Gravito-electromagnetism

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Abstract.

We develop and apply a fully covariant $1 + 3$ electromagnetic analogy for gravity. The free gravitational field is covariantly characterized by the Weyl gravito-electric and gravito-magnetic spatial tensor fields, whose dynamical equations are the Bianchi identities. Using a covariant generalization of spatial vector algebra and calculus to spatial tensor fields, we exhibit the covariant analogy between the tensor Bianchi equations and the vector Maxwell equations. We identify gravitational source terms, couplings and potentials with and without electromagnetic analogues. The nonlinear vacuum Bianchi equations are shown to be invariant under covariant spatial duality rotation of the gravito-electric and gravito-magnetic tensor fields. We construct the super-energy density and super-Poynting vector of the gravitational field as natural $U(1)$ group invariants, and derive their super-energy conservation equation. A covariant approach to gravito-electric/magnetic monopoles is also presented.

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1. Introduction

There is a surprisingly rich and detailed correspondence between electromagnetism and General Relativity, uncovered in a series of fundamental papers by Bel [1], Penrose [2] and others [3, 4, 5, 6, 7, 8, 9] (see [9, 10] for more references), and further developed recently (see, e.g., [11, 12, 13, 14, 15, 16, 17, 18, 19]). This correspondence is reflected in the Maxwell-like form of the gravitational field tensor (the Weyl tensor), the super-energy-momentum tensor (the Bel-Robinson tensor) and the dynamical equations (the

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Bianchi identities). Another form of the correspondence arises in the search for geons (localized, non-singular, topological solutions of Einstein's field equations with mass and angular momentum): in the known (approximate) solutions, the geometry of the electromagnetic geon is identical to that of the gravitational geon [20, 21].

Here we pursue the 'electromagnetic' properties of gravity in areas which have already proved useful for extensions of electromagnetism to non-Abelian gauge theories and string theory. Our emphasis is on a $1 + 3$ covariant, physically transparent, and non-perturbative approach, with the gravito-electric and gravito-magnetic spatial tensor fields as the fundamental physical variables. Using an improved covariant formalism, including a covariant generalization to spatial tensors of spatial vector algebra and calculus, we show in detailed and transparent form the correspondence between the electric/ magnetic parts of the gravitational field and of the Maxwell field. We identify gravitational source terms, couplings and potentials with and without electromagnetic analogues, thus providing further physical insight into the role of the kinematic quantities shear, vorticity and four-acceleration.

In the vacuum case, we show that the nonlinear (non-perturbative) Bianchi equations for the gravito-electric and gravito-magnetic fields are invariant under covariant spatial duality rotations, in exact analogy with the source-free Maxwell equations for the electric and magnetic fields. The analogy is of course limited by the fact that the Maxwell field propagates on a given spacetime, whereas the gravitational field itself generates the spacetime. The electromagnetic vectors fully characterize a Maxwell solution, and duality maps Maxwell solutions into Maxwell solutions. The gravito-electric/magnetic tensors are not sufficient to characterize covariantly a solution of Einstein's equations – one needs also the kinematic quantities which are subject to the Ricci identities [9]. Duality is an invariance only of the Bianchi identities, and not the Ricci identities, so that it does not map Einstein solutions into Einstein solutions. Nevertheless, the covariant gravito-electric/magnetic duality reveals important properties of the gravitational field.

The covariant $1 + 3$ duality has not to our knowledge been given before. Although duality invariance follows implicitly from Penrose's spinor formalism [2, 22], this is in terms of the 4-dimensional Weyl spinor, rather than its $1+3$ electric and magnetic tensor parts. Four-dimensional covariant tensor approaches to the electromagnetic analogy have been developed (see e.g. [12]), and non-covariant linearized Maxwell-type equations are well established, both in terms of gravito-electromagnetic vectors (see e.g. [23, 24]) and tensors (see e.g. [8]). In [25], a covariant and nonlinear vector approach is developed for stationary spacetimes. Our approach is fully covariant and non-perturbative, and in addition is centred on the gravito-electromagnetic spatial tensor fields, allowing for a more direct and transparent interpretation based on the Maxwell vector analogy. This approach is a development of the work by Trümper [5], Hawking [7] and Ellis [9], and is

related to recent work on a covariant approach to gravitational waves [16, 17, 18, 19] and to local freedom in the gravitational field [19]. A shadow of our general duality result arises in linearized gravitational wave theory, where for vacuum or de Sitter spacetime, there is an interchange symmetry between the gravito-electric and -magnetic tensors [18].

Duality invariance has important implications in field theory in general. It was essentially this symmetry of the Abelian theory, and attempts to extend it to include matter, which led to the Montonen-Olive electromagnetic duality conjecture that there exists a group transformation mapping electric monopoles into magnetic monopoles within the framework of a non-Abelian (specifically $SU(2)$) gauge theory [26]. This conjecture has proved particularly fruitful, stimulating work on S, T and U dualities in string theory (see e.g. [27, 28, 29]), the extension of the electromagnetic duality to magnetically charged black holes and nonlinear electrodynamics [31, 32] and leading to the Seiberg-Witten proof of quark confinement in supersymmetric Yang-Mills theory via monopole condensation [30].

We use the covariant spatial duality to find the gravitational super-energy density and super-Poynting vector as natural group invariants, and derive a new covariant super-energy conservation equation. Finally, we discuss gravito-electric/magnetic monopoles, providing a covariant characterization, in contrast to previous non-covariant treatments [33, 34, 24]. In the linearized case, we show that the Taub-NUT gravito-magnetic monopole given in [33] is related to the Schwarzschild gravito-electric monopole by a spatial duality rotation and an interchange of four-acceleration and vorticity. This provides a covariant form of the relation previously given in non-covariant approaches [35, 36]. It is well-known that the NUT metrics may be obtained from the Schwarzschild metric via the Ehlers-Geroch transformation [35]. This transformation is in fact the generator of T-duality in string theory, but it is not a duality transformation in the sense described here, since it maps Einstein solutions to Einstein solutions and thus necessarily involves kinematic and geometric conditions in addition to a duality rotation. Furthermore, the Ehlers-Geroch transformation requires the existence of a Killing vector field, whereas the general duality that we present does not require any spacetime symmetry.

2. Covariant spatial vector and tensor calculus

To elaborate the electromagnetic properties of the free gravitational field in General Relativity, we first present the required covariant formalism, which is based on [37], a streamlined and extended version of the Ehlers-Ellis $1 + 3$ formalism [9]. Then we give the covariant form of the Maxwell spatial duality in a general curved spacetime. In the following section we extend the treatment to the gravitational field.

Given a congruence of observers with four-velocity field u^a, then $h_{ab} = g_{ab} + u_a u_b$ projects into the local rest spaces, where g_{ab} is the spacetime metric.† The spatially projected part of a vector is

$$V_{\langle a \rangle} = h_a{}^b V_b \,,$$

and the spatially projected, symmetric and tracefree part of a rank-2 tensor is

$$A_{\langle ab \rangle} = h_{(a}{}^c h_{b)}{}^d A_{cd} - \tfrac{1}{3} h_{cd} A^{cd} h_{ab} \,.$$

The spatial alternating tensor is

$$\varepsilon_{abc} = \eta_{abcd} u^d = \varepsilon_{[abc]} \,,$$

where $\eta_{abcd} = \eta_{[abcd]}$ is the spacetime alternating tensor. Any spatial rank-2 tensor has the covariant irreducible decomposition:

$$A_{ab} = \tfrac{1}{3} h_{cd} A^{cd} h_{ab} + A_{\langle ab \rangle} + \varepsilon_{abc} A^c \,,$$

where

$$A_a = \tfrac{1}{2} \varepsilon_{abc} A^{[bc]}$$

is the vector that is the spatial dual to the skew part. Thus the skew part of a spatial tensor is vectorial, and the irreducibly tensor part is symmetric. In the $1 + 3$ covariant approach [9, 19], all physical and geometric variables split into scalars, spatial vectors or spatial tensors that satisfy $A_{ab} = A_{\langle ab \rangle}$. From now on, all rank-2 spatial tensors will be assumed to satisfy this condition.

The covariant spatial vector product is

$$[V, W]_a = \varepsilon_{abc} V^b W^c \,,$$

and the covariant generalization to spatial tensors is

$$[A, B]_a = \varepsilon_{abc} A^b{}_d B^{cd} \,,$$

which is the vector that is spatially dual to the covariant tensor commutator. The covariant time derivative is

$$\dot{A}^{a\cdots}{}_{b\cdots} = u^c \nabla_c A^{a\cdots}{}_{b\cdots} \,,$$

and the covariant spatial derivative is

$$\mathrm{D}_a A^{b\cdots}{}_{c\cdots} = h_a{}^p h^b{}_q \cdots h_c{}^r \cdots \nabla_p A^{q\cdots}{}_{r\cdots} \,.$$

† We follow the notation and conventions of [9, 37]. (Square) round brackets enclosing indices denote (anti-) symmetrization, while angled brackets denote the spatially projected, symmetric and tracefree part; $a, b, \cdots$ are spacetime indices.

Then the covariant spatial divergence and curl of vectors and rank-2 tensors are defined by [37, 19]:

$$\mathrm{div}\,V = \mathrm{D}^a V_a\,, \qquad \mathrm{curl}\,V_a = \varepsilon_{abc}\mathrm{D}^b V^c\,, \tag{1}$$

$$(\mathrm{div}\,A)_a = \mathrm{D}^b A_{ab}\,, \qquad \mathrm{curl}\,A_{ab} = \varepsilon_{cd(a}\mathrm{D}^c A_{b)}{}^d\,, \tag{2}$$

where $\mathrm{curl}\,A_{ab}$ is tracefree if $A_{ab} = A_{(ab)}$. The tensor curl and divergence are related by

$$\varepsilon_{abc}\mathrm{D}^b A_d{}^c = \mathrm{curl}\,A_{ad} + \tfrac{1}{2}\varepsilon_{adc}\mathrm{D}_b A^{bc}\,.$$

The kinematics of the u^a-congruence are described by the expansion $\Theta = \mathrm{D}^a u_a$, the shear $\sigma_{ab} = \mathrm{D}_{\langle a}u_{b\rangle}$, the vorticity $\omega_a = -\tfrac{1}{2}\mathrm{curl}\,u_a$, and the four-acceleration $\dot{u}_a = \dot{u}_{\langle a\rangle}$.

The above operators obey the covariant identities

$$(\mathrm{D}_a f)^{\cdot} \quad = \mathrm{D}_a \dot{f} - \tfrac{1}{3}\Theta \mathrm{D}_a f + \dot{u}_a \dot{f} - \sigma_a{}^b \mathrm{D}_b f - [\omega, \mathrm{D}f]_a + u_a \dot{u}^b \mathrm{D}_b f\,, \tag{3}$$

$$\mathrm{curl}\,\mathrm{D}_a f \quad = -2\dot{f}\omega_a\,, \tag{4}$$

$$\mathrm{D}^a[V,W]_a = W^a \mathrm{curl}\,V_a - V^a \mathrm{curl}\,W_a\,, \tag{5}$$

$$\mathrm{D}^a[A,B]_a = B^{ab}\mathrm{curl}\,A_{ab} - A^{ab}\mathrm{curl}\,B_{ab}\,, \tag{6}$$

together with far more complicated identities [37, 19]. In the case where spacetime is almost spatially isotropic and homogeneous, i.e. a linearized perturbation of a Friedmann-Lemaitre-Robertson-Walker (FLRW) background, some of the main further identities take the linearized form [38, 39]

$$(\mathrm{D}^a V_a)^{\cdot} \quad \approx \mathrm{D}^a \dot{V}_a - H\mathrm{D}^a V_a\,, \tag{7}$$

$$\left(\mathrm{D}^b A_{ab}\right)^{\cdot} \quad \approx \mathrm{D}^b \dot{A}_{ab} - H\mathrm{D}^b A_{ab}\,, \tag{8}$$

$$(\mathrm{curl}\,V_a)^{\cdot} \quad \approx \mathrm{curl}\,\dot{V}_a - H\mathrm{curl}\,V_a\,, \tag{9}$$

$$(\mathrm{curl}\,A_{ab})^{\cdot} \approx \mathrm{curl}\,\dot{A}_{ab} - H\mathrm{curl}\,A_{ab}\,, \tag{10}$$

$$\mathrm{D}^a \mathrm{curl}\,V_a \quad \approx 0\,, \tag{11}$$

$$\mathrm{D}^b \mathrm{curl}\,A_{ab} \quad \approx \tfrac{1}{2}\mathrm{curl}\left(\mathrm{D}^b A_{ab}\right)\,, \tag{12}$$

$$\mathrm{curl}\,\mathrm{curl}\,V_a \quad \approx -\mathrm{D}^2 V_a + \mathrm{D}_a\left(\mathrm{D}^b V_b\right) + \tfrac{2}{3}\left(\rho - 3H^2\right)V_a\,, \tag{13}$$

$$\mathrm{curl}\,\mathrm{curl}\,A_{ab} \approx -\mathrm{D}^2 A_{ab} + \tfrac{3}{2}\mathrm{D}_{\langle a}\mathrm{D}^c A_{b\rangle c} + \left(\rho - 3H^2\right)A_{ab}\,, \tag{14}$$

where H is the background Hubble rate, ρ is the background energy density and $\mathrm{D}^2 = \mathrm{D}^a \mathrm{D}_a$ is the covariant Laplacian.

The electric and magnetic fields measured by u^a observers are defined via the Maxwell tensor F_{ab} by

$$E_a = F_{ab}u^b = E_{\langle a\rangle}\,, \quad H_a = \tfrac{1}{2}\varepsilon_{abc}F^{bc} \equiv {}^*F_{ab}u^b = H_{\langle a\rangle}\,, \tag{15}$$

where * denotes the dual. These spatial physically measurable vectors are equivalent to the spacetime Maxwell tensor, since

$$F_{ab} = 2u_{[a}E_{b]} + \varepsilon_{abc}H^c\,. \tag{16}$$

Maxwell's equations $\nabla_{[a}F_{bc]} = 0$ and $\nabla^b F_{ab} = J_a$ are given in $1 + 3$ covariant form for E_a and H_a by Ellis [40]. In the streamlined formalism, these equations take the simplified form

$$\mathrm{D}^a E_a \qquad = -2\omega^a H_a + \varrho \,, \tag{17}$$

$$\mathrm{D}^a H_a \qquad = 2\omega^a E_a \,, \tag{18}$$

$$\dot{E}_{\langle a\rangle} - \operatorname{curl} H_a = -\tfrac{2}{3}\Theta E_a + \sigma_{ab}E^b - [\omega, E]_a + [\dot{u}, H]_a - j_a \,, \tag{19}$$

$$\dot{H}_{\langle a\rangle} + \operatorname{curl} E_a = -\tfrac{2}{3}\Theta H_a + \sigma_{ab}H^b - [\omega, H]_a - [\dot{u}, E]_a \,, \tag{20}$$

where $\varrho = -J_a u^a$ is the electric charge density and $j_a = J_{\langle a\rangle}$ is the electric current. In flat spacetime, relative to an inertial congruence ($\Theta = \dot{u}_a = \omega_a = \sigma_{ab} = 0$), these equations take their familiar non-covariant form.

Introducing the complex electromagnetic spatial vector field $\mathcal{I}_a = E_a + iH_a$, we see that in the source-free case ($J_a = 0$) Maxwell's equations become

$$\mathrm{D}^a \mathcal{I}_a \qquad = 2i\,\omega^a \mathcal{I}_a \,, \tag{21}$$

$$\dot{\mathcal{I}}_{\langle a\rangle} + i\operatorname{curl}\mathcal{I}_a = -\tfrac{2}{3}\Theta\mathcal{I}_a + \sigma_{ab}\mathcal{I}^b - [\omega, \mathcal{I}]_a - i\,[\dot{u}, \mathcal{I}]_a \,. \tag{22}$$

It follows that the source-free Maxwell equations in an arbitrary curved spacetime, relative to an arbitrary congruence of observers, are invariant under the covariant global spatial duality rotation $\mathcal{I}_a \rightarrow e^{i\phi}\mathcal{I}_a$, where ϕ is constant. The energy density and Poynting vector

$$U = \tfrac{1}{2}\mathcal{I}^a\overline{\mathcal{I}}_a = \tfrac{1}{2}(E_a E^a + H_a H^a) \,, \tag{23}$$

$$P_a = \frac{1}{2i}[\overline{\mathcal{I}}, \mathcal{I}]_a = [E, H]_a \,, \tag{24}$$

are natural group invariants. Their invariance also follows from the duality invariance of the energy-momentum tensor [22, 40]

$$M_a{}^b = \tfrac{1}{2}\left(F_{ac}F^{bc} + {}^*F_{ac}{}^*F^{bc}\right) \,, \tag{25}$$

since $U = M_{ab}u^a u^b$ and $P_a = -M_{\langle a\rangle b}u^b$. Using the identity (5), and the propagation equations (19) and (20), we find a covariant energy conservation equation:

$$\dot{U} + \mathrm{D}^a P_a = -\tfrac{4}{3}\Theta U - 2\dot{u}^a P_a + \sigma_{ab}\left(E^a E^b + H^a H^b\right) \,. \tag{26}$$

This reduces in flat spacetime for inertial observers to the well-known form $\partial_t U + \operatorname{div}\vec{P} = 0$.

A further natural group invariant is

$$\pi_{ab} = -\mathcal{I}_{\langle a}\overline{\mathcal{I}}_{b\rangle} = -E_{\langle a}E_{b\rangle} - H_{\langle a}H_{b\rangle} \,, \tag{27}$$

which is just the anisotropic electromagnetic pressure [40]. It occurs in the last term of the conservation equation (26), i.e. $-\sigma_{ab}\pi^{ab}$.

For later comparison with the gravitational case, we conclude this section by considering the propagation of source-free electromagnetic waves on an FLRW background, assuming that $E_a = 0 = H_a$ in the background. We linearize and take the curl of equation (19), evaluating $\operatorname{curl} \operatorname{curl} H_a$ by the identity (13) and equation (18). We eliminate $\operatorname{curl} \dot{E}_a$ by linearizing equation (20), taking its time derivative, and using identity (9). The result is the wave equation

$$\Box^2 H_a \equiv -\ddot{H}_a + \mathrm{D}^2 H_a \approx 5H\dot{H}_a + \left(2H^2 + \tfrac{1}{3}\rho - p\right)H_a \,, \tag{28}$$

where p is the background pressure, and we used the FLRW field equation $3\dot{H} = -3H^2 - \tfrac{1}{2}(\rho + 3p)$. A similar wave equation may be derived for E_a.

3. The Bianchi identities and nonlinear duality

The Maxwell analogy in General Relativity is based on the the correspondence $C_{abcd} \leftrightarrow F_{ab}$, where the Weyl tensor C_{abcd} is the free gravitational field (see [19]). For a given u^a, it splits irreducibly and covariantly into

$$E_{ab} = C_{acbd}u^c u^d = E_{\langle ab\rangle} \,, \;\; H_{ab} = {}^{*}C_{acbd}u^c u^d = H_{\langle ab\rangle} \,, \tag{29}$$

which are called its 'electric' and 'magnetic' parts by analogy with the Maxwell decomposition (15). These gravito-electric/magnetic spatial tensors are in principle physically measurable in the frames of comoving observers, and together they are equivalent to the spacetime Weyl tensor, since [37]

$$C_{ab}{}^{cd} = 4\left\{u_{[a}u^{[c} + h_{[a}{}^{[c}\right\}E_{b]}{}^{d]} + 2\varepsilon_{abe}u^{[c}H^{d]e} + 2u_{[a}H_{b]e}\varepsilon^{cde} \,. \tag{30}$$

This is the gravito-electromagnetic version of the expression (16). The electromagnetic interpretation of E_{ab} and H_{ab} is reinforced by the fact that these fields covariantly (and gauge-invariantly) describe gravitational waves on an FLRW background (including the special case of a flat vacuum background) [7, 38].

In the $1+3$ covariant approach to General Relativity [9], the fundamental quantities are not the metric (which in itself does not provide a covariant description), but the kinematic quantities of the fluid, its energy density ρ and pressure p, and the gravito-electric/magnetic tensors. The fundamental equations governing these quantities are the Bianchi identities and the Ricci identities for u^a, with Einstein's equations incorporated via the algebraic definition of the Ricci tensor R_{ab} in terms of the energy-momentum tensor T_{ab}. We assume that the source of the gravitational field is a perfect fluid (the generalization to imperfect fluids is straightforward). The Bianchi identities are

$$\nabla^d C_{abcd} = \nabla_{[a}\left(-R_{b]c} + \tfrac{1}{6}R g_{b]c}\right) \,, \tag{31}$$

where $R = R_a{}^a$ and $R_{ab} = T_{ab} - \tfrac{1}{2}T_c{}^c g_{ab}$. The contraction of (31) implies the conservation equations. The tracefree part of (31) gives the gravitational equivalents of the Maxwell

equations (17)–(20), via a covariant $1 + 3$ decomposition [5, 9]. In our notation, these take the simplified form:

$$D^b E_{ab} = -3\omega^b H_{ab} + \tfrac{1}{3}D_a\rho + [\sigma, H]_a \,, \tag{32}$$

$$D^b H_{ab} = 3\omega^b E_{ab} + (\rho + p)\omega_a - [\sigma, E]_a \,, \tag{33}$$

$$\dot{E}_{\langle ab\rangle} - \operatorname{curl} H_{ab} = -\Theta E_{ab} + 3\sigma_{c\langle a}E_{b\rangle}{}^c - \omega^c\varepsilon_{cd(a}E_{b)}{}^d$$
$$+ 2\dot{u}^c\varepsilon_{cd(a}H_{b)}{}^d - \tfrac{1}{2}(\rho + p)\sigma_{ab} \,, \tag{34}$$

$$\dot{H}_{\langle ab\rangle} + \operatorname{curl} E_{ab} = -\Theta H_{ab} + 3\sigma_{c\langle a}H_{b\rangle}{}^c - \omega^c\varepsilon_{cd(a}H_{b)}{}^d$$
$$- 2\dot{u}^c\varepsilon_{cd(a}E_{b)}{}^d \,. \tag{35}$$

These are the fully nonlinear equations in covariant form, and the analogy with the Maxwell equations (17)–(20) is made strikingly apparent in our formalism.

Vorticity couples to the fields to produce source terms in both cases, but gravity has additional sources from a *tensor coupling of the shear* to the field. The analogue of the charge density ϱ as a source for the electric field, is the energy density spatial gradient $D_a\rho$ as a source for the gravito-electric field. Since $D_a\rho$ covariantly describes *inhomogeneity* in the fluid, this is consistent with the fact that the gravito-electric field is the generalization of the Newtonian tidal tensor [9].

There is no magnetic charge source for H_a, but the gravito-magnetic field H_{ab} has the source $(\rho + p)\omega_a$. Since $\rho + p$ is the relativistic inertial mass-energy density [9], $(\rho + p)\omega_a$ is the *'angular momentum density'*, which we identify as a gravito-magnetic 'charge' density. Note however that angular momentum density does not always generate a gravito-magnetic field. The Gödel solution [9] provides a counter-example, where $H_{ab} = 0$ and the non-zero angular momentum density is exactly balanced by the vorticity/ gravito-electric coupling in equation (33), with $\sigma_{ab} = 0$.

For both electromagnetism and gravity, the propagation of the fields is determined by the spatial curls, together with a coupling of the expansion, shear, vorticity, and acceleration to the fields. The analogue of the electric current j_a is the gravito-electric 'current' $(\rho + p)\sigma_{ab}$, which is the *'density of the rate-of-distortion energy'* of the fluid. There is no magnetic current in either case.

If the Maxwell field is source-free, i.e. $\varrho = 0 = j_a$, and the gravitational field is source-free, i.e. $\rho = 0 = p$, then the similarity of the two sets of equations is even more apparent, and only the tensor shear coupling in the case of gravity lacks a direct electromagnetic analogue. (Note that these shear coupling terms govern the possibility of simultaneous diagonalization of the shear and E_{ab}, H_{ab} in tetrad formulations of general relativity [41, 42].)

To obtain the gravitational analogue of the complex equations (21) and (22), which lead to the Maxwell duality invariance, we consider the vacuum case $\rho = 0 = p$. In general, u^a is no longer uniquely defined in vacuum, although in particular cases

(such as stationary spacetimes), there may be a physically unique choice. However, our results hold for an *arbitrary* covariant choice of u^a, without any special conditions on the congruence. By analogy with the complex electromagnetic spatial vector $\mathcal{I}_a$, we define the complex gravito-electromagnetic spatial tensor

$$\mathcal{I}_{ab} = E_{ab} + i\,H_{ab}\,. \tag{36}$$

Then equations (32)–(35) reduce to:

$$\mathrm{D}^b\mathcal{I}_{ab} \qquad\quad = 3i\,\omega^b\mathcal{I}_{ab} - i\,[\sigma,\mathcal{I}]_a\,, \tag{37}$$

$$\dot{\mathcal{I}}_{\langle ab\rangle} + i\,\mathrm{curl}\,\mathcal{I}_{ab} = -\Theta\mathcal{I}_{ab} + 3\sigma_{c\langle a}\mathcal{I}_{b\rangle}{}^c$$
$$- \omega^c\varepsilon_{cd(a}\mathcal{I}_{b)}{}^d - 2i\,\dot{u}^c\varepsilon_{cd(a}\mathcal{I}_{b)}{}^d\,. \tag{38}$$

Apart from the increased economy, the system is now clearly seen to be invariant under the global $U(1)$ transformation:

$$\mathcal{I}_{ab} \;\rightarrow\; e^{i\phi}\mathcal{I}_{ab}\,, \tag{39}$$

which is precisely the tensor (spin-2) version of the vector symmetry of the source-free Maxwell equations. We have thus established the existence of the covariant spatial duality at the level of the physically relevant gravito-electric/magnetic fields, in the general (non-perturbative, arbitrary observer congruence) vacuum case. (As with electromagnetism, duality invariance breaks down in the presence of sources.)

A covariant super-energy density and super-Poynting vector arise naturally as invariants under spatial duality rotation, in direct analogy with the Maxwell invariants of equations (23) and (24):

$$U \;= \tfrac{1}{2}\mathcal{I}^{ab}\overline{\mathcal{I}}_{ab} = \tfrac{1}{2}\left(E_{ab}E^{ab} + H_{ab}H^{ab}\right)\,, \tag{40}$$

$$P_a = \frac{1}{2i}[\overline{\mathcal{I}},\mathcal{I}]_a = [E,H]_a \equiv \varepsilon_{abc}E^b{}_dH^{cd}\,. \tag{41}$$

This reflects the duality invariance of the Bel-Robinson tensor [1]

$$M_{ab}{}^{cd} = \tfrac{1}{2}\left(C_{aebf}C^{cedf} + {}^*C_{aebf}\,{}^*C^{cedf}\right)\,, \tag{42}$$

which is the natural covariant definition of the super-energy-momentum tensor for the free gravitational field, since [1, 10]

$$U \;= M_{abcd}u^au^bu^cu^d\,, \tag{43}$$

$$P_a = -\,M_{\langle a\rangle bcd}u^bu^cu^d\,. \tag{44}$$

The agreement between equations (43) and (40) follows obviously from equation (42) on using equation (29). However, it is not obvious that equation (44) agrees with our equation (41) for the super-Poynting vector, and one requires the identity (30) to show the agreement.

Our expression (40) for the gravitational super-energy density gives a direct and clear analogy with the electromagnetic energy density (23). Our expression (41) for the gravitational super-Poynting vector, in terms of the tensor generalization of the vector product, provides a clearer analogy with the electromagnetic Poynting vector (24). The analogy is reinforced by the fact that U and P_a obey a super-energy conservation equation which is the tensor version of the electromagnetic energy conservation equation (26). To show this, we need the new covariant identity (6). Using this and the Bianchi propagation equations (34) and (35), we find that

$$\dot{U} + D^a P_a = -2\Theta U - 4\dot{u}^a P_a + 3\sigma^c{}_{\langle a} \left[E_{b\rangle c} E^{ab} + H_{b\rangle c} H^{ab} \right] . \tag{45}$$

This is the non-perturbative and covariant generalization of Bel's linearized conservation equation [1, 10]: $\partial_t U = -\text{div } \vec{P}$.

The last term in the conservation equation (45) contains another natural group invariant

$$\pi_{ab} = -\mathcal{I}_{c\langle a} \overline{\mathcal{I}}_{b\rangle}{}^c = -E_{c\langle a} E_{b\rangle}{}^c - H_{c\langle a} H_{b\rangle}{}^c , \tag{46}$$

which we interpret as the anisotropic super-pressure of the gravito-electromagnetic field.

A further covariant quantity that may be naturally constructed from equation (36) is

$$\begin{aligned}
\mathcal{I}_{ab} \mathcal{I}^{ab} &= (E_{ab} E^{ab} - H_{ab} H^{ab}) + 2i\, E_{ab} H^{ab} \\
&= \tfrac{1}{8}(C_{abcd} C^{abcd} + iC_{abcd}{}^* C^{abcd})
\end{aligned} \tag{47}$$

which is *not* invariant under equation (39). However it proves very useful in categorizing spacetimes [44] and vanishes in Petrov type-III and type-N spacetimes [13] (supporting the existence of gravitational waves in these spacetimes, since the analogous quantities vanish for purely radiative electromagnetic fields). Further, the electromagnetic analogue of the real part of equation (47), namely $E_a E^a - H_a H^a = -\tfrac{1}{2} F_{ab} F^{ab}$, is just the Lagrangian density. The analogue of the imaginary part is $E_a H^a = \tfrac{1}{4} F_{ab}{}^* F^{ab}$ whose integral in non-Abelian gauge theories is proportional to the topological instanton number.

As pointed out in the introduction, duality rotations preserve the Bianchi identities in vacuum, but not the Ricci identities for u^a. This is clearly apparent from the spatial tensor parts of the Ricci identities [9], which in our formalism have the simplified form

$$E_{ab} = D_{\langle a} \dot{u}_{b\rangle} - \dot{\sigma}_{\langle ab\rangle} - \tfrac{2}{3}\Theta\sigma_{ab} - \sigma_{c\langle a}\sigma_{b\rangle}{}^c - \omega_{\langle a}\omega_{b\rangle} + \dot{u}_{\langle a}\dot{u}_{b\rangle} , \tag{48}$$

$$H_{ab} = \text{curl } \sigma_{ab} + D_{\langle a}\omega_{b\rangle} + 2\dot{u}_{\langle a}\omega_{b\rangle} . \tag{49}$$

In order to preserve the Ricci identities, and map Einstein solutions to Einstein solutions, one needs to perform kinematic transformations in addition to the duality rotation. An example is presented in the following section.

The electromagnetic analogy suggests a further interesting interpretation of the kinematic quantities arising from the Ricci equations (48) and (49).† In flat spacetime, relative to inertial observers, the electric and magnetic vectors may be written as

$$\vec{E} = \vec{\nabla} V - \partial_t \vec{\alpha}, \ \vec{H} = \operatorname{curl} \vec{\alpha},$$

where V is the electric scalar potential and $\vec{\alpha}$ is the magnetic vector potential.† Comparing now with the Ricci equations (48) and (49), we see that the four-acceleration is a covariant gravito-electric vector potential and the shear is a covariant gravito-magnetic tensor potential. The vorticity derivative in (49) has no electromagnetic analogue, and vorticity appears to be an additional gravito-magnetic vector potential. Furthermore, the gauge freedom in the electromagnetic potentials does not have a direct gravitational analogue in the Ricci gravito-potential equations (48) and (49), since the gravito-electric/magnetic potentials are invariantly defined kinematic quantities. (Note that the Lanczos potential for the Weyl tensor does have a gauge freedom analogous to that in the Maxwell four-potential [12].)

The remaining Ricci equations in $1 + 3$ covariant form are [19]

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 \quad = -\tfrac{1}{2}(\rho + 3p) + \mathrm{D}^a \dot{u}_a + \dot{u}^a \dot{u}_a + 2\omega^a \omega_a - \sigma^{ab}\sigma_{ab}, \tag{50}$$

$$\dot{\omega}_{\langle a \rangle} + \tfrac{2}{3}\Theta\omega_a = -\tfrac{1}{2}\operatorname{curl} \dot{u}_a + \sigma_{ab}\omega^b, \tag{51}$$

$$\tfrac{2}{3}\mathrm{D}_a\Theta \quad = -\operatorname{curl}\omega_a + \mathrm{D}^b\sigma_{ab} + 2[\omega, \dot{u}]_a, \tag{52}$$

$$\mathrm{D}^a\omega_a \quad = \dot{u}^a\omega_a, \tag{53}$$

and do not involve the gravito-electromagnetic field.

Finally in this section, we extend the analogy to wave propagation. The magnetic wave equation (28) has a simple gravito-magnetic analogue. In order to isolate the purely tensor perturbations of an FLRW background in a covariant (and gauge-invariant) way, one imposes $\omega_a = 0$ [17]. We linearize and take the curl of equation (34), using the linearizations of equations (35) and (33), and identities (10) and (14). This does not directly produce a wave equation, since the curl of the shear term in (34) has to be eliminated. (In the Maxwell case this feature did not arise, since we set $j_a = 0$.) The elimination is achieved via the Ricci equation (49), and we find that

$$\square^2 H_{ab} \equiv -\ddot{H}_{ab} + \mathrm{D}^2 H_{ab} \approx 7H\dot{H}_{ab} + 2\left(3H^2 - p\right)H_{ab}, \tag{54}$$

in agreement with [17, 18], and in striking analogy with the magnetic wave equation (28). Further discussion of covariant gravitational wave theory may be found in [17, 18, 19, 43].

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† Note that these Ricci equations have the same form in the non-vacuum case.

† The covariant form of these potentials is $V = u^a A_a$, $\alpha_a = A_{\langle a \rangle}$, where A_a is the four-potential.

4. Gravitational monopoles

The electromagnetic correspondence we have developed suggests a covariant characterization of gravito-electric (magnetic) *monopoles*, as stationary vacuum spacetimes outside isolated sources, with purely electric (magnetic) free gravitational field, i.e., $H_{ab} = 0$ ($E_{ab} = 0$). This is reinforced by the fact that monopoles do not radiate, and gravitational radiation necessarily involves both E_{ab} and H_{ab} nonzero (see [16, 18, 19], consistent with Bel's criterion $P_a \neq 0$ [1, 10]). Our identification in the previous section of density inhomogeneity and angular momentum density as sources of, respectively, gravito-electric and gravito-magnetic fields, suggests that the monopole sources will be respectively mass and angular momentum. However, as pointed out previously, it is possible that non-zero angular momentum is compatible with a purely gravito-electric field, as illustrated by the Gödel solution.

The four-velocity field u^a is not defined by a fluid, but is defined as the normalization of the stationary Killing vector field $\xi^a = \xi u^a$. As a consequence of Killing's equations, we have $\Theta = 0 = \sigma_{ab}$ [44], so that

$$\nabla_b u_a = \varepsilon_{abc}\omega^c - \dot{u}_a u_b \,.$$

The covariant equations governing non-perturbative monopoles are complicated. Some simplification arises from the Killing symmetry, which implies

$$\mathcal{L}_\xi \omega_a = \xi\dot{\omega}_a + u_a\omega^b D_b\xi = 0 \,,$$
$$\mathcal{L}_\xi H_{ab} = \xi\dot{H}_{ab} + 2\xi\omega^c\varepsilon_{cd(a}H_{b)}{}^d - 2\xi u_{(a}H_{b)c}\dot{u}^c = 0 \,,$$

and a similar equation for E_{ab}. Then it follows that

$$\dot{\omega}_{\langle a\rangle} = 0 \,, \tag{55}$$
$$\dot{H}_{\langle ab\rangle} = -2\omega^c\varepsilon_{cd(a}H_{b)}{}^d \,, \tag{56}$$
$$\dot{E}_{\langle ab\rangle} = -2\omega^c\varepsilon_{cd(a}E_{b)}{}^d \,. \tag{57}$$

Now equations (55)–(57), together with the basic monopole conditions, are applied to the Bianchi equations (32)–(35) and Ricci equations (48)–(53). We obtain:

Gravito-electric and -magnetic monopoles:

$$D^a\dot{u}_a = -\dot{u}^a\dot{u}_a - 2\omega^a\omega_a \,, \tag{58}$$
$$\dot{\omega}_{\langle a\rangle} = 0 \,, \tag{59}$$
$$\mathrm{curl}\,\dot{u}_a = 0 \,, \tag{60}$$
$$\mathrm{curl}\,\omega_a = -2[\dot{u},\omega]_a \,, \tag{61}$$
$$D^a\omega_a = \dot{u}^a\omega_a \,. \tag{62}$$

Gravito-electric monopole:

$$D^b E_{ab} = 0\,, \tag{63}$$

$$0 = E_{ab}\omega^b\,, \tag{64}$$

$$\dot{E}_{\langle ab\rangle} = 0\,, \tag{65}$$

$$0 = \omega^c \varepsilon_{cd(a} E_{b)}{}^d\,, \tag{66}$$

$$\mathrm{curl}\,E_{ab} = -2\dot{u}^c \varepsilon_{cd(a} E_{b)}{}^d\,, \tag{67}$$

$$E_{ab} - D_{\langle a}\dot{u}_{b\rangle} = \dot{u}_{\langle a}\dot{u}_{b\rangle} - \omega_{\langle a}\omega_{b\rangle}\,, \tag{68}$$

$$D_{\langle a}\omega_{b\rangle} = -2\dot{u}_{\langle a}\omega_{b\rangle}\,. \tag{69}$$

Gravito-magnetic monopole:

$$D^b H_{ab} = 0\,, \tag{70}$$

$$0 = H_{ab}\omega^b\,, \tag{71}$$

$$\dot{H}_{\langle ab\rangle} = 0\,, \tag{72}$$

$$0 = \omega^c \varepsilon_{cd(a} H_{b)}{}^d\,, \tag{73}$$

$$\mathrm{curl}\,H_{ab} = -2\dot{u}^c \varepsilon_{cd(a} H_{b)}{}^d\,, \tag{74}$$

$$D_{\langle a}\dot{u}_{b\rangle} = -\dot{u}_{\langle a}\dot{u}_{b\rangle} + \omega_{\langle a}\omega_{b\rangle}\,, \tag{75}$$

$$H_{ab} - D_{\langle a}\omega_{b\rangle} = 2\dot{u}_{\langle a}\omega_{b\rangle}\,. \tag{76}$$

Equation (60) implies that there exists an acceleration potential:

$$\dot{u}_a = D_a\Phi\,. \tag{77}$$

This holds even when $\omega_a \neq 0$, despite the identity (4), since Φ is invariant under ξ^a, so that $\dot{\Phi} = 0$. Equation (61) shows that $\mathrm{curl}\,\omega_a$ is orthogonal to the vorticity and four-acceleration:

$$\omega^a \mathrm{curl}\,\omega_a = 0 = \dot{u}^a \mathrm{curl}\,\omega_a\,.$$

Schwarzschild spacetime, where also $\omega_a = 0$ (since staticity implies u^a is hypersurface orthogonal), is clearly a non-perturbative gravito-electric monopole according to our covariant definition: it is a static vacuum spacetime satisfying $H_{ab} = 0$, by virtue of the Ricci equation (49). Equations (77) and (58) imply

$$D^2\Phi + D^a\Phi D_a\Phi = 0\,. \tag{78}$$

The solution Φ determines $\dot{u}_a$ and E_{ab}, and equation (78) ensures that the monopole conditions (58)–(69) are identically satisfied.

It is not clear whether there exist consistent non-perturbative gravito-magnetic monopoles, i.e. spacetimes satisfying the covariant equations (58)–(62) and (70)–(76).† However, linearized gravito-magnetic monopoles have been found, for example

† In [45] it is shown that non-flat vacuum solutions with purely magnetic Weyl tensor are a very restricted class, and it is suggested that there may be no such solutions.

the Demianski-Newman solution [33] (see below). It is also not clear whether there exist gravito-electric monopoles with angular momentum (i.e. $\omega_a \neq 0$).

In the case of linearization about a flat Minkowski spacetime, the right-hand sides of equations (58)–(76) may all be set to zero. In particular, equation (61) implies that there is a vorticity potential:

$$\omega_a \approx D_a \Psi \,. \tag{79}$$

The linearization of equations (68) and (76), together with the scalar potential equations (77) and (79), then imply that the curls vanish to linear order. Thus the linearized gravito-electric monopole is covariantly characterized by equations (77), (79) and

$$D^2 \Phi \approx 0 \,, \quad E_{ab} \approx D_a D_b \Phi \,, \quad D_{\langle a} D_{b\rangle} \Psi \approx 0 \,, \tag{80}$$

while for the linearized gravito-magnetic monopole

$$D^2 \Psi \approx 0 \,, \quad H_{ab} \approx D_a D_b \Psi \,, \quad D_{\langle a} D_{b\rangle} \Phi \approx 0 \,. \tag{81}$$

It follows in particular that a linearized non-rotating gravito-electric monopole is mapped to a linearized non-accelerating gravito-magnetic monopole via

$$\mathcal{I}_{ab} \to i \mathcal{I}_{ab} \,, \quad \omega_a \to \dot{u}_a \,, \quad \dot{u}_a \to -\omega_a \,. \tag{82}$$

Linearized Schwarzschild spacetime is readily seen to satisfy equation (80) with $\Phi = -M/r$, where M is the mass and r the area coordinate. Using the spatial duality rotation and kinematic interchange described by equation (82), this monopole is mapped to a linearized non-accelerating gravito-magnetic monopole with potential $\Psi = -M/r$. In comoving stationary coordinates, the metric of the linearized magnetic monopole follows from $\dot{u}_a = 0$ and $\omega_a = D_a \Psi$, using a theorem in [40] (p 24):

$$ds^2 = -dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) + 4M \cos \theta \, d\varphi \, dt \,. \tag{83}$$

This is a Taub-NUT solution with $m = 0$, $\ell = -M$ and linearized in ℓ ([44], p.133; see also [36]). In fact, this is precisely the linearized solution found in [33], so that we have a covariant characterization of that solution in the framework of gravitational duality. Clearly the magnetic 'charge' M is an angular momentum parameter, not a mass parameter, and the metric in equation (83) describes an isolated source with angular momentum but no mass.

5. Concluding remarks

A covariant $1 + 3$ approach, based on [9] and its extension [37, 19], is ideally suited to an analysis of the free gravitational field that is based on observable physical and geometric quantities, with a clear and transparent analogy in well-established electromagnetic theory. We have used such an approach, including in particular the

generalization of covariant spatial vector analysis to spatial tensor analysis, which involves developing a consistent covariant definition of the tensor curl and its properties. Via this approach, we showed the remarkably close analogy between the Maxwell equations for the electric/magnetic fields and the Bianchi identities for the gravito-electric/magnetic fields. Although this analogy has long been known in general terms, our approach reveals its properties at a physically transparent level, with a detailed accounting for each physical and geometric quantity. We found new interpretations of the role of the kinematic quantities – expansion, acceleration, vorticity and shear – in the source and coupling terms of gravito-electromagnetism. The tracefree part of the Ricci identities also reveals the role of the kinematic quantities as gravito-electric/magnetic potentials.

The analogy provides a simple interpretation of the super-energy density and super-Poynting vector as natural $U(1)$ invariants, and we derived the exact nonlinear conservation equation that governs these quantities, and which involves a further natural invariant, i.e. the anisotropic super-pressure. We also used the analogy to show that a covariant spatial duality invariance exists in vacuum gravito-electromagnetism, precisely as in source-free electromagnetism. Duality invariance has been important in some recent developments in field and string theory, and the gravito-electromagnetic invariance in the form found here may also facilitate new insights into gravity. A crucial feature in the gravitational case, arising from its intrinsic nonlinearity, is that the duality invariance does not map Einstein solutions to Einstein solutions, since the Ricci identities are not invariant. Further work is needed to investigate whether a simultaneous geometric or kinematic transformation can be found, so that the Bianchi and Ricci equations are invariant under the combined transformation.

We showed that in linearized vacuum gravity, there is a simple combined duality/ kinematic transformation that maps the Schwarzschild gravito-electric monopole to the Demianski-Newman gravito-magnetic monopole. This covariant characterization of the relation between these linearized solutions was based on our covariant definition of gravito-monopoles in the general nonlinear theory. Further work is needed on the governing equations for these monopoles, in particular to see whether nonlinear gravito-magnetic monopole solutions may be found. A better understanding of the relation between nonlinear gravito-electric/magnetic monopoles could, as in field theory, open up new approaches and insights.

Acknowledgments

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