7 goes as follows. We first observe that when we set $e_1 = e_4 = e_7 = 1$, the perturbation defined by Eqs. (3.3) or Eqs. (3.4) satisfies the equations of linearized theory for a perturbation of Minkowski spacetime. This exercise reveals that h_{vv} must be proportional to r^l, j_v to r^{l+1}, and G to r^l; the relative numerical coefficients between these fields are also determined by solving the perturbation equations in flat spacetime. The remaining absolute numerical coefficient that relates the perturbation to the tidal moment $\mathcal{E}_L$ is determined by the definition of the tidal moment in terms of the Weyl tensor of the perturbed spacetime; this coefficient — the factor $-2/[(l-1)l]$ in h_{vv} — can be read off Eq. (3.26a) of Ref. [9].

Inserting the functions e_1, e_4, and e_7 in Eqs. (3.3) allows the perturbation to be a solution to the Einstein field equations linearized about the Schwarzschild metric instead of the Minkowski metric. We impose the boundary conditions

$$e_1(r \to \infty) = e_4(r \to \infty) = e_7(r \to \infty) = 1. \qquad (3.5)$$

The field equations do not determine these functions uniquely. The light-cone gauge comes with a class of residual gauge transformations that preserve the light-cone nature of the coordinate system (see Preston and Poisson [8]). In the even-parity sector, and for static perturbations, the residual gauge freedom that keeps $K = 0$ is a one-parameter family described by

$$e_1 \to e_1 - la_1(2M/r)^{l+2}, \qquad (3.6a)$$

$$e_4 \to e_4 + a_1\big[(l-1)(l+2) + 4M/r\big](2M/r)^{l+1}, \quad (3.6b)$$

$$e_7 \to e_7 + 2la_1(2M/r)^{l+1}, \qquad (3.6c)$$

in which a_1 is the (dimensionless) parameter. The residual gauge freedom does not interfere with the boundary conditions of Eq. (3.5).

When K is allowed to change, the residual gauge freedom becomes a three-parameter family. In this case we

have

$$e_1 \to e_1 - la_1(2M/r)^{l+2} + a_3(2M/r)^{l+2}, \qquad (3.7a)$$

$$e_4 \to e_4 + a_1\big[(l-1)(l+2) + 4M/r\big](2M/r)^{l+1}$$
$$- (l+1)a_3(2M/r)^{l+1}, \qquad (3.7b)$$

$$e_7 \to e_7 + 2la_1(2M/r)^{l+1} + 2a_2(2M/r)^l, \qquad (3.7c)$$

and K becomes

$$K = \frac{4(2M)^l}{(l-1)l}\big[a_2 + a_3(2M/r)\big]\mathcal{E}_m^{(l)}. \qquad (3.8)$$

Here a_2 and a_3 are two additional gauge parameters.

The differential equations satisfied by e_1, e_4, and e_7 can be extracted from the perturbation equations. These equations are coupled, and some effort must be devoted to their decoupling before an attempt is made to find solutions. We shall not describe these routine steps here. We state simply that the solutions are the ones that were displayed in Eqs. (1.6) and (1.7). These are given in a minimal implementation of the light-cone gauge, in which all constants of integrations are set equal to zero. The most general form of the solution is obtained from this by effecting the shifts described by Eqs. (3.7) and (3.8). The functions A_n and B_n are displayed for selected values of l in Table I.

The metric perturbation can be represented in terms of gauge-invariant variables. We employ the set defined by Eqs. (4.10)–(4.12) of Martel and Poisson [17]. According to these equations, and as can be directly verified from Eq. (3.7), the variables

$$\tilde{h}_{vv} := h_{vv} + \frac{2M}{r^2}j_v - MfG', \qquad (3.9a)$$

$$\tilde{h}_{vr} := MG' - j_v', \qquad (3.9b)$$

$$\tilde{h}_{rr} := 2rG' + r^2G'', \qquad (3.9c)$$

$$\tilde{K} := -\frac{2}{r}j_v + \frac{1}{2}l(l+1)G + rfG' \qquad (3.9d)$$

are gauge-invariant; a prime indicates differentiation with respect to r. We express them as

$$\tilde{h}_{vv} := -\frac{2}{(l-1)l}r^l e_{vv}(r)\mathcal{E}_m^{(l)}, \qquad (3.10a)$$

$$\tilde{h}_{vr} := \frac{2}{(l-1)l}r^l e_{vr}(r)\mathcal{E}_m^{(l)}, \qquad (3.10b)$$

$$\tilde{h}_{rr} := -\frac{4}{(l-1)l}r^l e_{rr}(r)\mathcal{E}_m^{(l)}, \qquad (3.10c)$$

$$\tilde{K} := -\frac{2}{(l-1)l}r^l e_K(r)\mathcal{E}_m^{(l)}, \qquad (3.10d)$$

in terms of new radial functions e_{vv}, e_{vr}, e_{rr}, and e_K. Calculation reveals that these are given in terms of the old ones by

$$e_{vv} = e_1 + \frac{1}{l+1}\frac{2M}{r}e_4 - \frac{1}{l+1}\frac{2M}{r}fe_7$$

TABLE I: Functions A_n and B_n for selected values of l, expressed in terms of $z := 2M/r$. The numbers μ_l and λ_l are given by $\lambda_l = (2l)!(2l+1)!/[(l-2)!(l-1)!(l+1)!(l+2)!]$ and $\mu_l = (l+1)\lambda_l/l$.

$l = 2$	$\mu_2 = 30, \qquad \lambda_2 = 20$
$A_1 = (1-z)^2$	$z^5 B_1 = -\mu_2 A_1 \ln(1-z) - \frac{5}{2}z(2-z)(6-6z-z^2)$
$A_4 = 1 - z$	$z^5 B_4 = \lambda_2 A_4 \ln(1-z) + \frac{5}{3}z(12-6z-2z^2-z^3)$
$A_7 = 1 - \frac{1}{2}z^2$	$z^5 B_7 = -\mu_2 A_7 \ln(1-z) - 5z(6+3z-z^2)$
$l = 3$	$\mu_3 = 840, \qquad \lambda_3 = 630$
$A_1 = \frac{1}{2}(1-z)^2(2-z)$	$z^7 B_1 = -\mu_3 A_1 \ln(1-z) - 7z(120-240z+130z^2-10z^3-z^4)$
$A_4 = \frac{1}{3}(1-z)(3-2z)$	$z^7 B_4 = \lambda_3 A_4 \ln(1-z) + \frac{7}{2}z(180-210z+30z^2+5z^3+z^4)$
$A_7 = 1 - z + \frac{1}{10}z^3$	$z^7 B_7 = -\mu_3 A_7 \ln(1-z) - 14z(60-30z-10z^2+z^3)$
$l = 4$	$\mu_4 = 17\,640, \qquad \lambda_4 = 14\,112$
$A_1 = \frac{1}{14}(1-z)^2(14-14z+3z^2)$	$z^9 B_1 = -\mu_4 A_1 \ln(1-z) - 21z(2-z)(420-840z+440z^2-20z^3-z^4)$
$A_4 = \frac{1}{28}(1-z)(28-35z+10z^2)$	$z^9 B_4 = \lambda_4 A_4 \ln(1-z) + \frac{42}{5}z(1680-2940z+1370z^2-90z^3-9z^4-z^5)$
$A_7 = 1 - \frac{5}{3}z + \frac{5}{7}z^2 - \frac{1}{42}z^4$	$z^9 B_7 = -\mu_4 A_7 \ln(1-z) - 14z(1260-1470z+270z^2+65z^3-3z^4)$
$l = 5$	$\mu_5 = 332\,640, \qquad \lambda_5 = 277\,200$
$A_1 = \frac{1}{12}(1-z)^2(2-z)(6-6z+z^2)$	$z^{11} B_1 = -\mu_5 A_1 \ln(1-z) - 66z(5040-15120z+16380z^2-7560z^3+1288z^4-28z^5-z^6)$
$A_4 = \frac{1}{30}(1-z)(30-54z+30z^2-5z^3)$	$z^{11} B_4 = \lambda_5 A_4 \ln(1-z) + 22z(12600-28980z+21840z^2-5670z^3+210z^4+14z^5+z^6)$
$A_7 = 1 - \frac{9}{4}z + \frac{5}{3}z^2 - \frac{5}{12}z^3 + \frac{1}{168}z^5$	$z^{11} B_7 = -\mu_5 A_7 \ln(1-z) - 66z(5040-8820z+4410z^2-420z^3-77z^4+2z^5)$

$$- \frac{1}{l(l+1)}2Mfe_7', \tag{3.11a}$$

$$e_{vr} = e_4 + \frac{1}{l+1}re_4' - \frac{1}{l+1}\frac{2M}{r}e_7$$
$$- \frac{1}{l(l+1)}2Me_7', \tag{3.11b}$$

$$e_{rr} = e_7 + \frac{2}{l}re_7' + \frac{1}{l(l+1)}r^2e_7'', \tag{3.11c}$$

$$e_K = -\frac{2}{l+1}e_4 + \frac{1}{l+1}(l+3-4M/r)e_7$$
$$+ \frac{2}{l(l+1)}rfe_7'. \tag{3.11d}$$

It is easy to see that these functions, like the old ones, all go to one as r goes to infinity.

Substitution of our expressions for e_1, e_4, and e_7 into Eqs. (3.11) and repeated use of the properties of hypergeometric functions reveal that

$$e_{vv} = fe_{vr} = f^2 e_{rr} = A_1 + 2k_{\mathrm{el}}(R/r)^{2l+1}B_1 \tag{3.12}$$

and

$$e_K = A_7 + 2k_{\mathrm{el}}(R/r)^{2l+1}B_7. \tag{3.13}$$

Notice that e_{vv}, fe_{vr}, and $f^2 e_{rr}$ are all equal to the minimal implementation of e_1, and e_K is equal to the minimal implementation of e_7. All this shows that the relativistic Love numbers k_{el} possess gauge-invariant significance.

B. Odd-parity sector

In the light-cone gauge the odd-parity metric perturbation is given by

$$p_{vA} = \sum_m h_v^{lm}(r)X_A^{lm}(\theta^A), \tag{3.14a}$$

$$p_{AB} = \sum_m h_2^{lm}(r)X_{AB}^{lm}(\theta^A). \tag{3.14b}$$

We consider each l-mode separately, and we henceforth omit the label lm on the perturbation variables h_v and h_2, which depend on r only. To simplify the task of solving the field equations we set

$$h_v = \frac{2}{3(l-1)l}r^{l+1}b_4(r)\mathcal{B}_m^{(l)}, \tag{3.15a}$$

$$h_2 = \frac{4}{3(l-1)l^2}r^{l+2}b_7(r)\mathcal{B}_m^{(l)}, \tag{3.15b}$$

where the functions $b_4(r)$ and $b_7(r)$ are to be determined. Substitution of Eqs. (3.15) into Eq. (3.14) produces

$$p_{vA} = \frac{2}{3(l-1)}r^{l+1}b_4(r)\mathcal{B}_A^{(l)}, \tag{3.16a}$$

$$p_{AB} = \frac{2}{3l}r^{l+2}b_7(r)\mathcal{B}_{AB}^{(l)}, \tag{3.16b}$$

where $\mathcal{B}_A^{(l)}$ and $\mathcal{B}_{AB}^{(l)}$ are the tidal potentials first introduced in Eq. (2.7).

The motivation behind the introduction of the functions b_4 and b_7 is identical to what was done in the even-parity sector. When we set $b_4 = b_7 = 1$, the perturbation defined by Eqs. (3.15) or Eqs. (3.16) satisfies the equations of linearized theory for a perturbation of Minkowski spacetime. This exercise reveals the relative numerical coefficients between h_v and h_2. The remaining absolute numerical coefficient that relates the perturbation to the tidal moment $\mathcal{B}_L$ is determined by the definition of the tidal moment in terms of the Weyl tensor of the perturbed spacetime; this coefficient — the factor $2/[3(l-1)l]$ in h_v — can be read off Eq. (3.26b) of Ref. [9].

Inserting the functions b_4 and b_7 in Eqs. (3.15) allows the perturbation to be a solution to the Einstein field

equations linearized about the Schwarzschild metric instead of the Minkowski metric. We impose the boundary conditions

$$b_4(r \to \infty) = b_7(r \to \infty) = 1. \qquad (3.17)$$

The field equations do not determine these functions uniquely. As in the even-parity case we have a residual gauge freedom that preserves the nature of the light-cone coordinates. It is described by

$$b_4 \to b_4 \qquad (3.18a)$$

$$b_7 \to b_7 + \alpha \left(\frac{2M}{r} \right)^l, \qquad (3.18b)$$

in which α is a (dimensionless) parameter. The residual gauge freedom does not interfere with the boundary conditions of Eq. (3.17).

The differential equations satisfied by b_4 and b_7 can be extracted from the perturbation equations. The solutions are displayed in Eqs. (1.6) and (1.7). They are given in a minimal implementation of the light-cone gauge, in which all constants of integrations are set equal to zero. The most general form of the solution is obtained from this by effecting the shifts described by Eqs. (3.18).

The metric perturbation can be represented in terms of gauge-invariant variables. We employ the set defined by Eq. (5.7) of Martel and Poisson [17]. According to this, and as can be directly verified from Eq. (3.18), the variables

$$\tilde{h}_v := h_v, \qquad (3.19a)$$

$$\tilde{h}_r := \frac{1}{r} h_2 - \frac{1}{2} h_2' \qquad (3.19b)$$

are gauge-invariant. We express them as

$$\tilde{h}_v := \frac{2}{3(l-1)l} r^{l+1} b_v(r) \mathcal{B}_m^{(l)}, \qquad (3.20a)$$

$$\tilde{h}_r := -\frac{2}{3(l-1)l} r^{l+1} b_r(r) \mathcal{B}_m^{(l)}, \qquad (3.20b)$$

$$(3.20c)$$

in terms of new radial functions b_v and b_r. Calculation reveals that these are given in terms of the old ones by

$$b_v = b_4, \qquad (3.21a)$$

$$b_r = b_7 + \frac{r}{l} b_7'. \qquad (3.21b)$$

It is easy to see that these functions, like the old ones, all go to one as r goes to infinity.

Substitution of our expressions for b_4 and b_7 into Eqs. (3.21) and repeated use of the properties of the hypergeometric functions reveal that

$$b_v = f b_r = A_4 - 2 \frac{l+1}{l} k_{\mathrm{mag}} (R/r)^{2l+1} B_4. \qquad (3.22)$$

Notice that b_v and $f b_r$ are both equal to b_4, which is gauge-invariant. This shows that the relativistic Love numbers k_{mag} possess gauge-invariant significance.

IV. INTERNAL PROBLEM

A. Background metric for relativistic stellar models

We begin with an examination of the internal gravitational field of a body that is not yet perturbed by an external tidal field. The body is spherically-symmetric, and the matter consists of a perfect fluid. In light-cone coordinates (v, r, θ^A) the metric is expressed as

$$ds_0^2 = -e^{2\psi} f \, dv^2 + 2e^{\psi} \, dv dr + r^2 \, d\Omega^2, \qquad (4.1)$$

with $f = 1 - 2m(r)/r$ and $\psi = \psi(r)$. The Einstein field equations are

$$m' = 4\pi r^2 \rho, \qquad \psi' = \frac{4\pi r}{f} (\rho + p), \qquad (4.2)$$

and the equation of hydrostatic equilibrium is

$$p' = -\frac{m + 4\pi r^3 p}{r^2 f} (\rho + p). \qquad (4.3)$$

Here ρ is the fluid's proper energy density, and p is the pressure.

These equations can be integrated once an equation of state is specified. The boundary conditions are $m(r = 0) = 0$ and $\psi(r = 0) = \psi_0$, where ψ_0 is chosen so that ψ vanishes at the stellar surface: $\psi(r = R) = 0$.

B. Light-cone gauge

The internal light-cone gauge is a modified version of the external gauge constructed by Preston and Poisson [8]. We define it properly in this section.

The metric of Eq. (4.1) reveals the meaning of the coordinates (v, r, θ^A) in the background spacetime. We note first that $l_\alpha = -\partial_\alpha v$ is a null vector, so that the surfaces $v = $ constant are null hypersurfaces; they describe light cones that converge toward $r = 0$. The vector

$$l^\alpha = (0, -e^{-\psi}, 0, 0) \qquad (4.4)$$

is tangent to the null generators of these light cones, and the expression reveals that θ^A is constant along the generators. In addition, the affine parameter λ that runs along the generators is related to r by $d\lambda = -e^{\psi} dr$. In the interior portion of the spacetime, r is no longer an affine parameter on the null generators; but it still possesses the property of being an areal radius, in the sense that the area of a surface of constant (v, r) is given by $4\pi r^2$.

In the *internal light-cone gauge*, the metric of the perturbed spacetime is presented in coordinates (v, r, θ^A) that possess the same geometrical meaning as in the background spacetime. In particular, v continues to label null hypersurfaces, θ^A continues to be constant along

the null generators, and r continues to be related to the affine parameter by $d\lambda = -e^\psi \, dr$. It is easy to show that these statements imply the same conditions

$$p_{vr} = p_{rr} = p_{rA} = 0 \qquad (4.5)$$

that were employed in the external problem. The nonvanishing components of the metric perturbation are therefore p_{vv}, p_{vA}, and p_{AB}. The radial coordinate, however, will lose its meaning as an areal radius in the stellar interior.

In the even-parity sector the perturbation is decomposed as in Eq. (3.2), and the fields h_{vv}^{lm}, j_v^{lm}, K^{lm}, G^{lm} depend (in general) on the coordinates (v, r). An even-parity gauge transformation is generated by the vector field Ξ_α, with components

$$\Xi_v = \sum_{lm} \xi_v^{lm}(v, r) Y^{lm}(\theta^A), \qquad (4.6a)$$

$$\Xi_r = \sum_{lm} \xi_r^{lm}(v, r) Y^{lm}(\theta^A), \qquad (4.6b)$$

$$\Xi_A = \sum_{lm} \xi^{lm}(v, r) Y_A^{lm}(\theta^A). \qquad (4.6c)$$

It can be shown that the condition $h_{vr} = 0$ determines ξ_v, that $h_{rr} = 0$ determines ξ_r, and that $j_r = 0$ determines ξ. The gauge, however, is not determined uniquely. There exists a residual gauge freedom that preserves the geometrical meaning of the coordinates. In the case of v-independent perturbations, the residual gauge freedom is a three-parameter family described by

$$\xi_v = -a_1 e^{2\psi} f + a_2, \qquad (4.7a)$$

$$\xi_r = a_1 e^\psi, \qquad (4.7b)$$

$$\xi = -a_1 r^2 \int^r r'^{-2} e^{\psi(r')} \, dr' + a_3 r^2. \qquad (4.7c)$$

Here we suppressed the lm labels on ξ_v, ξ_r, and ξ, as well as the constants a_1, a_2, and a_3.

In the odd-parity sector the perturbation is decomposed as in Eq. (3.14), and the fields h_v^{lm}, h_2^{lm} depend (in general) on the coordinates (v, r). An odd-parity gauge transformation is generated by the vector field Ξ_α, with components

$$\Xi_v = \Xi_r = 0, \qquad \Xi_A = \sum_{lm} \xi^{lm}(v, r) X_A^{lm}(\theta^A). \quad (4.8)$$

It can be shown that the condition $h_r = 0$ determines ξ. In this case also there exists a residual gauge freedom that preserves the geometrical meaning of the coordinates. In the case of v-independent perturbations, the residual gauge freedom is a one-parameter family described by

$$\xi = \alpha r^2. \qquad (4.9)$$

Here also we suppressed the lm labels on ξ and the constant α.

The decompositions of Eq. (3.2) and (3.14) can be used to compute $\delta G_{\alpha\beta}$, the perturbation of the Einstein tensor inside the body. The even-parity sector decouples from the odd-parity sector, and the perturbation takes the form of

$$\delta G_{vv} = \sum_{lm} A_{vv}^{lm} Y^{lm}, \qquad (4.10a)$$

$$\delta G_{vr} = \sum_{lm} A_{vr}^{lm} Y^{lm}, \qquad (4.10b)$$

$$\delta G_{rr} = \sum_{lm} A_{rr}^{lm} Y^{lm}, \qquad (4.10c)$$

$$\delta G_{vA} = \sum_{lm} \left(A_v^{lm} Y_A^{lm} + B_v^{lm} X_A^{lm} \right), \qquad (4.10d)$$

$$\delta G_{rA} = \sum_{lm} \left(A_r^{lm} Y_A^{lm} + B_r^{lm} X_A^{lm} \right), \qquad (4.10e)$$

$$\delta G_{AB} = \sum_{lm} \left(A_\flat^{lm} \Omega_{AB} Y^{lm} + A_\sharp^{lm} Y_{AB} + B^{lm} X_{AB}^{lm} \right). \qquad (4.10f)$$

Here the even-parity fields A_{vv}, A_{vr}, A_{rr}, A_v, A_r, $A_\flat$, $A_\sharp$ and the odd-parity fields B_v, B_r, B depend on v and r only. In the case of a stationary perturbation, they depend on r only.

The expressions are too long to be displayed here. In practice they are easily generated with GRTensorII [18] by specializing the perturbation to an axisymmetric mode $m = 0$ with a specific multipole order l. With $Y^{lm} = Y(\theta)$ we have $Y_\theta = Y'$, $Y_\phi = 0$, $Y_{\theta\theta} = -\cos\theta Y'/\sin\theta - \frac{1}{2} l(l+1) Y$, $Y_{\theta\phi} = 0$, and $Y_{\phi\phi} = \sin\theta \cos\theta Y' + \frac{1}{2} l(l+1) \sin^2\theta Y$ in the even-parity case, and $X_\theta = 0$, $X_\phi = \sin\theta Y'$, $X_{\theta\theta} = 0$, $X_{\theta\phi} = -\cos\theta Y' - \frac{1}{2} l(l+1) \sin\theta Y$, and $X_{\phi\phi} = 0$ in the odd-parity case. The definition of the metric implements the constraint $Y'' = -\cos\theta Y'/\sin\theta - l(l+1) Y$ on the spherical-harmonic functions, and this simplifies the final expression for the perturbed Einstein tensor.

C. Energy-momentum tensor

We consider *stationary tides* raised by a tidal environment characterized by an electric-type tidal moment $\mathcal{E}_L$ and a magnetic-type tidal moment $\mathcal{B}_L$; these are actually time-dependent, but the dependence is sufficiently slow that it can be neglected in the process of integrating the Einstein field equations. The perturbed metric will therefore carry a parametric dependence upon v.

The fluid's velocity vector in the background configuration is given by $u^\alpha = (e^{-\psi} f^{-1/2}, 0, 0, 0)$. In the perturbed configuration it becomes $\hat{u}^\alpha = (\hat{u}^v, 0, 0, 0)$, reflecting the fact that the tide is stationary and does not create motion within the fluid. The time component of the vector changes by virtue of the fact that the metric changes; we have that $\hat{u}^v = e^{-\psi} f^{-1/2} + \delta u^v$, with $\delta u^v = \frac{1}{2} e^{-3\psi} f^{-3/2} p_{vv}$.

After lowering the index on $\hat{u}^\alpha$ with the perturbed metric $g^0_{\alpha\beta} + p_{\alpha\beta}$, we find that $\hat{u}_v = -e^\psi f^{1/2} + \delta u_v$, $\hat{u}_r = f^{-1/2} + \delta u_r$, and $\hat{u}_A = \delta u_A$, with

$$\delta u_v = \frac{1}{2} e^{-\psi} f^{-1/2} p_{vv}, \tag{4.11a}$$

$$\delta u_r = \frac{1}{2} e^{-2\psi} f^{-3/2} p_{vv}, \tag{4.11b}$$

$$\delta u_A = e^{-\psi} f^{-1/2} p_{vA}. \tag{4.11c}$$

These expressions are valid in the light-cone gauge. The perturbation δu_A can be decomposed into even-parity and odd-parity components; the perturbations δu_v and δu_r are necessarily of even parity.

The perturbation in the energy-momentum tensor is generated by the perturbation in u_α, but also by a perturbation in the density ρ and pressure p created by the tide; these are related by the equation of state. We have

$$\begin{aligned} \delta T_{\alpha\beta} &= (\rho + p)\big(u_\alpha \delta u_\beta + u_\beta \delta u_\alpha\big) + p\, p_{\alpha\beta} \\ &\quad + (\delta\rho + \delta p)u_\alpha u_\beta + (\delta p)g_{\alpha\beta}, \end{aligned} \tag{4.12}$$

and in the light-cone gauge this reads

$$\delta T_{vv} = -\rho\, p_{vv} + e^{2\psi} f\, \delta\rho, \tag{4.13a}$$

$$\delta T_{vr} = -e^\psi\, \delta\rho, \tag{4.13b}$$

$$\delta T_{vA} = -\rho\, p_{vA}, \tag{4.13c}$$

$$\delta T_{rr} = (\rho + p)e^{-2\psi} f^{-2} p_{vv} + f^{-1}(\delta\rho + \delta p), \tag{4.13d}$$

$$\delta T_{rA} = e^{-\psi} f^{-1}(\rho + p) p_{vA}, \tag{4.13e}$$

$$\delta T_{AB} = p\, p_{AB} + r^2 \delta p\, \Omega_{AB}. \tag{4.13f}$$

The perturbations δT_{vA}, δT_{rA}, and δT_{AB} can be decomposed into even-parity and odd-parity components; the perturbations δT_{vv}, δT_{vr}, and δT_{rr} are necessarily of even parity.

From Eqs. (4.13) we find that $\delta T_{\alpha\beta}$ is given by

$$\delta T_{vv} = \sum_{lm} Q_{vv}^{lm} Y^{lm}, \tag{4.14a}$$

$$\delta T_{vr} = \sum_{lm} Q_{vr}^{lm} Y^{lm}, \tag{4.14b}$$

$$\delta T_{rr} = \sum_{lm} Q_{rr}^{lm} Y^{lm}, \tag{4.14c}$$

$$\delta T_{vA} = \sum_{lm} \big(Q_v^{lm} Y_A^{lm} + P_v^{lm} X_A^{lm}\big), \tag{4.14d}$$

$$\delta T_{rA} = \sum_{lm} \big(Q_r^{lm} Y_A^{lm} + P_r^{lm} X_A^{lm}\big), \tag{4.14e}$$

$$\delta T_{AB} = \sum_{lm} \big(Q_\flat^{lm} \Omega_{AB} Y^{lm} + Q_\sharp^{lm} Y_{AB}^{lm} + P^{lm} X_{AB}^{lm}\big). \tag{4.14f}$$

The even-parity fields are

$$Q_{vv} = -\rho\, h_{vv} + e^{2\psi} f\, \sigma, \tag{4.15a}$$

$$Q_{vr} = -e^\psi\, \sigma, \tag{4.15b}$$

$$Q_{rr} = (\rho + p)e^{-2\psi} f^{-2} h_{vv} + f^{-1}(\sigma + q), \tag{4.15c}$$

$$Q_v = -\rho\, j_v, \tag{4.15d}$$

$$Q_r = e^{-\psi} f^{-1}(\rho + p)\, j_v, \tag{4.15e}$$

$$Q_\flat = r^2(p\, K + q), \tag{4.15f}$$

$$Q_\sharp = r^2 p\, G, \tag{4.15g}$$

and the perturbations in the density and pressure were also decomposed in spherical harmonics:

$$\delta\rho = \sum_{lm} \sigma^{lm} Y^{lm}, \qquad \delta p = \sum_{lm} q^{lm} Y^{lm}. \tag{4.16}$$

The odd-parity fields are

$$P_v = -\rho\, h_v, \tag{4.17a}$$

$$P_r = e^{-\psi} f^{-1}(\rho + p)\, h_v, \tag{4.17b}$$

$$P = p\, h_2. \tag{4.17c}$$

Information about $\delta\rho$ and δp, or σ and q, can be obtained from the equation of hydrostatic equilibrium. In the perturbed spacetime the equation states that $(\hat{\rho} + \hat{p})\hat{a}_\alpha + \partial_\alpha \hat{p} = 0$, where $\hat{\rho} = \rho + \delta\rho$ is the perturbed density, $\hat{p} = p + \delta p$ is the perturbed pressure, and $\hat{a}_\alpha$ is the perturbed acceleration of the fluid elements. The equation becomes

$$(\rho + p)\delta a_\alpha + (\delta\rho + \delta p)a_\alpha + \partial_\alpha \delta p = 0 \tag{4.18}$$

when expressed in terms of the perturbations $\delta\rho$, δp, and δa_α. The unperturbed acceleration has $a_r = \frac{1}{2} e^{-2\psi} f^{-1}(e^\psi f)'$ as its only nonvanishing component, and the perturbation has components

$$\delta a_v = 0, \tag{4.19a}$$

$$\delta a_r = -\frac{1}{2} e^{-2\psi} f^{-1} \partial_r p_{vv} + \frac{1}{2} e^{-4\psi} f^{-2}(e^{2\psi} f)'\, p_{vv}, \tag{4.19b}$$

$$\delta a_A = -\frac{1}{2} e^{-2\psi} f^{-1} \partial_A p_{vv}. \tag{4.19c}$$

Substitution of Eqs. (4.16) and (4.19), as well as $p_{vv} = \sum_{lm} h_{vv}^{lm} Y^{lm}$, into Eq. (4.18) reveals that

$$\begin{aligned} q' &= \frac{1}{2}(\rho + p)e^{-2\psi} f^{-1} h_{vv}' \\ &\quad - \frac{1}{2}(\rho + p)e^{-4\psi} f^{-2}(e^{2\psi} f)' h_{vv} \\ &\quad - \frac{1}{2} e^{-2\psi} f^{-1}(e^{2\psi} f)'(\sigma + q) \end{aligned} \tag{4.20}$$

and

$$q = \frac{1}{2}(\rho + p)e^{-2\psi} f^{-1} h_{vv}. \tag{4.21}$$

If we next differentiate Eq. (4.21) and insert the result within Eq. (4.20), we discover that

$$(\rho + p)' h_{vv} = -(e^{2\psi} f)'(\sigma + q). \tag{4.22}$$

The last two equations allow us to express σ^{lm} and q^{lm} directly in terms of h_{vv}^{lm}; hydrostatic equilibrium implies that these are not independent variables.

D. Perturbation equations: even-parity sector

The useful combinations of Einstein field equations are

$$E_1 := (A_{vv} - 8\pi Q_{vv}) + e^\psi f(A_{vr} - 8\pi Q_{vr}) = 0, \tag{4.23a}$$

$$E_2 := (A_{vv} - 8\pi Q_{vv}) + 2e^\psi f(A_{vr} - 8\pi Q_{vr})$$
$$+ e^{2\psi} f^2 (A_{rr} - 8\pi Q_{rr}) = 0, \tag{4.23b}$$

$$E_3 := (A_{rr} - 8\pi Q_{rr}) = 0, \tag{4.23c}$$

$$E_4 := e^{-\psi} r E_2 + 2f(A_v - 8\pi Q_v)$$
$$+ 2e^\psi f^2 (A_r - 8\pi Q_r) = 0. \tag{4.23d}$$

These are a set of coupled differential equations for the variables $h_{vv}(r)$, $j_v(r)$, $K(r)$, and $G(r)$; the remaining field equations are redundant by virtue of the Bianchi identities. The explicit forms reveal that $E_1 = 0$ is a first-order differential equation for j_v, $E_2 = 0$ is a first-order differential equation for h_{vv}, $E_3 = 0$ is a second-order differential equation for K, and $E_4 = 0$ is a first-order differential equation for G.

The field equations can be manipulated to yield a decoupled equation for the master function

$$\begin{aligned}\tilde{h}_{vv} &:= h_{vv} + e^{-\psi}(e^{2\psi}f)' j_v - \frac{1}{2}r^2 f(e^{2\psi}f)' G' \\ &= h_{vv} + \frac{2e^\psi}{r^2}(m + 4\pi r^3 p) j_v \\ &\quad - e^{2\psi} f(m + 4\pi r^3 p) G'. \end{aligned} \tag{4.24}$$

This function is gauge invariant, and it joins smoothly with the external version of Eq. (3.9) at $r = R$. The master equation is

$$r^2 \tilde{h}_{vv}'' + A r \tilde{h}_{vv}' - B \tilde{h}_{vv} = 0, \tag{4.25}$$

where

$$A = \frac{2}{f}\left[1 - \frac{3m}{r} - 2\pi r^2(\rho + 3p)\right], \tag{4.26a}$$

$$B = \frac{1}{f}\left[l(l+1) - 4\pi r^2(\rho + p)\left(3 + \frac{d\rho}{dp}\right)\right]. \tag{4.26b}$$

The master equation is equivalent to Eq. (27) of Ref. [12], in which $H := e^{-2\psi} f^{-1} \tilde{h}_{vv}$ is used as an alternative choice of dependent variable.

The master equation can be derived by the following procedure. First, integrate the field equation $E_\sharp := A_\sharp - 8\pi Q_\sharp = 0$ and obtain $j_v = \frac{1}{2}r^2 f e^\psi G'$. This implies that $\tilde{h}_{vv} = h_{vv}$. Second, make the substitution in the other field equations. The result is that E_1 now involves h_{vv}, G', and G''; E_2 involves h_{vv}, h_{vv}', K, K', and G'; E_3 involves h_{vv}, K', and K''; and E_4 involves h_{vv}, K, K', G, and G'. Third, differentiate E_2 with respect to r, and use E_1 to eliminate the terms in G'', and E_3 to eliminate the terms in K''. The result is that E_2' now involves h_{vv},

h_{vv}', h_{vv}'', K, K', G, and G'. Fourth, construct the linear combination $rE_2' + aE_2 + bE_4$ and determine the functions a and b that eliminate all terms involving K, K', G, G'. The solution is unique, and the final result is Eq. (4.25).

For numerical integration it is advantageous to make the same substitution as in Eq. (3.10),

$$\tilde{h}_{vv} = -\frac{2}{(l-1)l} r^l e_{vv}(r) \mathcal{E}_m^{(l)}, \tag{4.27}$$

and to rewrite Eq. (4.25) as a second-order differential equation for $e_{vv}(r)$. This function joins smoothly with the external version of Eq. (3.12), and $k_{\rm el}$ is determined by matching the values of the internal and external functions (along with their first derivatives) at $r = R$.

E. Perturbation equations: odd-parity sector

The useful combinations of field equations are

$$O_1 := (B_v - 8\pi P_v) = 0, \tag{4.28a}$$

$$O_2 := (B_v - 8\pi P_v) + e^\psi f(B_r - 8\pi P_r) = 0. \tag{4.28b}$$

The first is a second-order differential equation for h_v, while the second is a first-order differential equation for h_2.

The equation $O_1 = 0$ is fully decoupled, and the perturbation variable h_v is easily shown to be gauge invariant, as it was in the external problem. The master variable for the odd-parity sector is therefore $\tilde{h}_v := h_v$, and the master equation is

$$r^2 \tilde{h}_v'' - F r \tilde{h}_v' - G \tilde{h}_v = 0, \tag{4.29}$$

where

$$F = \frac{4\pi r^2}{f}(\rho + p), \tag{4.30a}$$

$$G = \frac{1}{f}\left[l(l+1) - \frac{4m}{r} + 8\pi r^2(\rho + p)\right]. \tag{4.30b}$$

This equation is equivalent to Eq. (31) of Ref. [12], in which $\psi := r\tilde{h}_v' - 2\tilde{h}_v$ is used as an alternative choice of dependent variable. The function $\tilde{h}_v$ joins smoothly with the external version of Eq. (3.19) at $r = R$.

For numerical integration it is advantageous to make the same substitution as in Eq. (3.20),

$$\tilde{h}_v = \frac{2}{3(l-1)l} r^{l+1} b_v(r) \mathcal{B}_m^{(l)}, \tag{4.31}$$

and to rewrite Eq. (4.29) as a second-order differential equation for $b_v(r)$. This function joins smoothly with the external version of Eq. (3.22), and $k_{\rm mag}$ is determined by matching the values of the internal and external functions (along with their first derivatives) at $r = R$.

V. IMPLEMENTATION FOR POLYTROPES

The relativistic Love numbers $k_{\rm el}$ and $k_{\rm mag}$ are determined by the numerical integration of Eqs. (4.25) and (4.29) and matching with the external solutions at $r = R$. This defines a simple computational procedure that can be implemented for any choice of equation of state. In this section we describe the steps that are involved when the polytropic form

$$p = K\rho^{1+1/n}, \qquad (5.1)$$

is adopted; here K and the polytropic index n are constants. We choose, however, to deviate from the procedure just outlined: Instead of integrating the master equations for the variables $\tilde{h}_{vv}$ and $\tilde{h}_v$, we integrate *the complete set of independent field equations*. This allows us to calculate all components of the metric perturbation, and matching them across $r = R$ determines, in addition to the Love numbers, the gauge parameters a_1, a_2, a_3, and α that are automatically selected by the internal solution.[1]

A. Unperturbed stellar model

The numerical integration of Eqs. (4.2) and (4.3) is conveniently accomplished by introducing the dimensionless variables θ, μ, and ξ defined by

$$\rho = \rho_c \theta^n, \quad p = p_c \theta^{n+1}, \quad m = m_0 \mu, \quad r = r_0 \xi. \quad (5.2)$$

Here $\rho_c := \rho(r = 0)$ is the central density, and $p_c := K\rho_c^{1+1/n}$ is the central pressure. The units of mass and radius are chosen to be

$$m_0 := 4\pi r_0^3 \rho_c, \qquad r_0^2 := \frac{(n+1)p_c}{4\pi \rho_c^2}, \qquad (5.3)$$

so as to simplify the form of the field equations.

It is useful to introduce also a "relativistic factor"

$$b := p_c/\rho_c, \qquad (5.4)$$

which determines the degree with which the stellar model is relativistic. In terms of this we have $\rho_c = b^n/K^n$,

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$p_c = b^{n+1}/K^n$, and b can be used in place of ρ_c to label a stellar model, given a choice (K, n) of equation of state. We also note the relation $m_0/r_0 = (n+1)b$. We find that the units m_0 and r_0 vary with b even when the equation of state is fixed. To eliminate this dependence it is useful to define the alternative units

$$M_0 = \frac{(n+1)^{3/2}}{\sqrt{4\pi}} K^{n/2}, \qquad R_0 = \sqrt{\frac{n+1}{4\pi}} K^{n/2}, \quad (5.5)$$

which do not depend on b. We have that $m_0 = M_0 b^{(3-n)/2}$ and $r_0 = R_0 b^{(1-n)/2}$.

In terms of the dimensionless variables the field equations (4.2) and (4.3) become

$$\frac{d\mu}{d\xi} = \xi^2 \theta^n, \qquad (5.6a)$$

$$\frac{d\psi}{d\xi} = (n+1)b\frac{\xi\theta^n(1+b\theta)}{f}, \qquad (5.6b)$$

$$\frac{d\theta}{d\xi} = -\frac{(\mu + b\xi^3\theta^{n+1})(1+b\theta)}{\xi^2 f}, \qquad (5.6c)$$

with $f = 1 - 2(n+1)b\mu/\xi$. The boundary conditions are $\theta(\xi = 0) = 1$, $\mu(\xi = 0) = 0$, and $\psi(\xi = 0) = \psi_0$. In the limit $b \to 0$ the model becomes nonrelativistic, and the equations for μ and θ can be combined into the well-known Lane-Emden equation; in the limit the equation for ψ becomes irrelevant.

The formulation of Eq. (5.6) is not optimal from a numerical point of view. For accurate integrations it is better to use the variable $\nu := \mu/\xi^3$ instead of μ, and $x := \ln \xi$ instead of ξ. The system of equations becomes

$$\frac{d\nu}{dx} = \theta^n - 3\nu, \qquad (5.7a)$$

$$\frac{d\psi}{dx} = (n+1)b\xi^2 f^{-1}\theta^n(1+b\theta), \qquad (5.7b)$$

$$\frac{d\theta}{dx} = -\xi^2 f^{-1}(\nu + b\theta^{n+1})(1+b\theta), \qquad (5.7c)$$

with $f = 1 - 2(n+1)b\xi^2\nu$. The integration begins at a large and negative value of x, so that $\xi = e^x$ is small, with the starting values

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$$\nu = \frac{1}{3} - \frac{n}{30}(1+b)(1+3b)\xi^2 + \frac{n}{2520}(1+b)(1+3b)\left[8n - 5 + (18n - 20)b + (15 + 30n)b^2\right]\xi^4 + O(\xi^6), \qquad (5.8a)$$

$$\theta = 1 - \frac{1}{6}(1+b)(1+3b)\xi^2 + \frac{1}{360}(1+b)(1+3b)\left[3n - 2nb + (30 + 15n)b^2\right]\xi^4 + O(\xi^6), \qquad (5.8b)$$

$$\psi = \psi_0 + \frac{1}{2}(n+1)b(1+b)\xi^2 - \frac{1}{24}(n+1)b(1+b)\left[n - 3b + (3 + 3n)b^2\right]\xi^4 + O(\xi^6). \qquad (5.8c)$$

The integration stops at $\xi = \xi_1$, where θ goes to zero, and ψ_0 is chosen so that $\psi(\xi_1) = 0$. The stellar mass and radius are then given by

$$M = M_0 b^{(3-n)/2}\xi_1^3\nu(\xi_1), \qquad R = R_0 b^{(1-n)/2}\xi_1, \quad (5.9)$$

in the units of Eq. (5.5). The compactness of the body is measured by $C := 2M/R = 2(n+1)b\xi_1^2\nu(\xi_1)$; this is dimensionless, and therefore independent of the units M_0 and R_0.

B. Perturbation: Even-parity sector

The perturbation equations (4.23) are simplified by involving the background field equations (4.2) and (4.3). They are also simplified by making the substitutions of Eqs. (5.2), (5.3), and (5.4); we therefore write $\rho = \rho_c\theta^n$, $p = p_c\theta^{n+1}$, $r = r_0\xi$, and $m = m_0\xi^3\nu$, where $\rho_c = (n+$ $1)b/(4\pi r_0^2)$, $p_c = (n+1)b^2/(4\pi r_0^2)$, and $m_0 = (n+1)br_0$, with θ and ν (as well as ψ) depending on ξ. Finally, we use the fact that a term ρ' in the perturbation equations can be related to p' by the equation $\rho' = (d\rho/dp)p'$, with $d\rho/dp$ determined by the equation of state.

Another useful set of substitutions is the one displayed in Eqs. (3.3), along with

$$r_0^2 K = \frac{2}{(l-1)l(l+2)(l+3)}r^{l+2}e_{10}(\xi)\mathcal{E}_m^{(l)}, \qquad (5.10)$$

in which we replace the original variables with the radial functions e_1, e_4, e_7, and e_{10}. These replacements are motivated by an analysis of the perturbation equations for small values of r, which reveals that h_{vv} behaves as r^l, j_v as r^{l+1}, G as r^l, and K as r^{l+2}. The numerical factors in front of e_{10} is inserted to simplify the form of the small-r expansion of K, as we shall see below.

The final expression of the perturbation equations is

$$0 = E_1 = -\xi e_4' + (l+1)e^{-\psi}f^{-1}\,e_1 - f^{-1}A_1\,e_4, \qquad (5.11a)$$

$$0 = E_2 = -\xi e_1' + \frac{1}{2}f^{-1}A_2\,e_1 + le^{\psi}f^{-1}B_2\,e_4 - \frac{1}{2}(l-1)(l+2)e^{2\psi}\,e_7$$
$$+ \frac{1}{2(l+3)}e^{2\psi}C_2\xi^2\,e_{10} - \frac{1}{(l+2)(l+3)}e^{2\psi}B_2\xi^3\,e_{10}', \qquad (5.11b)$$

$$0 = E_3 = -\xi^2 e_{10}'' + (l+2)(l+3)e^{-2\psi}f^{-2}A_3\,e_1 - (l+2)f^{-1}B_3\,e_{10} - f^{-1}C_3\xi\,e_{10}', \qquad (5.11c)$$

$$0 = E_4 = -\xi e_7' + \frac{l(l+1)}{2(l-1)(l+2)}e^{-2\psi}f^{-2}A_4\,e_1 + \frac{l}{(l-1)(l+2)}e^{-\psi}f^{-2}B_4\,e_4$$
$$- \frac{1}{2}l\big[l+3-4(n+1)b\xi^2\nu\big]f^{-1}\,e_7 + \frac{l(l+1)}{2(l-1)(l+2)(l+3)}f^{-1}C_4\xi^2\,e_{10}$$
$$- \frac{l(l+1)}{(l-1)(l+2)^2(l+3)}\big[(n+1)b(\nu+b\theta^{n+1})\big]f^{-1}\xi^5\,e_{10}', \qquad (5.11d)$$

where a prime indicates differentiation with respect to ξ, and

$$A_1 = l+1 - 2(n+1)b\xi^2\big[(l+2)\nu + b\theta^{n+1}\big], \qquad (5.12a)$$

$$A_2 = (l-2)(l+1) + 2(n+1)b\xi^2\big[2(l+1)\nu - \theta^n(1+b\theta)\big], \qquad (5.12b)$$

$$B_2 = 1 - (n+1)b\xi^2(\nu - b\theta^{n+1}), \qquad (5.12c)$$

$$C_2 = l-3 + 2(n+1)b\xi^2(\nu - b\theta^{n+1}), \qquad (5.12d)$$

$$A_3 = n\theta^{n-1} + (4n+3)b\theta^n + 3(n+1)b^2\theta^{n+1}, \qquad (5.12e)$$

$$B_3 = l+3 - (n+1)b\xi^2\big[2(l+3)\nu + \theta^n(1+b\theta)\big], \qquad (5.12f)$$

$$C_3 = 2(l+3) - (n+1)b\xi^2\big[4(l+3)\nu + \theta^n(1+b\theta)\big], \qquad (5.12g)$$

$$A_4 = (l-1)(l+2) + 2(n+1)b\xi^2\big[2\nu - \theta^n(1+b\theta)\big], \qquad (5.12h)$$

$$B_4 = (l-1)(l+2) - (n+1)b\xi^2\big[(l^2+l-4)\nu - l(l+1)b\theta^{n+1}\big], \qquad (5.12i)$$

$$C_4 = l-1 - 2(n+1)b\xi^2(\nu + b\theta^{n+1}). \qquad (5.12j)$$

A small-ξ expansion of these equations, using Eqs. (5.8), reveals that $e_1 = a_0 + O(\xi^2)$, $e_4 = a_0 e^{-\psi_0} + O(\xi^2)$, $e_7 =$ $a_0 e^{-2\psi_0} + O(\xi^2)$, and $e_{10} = a_0 e^{-2\psi_0}(1+b)[3(n+1)b+n] + O(\xi^2)$, where a_0 is a parameter that must be determined

by matching the internal and external perturbations at the stellar boundary.

The perturbation equations are easily written as a first-order dynamical system for the variables $u_1 := e_1$, $u_2 := e_4$, $u_3 = e_7$, $u_4 := e_{10}$, and $u_5 := \xi e'_{10}$. The numerical integration is carried out with $x := \ln \xi$ as the independent variable, and the differential equations are integrated simultaneously with Eqs. (5.7) to determine the unperturbed stellar model. The integration proceeds from a large and negative value of x, for which $\xi = e^x$ is small, and it stops at at $\xi = \xi_1$ where θ goes to zero.

The term $n\theta^{n-1}$ in A_3 originates from a term involving $d\rho/dp \propto \theta^{-1}$ that multiplies $\rho \propto \theta^n$ in the field equation for K (or e_{10}). This term diverges at the stellar boundary when $n < 1$. The singularity is integrable, however, and it can be shown that the solution for $K(r)$ (or e_{10}) is actually well-behaved at the boundary. The divergence of A_3 nevertheless causes issues in the numerical integration of the perturbation equations. For this reason, the accuracy achieved for $n < 1$ is limited compared with the accuracy obtained for $n > 1$.

The internal perturbation must match the external perturbation at $\xi = \xi_1$, or $r = R$, the position of the stellar boundary. The five internal functions e_1, e_4, e_7, e_{10}, and $\xi e'_{10}$ depend on one free parameter a_0. The external functions, on the other hand, depend on three gauge parameters a_1, a_2, and a_3, as well as the electric-type Love number $k_{\rm el}$. The five matching conditions determine the five parameters uniquely, including the Love number.

We suppose that the internal functions $u_1, \cdots, u_5$ are determined by setting $a_0 \equiv 1$ in the numerical integrations. The desired functions $e_1, \cdots, e_{10}$ then differ from these by an overall multiplicative factor that we denote λ^{-1}. We have

$$e_1^{\rm in} = \lambda^{-1} u_1, \tag{5.13a}$$

$$e_4^{\rm in} = \lambda^{-1} u_2, \tag{5.13b}$$

$$e_7^{\rm in} = \lambda^{-1} u_3, \tag{5.13c}$$

$$e_{10}^{\rm in} = \lambda^{-1} u_4, \tag{5.13d}$$

$$\xi \frac{de_{10}^{\rm in}}{d\xi} = \lambda^{-1} u_5, \tag{5.13e}$$

and the matching conditions are

$$e_1^{\rm in} = e_1^{\rm out}, \tag{5.14a}$$

$$e_4^{\rm in} = e_4^{\rm out}, \tag{5.14b}$$

$$e_7^{\rm in} = e_7^{\rm out}, \tag{5.14c}$$

$$e_{10}^{\rm in} = e_{10}^{\rm out}, \tag{5.14d}$$

$$\xi \frac{de_{10}^{\rm in}}{d\xi} = \xi \frac{de_{10}^{\rm out}}{d\xi}, \tag{5.14e}$$

where each side of the equation is evaluated at $\xi = \xi_1$. The external expressions for e_1, e_4, and e_7 are presented in Eqs. (1.6) and (1.7), and these must be modified by the gauge adjustments of Eqs. (3.7).

The function e_{10} is related to K by Eq. (5.10), and the external expression for K is given by Eq. (3.8). This

equation and its derivative with respect to r imply that at $\xi = \xi_1$,

$$e_{10}^{\rm out} = 2(l+2)(l+3)C^l \xi_1^{-2}\big[a_2 + a_3(2M/R)\big], \tag{5.15a}$$

$$\xi \frac{de_{10}^{\rm out}}{d\xi} = -2(l+2)(l+3)C^l \xi_1^{-2}\big[(l+2)a_2 + (l+3)a_3(2M/R)\big], \tag{5.15b}$$

where $C := 2M/R$ is the compactness factor. These equations can be solved for a_2 and a_3. Involving also the matching equations and Eqs. (5.13), we arrive at

$$\lambda a_2 = \frac{\xi_1^2}{2(l+2)(l+3)C^l}\Big[(l+3)u_4 + u_5\Big], \tag{5.16a}$$

$$\lambda a_3 = -\frac{\xi_1^2}{2(l+2)(l+3)C^{l+1}}\Big[(l+2)u_4 + u_5\Big]. \tag{5.16b}$$

We see that the gauge parameters a_2 and a_3, rescaled by the unknown coefficient λ, are determined by the numerical values obtained for u_4 and u_5.

To solve the remaining matching equations we transfer the a_2 and a_3 terms from the right-hand side of Eqs. (3.7) to the left-hand side. Taking Eqs. (5.16) into account, we form the combinations

$$w_1 := u_1 + \frac{C\xi_1^2}{2(l+2)(l+3)}\Big[(l+2)u_4 + u_5\Big], \tag{5.17a}$$

$$w_2 := u_2 - \frac{(l+1)\xi_1^2}{2(l+2)(l+3)}\Big[(l+2)u_4 + u_5\Big], \tag{5.17b}$$

$$w_3 := u_3 - \frac{\xi_1^2}{(l+2)(l+3)}\Big[(l+3)u_4 + u_5\Big], \tag{5.17c}$$

which can be determined numerically. Involving now Eqs. (1.6), the matching conditions take the explicit form

$$w_1 = A_1 \cdot \lambda + 2B_1 \cdot (\lambda k_{\rm el}) - lC \cdot (\lambda C^{l+1}a_1), \tag{5.18a}$$

$$w_2 = A_4 \cdot \lambda - 2\frac{l+1}{l}B_4 \cdot (\lambda k_{\rm el}) + \big[(l-1)(l+2)+2C\big] \cdot (\lambda C^{l+1}a_1), \tag{5.18b}$$

$$w_3 = A_7 \cdot \lambda + 2B_7 \cdot (\lambda k_{\rm el}) + 2l \cdot (\lambda C^{l+1}a_1); \tag{5.18c}$$

in these expressions the functions A_n and B_n are evaluated at $r = R$, or $2M/r = C$.

If we define a vector $\boldsymbol{w} = (w_1, w_2, w_3)$ of numerical quantities, and another vector $\boldsymbol{p} = (\lambda, \lambda k_{\rm rel}, \lambda C^{l+1}a_1)$ of unknown parameters, these equations take the form of the matrix equation $\boldsymbol{w} = \mathsf{M}\boldsymbol{p}$, with a matrix M that is known analytically. Solving for $\boldsymbol{p}$, the Love number is finally determined by $k_{\rm el} = p_2/p_1$.

C. Perturbation: Odd-parity sector

To arrive at the final form of the perturbation equations (4.28) we follow the same steps as in the even-parity sector. These include making the substitutions

of Eqs. (3.15), to replace the original variables h_v and h_2 with the radial functions b_4 and b_7.

The perturbation equations are

$$0 = O_1 = -\xi^2 b_4'' - f^{-1}F_1\xi\, b_4' + f^{-1}G_1\, b_4, \quad (5.19a)$$

$$0 = O_2 = -\xi b_7' - l\, b_7 + le^{-\psi}f^{-1}\, b_4, \quad (5.19b)$$

with

$$F_1 = 2(l+1) - (n+1)b\xi^2\big[4(l+1)\nu + \theta^n(1+b\theta)\big], \quad (5.20a)$$

$$G_1 = (n+1)b\xi^2\big[2(l-1)(l+2)\nu + (l+3)\theta^n(1+b\theta)\big]. \quad (5.20b)$$

A small-ξ expansion of these equations reveals that $b_4 = \alpha_0 + O(\xi^2)$ and $b_7 = \alpha_0 e^{-\psi_0} + O(\xi^2)$, where α_0 is a parameter that must be determined by matching the internal and external perturbations at the stellar boundary.

The perturbation equations are easily written as a first-order dynamical system for the variables $v_1 := b_4$, $v_2 := \xi b_4'$, and $v_3 := b_7$.

The internal perturbation must match the external perturbation at $\xi = \xi_1$, or $r = R$, the position of the stellar boundary. The three internal functions b_4, $\xi b_4'$, and b_7 depend on one free parameter α_0. The external functions, on the other hand, depend on one gauge parameter α as well as the magnetic-type Love number k_{mag}. The three matching conditions determine the three parameters uniquely, including the Love number.

We suppose that the perturbation equations for v_1, v_2, and v_3 are integrated with $\alpha_0 \equiv 1$. The desired internal functions b_4 and b_7 are then given by

$$b_4^{\mathrm{in}} = \lambda^{-1}v_1, \quad (5.21a)$$

$$\xi\frac{db_4^{\mathrm{in}}}{d\xi} = \lambda^{-1}v_2, \quad (5.21b)$$

$$b_7^{\mathrm{in}} = \lambda^{-1}v_3, \quad (5.21c)$$

where λ is an unknown constant. The matching conditions are

$$b_4^{\mathrm{in}} = b_4^{\mathrm{out}}, \quad (5.22a)$$

$$\xi\frac{db_4^{\mathrm{in}}}{d\xi} = \xi\frac{db_4^{\mathrm{out}}}{d\xi}, \quad (5.22b)$$

$$b_7^{\mathrm{in}} = b_7^{\mathrm{out}}, \quad (5.22c)$$

where each side of the equation is evaluated at $\xi = \xi_1$. The external expressions for b_4 and b_7 are presented in Eqs. (1.6), together with the gauge adjustment of Eq. (3.18). We observe that b_4 is gauge-invariant, and that the purpose of the matching equation for b_7 is to determine the (uninteresting) gauge parameter α.

We focus on the two equations involving b_4. Using Eqs. (1.6), we find that the explicit form of the matching conditions is

$$v_1 = A_4 \cdot \lambda - 2\frac{l+1}{l}B_4 \cdot (\lambda k_{\mathrm{mag}}), \quad (5.23a)$$

$$v_2 = -CA_4' \cdot \lambda + 2\frac{l+1}{l}\big[CB_4' + (2l+1)B_4\big] \cdot (\lambda k_{\mathrm{mag}}). \quad (5.23b)$$

In these expressions the functions A_4, $A_4' := dA_4/dz$, B_4, and $B_4' := dB_4/dz$ are evaluated at $z := 2M/r = C$.

If we define a vector $\boldsymbol{v} = (v_1, v_2)$ of numerical quantities, and another vector $\boldsymbol{p} = (\lambda, \lambda k_{\mathrm{mag}})$ of unknown parameters, these equations take the form of the matrix equation $\boldsymbol{v} = \mathsf{M}\,\boldsymbol{p}$, with a matrix M that is known analytically. Solving for $\boldsymbol{p}$, the Love number is finally determined by $k_{\mathrm{mag}} = p_2/p_1$.

To evaluate the derivatives of A_4 and B_4 with respect to z we use the well-known property of hypergeometric functions that $(d/dz)F(a,b;c;z) = (ab/c)F(a+1,b+1;c+1;z)$.

VI. NUMERICAL RESULTS

The computations presented in this work were generated with two independent codes, one written by each author. Consistency between our results provides evidence that each set of computations were carried our correctly, and the comparison allows us to estimate the numerical accuracy of our results.

The background spacetime is constructed by solving the Einstein field equations for a spherical matter configuration with a polytropic equation of state. The equations were formulated in Sec. V A, and the system of equations (5.7) is integrated numerically for selected values of the polytropic index n. The integration begins at a large and negative value of the radial variable $x = \ln\xi$, using the starting values listed in Eqs. (5.8). It proceeds until θ changes sign at the stellar boundary, $x = x_1$. In the first code, the integration is performed using the Bulirsh-Stoer method as implemented in the *Numerical Recipes* routine `bsstep`, which is embedded within `odeint`; we use the Second Edition of *Numerical Recipes* [19], and the code is written in C++. In the second code, the integration is performed using the embedded Runge-Kutta Prince-Dormand method as implemented in the *GNU Scientific Library* routine `rk8pd`, which is embedded within `odeiv`; we use version 1.9 of the libraries [20], and the code is written in C. In each code all floating-point operations are carried out with double precision. The accuracy of the integration is determined by the integrator's tolerance ϵ and the errors of order ξ^6 that are incorporated in the starting values. As Eqs. (5.7) are exceptionally well conditioned toward numerical integration, a high degree of accuracy can easily be achieved. We estimate that our stellar configurations are computed accurately to at least twelve significant digits.

The stellar boundary is identified with the help of a bisection search for the solution to $\theta(x) = 0$. In the first code this is carried out with the *Numerical Recipe* routine `zbrent`; the search is loosely bracketed between the values $x_0 < x_1$ (where θ is positive) and $x_2 > x_1$ (where θ is negative). In the second code this is carried out with

the *GNU Scientific Library* routine `brent`, using a similar bracketing method. The search is carried out with high accuracy, again of the order of twelve significant digits.

The even-parity perturbation equations (5.11) are next integrated for selected values of n and l, simultaneously with the background field equations (5.7). Once more the integration begins at a large and negative value of x, using the starting values derived in Sec. V B, and it proceeds up to $x = x_1$. In the first code we continue to use `bsstep` and `odeint`, and the caption of Table II discusses the accuracy of these integrations. In the second code we continue to use `rk8pd` and `odeiv`; the tolerance of the integrator is set uniformly to $\epsilon = 1.0e{-}12$, and all integrations begin at $x = -10.0$. Each code returns the values of u_1, u_2, u_3, u_4, and u_5 at the stellar boundary.

The odd-parity equations (5.19) are integrated in exactly the same way. Here the codes return the values of v_1, v_2, and v_3 at the stellar boundary.

The matching problem of Eqs. (5.18) requires the numerical solution of the matrix equation $\boldsymbol{w} = \mathsf{M}\,\boldsymbol{p}$, where $\boldsymbol{w}$ is constructed from the perturbations, M is known analytically, and $\boldsymbol{p}$ is the vector of unknown parameters, which include the electric-type Love number $k_{\rm el}$. In the first code the system of equations is solved by performing an LU decomposition of the matrix M, and this is handled by the *Numerical Recipes* routines `ludcmp` and `lubksb`. In the second code the LU decomposition is handled by the *GNU Scientific Library* routines `gsl_linalg_LU_decomp` and `gsl_linalg_LU_solve`. In view of the small number of equations involved (three), this task is essentially carried out at machine precision. The final output is $k_{\rm el}$.

The matching problem of Eqs. (5.23) is handled in exactly the same way. Here the final output is the magnetic-type Love number $k_{\rm mag}$.

Our results are presented in the figures displayed in Sec. I and in the tables provided in the Appendix. The electric-type and magnetic-type Love numbers are computed for selected values of n and l, as functions of the relativistic parameter $b := p_c/\rho_c$ and the compactness $C := 2M/R$. The allowed interval begins at $b = 0$ and $C = 0$, where the equations reduce to their Newtonian limit, and ends at $b = b_{\max}$ and $C = C_{\max}$, where the stellar configuration achieves its maximum mass. Each table caption discusses the estimated accuracy of our results. Overall we claim an approximate accuracy of nine significant digits for the Love numbers (with some exceptions, as detailed in the table captions).

TABLE II: Integration errors for even-parity perturbations. For each selected value of n the first row shows the value of ϵ, the integrator's tolerance. When $\epsilon = 1.0e{-}12$ the integrations are started at $x = -7.0$, so that the errors in the starting values are of the order of 1.0e-12. When $\epsilon > 1.0e{-}12$ the integrations are started at $x = -6.5$, so that the errors in the starting values are of the order of 1.0e-11. For the odd-parity equations the tolerance of the integrator is set uniformly to $\epsilon = 1.0e{-}12$, and all integrations begin at $x = -7.0$. The second column shows δ, an intrinsic measure of the accuracy of our results. This is defined as $\delta := |\nu_{\rm model} - \nu_{\rm pert}|/\nu_{\rm model}$, where $\nu_{\rm model}$ is the value of ν at the stellar boundary $\xi = \xi_1$ as determined with exquisite precision by integrating the stellar-model equations only, while $\nu_{\rm pert}$ is the value as determined by also integrating the perturbation equations. The least accurate determinations are for small values of b; the accuracy typically improves by two orders of magnitude at larger values of b. For reasons that were explained at the end of Sec. IV D, when $n < 1$ the accuracy that can be achieved for the even-parity perturbations is more limited than what is achieved for ν; for these cases δ gives an overestimate of the true accuracy. For $n > 1$, and for the odd-parity perturbations, δ should be an accurate measure of our accuracy.

	$l = 2$	$l = 3$	$l = 4$	$l = 5$
$n = 0.50$	$\epsilon = 1.0e{-}10$	$\epsilon = 1.0e{-}10$	$\epsilon = 1.0e{-}10$	$\epsilon = 1.0e{-}10$
	$\delta < 1.2e{-}10$	$\delta < 1.2e{-}10$	$\delta < 1.2e{-}10$	$\delta < 1.2e{-}10$
$n = 0.75$	$\epsilon = 3.0e{-}11$	$\epsilon = 3.0e{-}11$	$\epsilon = 3.0e{-}11$	$\epsilon = 3.0e{-}11$
	$\delta < 8.6e{-}11$	$\delta < 6.6e{-}11$	$\delta < 8.6e{-}11$	$\delta < 8.6e{-}11$
$n = 1.00$	$\epsilon = 1.0e{-}12$	$\epsilon = 3.0e{-}11$	$\epsilon = 3.0e{-}11$	$\epsilon = 3.0e{-}11$
	$\delta < 1.6e{-}09$	$\delta < 1.7e{-}09$	$\delta < 2.7e{-}10$	$\delta < 4.0e{-}11$
$n = 1.25$	$\epsilon = 1.0e{-}12$	$\epsilon = 3.0e{-}11$	$\epsilon = 3.0e{-}11$	$\epsilon = 3.0e{-}11$
	$\delta < 9.5e{-}11$	$\delta < 9.6e{-}11$	$\delta < 9.5e{-}11$	$\delta < 9.5e{-}11$
$n = 1.50$	$\epsilon = 1.0e{-}12$	$\epsilon = 3.0e{-}11$	$\epsilon = 3.0e{-}11$	$\epsilon = 3.0e{-}11$
	$\delta < 7.2e{-}11$	$\delta < 7.2e{-}11$	$\delta < 7.2e{-}11$	$\delta < 7.2e{-}11$
$n = 1.75$	$\epsilon = 1.0e{-}12$	$\epsilon = 1.0e{-}12$	$\epsilon = 7.0e{-}11$	$\epsilon = 7.0e{-}11$
	$\delta < 9.2e{-}11$	$\delta < 9.2e{-}11$	$\delta < 9.2e{-}11$	$\delta < 9.2e{-}11$
$n = 2.00$	$\epsilon = 1.0e{-}12$	$\epsilon = 1.0e{-}12$	$\epsilon = 1.0e{-}12$	$\epsilon = 3.0e{-}11$
	$\delta < 2.4e{-}12$	$\delta < 2.4e{-}12$	$\delta < 2.4e{-}12$	$\delta < 2.4e{-}12$

APPENDIX: TABLES OF RELATIVISTIC LOVE NUMBERS

TABLE III: Love numbers for $n = 0.50$ and $l = 2$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\rm el} = 4.491539995415$e-01. This provides evidence that our results for the electric-type Love numbers are accurate to five significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	$k_{\rm el}$	$k_{\rm mag}$
0.0000000000	0.0000000000	4.4915295584e-01	0.0000000000e+00
0.0162962963	0.0627859865	3.6857599573e-01	1.6896831556e-03
0.0651851852	0.2085406132	2.2117103643e-01	4.1596525372e-03
0.1466666667	0.3636165454	1.1528493484e-01	4.8522931593e-03
0.2607407407	0.4883414066	6.0195686393e-02	4.2583493395e-03
0.4074074074	0.5772867923	3.3825660571e-02	3.3759103748e-03
0.5866666667	0.6379537736	2.0879830949e-02	2.6271666539e-03
0.7985185185	0.6789539591	1.4107892752e-02	2.0809271632e-03
1.0429629630	0.7068171264	1.0311605798e-02	1.7004161103e-03
1.3200000000	0.7259502382	8.0453742292e-03	1.4372173622e-03

TABLE IV: Love numbers for $n = 0.75$ and $l = 2$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\rm el} = 3.434291771770$e-01. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	$k_{\rm el}$	$k_{\rm mag}$
0.0000000000	0.0000000000	3.4342917761e-01	0.0000000000e+00
0.0092592593	0.0363293144	3.0528456672e-01	8.4958217044e-04
0.0370370370	0.1294393332	2.2114677984e-01	2.5151747439e-03
0.0833333333	0.2455878442	1.4055702918e-01	3.6455944426e-03
0.1481481481	0.3564603549	8.4890119064e-02	3.8700558106e-03
0.2314814815	0.4485200887	5.1708383719e-02	3.5282781348e-03
0.3333333333	0.5194393410	3.2857454025e-02	3.0026963633e-03
0.4537037037	0.5719839827	2.2091374455e-02	2.4979305215e-03
0.5925925926	0.6101589410	1.5755967215e-02	2.0827696087e-03
0.7500000000	0.6376107260	1.1882367812e-02	1.7629062677e-03

TABLE V: Love numbers for $n = 1.00$ and $l = 2$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\rm el} = 2.599088771480$e-01. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	$k_{\rm el}$	$k_{\rm mag}$
0.0000000000	0.0000000000	2.5990887732e-01	0.0000000000e+00
0.0054320988	0.0211887760	2.4198937486e-01	4.1832500500e-04
0.0217283951	0.0788326459	1.9761362790e-01	1.3874544444e-03
0.0488888889	0.1586178173	1.4594601117e-01	2.3418562643e-03
0.0869135802	0.2449940757	1.0135201470e-01	2.9092025152e-03
0.1358024691	0.3264977638	6.8656738911e-02	3.0470778168e-03
0.1955555556	0.3971100356	4.6672564713e-02	2.8932146678e-03
0.2661728395	0.4550360296	3.2438798694e-02	2.6030597661e-03
0.3476543210	0.5008905693	2.3293063321e-02	2.2824210048e-03
0.4400000000	0.5363092473	1.7360105151e-02	1.9854445481e-03

TABLE VI: Love numbers for $n = 1.25$ and $l = 2$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\rm el} = 1.943393766752$e-01. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	$k_{\rm el}$	$k_{\rm mag}$
0.0000000000	0.0000000000	1.9433937665e-01	0.0000000000e+00
0.0037037037	0.0140910881	1.8487046323e-01	2.2908538860e-04
0.0148148148	0.0535218477	1.6007564892e-01	8.0226880316e-04
0.0333333333	0.1109782733	1.2818558203e-01	1.4646634398e-03
0.0592592593	0.1774670027	9.7029969481e-02	1.9894582440e-03
0.0925925926	0.2449789236	7.1049247827e-02	2.2762232723e-03
0.1333333333	0.3079051472	5.1376476586e-02	2.3393293198e-03
0.1814814815	0.3631764781	3.7288438980e-02	2.2472291456e-03
0.2370370370	0.4096934360	2.7477497915e-02	2.0722703848e-03
0.3000000000	0.4475972736	2.0708768325e-02	1.8682931012e-03

TABLE VII: Love numbers for $n = 1.50$ and $l = 2$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 1.432787706403$e-01. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	1.4327877058e-01	0.0000000000e+00
0.0025925926	0.0095061309	1.3824519472e-01	1.2516879778e-04
0.0103703704	0.0366144717	1.2455723004e-01	4.5461861960e-04
0.0233333333	0.0775387088	1.0568541028e-01	8.7668437452e-04
0.0414814815	0.1272141307	8.5491826720e-02	1.2719049787e-03
0.0648148148	0.1805262693	6.6864122654e-02	1.5603334794e-03
0.0933333333	0.2332124569	5.1266906792e-02	1.7154390472e-03
0.1270370370	0.2822627839	3.9011593697e-02	1.7510464094e-03
0.1659259259	0.3259042474	2.9758544257e-02	1.6998078860e-03
0.2100000000	0.3633567807	2.2929142045e-02	1.5963449703e-03

TABLE IX: Love numbers for $n = 2.00$ and $l = 2$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 7.393839192094$e-02. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	7.3938391925e-02	0.0000000000e+00
0.0013580247	0.0044806121	7.2500928560e-02	3.6722504616e-05
0.0054320988	0.0175403971	6.8415857342e-02	1.3908592972e-04
0.0122222222	0.0381005822	6.2293615431e-02	2.8628982295e-04
0.0217283951	0.0645673693	5.4948705679e-02	4.5110089827e-04
0.0339506173	0.0950758472	4.7195018104e-02	6.0738825840e-04
0.0488888889	0.1277344111	3.9692062306e-02	7.3574977114e-04
0.0665432099	0.1608206440	3.2876266347e-02	8.2584357166e-04
0.0869135802	0.1929044035	2.6967366952e-02	8.7572897640e-04
0.1100000000	0.2228982181	2.2017664632e-02	8.8947977626e-04

TABLE VIII: Love numbers for $n = 1.75$ and $l = 2$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 1.039154459896$e-01. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	1.0391544596e-01	0.0000000000e+00
0.0018518519	0.0064743298	1.0123582505e-01	6.7882541528e-05
0.0074074074	0.0251788386	9.3755924816e-02	2.5279292539e-04
0.0166666667	0.0541259986	8.2923108986e-02	5.0667193661e-04
0.0296296296	0.0904962838	7.0531274857e-02	7.7147592064e-04
0.0462962963	0.1311832131	5.8177787311e-02	9.9862752282e-04
0.0666666667	0.1732771212	4.6952247447e-02	1.1599057338e-03
0.0907407407	0.2143825635	3.7393783155e-02	1.2480829694e-03
0.1185185185	0.2527489948	2.9616069949e-02	1.2710091094e-03
0.1500000000	0.2872548584	2.3478510254e-02	1.2440214762e-03

TABLE X: Love numbers for $n = 0.50$ and $l = 3$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 2.033844048605$e-01. This provides evidence that our results for the electric-type Love numbers are accurate to five significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	2.0338399420e-01	0.0000000000e+00
0.0162962963	0.0627859865	1.5613095764e-01	7.6806695868e-04
0.0651851852	0.2085406132	7.9298498872e-02	1.5334802180e-03
0.1466666667	0.3636165454	3.3876635518e-02	1.4053484595e-03
0.2607407407	0.4883414066	1.4803140981e-02	1.0040207606e-03
0.4074074074	0.5772867923	7.2690559784e-03	6.8628867282e-04
0.5866666667	0.6379537736	4.0970043659e-03	4.8495681685e-04
0.7985185185	0.6789539591	2.6182028430e-03	3.6226354828e-04
1.0429629630	0.7068171264	1.8555610194e-03	2.8623307831e-04
1.3200000000	0.7259502382	1.4266248200e-03	2.3757226341e-04

TABLE XI: Love numbers for $n = 0.75$ and $l = 3$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\mathrm{el}} = 1.479565910794\text{e-}01$, and this value was copied in the first row of the Table. [We were not able to accurately compute the electric-type Love number for $b = 0$ for these specific values of n and l. The reason has to do with the fact that for these values, $\xi e'_{10} = O(\xi^4)$ instead of being of order ξ^2 near $\xi = 0$; the integrator then has difficulty moving out of the small-ξ region and the number of steps required exceeds the set limit.] We believe that our results for the electric-type Love numbers are accurate to nine significant digits, and that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	1.4795659108e-01	0.0000000000e+00
0.0092592593	0.0363293144	1.2656912474e-01	3.7606006609e-04
0.0370370370	0.1294393332	8.2729118342e-02	9.7442657049e-04
0.0833333333	0.2455878442	4.5802954407e-02	1.1847104807e-03
0.1481481481	0.3564603549	2.3980661301e-02	1.0533665372e-03
0.2314814815	0.4485200887	1.2870072782e-02	8.2451036345e-04
0.3333333333	0.5194393410	7.3958196272e-03	6.2355209034e-04
0.4537037037	0.5719839827	4.6219154497e-03	4.7654009722e-04
0.5925925926	0.6101589410	3.1388009813e-03	3.7510397112e-04
0.7500000000	0.6376107260	2.2971603620e-03	3.0589573877e-04

TABLE XIII: Love numbers for $n = 1.25$ and $l = 3$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\mathrm{el}} = 7.558993098406\text{e-}02$. This provides evidence that our results for the electric-type Love numbers are accurate to eight significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	7.5589930713e-02	0.0000000000e+00
0.0037037037	0.0140910881	7.0841000307e-02	9.1484438895e-05
0.0148148148	0.0535218477	5.8788456530e-02	3.0220575213e-04
0.0333333333	0.1109782733	4.4167684362e-02	5.0553940166e-04
0.0592592593	0.1774670027	3.0965107128e-02	6.1848609429e-04
0.0925925926	0.2449789236	2.0910331154e-02	6.3402270724e-04
0.1333333333	0.3079051472	1.3985888466e-02	5.8644352242e-04
0.1814814815	0.3631764781	9.4660656773e-03	5.1264600426e-04
0.2370370370	0.4096934360	6.5780656202e-03	4.3641911298e-04
0.3000000000	0.4475972736	4.7331767581e-03	3.6881598982e-04

TABLE XII: Love numbers for $n = 1.00$ and $l = 3$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\mathrm{el}} = 1.064540469774\text{e-}01$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	1.0645404707e-01	0.0000000000e+00
0.0054320988	0.0211887760	9.6920315090e-02	1.7712436105e-04
0.0217283951	0.0788326459	7.4354157385e-02	5.4046973625e-04
0.0488888889	0.1586178173	5.0164622016e-02	8.0927197289e-04
0.0869135802	0.2449940757	3.1412814100e-02	8.7805794150e-04
0.1358024691	0.3264977638	1.9197799362e-02	8.0536598779e-04
0.1955555556	0.3971100356	1.1894296947e-02	6.7961132670e-04
0.2661728395	0.4550360296	7.6520308738e-03	5.5466172884e-04
0.3476543210	0.5008905693	5.1742095921e-03	4.5069370183e-04
0.4400000000	0.5363092473	3.6915952107e-03	3.7043176559e-04

TABLE XIV: Love numbers for $n = 1.50$ and $l = 3$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\mathrm{el}} = 5.284852444148\text{e-}02$. This provides evidence that our results for the electric-type Love numbers are accurate to eight significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	5.2848524127e-02	0.0000000000e+00
0.0025925926	0.0095061309	5.0478328434e-02	4.6818602757e-05
0.0103703704	0.0366144717	4.4172676367e-02	1.6314544473e-04
0.0233333333	0.0775387088	3.5833784527e-02	2.9516806482e-04
0.0414814815	0.1272141307	2.7409928071e-02	3.9553177536e-04
0.0648148148	0.1805262693	2.0154523547e-02	4.4448482790e-04
0.0933333333	0.2332124569	1.4516418814e-02	4.4702358096e-04
0.1270370370	0.2822627839	1.0411153801e-02	4.1916508131e-04
0.1659259259	0.3259042474	7.5326749926e-03	3.7680447176e-04
0.2100000000	0.3633567807	5.5503018177e-03	3.3107508747e-04

TABLE XV: Love numbers for $n = 1.75$ and $l = 3$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 3.628620386492e\text{-}02$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	3.6286203851e-02	0.0000000000e+00
0.0018518519	0.0064743298	3.5106196263e-02	2.3637643620e-05
0.0074074074	0.0251788386	3.1863068261e-02	8.5448967297e-05
0.0166666667	0.0541259986	2.7304450667e-02	1.6345757744e-04
0.0296296296	0.0904962838	2.2304606782e-02	2.3443830054e-04
0.0462962963	0.1311832131	1.7570120169e-02	2.8338116595e-04
0.0666666667	0.1732771212	1.3509063981e-02	3.0609684764e-04
0.0907407407	0.2143825635	1.0255357455e-02	3.0629942762e-04
0.1185185185	0.2527489948	7.7654175911e-03	2.9105710804e-04
0.1500000000	0.2872548584	5.9143273246e-03	2.6737047585e-04

TABLE XVII: Love numbers for $n = 0.50$ and $l = 4$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 1.250625809919e\text{-}01$. This provides evidence that our results for the electric-type Love numbers are accurate to six significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	1.2506232752e-01	0.0000000000e+00
0.0162962963	0.0627859865	8.9880099035e-02	4.1259713417e-04
0.0651851852	0.2085406132	3.8670819375e-02	6.8285490951e-04
0.1466666667	0.3636165454	1.3511973834e-02	4.9948942099e-04
0.2607407407	0.4883414066	4.9038796705e-03	2.9045885185e-04
0.4074074074	0.5772867923	2.0751263151e-03	1.6878093386e-04
0.5866666667	0.6379537736	1.0476762526e-03	1.0600793048e-04
0.7985185185	0.6789539591	6.1942259351e-04	7.2980383110e-05
1.0429629630	0.7068171264	4.1610976670e-04	5.4565032992e-05
1.3200000000	0.7259502382	3.0850496903e-04	4.3643093032e-05

TABLE XVI: Love numbers for $n = 2.00$ and $l = 3$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 2.439399851849e\text{-}02$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	2.4393998521e-02	0.0000000000e+00
0.0013580247	0.0044806121	2.3804409363e-02	1.1834173013e-05
0.0054320988	0.0175403971	2.2147566853e-02	4.3858418141e-05
0.0122222222	0.0381005822	1.9717948676e-02	8.7207170874e-05
0.0217283951	0.0645673693	1.6892301431e-02	1.3133222716e-04
0.0339506173	0.0950758472	1.4022630411e-02	1.6767660885e-04
0.0488888889	0.1277344111	1.1366324623e-02	1.9163622562e-04
0.0665432099	0.1608206440	9.0665499267e-03	2.0252062134e-04
0.0869135802	0.1929044035	7.1697064326e-03	2.0229022040e-04
0.1100000000	0.2228982181	5.6581213050e-03	1.9405485402e-04

TABLE XVIII: Love numbers for $n = 0.75$ and $l = 4$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 8.731859904775e\text{-}02$. This provides evidence that our results for the electric-type Love numbers are accurate to eight significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	8.7318599147e-02	0.0000000000e+00
0.0092592593	0.0363293144	7.1916151340e-02	1.9820540551e-04
0.0370370370	0.1294393332	4.2448736557e-02	4.5682358629e-04
0.0833333333	0.2455878442	2.0467648174e-02	4.7418211732e-04
0.1481481481	0.3564603549	9.2636229108e-03	3.5718798449e-04
0.2314814815	0.4485200887	4.3517890992e-03	2.4045653422e-04
0.3333333333	0.5194393410	2.2374880023e-03	1.6046670110e-04
0.4537037037	0.5719839827	1.2812234382e-03	1.1123202446e-04
0.5925925926	0.6101589410	8.1468581955e-04	8.1394894614e-05
0.7500000000	0.6376107260	5.6833824831e-04	6.2954932476e-05

TABLE XIX: Love numbers for $n = 1.00$ and $l = 4$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\text{el}} = 6.024125532418\text{e-}02$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	6.0241255395e-02	0.0000000000e+00
0.0054320988	0.0211887760	5.3646913671e-02	9.0169742307e-05
0.0217283951	0.0788326459	3.8686876721e-02	2.5599084572e-04
0.0488888889	0.1586178173	2.3850649236e-02	3.4511609518e-04
0.0869135802	0.2449940757	1.3456371026e-02	3.3171170926e-04
0.1358024691	0.3264977638	7.3994944976e-03	2.6913831658e-04
0.1955555556	0.3971100356	4.1569144571e-03	2.0272875503e-04
0.2661728395	0.4550360296	2.4561332329e-03	1.4990535283e-04
0.3476543210	0.5008905693	1.5478883581e-03	1.1224767948e-04
0.4400000000	0.5363092473	1.0442398816e-03	8.6440104267e-05

TABLE XXI: Love numbers for $n = 1.50$ and $l = 4$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\text{el}} = 2.739306738271\text{e-}02$. This provides evidence that our results for the electric-type Love numbers are accurate to eight significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	2.7393067294e-02	0.0000000000e+00
0.0025925926	0.0095061309	2.5904136864e-02	2.1590905746e-05
0.0103703704	0.0366144717	2.2022860127e-02	7.2661900733e-05
0.0233333333	0.0775387088	1.7085103298e-02	1.2456848415e-04
0.0414814815	0.1272141307	1.2356917233e-02	1.5600796417e-04
0.0648148148	0.1805262693	8.5360694947e-03	1.6255526538e-04
0.0933333333	0.2332124569	5.7656653837e-03	1.5121350651e-04
0.1270370370	0.2822627839	3.8858798086e-03	1.3142526538e-04
0.1659259259	0.3259042474	2.6550241954e-03	1.1010882007e-04
0.2100000000	0.3633567807	1.8598572889e-03	9.0858200272e-05

TABLE XX: Love numbers for $n = 1.25$ and $l = 4$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\text{el}} = 4.096746123839\text{e-}02$. This provides evidence that our results for the electric-type Love numbers are accurate to eight significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	4.0967461120e-02	0.0000000000e+00
0.0037037037	0.0140910881	3.7831532658e-02	4.4448609345e-05
0.0148148148	0.0535218477	3.0102008525e-02	1.3969326868e-04
0.0333333333	0.1109782733	2.1224824306e-02	2.1676154291e-04
0.0592592593	0.1774670027	1.3778277197e-02	2.4208766237e-04
0.0925925926	0.2449789236	8.5688261330e-03	2.2508402313e-04
0.1333333333	0.3079051472	5.2857027758e-03	1.8910615297e-04
0.1814814815	0.3631764781	3.3202258257e-03	1.5123549667e-04
0.2370370370	0.4096934360	2.1611153793e-03	1.1903913573e-04
0.3000000000	0.4475972736	1.4718380690e-03	9.4141547749e-05

TABLE XXII: Love numbers for $n = 1.75$ and $l = 4$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\text{el}} = 1.795919608352\text{e-}02$. This provides evidence that our results for the electric-type Love numbers are accurate to eight significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	1.7959195798e-02	0.0000000000e+00
0.0018518519	0.0064743298	1.7256414891e-02	1.0299492462e-05
0.0074074074	0.0251788386	1.5352372303e-02	3.6329038871e-05
0.0166666667	0.0541259986	1.2748788134e-02	6.6858496307e-05
0.0296296296	0.0904962838	1.0001909468e-02	9.1232683035e-05
0.0462962963	0.1311832131	7.5212893015e-03	1.0412456086e-04
0.0666666667	0.1732771212	5.5036193453e-03	1.0576629570e-04
0.0907407407	0.2143825635	3.9751203447e-03	9.9452555069e-05
0.1185185185	0.2527489948	2.8695308743e-03	8.8982268992e-05
0.1500000000	0.2872548584	2.0913685076e-03	7.7283983050e-05

TABLE XXIII: Love numbers for $n = 2.00$ and $l = 4$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 1.150774963254e\text{-}02$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	1.1507749634e-02	0.0000000000e+00
0.0013580247	0.0044806121	1.1175986242e-02	4.8517346178e-06
0.0054320988	0.0175403971	1.0253199166e-02	1.7666877492e-05
0.0122222222	0.0381005822	8.9267030374e-03	3.4155212233e-05
0.0217283951	0.0645673693	7.4272319821e-03	4.9576781437e-05
0.0339506173	0.0950758472	5.9573506314e-03	6.0605384193e-05
0.0488888889	0.1277344111	4.6508432777e-03	6.6035419880e-05
0.0665432099	0.1608206440	3.5682657166e-03	6.6396107161e-05
0.0869135802	0.1929044035	2.7150400670e-03	6.3100849384e-05
0.1100000000	0.2228982181	2.0653404463e-03	5.7696936776e-05

TABLE XXV: Love numbers for $n = 0.75$ and $l = 5$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 5.904211079675e\text{-}02$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	5.9042110830e-02	0.0000000000e+00
0.0092592593	0.0363293144	4.6830270753e-02	1.1653809273e-04
0.0370370370	0.1294393332	2.4976158762e-02	2.4051097139e-04
0.0833333333	0.2455878442	1.0491664768e-02	2.1488760292e-04
0.1481481481	0.3564603549	4.1030598266e-03	1.3812306450e-04
0.2314814815	0.4485200887	1.6845383073e-03	8.0340315047e-05
0.3333333333	0.5194393410	7.7279612957e-04	4.7382520827e-05
0.4537037037	0.5719839827	4.0394347591e-04	2.9755158176e-05
0.5925925926	0.6101589410	2.3943047563e-04	2.0179431020e-05
0.7500000000	0.6376107260	1.5846629836e-04	1.4743433223e-05

TABLE XXIV: Love numbers for $n = 0.50$ and $l = 5$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 8.758378097872e\text{-}02$. This provides evidence that our results for the electric-type Love numbers are accurate to five significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	8.7583597477e-02	0.0000000000e+00
0.0162962963	0.0627859865	5.8953726923e-02	2.4566625412e-04
0.0651851852	0.2085406132	2.1502135233e-02	3.4017649539e-04
0.1466666667	0.3636165454	6.1438638016e-03	2.0040145035e-04
0.2607407407	0.4883414066	1.8484475878e-03	9.5309205703e-05
0.4074074074	0.5772867923	6.7146633825e-04	4.7055290188e-05
0.5866666667	0.6379537736	3.0201770900e-04	2.6143552436e-05
0.7985185185	0.6789539591	1.6415053975e-04	1.6470651693e-05
1.0429629630	0.7068171264	1.0383693027e-04	1.1563913004e-05
1.3200000000	0.7259502382	7.3777009618e-05	8.8490411630e-06

TABLE XXVI: Love numbers for $n = 1.00$ and $l = 5$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 3.929250022713e\text{-}02$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	3.9292500283e-02	0.0000000000e+00
0.0054320988	0.0211887760	3.4232186798e-02	5.1595279179e-05
0.0217283951	0.0788326459	2.3214946910e-02	1.3691347868e-04
0.0488888889	0.1586178173	1.3083821677e-02	1.6725789364e-04
0.0869135802	0.2449940757	6.6517658359e-03	1.4339451212e-04
0.1358024691	0.3264977638	3.2897043734e-03	1.0355791045e-04
0.1955555556	0.3971100356	1.6738586645e-03	6.9962844411e-05
0.2661728395	0.4550360296	9.0663862820e-04	4.7012138969e-05
0.3476543210	0.5008905693	5.3118742797e-04	3.2481406675e-05
0.4400000000	0.5363092473	3.3782253209e-04	2.3432821346e-05

TABLE XXVII: Love numbers for $n = 1.25$ and $l = 5$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\mathrm{el}} = 2.574776897544\mathrm{e}{-}02$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	2.5747768892e-02	0.0000000000e+00
0.0037037037	0.0140910881	2.3431732289e-02	2.4484594391e-05
0.0148148148	0.0535218477	1.7882343045e-02	7.3469913671e-05
0.0333333333	0.1109782733	1.1838067300e-02	1.0630025814e-04
0.0592592593	0.1774670027	7.1175929856e-03	1.0903958368e-04
0.0925925926	0.2449789236	4.0764785256e-03	9.2520830998e-05
0.1333333333	0.3079051472	2.3180122367e-03	7.1006099307e-05
0.1814814815	0.3631764781	1.3500205542e-03	5.2196651860e-05
0.2370370370	0.4096934360	8.2182992059e-04	3.8121417411e-05
0.3000000000	0.4475972736	5.2873946327e-04	2.8280143688e-05

TABLE XXX: Love numbers for $n = 2.00$ and $l = 5$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\mathrm{el}} = 6.419966834096\mathrm{e}{-}03$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	6.4199668350e-03	0.0000000000e+00
0.0013580247	0.0044806121	6.2054683813e-03	2.3273253180e-06
0.0054320988	0.0175403971	5.6146736603e-03	8.3410581918e-06
0.0122222222	0.0381005822	4.7814250834e-03	1.5722362491e-05
0.0217283951	0.0645673693	3.8647546556e-03	2.2075823575e-05
0.0339506173	0.0950758472	2.9960238568e-03	2.5950234182e-05
0.0488888889	0.1277344111	2.2531557240e-03	2.7084179213e-05
0.0665432099	0.1608206440	1.6628187985e-03	2.6037432070e-05
0.0869135802	0.1929044035	1.2172373651e-03	2.3660856250e-05
0.1100000000	0.2228982181	8.9227889200e-04	2.0720823774e-05

TABLE XXVIII: Love numbers for $n = 1.50$ and $l = 5$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\mathrm{el}} = 1.656876321404\mathrm{e}{-}02$. This provides evidence that our results for the electric-type Love numbers are accurate to eight significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	1.6568763135e-02	0.0000000000e+00
0.0025925926	0.0095061309	1.5513747923e-02	1.1396769321e-05
0.0103703704	0.0366144717	1.2817102827e-02	3.7144252136e-05
0.0233333333	0.0775387088	9.5123354123e-03	6.0593529815e-05
0.0414814815	0.1272141307	6.5071687257e-03	7.1292599566e-05
0.0648148148	0.1805262693	4.2236890233e-03	6.9268626197e-05
0.0933333333	0.2332124569	2.6751245641e-03	5.9938429831e-05
0.1270370370	0.2822627839	1.6935347048e-03	4.8543637402e-05
0.1659259259	0.3259042474	1.0918134805e-03	3.8082191743e-05
0.2100000000	0.3633567807	7.2625290353e-04	2.9627611402e-05

TABLE XXIX: Love numbers for $n = 1.75$ and $l = 5$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\mathrm{el}} = 1.043995446810\mathrm{e}{-}02$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.0000000000	0.0000000000	1.0439954387e-02	0.0000000000e+00
0.0018518519	0.0064743298	9.9635882364e-03	5.1907364491e-06
0.0074074074	0.0251788386	8.6906171854e-03	1.7903644602e-05
0.0166666667	0.0541259986	6.9954658202e-03	3.1806852682e-05
0.0296296296	0.0904962838	5.2724767209e-03	4.1473311159e-05
0.0462962963	0.1311832131	3.7857836403e-03	4.4914541070e-05
0.0666666667	0.1732771212	2.6366800282e-03	4.3129384841e-05
0.0907407407	0.2143825635	1.8116921836e-03	3.8310103373e-05
0.1185185185	0.2527489948	1.2463184219e-03	3.2437461311e-05
0.1500000000	0.2872548584	8.6864853073e-04	2.6760732768e-05

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