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PATTERN RECOGNITION ON ORIENTED MATROIDS

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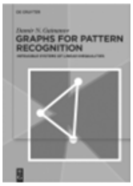

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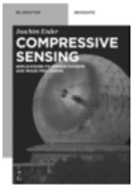

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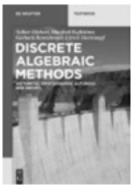

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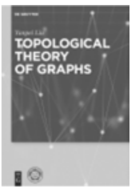

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To the memory of

Alexander Semënovich Antonov
1866–1919
Elizaveta Vasilyevna Antonova–Gryaznykh ⟨Chistyakova⟩
1880–1948

Grigory Alexeevich Matveev
1865–1920
Yulia Nikiforovna Matveeva ⟨Batmanova⟩
1872–1953

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Preface

Pattern Recognition on Oriented Matroids is a range of problems in combinatorics, combinatorial optimization, posets, graphs, in elementary number theory, and other areas that represent a far-reaching extension of the arsenal of *committee methods in pattern recognition*.

The groundwork for the modern *committee theory* was laid in the mid-1960s, when it was shown that the familiar notion of *solution to a feasible system* of *linear inequalities* has ingenious analogues which can serve as *collective solutions to infeasible systems*.

A hierarchy of dialects in the language of mathematics, for instance, *open cones* in the context of *linear inequality systems*, *regions* of *hyperplane arrangements*, and *maximal covectors* (or *topes*) of *oriented matroids*, provides an excellent opportunity to take a fresh look at the *infeasible system* of *homogeneous strict linear inequalities* which is the standard working model for the contradictory two-class *pattern recognition problem* in its *geometric setting*.

Versatile tools provided by *oriented matroid theory* considerably simplify a structural and enumerative analysis of applied aspects of the *infeasibility phenomenon*.

This book is devoted to several selected topics in the emerging field of *Pattern Recognition on Oriented Matroids*. In Chapter 1, we state the two-class *pattern recognition problem* in the language of oriented matroids and show that oriented matroids with very weak restrictions on their properties always have certain subsets of maximal covectors that can serve as building blocks of collective decision rules; these are so-called tope committees.

A *tope committee* $\mathscr{K}^*$ for an oriented matroid is defined to be a subset of its *maximal covectors* such that for every element of the *ground set*, the corresponding *positive halfspace* of the oriented matroid contains *more than half of the covectors of* $\mathscr{K}^*$.

Solving the existence problem on tope committees, throughout the book we apply a simple but useful concept of *symmetric cycles* in the *tope graph* of an oriented matroid, which is a matroidal analogue of the concept of *centrally symmetric cycles* of *adjacent regions* in an *arrangement of oriented linear hyperplanes*.

If we consider a graph on the set of topes of a simple oriented matroid, whose edges connect topes with *nonintersecting negative parts*, then the vertex set of any *odd cycle* in this *connected* graph is a *tope committee*. Furthermore, the induced connected subgraph on the subset of *topes with inclusion-maximal positive parts*, which is a natural generalization of the basic concept of the *graph of maximal feasible subsystems of an infeasible system of linear inequalities*, also contains at least one *odd cycle*.

In Chapters 2–6 we discuss mathematical concepts and tools which can be useful in an analysis of *infeasible systems of constraints*.

In Chapter 2 we describe an approach toward investigating the structural and combinatorial properties of *face systems* in the power set of a finite set, based on their partitioning into *Boolean intervals*; face systems of applied significance include the

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abstract simplicial complex of the multi-indices of feasible subsystems of an infeasible system of linear inequalities, and the *complex of acyclic subsets* of the ground set of an oriented matroid which is not acyclic.

In Chapter 3 we are concerned with interesting face systems that satisfy *Dehn–Sommerville type relations*; we investigate them from the linear algebraic viewpoint, and we enumerate vector descriptions of these face systems regarded as *lattice points* in certain *convex polytopes*.

In an investigation of the family of committees, certain subsequences of the number-theoretic *Farey sequences* occur naturally; their features are briefly discussed in Chapter 4.

In Chapters 5 and 6 we discuss a poset-theoretic perspective on such fundamental constructs of combinatorial optimization as *blocking sets* and *committees* of *set families*. Various results on the families of tope committees for oriented matroids will later be derived in the book from the general discrete mathematical results of these two chapters.

In Chapter 7 we consider the *layers of tope committees* for an oriented matroid that consist of the committees of the same size. We also investigate the layers consisting of the committees that contain *no pairs of opposites*.

Chapter 8 is devoted to the *three-tope committees* which are the most preferred approximation to the notion of *solution to an infeasible system of linear constraints*.

In Chapter 9, for enumerating tope committees, the combinatorial *Principle of Inclusion–Exclusion* is applied to subsets of topes contained in the *halfspaces* of an oriented matroid, and to *convex subsets* of its ground set.

In Chapter 10 we clarify how elementary changes in an oriented matroid, namely *reorientations on one-element subsets* of the ground set, affect the family of its tope committees.

In Chapter 11 we show that the *vertex set* of any *symmetric* $2t$*-cycle* in the *tope graph* of an oriented matroid on a t-element ground set, regarded as a subset of the *real Euclidean vector space* $\mathbb{R}^t$, is a *maximal positive basis* of $\mathbb{R}^t$; as a consequence, every tope can be represented in a *unique way* as the *unweighted sum* of the vertices in a certain *inclusion-minimal subset* of the vertex set of the cycle.

Chapter 12 provides a *discrete Fourier analysis* of the important family of *critical tope committees* through *rank* and *graph distance* relations in the *tope poset* and in the *tope graph*.

In Chapter 13 we are concerned with the characterization of a key discrete mathematical role played by the *symmetric cycles* in the *hypercube graphs*.

This monograph is a follow-up to the book *Graphs for Pattern Recognition — Infeasible Systems of Linear Inequalities* by *D.N. Gainanov* published with *De Gruyter* in 2016.

October 2017 *Andrey O. Matveev*

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Committees for Pattern Recognition: Infeasible Systems of Linear Inequalities, Hyperplane Arrangements, and Realizable Oriented Matroids

In *supervised learning* to recognize, through the synthesis of *decision rules* that use *separating surfaces* in the *feature space*, a *recognition complex* (whose concept is relevant to the mathematical constructs discussed in this book) is provided with the *training patterns* contained in a *training set*, together with *class labels* which are associated with the patterns. The training patterns from the same class compose a *training sample*. Thus, in the *two-class pattern recognition problem*, the training set is a disjoint union of two training samples. The recognition complex *classifies* new patterns as representatives of the two classes partially represented by the corresponding samples.

Infeasible systems of *homogeneous strict linear inequalities* over $\mathbb{R}^n$, *arrangements* of *oriented real linear hyperplanes* in which the intersections of the corresponding *positive halfspaces* of $\mathbb{R}^n$ are *empty*, and *realizable simple oriented matroids* that are *not acyclic*, are all closely related mathematical constructs that allow one to model mechanisms of collective decision-making which are described in the language of *infeasible systems* of *linear constraints*.

Let us briefly discuss these three viewpoints on the committee methods of solving the two-class pattern recognition problem in its geometric setting.

Infeasible Systems of Homogeneous Strict Linear Inequalities and Their Committees

The effect of a decision rule is to divide the feature space into *decision regions* each of which contains training patterns of *at most one* class. The recognition complex *classifies* a new pattern on the basis of its inclusion into a decision region, thus assigning to the new pattern the label of one of the two classes partially represented by the training samples; the *decision rule* can also force the complex to leave the new pattern *unclassified*.

If the allowed separating surfaces in a *real Euclidean* feature *space* are oriented *hyperplanes*, then the decision regions are *open convex polyhedra*.

Let $\widetilde{\boldsymbol{B}} \subset \mathbb{R}^{n-1}$ and $\widetilde{\boldsymbol{C}} \subset \mathbb{R}^{n-1}$ be two disjoint finite and nonempty training samples in the feature space $\mathbb{R}^{n-1}$. It is customary to augment each vector in the samples $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{C}}$ by the additional nth component 1. For the resulting subsets $\boldsymbol{B}, \boldsymbol{C} \subset \mathbb{R}^n$, the recognition complex seeks a vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$\begin{cases} \langle \boldsymbol{b}, \mathbf{x}\rangle > 0 \,, & \boldsymbol{b} \in \boldsymbol{B} \,, \\ \langle \boldsymbol{c}, \mathbf{x}\rangle < 0 \,, & \boldsymbol{c} \in \boldsymbol{C} \,, \end{cases} \tag{0.1}$$

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where $\langle \boldsymbol{b}, \mathbf{x}\rangle$ denotes the standard scalar product $\sum_{j\in[n]} b_j x_j$, and $[n] := \{1, 2, \dots, n\}$. In a typical situation, system (0.1) has no solution or, in other words, it is *infeasible*. Let us regard the derived infeasible system

$$\begin{cases} \langle \boldsymbol{b}, \mathbf{x}\rangle > 0 \,, & \boldsymbol{b} \in \boldsymbol{B} \,, \\ \langle -\boldsymbol{c}, \mathbf{x}\rangle > 0 \,, & \boldsymbol{c} \in \boldsymbol{C} \,, \end{cases}$$

as a certain system

$$\{\langle \boldsymbol{a}, \mathbf{x}\rangle > 0 \colon \boldsymbol{a} \in \boldsymbol{A}\} \,. \tag{0.2}$$

An attractive generalization, to infeasible systems, of the notion of solution to a feasible system of linear inequalities is that of majority committee.

Recall that a *committee* of the infeasible system (0.2) is defined to be a finite subset of vectors $\mathcal{K} \subset \mathbb{R}^n$ such that for each inequality more than half of the vectors of $\mathcal{K}$ are solutions to the inequality, that is,

$$|\{\boldsymbol{x} \in \mathcal{K} \colon \langle \boldsymbol{a}, \boldsymbol{x}\rangle > 0\}| > \tfrac{1}{2}|\mathcal{K}| \,,$$

for each vector $\boldsymbol{a} \in \boldsymbol{A}$.

It is well known that system (0.2), in which every subsystem with two inequalities is feasible, has a committee.

If the recognition complex has successfully constructed a committee $\mathcal{K}$ of system (0.2), then it classifies a new pattern in the feature space $\mathbb{R}^{n-1}$, augmented by the additional nth component 1, as a representative of one of the two classes partially described by the samples $\tilde{\boldsymbol{B}}$ and $\tilde{\boldsymbol{C}}$, on the basis of the majority voting procedure performed by the members of the committee $\mathcal{K}$.

An elegant applied approach to committees of the infeasible system of homogeneous strict linear inequalities (0.2) with the *feasible subsystems of size* 2 consists in a structural and combinatorial analysis of the family of its *maximal feasible subsystems* (*MFSs*) and in an investigation of the corresponding graph of MFSs which is naturally associated with these maximal subsystems.

If $[m]$ is the set of *indices* with which the inequalities of the system (0.2) are marked, then the *graph of MFSs* of the system (0.2), on the *vertex set* $\mathbf{J}$ which is the family of the *multi-indices of MFSs* of system (0.2), by definition has the *edges* $\{J, J'\} \subset \mathbf{J}$ connecting multi-indices J and J' if and only if $J \cup J' = [m]$.

The *connectedness* and the *existence of an odd cycle* in the graph of MFSs of system (0.2) explain the applied significance of the graph. Indeed, if the multi-indices of certain maximal feasible subsystems represent the vertex set of a cycle of odd length in the graph of MFSs, then for constructing a committee of system (0.2) it suffices to take one vector from the open cone of solutions to each MFS whose multi-index is a vertex of the cycle.

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Arrangements of Oriented Linear Hyperplanes and Committees of Regions

A finite collection $\mathscr{H}$ of pairwise distinct linear hyperplanes in the *feature space* $\mathbb{R}^n$ is said to be a *training set* if it is partitioned into two nonempty subsets $\mathscr{A}$ and $\mathscr{B}$. A codimension one subspace $\boldsymbol{H} := \{\mathbf{x} \in \mathbb{R}^n : \langle \boldsymbol{p}, \mathbf{x}\rangle := \sum_{j\in[n]} p_j x_j = 0\}$ of the *hyperplane arrangement* $\mathscr{H}$ is determined by its normal vector $\boldsymbol{p} \in \mathbb{R}^n$, and this hyperplane is oriented: a vector $\boldsymbol{v}$ lies on the *positive side* of $\boldsymbol{H}$ if $\langle \boldsymbol{h}, \boldsymbol{v}\rangle > 0$, where, by convention, $\boldsymbol{h} := -\boldsymbol{p}$ if $\boldsymbol{H} \in \mathscr{A}$, and $\boldsymbol{h} := \boldsymbol{p}$ if $\boldsymbol{H} \in \mathscr{B}$. In a similar manner, a *region* $\boldsymbol{T}$ of the hyperplane arrangement $\mathscr{H}$, that is, a connected component of the *complement* $\mathscr{T} := \mathbb{R}^n - \mathscr{H}$, lies on the *positive side* of the hyperplane $\boldsymbol{H}$ if $\langle \boldsymbol{h}, \boldsymbol{v}\rangle > 0$ for some vector $\boldsymbol{v} \in \boldsymbol{T}$. Let $\mathscr{T}^+_{\boldsymbol{H}}$ denote the set of all regions lying on the positive side of $\boldsymbol{H}$.

The oriented hyperplanes of the arrangement $\mathscr{H}$ are called the *training patterns*. The *training samples* $\mathscr{A}$ and $\mathscr{B}$ provide a partial description of two disjoint *classes* $\mathbf{A}$ and $\mathbf{B}$, respectively: *a priori*, we have $\mathbf{A} \supseteq \mathscr{A}$ and $\mathbf{B} \supseteq \mathscr{B}$.

A subset $\mathscr{K}^* \subset \mathscr{T}$ is by definition a *committee of regions* for the arrangement $\mathscr{H}$ if

$$|\mathscr{K}^* \cap \mathscr{T}^+_{\boldsymbol{H}}| > \tfrac{1}{2}|\mathscr{K}^*| ,$$

for each hyperplane $\boldsymbol{H} \in \mathscr{H}$.

Consider a new pattern $\boldsymbol{G} := \{\mathbf{x} \in \mathbb{R}^n : \langle \boldsymbol{g}, \mathbf{x}\rangle = 0\} \notin \mathscr{H}$, determined by its normal vector $\boldsymbol{g} \in \mathbb{R}^n$. If a system of distinct representatives $\boldsymbol{W} := \{\boldsymbol{w} \in \boldsymbol{K} : \boldsymbol{K} \in \mathscr{K}^*\}$, of cardinality $|\mathscr{K}^*|$, for the committee of regions $\mathscr{K}^*$ is fixed, then the corresponding *committee decision rule* allows a recognition complex to classify the pattern $\boldsymbol{G}$ as an element of the class $\mathbf{A}$ if $|\{\boldsymbol{w} \in \boldsymbol{W} : \langle \boldsymbol{g}, \boldsymbol{w}\rangle > 0\}| < \frac{1}{2}|\boldsymbol{W}|$; the pattern $\boldsymbol{G}$ is recognized as an element of the class $\mathbf{B}$ if $|\{\boldsymbol{w} \in \boldsymbol{W} : \langle \boldsymbol{g}, \boldsymbol{w}\rangle > 0\}| > \frac{1}{2}|\boldsymbol{W}|$.

Realizable Simple Oriented Matroids and Tope Committees

Let $\mathscr{M} := (E_t, \mathscr{T})$ be a *simple oriented matroid* (by convention, it has no *loops*, *parallel elements* or *antiparallel elements*) which is *realizable* as an arrangement $\mathscr{H}$ of t pairwise distinct oriented linear hyperplanes in the *feature space* $\mathbb{R}^n$; the set $E_t := [t]$ of the *indices* of hyperplanes is called the *ground set* of $\mathscr{M}$, and $\mathscr{T}$ denotes the set of *maximal covectors* (or *topes*) of $\mathscr{M}$.

More precisely, according to a standard argument in oriented matroid theory, the arrangement $\mathscr{H}$ decomposes the space $\mathbb{R}^n$ in a collection of relatively open topological *cells* of various dimensions, for each of which we can collect the information whether for the hyperplane $\boldsymbol{H}_e \in \mathscr{H}$, the cell is on its *negative side*, *on the hyperplane itself*, or on the *positive side*. The resulting family of tuples of *signs*, of length t, is the set of *covectors* $Y \in \{-1, 0, 1\}^{E_t}$ of the oriented matroid $\mathscr{M}$. The sign tuples corresponding to the cells of the *complement* $\mathbb{R}^n - \mathscr{H}$ compose the *set of topes* $\mathscr{T} \subseteq \{-1, 1\}^{E_t}$ of $\mathscr{M}$.

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Now let $\mathscr{M} := (E_t, \mathscr{T})$ be an arbitrary simple oriented matroid (*realizable* or *not*), on the ground set E_t, with set of topes $\mathscr{T}$.

The *positive halfspace* $\mathscr{T}_e^+$ of the oriented matroid $\mathscr{M}$ associated with an element $e \in E_t$ of its ground set is defined to be the subset of topes $\{T \in \mathscr{T} : T(e) = 1\}$. We say that a subset $\mathscr{K}^* \subset \mathscr{T}$ is a *tope committee* for $\mathscr{M}$ if we have

$$|\mathscr{K}^* \cap \mathscr{T}_e^+| > \tfrac{1}{2}|\mathscr{K}^*| ,$$

for each element $e \in E_t$.

Let $\widetilde{\mathscr{M}}$ be (in the terminology which will be recalled in Section 1.1) the *nontrivial extension* of $\mathscr{M}$ by a new element g which is *not a loop*, and which is *parallel* or *antiparallel* to neither of the elements of the ground set E_t. Let σ be the corresponding *localization*. Fix a tope committee $\mathscr{K}^*$ for $\mathscr{M}$. Let $\mathscr{C}^*$ denote the set of *cocircuits* of $\mathscr{M}$, and suppose that the sets $\{(X, \Sigma_K := \sigma(X)) : X \in \mathscr{C}^*,\ X \text{ restriction of } K\}$ are *conformal*, for all topes $K \in \mathscr{K}^*$. The *committee decision rule* corresponding to the committee $\mathscr{K}^*$ allows a recognition complex to classify the element g as a representative of a *class* **A** if $|\{K \in \mathscr{K}^* : \Sigma_K = 1\}| < \frac{1}{2}|\mathscr{K}^*|$; on the contrary, g is *recognized* as an element of the other *class* **B** if $|\{K \in \mathscr{K}^* : \Sigma_K = 1\}| > \frac{1}{2}|\mathscr{K}^*|$.

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1 Oriented Matroids, the Pattern Recognition Problem, and Tope Committees

In this chapter we state the two-class pattern recognition problem in the language of oriented matroids and show that oriented matroids with very weak restrictions on their properties always have certain subsets of maximal covectors that can serve as building blocks of collective decision rules.

We present a natural generalization, in terms of maximal covectors of an oriented matroid, of the notion of committee of a linear inequality system: a *tope committee* $\mathscr{K}^*$ for an oriented matroid is defined to be a subset of its maximal covectors such that for every element of the ground set the corresponding *positive halfspace* of the oriented matroid contains *more than half of the covectors* of the set $\mathscr{K}^*$.

In Section 1.1, we recall the particular terminology of oriented matroid theory.

In Section 1.2, the setting of the pattern recognition problem is reviewed; we define the *committees of maximal* (*co*)*vectors* and discuss the *committee decision rules* of recognition.

It is shown in Section 1.3 that any *simple* oriented matroid on its ground t-set has a tope committee of size *at most* t. The argument is based on an analysis of consecutive *reorientations* of an initial *acyclic* oriented matroid on *one-element subsets* of its ground set.

Section 1.4 is devoted to graphs that are naturally associated with the sets of topes of simple oriented matroids, and with the subsets of their topes whose *positive parts* are *maximal* with respect to inclusion. The vertex sets of any *odd cycles* in these graphs are *tope committees*. We also apply this efficient graph-theoretic technique, used in the construction of *committees of infeasible systems of linear inequalities* through the *odd cycles* in their *graphs of maximal feasible subsystems*, to a matroidal analogue of the concept of *centrally symmetric cycles of adjacent regions in a hyperplane arrangement*; and vice versa, we use "*symmetric cycles*," inspired by the above-mentioned cycles of regions, in the *tope graph* of an oriented matroid in order to give additional graph-theoretic results on tope committees.

Section 1.5 surveys selected mathematical concepts and tools which can be useful in an analysis of *infeasible systems of constraints*.

1.1 Preliminaries

All oriented matroids considered in the book are of rank at least 2.

Using a somewhat nonstandard terminology, we say that an oriented matroid is *simple* if it has *no loops*, *parallel elements*, or *antiparallel elements*. The nonzero components of row *sign vectors* appearing in the book will sometimes be *real numbers* -1 and 1, instead of the familiar characters "–" and "+," respectively.

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Let $[j, t]$ denote the set of *consecutive positive integers* $\{j, j+1, \dots, t\}$. We will often denote the sets $[t] := [1, t]$ by E_t. By convention, the components of the row vector $\mathrm{T}^{(+)} := (1, 1, \dots, 1) \in \mathbb{R}^t$ are all 1, and similarly $\mathrm{T}^{(-)} := -\mathrm{T}^{(+)} = (-1, -1, \dots, -1)$.

If V is a subset of a *poset* P, then $\mathbf{min}V$ and $\mathbf{max}V$ denote the set of all *minimal elements* and the set of all *maximal elements* of V, respectively. The *order ideal* $\mathfrak{I}(V)$ and the *order filter* $\mathfrak{F}(V)$ of P generated by the subset V are defined by

$$\mathfrak{I}(V) := \mathfrak{I}_P(V) := \{a \in P : \exists v \in V,\ a \preceq v\}\ ,$$

$$\mathfrak{F}(V) := \mathfrak{F}_P(V) := \{a \in P : \exists v \in V,\ v \preceq a\}\ .$$

The ideals and filters generated by one-element subsets $\{v\}$ are called *principal*. If P is *graded*, then $P^{(k)}$ denotes the kth *layer* of the poset P, that is, the subset of all its elements of *rank* k.

If $C \subseteq P$, and for all elements $a \in C$ and $c \in C$, such that $a \preceq c$, the implications $a \preceq b \preceq c \Rightarrow b \in C$ are valid, then the subposet C is called *convex*.

Given a set family $\mathscr{F} := \{F_i : i \in [t]\}$, we denote by $\mathbf{min}\mathscr{F}$ and $\mathbf{max}\mathscr{F}$ the subfamilies of all *inclusion-minimal* and *inclusion-maximal* sets in $\mathscr{F}$, respectively.

The *nerve* of the family $\mathscr{F}$ is defined to be an *abstract simplicial complex* on the vertex set $\mathscr{F}$ such that a subset $K \subseteq [t]$ is a *face* of the complex if and only if $|\bigcap_{k\in K} F_k| > 0$.

Throughout the book, we deal with undirected graphs; they have no loops and multiple edges. For a graph $\mathbf{G}$, its *vertex set* and the *family of edges* are denoted by $\mathrm{V}(\mathbf{G})$ and $\mathscr{E}(\mathbf{G})$, respectively. All vertices of *paths* in graphs are distinct.

The *neighborhood complex* $\mathtt{NC}(\mathbf{G})$ of a graph $\mathbf{G}$ is an *abstract simplicial complex* on the *vertex set* $\mathrm{V}(\mathbf{G})$ defined as follows: a subset $N := \{n_1, \dots, n_k\} \subset \mathrm{V}(\mathbf{G})$ is a *face* of the complex $\mathtt{NC}(\mathbf{G})$ if and only if there is $v \in \mathrm{V}(\mathbf{G})$ such that $\{v, n_1\}, \dots, \{v, n_k\} \in \mathscr{E}(\mathbf{G})$.

Recall that for graphs $\mathbf{G}'$ and $\mathbf{G}''$, a *simplicial map* from $\mathbf{G}'$ to $\mathbf{G}''$ is defined to be a map $\eta : \mathrm{V}(\mathbf{G}') \to \mathrm{V}(\mathbf{G}'')$ such that $\{u, v\} \in \mathscr{E}(\mathbf{G}')$ implies $\{\eta(u), \eta(v)\} \in \mathscr{E}(\mathbf{G}'')$ or $\eta(u) = \eta(v)$.

If $\mathscr{F}$ is a set family, then the *Kneser graph* $\mathtt{KG}(\mathscr{F})$ of $\mathscr{F}$ is defined to be the graph on the vertex set $\mathrm{V}(\mathtt{KG}(\mathscr{F})) := \mathscr{F}$; if $F', F'' \in \mathscr{F}$, then $\{F', F''\} \in \mathscr{E}(\mathtt{KG}(\mathscr{F}))$ if and only if $|F' \cap F''| = 0$.

Given a *ground set* E_t, and the set

$$\{-1, 0, 1\}$$

of allowed *sign components*, which is, in the traditional notation, the set

$$\{-, 0, +\}\ ,$$

let us consider the corresponding entire set $\{-1, 0, 1\}^{E_t} = \{-1, 0, 1\}^t \subset \mathbb{R}^t$ of row *sign vectors*. The *support* $\underline{X}$ of a sign vector $X \in \{-1, 0, 1\}^t$ is defined by $\underline{X} := \{e \in E_t : X(e) \neq 0\}$, where $X(e)$ denotes the eth sign component of X. Further,

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$$X^- := \{e \in E_t : X(e) = -1\}$$

is the set of *negative elements* of X; similarly,

$$X^+ := \{e \in E_t : X(e) = 1\}$$

is the set of *positive elements* of X. Thus, $\underline{X} := X^- \dot\cup X^+$. The sets X^- and X^+ are also called the *negative* and *positive parts* of the sign vector X, respectively. An inclusion $e \in X$ means $e \in \underline{X}$. The *zero* sign vector $(0, 0, \ldots, 0) \in \mathbb{R}^t$, with the *empty support*, is denoted by $\mathbf{0}$. The *zero set* $\mathbf{z}(X)$ of the sign vector X is defined by $\mathbf{z}(X) := \{e \in E_t : X(e) = 0\}$.

If $\mathcal{P}$ is a set of sign vectors, then

$$\begin{aligned}\mathbf{max}^+(\mathcal{P}) &:= \left\{R \in \mathcal{P} : R^+ \in \mathbf{max}\{P^+ : P \in \mathcal{P}\}\right\}\,,\\ \mathbf{min}^+(\mathcal{P}) &:= \left\{R \in \mathcal{P} : R^+ \in \mathbf{min}\{P^+ : P \in \mathcal{P}\}\right\}\,.\end{aligned}$$

If $A \subseteq E_t$, then the sign vector ${}_{-A}X$ is defined by

$$({}_{-A}X)(e) := \begin{cases} 1\,, & \text{if } e \in A \text{ and } X(e) = -1\,,\\ -1\,, & \text{if } e \in A \text{ and } X(e) = 1\,,\\ X(e)\,, & \text{otherwise}\,; \end{cases}$$

if $e \in E_t$ then we write ${}_{-e}X$ instead of ${}_{-\{e\}}X$. In particular, the *opposite* of X is the sign vector $-X := {}_{-E_t}X$, that is,

$$(-X)(e) := \begin{cases} 1\,, & \text{if } X(e) = -1\,,\\ -1\,, & \text{if } X(e) = 1\,,\\ 0\,, & \text{if } X(e) = 0\,, \end{cases}$$

for all $e \in E_t$. If $\mathcal{P} \subseteq \{-1, 0, 1\}^t$ and $A \subseteq E_t$, then

$${}_{-A}\mathcal{P} := \{{}_{-A}X : X \in \mathcal{P}\}\,;$$

in particular,

$$-\mathcal{P} := {}_{-E_t}\mathcal{P} = \{-X : X \in \mathcal{P}\}\,.$$

If $X \in \{-1, 0, 1\}^t$ then the sign vector X is called *nonpositive* (resp., *negative*) if $X(e) \in \{-1, 0\}$ (resp., $X(e) = -1$), for all $e \in E_t$. Similarly, X is *nonnegative* (resp., *positive*) if $-X$ is *nonpositive* (resp., *negative*).

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The *composition* of two sign vectors X and Y is the sign vector $X \circ Y$ defined by

$$(X \circ Y)(e) := \begin{cases} X(e)\,, & \text{if } X(e) \neq 0\,, \\ Y(e)\,, & \text{otherwise}\,. \end{cases}$$

The *separation set* $\mathbf{S}(X, Y)$ of X and Y is defined by

$$\mathbf{S}(X, Y) := \{e \in E_t \colon X(e) = -Y(e) \neq 0\}\,.$$

If $|\mathbf{S}(X, Y)| = 0$, then the sign vectors X and Y are said to be *conformal*; in this case $X \circ Y = Y \circ X$. If sign vectors $X^1, X^2, \ldots, X^k \in \{-1, 0, 1\}^t$ are pairwise conformal, then $\bigcirc_{i \in [k]} X^i$ is a short notation for the *conformal composition* $X^1 \circ X^2 \circ \cdots \circ X^k$.

The *partial order* $\prec$ on the set of sign components $\{-1, 0, 1\}$"="$\{-, 0, +\}$ is defined by the relations $0 \prec -$, and $0 \prec +$; the sign components "$-$" and "$+$" are by convention *incomparable*. This induces the *product partial order* on $\{-, 0, +\}^t$, in which sign vectors are compared *componentwise*. Thus $X \preceq Y$ if and only if $X(e) \in \{0, Y(e)\}$, for all $e \in E_t$.

Oriented matroids are defined by several equivalent axiom systems.

If $\mathscr{C} \subseteq \{-1, 0, 1\}^t$, then $\mathscr{C}$ is by definition the set of *circuits* of an oriented matroid on the ground set E_t if and only if it satisfies the following *Circuit Axioms*:

(C0) $\mathbf{0} \notin \mathscr{C}$;
(C1) $X \in \mathscr{C}$ implies $-X \in \mathscr{C}$;
(C2) $X, Y \in \mathscr{C}$ and $\underline{X} \subseteq \underline{Y}$ imply $X = Y$ or $X = -Y$;
(C3) if $X, Y \in \mathscr{C}$, $X \neq -Y$, and $e \in X^+ \cap Y^-$, then there is $Z \in \mathscr{C}$ such that $Z^- \subseteq (X^- \cup Y^-) - \{e\}$ and $Z^+ \subseteq (X^+ \cup Y^+) - \{e\}$.

An *oriented matroid* on E_t, with set of circuits $\mathscr{C}$, is denoted by $(E_t, \mathscr{C})$.

The circuit supports $\underline{\mathscr{C}} := \{\underline{C} \colon C \in \mathscr{C}\}$ in an oriented matroid $\mathscr{M} := (E_t, \mathscr{C})$ constitute the circuits of the *underlying matroid of* $\mathscr{M}$, denoted by $\underline{\mathscr{M}}$. The *rank* of $\mathscr{M}$ is by definition the rank of the matroid $\underline{\mathscr{M}}$.

A *vector* of an oriented matroid is any *composition* of its *circuits*. An *oriented matroid* on E_t given by set of its vectors $\mathscr{V}$ is denoted by $(E_t, \mathscr{V})$. A *maximal vector* of an oriented matroid is a vector whose *support* is *maximal* with respect to inclusion. An *oriented matroid* on the ground set E_t, with set of maximal vectors $\mathscr{W}$, is denoted by $(E_t, \mathscr{W})$.

If $\mathscr{L} \subseteq \{-1, 0, 1\}^t$, then the pair $(E_t, \mathscr{L})$ is an *oriented matroid* on the ground set E_t, with the set of *covectors* $\mathscr{L}$, if and only if $\mathscr{L}$ satisfies the following *Covector Axioms*:

(L0) $\mathbf{0} \in \mathscr{L}$;
(L1) $X \in \mathscr{L}$ implies $-X \in \mathscr{L}$;
(L2) $X, Y \in \mathscr{L}$ implies $X \circ Y \in \mathscr{L}$;
(L3) if $X, Y \in \mathscr{L}$ and $e \in \mathbf{S}(X, Y)$, then there exists $Z \in \mathscr{L}$ such that $Z(e) = 0$ and $Z(f) = (X \circ Y)(f) = (Y \circ X)(f)$ for all $f \notin \mathbf{S}(X, Y)$.

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For an oriented matroid $(E_t, \mathscr{L})$ of rank r, the poset $\widehat{\mathscr{L}} := \mathscr{L} \dot\cup \{\hat{1}\}$, with a greatest element $\hat{1}$ adjoined, is a *graded lattice* (the so-called *"big" face lattice*) of poset rank $r+1$; the least element $\hat{0}$ of $\widehat{\mathscr{L}}$ is the sign vector $\mathbf{0}$.

A *maximal covector* (or a *tope*) of an oriented matroid is defined to be a covector whose *support* is *maximal* with respect to inclusion. An *oriented matroid* $\mathscr{M}$ on the ground set E_t, with set of topes $\mathscr{T}$, is denoted as $(E_t, \mathscr{T})$.

The set $\mathscr{C}^*$ of *nonzero covectors* of an oriented matroid $\mathscr{M}$, with *inclusion-minimal supports*, is the set of *cocircuits* of $\mathscr{M}$. This *oriented matroid* $\mathscr{M}$ on the ground set E_t, with the set of cocircuits $\mathscr{C}^*$, is denoted as $(E_t, \mathscr{C}^*)$.

For every $e \in E_t$, the corresponding *positive halfspace* $\mathscr{T}_e^+$ of an oriented matroid $(E_t, \mathscr{T})$ with set of topes $\mathscr{T}$ is defined by

$$\mathscr{T}_e^+ := \{T \in \mathscr{T} : T(e) = 1\} ;$$

the *negative halfspace* $\mathscr{T}_e^-$ of $(E_t, \mathscr{T})$ is by definition the set $-\mathscr{T}_e^+$.

Given an oriented matroid $\mathscr{M} := (E_t, \mathscr{L}) = (E_t, \mathscr{T})$, the *vertex set* $\mathrm{V}(\mathscr{T}(\mathscr{L}))$ of the *tope graph* $\mathscr{T}(\mathscr{L}) := \mathscr{T}(\mathscr{L}(\mathscr{M}))$ is by definition the *set of topes* $\mathscr{T}$; two topes T' and T'' are connected by an *edge* in $\mathscr{T}(\mathscr{L})$, that is, $\{T', T''\} \in \mathscr{E}(\mathscr{T}(\mathscr{L}))$, if these topes are *adjacent*, that is, if T' and T'' *cover* the same element (called a *subtope*) of poset rank $r-1$ in the *big face lattice* $\widehat{\mathscr{L}}$, where r is the *rank* of the oriented matroid $\mathscr{M}$.

If $B \in \mathscr{T}$, then the *tope poset* $\mathscr{T}(\mathscr{L}, B)$, based at B, is determined by the partial order on the set of topes:

$$T' \preceq T'' \quad \Longleftrightarrow \quad \mathbf{S}(B, T') \subseteq \mathbf{S}(B, T'') .$$

A subset of topes $\mathscr{Q} \subset \mathscr{T}$ is said to be T-convex if the following implication is valid:

$$T', T'' \in \mathscr{Q}\ , \quad T \in \mathscr{T}\ , \quad |\mathbf{S}(T', T'')| = |\mathbf{S}(T', T)| + |\mathbf{S}(T, T'')| \quad \Longrightarrow \quad T \in \mathscr{Q}\ ,$$

that is, if $\mathscr{Q}$ contains every *shortest path* in the *tope graph* $\mathscr{T}(\mathscr{L})$ between any two of its members. The T-convex hull $\mathrm{conv}_{\mathrm{T}}(\mathscr{Q})$ of $\mathscr{Q} \subset \mathscr{T}$ is the intersection of all halfspaces that contain $\mathscr{Q}$.

All topes $T \in \mathscr{T}$ have the same *support* and the same *zero set* $E_{t,\circ} := \mathbf{z}(T)$. The elements in $E_{t,\circ}$ are called the *loops* of $\mathscr{M}$. Thus, $e \in E_t$ is a loop of $\mathscr{M} := (E_t, \mathscr{C})$ if and only if there is a *circuit* $(0, \dots, 0, \underset{\substack{\uparrow \\ e}}{1}, 0, \dots, 0) \in \mathscr{C}$.

If $e \notin C$, for every circuit $C \in \mathscr{C}$ of an oriented matroid $\mathscr{M} := (E_t, \mathscr{C})$, then e is called a *coloop* of $\mathscr{M}$.

Elements $e, f \in E_t$, $e \neq f$, are called *parallel* if $X(e) = X(f)$ for all $X \in \mathscr{L}$; they are called *antiparallel*, if $X(e) = -X(f)$ for all $X \in \mathscr{L}$.

Recall that by a *simple oriented matroid* we mean, throughout the book, an oriented matroid without *loops*, *parallel* elements, or *antiparallel* elements.

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The *restriction* of a sign vector $X \in \{-1,0,1\}^t$ to a subset $A \subseteq E_t$ is the sign vector $X|_A \in \{-1,0,1\}^A$ defined by $(X|_A)(e) := X(e)$ for all $e \in A$.

For an oriented matroid $\mathcal{M} := (E_t, \mathcal{L})$, the oriented matroid $(E_t - A, \mathcal{L}\backslash A)$ on the ground set $E_t - A$, given by its set of *covectors* $\mathcal{L}\backslash A := \{X|_{E_t - A} : X \in \mathcal{L}\} \subseteq \{-1,0,1\}^{E_t - A}$ is called the *deletion* $\mathcal{M}\backslash A$ or the *restriction* $\mathcal{M}|_{E_t - A}$.

The oriented matroid $(E_t, {}_{-A}\mathcal{L})$ on E_t, given by its set of *covectors* ${}_{-A}\mathcal{L} \subseteq \{-1,0,1\}^t$ is called the *reorientation* ${}_{-A}\mathcal{M}$.

An oriented matroid $\mathcal{M} := (E_t, \mathcal{C}) = (E_t, \mathcal{T})$ is said to be *acyclic* if there is no nonnegative circuit in the set $\mathcal{C}$ or, equivalently, if there exists a *nonnegative tope* in $\mathcal{T}$. A subset $A \subseteq E_t$ of the ground set is called *acyclic* if the *restriction* $\mathcal{M}|_A$ is *acyclic*. The oriented matroid $\mathcal{M}$ is said to be *totally cyclic* if for each element $e \in E_t$, there exists a *nonnegative circuit* $C \in \mathcal{C}$ such that $e \in C$.

The *circuits*, *vectors*, and *maximal vectors* of an oriented matroid $\mathcal{M}$ are defined to be the *cocircuits*, *covectors*, and *topes*, respectively, of the oriented matroid $\mathcal{M}^*$, the *dual* (or *orthogonal*) of $\mathcal{M}$. The *loops* of $\mathcal{M}$ are the *coloops* of $\mathcal{M}^*$; the oriented matroid $\mathcal{M}$ is *acyclic* if and only if $\mathcal{M}^*$ is *totally cyclic*.

For an oriented matroid $\mathcal{M} := (E_t, \mathcal{C}^*)$, a *single element extension* $\widetilde{\mathcal{M}} := (\widetilde{E_t}, \widetilde{\mathcal{C}^*})$ of $\mathcal{M}$ is defined to be an oriented matroid on a set $\widetilde{E_t} = E_t \dot\cup \{g\}$, with set of *cocircuits* $\widetilde{\mathcal{C}^*}$. If g is *not a coloop* of $\widetilde{\mathcal{M}}$, then the oriented matroid $\widetilde{\mathcal{M}}$ is called a *nontrivial extension* of $\mathcal{M}$. The set $\widetilde{\mathcal{C}^*}$ of *cocircuits* of the extension $\widetilde{\mathcal{M}}$ is described as follows:

(i) Let $\widetilde{\mathcal{M}}$ be a *nontrivial single element extension* of $\mathcal{M} := (E_t, \mathcal{C}^*) = (E_t, \mathcal{L})$. Then for every cocircuit $Y \in \mathcal{C}^*$ there is a *unique way* to extend Y to a cocircuit of $\widetilde{\mathcal{M}}$: there is a *unique function* $\sigma : \mathcal{C}^* \to \{-1,0,1\}$, called the *localization*, such that $\{(Y, \sigma(Y)) : Y \in \mathcal{C}^*\} \subseteq \widetilde{\mathcal{C}^*}$, that is, $(Y, \sigma(Y))$ is a *cocircuit* of $\widetilde{\mathcal{M}}$ for *every cocircuit* Y of $\mathcal{M}$. Furthermore, this σ satisfies $\sigma(-Y) = -\sigma(Y)$ for all $Y \in \mathcal{C}^*$.

(ii) $\widetilde{\mathcal{M}}$ is *uniquely* determined by σ, with

$$\begin{aligned}\widetilde{\mathcal{C}^*} = \Big\{\big(Y, \sigma(Y)\big) : Y \in \mathcal{C}^*\Big\} \\ \dot\cup \Big\{(Y' \circ Y'', 0) : Y', Y'' \in \mathcal{C}^*\ ,\ \sigma(Y') = -\sigma(Y'') \neq 0\ , \\ |\mathbf{S}(Y', Y'')| = 0\ ,\ \rho(Y' \circ Y'') = 2\Big\}\ ,\end{aligned}$$

where ρ is the poset *rank function* of the *big face lattice* $\widehat{\mathcal{L}}$.

1.2 Pattern Recognition on Oriented Matroids

In this section we state the *two-class pattern recognition problem* in the language of *oriented matroids*, define *committees of maximal (co)vectors*, and discuss *committee decision rules* of recognition.

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The Two-class Pattern Recognition Problem

Let $\mathscr{S}$ be a *simple oriented matroid*. Its ground set E_t is said to be a *training set* if it is equipped with a map $\lambda: E_t \to \{-1, 1\}$ such that the *training samples* $\lambda^{-1}(-1)$ and $\lambda^{-1}(1)$ are *nonempty*. The elements of the set E_t are called the *training patterns*.

By *classes* **A** and **B**, we mean certain disjoint sets which are partially characterized by the prescribed inclusions $\mathbf{A} \supseteq \lambda^{-1}(-1)$ and $\mathbf{B} \supseteq \lambda^{-1}(1)$. In other words, a pattern $e \in E_t$ *a priori* belongs to the class **A** if and only if $\lambda(e) = -1$; it *a priori* belongs to the class **B** if and only if $\lambda(e) = 1$.

Let $\widetilde{\mathscr{S}}$ be the *nontrivial single element extension* of $\mathscr{S}$ by a new unclassified pattern g such that g is *not a loop*, and every element of E_t is neither *parallel* nor *antiparallel* to g.

A *decision rule* is defined to be any map

$$\mathfrak{r}: E_t \,\dot{\cup}\, \{g\} \to \{-1, 0, 1\}$$

such that

$$e \in E_t \implies \mathfrak{r}(e) = \lambda(e) .$$

If $f \in E_t \dot{\cup} \{g\}$ and $\mathfrak{r}(f) = -1$, then a recognition complex using the rule $\mathfrak{r}$ relates the pattern f to the class **A**. If $\mathfrak{r}(f) = 1$, then the pattern f is recognized as an element of the class **B**. If $\mathfrak{r}(f) = 0$, then the recognition complex leaves the pattern f *unclassified*.

Committees of Maximal (Co)vectors for Oriented Matroids

Let us define *committees of maximal* (*co*)*vectors* for oriented matroids.

Definition 1.1. Let $\mathscr{M} := (E_t, \mathscr{T}) = (E_t, \mathscr{W})$ be an oriented matroid, on the ground set E_t, with set of topes $\mathscr{T}$, and with set of maximal vectors $\mathscr{W}$.

Let p be a rational number such that $0 \le p < 1$.

(i) A subset $\mathscr{K}^* \subset \mathscr{T}$ is a *tope p-committee* (or a *p-committee of maximal covectors*) *for $\mathscr{M}$*, if for every $e \in E_t$ we have

$$|\{K \in \mathscr{K}^* : K(e) = 1\}| > p|\mathscr{K}^*| .$$

A tope $\frac{1}{2}$-committee $\mathscr{K}^*$ for $\mathscr{M}$, that is, a subset $\mathscr{K}^* \subset \mathscr{T}$ with the relation

$$\sum_{K \in \mathscr{K}^*} K \ge \mathrm{T}^{(+)}$$

interpreted componentwise is called a *tope committee for $\mathscr{M}$*.

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(ii) A subset $\mathscr{K} \subset \mathscr{W}$ is a *p-committee of maximal vectors for* $\mathscr{M}$, if for every $e \in E_t$ we have

$$|\{K \in \mathscr{K} : K(e) = 1\}| > p|\mathscr{K}| .$$

A $\frac{1}{2}$-committee of maximal vectors for $\mathscr{M}$ is called a *committee of maximal vectors for* $\mathscr{M}$.

Remark 1.2.

(i) A set $\mathscr{K}^*$ is a tope p-committee for an oriented matroid $\mathscr{M}$ if and only if $\mathscr{K}^*$ is a p-committee of maximal vectors for the dual oriented matroid $\mathscr{M}^*$.

(ii) A subset $\mathscr{K}^* \subset \mathscr{T}$ is a committee for $\mathscr{M}$ if and only if the set $-(\mathscr{T} - \mathscr{K}^*) := \{-T : T \in \mathscr{T} - \mathscr{K}^*\}$ is also a committee for $\mathscr{M}$.

(iii) Suppose $\mathscr{K}^*$ is a tope committee for $\mathscr{M}$. If $\widetilde{\mathscr{M}}$ is the *trivial single element extension* of $\mathscr{M}$ by a *coloop*, then the set $\{(K, 1) : K \in \mathscr{K}^*\}$ is a tope committee for $\widetilde{\mathscr{M}}$.

Given an oriented matroid $\mathscr{M}$, we denote the family of all its *tope committees* by $\mathbf{K}^*(\mathscr{M})$.

Definition 1.3. Let $\mathscr{M} := (E_t, \mathscr{T})$ be an oriented matroid, and $\mathscr{K}^*$ a tope committee for $\mathscr{M}$.

(i) The committee $\mathscr{K}^*$ is *minimal* if any proper subset of the set $\mathscr{K}^*$ is not a tope committee for $\mathscr{M}$.

(ii) If the committee $\mathscr{K}^*$ is minimal, then it is a *critical committee for* $\mathscr{M}$ if

$$\sum_{K \in \mathscr{K}^*} K = \mathrm{T}^{(+)} .$$

(iii) The committee $\mathscr{K}^*$ is a *committee of minimal size* if $|\mathscr{K}^*| \le |\mathscr{Q}^*|$, for any committee $\mathscr{Q}^* \in \mathbf{K}^*(\mathscr{M})$.

It follows directly from Definition 1.3 that minimal committees and critical committees, as well as committees of minimal size, have *no pairs of opposites*. Note that if an oriented matroid $\mathscr{M}$ is *acyclic*, then the one-element set $\{\mathrm{T}^{(+)}\}$ is a *critical tope committee of minimal size* for $\mathscr{M}$.

Example 1.4. Consider the realizable acyclic oriented matroid $\mathscr{N}^0 := (E_6, \mathscr{T}^0)$ given by the linear hyperplane arrangement of Figure 1.1.

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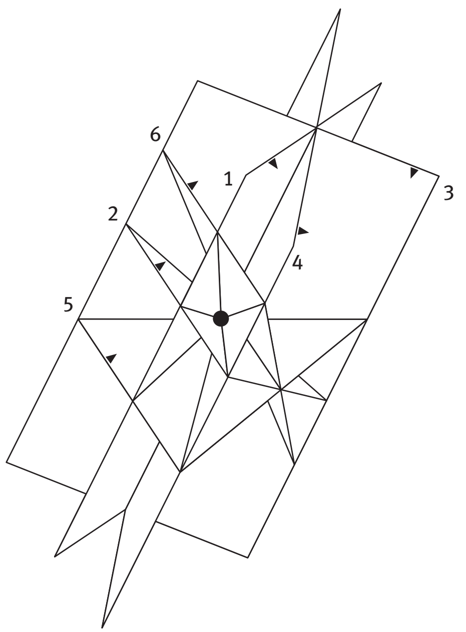

Figure 1.1: A linear hyperplane arrangement that realizes a simple acyclic oriented matroid $\mathcal{N}^0 := (E_6, \mathcal{T}^0)$ with 28 topes. The positive halfspaces of $\mathbb{R}^3$ are marked by arrows.

The sets of topes $\mathcal{T}^0$ and $\mathcal{T}^2$ of the oriented matroids $\mathcal{N}^0$ and $\mathcal{N}^2 := {}_{-[2]}\mathcal{N}^0 = (E_6, \mathcal{T}^2)$, respectively, are as follows:

$$\mathcal{T}^0 := \left\{ \begin{array}{cccccc}
+ & + & + & + & + & + \\
+ & + & + & - & + & + \\
- & + & + & - & + & + \\
- & + & + & - & + & - \\
+ & + & + & - & + & - \\
+ & + & + & + & + & - \\
+ & + & + & + & - & + \\
+ & - & + & + & - & + \\
+ & - & + & + & + & - \\
+ & - & + & - & + & - \\
- & - & + & - & + & - \\
- & - & + & - & - & - \\
+ & - & + & - & - & - \\
+ & - & + & + & - & - \\
- & + & - & - & + & + \\
- & + & - & + & + & +
\end{array} \right. \qquad
\mathcal{T}^2 = \left\{ \begin{array}{cccccc}
- & - & + & + & + & + \\
- & - & + & - & + & + \\
+ & - & + & - & + & + \\
+ & - & + & - & + & - \\
- & - & + & - & + & - \\
- & - & + & + & + & - \\
- & - & + & + & - & + \\
- & + & + & + & - & + \\
- & + & + & + & + & - \\
- & + & + & - & + & - \\
+ & + & + & - & + & - \\
+ & + & + & - & - & - \\
- & + & + & - & - & - \\
- & + & + & + & - & - \\
+ & - & - & - & + & + \\
+ & - & - & + & + & +
\end{array} \right.$$

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$$
\begin{array}{cccccc}
+ & + & - & + & + & + \\
+ & + & - & + & - & + \\
- & + & - & + & - & + \\
- & + & - & - & - & + \\
- & + & - & - & + & - \\
- & - & - & - & + & - \\
- & - & - & - & - & + \\
- & - & - & + & - & + \\
+ & - & - & + & - & + \\
+ & - & - & + & - & - \\
- & - & - & + & - & - \\
- & - & - & - & - & - \;\}
\end{array}
\qquad\qquad
\begin{array}{cccccc}
- & - & - & + & + & + \\
- & - & - & + & - & + \\
+ & - & - & + & - & + \\
+ & - & - & - & - & + \\
+ & - & - & - & + & - \\
+ & + & - & - & + & - \\
+ & + & - & - & - & + \\
+ & + & - & + & - & + \\
- & + & - & + & - & + \\
- & + & - & + & - & - \\
+ & + & - & + & - & - \\
+ & + & - & - & - & - \;\}
\end{array}
$$

The tope committee

$$
\begin{array}{ccccccc}
\{ & + & + & + & - & + & - \\
 & - & + & + & + & - & + \\
 & + & - & - & + & + & + \\
 & + & + & - & + & - & + \\
 & + & - & + & - & + & + \\
 & - & + & + & + & + & - \;\}
\end{array}
$$

for $\mathscr{N}^2$, of even size, is not minimal; indeed, it splits up into a disjoint union of two critical committees,

$$
\begin{array}{ccccccc}
\{ & + & + & + & - & + & - \\
 & - & + & + & + & - & + \\
 & + & - & - & + & + & + \;\}
\end{array}
\quad \text{and} \quad
\begin{array}{ccccccc}
\{ & + & + & - & + & - & + \\
 & + & - & + & - & + & + \\
 & - & + & + & + & + & - \;\}\,,
\end{array}
$$

of minimal size; note that $\mathscr{N}^2$ is not acyclic.

Since the topes K in the committee

$$
\begin{array}{ccccccc}
\{ & + & + & + & - & + & - \\
 & - & + & + & + & - & + \\
 & + & + & - & + & - & + \\
 & + & - & - & + & + & + \\
 & + & - & + & - & + & + \;\}
\end{array}
$$

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for $\mathscr{N}^2$ have the positive parts K^+ that are maximal with respect to inclusion, this committee, intuitively, is of "better quality" than the critical committee

$$\{\begin{array}{cccccc} - & + & + & - & + & - \\ - & + & + & + & - & - \\ + & + & - & + & - & + \\ + & - & - & + & + & + \\ + & - & + & - & + & + \end{array}\} .$$

Committee Decision Rules of Recognition

Let $\mathscr{S}$ be a *simple* oriented matroid whose ground set E_t is regarded as a *training set* in the two-class pattern recognition problem. Denote by $\mathscr{M}$ the *reorientation*

$$\mathscr{M} := {}_{-\lambda^{-1}(-1)}\mathscr{S} .$$

Let $\widetilde{\mathscr{M}} := (E_t \dot\cup \{g\}, \widetilde{\mathscr{C}^*})$ denote the *nontrivial single element extension* of $\mathscr{M} := (E_t, \mathscr{C}^*) = (E_t, \mathscr{L})$ by a new pattern g such that the oriented matroid $\widetilde{\mathscr{M}}$ is *simple*. Let $\sigma: \mathscr{C}^* \to \{-1, 0, 1\}$ be the corresponding *localization*.

If $\mathscr{K}^*$ is a *tope committee for* $\mathscr{M}$, then assign to each tope $K \in \mathscr{K}^*$ the set of *cocircuits*

$$\mathscr{C}_K{}^* := \{D \in \mathscr{C}^* : D \text{ conforms to } K\} .$$

– If the sets $\{(D, \sigma(D)) : D \in \mathscr{C}_K{}^*\}$ are *conformal*, for all topes $K \in \mathscr{K}^*$, then define a subset of topes $\widetilde{\mathscr{K}^*}$ of $\widetilde{\mathscr{M}}$ as follows:

$$\widetilde{\mathscr{K}^*} := \left\{ \bigcirc_{D \in \mathscr{C}_K{}^*} \left({}_{-\lambda^{-1}(-1)}D, \sigma(D)\right) : K \in \mathscr{K}^* \right\} . \qquad (1.1)$$

The *committee decision rule* corresponding to the committee $\mathscr{K}^*$ is defined to be the map $\mathfrak{r}: E_t \dot\cup \{g\} \to \{-1, 0, 1\}$ such that

$$\mathfrak{r}: f \mapsto \begin{cases} -1 , & \text{if } \left|\{\widetilde{K} \in \widetilde{\mathscr{K}^*} : \widetilde{K}(f) = -1\}\right| > \left|\{\widetilde{K} \in \widetilde{\mathscr{K}^*} : \widetilde{K}(f) = 1\}\right| , \\ 1 , & \text{if } \left|\{\widetilde{K} \in \widetilde{\mathscr{K}^*} : \widetilde{K}(f) = -1\}\right| < \left|\{\widetilde{K} \in \widetilde{\mathscr{K}^*} : \widetilde{K}(f) = 1\}\right| , \\ 0 , & \text{otherwise} . \end{cases} \qquad (1.2)$$

– If the set $\{(D, \sigma(D)) : D \in \mathscr{C}_K{}^*\}$ is *not conformal*, for some $K \in \mathscr{K}^*$, then the *committee decision rule* $\mathfrak{r}$ corresponding to the committee $\mathscr{K}^*$ is defined by

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$$\mathfrak{r}\colon e \mapsto \begin{cases} -1\,, & \text{if } \left|\{K \in \mathcal{K}^*\colon ({}_{-\lambda^{-1}(-1)}K)(e) = -1\}\right| \\ & \qquad > \left|\{K \in \mathcal{K}^*\colon ({}_{-\lambda^{-1}(-1)}K)(e) = 1\}\right|\,, \\ 1\,, & \text{if } \left|\{K \in \mathcal{K}^*\colon ({}_{-\lambda^{-1}(-1)}K)(e) = -1\}\right| \\ & \qquad < \left|\{K \in \mathcal{K}^*\colon ({}_{-\lambda^{-1}(-1)}K)(e) = 1\}\right|\,, \end{cases}$$

for all $e \in E_t$; by convention, $\mathfrak{r}\colon g \mapsto 0$.

Example 1.5. Figure 1.2(a) depicts a realization of a rank 2 simple oriented matroid $\mathcal{S}$ on the *training set* E_4. A realization of its reorientation $\mathcal{M} := {}_{-\lambda^{-1}(-1)}\mathcal{S} = {}_{-4}\mathcal{S}$ is shown in Figure 1.2(b). The set of topes

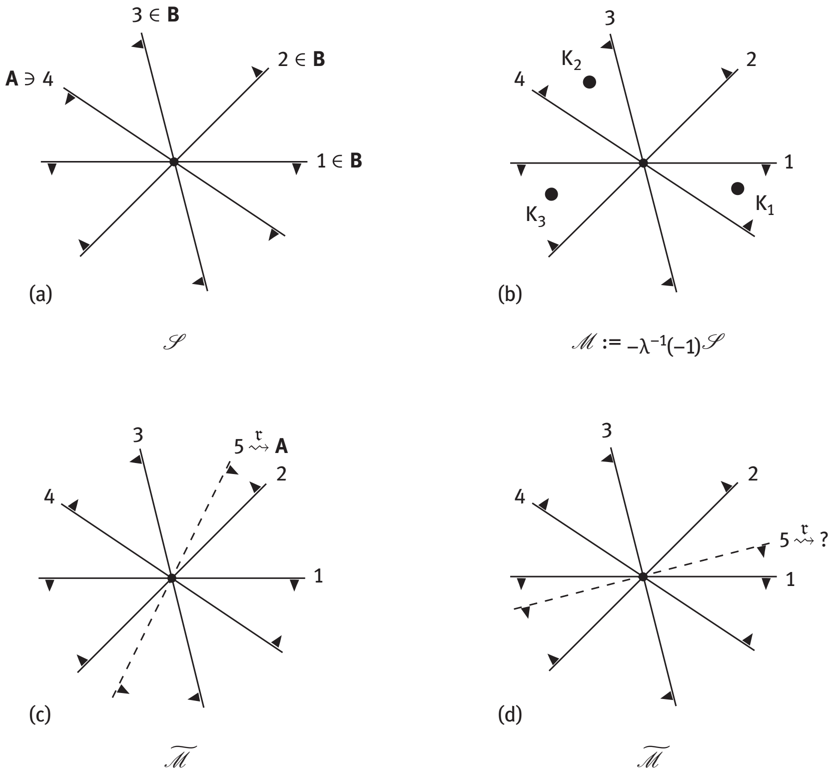

Figure 1.2: (a) A central line arrangement that realizes a rank 2 simple oriented matroid $\mathcal{S}$ on the *training set* E_4. The positive halfplanes of $\mathbb{R}^2$ are marked by arrows; (b) A realization of the reorientation $\mathcal{M} := {}_{-\lambda^{-1}(-1)}\mathcal{S}$. The set $\{K_1, K_2, K_3\}$ of regions marked by discs corresponds to a *tope committee for* $\mathcal{M}$; (c) The new pattern 5 is recognized as an element of the class **A**; (d) The new pattern 5 is not recognized.

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$$\mathscr{K}^* := \begin{array}{l} \{\ K_1 := + - - + \\ \ \ K_2 := - + + + \\ \ \ K_3 := + + + - \ \} \end{array}$$

is a *committee for* $\mathscr{M}$.

– Let $\widetilde{\mathscr{M}}$ be a nontrivial single element extension of $\mathscr{M}$ by the new pattern 5, as shown in Figure 1.2(c).

  Each of the cocircuit sets

$$\big\{\big(D, \sigma(D)\big) : D \in \mathscr{C}_{K_1}{}^*\big\} = \begin{array}{l} \{\ 0 - - + + \\ \ \ + - - 0 + \ \}\,, \end{array}$$

$$\big\{\big(D, \sigma(D)\big) : D \in \mathscr{C}_{K_2}{}^*\big\} = \begin{array}{l} \{\ - + 0 + - \\ \ \ - + + 0 - \ \} \end{array}$$

  and

$$\big\{\big(D, \sigma(D)\big) : D \in \mathscr{C}_{K_3}{}^*\big\} = \begin{array}{l} \{\ 0 + + - - \\ \ \ + 0 + - - \ \}\,, \end{array}$$

  is *conformal*. The set of topes $\widetilde{\mathscr{K}^*}$ defined by eq. (1.1) is

$$\widetilde{\mathscr{K}^*} = \begin{array}{l} \{\ + - - - + \\ \ \ - + + - - \\ \ \ + + + + - \ \}\,. \end{array}$$

  Therefore, the decision rule $\mathfrak{r}$ corresponding to the committee $\mathscr{K}^*$ and defined by (1.2) recognizes the pattern 5 as an element of the class **A**.

– If $\widetilde{\mathscr{M}}$ is a nontrivial single element extension of $\mathscr{M}$ by the new pattern 5, as shown in Figure 1.2(d), then the set of cocircuits

$$\big\{\big(D, \sigma(D)\big) : D \in \mathscr{C}_{K_3}{}^*\big\} = \begin{array}{l} \{\ 0 + + - - \\ \ \ + 0 + - + \ \} \end{array}$$

  is *not conformal*. As a consequence, $\mathfrak{r}(g) := 0$, that is, the decision rule $\mathfrak{r}$ corresponding to $\mathscr{K}^*$ leaves the pattern 5 unclassified.

1.3 The Existence of a Tope Committee: Reorientations

In this section we show that any *simple* oriented matroid has a *tope committee*. In fact, a *critical* tope committee for a simple oriented matroid can be built up from information on an arbitrary *maximal chain* in its *tope poset*.

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Our argument uses the technique of *consecutive reorientations* of an initial *acyclic* oriented matroid on *one-element subsets* of its ground set.

The following basic construct is a direct generalization of the *centrally symmetric cycles* of *adjacent regions* in the *arrangements* of *oriented linear hyperplanes*:

Definition 1.6. Let $\mathscr{M} := (E_t, \mathscr{L}) = (E_t, \mathscr{T})$ be a simple oriented matroid. A cycle $\boldsymbol{R} := (T^0, T^1, \ldots, T^{2t-1}, T^0)$ in its tope graph $\mathscr{T}(\mathscr{L})$ is a *symmetric* $2t$*-cycle* (or a *symmetric cycle* for short) if

$$T^{k+t} = -T^k\ , \quad 0 \le k \le t-1\ .$$

Remark 1.7. Let $\mathscr{M} := (E_t, \mathscr{T})$ be a simple oriented matroid, and $\boldsymbol{R}$ a symmetric cycle in its tope graph.

The vertex set $\mathrm{V}(\boldsymbol{R})$ of the cycle $\boldsymbol{R}$ is the set of *topes* of a *rank* 2 *simple oriented matroid*, denoted as $\mathscr{M}_{\boldsymbol{R}} := (E_t, \mathrm{V}(\boldsymbol{R}))$; the *ground set* of $\mathscr{M}_{\boldsymbol{R}}$ is the ground set E_t of $\mathscr{M}$.

The set of *cocircuits*, denoted $\mathscr{C}_{\boldsymbol{R}}{}^*$, of the oriented matroid $\mathscr{M}_{\boldsymbol{R}} := (E_t, \mathscr{C}_{\boldsymbol{R}}{}^*)$ is the subset of *subtopes* of $\mathscr{M}$, each of which is covered in the big face lattice of $\mathscr{M}$ by a pair of adjacent vertices of the cycle $\boldsymbol{R}$.

The oriented matroid $\mathscr{M}_{\boldsymbol{R}} := (E_t, \mathrm{V}(\boldsymbol{R})) = (E_t, \mathscr{C}_{\boldsymbol{R}}{}^*)$ can be represented by a central line arrangement in the plane; the set of topes $\mathrm{V}(\boldsymbol{R})$ of $\mathscr{M}_{\boldsymbol{R}}$ corresponds to the centrally symmetric set of regions of the arrangement, and the set $\mathscr{C}_{\boldsymbol{R}}{}^*$ of cocircuits of $\mathscr{M}_{\boldsymbol{R}}$ corresponds to the centrally symmetric set of rays emanating from the origin.

Remark 1.8. Let $\boldsymbol{R}$ be a symmetric cycle in the tope graph of a simple oriented matroid.

(i) For every element $e \in E_t$, the set of topes

$$\{T \in \mathrm{V}(\boldsymbol{R}) \colon T(e) = 1\}$$

is the vertex set of a path of length $t-1$ in $\boldsymbol{R}$; if a tope X is an endvertex of this path, then the other endvertex is the tope ${}_{-e}(-X)$.

(ii) Let $(T^{k_1}, T^{k_2}, T^{k_3})$ be a 2-path in $\boldsymbol{R}$. We have

$$T^{k_2} \in \mathbf{max}^+(\mathrm{V}(\boldsymbol{R})) \tag{1.3}$$

if and only if $(T^{k_1})^+ \subsetneqq (T^{k_2})^+ \supsetneqq (T^{k_3})^+$ or, equivalently, $|(T^{k_1})^+| = |(T^{k_3})^+| = |(T^{k_2})^+| - 1$.

In other words, let $\{f\} := \mathbf{S}(T^{k_1}, T^{k_2})$ and $\{g\} := \mathbf{S}(T^{k_2}, T^{k_3})$; inclusion (1.3) holds if and only if $T^{k_2}(f) = T^{k_2}(g) = 1$.

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Rank 2

In the theory of oriented matroids, the rank 2 case is instructive. Note that the *tope graph* of a *simple rank* 2 oriented matroid $\mathscr{M}$ is just a *symmetric cycle*. In this section we show that $\mathscr{M}$ has a *unique critical* tope committee.

Lemma 1.9. *Let $\mathscr{N}^0 := (E_t, \mathscr{T}^0)$ be a simple* acyclic *oriented matroid of rank* 2*, on the ground set E_t, with set of topes $\mathscr{T}^0$.*

Let $(j_1, \dots, j_s)$ be a nonempty sequence of integers with $j_i \in E_t$, $1 \le i \le s$. Define the reorientation $\mathscr{N}^i := (E_t, \mathscr{T}^i) := {}_{-j_i}\mathscr{N}^{i-1}$ whose set of topes is denoted as $\mathscr{T}^i$.

The reorientation $\mathscr{N}^s$ of $\mathscr{N}^0$ has a critical *tope committee.*

Three types of committee transformations are carried out by Algorithm 1.10 that will underlie the constructive proof of Lemma 1.9 given on page 21. Let us illustrate these transformations by considering several rank 2 oriented matroids represented by central line arrangements in the plane. The regions corresponding to the topes which are members of committees will be marked in figures by discs or circles.

We begin with the one-tope *critical* committee $\mathscr{K}_0{}^* := \{\mathrm{T}^{(+)}\}$ for the oriented matroid $\mathscr{N}^0$.

Let $\mathscr{K}_i^*$ denote the tope committee built up by Algorithm 1.10 for the reorientation $\mathscr{N}^i$.

1. If there is a tope $K \in \mathscr{K}_{i-1}{}^*$ such that
$$K(j_i) = 1 \quad\text{and}\quad {}_{-j_i}(-K) \in \mathscr{T}^{i-1}$$
(or, equivalently, $K(j_i) = 1$, and there is a subtope $H \lessdot K$ such that $\mathbf{z}(H) = \{j_i\}$), but there is no tope S in $\mathscr{K}_{i-1}{}^*$ such that $S(j_i) = 1$ and ${}_{-j_i}S = -K$, then the set
$$\mathscr{K}_i^* := {}_{-j_i}\big(\mathscr{K}_{i-1}{}^* - \{K\}\big) \,\dot\cup\, \{K\}$$
is a tope committee for $\mathscr{N}^i$; see Figure 1.3.
2. If there is no tope K in $\mathscr{K}_{i-1}{}^*$ such that $K(j_i) = 1$ and ${}_{-j_i}(-K) \in \mathscr{T}^{i-1}$, then pick the pair of topes $\{T'', T'''\} \subset \mathscr{T}^{i-1}$ such that $T''(j_i) = T'''(j_i) = -1$ and ${}_{-j_i}T'' = -T'''$. The set
$$\mathscr{K}_i^* := {}_{-j_i}\big(\mathscr{K}_{i-1}{}^* \,\dot\cup\, \{T'', T'''\}\big)$$
is a tope committee for $\mathscr{N}^i$; see Figure 1.4.
3. If there is a tope $K \in \mathscr{K}_{i-1}{}^*$ such that $K(j_i) = 1$ and ${}_{-j_i}(-K) \in \mathscr{T}^{i-1}$, and if there is a tope $S \in \mathscr{K}_{i-1}{}^*$ such that ${}_{-j_i}S = -K$, then the set
$$\mathscr{K}_i^* := {}_{-j_i}\big(\mathscr{K}_{i-1}{}^* - \{K, S\}\big)$$
is a tope committee for $\mathscr{N}^i$; see Figure 1.5.

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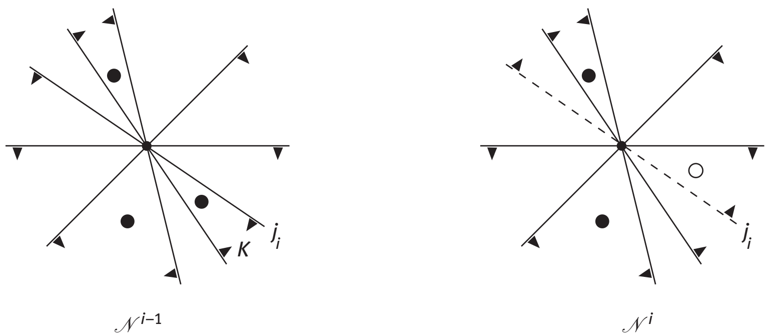

Figure 1.3: A transformation of a tope committee under a reorientation: $\mathscr{K}_i^* := {}_{-j_i}(\mathscr{K}_{i-1}^* - \{K\}) \dot\cup \{K\}$; here $|\mathscr{K}_i^*| = |\mathscr{K}_{i-1}^*|$.

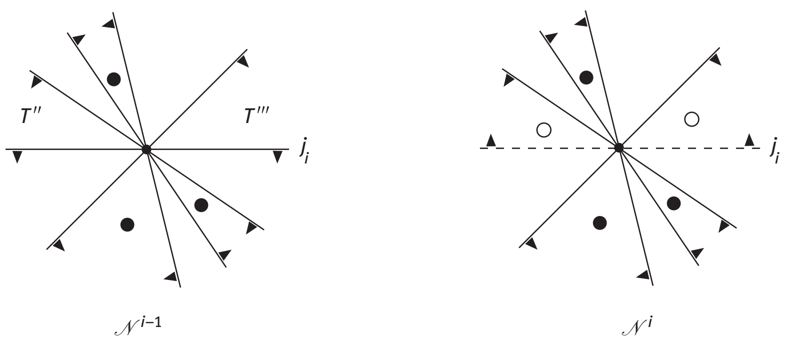

Figure 1.4: A transformation of a tope committee under a reorientation: $\mathscr{K}_i^* := {}_{-j_i}(\mathscr{K}_{i-1}^* \dot\cup \{T'', T'''\})$; here $|\mathscr{K}_i^*| = |\mathscr{K}_{i-1}^*| + 2$.

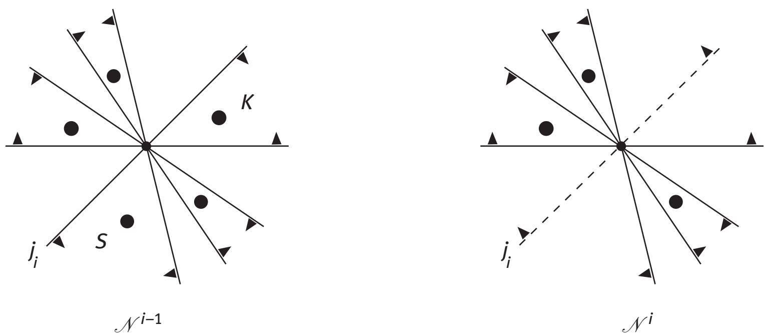

Figure 1.5: A transformation of a tope committee under a reorientation: $\mathscr{K}_i^* := {}_{-j_i}\Big(\mathscr{K}_{i-1}^* - \{K, S\}\Big)$; here $|\mathscr{K}_i^*| = |\mathscr{K}_{i-1}^*| - 2$.

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Algorithm 1.10.

$\mathcal{K}_0{}^* \leftarrow \{\mathrm{T}^{(+)}\}$;
for $i \leftarrow 1$ **to** s
  do $\mathcal{K}_i^* \leftarrow$ `emptyset`;
    FOUND $\leftarrow$ `false`;
    while $|\mathcal{K}_{i-1}{}^*| > 0$
      do *pick a tope* $K \in \mathcal{K}_{i-1}{}^*$;
        if

$$K(j_i) = 1 \text{ and } {}_{-j_i}(-K) \in \mathcal{T}^{i-1} \tag{1.4}$$

          then FOUND $\leftarrow$ `true`;
            if *there is a tope* $S \in \mathcal{K}_{i-1}{}^*$ *such that*

$$S(j_i) = 1 \text{ and } {}_{-j_i}S = -K \tag{1.5}$$

              then $\mathcal{K}_{i-1}{}^* \leftarrow \mathcal{K}_{i-1}{}^* - \{S\}$;
              else $\mathcal{K}_i^* \leftarrow \mathcal{K}_i^* \dot\cup \{K\}$;
          else $\mathcal{K}_i^* \leftarrow \mathcal{K}_i^* \dot\cup \left\{{}_{-j_i}K\right\}$;
        $\mathcal{K}_{i-1}{}^* \leftarrow \mathcal{K}_{i-1}{}^* - \{K\}$;
    if FOUND = `false`
      then *pick the pair of topes* $\{T'', T'''\} \subset \mathcal{T}^{i-1}$ *such that*

$$T''(j_i) = T'''(j_i) = -1 \text{ and } {}_{-j_i}T'' = -T''';$$

$$\mathcal{K}_i^* \leftarrow \mathcal{K}_i^* \dot\cup \left\{{}_{-j_i}T'', {}_{-j_i}T'''\right\};$$

We will now prove Lemma 1.9.

Proof. In order to show that the set $\mathcal{K}_s{}^* \subset \mathcal{T}^s$ built up by Algorithm 1.10 is a critical tope committee for $\mathcal{N}^s$, let us verify two assertions:
Claim 1: *For any i*, $1 \le i \le s$, *the set* $\mathcal{K}_i^*$ *is of* odd size, *and for any* $e \in E_t$, *we have*

$$|\{K \in \mathcal{K}_i{}^* : K(e) = 1\}| = \left\lceil \tfrac{|\mathcal{K}_i{}^*|}{2} \right\rceil ; \tag{1.6}$$

as a consequence,

$$\sum_{K \in \mathcal{K}_i{}^*} K = \mathrm{T}^{(+)} .$$

Claim 2: *For any i*, $1 \le i \le s$, *we have*

$$\mathcal{K}_i^* = \left\{K \in \mathcal{T}^i : T \in \mathcal{T}^i,\ \mathbf{S}(K, T) =: \{e\} \implies K(e) = 1\right\} ; \tag{1.7}$$

as a consequence, the committee $\mathcal{K}_i^*$ *is minimal.*

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Indeed, let $i := 1$.

– If conditions (1.4) are satisfied for $K := \mathrm{T}^{(+)}$ in $\mathcal{N}^{i-1}$, then pick the tope $T \in \mathcal{T}^{i-1}$ such that $\mathbf{S}(K, T) = \{j_i\}$ in $\mathcal{N}^{i-1}$; the one-element set $\mathcal{K}_i^* := \left\{{}_{-j_i}T\right\} = \{K\} = \{\mathrm{T}^{(+)}\}$, formed at Step *10* of the algorithm, is a tope committee for $\mathcal{N}^i$.
– If conditions (1.4) are not satisfied for $K := \mathrm{T}^{(+)}$, then the three-element set of maximal covectors $\mathcal{K}_i^* := \{K', K'', K'''\}$, formed at Step *14* of the algorithm, with

$$K' := {}_{-j_i}K\,, \quad K'' := {}_{-j_i}T''\,, \quad K''' := {}_{-j_i}T'''\,,$$

is a tope committee for $\mathcal{N}^i$, since we have

$$\mathcal{K}_i^* = \left\{ \begin{array}{lccccccc} K' = & 1 & \dots & 1 & -1 & 1 & \dots & 1 \\ & & & \uparrow & \uparrow & \uparrow & & \\ & & & (j_i-1) & j_i & (j_i+1) & & \\ K'' = & ? & \dots & ? & 1 & ? & \dots & ? \\ K''' = & -(K''(1)) & \dots & -(K''(j-1)) & 1 & -(K''(j+1)) & \dots & -(K''(t)) \end{array} \right\},$$

that is, for every $e \in E_t$, we have $|\{K \in \mathcal{K}_i^* : K(e) = 1\}| = 2 = \left\lceil \frac{|\mathcal{K}_i^*|}{2} \right\rceil$.

Note that conditions (1.6) and (1.7) are satisfied for $i = 1$.

Now let $i > 1$.

– If there is no tope K in $\mathcal{K}_{i-1}{}^*$ such that conditions (1.4) are satisfied, then for the topes ${}_{-j_i}T''$ and ${}_{-j_i}T'''$ added to the set $\mathcal{K}_i^*$ at Step *14* of the algorithm, we have $\left({}_{-j_i}T''\right)(j_i) = \left({}_{-j_i}T'''\right)(j_i) = 1$. Assume that for the tope $T \in \mathcal{T}^{i-1}$ such that $\mathbf{S}(T'', T) =: \{f\}$ and $f \neq j_i$, it holds $T''(f) = -1$. Then the tope $K := -T''$, with $K(j_i) = K(f) = 1$, must belong to the committee $\mathcal{K}_{i-1}{}^*$ and satisfy conditions (1.4), but this contradicts the negative decision made at Step *06*. Hence, $T''(f) = 1$. For the tope $Q \in \mathcal{T}^{i-1}$ such that $\mathbf{S}(T''', Q) =: \{g\}$ and $g \neq j_i$, we also have $T'''(g) = 1$. As a result, eqs. (1.6) and (1.7) are satisfied; hence, $\mathcal{K}_i^*$ is a tope committee for $\mathcal{N}^i$, of size $|\mathcal{K}_i^*| = |\mathcal{K}_{i-1}{}^*| + 2$.
– If there are topes $K, S \in \mathcal{K}_{i-1}{}^*$ such that conditions (1.4) and (1.5) are satisfied, then the algorithm excludes them from consideration at Steps *09* and *12*, and we obtain $|\mathcal{K}_i^*| = |\mathcal{K}_{i-1}{}^*| - 2$.

  If there is a tope $K \in \mathcal{K}_{i-1}{}^*$ such that conditions (1.4) are satisfied, but there is no tope S in $\mathcal{K}_{i-1}{}^*$ satisfying eq. (1.5), then we have $|\mathcal{K}_i^*| = |\mathcal{K}_{i-1}{}^*|$.

  In any case, eq. (1.6) is satisfied for all $e \in E_t$; therefore, $\mathcal{K}_i^*$ is a tope committee for $\mathcal{N}^i$.

If $\mathrm{T}^{(+)} \notin \mathcal{K}_s{}^*$, then assume that $\mathcal{K}_s{}^*$ is not minimal, that is, there is a proper subset $\mathcal{Q}^*$ of the set $\mathcal{K}_s{}^*$ such that $\mathcal{K}_s{}^* - \mathcal{Q}^*$ is a committee for $\mathcal{N}^s$. Since $\mathrm{T}^{(+)} \notin \mathcal{K}_s{}^*$, we have $|\mathcal{K}_s{}^* - \mathcal{Q}^*| > 1$.

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Denote by $\boldsymbol{R} := (T^0, T^1, \dots, T^{2t-1}, T^0)$ the cycle which is the tope graph of $\mathscr{N}^s$, and suppose without loss of generality that $T^0 \in \mathscr{K}_s{}^*$ and $T^0 \notin \mathscr{Q}^*$. Recall that $\mathscr{K}_s{}^*$ is exactly the set $\mathbf{max}^+(\mathrm{V}(\boldsymbol{R})) = \mathbf{max}^+(\mathscr{T}^s)$, see Remark 1.8(ii).

Let $\{g\} := \mathbf{S}(T^0, T^1)$ and $\{f\} := \mathbf{S}(T^{2t-1}, T^0)$; note that $f \neq g$. We have $T^0(f) = T^0(g) = 1$. For every k, $0 < k < t$, we have $T^k(f) = 1$; for every l, $t < l < 2t$, we have $T^l(g) = 1$, see Remark 1.8(i).

Claim 1 implies that for every $e \in \{f, g\}$, we have $|\{Q \in \mathscr{Q}^* : Q(e) = -1\}| = |\{R \in \mathscr{Q}^* : R(e) = 1\}|$, and we have

$$|\{T^1, T^2, \dots, T^{t-1}\} \cap \mathscr{Q}^*| = |\{T^{t+1}, T^{t+2}, \dots, T^{2t-1}\} \cap \mathscr{Q}^*| = \frac{|\mathscr{Q}^*|}{2} ;$$

thus, $\mathscr{Q}^*$ is of even size.

Let T^{k_2} be a tope contained in the set $\mathscr{Q}^*$, and $(T^{k_1}, T^{k_2}, T^{k_3})$ a 2-path in $\boldsymbol{R}$. If $\{p\} := \mathbf{S}(T^{k_1}, T^{k_2})$ and $\{q\} := \mathbf{S}(T^{k_2}, T^{k_3})$, then there is an element $h \in \{p, q\}$ such that $|\{K \in \mathscr{K}_s{}^* - \mathscr{Q}^* : K(h) = 1\}| = \left\lfloor \frac{|\mathscr{K}_s{}^* - \mathscr{Q}^*|}{2} \right\rfloor$, that is, the set of topes $\mathscr{K}_s{}^* - \mathscr{Q}^*$ is not a committee for $\mathscr{N}^s$, a contradiction. Thus, $\mathscr{K}_s{}^*$ is minimal.

We conclude from Claims 1 and 2 that the committee $\mathscr{K}_s{}^*$ for the oriented matroid $\mathscr{N}^s$ is critical. □

Remark 1.11. In Lemma 1.9 we did not touch on the question of the uniqueness, but a linear algebraic argument appearing later in Chapter 11 guarantees that any rank 2 simple oriented matroid has *exactly one* critical tope committee.

Proposition 1.12. *Let $\mathscr{M} := (E_t, \mathscr{T})$ be a simple oriented matroid of rank 2.*

(i) *The only critical tope committee for $\mathscr{M}$ is the set*

$$\mathscr{K}^* := \left\{K \in \mathscr{T} : T \in \mathscr{T},\ \mathbf{S}(K, T) =: \{e\} \implies K(e) = 1\right\} . \tag{1.8}$$

(ii) *The critical committee (1.8) is the set*

$$\mathscr{K}^* = \mathbf{max}^+(\mathscr{T}) .$$

Proof. (i) If $\mathscr{M}$ is *acyclic*, then the *one-element* set $\{\mathrm{T}^{(+)}\}$ is its critical tope committee.

Suppose that $\mathscr{M}$ is *not acyclic*. If $\mathscr{N}^0$ is an *acyclic reorientation* of $\mathscr{M}$, and $J := (j_1, \dots, j_s) \subset E_t$ is an ordered subset of integers such that $\mathscr{M} = {}_{-J}\mathscr{N}^0$, then the proof of the assertion follows from (the proof of) Lemma 1.9 and Remark 1.11.

The proof of the assertion (ii) follows from the assertion (i) and Remark 1.8(ii). □

Arbitrary Rank

In this section we show, without explicit using Remark 1.7, that a *simple* oriented matroid of arbitrary rank has a *tope committee*. We again use the technique of

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consecutive reorientations of an initial *acyclic* oriented matroid $\mathcal{N}^0$ on *one-element subsets* of its ground set.

In order to simplify the exposition, we will define the ith reorientation of $\mathcal{N}^0$ to be the oriented matroid ${}_{-[i]}\mathcal{N}^0$.

Lemma 1.13. *Let* $\mathcal{N}^0 := (E_t, \mathcal{L}^0) = (E_t, \mathcal{T}^0)$ *be a simple* acyclic *oriented matroid on the ground set* E_t*, with set of covectors* $\mathcal{L}^0$*, and with set of topes* $\mathcal{T}^0$*.*

For each $i \in [t]$*, define the reorientation* $\mathcal{N}^i := {}_{-[i]}\mathcal{N}^0$*.*

For any $s \in [t]$*, the reorientation* $\mathcal{N}^s$ *of* $\mathcal{N}^0$ *has a tope committee.*

Before giving a constructive proof of this lemma, let us note that Proposition 1.27 on page 34 asserts that Algorithm 1.14 (and, as a consequence, Algorithm 1.15) underlying the proof constructs for the reorientation $\mathcal{N}^s$ a committee which is *critical*.

Algorithm 1.14.

01 $\mathcal{K}_0{}^* \leftarrow \{\mathrm{T}^{(+)}\}$;
  $\mathbf{m} \leftarrow$ *maximal chain* $(R^0 \lessdot R^1 \lessdot \cdots \lessdot R^t)$ *in* $\mathcal{T}^0(\mathcal{L}^0, \mathrm{T}^{(+)})$;
  $(\{\ell_1\}, \ldots, \{\ell_t\}) \leftarrow (\mathbf{S}(R^0, R^1), \ldots, \mathbf{S}(R^{t-1}, R^t))$;
02 **for** $i \leftarrow 1$ **to** s
03   **do** $\mathcal{K}_i^* \leftarrow {}_{-i}(\mathcal{K}_{i-1}{}^*)$;
    $R \leftarrow R^k$ *such that* $\ell_k = i$;
    $\mathcal{K}_i^* \leftarrow \mathcal{K}_i^* \,\dot\cup\, \{{}_{-[i]}R, {}_{-[i,t]}R\}$;
04   **while** $\mathcal{K}_i^* \supset \{K, T\}$ *such that* $T = -K$
05   **do** $\mathcal{K}_i^* \leftarrow \mathcal{K}_i^* - \{K, T\}$;

Algorithm 1.15.

01 $\mathcal{K}_0{}^* \leftarrow \{\mathrm{T}^{(+)}\}$;
  $\mathbf{m} \leftarrow$ *maximal chain* $(R^0 \lessdot R^1 \lessdot \cdots \lessdot R^t)$ *in* $\mathcal{T}^0(\mathcal{L}^0, \mathrm{T}^{(+)})$;
  $(\{\ell_1\}, \ldots, \{\ell_t\}) \leftarrow (\mathbf{S}(R^0, R^1), \ldots, \mathbf{S}(R^{t-1}, R^t))$;
02 **for** $i \leftarrow 1$ **to** s
03   **do** `multiset` $\mathcal{K}_i^* \leftarrow \{{}_{-i}K : K \in \mathcal{K}_{i-1}{}^*\}$;
    $R \leftarrow R^k$ *such that* $\ell_k = i$;
    $\mathcal{K}_i^* \leftarrow \{\mathcal{K}_i^*, {}_{-[i]}R, {}_{-[i,t]}R\}$;
04 **while** $\mathcal{K}_s{}^* \supset \{K, T\}$ *such that* $T = -K$
05   **do** $\mathcal{K}_s{}^* \leftarrow \mathcal{K}_s{}^* - \{K, T\}$;

Proof. Let us verify that the set of maximal covectors $\mathcal{K}_s{}^* \subset \mathcal{T}^s$ built up by Algorithms 1.14 and 1.15 is indeed a tope committee for $\mathcal{N}^s$.

Claim: *The set* $\mathcal{K}_s{}^*$ *is of* odd size*, and for any* $e \in E_t$*, we have*

$$|\{K \in \mathcal{K}_s{}^* : K(e) = 1\}| = \left\lceil \tfrac{|\mathcal{K}_s{}^*|}{2} \right\rceil ; \tag{1.9}$$

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as a consequence,

$$\sum_{K\in\mathcal{K}_s{}^*} K = \mathrm{T}^{(+)} .$$

Fix a *maximal chain* $\mathbf{m} := (R^0 := \mathrm{T}^{(+)} \prec\!\cdot\, R^1 \prec\!\cdot \cdots \prec\!\cdot\, R^t := \mathrm{T}^{(-)})$ in the *tope poset* $\mathcal{T}^0(\mathcal{L}^0, \mathrm{T}^{(+)})$. Associate with every tope $R^i \in \{R^1, \ldots, R^t\}$ a label $\ell_i \in [t]$ defined by

$$\{\ell_i\} := \mathbf{S}(R^{i-1}, R^i) . \tag{1.10}$$

Note that
- $|\{R^i : i \in [t-1]\} \cap \{{}_{-\ell_i}(-R^i) : i \in [2,t]\}| = 0$, since in the poset $\mathcal{T}^0(\mathcal{L}^0, \mathrm{T}^{(+)}) - \{\mathrm{T}^{(+)}, \mathrm{T}^{(-)}\}$ its maximal chains

$$\mathbf{m} - \{R^0, R^t\}$$

  and

$$\left({}_{-\ell_t}(-R^t) \prec\!\cdot\, {}_{-\ell_{t-1}}(-R^{t-1}) \prec\!\cdot \cdots \prec\!\cdot\, {}_{-\ell_2}(-R^2) \right)$$

  are disjoint;
- for every $i \in [t-1]$, we have

$$-R^i = {}_{-\ell_{i+1}}(-R^{i+1}) ;$$

  as a consequence, the set of topes

$$\left\{R^i : i \in [t-1]\right\} \,\dot\cup\, \left\{{}_{-\ell_i}(-R^i) : i \in [2,t]\right\}$$

  contains exactly $t-1$ pairs of opposites, see Figure 1.6;
- the multiset $\{R^0, {}_{-\ell_1}(-R^1), R^t\} = \{\mathrm{T}^{(+)}, \mathrm{T}^{(-)}, \mathrm{T}^{(-)}\}$ contains exactly two pairs of opposites, namely $\{R^0, {}_{-\ell_1}(-R^1)\}$ and $\{R^0, R^t\}$, see Figure 1.6;
- the sequence of topes

$$\begin{aligned}&\left(\mathbf{m}, {}_{-\ell_2}(-R^2), {}_{-\ell_3}(-R^3), \ldots, {}_{-\ell_t}(-R^t), R^0\right)\\ &\qquad = \left(R^0, R^1, \ldots, R^t, {}_{-\ell_2}(-R^2), {}_{-\ell_3}(-R^3), \ldots, {}_{-\ell_t}(-R^t), R^0\right)\end{aligned}$$

  is a symmetric cycle in the tope graph $\mathcal{T}^0(\mathcal{L}^0)$ of $\mathcal{N}^0$.

Now consider Algorithm 1.15, a slight modification of Algorithm 1.14. Both algorithms construct the same committee $\mathcal{K}_s{}^*$, provided the same maximal chain of topes $\mathbf{m}$ is chosen at their Steps *01*. Algorithm 1.14 immediately removes pairs of opposites if

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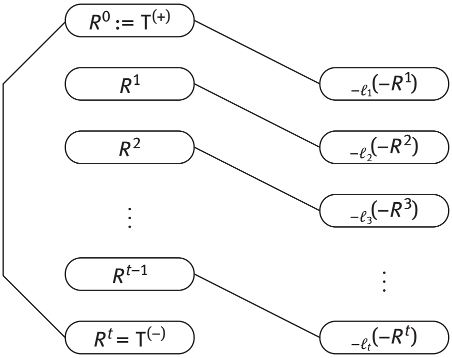

Figure 1.6: A multiset consisting of the elements $R^0 := \mathrm{T}^{(+)} \lessdot R^1 \lessdot \cdots \lessdot R^{t-1}$ of the tope poset $\mathcal{T}^0(\mathcal{L}^0, \mathrm{T}^{(+)})$ associated with a simple acyclic oriented matroid $\mathcal{N}^0 := (E_t, \mathcal{L}^0) = (E_t, \mathcal{T}^0)$, and of their opposites. Every pair of opposites is connected by an edge.

they appear, while Algorithm 1.15 removes pairs of opposites when it completes its operations at Steps *04–05*.

At Step *03*, Algorithm 1.15 constructs a multiset $\mathcal{K}_s{}^*$ of odd cardinality that satisfies eq. (1.9). At Steps *04–05*, pairs of opposites are deleted; the resulting set $\mathcal{K}_s{}^*$ still satisfies eq. (1.9). □

Example 1.16. Consider the simple *acyclic* oriented matroid $\mathcal{N}^0 := (E_6, \mathcal{T}^0)$ given by the linear hyperplane arrangement of Figure 1.1.

If the *maximal chain*

$$\mathbf{m} := \quad \begin{array}{rcrrrrrr} (R^0 & := & + & + & + & + & + & + \\ R^1 & := & + & + & - & + & + & + \\ R^2 & := & - & + & - & + & + & + \\ R^3 & := & - & + & - & - & + & + \\ R^4 & := & - & + & - & - & + & - \\ R^5 & := & - & - & - & - & + & - \\ R^6 & := & - & - & - & - & - & - \,) \end{array}$$

in its *tope poset* $\mathcal{T}^0(\mathcal{L}^0, \mathrm{T}^{(+)})$ is chosen at Step *01* of Algorithm 1.14, with the corresponding labels

$$\ell_1 = 3\,, \quad \ell_2 = 1\,, \quad \ell_3 = 4\,, \quad \ell_4 = 6\,, \quad \ell_5 = 2\,, \quad \ell_6 = 5\,,$$

then the algorithm constructs, when applied to the reorientation $\mathcal{N}^6 := {}_{-[6]}\mathcal{N}^0$, the following sequence of *critical tope committees*:

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$$\mathcal{K}_1^* = \begin{matrix}\{ & - & + & + & + & + & + & \\ & + & + & - & + & + & + & \\ & + & - & + & - & - & - & \};\end{matrix} \qquad \mathcal{K}_2^* = \begin{matrix}\{ & - & - & + & + & + & + & \\ & + & - & - & + & + & + & \\ & + & + & + & - & - & - & \\ & + & + & - & - & + & - & \\ & - & + & + & + & - & + & \};\end{matrix}$$

$$\mathcal{K}_3^* = \begin{matrix}\{ & + & - & + & + & + & + & \\ & + & + & + & - & + & - & \\ & - & + & - & + & - & + & \};\end{matrix} \qquad \mathcal{K}_4^* = \begin{matrix}\{ & + & + & + & + & + & - & \\ & - & + & - & - & - & + & \\ & + & - & + & + & + & + & \};\end{matrix}$$

$$\mathcal{K}_5^* = \begin{matrix}\{ & - & + & - & - & + & + & \\ & + & - & + & + & - & + & \\ & + & + & + & + & + & - & \};\end{matrix} \qquad \mathcal{K}_6^* = \{ + + + + + + \}.$$

Algorithm 1.14 does not necessarily construct a tope committee of minimal size. For example, it constructs for the reorientation $\mathcal{N}^5 := {}_{-[5]}\mathcal{N}^0$ of the oriented matroid $\mathcal{N}^0$ (which is realized by the hyperplane arrangement of Figure 1.1) a three-tope committee, while the oriented matroid $\mathcal{N}^5$ is acyclic, that is, the one-tope set $\{\mathrm{T}^{(+)}\}$ is a committee of minimal size for $\mathcal{N}^5$.

Any simple oriented matroid has a tope committee of size that is less than or equal to the cardinality of its ground set:

Theorem 1.17. *Let $\mathcal{N}^0 := (E_t, \mathcal{T}^0)$ be a simple* acyclic *oriented matroid on the ground set E_t, with set of topes $\mathcal{T}^0$. Let s be an integer, $s \le t$. Define the reorientation $\mathcal{N}^s := {}_{-[s]}\mathcal{N}^0$.*

(i) *The oriented matroid $\mathcal{N}^s$ has a tope committee $\mathcal{K}_s^*$ such that $|\mathcal{K}_s^*| \le t$ when t is odd, and $|\mathcal{K}_s^*| \le t-1$ when t is even.*

As a consequence, any simple oriented matroid $\mathcal{M}$ on the ground set E_t has a tope committee $\mathcal{K}^*$ such that

$$|\mathcal{K}^*| \le \begin{cases} t\,, & \textit{if } t \textit{ is odd}\,, \\ t-1\,, & \textit{if } t \textit{ is even}\,. \end{cases}$$

(ii) *If $s \le \lfloor t/2 \rfloor$, then $\mathcal{N}^s$ has a tope committee $\mathcal{K}_s^*$ such that $|\mathcal{K}_s^*| \le 2s+1$.*

Proof. In order to construct a tope committee for $\mathcal{N}^s$, let us apply Algorithm 1.14 to the reorientation $\mathcal{N}^s$; see the proof of Lemma 1.13.

(i) – If t is *odd*, then the tope committee of maximal size, which can be constructed by Algorithm 1.14, is the set

$$\mathcal{K}_s^* = \{{}_{-[s]}R^0\} \,\dot\cup\, \{{}_{-[s]}R^k,\ {}_{-[s,t]}R^k : 2 \le k \le t-1,\ k \text{ even}\} \tag{1.11}$$

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of cardinality t. Figure 1.7 depicts such an arrangement of topes; cf. Figure 1.6.
- If t is *even*, then the tope committee of maximal size, which can be constructed by Algorithm 1.14, is either the set

$$\mathcal{K}_s^* = \{{}_{-[s]}R^k,\ {}_{-[s,t]}R^k : 2 \le k \le t-2,\ k \text{ even}\} \ \dot\cup\ \{{}_{-[s,t]}R^t\} \tag{1.12}$$

or the set

$$\mathcal{K}_s^* = \{{}_{-[s]}R^1\} \ \dot\cup\ \{{}_{-[s]}R^k,\ {}_{-[s,t]}R^k : 3 \le k \le t-1,\ k \text{ odd}\}\,, \tag{1.13}$$

see Figures 1.8(a) and (b), respectively; cf. Figure 1.6. We have $|\mathcal{K}_s^*| = t-1$.

Figure 1.6 suggests the following: if Algorithm 1.14 constructs, for $s = \lfloor t/2 \rfloor$, a tope committee of the form (1.11), (1.12), or (1.13), of maximal possible size, then

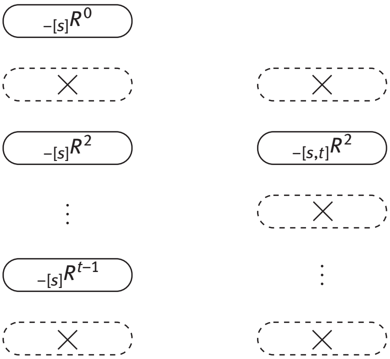

Figure 1.7: The tope committee (1.11) of maximal size, which can be constructed by Algorithm 1.14 in the case of *t odd*. Cf. Figure 1.6.

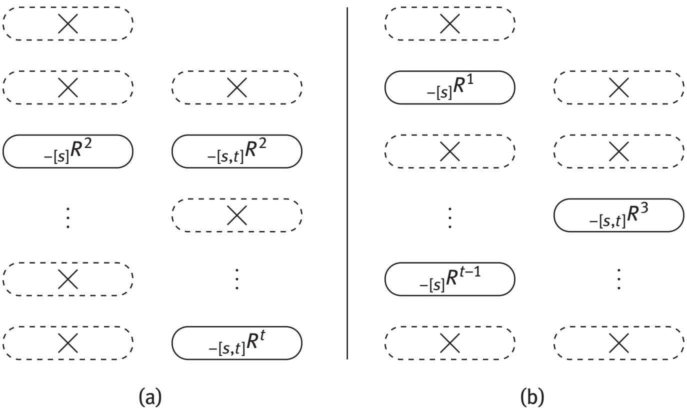

Figure 1.8: The tope committees of maximal size, which can be constructed by Algorithm 1.14 in the case of *t even*. (a) The committee (1.12). (b) The committee (1.13). Cf. Figure 1.6.

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for every $s > \lfloor t/2 \rfloor$, the cardinality of the set $\mathscr{K}_s^*$ will decrease, since pairs of opposites will be deleted.

(ii) The committees of maximal size which we have considered in the proof of the assertion (i) were built up under $s := \lfloor t/2 \rfloor$. One can argue analogously, by using Algorithm 1.14, in order to prove this assertion (ii) in the case of smaller values of s. □

1.4 Graphs Related to Tope Committees

In this section we consider two "extremal" committee constructs for simple oriented matroids: on the one hand, these are the *critical* committees (which we have already dealt with in the previous Section 1.3) with the topes that belong to the *vertex sets* of the *symmetric cycles* in the *tope graph* and, on the other hand, the committees whose topes have *inclusion-maximal* (among all topes) *positive parts*. In the rank 2 case these committee constructs coincide.

The committees whose topes have inclusion-maximal positive parts are arguably the best-quality building blocks of decision rules in the two-class pattern recognition problem.

Although critical committees can be regarded as worst-quality prerequisites for committee decision rules, they play a fundamental discrete mathematical role, as explained later in Chapter 13.

Given a simple oriented matroid $\mathscr{M} := (E_t, \mathscr{T})$, let us consider a graph isomorphic to the *Kneser graph* $\mathrm{KG}(\{T^-: T \in \mathscr{T}\})$ *of the family of the negative parts of topes* of $\mathscr{M}$:

Definition 1.18. Given a simple oriented matroid $\mathscr{M} := (E_t, \mathscr{T})$, the graph $\mathbf{\Gamma} := \mathbf{\Gamma}(\mathscr{M})$ is defined as follows:

$$\begin{aligned} \mathrm{V}(\mathbf{\Gamma}) &:= \mathscr{T} , \\ \{T', T''\} \in \mathscr{E}(\mathbf{\Gamma}) &\iff (T')^+ \cup (T'')^+ = E_t . \end{aligned} \tag{1.14}$$

Let us note a useful property of the graph $\mathbf{\Gamma}(\mathscr{M})$:

Lemma 1.19. *Let $\mathscr{M} := (E_t, \mathscr{T})$ be a simple oriented matroid.*

If $\boldsymbol{C}$ is an odd cycle *in the graph $\mathbf{\Gamma}(\mathscr{M})$, then the set of its* vertices $\mathrm{V}(\boldsymbol{C})$ *is a* tope committee *for $\mathscr{M}$.*

Proof. Assume that there is an element $e \in E_t$ such that $|\{K \in \mathrm{V}(\boldsymbol{C}): K(e) = -1\}| \geq \lceil |\mathrm{V}(\boldsymbol{C})|/2 \rceil$. Then there exists an edge $\{T^k, T^l\} \in \mathscr{E}(\boldsymbol{C})$ such that $(T^k)^+ \cup (T^l)^+ \not\ni e$; hence, $(T^k)^+ \cup (T^l)^+ \neq E_t$, a contradiction. □

Example 1.20. Consider the reorientation $\mathscr{N}^3 := {}_{-[3]}\mathscr{N}^0$ of the oriented matroid $\mathscr{N}^0$ which is realized by the hyperplane arrangement of Figure 1.1. The set of *vertices* of the 5-cycle in $\mathbf{\Gamma}(\mathscr{N}^3)$, shown in Figure 1.9, is a *tope committee* for $\mathscr{N}^3$.

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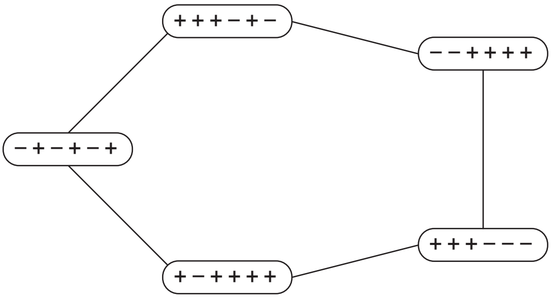

Figure 1.9: A 5-cycle in the graph $\mathbf{\Gamma}(\mathscr{N}^3)$, defined by (1.14), which is associated with the reorientation $\mathscr{N}^3 := {}_{-[3]}\mathscr{N}^0$. The oriented matroid $\mathscr{N}^0$ is realized by the hyperplane arrangement of Figure 1.1. The *vertex set* of the cycle is a *tope committee* for $\mathscr{N}^3$.

Symmetric Cycles in the Tope Graph

In this section we show that the *symmetric cycles* in the *tope graphs* (a straightforward graph-theoretic generalization of the concept of *centrally symmetric cycles of adjacent regions* in the linear hyperplane arrangements) of simple oriented matroids $\mathscr{M}$ given in Definition 1.6 lead to particular *odd*-length cyclic constructs in the graphs $\mathbf{\Gamma}(\mathscr{M})$ defined by (1.14) and, as a consequence, to particular *tope committees* for the oriented matroids $\mathscr{M}$, as explained in Lemma 1.19. Recall that the *vertex set* of any *symmetric cycle* is the *set of topes of a rank* 2 *simple oriented matroid*, see Remark 1.7.

Proposition 1.21. *Let* $\mathscr{M} := (E_t, \mathscr{L}) = (E_t, \mathscr{T})$ *be a simple oriented matroid. Let* $\boldsymbol{R} := (T^0, T^1, \ldots, T^{2t-1}, T^0)$ *be a symmetric cycle (that does* not *contain the* positive tope $\mathrm{T}^{(+)}$*) in the tope graph* $\mathscr{T}(\mathscr{L})$ *of* $\mathscr{M}$.

Consider the graph $\boldsymbol{G}$ *defined by*

$$\begin{aligned} \mathrm{V}(\boldsymbol{G}) &:= \mathrm{V}(\boldsymbol{R}) := \{T^0, T^1, \ldots, T^{2t-1}\}\,, \\ \{T', T''\} \in \mathscr{E}(\boldsymbol{G}) &\iff (T')^+ \cup (T'')^+ = E_t\,. \end{aligned} \tag{1.15}$$

The set $\mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))$ *is the* vertex set *of an* odd cycle *in* $\boldsymbol{G}$.

Proof. Example 1.22 and Figure 1.10 will illustrate the proof.

Suppose without loss of generality that $T^0 \in \mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))$.

Note that the path $(T^1, T^2, \ldots, T^t)$ contains at least one vertex T^j such that $T^j \in \mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))$. This follows from the observation that $|(T^0)^+| < t = |\mathbf{S}(T^0, T^t)|$ and $|(T^t)^+| > 0$, and from Remark 1.8(ii).

Let T^l be a vertex of $\boldsymbol{R}$ such that $1 < l < t$, $T^l \in \mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))$, and $(T^l)^+ \supsetneqq (T^t)^+ = (-T^0)^+$. The pair $\{T^0, T^l\}$ is an edge of $\boldsymbol{G}$.

We have $\{T^0, T^j\} \in \mathscr{E}(\boldsymbol{G})$ for all j, $l \le j \le t$.

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On the contrary, if $0 < j < l$, then $\{T^0, T^j\} \notin \mathscr{E}(\boldsymbol{G})$. Indeed, let $\{e\}$ be the one-element separation set of the topes T^l and T^{l-1}. Then we have $e \notin (T^j)^+$ and $e \notin (T^0)^+$.

Similarly, there is a unique vertex T^p of the cycle $\boldsymbol{R}$ such that $p > t$, $T^p \in \mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))$, and $\{T^0, T^p\} \in \mathscr{E}(\boldsymbol{G})$. Here $(T^p)^+ \supsetneqq (T^t)^+ = (-T^0)^+$. For all j, $t \leq j \leq p$, we have $\{T^0, T^j\} \in \mathscr{E}(\boldsymbol{G})$. If $p < j \leq 2t-1$, then $\{T^0, T^j\} \notin \mathscr{E}(\boldsymbol{G})$.

Thus, we have

$$\begin{aligned}\Big\{\{T', T''\} \in \mathscr{E}(\boldsymbol{G})\colon \{T', T''\} \ni T^0\Big\} \\ = \Big\{\{T^0, T^l\}, \{T^0, T^{l+1}\}, \ldots, \{T^0, T^{t-1}\}, \{T^0, T^t\}, \\ \{T^0, T^{t+1}\}, \ldots, \{T^0, T^{p-1}\}, \{T^0, T^p\}\Big\} \,.\end{aligned}$$

Note that for all j, $l < j < p$, we have $T^j \notin \mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))$.

Let T^i be a vertex of $\boldsymbol{R}$ such that $1 < i \leq l$, $T^i \in \mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))$, and $(T^{i-1})^+ \subsetneqq (T^0)^+$. The pair $\{T^p, T^i\}$ is an edge of $\boldsymbol{G}$.

We have the inclusions

$$\{T^0, T^l\},\ \{T^0, T^p\},\ \{T^p, T^i\} \in \mathscr{E}(\boldsymbol{G}) \,.$$

If $i = l$, then the sequence of vertices (T^0, T^l, T^p, T^0) is a triangle in $\boldsymbol{G}$ with the property $\{T^0, T^l, T^p\} = \mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))$.

In the general case, when $i \leq l$, let us consider consecutively all the vertices T^j, with $i \leq j \leq l$, in order to see that the set $\mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))$ is of odd cardinality, since

$$\Big|\{i\colon 1 < i < t,\ T^i \in \mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))\Big| = \Big|\{i\colon t < i < 2t-1,\ T^i \in \mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))\Big| \,, \tag{1.16}$$

and to see that this set is the vertex set $\mathrm{V}(\boldsymbol{C})$ of a cycle $\boldsymbol{C}$ in $\boldsymbol{G}$: if

$$\mathbf{max}^+(\mathrm{V}(\boldsymbol{R})) = \{T^0, T^{k_1}, \ldots, T^{k_d}, T^{k_{d+1}}, \ldots, T^{k_{2d}}\} \,, \quad 0 < k_1 < \cdots < k_{2d} \,,$$

then the family of edges of this cycle is

$$\begin{aligned}\mathscr{E}(\boldsymbol{C}) = \Big\{ &\{T^0, T^{k_d}\} \,,\ \{T^0, T^{k_{d+1}}\} \,, \\ &\{T^{k_1}, T^{k_{d+1}}\} \,,\ \{T^{k_1}, T^{k_{d+2}}\} \,,\ \ldots \,, \\ &\{T^{k_{d-1}}, T^{k_{2d-1}}\} \,,\ \{T^{k_{d-1}}, T^{k_{2d}}\} \,, \\ &\{T^{k_d}, T^{k_{2d}}\} \Big\} \,.\end{aligned} \tag{1.17}$$

□

Example 1.22. Consider the reorientation $\mathscr{M} := {}_{-[2]}\mathscr{N}^0$ of the simple oriented matroid $\mathscr{N}^0$ which is realized by the linear hyperplane arrangement of Figure 1.1. Figure 1.10 depicts the set of vertices of a symmetric cycle $\boldsymbol{R}$ in the tope graph of $\mathscr{M}$, and the corresponding graph $\boldsymbol{G}$ defined by eq. (1.15).

The set $\mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))$ is the vertex set of the unique *odd cycle* in $\boldsymbol{G}$.

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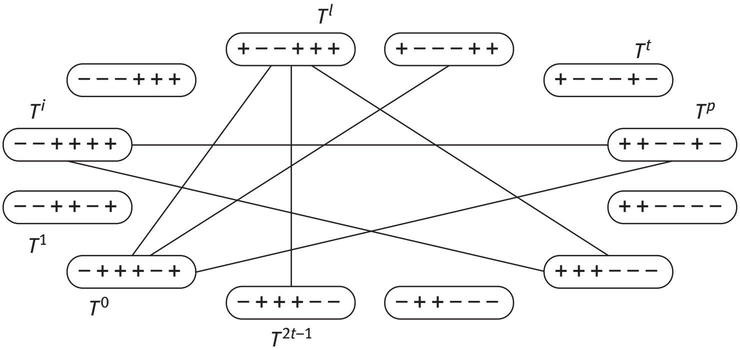

Figure 1.10: The graph $\boldsymbol{G}$ defined by (1.15) that corresponds to a *symmetric cycle* $\boldsymbol{R} := (T^0, T^1, \dots, T^{2t-1}, T^0)$ in the *tope graph* of the reorientation $\mathscr{M} := {}_{-[2]}\mathscr{N}^0$, where the oriented matroid $\mathscr{N}^0$ is realized by the hyperplane arrangement of Figure 1.1; $t = 6$. The edges that connect opposites are ***not*** depicted. The set $\mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))$ is the vertex set of the 5-cycle in $\boldsymbol{G}$.

Lemma 1.23. *Let $\mathscr{M} := (E_t, \mathscr{L}) = (E_t, \mathscr{T})$ be a simple oriented matroid. Let $\boldsymbol{R} := (T^0, T^1, \dots, T^{2t-1}, T^0)$ be a symmetric cycle (that does* not *contain the* positive tope $\mathrm{T}^{(+)}$*) in the tope graph $\mathscr{T}(\mathscr{L})$ of $\mathscr{M}$. For every $e \in E_t$, we have*

$$\left|\left\{T \in \mathbf{max}^+(\mathrm{V}(\boldsymbol{R})) \colon T(e) = 1\right\}\right| = \left\lceil \frac{|\mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))|}{2} \right\rceil .$$

Proof. Let $\boldsymbol{G}$ and $\boldsymbol{C}$ be the graph and the odd cycle, respectively, which were constructed in the hypothesis and in the proof of Proposition 1.21; see their descriptions (1.15) and (1.17).

Let $\boldsymbol{P} := (T^{l_0}, \dots, T^{l_{t-1}})$ be the $(t-1)$-path in $\boldsymbol{R}$ such that $T^{l_0}(e) = \cdots = T^{l_{t-1}}(e) = 1$; see Remark 1.8(i).

Suppose without loss of generality that $(T^0)^+ \supseteq (T^{l_0})^+ \supsetneq (T^{k_d})^-$. We then have $T^{k_d} \in \mathrm{V}(\boldsymbol{P})$, and the proof of the assertion follows from (1.16), since $\left\{T \in \mathbf{max}^+(\mathrm{V}(\boldsymbol{R})) \colon T(e) = 1\right\} = \mathrm{V}(\boldsymbol{P}) \cap \mathbf{max}^+(\mathrm{V}(\boldsymbol{R})) = \{T^0, T^{k_1}, T^{k_2}, \dots, T^{k_d}\}$. □

The higher-rank analogue of Proposition 1.12 is as follows:

Proposition 1.24. *Let $\mathscr{M} := (E_t, \mathscr{L}) = (E_t, \mathscr{T})$ be a simple oriented matroid. Let $\boldsymbol{R} := (T^0, T^1, \dots, T^{2t-1}, T^0)$ be a symmetric cycle in the tope graph $\mathscr{T}(\mathscr{L})$ of $\mathscr{M}$. The set*

$$\mathscr{K}^* := \mathbf{max}^+(\mathrm{V}(\boldsymbol{R})) \tag{1.18}$$

or, equivalently,

$$\mathscr{K}^* = \left\{T \in \mathrm{V}(\boldsymbol{R}) \colon S \in \mathrm{V}(\boldsymbol{R}) \ , \ \mathbf{S}(S, T) =: \{e\} \implies T(e) = 1\right\} \tag{1.19}$$

is a critical tope committee *for $\mathscr{M}$.*

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Proof. By Remark 1.8(ii), descriptions (1.18) and (1.19) are equivalent.

If $\mathrm{T}^{(+)} \in \mathrm{V}(\boldsymbol{R})$, then eq. (1.18) is the one-element set $\{\mathrm{T}^{(+)}\}$, that is, a *critical* committee for $\mathscr{M}$; we are done.

If $\mathrm{T}^{(+)} \notin \mathrm{V}(\boldsymbol{R})$, then Lemma 1.23 implies that $\mathscr{K}^*$ is a tope committee that satisfies

$$|\{K \in \mathscr{K}^* : K(e) = 1\}| = \left\lceil \tfrac{|\mathscr{K}^*|}{2} \right\rceil , \tag{1.20}$$

for each $e \in E_t$. We need to show that $\mathscr{K}^*$ is critical.

Assume that there is a proper subset $\mathscr{Q}^*$ of the set $\mathscr{K}^*$ such that $\mathscr{K}^* - \mathscr{Q}^*$ is a committee for $\mathscr{M}$. Since $\mathrm{T}^{(+)} \notin \mathrm{V}(\boldsymbol{R})$, we have $|\mathscr{K}^* - \mathscr{Q}^*| > 1$.

We suppose without loss of generality that $T^0 \in \mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))$ and $T^0 \notin \mathscr{Q}^*$.

Let $\{g\} := \mathbf{S}(T^0, T^1)$ and $\{f\} := \mathbf{S}(T^{2t-1}, T^0)$; note that $f \neq g$. We have $T^0(f) = T^0(g) = 1$. For every k, $0 < k < t$, we have $T^k(f) = 1$; for every l, $t < l < 2t$, we have $T^l(g) = 1$, see Remark 1.8(i).

By Lemma 1.23, for every $e \in \{f, g\}$, we have $|\{Q \in \mathscr{Q}^* : Q(e) = -1\}| = |\{R \in \mathscr{Q}^* : R(e) = 1\}|$, and we have

$$|\{T^1, T^2, \ldots, T^{t-1}\} \cap \mathscr{Q}^*| = |\{T^{t+1}, T^{t+2}, \ldots, T^{2t-1}\} \cap \mathscr{Q}^*| = \tfrac{|\mathscr{Q}^*|}{2} ;$$

the set $\mathscr{Q}^*$ is of even cardinality.

Let T^{k_2} be a tope in the set $\mathscr{Q}^*$, and $(T^{k_1}, T^{k_2}, T^{k_3})$ a 2-path in $\boldsymbol{R}$. If $\{p\} := \mathbf{S}(T^{k_1}, T^{k_2})$ and $\{q\} := \mathbf{S}(T^{k_2}, T^{k_3})$, then there is an element $h \in \{p, q\}$ such that $|\{K \in \mathscr{K}^* - \mathscr{Q}^* : K(h) = 1\}| = \left\lfloor \tfrac{|\mathscr{K}^* - \mathscr{Q}^*|}{2} \right\rfloor$, that is, the set of topes $\mathscr{K}^* - \mathscr{Q}^*$ is not a committee for $\mathscr{M}$, a contradiction. Thus, $\mathscr{K}^*$ is *minimal* and, as a consequence, it is *critical*, in view of eq. (1.20). □

We now discuss some poset-theoretic properties of topes which are useful for an analysis of the coverings of the ground sets of oriented matroids by pairs of the positive parts of topes.

Corollary 1.25. *Let* $\mathscr{M} := (E_t, \mathscr{L}) := (E_t, \mathscr{T})$ *be a simple oriented matroid that is* not acyclic. *Let* $\mathbf{m}$ *be an arbitrary* maximal chain *in the tope poset* $\mathscr{T}(\mathscr{L}, B)$ *with base tope* $B \in \mathbf{max}^+(\mathscr{T})$.

(i) *Let* $c := \max\{|T^+| : T \in \mathscr{T}\}$.
The subchain $\mathbf{max}^+(\mathbf{m})$ *contains a* unique *tope* K *such that* $B^+ \cup K^+ = E_t$. *The poset rank* $\rho(K)$ *of* K *satisfies*

$$\rho(K) \geq 2t - c - |B^+| . \tag{1.21}$$

For a tope $R \in \mathbf{m}$, *we have*

$$B^+ \cup R^+ = E_t \iff R \succeq K . \tag{1.22}$$

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(ii) *For all topes* $T', T'' \in \mathbf{m} - \{B\}$, *we have* $(T')^+ \cup (T'')^+ \neq E_t$.

(iii) – *The subposet*

$$\mathscr{O}(B) := \{T \in \mathscr{T}(\mathscr{L}, B) \colon B^+ \cup T^+ = E_t\} = \bigcap_{e \in B^-} \mathscr{T}_e^+ \tag{1.23}$$

is an order filter *in the tope poset* $\mathscr{T}(\mathscr{L}, B)$, *with*

$$\mathbf{min}\,\mathscr{O}(B) = \mathscr{G}(B)\ ,$$

where the antichain $\mathscr{G}(B)$ *is defined by*

$$\mathscr{G}(B) := \left\{T \in \mathscr{T}(\mathscr{L}, B) \colon T \in \mathbf{max}^+(\mathscr{T}),\ B^+ \cup T^+ = E_t\right\}. \tag{1.24}$$

Furthermore, if $\mathscr{M}$ *is* totally cyclic, *then we have*

$$\mathscr{O}(B) = \mathrm{conv}_{\mathrm{T}}\big(\mathscr{G}(B)\big)\ .$$

– *The union* $\bigcup_{B \in \mathbf{max}^+(\mathscr{T})} \mathscr{O}(B)$ *covers the set of topes* $\mathscr{T}$ *of* $\mathscr{M}$.

– *For any topes* $B', B'' \in \mathbf{max}^+(\mathscr{T})$, *we have*

$$|\mathscr{O}(B') \cap \mathscr{O}(B'')| > 0 \iff |\mathscr{G}(B') \cap \mathscr{G}(B'')| > 0\ .$$

Proof.

(i) The uniqueness of the tope $K \in \mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))$ such that $B^+ \cup K^+ = E_t$, and relation (1.22) are discussed in the proof of Proposition 1.21 (substitute T^0, T^l and T^t in that proof by B, K and $-B$, respectively).

We have $|(-B)^+| = t - |B^+|$, and $|K^+| = |(-B)^+| + (t - \rho(K))$; hence, $|K^+| = 2t - |B^+| - \rho(K) \leq c$, and eq. (1.21) follows.

(ii) This assertion is also inspired by the proof of Proposition 1.21: if $\mathbf{m} = (T^0, T^1, \dots, T^t)$, then for all topes $T', T'' \in \mathbf{m} - \{B\}$, we have $(T')^+ \cup (T'')^+ \not\ni e$, where $\{e\} := \mathbf{S}(T^0, T^1)$.

(iii) The proof of the assertion follows from (i). □

Example 1.26. Figure 1.11 depicts the Hasse diagram of a subposet $\mathscr{O}(B)$ given in Corollary 1.25(iii) and related to a reorientation of the oriented matroid realized by the linear hyperplane arrangement of Figure 1.1.

The following statement, the proof of which we sketch on page 44, shows that Algorithm 1.14 always constructs critical committees.

Proposition 1.27. *Let* $\mathscr{N}^0 := (E_t, \mathscr{L}^0) = (E_t, \mathscr{T}^0)$ *be a simple* acyclic *oriented matroid whose set of covectors and the set of topes are denoted by* $\mathscr{L}^0$ *and* $\mathscr{T}^0$, *respectively.*

Let $\mathbf{m} := (R^0 := \mathrm{T}^{(+)} \lessdot R^1 \lessdot \cdots \lessdot R^t := \mathrm{T}^{(-)})$ *be a* maximal chain *in the tope poset* $\mathscr{T}(\mathscr{L}^0, \mathrm{T}^{(+)})$.

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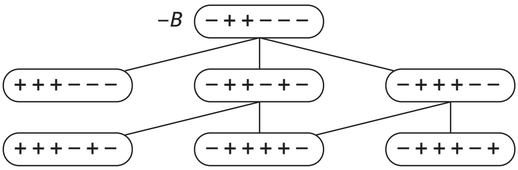

Figure 1.11: The Hasse diagram of the order filter $\mathscr{O}(B)$ in the tope poset $\mathscr{T}(\mathscr{L}, B)$, defined by (1.23), for the reorientation $\mathscr{M} := {}_{-[2]}\mathscr{N}^0$; the oriented matroid $\mathscr{N}^0$ is realized by the hyperplane arrangement of Figure 1.1, $B := +--+++$. The antichain $\mathscr{G}(B) := \mathbf{min}\,\mathscr{O}(B)$ is the subset $\{+++-+-, -++++-, -+++-+\} \subset \mathbf{max}^+(\mathscr{T})$.

For an integer s, $1 \le s \le t$, denote by $\mathscr{L}^s$ and $\mathscr{T}^s$ the set of covectors and the set of topes, respectively, of the reorientation $\mathscr{N}^s := {}_{-[s]}\mathscr{N}^0$.

Let $\boldsymbol{R} := (T^0, T^1, \ldots, T^{2t-1}, T^0)$ be a symmetric cycle *in the* tope graph $\mathscr{T}^s(\mathscr{L}^s)$ *of $\mathscr{N}^s$ such that*

$$T^k := {}_{-[s]}R^k\,, \quad 0 \le k \le t\,.$$

Algorithm 1.14 constructs the set $\mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))$ *which is a* critical tope committee *for $\mathscr{N}^s$.*

Remark 1.28. (cf. Remark 1.11). Let $\mathscr{M}$ be a simple oriented matroid, and $\boldsymbol{R}$ a symmetric cycle in its tope graph. A linear algebraic argument appearing later in Chapter 11 guarantees, according to Proposition 1.27, that the subset $\mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))$ is the *only* critical tope committee for $\mathscr{M}$ whose topes all belong to the vertex set $\mathrm{V}(\boldsymbol{R})$ of the cycle $\boldsymbol{R}$.

The Graph of Topes with Maximal Positive Parts

In this section we discuss a matroidal extension of the concept of the *graph of maximal feasible subsystems* of a linear inequality system.

Definition 1.29. (cf. Definition 1.18). Let $\mathscr{M} := (E_t, \mathscr{T})$ be a simple oriented matroid. The *graph of topes with inclusion-maximal positive parts* $\boldsymbol{\Gamma}^+_{\mathbf{max}} := \boldsymbol{\Gamma}^+_{\mathbf{max}}(\mathscr{M})$ is the induced subgraph of the graph $\boldsymbol{\Gamma}(\mathscr{M})$ on its vertex subset $\mathbf{max}^+(\mathscr{T})$:

$$\begin{aligned} \mathrm{V}(\boldsymbol{\Gamma}^+_{\mathbf{max}}) &:= \mathbf{max}^+(\mathscr{T})\,, \\ \{T', T''\} \in \mathscr{E}(\boldsymbol{\Gamma}^+_{\mathbf{max}}) &\iff (T')^+ \cup (T'')^+ = E_t\,. \end{aligned}$$

Example 1.30. The graph $\boldsymbol{\Gamma}^+_{\mathbf{max}}$ associated with a reorientation of the oriented matroid, which is realized by the linear hyperplane arrangement of Figure 1.1, is given in Figure 1.12.

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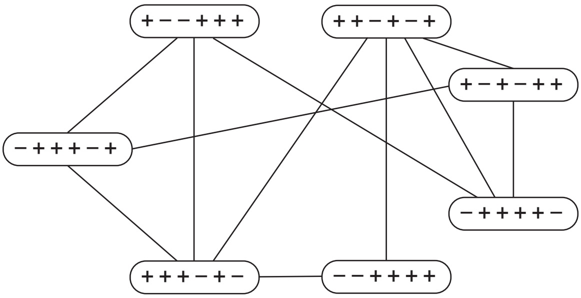

Figure 1.12: The *graph of topes with maximal positive parts* $\mathbf{\Gamma}^+_{\mathbf{max}}({}_{-[2]}\mathcal{N}^0)$; the oriented matroid $\mathcal{N}^0$ is realized by the hyperplane arrangement of Figure 1.1.

Basic Properties of the Graph $\mathbf{\Gamma}^+_{\mathbf{max}}$

Many properties of the graph $\mathbf{\Gamma}^+_{\mathbf{max}}$, among which the most important are the *connectedness* and the existence of an *odd cycle*, are inherited from the realizable case, and they are very useful for graph-theoretic procedures of constructing "high-quality" tope committees.

Proposition 1.31. *Let* $\mathcal{M} := (E_t, \mathcal{L}) = (E_t, \mathcal{T})$ *be a simple oriented matroid that is* not acyclic.

(i) *The graph* $\mathbf{\Gamma}^+_{\mathbf{max}} := \mathbf{\Gamma}^+_{\mathbf{max}}(\mathcal{M})$ *is* connected. *The* degree *of every its vertex is at least* 2.

Any edge of $\mathbf{\Gamma}^+_{\mathbf{max}}$ *is an* edge of a cycle.

(ii) *If for any 2-path* (R, B, S) *in* $\mathbf{\Gamma}^+_{\mathbf{max}}$ *there exist topes* $R', S' \in \mathcal{T} - \{-B\}$ *such that*

$$\begin{aligned} (R')^+ \subseteq R^+ \ , \quad (S')^+ \subseteq S^+ \ , \\ B^+ \cup (R')^+ = B^+ \cup (S')^+ = E_t \ , \\ |(R')^+ \cap B^+ \cap (S')^+| = 0 \ , \end{aligned} \tag{1.25}$$

then B is not a cutvertex *in* $\mathbf{\Gamma}^+_{\mathbf{max}}$.

(iii) *The graph* $\mathbf{\Gamma}^+_{\mathbf{max}}$ *has at least one* odd cycle *whose* vertex set *is a* tope committee *for* $\mathcal{M}$.

Proof. (i) Let B and R be any two distinct topes in the set $\mathbf{max}^+(\mathcal{T})$.

Let $\boldsymbol{R} := (T^0 := B, T^1, \dots, T^k := R, \dots, T^{2t-1}, T^0)$ be a symmetric cycle in the tope graph $\mathcal{T}(\mathcal{L})$ of $\mathcal{M}$. By Proposition 1.21, the set $\mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))$ is the vertex set of an odd cycle $\boldsymbol{C}$ defined as follows: for $T', T'' \in \mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))$, we have $\{T', T''\} \in \mathcal{E}(\boldsymbol{C})$ if and only if $(T')^+ \cup (T'')^+ = E_t$.

Let $\phi\colon \mathrm{V}(\boldsymbol{C}) \to \mathrm{V}(\mathbf{\Gamma}^+_{\mathbf{max}}) := \mathbf{max}^+(\mathcal{T})$ be any map such that $T^+ \subseteq (\phi(T))^+$, for all topes $T \in \mathrm{V}(\boldsymbol{C})$. It is a simplicial map from $\boldsymbol{C}$ to $\mathbf{\Gamma}^+_{\mathbf{max}}$. Since $\boldsymbol{C}$ is (2-)connected, there is a

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path in $\mathbf{\Gamma}^+_{\mathbf{max}}$ between the vertices $\phi(T')$ and $\phi(T'')$, for all $T', T'' \in \mathrm{V}(\boldsymbol{C})$. In particular, there is a path in $\mathbf{\Gamma}^+_{\mathbf{max}}$ between B and R, since $\phi(B) = B$ and $\phi(R) = R$.

Now suppose that $B^+ \cup R^+ = E_t$, that is, $\{B, R\} \in \mathscr{E}(\boldsymbol{C})$. Let $\{B, T\}$ be the edge of $\boldsymbol{C}$ such that $T \neq R$; then we have $\{B, R\}, \{B, \phi(T)\} \in \mathscr{E}(\mathbf{\Gamma}^+_{\mathbf{max}})$, where $\phi(T) \neq B$ and $\phi(T) \neq R$; therefore, the degree of B in $\mathbf{\Gamma}^+_{\mathbf{max}}$ is at least 2.

Let $\boldsymbol{D}$ denote the path in the cycle $\boldsymbol{C}$ between the vertices B and R such that $T \in \mathrm{V}(\boldsymbol{D})$. The image of $\boldsymbol{D}$ under the simplicial map ϕ is a connected subgraph of $\mathbf{\Gamma}^+_{\mathbf{max}}$ whose family of edges does not contain the edge $\{B, R\} = \{\phi(B), \phi(R)\}$. Hence, the edge $\{B, R\} \in \mathscr{E}(\mathbf{\Gamma}^+_{\mathbf{max}})$ is an edge of a cycle.

(ii) Assume that R and S belong to different blocks of the graph $\mathbf{\Gamma}^+_{\mathbf{max}}$, that is, B is a cutvertex.

Since $B^+ \cup (R')^+ = B^+ \cup (S')^+ = E_t$, by the hypothesis of the assertion, the condition (1.25) implies

$$\mathbf{S}(B, -R') \subset \mathbf{S}(B, S') , \quad \mathbf{S}(B, -S') \subset \mathbf{S}(B, R') ,$$

and, as a consequence,

$$S' \succ -R' , \quad R' \succ -S'$$

in the tope poset $\mathscr{T}(\mathscr{L}, B)$. This implies that in the tope graph $\mathscr{T}(\mathscr{L})$ there exists a symmetric cycle $\boldsymbol{R}$ such that $\{R', B, S'\} \subset \mathrm{V}(\boldsymbol{R})$. Let R'' and S'' be the topes with $(R')^+ \subseteq (R'')^+$ and $(S')^+ \subseteq (S'')^+$ such that $R'', S'' \in \mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))$.

Let $\boldsymbol{C}$ and $\phi\colon \mathrm{V}(\boldsymbol{C}) \to \mathrm{V}(\mathbf{\Gamma}^+_{\mathbf{max}})$ be an odd cycle and a simplicial map, respectively, which were defined in the proof of the assertion (i), with $\phi(R'') := R$ and $\phi(S'') := S$.

By Proposition 1.21 the sets $\{R'', B\}$ and $\{B, S''\}$ are edges of $\boldsymbol{C}$ and, as a consequence, $(\phi(R''), \phi(B), \phi(S'')) = (R, B, S)$ is a 2-path in $\mathbf{\Gamma}^+_{\mathbf{max}}$. Let $\boldsymbol{D}$ denote the path in the cycle $\boldsymbol{C}$ between the vertices R'' and S'', that does not contain B. The image of $\boldsymbol{D}$ under the simplicial map ϕ is a connected subgraph of $\mathbf{\Gamma}^+_{\mathbf{max}}$ whose set of vertices does not contain the tope $B = \phi(B)$. Hence, there is a cycle in $\mathbf{\Gamma}^+_{\mathbf{max}}$ such that $\{R, B, S\}$ is a subset of its vertices. This contradicts our assumption that B is a cutvertex in $\mathbf{\Gamma}^+_{\mathbf{max}}$.

(iii) Let $\boldsymbol{R} := (T^0, T^1, \ldots, T^{2t-1}, T^0)$ be a symmetric cycle in the tope graph $\mathscr{T}(\mathscr{L})$ such that $T^0 \in \mathbf{max}^+(\mathscr{T})$. Let $\boldsymbol{G}$ be the graph defined by (1.15). Recall that by Proposition 1.21 the set $\mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))$ is the vertex set $\mathrm{V}(\boldsymbol{C})$ of an odd cycle $\boldsymbol{C}$ in the graph $\boldsymbol{G}$. The edge family $\mathscr{E}(\boldsymbol{C})$ of the cycle $\boldsymbol{C}$ is described by eq. (1.17).

Assume that $\mathbf{\Gamma}^+_{\mathbf{max}}$ is bipartite, with partition classes V' and V''. Suppose that $T^0 \in \mathrm{V}'$. Then for a simplicial map

$$\phi\colon \mathrm{V}(\boldsymbol{C}) := \mathbf{max}^+(\mathrm{V}(\boldsymbol{R})) \;\to\; \mathrm{V}(\mathbf{\Gamma}^+_{\mathbf{max}}) := \mathbf{max}^+(\mathscr{T})$$

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from $\boldsymbol{C}$ to $\boldsymbol{\Gamma}^+_{\mathbf{max}}$ such that $T^+ \subseteq (\phi(T))^+$ for all $T \in \mathrm{V}(\boldsymbol{C})$, we have

$$\begin{aligned} \phi(T^0) = T^0 \in \mathrm{V}' \;,\; \{T^0, T^{k_{d+1}}\}, \{T^0, T^{k_d}\} \in \mathscr{E}(\boldsymbol{C}) \quad &\Longrightarrow \quad \phi(T^{k_{d+1}}) \in \mathrm{V}'' \;, \\ &\phi(T^{k_d}) \in \mathrm{V}'' \;; \end{aligned} \tag{1.26}$$

$$\begin{aligned} \phi(T^{k_{d+1}}) \in \mathrm{V}'' \;,\; \{T^{k_1}, T^{k_{d+1}}\} \in \mathscr{E}(\boldsymbol{C}) \quad &\Longrightarrow \quad \phi(T^{k_1}) \in \mathrm{V}' \;; \\ \phi(T^{k_1}) \in \mathrm{V}' \;,\; \{T^{k_1}, T^{k_{d+2}}\} \in \mathscr{E}(\boldsymbol{C}) \quad &\Longrightarrow \quad \phi(T^{k_{d+2}}) \in \mathrm{V}'' \;; \\ &\vdots \\ \phi(T^{k_{2d}}) \in \mathrm{V}'' \;,\; \{T^{k_d}, T^{k_{2d}}\} \in \mathscr{E}(\boldsymbol{C}) \quad &\Longrightarrow \\ &\phi(T^{k_d}) \in \mathrm{V}' . \end{aligned} \tag{1.27}$$

Since eq. (1.27) contradicts eq. (1.26), the graph $\boldsymbol{\Gamma}^+_{\mathbf{max}}$ is not bipartite; as a consequence, it contains an odd cycle whose vertex set by Lemma 1.19 is a tope committee for $\mathscr{M}$. □

Suppose that $\boldsymbol{\Gamma}^+_{\mathbf{max}}$ has no cutvertices, that is, $\boldsymbol{\Gamma}^+_{\mathbf{max}}$ is 2-*connected*. Since $\boldsymbol{\Gamma}^+_{\mathbf{max}}$ contains an odd cycle, the basic features of 2-connected graphs guarantee that every vertex of $\boldsymbol{\Gamma}^+_{\mathbf{max}}$ is contained in an odd cycle.

The Neighborhood Complex of the Graph $\boldsymbol{\Gamma}^+_{\mathbf{max}}$

Let $\mathscr{M} := (E_t, \mathscr{L}) = (E_t, \mathscr{T})$ be a simple oriented matroid that is *not acyclic*. Recall that the neighborhood of a vertex B in the graph $\boldsymbol{\Gamma}^+_{\mathbf{max}} := \boldsymbol{\Gamma}^+_{\mathbf{max}}(\mathscr{M})$ is by definition the set $\{T \in \mathbf{max}^+(\mathscr{T}) \colon B^+ \cup T^+ = E_t\}$. Equivalently, the neighborhood of B is the antichain $\mathscr{G}(B)$ in the tope poset $\mathscr{T}(\mathscr{L}, B)$, defined by eq. (1.24).

In accordance with the *Topological Representation Theorem*, one can consider a representation of $\mathscr{M}$ by an arrangement of *oriented pseudospheres*

$$\{S_e \colon e \in E_t\}$$

lying on the standard $(r(\mathscr{M})-1)$-dimensional sphere $\mathbb{S}^{r(\mathscr{M})-1}$, where $r(\mathscr{M})$ denotes the rank of $\mathscr{M}$.

Recall that the abstract simplicial *complex of acyclic subsets* of the ground set E_t, denoted by $\Delta_{\text{acyclic}}(\mathscr{M})$, is by definition the *nerve* of the family

$$\{P_e^+ \colon e \in E_t\} \;, \tag{1.28}$$

where P_e^+ denotes the *open positive pseudohemisphere* corresponding to the pseudosphere S_e, that is, the *positive side* of S_e.

- Suppose that $\mathscr{M}$ is *totally cyclic*. Recall that in this case the complex Δ_{acyclic} is *homotopy equivalent* to $\mathbb{S}^{r(\mathscr{M})-1}$, since the union of the sets from (1.28) is an *open*

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cover of the sphere by subspaces whose nonempty intersections are *contractible*:

$$\bigcup_{e \in E_t} P_e^+ = \mathbb{S}^{r(\mathscr{M})-1} .$$

By Corollary 1.25(iii) the *neighborhood complex* $\mathtt{NC}(\mathbf{\Gamma}^+_{\mathbf{max}})$ of $\mathbf{\Gamma}^+_{\mathbf{max}}$ is the *nerve* of the family of open subspaces

$$\Big\{ \bigcap_{e \in B^-} P_e^+ : B \in \mathbf{max}^+(\mathscr{T}) \Big\} \tag{1.29}$$

such that every their nonempty intersection is *contractible*.

The *Nerve Theorem* implies that $\mathtt{NC}(\mathbf{\Gamma}^+_{\mathbf{max}})$ is *homotopy equivalent* to the subset of $\mathbb{S}^{r(\mathscr{M})-1}$ covered by the family (1.29).

– Suppose that $\mathscr{M}$ is *neither acyclic nor totally cyclic*. Recall that there exists a unique nonnegative covector $F \in \mathscr{L}$ with its inclusion-maximal positive part. Denote by $\mathbf{\Gamma}^+_{\mathbf{max}}(\mathscr{M}\backslash F^+)$ the graph of topes with maximal positive parts, which is associated with the (*totally cyclic*) deletion $\mathscr{M}\backslash F^+$. The graph $\mathbf{\Gamma}^+_{\mathbf{max}}(\mathscr{M})$ is isomorphic to the graph $\mathbf{\Gamma}^+_{\mathbf{max}}(\mathscr{M}\backslash F^+)$.

1.5 Selected Mathematical Concepts and Tools for an Analysis of Infeasible Systems of Constraints

In this section we survey several mathematical concepts and tools discussed in Chapters 2–6, which can be useful in an analysis of infeasible systems of constraints.

Recall that an *infeasible monotone system* is defined to be a finite nonempty collection $\mathfrak{S} := \{\mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_m\}$ of *constraints*, together with a map $\pi : \mathbb{B}(m) \to \mathbf{2}^{\Gamma}$ from the Boolean lattice $\mathbb{B}(m)$ of subsets of the *index* set $[m]$ to the *power set* of a nonempty set Γ (that is, to the unordered family of all subsets of Γ) such that

$$\pi(\hat{0}) \neq \emptyset ;$$

$$B \in \mathbb{B}(m)^{(1)} \Longrightarrow \pi(B) \neq \emptyset ;$$

$$A, B \in \mathbb{B}(m) , \quad A \preceq B \Longrightarrow \pi(A) \supseteq \pi(B) ;$$

$$\pi(\hat{1}) = \emptyset .$$

The elements of the lattice $\mathbb{B}(m)$ are called the *multi-indices of subsystems* of the system $\mathfrak{S}$. The multi-indices $\hat{0}$ and $\hat{1}$ are by convention the empty set and the entire set $[m]$, respectively; $\mathbb{B}(m)^{(1)}$ denotes the set of atoms of the lattice $\mathbb{B}(m)$, that is,

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the family $\{\{1\},\{2\},\ldots,\{m\}\}$ of the one-element subsets of the indices with which the individual constraints are marked.

Boolean Intervals

Consider a finite *infeasible system* of *homogeneous strict linear inequalities*

$$\mathfrak{S} := \left\{\langle \boldsymbol{a}_i, \mathbf{x}\rangle > 0 \colon \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^r;\ \|\boldsymbol{a}_i\| = 1,\ i \in [m];\ i_1 \neq i_2 \Rightarrow \boldsymbol{a}_{i_1} \neq -\boldsymbol{a}_{i_2}\right\} \tag{1.30}$$

over the real Euclidean space $\mathbb{R}^r$. The family of all the multi-indices $J \in \mathbb{B}(m)$, for which the corresponding subsystems of $\mathfrak{S}$ are *feasible*, is an *abstract simplicial complex* on the set of indices $[m]$.

Recall that an arbitrary *abstract simplicial complex* on the *vertex set* $[m]$ is defined to be a family $\Delta \subseteq \mathbf{2}^{[m]}$ such that $\{v\} \in \Delta$, for every vertex $v \in [m]$, and

$$A, B \subseteq [m]\ , \quad A \subseteq B \in \Delta \Longrightarrow A \in \Delta\ .$$

One natural approach toward investigating the structural and enumerative properties of any *face system* $\Phi \subseteq \mathbf{2}^{[m]}$, given in Chapter 2, involves a partitioning of Φ into several *Boolean intervals* $[A, C] := \{B \in \mathbf{2}^{[m]} \colon A \subseteq B \subseteq C\}$ determined by pairs of subsets $A \subseteq C$ of the vertex set $[m]$.

The Dehn–Sommerville Relations

Consider a rank r *infeasible system of homogeneous strict linear inequalities*

$$\mathsf{S}_2 := \left\{\langle \boldsymbol{a}_i, \mathbf{x}\rangle > 0 \colon \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^r;\ \|\boldsymbol{a}_i\| = 1,\ i \in [m]\right\} \tag{1.31}$$

over $\mathbb{R}^r$, with the set of determining vectors $\boldsymbol{A}(\mathsf{S}_2) := \{\boldsymbol{a}_i\ :\ i \in [m]\}$ that satisfies two conditions: *for any open half-space $\mathbf{C}_> \subset \mathbb{R}^r$, bounded by a codimension one linear subspace, we have*

$$|\{\boldsymbol{a} \in \boldsymbol{A}(\mathsf{S}_2) \colon \boldsymbol{a} \in \mathbf{C}_>\}| \geq 2\ , \tag{1.32}$$

and

$$\text{every subsystem of } \mathsf{S}_2\text{, of rank at most } r-1\text{, is feasible} \tag{1.33}$$

or, equivalently,

every minimal infeasible subsystem of S_2 consists of $r + 1$ inequalities.

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Remark 1.32. If in the set $\boldsymbol{A}(\mathsf{S}_2)$ of vectors, that determine a rank 2 system S_2 over $\mathbb{R}^2$, defined by eqs. (1.31) and (1.32), there are no pairs of antipodes, then the system S_2 satisfies condition (1.33).

Proposition 1.33. *Suppose that the set $\boldsymbol{A}(\mathsf{S}_2) := \{\boldsymbol{a}_i : i \in [m]\}$ of vectors determining a rank r infeasible system*

$$\mathsf{S}_2 := \left\{\langle \boldsymbol{a}_i, \mathbf{x}\rangle > 0 \colon \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^r;\ \|\boldsymbol{a}_i\| = 1,\ i \in [m];\ i_1 \neq i_2 \Rightarrow \boldsymbol{a}_{i_1} \neq -\boldsymbol{a}_{i_2}\right\} \tag{1.34}$$

over $\mathbb{R}^r$ satisfies conditions (1.32) and (1.33).

Let v_i be the number of feasible subsystems *of S_2, of cardinality i. The following relations (where* x *is a formal variable), called the* Dehn–Sommerville equations for the feasible subsystems of the system S_2, *hold:*

$$\begin{cases} v_j = \binom{m}{j}, \text{ if } 0 \le j \le r\ , \\ v_{m-1} = v_m = 0\ , \\ \sum_{j=r+1}^{m}\left(\binom{m}{j}-v_j\right)(\mathrm{x}-1)^{m-j} = \sum_{j=r+1}^{m}(-1)^{j-r-1}\left(\binom{m}{j}-v_j\right)\mathrm{x}^{m-j}\ . \end{cases}$$

In particular,

– *if $m = r + 3$, then $v_{r+1} = \binom{m}{2} - m$;*
– *if $m = r + 4$, then*

$$\begin{aligned} v_{r+1} &= \tbinom{m}{3} - 2m + 4\ , \\ v_{r+2} &= \tbinom{m}{2} - 3m + 6\ ; \end{aligned}$$

– *if $m = r + 5$, then*

$$\begin{aligned} v_{r+2} &= 2v_{r+1} - 2\tbinom{m}{4} + \tbinom{m}{3}\ , \\ v_{r+3} &= v_{r+1} - \tbinom{m}{4} + \tbinom{m}{2} - m\ . \end{aligned}$$

A linear algebraic perspective on the Dehn–Sommerville relations is discussed in Chapter 3.

Farey Subsequences

Given a rank *r infeasible system of homogeneous strict linear inequalities* $\mathfrak{S}$ over $\mathbb{R}^r$, defined in eq. (1.30), consider the associated arrangement

$$\mathscr{H} := \left\{\mathbf{H}(i) \colon i \in [m]\right\} \tag{1.35}$$

of oriented linear hyperplanes $\mathbf{H}(i)$ determined by their normal vectors $\boldsymbol{a}_i$ as follows: $\mathbf{H}(i) := \left\{\mathbf{x} \in \mathbb{R}^r \colon \langle \boldsymbol{a}_i, \mathbf{x}\rangle := \sum_{j\in[r]} a_{ij}\mathrm{x}_j = 0\right\}$. By convention, a vector $\boldsymbol{v} \in \mathbb{R}^r$ lies on

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the *positive side* of a hyperplane $\mathbf{H}(i)$ if $\langle \boldsymbol{a}_i, \boldsymbol{v}\rangle > 0$. In an analogous manner, a *region* $\boldsymbol{T}$ of the hyperplane arrangement $\mathscr{H}$, that is, a connected component of the *complement* $\mathscr{T} := \mathbb{R}^n - \mathscr{H}$, lies on the *positive side* of the hyperplane $\mathbf{H}(i)$ if $\langle \boldsymbol{a}_i, \boldsymbol{v}\rangle > 0$ for some vector $\boldsymbol{v} \in \boldsymbol{T}$. Let $\mathscr{T}_i^+$ denote the set of all regions lying on the positive side of the hyperplane $\mathbf{H}(i)$.

Define a *committee of regions* for the arrangement $\mathscr{H}$ to be a subset $\mathscr{K}^* \subset \mathscr{T}$ such that

$$|\mathscr{K}^* \cap \mathscr{T}_i^+| > \tfrac{1}{2}|\mathscr{K}^*| ,$$

for each hyperplane $\mathbf{H}(i) \in \mathscr{H}$.

For any hyperplane $\mathbf{H}(i)$ of the arrangement $\mathscr{H}$, the increasing collection of irreducible fractions

$$\left(\frac{|\mathscr{R} \cap \mathscr{T}_i^+|}{\gcd(|\mathscr{R} \cap \mathscr{T}_i^+|, |\mathscr{R}|)} \Big/ \frac{|\mathscr{R}|}{\gcd(|\mathscr{R} \cap \mathscr{T}_i^+|, |\mathscr{R}|)} : \mathscr{R} \subseteq \mathscr{T},\ |\mathscr{R}| > 0 \right)$$

is the so-called *Farey subsequence* $\mathscr{F}(\mathbb{B}(|\mathscr{T}|), |\mathscr{T}|/2)$. From the number-theoretic point of view, a subset of regions $\mathscr{K}^* \subset \mathscr{T}$ is a *committee* for the arrangement $\mathscr{H}$ if and only if for each hyperplane $\mathbf{H}(i) \in \mathscr{H}$, we have

$$\frac{|\mathscr{K}^* \cap \mathscr{T}_i^+|}{\gcd(|\mathscr{K}^* \cap \mathscr{T}_i^+|, |\mathscr{K}^*|)} \Big/ \frac{|\mathscr{K}^*|}{\gcd(|\mathscr{K}^* \cap \mathscr{T}_i^+|, |\mathscr{K}^*|)} \in \left\{ \frac{h}{k} \in \mathscr{F}\big(\mathbb{B}(|\mathscr{T}|), |\mathscr{T}|/2\big) : \frac{h}{k} > \frac{1}{2} \right\} .$$

Recall that the standard *Farey sequence* $\mathscr{F}_n$ of *order* $n \geq 1$ is defined to be the increasing sequence of irreducible fractions $\frac{h}{k}$ such that $\frac{0}{1} \leq \frac{h}{k} \leq \frac{1}{1}$ and $k \leq n$.

The *Farey subsequence* $\mathscr{F}(\mathbb{B}(2m), m)$, discussed in Chapter 4, is by definition the sequence of fractions

$$\mathscr{F}\big(\mathbb{B}(2m), m\big) := \left(\tfrac{h}{k} \in \mathscr{F}_{2m} : k - m \leq h \leq m \right) .$$

Blocking Sets of Set Families, and Absolute Blocking Constructions in Posets

Let $\mathscr{A} := \{A_1, \ldots, A_\alpha\}$ be a nonempty family of nonempty and pairwise distinct subsets of a finite *ground set* $\mathrm{V}(\mathscr{A}) := \bigcup_{i=1}^{\alpha} A_i$. The family $\mathscr{A}$ is called a *Sperner family* (or a *clutter*) if

$$A_i \not\subseteq A_j ,$$

for all $i, j \in [\alpha]$, $i \neq j$.

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A *blocking set* of the family $\mathscr{A}$ is defined to be a subset $B \subseteq S$ of a set $S \supseteq \mathrm{V}(\mathscr{A})$ such that

$$|B \cap A_i| > 0 \ , \tag{1.36}$$

for all $i \in [\alpha]$.

The *blocker* $\mathfrak{B}(\mathscr{A})$ of the family $\mathscr{A}$ is by definition the family of all *inclusion-minimal blocking sets* of $\mathscr{A}$.

Proposition 1.34. *If $\mathscr{A}$ is a* Sperner family, *then*

$$\mathfrak{B}\big(\mathfrak{B}(\mathscr{A})\big) = \mathscr{A} \ .$$

A subsystem, with its multi-index I, of an *infeasible* system of linear inequalities $\mathfrak{S}$ defined in (1.30), is by definition a *minimal* (by inclusion) *infeasible subsystem* if the subsystem with the multi-index $I - \{k\}$ is *feasible*, for any index $k \in I$. Dually, a subsystem, with its multi-index J, of the system $\mathfrak{S}$ is by definition a *maximal* (by inclusion) *feasible subsystem* if the subsystem with the multi-index $J\dot{\cup}\{k\}$ is *infeasible*, for any index $k \in [m] - J$.

Let $\mathbf{I}$ and $\mathbf{J}$ denote the Sperner family of the *multi-indices* of all *minimal infeasible subsystems* and the Sperner family of the *multi-indices* of all *maximal feasible subsystems* of the system $\mathfrak{S}$, respectively. In view of the fundamental *blocker duality* characterized in Proposition 1.34, the key observation on the families $\mathbf{I}$ and $\mathbf{J}$ is as follows: *the family* $\mathbf{I}$ *of the multi-indices of minimal infeasible subsystems is the blocker of the family* $\mathbf{J}^{\perp}$ *of complements of the multi-indices of maximal feasible subsystems*:

$$\mathbf{I} = \mathfrak{B}(\mathbf{J}^{\perp}) \ , \tag{1.37}$$

where $\mathbf{J}^{\perp} := \{[m] - J : J \in \mathbf{J}\}$. Dually, we have

$$\mathbf{J} = \mathfrak{B}(\mathbf{I})^{\perp} \ . \tag{1.38}$$

Poset-theoretic constructions that extend the concept of *absolute blocking*, explained in eq. (1.36), where the *cardinalities of blocking sets are irrelevant*, are discussed in Chapter 5.

Committees of Set Families, and Relative Blocking Constructions in Posets

Again let $\mathscr{A} := \{A_1, \ldots, A_\alpha\}$ be a nonempty family of nonempty and pairwise distinct subsets of a finite set. In the context of *relative blocking*, given a rational number p

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such that $0 \le p < 1$, a subset $B \subseteq S$ of a set $S \supseteq \mathrm{V}(\mathscr{A})$ is by definition a *p-committee* of the family $\mathscr{A}$, if for each $i \in [\alpha]$, we have

$$|B \cap A_i| > p \cdot |B| \; ; \tag{1.39}$$

note that the *cardinality of the blocking set B* and the *cardinality of the intersection* $B \cap A_i$ both have combinatorial significance.

A $\frac{1}{2}$-committee B is called, for short, a *committee* of the family $\mathscr{A}$.

Let $\mathscr{H}$ be the *linear hyperplane arrangement* (1.35) associated with a rank *r infeasible system of linear inequalities* $\mathfrak{S}$ defined in eq. (1.30). Recall that a *committee of regions* for $\mathscr{H}$ was defined on page 42 to be a subset $\mathscr{K}^* \subset \mathscr{T} := \mathbb{R}^r - \mathscr{H}$ such that

$$|\mathscr{K}^* \cap \mathscr{T}_i^+| > \tfrac{1}{2}|\mathscr{K}^*| \, ,$$

for each hyperplane $\mathbf{H}(i) \in \mathscr{H}$.

Let $\boldsymbol{I}_{\frac{1}{2}}(\mathscr{T}_1^+, \ldots, \mathscr{T}_m^+)$ be the family of all set-theoretic committees of the family $\{\mathscr{T}_1^+, \ldots, \mathscr{T}_m^+\}$. Denoting by $\mathbf{K}^*(\mathscr{H})$ the family of all committees of regions for the hyperplane arrangement $\mathscr{H}$, we have

$$\mathbf{K}^*(\mathscr{H}) = \boldsymbol{I}_{\frac{1}{2}}(\mathscr{T}_1^+, \ldots, \mathscr{T}_m^+) \, ,$$

that is, a subset $\mathscr{K}^* \subset \mathscr{T}$ is a *committee of regions* for $\mathscr{H}$ if and only if $\mathscr{K}^*$ is a set-theoretic committee of the family $\{\mathscr{T}_1^+, \ldots, \mathscr{T}_m^+\}$.

Let $\mathbb{B}(\mathscr{T})$ denote the Boolean lattice of all subsets of the set of regions $\mathscr{T}$. For any index $i \in [m]$, associate with the set $\mathscr{T}_i^+$ the element $\boldsymbol{v}_i := \bigvee_{\boldsymbol{T} \in \mathscr{T}_i^+} \boldsymbol{T} \in \mathbb{B}(\mathscr{T})$, the join of those atoms of $\mathbb{B}(\mathscr{T})$ that form the set of regions $\mathscr{T}_i^+$.

Now regard each committee of regions for the arrangement $\mathscr{H}$ as an element of the Boolean lattice $\mathbb{B}(\mathscr{T})$. The family $\mathbf{K}^*(\mathscr{H})$ is then exactly the subposet $\mathbf{I}_{\frac{1}{2}}(\mathbb{B}(\mathscr{T}), \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\})$ of the so-called *relatively* $\frac{1}{2}$*-blocking elements* of the antichain $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ in the lattice $\mathbb{B}(\mathscr{T})$, with respect to the poset rank function of $\mathbb{B}(\mathscr{T})$. The antichain $\mathbf{min}\; \mathbf{I}_{\frac{1}{2}}(\mathbb{B}(\mathscr{T}), \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\})$, called the *relative* $\frac{1}{2}$*-blocker of* $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ *in the lattice* $\mathbb{B}(\mathscr{T})$, is the family of all *minimal* (by inclusion) *committees of regions* for the arrangement $\mathscr{H}$.

Relative blocking in posets is discussed in Chapter 6 which is devoted to an investigation of structural and combinatorial properties of poset-theoretic generalizations of the set-theoretic committees.

Appendix: Proof of Proposition 1.27 (Sketch)

Sketch of proof. The proof is by induction on i, $1 \le i \le s$.

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We begin with the following useful observation on the topes of a maximal chain $\mathbf{m}$ in the tope poset $\mathscr{T}(\mathscr{L}^0, \mathrm{T}^{(+)})$:
Claim: *For any t' and t'' such that $0 \le t' < t'' \le t$, we have*

$$\big((_{-[i]}R^{t'})|_{[i]}\big)^+ \subseteq \big((_{-[i]}R^{t''})|_{[i]}\big)^+ ,$$

$$\big((_{-[i]}R^{t'})|_{[i+1,t]}\big)^+ \supseteq \big((_{-[i]}R^{t''})|_{[i+1,t]}\big)^+ ,$$

and

$$\big((_{-[i,t]}R^{t'})|_{[i]}\big)^+ \supseteq \big((_{-[i,t]}R^{t''})|_{[i]}\big)^+ ,$$

$$\big((_{-[i,t]}R^{t'})|_{[i+1,t]}\big)^+ \subseteq \big((_{-[i,t]}R^{t''})|_{[i+1,t]}\big)^+ .$$

Associate with every tope $R^i \in \{R^1, \dots, R^t\}$ the label ℓ_i defined by eq. (1.10).

Suppose $i := 1$.

- If $\ell_1 = i$, then we have $\mathscr{K}_i^* = \{_{-[i]}R^1\} = \{\mathrm{T}^{(+)}\}$, since Algorithm 1.14 constructs at Step *03* the set $\mathscr{K}_i^* = \{_{-[i]}R^1, {}_{-[i]}R^0, {}_{-[i,t]}R^1\}$, and the pair of opposites $\{_{-[i]}R^0, {}_{-[i,t]}R^1\}$ is deleted at Steps *04–05*.
- If $\ell_t = i$, then we have $\mathscr{K}_i^* = \{_{-[i,t]}R^t\} = \{\mathrm{T}^{(+)}\}$, since the algorithm constructs at Step *03* the set $\mathscr{K}_i^* = \{_{-[i,t]}R^t, {}_{-[i]}R^0, {}_{-[i]}R^t\}$, and the pair of opposites $\{_{-[i]}R^0, {}_{-[i]}R^t\}$ is deleted at Steps *04–05*.
- If $\ell_j = i$, for some j, $1 < j < t$, then we have $\mathscr{K}_i^* = \{_{-[i]}R^0, {}_{-[i]}R^k, {}_{-[i,t]}R^k\}$. Since

$$(_{-\ell_t}(-R^t))(i) = R^0(i) = R^1(i) = 1\,, \quad (_{-\ell_t}(-R^t))^+ \subsetneq (R^0)^+ \supsetneq (R^1)^+ ,$$

$$R^{j-1}(i) = 1\,, \quad R^j(i) = R^{j+1}(i) = -1\,, \quad (R^{j-1})^+ \supsetneq (R^j)^+ \supsetneq (R^{j+1})^+ ,$$

and

$$(_{-\ell_{j-1}}(-R^{j-1}))(i) = (_{-\ell_j}(-R^j))(i) = -1\,, \quad (_{-\ell_{j+1}}(-R^{j+1}))(i) = 1\,,$$
$$(_{-\ell_{j-1}}(-R^{j-1}))^+ \subsetneq (_{-\ell_j}(-R^j))^+ \subsetneq (_{-\ell_{j+1}}(-R^{j+1}))^+ ,$$

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we have

$$({}_{-[i,t]}R^t)^+ \subsetneqq (R^0)^+ \supsetneqq (R^1)^+ \ ,$$

$$({}_{-[i]}R^{j-1})^+ \subsetneqq ({}_{-[i]}R^j)^+ \supsetneqq ({}_{-[i]}R^{j+1})^+ ,$$

$$({}_{-[i,t]}R^{j-1})^+ \subsetneqq ({}_{-[i,t]}R^j)^+ \supsetneqq ({}_{-[i,t]}R^{j+1})^+ \ ;$$

therefore,

$$(R^0)^+, ({}_{-[i]}R^j)^+, ({}_{-[i,t]}R^j)^+ \in \mathbf{max}\{({}_{-[i]}R^k)^+ : 0 \le k \le 2t-1\} \ ,$$

see Remark 1.8(ii).

A similar argument shows that for every $R \in \mathrm{V}(\boldsymbol{R})$ such that $R \notin \{R^0, R^j, {}_{-\ell_j}(-R^j)\}$, we have $R^+ \notin \mathbf{max}\{({}_{-[i]}R^k)^+ : 0 \le k \le 2t-1\}$.

Thus,

$$\{K^+ : K \in \mathscr{K}_i{}^*\} = \mathbf{max}\{({}_{-[i]}R^k)^+ : 0 \le k \le 2t-1\} \ .$$

If $i = s$, then we are done.
Suppose $i > 1$.
By the induction hypothesis, we have

$$\{K^+ : K \in \mathscr{K}_{i-1}{}^*\} = \mathbf{max}\{({}_{-[i-1]}R^k)^+ : 0 \le k \le 2t-1\} \ .$$

Suppose that $\ell_j = i$, for some j, $1 \le j \le t$.
Consider the set

$$\overline{\mathscr{K}_i{}^*} := \{{}_{-i}K : K \in \mathscr{K}_{i-1}{}^*\} \,\dot\cup\, \{{}_{-[i]}R^j, {}_{-[i,t]}R^j\} \ .$$

– Suppose $j = 1$.
  (a) Suppose that

$$ {}_{-[i]}R^0 \notin \overline{\mathscr{K}_i{}^*} \ , \quad {}_{-[i,t]}R^{j+1} \notin \overline{\mathscr{K}_i{}^*} \ . \tag{1.40}$$

For $i > 1$, we describe only the induction step for the case where $j = 1$ and (1.40) holds. An analysis of other situations is similar.

Condition (1.40) implies that

$$ {}_{-[i]}R^{t-1} \notin \overline{\mathscr{K}_i{}^*} \ , \quad {}_{-[i]}R^t \notin \overline{\mathscr{K}_i{}^*} \ , \quad {}_{-[i]}R^{j+1} \notin \overline{\mathscr{K}_i{}^*} \ ,$$

and we have

$$({}_{-[i-1,t]}R^t)^+ \supsetneqq ({}_{-[i-1]}R^0)^+ \supsetneqq ({}_{-[i-1]}R^j)^+ \supsetneqq ({}_{-[i-1]}R^{j+1})^+ \ ;$$

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$$({}_{-[i-1]}R^{t-1})^+ \subsetneqq ({}_{-[i-1]}R^{t})^+ = ({}_{-[i-1,t]}R^{j})^+ \subsetneqq ({}_{-[i-1,t]}R^{j+1})^+ \subsetneqq ({}_{-[i-1,t]}R^{j+2})^+ \ .$$

Note that

$$({}_{-[i-1]}R^0)(e) = -1\ ,\ 1 \le e \le i-1\ ;\quad ({}_{-[i-1]}R^0)(e) = 1\ ,\ i \le e \le t\ ;$$
$$({}_{-[i-1]}R^j)(e) = -1\ ,\ 1 \le e \le i\ ;\quad ({}_{-[i-1]}R^j)(e) = 1\ ,\ i+1 \le e \le t\ ;$$
$$({}_{-[i-1]}R^{j+1})(e) = -1\ ,\ 1 \le e \le i\ .$$

As a consequence, we have

$$({}_{-[i]}R^0)^+ \subsetneqq ({}_{-[i]}R^j)^+ \supsetneqq ({}_{-[i]}R^{j+1})^+\ ,$$

that is,

$$({}_{-[i]}R^j)^+ \in \mathbf{max}\{({}_{-[i]}R^k)^+ : 0 \le k \le 2t-1\}\ , \tag{1.41}$$

see Remark 1.8(ii).

Since

$$({}_{-[i-1]}R^{t-1})(e) = -1\ ,\ i \le e \le t\ ;$$
$$({}_{-[i-1]}R^t)(e) = 1\ ,\ 1 \le e \le i-1\ ;\quad ({}_{-[i-1]}R^t)(e) = -1\ ,\ i \le e \le t\ ,$$

and $({}_{-[i,t]}R^{j+1})^+ = (-{}_{-[i]}R^j)^+ \in \mathbf{min}\{({}_{-[i]}R^k)^+ : 0 \le k \le 2t-1\}$, by (1.41), we obtain

$$({}_{-[i]}R^{t-1})^+ \subsetneqq ({}_{-[i]}R^t)^+ = ({}_{-[i,t]}R^j)^+ \supsetneqq ({}_{-[i,t]}R^{j+1})^+\ ,$$

that is,

$$({}_{-[i,t]}R^j)^+ \in \mathbf{max}\{({}_{-[i]}R^k)^+ : 0 \le k \le 2t-1\}\ ,$$

by Remark 1.8(ii).

Note that for all k, $1 \le k \le t$, we have $({}_{-[i]}R^k)(i) = 1$, cf. Remark 1.8(i).

Let ${}_{-i}K \in \overline{\mathscr{K}_i^{\,*}} - \{{}_{-[i]}R^j, {}_{-[i,t]}R^j\}$, that is, $K \in \mathscr{K}_{i-1}{}^*$. Let (R', K, R'') be a 2-path in the cycle $({}_{-[i-1]}R^0, {}_{-[i-1]}R^1, \dots, {}_{-[i-1]}R^{2t-1}, {}_{-[i-1]}R^0)$; by Remark 1.8(ii) we have the inclusions $(R')^+ \subsetneqq K^+ \supsetneqq (R'')^+$ for the vertices of the path. Remark 1.8(i) implies that $R'(i) = K(i) = R''(i)$. As a consequence, the equality $({}_{-i}R')(i) = ({}_{-i}K)(i) = ({}_{-i}R'')(i)$ also holds, and we have $({}_{-i}R')^+ \subsetneqq ({}_{-i}K)^+ \supsetneqq ({}_{-i}R'')^+$. Thus, we have $\{({}_{-i}K)^+ : K \in \mathscr{K}_{i-1}{}^*\} \subset \mathbf{max}\{({}_{-[i]}R^k)^+ : 0 \le k \le 2t-1\}$.

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In an analogous manner, one can show that for any $R \in \{{}_{-[i]}R^0, {}_{-[i]}R^1, \dots, {}_{-[i]}R^{2t-1}\}$ such that ${}_{-[i]}R \notin \overline{\mathcal{K}_i{}^*}$, we have $({}_{-[i]}R)^+ \notin \mathbf{max}\{({}_{-[i]}R^k)^+ : 0 \le k \le 2t-1\}$.

The algorithm constructs the set

$$\mathcal{K}_i{}^* = \overline{\mathcal{K}_i{}^*} \ ;$$

we have seen that

$$\{K^+ : K \in \mathcal{K}_i{}^*\} = \mathbf{max}\{({}_{-[i]}R^k)^+ : 0 \le k \le 2t-1\} \ .$$

(b) If

$${}_{-[i]}R^0 \in \overline{\mathcal{K}_i{}^*} \ , \quad {}_{-[i,t]}R^{j+1} \notin \overline{\mathcal{K}_i{}^*} \ ,$$

then

$$\begin{aligned}\mathcal{K}_i{}^* &= \overline{\mathcal{K}_i{}^*} \ - \ \{{}_{-[i]}R^0, {}_{-[i,t]}R^j\} \\ &= \Big(\{{}_{-i}K : K \in \mathcal{K}_{i-1}{}^*\} - \{{}_{-[i]}R^0\}\Big) \ \dot{\cup} \ \{{}_{-[i]}R^j\} \ .\end{aligned}$$

(c) If

$${}_{-[i]}R^0 \in \overline{\mathcal{K}_i{}^*} \ , \quad {}_{-[i,t]}R^{j+1} \in \overline{\mathcal{K}_i{}^*} \ ,$$

then

$$\begin{aligned}\mathcal{K}_i{}^* &= \overline{\mathcal{K}_i{}^*} \ - \ \{{}_{-[i]}R^0, {}_{-[i]}R^j, {}_{-[i,t]}R^j, {}_{-[i,t]}R^{j+1}\} \\ &= \{{}_{-i}K : K \in \mathcal{K}_{i-1}{}^*\} \ - \ \{{}_{-[i]}R^0, {}_{-[i,t]}R^{j+1}\} \ .\end{aligned}$$

If $i = s$, then we are done.

– Suppose $j = t$.

(a) If

$${}_{-[i]}R^0 \notin \overline{\mathcal{K}_i{}^*} \ , \quad {}_{-[i]}R^{j-1} \notin \overline{\mathcal{K}_i{}^*} \ ,$$

then

$$\mathcal{K}_i{}^* = \overline{\mathcal{K}_i{}^*} \ .$$

(b) If

$${}_{-[i]}R^0 \in \overline{\mathcal{K}_i{}^*} \ , \quad {}_{-[i]}R^{j-1} \notin \overline{\mathcal{K}_i{}^*} \ ,$$

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then

$$\begin{aligned}\mathscr{K}_i^* &= \overline{\mathscr{K}_i^*} - \{{}_{-[i]}R^0, {}_{-[i]}R^j\}\\ &= \left(\{{}_{-i}K\colon K \in \mathscr{K}_{i-1}{}^*\} - \{{}_{-[i]}R^0\}\right) \dot\cup \{{}_{-[i,t]}R^j\}\,.\end{aligned}$$

(c) If

$${}_{-[i]}R^0 \in \overline{\mathscr{K}_i^*}\,, \quad {}_{-[i]}R^{j-1} \in \overline{\mathscr{K}_i^*}\,,$$

then

$$\begin{aligned}\mathscr{K}_i^* &= \overline{\mathscr{K}_i^*} - \{{}_{-[i]}R^0, {}_{-[i]}R^{j-1}, {}_{-[i]}R^t, {}_{-[i,t]}R^j\}\\ &= \{{}_{-i}K\colon K \in \mathscr{K}_{i-1}{}^*\} - \{{}_{-[i]}R^0, {}_{-[i]}R^{j-1}\}\,.\end{aligned}$$

– Suppose $1 < j < t$.

(a) If

$${}_{-[i]}R^{j-1} \notin \overline{\mathscr{K}_i^*}\,, \quad {}_{-[i,t]}R^{j+1} \notin \overline{\mathscr{K}_i^*}\,,$$

then

$$\mathscr{K}_i^* = \overline{\mathscr{K}_i^*}\,.$$

(b) If

$${}_{-[i]}R^{j-1} \in \overline{\mathscr{K}_i^*}\,, \quad {}_{-[i,t]}R^{j+1} \notin \overline{\mathscr{K}_i^*}\,,$$

then

$$\begin{aligned}\mathscr{K}_i^* &= \overline{\mathscr{K}_i^*} - \{{}_{-[i]}R^{j-1}, {}_{-[i,t]}R^j\}\\ &= \left(\{{}_{-i}K\colon K \in \mathscr{K}_{i-1}{}^*\} - \{{}_{-[i]}R^{j-1}\}\right) \dot\cup \{{}_{-[i]}R^j\}\,.\end{aligned}$$

(c) If

$${}_{-[i]}R^{j-1} \in \overline{\mathscr{K}_i^*}\,, \quad {}_{-[i,t]}R^{j+1} \in \overline{\mathscr{K}_i^*}\,,$$

then

$$\begin{aligned}\mathscr{K}_i^* &= \overline{\mathscr{K}_i^*} - \{{}_{-[i]}R^{j-1}, {}_{-[i]}R^j, {}_{-[i,t]}R^j, {}_{-[i,t]}R^{j+1}\}\\ &= \{{}_{-i}K\colon K \in \mathscr{K}_{i-1}{}^*\} - \{{}_{-[i]}R^{j-1}, {}_{-[i,t]}R^{j+1}\}\,.\end{aligned}$$

By induction, we have

$$\mathscr{K}_s{}^* = \mathbf{max}^+(\mathrm{V}(\boldsymbol{R}))\,.$$

According to Proposition 1.24, this is a *critical committee* for $\mathscr{N}^s$. □

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Notes

There is a surprising multitude of definitions for *oriented matroids* [30, p. 1]. They are defined by various equivalent *axiom systems*, and they can be thought of as a combinatorial abstraction of *point configurations* over the reals, of real *hyperplane arrangements*, of *convex polytopes*, and of *directed graphs* [27, p. 1].

Oriented matroids are surveyed, for example, in Refs. [25, 31, 78, 166]; they are extensively studied in the books [13, 27, 30, 34, 57, 59, 170, 186] and in numerous research articles.

In the very brief survey of oriented matroids given in Section 1.1, we mainly adopt the terminology of Ref. [27, Ch. 3, 4, 7].

Recall that the standard definition of a *simple* oriented matroid is that it has *no loops* and *parallel* elements, see for example [27, p. 166].

We refer the reader, for example, to [12, Sect. 1.5], [27, Sect. 4.1], [179, Ch. 3] for basic information on *posets*. See also, for example, the books [43, 167, 173].

The *neighborhood complex* of a graph appears in [127]. See also [114, 130].

Recall that unlike a *simplicial map* between graphs, a *graph homomorphism* from a graph $\mathbf{G}'$ to a graph $\mathbf{G}''$ is defined to be a map $\varphi\colon \mathrm{V}(\mathbf{G}') \to \mathrm{V}(\mathbf{G}'')$ such that $\{u, v\} \in \mathscr{E}(\mathbf{G}')$ implies $\{\varphi(u), \varphi(v)\} \in \mathscr{E}(\mathbf{G}'')$; see for example [114, Sect. 9.2.3].

The *Kneser graph* of a *set family* is considered, for example, in [130, Sect. 3.3], [131].

The *Circuit Axioms* of oriented matroids (C0)–(C3) given on page 8 are taken from [27, Definition 3.2.1]. The *Covector Axioms* (L0)–(L3) are taken from [27, Proposition 4.1.1].

It is shown in [27, Theorem 4.1.14(i)], following the works [69, 79, 118], that the poset $\widehat{\mathscr{L}}$ (see page 9) of *covectors* of an oriented matroid, together with a greatest element adjoined, is a *graded lattice*.

The sets of *covectors* of the *deletion* $\mathscr{M}\setminus A$ and of the *reoriented matroid* ${}_{-A}\mathscr{M}$ are described in [27, Lemma 4.1.8(i),(iii)].

Note that "most" oriented matroids are *neither acyclic nor totally cyclic* [166, Theorem 6.3.1].

It is shown in [27, Proposition 3.4.8(a)(i),(a)(iv)] that an oriented matroid $\mathscr{M}$ is *acyclic* if and only if its *dual oriented matroid* $\mathscr{M}^*$ is *totally cyclic*.

Following the work [119], the set $\widetilde{\mathscr{C}^*}$ of *cocircuits* of a *single element extension* $\widetilde{\mathscr{M}}$ of an oriented matroid $\mathscr{M}$ (see page 10) is described in [27, Proposition 7.1.4].

Important aspects of the *infeasibility phenomenon* are discussed in the monograph [47].

Various properties of *infeasible systems* of *linear constraints* have been studied in depth, see for example [9, 48, 72–76, 84, 86, 142, 158, 160].

The theory of *hyperplane arrangements* is the area of an active cross-disciplinary study [55, 155, 156, 177]. See also for example [12, Sect. 1.7], [164], [179, Sect. 3.11].

Books [24] and [62] are standard texts on *pattern recognition*; see also the surveys [45] and [169].

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Recall that a *committee* of an *infeasible system* of *strict linear inequalities* over $\mathbb{R}^n$ is a finite subset of elements of $\mathbb{R}^n$ such that for every inequality of the system, *more than half* of the elements are its *solutions*. Such committees were introduced in the seminal short notes [1, 2]; in the former, an application of committees to the *pattern recognition* problem was discussed. See also [152].

Various *committee constructions* for contradictory problems raised in pure and applied areas have later been invented and explored in detail; some of the surveys in this subject are the works [86, 109, 142–145].

One basic construct discussed extensively in the book is the *symmetric cycle* in the *tope graph* of a simple oriented matroid, see Definition 1.6. It is a direct generalization of the notion of *centrally symmetric cycle* of *adjacent regions* in an *arrangement* of *oriented linear hyperplanes* that appears in [1, p. 453].

The planar interpretation of the *symmetric cycle* in Remark 1.7 is influenced by the argument also given in [1, p. 453].

The transformations of tope committees under *reorientations* of a rank 2 simple-oriented matroid on one-element subsets of its ground set, illustrated by Figures 1.3–1.5, reflect implicitly the changes in committees under the *insertions* of new lines into an arrangement of oriented central lines in the plane, illustrated in Ref. [1, Figure 1].

Following a remark given in the working example for Ref. [1], we note on page 27 that a symmetric cycle in the tope graph leads to a critical tope committee of size which is not necessarily minimal. In Theorem 1.17 we adopt an estimate for the size of a tope committee from an argument given in [1, p. 454].

One approach to the combinatorial analysis of *committees* of *infeasible systems of linear inequalities* consists in exploring the families of their *maximal feasible subsystems* (*MFSs*), and in investigating the properties of *graphs* which are naturally associated with those subsystems; see the monograph [86]. Certain properties of the graphs associated with the MFSs, such as their *connectedness* and the existence of *odd cycles*, are important in the context of graph-theoretic algorithms for constructing efficient committees.

Consider a *finite infeasible system* (1.30)

$$\mathfrak{S} := \left\{ \langle \boldsymbol{a}_i, \mathbf{x} \rangle > 0 \colon \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^r;\ \|\boldsymbol{a}_i\| = 1,\ i \in [m];\ i_1 \neq i_2 \Rightarrow \boldsymbol{a}_{i_1} \neq -\boldsymbol{a}_{i_2} \right\}$$

of *homogeneous strict linear inequalities*, of rank r, over the real Euclidean space $\mathbb{R}^r$, whose set of determining vectors $\boldsymbol{A}(\mathfrak{S}) := \{\boldsymbol{a}_i \colon i \in [m]\}$ contains *no pairs of antipodes*.

Definition 1.35. (see [86, p. 34]). The graph $\mathrm{MFSG}(\mathfrak{S})$ of maximal feasible subsystems (the graph of MFSs) of the system $\mathfrak{S}$ is defined as follows:

– the vertex set of the graph $\mathrm{MFSG}(\mathfrak{S})$ is the family $\mathbf{J}$ of the multi-indices of MFSs of the system $\mathfrak{S}$;

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– the edge family of the graph MFSG($\mathfrak{S}$) is the family of all the unordered pairs $\{J, J'\}$ of the multi-indices of MFSs of the system $\mathfrak{S}$ that cover the index set of the inequalities of the system:

$$J \cup J' = [m] .$$

Theorem 1.36.

(i) (see [86, Theorem 2.20]). *The graph* MFSG($\mathfrak{S}$) *is* connected.

(ii) (see [86, Theorem 2.28]). *The graph* MFSG($\mathfrak{S}$) *contains at least one* cycle *of* odd length.

(iii) (see [86, Theorem 5.4]). *Let a sequence* $(J_{i_1}, J_{i_2}, \ldots, J_{i_{2k+1}}, J_{i_1})$ *compose an* odd cycle *in the graph* MFSG($\mathfrak{S}$). *Suppose that pairwise distinct vectors* $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{2k+1}$ *are solutions to the MFSs with the multi-indices* $J_1, J_2, \ldots, J_{2k+1}$, *respectively. Then the collection of vectors* $\mathscr{K} := \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{2k+1}\}$ *is a* committee *of the system* $\mathfrak{S}$.

Section 1.4 represents a combined survey of central ideas from [1], [86, Sect. 2.3, 5.1], [87], rephrased in the language of *oriented matroids*.

Note that Lemma 1.19 is a relaxed (unlike Proposition 1.31(iii)) matroidal version of Theorem 1.36(iii), since in the lemma the *maximality* of the *positive parts* of all vertices of an *odd cycle* is not expected.

In the case of a *realizable* simple oriented matroid $\mathscr{M}$ which is *not acyclic*, Definition 1.29 of the *graph of topes with maximal positive parts* $\boldsymbol{\Gamma}^+_{\mathbf{max}}(\mathscr{M})$ merely rephrases Definition 1.35 of the *graph* MFSG($\mathfrak{S}$) *of MFSs* of a corresponding *infeasible system* $\mathfrak{S}$ of homogeneous strict *linear inequalities*. From this point of view, Proposition 1.31(i) is a matroidal analogue of Theorem 1.36(i) and of the assertions given in [86, Theorem 2.25(i), Prop. 2.35(i)]. Proposition 1.31(iii) is a straightforward matroidal restatement of Theorem 1.36(ii, iii).

Recall that according to [154, p. 89, Problem 2], every vertex in a graph that contains an *odd cycle* and has *no cutvertices*, is a *vertex* of an *odd cycle*; cf. the concluding remark of the section *Basic Properties of the Graph* $\Gamma^+_{\mathbf{max}}$ on page 38.

By the *Topological Representation Theorem* [79], every oriented matroid has a *pseudosphere representation* [27, p. 17]. See for example [27, Sect. 5.2], [30, Ch. 3, 4] and [32, 33] on this fundamental result of oriented matroid theory.

Abstract simplicial complexes whose definition is recalled on page 40 are central constructs of combinatorial algebraic topology; several references will be given in the Notes to Chapter 2 on page 65.

See for example [65, 67] on the *complex* $\Delta_{\text{acyclic}}(\mathscr{M})$ of *acyclic subsets* of the ground set of an oriented matroid $\mathscr{M}$.

Combinatorial homotopy is discussed e.g. in [16, 25, 107, 114, 126, 130, 161].

See for example [16, Sect. 5.3], [25, Sect. 10], [107, Sect. 6.1], [114, Sect. 15.4], [130, Sect. 4.4], [161, Sect. 3.2.5] on the combinatorial topology of the *nerve*.

Considering the neighborhood complex $\mathtt{NC}(\boldsymbol{\Gamma}^+_{\mathbf{max}})$ of the graph $\boldsymbol{\Gamma}^+_{\mathbf{max}}$ on page 39, we adopt the argument given in [67, p. 103, Proof of Theorem 1] and we apply the *Nerve*

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Theorem stated in [25, Theorem 10.7]; note that if the simple oriented matroid $\mathscr{M}$ considered on page 38 is *totally cyclic*, then the complex $\Delta_{\mathrm{acyclic}}(\mathscr{M})$ is *homotopy equivalent* to the *sphere* $\mathbb{S}^{r(\mathscr{M})-1}$. If $\mathscr{M}$ is *neither acyclic nor totally cyclic*, then by [67, Prop. 7] there exists a unique *nonnegative* covector F of $\mathscr{M}$, with its *inclusion-maximal positive part*.

We recall the notion of *infeasible monotone system* of *constraints* in the introduction to Section 1.5 by adopting the terminology of [86, Ch. 1].

The *Dehn–Sommerville equations* for the numbers of *feasible subsystems* of infeasible systems S_2 of linear inequalities, defined by eqs. (1.32)–(1.34), together with several their solutions given in Proposition 1.33, are taken from [86, Prop. 3.53, Cor. 3.54] and can be found in [141]. Several references on the *Dehn–Sommerville relations* will be given in the Notes to Chapter 3 on page 81.

The *Farey subsequences* $\mathscr{F}(\mathbb{B}(2m), m)$ mentioned on page 42 are discussed, for example, in [132, 138]. See for example [39, Ch. 27], [90, Ch. 4], [95, Ch. III] on the standard *Farey sequence* $\mathscr{F}_n$. See also the Notes to Chapter 4. Monograph [137] is devoted entirely to Farey sequences and subsequences.

The *blocking sets* of *set families* defined by eq. (1.36) play an important role in combinatorial optimization, see for example [52, Ch. 1], [92, Ch. 8], [171, Ch. 77]. For further references, see also the Notes to Chapter 5 on page 116.

The fundamental result of Proposition 1.34 appears in [68, 121, 122].

The *blocker* connection (1.37), (1.38) between the families of the multi-indices of extremal subsystems of an infeasible system of linear inequalities is discussed, for example, in [86, pp. 12–13].

It is shown in [142] that various problems of collective decision-making may often be reduced to the problem of constructing a *p-committee* of a certain family of sets, defined by eq. (1.39).

For a review of the poset-theoretic properties of the *regions* of *hyperplane arrangements*, see, for example, [164] and [179, Sect. 3.11.3].

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2 Boolean Intervals

Let V be a finite set, and $\mathbf{2}^V$ the *simplex* $\{F: F \subseteq V\}$, that is, the *power set* of the set V. Recall that a family $\Delta \subseteq \mathbf{2}^V$ is called an *abstract simplicial complex* (or a *complex* for short) on the *vertex set* V if, given subsets A and B of the set V, the inclusions $A \subseteq B \in \Delta$ imply $A \in \Delta$, and if $\{v\} \in \Delta$, for any $v \in V$. If Γ is a complex such that $\Gamma \subset \Delta$ (that is, Γ is a *subcomplex* of Δ), then the family $\Delta - \Gamma$ is called a *relative simplicial complex*.

If Ψ is a relative complex, then the sets $F \in \Psi$ are called the *faces* of Ψ. The *dimension* $\dim(F)$ of a face F is by convention equal to $|F| - 1$; the cardinality $|F|$ is also called the *size* of F. If we let # denote the number of sets in a family, and if $\#\Psi > 0$, then the *size* $d(\Psi)$ of Ψ is defined by $d(\Psi) := \max_{F \in \Psi} |F|$, and the *dimension* $\dim(\Psi)$ of Ψ is by convention $d(\Psi) - 1$.

The row vector

$$\boldsymbol{f}(\Psi) := (f_0(\Psi), f_1(\Psi), \ldots, f_{\dim(\Psi)}(\Psi)) \in \mathbb{N}^{d(\Psi)} ,$$

where

$$f_i(\Psi) := \#\{F \in \Psi: |F| = i + 1\} ,$$

is called the *f-vector* of Ψ. The row *h-vector*

$$\boldsymbol{h}(\Psi) := (h_0(\Psi), h_1(\Psi), \ldots, h_{d(\Psi)}(\Psi)) \in \mathbb{Z}^{d(\Psi)+1}$$

of Ψ is defined by

$$\sum_{i=0}^{d(\Psi)} h_i(\Psi) \cdot \mathrm{y}^{d(\Psi)-i} := \sum_{i=0}^{d(\Psi)} f_{i-1}(\Psi) \cdot (\mathrm{y} - 1)^{d(\Psi)-i} ,$$

where y is a formal variable.

In this chapter we consider redundant analogues $\boldsymbol{f}(\Phi; |V|) \in \mathbb{N}^{|V|+1}$ and $\boldsymbol{h}(\Phi; |V|) \in \mathbb{Z}^{|V|+1}$ of the f- and h-vectors that can be used to describe the combinatorial properties of arbitrary *face systems* $\Phi \subseteq \mathbf{2}^V$ in *decomposition problems*.

In Section 2.1 we define these "long" f- and h-vectors $\boldsymbol{f}(\Phi; |V|)$ and $\boldsymbol{h}(\Phi; |V|)$, and discuss their basic properties.

In Section 2.2, several distinguished bases of the real Euclidean spaces occurring naturally in a combinatorial analysis of face systems are described; we list all explicitly described *change of basis matrices*.

In Section 2.3 we investigate the *partitions of face systems into Boolean intervals*, and ascertain restrictions imposed on the *partition profiles* of the face systems that satisfy *Dehn–Sommerville type relations*.

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2.1 Long *f*- and *h*-Vectors of Face Systems

In this section we give linear algebraic descriptions of face systems which, being redundant, can nevertheless be useful for *decomposition problems*, since they provide *valuations* on the Boolean lattices of all face systems.

Given a positive integer m, we associate with a *face system* $\Phi \subseteq \mathbf{2}^{[m]}$ the row vectors

$$\boldsymbol{f}(\Phi; m) := \big(f_0(\Phi; m), f_1(\Phi; m), \ldots, f_m(\Phi; m)\big) \in \mathbb{N}^{m+1} , \tag{2.1}$$

$$\boldsymbol{h}(\Phi; m) := \big(h_0(\Phi; m), h_1(\Phi; m), \ldots, h_m(\Phi; m)\big) \in \mathbb{Z}^{m+1} , \tag{2.2}$$

where

$$f_i(\Phi; m) := \#\{F \in \Phi\colon |F| = i\} ,$$

for $0 \le i \le m$, and the vector $\boldsymbol{h}(\Phi; m)$ is defined by

$$\sum_{i=0}^{m} h_i(\Phi; m) \cdot \mathrm{y}^{m-i} := \sum_{i=0}^{m} f_i(\Phi; m) \cdot (\mathrm{y} - 1)^{m-i} .$$

Note that if $\Psi \subset \mathbf{2}^{[m]}$ is a relative complex, then we set

$$f_0(\Psi; m) := f_{-1}(\Psi) := \#\{F \in \Psi\colon |F| = 0\} \in \{0, 1\} ,$$
$$f_i(\Psi; m) := f_{i-1}(\Psi) , \quad 1 \le i \le d(\Psi) ,$$

and finally

$$f_i(\Psi; m) := 0 , \quad d(\Psi) + 1 \le i \le m .$$

In certain situations the "long" f- and h-vectors defined by eqs. (2.1) and (2.2) can be used as either an intermediate description of face systems or they can independently be involved in combinatorial computations, see for example Chapter 3. Since the mappings $\Phi \mapsto \boldsymbol{f}(\Phi; m)$ and $\Phi \mapsto \boldsymbol{h}(\Phi; m)$ of the elements of the *Boolean lattice* $\mathscr{D}(m)$ *of all face systems* (ordered by inclusion) to row vectors in $\mathbb{Z}^{m+1}$ provide *valuations* on $\mathscr{D}(m)$, the long f- and h-vectors can also be useful for *decomposition problems*. A basic construct is a *Boolean interval*, that is, the family $[A, C] := \{B \in \mathbf{2}^{[m]}\colon A \subseteq B \subseteq C\}$, for some faces $A \subseteq C \subseteq [m]$.

We regard the vectors $\boldsymbol{f}(\Phi; m)$ and $\boldsymbol{h}(\Phi; m)$ as elements of the real Euclidean space $\mathbb{R}^{m+1}$ of row vectors. We will give several bases of $\mathbb{R}^{m+1}$ that occur naturally in the context of face systems, and we will describe the corresponding change of basis matrices.

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Throughout the chapter, all vectors are of dimension $m + 1$, and all matrices are square matrices of order $m + 1$. The components of vectors, as well as the rows and columns of matrices, are indexed starting with *zero*.

$\mathbf{I}(m)$ denotes the *identity matrix*. $\mathbf{U}(m)$ is the *backward identity matrix* whose (i, j)th entry is the Kronecker delta $\delta_{i+j,m}$. $\mathbf{T}(m)$ is the *forward shift matrix* whose (i, j)th entry is $\delta_{j-i,1}$.

Given a vector $\boldsymbol{w}$, we denote by $\boldsymbol{w}^\top$ its transpose.

If Φ is a face system with $\#\Phi > 0$, then its *size* $d(\Phi)$ is defined by $d(\Phi) := \max_{F\in\Phi} |F|$.

We denote the *empty set* by $\hat{0}$, and we use the notation $\emptyset$ for the *void family* containing *no sets*. Thus, we have $\#\emptyset = 0$, $\#\{\hat{0}\} = 1$, and

$$\boldsymbol{f}(\emptyset; m) = \boldsymbol{h}(\emptyset; m) = (0, 0, \dots, 0) \, ,$$

$$\boldsymbol{f}(\{\hat{0}\}; m) = \boldsymbol{h}(\mathbf{2}^{[m]}; m) = (1, 0, \dots, 0) \, .$$

Two distinguished row vectors of $\mathbb{R}^{m+1}$ are defined by

$$\boldsymbol{\iota}(m) := (1, 1, \dots, 1) \, ; \qquad \boldsymbol{\tau}(m) := (2^m, 2^{m-1}, \dots, 1) \, .$$

If $\boldsymbol{B} := (\boldsymbol{b}_0, \dots, \boldsymbol{b}_m)$ is an *ordered basis* of the space $\mathbb{R}^{m+1}$, then for a vector $\boldsymbol{w} \in \mathbb{R}^{m+1}$, we denote by $[\boldsymbol{w}]_{\boldsymbol{B}} := (\kappa_0(\boldsymbol{w}, \boldsymbol{B}), \dots, \kappa_m(\boldsymbol{w}, \boldsymbol{B})) \in \mathbb{R}^{m+1}$ the $(m+1)$-tuple such that $\sum_{i=0}^{m} \kappa_i(\boldsymbol{w}, \boldsymbol{B}) \cdot \boldsymbol{b}_i = \boldsymbol{w}$.

Recall several basic properties of the long f- and h-vectors (2.1) and (2.2):

Proposition 2.1.

(i) *The mappings $\Phi \mapsto \boldsymbol{f}(\Phi; m)$ and $\Phi \mapsto \boldsymbol{h}(\Phi; m)$ determine* valuations $\mathscr{D}(m) \to \mathbb{Z}^{m+1}$ *on the Boolean lattice $\mathscr{D}(m)$ of all face systems contained in $\mathbf{2}^{[m]}$ and ordered by inclusion.*

(ii) *Let $\Psi \subseteq \mathbf{2}^{[m]}$ be a* relative complex. *The standard h-vectors and the long h-vectors are related by*

$$h_l(\Psi) = \sum_{k=0}^{l} \binom{m - d(\Psi) + l - k - 1}{l - k} h_k(\Psi; m) \, , \qquad 0 \le l \le d(\Psi) \, ;$$

$$h_l(\Psi; m) = (-1)^l \sum_{k=0}^{l} (-1)^k \binom{m - d(\Psi)}{l - k} h_k(\Psi) \, , \qquad 0 \le l \le m \, .$$

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(iii) *Let $\Phi \subseteq \mathbf{2}^{[m]}$ be a face system.*

(a)

$$h_l(\Phi; m) = (-1)^l \sum_{k=0}^{l} (-1)^k \binom{m-k}{l-k} f_k(\Phi; m) \ ,$$
$$f_l(\Phi; m) = \sum_{k=0}^{l} \binom{m-k}{l-k} h_k(\Phi; m) \ , \quad 0 \le l \le m \ .$$

(b)

$$\begin{aligned} h_0(\Phi; m) &= f_0(\Phi; m) \ , \\ h_1(\Phi; m) &= f_1(\Phi; m) - m f_0(\Phi; m) \ , \\ h_m(\Phi; m) &= (-1)^m \sum_{k=0}^{m} (-1)^k f_k(\Phi; m) \ , \\ \boldsymbol{h}(\Phi; m) \cdot \boldsymbol{\iota}(m)^\top &= f_m(\Phi; m) \ . \end{aligned} \tag{2.3}$$

(c)

$$\boldsymbol{h}(\Phi; m) \cdot \boldsymbol{\tau}(m)^\top = \boldsymbol{f}(\Phi; m) \cdot \boldsymbol{\iota}(m)^\top = \#\Phi \ .$$

(d) *Consider the face system*

$$\Phi^* := \left\{ [m] - F \colon F \in \mathbf{2}^{[m]}, \ F \notin \Phi \right\}$$

"dual" to the system Φ. We have

$$h_l(\Phi; m) + (-1)^l \sum_{k=l}^{m} \binom{k}{l} h_k(\Phi^*; m) = \delta_{l,0} \ , \quad 0 \le l \le m \ ;$$
$$h_m(\Phi; m) = (-1)^{m+1} h_m(\Phi^*; m) \ .$$

If Δ is a complex on the vertex set $[m]$, and $\#\Delta > 0$, $\#\Delta^* > 0$, then we have

$$h_l(\Delta; m) = 0 \ , \quad 1 \le l \le m - d(\Delta^*) - 1 \ ,$$
$$h_{m-d(\Delta^*)}(\Delta; m) = -f_{d(\Delta^*)}(\Delta^*; m) \ .$$

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2.2 Face Systems, Distinguished Bases, and Change of Basis Matrices

In this section we first choose certain bases of the real Euclidean spaces, suitable for linear algebraic descriptions of the face systems contained in a simplex, and then we list the corresponding change of basis matrices.

Let $\{F_0, F_1, \ldots, F_m\} \subseteq \mathbf{2}^{[m]}$ be an arbitrary face system such that $|F_k| = k$, $0 \le k \le m$; in particular, $F_0 := \hat{0}$ and $F_m := [m]$.

Considering face systems in the simplex $\mathbf{2}^{[m]}$, we fix three pairs of distinguished *ordered bases* of the space $\mathbb{R}^{m+1}$:

1) A first pair consists of the bases $(\boldsymbol{f}(\{F_0\}; m), \boldsymbol{f}(\{F_1\}; m), \ldots, \boldsymbol{f}(\{F_m\}; m))$ and $(\boldsymbol{h}(\{F_0\}; m), \boldsymbol{h}(\{F_1\}; m), \ldots, \boldsymbol{h}(\{F_m\}; m))$.
   The basis

$$\begin{aligned}\mathfrak{S}_m &:= \left(\boldsymbol{\sigma}(i; m) \colon 0 \le i \le m\right) \\ &= \left(\boldsymbol{f}(\{F_0\}; m), \boldsymbol{f}(\{F_1\}; m), \ldots, \boldsymbol{f}(\{F_m\}; m)\right),\end{aligned}$$

   where

$$\boldsymbol{\sigma}(i; m) := (1, 0, \ldots, 0) \cdot \mathbf{T}(m)^i, \quad 0 \le i \le m,$$

   is just the *standard basis* of $\mathbb{R}^{m+1}$. The other basis of $\mathbb{R}^{m+1}$ is

$$\begin{aligned}\mathfrak{H}^{\bullet}_m &:= \left(\boldsymbol{\vartheta}^{\bullet}(i; m) \colon 0 \le i \le m\right) \\ &= \left(\boldsymbol{h}(\{F_0\}; m), \boldsymbol{h}(\{F_1\}; m), \ldots, \boldsymbol{h}(\{F_m\}; m)\right)\end{aligned}$$

   whose vectors

$$\boldsymbol{\vartheta}^{\bullet}(i; m) := \left(\vartheta^{\bullet}_0(i; m), \vartheta^{\bullet}_1(i; m), \ldots, \vartheta^{\bullet}_m(i; m)\right) \in \mathbb{Z}^{m+1}$$

   are defined by

$$\vartheta^{\bullet}_j(i; m) := (-1)^{j-i}\tbinom{m-i}{j-i}, \quad 0 \le j \le m.$$

2) The bases $(\boldsymbol{f}([F_0, F_0]; m), \boldsymbol{f}([F_0, F_1]; m), \ldots, \boldsymbol{f}([F_0, F_m]; m))$ and $(\boldsymbol{h}([F_0, F_0]; m), \boldsymbol{h}([F_0, F_1]; m), \ldots, \boldsymbol{h}([F_0, F_m]; m))$ form a second pair.
   One basis

$$\begin{aligned}\mathfrak{F}^{\blacktriangle}_m &:= \left(\boldsymbol{\varphi}^{\blacktriangle}(i; m) \colon 0 \le i \le m\right) \\ &:= \left(\boldsymbol{f}([F_0, F_0]; m), \boldsymbol{f}([F_0, F_1]; m), \ldots, \boldsymbol{f}([F_0, F_m]; m)\right)\end{aligned}$$

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of $\mathbb{R}^{m+1}$ consists of the vectors

$$\boldsymbol{\varphi}^{\blacktriangle}(i;m) := \big(\varphi_0^{\blacktriangle}(i;m), \varphi_1^{\blacktriangle}(i;m), \ldots, \varphi_m^{\blacktriangle}(i;m)\big) \in \mathbb{N}^{m+1} ,$$

where

$$\varphi_j^{\blacktriangle}(i;m) := \tbinom{i}{j} , \quad 0 \le j \le m .$$

The other basis

$$\begin{aligned} \mathfrak{H}_m^{\blacktriangle} &:= \big(\boldsymbol{\vartheta}^{\blacktriangle}(i;m) \colon 0 \le i \le m\big) \\ &:= \big(\, \boldsymbol{h}([F_0, F_0]; m), \boldsymbol{h}([F_0, F_1]; m), \ldots, \boldsymbol{h}([F_0, F_m]; m) \,\big) \end{aligned}$$

of $\mathbb{R}^{m+1}$ consists of the vectors

$$\boldsymbol{\vartheta}^{\blacktriangle}(i;m) := \big(\vartheta_0^{\blacktriangle}(i;m), \vartheta_1^{\blacktriangle}(i;m), \ldots, \vartheta_m^{\blacktriangle}(i;m)\big) \in \mathbb{Z}^{m+1}$$

defined by

$$\vartheta_j^{\blacktriangle}(i;m) := (-1)^j \tbinom{m-i}{j} , \quad 0 \le j \le m .$$

3) A third pair consists of the bases $(\boldsymbol{f}([F_m, F_m]; m), \boldsymbol{f}([F_{m-1}, F_m]; m), \ldots, \boldsymbol{f}([F_0, F_m]; m))$ and $(\boldsymbol{h}([F_m, F_m]; m), \boldsymbol{h}([F_{m-1}, F_m]; m), \ldots, \boldsymbol{h}([F_0, F_m]; m))$. These are the basis

$$\begin{aligned} \mathfrak{F}_m^{\blacktriangledown} &:= \big(\boldsymbol{\varphi}^{\blacktriangledown}(i;m) \colon 0 \le i \le m\big) \\ &:= \big(\boldsymbol{f}([F_m, F_m]; m), \boldsymbol{f}([F_{m-1}, F_m]; m), \ldots, \boldsymbol{f}([F_0, F_m]; m) \,\big) \end{aligned}$$

whose vectors

$$\boldsymbol{\varphi}^{\blacktriangledown}(i;m) := \big(\varphi_0^{\blacktriangledown}(i;m), \varphi_1^{\blacktriangledown}(i;m), \ldots, \varphi_m^{\blacktriangledown}(i;m)\big) \in \mathbb{N}^{m+1}$$

are defined by

$$\varphi_j^{\blacktriangledown}(i;m) := \tbinom{i}{m-j} , \quad 0 \le j \le m ,$$

and the basis

$$\begin{aligned} \mathfrak{H}_m^{\blacktriangledown} &:= \big(\boldsymbol{\vartheta}^{\blacktriangledown}(i;m) \colon 0 \le i \le m\big) \\ &:= \big(\, \boldsymbol{h}([F_m, F_m]; m), \boldsymbol{h}([F_{m-1}, F_m]; m), \ldots, \boldsymbol{h}([F_0, F_m]; m) \,\big) \end{aligned}$$

with the vectors

$$\boldsymbol{\vartheta}^{\blacktriangledown}(i;m) := \big(\vartheta_0^{\blacktriangledown}(i;m), \vartheta_1^{\blacktriangledown}(i;m), \ldots, \vartheta_m^{\blacktriangledown}(i;m)\big) \in \mathbb{Z}^{m+1} ,$$

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where

$$\vartheta_j^{\blacktriangledown}(i;m) := \delta_{m-i,j}\ , \quad 0 \le j \le m\ .$$

Note that $\mathfrak{H}_m^{\blacktriangledown}$ is (up to rearrangement) the standard basis $\mathfrak{S}_m$.

Let $\mathbf{S}(m)$ be the change of basis matrix from $\mathfrak{S}_m$ to $\mathfrak{H}_m^{\bullet}$:

$$\mathbf{S}(m) := \begin{pmatrix} \boldsymbol{\vartheta}^{\bullet}(0;m) \\ \vdots \\ \boldsymbol{\vartheta}^{\bullet}(m;m) \end{pmatrix}\ ;$$

the (i,j)th entry of the inverse matrix $\mathbf{S}(m)^{-1}$ is $\binom{m-i}{j-i}$.

For any $i \in \mathbb{N}$, $i \le m$, we have

$$\boldsymbol{\vartheta}^{\bullet}(i;m) = \boldsymbol{\sigma}(i;m) \cdot \mathbf{S}(m)\ ,$$

$$\boldsymbol{\vartheta}^{\blacktriangle}(i;m) = \boldsymbol{\varphi}^{\blacktriangle}(i;m) \cdot \mathbf{S}(m)\ ,$$

$$\boldsymbol{\vartheta}^{\blacktriangledown}(i;m) = \boldsymbol{\varphi}^{\blacktriangledown}(i;m) \cdot \mathbf{S}(m)\ .$$

For any face system $\Phi \subseteq \mathbf{2}^{[m]}$, we have

$$\begin{aligned} \boldsymbol{h}(\Phi;m) &= \boldsymbol{f}(\Phi;m) \cdot \mathbf{S}(m) = \sum_{l=0}^{m} f_l(\Phi;m) \cdot \boldsymbol{\vartheta}^{\bullet}(l;m)\ , \\ \boldsymbol{f}(\Phi;m) &= \boldsymbol{h}(\Phi;m) \cdot \mathbf{S}(m)^{-1}\ . \end{aligned} \tag{2.4}$$

The change of basis matrices corresponding to the bases defined earlier are presented in Table 2.1.

If $\Phi \subseteq \mathbf{2}^{[m]}$, then by eq. (2.4) we have

$$\boldsymbol{f}(\Phi;m) = [\boldsymbol{h}(\Phi;m)]_{\mathfrak{H}_m^{\bullet}}\ ;$$

note also that

$$\begin{aligned} [\boldsymbol{h}(\Phi;m)]_{\mathfrak{H}_m^{\blacktriangle}} &= [\boldsymbol{f}(\Phi;m)]_{\mathfrak{F}_m^{\blacktriangle}}\ ; \\ [\boldsymbol{h}(\Phi;m)]_{\mathfrak{H}_m^{\blacktriangledown}} &= [\boldsymbol{f}(\Phi;m)]_{\mathfrak{F}_m^{\blacktriangledown}} \\ &= \boldsymbol{h}(\Phi;m) \cdot \mathbf{U}(m)\ ; \\ [\boldsymbol{f}(\Phi;m)]_{\mathfrak{H}_m^{\blacktriangledown}} &= \boldsymbol{f}(\Phi;m) \cdot \mathbf{U}(m) \\ &= [\boldsymbol{h}(\Phi;m)]_{\mathfrak{H}_m^{\bullet}} \cdot \mathbf{U}(m)\ . \end{aligned}$$

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Table 2.1: Change of basis matrices.

Change of basis matrix	**(i,j)th entry**	**Notation**	**Case $m = 3$**
from $\mathfrak{S}_m$ to $\mathfrak{F}_m^{\blacktriangle}$ from $\mathfrak{H}_m^{\bullet}$ to $\mathfrak{H}_m^{\blacktriangle}$ from $\mathfrak{H}_m^{\blacktriangledown}$ to $\mathfrak{F}_m^{\blacktriangledown}$	$\binom{i}{j}$	$\begin{pmatrix} \varphi^{\blacktriangle}(0;m) \\ \vdots \\ \varphi^{\blacktriangle}(m;m) \end{pmatrix}$	$\begin{pmatrix} 1&0&0&0 \\ 1&1&0&0 \\ 1&2&1&0 \\ 1&3&3&1 \end{pmatrix}$
from $\mathfrak{F}_m^{\blacktriangle}$ to $\mathfrak{S}_m$ from $\mathfrak{H}_m^{\blacktriangle}$ to $\mathfrak{H}_m^{\bullet}$ from $\mathfrak{F}_m^{\blacktriangledown}$ to $\mathfrak{H}_m^{\blacktriangledown}$	$(-1)^{i+j}\binom{i}{j}$	$\begin{pmatrix} \varphi^{\blacktriangle}(0;m) \\ \vdots \\ \varphi^{\blacktriangle}(m;m) \end{pmatrix}^{-1}$	$\begin{pmatrix} 1&0&0&0 \\ -1&1&0&0 \\ 1&-2&1&0 \\ -1&3&-3&1 \end{pmatrix}$
from $\mathfrak{S}_m$ to $\mathfrak{F}_m^{\blacktriangledown}$ from $\mathfrak{H}_m^{\bullet}$ to $\mathfrak{H}_m^{\blacktriangledown}$ from $\mathfrak{H}_m^{\blacktriangledown}$ to $\mathfrak{F}_m^{\blacktriangle}$	$\binom{i}{m-j}$	$\begin{pmatrix} \varphi^{\blacktriangledown}(0;m) \\ \vdots \\ \varphi^{\blacktriangledown}(m;m) \end{pmatrix}$	$\begin{pmatrix} 0&0&0&1 \\ 0&0&1&1 \\ 0&1&2&1 \\ 1&3&3&1 \end{pmatrix}$
from $\mathfrak{F}_m^{\blacktriangledown}$ to $\mathfrak{S}_m$ from $\mathfrak{H}_m^{\blacktriangledown}$ to $\mathfrak{H}_m^{\bullet}$ from $\mathfrak{F}_m^{\blacktriangle}$ to $\mathfrak{H}_m^{\blacktriangledown}$	$(-1)^{m-j-i}\binom{m-i}{j}$	$\begin{pmatrix} \varphi^{\blacktriangledown}(0;m) \\ \vdots \\ \varphi^{\blacktriangledown}(m;m) \end{pmatrix}^{-1}$	$\begin{pmatrix} -1&3&-3&1 \\ 1&-2&1&0 \\ -1&1&0&0 \\ 1&0&0&0 \end{pmatrix}$
from $\mathfrak{S}_m$ to $\mathfrak{H}_m^{\blacktriangle}$	$(-1)^{j}\binom{m-i}{j}$	$\begin{pmatrix} \vartheta^{\blacktriangle}(0;m) \\ \vdots \\ \vartheta^{\blacktriangle}(m;m) \end{pmatrix}$	$\begin{pmatrix} 1&-3&3&-1 \\ 1&-2&1&0 \\ 1&-1&0&0 \\ 1&0&0&0 \end{pmatrix}$
from $\mathfrak{H}_m^{\blacktriangle}$ to $\mathfrak{S}_m$	$(-1)^{m-j}\binom{i}{m-j}$	$\begin{pmatrix} \vartheta^{\blacktriangle}(0;m) \\ \vdots \\ \vartheta^{\blacktriangle}(m;m) \end{pmatrix}^{-1}$	$\begin{pmatrix} 0&0&0&1 \\ 0&0&-1&1 \\ 0&1&-2&1 \\ -1&3&-3&1 \end{pmatrix}$
from $\mathfrak{S}_m$ to $\mathfrak{H}_m^{\blacktriangledown}$ from $\mathfrak{H}_m^{\blacktriangledown}$ to $\mathfrak{S}_m$	$\delta_{m-i,j}$	$\begin{pmatrix} \vartheta^{\blacktriangledown}(0;m) \\ \vdots \\ \vartheta^{\blacktriangledown}(m;m) \end{pmatrix}$ or $\mathbf{U}(m)$, or $\begin{pmatrix} \vartheta^{\blacktriangledown}(0;m) \\ \vdots \\ \vartheta^{\blacktriangledown}(m;m) \end{pmatrix}^{-1}$	$\begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix}$
from $\mathfrak{S}_m$ to $\mathfrak{H}_m^{\bullet}$	$(-1)^{j-i}\binom{m-i}{j-i}$	$\begin{pmatrix} \vartheta^{\bullet}(0;m) \\ \vdots \\ \vartheta^{\bullet}(m;m) \end{pmatrix}$ or $\mathbf{S}(m)$	$\begin{pmatrix} 1&-3&3&-1 \\ 0&1&-2&1 \\ 0&0&1&-1 \\ 0&0&0&1 \end{pmatrix}$

(*continued*)

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Table 2.1: (Continued)

Change of basis matrix	**(i,j)th entry**	**Notation**	**Case $m = 3$**
from $\mathfrak{H}^{\bullet}_m$ to $\mathfrak{S}_m$	$\binom{m-i}{j-i}$	$\begin{pmatrix} \vartheta^{\bullet}(0;m) \\ \vdots \\ \vartheta^{\bullet}(m;m) \end{pmatrix}^{-1}$ or $\mathbf{S}(m)^{-1}$	$\begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
from $\mathfrak{F}^{\blacktriangle}_m$ to $\mathfrak{H}^{\blacktriangle}_m$	$(-1)^j 2^{m-j-i}\binom{m-i}{j}$		$\begin{pmatrix} 8 & -12 & 6 & -1 \\ 4 & -4 & 1 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$
from $\mathfrak{H}^{\blacktriangle}_m$ to $\mathfrak{F}^{\blacktriangle}_m$	$(-1)^{m-j} 2^{i+j-m}\binom{i}{m-j}$		$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -4 & 4 \\ -1 & 6 & -12 & 8 \end{pmatrix}$
from $\mathfrak{F}^{\blacktriangle}_m$ to $\mathfrak{F}^{\blacktriangledown}_m$ from $\mathfrak{F}^{\blacktriangledown}_m$ to $\mathfrak{F}^{\blacktriangle}_m$ from $\mathfrak{H}^{\blacktriangle}_m$ to $\mathfrak{H}^{\blacktriangledown}_m$ from $\mathfrak{H}^{\blacktriangledown}_m$ to $\mathfrak{H}^{\blacktriangle}_m$	$(-1)^{m-j}\binom{m-i}{m-j}$		$\begin{pmatrix} -1 & 3 & -3 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
from $\mathfrak{H}^{\blacktriangle}_m$ to $\mathfrak{F}^{\blacktriangledown}_m$	$(-1)^{m-j}\sum_s \binom{i}{m-s}\binom{s}{m-j}$		$\begin{pmatrix} -1 & 3 & -3 & 1 \\ -1 & 4 & -5 & 2 \\ -1 & 5 & -8 & 4 \\ -1 & 6 & -12 & 8 \end{pmatrix}$
from $\mathfrak{F}^{\blacktriangledown}_m$ to $\mathfrak{H}^{\blacktriangle}_m$	$(-1)^{m-j}\sum_s \binom{m-i}{s}\binom{m-s}{j}$		$\begin{pmatrix} -8 & 12 & -6 & 1 \\ -4 & 8 & -5 & 1 \\ -2 & 5 & -4 & 1 \\ -1 & 3 & -3 & 1 \end{pmatrix}$
from $\mathfrak{H}^{\bullet}_m$ to $\mathfrak{F}^{\blacktriangle}_m$	$\sum_s \binom{i}{s}\binom{m-s}{m-j}$		$\begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 4 & 5 & 2 \\ 1 & 5 & 8 & 4 \\ 1 & 6 & 12 & 8 \end{pmatrix}$
from $\mathfrak{F}^{\blacktriangle}_m$ to $\mathfrak{H}^{\bullet}_m$	$(-1)^{i+j}\sum_s \binom{m-i}{m-s}\binom{s}{j}$		$\begin{pmatrix} 8 & -12 & 6 & -1 \\ -4 & 8 & -5 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 3 & -3 & 1 \end{pmatrix}$
from $\mathfrak{H}^{\bullet}_m$ to $\mathfrak{F}^{\blacktriangledown}_m$	$2^{i+j-m}\binom{i}{m-j}$		$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 4 & 4 \\ 1 & 6 & 12 & 8 \end{pmatrix}$
from $\mathfrak{F}^{\blacktriangledown}_m$ to $\mathfrak{H}^{\bullet}_m$	$(-2)^{m-j-i}\binom{m-i}{j}$		$\begin{pmatrix} -8 & 12 & -6 & 1 \\ 4 & -4 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

2.3 Partitions of Face Systems into Boolean Intervals, and the Long *f*- and *h*-Vectors. Dehn–Sommerville Type Relations

In this section we discuss a natural approach toward investigating the structural and combinatorial properties of face systems, which constitutes a study of their *decompositions* into *Boolean intervals*, the simplest building blocks of the systems.

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We also describe restrictions imposed on the *partition profiles* of the face systems satisfying *Dehn–Sommerville type relations*.
If

$$\Phi = [A_1, B_1]\dot{\cup}\cdots\dot{\cup}[A_\theta, B_\theta] \tag{2.5}$$

is a partition of a nonempty face system $\Phi \subseteq \mathbf{2}^{[m]}$ into *Boolean intervals* $[A_k, B_k]$, $1 \le k \le \theta$, then we call the collection P of *positive* integers p_{ij}, defined by

$$p_{ij} := \#\big\{[A_k, B_k]\colon |B_k - A_k| = i,\ |A_k| = j\big\} > 0 \ ,$$

the *profile* of partition (2.5). If $\theta = \#\Phi$, then $p_{0l} = f_l(\Phi; m)$ whenever $f_l(\Phi; m) > 0$.

Table 2.2 lists the representations of the vectors $\boldsymbol{f}(\Phi; m)$ and $\boldsymbol{h}(\Phi; m)$ with respect to distinguished bases.

Recall that the standard h-vector of an abstract simplicial complex Δ is said to satisfy the *Dehn–Sommerville relations* if we have

$$h_l(\Delta) = h_{d(\Delta)-l}(\Delta) \ , \qquad 0 \le l \le d(\Delta) \ ,$$

or, equivalently, in the language of the long h-vectors,

$$h_l(\Delta; m) = (-1)^{m-d(\Delta)} h_{m-l}(\Delta; m) \ , \qquad 0 \le l \le m \ . \tag{2.6}$$

Table 2.2: Representations (based on the profile of a partition of a face system $\Phi \subseteq \mathbf{2}^{[m]}$, $\#\Phi > 0$, into Boolean intervals) of the long f- and h-vectors $\boldsymbol{f}(\Phi; m)$ and $\boldsymbol{h}(\Phi; m)$ with respect to distinguished bases of $\mathbb{R}^{m+1}$.

lth component	Expression
$f_l(\Phi; m)$	$\sum_{i,j} p_{ij} \cdot \binom{i}{l-j}$
$\kappa_l\big(\boldsymbol{f}(\Phi; m), \mathfrak{H}^{\bullet}_m\big)$	$\sum_s \binom{m-s}{m-l} \sum_{i,j} p_{ij} \cdot \binom{i}{s-j}$
$\kappa_l\big(\boldsymbol{f}(\Phi; m), \mathfrak{F}^{\blacktriangle}_m\big)$	$(-1)^l \sum_{i,j} p_{ij} \cdot (-1)^{i+j} \binom{j}{l-i}$
$\kappa_l\big(\boldsymbol{f}(\Phi; m), \mathfrak{H}^{\blacktriangle}_m\big)$	$(-1)^{m-l} \sum_s \binom{s}{m-l} \sum_{i,j} p_{ij} \cdot \binom{i}{s-j}$
$\kappa_l\big(\boldsymbol{f}(\Phi; m), \mathfrak{F}^{\blacktriangledown}_m\big)$	$(-1)^{m-l} \sum_{i,j} p_{ij} \cdot (-1)^j \binom{m-i-j}{l-i}$
$\kappa_l\big(\boldsymbol{f}(\Phi; m), \mathfrak{H}^{\blacktriangledown}_m\big)$	$\sum_{i,j} p_{ij} \cdot \binom{i}{m-l-j}$
$h_l(\Phi; m)$	$(-1)^l \sum_{i,j} p_{ij} \cdot (-1)^j \binom{m-i-j}{l-j}$
$\kappa_l\big(\boldsymbol{h}(\Phi; m), \mathfrak{H}^{\bullet}_m\big)$	$\sum_{i,j} p_{ij} \cdot \binom{i}{l-j}$
$\kappa_l\big(\boldsymbol{h}(\Phi; m), \mathfrak{F}^{\blacktriangle}_m\big)$	$(-1)^l \sum_s \binom{s}{l} \sum_{i,j} p_{ij} \cdot (-1)^j \binom{m-i-j}{s-j}$
$\kappa_l\big(\boldsymbol{h}(\Phi; m), \mathfrak{H}^{\blacktriangle}_m\big)$	$(-1)^l \sum_{i,j} p_{ij} \cdot (-1)^{i+j} \binom{j}{l-i}$
$\kappa_l\big(\boldsymbol{h}(\Phi; m), \mathfrak{F}^{\blacktriangledown}_m\big)$	$(-1)^{m-l} \sum_s \binom{m-s}{l} \sum_{i,j} p_{ij} \cdot (-1)^j \binom{m-i-j}{s-j}$
$\kappa_l\big(\boldsymbol{h}(\Phi; m), \mathfrak{H}^{\blacktriangledown}_m\big)$	$(-1)^{m-l} \sum_{i,j} p_{ij} \cdot (-1)^j \binom{m-i-j}{l-i}$

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We will say for brevity that a face system $\Phi \subset \mathbf{2}^{[m]}$ is a *DS-system* if the Dehn–Sommerville type relations

$$h_l(\Phi; m) = (-1)^{m-d(\Phi)} h_{m-l}(\Phi; m) , \quad 0 \le l \le m , \tag{2.7}$$

are valid, in exact analogy to eq. (2.6). Note that the systems $\emptyset$ and $\{\hat{0}\}$ are both DS-systems.

If $\#\Phi > 0$, then define the integer

$$\eta(\Phi) := \begin{cases} |\bigcup_{F\in\Phi} F| , & \text{if } |\bigcup_{F\in\Phi} F| \equiv d(\Phi) \pmod 2 , \\ |\bigcup_{F\in\Phi} F| + 1 , & \text{if } |\bigcup_{F\in\Phi} F| \not\equiv d(\Phi) \pmod 2 . \end{cases} \tag{2.8}$$

Given a complex Δ with v vertices, $v > 0$, we thus have

$$\eta(\Delta) = \begin{cases} v , & \text{if } v \equiv d(\Delta) \pmod 2 , \\ v + 1 , & \text{if } v \not\equiv d(\Delta) \pmod 2 . \end{cases}$$

Relations (2.7) and definition (2.8) lead to the following observation: A face system Φ, with $\#\Phi > 0$, is a DS-system if and only if for any $n \in \mathbb{P}$ such that

$$\begin{aligned} \eta(\Phi) &\le n , \\ n &\equiv d(\Phi) \pmod 2 , \end{aligned} \tag{2.9}$$

we have

$$h_l(\Phi; n) = h_{n-l}(\Phi; n) , \quad 0 \le l \le n ,$$

or, equivalently,

$$\boldsymbol{h}(\Phi; n) = \boldsymbol{h}(\Phi; n) \cdot \mathbf{U}(n) ,$$

that is, $\boldsymbol{h}(\Phi; n)$ is a *left eigenvector* of the *backward identity matrix* of order $n+1$, which corresponds to its eigenvalue 1.

We deduce the following result:

Proposition 2.2. *Let Φ be a* DS-system *such that $\#\Phi > 0$, and let n be any positive integer satisfying conditions (2.9). Let $l \in \mathbb{N}$, $l \le n$.*

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(i) *We have*

$$\begin{aligned}\kappa_l\big(\boldsymbol{h}(\Phi;n),\mathfrak{H}_n^{\blacktriangle}\big)&=\kappa_l\big(\boldsymbol{f}(\Phi;n),\mathfrak{F}_n^{\blacktriangle}\big)\\&=(-1)^{n-l}f_l(\Phi;n)\ ;\\\kappa_l\big(\boldsymbol{h}(\Phi;n),\mathfrak{H}_n^{\blacktriangledown}\big)&=\kappa_l\big(\boldsymbol{f}(\Phi;n),\mathfrak{F}_n^{\blacktriangledown}\big)\\&=h_l(\Phi;n)=h_{n-l}(\Phi;n)\ ;\\\kappa_l\big(\boldsymbol{h}(\Phi;n),\mathfrak{F}_n^{\blacktriangle}\big)&=\kappa_l\big(\boldsymbol{h}(\Phi;n),\mathfrak{F}_n^{\blacktriangledown}\big)\ .\end{aligned}$$

(ii) *If P is the profile of a partition of the* DS-system Φ *into Boolean intervals, then we have*

$$\sum_{i,j}p_{ij}\cdot(-1)^{i+j}\binom{j}{l-i}=(-1)^n\sum_{i,j}p_{ij}\cdot\binom{i}{l-j}\ ;$$

$$\sum_{i,j}p_{ij}\cdot(-1)^{j}\binom{n-i-j}{l-i}=(-1)^n\sum_{i,j}p_{ij}\cdot(-1)^{j}\binom{n-i-j}{l-j}\ ;$$

$$\begin{aligned}&\sum_{s}\binom{s}{l}\sum_{i,j}p_{ij}\cdot(-1)^{j}\binom{n-i-j}{s-j}\\&=(-1)^n\sum_{s}\binom{n-s}{l}\sum_{i,j}p_{ij}\cdot(-1)^{j}\binom{n-i-j}{s-j}\ .\end{aligned}$$

Notes

Abstract simplicial complexes are discussed, for example, in [16, 22, 25, 38, 40, 41, 77, 81, 93, 102–104, 107, 111, 114, 126, 130, 148, 159, 161, 165, 168, 175, 178, 179, 181, 186].

The f-vectors of abstract simplicial complexes are characterized by the *Schützenberger–Kruskal–Katona theorem*, see for example [178, Theorem 2.1].

See, for example, [178, Sect. III.7] on *relative simplicial complexes*.

We refer the reader to the works [63, 64] for a review of central combinatorial decomposition problems and methods.

The poset-theoretic properties of the family of intervals of the Boolean lattices are investigated, for example, in Ref. [4].

Vectors (2.1) and (2.2) are closely related to analogous redundant descriptions of combinatorial objects that appear, for example, in Ref. [146, 147].

Valuations on *distributive lattices* are discussed, for example, in [6, Sect. IV.4], [17, 88], [115, Ch. 3].

The properties of the long f- and h-vectors given in Proposition 2.1 are described in Ref. [135]. See, for example, [3, 135, 141] on an inclusion–exclusion approach to enumerating the faces of set systems and of their “duals.”

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Recall that if the complexes Δ and Δ^* appearing in Proposition 2.1(d) have the same vertex set $[m]$, then the complex $\Delta^* =: \Delta^\vee$ is called its *Alexander dual*. Combinatorial *Alexander duality* is discussed, for example, in [29], [40, Sect. 2.4], [41, Sect. 2.2], [80, 81, 103], [148, Ch. 5], [159, Ch. 3],[181, Ch. 6, 7].

Book [105] is a standard text on *matrix analysis*.

Several references on the *Dehn–Sommerville relations* will be given in the Notes to Chapter 3 on page 81.

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3 Dehn–Sommerville Type Relations

In this chapter we continue to consider face systems that satisfy the Dehn–Sommerville type relations discussed in Section 2.3.

Throughout the chapter, m denotes an integer at least 2. Let $\Delta \subseteq \mathbf{2}^{[m]}$ be an abstract simplicial complex of size $d(\Delta) > 0$. We denote by $\mathbf{U}(d(\Delta))$ the *backward identity matrix* of order $d(\Delta) + 1$ whose rows and columns are indexed starting with zero, and whose (i, j)th entry is the Kronecker delta $\delta_{i+j,d(\Delta)}$.

Recall that the h-vector of the complex Δ is said to satisfy the *Dehn–Sommerville relations* if $\boldsymbol{h}(\Delta)$ is a *left eigenvector* of the matrix $\mathbf{U}(d(\Delta))$ corresponding to its eigenvalue 1, that is,

$$h_l(\Delta) = h_{d(\Delta)-l}(\Delta) \,, \qquad\qquad 0 \le l \le d(\Delta) \,, \tag{3.1}$$

or, equivalently, in the language of the long h-vectors,

$$h_l(\Delta; m) = (-1)^{m-d(\Delta)} h_{m-l}(\Delta; m) \,, \qquad\qquad 0 \le l \le m \,. \tag{3.2}$$

We investigate the set of all the row vectors $\mathfrak{f} \in \mathbb{N}^{m+1}$ such that for each of them there exists a face system $\Phi \subset \mathbf{2}^{[m]}$ satisfying Dehn–Sommerville type relations analogous to eq. (3.2), and $\boldsymbol{f}(\Phi; m) = \mathfrak{f}$; we interpret the set of those vectors $\mathfrak{f}$ as a set of integer points contained in certain rational convex polytopes.

3.1 Dehn–Sommerville Type Relations for the Long *h*-Vectors

In this section we characterize the long h-vectors of the face systems $\Phi \subset \mathbf{2}^{[m]}$ satisfying Dehn–Sommerville type relations, describe their linear spans, and give the representations of these vectors with respect to distinguished bases of the space $\mathbb{R}^{m+1}$.

Recall that a *DS-system* was defined in Section 2.3 to be a face system $\Phi \subset \mathbf{2}^{[m]}$ such that its long h-vector $\boldsymbol{h}(\Phi; m)$ satisfies the *Dehn–Sommerville type relations*

$$h_l(\Phi; m) = (-1)^{m-d(\Phi)} h_{m-l}(\Phi; m) \,, \quad 0 \le l \le m \,. \tag{3.3}$$

A face system Φ with $\#\Phi > 0$ is a DS-system if and only if for any $n \in \mathbb{P}$ such that

$$\begin{aligned} \eta(\Phi) &\le n \,, \\ n &\equiv d(\Phi) \pmod 2 \,, \end{aligned} \tag{3.4}$$

where the integer $\eta(\Phi)$ was defined by eq. (2.8), the vector $\boldsymbol{h}(\Phi; n)$ is a *left eigenvector* (corresponding to the *eigenvalue* 1) of the *backward identity matrix* $\mathbf{U}(n)$ of order $n + 1$:

$$\boldsymbol{h}(\Phi; n) = \boldsymbol{h}(\Phi; n) \cdot \mathbf{U}(n) \,.$$

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The Spectrum of a Backward Identity Matrix

Since the backward identity matrix $\mathbf{U}(m)$ is a *permutation matrix*, its *eigenvalues* are -1 and 1; the matrix $\mathbf{U}(m)$ is *unimodular*.

The *algebraic multiplicity* of the eigenvalue 1 is $\lceil \frac{m+1}{2} \rceil$: the *characteristic polynomial* $\wp(\mathbf{U}(m))$, in the variable λ, of this matrix is

$$\wp\big(\mathbf{U}(m)\big) = \begin{cases} (\lambda-1)^{\frac{m+2}{2}}(\lambda+1)^{\frac{m}{2}} , & \text{if } m \text{ is even} , \\ (\lambda-1)^{\frac{m+1}{2}}(\lambda+1)^{\frac{m+1}{2}} , & \text{if } m \text{ is odd} . \end{cases}$$

In other words,

$$\wp\big(\mathbf{U}(m)\big) = \begin{cases} \sum_{s=0}^{m+1} \mathbf{K}_s(m+1, \frac{m+2}{2}) \cdot \lambda^s , & \text{if } m \text{ is even} , \\ \sum_{s=0}^{m+1} \mathbf{K}_s(m+1, \frac{m+1}{2}) \cdot \lambda^s , & \text{if } m \text{ is odd} , \end{cases}$$

where $\mathbf{K}_s(t, i)$ is the *Krawtchouk polynomial* defined by

$$\sum_{s=0}^{t} \mathbf{K}_s(t,i) \cdot \lambda^s := (-\lambda+1)^i(\lambda+1)^{t-i} .$$

The *geometric multiplicity* of the eigenvalue 1 equals its algebraic multiplicity.

The Linear Spans of the Long *h*-Vectors of DS-Systems

For the positive integers k, define abstract simplicial complexes $\overline{\mathbf{2}^{[k]}}$ by

$$\overline{\mathbf{2}^{[k]}} := \mathbf{2}^{[k]} - \{[k]\} = [\hat{0}, [k]] - \{[k]\} . \tag{3.5}$$

We denote the *eigenspace* of the *backward identity matrix* $\mathbf{U}(m)$, that corresponds to its *eigenvalue* 1, by $\mathscr{E}^{\mathrm{h}}(m)$; span$(\cdot)$ will denote the *linear span* of a set of vectors over the reals.

Proposition 3.1.

(i) *If m is* even, *then*

$$\mathscr{E}^{\mathrm{h}}(m) = \mathrm{span}\Big(\boldsymbol{h}(\overline{\mathbf{2}^{[1]}}; m),\ \boldsymbol{h}(\overline{\mathbf{2}^{[3]}}; m),\ \ldots,\ \boldsymbol{h}(\overline{\mathbf{2}^{[m-1]}}; m),\ \boldsymbol{\iota}(m)\Big) .$$

If m is odd, *then*

$$\mathscr{E}^{\mathrm{h}}(m) = \mathrm{span}\Big(\boldsymbol{h}(\overline{\mathbf{2}^{[2]}}; m),\ \boldsymbol{h}(\overline{\mathbf{2}^{[4]}}; m),\ \ldots,\ \boldsymbol{h}(\overline{\mathbf{2}^{[m-1]}}; m),\ \boldsymbol{\iota}(m)\Big) .$$

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(ii) *If* $\Phi \subset \mathbf{2}^{[m]}$ *is a* DS-system *such that* $\Phi \not\ni [m]$ *(in particular, if* Φ *is an abstract simplicial complex* Δ *whose h-vector satisfies eq.* (3.1)*), then*

$$\boldsymbol{h}(\Phi; m) \cdot \boldsymbol{\iota}(m)^{\top} = 0 \ .$$

Proof. It is straightforward to check assertion (i), thanks to the simple characterization of the spectrum of the matrix $\mathbf{U}(m)$.

Assertion (ii) is an immediate consequence of identity (2.3). □

Define a linear hyperplane $\boldsymbol{H}(m)$ in the space $\mathscr{E}^{\mathrm{h}}(m)$ as follows: if m is *even*, then

$$\boldsymbol{H}(m) := \operatorname{span}\left(\boldsymbol{h}(\overline{\mathbf{2}^{[1]}}; m),\ \boldsymbol{h}(\overline{\mathbf{2}^{[3]}}; m),\ \ldots,\ \boldsymbol{h}(\overline{\mathbf{2}^{[m-1]}}; m)\right) \ ,$$

with $\dim \boldsymbol{H}(m) = \frac{m}{2}$; if m is *odd*, then

$$\boldsymbol{H}(m) := \operatorname{span}\left(\boldsymbol{h}(\overline{\mathbf{2}^{[2]}}; m),\ \boldsymbol{h}(\overline{\mathbf{2}^{[4]}}; m),\ \ldots,\ \boldsymbol{h}(\overline{\mathbf{2}^{[m-1]}}; m)\right) \ ,$$

with $\dim \boldsymbol{H}(m) = \frac{m-1}{2}$.

Note that the one-dimensional subspace $\operatorname{span}(\boldsymbol{\iota}(m))$ is the *orthogonal complement* of the subspace $\boldsymbol{H}(m)$ of the space $\mathscr{E}^{\mathrm{h}}(m)$, with respect to the standard scalar product.

Distinguished Bases of $\mathbb{R}^{m+1}$

Let $\{F_0, \ldots, F_m\} \subset \mathbf{2}^{[m]}$ be an arbitrary face system such that $|F_k| = k$, for $0 \le k \le m$; note that $F_0 := \hat{0}$ and $F_m := [m]$.

The following six bases of $\mathbb{R}^{m+1}$ were defined in Section 2.2:

$$\begin{aligned}
\mathfrak{S}_m &= \big(\boldsymbol{\sigma}(0; m), \ldots, \boldsymbol{\sigma}(m; m)\big) &&:= \big(\, \boldsymbol{f}(\{F_0\}; m), \ldots, \boldsymbol{f}(\{F_m\}; m)\, \big)\,,\\
\mathfrak{H}^{\bullet}_m &= \big(\boldsymbol{\vartheta}^{\bullet}(0; m), \ldots, \boldsymbol{\vartheta}^{\bullet}(m; m)\big) &&:= \big(\, \boldsymbol{h}(\{F_0\}; m), \ldots, \boldsymbol{h}(\{F_m\}; m)\, \big)\,,\\
\mathfrak{F}^{\blacktriangle}_m &= \big(\boldsymbol{\varphi}^{\blacktriangle}(0; m), \ldots, \boldsymbol{\varphi}^{\blacktriangle}(m; m)\big) &&:= \big(\, \boldsymbol{f}([F_0, F_0]; m), \ldots, \boldsymbol{f}([F_0, F_m]; m)\, \big)\,,\\
\mathfrak{H}^{\blacktriangle}_m &= \big(\boldsymbol{\vartheta}^{\blacktriangle}(0; m), \ldots, \boldsymbol{\vartheta}^{\blacktriangle}(m; m)\big) &&:= \big(\, \boldsymbol{h}([F_0, F_0]; m), \ldots, \boldsymbol{h}([F_0, F_m]; m)\, \big)\,,\\
\mathfrak{F}^{\blacktriangledown}_m &= \big(\boldsymbol{\varphi}^{\blacktriangledown}(0; m), \ldots, \boldsymbol{\varphi}^{\blacktriangledown}(m; m)\big) &&:= \big(\, \boldsymbol{f}([F_m, F_m]; m), \ldots, \boldsymbol{f}([F_0, F_m]; m)\, \big)\,,\\
\mathfrak{H}^{\blacktriangledown}_m &= \big(\boldsymbol{\vartheta}^{\blacktriangledown}(0; m), \ldots, \boldsymbol{\vartheta}^{\blacktriangledown}(m; m)\big) &&:= \big(\, \boldsymbol{h}([F_m, F_m]; m), \ldots, \boldsymbol{h}([F_0, F_m]; m)\, \big)\,;
\end{aligned}$$

the basis $\mathfrak{H}^{\blacktriangledown}_m$ is up to rearrangement the standard basis $\mathfrak{S}_m$.

Table 3.1 lists the representations of the long h-vectors $\boldsymbol{h}(\overline{\mathbf{2}^{[k]}}; m)$ of the face systems (3.5) with respect to the above bases.

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Table 3.1: Representations of $\boldsymbol{h}(\overline{\mathbf{2}^{[k]}}; m)$ and $\boldsymbol{f}(\overline{\mathbf{2}^{[k]}}; m)$, $1 \le k \le m$, with respect to distinguished bases.

l**th component**	**Expression**
$h_l(\overline{\mathbf{2}^{[k]}}; m)$	$\vartheta_l^{\blacktriangle}(k;m) - \vartheta_l^{\bullet}(k;m) = (-1)^l\left(\binom{m-k}{l} - (-1)^k\binom{m-k}{l-k}\right)$
$\kappa_l\left(\boldsymbol{h}(\overline{\mathbf{2}^{[k]}}; m), \mathfrak{H}_m^{\bullet}\right)$	$\binom{k}{l} - \delta_{k,l}$
$\kappa_l\left(\boldsymbol{h}(\overline{\mathbf{2}^{[k]}}; m), \mathfrak{F}_m^{\blacktriangle}\right)$	$(-1)^l\left(2^{m-k-l}\binom{m-k}{l} - (-1)^k\sum_{s=l}^{m}\binom{m-k}{s-k}\binom{s}{l}\right)$
$\kappa_l\left(\boldsymbol{h}(\overline{\mathbf{2}^{[k]}}; m), \mathfrak{H}_m^{\blacktriangle}\right)$	$(-1)^{k-l}\left(\delta_{k,l} - \binom{k}{l}\right)$
$\kappa_l\left(\boldsymbol{h}(\overline{\mathbf{2}^{[k]}}; m), \mathfrak{F}_m^{\blacktriangledown}\right)$	$(-1)^{m-l}\sum_{s=0}^{m}\left(\binom{m-k}{s} - (-1)^k\binom{m-k}{s-k}\right)\binom{m-s}{l}$
$\kappa_l\left(\boldsymbol{h}(\overline{\mathbf{2}^{[k]}}; m), \mathfrak{H}_m^{\blacktriangledown}\right)$	$(-1)^{m-l}\left(\binom{m-k}{l-k} - (-1)^k\binom{m-k}{l}\right)$
$f_l(\overline{\mathbf{2}^{[k]}}; m)$	$\varphi_l^{\blacktriangle}(k;m) - \sigma_l(k;m) = \varphi_l^{\blacktriangle}(k;m) - \delta_{k,l} = \binom{k}{l} - \delta_{k,l}$
$\kappa_l\left(\boldsymbol{f}(\overline{\mathbf{2}^{[k]}}; m), \mathfrak{H}_m^{\bullet}\right)$	$\sum_{s=0}^{\min\{k,l\}}\binom{k}{s}\binom{m-s}{m-l} - \binom{m-k}{m-l}$
$\kappa_l\left(\boldsymbol{f}(\overline{\mathbf{2}^{[k]}}; m), \mathfrak{F}_m^{\blacktriangle}\right)$	$(-1)^{k-l}\left(\delta_{k,l} - \binom{k}{l}\right)$
$\kappa_l\left(\boldsymbol{f}(\overline{\mathbf{2}^{[k]}}; m), \mathfrak{H}_m^{\blacktriangle}\right)$	$(-1)^{m-l}\binom{k}{m-l}\left(2^{k+l-m} - 1\right)$
$\kappa_l\left(\boldsymbol{f}(\overline{\mathbf{2}^{[k]}}; m), \mathfrak{F}_m^{\blacktriangledown}\right)$	$\binom{m-k}{m-l} - (-1)^{m-k-l}\binom{m-k}{l}$
$\kappa_l\left(\boldsymbol{f}(\overline{\mathbf{2}^{[k]}}; m), \mathfrak{H}_m^{\blacktriangledown}\right)$	$\binom{k}{m-l} - \delta_{k,m-l}$

Recall that the matrix $\mathbf{S}(m)$ whose (i,j)th entry is $(-1)^{j-i}\binom{m-i}{j-i}$ was defined on page 60 to be the change of basis matrix from $\mathfrak{S}_m$ to $\mathfrak{H}_m^{\bullet}$:

$$\mathbf{S}(m) := \begin{pmatrix} \boldsymbol{\vartheta}^{\bullet}(0;m) \\ \vdots \\ \boldsymbol{\vartheta}^{\bullet}(m;m) \end{pmatrix} .$$

We denote by $\mathrm{S}_m \colon \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ the *automorphism* such that $\boldsymbol{v} \mapsto \boldsymbol{v} \cdot \mathbf{S}(m)$.

3.2 Dehn–Sommerville Type Relations for the Long f-Vectors

In this section we show how the *Dehn–Sommerville* type *relations* for the long h-vectors, discussed in Section 3.1, are restated in the language of the long f-vectors.

Define a *unimodular* matrix $\mathbf{D}(m)$ by

$$\mathbf{D}(m) := \mathbf{S}(m)\cdot\mathbf{U}(m)\cdot\mathbf{S}(m)^{-1} = \mathbf{S}(m)\cdot\begin{pmatrix} \boldsymbol{\varphi}^{\blacktriangledown}(0;m) \\ \vdots \\ \boldsymbol{\varphi}^{\blacktriangledown}(m;m) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\varphi}^{\blacktriangledown}(0;m) \\ \vdots \\ \boldsymbol{\varphi}^{\blacktriangledown}(m;m) \end{pmatrix}^{-1}\cdot\mathbf{S}(m)^{-1} ;$$

its (i,j)th entry is $(-1)^{m-i}\binom{i}{j}$.

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Since the matrices $\mathbf{D}(m)$ and $\mathbf{U}(m)$ are *similar*, the properties of $\mathbf{D}(m)$ coincide with those of $\mathbf{U}(m)$ mentioned on page 68 in section recalling the spectrum of $\mathbf{U}(m)$. Further, for any DS-system Φ such that $\#\Phi > 0$, and for any integer n satisfying eq. (3.4), we have

$$\boldsymbol{f}(\Phi; n) = \boldsymbol{f}(\Phi; n) \cdot \mathbf{D}(n) \; ; \tag{3.6}$$

in other words, we have

$$f_l(\Phi; n) = (-1)^n \sum_{i=l}^{d(\Phi)} (-1)^i \binom{i}{l} f_i(\Phi; n) \; , \quad 0 \le l \le d(\Phi) \; . \tag{3.7}$$

Let $\mathrm{S}_m^{-1} \colon \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ be the *automorphism* such that $\boldsymbol{v} \mapsto \boldsymbol{v} \cdot \mathbf{S}(m)^{-1}$. We define subspaces $\boldsymbol{F}(m) \subset \mathscr{E}^{\mathrm{f}}(m)$ of $\mathbb{R}^{m+1}$ by

$$\boldsymbol{F}(m) := \mathrm{S}_m^{-1}\big(\boldsymbol{H}(m)\big)$$

and

$$\mathscr{E}^{\mathrm{f}}(m) := \mathrm{S}_m^{-1}\big(\mathscr{E}^{\mathrm{h}}(m)\big) \; .$$

Thus, if m is *even*, then

$$\boldsymbol{F}(m) = \operatorname{span}\Big(\boldsymbol{f}(\overline{\mathbf{2}^{[1]}}; m),\ \boldsymbol{f}(\overline{\mathbf{2}^{[3]}}; m),\ \ldots,\ \boldsymbol{f}(\overline{\mathbf{2}^{[m-1]}}; m)\Big) \; ; \tag{3.8}$$

if m is *odd*, then

$$\boldsymbol{F}(m) = \operatorname{span}\Big(\boldsymbol{f}(\overline{\mathbf{2}^{[2]}}; m),\ \boldsymbol{f}(\overline{\mathbf{2}^{[4]}}; m),\ \ldots,\ \boldsymbol{f}(\overline{\mathbf{2}^{[m-1]}}; m)\Big) \; . \tag{3.9}$$

Define the row vector

$$\boldsymbol{\pi}(m) := \left(\tbinom{m+1}{0}, \tbinom{m+1}{1}, \ldots, \tbinom{m+1}{m}\right) = \boldsymbol{\iota}(m) \cdot \mathbf{S}(m)^{-1} \in \mathbb{N}^{m+1} \; .$$

For any m, we have

$$\mathscr{E}^{\mathrm{f}}(m) = \boldsymbol{F}(m) \oplus \operatorname{span}(\boldsymbol{\pi}(m)) \; ,$$

where $\oplus$ denotes the *direct sum of subspaces*; it is the *eigenspace* of the matrix $\mathbf{D}(m)$ corresponding to its *eigenvalue* 1.

All the long f-vectors that appear in expressions (3.8) and (3.9), as well as the vector $\boldsymbol{\pi}(m)$, lie in the affine hyperplane $\{\mathbf{z} \in \mathbb{R}^{m+1} \colon \mathrm{z}_0 = 1\}$. We also have

$$(\boldsymbol{\pi}(m), 0, 0) = \boldsymbol{f}(\overline{\mathbf{2}^{[m+1]}}; m+2) \; .$$

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The representations of the vectors $\boldsymbol{f}(\overline{\mathbf{2}^{[k]}}; m)$ with respect to distinguished bases of $\mathbb{R}^{m+1}$ are listed in Table 3.1.

3.3 The Long f-Vectors of DS-Systems, and Integer Points in Rational Convex Polytopes

In this section we regard the long f-vectors of face systems satisfying *Dehn–Sommerville* type *relations* as *integer points* contained in *rational convex polytopes*, and we give *generating functions* of these vectors.

We denote the *origin* $(0, \dots, 0) \in \mathbb{R}^{m+1}$ by $\mathbf{0}$; cone$(\cdot)$ and conv$(\cdot)$ denote the *conical hull* and the *convex hull*, respectively.

Investigating the long f-vectors of the DS-systems $\Phi \subset \mathbf{2}^{[m]}$ such that $\#\Phi > 0$ and $m \equiv d(\Phi) \pmod 2$, and considering the relation (3.6), we are interested in the solutions $\mathbf{z} \in \mathbf{N}^{m+1}$, $\mathbf{z} \neq \mathbf{0}$, to the system

$$\mathbf{z} \cdot \big(\mathbf{I}(m) - \mathbf{D}(m)\big) = \mathbf{0}\ , \quad \mathbf{0} \le \mathbf{z} \le \left(\tbinom{m}{0}, \tbinom{m}{1}, \dots, \tbinom{m}{m}\right) .$$

The matrix $\mathbf{I}(m) - \mathbf{D}(m)$ is *lower-triangular*, of *rank* $\left\lfloor \frac{m+1}{2} \right\rfloor$, with the *eigenvalues* 0 and 2.

Define the polytope

$$\mathscr{Q}^{\mathrm{f}}(m) := \left\{\mathbf{x} \in \mathbb{R}^{m+1} \colon \mathbf{x} \cdot \big(\mathbf{I}(m) - \mathbf{D}(m)\big) = \mathbf{0},\ \mathbf{0} \le \mathbf{x} \le \left(\tbinom{m}{0}, \tbinom{m}{1}, \dots, \tbinom{m}{m}\right)\right\} .$$

If m is *even*, then another description of $\mathscr{Q}^{\mathrm{f}}(m)$ is

$$\begin{aligned}\mathscr{Q}^{\mathrm{f}}(m) = \operatorname{span}\Big(&\boldsymbol{f}(\overline{\mathbf{2}^{[1]}}; m),\ \boldsymbol{f}(\overline{\mathbf{2}^{[3]}}; m),\ \dots,\ \boldsymbol{f}(\overline{\mathbf{2}^{[m-1]}}; m),\ \boldsymbol{\pi}(m)\Big) \\ &\cap\ \left\{\mathbf{x} \in \mathbb{R}^{m+1} \colon \mathbf{0} \le \mathbf{x} \le \left(\tbinom{m}{0}, \tbinom{m}{1}, \dots, \tbinom{m}{m}\right)\right\}\ ;\end{aligned}$$

if m is *odd*, then we have

$$\begin{aligned}\mathscr{Q}^{\mathrm{f}}(m) = \operatorname{span}\Big(&\boldsymbol{f}(\overline{\mathbf{2}^{[2]}}; m),\ \boldsymbol{f}(\overline{\mathbf{2}^{[4]}}; m),\ \dots,\ \boldsymbol{f}(\overline{\mathbf{2}^{[m-1]}}; m),\ \boldsymbol{\pi}(m)\Big) \\ &\cap\ \left\{\mathbf{x} \in \mathbb{R}^{m+1} \colon \mathbf{0} \le \mathbf{x} \le \left(\tbinom{m}{0}, \tbinom{m}{1}, \dots, \tbinom{m}{m}\right)\right\}\ .\end{aligned}$$

In other words, for any m, we have

$$\mathscr{Q}^{\mathrm{f}}(m) = \mathscr{E}^{\mathrm{f}}(m)\ \cap\ \boldsymbol{\Pi}(m)\ ,$$

where

$$\boldsymbol{\Pi}(m) := \operatorname{conv}\left(\sum_{k=0}^{m} a_k \binom{m}{k} \boldsymbol{\sigma}(k; m) \colon (a_0, \dots, a_m) \in \{0, 1\}^{m+1} \right) .$$

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On the one hand, the points $\{\mathbf{z} \in \mathscr{Q}^{\mathrm{f}}(m) \cap \mathbb{N}^{m+1} : \mathbf{z} \neq \mathbf{0}\}$ are exactly the vectors $\boldsymbol{f}(\Phi; m)$ of the DS-systems $\Phi \subset \mathbf{2}^{[m]}$ such that $\#\Phi > 0$ and $m \equiv d(\Phi) \pmod 2$. On the other hand, if $\mathbf{z}' \in \mathscr{Q}^{\mathrm{f}}(m)$, then there are $\prod_{k=0}^{m} \binom{\binom{m}{k}}{z'_k}$ DS-systems $\Phi \subset \mathbf{2}^{[m]}$ corresponding to the point $\mathbf{z}'$.

Example 3.2. We have $\mathscr{Q}^{\mathrm{f}}(2) = \operatorname{conv}(\mathbf{0}, (1,0,0), (0,1,1), (1,1,1))$, see Figure 3.1(a).

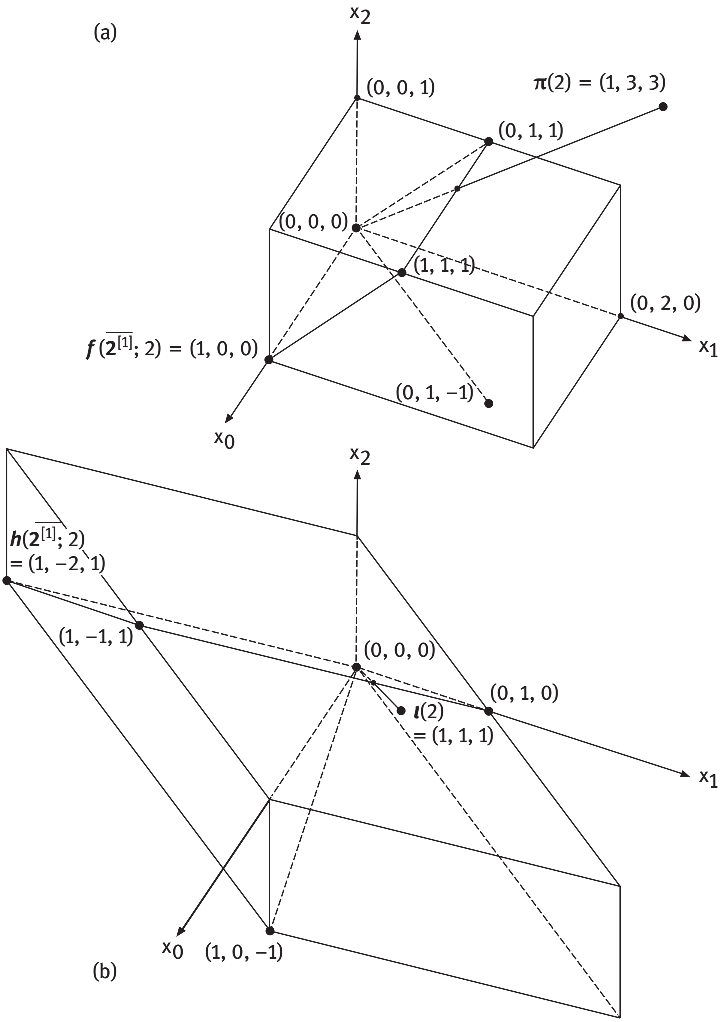

Figure 3.1: (a) The polytope $\mathscr{Q}^{\mathrm{f}}(2)$ of Example 3.2 is $\operatorname{conv}(\mathbf{0}, (1,0,0), (0,1,1), (1,1,1))$. (b) The polytope $\mathscr{Q}^{\mathrm{h}}(2)$ of Example 3.6 is $\operatorname{conv}(\mathbf{0}, (1,-2,1), (0,1,0), (1,-1,1))$.

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Yet another description of the polytope $\mathscr{Q}^{\mathrm{f}}(m)$ is

$$\mathscr{Q}^{\mathrm{f}}(m) = \left\{\mathbf{x} \in \mathscr{C}^{\mathrm{f}}(m)\colon \mathbf{x} \le (\tbinom{m}{0}, \tbinom{m}{1}, \dots, \tbinom{m}{m})\right\} ,$$

where a *convex pointed polyhedral cone* $\mathscr{C}^{\mathrm{f}}(m)$ is defined by

$$\mathscr{C}^{\mathrm{f}}(m) := \mathscr{E}^{\mathrm{f}}(m) \cap \operatorname{cone}\big(\boldsymbol{\sigma}(0; m),\ \boldsymbol{\sigma}(1; m),\ \dots,\ \boldsymbol{\sigma}(m; m)\big) .$$

Proposition 3.3.

(i) *If m is* even, *then*

$$\mathscr{C}^{\mathrm{f}}(m) \supseteq \operatorname{cone}\left(\boldsymbol{\varphi}^{\blacktriangle}(i; m) \cdot \mathbf{T}(m)^i\colon 0 \le i \le \tfrac{m}{2}\right) .$$

For any $i \in \mathbb{N}$, $i \le \frac{m}{2}$, the ray $\operatorname{cone}\big(\boldsymbol{\varphi}^{\blacktriangle}(i; m) \cdot \mathbf{T}(m)^i\big)$ *is an* extreme ray *of the $\frac{m+2}{2}$-dimensional* unimodular cone

$$\operatorname{cone}\big(\boldsymbol{\varphi}^{\blacktriangle}(i; m) \cdot \mathbf{T}(m)^i\colon 0 \le i \le \tfrac{m}{2}\big) . \tag{3.10}$$

(ii) *If m is* odd, *then*

$$\mathscr{C}^{\mathrm{f}}(m) \supseteq \operatorname{cone}\Big(\boldsymbol{\varphi}^{\blacktriangle}(i+1; m) \cdot \mathbf{T}(m)^i + \boldsymbol{\varphi}^{\blacktriangle}(i; m) \cdot \mathbf{T}(m)^{i+1}\colon 0 \le i \le \tfrac{m-1}{2}\Big) .$$

For any $i \in \mathbb{N}$, $i \le \frac{m-1}{2}$, the ray $\operatorname{cone}\big(\boldsymbol{\varphi}^{\blacktriangle}(i+1; m) \cdot \mathbf{T}(m)^i + \boldsymbol{\varphi}^{\blacktriangle}(i; m) \cdot \mathbf{T}(m)^{i+1}\big)$ *is an* extreme ray *of the $\frac{m+1}{2}$-dimensional* unimodular cone

$$\operatorname{cone}\big(\boldsymbol{\varphi}^{\blacktriangle}(i+1; m) \cdot \mathbf{T}(m)^i + \boldsymbol{\varphi}^{\blacktriangle}(i; m) \cdot \mathbf{T}(m)^{i+1}\colon 0 \le i \le \tfrac{m-1}{2}\big) .$$

Proof. (i) The vectors of the sequence

$$\big(\boldsymbol{\varphi}^{\blacktriangle}(i; m) \cdot \mathbf{T}(m)^i\colon 0 \le i \le \tfrac{m}{2}\big) \tag{3.11}$$

are *linearly independent*, and for $i, j \in \mathbb{N}$ such that $i \le \frac{m}{2}$ and $j \le m$, we have

$$\underbrace{\kappa_j}_{\text{or } \kappa_{m-j}}\big(\boldsymbol{\varphi}^{\blacktriangle}(i; m) \cdot \mathbf{T}(m)^i \cdot \mathbf{S}(m)\big) = (-1)^{j-i}\binom{m-2i}{j-i} ;$$

hence, sequence (3.11) is a *basis* of the space $\mathscr{E}^{\mathrm{f}}(m)$, and the cone generated by this basis is *simplicial*. The matrix

$$\begin{pmatrix} \kappa_0\big(\boldsymbol{\varphi}^{\blacktriangle}(0; m) \cdot \mathbf{T}(m)^0\big) & \cdots & \kappa_{\frac{m}{2}}\big(\boldsymbol{\varphi}^{\blacktriangle}(0; m) \cdot \mathbf{T}(m)^0\big) \\ \cdots & \cdots & \cdots \\ \kappa_0\big(\boldsymbol{\varphi}^{\blacktriangle}(\tfrac{m}{2}; m) \cdot \mathbf{T}(m)^{\frac{m}{2}}\big) & \cdots & \kappa_{\frac{m}{2}}\big(\boldsymbol{\varphi}^{\blacktriangle}(\tfrac{m}{2}; m) \cdot \mathbf{T}(m)^{\frac{m}{2}}\big) \end{pmatrix}$$

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is *upper-triangular*; its diagonal entries are all 1. This implies that sequence (3.11) is an *integral basis* of the intersection of the linear span of eq. (3.11) with $\mathbb{Z}^{m+1}$. In other words, the *cone* (3.10) is *unimodular*.

Pick a vector $\boldsymbol{v} \in \mathscr{E}^{\mathrm{f}}(m)$ such that $\boldsymbol{v} \geq \mathbf{0}$ and $\boldsymbol{v} \neq \mathbf{0}$. If $\boldsymbol{v} = \sum_{i=0}^{m/2} a_i \cdot \boldsymbol{\varphi}^{\blacktriangle}(i;m) \cdot \mathbf{T}(m)^i$, for some multipliers $a_0, \ldots, a_{\frac{m}{2}} \in \mathbb{R}$, then the equalities $\kappa_0(\boldsymbol{v}) = a_0$, $\kappa_1(\boldsymbol{v}) = a_1$ and $\kappa_{\frac{m}{2}}(\boldsymbol{v}) = a_{\frac{m}{2}}$ imply that $a_0, a_1, a_{\frac{m}{2}} \geq 0$. As a consequence, if $m \in \{2, 4\}$, then

$$\mathscr{C}^{\mathrm{f}}(m) = \operatorname{cone}\left(\boldsymbol{\varphi}^{\blacktriangle}(i;m) \cdot \mathbf{T}(m)^i \colon 0 \leq i \leq \tfrac{m}{2}\right) .$$

The proof of the assertion (ii) is analogous to that of the assertion (i). In particular, for $i, j \in \mathbb{N}$ such that $i \leq \frac{m-1}{2}$ and $j \leq m$, we have

$$\underbrace{\kappa_j}_{\text{or } \kappa_{m-j}}\Big(\left(\boldsymbol{\varphi}^{\blacktriangle}(i+1;m) \cdot \mathbf{T}(m)^i + \boldsymbol{\varphi}^{\blacktriangle}(i;m) \cdot \mathbf{T}(m)^{i+1}\right) \cdot \mathbf{S}(m) \Big)$$

$$= (-1)^{j-i}\left(\binom{m-2i-1}{j-i} - \binom{m-2i-1}{j-i-1}\right) ;$$

the sequence

$$\left(\boldsymbol{\varphi}^{\blacktriangle}(i+1;m) \cdot \mathbf{T}(m)^i + \boldsymbol{\varphi}^{\blacktriangle}(i;m) \cdot \mathbf{T}(m)^{i+1} \colon 0 \leq i \leq \tfrac{m-1}{2}\right) \tag{3.12}$$

is a *basis* of $\mathscr{E}^{\mathrm{f}}(m)$.

If $\boldsymbol{v} := \sum_{i=0}^{(m-1)/2} a_i \cdot \left(\boldsymbol{\varphi}^{\blacktriangle}(i+1;m) \cdot \mathbf{T}(m)^i + \boldsymbol{\varphi}^{\blacktriangle}(i;m) \cdot \mathbf{T}(m)^{i+1}\right) \in \mathscr{E}^{\mathrm{f}}(m)$ and $\boldsymbol{v} \geq \mathbf{0}$ and $\boldsymbol{v} \neq \mathbf{0}$, for some multipliers $a_0, \ldots, a_{\frac{m-1}{2}} \in \mathbb{R}$, then the equalities $\kappa_0(\boldsymbol{v}) = a_0$ and $\kappa_{\frac{m-1}{2}}(\boldsymbol{v}) = a_{\frac{m-1}{2}}$ imply that $a_0, a_{\frac{m-1}{2}} \geq 0$ and, as a consequence, we have

$$\mathscr{C}^{\mathrm{f}}(3) = \operatorname{cone}\Big(\boldsymbol{\varphi}^{\blacktriangle}(i+1;3) \cdot \mathbf{T}(3)^i + \boldsymbol{\varphi}^{\blacktriangle}(i;3) \cdot \mathbf{T}(3)^{i+1} \colon 0 \leq i \leq 1\Big) . \qquad \square$$

The structure of the bases (3.11) and (3.12) of the subspace $\mathscr{E}^{\mathrm{f}}(m)$ allows us to conclude the following result:

Corollary 3.4.

(i) *Let* $\Phi \subset \mathbf{2}^{[m]}$ *be a* DS-system.

If m *is* even, *then either* $f_{m-1}(\Phi;m) = \frac{m}{2}$ *and* $f_m(\Phi;m) = 1$, *or* $f_{m-1}(\Phi;m) = f_m(\Phi;m) = 0$.

If m *is* odd, *then we have* $f_{m-1}(\Phi;m) = f_m(\Phi;m) = 0$.

(ii) *If* m *is* even, *then for any* $t \in \mathbb{N}$, $t \leq \frac{m-2}{2}$, *we have*

$$\operatorname{span}\left(\boldsymbol{f}(\overline{\mathbf{2}^{[2i+1]}};m) \colon 0 \leq i \leq t\right) = \operatorname{span}\left(\boldsymbol{\varphi}^{\blacktriangle}(i;m) \cdot \mathbf{T}(m)^i \colon 0 \leq i \leq t\right) ;$$

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if m is odd, *then for any $t \in \mathbb{N}$, $t \le \frac{m-3}{2}$, we have*

$$\operatorname{span}\Big(\boldsymbol{f}(\overline{\mathbf{2}^{[2(i+1)]}}; m) \colon 0 \le i \le t\Big) \\ = \operatorname{span}\Big(\boldsymbol{\varphi}^{\blacktriangle}(i+1; m) \cdot \mathbf{T}(m)^{i} + \boldsymbol{\varphi}^{\blacktriangle}(i; m) \cdot \mathbf{T}(m)^{i+1} \colon 0 \le i \le t\Big) \ .$$

(iii) *We have*

$$\operatorname{span}\Big(\boldsymbol{f}(\Phi; m) \colon \Phi \subset \mathbf{2}^{[m]},\ m \equiv d(\Phi) \pmod 2,\ \Phi \text{ is DS}\Big) = \begin{cases} \mathscr{E}^{\mathrm{f}}(m)\ , & \text{if } m \text{ is even}\ , \\ \boldsymbol{F}(m)\ , & \text{if } m \text{ is odd}\ . \end{cases}$$

See Tables 3.2 and 3.3 on the representations of the linearly independent vectors in sequences (3.11) and (3.12) that span the subspace $\mathscr{E}^{\mathrm{f}}(m)$, with respect to distinguished bases of the space $\mathbb{R}^{m+1}$.

If m is *even*, define the $\frac{m}{2}$-dimensional polytope

$$\mathscr{P}^{\mathrm{f}}(m) := \mathscr{E}^{\mathrm{f}}(m) \cap \Big\{\mathbf{z} \in \mathbb{R}^{m+1} \colon \mathrm{z}_0 = 0\ ,\ \ \mathbf{0} \le (\mathrm{z}_1, \ldots, \mathrm{z}_m) \le (\tbinom{m}{1}, \tbinom{m}{2}, \ldots, \tbinom{m}{m})\Big\}\ ,$$

and note that the polytope $\mathscr{Q}^{\mathrm{f}}(m)$ is a *prism* whose *basis* is $\mathscr{P}^{\mathrm{f}}(m)$:

$$\mathscr{Q}^{\mathrm{f}}(m) = \mathscr{P}^{\mathrm{f}}(m) \boxplus \boldsymbol{\varphi}^{\blacktriangle}(0; m) \cdot \mathbf{T}(m)^{0} = \mathscr{P}^{\mathrm{f}}(m) \boxplus \boldsymbol{\sigma}(0; m)\ ,$$

Table 3.2: Representations of the vectors $\boldsymbol{w} := \boldsymbol{\varphi}^{\blacktriangle}(i; m) \cdot \mathbf{T}(m)^{i}$, where m is *even*, and $0 \le i \le \frac{m}{2}$, with respect to distinguished bases.

lth component	Expression
$\kappa_l\big(\boldsymbol{w}, \mathfrak{S}_m\big)$	$\binom{i}{l-i}$
$\kappa_l\big(\boldsymbol{w}, \mathfrak{H}^{\bullet}_m\big)$	$\sum_{s=i}^{\min\{2i,l\}} \binom{i}{s-i}\binom{m-s}{m-l}$
$\kappa_l\big(\boldsymbol{w}, \mathfrak{F}^{\blacktriangle}_m\big)$	$(-1)^{l}\binom{i}{l-i}$
$\kappa_l\big(\boldsymbol{w}, \mathfrak{H}^{\blacktriangle}_m\big)$	$(-1)^{l}\sum_{s=\max\{i,m-l\}}^{2i} \binom{i}{s-i}\binom{s}{m-l}$
$\kappa_l\big(\boldsymbol{w}, \mathfrak{F}^{\blacktriangledown}_m\big)$	$(-1)^{l-i}\binom{m-2i}{l-i}$
$\kappa_l\big(\boldsymbol{w}, \mathfrak{H}^{\blacktriangledown}_m\big)$	$\binom{i}{m-l-i}$
$\kappa_l\big(\boldsymbol{w} \cdot \mathbf{S}(m), \mathfrak{S}_m\big)$	$(-1)^{l-i}\binom{m-2i}{l-i}$
$\kappa_l\big(\boldsymbol{w} \cdot \mathbf{S}(m), \mathfrak{H}^{\bullet}_m\big)$	$\binom{i}{l-i}$
$\kappa_l\big(\boldsymbol{w} \cdot \mathbf{S}(m), \mathfrak{F}^{\blacktriangle}_m\big)$	$(-1)^{l-i}\sum_{s=\max\{i,l\}}^{m-i} \binom{s}{l}\binom{m-2i}{s-i}$
$\kappa_l\big(\boldsymbol{w} \cdot \mathbf{S}(m), \mathfrak{H}^{\blacktriangle}_m\big)$	$(-1)^{l}\binom{i}{l-i}$
$\kappa_l\big(\boldsymbol{w} \cdot \mathbf{S}(m), \mathfrak{F}^{\blacktriangledown}_m\big)$	$(-1)^{l-i}\sum_{s=\max\{i,l\}}^{m-i} \binom{s}{l}\binom{m-2i}{s-i}$
$\kappa_l\big(\boldsymbol{w} \cdot \mathbf{S}(m), \mathfrak{H}^{\blacktriangledown}_m\big)$	$(-1)^{l-i}\binom{m-2i}{l-i}$

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Table 3.3: Representations of the vectors $\boldsymbol{w} := \boldsymbol{\varphi}^{\blacktriangle}(i+1;m)\cdot\mathbf{T}(m)^{i} + \boldsymbol{\varphi}^{\blacktriangle}(i;m)\cdot\mathbf{T}(m)^{i+1}$, where m is *odd*, and $0 \le i \le \frac{m-1}{2}$, with respect to distinguished bases.

lth component	Expression
$\kappa_l\big(\boldsymbol{w}, \mathfrak{S}_m\big)$	$\binom{i+1}{l-i} + \binom{i}{l-i-1}$
$\kappa_l\big(\boldsymbol{w}, \mathfrak{H}^{\bullet}_m\big)$	$\binom{m-i}{m-l} + \sum_{s=i+1}^{\min\{2i+1,l\}}\big(\binom{i+1}{s-i} + \binom{i}{s-i-1}\big)\binom{m-s}{m-l}$
$\kappa_l\big(\boldsymbol{w}, \mathfrak{F}^{\blacktriangle}_m\big)$	$(-1)^{l-1}\big(\binom{i}{l-i-1} + \binom{i+1}{l-i}\big)$
$\kappa_l\big(\boldsymbol{w}, \mathfrak{H}^{\blacktriangle}_m\big)$	$(-1)^{l-1}\sum_{s=\max\{i,m-l\}}^{2i+1}\big(\binom{i+1}{s-i} + \binom{i}{s-i-1}\big)\binom{s}{m-l}$
$\kappa_l\big(\boldsymbol{w}, \mathfrak{F}^{\blacktriangledown}_m\big)$	$(-1)^{l-i-1}\big(\binom{m-2i-1}{m-l-i} - \binom{m-2i-1}{m-l-i-1}\big)$
$\kappa_l\big(\boldsymbol{w}, \mathfrak{H}^{\blacktriangledown}_m\big)$	$\binom{i+1}{m-l-i} + \binom{i}{m-l-i-1}$
$\kappa_l\big(\boldsymbol{w}\cdot\mathbf{S}(m), \mathfrak{S}_m\big)$	$(-1)^{l-i-1}\big(\binom{m-2i-1}{m-l-i} - \binom{m-2i-1}{m-l-i-1}\big)$
$\kappa_l\big(\boldsymbol{w}\cdot\mathbf{S}(m), \mathfrak{H}^{\bullet}_m\big)$	$\binom{i}{l-i-1} + \binom{i+1}{l-i}$
$\kappa_l\big(\boldsymbol{w}\cdot\mathbf{S}(m), \mathfrak{F}^{\blacktriangle}_m\big)$	$(-1)^{l-i}\sum_{s=\max\{i,l\}}^{m-i}\big(\binom{m-2i-1}{s-i} - \binom{m-2i-1}{s-i-1}\big)\binom{s}{l}$
$\kappa_l\big(\boldsymbol{w}\cdot\mathbf{S}(m), \mathfrak{H}^{\blacktriangle}_m\big)$	$(-1)^{l-1}\big(\binom{i}{l-i-1} + \binom{i+1}{l-i}\big)$
$\kappa_l\big(\boldsymbol{w}\cdot\mathbf{S}(m), \mathfrak{F}^{\blacktriangledown}_m\big)$	$(-1)^{l-i}\sum_{s=\max\{i,l\}}^{m-i}\big(\binom{m-2i-1}{s-i} - \binom{m-2i-1}{s-i-1}\big)\binom{s}{l}$
$\kappa_l\big(\boldsymbol{w}\cdot\mathbf{S}(m), \mathfrak{H}^{\blacktriangledown}_m\big)$	$(-1)^{l-i-1}\big(\binom{m-2i-1}{m-l-i} - \binom{m-2i-1}{m-l-i-1}\big)$

where $\boxplus$ denotes the *Minkowski addition*; as a consequence, we have

$$\sum_{\boldsymbol{\alpha}\in\mathscr{Q}^{\mathfrak{f}}(m)\cap\mathbb{N}^{m+1}} \mathbf{x}^{\boldsymbol{\alpha}} = (1+\mathrm{x}_0)\sum_{\boldsymbol{\alpha}\in\mathscr{P}^{\mathfrak{f}}(m)\cap\mathbb{N}^{m+1}} \mathbf{x}^{\boldsymbol{\alpha}}\,,$$

where $\mathbf{x}^{\boldsymbol{\alpha}} := \mathrm{x}_0{}^{\alpha_0}\cdots\mathrm{x}_m{}^{\alpha_m}$.

For any m, the generating function of the long f-vectors of DS-systems contained in a simplex $\mathbf{2}^{[m]}$ is

$$\sum_{\substack{\mathfrak{f}\in\mathbb{N}^{m+1}:\\ \exists\Phi\subset\mathbf{2}^{[m]},\\ \Phi\text{ is DS},\,\boldsymbol{f}(\Phi;m)=\mathfrak{f}}} \mathbf{x}^{\mathfrak{f}} = -1 + \sum_{\boldsymbol{\alpha}\in\mathscr{Q}^{\mathfrak{f}}(m)\cap\mathbb{N}^{m+1}} \mathbf{x}^{\boldsymbol{\alpha}} + \sum_{\substack{(\boldsymbol{\alpha},0)\in\mathscr{Q}^{\mathfrak{f}}(m+1)\cap\mathbb{N}^{m+2}:\\ \boldsymbol{\alpha}\le(\binom{m}{0},\binom{m}{1},\ldots,\binom{m}{m})}} \mathbf{x}^{\boldsymbol{\alpha}}\,,$$

and we conclude the following result:

Proposition 3.5.

(i) *If m is* even, *then*

$$\sum_{\substack{\mathfrak{f}\in\mathbb{N}^{m+1}:\\ \exists\Phi\subset\mathbf{2}^{[m]},\\ \Phi\text{ is DS},\,\boldsymbol{f}(\Phi;m)=\mathfrak{f}}} \mathbf{x}^{\mathfrak{f}} = -1 + (1+\mathrm{x}_0)\sum_{(0,\boldsymbol{\beta})\in\mathscr{P}^{\mathfrak{f}}(m)\cap\mathbb{N}^{m+1}} \mathbf{x}^{\boldsymbol{\beta}} + \sum_{\substack{(\boldsymbol{\gamma},0,0)\in\mathscr{Q}^{\mathfrak{f}}(m+1)\cap\mathbb{N}^{m+2}:\\ \boldsymbol{\gamma}\le(\binom{m}{0},\binom{m}{1},\ldots,\binom{m}{m-1})}} \mathbf{x}^{\boldsymbol{\gamma}}\,.$$

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Table 3.4: The number of the long f-vectors of the *DS-systems* contained in the simplices $\mathbf{2}^{[m]}$, $2 \le m \le 10$.

m	$\lvert\{ \mathfrak{f} \in \mathbb{N}^{m+1}: \exists\Phi \subset \mathbf{2}^{[m]}, \#\Phi > 0, m \equiv d(\Phi) \pmod 2, \Phi \text{ is DS}, \boldsymbol{f}(\Phi; m) = \mathfrak{f} \}\rvert$	$\lvert\{ \mathfrak{f} \in \mathbb{N}^{m+1}: \exists\Phi \subset \mathbf{2}^{[m]}, \#\Phi > 0, m \not\equiv d(\Phi) \pmod 2, \Phi \text{ is DS}, \boldsymbol{f}(\Phi; m) = \mathfrak{f} \}\rvert$	$\lvert\{ \mathfrak{f} \in \mathbb{N}^{m+1}: \exists\Phi \subset \mathbf{2}^{[m]}, \Phi \text{ is DS}, \boldsymbol{f}(\Phi; m) = \mathfrak{f} \}\rvert$
2	3	1	5
3	1	7	9
4	19	5	25
5	7	71	79
6	291	41	333
7	103	2223	2327
8	17465	1107	18573
9	4905	271619	276525
10	3959091	103057	4062149

(ii) *If m is* odd, *then*

$$\sum_{\substack{\mathfrak{f}\in\mathbb{N}^{m+1}:\\ \exists\Phi\subset\mathbf{2}^{[m]},\\ \Phi \text{ is DS } \boldsymbol{f}(\Phi;m)=\mathfrak{f}}} \mathbf{x}^{\mathfrak{f}} = -1 + \sum_{(\boldsymbol{\beta},0,0)\in\mathscr{Q}^{\mathrm{f}}(m)\cap\mathbb{N}^{m+1}} \mathbf{x}^{\boldsymbol{\beta}} + (1+\mathrm{x}_0) \sum_{\substack{(0,\boldsymbol{\gamma},0,0)\in\mathscr{P}^{\mathrm{f}}(m+1)\cap\mathbb{N}^{m+2}:\\ \boldsymbol{\gamma}\le(\binom{m}{1},\binom{m}{2},\dots,\binom{m}{m-1})}} \mathbf{x}^{\boldsymbol{\gamma}} .$$

Table 3.4 provides information on the number of the long f-vectors of DS-systems of small sizes.

3.4 The Long h-Vectors of DS-Systems, and Integer Points in Rational Convex Polytopes

In this section we first discuss those integer points in certain rational convex polytopes which are the long h-vectors of face systems that satisfy *Dehn–Sommerville* type *relations*. Then we give several additional linear algebraic results on the long f- and h-vectors of DS-systems.
We begin with the polytope

$$\mathscr{Q}^{\mathrm{h}}(m) := \mathrm{S}_m\bigl(\mathscr{Q}^{\mathrm{f}}(m)\bigr) := \bigl\{\mathbf{z}\cdot\mathbf{S}(m) \colon \mathbf{z}\in\mathscr{Q}^{\mathrm{f}}(m)\bigr\} .$$

The relation $\boldsymbol{h}(\Phi;m) = \boldsymbol{h}(\Phi;m)\cdot\mathbf{U}(m)$, $\boldsymbol{h}(\Phi;m) \ne \mathbf{0}$, describes the long h-vectors of the *DS-systems* $\Phi \subset \mathbf{2}^{[m]}$ such that $\#\Phi > 0$ and $m \equiv d(\Phi) \pmod 2$, and we are interested in the solutions $\mathbf{z} \in \mathbb{Z}^{m+1}$, $\mathbf{z} \ne \mathbf{0}$, to the system

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$$\mathbf{z}\cdot\big(\mathbf{I}(m)-\mathbf{U}(m)\big)=\mathbf{0}\ , \tag{3.13}$$

$$\mathbf{z}\in \mathrm{S}_m\big(\boldsymbol{\Pi}(m)\big)=\mathrm{conv}\left(\sum_{k=0}^{m} a_k\binom{m}{k}\boldsymbol{\vartheta}^{\bullet}(k;m)\colon (a_0,\ldots,a_m)\in\{0,1\}^{m+1}\right).$$

The matrix $\mathbf{I}(m)-\mathbf{U}(m)$ of *rank* $\lfloor\frac{m+1}{2}\rfloor$ is *totally unimodular*, that is, any of its *minor* is either -1, 0, or 1; in system (3.13), this matrix can be substituted by its submatrix composed of the first $\lfloor\frac{m+1}{2}\rfloor$ columns.

System (3.13) describes the intersection

$$\mathscr{Q}^{\mathrm{h}}(m):=\bigcap_{1\le k\le\lfloor(m+1)/2\rfloor}\big\{\mathbf{x}\in\mathbb{R}^{m+1}\colon \mathrm{x}_{k-1}=\mathrm{x}_{m-k+1}\big\}\ \cap\ \mathrm{S}_m\big(\boldsymbol{\Pi}(m)\big) \tag{3.14}$$

of the $\lceil\frac{m+1}{2}\rceil$-dimensional *center* of a *graphical hyperplane arrangement* with an $(m+1)$-dimensional parallelepiped. The *intersection poset* of the arrangement $\big\{\{\mathbf{x}\in\mathbb{R}^{m+1}\colon \mathrm{x}_{k-1}=\mathrm{x}_{m-k+1}\}\colon 1\le k\le\lfloor\frac{m+1}{2}\rfloor\big\}$ is a Boolean lattice of rank $\lfloor\frac{m+1}{2}\rfloor$.

Example 3.6. (cf. Example 3.2). We have $\mathscr{Q}^{\mathrm{h}}(2)=\mathrm{conv}(\mathbf{0},(1,-2,1),(0,1,0),(1,-1,1))$, see Figure 3.1(b).

Another description of polytope (3.14) is

$$\mathscr{Q}^{\mathrm{h}}(m)=\mathrm{span}\Big(\boldsymbol{h}(\overline{\mathbf{2}^{[1]}};m),\ \boldsymbol{h}(\overline{\mathbf{2}^{[3]}};m),\ \ldots,\ \boldsymbol{h}(\overline{\mathbf{2}^{[m-1]}};m),\ \boldsymbol{\iota}(m)\Big)\ \cap\ \mathrm{S}_m\big(\boldsymbol{\Pi}(m)\big)$$

in the case of m *even*, and

$$\mathscr{Q}^{\mathrm{h}}(m)=\mathrm{span}\Big(\boldsymbol{h}(\overline{\mathbf{2}^{[2]}};m),\ \boldsymbol{h}(\overline{\mathbf{2}^{[4]}};m),\ \ldots,\ \boldsymbol{h}(\overline{\mathbf{2}^{[m-1]}};m),\ \boldsymbol{\iota}(m)\Big)\ \cap\ \mathrm{S}_m\big(\boldsymbol{\Pi}(m)\big)\ ,$$

if m is *odd*. In other words, for any m, we have

$$\mathscr{Q}^{\mathrm{h}}(m)=\mathscr{E}^{\mathrm{h}}(m)\ \cap\ \mathrm{S}_m\big(\boldsymbol{\Pi}(m)\big)\ .$$

Biorthogonality

By the *principle of biorthogonality*, we have

$$\boldsymbol{h}(\overline{\mathbf{2}^{[i]}};m)\cdot\boldsymbol{h}(\overline{\mathbf{2}^{[j]}};m)^{\top}=0\ ,$$

for all positive integers i and j, less than or equal to m, and such that $i\not\equiv j \pmod 2$. Indeed, one of the vectors $\boldsymbol{h}(\overline{\mathbf{2}^{[i]}};m)$ and $\boldsymbol{h}(\overline{\mathbf{2}^{[j]}};m)$ is a *left eigenvector* of the backward identity matrix $\mathbf{U}(m)$ corresponding to an eigenvalue $\lambda\in\{-1,1\}$, while the other vector is a *right eigenvector* corresponding to the other eigenvalue $\mu\in\{-1,1\}$.

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The Norms

Given a vector $\boldsymbol{w} := (w_0, \ldots, w_m) \in \mathbb{R}^{m+1}$, we set $\|\boldsymbol{w}\|^2 := \sum_{i=0}^{m} w_i^2$.

For a positive integer k such that $k \le m$, we have

$$\|\boldsymbol{h}(\overline{\mathbf{2}^{[k]}}; m)\|^2 = 2\left(\binom{2(m-k)}{m-k} - (-1)^k \binom{2(m-k)}{m}\right)$$

and

$$\|\boldsymbol{f}(\overline{\mathbf{2}^{[k]}}; m)\|^2 = \binom{2k}{k} - 1\,.$$

Orthogonal Projectors

– If m is *even*, then the matrix of the *orthogonal projector* into $\boldsymbol{H}(m)$, relative either to the standard basis $\mathfrak{S}_m$, or to the basis $\mathfrak{H}_m^{\blacktriangledown}$, is

$$\left(\boldsymbol{h}(\overline{\mathbf{2}^{[1]}}; m)^\top \; \boldsymbol{h}(\overline{\mathbf{2}^{[3]}}; m)^\top \; \cdots \; \boldsymbol{h}(\overline{\mathbf{2}^{[m-1]}}; m)^\top\right)$$
$$\cdot \left(\begin{pmatrix} \boldsymbol{h}(\overline{\mathbf{2}^{[1]}}; m) \\ \boldsymbol{h}(\overline{\mathbf{2}^{[3]}}; m) \\ \vdots \\ \boldsymbol{h}(\overline{\mathbf{2}^{[m-1]}}; m) \end{pmatrix} \cdot \left(\boldsymbol{h}(\overline{\mathbf{2}^{[1]}}; m)^\top \; \boldsymbol{h}(\overline{\mathbf{2}^{[3]}}; m)^\top \; \cdots \; \boldsymbol{h}(\overline{\mathbf{2}^{[m-1]}}; m)^\top\right) \right)^{-1} \cdot \begin{pmatrix} \boldsymbol{h}(\overline{\mathbf{2}^{[1]}}; m) \\ \boldsymbol{h}(\overline{\mathbf{2}^{[3]}}; m) \\ \vdots \\ \boldsymbol{h}(\overline{\mathbf{2}^{[m-1]}}; m) \end{pmatrix} .$$

The matrix of the *orthogonal projector* (relative to the standard basis $\mathfrak{S}_m$) into the subspace $\boldsymbol{F}(m)$ is

$$\left(\boldsymbol{f}(\overline{\mathbf{2}^{[1]}}; m)^\top \; \boldsymbol{f}(\overline{\mathbf{2}^{[3]}}; m)^\top \; \cdots \; \boldsymbol{f}(\overline{\mathbf{2}^{[m-1]}}; m)^\top\right)$$
$$\cdot \left(\begin{pmatrix} \boldsymbol{f}(\overline{\mathbf{2}^{[1]}}; m) \\ \boldsymbol{f}(\overline{\mathbf{2}^{[3]}}; m) \\ \vdots \\ \boldsymbol{f}(\overline{\mathbf{2}^{[m-1]}}; m) \end{pmatrix} \cdot \left(\boldsymbol{f}(\overline{\mathbf{2}^{[1]}}; m)^\top \; \boldsymbol{f}(\overline{\mathbf{2}^{[3]}}; m)^\top \; \cdots \; \boldsymbol{f}(\overline{\mathbf{2}^{[m-1]}}; m)^\top\right) \right)^{-1} \cdot \begin{pmatrix} \boldsymbol{f}(\overline{\mathbf{2}^{[1]}}; m) \\ \boldsymbol{f}(\overline{\mathbf{2}^{[3]}}; m) \\ \vdots \\ \boldsymbol{f}(\overline{\mathbf{2}^{[m-1]}}; m) \end{pmatrix} .$$

The matrices of the orthogonal projectors in the case of m *odd* are constructed in the analogous way.

– Let k be a positive integer such that $k \le m$. The (i, j)th entry of the matrix of the *orthogonal projector* into the *one-dimensional subspace* $\operatorname{span}\big(\boldsymbol{h}(\overline{\mathbf{2}^{[k]}}; m)\big)$ of the space $\mathbb{R}^{m+1}$, relative to either of the bases $\mathfrak{S}_m$ and $\mathfrak{H}_m^{\blacktriangledown}$, is

$$\frac{(-1)^{i+j}\left(\binom{m-k}{i} - (-1)^k\binom{m-k}{i-k}\right)\left(\binom{m-k}{j} - (-1)^k\binom{m-k}{j-k}\right)}{2\left(\binom{2(m-k)}{m-k} - (-1)^k\binom{2(m-k)}{m}\right)}\;;$$

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the (i, j)th entry of the matrix of the *orthogonal projector* into the *one-dimensional subspace* $\operatorname{span}\big(\boldsymbol{f}(\overline{\mathbf{2}^{[k]}}; m)\big)$ of $\mathbb{R}^{m+1}$, relative to $\mathfrak{S}_m$, is

$$\frac{\binom{k}{i}\binom{k}{j}}{\binom{2k}{k} - 1} \, .$$

Notes

The *Dehn–Sommerville relations* are discussed, for example, in [18, Sect. VI.6], [21, Ch. 5], [38, Sect. II.5],[40, Sect. 1.3, 3.4, 3.8], [41, Sect. 1.2, 3.6, 8.6], [104, Sect. III.11], [178, Sect. II.3, II.6, III.6], [179, Sect. 3.16], [186, Sect. 8.3].

The defining relations (3.2) and (3.3) merely restate relations given in [146, p. 171].

See, for example, [172, Ch. 4], [180, Ch. 8] on *unimodular matrices*, and [172, Sect. 19] on *totally unimodular matrices*.

The *Krawtchouk polynomials* are discussed, for example, in [123, Sect. 1.2].

Relations (3.7) mimic relations given in [186, p. 253].

The theory and algorithmics of the *lattice-point counting* in *rational convex polytopes* are discussed, for example, in [18, Ch. VIII], [19–21, 35], [56, Part III], [117], [148, Ch. 12].

The information given in Table 3.4 is collected using the software `LattE`[1] dedicated to the problems of counting lattice points and integration inside convex polytopes.

See, for example, [155, Sect. 2.4], [177, Lecture 2] on *graphical hyperplane arrangements*.

The *principle of biorthogonality* is explained, for example, in [105, Theorem 1.4.7(a)].

1 The software `LattE` is available at `www.math.ucdavis.edu/~latte/`

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4 Farey Subsequences

The *Farey sequence* $\mathscr{F}_n$ *of order n* is defined to be the *increasing* sequence of *irreducible* fractions $\frac{h}{k} \in \mathbb{Q}$ such that $\frac{0}{1} \leq \frac{h}{k} \leq \frac{1}{1}$ and $k \leq n$. For example,

$$\mathscr{F}_6 = \left(\tfrac{0}{1} < \tfrac{1}{6} < \tfrac{1}{5} < \tfrac{1}{4} < \tfrac{1}{3} < \tfrac{2}{5} < \tfrac{1}{2} < \tfrac{3}{5} < \tfrac{2}{3} < \tfrac{3}{4} < \tfrac{4}{5} < \tfrac{5}{6} < \tfrac{1}{1}\right) .$$

Let C be a finite set of cardinality $n := |C| > 1$, and A its proper subset; $m := |A|$. The increasing sequence of fractions

$$\begin{aligned}\mathscr{F}\big(\mathbb{B}(n), m\big) &:= \left(\tfrac{|B\cap A|}{\gcd(|B\cap A|,|B|)}\Big/\tfrac{|B|}{\gcd(|B\cap A|,|B|)} : B \subseteq C,\ |B| > 0\right) \\ &= \left(\tfrac{h}{k} \in \mathscr{F}_n \colon m + k - n \leq h \leq m\right) ,\end{aligned} \tag{4.1}$$

where $\mathbb{B}(n)$ as earlier denotes the Boolean lattice of rank n, has the properties very similar to those of the standard Farey sequence $\mathscr{F}_n$.

In this chapter we briefly review the *Farey subsequences*

$$\mathscr{F}\big(\mathbb{B}(2m), m\big) := \left(\tfrac{h}{k} \in \mathscr{F}_{2m} :\ h \leq m,\ k - h \leq m\right)$$

that appear naturally in an analysis of contradictory decision-making problems. One such subsequence is

$$\begin{aligned}\mathscr{F}\big(\mathbb{B}(12), 6\big) = \Big(&\tfrac{0}{1} < \tfrac{1}{7} < \tfrac{1}{6} < \tfrac{1}{5} < \tfrac{1}{4} < \tfrac{2}{7} < \tfrac{1}{3} < \tfrac{3}{8} < \tfrac{2}{5} < \tfrac{3}{7} < \tfrac{4}{9} < \tfrac{5}{11} < \tfrac{1}{2} \\ &< \tfrac{6}{11} < \tfrac{5}{9} < \tfrac{4}{7} < \tfrac{3}{5} < \tfrac{5}{8} < \tfrac{2}{3} < \tfrac{5}{7} < \tfrac{3}{4} < \tfrac{4}{5} < \tfrac{5}{6} < \tfrac{6}{7} < \tfrac{1}{1}\Big) .\end{aligned}$$

In Section 4.1 we deal with arbitrary Farey subsequences $\mathscr{F}(\mathbb{B}(n), m)$.

Basic properties of the Farey subsequences of the form $\mathscr{F}(\mathbb{B}(2m), m)$ are surveyed in Section 4.2.

By *Farey duality*, recalled in Section 4.3, the *left-half sequence*

$$\mathscr{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) := \left(\tfrac{h}{k} \in \mathscr{F}\big(\mathbb{B}(2m), m\big) \colon \tfrac{h}{k} \leq \tfrac{1}{2}\right)$$

and the *right-half sequence*

$$\mathscr{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) := \left(\tfrac{h}{k} \in \mathscr{F}\big(\mathbb{B}(2m), m\big) \colon \tfrac{h}{k} \geq \tfrac{1}{2}\right)$$

of the sequence $\mathscr{F}(\mathbb{B}(2m), m)$ are in one-to-one correspondence

$$\mathscr{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) \longleftrightarrow \mathscr{F}_m \longleftrightarrow \mathscr{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) \tag{4.2}$$

with the Farey sequence $\mathscr{F}_m$.

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4.1 The Farey Subsequence $\mathscr{F}(\mathbb{B}(n), m)$

In this section we consider general Farey subsequences of the form (4.1).

Recall that the map $\mathscr{F}_n \to \mathscr{F}_n$ that sends a fraction $\frac{h}{k}$ to $\frac{k-h}{k}$ is *order-reversing* and *bijective*. The sequences $\mathscr{F}(\mathbb{B}(n), m)$ and $\mathscr{F}(\mathbb{B}(n), n-m)$ have the analogous property:

Lemma 4.1. *The map*

$$\mathscr{F}\big(\mathbb{B}(n), m\big) \to \mathscr{F}\big(\mathbb{B}(n), n-m\big)\ , \quad \tfrac{h}{k} \mapsto \tfrac{k-h}{k}\ ,$$

is order-reversing *and* bijective.

If we write the fractions $\frac{h}{k} \in \mathbb{Q}$ as the column vectors $\left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \in \mathbb{Z}^2$, then the mapping $\frac{h}{k} \mapsto \frac{k-h}{k}$ can also be thought of as the mapping

$$\left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right]\ .$$

Recall that, given a subset $\mathrm{A} \subset \mathbb{B}(n)$, the *order ideal* $\mathfrak{I}(\mathrm{A})$ and the *order filter* $\mathfrak{F}(\mathrm{A})$ of the lattice $\mathbb{B}(n)$, generated by A, are defined by $\mathfrak{I}(\mathrm{A}) := \{e \in \mathbb{B}(n) \colon \exists a \in \mathrm{A},\ e \preceq a\}$ and $\mathfrak{F}(\mathrm{A}) := \{e \in \mathbb{B}(n) \colon \exists a \in \mathrm{A},\ a \preceq e\}$. Recall also that the set $\mathbb{B}(n)^{(1)}$ of *atoms* of the lattice $\mathbb{B}(n)$ consists of the elements that cover the least element $\hat{0}$. The *poset rank* $\rho(a)$ of an element $a \in \mathbb{B}(n)$ is equal to the *cardinality* $|\mathfrak{I}(a) \cap \mathbb{B}(n)^{(1)}|$; the set of all rank l elements of $\mathbb{B}(n)$ is denoted by $\mathbb{B}(n)^{(l)}$.

Let $a' \in \mathbb{B}(n)$, and $0 < m := \rho(a') < n$. The filter $\mathfrak{F}(\mathfrak{I}(a') \cap \mathbb{B}(n)^{(1)})$, of cardinality $2^n - 2^{n-m}$, can be partitioned in the following way:

$$\begin{aligned}\mathfrak{F}\big(\mathfrak{I}(a') \cap \mathbb{B}(n)^{(1)}\big) = \big(\mathfrak{I}(a') - \{\hat{0}\}\big)\ \dot\cup \\ \dot{\bigcup_{\substack{\frac{h}{k} \in \mathscr{F}(\mathbb{B}(n),m):\\ \frac{0}{1} < \frac{h}{k} < \frac{1}{1}}}}\ \dot{\bigcup_{1 \le s \le \left\lfloor \min\left\{\frac{m}{h}, \frac{n-m}{k-h}\right\}\right\rfloor}} \Big(\mathbb{B}(n)^{(s\cdot k)} \cap \\ \Big(\mathfrak{F}\big(\mathfrak{I}(a') \cap \mathbb{B}(n)^{(s\cdot h)}\big) - \mathfrak{F}\big(\mathfrak{I}(a') \cap \mathbb{B}(n)^{(s\cdot h+1)}\big)\Big)\Big)\ ,\end{aligned}$$

where

$$\Big|\mathbb{B}(n)^{(s\cdot k)} \cap \Big(\mathfrak{F}\big(\mathfrak{I}(a') \cap \mathbb{B}(n)^{(s\cdot h)}\big) - \mathfrak{F}\big(\mathfrak{I}(a') \cap \mathbb{B}(n)^{(s\cdot h+1)}\big)\Big)\Big| = \tbinom{m}{s\cdot h}\tbinom{n-m}{s\cdot(k-h)}\ .$$

Since $|\mathfrak{I}(a') - \{\hat{0}\}| = 2^m - 1$, we obtain

$$2^n - 2^{n-m} = 2^m - 1 + \sum_{\substack{\frac{h}{k} \in \mathscr{F}(\mathbb{B}(n),m):\\ \frac{0}{1} < \frac{h}{k} < \frac{1}{1}}}\ \sum_{1 \le s \le \left\lfloor \min\left\{\frac{m}{h}, \frac{n-m}{k-h}\right\}\right\rfloor} \tbinom{m}{s\cdot h}\tbinom{n-m}{s\cdot(k-h)}\ .$$

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If $a'' \in \mathbb{B}(n)$ and $\rho(a'') = n - m$, then Lemma 4.1 implies

$$2^n - 2^m = 2^{n-m} - 1 + \sum_{\substack{\frac{h}{k} \in \mathscr{F}(\mathbb{B}(n), n-m):\\ \frac{0}{1} < \frac{h}{k} < \frac{1}{1}}} \ \sum_{1 \le s \le \lfloor \min\{\frac{n-m}{h}, \frac{m}{k-h}\} \rfloor} \binom{n-m}{s \cdot h} \binom{m}{s \cdot (k-h)} ,$$

and we conclude the following result:

Proposition 4.2. *Given a Farey subsequence $\mathscr{F}(\mathbb{B}(n), m)$, the following identity is valid:*

$$\begin{aligned} &\sum_{\substack{\frac{h}{k} \in \mathscr{F}(\mathbb{B}(n), m):\\ \frac{0}{1} < \frac{h}{k} < \frac{1}{1}}} \ \sum_{1 \le s \le \lfloor \min\{\frac{m}{h}, \frac{n-m}{k-h}\} \rfloor} \binom{m}{s \cdot h} \binom{n-m}{s \cdot (k-h)} \\ &= \sum_{\substack{\frac{h}{k} \in \mathscr{F}(\mathbb{B}(n), n-m):\\ \frac{0}{1} < \frac{h}{k} < \frac{1}{1}}} \ \sum_{1 \le s \le \lfloor \min\{\frac{n-m}{h}, \frac{m}{k-h}\} \rfloor} \binom{n-m}{s \cdot h} \binom{m}{s \cdot (k-h)} \\ &= 2^n - 2^m - 2^{n-m} + 1 . \end{aligned}$$

4.2 The Farey Subsequence $\mathscr{F}(\mathbb{B}(2m), m)$

In this section we survey interesting properties of the Farey subsequences $\mathscr{F}(\mathbb{B}(2m), m)$. These sequences appear naturally in decision-making problems stated in terms of systems of homogeneous strict linear inequalities, arrangements of oriented linear hyperplanes, or simple oriented matroids.

We begin with *monotone* and *bijective* maps between the *half sequences* of the sequences $\mathscr{F}(\mathbb{B}(2m), m)$:

Proposition 4.3.

(i) *The maps*

$$\mathscr{F}\big(\mathbb{B}(2m), m\big) \to \mathscr{F}\big(\mathbb{B}(2m), m\big) , \qquad \tfrac{h}{k} \mapsto \tfrac{k-h}{k} , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] ,$$
$$\mathscr{F}^{\le \frac{1}{2}}\big(\mathbb{B}(2m), m\big) \to \mathscr{F}^{\le \frac{1}{2}}\big(\mathbb{B}(2m), m\big) , \qquad \tfrac{h}{k} \mapsto \tfrac{k-2h}{2k-3h} , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -2 & 1 \\ -3 & 2 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] ,$$

and

$$\mathscr{F}^{\ge \frac{1}{2}}\big(\mathbb{B}(2m), m\big) \to \mathscr{F}^{\ge \frac{1}{2}}\big(\mathbb{B}(2m), m\big) , \qquad \tfrac{h}{k} \mapsto \tfrac{h}{3h-k} , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1 & 0 \\ 3 & -1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] ,$$

are order-reversing and bijective.

(ii) *The maps*

$$\mathscr{F}^{\le \frac{1}{2}}\big(\mathbb{B}(2m), m\big) \to \mathscr{F}^{\ge \frac{1}{2}}\big(\mathbb{B}(2m), m\big) , \qquad \tfrac{h}{k} \mapsto \tfrac{k-h}{2k-3h} , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -1 & 1 \\ -3 & 2 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] ,$$

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and

$$\mathscr{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)\to\mathscr{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)\,,\qquad \tfrac{h}{k}\mapsto\tfrac{2h-k}{3h-k}\,,\qquad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2&-1\\3&-1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\,,$$

are order-preserving and bijective.

A neighborhood of the fraction $\frac{1}{2}$, that plays a decisive role in contradictory problems, has the following surprisingly simple description:

Proposition 4.4. *In the Farey subsequence $\mathscr{F}(\mathbb{B}(2m),m)$, the fractions*

$$\begin{aligned}&\tfrac{\lceil m/2\rceil-1}{2\lceil m/2\rceil-1}<\cdots<\tfrac{m-3}{2m-5}<\tfrac{m-2}{2m-3}<\tfrac{m-1}{2m-1}<\tfrac{1}{2}\\&<\tfrac{m}{2m-1}<\tfrac{m-1}{2m-3}<\tfrac{m-2}{2m-5}<\cdots<\tfrac{\lceil m/2\rceil}{2\lceil m/2\rceil-1}\end{aligned}$$

are consecutive.

4.3 Farey Duality

In this section we recall the characterization of *Farey duality*, by which we mean the existence of *monotone* and *bijective* maps (4.2) between the *Farey sequence* $\mathscr{F}_m$ and the *half sequences* of the sequence $\mathscr{F}(\mathbb{B}(2m),m)$.

Theorem 4.5. *Consider a Farey subsequence $\mathscr{F}(\mathbb{B}(2m),m)$. The maps*

$$\begin{aligned}\mathscr{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)&\to\mathscr{F}_m\,, & \tfrac{h}{k}&\mapsto\tfrac{h}{k-h}\,, & \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]&\mapsto\left[\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\,,\\ \mathscr{F}_m&\to\mathscr{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)\,, & \tfrac{h}{k}&\mapsto\tfrac{h}{h+k}\,, & \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]&\mapsto\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\,,\\ \mathscr{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)&\to\mathscr{F}_m\,, & \tfrac{h}{k}&\mapsto\tfrac{2h-k}{h}\,, & \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]&\mapsto\left[\begin{smallmatrix}2&-1\\1&0\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\,,\end{aligned}$$

and

$$\mathscr{F}_m\to\mathscr{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)\,,\qquad \tfrac{h}{k}\mapsto\tfrac{k}{2k-h}\,,\qquad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\,,$$

are order-preserving *and* bijective. *The maps*

$$\begin{aligned}\mathscr{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)&\to\mathscr{F}_m\,, & \tfrac{h}{k}&\mapsto\tfrac{k-2h}{k-h}\,, & \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]&\mapsto\left[\begin{smallmatrix}-2&1\\-1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\,,\\ \mathscr{F}_m&\to\mathscr{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)\,, & \tfrac{h}{k}&\mapsto\tfrac{k-h}{2k-h}\,, & \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]&\mapsto\left[\begin{smallmatrix}-1&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\,,\\ \mathscr{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)&\to\mathscr{F}_m\,, & \tfrac{h}{k}&\mapsto\tfrac{k-h}{h}\,, & \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]&\mapsto\left[\begin{smallmatrix}-1&1\\1&0\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\,,\end{aligned}$$

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and

$$\mathscr{F}_m \to \mathscr{F}^{\geq\frac{1}{2}}\bigl(\mathbb{B}(2m), m\bigr)\,, \qquad \tfrac{h}{k} \mapsto \tfrac{k}{h+k}\,, \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right]\,,$$

are order-reversing *and* bijective.

Proposition 4.2 can be restated, in the case where $n := 2m$, by means of the bijections given in Proposition 4.3:

Proposition 4.6. *The following identities hold for fractions in the Farey subsequences* $\mathscr{F}(\mathbb{B}(2m), m)$ *with* $m > 1$*:*

(i)

$$\sum_{\substack{\frac{h}{k} \in \mathscr{F}(\mathbb{B}(2m),m):\\ \frac{0}{1} < \frac{h}{k} < \frac{1}{1}}} \ \sum_{1 \leq s \leq \lfloor \min\{\frac{m}{h}, \frac{m}{k-h}\} \rfloor} \binom{m}{s \cdot h}\binom{m}{s \cdot (k-h)} = 2^{2m} - 2^{m+1} + 1\,.$$

(ii)

$$\begin{aligned}\sum_{\substack{\frac{h}{k} \in \mathscr{F}(\mathbb{B}(2m),m):\\ \frac{0}{1} < \frac{h}{k} < \frac{1}{2}}} \ \sum_{1 \leq s \leq \lfloor \frac{m}{k-h} \rfloor} \binom{m}{s \cdot h}\binom{m}{s \cdot (k-h)} &= \sum_{\substack{\frac{h}{k} \in \mathscr{F}(\mathbb{B}(2m),m):\\ \frac{1}{2} < \frac{h}{k} < \frac{1}{1}}} \ \sum_{1 \leq s \leq \lfloor \frac{m}{h} \rfloor} \binom{m}{s \cdot h}\binom{m}{s \cdot (k-h)} \\ &= 2^{2m-1} - 2^m - \tfrac{1}{2}\tbinom{2m}{m} + 1\,.\end{aligned}$$

(iii)

$$\begin{aligned}&\sum_{\substack{\frac{h}{k} \in \mathscr{F}(\mathbb{B}(2m),m):\\ \frac{0}{1} < \frac{h}{k} < \frac{1}{3}}} \ \sum_{1 \leq s \leq \lfloor \frac{m}{k-h} \rfloor} \binom{m}{s \cdot (k-h)}\left(\binom{m}{s \cdot h} + \binom{m}{s \cdot (k-2h)}\right) \\ &= \sum_{\substack{\frac{h}{k} \in \mathscr{F}(\mathbb{B}(2m),m):\\ \frac{1}{3} < \frac{h}{k} < \frac{1}{2}}} \ \sum_{1 \leq s \leq \lfloor \frac{m}{k-h} \rfloor} \binom{m}{s \cdot (k-h)}\left(\binom{m}{s \cdot h} + \binom{m}{s \cdot (k-2h)}\right) \\ &= \sum_{\substack{\frac{h}{k} \in \mathscr{F}(\mathbb{B}(2m),m):\\ \frac{1}{2} < \frac{h}{k} < \frac{2}{3}}} \ \sum_{1 \leq s \leq \lfloor \frac{m}{h} \rfloor} \binom{m}{s \cdot h}\left(\binom{m}{s \cdot (k-h)} + \binom{m}{s \cdot (2h-k)}\right) \\ &= \sum_{\substack{\frac{h}{k} \in \mathscr{F}(\mathbb{B}(2m),m):\\ \frac{2}{3} < \frac{h}{k} < \frac{1}{1}}} \ \sum_{1 \leq s \leq \lfloor \frac{m}{h} \rfloor} \binom{m}{s \cdot h}\left(\binom{m}{s \cdot (k-h)} + \binom{m}{s \cdot (2h-k)}\right) \\ &= 2^{2m-1} - 2^m - \tfrac{1}{2}\tbinom{2m}{m} - \sum_{1 \leq t \leq \lfloor \frac{m}{2} \rfloor} \binom{m}{2t}\binom{m}{t} + 1\,.\end{aligned}$$

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The bijections between the Farey sequence $\mathcal{F}_m$ and the half sequences of $\mathcal{F}(\mathbb{B}(2m), m)$ given in Theorem 4.5 allow us to describe the properties of fractions in $\mathcal{F}_m$, analogous to those of fractions in $\mathcal{F}(\mathbb{B}(2m), m)$, given in Proposition 4.6(ii,iii):

Corollary 4.7. *The following identities hold for fractions in the Farey sequences $\mathcal{F}_m$ with $m > 1$:*

(i)

$$\sum_{\substack{\frac{h}{k} \in \mathcal{F}_m: \\ \frac{0}{1} < \frac{h}{k} < \frac{1}{1}}} \sum_{1 \le s \le \lfloor \frac{m}{k} \rfloor} \binom{m}{s \cdot h}\binom{m}{s \cdot k} = 2^{2m-1} - 2^m - \tfrac{1}{2}\binom{2m}{m} + 1 \, .$$

(ii)

$$\begin{aligned}
&\sum_{\substack{\frac{h}{k} \in \mathcal{F}_m: \\ \frac{0}{1} < \frac{h}{k} < \frac{1}{2}}} \sum_{1 \le s \le \lfloor \frac{m}{k} \rfloor} \binom{m}{s \cdot k}\left(\binom{m}{s \cdot h} + \binom{m}{s \cdot (k-h)}\right) \\
&= \sum_{\substack{\frac{h}{k} \in \mathcal{F}_m: \\ \frac{1}{2} < \frac{h}{k} < \frac{1}{1}}} \sum_{1 \le s \le \lfloor \frac{m}{k} \rfloor} \binom{m}{s \cdot k}\left(\binom{m}{s \cdot h} + \binom{m}{s \cdot (k-h)}\right) \\
&= 2^{2m-1} - 2^m - \tfrac{1}{2}\binom{2m}{m} - \sum_{1 \le t \le \lfloor \frac{m}{2} \rfloor} \binom{m}{2t}\binom{m}{t} + 1 \, .
\end{aligned}$$

Notes

The books [11, 39, 90, 95, 106, 116, 153, 162] are well-known standard references on the *Farey sequences*. Monograph [137] is a text on Farey sequences and subsequences.

The recent books [7, 98, 99] provide much information on the *Farey table*, *Farey sequences*, and the *Farey diagram*.

The *Farey subsequences* of the general form $\mathcal{F}(\mathbb{B}(n), m)$ defined by eq. (4.1) are considered, for example, in [132, 133, 140]. The *Farey subsequences* $\mathcal{F}(\mathbb{B}(2m), m)$ are discussed, for example, in [132, 138].

Lemma 4.1 is taken from [140, Prop. 7.4], and Proposition 4.2 reproduces [132, Prop. 2]. Proposition 4.3(i) and Proposition 4.3(ii) appear as [132, Lem. 3] and [132, Cor. 4], respectively. The statement of Proposition 4.4 is explained in [138, Rem. 5.1]. Theorem 4.5 is an extended version of [132, Theorem 5]. Proposition 4.6 reproduces [132, Prop. 7]. Corollary 4.7 appears as [132, Cor. 8].

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5 Blocking Sets of Set Families, and Absolute Blocking Constructions in Posets

Let $\mathscr{A} := \{A_1, \ldots, A_\alpha\}$ be a nonempty *family* of nonempty and pairwise distinct *subsets* of its finite *ground set* $\mathrm{V}(\mathscr{A}) := \bigcup_{i=1}^{\alpha} A_i$. Recall that a subset $B \subseteq S$ of a set $S \supseteq \mathrm{V}(\mathscr{A})$ is called a *blocking set* of the family $\mathscr{A}$ if for each index $i \in [\alpha]$, we have

$$|B \cap A_i| > 0 \; . \tag{5.1}$$

The *blocker* $\mathfrak{B}(\mathscr{A})$ of the family $\mathscr{A}$ is defined to be the family of all *inclusion-minimal blocking sets of* $\mathscr{A}$.

A family of subsets of a finite set is called a *clutter* (or a *Sperner family*) if *no set* from it contains another. Recall that the clutters on a set S are in natural one-to-one correspondence with the *antichains* of the Boolean lattice of subsets of S. Two distinguished clutters are called *trivial*: these are the *void family* $\emptyset$ containing *no sets*, and the family $\{\hat{0}\}$ whose *only* member is the *empty set* $\hat{0}$.

The *blockers* of the *trivial clutters* are defined by

$$\mathfrak{B}(\emptyset) := \{\hat{0}\} \quad \text{and} \quad \mathfrak{B}\big(\{\hat{0}\}\big) := \emptyset \; .$$

Recall a fundamental result in combinatorial optimization given in Proposition 1.34: *If $\mathscr{A}$ is a clutter, then we have*

$$\mathfrak{B}\big(\mathfrak{B}(\mathscr{A})\big) = \mathscr{A} \; . \tag{5.2}$$

Deletion and contraction are basic operators on clutters. Consider a *nontrivial* clutter $\mathscr{A}$ on its ground set $\mathrm{V}(\mathscr{A})$. Let $X \subseteq S \supseteq \mathrm{V}(\mathscr{A})$. Recall that the *deletion* $\mathscr{A} \backslash X$ is defined to be the clutter

$$\mathscr{A} \backslash X := \{A \in \mathscr{A} : |A \cap X| = 0\}$$

on the *ground set* $\mathrm{V}(\mathscr{A}) - X$; the *contraction* $\mathscr{A}/X$ is defined to be the clutter

$$\mathscr{A}/X := \mathbf{min}\,\{A - X : A \in \mathscr{A}\} \; ,$$

that consists of the *inclusion-minimal* sets of the family $\{A - X : A \in \mathscr{A}\}$, on the *ground set* $\mathrm{V}(\mathscr{A}) - X$.

Deletion and *contraction* are also defined for the *trivial* clutters:

$$\emptyset \backslash X = \emptyset / X := \emptyset \quad \text{and} \quad \{\hat{0}\} \backslash X = \{\hat{0}\} / X := \{\hat{0}\} \; .$$

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The operators of deletion and contraction applied sequentially to a clutter yield its *minors*.

In addition to Proposition 1.34, another fundamental result in combinatorial optimization is as follows:

Proposition 5.1. *Let* $\mathscr{A}$ *be a clutter on its ground set* $\mathrm{V}(\mathscr{A})$. *For any subset* $X \subseteq S \supseteq \mathrm{V}(\mathscr{A})$, *we have*

$$\mathfrak{B}(\mathscr{A})\backslash X = \mathfrak{B}(\mathscr{A}/X) \quad \textit{and} \quad \mathfrak{B}(\mathscr{A})/X = \mathfrak{B}(\mathscr{A}\backslash X)\,. \tag{5.3}$$

In this chapter we discuss poset-theoretic generalizations of the concepts of the *blocking set*, the *blocker*, *deletion* and *contraction*. The result of Proposition 5.1 can be interpreted as the relations

$$\mathfrak{B}(\mathscr{A})\backslash X = \mathfrak{B}(\mathscr{A}/X) \preceq \mathfrak{B}(\mathscr{A}) \preceq \mathfrak{B}(\mathscr{A})/X = \mathfrak{B}(\mathscr{A}\backslash X) \tag{5.4}$$

in the *free distributive lattice* of *antichains* of the *Boolean lattice* of subsets of a set $S \supseteq \mathrm{V}(\mathscr{A})$.

Recall that a poset P is called *bounded* if it has a unique minimal element denoted by $\hat{0}$ or $\hat{0}_P$, and a unique maximal element denoted by $\hat{1}$ or $\hat{1}_P$. Throughout the chapter, P denotes a finite bounded poset with $|P| > 1$.

In Section 5.1 we show that one can associate in a natural way with any subposet A of the poset P the order filter of *blocking elements* of A playing a role analogous to that played by the family of *blocking sets* of a *set family*.

In Section 5.2 we investigate the structure of the order filters of blocking elements in *direct products* of two posets.

The set of *minimal elements* of the order filter of blocking elements is the poset-theoretic analogue of the set-theoretic *blocker*. In Section 5.3 the *antichains* of the poset P, as well as the corresponding *blockers*, are regarded as the elements of the distributive *lattice* $\mathfrak{A}(P)$ *of antichains of* P and as their images under the poset-theoretic *blocker map* $\mathfrak{b} : \mathfrak{A}(P) \to \mathfrak{A}(P)$.

The *poset* $\mathscr{B}(P) := \mathfrak{b}(\mathfrak{A}(P))$ *of blockers in* P, that is, the image of the lattice $\mathfrak{A}(P)$ under the blocker map, discussed in Section 5.4, is a *self-dual lattice* with its *anti-automorphism* established by the restriction $\mathfrak{b} : \mathscr{B}(P) \to \mathscr{B}(P)$ of the *blocker map*.

In Section 5.5 we consider the *deletion operator* $(\backslash X) : \mathfrak{A}(P) \to \mathfrak{A}(P)$ and the *contraction operator* $(/X) : \mathfrak{A}(P) \to \mathfrak{A}(P)$ on the *antichains* of the poset P; we verify that these maps are *(co)closure operators* on the lattice of antichains $\mathfrak{A}(P)$, and we present the general form

$$\mathfrak{b}(\mathrm{A})\backslash X \preceq \mathfrak{b}(\mathrm{A}/X) \preceq \mathfrak{b}(\mathrm{A}) \preceq \mathfrak{b}(\mathrm{A})/X \preceq \mathfrak{b}(\mathrm{A}\backslash X) \tag{5.5}$$

of the relations similar to eq. (5.4).

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In Section 5.6 we again derive relations (5.5) in the context of *order-preserving* and *order-reversing* maps on posets.

Sections 5.8 and 5.9 are devoted to the families of the so-called (X, k)-*blocker maps* on the lattice $\mathfrak{A}(P)$, as well as to the families of (X, k)-*deletion operators* and (X, k)-*contraction operators* on $\mathfrak{A}(P)$, which are parameterized by the subsets X of the set of atoms P^{a} of the poset P, and by the integers k such that $0 \le k < |P^{\mathrm{a}}|$.

5.1 Blocking Elements and Complementing Elements of Subposets

In this section we discuss a natural poset-theoretic generalization of the *blocking sets* of *set families*.

Given a finite bounded poset Q, we as earlier denote by Q^{a} its set of atoms; $\mathbf{min}\ Q$ and $\mathbf{max}\ Q$ denote the sets of all minimal elements and all maximal elements of Q, respectively; $\mathfrak{I}(X) := \mathfrak{I}_Q(X)$ and $\mathfrak{F}(X) := \mathfrak{F}_Q(X)$ denote the order ideal and order filter of Q generated by a subset $X \subseteq Q$, respectively. If x and y are elements of Q, and $x \prec y$ (or $x \preceq y$), then we often write $x \prec_Q y$ (or $x \preceq_Q y$). Similarly, we denote by $\vee$ or $\vee_Q$ the operation of *join* in a join-semilattice Q, and we denote by $\wedge$ or $\wedge_Q$ the operation of *meet* in a meet-semilattice Q. We use $\times$ to denote the operation of *direct product* of posets.

Definition 5.2. Let A be a subset of a finite bounded poset P.

(i) If $\mathrm{A} \neq \emptyset$ and $\mathrm{A} \neq \{\hat{0}\}$, then an element $b \in P$ is a *blocking element* of A in P, if for every $a \in \mathrm{A} - \{\hat{0}\}$, we have

$$|\mathfrak{I}(b) \cap \mathfrak{I}(a) \cap P^{\mathrm{a}}| > 0 \ .$$

(ii) If $\mathrm{A} = \{\hat{0}\}$, then A *has no blocking elements* in P.

(iii) If $\mathrm{A} = \emptyset$, then every element of P is a *blocking element* of A in P.

(iv) Every nonblocking element of A in P is a *complementing element* of A in P.

Remark 5.3. If A is a nonempty subset of the subposet $\mathbb{B}(n) - \{\hat{0}\}$ of the Boolean lattice $\mathbb{B}(n)$, then an element $b \in \mathbb{B}(n)$ is a *blocking element* of A if and only if $\mathfrak{I}(b) \cap \mathbb{B}(n)^{(1)}$ is a *blocking set* of the family $\{\mathfrak{I}(a) \cap \mathbb{B}(n)^{(1)} : a \in \mathrm{A}\}$.

We denote by $\mathbf{I}(P, \mathrm{A})$ and $\mathbf{C}(P, \mathrm{A})$ the subposet of all *blocking elements* and the subposet of all *complementing elements* of A in P, respectively. For a one-element subset $\{a\} \subset P$, we write $\mathbf{I}(P, a)$ instead of $\mathbf{I}(P, \{a\})$, and $\mathbf{C}(P, a)$ instead of $\mathbf{C}(P, \{a\})$.

For a nonempty subset $\mathrm{A} \subseteq P - \{\hat{0}\}$, the subposets of its blocking elements and complementing elements are nonempty; indeed, we have $\mathbf{I}(P, \mathrm{A}) \ni \hat{1}$ and $\mathbf{C}(P, \mathrm{A}) \ni \hat{0}$.

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It follows from Definition 5.2 that for any subset $\mathrm{A} \subseteq P - \{\hat{0}\}$, we have the partition

$$P = \mathbf{I}(P, \mathrm{A}) \mathbin{\dot{\cup}} \mathbf{C}(P, \mathrm{A}) ,$$

and we have

$$\mathbf{I}(P, \mathrm{A}) = \mathbf{I}(P, \mathbf{min}\, \mathrm{A}) , \quad \mathbf{C}(P, \mathrm{A}) = \mathbf{C}(P, \mathbf{min}\, \mathrm{A}) ;$$

therefore, in many cases, we can restrict our attention to the blocking elements and the complementing elements of *antichains*. Further, note that

$$\mathbf{I}(P, \mathrm{A}) = \bigcap_{a \in \mathrm{A}} \mathbf{I}(P, a) , \quad \mathbf{C}(P, \mathrm{A}) = \bigcup_{a \in \mathrm{A}} \mathbf{C}(P, a) . \tag{5.6}$$

For any antichains A_1 and A_2 of P such that $\mathfrak{F}(\mathrm{A}_1) \subseteq \mathfrak{F}(\mathrm{A}_2)$, we have

$$\mathbf{I}(P, \mathrm{A}_1) \supseteq \mathbf{I}(P, \mathrm{A}_2) , \quad \mathbf{C}(P, \mathrm{A}_1) \subseteq \mathbf{C}(P, \mathrm{A}_2) . \tag{5.7}$$

The subposet $\mathbf{I}(P, a)$ of blocking elements of a one-element antichain $\{a\} \subset P$ is clearly the order filter $\mathfrak{F}(\mathfrak{I}(a) \cap P^{\mathrm{a}})$; hence, in view of eq. (5.6), identity (5.8) in the following statement is valid.

Lemma 5.4. *Let* A *be a nonempty subset of* $P - \{\hat{0}\}$. *The subposet* $\mathbf{I}(P, \mathrm{A})$ *of* blocking elements *of* A *in* P *can be described in two equivalent ways:*

$$\mathbf{I}(P, \mathrm{A}) = \bigcap_{a \in \mathrm{A}} \mathfrak{F}\big(\mathfrak{I}(a) \cap P^{\mathrm{a}}\big) ; \tag{5.8}$$

$$\mathbf{I}(P, \mathrm{A}) = \bigcup_{E \in \mathfrak{B}(\{\mathfrak{I}(a) \cap P^{\mathrm{a}} : a \in \mathrm{A}\})} \bigcap_{e \in E} \mathfrak{F}(e) . \tag{5.9}$$

Proof. To prove eq. (5.9), note that the reverse inclusion

$$\mathbf{I}(P, \mathrm{A}) \supseteq \bigcup_{E \in \mathfrak{B}(\{\mathfrak{I}(a) \cap P^{\mathrm{a}} : a \in \mathrm{A}\})} \bigcap_{e \in E} \mathfrak{F}(e)$$

follows from the definition of blocking elements.

It remains to prove the inclusion

$$\mathbf{I}(P, \mathrm{A}) \subseteq \bigcup_{E \in \mathfrak{B}(\{\mathfrak{I}(a) \cap P^{\mathrm{a}} : a \in \mathrm{A}\})} \bigcap_{e \in E} \mathfrak{F}(e) .$$

Suppose to the contrary that there exists a blocking element b of A such that $b \notin \bigcup_{E \in \mathfrak{B}(\{\mathfrak{I}(a) \cap P^{\mathrm{a}} : a \in \mathrm{A}\})} \bigcap_{e \in E} \mathfrak{F}(e)$. Then the inclusion $b \in \bigcap_{e \in E} \mathfrak{F}(e)$ holds not for all sets E in the family $\mathfrak{B}(\{\mathfrak{I}(a) \cap P^{\mathrm{a}} : a \in \mathrm{A}\})$; hence, there exists an element $a \in \mathrm{A}$ such that

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$|\mathfrak{I}(b) \cap \mathfrak{I}(a) \cap P^{\mathrm{a}}| = 0$. Therefore, b is not a blocking element of A, but this contradicts our choice of b. □

Thus, for any antichain A of the poset P, the subposet

$$\mathbf{I}(P, \mathrm{A}) = \mathfrak{F}\big(\mathbf{min}\, \mathbf{I}(P, \mathrm{A})\big)$$

of all *blocking elements* of A in P is an *order filter of* P. As a consequence, the subposet $\mathbf{C}(P, \mathrm{A})$ of all *complementing elements* of A in P is the *order ideal* $\mathfrak{I}(\mathbf{max}\, \mathbf{C}(P, \mathrm{A}))$.

Definition 5.5. Let A be a subset of the finite bounded poset P.

(i) The *blocker of* A *in* P is the antichain

$$\mathfrak{b}(\mathrm{A}) := \mathbf{min}\, \mathbf{I}(P, \mathrm{A}) .$$

(ii) The *minimal blocking elements of* A *in* P are the elements of its blocker $\mathfrak{b}(\mathrm{A})$.
The *maximal complementing elements of* A *in* P are the elements of the antichain $\mathfrak{c}(\mathrm{A}) := \mathbf{max}\, \mathbf{C}(P, \mathrm{A})$.

The image of any *blocking element* under a suitable *order-preserving* map is also a blocking element:

Proposition 5.6. *Let P_1 and P_2 be disjoint finite bounded posets with $|P_1|, |P_2| > 1$. Let $\psi : P_1 \to P_2$ be an* order-preserving *map such that*

$$\begin{aligned} &\psi(\hat{0}_{P_1}) = \hat{0}_{P_2} \; ; \\ &\psi(x_1) \succ \hat{0}_{P_2} \, , \quad \forall x_1 \succ \hat{0}_{P_1} \, . \end{aligned} \tag{5.10}$$

For any subset A_1 *of* P_1, *we have*

$$\psi\big(\mathbf{I}(P_1, \mathrm{A}_1)\big) \subseteq \mathbf{I}\big(P_2, \psi(\mathrm{A}_1)\big) .$$

Proof. If $\mathrm{A}_1 := \emptyset \subset P_1$, or $\mathrm{A}_1 := \{\hat{0}_{P_1}\}$, then there is nothing to prove.

Suppose that $\mathrm{A}_1 \neq \emptyset \subset P_1$ and $\mathrm{A}_1 \neq \{\hat{0}_{P_1}\}$. Let b_1 be a blocking element of A_1. According to Definition 5.2, for all $a_1 \in \mathrm{A}_1$, $a_1 \succ \hat{0}_{P_1}$, we have $|\mathfrak{I}_{P_1}(b_1) \cap \mathfrak{I}_{P_1}(a_1) \cap {P_1}^{\mathrm{a}}| \geq 1$ and, in view of (5.10), for every atom $z_1 \in \mathfrak{I}_{P_1}(b_1) \cap \mathfrak{I}_{P_1}(a_1) \cap {P_1}^{\mathrm{a}}$, we have the inclusion

$$\mathfrak{I}_{P_2}\big(\psi(z_1)\big) \cap {P_2}^{\mathrm{a}} \subseteq \mathfrak{I}_{P_2}\big(\psi(a_1)\big) \cap {P_2}^{\mathrm{a}} ,$$

the left-hand part of which is nonempty. Hence, for all $a_2 \in \psi(\mathrm{A}_1)$ the inclusion $b_1 \in \mathbf{I}(P_1, \mathrm{A}_1)$ implies that

$$\Big|\mathfrak{I}_{P_2}\big(\psi(b_1)\big) \cap \mathfrak{I}_{P_2}\big(\psi(a_1)\big) \cap {P_2}^{\mathrm{a}}\Big| > 0 .$$

This means that $\psi(b_1) \in \mathbf{I}(P_2, \psi(\mathrm{A}_1))$, and completes the proof. □

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5.2 Blocking Elements in Direct Products of Posets

In this section we investigate the structure of the order filters of *blocking elements* in the *direct products* of two posets.

Proposition 5.7. *Let P_1 and P_2 be disjoint finite bounded posets with $|P_1|, |P_2| > 2$. Define a new poset Q by*

$$Q := (P_1 - \{\hat{0}_{P_1}, \hat{1}_{P_1}\}) \times (P_2 - \{\hat{0}_{P_2}, \hat{1}_{P_2}\}) \,\dot{\cup}\, \{\hat{0}_Q, \hat{1}_Q\} ,$$

where $\hat{0}_Q$ and $\hat{1}_Q$ are new least and greatest elements adjoined. Let A be a nonempty subset of the subposet $Q - \{\hat{0}_Q, \hat{1}_Q\}$, and define subsets $\mathrm{A}\lfloor_{P_1} \subset P_1$ and $\mathrm{A}\lfloor_{P_2} \subset P_2$ by

$$\mathrm{A}\lfloor_{P_1} := \{a_1 \in P_1 : (a_1; a_2) \in \mathrm{A}\} \quad \textit{and} \quad \mathrm{A}\lfloor_{P_2} := \{a_2 \in P_2 : (a_1; a_2) \in \mathrm{A}\} ,$$

respectively.

(i) *If* $\mathbf{min}\,\mathbf{I}(P_1, \mathrm{A}\lfloor_{P_1}) = \{\hat{1}_{P_1}\}$ *or* $\mathbf{min}\,\mathbf{I}(P_2, \mathrm{A}\lfloor_{P_2}) = \{\hat{1}_{P_2}\}$, *then*

$$\mathbf{I}(Q, \mathrm{A}) = \mathbf{min}\,\mathbf{I}(Q, \mathrm{A}) = \{\hat{1}_Q\} .$$

(ii) *If* $\mathbf{min}\,\mathbf{I}(P_1, \mathrm{A}\lfloor_{P_1}) \neq \{\hat{1}_{P_1}\}$ *and* $\mathbf{min}\,\mathbf{I}(P_2, \mathrm{A}\lfloor_{P_2}) \neq \{\hat{1}_{P_2}\}$, *then*

$$\mathbf{I}(Q, \mathrm{A}) = \Big(\mathbf{I}(P_1, \mathrm{A}\lfloor_{P_1}) - \{\hat{1}_{P_1}\}\Big) \times \Big(\mathbf{I}(P_2, \mathrm{A}\lfloor_{P_2}) - \{\hat{1}_{P_2}\}\Big) \,\dot{\cup}\, \{\hat{1}_Q\} ,$$

and $\mathbf{min}\,\mathbf{I}(Q, \mathrm{A}) = \mathbf{min}\,\mathbf{I}(P_1, \mathrm{A}\lfloor_{P_1}) \times \mathbf{min}\,\mathbf{I}(P_2, \mathrm{A}\lfloor_{P_2})$.

Proof. The atom set Q^{a} of the poset Q is ${P_1}^{\mathrm{a}} \times {P_2}^{\mathrm{a}}$; therefore, by eq. (5.8) the subposet of blocking elements of A in Q is

$$\begin{aligned}\mathbf{I}(Q, \mathrm{A}) &= \bigcap_{a=(a_1;a_2)\in \mathrm{A}} \mathfrak{F}_Q\Big(\Big(\mathfrak{I}_{P_1}(a_1) \times \mathfrak{I}_{P_2}(a_2)\Big) \cap \Big({P_1}^{\mathrm{a}} \times {P_2}^{\mathrm{a}}\Big)\Big) \\ &= \Big(\mathbf{I}(P_1, \mathrm{A}\lfloor_{P_1}) - \{\hat{1}_{P_1}\}\Big) \times \Big(\mathbf{I}(P_2, \mathrm{A}\lfloor_{P_2}) - \{\hat{1}_{P_2}\}\Big) \,\dot{\cup}\, \{\hat{1}_Q\} ,\end{aligned}$$

and the proof follows. □

Proposition 5.8. *Let P_1 and P_2 be disjoint finite bounded posets with $|P_1|, |P_2| > 1$. Define a new poset Q by $Q := P_1 \times P_2$, and let A be a nonempty subset of the poset $Q - \{\hat{0}_Q\}$. We have*

$$\mathbf{I}(Q, \mathrm{A}) = \bigcap_{(a_1;a_2)\in \mathrm{A}} \Big(\Big(P_1 \times \mathbf{I}(P_2, a_2)\Big) \,\cup\, \Big(\mathbf{I}(P_1, a_1) \times P_2\Big)\Big) .$$

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Proof. Since the atom set Q^{a} of the poset Q is $(\{\hat{0}_{P_1}\} \times P_2{}^{\mathrm{a}}) \dot{\cup} (P_1{}^{\mathrm{a}} \times \{\hat{0}_{P_2}\})$, by eq. (5.8) we have

$$\mathbf{I}(Q, \mathrm{A}) = \bigcap_{(a_1;a_2)\in \mathrm{A}} \mathfrak{F}_Q\Big(\big(\mathfrak{I}_{P_1}(a_1) \times \mathfrak{I}_{P_2}(a_2)\big) \cap \Big(\big(\{\hat{0}_{P_1}\} \times P_2{}^{\mathrm{a}}\big) \dot{\cup} \big(P_1{}^{\mathrm{a}} \times \{\hat{0}_{P_2}\}\big)\Big)\Big)$$

$$= \bigcap_{(a_1;a_2)\in \mathrm{A}} \mathfrak{F}_Q\Big(\big(\{\hat{0}_{P_1}\} \times (\mathfrak{I}_{P_2}(a_2) \cap P_2{}^{\mathrm{a}})\big) \dot{\cup} \big((\mathfrak{I}_{P_1}(a_1) \cap P_1{}^{\mathrm{a}}) \times \{\hat{0}_{P_2}\}\big)\Big) ,$$

and the proof follows. □

5.3 The Blocker Map and the Complementary Map on Antichains

In this section we regard the *antichains* of a finite bounded poset and their *blockers* as the elements of the *lattice of antichains* and as their images under the *blocker map*.

Let $\mathscr{F}(P)$ denote the *distributive* lattice of all *order filters* (ordered by *inclusion*) of a finite bounded poset P, and let $\mathfrak{A}(P)$ denote the *lattice* of all *antichains* of P, where for two antichains $\mathrm{A}', \mathrm{A}'' \in \mathfrak{A}(P)$ we set

$$\mathrm{A}' \preceq \mathrm{A}'' \iff \mathfrak{F}_P(\mathrm{A}') \subseteq \mathfrak{F}_P(\mathrm{A}'') ;$$

in other words, we use the *isomorphism* $\mathscr{F}(P) \to \mathfrak{A}(P)$, $F \mapsto \mathbf{min}\, F$. We call the least element $\hat{0}_{\mathfrak{A}(P)} := \emptyset \subset P$ and the greatest element $\hat{1}_{\mathfrak{A}(P)} := \{\hat{0}_P\}$ of the lattice $\mathfrak{A}(P)$ the *trivial antichains* of P, since they are the poset-theoretic analogues of the *trivial clutters*.

Recall that for two antichains $\mathrm{A}', \mathrm{A}'' \in \mathfrak{A}(P)$, we have

$$\mathrm{A}' \vee \mathrm{A}'' = \mathbf{min}(\mathrm{A}' \cup \mathrm{A}'') \text{ and } \mathrm{A}' \wedge \mathrm{A}'' = \mathbf{min}\big(\mathfrak{F}_P(\mathrm{A}') \cap \mathfrak{F}_P(\mathrm{A}'')\big) .$$

Let $\mathfrak{b} : \mathfrak{A}(P) \to \mathfrak{A}(P)$ be the *blocker map on* $\mathfrak{A}(P)$ defined by

$$\mathfrak{b} : \mathrm{A} \mapsto \mathbf{min}\, \mathbf{I}(P, \mathrm{A}) . \tag{5.11}$$

For every element $a \in P - \{\hat{0}\}$, we have $\mathfrak{b}(\{a\}) = \mathfrak{I}(a) \cap P^{\mathrm{a}}$. We also have

$$\mathfrak{b}(\emptyset \subset P) = \{\hat{0}\} , \quad \mathfrak{b}\big(\{\hat{0}\}\big) = \emptyset \subset P .$$

For a one-element antichain $\{a\}$, we will write $\mathfrak{b}(a)$ instead of $\mathfrak{b}(\{a\})$.

If A is a nontrivial antichain of P, then Lemma 5.4 implicitly gives the following identities in $\mathfrak{A}(P)$:

$$\mathfrak{b}(\mathrm{A}) = \bigwedge_{a\in \mathrm{A}} \bigvee_{e\in \mathfrak{b}(a)} \{e\} = \bigvee_{E\in \mathfrak{B}(\{\mathfrak{b}(a)\,:\, a\in \mathrm{A}\})} \bigwedge_{e\in E} \{e\} . \tag{5.12}$$

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Let the subposet

$$\mathscr{B}(P) := \mathfrak{b}(\mathfrak{A}(P)) \subseteq \mathfrak{A}(P)$$

be the image of the lattice of antichains $\mathfrak{A}(P)$ under the blocker map, with the induced order. The blockers $\mathrm{B} \in \mathscr{B}(P)$ determine the partition of the lattice $\mathfrak{A}(P)$ into its preimages under the blocker map:

$$\mathfrak{A}(P) = \dot{\bigcup_{\mathrm{B}\in\mathscr{B}(P)}} \mathfrak{b}^{-1}(\mathrm{B}) = \dot{\bigcup_{\mathrm{B}\in\mathscr{B}(P)}} \{\mathrm{A} \in \mathfrak{A}(P) : \mathfrak{b}(\mathrm{A}) = \mathrm{B}\} \ .$$

The following statement follows directly from (5.7):

Lemma 5.9. *The* blocker map *on the antichains is* order-reversing*: If* $\mathrm{A}', \mathrm{A}'' \in \mathfrak{A}(P)$ *and* $\mathrm{A}' \preceq \mathrm{A}''$*, then* $\mathfrak{b}(\mathrm{A}') \succeq \mathfrak{b}(\mathrm{A}'')$.

Note that for a one-element antichain $\{a\}$ of P, we have in $\mathfrak{A}(P)$ the relations

$$\{a\} \preceq \mathfrak{b}(\mathfrak{b}(a)) \preceq \mathfrak{b}(a) \ .$$

Definition 5.2 implies the following *reciprocity property* of *blocking elements*: For any antichain A of P, we have

$$\mathrm{A} \subseteq \mathbf{I}\big(P, \mathfrak{b}(\mathrm{A})\big) \ ,$$

that is, *every element of the antichain* A *is a blocking element of the blocker* $\mathfrak{b}(\mathrm{A})$.

Proposition 1.34 can be generalized in the following way:

Theorem 5.10. *The restriction map* $\mathfrak{b}|_{\mathscr{B}(P)}$ *is an* involution*, that is, for any blocker* $\mathrm{B} \in \mathscr{B}(P)$*, we have*

$$\mathfrak{b}\big(\mathfrak{b}(\mathrm{B})\big) = \mathrm{B} \ .$$

In other words, for any antichain A *of* P*, we have*

$$\mathfrak{b}\Big(\mathfrak{b}\big(\mathfrak{b}(\mathrm{A})\big)\Big) = \mathfrak{b}(\mathrm{A}) \ .$$

Proof. There is nothing to prove for the *trivial* blockers $\mathrm{B} := \hat{0}_{\mathscr{B}(P)} = \emptyset \subset P$ and $\mathrm{B} := \hat{1}_{\mathscr{B}(P)} = \{\hat{0}_P\}$.

Suppose that B is a *nontrivial* blocker, and pick an arbitrary antichain $\mathrm{A}' \in \mathfrak{b}^{-1}(\mathrm{B})$. Because of the reciprocity property, every element of A' is a blocking element of the antichain $\mathrm{B} = \mathfrak{b}(\mathrm{A}')$. In other words, for each element $a' \in \mathrm{A}'$, we have the inclusion $a' \in \mathbf{I}(P, \mathrm{B}) = \bigcap_{b\in\mathrm{B}} \mathfrak{F}_P(\mathfrak{b}(b))$. Taking into account this inclusion, let us assign to the antichain A' the antichain

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$$\mathrm{A} = \mathbf{min} \bigcap_{b \in \mathrm{B}} \mathfrak{F}_P\big(\mathfrak{b}(b)\big) \in \mathfrak{b}^{-1}(\mathrm{B}) \ ,$$

which by eq. (5.8) is the blocker of B. The equalities $\mathfrak{b}(\mathrm{A}) = \mathrm{B}$ and $\mathfrak{b}(\mathrm{B}) = \mathrm{A}$ complete the proof. □

The finite bounded posets P with *strong blocker duality* in which we have

$$\mathfrak{b}\big(\mathfrak{b}(\mathrm{A})\big) = \mathrm{A} \ ,$$

for *all* nontrivial antichains $\mathrm{A} \subset P$, are characterized in Theorem 5.45.

By Lemma 5.4 a nontrivial antichain A of P, considered as an element of the lattice $\mathfrak{A}(P)$, is a *fixed point* of the blocker map on $\mathfrak{A}(P)$ if and only if $\mathrm{A} = \bigwedge_{a \in \mathrm{A}} \bigvee_{e \in \mathfrak{b}(a)} \{e\}$ or, equivalently, $\mathrm{A} = \bigvee_{E \in \mathfrak{B}(\{\mathfrak{b}(a) : a \in \mathrm{A}\})} \bigwedge_{e \in E} \{e\}$.

Let us now investigate the structure of the preimages of blockers under the blocker map.

Theorem 5.11. *For any blocker* $\mathrm{B} \in \mathscr{B}(P)$*, its preimage* $\mathfrak{b}^{-1}(\mathrm{B})$ *is a* sub-join-semilattice *of the lattice* $\mathfrak{A}(P)$.

Proof. There is nothing to prove for a *trivial* blocker B.

Suppose that B is a *nontrivial* blocker, and pick two antichains $\mathrm{A}', \mathrm{A}'' \in \mathfrak{b}^{-1}(\mathrm{B})$. According to eq. (5.12), we have the following identities in the lattice $\mathfrak{A}(P)$:

$$\begin{aligned} \mathrm{B} = \mathfrak{b}(\mathrm{A}') &= \bigwedge_{a' \in \mathrm{A}'} \ \bigvee_{e \in \mathfrak{b}(a')} \{e\} \\ = \mathfrak{b}(\mathrm{A}'') &= \bigwedge_{a'' \in \mathrm{A}''} \ \bigvee_{e \in \mathfrak{b}(a'')} \{e\} \ . \end{aligned}$$

Therefore,

$$\mathrm{B} = \bigwedge_{a \in \mathrm{A}' \vee \mathrm{A}''} \ \bigvee_{e \in \mathfrak{b}(a)} \{e\} = \mathfrak{b}(\mathrm{A}' \vee \mathrm{A}'') \ .$$

Hence, $\mathrm{A}' \vee \mathrm{A}'' \in \mathfrak{b}^{-1}(\mathrm{B})$. □

Note that the greatest element of the preimage $\mathfrak{b}^{-1}(\mathrm{B})$ is the antichain $\mathfrak{b}(\mathrm{B})$.

Let $\mathfrak{c} : \mathfrak{A}(P) \to \mathfrak{A}(P)$ be the *complementary map on* $\mathfrak{A}(P)$ defined by

$$\mathfrak{c} : \mathrm{A} \mapsto \mathbf{max}\, \mathbf{C}(P, \mathrm{A}) \ .$$

In particular, we have $\mathfrak{c}(\emptyset \subset P) = \emptyset \subset P$ and $\mathfrak{c}(\{\hat{0}_P\}) = \{\hat{1}_P\}$.

Let the poset

$$\mathscr{C}(P) := \mathfrak{c}(\mathfrak{A}(P))$$

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be the image of the lattice $\mathfrak{A}(P)$ under the complementary map, with the *order inherited from the distributive lattice of ideals of P*: for antichains $\mathrm{C}', \mathrm{C}'' \in \mathscr{C}(P)$, we set

$$\mathrm{C}' \preceq \mathrm{C}'' \iff \mathfrak{I}_P(\mathrm{C}') \subseteq \mathfrak{I}_P(\mathrm{C}'') \ .$$

We conclude this section by revisiting the concept of the blocker of an antichain.

Definition 5.12.

(i) If $\{a\} \in \mathfrak{A}(P) - \{\hat{0}_{\mathfrak{A}(P)}, \hat{1}_{\mathfrak{A}(P)}\}$ is a nontrivial one-element antichain of P, then the *blocker* $\mathfrak{b}(a)$ *of* $\{a\}$ *in* P is the antichain

$$\mathfrak{b}(a) := \mathfrak{I}(a) \cap P^{\mathrm{a}} \ .$$

(ii) If $\mathrm{A} \in \mathfrak{A}(P) - \{\hat{0}_{\mathfrak{A}(P)}, \hat{1}_{\mathfrak{A}(P)}\}$ is a nontrivial antichain of P, then the *blocker* $\mathfrak{b}(\mathrm{A})$ *of* A *in* P is the meet

$$\mathfrak{b}(\mathrm{A}) := \bigwedge_{a \in \mathrm{A}} \mathfrak{b}(a) \tag{5.13}$$

in $\mathfrak{A}(P)$.

(iii) The *blockers* $\mathfrak{b}(\hat{0}_{\mathfrak{A}(P)})$ and $\mathfrak{b}(\hat{1}_{\mathfrak{A}(P)})$ of the trivial antichains $\hat{0}_{\mathfrak{A}(P)}$ and $\hat{1}_{\mathfrak{A}(P)}$ of P are the trivial antichains

$$\mathfrak{b}(\hat{0}_{\mathfrak{A}(P)}) := \hat{1}_{\mathfrak{A}(P)} \ , \quad \mathfrak{b}(\hat{1}_{\mathfrak{A}(P)}) := \hat{0}_{\mathfrak{A}(P)} \ .$$

(iv) The map $\mathfrak{b} : \mathfrak{A}(P) \to \mathfrak{A}(P)$ is the *blocker map on* $\mathfrak{A}(P)$.

5.4 The Lattice of Blockers

In this section we investigate the structure of the *poset of blockers* in a finite bounded poset.

Lemma 5.13. *The poset $\mathscr{B}(P)$ of blockers in a finite bounded poset P is a* sub-meet-semilattice *of the lattice of antichains $\mathfrak{A}(P)$.*

Proof. We need to show that for any two blockers $\mathrm{B}', \mathrm{B}'' \in \mathscr{B}(P)$, we have $\mathrm{B}' \wedge_{\mathfrak{A}(P)} \mathrm{B}'' \in \mathscr{B}(P)$, where $\wedge_{\mathfrak{A}(P)}$ denotes the operation of meet in $\mathfrak{A}(P)$. If one of the blockers B' and B'' is *trivial*, then there is nothing to prove.

Suppose that B' and B'' are both *nontrivial* blockers. In view of Theorem 5.10, we can write

$$\mathrm{B}' \wedge_{\mathfrak{A}(P)} \mathrm{B}'' = \mathfrak{b}\big(\mathfrak{b}(\mathrm{B}')\big) \wedge_{\mathfrak{A}(P)} \mathfrak{b}\big(\mathfrak{b}(\mathrm{B}'')\big)$$

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and, according to eq. (5.12), we have the following identities in the lattice $\mathfrak{A}(P)$:

$$
\begin{aligned}
\mathrm{B}' \wedge_{\mathfrak{A}(P)} \mathrm{B}'' &= \Big(\bigwedge_{a' \in \mathfrak{b}(\mathrm{B}')} \ \bigvee_{e \in \mathfrak{b}(a')} \{e\} \Big) \wedge_{\mathfrak{A}(P)} \Big(\bigwedge_{a'' \in \mathfrak{b}(\mathrm{B}'')} \ \bigvee_{e \in \mathfrak{b}(a'')} \{e\} \Big) \\
&= \bigwedge_{a \in \mathfrak{b}(\mathrm{B}') \vee_{\mathfrak{A}(P)} \mathfrak{b}(\mathrm{B}'')} \ \bigvee_{e \in \mathfrak{b}(a)} \{e\} \\
&= \mathfrak{b}\Big(\mathfrak{b}(\mathrm{B}') \vee_{\mathfrak{A}(P)} \mathfrak{b}(\mathrm{B}'')\Big) \in \mathscr{B}(P) \, . \qquad \square
\end{aligned}
$$

Lemma 5.14. *The meet-semilattice $\mathscr{B}(P)$ of blockers in P is* self-dual.

Proof. Let $\mathrm{B}', \mathrm{B}'' \in \mathscr{B}(P)$. If $\mathrm{B}' \preceq_{\mathscr{B}(P)} \mathrm{B}''$, then $\mathrm{B}' \preceq_{\mathfrak{A}(P)} \mathrm{B}''$, where $\preceq_{\mathfrak{A}(P)}$ and $\preceq_{\mathscr{B}(P)}$ denote the orderings in the posets $\mathfrak{A}(P)$ and $\mathscr{B}(P)$, respectively. It follows from Lemma 5.9 that $\mathfrak{b}(\mathrm{B}') \succeq_{\mathscr{B}(P)} \mathfrak{b}(\mathrm{B}'')$.

Conversely, the relation $\mathfrak{b}(\mathrm{B}') \succeq_{\mathscr{B}(P)} \mathfrak{b}(\mathrm{B}'')$ implies the relation $\mathrm{B}' = \mathfrak{b}(\mathfrak{b}(\mathrm{B}')) \preceq_{\mathscr{B}(P)} \mathrm{B}'' = \mathfrak{b}(\mathfrak{b}(\mathrm{B}''))$, in view of Theorem 5.10 and Lemma 5.9.

Since the restriction map $\mathfrak{b}|_{\mathscr{B}(P)}$ is bijective, it is an *anti-automorphism* of the poset $\mathscr{B}(P)$. □

We can sum up the information of this section in the following result:

Theorem 5.15. *The poset $\mathscr{B}(P)$ of blockers in a finite bounded poset P is a* lattice, *with the* least element $\hat{0}_{\mathscr{B}(P)} = \emptyset \subset P$ *and the* greatest element $\hat{1}_{\mathscr{B}(P)} = \{\hat{0}_P\}$. *The unique* atom *of $\mathscr{B}(P)$ is the blocker $\mathfrak{b}(P^{\mathrm{a}})$, and the unique* coatom *of $\mathscr{B}(P)$ is the antichain P^{a}.*

(i) *The lattice $\mathscr{B}(P)$ is a* sub-meet-semilattice *of the lattice $\mathfrak{A}(P)$.*

(ii) *The lattice $\mathscr{B}(P)$ is* self-dual. *The restriction map $\mathfrak{b}|_{\mathscr{B}(P)}$ is an* anti-automorphism of $\mathscr{B}(P)$.

(iii) *The operation of* meet $\wedge_{\mathscr{B}(P)}$ *and the operation of* join $\vee_{\mathscr{B}(P)}$ *in the lattice $\mathscr{B}(P)$ are determined as follows: given blockers $\mathrm{B}', \mathrm{B}'' \in \mathscr{B}(P)$, we have*

$$\mathrm{B}' \wedge_{\mathscr{B}(P)} \mathrm{B}'' = \mathrm{B}' \wedge_{\mathfrak{A}(P)} \mathrm{B}'' \, , \tag{5.14}$$

and

$$\mathrm{B}' \vee_{\mathscr{B}(P)} \mathrm{B}'' = \mathfrak{b}\Big(\mathfrak{b}(\mathrm{B}') \wedge_{\mathfrak{A}(P)} \mathfrak{b}(\mathrm{B}'')\Big) \, , \tag{5.15}$$

where $\wedge_{\mathfrak{A}(P)}$ and $\vee_{\mathfrak{A}(P)}$ denote the operations of meet and join, respectively, in the lattice $\mathfrak{A}(P)$.

Proof. It remains to prove the identity $\mathrm{B}' \vee_{\mathscr{B}(P)} \mathrm{B}'' = \mathfrak{b}(\mathfrak{b}(\mathrm{B}') \wedge_{\mathscr{B}(P)} \mathfrak{b}(\mathrm{B}''))$ given in eq. (5.15) and restated in accordance with eq. (5.14), but this identity follows immediately from the self-duality of the lattice $\mathscr{B}(P)$, in view of the existence of its anti-automorphism $\mathfrak{b}|_{\mathscr{B}(P)}$. □

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Definition 5.16. Given a finite bounded poset P, the poset $\mathscr{B}(P) := \mathfrak{b}(\mathfrak{A}(P))$ is the *lattice of blockers in P*.

It follows immediately from the definition of the complementary map that its restriction $\mathfrak{c}|_{\mathscr{B}(P)} : \mathscr{B}(P) \to \mathscr{C}(P)$, $\mathrm{B} \mapsto \mathfrak{c}(\mathrm{B})$, is an *isomorphism* of $\mathscr{B}(P)$ onto the self-dual *lattice* $\mathscr{C}(P)$.

We conclude this section by mentioning a natural poset-theoretic viewpoint on the above discussed properties of the blocker map.

Remark 5.17. We have (see, e.g., Lemma 5.33)

$$\mathrm{A} \preceq \mathfrak{b}(\mathfrak{b}(\mathrm{A})) ,$$

for any element A of the lattice of antichains $\mathfrak{A}(P)$. Together with Lemma 5.9 and Theorem 5.10, these relations show that the composite map

$$\mathfrak{b} \circ \mathfrak{b} : \mathfrak{A}(P) \to \mathfrak{A}(P)$$

is a *closure operator on* $\mathfrak{A}(P)$.

5.5 Deletion and Contraction

In this section, the set-theoretic *deletion* operator and the *contraction* operator on *clutters* are generalized to those on *antichains* of finite bounded posets.

Let $X \subseteq P^{\mathrm{a}}$ be a subset of the set of atoms P^{a} of a finite bounded poset P.

Definition 5.18.

(i) If $\{a\} \in \mathfrak{A}(P) - \{\hat{0}_{\mathfrak{A}(P)}, \hat{1}_{\mathfrak{A}(P)}\}$ is a nontrivial one-element antichain of P, then the *deletion* $\{a\}\backslash X$ and *contraction* $\{a\}/X$ are the antichains

$$\{a\}\backslash X := \begin{cases} \{a\} , & \text{if } |\mathfrak{b}(a) \cap X| = 0 , \\ \hat{0}_{\mathfrak{A}(P)} , & \text{if } |\mathfrak{b}(a) \cap X| > 0 , \end{cases}$$

$$\{a\}/X := \begin{cases} \{a\} , & \text{if } |\mathfrak{b}(a) \cap X| = 0 , \\ \mathfrak{b}\big(\mathfrak{b}(a) - X\big) , & \text{if } |\mathfrak{b}(a) \cap X| > 0 \text{ and } \mathfrak{b}(a) \not\subseteq X , \\ \hat{1}_{\mathfrak{A}(P)} , & \text{if } \mathfrak{b}(a) \subseteq X . \end{cases}$$

(ii) If $\mathrm{A} \in \mathfrak{A}(P) - \{\hat{0}_{\mathfrak{A}(P)}, \hat{1}_{\mathfrak{A}(P)}\}$ is a nontrivial antichain of P, then the *deletion* $\mathrm{A}\backslash X$ and *contraction* A/X are the antichains

$$\mathrm{A}\backslash X := \bigvee_{a \in \mathrm{A}} \big(\{a\}\backslash X\big) , \quad \mathrm{A}/X := \bigvee_{a \in \mathrm{A}} \big(\{a\}/X\big) . \tag{5.16}$$

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(iii) The *deletion* and *contraction* of the trivial antichains of P are

$$\hat{0}_{\mathfrak{A}(P)}\backslash X = \hat{0}_{\mathfrak{A}(P)}/X := \hat{0}_{\mathfrak{A}(P)} ,$$
$$\hat{1}_{\mathfrak{A}(P)}\backslash X = \hat{1}_{\mathfrak{A}(P)}/X := \hat{1}_{\mathfrak{A}(P)} .$$

(iv) The map

$$(\backslash X) : \mathfrak{A}(P) \to \mathfrak{A}(P) , \quad \mathrm{A} \mapsto \mathrm{A}\backslash X ,$$

is the *deletion operator on* $\mathfrak{A}(P)$.

The map

$$(/X) : \mathfrak{A}(P) \to \mathfrak{A}(P) , \quad \mathrm{A} \mapsto \mathrm{A}/X ,$$

is the *contraction operator on* $\mathfrak{A}(P)$.

Let A be an antichain of P; in general, if X and Y are disjoint nonempty subsets of the atom set P^{a}, then we have $(\mathrm{A}/X)/Y \neq (\mathrm{A}/Y)/X$, but we always have $(\mathrm{A}\backslash X)\backslash Y = (\mathrm{A}\backslash Y)\backslash X$.

The following conclusions follow immediately from Definition 5.18: if $\{a\}$ is a one-element antichain of P, then we have

$$\{a\}\backslash X \preceq \{a\} \preceq \{a\}/X , \tag{5.17}$$

and

$$\mathfrak{b}(a)\backslash X \preceq \mathfrak{b}\big(\{a\}/X\big) \preceq \mathfrak{b}(a) \preceq \mathfrak{b}(a)/X = \mathfrak{b}\big(\{a\}\backslash X\big) . \tag{5.18}$$

Yet another conclusion is the following statement:

Lemma 5.19. *If* $\mathrm{A}', \mathrm{A}'' \in \mathfrak{A}(P)$, *and* $\mathrm{A}' \preceq \mathrm{A}''$, *then we have*

$$\mathrm{A}'\backslash X \preceq \mathrm{A}''\backslash X \quad \textit{and} \quad \mathrm{A}'/X \preceq \mathrm{A}''/X .$$

If $\mathrm{A} \in \mathfrak{A}(P)$, then the elements $\mathrm{A}\backslash X$, A and A/X of the lattice $\mathfrak{A}(P)$ are all comparable:

Lemma 5.20. *If* $\mathrm{A} \in \mathfrak{A}(P)$, *then*

$$\mathrm{A}\backslash X \preceq \mathrm{A} \preceq \mathrm{A}/X .$$

Proof. If A is a *trivial* antichain, then there is nothing to prove.

Suppose that the antichain A is *nontrivial*. Because of eqs. (5.16) and (5.17), we have $\mathrm{A}\backslash X = \bigvee_{a\in\mathrm{A}}(\{a\}\backslash X) \preceq \bigvee_{a\in\mathrm{A}}\{a\} = \mathrm{A} \preceq \bigvee_{a\in\mathrm{A}}(\{a\}/X) = \mathrm{A}/X$. □

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If $\{a\}$ is a nontrivial one-element antichain of P, then we clearly have $\{a\}\backslash X = (\{a\}\backslash X)\backslash X$. The antichain $\{a\}/X$ has an analogous property. Indeed, it follows from the definition of contraction that if either $|\mathfrak{b}(a) \cap X| = 0$, or $\mathfrak{b}(a) \subseteq X$, then $(\{a\}/X)/X = \{a\}/X$. Further, on the one hand, if $|\mathfrak{b}(a) \cap X| > 0$ and $\mathfrak{b}(a) \nsubseteq X$, then by Lemma 5.20 we have $(\{a\}/X)/X \succeq \{a\}/X$. On the other hand, for every $b \in \{a\}/X = \mathfrak{b}(\mathfrak{b}(a) - X)$ we have $\mathfrak{b}(b) - X \succeq \mathfrak{b}(a) - X$ and, as a consequence, we have $(\{a\}/X)/X = \bigvee_{b \in \{a\}/X} \mathfrak{b}(\mathfrak{b}(b) - X) \preceq \mathfrak{b}(\mathfrak{b}(a) - X) = \{a\}/X$. We conclude that $(\{a\}/X)/X = \{a\}/X$. In view of eq. (5.16), we obtain the following result:

Lemma 5.21. *If* $\mathrm{A} \in \mathfrak{A}(P)$*, then*

$$(\mathrm{A}\backslash X)\backslash X = \mathrm{A}\backslash X \quad \textit{and} \quad (\mathrm{A}/X)/X = \mathrm{A}/X\ .$$

Together with Lemmas 5.19 and 5.20, Lemma 5.21 establishes a connection between the maps $(\backslash X), (/X) : \mathfrak{A}(P) \to \mathfrak{A}(P)$ and (co)closure operators:

Theorem 5.22. *The map* $(\backslash X)$ *is a* coclosure operator on $\mathfrak{A}(P)$.
The map $(/X)$ *is a* closure operator on $\mathfrak{A}(P)$.

Given a nonempty subset $X \subseteq P^{\mathrm{a}}$ of the set of atoms of a finite bounded poset P, we are slightly abusing notation by letting $\mathfrak{A}(P)\backslash X$ and $\mathfrak{A}(P)/X$ denote the images $(\backslash X)(\mathfrak{A}(P)) := \{\mathrm{A}\backslash X : \mathrm{A} \in \mathfrak{A}(P)\}$ and $(/X)(\mathfrak{A}(P)) := \{\mathrm{A}/X : \mathrm{A} \in \mathfrak{A}(P)\}$, respectively.

Several well-known properties of (semi)lattice maps and (co)closure operators on lattices can be interpreted, in the case of the maps $(\backslash X)$ and $(/X)$, in the following manner:

Definition 5.18 implies that the maps $(\backslash X), (/X) : \mathfrak{A}(P) \to \mathfrak{A}(P)$ are *upper* $\{\hat{0}_{\mathfrak{A}(P)}, \hat{1}_{\mathfrak{A}(P)}\}$*-homomorphisms*, that is, for antichains $\mathrm{A}', \mathrm{A}'' \in \mathfrak{A}(P)$, we have $(\mathrm{A}' \vee \mathrm{A}'')\backslash X = (\mathrm{A}'\backslash X) \vee (\mathrm{A}''\backslash X)$, $(\mathrm{A}' \vee \mathrm{A}'')/X = (\mathrm{A}'/X) \vee (\mathrm{A}''/X)$, and we have $\hat{0}_{\mathfrak{A}(P)}\backslash X = \hat{0}_{\mathfrak{A}(P)}/X = \hat{0}_{\mathfrak{A}(P)}$, and $\hat{1}_{\mathfrak{A}(P)}\backslash X = \hat{1}_{\mathfrak{A}(P)}/X = \hat{1}_{\mathfrak{A}(P)}$.

The posets $\mathfrak{A}(P)\backslash X$ and $\mathfrak{A}(P)/X$, with the partial orders induced by the partial order on $\mathfrak{A}(P)$, are *lattices*.

Definition 5.23. Given a finite bounded poset P, the poset

$$\mathfrak{A}(P)\backslash X := \left\{\mathrm{A}\backslash X : \mathrm{A} \in \mathfrak{A}(P)\right\}$$

is the *lattice of deletions in P*.

The poset

$$\mathfrak{A}(P)/X := \left\{\mathrm{A}/X : \mathrm{A} \in \mathfrak{A}(P)\right\}$$

is the *lattice of contractions in P*.

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The lattice $\mathfrak{A}(P)\backslash X$ is a *sub-join-semilattice of* $\mathfrak{A}(P)$. Denote by $\wedge_{\mathfrak{A}(P)\backslash X}$ the operation of *meet* in $\mathfrak{A}(P)\backslash X$. If $\mathrm{D}', \mathrm{D}'' \in \mathfrak{A}(P)\backslash X$, then we have $\mathrm{D}' \wedge_{\mathfrak{A}(P)\backslash X} \mathrm{D}'' = (\mathrm{D}' \wedge \mathrm{D}'')\backslash X$.

The lattice $\mathfrak{A}(P)/X$ is a *sublattice of* $\mathfrak{A}(P)$.

If $\mathrm{D} \in \mathfrak{A}(P)\backslash X$, then the *preimage* $(\backslash X)^{-1}(\mathrm{D})$ of the antichain D under the map $(\backslash X)$ is the *closed interval* $[\mathrm{D}, \mathrm{D} \vee X]$ *of* $\mathfrak{A}(P)$.

If $\mathrm{D} \in \mathfrak{A}(P)/X$, then the *preimage* $(/X)^{-1}(\mathrm{D})$ of the antichain D under the map $(/X)$ is a *convex sub-join-semilattice of the lattice* $\mathfrak{A}(P)$, with the *greatest element* D.

Identities (5.3) for *clutters* can be generalized to *antichains* of arbitrary finite bounded posets. Indeed, let $\mathrm{A} \in \mathfrak{A}(P)$. From Lemma 5.20 we conclude that the relations

$$\mathfrak{b}(\mathrm{A})\backslash X \preceq \mathfrak{b}(\mathrm{A}) \preceq \mathfrak{b}(\mathrm{A})/X \quad \text{and} \quad \mathfrak{b}(\mathrm{A}/X) \preceq \mathfrak{b}(\mathrm{A}) \preceq \mathfrak{b}(\mathrm{A}\backslash X)$$

are valid, and an additional result is as follows:

Theorem 5.24. *If* $\mathrm{A} \in \mathfrak{A}(P)$, *then*

$$\mathfrak{b}(\mathrm{A})\backslash X \preceq \mathfrak{b}(\mathrm{A}/X) \preceq \mathfrak{b}(\mathrm{A}) \preceq \mathfrak{b}(\mathrm{A})/X \preceq \mathfrak{b}(\mathrm{A}\backslash X)\ . \tag{5.19}$$

Proof. If the antichain A is *trivial*, then there is nothing to prove.

If A' and A'' are *arbitrary* antichains of P, then it is routine to check that

$$(\mathrm{A}' \wedge \mathrm{A}'')\backslash X \preceq (\mathrm{A}'\backslash X) \wedge (\mathrm{A}''\backslash X)\ , \tag{5.20}$$

$$(\mathrm{A}' \wedge \mathrm{A}'')/X \preceq (\mathrm{A}'/X) \wedge (\mathrm{A}''/X)\ . \tag{5.21}$$

Suppose that A is a *nontrivial* antichain of P. To verify the relation $\mathfrak{b}(\mathrm{A})\backslash X \preceq \mathfrak{b}(\mathrm{A}/X)$, we use relations (5.20) and (5.18); we see that

$$\begin{aligned}\mathfrak{b}(\mathrm{A})\backslash X &= \Big(\bigwedge_{a\in\mathrm{A}} \mathfrak{b}(a)\Big)\backslash X \preceq \bigwedge_{a\in\mathrm{A}} (\mathfrak{b}(a)\backslash X)\\ &\preceq \bigwedge_{a\in\mathrm{A}} \mathfrak{b}(\{a\}/X) = \mathfrak{b}\Big(\bigvee_{a\in\mathrm{A}} (\{a\}/X)\Big) = \mathfrak{b}(\mathrm{A}/X)\ .\end{aligned}$$

Let us verify the relation $\mathfrak{b}(\mathrm{A})/X \preceq \mathfrak{b}(\mathrm{A}\backslash X)$. Looking at expressions (5.21) and (5.18), we see that

$$\begin{aligned}\mathfrak{b}(\mathrm{A})/X &= \Big(\bigwedge_{a\in\mathrm{A}} \mathfrak{b}(a)\Big)/X \preceq \bigwedge_{a\in\mathrm{A}} (\mathfrak{b}(a)/X)\\ &= \bigwedge_{a\in\mathrm{A}} \mathfrak{b}(\{a\}\backslash X) = \mathfrak{b}\Big(\bigvee_{a\in\mathrm{A}} (\{a\}\backslash X)\Big) = \mathfrak{b}(\mathrm{A}\backslash X)\ .\end{aligned}$$

□

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5.6 The Blocker, Deletion, Contraction, and Maps on Posets

In this section we give yet another poset-theoretic interpretation of the fundamental identities (5.3) which establish the connection between the set-theoretic *blocker*, *deletion*, and *contraction* maps on clutters.

Theorem 5.25. *Let L be a finite poset. Let $\delta, \gamma : L \to L$ be* order-preserving maps *on L such that*

$$\gamma(x) \succeq x \ , \tag{5.22}$$

for all $x \in L$.

Let $\beta : L \to L$ be an order-reversing map *such that*

$$\beta(\beta(x)) \succeq x \ , \tag{5.23}$$

for all $x \in L$.

If the relations

$$\beta(\delta(\beta(x))) \succeq \gamma(x) \tag{5.24}$$

and

$$\beta(\gamma(\beta(x))) \succeq \delta(x) \tag{5.25}$$

are valid for all $x \in L$, then either of these relations implies

$$\delta(\beta(z)) \preceq \beta(\gamma(z)) \preceq \beta(z) \preceq \gamma(\beta(z)) \preceq \beta(\delta(z)) \ , \tag{5.26}$$

for any $z \in L$.

Moreover, suppose that $\beta(\beta(x)) = x$, for all $x \in L$. If the identities $\beta(\delta(\beta(x))) = \gamma(x)$ and $\beta(\gamma(\beta(x))) = \delta(x)$ are valid for all $x \in L$, then either of these identities implies

$$\delta(\beta(z)) = \beta(\gamma(z)) \preceq \beta(z) \preceq \gamma(\beta(z)) = \beta(\delta(z)) \ , \tag{5.27}$$

for any $z \in L$.

Proof. Relation (5.22) implies

$$\beta(\gamma(z)) \preceq \beta(z) \ ,$$

since the map β is order-reversing; moreover, we have

$$\beta(z) \preceq \gamma(\beta(z)) \ .$$

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We now verify implication (5.24)⇒(5.26).

On the one hand, in view of eq. (5.23), we have $\delta(\beta(z)) \preceq \beta(\beta(\delta(\beta(z))))$. On the other hand, since β is order-reversing, relation (5.24) implies $\beta(\beta(\delta(\beta(z)))) \preceq \beta(\gamma(z))$. We obtain

$$\delta(\beta(z)) \preceq \beta(\gamma(z)) \ . \tag{5.28}$$

Further, on the one hand, relation (5.24) implies $\gamma(\beta(z)) \preceq \beta(\delta(\beta(\beta(z))))$. On the other hand, since $\beta(\beta(z)) \succeq z$, by eq. (5.23), and δ is order-preserving, and β is order-reversing, we obtain $\beta(\delta(\beta(\beta(z)))) \preceq \beta(\delta(z))$. We conclude that

$$\gamma(\beta(z)) \preceq \beta(\delta(z)) \ , \tag{5.29}$$

and we are done.

We now verify implication (5.25)⇒(5.26).

On the one hand, in view of eq. (5.25), we have $\delta(\beta(z)) \preceq \beta(\gamma(\beta(\beta(z))))$. On the other hand, since β is order-reversing, and γ is order-preserving, relation (5.23) implies $\beta(\gamma(\beta(\beta(z)))) \preceq \beta(\gamma(z))$. We obtain eq. (5.28).

Further, on the one hand, relation (5.23) implies $\gamma(\beta(z)) \preceq \beta(\beta(\gamma(\beta(z))))$. On the other hand, since β is order-reversing, the relation (5.25) implies $\beta(\beta(\gamma(\beta(z)))) \preceq \beta(\delta(z))$. We come to eq. (5.29), and we are done.

The proof of relation (5.27) is now straightforward. □

Note that since the map β in Theorem 5.25 is order-reversing, and eq. (5.23) holds, it is a consequence of the well-known results on closure operators that we have

$$\beta\big(\beta\big(\beta(x)\big)\big) = \beta(x) \ , \tag{5.30}$$

for any $x \in L$.

To illustrate Theorem 5.25, we will comment on identities (5.3).

Let us turn to the poset-theoretic *blocker map* $\mathfrak{b} : \mathfrak{A}(P) \to \mathfrak{A}(P)$ on the lattice of antichains $\mathfrak{A}(P)$ of a finite bounded poset P; see Definition 5.12. Recall that the blocker map is *order-reversing*, with the property $\mathfrak{b}(\mathfrak{b}(\mathrm{A})) \succeq \mathrm{A}$, for all $\mathrm{A} \in \mathfrak{A}(P)$. Thus, identity (5.30) implies

$$\mathfrak{b}\Big(\mathfrak{b}\Big(\mathfrak{b}(\mathrm{A})\Big)\Big) = \mathfrak{b}(\mathrm{A}) \ ,$$

as stated in Theorem 5.10; cf. eq. (5.2).

Recall also that the *deletion operator* $(\backslash X) : \mathfrak{A}(P) \to \mathfrak{A}(P)$, $\mathrm{A} \mapsto \mathrm{A} \backslash X$, is a *coclosure operator on* $\mathfrak{A}(P)$. The *contraction operator* $(/X) : \mathfrak{A}(P) \to \mathfrak{A}(P)$, $\mathrm{A} \mapsto \mathrm{A}/X$, is a *closure operator on* $\mathfrak{A}(P)$; see Definition 5.18 and Theorem 5.22.

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Now let the poset L of Theorem 5.25 be the lattice $\mathfrak{A}(P)$. In this context, the maps $\mathfrak{b}, (\backslash X), (/X) : \mathfrak{A}(P) \to \mathfrak{A}(P)$ are instances of the maps β, δ, and γ of Theorem 5.25, respectively. In particular, eqs. (5.24) and (5.25) are interpreted in the following way:

Lemma 5.26. *For any antichain* A *of* P, *the relations*

$$\mathfrak{b}(\mathfrak{b}(\mathrm{A})\backslash X) \succeq \mathrm{A}/X \tag{5.31}$$

and

$$\mathfrak{b}(\mathfrak{b}(\mathrm{A})/X) \succeq \mathrm{A}\backslash X \tag{5.32}$$

are valid in $\mathfrak{A}(P)$.

Proof. If the antichain A is *trivial*, then there is nothing to prove.

Let $\{a'\}$ be a *nontrivial one-element* antichain of P.

(i) Suppose that $|\mathfrak{b}(a') \cap X| = 0$. In this case we have

$$\mathfrak{b}\Big(\mathfrak{b}(a')\backslash X\Big) = \mathfrak{b}\Big(\mathfrak{b}(a')\Big) \succeq \{a'\} = \{a'\}/X$$

and

$$\mathfrak{b}\Big(\mathfrak{b}(a')/X\Big) = \mathfrak{b}\Big(\mathfrak{b}(a')\Big) \succeq \{a'\} = \{a'\}\backslash X\,.$$

(ii) Suppose that $|\mathfrak{b}(a') \cap X| > 0$ and $\mathfrak{b}(a') \not\subseteq X$. In this case we have

$$\mathfrak{b}\Big(\mathfrak{b}(a')\backslash X\Big) = \mathfrak{b}\Big(\mathfrak{b}(a') - X\Big) = \{a'\}/X$$

and

$$\mathfrak{b}\Big(\mathfrak{b}(a')/X\Big) = \mathfrak{b}\Big(\hat{1}_{\mathfrak{A}(P)}\Big) = \hat{0}_{\mathfrak{A}(P)} = \{a'\}\backslash X\,.$$

(iii) If $\mathfrak{b}(a') \subseteq X$, then we have

$$\mathfrak{b}\Big(\mathfrak{b}(a')\backslash X\Big) = \mathfrak{b}\Big(\mathfrak{b}(a') - X\Big) = \mathfrak{b}(\hat{0}_{\mathfrak{A}(P)}) = \hat{1}_{\mathfrak{A}(P)} = \{a'\}/X$$

and

$$\mathfrak{b}\Big(\mathfrak{b}(a')/X\Big) = \mathfrak{b}\Big(\hat{1}_{\mathfrak{A}(P)}\Big) = \hat{0}_{\mathfrak{A}(P)} = \{a'\}\backslash X\,.$$

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Now let A be an *arbitrary nontrivial antichain* of P. On the one hand, by definition (5.13) we have

$$\mathfrak{b}\big(\mathfrak{b}(\mathrm{A})\backslash X\big) = \mathfrak{b}\Big(\Big(\bigwedge_{a\in\mathrm{A}} \mathfrak{b}(a)\Big)\backslash X\Big) \text{ and } \mathfrak{b}\big(\mathfrak{b}(\mathrm{A})/X\big) = \mathfrak{b}\Big(\Big(\bigwedge_{a\in\mathrm{A}} \mathfrak{b}(a)\Big)/X\Big)$$

in $\mathfrak{A}(P)$. On the other hand, for any element $a' \in \mathrm{A}$, we have

$$\mathfrak{b}\Big(\Big(\bigwedge_{a\in\mathrm{A}} \mathfrak{b}(a)\Big)\backslash X\Big) \succeq \mathfrak{b}\big(\mathfrak{b}(a')\backslash X\big) \succeq \{a'\}/X,$$

and

$$\mathfrak{b}\Big(\Big(\bigwedge_{a\in\mathrm{A}} \mathfrak{b}(a)\Big)/X\Big) \succeq \mathfrak{b}\big(\mathfrak{b}(a')/X\big) \succeq \{a'\}\backslash X$$

in $\mathfrak{A}(P)$. Now Definition 5.18(ii) implies relations (5.31) and (5.32). □

In view of relation (5.26) and Lemma 5.26, we come to the result of Theorem 5.24: *for any antichain* A *of P, and for any subset $X \subseteq P^{\mathrm{a}}$, the relations*

$$\mathfrak{b}(\mathrm{A})\backslash X \preceq \mathfrak{b}(\mathrm{A}/X) \preceq \mathfrak{b}(\mathrm{A}) \preceq \mathfrak{b}(\mathrm{A})/X \preceq \mathfrak{b}(\mathrm{A}\backslash X)$$

are valid in $\mathfrak{A}(P)$.

5.7 The Blocker, Deletion, Contraction, Powers of 2, and the Self-Dual Clutters

In this section we discuss simple but important properties of the *set-theoretic blocker* map, as well as of the *deletion* operator and the *contraction* operator on *clutters*.

Theorem 5.27. *For any antichain* A *of the Boolean lattice* $\mathbb{B}(n)$*, its* blocker $\mathfrak{b}(\mathrm{A})$ *satisfies*

$$|\mathfrak{F}_{\mathbb{B}(n)}(\mathrm{A})| + |\mathfrak{F}_{\mathbb{B}(n)}(\mathfrak{b}(\mathrm{A}))| = 2^n\ . \tag{5.33}$$

As a consequence, we have

$$\mathfrak{A}(\mathbb{B}(n)) \ni \mathrm{A} = \mathfrak{b}(\mathrm{A}) \iff |\mathfrak{F}_{\mathbb{B}(n)}(\mathrm{A})| = 2^{n-1}\ .$$

In other words, the middle layer $\mathfrak{A}(\mathbb{B}(n))^{(2^{n-1})}$ *of the rank* 2^n free distributive lattice $\mathfrak{A}(\mathbb{B}(n))$ *is the* set of fixed points *of the* blocker map $\mathfrak{b} : \mathfrak{A}(\mathbb{B}(n)) \to \mathfrak{A}(\mathbb{B}(n))$.

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Proof. We have $\mathscr{B}(\mathfrak{A}(\mathbb{B}(n))) := \mathfrak{b}(\mathfrak{A}(\mathbb{B}(n))) = \mathfrak{A}(\mathbb{B}(n))$, and our observations follow immediately from Theorem 5.15(ii). □

Given a positive integer n, and a *clutter* $\mathscr{A}$ on its ground set $\mathrm{V}(\mathscr{A}) \subseteq [n]$, let us denote by $\mathscr{A}^{\triangledown}$ the family of all subsets of the set $[n]$ containing at least one member of the clutter $\mathscr{A}$. In other words, if A is the *antichain* of the *Boolean lattice* $\mathbb{B}(n)$ of subsets of the set $[n]$ such that

$$\left\{\mathfrak{I}_{\mathbb{B}(n)}(a) \cap \mathbb{B}(n)^{(1)} : a \in \mathrm{A}\right\} = \mathscr{A} ,$$

then

$$\mathscr{A}^{\triangledown} := \left\{\mathfrak{I}_{\mathbb{B}(n)}(f) \cap \mathbb{B}(n)^{(1)} : f \in \mathfrak{F}_{\mathbb{B}(n)}(\mathrm{A})\right\} ;$$

in particular, we have $\#\mathscr{A}^{\triangledown} := |\mathfrak{F}_{\mathbb{B}(n)}(\mathrm{A})|$.

Corollary 5.28.

(i) *For any clutter $\mathscr{A}$ on its ground set $\mathrm{V}(\mathscr{A}) \subseteq [n]$, we have*

$$\#\mathscr{A}^{\triangledown} + \#\mathfrak{B}(\mathscr{A})^{\triangledown} = 2^n . \tag{5.34}$$

Thus, the clutter $\mathscr{A}$ is self-dual, *that is,*

$$\mathscr{A} = \mathfrak{B}(\mathscr{A}) ,$$

if and only if

$$\#\mathscr{A}^{\triangledown} = 2^{n-1} .$$

(ii) *As a consequence, for any subset $X \subseteq [n] \supseteq \mathrm{V}(\mathscr{A})$, we have*

$$\#(\mathscr{A} \backslash X)^{\triangledown} + \#(\mathfrak{B}(\mathscr{A})/X)^{\triangledown} = 2^n , \tag{5.35}$$

$$\#(\mathscr{A}/X)^{\triangledown} + \#(\mathfrak{B}(\mathscr{A}) \backslash X)^{\triangledown} = 2^n , \tag{5.36}$$

and

$$\#\mathscr{A}^{\triangledown} - \#(\mathscr{A} \backslash X)^{\triangledown} = \#(\mathfrak{B}(\mathscr{A})/X)^{\triangledown} - \#\mathfrak{B}(\mathscr{A})^{\triangledown} \geq 0 , \tag{5.37}$$

$$\#(\mathscr{A}/X)^{\triangledown} - \#\mathscr{A}^{\triangledown} = \#\mathfrak{B}(\mathscr{A})^{\triangledown} - \#(\mathfrak{B}(\mathscr{A}) \backslash X)^{\triangledown} \geq 0 , \tag{5.38}$$

$$\#(\mathscr{A}/X)^{\triangledown} - \#(\mathscr{A} \backslash X)^{\triangledown} = \#(\mathfrak{B}(\mathscr{A})/X)^{\triangledown} - \#(\mathfrak{B}(\mathscr{A}) \backslash X)^{\triangledown} \geq 0 . \tag{5.39}$$

Proof. Identity (5.34) is just a restatement of eq. (5.33).

In view of Proposition 5.1, identities (5.35) and (5.36) follow immediately from eq. (5.34).

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Implications (5.34) & (5.35) ⇒ (5.37), (5.34) & (5.36) ⇒ (5.38), and (5.35) & (5.36) ⇒ (5.39) are valid in view of Lemma 5.20. □

5.8 The (X,k)-Blocker Map

In this section we consider a family of maps on the antichains of a finite bounded poset that generalize the *set-theoretic blocker map* on *clutters*, as well as the *poset-theoretic blocker map* on *antichains*, defined by eq. (5.11).

Let $X \subseteq P^{\mathrm{a}}$ be a subset of the set of atoms of a finite bounded poset P, and k a nonnegative integer, $k \leq |P^{\mathrm{a}}|$.

Definition 5.29.

(i) The (X,k)-*blocker map* on the lattice of antichains of a finite bounded poset P is the map $\mathfrak{b}_k^X : \mathfrak{A}(P) \to \mathfrak{A}(P)$ given by

$$\mathrm{A} \mapsto \mathbf{min}\Big\{b \in P : |\mathfrak{I}(b) \cap \mathfrak{I}(a) \cap (P^{\mathrm{a}} - X)| > k, \quad \forall a \in \mathrm{A}\Big\}$$

if the antichain A is *nontrivial*, and

$$\hat{0}_{\mathfrak{A}(P)} \mapsto \hat{1}_{\mathfrak{A}(P)}\ , \quad \hat{1}_{\mathfrak{A}(P)} \mapsto \hat{0}_{\mathfrak{A}(P)}\ .$$

(ii) Given an antichain $\mathrm{A} \in \mathfrak{A}(P)$, the (X,k)-*blocker of* A in P is the antichain $\mathfrak{b}_k^X(\mathrm{A})$.

We use the notation $\mathfrak{b}_k$ and $\mathfrak{b}^X$ instead of the notation $\mathfrak{b}_k^{\emptyset^{\mathrm{a}}}$ and $\mathfrak{b}_0^X$, respectively. The $(\emptyset^{\mathrm{a}}, 0)$-blocker map $\mathfrak{b} := \mathfrak{b}_0^{\emptyset^{\mathrm{a}}}$ is the *blocker map* on the lattice $\mathfrak{A}(P)$ considered in Section 5.3 and defined by eq. (5.11); given $\mathrm{A} \in \mathfrak{A}(P)$, the antichain $\mathfrak{b}(\mathrm{A})$ is called the *blocker of* A in P.

If $\{a\}$ is a one-element antichain of P, then we write $\mathfrak{b}_k^X(a)$ instead of $\mathfrak{b}_k^X(\{a\})$.

Since the blocker map on the lattice $\mathfrak{A}(P)$ is order-reversing, we have for any element $a \in P - \{\hat{0}_P\}$ and for any subset $E \subseteq \mathfrak{b}(a) - X$ the relations

$$\{a\} \preceq \mathfrak{b}(\mathfrak{b}(a)) \preceq \mathfrak{b}(\mathfrak{b}(a) - X) \preceq \mathfrak{b}(E) \preceq \mathfrak{b}(a)\ .$$

The following statement follows immediately from Definition 5.29:

Lemma 5.30. *Let* A *be a* nontrivial *antichain of* P. *If* $\mathfrak{b}_k^X(\mathrm{A}) \neq \hat{0}_{\mathfrak{A}(P)}$, *then for each* $a \in \mathrm{A}$ *and for all* $b \in \mathfrak{b}_k^X(\mathrm{A})$, *we have*

$$|\mathfrak{I}(a) \cap \mathfrak{I}(b) \cap (P^{\mathrm{a}} - X)| > k\ .$$

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Given an element $a \in P - \{\hat{0}_P\}$, let $\mathscr{S}_a$ be a family of subsets of the atom set P^{a}, defined by

$$\mathscr{S}_a := \{E \subseteq \mathfrak{b}(a) - X : |E| = k+1\} .$$

Let $\mathfrak{L}(P^{\mathrm{a}})$ be the Boolean lattice of all subsets of the atom set P^{a}, and let $\mathfrak{L}(P^{\mathrm{a}})^{(k+1)}$ be the subset of all elements of rank $k+1$ in $\mathfrak{L}(P^{\mathrm{a}})$. Given a $(k+1)$-subset $E \subseteq P^{\mathrm{a}}$, we denote by $\varepsilon(E)$ the least upper bound of E in $\mathfrak{L}(P^{\mathrm{a}})$; conversely, given an element $\mathfrak{e} \in \mathfrak{L}(P^{\mathrm{a}})^{(k+1)}$, we denote by $\varepsilon^{-1}(\mathfrak{e})$ the $(k+1)$-subset of atoms of $\mathfrak{L}(P^{\mathrm{a}})$ that are all comparable with $\mathfrak{e}$.

Let A be a nontrivial antichain of P. If $|\mathfrak{b}(a) - X| \le k$, for some $a \in \mathrm{A}$, then Definition 5.29 implies that $\mathfrak{b}_k^X(\mathrm{A}) = \hat{0}_{\mathfrak{A}(P)}$. In the case where $|\mathfrak{b}(a) - X| > k$, for all $a \in \mathrm{A}$, Proposition 5.31 describes two ways of finding the (X,k)-blocker of the antichain A, cf. eq. (5.12); this statement involves the set-theoretic blocker $\mathfrak{B}(\cdot)$ of a set family.

Proposition 5.31. *Let* A *be a* nontrivial *antichain of a finite bounded poset P. If $|\mathfrak{b}(a) - X| > k$, for all $a \in$* A, *then we have*

$$\mathfrak{b}_k^X(\mathrm{A}) = \bigwedge_{a\in\mathrm{A}} \bigvee_{E\in\mathscr{S}_a} \mathfrak{b}(E) = \bigvee_{\mathfrak{E}\in\mathfrak{B}(\{\{\varepsilon(E):E\in\mathscr{S}_a\}:a\in\mathrm{A}\})} \bigwedge_{\mathfrak{e}\in\mathfrak{E}} \mathfrak{b}\big(\varepsilon^{-1}(\mathfrak{e})\big) .$$

Proof. We have

$$\mathfrak{b}_k^X(\mathrm{A}) = \bigwedge_{a\in\mathrm{A}} \mathfrak{b}_k^X(a) , \tag{5.40}$$

and a poset-theoretic argument leads to the conclusion that for any $a \in \mathrm{A}$, we have

$$\mathfrak{b}_k^X(a) = \bigvee_{E\in\mathscr{S}_a} \mathfrak{b}(E) , \tag{5.41}$$

where $\mathfrak{b}(E) = \bigwedge_{e\in E}\{e\}$.

The reverse inclusion $\mathfrak{b}_k^X(\mathrm{A}) \supseteq \bigvee_{\mathfrak{E}\in\mathfrak{B}(\{\{\varepsilon(E):E\in\mathscr{S}_a\}:a\in\mathrm{A}\})} \bigwedge_{\mathfrak{e}\in\mathfrak{E}} \mathfrak{b}(\varepsilon^{-1}(\mathfrak{e}))$ follows from Definition 5.29. Suppose that the inclusion

$$\mathfrak{b}_k^X(\mathrm{A}) \subseteq \bigvee_{\mathfrak{E}\in\mathfrak{B}(\{\{\varepsilon(E):E\in\mathscr{S}_a\}:a\in\mathrm{A}\})} \bigwedge_{\mathfrak{e}\in\mathfrak{E}} \mathfrak{b}\big(\varepsilon^{-1}(\mathfrak{e})\big) \tag{5.42}$$

is not valid. Consider an element $b \in \mathfrak{b}_k^X(\mathrm{A})$ that does not belong to the right-hand side of eq. (5.42). In this case there is an element $a \in \mathrm{A}$ such that $|\mathfrak{I}(b) \cap \mathfrak{I}(a) \cap (P^{\mathrm{a}} - X)| \le k$. This means that the left-hand side of eq. (5.42) is not an (X,k)-blocker of A, a contradiction. □

The following statement specifies how the parameters X and k of the (X,k)-blocker map affect the corresponding images of the lattice $\mathfrak{A}(P)$, and it shows that the map $\mathfrak{b}_k^X$ is *order-reversing*.

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Lemma 5.32. *Let P be a finite bounded poset.*

(i) *Let $X, Y \subseteq P^{\mathrm{a}}$, $Y \supseteq X$, and let j be a nonnegative integer, $j \leq k$. Given an antichain $\mathrm{A} \in \mathfrak{A}(P)$, we have*

$$\mathfrak{b}_j^X(\mathrm{A}) \succeq \mathfrak{b}_k^X(\mathrm{A}) \succeq \mathfrak{b}_k^Y(\mathrm{A}) .$$

(ii) *The (X,k)-blocker map on the antichains is* order-reversing*: If $\mathrm{A}', \mathrm{A}'' \in \mathfrak{A}(P)$ and $\mathrm{A}' \preceq \mathrm{A}''$, then*

$$\mathfrak{b}_k^X(\mathrm{A}') \succeq \mathfrak{b}_k^X(\mathrm{A}'') .$$

Proof.

(i) If the antichain A is *trivial*, then there is nothing to prove.

Suppose that A is a *nontrivial* antichain of P. For each element $a \in \mathrm{A}$, by eq. (5.41) we have

$$\mathfrak{b}_k^X(a) = \bigvee_{E \in \mathscr{S}_a} \mathfrak{b}(E) \succeq \bigvee_{\substack{E \subseteq \mathfrak{b}(a) - Y : \\ |E| = k+1}} \mathfrak{b}(E) = \mathfrak{b}_k^Y(a) .$$

In view of eq. (5.40), this yields

$$\mathfrak{b}_k^X(\mathrm{A}) = \bigwedge_{a \in \mathrm{A}} \mathfrak{b}_k^X(a) \succeq \bigwedge_{a \in \mathrm{A}} \mathfrak{b}_k^Y(a) = \mathfrak{b}_k^Y(\mathrm{A}) .$$

The relation $\mathfrak{b}_j^X(\mathrm{A}) \succeq \mathfrak{b}_k^X(\mathrm{A})$ is proved analogously.

(ii) If A' is *trivial*, then the assertion follows immediately from Definition 5.29.

Suppose that A' is a *nontrivial* antichain. For any $a' \in \mathrm{A}'$, there is an element $a'' \in \mathrm{A}''$ such that $\{a'\} \preceq \{a''\}$ and, as a consequence, we have the reverse inclusion $\mathfrak{b}(a') \supseteq \mathfrak{b}(a'')$. Looking at expression (5.41), we see that

$$\mathfrak{b}_k^X(a') = \bigvee_{E \in \mathscr{S}_{a'}} \mathfrak{b}(E) \succeq \bigvee_{E \in \mathscr{S}_{a''}} \mathfrak{b}(E) = \mathfrak{b}_k^X(a'') ,$$

and identity (5.40) yields

$$\mathfrak{b}_k^X(\mathrm{A}') = \bigwedge_{a' \in \mathrm{A}'} \mathfrak{b}_k^X(a') \succeq \bigwedge_{a'' \in \mathrm{A}''} \mathfrak{b}_k^X(a'') = \mathfrak{b}_k^X(\mathrm{A}'') .$$

□

In addition to Lemma 5.32(ii), we need the following statement for describing the structure of the image of the lattice of antichains $\mathfrak{A}(P)$ under the (X,k)-blocker map.

Lemma 5.33. *For any antichain $\mathrm{A} \in \mathfrak{A}(P)$, we have*

$$\mathrm{A} \preceq \mathfrak{b}_k^X\big(\mathfrak{b}_k^X(\mathrm{A})\big) . \tag{5.43}$$

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Proof. If the antichain A is *trivial*, then the lemma follows from Definition 5.29, since in this case we have $\mathfrak{b}_k^X(\mathfrak{b}_k^X(\mathrm{A})) = \mathrm{A}$.

Suppose that A is a *nontrivial* antichain of P. If $\mathfrak{b}_k^X(\mathrm{A}) = \hat{0}_{\mathfrak{A}(P)}$, then we have $\mathfrak{b}_k^X(\mathfrak{b}_k^X(\mathrm{A})) = \hat{1}_{\mathfrak{A}(P)} \succ \mathrm{A}$, and we are done. Finally, suppose that $\mathfrak{b}_k^X(\mathrm{A})$ is a nontrivial antichain. On the one hand, according to Lemma 5.30, for each $a \in \mathrm{A}$ and for all $b \in \mathfrak{b}_k^X(\mathrm{A})$, we have

$$|\mathfrak{I}(a) \cap \mathfrak{I}(b) \cap (P^{\mathrm{a}} - X)| > k\ .$$

On the other hand, by Definition 5.29 we have

$$\mathfrak{b}_k^X\big(\mathfrak{b}_k^X(\mathrm{A})\big) = \mathbf{min}\big\{g \in P : |\mathfrak{I}(g) \cap \mathfrak{I}(b) \cap (P^{\mathrm{a}} - X)| > k,\ \ \forall b \in \mathfrak{b}_k^X(\mathrm{A})\big\}\ . \tag{5.44}$$

Hence, eq. (5.43) follows. □

We conclude this section by applying standard techniques from the poset theory to the lattice $\mathfrak{A}(P)$ and to the (X, k)-blocker map.

Proposition 5.34. *Let P be a finite bounded poset.*

(i) *The composite map*

$$\mathfrak{b}_k^X \circ \mathfrak{b}_k^X : \mathfrak{A}(P) \to \mathfrak{A}(P)$$

is a closure operator *on the lattice of antichains* $\mathfrak{A}(P)$.

(ii) *The poset* $\mathscr{B}_k^X(P) := \{\mathfrak{b}_k^X(\mathrm{A}) : \mathrm{A} \in \mathfrak{A}(P)\}$ *is a* self-dual lattice*; the restriction map* $\mathfrak{b}_k^X|_{\mathscr{B}_k^X(P)}$ *is an* anti-automorphism of $\mathscr{B}_k^X(P)$. *The lattice* $\mathscr{B}_k^X(P)$ *is a* sub-meet-semilattice *of the lattice* $\mathfrak{A}(P)$.

(iii) *For any* $\mathrm{B} \in \mathscr{B}_k^X(P)$*, its preimage* $(\mathfrak{b}_k^X)^{-1}(\mathrm{B})$ *under the* (X, k)*-blocker map is a* convex sub-join-semilattice *of the lattice* $\mathfrak{A}(P)$. *The* greatest element *of* $(\mathfrak{b}_k^X)^{-1}(\mathrm{B})$ *is* $\mathfrak{b}_k^X(\mathrm{B})$.

Proof. In view of Lemma 5.32(ii) and Lemma 5.33, the assertions (i) and (ii) follow from the well-known results on closure operators.

To prove assertion (iii), pick arbitrary elements $\mathrm{A}', \mathrm{A}'' \in (\mathfrak{b}_k^X)^{-1}(\mathrm{B})$, where $\mathrm{B} = \mathfrak{b}_k^X(\mathrm{A})$ for some $\mathrm{A} \in \mathfrak{A}(P)$, and note that $\mathfrak{b}_k^X(\mathrm{A}' \vee \mathrm{A}'') = \mathfrak{b}_k^X(\mathrm{A}') \wedge \mathfrak{b}_k^X(\mathrm{A}'') = \mathrm{B}$. If $\mathrm{B} = \hat{0}_{\mathfrak{A}(P)}$, then $\mathfrak{b}_k^X(\mathrm{B}) = \hat{1}_{\mathfrak{A}(P)}$ is the greatest element of $(\mathfrak{b}_k^X)^{-1}(\mathrm{B})$. If $\mathrm{B} = \hat{1}_{\mathfrak{A}(P)}$, then $(\mathfrak{b}_k^X)^{-1}(\mathrm{B})$ is the one-element subposet $\{\hat{0}_{\mathfrak{A}(P)}\}$ of $\mathfrak{A}(P)$. Finally, if B is a nontrivial antichain of P, then by eq. (5.44) the element $\mathfrak{b}_k^X(\mathrm{B}) = \mathfrak{b}_k^X(\mathfrak{b}_k^X(\mathrm{A}))$ is the greatest element of $(\mathfrak{b}_k^X)^{-1}(\mathrm{B})$. Since the (X, k)-blocker map is order-reversing, we see that the subposet $(\mathfrak{b}_k^X)^{-1}(\mathrm{B})$ of $\mathfrak{A}(P)$ is convex. □

We now extend Definition 5.16.

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Definition 5.35. Given a finite bounded poset P, the poset

$$\mathscr{B}_k^X(P) := \mathfrak{b}_k^X\big(\mathfrak{A}(P)\big) = \big\{\mathfrak{b}_k^X(\mathrm{A}) : \mathrm{A} \in \mathfrak{A}(P)\big\}$$

is the *lattice of* (X,k)*-blockers in* P.

The poset $\mathscr{B}(P) := \mathscr{B}_0^{\emptyset^{\mathrm{a}}}(P)$ is the *lattice of blockers in* P.

5.9 (X,k)-Deletion and (X,k)-Contraction

In this section we consider further generalizations of the *set-theoretic deletion* operator and the *contraction* operator on *clutters* that extend their poset-theoretic analogues given in Section 5.5.

Again let $X \subseteq P^{\mathrm{a}}$ be a subset of the set of atoms of a finite bounded poset P, and k a nonnegative integer, $k \le |P^{\mathrm{a}}|$.

Definition 5.36.

(i) If $\{a\} \in \mathfrak{A}(P) - \{\hat{0}_{\mathfrak{A}(P)}, \hat{1}_{\mathfrak{A}(P)}\}$ is a nontrivial one-element antichain of a finite bounded poset P, then the (X,k)-*deletion* $\{a\}\setminus_k X$ and the (X,k)-*contraction* $\{a\}/\!\!/_k X$ are the antichains

$$\{a\}\setminus_k X := \begin{cases} \{a\}\,, & \text{if } |\mathfrak{b}(a) \cap X| \le k\,,\\ \hat{0}_{\mathfrak{A}(P)}\,, & \text{if } |\mathfrak{b}(a) \cap X| > k\,, \end{cases} \tag{5.45}$$

$$\{a\}/\!\!/_k X := \begin{cases} \{a\}\,, & \text{if } |\mathfrak{b}(a) \cap X| \le k\,,\\ \mathfrak{b}_k^X(\mathfrak{b}_k^X(a))\,, & \text{if } |\mathfrak{b}(a) \cap X| > k \text{ and } \mathfrak{b}(a) \not\subseteq X\,,\\ \hat{1}_{\mathfrak{A}(P)}\,, & \text{if } |\mathfrak{b}(a) \cap X| > k \text{ and } \mathfrak{b}(a) \subseteq X\,. \end{cases} \tag{5.46}$$

(ii) If $\mathrm{A} \in \mathfrak{A}(P) - \{\hat{0}_{\mathfrak{A}(P)}, \hat{1}_{\mathfrak{A}(P)}\}$ is a nontrivial antichain of P, then the (X,k)-*deletion* $\mathrm{A}\setminus_k X$ and the (X,k)-*contraction* $\mathrm{A}/\!\!/_k X$ are the antichains

$$\mathrm{A}\setminus_k X := \bigvee_{a\in\mathrm{A}}\big(\{a\}\setminus_k X\big)\,, \quad \mathrm{A}/\!\!/_k X := \bigvee_{a\in\mathrm{A}}\big(\{a\}/\!\!/_k X\big)\,. \tag{5.47}$$

(iii) The (X,k)-*deletion* and the (X,k)-*contraction* of the trivial antichains $\hat{0}_{\mathfrak{A}(P)}$ and $\hat{1}_{\mathfrak{A}(P)}$ of P are the trivial antichains

$$\begin{aligned} \hat{0}_{\mathfrak{A}(P)}\setminus_k X = \hat{0}_{\mathfrak{A}(P)}/\!\!/_k X := \hat{0}_{\mathfrak{A}(P)}\,,\\ \hat{1}_{\mathfrak{A}(P)}\setminus_k X = \hat{1}_{\mathfrak{A}(P)}/\!\!/_k X := \hat{1}_{\mathfrak{A}(P)}\,. \end{aligned} \tag{5.48}$$

(iv) The map

$$(\setminus_k X) : \mathfrak{A}(P) \to \mathfrak{A}(P)\,, \quad \mathrm{A} \mapsto \mathrm{A}\setminus_k X\,,$$

is the (X,k)-*deletion operator on* $\mathfrak{A}(P)$.

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The map

$$(/\!\!/_k X) : \mathfrak{A}(P) \to \mathfrak{A}(P) , \quad \mathrm{A} \mapsto \mathrm{A}/\!\!/_k X ,$$

is the (X, k)-*contraction operator on* $\mathfrak{A}(P)$.

Given an antichain $\mathrm{A} \in \mathfrak{A}(P)$, we use the notation $\mathrm{A}\backslash X$ and A/X instead of the notation $\mathrm{A}\backslash_0 X$ and $\mathrm{A}/_0 X$, respectively. The $(X, 0)$-deletion operator $(\backslash X) : \mathfrak{A}(P) \to \mathfrak{A}(P)$ and the $(X, 0)$-contraction operator $(/X) : \mathfrak{A}(P) \to \mathfrak{A}(P)$ are the *deletion operator* and the *contraction operator* on $\mathfrak{A}(P)$, respectively, considered in Section 5.5.

The following remark is an immediate consequence of Definition 5.36: If $a', a'' \in P$ and $\{a'\} \preceq \{a''\}$, then

$$\{a'\}\backslash_k X \preceq \{a''\}\backslash_k X \text{ and } \{a'\}/\!\!/_k X \preceq \{a''\}/\!\!/_k X ;$$

hence, in view of (5.47) and (5.48), we obtain the following result:

Lemma 5.37. *If* $\mathrm{A}', \mathrm{A}'' \in \mathfrak{A}(P)$ *and* $\mathrm{A}' \preceq \mathrm{A}''$, *then we have*

$$\mathrm{A}'\backslash_k X \preceq \mathrm{A}''\backslash_k X \quad \textit{and} \quad \mathrm{A}'/\!\!/_k X \preceq \mathrm{A}''/\!\!/_k X .$$

Moreover, if $\{a\}$ is a one-element antichain of P, then we have

$$\{a\}\backslash_k X \preceq \{a\} \preceq \{a\}/\!\!/_k X ,$$

and a more general statement is valid:

Lemma 5.38. *If* $\mathrm{A} \in \mathfrak{A}(P)$, *then we have*

$$\mathrm{A}\backslash_k X \preceq \mathrm{A} \preceq \mathrm{A}/\!\!/_k X .$$

Another direct consequence of Definition 5.36 is that for a one-element antichain $\{a\}$ of P we have

$$\mathfrak{b}_k^X(a)\backslash_k X \preceq \mathfrak{b}_k^X\big(\{a\}/\!\!/_k X\big) \preceq \mathfrak{b}_k^X(a) \preceq \mathfrak{b}_k^X(a)/\!\!/_k X \preceq \mathfrak{b}_k^X\big(\{a\}\backslash_k X\big) . \tag{5.49}$$

Let $\{a\}$ be a nontrivial one-element antichain of P. We clearly have

$$\big(\{a\}\backslash_k X\big)\backslash_k X = \{a\}\backslash_k X .$$

Let us now show that $(\{a\}/\!\!/_k X)/\!\!/_k X = \{a\}/\!\!/_k X$. If $|\mathfrak{b}(a) \cap X| \le k$, then Definition 5.36 implies $(\{a\}/\!\!/_k X)/\!\!/_k X = \{a\}/\!\!/_k X = \{a\}$; further, if $|\mathfrak{b}(a) \cap X| > k$ and $\mathfrak{b}(a) \subseteq X$, then Definition 5.36 implies $(\{a\}/\!\!/_k X)/\!\!/_k X = \{a\}/\!\!/_k X = \hat{1}_{\mathfrak{A}(P)}$. Suppose that $|\mathfrak{b}(a) \cap X| > k$ and $\mathfrak{b}(a) \not\subseteq X$. In this

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case, on the one hand, by Lemma 5.38 we have $(\{a\}/\!_k X)/\!_k X \succeq \{a\}/\!_k X$. On the other hand, for any element $b \in \{a\}/\!_k X = \mathfrak{b}_k^X(\mathfrak{b}_k^X(a))$, we have $\mathfrak{b}_k^X(b) \succeq \mathfrak{b}_k^X(a)$ and, as a consequence, we have $(\{a\}/\!_k X)/\!_k X = \bigvee_{b\in\{a\}/\!_k X}(\{b\}/\!_k X) \preceq \mathfrak{b}_k^X(\mathfrak{b}_k^X(a)) = \{a\}/\!_k X$. We conclude that indeed

$$\big(\{a\}/\!_k X\big)/\!_k X = \{a\}/\!_k X \ .$$

Looking at eq. (5.47), we come to the following result:

Lemma 5.39. *Given an antichain* $\mathrm{A} \in \mathfrak{A}(P)$, *we have*

$$(\mathrm{A}\backslash_k X)\backslash_k X = \mathrm{A}\backslash_k X \ \textit{ and } \ (\mathrm{A}/\!_k X)/\!_k X = \mathrm{A}/\!_k X \ .$$

Lemmas 5.37, 5.38 and 5.39 lead to a characterization of the (X, k)-deletion map and of the (X, k)-contraction map in terms of (co)closure operators:

Proposition 5.40. *The map* $(\backslash_k X) : \mathfrak{A}(P) \to \mathfrak{A}(P)$ *is a* coclosure operator on $\mathfrak{A}(P)$.

The map $(/\!_k X) : \mathfrak{A}(P) \to \mathfrak{A}(P)$ *is a* closure operator on $\mathfrak{A}(P)$.

The following statement is an analogue of Lemma 5.32(i):

Proposition 5.41. *Let P be a finite bounded poset. Let $Y \subseteq P^{\mathrm{a}}$, $Y \supseteq X$, and let m be an integer, $k \le m < |P^{\mathrm{a}}|$. Given an antichain* $\mathrm{A} \in \mathfrak{A}(P)$, *we have*

$$\mathrm{A}\backslash_m X \succeq \mathrm{A}\backslash_k X \succeq \mathrm{A}\backslash_k Y$$

and

$$\mathrm{A}/\!_m X \preceq \mathrm{A}/\!_k X \preceq \mathrm{A}/\!_k Y \ .$$

Proof. If the antichain A is *trivial*, then the proposition follows from eq. (5.48).

Suppose that A is a *nontrivial* antichain of P. For each $a \in \mathrm{A}$, eq. (5.45) implies $\{a\}\backslash_k X \succeq \{a\}\backslash_k Y$; eq. (5.46) implies $\{a\}/\!_k X \preceq \{a\}/\!_k Y$, and eq. (5.47) yields

$$\mathrm{A}\backslash_k X = \bigvee_{a\in\mathrm{A}}\big(\{a\}\backslash_k X\big) \succeq \bigvee_{a\in\mathrm{A}}\big(\{a\}\backslash_k Y\big) = \mathrm{A}\backslash_k Y \ ,$$
$$\mathrm{A}/\!_k X = \bigvee_{a\in\mathrm{A}}\big(\{a\}/\!_k X\big) \preceq \bigvee_{a\in\mathrm{A}}\big(\{a\}/\!_k Y\big) = \mathrm{A}/\!_k Y \ .$$

The remaining relations are proved analogously. □

We denote the images $(\backslash_k X)(\mathfrak{A}(P)) := \{\mathrm{A}\backslash_k X : \mathrm{A} \in \mathfrak{A}(P)\}$ and $(/\!_k X)(\mathfrak{A}(P)) := \{\mathrm{A}/\!_k X : \mathrm{A} \in \mathfrak{A}(P)\}$ by $\mathfrak{A}(P)\backslash_k X$ and $\mathfrak{A}(P)/\!_k X$, respectively. In analogy to the list of properties of the maps $(\backslash X)$ and $(/X)$ given on page 101, we can interpret some well-known properties

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of (semi)lattice maps and (co)closure operators on lattices in the case of the (X,k)-deletion map and of the (X,k)-contraction map:

Definition 5.36 implies that the maps $(\backslash_k X), (/\!\!_k X) : \mathfrak{A}(P) \to \mathfrak{A}(P)$ are *upper* $\{\hat{0}_{\mathfrak{A}(P)}, \hat{1}_{\mathfrak{A}(P)}\}$*-homomorphisms*, that is, for any antichains $\mathrm{A}', \mathrm{A}'' \in \mathfrak{A}(P)$ we have $(\mathrm{A}' \vee \mathrm{A}'')\backslash_k X = (\mathrm{A}'\backslash_k X) \vee (\mathrm{A}''\backslash_k X)$, $(\mathrm{A}' \vee \mathrm{A}'')/\!\!_k X = (\mathrm{A}'/\!\!_k X) \vee (\mathrm{A}''/\!\!_k X)$, and we have $\hat{0}_{\mathfrak{A}(P)}\backslash_k X = \hat{0}_{\mathfrak{A}(P)}/\!\!_k X = \hat{0}_{\mathfrak{A}(P)}$ and $\hat{1}_{\mathfrak{A}(P)}\backslash_k X = \hat{1}_{\mathfrak{A}(P)}/\!\!_k X = \hat{1}_{\mathfrak{A}(P)}$.

The posets $\mathfrak{A}(P)\backslash_k X$ and $\mathfrak{A}(P)/\!\!_k X$, with the partial orders induced by the partial order on $\mathfrak{A}(P)$, are *lattices*. Let us extend Definition 5.23:

Definition 5.42. Given a finite bounded poset P, the poset

$$\mathfrak{A}(P)\backslash_k X := \left\{\mathrm{A}\backslash_k X : \mathrm{A} \in \mathfrak{A}(P)\right\}$$

is the *lattice of* (X,k)*-deletions in* P.

The poset $\mathfrak{A}(P)\backslash X := \mathfrak{A}(P)\backslash_0 X$ is the *lattice of deletions in* P.

The poset

$$\mathfrak{A}(P)/\!\!_k X := \left\{\mathrm{A}/\!\!_k X : \mathrm{A} \in \mathfrak{A}(P)\right\}$$

is the *lattice of* (X,k)*-contractions in* P.

The poset $\mathfrak{A}(P)/X := \mathfrak{A}(P)/_0 X$ is the *lattice of contractions in* P.

The lattice $\mathfrak{A}(P)\backslash_k X$ is a *sub-join-semilattice of* $\mathfrak{A}(P)$. Denote by $\wedge_{\mathfrak{A}(P)\backslash_k X}$ the operation of *meet* in $\mathfrak{A}(P)\backslash_k X$. If $\mathrm{D}', \mathrm{D}'' \in \mathfrak{A}(P)\backslash_k X$, then we have $\mathrm{D}' \wedge_{\mathfrak{A}(P)\backslash_k X} \mathrm{D}'' = (\mathrm{D}' \wedge \mathrm{D}'')\backslash_k X$.

The lattice $\mathfrak{A}(P)/\!\!_k X$ is a *sublattice of* $\mathfrak{A}(P)$.

If $\mathrm{D} \in \mathfrak{A}(P)\backslash_k X$, then the *preimage* $(\backslash_k X)^{-1}(\mathrm{D})$ of D under the (X,k)-deletion map is the *closed interval* $[\mathrm{D}, \mathrm{D} \vee \bigvee_{E \subseteq X : |E|=k+1} \mathfrak{b}(E)]$ *of* $\mathfrak{A}(P)$.

If $\mathrm{D} \in \mathfrak{A}(P)/\!\!_k X$, then the *preimage* $(/\!\!_k X)^{-1}(\mathrm{D})$ of D under the (X,k)-contraction map is a *convex sub-join-semilattice of the lattice* $\mathfrak{A}(P)$, with the *greatest element* D.

Relations (5.3) and (5.19) can be generalized in the following way:

Theorem 5.43. *If* $\mathrm{A} \in \mathfrak{A}(P)$*, then*

$$\mathfrak{b}_k^X(\mathrm{A})\backslash_k X \preceq \mathfrak{b}_k^X(\mathrm{A}/\!\!_k X) \preceq \mathfrak{b}_k^X(\mathrm{A}) \preceq \mathfrak{b}_k^X(\mathrm{A})/\!\!_k X \preceq \mathfrak{b}_k^X(\mathrm{A}\backslash_k X)\ .$$

Proof. If the antichain A is *trivial*, then there is nothing to prove.

Suppose that A is a *nontrivial* antichain of P. The relations

$$\mathfrak{b}_k^X(\mathrm{A})\backslash_k X \preceq \mathfrak{b}_k^X(\mathrm{A}) \preceq \mathfrak{b}_k^X(\mathrm{A})/\!\!_k X \quad \text{and} \quad \mathfrak{b}_k^X(\mathrm{A}/\!\!_k X) \preceq \mathfrak{b}_k^X(\mathrm{A}) \preceq \mathfrak{b}_k^X(\mathrm{A}\backslash_k X)$$

follow from Lemma 5.38 and Lemma 5.32(ii).

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We need the following auxiliary relations: if A' and A'' are arbitrary antichains of P, then

$$(\mathrm{A}' \wedge \mathrm{A}'')\setminus_k X \preceq (\mathrm{A}'\setminus_k X) \wedge (\mathrm{A}''\setminus_k X)\ , \tag{5.50}$$

$$(\mathrm{A}' \wedge \mathrm{A}'')/_k X \preceq (\mathrm{A}'/_k X) \wedge (\mathrm{A}''/_k X)\ . \tag{5.51}$$

To verify the relation $\mathfrak{b}_k^X(\mathrm{A})\setminus_k X \ \preceq\ \mathfrak{b}_k^X(\mathrm{A}/_k X)$, we use eqs. (5.50) and (5.49), and we see that

$$\begin{aligned}\mathfrak{b}_k^X(\mathrm{A})\setminus_k X &= \Big(\bigwedge_{a\in\mathrm{A}} \mathfrak{b}_k^X(a)\Big)\setminus_k X \preceq \bigwedge_{a\in\mathrm{A}}\big(\mathfrak{b}_k^X(a)\setminus_k X\big)\\ &\preceq \bigwedge_{a\in\mathrm{A}} \mathfrak{b}_k^X\big(\{a\}/_k X\big) = \mathfrak{b}_k^X\Big(\bigvee_{a\in\mathrm{A}}\big(\{a\}/_k X\big)\Big) = \mathfrak{b}_k^X(\mathrm{A}/_k X)\ .\end{aligned}$$

To verify the relation $\mathfrak{b}_k^X(\mathrm{A})/_k X \ \preceq\ \mathfrak{b}_k^X(\mathrm{A}\setminus_k X)$, we use eqs. (5.51) and (5.49), and we see that

$$\begin{aligned}\mathfrak{b}_k^X(\mathrm{A})/_k X &= \Big(\bigwedge_{a\in\mathrm{A}} \mathfrak{b}_k^X(a)\Big)/_k X \preceq \bigwedge_{a\in\mathrm{A}}\big(\mathfrak{b}_k^X(a)/_k X\big)\\ &\preceq \bigwedge_{a\in\mathrm{A}} \mathfrak{b}_k^X\big(\{a\}\setminus_k X\big) = \mathfrak{b}_k^X\Big(\bigvee_{a\in\mathrm{A}}\big(\{a\}\setminus_k X\big)\Big) = \mathfrak{b}_k^X(\mathrm{A}\setminus_k X)\ .\end{aligned}$$

□

Notes

The *blocking sets* (or *systems of representatives*, *transversals*) of *set families*, or the *vertex covers* of *(hyper)graphs*, defined by eq. (5.1) play an important role in combinatorial optimization and in other areas, see for example [8, 46], [49, Ch. 2], [52, Ch. 1], [53, Ch. 4], [54, 80, 85, 86], [92, Ch. 8], [97, 101], [108, Ch. 9], [113, Ch. 13], [150, Ch. 10], [151, Sect. III.1], [171, Ch. 77], [181, Ch. 6].

The fundamental result (5.2) of Proposition 1.34 appears in Refs. [68, 121, 122].

The fundamental result given in Proposition 5.1 appears in Ref. [174]; see also for example [52, Ch. 1], [53, Application 4.3].

The concepts of the *blocker map* and of the *complementary map* on *clutters* made it possible to clarify a connection between families of sets appearing in matroid theory with maps on them, see Ref. [51].

The *blockers* of *antichains* in posets are considered, for example, in the works [26, 28].

Strong blocker duality is discussed in the work [26], where the (*self-dual*) posets P with the property

$$\mathfrak{b}\big(\mathfrak{b}(\mathrm{A})\big) = \mathrm{A}\ ,$$

for *all* nontrivial antichains $\mathrm{A} \subset P$, are characterized.

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Definition 5.44. (see [26, Definition 2.6]). Let V be a finite set. A subposet of the Boolean lattice $\mathbb{B}(V)$ of subsets of the set V, induced by a family $\mathscr{S} \subseteq \mathbf{2}^V$, is called *well complemented* if (i) the empty set and all singletons belong to $\mathscr{S}$, and (ii) $\mathscr{S}$ is closed under taking complements in V.

Theorem 5.45. (see [26, Theorem 2.7]). *Let P be a finite bounded poset. Then the following are equivalent:*

(i) $\mathfrak{b}(\mathfrak{b}(\mathrm{A})) = \mathrm{A}$ *for* all *nontrivial antichains* A *in P.*
(ii) *P is isomorphic to a* well-complemented *subposet of a Boolean lattice.*
(iii) *P satisfies*
  (a) *if $\mathfrak{I}(x) \cap P^{\mathrm{a}} \subseteq \mathfrak{I}(y) \cap P^{\mathrm{a}}$, then $x \preceq y$, for all $x, y \in P$;*
  (b) *for all $x \in P$, there exists $y \in P$ such that $P^{\mathrm{a}} - (\mathfrak{I}(x) \cap P^{\mathrm{a}}) = \mathfrak{I}(y) \cap P^{\mathrm{a}}$.*

Proposition 5.46. (see [26, Cor. 2.9]). *Let L be a finite lattice. Then the following are equivalent:*

(i) $\mathfrak{b}(\mathfrak{b}(\mathrm{A})) = \mathrm{A}$ *for* all *nontrivial antichains* A *in L.*
(ii) *L is Boolean.*

In Sections 5.1–5.4 we follow the article in Ref. [139], where the *blocking elements* of subposets were called the *intersecters*, and the *complementing elements* were called the *complementers*.

Closure operators and *coclosure operators* on *posets* are discussed, for example, in [6, Ch. IV],[114, Sect. 13.2].

In Section 5.5 we follow the article [134].

The statement of Theorem 5.27 is mentioned in Ref. [140, Rem. 3.2].

In Sections 5.8 and 5.9 we follow the article [136].

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6 Committees of Set Families, and Relative Blocking Constructions in Posets

Let r be a rational number, $0 \le r < 1$. Given a nonempty *family* $\mathscr{A} := \{A_1, \dots, A_\alpha\}$ of nonempty and pairwise distinct *subsets* of its finite *ground set* $\mathrm{V}(\mathscr{A}) := \bigcup_{i=1}^{\alpha} A_i$, an *r-committee* of $\mathscr{A}$ is defined to be a subset $B \subseteq S$ of a set $S \supseteq \mathrm{V}(\mathscr{A})$ such that

$$|B \cap A_i| > r \cdot |B| \;, \tag{6.1}$$

for each $i \in [\alpha]$. Thus, a 0-*committee* of $\mathscr{A}$ is nothing else than a *blocking set* of $\mathscr{A}$.

In this chapter we consider *poset-theoretic* analogues of the above-defined *set-theoretic* r-committees.

These analogues called *relatively r-blocking elements* of subposets of finite bounded posets are introduced in Section 6.1, where their basic properties are discussed.

In Section 6.2 we give further generalizations of *absolute blocking constructions* in posets that were investigated in Chapter 5.

A connection between absolute and relative blocking constructions in the same finite bounded poset is mentioned in Section 6.3.

In Section 6.4 we describe the structure of the subposets of relatively blocking elements in general finite bounded posets, and we make attempts to count the relative blocking elements of a given rank in graded posets, in the principal order ideals of binomial posets and, in particular, in the Boolean lattices.

It is worth noting once again that the *enumeration of the relatively r-blocking elements*, with respect to the rank function, in the Boolean lattices is equivalent, in the case $r := 0$, to the *enumeration of the blocking sets of set families*.

In Section 6.5 we associate with the elements of finite bounded posets the corresponding *Farey subsequences*. One such important sequence $\mathscr{F}(\mathbb{B}(n), m)$ was earlier discussed in Chapter 4 in the context of the Boolean lattices.

An application of Farey subsequences to the existence problem on relatively blocking elements is given in Section 6.6.

In Section 6.7 we discuss several ways to count the *relatively blocking elements* of a given rank in the *Boolean lattices*.

In Section 6.8 we are concerned with counting the *relatively blocking elements* b with the property $b \wedge -b = \hat{0}$ in the *Boolean lattices* of subsets of the sets $\pm[m] := \{-m, \dots, -1, 1, \dots, m\}$. This enumerative problem is equivalent to the counting of the *committees*, with *no pairs of opposites* $\{-i, i\}$, of *clutters* on the ground sets $S \subseteq \pm[m]$.

Section 6.9 is devoted to the similar problem on the enumeration of *relatively blocking elements* in the posets isomorphic to the *face lattices of crosspolytopes*; recall that the face meet-semilattice of the boundary of an m-dimensional crosspolytope

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is defined to be the family of the subsets, ordered by inclusion, of the set $\pm[m]$ that contain *no pairs of opposites* $\{-i, i\}$.

Further enumerative results on the *relatively blocking elements* in the principal order ideals of *binomial posets* are given in Section 6.10.

6.1 Relatively Blocking Elements of Antichains

Throughout the chapter, we denote by $\mathfrak{A}_\triangle(P)$ and $\mathfrak{A}_\triangledown(P) =: \mathfrak{A}(P)$ *distributive lattices* of all *antichains* of a finite bounded poset P, with $|P| > 1$, defined as follows. If A' and A'' are antichains of P, then we set $\mathrm{A}' \preceq \mathrm{A}''$ in $\mathfrak{A}_\triangle(P)$ if and only if $\mathfrak{I}(\mathrm{A}') \subseteq \mathfrak{I}(\mathrm{A}'')$, and we set $\mathrm{A}' \preceq \mathrm{A}''$ in $\mathfrak{A}_\triangledown(P) =: \mathfrak{A}(P)$ as earlier if and only if $\mathfrak{F}(\mathrm{A}') \subseteq \mathfrak{F}(\mathrm{A}'')$. We use the notation $\hat{0}_{\mathfrak{A}_\triangle(P)}$ and $\hat{0}_{\mathfrak{A}_\triangledown(P)}$ to denote the *least* elements of the lattices $\mathfrak{A}_\triangle(P)$ and $\mathfrak{A}_\triangledown(P)$, respectively. We use the similar notation $\hat{1}_{\mathfrak{A}_\triangle(P)}$ and $\hat{1}_{\mathfrak{A}_\triangledown(P)}$ for the *greatest* elements. The operations of *meet* in $\mathfrak{A}_\triangle(P)$ and $\mathfrak{A}_\triangledown(P)$ are denoted by $\wedge_\triangle$ and $\wedge_\triangledown$, respectively; similarly, $\vee_\triangle$ and $\vee_\triangledown$ denote the operations of *join*. If A' and A'' are antichains of P, then we have $\mathrm{A}' \wedge_\triangle \mathrm{A}'' = \mathbf{max}(\mathfrak{I}(\mathrm{A}') \cap \mathfrak{I}(\mathrm{A}''))$, $\mathrm{A}' \vee_\triangle \mathrm{A}'' = \mathbf{max}(\mathrm{A}' \cup \mathrm{A}'')$ and, dually, $\mathrm{A}' \wedge_\triangledown \mathrm{A}'' = \mathbf{min}(\mathfrak{F}(\mathrm{A}') \cap \mathfrak{F}(\mathrm{A}''))$, $\mathrm{A}' \vee_\triangledown \mathrm{A}'' = \mathbf{min}(\mathrm{A}' \cup \mathrm{A}'')$.

Recall that the *least* and *greatest* elements of the lattice $\mathfrak{A}_\triangledown(P)$ are called the *trivial antichains* of P: $\hat{0}_{\mathfrak{A}_\triangledown(P)}$ is the empty subset of P, and $\hat{1}_{\mathfrak{A}_\triangledown(P)}$ is the one-element antichain $\{\hat{0}_P\}$.

The *Möbius function* $\mu_P : P \times P \to \mathbb{Z}$ of the poset P is defined as follows: $\mu_P(x, x) := 1$, for any $x \in P$; further, if $z \in P$ and $x \prec z$ in P, then

$$\mu_P(x, z) := -\sum_{y \in P \,:\, x \preceq y \prec z} \mu_P(x, y);$$

finally, if $x \npreceq z$ in P, then $\mu_P(x, z) := 0$.

We denote by $\mathbb{V}_q(n)$ the *lattice* of all *subspaces* of an n-dimensional *vector space* over a *finite field* of q elements; $\binom{j}{i}_q$ is the *q-binomial coefficient*.

In this section we consider poset-theoretic generalizations of set-theoretic *committees* of *set families*.

Let

$$\omega : \mathfrak{A}_\triangle(P) \to \{-1\} \cup \mathbb{N} \tag{6.2}$$

be a map such that

$$\hat{0}_{\mathfrak{A}_\triangle(P)} \mapsto -1 \,, \quad \{\hat{0}_P\} \mapsto 0 \,, \tag{6.3}$$

and for antichains A' and A'' of P such that $\{\hat{0}_P\} \prec \mathrm{A}' \preceq \mathrm{A}''$ in $\mathfrak{A}_\triangle(P)$, we have

$$0 < \omega(\mathrm{A}') \leq \omega(\mathrm{A}'') \,. \tag{6.4}$$

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Let us mention just three mappings that satisfy conditions (6.3) and (6.4):

$$\omega_{\mathfrak{I}} : \mathrm{A} \mapsto \rho_{\triangle}(\mathrm{A}) - 1 = |\mathfrak{I}(\mathrm{A})| - 1 ,$$

where $\rho_{\triangle}(\mathrm{A})$ denotes the *poset rank* of an element A in the lattice $\mathfrak{A}_{\triangle}(P)$;

$$\omega_{\mathrm{a}} : \mathrm{A} \mapsto \rho_{\triangle}(\mathrm{A} \wedge_{\triangle} P^{\mathrm{a}}) - 1 = \begin{cases} -1 , & \text{if } \mathrm{A} = \hat{0}_{\mathfrak{A}_{\triangle}(P)} , \\ |\mathfrak{I}(\mathrm{A}) \cap P^{\mathrm{a}}| , & \text{if } \mathrm{A} \neq \hat{0}_{\mathfrak{A}_{\triangle}(P)} ; \end{cases} \tag{6.5}$$

$$\omega_{\rho} : \mathrm{A} \mapsto \begin{cases} -1 , & \text{if } \mathrm{A} = \hat{0}_{\mathfrak{A}_{\triangle}(P)} , \\ \max_{a \in \mathrm{A}} \rho(a) , & \text{if } \mathrm{A} \neq \hat{0}_{\mathfrak{A}_{\triangle}(P)} , \end{cases} \tag{6.6}$$

if the poset P is *graded*, with the *rank function* ρ.

If $\{a\}$ is a one-element antichain of P, then we write $\omega(a)$ instead of $\omega(\{a\})$, and we write $\omega(P)$ instead of $\omega(\hat{1}_P) = \omega(\hat{1}_{\mathfrak{A}_{\triangle}(P)})$.

Definition 6.1. Let A be a subset of a finite bounded poset P.

(i) If $\mathrm{A} \neq \emptyset$ and $\mathrm{A} \neq \{\hat{0}_P\}$, then an element $b \in P - \{\hat{0}_P\}$ is a *relatively r-blocking element* of A in P (with respect to a map ω), if for every $a \in \mathrm{A} - \{\hat{0}_P\}$, we have

$$\frac{\omega(\{b\} \wedge_{\triangle} \{a\})}{\omega(b)} > r . \tag{6.7}$$

(ii) If $\mathrm{A} = \{\hat{0}_P\}$, then A has *no relatively r-blocking elements* in P.

(iii) If $\mathrm{A} = \emptyset$, then every element of P is a *relatively r-blocking element of* A in P.

Remark 6.2. If the poset P is a *meet-semilattice*, then of course eq. (6.7) is equivalent to

$$\frac{\omega(b \wedge_P a)}{\omega(b)} > r .$$

If in addition the *meet-semilattice* P is *graded*, then the condition $\frac{\omega_\rho(b \wedge_P a)}{\omega_\rho(b)} > r$ is simply the condition $\frac{\rho(b \wedge_P a)}{\rho(b)} > r$.

Remark 6.3. Let A be a nonempty subset of $\mathbb{B}(n) - \{\hat{0}_{\mathbb{B}(n)}\}$. An element $b \in \mathbb{B}(n) - \{\hat{0}_{\mathbb{B}(n)}\}$ is a *relatively r-blocking element of* A *in* $\mathbb{B}(n)$, with respect to either of the maps ω defined by eqs. (6.5) and (6.6), if and only if the set $\mathfrak{I}(b) \cap \mathbb{B}(n)^{(1)}$ is an *r-committee of the family* $\{\mathfrak{I}(a) \cap \mathbb{B}(n)^{(1)} : a \in \mathrm{A}\}$, that is,

$$|\mathfrak{I}(b) \cap \mathfrak{I}(a) \cap \mathbb{B}(n)^{(1)}| > r \cdot |\mathfrak{I}(b) \cap \mathbb{B}(n)^{(1)}| ,$$

for all $a \in \mathrm{A}$.

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We denote the subposet of P consisting of all relatively r-blocking elements of A, with respect to a map ω, by $\mathbf{I}_r(P, \mathrm{A}; \omega)$. Given an element $a \in P$, we write $\mathbf{I}_r(P, a; \omega)$ instead of $\mathbf{I}_r(P, \{a\}; \omega)$. If $k \in [\omega(P)]$, then we denote by $\mathbf{I}_{r,k}(P, \mathrm{A}; \omega)$ the subposet $\{b \in \mathbf{I}_r(P, \mathrm{A}; \omega) : \omega(b) = k\}$.

If A is a nonempty subset of $P - \{\hat{0}_P\}$, then Definition 6.1 implies $\mathbf{I}_r(P, \mathrm{A}; \omega) = \mathbf{I}_r(P, \mathbf{min}\ \mathrm{A}; \omega)$; for this reason we concern ourselves primarily with the relatively r-blocking elements of *antichains*.

Let us list three observations:

Proposition 6.4.

(i) *If* A *is a nontrivial antichain of P, then it holds*

$$\mathbf{I}_r(P, \mathrm{A}; \omega) = \bigcap_{a \in \mathrm{A}} \mathbf{I}_r(P, a; \omega) ,$$

for any map ω.

(ii) *If* A' *and* A'' *are antichains of P, and* $\mathrm{A}' \preceq \mathrm{A}''$ *in* $\mathfrak{A}_{\triangledown}(P)$*, then*

$$\mathbf{I}_r(P, \mathrm{A}'; \omega) \supseteq \mathbf{I}_r(P, \mathrm{A}''; \omega) ,$$

for any map ω.

(iii) *Let* $r', r'' \in \mathbb{Q}$, $0 \le r' \le r'' < 1$*. For any antichain* A *of P, and for any map ω, we have*

$$\mathbf{I}_{r'}(P, \mathrm{A}; \omega) \supseteq \mathbf{I}_{r''}(P, \mathrm{A}; \omega) .$$

The *minimal elements* of the subposets $\mathbf{I}_r(P, \mathrm{A}; \omega)$ are the poset-theoretic analogues of the set-theoretic *inclusion-minimal r-committees* of set families:

Definition 6.5.

(i) The *relative r-blocker map* on the lattice of antichains $\mathfrak{A}_{\triangledown}(P)$, *with respect to a map* ω, is the map $\mathfrak{y}_r : \mathfrak{A}_{\triangledown}(P) \to \mathfrak{A}_{\triangledown}(P)$ given by

$$\mathrm{A} \mapsto \mathbf{min}\,\mathbf{I}_r(P, \mathrm{A}; \omega) = \mathbf{min}\left\{ b \in P - \{\hat{0}_P\} : \frac{\omega(\{b\} \wedge_{\vartriangle} \{a\})}{\omega(b)} > r,\ \ \forall a \in \mathrm{A} \right\} ,$$

if the antichain A is nontrivial, and

$$\hat{0}_{\mathfrak{A}_{\triangledown}(P)} \mapsto \hat{1}_{\mathfrak{A}_{\triangledown}(P)} , \quad \hat{1}_{\mathfrak{A}_{\triangledown}(P)} \mapsto \hat{0}_{\mathfrak{A}_{\triangledown}(P)} .$$

(ii) Given an antichain A of P, the *relative r-blocker*, w.r.t. the map ω, of A in P is the antichain $\mathfrak{y}_r(\mathrm{A})$.

The *minimal relatively r-blocking elements*, w.r.t. the map ω, of A in P are the elements of the relative r-blocker $\mathfrak{y}_r(\mathrm{A})$.

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The relatively r-blocking elements b having the minimal possible values of $\omega(b)$ are the poset-theoretic analogues of the set-theoretic *r-committees* of set families, of *minimal size*.

The following statement is a consequence of Proposition 6.4(ii, iii). Note in particular that the *relative r-blocker map* $\mathfrak{y}_r : \mathfrak{A}_\triangledown(P) \to \mathfrak{A}_\triangledown(P)$ is *order-reversing*.

Corollary 6.6. *Let $r', r'' \in \mathbb{Q}$, $0 \le r' \le r'' < 1$. Given antichains A' and A'' of P such that $\mathrm{A}' \preceq \mathrm{A}''$ in $\mathfrak{A}_\triangledown(P)$, we have*

$$\mathfrak{y}_{r''}(\mathrm{A}'') \preceq \mathfrak{y}_{r'}(\mathrm{A}'') \preceq \mathfrak{y}_{r'}(\mathrm{A}') \ .$$

Let A be a nontrivial antichain of P. If the relative r-blocker $\mathfrak{y}_r(\mathrm{A})$ of A in P, with respect to a map ω, is not the trivial antichain $\hat{0}_{\mathfrak{A}_\triangledown(P)}$, then A is a subset of relatively r'-blocking elements for the antichain $\mathfrak{y}_r(\mathrm{A})$, for some rational number r'. Indeed, for each $a \in \mathrm{A}$ and for all $b \in \mathfrak{y}_r(\mathrm{A})$, by eq. (6.7) we have

$$\frac{\omega(\{a\} \wedge_\vartriangle \{b\})}{\omega(a)} > r \cdot \frac{\omega(b)}{\omega(a)} \ge r \cdot \frac{\min_{p \in \mathfrak{y}_r(\mathrm{A})} \omega(p)}{\max_{p \in \mathrm{A}} \omega(p)} \ ,$$

and we obtain the following result.

Proposition 6.7. *If A is a nontrivial antichain of P, and $\mathfrak{y}_r(\mathrm{A}) \neq \hat{0}_{\mathfrak{A}_\triangledown(P)}$, with respect to a map ω, then*

$$\mathrm{A} \subseteq \mathbf{I}_{r'}\Big(P, \mathfrak{y}_r(\mathrm{A}); \omega\Big) \ ,$$

where

$$r' := r \cdot \frac{\min_{p \in \mathfrak{y}_r(\mathrm{A})} \omega(p)}{\max_{p \in \mathrm{A}} \omega(p)} \ .$$

6.2 Absolutely Blocking Elements of Antichains, and Convex Subposets

In this section we discuss further poset-theoretic generalizations of the *blocking sets* and the *blockers* of *set families*.

Let A be a nontrivial antichain of a finite bounded poset P. Let h and k be positive integers such that $h \le k \le \omega(P)$, for some map ω. We will often turn to auxiliary subposets of the form

$$\Big\{b \in P : \omega(b) = k,\ \ \omega\Big(\{b\} \wedge_\vartriangle \{a\}\Big) \ge h,\ \ \forall a \in \mathrm{A}\Big\} \ , \tag{6.8}$$

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which are clearly the subposets

$$\Big(\big\{b \in P : \omega(b) > k-1\big\} - \big\{b \in P : \omega(b) > k\big\}\Big) \\ \cap \ \Big\{b \in P : \omega\big(\{b\} \wedge_{\triangle} \{a\}\big) > h-1,\ \ \forall a \in \mathrm{A}\Big\} . \quad (6.9)$$

Each component of the expression (6.9) can be described in terms of *absolute blocking*.

Definition 6.8.

(i) If $\mathrm{A} \in \mathfrak{A}_{\triangledown}(P) - \{\hat{0}_{\mathfrak{A}_{\triangledown}(P)}, \hat{1}_{\mathfrak{A}_{\triangledown}(P)}\}$ is a nontrivial antichain of P, and $j \in \mathbb{N}$, $j < \omega(P)$, then the *absolute j-blocker* $\mathbf{b}_j(\mathrm{A})$, with respect to a map ω, of A in P is the antichain

$$\mathbf{b}_j(\mathrm{A}) := \mathbf{min}\Big\{b \in P : \omega\big(\{b\} \wedge_{\triangle} \{a\}\big) > j,\ \ \forall a \in \mathrm{A}\Big\} , \quad (6.10)$$

that is, the meet

$$\mathbf{b}_j(\mathrm{A}) := \bigwedge_{a \in \mathrm{A}} \mathbf{b}_j(a) \quad (6.11)$$

in $\mathfrak{A}_{\triangledown}(P)$.

(ii) The *absolute j-blockers* $\mathbf{b}_j(\hat{0}_{\mathfrak{A}_{\triangledown}(P)})$ and $\mathbf{b}_j(\hat{1}_{\mathfrak{A}_{\triangledown}(P)})$ of the trivial antichains $\hat{0}_{\mathfrak{A}_{\triangledown}(P)}$ and $\hat{1}_{\mathfrak{A}_{\triangledown}(P)}$ of P are the trivial antichains

$$\mathbf{b}_j\big(\hat{0}_{\mathfrak{A}_{\triangledown}(P)}\big) := \hat{1}_{\mathfrak{A}_{\triangledown}(P)} , \quad \mathbf{b}_j\big(\hat{1}_{\mathfrak{A}_{\triangledown}(P)}\big) := \hat{0}_{\mathfrak{A}_{\triangledown}(P)} . \quad (6.12)$$

(iii) The *absolute j-blocker map,* w.r.t. *the map ω, on* $\mathfrak{A}_{\triangledown}(P)$ is the map $\mathbf{b}_j : \mathfrak{A}_{\triangledown}(P) \to \mathfrak{A}_{\triangledown}(P)$.

(iv) The absolutely j-blocking elements, w.r.t. *the map ω,* of an *antichain* A in P are the elements of the subposet

$$\Big\{b \in P : \omega\big(\{b\} \wedge_{\triangle} \{a\}\big) > j,\ \ \forall a \in \mathrm{A}\Big\} ,$$

which is the order filter $\mathfrak{F}_P(\mathbf{b}_j(\mathrm{A}))$.

In definition (6.11) we write $\mathbf{b}_j(a)$ instead of $\mathbf{b}_j(\{a\})$. We set

$$\mathbf{b}_{\omega(P)}(\mathrm{A}) := \hat{0}_{\mathfrak{A}_{\triangledown}(P)} .$$

Recall that in the language of Chapter 5 the elements of the order filter $\mathfrak{F}_P(\mathfrak{b}(\mathrm{A}))$, where the *blocker* $\mathfrak{b}(\mathrm{A}) := \mathfrak{b}_0(\mathrm{A}) := \mathfrak{b}_0^{\emptyset^{\mathrm{a}}}(\mathrm{A})$ is given by Definitions 5.5, 5.12, and 5.29, are just the *blocking elements of* A *in* P.

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If the poset P is *graded*, then for the map ω_ρ defined by eq. (6.6), and for any nontrivial one-element antichain $\{a\}$ of P, we have

$$\mathbf{b}_j(a) = \mathfrak{I}(a) \cap P^{(j+1)} .$$

The *absolute j-blocker map* $\mathbf{b}_j : \mathfrak{A}_\triangledown(P) \to \mathfrak{A}_\triangledown(P)$, with respect to any map ω, is *order-reversing*: for any elements $\mathrm{A}', \mathrm{A}'' \in \mathfrak{A}_\triangledown(P)$ with $\mathrm{A}' \preceq \mathrm{A}''$, we have

$$\mathbf{b}_j(\mathrm{A}') \succeq \mathbf{b}_j(\mathrm{A}'') .$$

Note also that for any antichain A of P, and for nonnegative integers i and j such that $i \le j < \omega(P)$, we have

$$\mathbf{b}_i(\mathrm{A}) \succeq \mathbf{b}_j(\mathrm{A}) \tag{6.13}$$

in $\mathfrak{A}_\triangledown(P)$.

If A is a *trivial* antichain of P, then the convention (6.12) implies $\mathbf{b}_j(\mathbf{b}_j(\mathrm{A})) = \mathrm{A}$.

Now let A be a *nontrivial* antichain of P. If $\mathbf{b}_j(\mathrm{A}) = \hat{0}_{\mathfrak{A}_\triangledown(P)}$, then we have $\mathbf{b}_j(\mathbf{b}_j(\mathrm{A})) = \hat{1}_{\mathfrak{A}_\triangledown(P)} \succ \mathrm{A}$ in $\mathfrak{A}_\triangledown(P)$. Finally, suppose that $\mathbf{b}_j(\mathrm{A})$ is a nontrivial antichain of P. On the one hand, for each $a \in \mathrm{A}$, and for all $b \in \mathbf{b}_j(\mathrm{A})$, we have $\omega\big(\{a\} \wedge_\vartriangle \{b\}\big) > j$. On the other hand, the definition (6.10) implies

$$\mathbf{b}_j\big(\mathbf{b}_j(\mathrm{A})\big) = \mathbf{min}\Big\{g \in P : \omega\big(\{g\} \wedge_\vartriangle \{b\}\big) > j\ ,\ \ \forall b \in \mathbf{b}_j(\mathrm{A})\Big\} . \tag{6.14}$$

Thus, we have

$$\mathrm{A} \preceq \mathbf{b}_j\big(\mathbf{b}_j(\mathrm{A})\big) \tag{6.15}$$

in $\mathfrak{A}_\triangledown(P)$, for any $\mathrm{A} \in \mathfrak{A}_\triangledown(P)$.

Since the absolute j-blocker map $\mathbf{b}_j$ on $\mathfrak{A}_\triangledown(P)$ is order-reversing, and eq. (6.15) holds, we can use the technique of *Galois connections* to obtain the following result:

Proposition 6.9. *Consider the* absolute j-blocker map $\mathbf{b}_j : \mathfrak{A}_\triangledown(P) \to \mathfrak{A}_\triangledown(P)$, *with respect to a map ω, on the lattice of antichains $\mathfrak{A}_\triangledown(P)$.*

(i) *The composite map*

$$\mathbf{b}_j \circ \mathbf{b}_j : \mathfrak{A}_\triangledown(P) \to \mathfrak{A}_\triangledown(P)$$

is a closure operator *on $\mathfrak{A}_\triangledown(P)$.*

(ii) *The image $\mathbf{b}_j(\mathfrak{A}_\triangledown(P))$ of the lattice $\mathfrak{A}_\triangledown(P)$ under the map $\mathbf{b}_j$ is a* self-dual lattice.

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The restriction of the map $\mathbf{b}_j$ *to* $\mathbf{b}_j(\mathfrak{A}_\triangledown(P))$ *is an* anti-automorphism *of* $\mathbf{b}_j(\mathfrak{A}_\triangledown(P))$. *As a consequence, for any antichain* $\mathrm{B} \in \mathbf{b}_j(\mathfrak{A}_\triangledown(P))$, *we have*

$$\mathbf{b}_j\big(\mathbf{b}_j(\mathrm{B})\big) = \mathrm{B} \,.$$

The lattice $\mathbf{b}_j(\mathfrak{A}_\triangledown(P))$ *is a* sub-meet-semilattice *of* $\mathfrak{A}_\triangledown(P)$.

The operation of meet $\wedge$ *and the operation of* join $\vee$ *in the lattice* $\mathbf{b}_j(\mathfrak{A}_\triangledown(P))$ *are determined as follows: given antichains* $\mathrm{B}', \mathrm{B}'' \in \mathbf{b}_j(\mathfrak{A}_\triangledown(P))$, *we have*

$$\mathrm{B}' \wedge \mathrm{B}'' = \mathrm{B}' \wedge_\triangledown \mathrm{B}''$$

and

$$\mathrm{B}' \vee \mathrm{B}'' = \mathbf{b}_j\big(\mathbf{b}_j(\mathrm{B}') \wedge_\triangledown \mathbf{b}_j(\mathrm{B}'')\big) \,,$$

where $\wedge_\triangledown$ *and* $\vee_\triangledown$ *denote the operations of meet and join, respectively, in the lattice* $\mathfrak{A}_\triangledown(P)$.

(iii) *For any* $\mathrm{B} \in \mathbf{b}_j(\mathfrak{A}_\triangledown(P))$, *its preimage* $(\mathbf{b}_j)^{-1}(\mathrm{B})$ *in* $\mathfrak{A}_\triangledown(P)$ *under the map* $\mathbf{b}_j$ *is a* convex sub-join-semilattice *of* $\mathfrak{A}_\triangledown(P)$; *the* greatest element *of* $(\mathbf{b}_j)^{-1}(\mathrm{B})$ *is* $\mathbf{b}_j(\mathrm{B})$.

Proof. The assertions (i) and (ii) are direct consequences of the standard results on Galois connections.

To prove assertion (iii), pick arbitrary elements $\mathrm{A}', \mathrm{A}'' \in (\mathbf{b}_j)^{-1}(\mathrm{B})$, where $\mathrm{B} = \mathbf{b}_j(\mathrm{A})$, for some $\mathrm{A} \in \mathfrak{A}_\triangledown(P)$, and note that $\mathbf{b}_j(\mathrm{A}' \vee_\triangledown \mathrm{A}'') = \mathbf{b}_j(\mathrm{A}') \wedge_\triangledown \mathbf{b}_j(\mathrm{A}'') = \mathrm{B}$. Thus, $(\mathbf{b}_j)^{-1}(\mathrm{B})$ is a sub-join-semilattice of $\mathfrak{A}_\triangledown(P)$. If $\mathrm{B} = \hat{0}_{\mathfrak{A}_\triangledown(P)}$, then $\mathbf{b}_j(\mathrm{B}) = \hat{1}_{\mathfrak{A}_\triangledown(P)}$ is the greatest element of $(\mathbf{b}_j)^{-1}(\mathrm{B})$. If $\mathrm{B} = \hat{1}_{\mathfrak{A}_\triangledown(P)}$, then $(\mathbf{b}_j)^{-1}(\mathrm{B})$ is the one-element subposet $\{\hat{0}_{\mathfrak{A}_\triangledown(P)}\} \subset \mathfrak{A}_\triangledown(P)$. Finally, if B is a nontrivial antichain of P then the element $\mathbf{b}_j(\mathrm{B}) = \mathbf{b}_j(\mathbf{b}_j(\mathrm{A}))$ is by eq. (6.14) the greatest element of $(\mathbf{b}_j)^{-1}(\mathrm{B})$. Since the map $\mathbf{b}_j$ is order-reversing, the subposet $(\mathbf{b}_j)^{-1}(\mathrm{B})$ of $\mathfrak{A}_\triangledown(P)$ is convex. □

Definition 6.10. The *lattice of absolute j-blockers* (with respect to a map ω) *in* a finite bounded poset P is the lattice $\mathbf{b}_j(\mathfrak{A}_\triangledown(P))$.

Let us now return to consider the posets (6.8) and (6.9). Since

$$\begin{aligned}
\big\{b \in P : \omega(b) > k-1\big\} &= \mathfrak{F}\big(\mathbf{b}_{k-1}(\hat{1}_P)\big) \,,\\
\big\{b \in P : \omega(b) > k\big\} &= \mathfrak{F}\big(\mathbf{b}_{k}(\hat{1}_P)\big) \,,\\
\big\{b \in P : \omega\big(\{b\} \wedge_\vartriangle \{a\}\big) > h-1 \,,\ \ \forall a \in \mathrm{A}\big\} &= \mathfrak{F}\big(\mathbf{b}_{h-1}(\mathrm{A})\big) \,,
\end{aligned}$$

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we obtain

$$\begin{aligned}\Big\{b \in P : \omega(b) = k,\ \omega\big(\{b\} \wedge_{\triangle} \{a\}\big) \geq h\ ,\ \ \forall a \in \mathrm{A}\Big\}\\ = \Big(\mathfrak{F}\big(\mathbf{b}_{k-1}(\hat{1}_P)\big) - \mathfrak{F}\big(\mathbf{b}_k(\hat{1}_P)\big)\Big) \cap \mathfrak{F}\big(\mathbf{b}_{h-1}(\mathrm{A})\big)\\ = \mathfrak{F}\big(\mathbf{b}_{k-1}(\hat{1}_P) \wedge_{\triangledown} \mathbf{b}_{h-1}(\mathrm{A})\big) - \mathfrak{F}\big(\mathbf{b}_k(\hat{1}_P)\big)\ .\end{aligned} \tag{6.16}$$

Since $\mathbf{b}_{k-1}(\hat{1}_P) \geq \mathbf{b}_k(\hat{1}_P)$ in $\mathfrak{A}_{\triangledown}(P)$, according to eq. (6.13), the second line in the expression (6.16) describes an intersection of convex subposets of P; hence, the subposet given in the first line of (6.16) is *convex*.

Again let h and k be positive integers such that $h \leq k \leq \omega(P)$. Let $\{a\}$ be a nontrivial one-element antichain of P. In addition to the subposet (6.8),(6.9),(6.16), we will also need the *convex* subposet

$$\begin{aligned}\Big\{b \in P : \omega(b) = k,\ \omega\big(\{b\} \wedge_{\triangle} \{a\}\big) = h\Big\}\\ = \Big(\mathfrak{F}\big(\mathbf{b}_{k-1}(\hat{1}_P)\big) - \mathfrak{F}\big(\mathbf{b}_k(\hat{1}_P)\big)\Big) \cap \Big(\mathfrak{F}\big(\mathbf{b}_{h-1}(a)\big) - \mathfrak{F}\big(\mathbf{b}_h(a)\big)\Big)\\ = \mathfrak{F}\big(\mathbf{b}_{k-1}(\hat{1}_P) \wedge_{\triangledown} \mathbf{b}_{h-1}(a)\big) - \mathfrak{F}\big(\mathbf{b}_k(\hat{1}_P) \vee_{\triangledown} \mathbf{b}_h(a)\big)\ .\end{aligned} \tag{6.17}$$

Remark 6.11. Let h, k, m, and n be positive integers such that $m \leq n$ and $h \leq k \leq n$. Recall that if $\{a\}$ is a nontrivial one-element antichain of the lattice of subspaces $\mathbb{V}_q(n)$, with $\rho(a) =: m$, then we have

$$\Big\{b \in \mathbb{V}_q(n) : \rho(b) = k,\ \rho(b \wedge a) \geq h\Big\} = \mathfrak{F}\big(\mathfrak{I}(a) \cap \mathbb{V}_q(n)^{(h)}\big) \cap \mathbb{V}_q(n)^{(k)}$$

and

$$\Big|\Big\{b \in \mathbb{V}_q(n) : \rho(b) = k,\ \rho(b \wedge a) \geq h\Big\}\Big| = \sum_{j \in [h,k]} \binom{m}{j}_q \binom{n-m}{k-j}_q q^{(m-j)(k-j)}\ .$$

Similarly, we have

$$\begin{aligned}\Big\{b \in \mathbb{V}_q(n) : \rho(b) = k,\ \rho(b \wedge a) = h\Big\}\\ = \Big(\mathfrak{F}\big(\mathfrak{I}(a) \cap \mathbb{V}_q(n)^{(h)}\big) - \mathfrak{F}\big(\mathfrak{I}(a) \cap \mathbb{V}_q(n)^{(h+1)}\big)\Big) \cap \mathbb{V}_q(n)^{(k)}\end{aligned}$$

and

$$\Big|\Big\{b \in \mathbb{V}_q(n) : \rho(b) = k,\ \rho(b \wedge a) = h\Big\}\Big| = \binom{m}{h}_q \binom{n-m}{k-h}_q q^{(m-h)(k-h)}\ .$$

These expressions for the cardinalities of subposets have a direct connection with the (q-) Chu–Vandermonde identity.

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6.3 A Connection Between Absolute and Relative Blocking Constructions

It follows from Definition 6.5 that the *relative* 0-*blocker* $\mathfrak{y}_0(\mathrm{A})$ of a nontrivial antichain A in P, with respect to an arbitrary map ω, is nothing else than the *absolute* 0-*blocker* $\mathfrak{b}(\mathrm{A}) := \mathfrak{b}_0(\mathrm{A}) := \mathfrak{b}_0^{\emptyset^{\mathrm{a}}}(\mathrm{A})$, or just the *blocker* for short, of A in P, considered in Chapter 5. Moreover, if $\mathfrak{b}(\mathrm{A}) \subseteq P^{\mathrm{a}}$, then $\bigcap_{a \in \mathrm{A}} \mathfrak{I}(a) - \{\hat{0}_P\} \subseteq \mathbf{I}_r(P, \mathrm{A}; \omega)$, and $\mathfrak{y}_r(\mathrm{A}) = \mathfrak{b}(\mathrm{A})$, for any value of the parameter r.

Again let A be a nontrivial antichain of P, and let $j \in \mathbb{N}$, $j < \omega(P)$, for some map ω. If $\mathbf{b}_j(\mathrm{A}) \neq \hat{0}_{\mathfrak{A}_\triangle(P)}$, then for all $b \in \mathbf{b}_j(\mathrm{A})$ and for all $a \in \mathrm{A}$, by eq. (6.10) we have

$$\frac{\omega(\{b\} \wedge_\triangle \{a\})}{\omega(b)} > \frac{j}{\max_{p \in \mathbf{b}_j(\mathrm{A})} \omega(p)} ,$$
$$\frac{\omega(\{a\} \wedge_\triangle \{b\})}{\omega(a)} > \frac{j}{\max_{p \in \mathrm{A}} \omega(p)} .$$

If $\mathfrak{y}_r(\mathrm{A}) \neq \hat{0}_{\mathfrak{A}_\triangle(P)}$, then for each $b \in \mathfrak{y}_r(\mathrm{A})$ and for all $a \in \mathrm{A}$, by eq. (6.7) we have

$$\omega\big(\{b\} \wedge_\triangle \{a\}\big) > r \cdot \omega(b) \geq r \cdot \min_{p \in \mathfrak{y}_r(\mathrm{A})} \omega(p) .$$

6.4 The Structure of the Subposets of Relatively Blocking Elements, and Their Enumeration

We now turn to an investigation of the structure of the subposets of relatively r-blocking elements, and we touch on the question of their enumeration.

For $k \in \mathbb{P}$ such that $k \leq \omega(P)$, define the integer

$$v(r \cdot k) := \lfloor r \cdot k \rfloor + 1 . \tag{6.18}$$

If A is a nontrivial antichain of P, then it follows from Definition 6.1(i) that

$$\mathbf{I}_r(P, \mathrm{A}; \omega) = \bigcup_{1 \leq k \leq \omega(P)} \bigcup_{v(r \cdot k) \leq h \leq \max_{a \in \mathrm{A}} \omega(a)} \Big\{b \in P : \omega(b) = k,\ \omega\big(\{b\} \wedge_\triangle \{a\}\big) \geq h,\ \ \forall a \in \mathrm{A}\Big\} . \tag{6.19}$$

Recall that for any values of h and k appearing in this expression, the structure of the subposet $\Big\{b \in P : \omega(b) = k,\ \omega\big(\{b\} \wedge_\triangle \{a\}\big) \geq h,\ \ \forall a \in \mathrm{A}\Big\}$ is described in eq. (6.16).

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Further, for any $h \geq \nu(r \cdot k)$, by eq. (6.13) we have $\mathfrak{F}\big(\mathbf{b}_{\nu(r\cdot k)-1}(\mathrm{A})\big) \supseteq \mathfrak{F}\big(\mathbf{b}_{h-1}(\mathrm{A})\big)$; therefore, eq. (6.19) reduces to

$$\mathbf{I}_r(P, \mathrm{A}; \omega) = \bigcup_{1 \leq k \leq \omega(P)} \Big\{ b \in P : \omega(b) = k,\ \omega\big(\{b\} \wedge_\vartriangle \{a\}\big) \geq \nu(r \cdot k),\ \ \forall a \in \mathrm{A} \Big\} ,$$

and we obtain the following result.

Proposition 6.12. *Let* A *be a nontrivial antichain of a finite bounded poset P.*

(i) *For any map ω, we have*

$$\begin{aligned}\mathbf{I}_r(P, \mathrm{A}; \omega) &= \bigcup_{1 \leq k \leq \omega(P)} \Big(\Big(\mathfrak{F}\big(\mathbf{b}_{k-1}(\hat{1}_P)\big) - \mathfrak{F}\big(\mathbf{b}_k(\hat{1}_P)\big)\Big) \cap \mathfrak{F}\big(\mathbf{b}_{\nu(r\cdot k)-1}(\mathrm{A})\big)\Big) \\ &= \bigcup_{1 \leq k \leq \omega(P)} \Big(\mathfrak{F}\big(\mathbf{b}_{k-1}(\hat{1}_P) \wedge_\triangledown \mathbf{b}_{\nu(r\cdot k)-1}(\mathrm{A})\big) - \mathfrak{F}\big(\mathbf{b}_k(\hat{1}_P)\big)\Big) .\end{aligned}$$

(ii) *If P is* graded, *then*

$$\mathbf{I}_r(P, \mathrm{A}; \omega_\rho) = \bigcup_{k \in [\rho(P)] \,:\, \nu(r\cdot k) \leq \min_{a \in \mathrm{A}} \rho(a)} \Big(P^{(k)} \cap \mathfrak{F}\big(\mathbf{b}_{\nu(r\cdot k)-1}(\mathrm{A})\big)\Big) . \tag{6.20}$$

To find the cardinality of subposet (6.20), we can use the combinatorial *Principle of Inclusion–Exclusion*: We have

$$\begin{aligned}|\mathbf{I}_r(P, \mathrm{A}; \omega_\rho)| &= \sum_{k \in [\rho(P)] \,:\, \nu(r\cdot k) \leq \min_{a \in \mathrm{A}} \rho(a)} \\ &\quad \sum_{\mathrm{C} \subseteq \mathrm{A} \,:\, |\mathrm{C}| > 0} (-1)^{|\mathrm{C}|-1} \cdot \Big| P^{(k)} \cap \mathfrak{F}\big(\mathfrak{I}(\mathrm{C}) \cap P^{(\nu(r\cdot k))}\big)\Big| \\ &= \sum_{k \in [\rho(P)] \,:\, \nu(r\cdot k) \leq \min_{a \in \mathrm{A}} \rho(a)} \\ &\quad \sum_{\mathrm{E} \subseteq P^{(\nu(r\cdot k))} \cap \mathfrak{I}(\mathrm{A}) \,:\, |\mathrm{E}| > 0} \Bigg(\sum_{\mathrm{C} \subseteq \mathrm{A} \,:\, \mathrm{E} \subseteq \mathfrak{I}(\mathrm{C})} (-1)^{|\mathrm{C}|-1} \Bigg) \cdot \Big| P^{(k)} \cap \mathfrak{F}(\mathrm{E}) \Big| .\end{aligned}$$

For the remainder of this section, let A be a *nontrivial* antichain of a *graded lattice P* of rank n, with the property: each interval of length k in P contains the same number $\mathrm{B}(k)$ of *maximal chains*; in other words, we suppose P to be a *principal order ideal* of some *binomial poset*. The function $\mathrm{B}(k)$ is called the *factorial function* of P; we have $\mathrm{B}(0) = \mathrm{B}(1) = 1$. The number of elements of rank i in any interval of length j is denoted by $\left[{j \atop i} \right]$; we have $\left[{j \atop i} \right] = \frac{\mathrm{B}(j)}{\mathrm{B}(i) \cdot \mathrm{B}(j-i)}$. If P is the lattice $\mathbb{B}(n)$ or the lattice $\mathbb{V}_q(n)$, then $\left[{j \atop i} \right] = \binom{j}{i}$ or $\left[{j \atop i} \right] = \binom{j}{i}_q$, respectively.

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Given an integer $k \in [n]$ such that $v(r \cdot k) \le \min_{a \in \mathrm{A}} \rho(a)$, we have

$$\begin{aligned}|\mathbf{I}_{r,k}(P, \mathrm{A}; \omega_\rho)| &= \sum_{\mathrm{C} \subseteq \mathrm{A}\,:\,|\mathrm{C}|>0} (-1)^{|\mathrm{C}|-1} \\ &\quad \cdot \sum_{\mathrm{E} \subseteq P^{(v(r\cdot k))} \cap \mathfrak{I}(\mathrm{C})\,:\,|\mathrm{E}|>0} (-1)^{|\mathrm{E}|-1} \cdot \begin{bmatrix} n-\rho\left(\bigvee_{e\in\mathrm{E}} e\right) \\ n-k \end{bmatrix} \\ &= \sum_{\mathrm{E} \subseteq P^{(v(r\cdot k))} \cap \mathfrak{I}(\mathrm{A})\,:\,|\mathrm{E}|>0} (-1)^{|\mathrm{E}|} \\ &\quad \cdot \left(\sum_{\mathrm{C} \subseteq \mathrm{A}\,:\,\mathrm{E} \subseteq \mathfrak{I}(\mathrm{C})} (-1)^{|\mathrm{C}|} \right) \cdot \begin{bmatrix} n-\rho\left(\bigvee_{e\in\mathrm{E}} e\right) \\ n-k \end{bmatrix} .\end{aligned} \tag{6.21}$$

Indeed, for example, the sum

$$\sum_{\mathrm{E} \subseteq P^{(v(r\cdot k))} \cap \mathfrak{I}(\mathrm{C})\,:\,|\mathrm{E}|>0} (-1)^{|\mathrm{E}|-1} \cdot \begin{bmatrix} n-\rho\left(\bigvee_{e\in\mathrm{E}} e\right) \\ n-k \end{bmatrix} \tag{6.22}$$

counts the number of elements of the layer $P^{(k)}$ comparable with at least one element of the antichain $P^{(v(r\cdot k))} \cap \mathfrak{I}(\mathrm{C})$.

To refine expression (6.21) by the application of the technique of the *Möbius function*, let us consider certain auxiliary lattices which can be associated with the antichain A.

A first *lattice* denoted by $\mathscr{C}_{r,k}(P, \mathrm{A})$ is the lattice that consists of all sets of the family $\{P^{(v(r\cdot k))} \cap \mathfrak{I}(\mathrm{C}) : \mathrm{C} \subseteq \mathrm{A}\}$ ordered by inclusion. The *greatest* element of $\mathscr{C}_{r,k}(P, \mathrm{A})$ is the set $P^{(v(r\cdot k))} \cap \mathfrak{I}(\mathrm{A})$. The *least* element of $\mathscr{C}_{r,k}(P, \mathrm{A})$ denoted by $\hat{0}$ is the empty subset of $P^{(v(r\cdot k))}$.

The remaining *lattices*, denoted as $\mathscr{E}_{r,k}(P, \mathrm{X})$, where X are *nonempty* subsets of $P^{(v(r\cdot k))} \cap \mathfrak{I}(\mathrm{A})$, are defined as follows. Given an antichain $\mathrm{X} \subseteq P^{(v(r\cdot k))} \cap \mathfrak{I}(\mathrm{A})$, the poset $\mathscr{E}_{r,k}(P, \mathrm{X})$ is the *sub-join-semilattice* of the lattice P generated by X and augmented by a new *least* element, denoted by $\hat{0}$ (it is regarded as the empty subset of P). The *greatest* element of $\mathscr{E}_{r,k}(P, \mathrm{X})$ is the join $\bigvee_{x\in\mathrm{X}} x$ in P.

We have

$$\begin{aligned}|\mathbf{I}_{r,k}(P, \mathrm{A}; \omega_\rho)| &= \sum_{\mathrm{X} \in \mathscr{C}_{r,k}(P,\mathrm{A})\,:\,\hat{0}<\mathrm{X}} \mu_{\mathscr{C}_{r,k}(P,\mathrm{A})}(\hat{0}, \mathrm{X}) \\ &\quad \cdot \sum_{z \in \mathscr{E}_{r,k}(P,\mathrm{X})\,:\,\hat{0}<z,\ \rho(z)\le k} \mu_{\mathscr{E}_{r,k}(P,\mathrm{X})}(\hat{0}, z) \cdot \begin{bmatrix} n-\rho(z) \\ n-k \end{bmatrix} ,\end{aligned} \tag{6.23}$$

where $\rho(\cdot)$ means the poset rank of an element in P, and where, for example, the sum

$$- \sum_{z \in \mathscr{E}_{r,k}(P,\mathrm{X})\,:\,\hat{0}<z,\ \rho(z)\le k} \mu_{\mathscr{E}_{r,k}(P,\mathrm{X})}(\hat{0}, z) \cdot \begin{bmatrix} n-\rho(z) \\ n-k \end{bmatrix}$$

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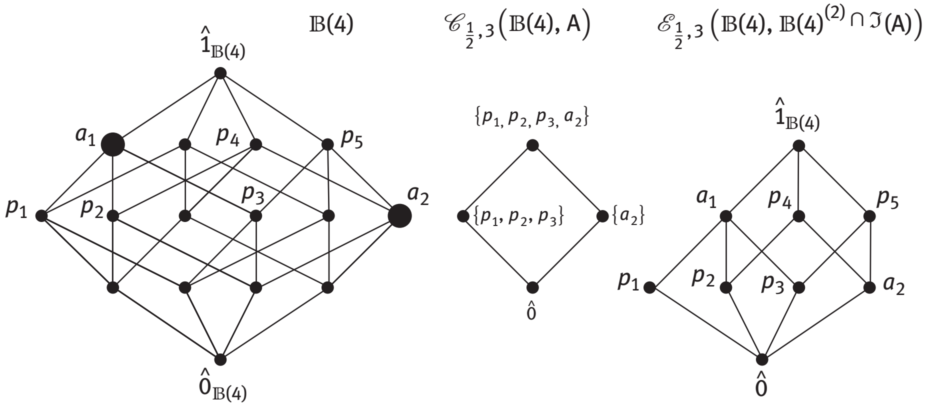

Figure 6.1: An antichain $\mathrm{A} := \{a_1, a_2\}$ of the Boolean lattice $\mathbb{B}(4)$, and auxiliary lattices involved in the enumeration of relatively blocking elements.

is equivalent to sum (6.22) when $\mathrm{X} = P^{(\nu(r\cdot k))} \cap \mathfrak{I}(\mathrm{C})$.

If P is the Boolean lattice $\mathbb{B}(n)$, then because of Remark 6.3, formulas (6.21) and (6.23) give for a *nontrivial clutter* the number of all its *r-committees of size k*.

Example 6.13. Figure 6.1 depicts the Hasse diagram of the Boolean lattice $\mathbb{B}(4)$, its antichain $\mathrm{A} := \{a_1, a_2\}$, and the lattices $\mathscr{C} := \mathscr{C}_{\frac{1}{2},3}(\mathbb{B}(4), \mathrm{A})$ and $\mathscr{E} := \mathscr{E}_{\frac{1}{2},3}(\mathbb{B}(4), \mathbb{B}(4)^{(2)} \cap \mathfrak{I}(\mathrm{A}))$. To find the number of elements in $\mathbf{I}_{\frac{1}{2},3}(\mathbb{B}(4), \mathrm{A}; \omega_p)$, note that $\mu_{\mathscr{C}}(\hat{0}, \{p_1, p_2, p_3\}) = \mu_{\mathscr{C}}(\hat{0}, \{a_2\}) = -1$ and $\mu_{\mathscr{C}}(\hat{0}, \{p_1, p_2, p_3, a_2\}) = 1$. Further, we have $\mu_{\mathscr{E}}(\hat{0}, p_1) = \mu_{\mathscr{E}}(\hat{0}, p_2) = \mu_{\mathscr{E}}(\hat{0}, p_3) = \mu_{\mathscr{E}}(\hat{0}, a_2) = -1$, $\mu_{\mathscr{E}}(\hat{0}, a_1) = 2$, $\mu_{\mathscr{E}}(\hat{0}, p_4) = \mu_{\mathscr{E}}(\hat{0}, p_5) = 1$ and $\mu_{\mathscr{E}}(\hat{0}, \hat{1}_{\mathbb{B}(4)}) = -1$.

By eq. (6.23) we have $|\mathbf{I}_{\frac{1}{2},3}(\mathbb{B}(4), \mathrm{A}; \omega_p)| = |\{p_4, p_5\}| = 2$.

Proposition 6.12(ii) gives a general description of the subposets of relatively r-blocking elements in graded posets. The aim of Section 6.6 is to investigate the structure of those subposets in more detail.

6.5 Principal Order Ideals and Farey Subsequences

In this section we define Farey subsequences which can be useful for the existence problems on the relatively blocking elements in posets.

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Let $\{a\}$ be a nontrivial one-element antichain of a finite bounded poset P. Consider the sequence of irreducible fractions

$$\mathscr{F}(P,a;\omega) := \left\{\frac{0}{1},\frac{1}{1}\right\} \cup \left(\frac{\omega(\{b\}\wedge_{\triangle}\{a\})}{\gcd\bigl(\omega(\{b\}\wedge_{\triangle}\{a\}),\omega(b)\bigr)} \Big/ \frac{\omega(b)}{\gcd\bigl(\omega(\{b\}\wedge_{\triangle}\{a\}),\omega(b)\bigr)} : b\in P-\{\hat{0}_P\}\right) \quad (6.24)$$

arranged in *increasing* order. Note that $\mathscr{F}(P,a;\omega)$ is a subsequence of the *Farey sequence* of order $\omega(P)$.

We always index the fractions of $\mathscr{F}(P,a;\omega)$ starting with *zero*: $\mathscr{F}(P,a;\omega) = \bigl(f_0 := \frac{0}{1} < f_1 < f_2 \cdots < f_{|\mathscr{F}(P,a;\omega)|-1} := \frac{1}{1}\bigr)$.

Let a be an arbitrary element of the Boolean lattice $\mathbb{B}(n)$, of rank $m := \rho(a)$. Consider the Farey subsequence $\mathscr{F}(\mathbb{B}(n),a;\omega_\rho)$ associated with the principal order ideal $\mathfrak{I}(a)$ of $\mathbb{B}(n)$. The sequences $\mathscr{F}(\mathbb{B}(n),a;\omega_\rho)$ are the same for all elements a of rank m in the lattice $\mathbb{B}(n)$, and we write $\mathscr{F}(\mathbb{B}(n),m;\omega_\rho)$ instead of $\mathscr{F}(\mathbb{B}(n),a;\omega_\rho)$. Thus, we have

$$\mathscr{F}\bigl(\mathbb{B}(n),m;\omega_\rho\bigr) := \begin{cases}\mathscr{F}(\mathbb{B}(n),m)\,, & \text{if } 0<m<n\,,\\ (\frac{0}{1}<\frac{1}{1})\,, & \text{if } m\in\{0,n\}\,,\end{cases}$$

where the *Farey subsequences*

$$\mathscr{F}\bigl(\mathbb{B}(n),m\bigr) := \bigl(\tfrac{h}{k}\in\mathscr{F}_n : m+k-n\le h\le m\bigr)\,,\quad 0<m<n\,,$$

were defined set-theoretically by eq. (4.1) and surveyed in Chapter 4.

Example 6.14.

$$\mathscr{F}_6 = \bigl(\tfrac{0}{1}<\tfrac{1}{6}<\tfrac{1}{5}<\tfrac{1}{4}<\tfrac{1}{3}<\tfrac{2}{5}<\tfrac{1}{2}<\tfrac{3}{5}<\tfrac{2}{3}<\tfrac{3}{4}<\tfrac{4}{5}<\tfrac{5}{6}<\tfrac{1}{1}\bigr)\,,$$

$$\mathscr{F}(\mathbb{B}(6),6;\omega_\rho) = \bigl(\tfrac{0}{1}<\tfrac{1}{1}\bigr)\,,$$

$$\mathscr{F}(\mathbb{B}(6),5;\omega_\rho) = \bigl(\tfrac{0}{1}<\tfrac{1}{2}<\tfrac{2}{3}<\tfrac{3}{4}<\tfrac{4}{5}<\tfrac{5}{6}<\tfrac{1}{1}\bigr)\,,$$

$$\mathscr{F}(\mathbb{B}(6),4;\omega_\rho) = \bigl(\tfrac{0}{1}<\tfrac{1}{3}<\tfrac{1}{2}<\tfrac{3}{5}<\tfrac{2}{3}<\tfrac{3}{4}<\tfrac{4}{5}<\tfrac{1}{1}\bigr)\,,$$

$$\mathscr{F}(\mathbb{B}(6),3;\omega_\rho) = \bigl(\tfrac{0}{1}<\tfrac{1}{4}<\tfrac{1}{3}<\tfrac{2}{5}<\tfrac{1}{2}<\tfrac{3}{5}<\tfrac{2}{3}<\tfrac{3}{4}<\tfrac{1}{1}\bigr)\,,$$

$$\mathscr{F}(\mathbb{B}(6),2;\omega_\rho) = \bigl(\tfrac{0}{1}<\tfrac{1}{5}<\tfrac{1}{4}<\tfrac{1}{3}<\tfrac{2}{5}<\tfrac{1}{2}<\tfrac{2}{3}<\tfrac{1}{1}\bigr)\,,$$

$$\mathscr{F}(\mathbb{B}(6),1;\omega_\rho) = \bigl(\tfrac{0}{1}<\tfrac{1}{6}<\tfrac{1}{5}<\tfrac{1}{4}<\tfrac{1}{3}<\tfrac{1}{2}<\tfrac{1}{1}\bigr)\,,$$

$$\mathscr{F}(\mathbb{B}(6),0;\omega_\rho) = \bigl(\tfrac{0}{1}<\tfrac{1}{1}\bigr)\,.$$

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Remark 6.15. For all elements $a \in \mathbb{V}_q(n)$ of rank $m := \rho(a)$, the Farey subsequences $\mathscr{F}(\mathbb{V}_q(n), a; \omega_\rho)$ are the same, and we write $\mathscr{F}(\mathbb{V}_q(n), m; \omega_\rho)$ instead of $\mathscr{F}(\mathbb{V}_q(n), a; \omega_\rho)$. We have

$$\mathscr{F}\left(\mathbb{V}_q(n), m; \omega_\rho\right) = \mathscr{F}\left(\mathbb{B}(n), m; \omega_\rho\right) , \quad 0 \le m \le n .$$

See also Remark 6.11.

6.6 Relatively Blocking Elements in Graded Posets, and Farey Subsequences

In this section we show how Farey subsequences can be involved in the existence problems on the relatively blocking elements in posets.

Let $\{a\}$ be a nontrivial one-element antichain of a finite bounded poset P. Given a map ω, define the map

$$\mathfrak{f}_{P,a;\omega} : \{r \in \mathbb{Q} : 0 \le r < 1\} \to \mathscr{F}(P, a; \omega)$$

by

$$r \mapsto \max\left\{f \in \mathscr{F}(P, a; \omega) : f \le r\right\} .$$

Given an element a of the Boolean lattice $\mathbb{B}(n)$, of rank $m := \rho(a) > 0$, we write $\mathfrak{f}_{\mathbb{B}(n),m;\omega_\rho}(r)$ instead of $\mathfrak{f}_{\mathbb{B}(n),a;\omega_\rho}(r)$.

The Farey subsequences $\mathscr{F}(P, a; \omega)$ are in particular useful because for a nontrivial antichain A of P, and for a map ω, we have

$$\mathbf{I}_r(P, \mathrm{A}; \omega) = \bigcap_{a \in \mathrm{A}} \mathbf{I}_{\mathfrak{f}_{P,a;\omega}(r)}(P, a; \omega) ;$$

cf. Proposition 6.4(i).

For a fraction $f := \frac{h}{k} \in \mathbb{Q}$, we denote by $\underline{f} := h$ the *numerator* of f, and we denote by $\overline{f} := k$ its *denominator*.

Given a nontrivial antichain A of P, and a map ω, define a set $\mathscr{D}_r(P, \mathrm{A}; \omega) \subset \mathbb{P}$ by

$$\mathscr{D}_r(P, \mathrm{A}; \omega) := \bigcap_{a \in \mathrm{A}} \left(\bigcup_{f \in \mathscr{F}(P,a;\omega) : \, \mathfrak{f}_{P,a;\omega}(r) < f < \frac{1}{1}} \left\{ s \cdot \overline{f} : 1 \le s \le \min\left\{ \left\lfloor \omega(a)/\underline{f} \right\rfloor, \left\lfloor \omega(P)/\overline{f} \right\rfloor \right\} \right\} \cup \left\{ \omega(e) : e \in \mathfrak{I}(a) - \{\hat{0}\} \right\} \right) .$$

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This set of positive integers allows us to give the following commentary on Proposition 6.12(i).

Proposition 6.16. *Let a finite bounded poset P and a map ω satisfy the condition: for any elements $a', a'' \in P$, we have*

$$\omega\big(\{a'\} \wedge_{\triangle} \{a''\}\big) = \omega(a') \iff a' \preceq a'' .$$

Let A *be a nontrivial antichain of P, and let $k \in [\omega(P)]$. Suppose that $|\bigcap_{a \in \mathrm{A}} \mathfrak{I}(a) - \{\hat{0}_P\}| = 0$. If $k \notin \mathscr{D}_r(P, \mathrm{A}; \omega)$, then $|\mathbf{I}_{r,k}(P, \mathrm{A}; \omega)| = 0$.*

Example 6.17. If A is an antichain of $\mathbb{B}(6)$ such that $\{\rho(a) : a \in \mathrm{A}\} = \{2, 3\}$, then $\mathscr{D}_{\frac{1}{2}}(\mathbb{B}(6), \mathrm{A}; \omega_\rho) = \{1, 2, 3\}$. Thus, if the set $\mathbf{I}_{\frac{1}{2}}(\mathbb{B}(6), \mathrm{A}; \omega_\rho)$ is nonempty, and $\mathbf{I}_{\frac{1}{2}}(\mathbb{B}(6), \mathrm{A}; \omega_\rho) \ni b$, then either $\{b\} = \bigcap_{a \in \mathrm{A}} \mathfrak{I}(a) - \{\hat{0}_{\mathbb{B}(6)}\}$, and b is of rank 1, or b is of rank 3.

We conclude this section with a refinement of Proposition 6.12(ii); recall that the integers $\nu(\cdot)$ are defined by eq. (6.18).

Theorem 6.18. *Let P be a finite bounded* graded *poset. If* A *is a nontrivial antichain of P, then, on the one hand,*

$$\mathbf{I}_r(P, \mathrm{A}; \omega_\rho) = \Big(\bigcap_{a \in \mathrm{A}} \mathfrak{I}(a) - \{\hat{0}_P\} \Big) \cup \bigcup_{k \in \mathscr{D}_r(P, \mathrm{A}; \omega_\rho)} \Big(P^{(k)} \cap \mathfrak{F}\big(\mathbf{b}_{\nu(r \cdot k) - 1}(\mathrm{A})\big) \Big) . \tag{6.25}$$

On the other hand,

$$\begin{aligned}\mathbf{I}_r(P, \mathrm{A}; \omega_\rho) = &\Big(\bigcap_{a \in \mathrm{A}} \mathfrak{I}(a) - \{\hat{0}_P\} \Big) \\ &\cup \bigcap_{a \in \mathrm{A}} \; \bigcup_{f \in \mathscr{F}(P, a; \omega_\rho) : \mathfrak{f}_{P, a; \omega_\rho}(r) \lessdot f} \; \bigcup_{s \in [\min\{\lfloor \rho(a)/\underline{f} \rfloor, \lfloor \rho(P)/\overline{f} \rfloor\}] : s \cdot \overline{f} \in \mathscr{D}_r(P, \mathrm{A}; \omega_\rho)} \\ &\qquad \Big(P^{(s \cdot \overline{f})} \cap \Big(\mathfrak{F}\big(\mathbf{b}_{s \cdot \underline{f} - 1}(a)\big) - \mathfrak{F}\big(\mathbf{b}_{s \cdot \underline{f}}(a)\big) \Big) \Big) .\end{aligned} \tag{6.26}$$

Proof. First, in both expressions (6.25) and (6.26), we consider the component $\bigcap_{a \in \mathrm{A}} \mathfrak{I}(a) - \{\hat{0}_P\}$; if this component corresponding to the terminal fraction $\frac{1}{1}$ of the Farey subsequences $\mathscr{F}(P, a; \omega_\rho)$, $a \in \mathrm{A}$, is nonempty, then any element of the component is a relatively r-blocking element of A in P. Further, if $k \notin \mathscr{D}_r(P, \mathrm{A}; \omega_\rho)$, then the set $\mathbf{I}_{r,k}(P, \mathrm{A}; \omega_\rho) - \big(\bigcap_{a \in \mathrm{A}} \mathfrak{I}(a) - \{\hat{0}_P\}\big)$ is empty, see Proposition 6.16. Equation (6.25) now follows from Proposition 6.12(ii).

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If $a \in \mathrm{A}$, then we have

$$\mathbf{I}_r(P, a; \omega_\rho) = \bigcup_{f \in \mathscr{F}(P,a;\omega_\rho) : \mathfrak{f}_{P,a;\omega_\rho}(r) < f} \ \bigcup_{1 \le s \le \min\{\lfloor \rho(a)/\underline{f} \rfloor, \lfloor \rho(P)/\overline{f} \rfloor\}} \Big\{ b \in P : \rho(b) = s \cdot \overline{f},\ \rho\big(\{b\} \wedge_\triangle \{a\}\big) = s \cdot \underline{f} \Big\} .$$

Further, by eq. (6.17) we have

$$\Big\{ b \in P : \rho(b) = s \cdot \overline{f} \Big\} = P^{(s \cdot \overline{f})} ,$$
$$\Big\{ b \in P : \rho\big(\{b\} \wedge_\triangle \{a\}\big) = s \cdot \underline{f} \Big\} = \mathfrak{F}\big(\mathbf{b}_{s \cdot \underline{f} - 1}(a)\big) - \mathfrak{F}\big(\mathbf{b}_{s \cdot \underline{f}}(a)\big) ;$$

Equation (6.26) now follows, in view of Proposition 6.4(i). □

6.7 Relatively Blocking Elements in the Boolean Lattices

In this section we count in several ways the rank k *relatively r-blocking elements* of *antichains* in the *Boolean lattices*, and from the viewpoint of combinatorial optimization we thus implicitly count the *r-committees*, of size k, of *clutters*; note that the counting of the *relatively* 0*-blocking elements* is in fact equivalent to the counting of the *blocking sets*.

Let A be a nontrivial antichain of the Boolean lattice $\mathbb{B}(n)$ of rank n, and $\mathrm{A}^\perp$ the set of *lattice complements* of the elements of A in $\mathbb{B}(n)$. As earlier ρ denotes the *rank function*, $\mathbb{B}(n)^{(i)} := \{b \in \mathbb{B}(n) : \rho(b) = i\}$ is the ith *layer* of $\mathbb{B}(n)$, and $\mathfrak{I}(\mathrm{C})$ denotes the *order ideal* of the lattice $\mathbb{B}(n)$ generated by its antichain C.

For a rational number r, $0 \le r < 1$, and for a positive integer number k, consider the subposet $\mathbf{I}_{r,k}(\mathbb{B}(n), \mathrm{A}) := \mathbf{I}_{r,k}(\mathbb{B}(n), \mathrm{A}; \omega_\rho)$ given by

$$\mathbf{I}_{r,k}\big(\mathbb{B}(n), \mathrm{A}\big) := \Big\{ b \in \mathbb{B}(n) : \rho(b) = k,\ \rho(b \wedge a) > r \cdot k,\ \ \forall a \in \mathrm{A} \Big\} \subset \mathbb{B}(n)^{(k)} , \qquad (6.27)$$

which by definition consists of the rank k *relatively r-blocking elements* (with respect to either of the maps ω_{a} and ω_ρ) of the antichain A; see Remarks 6.2 and 6.3.

Following eq. (6.18), set

$$\nu(r \cdot k) := \lfloor r \cdot k \rfloor + 1 ,$$

and consider an arbitrary antichain $\mathrm{A} \subset \mathbb{B}(n)$ such that $\rho(a) \ge \nu(r \cdot k)$ and $n - \rho(a) \ge k - \nu(r \cdot k) + 1$, for each element $a \in \mathrm{A}$, that is,

$$\lfloor r \cdot k \rfloor + 1 \le \min_{a \in \mathrm{A}} \rho(a) \quad \text{and} \quad \max_{a \in \mathrm{A}} \rho(a) \le n + \lfloor r \cdot k \rfloor - k . \qquad (6.28)$$

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Since the antichain A satisfies the conditions (6.28), we have $b' \notin \mathbf{I}_{r,k}(\mathbb{B}(n), \mathrm{A})$ for an element $b' \in \mathbb{B}(n)^{(k)}$ if and only if $b' > d'$ for at least one element d' of rank $k - \nu(r \cdot k) + 1 = k - \lfloor r \cdot k \rfloor$ such that $d' \in \mathfrak{I}(\mathrm{A}^{\perp})$; therefore, on the one hand,

$$\left|\mathbf{I}_{r,k}\big(\mathbb{B}(n), \mathrm{A}\big)\right| = \binom{n}{k} + \sum_{\mathrm{D}' \subseteq \mathbb{B}(n)^{(k - \lfloor r \cdot k \rfloor)} \cap \mathfrak{I}(\mathrm{A}^{\perp}) \,:\, |\mathrm{D}'| > 0} (-1)^{|\mathrm{D}'|} \cdot \binom{n - \rho\left(\bigvee_{d' \in \mathrm{D}'} d'\right)}{n - k} . \tag{6.29}$$

On the other hand, the inclusion $b \in \mathbf{I}_{r,k}(\mathbb{B}(n), \mathrm{A})$ for an element $b \in \mathbb{B}(n)$ holds if and only if for each element $a \in \mathrm{A}$, we have $\rho(b \wedge \theta_a) > 0$, for any element $\theta_a \in \mathbb{B}(n)^{(\rho(a) - \nu(r \cdot k) + 1)} \cap \mathfrak{I}(a)$, that is,

$$b \in \mathbf{I}_{r,k}\big(\mathbb{B}(n), \mathrm{A}\big) \iff \rho(b \wedge \theta_a) > 0, \quad \forall \theta_a \in \mathbb{B}(n)^{(\rho(a) - \lfloor r \cdot k \rfloor)} \cap \mathfrak{I}(a), \quad \forall a \in \mathrm{A} ,$$

and we have

$$\left|\mathbf{I}_{r,k}\big(\mathbb{B}(n), \mathrm{A}\big)\right| = \binom{n}{k} + \sum_{\mathrm{D} \subseteq \mathbf{min} \bigcup_{a \in \mathrm{A}} (\mathbb{B}(n)^{(\rho(a) - \lfloor r \cdot k \rfloor)} \cap \mathfrak{I}(a)) \,:\, |\mathrm{D}| > 0} (-1)^{|\mathrm{D}|} \cdot \binom{n - \rho(\bigvee_{d \in \mathrm{D}} d)}{k} \tag{6.30}$$

or, via the *Chu–Vandermonde identity*,

$$\left|\mathbf{I}_{r,k}\big(\mathbb{B}(n), \mathrm{A}\big)\right| = - \sum_{\mathrm{D} \subseteq \mathbf{min} \bigcup_{a \in \mathrm{A}} (\mathbb{B}(n)^{(\rho(a) - \lfloor r \cdot k \rfloor)} \cap \mathfrak{I}(a)) \,:\, |\mathrm{D}| > 0} (-1)^{|\mathrm{D}|} \cdot \sum_{1 \le h \le k} \binom{\rho(\bigvee_{d \in \mathrm{D}} d)}{h} \binom{n - \rho(\bigvee_{d \in \mathrm{D}} d)}{k - h} . \tag{6.31}$$

Yet another inclusion–exclusion type formula for the cardinality of the set $\mathbf{I}_{r,k}(\mathbb{B}(n), \mathrm{A})$, for an antichain A such that $\rho(a) \ge \nu(r \cdot k)$, for all $a \in \mathrm{A}$, is given in eq. (6.21): If

$$\lfloor r \cdot k \rfloor + 1 \le \min_{a \in \mathrm{A}} \rho(a) , \tag{6.32}$$

then

$$\left|\mathbf{I}_{r,k}\big(\mathbb{B}(n), \mathrm{A}\big)\right| = \sum_{\mathrm{D} \subseteq \mathbb{B}(n)^{(\lfloor r \cdot k \rfloor + 1)} \cap \mathfrak{I}(\mathrm{A}) \,:\, |\mathrm{D}| > 0} (-1)^{|\mathrm{D}|} \cdot \left(\sum_{\mathrm{C} \subseteq \mathrm{A} \,:\, \mathrm{D} \subseteq \mathfrak{I}(\mathrm{C})} (-1)^{|\mathrm{C}|} \right) \binom{n - \rho(\bigvee_{d \in \mathrm{D}} d)}{n - k} . \tag{6.33}$$

We will now refine the formulas (6.29), (6.30) and (6.31) by involving the *Möbius function*; see the expressions (6.34), (6.35) and (6.36) below, respectively.

Let X be a nontrivial antichain of the Boolean lattice $\mathbb{B}(n)$. Denote by $\mathscr{E}(\mathbb{B}(n), \mathrm{X})$ the sub-join-semilattice of $\mathbb{B}(n)$ generated by the set X and augmented by a new *least*

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element $\hat{0}$; the *greatest* element $\hat{1}$ of the lattice $\mathscr{E}(\mathbb{B}(n), \mathrm{X})$ is the join $\bigvee_{x\in\mathrm{X}} x$ in $\mathbb{B}(n)$. We denote the *Möbius function* of the lattice $\mathscr{E}(\mathbb{B}(n), \mathrm{X})$ by $\mu_{\mathscr{E}}(\cdot,\cdot)$.

Let A be a nontrivial antichain of the Boolean lattice $\mathbb{B}(n)$ satisfying conditions (6.28). We have

$$\left|\mathbf{I}_{r,k}\big(\mathbb{B}(n),\mathrm{A}\big)\right| = \binom{n}{k} + \sum_{z\in\mathscr{E}(\mathbb{B}(n)^{(k-\lfloor r\cdot k\rfloor)}\cap\mathfrak{I}(\mathrm{A}^{\perp}))\,:\,z>\hat{0}} \mu_{\mathscr{E}}(\hat{0},z)\cdot\binom{n-\rho(z)}{n-k}\,, \tag{6.34}$$

$$\left|\mathbf{I}_{r,k}\big(\mathbb{B}(n),\mathrm{A}\big)\right| = \binom{n}{k} + \sum_{z\in\mathscr{E}(\mathbf{min}\bigcup_{a\in\mathrm{A}}(\mathbb{B}(n)^{(\rho(a)-\lfloor r\cdot k\rfloor)}\cap\mathfrak{I}(a)))\,:\,z>\hat{0}} \mu_{\mathscr{E}}(\hat{0},z)\cdot\binom{n-\rho(z)}{k} \tag{6.35}$$

and

$$\left|\mathbf{I}_{r,k}\big(\mathbb{B}(n),\mathrm{A}\big)\right| = -\sum_{z\in\mathscr{E}(\mathbf{min}\bigcup_{a\in\mathrm{A}}(\mathbb{B}(n)^{(\rho(a)-\lfloor r\cdot k\rfloor)}\cap\mathfrak{I}(a)))\,:\,z>\hat{0}} \mu_{\mathscr{E}}(\hat{0},z) \cdot \sum_{1\le h\le k}\binom{\rho(z)}{h}\binom{n-\rho(z)}{k-h}\,. \tag{6.36}$$

A companion formula to eq. (6.33) is eq. (6.23): Let $\mathrm{A}\subset\mathbb{B}(n)$ be an antichain satisfying condition (6.32), and let $\mathscr{C}_{r,k}(\mathbb{B}(n),\mathrm{A})$ be the *join-semilattice* of all sets of the family $\{\mathbb{B}(n)^{(\lfloor r\cdot k\rfloor+1)}\cap\mathfrak{I}(\mathrm{C}) : \mathrm{C}\subseteq\mathrm{A},\ |\mathrm{C}|>0\}$ ordered by inclusion and augmented by a new *least* element $\hat{0}$; the *greatest* element $\hat{1}$ of the *lattice* $\mathscr{C}_{r,k}(\mathbb{B}(n),\mathrm{A})$ is the set $\mathbb{B}(n)^{(\lfloor r\cdot k\rfloor+1)}\cap\mathfrak{I}(\mathrm{A})$. Denoting the *Möbius function* of $\mathscr{C}_{r,k}(\mathbb{B}(n),\mathrm{A})$ by $\mu_{\mathscr{C}}(\cdot,\cdot)$, we have

$$\left|\mathbf{I}_{r,k}\big(\mathbb{B}(n),\mathrm{A}\big)\right| = \sum_{\mathrm{X}\in\mathscr{C}_{r,k}(\mathbb{B}(n),\mathrm{A})\,:\,\mathrm{X}>\hat{0}} \mu_{\mathscr{C}}(\hat{0},\mathrm{X}) \cdot \sum_{z\in\mathscr{E}(\mathbb{B}(n),\mathrm{X})\,:\,z>\hat{0}} \mu_{\mathscr{E}}(\hat{0},z)\cdot\binom{n-\rho(z)}{n-k}\,. \tag{6.37}$$

6.8 Relatively Blocking Elements b with the Property $b\wedge -b=\hat{0}$ in the Boolean Lattices of Subsets of the Sets $\pm\{1,\dots,m\}$

Let m be a positive integer, and let $\pm[m]$ be the $2m$-set

$$\pm[m] := \{-m,\dots,-1,1,\dots,m\}\,.$$

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In this section we count the rank k *relatively r-blocking elements* b, with the property $b \wedge -b = \hat{0}$, of *antichains* of the *Boolean lattice* of subsets of the set $\pm[m]$. From another viewpoint, we implicitly count the *r-committees* K, of size k, of *clutters* on their *ground sets* $S \subseteq \pm[m]$, for which we by convention have $K \not\supset \{-i, i\}$, for all elements $i \in [m]$.

If we fix a subset $W \subseteq \pm[m]$, and define $-W$ to be the set

$$-W := \{-w : w \in W\} ,$$

then we have

$$|\pm [m]| - |W| - 2\#\Big\{\{i, -i\} \subseteq \pm[m] : |\{i, -i\} \cap W| = 0\Big\} = |W \cup -W| - |W| \quad (6.38)$$

and

$$\#\Big\{\{i, -i\} \subseteq \pm[m] : |\{i, -i\} \cap W| = 0\Big\} = m - \tfrac{1}{2}|W \cup -W| . \quad (6.39)$$

Recall that the number of k-subsets $V \subset \pm[m]$ such that

$$v \in V \implies -v \notin V , \quad (6.40)$$

is $\binom{m}{k}2^k$; this is the number of $(k-1)$-dimensional faces of an m-dimensional *crosspolytope*.

If $W \neq \pm[m]$, then consider a nonempty k-set $V \subset \pm[m]$ such that $|V \cap W| = 0$, and implication (6.40) is valid. Let $V = V'\dot{\cup}V''$ be a partition of V into two subsets with the following properties:

$$v' \in V' \implies -v' \in W , \quad (6.41)$$
$$v'' \in V'' \implies -v'' \notin W . \quad (6.42)$$

Let $|V'| =: j$ and $|V''| =: k - j$, for some j. In fact, eqs. (6.38) and (6.39) imply that there are $\binom{|W\cup -W|-|W|}{j}$ sets $V' \subset \pm[m]$ such that $|V'| = j$, $|V' \cap W| = 0$, and implication (6.41) is valid; there are $\binom{m-\frac{1}{2}|W\cup -W|}{k-j}2^{k-j}$ sets $V'' \subset \pm[m]$ such that $|V''| = k - j$, $|V'' \cap W| = 0$, and eq. (6.42) is valid.

Let $\mathbb{B}(2m)$ denote the Boolean lattice of subsets of the set $\pm[m]$. We denote the empty subset of $\pm[m]$ by $\hat{0}$. If $b \in \mathbb{B}(2m) - \{\hat{0}\}$, then we denote by $-b$ the set of the *negations* of elements in b.

Let r be a rational number, $0 \le r < 1$, and k an integer such that $1 \le k \le m$. If A is an antichain of $\mathbb{B}(2m)$ such that $v(r \cdot k) := \lfloor r \cdot k \rfloor + 1 \le \min_{a \in \mathrm{A}} \rho(a)$, then consider a subposet $\mathring{\mathbf{I}}_{r,k}(\mathbb{B}(2m), \mathrm{A}) := \mathring{\mathbf{I}}_{r,k}(\mathbb{B}(2m), \mathrm{A}; \omega_\rho)$ defined by

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$$\mathring{\mathbf{I}}_{r,k}\big(\mathbb{B}(2m),\mathrm{A}\big) := \big\{b \in \mathbb{B}(2m):\\ b \wedge -b = \hat{0},\ \rho(b) = k,\ \rho(b \wedge a) > r \cdot k,\quad \forall a \in \mathrm{A}\big\} \subset \mathbb{B}(2m)^{(k)}\ ,$$

where $\rho(\cdot)$ denotes the poset rank of an element in the lattice $\mathbb{B}(2m)$, and $\mathbb{B}(2m)^{(k)} := \{b \in \mathbb{B}(2m) : \rho(b) = k\}$. The collection $\mathring{\mathbf{I}}_{r,k}(\mathbb{B}(2m),\mathrm{A})$ is by definition the subposet of *relatively r-blocking elements* $b \in \mathbb{B}(2m)^{(k)}$, with respect to either of the maps ω_{a} and ω_{ρ}, and *with the additional property* $b \wedge -b = \hat{0}$, of the antichain A in the lattice $\mathbb{B}(2m)$.

As earlier, denote by $\mathfrak{I}(a)$ the *principal order ideal* of the lattice $\mathbb{B}(2m)$ generated by an element $a \in \mathrm{A}$. Using the *Principle of Inclusion–Exclusion*, we obtain

$$\Big|\mathring{\mathbf{I}}_{r,k}\big(\mathbb{B}(2m),\mathrm{A}\big)\Big| = \binom{m}{k}2^k + \sum_{\mathrm{D}\subseteq\mathbf{min}\bigcup_{a\in\mathrm{A}}(\mathbb{B}(2m)^{(\rho(a)-\lfloor r\cdot k\rfloor)}\cap\mathfrak{I}(a)):|\mathrm{D}|>0} (-1)^{|\mathrm{D}|}\cdot\sum_{0\le j\le k}\binom{\rho(\bigvee_{d\in\mathrm{D}} d \vee -\bigvee_{d\in\mathrm{D}} d) - \rho(\bigvee_{d\in\mathrm{D}} d)}{j} \cdot\binom{m-\frac{1}{2}\rho(\bigvee_{d\in\mathrm{D}} d\vee -\bigvee_{d\in\mathrm{D}} d)}{k-j}2^{k-j}\ . \tag{6.43}$$

Consider the lattice

$$\mathscr{E} := \Big\{\bigvee_{d\in\mathrm{D}} d : \mathrm{D} \subseteq \mathbf{min}\bigcup_{a\in\mathrm{A}}\big(\mathbb{B}(2m)^{(\rho(a)-\lfloor r\cdot k\rfloor)}\cap\mathfrak{I}(a)\big),\ |\mathrm{D}|>0\Big\}\ \dot\cup\ \{\hat{0}\}\ ,$$

where $\hat{0}$ is a new *least* element adjoined. If we let $\mu_{\mathscr{E}}(\cdot,\cdot)$ denote the *Möbius function* of the lattice $\mathscr{E}$, then we have

$$\Big|\mathring{\mathbf{I}}_{r,k}\big(\mathbb{B}(2m),\mathrm{A}\big)\Big| = \binom{m}{k}2^k + \sum_{z\in\mathscr{E}:z>\hat{0}}\mu_{\mathscr{E}}(\hat{0},z) \cdot\sum_{0\le j\le k}\binom{\rho(z\vee -z)-\rho(z)}{j}\binom{m-\frac{1}{2}\rho(z\vee -z)}{k-j}2^{k-j}\ , \tag{6.44}$$

where again $\rho(\cdot)$ means the poset rank of an element in the lattice $\mathbb{B}(2m)$.

6.9 Relatively Blocking Elements in the Posets Isomorphic to the Face Lattices of Crosspolytopes

In this section we are concerned with the enumeration of the *relatively blocking elements* in the posets isomorphic to the *face lattices of crosspolytopes*.

Consider a poset $\mathbf{O}'(m)$ isomorphic to the *face meet-semilattice* of the *boundary of* an m-dimensional *crosspolytope*, which is defined as follows: the semilattice $\mathbf{O}'(m)$ consists of all the subsets of the set $\pm[m]$, free of pairs of opposites $\{-i,i\}$, and ordered

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by inclusion. We define a rank $(m+1)$ *lattice* $\mathbf{O}(m)$, with the rank function $\rho(\cdot)$, isomorphic to the *face lattice* of an m-dimensional *crosspolytope* by

$$\mathbf{O}(m) := \mathbf{O}'(m) \dot{\cup} \{\hat{1}\} ,$$

where $\hat{1}$ is a *greatest* element adjoined.

Let a subposet $\mathrm{A} \subset \mathbf{O}(m) - \{\hat{0}, \hat{1}\}$ be an antichain. For a rational number r, $0 \le r < 1$, and for a positive integer $k \le m$, consider the subposet $\mathbf{I}_{r,k}(\mathbf{O}(m), \mathrm{A}) := \mathbf{I}_{r,k}(\mathbf{O}(m), \mathrm{A}; \omega_\rho)$ of the rank k *relatively r-blocking elements*, with respect to the map ω_ρ, of the antichain A, in analogy with the sets $\mathbf{I}_{r,k}(\mathbb{B}(n), \cdot)$ associated with the antichains of the Boolean lattices, cf. eq. (6.27):

$$\mathbf{I}_{r,k}\big(\mathbf{O}(m), \mathrm{A}\big) := \big\{b \in \mathbf{O}(m) : \rho(b) = k,\ \rho(b \wedge a) > r \cdot k,\quad \forall a \in \mathrm{A}\big\} \subset \mathbf{O}(m)^{(k)} ,$$

where $\mathbf{O}(m)^{(k)}$ is the kth layer of the lattice $\mathbf{O}(m)$.

On the one hand, we have

$$\big|\mathbf{I}_{r,k}\big(\mathbf{O}(m), \mathrm{A}\big)\big| = \sum_{\mathrm{X} \in \mathring{\mathscr{C}}_{r,k}(\mathbf{O}(m),\mathrm{A}) :\, \mathrm{X} > \hat{0}} \mu_{\mathring{\mathscr{C}}}(\hat{0}, \mathrm{X}) \cdot \sum_{z \in \mathring{\mathscr{E}}(\mathbf{O}(m),\mathrm{X}) :\, z > \hat{0}} \mu_{\mathring{\mathscr{E}}}(\hat{0}, z) \cdot 2^{k-\rho(z)} \cdot \binom{m-\rho(z)}{m-k} , \qquad (6.45)$$

cf. eq. (6.37), where $\mathring{\mathscr{C}}_{r,k}(\mathbf{O}(m), \mathrm{A})$ denotes the *join-semilattice* of all sets from the family $\{\mathbf{O}(m)^{(\lfloor r\cdot k\rfloor+1)} \cap \mathfrak{I}(\mathrm{C}) : \mathrm{C} \subseteq \mathrm{A},\ |\mathrm{C}| > 0\}$ ordered by inclusion and augmented by a new *least* element $\hat{0}$; the *greatest* element $\hat{1}$ of the *lattice* $\mathring{\mathscr{C}}_{r,k}(\mathbf{O}(m), \mathrm{A})$ is the set $\mathbf{O}(m)^{(\lfloor r\cdot k\rfloor+1)} \cap \mathfrak{I}(\mathrm{A})$. For an element $\mathrm{X} \in \mathring{\mathscr{C}}_{r,k}(\mathbf{O}(m), \mathrm{A})$, the notation $\mathring{\mathscr{E}}(\mathbf{O}(m), \mathrm{X})$ is used to denote the *sub-join-semilattice* of the lattice $\mathbf{O}(m)$ generated by the set $\mathrm{X} \subset \mathbf{O}(m)$, with the greatest element of $\mathbf{O}(m)$ deleted from it, and augmented by a new *least element* $\hat{0}$. The *Möbius functions* of the posets $\mathring{\mathscr{C}}_{r,k}(\mathbf{O}(m), \mathrm{A})$ and $\mathring{\mathscr{E}}(\mathbf{O}(m), \mathrm{X})$ are denoted by $\mu_{\mathring{\mathscr{C}}}(\cdot,\cdot)$ and $\mu_{\mathring{\mathscr{E}}}(\cdot,\cdot)$, respectively; $\rho(\cdot)$ means the rank of an element in the poset $\mathbf{O}(m)$.

On the other hand, we have

$$\big|\mathbf{I}_{r,k}\big(\mathbf{O}(m), \mathrm{A}\big)\big| = \sum_{\substack{\mathrm{D} \subseteq \mathbf{O}(m)^{(\lfloor r\cdot k\rfloor+1)} \cap \mathfrak{I}(\mathrm{A}) :\\ |\mathrm{D}|>0,\ \bigvee_{d \in \mathrm{D}} d \neq \hat{1}}} (-1)^{|\mathrm{D}|} \cdot \Bigg(\sum_{\mathrm{C} \subseteq \mathrm{A} :\, \mathrm{D} \subseteq \mathfrak{I}(\mathrm{C})} (-1)^{|\mathrm{C}|} \Bigg) \cdot 2^{k-\rho(\bigvee_{d\in\mathrm{D}} d)} \cdot \binom{m-\rho(\bigvee_{d\in\mathrm{D}} d)}{m-k} , \qquad (6.46)$$

cf. eq. (6.33).

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6.10 Relatively Blocking Elements in the Principal Order Ideals of Binomial Posets

Let P be a *graded lattice* of rank n which is a *principal order ideal* of some *binomial poset*. In this section we give further enumerative results on the *relatively blocking elements* of antichains of these lattices which include the Boolean lattices $\mathbb{B}(n)$ and the lattices $\mathbb{V}_q(n)$ of subspaces of the vector spaces over finite fields. Recall that the *factorial function* $\mathrm{B}(k)$ of P counts the number of *maximal chains* in any *interval* of length k in P. The number $\left[{j \atop i} \right]$ of elements of rank i in any interval of length j of P is equal to $\frac{\mathrm{B}(j)}{\mathrm{B}(i)\cdot \mathrm{B}(j-i)}$; see Section 6.4.

Let $\mathrm{A} := \{a_1, \ldots, a_\alpha\}$ be a nontrivial antichain of the lattice P. If r is a rational number, $0 \le r < 1$, and k is a positive integer, then the subposet $\mathbf{I}_{r,k}(P, \mathrm{A}) := \mathbf{I}_{r,k}(P, \mathrm{A}; \omega_\rho)$ of the rank k *relatively r-blocking elements*, with respect to the map ω_ρ, for the antichain A in P, is defined as earlier by

$$\mathbf{I}_{r,k}\big(P, \mathrm{A}\big) := \Big\{b \in P : \rho(b) = k,\ \rho(b \wedge a) > r \cdot k,\ \ \forall a \in \mathrm{A}\Big\} \subset P^{(k)} ,$$

where $\rho(\cdot)$ is the poset rank of an element in P, and $P^{(k)}$ is the kth *layer* of P.

Let $\mathscr{N}(\mathrm{A})$ be an *abstract simplicial complex* whose *facets* are the *inclusion-maximal* sets of indices $\{i_1, \ldots, i_j\}$ such that for the corresponding antichains $\{a_{i_1}, \ldots, a_{i_j}\} \subseteq \mathrm{A}$ we have $a_{i_1} \wedge \cdots \wedge a_{i_j} \succ \hat{0}$, where $\hat{0}$ is the *least* element of P. If the poset P is the *Boolean lattice* $\mathbb{B}(n)$, then the complex $\mathscr{N}(\mathrm{A})$ is the *nerve* of the corresponding *clutter* $\{\mathfrak{I}(a) \cap \mathbb{B}(n)^{(1)} : a \in \mathrm{A}\}$.

Again set $\nu(r \cdot k) := \lfloor r \cdot k \rfloor + 1$, and suppose that $\nu(r \cdot k) \le \min_{a \in \mathrm{A}} \rho(a)$.

Let

$$\begin{aligned}\mathfrak{c}_{r,k} : P^{(\nu(r\cdot k))} \cap \mathfrak{I}(\mathrm{A}) &\to \mathscr{N}(\mathrm{A}) ,\\ d &\mapsto \mathbf{max}\Big\{N \in \mathscr{N}(\mathrm{A}) : d \preceq \bigwedge_{i \in N} a_i\Big\}\end{aligned}$$

be the map that sends a rank $\nu(r \cdot k)$ element d of the order ideal $\mathfrak{I}(\mathrm{A})$ generated by the antichain A to the *inclusion-maximal face* of the complex $\mathscr{N}(\mathrm{A})$ *with the property* $d \preceq \bigwedge_{i \in N} a_i$.

Pick an arbitrary nonempty subset $\mathrm{D} \subseteq P^{(\nu(r\cdot k))} \cap \mathfrak{I}(\mathrm{A})$, and consider the set-theoretic *blocker* $\mathfrak{B}(\mathfrak{c}_{r,k}(\mathrm{D}))$ of the image $\mathfrak{c}_{r,k}(\mathrm{D})$; if we let $\mathbf{min}\ \mathfrak{c}_{r,k}(\mathrm{D})$ denote the subfamily of all *inclusion-minimal sets* of the family $\mathfrak{c}_{r,k}(\mathrm{D})$, then we know that $\mathfrak{B}(\mathfrak{c}_{r,k}(\mathrm{D})) = \mathfrak{B}(\mathbf{min}\ \mathfrak{c}_{r,k}(\mathrm{D}))$.

Let $\Delta^*(\mathrm{D})$ be an *abstract simplicial complex* whose *facets* are the *complements* $[\alpha] - B$ *of* the *minimal blocking sets* $B \in \mathfrak{B}(\mathbf{min}\ \mathfrak{c}_{r,k}(\mathrm{D}))$ in the *blocker* of the *clutter* $\mathbf{min}\ \mathfrak{c}_{r,k}(\mathrm{D})$, and let $\Delta(\mathrm{D})$ be the complex whose *facets* are the *complements* $[\alpha] - G$ *of* the sets $G \in \mathbf{min}\ \mathfrak{c}_{r,k}(\mathrm{D})$; if the complexes $\Delta(\mathrm{D})$ and $\Delta^*(\mathrm{D})$ have the same vertex set, then $\Delta^*(\mathrm{D})$ is by definition the *Alexander dual* of $\Delta(\mathrm{D})$. It is well known that the *reduced*

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Euler characteristics $\widetilde{\chi}(\cdot)$ of such complexes satisfy the equality

$$\widetilde{\chi}\big(\Delta^*(\mathrm{D})\big) = (-1)^{\alpha-1}\widetilde{\chi}\big(\Delta(\mathrm{D})\big)\,. \tag{6.47}$$

For a subset $\mathrm{C} := \{a_{i_1}, \ldots, a_{i_j}\} \subseteq \mathrm{A}$, we have $\mathrm{D} \subseteq \mathfrak{I}(\mathrm{C})$ if and only if the collection of indices $\{i_1, \ldots, i_j\}$ is a *blocking set* of the family $\mathbf{min}\ \mathfrak{c}_{r,k}(\mathrm{D})$; therefore,

$$\begin{aligned}\sum_{\mathrm{C}\subseteq\mathrm{A}\,:\,\mathrm{D}\subseteq\mathfrak{I}(\mathrm{C})} (-1)^{|\mathrm{C}|} &= (-1)^{\alpha-1}\widetilde{\chi}\big(\Delta^*(\mathrm{D})\big)\\ &= \widetilde{\chi}\big(\Delta(\mathrm{D})\big)\,.\end{aligned} \tag{6.48}$$

If $\bigcup_{F\in\mathbf{min}\ \mathfrak{c}_{r,k}(\mathrm{D})} F \subsetneqq [\alpha]$, then the complex $\Delta^*(\mathrm{D})$ is a *cone* and, as a consequence, $\widetilde{\chi}(\Delta^*(\mathrm{D})) = \widetilde{\chi}(\Delta(\mathrm{D})) = 0$.

According to eq. (6.21), we have

$$\begin{aligned}|\mathbf{I}_{r,k}(P,\mathrm{A})| = &\sum_{\mathrm{D}\subseteq P^{(\nu(r\cdot k))}\cap\mathfrak{I}(\mathrm{A})\,:\,|\mathrm{D}|>0} (-1)^{|\mathrm{D}|}\\ &\cdot\Bigg(\sum_{\mathrm{C}\subseteq\mathrm{A}\,:\,\mathrm{D}\subseteq\mathfrak{I}(\mathrm{C})} (-1)^{|\mathrm{C}|}\Bigg)\cdot\begin{bmatrix} n-\rho(\bigvee_{e\in\mathrm{D}} d)\\ n-k\end{bmatrix}\,,\end{aligned} \tag{6.49}$$

and relations (6.48) yield

$$|\mathbf{I}_{r,k}(P,\mathrm{A})| = \sum_{\substack{\mathrm{D}\subseteq P^{(\nu(r\cdot k))}\cap\mathfrak{I}(\mathrm{A}):\\ \bigcup_{F\in\mathbf{min}\ \mathfrak{c}_{r,k}(\mathrm{D})} F=[\alpha]}} (-1)^{|\mathrm{D}|}\cdot\widetilde{\chi}\big(\Delta(D)\big)\cdot\begin{bmatrix} n-\rho(\bigvee_{e\in\mathrm{D}} d)\\ k-\rho(\bigvee_{e\in\mathrm{D}} d)\end{bmatrix}\,.$$

Given a subset $\mathrm{D} \subseteq P^{(\nu(r\cdot k))} \cap \mathfrak{I}(\mathrm{A})$ such that $\bigcup_{F\in\mathbf{min}\mathfrak{c}_{r,k}(\mathrm{D})} F = [\alpha]$, let $\mathscr{S}(\mathrm{D})$ be the family of unions $\{\bigcup_{F\in\mathscr{F}} F : \mathscr{F} \subseteq \mathbf{min}\mathfrak{c}_{r,k}(\mathrm{D}),\ \#\mathscr{F} > 0\}$ ordered by inclusion and augmented by a new *least* element $\hat{0}$; the *greatest* element $\hat{1}$ of the *lattice* $\mathscr{S}(\mathrm{D})$ is the index set $[\alpha]$. The reduced Euler characteristic

$$\widetilde{\chi}\big(\Delta(\mathrm{D})\big) = \sum_{\substack{\mathscr{F}\subseteq\mathbf{min}\ \mathfrak{c}_{r,k}(\mathrm{D}):\\ \bigcup_{F\in\mathscr{F}} F=[\alpha]}} (-1)^{\#\mathscr{F}}$$

of the complex $\Delta(\mathrm{D})$ is equal to the *Möbius number* $\mu_{\mathscr{S}(\mathrm{D})}(\hat{0},\hat{1})$ *of* the lattice $\mathscr{S}(\mathrm{D})$. Note that $\widetilde{\chi}(\Delta(\mathrm{D})) = \mu_{\mathscr{S}(\mathrm{D})}(\hat{0},\hat{1}) = (-1)^{\#\mathbf{min}\ \mathfrak{c}_{r,k}(\mathrm{D})}$ when the sets in the family $\mathbf{min}\ \mathfrak{c}_{r,k}(\mathrm{D})$ are *pairwise disjoint*.

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We conclude this section by restating eq. (6.49) as follows:

$$|\mathbf{I}_{r,k}(P,\mathrm{A})| \quad = \sum_{\substack{\mathrm{D}\subseteq P^{(\nu(r\cdot k))}\cap\mathfrak{I}(\mathrm{A}):\\ 1\le|\mathrm{D}|\le\left[{k \atop \nu(r\cdot k)}\right],\\ |\bigcup_{F\in\mathbf{min}\ \mathfrak{c}_{r,k}(\mathrm{D})}F|=|\mathrm{A}|=:\alpha,\\ \rho(\bigvee_{d\in\mathrm{D}}d)\le k}} (-1)^{|\mathrm{D}|}\cdot\mu_{\mathscr{S}(\mathrm{D})}(\hat{0},\hat{1})\cdot\begin{bmatrix} n-\rho(\bigvee_{d\in\mathrm{D}}d)\\ k-\rho(\bigvee_{d\in\mathrm{D}}d)\end{bmatrix}.$$

Notes

Committee constructions for contradictory problems are surveyed, for example, in Refs. [86, 109, 142–145].

It is shown in the monograph [142] that various problems of collective decision-making may often be reduced to the problem of constructing an *r-committee* of a certain *family of sets*, as defined by eq. (6.1); this concept is given, for example, in Ref. [142, Definition 2.1.4].

The *Principle of Inclusion–Exclusion* and *Möbius inversion* on *posets* are important subjects in combinatorial theory, explained, for example, in [5, Ch. 5], [6, Ch. IV], [12, Sect. 1.5], [37, Ch. 6], [91], [115, Ch. 3], [124, Ch. 10], [125, Ch. 4], [129, Ch. 4], [179, Ch. 3].

The maps ω defined by eqs. (6.2)–(6.4) are sometimes well expressed in terms of incidence functions; see, for example, [6, Ch. IV] and [179, Ch. 3] on *incidence functions* of *posets*.

If A is a nontrivial antichain of the Boolean lattice $\mathbb{B}(n)$ then its order ideal $\mathfrak{I}(\mathrm{A})$ is assigned the isomorphic *face poset* of the *abstract simplicial complex* whose *facets* are the sets of the family $\{\mathfrak{I}(a)\cap\mathbb{B}(n)^{(1)} : a\in\mathrm{A}\}$; for references, see the Notes to Chapter 2 on page 65.

Galois connections are discussed, for example, in Refs. [6, Sections IV.3.A,B], [115, Sect. 1.5]. Proposition 6.9(i) follows from [6, Prop. 4.36], and Proposition 6.9(ii) follows from Ref. [6, Prop. 4.26].

On the q-analogue $\sum_{k=0}^{n}\binom{a}{k}_q\binom{b}{n-k}_q q^{k(k+b-n)}=\binom{a+b}{n}_q$ of the *Chu–Vandermonde identity* $\sum_{k=0}^{n}\binom{a}{k}\binom{b}{n-k}=\binom{a+b}{n}$, see for example [10, Sect. 4] and [179, p. 167].

The *binomial posets* are discussed for example in Refs. [61] and [179, Sect. 3.18].

For properties of *crosspolytopes* (or *hyperoctahedra*, *orthoplexes*), see for example [21, Sect. 2.5], [23, 96, 100, 130, 163].

For references on the combinatorial topology of the *nerve*, and on the *Nerve Theorem*, see the Notes to Chapter 1 on page 52.

Several references on combinatorial *Alexander duality* can be found in Notes to Chapter 2 on page 66.

Recall that the *reduced Euler characteristic* $\tilde{\chi}(\Delta)$ of an abstract simplicial complex Δ is defined by $\tilde{\chi}(\Delta):=-1+\sum_{i\ge 0}(-1)^i f_i(\Delta)$. Equality (6.47) is a consequence of the basic statement given in [179, Prop. 3.16.5]. By [114, Prop. 13.7] the *reduced Euler characteristic* of an abstract *cone* is equal to zero.

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7 Layers of Tope Committees

Let $\mathscr{M} := (E_t, \mathscr{T})$ be a *simple oriented matroid* (recall that throughout the book "*simple*" means that $\mathscr{M}$ has *no loops*, *parallel elements*, or *antiparallel elements*) of rank $r(\mathscr{M}) \geq 2$, on its *ground set* $E_t := [t]$, with set of *topes* $\mathscr{T} \subset \{-1,1\}^{E_t} := \{-1,1\}^t$.

It was shown in Chapter 1 that the family of *tope committees* for $\mathscr{M}$, denoted by $\mathbf{K}^*(\mathscr{M})$, is nonempty.

In this chapter, we give a poset-theoretic description of the structure of the family $\mathbf{K}^*(\mathscr{M})$ for a *simple oriented matroid* $\mathscr{M} := (E_t, \mathscr{T})$ which is *not acyclic*. The cardinality of any halfspace of the oriented matroid $\mathscr{M}$ is $|\mathscr{T}|/2$, and our structural description of the family of tope committees for $\mathscr{M}$ involves the *Farey subsequence* $\mathscr{F}(\mathbb{B}(|\mathscr{T}|), |\mathscr{T}|/2)$ and the standard *Farey sequence* $\mathscr{F}_{|\mathscr{T}|/2}$; see Chapter 4 on these sequences of fractions.

If $\mathscr{K}^*$ is a tope committee for the oriented matroid $\mathscr{M}$, then a disjoint union $\mathscr{K}^* \dot\cup \{T, -T\}$ of $\mathscr{K}^*$ with a pair of opposite topes $\{T, -T\} \subset \mathscr{T}$ is also a committee for $\mathscr{M}$. Similarly, if $\mathscr{K}'^*$ and $\mathscr{K}''^*$ are disjoint tope committees for $\mathscr{M}$, then their union is also a committee. Such redundant committees have no applied significance, because in practice one seeks inclusion-minimal committees: if $\mathscr{K}'^*$ and $\mathscr{K}''^*$ are tope committees for $\mathscr{M}$ such that $\mathscr{K}'^* \subsetneq \mathscr{K}''^*$, then the committee $\mathscr{K}'^*$ is preferred. Recall that a committee $\mathscr{K}^*$ is said to be *minimal* if any its proper subset is not a committee. The minimal committees clearly contain no pairs of opposites.

In Section 7.1 we regard the *tope committees* for the oriented matroid $\mathscr{M}$ as the *relatively $\frac{1}{2}$-blocking elements* of the *antichain of positive halfspaces* of $\mathscr{M}$ in the *Boolean lattice* $\mathbb{B}(\mathscr{T})$ of subsets of the set of topes. We describe the subfamilies of committees for $\mathscr{M}$ having fixed sizes and thus lying on certain *layers* of the lattice $\mathbb{B}(\mathscr{T})$.

In Section 7.2 we investigate the layered structure of the subfamily $\mathring{\mathbf{K}}^*(\mathscr{M}) \subset \mathbf{K}^*(\mathscr{M})$ of tope committees for $\mathscr{M}$ which contain *no pairs* of *opposites*.

7.1 Layers of Tope Committees, and Relatively Blocking Elements in the Boolean Lattice of Tope Subsets

In this section we consider the layered families of all *tope committees* for oriented matroids.

Let $\mathscr{M} = (E_t, \mathscr{T})$ be a *simple oriented matroid* on the ground set E_t, with set of topes $\mathscr{T}$; we suppose that $\mathscr{M}$ is *not acyclic*. We denote the Boolean lattice of subsets of the set $\mathscr{T}$ by $\mathbb{B}(\mathscr{T})$. Let Υ be the *antichain* $\{v_1, \ldots, v_t\} \subset \mathbb{B}(\mathscr{T})^{(|\mathscr{T}|/2)}$ corresponding to the *family of positive halfspaces* $\{\mathscr{T}_1^+ \subset \mathbb{B}(\mathscr{T})^{(1)}, \ldots, \mathscr{T}_t^+ \subset \mathbb{B}(\mathscr{T})^{(1)}\}$ of $\mathscr{M}$:

$$\mathbb{B}(\mathscr{T})^{(|\mathscr{T}|/2)} \supset \Upsilon \ni v_e \iff \mathfrak{I}(v_e) \cap \mathbb{B}(\mathscr{T})^{(1)} = \mathscr{T}_e^+ , \quad \forall e \in E_t ; \tag{7.1}$$

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indeed, the poset rank $\rho(v_e)$ of any element $v_e \in \Upsilon$ in $\mathbb{B}(\mathscr{T})$ is $|\mathscr{T}|/2$; therefore, the antichain Υ lies entirely on the $(|\mathscr{T}|/2)$th *layer* of the graded lattice $\mathbb{B}(\mathscr{T})$.

In exact analogy with the convention (7.1), we will interpret the tope committees $\mathscr{K}^* \subset \mathscr{T}$ for $\mathscr{M}$ as elements $b \in \mathbb{B}(\mathscr{T})$. More precisely, we establish a one-to-one correspondence between the family

$$\mathbf{K}^*(\mathscr{M}) := \left\{\mathscr{K}^* \subset \mathscr{T} : |\mathscr{K}^* \cap \mathscr{T}_e^+| > \tfrac{1}{2}|\mathscr{K}^*|,\ \forall e \in E_t\right\}$$

of all tope committees for $\mathscr{M}$ and the subposet

$$\mathbf{I}_{\frac{1}{2}}\left(\mathbb{B}(\mathscr{T}), \Upsilon\right) := \left\{b \in \mathbb{B}(\mathscr{T}) : \rho(b) > 0,\ \tfrac{\rho(b \wedge v_e)}{\rho(b)} > \tfrac{1}{2},\ \forall e \in E_t\right\}$$

of all *relatively* $\frac{1}{2}$*-blocking elements* of the antichain Υ in $\mathbb{B}(\mathscr{T})$ in the following way:

$$\mathbf{K}^*(\mathscr{M}) \ni \mathscr{K}^* \quad \Longleftrightarrow \quad \bigvee_{K \in \mathscr{K}^*} K =: b \in \mathbf{I}_{\frac{1}{2}}\left(\mathbb{B}(\mathscr{T}), \Upsilon\right) \subset \mathbb{B}(\mathscr{T}) . \tag{7.2}$$

Relative blocking constructions in finite bounded posets were discussed in Chapter 6. The antichain $\mathfrak{y}_{\frac{1}{2}}\left(\mathbb{B}(\mathscr{T}), \Upsilon\right) := \mathbf{min}\, \mathbf{I}_{\frac{1}{2}}\left(\mathbb{B}(\mathscr{T}), \Upsilon\right)$ called, according to Definition 6.5(ii), the *relative* $\frac{1}{2}$*-blocker of the antichain* Υ *in* $\mathbb{B}(\mathscr{T})$, is regarded via eq. (7.2) as the family of all *minimal tope committees* for the oriented matroid $\mathscr{M}$.

Since we consider the oriented matroid $\mathscr{M}$ which is *not acyclic*, the family of its tope committees $\mathbf{K}^*(\mathscr{M})$ does not contain the singleton set $\{\mathrm{T}^{(+)}\}$. For any k, $3 \le k \le |\mathscr{T}| - 3$, by Proposition 6.12(ii) the subposet

$$\mathbf{I}_{\frac{1}{2},k}\left(\mathbb{B}(\mathscr{T}), \Upsilon\right) := \mathbb{B}(\mathscr{T})^{(k)} \cap \mathbf{I}_{\frac{1}{2}}\left(\mathbb{B}(\mathscr{T}), \Upsilon\right) \tag{7.3}$$

of the rank k elements in $\mathbf{I}_{\frac{1}{2}}\, \mathbb{B}(\mathscr{T}), \Upsilon)$ is the antichain

$$\mathbb{B}(\mathscr{T})^{(k)} \cap \bigcap_{e \in E_t} \mathfrak{F}\left(\mathfrak{I}(v_e) \cap \mathbb{B}(\mathscr{T})^{(\lceil (k+1)/2 \rceil)}\right) . \tag{7.4}$$

We conclude this section with a result on the layered structure of the family of tope committees that combines the descriptions of layers (7.4) with a direct consequence of Theorem 6.18.

Proposition 7.1. *Let* $\mathscr{M}$ *be a simple oriented matroid, which is* not acyclic, *on the ground set* E_t, *with set of topes* $\mathscr{T}$.

On the one hand,

$$\mathbf{I}_{\frac{1}{2}}\left(\mathbb{B}(\mathscr{T}), \Upsilon\right) = \dot{\bigcup_{3 \le k \le |\mathscr{T}|-3}} \left(\mathbb{B}(\mathscr{T})^{(k)} \cap \bigcap_{e \in E_t} \mathfrak{F}\left(\mathfrak{I}(v_e) \cap \mathbb{B}(\mathscr{T})^{(\lceil (k+1)/2 \rceil)}\right)\right)$$

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and, in particular,

$$\mathbf{I}_{\frac{1}{2},3}\Big(\mathbb{B}(\mathscr{T}),\Upsilon\Big) = \mathbb{B}(\mathscr{T})^{(3)} \cap \bigcap_{e\in E_t} \mathfrak{F}\Big(\mathfrak{I}(v_e)\cap \mathbb{B}(\mathscr{T})^{(2)}\Big) .$$

On the other hand,

$$\mathbf{I}_{\frac{1}{2}}\Big(\mathbb{B}(\mathscr{T}),\Upsilon\Big) = \bigcap_{e\in E_t} \ \bigcup_{\substack{f\in\mathscr{F}(\mathbb{B}(|\mathscr{T}|),|\mathscr{T}|/2):\\ \frac{1}{2}<f}} \ \bigcup_{1\le s\le\lfloor|\mathscr{T}|/(2\underline{f})\rfloor} \Bigg(\mathbb{B}(\mathscr{T})^{(s\cdot\bar{f})} \cap \bigg(\mathfrak{F}\Big(\mathfrak{I}(v_e)\cap \mathbb{B}(\mathscr{T})^{(s\cdot\underline{f})}\Big) - \mathfrak{F}\Big(\mathfrak{I}(v_e)\ \cap\ \mathbb{B}(\mathscr{T})^{(s\cdot\underline{f}+1)}\Big)\bigg)\Bigg) .$$

Recall that one construct refining the description of the layers $\mathbb{B}(n)^{(d)}$, $1\le d\le\lfloor n/2\rfloor$, of the Boolean lattice $\mathbb{B}(n)$ of subsets of an n-set, is the Johnson association scheme.

The *Johnson scheme* $\mathbf{J}(n,d)$ is defined to be the pair $(\boldsymbol{X},\mathscr{R})$, where $\boldsymbol{X} := \mathbb{B}(n)^{(d)}$, with $|\boldsymbol{X}| = \binom{n}{d}$, and $\mathscr{R} := (\boldsymbol{R}_0,\boldsymbol{R}_1,\ldots,\boldsymbol{R}_d)$ is a partition of $\boldsymbol{X}\times\boldsymbol{X}$ given by

$$\boldsymbol{R}_i := \Big\{(x,y):\partial(x,y) := d-\rho(x\wedge y) = i\Big\} ,\quad 0\le i\le d .$$

For any $x,y\in\boldsymbol{X}$ with $\partial(x,y)=k$, and for any integers i and j, $0\le i,j\le d$, the *intersection numbers*

$$\begin{aligned}
\mathrm{p}_{ij}^{k} &:= \Big|\{z\in\boldsymbol{X}:\partial(z,x)=i,\ \partial(z,y)=j\}\Big|\\
&= \bigg|\mathbb{B}(n)^{(d)}\cap\Big(\mathfrak{F}\big(\mathfrak{I}(x)\cap\mathbb{B}(n)^{(d-i)}\big)-\mathfrak{F}\big(\mathfrak{I}(x)\cap\mathbb{B}(n)^{(d-i+1)}\big)\Big)\\
&\qquad \cap\Big(\mathfrak{F}\big(\mathfrak{I}(y)\cap\mathbb{B}(n)^{(d-j)}\big)-\mathfrak{F}\big(\mathfrak{I}(y)\cap\mathbb{B}(n)^{(d-j+1)}\big)\Big)\bigg|
\end{aligned}$$

are the same. We have

$$\mathrm{p}_{ij}^{k} = \sum_{c}\binom{d-k}{c}\binom{k}{d-i-c}\binom{k}{d-j-c}\binom{n-d-k}{i+j-d+c} ;$$

the quantity $\mathrm{n}_i := \mathrm{p}_{ii}^{0} = |\{z\in\boldsymbol{X}:\partial(z,x)=i\}|$, for any $x\in\boldsymbol{X}$, called the *valency* of $\boldsymbol{R}_i$, is $\mathrm{n}_i = \binom{d}{i}\binom{n-d}{i}$.

We conclude this section by noting that the family $\mathbf{K}^*(\mathscr{M})$ of all tope committees for any simple oriented matroid $\mathscr{M} := (E_t,\mathscr{T})$ can be regarded as the poset $\mathbf{I}_{\frac{1}{2}}\Big(\mathbb{B}(\mathscr{T}),\Upsilon\Big)$ of relatively $\frac{1}{2}$-blocking elements of the *subset* Υ of the Johnson scheme $\mathbf{J}(|\mathscr{T}|,|\mathscr{T}|/2) := (\boldsymbol{X},\mathscr{R})$ on the set $\boldsymbol{X} := \mathbb{B}(\mathscr{T})^{(|\mathscr{T}|/2)}$, with the partition $\mathscr{R} := (\boldsymbol{R}_0,\boldsymbol{R}_1,\ldots,\boldsymbol{R}_{|\mathscr{T}|/2})$ of $\boldsymbol{X}\times\boldsymbol{X}$ defined by $\boldsymbol{R}_i := \{(x,y):\partial(x,y) := \frac{1}{2}|\mathscr{T}|-\rho(x\wedge y)=i\}$, for all $0\le i\le|\mathscr{T}|/2$.

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7.2 Layers of Tope Committees, and Relatively Blocking Elements in the Poset of Tope Subsets Containing No Pairs of Opposites

In this section we are concerned with the layers of *tope committees* for oriented matroids, which contain *no pairs* of *opposites*.

For any element $e \in E_t$, the corresponding positive halfspace $\mathcal{T}_e^+$ of a simple oriented matroid $\mathcal{M} := (E_t, \mathcal{T})$ clearly contains *no pairs* of *opposites*. On the other hand, the tope committees containing pairs of opposites have no applied significance. We will now describe the layered structure of the family

$$\mathring{\mathbf{K}}^*(\mathcal{M}) := \{\mathcal{K}^* \in \mathbf{K}^*(\mathcal{M}) : T \in \mathcal{K}^* \Longrightarrow -T \notin \mathcal{K}^*\}$$

of the committees for $\mathcal{M}$, which contain *no pairs* of *opposites*.

Denote by $\mathbf{O}'(\mathcal{T})$ a graded *meet-sub-semilattice* of the lattice $\mathbb{B}(\mathcal{T})$ defined as follows: the elements of $\mathbf{O}'(\mathcal{T})$ are the subsets of topes with *no pairs* of *opposites*, ordered by inclusion. This poset is isomorphic to the *face poset* of the *boundary* of a $(|\mathcal{T}|/2)$-dimensional *crosspolytope*. The rank $(|\mathcal{T}|/2+1)$ graded poset $\mathbf{O}(\mathcal{T}) := \mathbf{O}'(\mathcal{T})\dot{\cup}\{\hat{1}\}$, that is, the semilattice $\mathbf{O}'(\mathcal{T})$ with a greatest element adjoined is isomorphic to the *face lattice* of a $(|\mathcal{T}|/2)$-dimensional *crosspolytope*.

In this section, we define Υ to be the coatomic *antichain* $\{v_1, \ldots, v_t\} \subset \mathbf{O}(\mathcal{T})^{(|\mathcal{T}|/2)}$ corresponding to the *family of positive halfspaces* $\{\mathcal{T}_1^+ \subset \mathbf{O}(\mathcal{T})^{(1)}, \ldots, \mathcal{T}_t^+ \subset \mathbf{O}(\mathcal{T})^{(1)}\}$ of $\mathcal{M}$:

$$\mathbf{O}(\mathcal{T})^{(|\mathcal{T}|/2)} \supset \Upsilon \ni v_e \quad \Longleftrightarrow \quad \mathfrak{I}(v_e) \cap \mathbf{O}(\mathcal{T})^{(1)} = \mathcal{T}_e^+ \,, \quad \forall e \in E_t \;;$$

contrast this viewpoint with the definition of the antichain Υ used in Section 7.1.

We will interpret the tope committees $\mathcal{K}^* \subset \mathcal{T}$ for $\mathcal{M}$, free of pairs of opposites, as elements $b \in \mathbf{O}(\mathcal{T})$ by establishing a one-to-one correspondence between the family

$$\mathring{\mathbf{K}}^*(\mathcal{M}) := \left\{\mathcal{K}^* \subset \mathcal{T} : T \in \mathcal{K}^* \Longrightarrow -T \notin \mathcal{K}^*,\ |\mathcal{K}^* \cap \mathcal{T}_e^+| > \tfrac{1}{2}|\mathcal{K}^*|,\ \ \forall e \in E_t\right\}$$

of all tope committees for $\mathcal{M}$, free of pairs of opposites, and the subposet

$$\mathbf{I}_{\frac{1}{2}}\Big(\mathbf{O}(\mathcal{T}), \Upsilon\Big) := \left\{b \in \mathbf{O}(\mathcal{T}) : \rho(b) > 0,\ \tfrac{\rho(b \wedge v_e)}{\rho(b)} > \tfrac{1}{2},\ \ \forall e \in E_t\right\}$$

of all *relatively* $\frac{1}{2}$*-blocking elements* of the antichain Υ in $\mathbf{O}(\mathcal{T})$, as follows:

$$\mathring{\mathbf{K}}^*(\mathcal{M}) \ni \mathcal{K}^* \quad \Longleftrightarrow \quad \bigvee_{K \in \mathcal{K}^*} K =: b \in \mathbf{I}_{\frac{1}{2}}\Big(\mathbf{O}(\mathcal{T}), \Upsilon\Big) \subset \mathbf{O}(\mathcal{T}) \,.$$

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Given a simple oriented matroid $\mathscr{M}$ which is *not acyclic*, for any k, $3 \le k \le |\mathscr{T}|/2$, by Proposition 6.12(ii) the subposet

$$\mathbf{I}_{\frac{1}{2},k}\big(\mathbf{O}(\mathscr{T}),\Upsilon\big) := \mathbf{O}(\mathscr{T})^{(k)} \cap \mathbf{I}_{\frac{1}{2}}\big(\mathbf{O}(\mathscr{T}),\Upsilon\big)$$

of the rank k elements in $\mathbf{I}_{\frac{1}{2}}\big(\mathbf{O}(\mathscr{T}),\Upsilon\big)$ is the antichain

$$\mathbf{O}(\mathscr{T})^{(k)} \cap \bigcap_{e \in E_t} \mathfrak{F}\Big(\mathfrak{I}(v_e) \cap \mathbf{O}(\mathscr{T})^{(\lceil (k+1)/2 \rceil)}\Big)\,, \tag{7.5}$$

cf. eqs. (7.3) and (7.4).

Associate with the elements $a \in \mathbf{O}(\mathscr{T}) - \{\hat{0}, \hat{1}\}$ increasing sequences $\mathscr{F}(\mathbf{O}(\mathscr{T}), \rho(a))$ of irreducible fractions defined by

$$\mathscr{F}\big(\mathbf{O}(\mathscr{T}),\rho(a)\big) := \left(\frac{\rho(b \wedge a)}{\gcd(\rho(b \wedge a),\rho(b))} \Big/ \frac{\rho(b)}{\gcd(\rho(b \wedge a),\rho(b))} : b \in \mathbf{O}(\mathscr{T}) - \{\hat{0},\hat{1}\} \right) .$$

If $\rho(a) < |\mathscr{T}|/2$, then

$$\mathscr{F}\big(\mathbf{O}(\mathscr{T}),\rho(a)\big) = \big(\tfrac{h}{k} \in \mathscr{F}_{|\mathscr{T}|/2} : h \le \rho(a)\big)\,,$$

where $\mathscr{F}_{|\mathscr{T}|/2}$ is the standard *Farey sequence* of order $|\mathscr{T}|/2$. In the case where $\rho(a) = |\mathscr{T}|/2$, we have

$$\mathscr{F}\big(\mathbf{O}(\mathscr{T}),\rho(a)\big) = \mathscr{F}_{|\mathscr{T}|/2}\,.$$

Let us give an analogue of Proposition 7.1, which is based on description (7.5) and follows from Theorem 6.18; it describes the layered structure of the family of tope committees containing *no pairs* of *opposites*.

Proposition 7.2. *Let $\mathscr{M}$ be a simple oriented matroid, which is* not acyclic*, on the ground set E_t, with set of topes $\mathscr{T}$.*

On the one hand,

$$\mathbf{I}_{\frac{1}{2}}\big(\mathbf{O}(\mathscr{T}),\Upsilon\big) = \dot{\bigcup_{3 \le k \le |\mathscr{T}|/2}} \Big(\mathbf{O}(\mathscr{T})^{(k)} \cap \bigcap_{e \in E_t} \mathfrak{F}\Big(\mathfrak{I}(v_e) \cap \mathbf{O}(\mathscr{T})^{(\lceil (k+1)/2 \rceil)}\Big)\Big)$$

and, in particular,

$$\mathbf{I}_{\frac{1}{2},3}\big(\mathbf{O}(\mathscr{T}),\Upsilon\big) = \mathbf{O}(\mathscr{T})^{(3)} \cap \bigcap_{e \in E_t} \mathfrak{F}\Big(\mathfrak{I}(v_e) \cap \mathbf{O}(\mathscr{T})^{(2)}\Big)\,.$$

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On the other hand,

$$\mathbf{I}_{\frac{1}{2}}\big(\mathbf{O}(\mathscr{T}),\Upsilon\big) = \bigcap_{e\in E_t}\ \bigcup_{\substack{f\in\mathscr{F}_{|\mathscr{T}|/2}:\\ \frac{1}{2}<f}}\ \bigcup_{1\le s\le\lfloor|\mathscr{T}|/(2\bar{f})\rfloor} \Big(\mathbf{O}(\mathscr{T})^{(s\cdot\bar{f})}\cap\Big(\mathfrak{F}\big(\mathfrak{I}(v_e)\cap\mathbf{O}(\mathscr{T})^{(s\cdot\underline{f})}\big)-\mathfrak{F}\big(\mathfrak{I}(v_e)\cap\mathbf{O}(\mathscr{T})^{(s\cdot\underline{f}+1)}\big)\Big)\Big)\,.$$

The *three-tope committees* for $\mathscr{M}$, regarded as the antichains $\mathbf{I}_{\frac{1}{2},3}\big(\mathbb{B}(\mathscr{T}),\Upsilon\big)$ and $\mathbf{I}_{\frac{1}{2},3}\big(\mathbf{O}(\mathscr{T}),\Upsilon\big)$, and mentioned in Propositions 7.1 and 7.2, are further discussed and enumerated in Chapter 8.

Let us consider, as earlier in Section 6.9, the family $\mathbf{O}'(m)$ of all the subsets $V\subset\pm[m]$ of the set $\pm[m] := \{-m,\ldots,-1,1,\ldots,m\}$ satisfying the condition

$$v\in V \implies -v\notin V\,, \tag{7.6}$$

and ordered by inclusion. The poset $\mathbf{O}'(m)\dot\cup\{\hat{1}\}$, that is, the *meet-semilattice* $\mathbf{O}'(m)$ augmented by a greatest element $\hat{1}$, is a graded lattice, of rank $(m+1)$, isomorphic to the *face lattice* of an m-dimensional *crosspolytope*; the *least* element $\hat{0}$ of $\mathbf{O}(m)$ is the empty subset of the set $\pm[m]$.

Fix an integer $d\in[m]$, and consider the dth layer $\boldsymbol{X} := \mathbf{O}(m)^{(d)}$ of the lattice $\mathbf{O}(m)$, that is, the family of all d-subsets $V\subset\pm[m]$ satisfying the condition (7.6) and viewed as elements of $\mathbf{O}(m)$. Recall that $|\boldsymbol{X}| := |\mathbf{O}(m)^{(d)}| = \binom{m}{d}2^d$.

Define a partition $\mathscr{R} := (\boldsymbol{R}_0,\boldsymbol{R}_1,\ldots,\boldsymbol{R}_d)$ of $\boldsymbol{X}\times\boldsymbol{X}$ as follows:

$$\boldsymbol{R}_i := \big\{(x,y):\partial(x,y) := d-\rho(x\wedge y) = i\big\}\,,\quad 0\le i\le d\,,$$

where $\rho(\cdot)$ is the poset rank of an element in $\mathbf{O}(m)$.

For all elements $x\in\boldsymbol{X}$, the quantities $\mathrm{n}_i := |\{z\in\boldsymbol{X}:\partial(z,x)=i\}|$ are the same: $\mathrm{n}_i = \binom{d}{i}\sum_{c=0}^{i}\binom{i}{c}\binom{m-d}{c}2^c = \binom{d}{i}\sum_{c=0}^{i}\binom{i}{c}\binom{m-d+c}{i}$, for any i, $0\le i\le d$.

Recall that in the case where $d := m$ and $\mathrm{n}_i = \binom{m}{i}$, for any i, $0\le i\le m$, the pair $(\boldsymbol{X},\mathscr{R})$ is an association scheme. Indeed, for any $x,y\in\boldsymbol{X}$ with $\partial(x,y) := m-\rho(x\wedge y) = k$, and for any integers i and j, $0\le i,j\le m$, the quantities

$$\begin{aligned}\mathrm{p}_{ij}^{k} &:= \big|\{z\in\boldsymbol{X}:\partial(z,x)=i,\ \partial(z,y)=j\}\big|\\ &= \Big|\mathbf{O}(m)^{(m)}\cap\Big(\mathfrak{F}\big(\mathfrak{I}(x)\cap\mathbf{O}(m)^{(m-i)}\big)-\mathfrak{F}\big(\mathfrak{I}(x)\cap\mathbf{O}(m)^{(m-i+1)}\big)\Big)\\ &\qquad\cap\Big(\mathfrak{F}\big(\mathfrak{I}(y)\cap\mathbf{O}(m)^{(m-j)}\big)-\mathfrak{F}\big(\mathfrak{I}(y)\cap\mathbf{O}(m)^{(m-j+1)}\big)\Big)\Big|\end{aligned}$$

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are the same; the *intersection numbers*

$$\mathrm{p}_{ij}^k = \sum_c \binom{m-k}{c}\binom{k}{m-i-c}\binom{i+k-m+c}{m-j-c}\binom{m-k-c}{i+j-m+c}$$
$$= \begin{cases} \binom{m-k}{\frac{i+j-k}{2}}\binom{k}{\frac{i-j+k}{2}}\,, & \text{if } i+j+k \text{ is even}\,, \\ 0\,, & \text{if } i+j+k \text{ is odd}\,, \end{cases} \tag{7.7}$$

are those of the *Hamming* association *scheme* $\mathbf{H}(m,2)$ which is defined to be the family $\mathbf{X}$, of cardinality 2^m, of all words $\mathbf{x} \in \{-1,1\}^m$, together with the partition $\mathscr{R} := (\mathbf{R}_0, \mathbf{R}_1, \dots, \mathbf{R}_m)$ of $\mathbf{X}\times\mathbf{X}$ defined by $\mathbf{R}_i := \big\{(\mathbf{x},\mathbf{y}) : \partial(\mathbf{x},\mathbf{y}) := |\{\ell \in [m] : \mathrm{x}_\ell \neq \mathrm{y}_\ell\}| = i\big\}$, $0 \le i \le m$.

Let $\mathscr{M} := (E_t, \mathscr{T})$ be a simple oriented matroid. The family $\mathring{\mathbf{K}}^*(\mathscr{M})$ of its tope committees containing *no pairs* of *opposites* can be regarded as the subposet $\mathbf{I}_{\frac{1}{2}}\big(\mathbf{O}(\mathscr{T}), \Upsilon\big)$ of relatively $\frac{1}{2}$-blocking elements of the *subset* Υ *of the association scheme* $(\boldsymbol{X}, \mathscr{R})$ on the set $\boldsymbol{X} := \mathbf{O}(\mathscr{T})^{(|\mathscr{T}|/2)}$, with the partition $\mathscr{R} := (\boldsymbol{R}_0, \boldsymbol{R}_1, \dots, \boldsymbol{R}_{|\mathscr{T}|/2})$ of $\boldsymbol{X} \times \boldsymbol{X}$ defined by $\boldsymbol{R}_i := \big\{(x,y) : \partial(x,y) := \frac{1}{2}|\mathscr{T}| - \rho(x \wedge y) = i\big\}$, for all $0 \le i \le |\mathscr{T}|/2$; the parameters of $(\boldsymbol{X}, \mathscr{R})$ are those of the *Hamming scheme* $\mathbf{H}(|\mathscr{T}|/2, 2)$.

Notes

In oriented matroid theory the principal technique of enumeration for finding the number $|\mathscr{T}|$ of *topes* of an oriented matroid is that presented independently in the works [118, 120] and [183–185]; see for example [27, Sect. 4.6]. In particular, the enumeration of *regions* of *hyperplane arrangements* is discussed, for example, in Refs. [12, Sect. 1.7.2], [177, Sect. 2.2], [179, Sect. 3.11.3].

The *Johnson* association *scheme* is a central object in algebraic combinatorics; see, for example, Refs. [15, Sect. 3.2], [36, Sect. 2.7, 9.1], [58, Sect. 4.2], [128, Sect. 21.6], [149].

See, for example, Refs. [23, 163] on the structural and combinatorial properties of the face lattice $\mathbf{O}(m)$ of an m-dimensional *crosspolytope*, which is dual to the face lattice of an m-dimensional geometric *hypercube*. See also the Notes to Chapter 6 on page 142.

The graded lattice $\mathbf{O}(m)$ is *Eulerian*. Fundamental properties of the *Eulerian posets* are investigated, for example, in [179, Sect. 3.16]; see also [12, Sect. 1.5.6]. Note that the poset $\mathbf{O}'(m) := \mathbf{O}(m) - \{\hat{1}\}$ is *simplicial*.

In addition to the *Johnson scheme*, the *Hamming scheme* is another important object in algebraic combinatorics; see, for example, [15, Sect. 3.2], [36, Sect. 2.5, 9.2], [58, Sect. 4.1], [70, 71], [128, Sect. 21.3].

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8 Three-Tope Committees

When the ground set of an oriented matroid $\mathscr{M}$ is interpreted in the contradictory pattern recognition problem as (a reorientation of) an abstract *training set*, the *three-tope committee* becomes the most preferred approximation to the notion of *solution to an infeasible system of linear constraints*. In this chapter we are concerned exclusively with the three-tope committees.

General tope committees of size 3 are considered and enumerated in Section 8.1.

Section 8.2 is devoted to the *three-tope committees* having an applied significance, whose topes have *inclusion-maximal positive parts*.

8.1 Three-Tope Committees and Anti-committees

Given a finite bounded poset Q, we denote by Q^{c} the set of its coatoms. The singleton has no atoms and coatoms. For a chain $x \prec y$ of unit length, we set $\{x, y\}^{\mathrm{c}} := \{x\}$. $[x, z] := \{y \in Q : x \le y \le z\}$ is a *closed interval* of Q, and $\mu_Q(\cdot, \cdot)$ as earlier denotes the *Möbius function* of Q.

Recall that a subset $A \subset E_t$ of the ground set of an oriented matroid $\mathscr{M} := (E_t, \mathscr{T})$ is said to be *acyclic* if the restriction of $\mathscr{M}$ to the set A is an *acyclic* oriented matroid. The *convex hull* $\mathrm{conv}(A)$ of an *acyclic* set A is defined to be the set $B \supseteq A$ such that for every covector F of $\mathscr{M}$, and for every element $b \in B$, the implication $F(a) = 1, \forall a \in A \Rightarrow F(b) = 1$ is valid. An *acyclic set* A is *convex* if $\mathrm{conv}(A) = A$; its subset of *extreme points* $\mathrm{ex}(A)$ is defined by $\mathrm{ex}(A) := \{a \in A : a \notin \mathrm{conv}(A - \{a\})\}$. For any acyclic set A, we have $\mathrm{conv}(\mathrm{ex}(A)) = \mathrm{conv}(A)$. A convex set $A \subset E$ is *free* if $\mathrm{ex}(A) = \mathrm{conv}(A) = A$. The *meet-semilattice* $L_{\mathrm{conv}}(\mathscr{M})$ is defined to be the family of all convex subsets of E_t, ordered by inclusion.

We regard the *least* element $\hat{0}$ of $L_{\mathrm{conv}}(\mathscr{M})$ as the empty subset of E_t. If a subset $H \subseteq E_t$ is *not acyclic*, then we set $\mathrm{conv}(H) := E_t$; and we set $\mathrm{ex}(E_t) := E_t$.

In this section we deal with general tope committees of size 3, and we begin with constructs playing a role opposite to that played by committees.

Definition 8.1. A *tope anti-committee* for a simple oriented matroid $\mathscr{M} := (E_t, \mathscr{T})$ is a subset $\mathscr{A}^* \subset \mathscr{T}$ of its maximal covectors such that the set $-\mathscr{A}^* := \{-T : T \in \mathscr{A}^*\}$ is a *tope committee* for $\mathscr{M}$.

We denote by $\mathbf{K}_k^*(\mathscr{M})$ and $\mathbf{A}_k^*(\mathscr{M})$ the families of all tope committees and anti-committees of size k for $\mathscr{M}$, respectively. Recall that by axiomatic symmetry we have

$$\mathscr{T} = -\mathscr{T} \ , \tag{8.1}$$

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and it then follows from Definition 8.1 that

$$\#\mathbf{K}_k^*(\mathscr{M}) = \#\mathbf{A}_k^*(\mathscr{M}) = \#\mathbf{K}_{|\mathscr{T}|-k}^*(\mathscr{M}) = \#\mathbf{A}_{|\mathscr{T}|-k}^*(\mathscr{M}) ,$$

since for any tope committee $\mathscr{K}^*$ of size k, its complement $\mathscr{T} - \mathscr{K}^*$ is an anti-committee of size $|\mathscr{T}| - k$.

Since any three-tope anti-committee meets each positive halfspace of $\mathscr{M}$ in at most one tope, we have

$$\#\mathbf{A}_3^*(\mathscr{M}) = \sum_{\substack{S_1,S_2,S_3 \subset E_t:\\ |S_1|,|S_2|,|S_3|>0,\\ |S_1\cap S_2|=|S_1\cap S_3|=|S_2\cap S_3|=0}} \prod_{k\in[3]} \Big| \bigcap_{s\in S_k} \mathscr{T}_s^+ - \bigcup_{s\in E_t - S_k} \mathscr{T}_s^+ \Big| ; \tag{8.2}$$

note that

$$\bigcap_{s\in S_k} \mathscr{T}_s^+ - \bigcup_{s\in E_t-S_k} \mathscr{T}_s^+ = \bigcap_{s\in S_k} \mathscr{T}_s^+ \cap \Big(- \bigcap_{s\in E_t-S_k} \mathscr{T}_s^+ \Big) .$$

On the right-hand side of eq. (8.2) the (*extreme points* of) *convex sets* S_1, S_2 and S_3 only contribute to the total sum, and we therefore turn to the *poset* $L_{\mathrm{conv}}(\mathscr{M})$ *of convex subsets* of the ground set of $\mathscr{M}$.

Let $\widehat{L}_{\mathrm{conv}}(\mathscr{M})$ be the poset $L_{\mathrm{conv}}(\mathscr{M})$ augmented by a *greatest* element $\hat{1}$; it is convenient to set $\hat{1} := E_t$. We will write $\widehat{L}$ instead of $\widehat{L}_{\mathrm{conv}}(\mathscr{M})$ for short.

If we let $\mathscr{T}_B^+ := \bigcap_{b\in B} \mathscr{T}_b^+$ denote the intersection of the *positive halfspaces* corresponding to the elements of a subset $B \subseteq E_t$, then expression (8.2) can be rewritten as follows:

$$\#\mathbf{K}_3^*(\mathscr{M}) = \#\mathbf{A}_3^*(\mathscr{M}) = \sum_{\substack{\{A_1,A_2,A_3\}\subset \widehat{L}-\{\hat{0},\hat{1}\}:\\ A_1\wedge A_2 = A_1\wedge A_3 = A_2\wedge A_3 = \hat{0}}} \prod_{k\in\{1,2,3\}} \Big| \mathscr{T}_{\mathrm{ex}(A_k)}^+ \cap \big(-\mathscr{T}_{\mathrm{ex}(\mathrm{conv}(E_t - A_k))}^+ \big) \Big| . \tag{8.3}$$

A related quantity

$$2^3\binom{|\mathscr{T}|/2}{3} + \sum_{A\in\widehat{L}-\{\hat{0},\hat{1}\}} \mu_{\widehat{L}}(\hat{0},A)\binom{|\mathscr{T}_{\mathrm{ex}(A)}^+|}{3} = 8\binom{|\mathscr{T}|/2}{3} + \sum_{\substack{A\in\widehat{L}-\{\hat{0},\hat{1}\}:\\ A \text{ free}}} (-1)^{|A|}\binom{|\mathscr{T}_A^+|}{3}$$

is the number of 3-subsets of topes containing no pairs of opposites and having a nonempty intersection with every positive halfspace of $\mathscr{M}$. Recall that if $A \in \widehat{L} - \{\hat{0}, \hat{1}\}$, then $\mu_{\widehat{L}}(\hat{0}, A) = (-1)^{|A|}$ whenever the convex set A is *free* and, as a consequence, the interval $[\hat{0}, A]$ is isomorphic to the Boolean lattice of rank $|A|$; otherwise, $\mu_{\widehat{L}}(\hat{0}, A) = 0$.

Let $\mathbf{G} := \mathbf{G}(\mathscr{M})$ be a graph with the vertex set $\mathrm{V}(\mathbf{G}) := \mathscr{T}$; a pair $\{T', T''\} \subset \mathscr{T}$ by definition belongs to the edge family $\mathscr{E}(\mathbf{G})$ of $\mathbf{G}$ if and only if no positive halfspace

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of $\mathcal{M}$ contains this pair, that is, $|(T')^+ \cap (T'')^+| = 0$ or, in other words, $\{T', T''\}$ is a 1-dimensional *missing face* of the abstract simplicial *complex* $\Delta_{\text{acyclic}}(\mathcal{M})$ *of acyclic subsets* of the ground set E_t. Thus, $\mathbf{G}$ is isomorphic to the *Kneser graph* $\mathrm{KG}(\{T^+ : T \in \mathcal{T}\})$ of the family of the positive parts of topes of $\mathcal{M}$.

Let $\mathbf{\Gamma} := \mathbf{\Gamma}(\mathcal{M})$ be the graph given in Definition 1.18 by

$$\mathrm{V}(\mathbf{\Gamma}) := \mathcal{T} \ ,$$

$$\{T', T''\} \in \mathcal{E}(\mathbf{\Gamma}) \iff (T')^+ \cup (T'')^+ = E_t \ .$$

By Lemma 1.19 the vertex set of any *odd cycle* in $\mathbf{\Gamma}$ is a *tope committee* for $\mathcal{M}$. Since $\{T', T''\} \in \mathcal{E}(\mathbf{G})$ if and only if $(T')^- \cup (T'')^- = E_t$, for any pair of topes $\{T', T''\}$, the four graphs $\mathbf{\Gamma}$, $\mathbf{G}$, $\mathrm{KG}(\{T^+ : T \in \mathcal{T}\})$ and $\mathrm{KG}(\{T^- : T \in \mathcal{T}\})$ are all isomorphic because of symmetry (8.1). For example, the map $\mathrm{V}(\mathbf{G}) \to \mathrm{V}(\mathbf{\Gamma})$, $T \mapsto -T$, is an isomorphism between the graphs $\mathbf{G}$ and $\mathbf{\Gamma}$. Since a subset $\{T', T'', T'''\} \subset \mathcal{T}$ is a three-tope anti-committee for $\mathcal{M}$ if and only if it is the vertex set of a triangle in $\mathbf{G}$ (or, in other terms, this subset is a 2-dimensional face of the *independence complex* of the graph whose edges are the 1-dimensional faces of $\Delta_{\text{acyclic}}(\mathcal{M})$), the three-tope committees for $\mathcal{M}$ are precisely the vertex sets of the triangles in $\mathbf{\Gamma}$. From the poset-theoretic point of view, the family $\mathbf{K}_3^*(\mathcal{M})$ is regarded in Propositions 7.1 and 7.2 as antichains of certain posets associated with the topes.

For a simple oriented matroid $\mathcal{M} := (E_t, \mathcal{T})$ which is *not acyclic*, we have

$$\#\mathcal{E}(\mathbf{\Gamma}) = \binom{|\mathcal{T}|}{2} + \sum_{\substack{A \in \hat{L} - \{\hat{0}, \hat{1}\} : \\ A \text{ free}}} (-1)^{|A|} \binom{|\mathcal{T}_A^+|}{2} \tag{8.4}$$

and

$$|\mathrm{V}(\mathbf{\Gamma})| := |\mathcal{T}| = \sum_{\substack{A \in \hat{L} - \{\hat{0}, \hat{1}\} : \\ A \text{ free}}} (-1)^{|A|-1} \left|\mathcal{T}_A^+\right| . \tag{8.5}$$

Now let $\mathbf{R}$ be a graph with the vertex set $\mathrm{V}(\mathbf{R}) := [|\mathcal{T}|]$, isomorphic to either of the graphs $\mathbf{\Gamma}$, $\mathbf{G}$, $\mathrm{KG}(\{T^+ : T \in \mathcal{T}\})$ and $\mathrm{KG}(\{T^- : T \in \mathcal{T}\})$. If we let A and $\mathcal{N}(i)$ denote the *adjacency matrix* of $\mathbf{R}$ and the *neighborhood* of the vertex i in $\mathbf{R}$, respectively, then well-known results of graph theory imply, for example, that $\#\mathbf{K}_3^*(\mathcal{M}) = \frac{1}{6}\mathrm{trace}(\mathrm{A}^3)$, and

$$\#\mathbf{K}_3^*(\mathcal{M}) = \frac{1}{3} \sum_{\{i,j\} \in \mathcal{E}(\mathbf{R})} |\mathcal{N}(i) \cap \mathcal{N}(j)| \ , \tag{8.6}$$

since the quantity $\#\mathbf{K}_3^*(\mathcal{M}) = \#\mathbf{A}_3^*(\mathcal{M})$ is the number of *triangles* in $\mathbf{R}$.

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8.2 Committees of Size 3 Whose Topes Have Maximal Positive Parts

In this section we discuss and enumerate *three-tope committees* $\mathscr{K}^* \subseteq \mathbf{max}^+(\mathscr{T})$, where $\mathbf{max}^+(\mathscr{T})$ is the subset of all topes of an oriented matroid $\mathscr{M} := (E_t, \mathscr{T})$, which is not acyclic, whose *positive parts are inclusion-maximal*; these parts are the *inclusion-maximal convex subsets* of the ground set E_t; thus, they are the *coatoms* of the lattice $\widehat{L}$; in other words, $\widehat{L}^{\mathrm{c}} := \{T^+ : T \in \mathbf{max}^+(\mathscr{T})\}$. If $\mathscr{M}$ is *realizable* as an *arrangement of oriented linear hyperplanes*, then the *coatoms* of $\widehat{L}$ are the *multi-indices of maximal feasible subsystems* of a certain *infeasible system of homogeneous strict linear inequalities*. By Definition 1.29 the *graph of topes with maximal positive parts* $\mathbf{\Gamma}^+_{\mathbf{max}} := \mathbf{\Gamma}^+_{\mathbf{max}}(\mathscr{M})$ is the induced subgraph of $\mathbf{\Gamma}(\mathscr{M})$ on its vertex subset $\mathbf{max}^+(\mathscr{T})$, that is,

$$\mathrm{V}(\mathbf{\Gamma}^+_{\mathbf{max}}) := \mathbf{max}^+(\mathscr{T}) ,$$
$$\{T', T''\} \in \mathscr{E}(\mathbf{\Gamma}^+_{\mathbf{max}}) \iff (T')^+ \cup (T'')^+ = E_t ;$$

by Proposition 1.31(i) the graph $\mathbf{\Gamma}^+_{\mathbf{max}}$ is *connected*.

Recall that if $T \in \mathbf{max}^+(\mathscr{T})$, then the symmetry (8.1) and the maximality of the positive part $T^+ \in \widehat{L}^{\mathrm{c}}$ imply that the set $E_t - T^+ = T^-$ is *acyclic* and *convex*; as a consequence, we have

$$\begin{aligned}\#\{\mathscr{K}^* \in \mathbf{K}^*_3(\mathscr{M}) : \mathscr{K}^* \subseteq \mathbf{max}^+(\mathscr{T})\} = \#\Big\{&\{D_1, D_2, D_3\} \subset \widehat{L} : \\ &\{E_t - D_1, E_t - D_2, E_t - D_3\} \subseteq \widehat{L}^{\mathrm{c}},\ D_1 \wedge D_2 = D_1 \wedge D_3 = D_2 \wedge D_3 = \hat{0}\Big\} ,\end{aligned}$$

cf. eq. (8.3).

The *degree* of a vertex T in the graph $\mathbf{\Gamma}^+_{\mathbf{max}}$ equals the number of coatoms in the interval $[T^-, \hat{1}]$ of the lattice $\widehat{L}$. As a consequence, the number of edges in $\mathbf{\Gamma}^+_{\mathbf{max}}$ is

$$\#\mathscr{E}(\mathbf{\Gamma}^+_{\mathbf{max}}) = \frac{1}{2} \sum_{D \in \widehat{L} : E_t - D \in \widehat{L}^{\mathrm{c}}} \left|[D, \hat{1}]^{\mathrm{c}}\right| ,$$

and the *cyclomatic number* of the graph $\mathbf{\Gamma}^+_{\mathbf{max}}$ equals

$$1 + \frac{1}{2} \sum_{D \in \widehat{L} : E_t - D \in \widehat{L}^{\mathrm{c}}} \left|[D, \hat{1}]^{\mathrm{c}}\right| - |\widehat{L}^{\mathrm{c}}| .$$

If a tope $T' \in \mathscr{T}$ does not belong to the set $\mathbf{max}^+(\mathscr{T})$, then by the symmetry (8.1) there exists a tope $T'' \in \mathbf{max}^+(\mathscr{T})$ such that the pair $\{T', T''\}$ is an edge of $\mathbf{\Gamma}$; since the graph $\mathbf{\Gamma}^+_{\mathbf{max}}$ is connected, this implies that the graph $\mathbf{\Gamma}$ is also *connected* and, according to eqs. (8.4) and (8.5), the *cyclomatic number* of the graph $\mathbf{\Gamma}$ equals

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$$1+\binom{|\mathscr{T}|}{2}+\sum_{\substack{A\in\widehat{L}-\{\hat{0},\hat{1}\}:\\ A\text{ free}}}(-1)^{|A|}\binom{1+|\mathscr{T}_A^+|}{2}\ .$$

We conclude this section with a result which is a direct consequence of the expression (8.6):

Proposition 8.2. *Given a simple oriented matroid $\mathscr{M}$ which is not acyclic, the number of its committees of size* 3*, whose topes have inclusion-maximal positive parts, is*

$$\#\{\mathscr{K}^*\in\mathbf{K}_3^*(\mathscr{M}):\mathscr{K}^*\subseteq\mathbf{max}^+(\mathscr{T})\}=\frac{1}{3}\sum_{\substack{\{D_1,D_2\}\subset\widehat{L}:\\ \{E_t-D_1,E_t-D_2\}\subset\widehat{L}^{\mathrm{c}},\\ D_1\wedge D_2=\hat{0}}}\left|[D_1\vee D_2,\hat{1}]^{\mathrm{c}}\right|\ . \tag{8.7}$$

Example 8.3. Let $\mathscr{M}:=(E_6,\mathscr{T})$ be the rank 3 simple oriented matroid on the ground set $E_6:=[6]$, with the set of topes

$$\mathscr{T}:=\begin{array}{l}\{\\ \end{array}\begin{array}{cccccc}
- & - & + & + & + & +\\
- & - & + & - & + & +\\
+ & - & + & - & + & +\\
+ & - & + & - & + & -\\
- & - & + & - & + & -\\
- & - & + & + & + & -\\
- & - & + & + & - & +\\
- & + & + & + & - & +\\
- & + & + & + & + & -\\
- & + & + & - & + & -\\
+ & + & + & - & + & -\\
+ & + & + & - & - & -\\
- & + & + & - & - & -\\
- & + & + & + & - & -\\
+ & - & - & - & + & +\\
+ & - & - & + & + & +\\
- & - & - & + & + & +\\
- & - & - & + & - & +\\
+ & - & - & + & - & +\\
+ & - & - & - & - & +\\
+ & - & - & - & + & -\\
+ & + & - & - & + & -
\end{array}$$

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$$\begin{array}{cccccc}
+ & + & - & - & - & + \\
+ & + & - & + & - & + \\
- & + & - & + & - & + \\
- & + & - & + & - & - \\
+ & + & - & + & - & - \\
+ & + & - & - & - & - \,\} \,.
\end{array}$$

A realization of the acyclic reorientation ${}_{-[2]}\mathscr{M}$ of $\mathscr{M}$ by an arrangement of oriented linear hyperplanes in $\mathbb{R}^3$ is shown in Figure 1.1. We have

$$\{D \in \widehat{L} : E_6 - D \in \widehat{L}^{\mathrm{c}}\} = \big\{\{12\}, \{15\}, \{16\}, \{23\}, \{24\}, \{35\}, \{46\}\big\}$$

and

$$\big|[\{12\} \vee \{35\}, \hat{1}]^{\mathrm{c}}\big| = \big|[\{1235\}, \hat{1}]^{\mathrm{c}}\big| = \#\big\{\{1235\}\big\} = 1\,,$$

$$\big|[\{12\} \vee \{46\}, \hat{1}]^{\mathrm{c}}\big| = \big|[\{1246\}, \hat{1}]^{\mathrm{c}}\big| = \#\big\{\{1246\}\big\} = 1\,,$$

$$\big|[\{15\} \vee \{23\}, \hat{1}]^{\mathrm{c}}\big| = \big|[\{1235\}, \hat{1}]^{\mathrm{c}}\big| = \#\big\{\{1235\}\big\} = 1\,,$$

$$\big|[\{15\} \vee \{24\}, \hat{1}]^{\mathrm{c}}\big| = \big|\{\hat{1}\}^{\mathrm{c}}\big| = 0\,,$$

$$\big|[\{15\} \vee \{46\}, \hat{1}]^{\mathrm{c}}\big| = \big|[\{1456\}, \hat{1}]^{\mathrm{c}}\big| = \#\big\{\{1456\}\big\} = 1\,,$$

$$\big|[\{16\} \vee \{23\}, \hat{1}]^{\mathrm{c}}\big| = \big|\{\hat{1}\}^{\mathrm{c}}\big| = 0\,,$$

$$\big|[\{16\} \vee \{24\}, \hat{1}]^{\mathrm{c}}\big| = \big|[\{1246\}, \hat{1}]^{\mathrm{c}}\big| = \#\big\{\{1246\}\big\} = 1\,,$$

$$\big|[\{16\} \vee \{35\}, \hat{1}]^{\mathrm{c}}\big| = \big|[\{1356\}, \hat{1}]^{\mathrm{c}}\big| = \#\big\{\{1356\}\big\} = 1\,,$$

$$\big|[\{23\} \vee \{46\}, \hat{1}]^{\mathrm{c}}\big| = \big|[\{2346\}, \hat{1}]^{\mathrm{c}}\big| = \#\big\{\{2346\}\big\} = 1\,,$$

$$\big|[\{24\} \vee \{35\}, \hat{1}]^{\mathrm{c}}\big| = \big|[\{2345\}, \hat{1}]^{\mathrm{c}}\big| = \#\big\{\{2345\}\big\} = 1\,,$$

$$\big|[\{35\} \vee \{46\}, \hat{1}]^{\mathrm{c}}\big| = \big|[\{3456\}, \hat{1}]^{\mathrm{c}}\big| = \#\big\{\{3456\}\big\} = 1\,.$$

Thus, by eq. (8.7) the family $\{\mathscr{K}^* \in \mathbf{K}_3^*(\mathscr{M}) : \mathscr{K}^* \subseteq \mathbf{max}^+(\mathscr{T})\}$ consists of $9/3 = 3$ committees which can be seen as the vertex sets of the triangles in the graph depicted in Figure 1.12.

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Notes

See, for example, [6, IV.4.B] on a connection between the number of *coatoms* in a finite bounded poset P and *incidence functions* on P.

Convexity in *oriented matroids* is discussed, for example, in Refs. [27, Ch. 9], [65–67, 118, 170].

In this chapter we adopt terms and results of Ref. [67]; in particular, following the argument given in Ref. [67, p. 115], we recall on page 151 that if $A \in \widehat{L} - \{\hat{0}, \hat{1}\}$, then $\mu_{\widehat{L}}(\hat{0}, A) = (-1)^{|A|}$ whenever the *convex* set A is *free*, and $\mu_{\widehat{L}}(\hat{0}, A) = 0$ otherwise.

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9 Halfspaces, Convex Sets, and Tope Committees

In this chapter, for enumerating tope committees, we apply the *Principle of Inclusion–Exclusion* to subsets of maximal covectors contained in the *halfspaces* of a simple oriented matroid, and to *convex subsets* of its ground set.

Let $\mathscr{M} = (E_t, \mathscr{T})$ be a simple oriented matroid which is *not acyclic*. Recall that the family $\mathbf{K}_k^*(\mathscr{M})$ of *tope committees* of size k, $3 \le k \le |\mathscr{T}| - 3$, for $\mathscr{M}$ is defined to be the collection

$$\mathbf{K}_k^*(\mathscr{M}) := \left\{\mathscr{K}^* \subset \mathscr{T} : |\mathscr{K}^*| = k,\ |\mathscr{K}^* \cap \mathscr{T}_e^+| > \tfrac{k}{2},\ \forall e \in E_t\right\} ,$$

where $\mathscr{T}_e^+ := \{T \in \mathscr{T} : T(e) = 1\}$ is the *positive halfspace* of $\mathscr{M}$ that corresponds to the element e. The family of *tope anti-committees* of size k for $\mathscr{M}$ is denoted as $\mathbf{A}_k^*(\mathscr{M})$; see Definition 8.1.

Denote by $\binom{\mathscr{T}}{k}$ the family of all k-subsets of the set of topes $\mathscr{T}$, and consider the families of tope subsets $\mathbf{N}_k^*(\mathscr{M}) := \binom{\mathscr{T}}{k} - (\mathbf{K}_k^* \dot\cup \mathbf{A}_k^*)$, $3 \le k \le |\mathscr{T}| - 3$, that is, the families

$$\begin{aligned}\mathbf{N}_k^*(\mathscr{M}) := \{&\mathscr{N}^* \subset \mathscr{T} : |\mathscr{N}^*| = k,\\ &\mathscr{N}^* \text{ neither a committee nor an anti-committee for } \mathscr{M}\} .\end{aligned}$$

We have

$$\#\mathbf{K}_k^*(\mathscr{M}) = \#\mathbf{A}_k^*(\mathscr{M}) = \frac{1}{2}\Big(\tbinom{|\mathscr{T}|}{k} - \#\mathbf{N}_k^*(\mathscr{M}) \Big) , \quad 3 \le k \le |\mathscr{T}| - 3 .$$

For an element $e \in E_t$, as earlier we denote by $\mathscr{T}_e^- := \{T \in \mathscr{T} : T(e) = -1\}$ the *negative halfspace* of $\mathscr{M}$ that corresponds to the element e. The family of all subsets, of cardinality j, of the positive halfspace $\mathscr{T}_e^+$ will be denoted by $\binom{\mathscr{T}_e^+}{j}$, and similarly we will denote by $\binom{\mathscr{T}_e^-}{i}$ the family of i-subsets of the negative halfspace $\mathscr{T}_e^-$. Define a collection $\binom{\mathscr{T}_e^-}{i} \boxplus \binom{\mathscr{T}_e^+}{j}$ of $(i+j)$-subsets of topes of $\mathscr{M}$ to be the family $\{A \dot\cup B : A \in \binom{\mathscr{T}_e^-}{i},\ B \in \binom{\mathscr{T}_e^+}{j}\}$.

On the one hand,

$$\mathbf{K}_k^*(\mathscr{M}) = \bigcap_{e \in E_t} \bigcup_{\lceil (k+1)/2 \rceil \le j \le k} \Big(\tbinom{\mathscr{T}_e^-}{k-j} \boxplus \tbinom{\mathscr{T}_e^+}{j} \Big) , \quad 3 \le k \le |\mathscr{T}| - 3 .$$

On the other hand, a k-subset $\mathscr{K}^* \subset \mathscr{T}$ is clearly a *committee* for $\mathscr{M}$ if and only if

– the set $\mathscr{K}^*$ contains *no set* of the family $\bigcup_{e \in E_t} \binom{\mathscr{T}_e^-}{\lfloor (k+1)/2 \rfloor}$;
– the set $\mathscr{K}^*$ contains *at least one set* of each family $\binom{\mathscr{T}_e^+}{\lceil (k+1)/2 \rceil}$, $e \in E_t$; in other words, the collection $\binom{\mathscr{K}^*}{\lceil (k+1)/2 \rceil}$ is a *blocking family* of the family $\{\binom{\mathscr{T}_1^+}{\lceil (k+1)/2 \rceil}, \ldots, \binom{\mathscr{T}_t^+}{\lceil (k+1)/2 \rceil}\}$, that is,

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$$\#\Big(\binom{\mathscr{K}^*}{\lceil (k+1)/2\rceil}\cap\binom{\mathscr{T}_e^+}{\lceil (k+1)/2\rceil}\Big)>0\,,\quad \forall e\in E_t\,.$$

As a consequence, the collection $\mathbf{K}_k^*(\mathscr{M})$ of *tope committees of size k for* $\mathscr{M}$ is the family of all *blocking k-sets* of the family

$$\bigcup_{e\in E_t}\binom{\mathscr{T}_e^+}{\lfloor(|\mathscr{T}|-k+1)/2\rfloor}\,,$$

and a tope committee $\mathscr{K}^*\in\mathbf{K}_k^*(\mathscr{M})$ is *minimal* if any its proper i-subset $\mathscr{I}^*\subset\mathscr{K}^*$ is *not a blocking set* of the family $\bigcup_{e\in E_t}\binom{\mathscr{T}_e^+}{\lfloor(|\mathscr{T}|-i+1)/2\rfloor}$.

In Sections 9.1 and 9.2 we use the above observations to count in several ways the numbers $\#\mathbf{K}_k^*(\mathscr{M})$ of committees of size k by applying the *Principle of Inclusion–Exclusion* to subsets of maximal covectors contained in the halfspaces of the oriented matroid $\mathscr{M}$. In Section 9.2, our computations involve the *convex subsets* of the ground set of $\mathscr{M}$.

In Section 9.3 we count the numbers $\#\mathring{\mathbf{K}}_k^*(\mathscr{M})$ of tope committees of size k that contain *no pairs* of *opposites*.

With the oriented matroid $\mathscr{M}$, we can associate various “κ^*-*vectors*” (and their flag generalizations) whose components are by convention the numbers of tope committees for $\mathscr{M}$ of the corresponding size, for example:

– the vector $\boldsymbol{\kappa}^*(\mathscr{M}):=(\kappa_1^*(\mathscr{M}),\ldots,\kappa_{|\mathscr{T}|/2}^*(\mathscr{M}))\in\mathbb{N}^{|\mathscr{T}|/2}$, where

$$\kappa_k^*(\mathscr{M}):=\#\mathbf{K}_k^*(\mathscr{M})\,;$$

  recall that $\#\mathbf{K}_k^*(\mathscr{M})=\#\mathbf{K}_{|\mathscr{T}|-k}^*(\mathscr{M})$, $1\le k\le|\mathscr{T}|-1$;

– the vector $\mathring{\boldsymbol{\kappa}}^*(\mathscr{M}):=(\mathring{\kappa}_1^*(\mathscr{M}),\ldots,\mathring{\kappa}_{|\mathscr{T}|/2}^*(\mathscr{M}))\in\mathbb{N}^{|\mathscr{T}|/2}$, where

$$\mathring{\kappa}_k^*(\mathscr{M}):=\#\mathring{\mathbf{K}}_k^*(\mathscr{M})\,;$$

  note that $\#\mathring{\mathbf{K}}_k^*(\mathscr{M})=0$ whenever $\frac{1}{2}|\mathscr{T}|<k\le|\mathscr{T}|-1$;

– the vector $\boldsymbol{\kappa}_{\min}^*(\mathscr{M}):=(\kappa_{\min\,1}^*(\mathscr{M}),\ldots,\kappa_{\min\,|\mathscr{T}|/2}^*(\mathscr{M}))\in\mathbb{N}^{|\mathscr{T}|/2}$, where $\kappa_{\min\,k}^*(\mathscr{M}):=\#\{\mathscr{K}^*\in\mathbf{K}_k^*(\mathscr{M}):\mathscr{K}^*\text{ minimal}\}$;

– the vector $\boldsymbol{\kappa}_{\mathbf{max}^+}^*(\mathscr{M}):=\big(\kappa_{\mathbf{max}^+\,1}^*(\mathscr{M}),\ldots,\kappa_{\mathbf{max}^+\,|\mathscr{T}|/2}^*(\mathscr{M})\big)\in\mathbb{N}^{|\mathscr{T}|/2}$, where $\kappa_{\mathbf{max}^+\,k}^*(\mathscr{M}):=\#\{\mathscr{K}^*\in\mathbf{K}_k^*(\mathscr{M}):\mathscr{K}^*\subseteq\mathbf{max}^+(\mathscr{T})\}$.

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Example 9.1. Let $\mathscr{M} := (E_6, \mathscr{T})$ be the simple oriented matroid represented by its third positive halfspace

$$\mathscr{T}_3^+ := \left\{ \begin{array}{cccccc} - & - & + & + & + & + \\ - & - & + & - & + & + \\ + & - & + & - & + & + \\ + & - & + & - & + & - \\ - & - & + & - & + & - \\ - & - & + & + & + & - \\ - & - & + & + & - & + \\ - & + & + & + & - & + \\ - & + & + & + & + & - \\ - & + & + & - & + & - \\ + & + & + & - & + & - \\ + & + & + & - & - & - \\ - & + & + & - & - & - \\ - & + & + & + & - & - \end{array} \right\};$$

a realization of its reorientation ${}_{-[2]}\mathscr{M}$ by an arrangement of oriented linear hyperplanes in $\mathbb{R}^3$ is shown in Figure 1.1.

The oriented matroid $\mathscr{M}$ with 28 maximal covectors has 238012 tope committees; namely,

$$\boldsymbol{\kappa}^*(\mathscr{M}) = (\underset{1}{0}, \overset{2}{0}, \underset{3}{3}, \overset{4}{0}, \underset{5}{144}, \overset{6}{1}, \underset{7}{1942}, \overset{8}{22}, \underset{9}{11872}, \overset{10}{136}, \underset{11}{37775}, \overset{12}{386}, \underset{13}{66454}, \overset{14}{542}),$$

and 4496 tope committees contain *no pairs* of *opposites*:

$$\mathring{\boldsymbol{\kappa}}^*(\mathscr{M}) = (\underset{1}{0}, \overset{2}{0}, \underset{3}{3}, \overset{4}{0}, \underset{5}{111}, \overset{6}{1}, \underset{7}{778}, \overset{8}{14}, \underset{9}{1935}, \overset{10}{24}, \underset{11}{1448}, \overset{12}{24}, \underset{13}{158}, \overset{14}{0}).$$

9.1 Halfspaces and Tope Committees

In this section we count general tope committees for simple oriented matroids.

Let $\mathbb{B}(\mathscr{T})$ be the Boolean lattice of all subsets of the set of topes $\mathscr{T}$ of an oriented matroid $\mathscr{M}$ which is *not acyclic*, and $\Upsilon := \{v_1, \dots, v_t\} \subset \mathbb{B}(\mathscr{T})^{(|\mathscr{T}|/2)}$ its antichain whose elements v_e, $e \in E_t$, represent in $\mathbb{B}(\mathscr{T})$ the positive halfspaces $\mathscr{T}_e^+$; see eq. (7.1). Via the natural correspondence (7.2), the family $\mathbf{K}_k^*(\mathscr{M})$ of tope committees of size k, $3 \le k \le |\mathscr{T}| - 3$, for $\mathscr{M}$ is represented in the lattice $\mathbb{B}(\mathscr{T})$ by the antichain

$$\mathbf{I}_{\frac{1}{2},k}\big(\mathbb{B}(\mathscr{T}), \Upsilon\big) := \Big\{b \in \mathbb{B}(\mathscr{T}) : \rho(b) = k,\ \rho(b \wedge v_e) > \tfrac{k}{2},\ \ \forall e \in E_t\Big\} \subset \mathbb{B}(\mathscr{T})^{(k)}\ .$$

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In view of the axiomatic symmetry $\mathscr{T} = -\mathscr{T}$, and by eq. (6.29) the cardinality of this antichain is

$$\left|\mathbf{I}_{\frac{1}{2},k}\big(\mathbb{B}(\mathscr{T}),\Upsilon\big)\right| = \binom{|\mathscr{T}|}{k} + \sum_{\mathrm{D}\subseteq\mathbb{B}(\mathscr{T})^{(\lfloor(k+1)/2\rfloor)}\cap\mathfrak{J}(\Upsilon)\,:\,|\mathrm{D}|>0} (-1)^{|\mathrm{D}|}\cdot\binom{|\mathscr{T}|-\rho\left(\bigvee_{d\in\mathrm{D}} d\right)}{|\mathscr{T}|-k}\,. \tag{9.1}$$

Note that for an integer j, $1 \le j \le |\mathscr{T}|/2$, we have

$$\left|\mathbb{B}(\mathscr{T})^{(j)}\cap\mathfrak{J}(\Upsilon)\right| = -\sum_{\substack{A\in L_{\mathrm{conv}}(\mathscr{M})-\{\hat{0}\}\,:\\ A\text{ free}}} (-1)^{|A|}\cdot\binom{|\mathscr{T}_A^+|}{j}\,,$$

where $L_{\mathrm{conv}}(\mathscr{M})$ denotes the *meet-semilattice* of *convex subsets* of the ground set E_t, and $\mathscr{T}_A^+ := \bigcap_{a\in A}\mathscr{T}_a^+$; $\hat{0}$ denotes the *least* element of $L_{\mathrm{conv}}(\mathscr{M})$.

Expression (9.1) can be restated as follows:

$$\begin{aligned}\#\mathbf{K}_k^*(\mathscr{M}) &= \#\mathbf{K}_{|\mathscr{T}|-k}^*(\mathscr{M})\\ &= \binom{|\mathscr{T}|}{|\mathscr{T}|-\ell} + \sum_{\substack{\mathscr{G}\subseteq\bigcup_{e\in E_t}\binom{\mathscr{T}_e^+}{\lfloor(\ell+1)/2\rfloor}\,:\\ 1\le\#\mathscr{G}\le\binom{\ell}{\lfloor(\ell+1)/2\rfloor},\\ |\bigcup_{G\in\mathscr{G}} G|\le\ell}} (-1)^{\#\mathscr{G}}\cdot\binom{|\mathscr{T}|-|\bigcup_{G\in\mathscr{G}} G|}{|\mathscr{T}|-\ell}\,,\end{aligned} \tag{9.2}$$

where $\ell \in \{k, |\mathscr{T}|-k\}$; this formula counts the number of all *blocking k-sets* of the family $\bigcup_{e\in E_t}\binom{\mathscr{T}_e^+}{\lfloor(|\mathscr{T}|-k+1)/2\rfloor}$, cf. eq. (6.30), and it counts the number of all *blocking* ($|\mathscr{T}|-k$)*-sets* of the family $\bigcup_{e\in E_t}\binom{\mathscr{T}_e^+}{\lfloor(k+1)/2\rfloor}$. We can also rewrite eq. (9.1) by means of the *Chu–Vandermonde identity* in the form

$$\left|\mathbf{I}_{\frac{1}{2},k}\big(\mathbb{B}(\mathscr{T}),\Upsilon\big)\right| = -\sum_{\mathrm{D}\subseteq\mathbb{B}(\mathscr{T})^{(\lfloor(|\mathscr{T}|-k+1)/2\rfloor)}\cap\mathfrak{J}(\Upsilon)\,:\,|\mathrm{D}|>0} (-1)^{|\mathrm{D}|} \cdot\sum_{1\le h\le k}\binom{\rho(\bigvee_{d\in\mathrm{D}} d)}{h}\binom{|\mathscr{T}|-\rho(\bigvee_{d\in\mathrm{D}} d)}{k-h}\,,$$

cf. eq. (6.31), that is,

$$\#\mathbf{K}_k^*(\mathscr{M}) = \#\mathbf{K}_{|\mathscr{T}|-k}^*(\mathscr{M}) = -\sum_{\mathscr{G}\subseteq\bigcup_{e\in E_t}\binom{\mathscr{T}_e^+}{\lfloor(|\mathscr{T}|-\ell+1)/2\rfloor}\,:\,\#\mathscr{G}>0} (-1)^{\#\mathscr{G}} \cdot\sum_{\max\{1,\ell-|\mathscr{T}|+|\bigcup_{G\in\mathscr{G}} G|\}\le h\le\min\{\ell,|\bigcup_{G\in\mathscr{G}} G|\}}\binom{|\bigcup_{G\in\mathscr{G}} G|}{h}\binom{|\mathscr{T}|-|\bigcup_{G\in\mathscr{G}} G|}{\ell-h}\,, \tag{9.3}$$

where $\ell \in \{k, |\mathscr{T}|-k\}$.

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If $\mathscr{G}$ is a family of tope subsets, then we denote by $\mathscr{E}(\mathscr{G})$ the *join-semilattice* $\{\bigcup_{F\in\mathscr{F}} F : \mathscr{F} \subseteq \mathscr{G},\ \#\mathscr{F} > 0\}$ consisting of the unions of the sets from the family $\mathscr{G}$, ordered by inclusion and augmented by a new *least* element $\hat{0}$; the *greatest* element $\hat{1}$ of the *lattice* $\mathscr{E}(\mathscr{G})$ is the set $\bigcup_{G\in\mathscr{G}} G$. We denote the *Möbius function* of the poset $\mathscr{E}(\mathscr{G})$ by $\mu_{\mathscr{E}}(\cdot,\cdot)$.

Expressions (9.4) and (9.5) refine formulas (9.2) and (9.3), respectively.

Proposition 9.2. *The number $\#\mathbf{K}_k^*(\mathscr{M})$ of tope committees of size k, $3 \le k \le |\mathscr{T}| - 3$, for a simple oriented matroid $\mathscr{M} := (E_t, \mathscr{T})$ which is* not acyclic, *can be computed as follows:*

(i)

$$\begin{aligned}\#\mathbf{K}_k^*(\mathscr{M}) &= \#\mathbf{K}_{|\mathscr{T}|-k}^*(\mathscr{M})\\ &= \binom{|\mathscr{T}|}{|\mathscr{T}|-\ell} + \sum_{G\in\mathscr{E}(\bigcup_{e\in E_t}\binom{\mathscr{T}_e^+}{\lfloor(\ell+1)/2\rfloor}) : 0<|G|\le\ell} \mu_{\mathscr{E}}(\hat{0},G)\cdot\binom{|\mathscr{T}|-|G|}{|\mathscr{T}|-\ell}, \qquad (9.4)\end{aligned}$$

where $\ell \in \{k, |\mathscr{T}| - k\}$.

(ii)

$$\begin{aligned}\#\mathbf{K}_k^*(\mathscr{M}) = \#\mathbf{K}_{|\mathscr{T}|-k}^*(\mathscr{M}) = -&\sum_{G\in\mathscr{E}(\bigcup_{e\in E_t}\binom{\mathscr{T}_e^+}{\lfloor(|\mathscr{T}|-\ell+1)/2\rfloor}) : |G|>0} \mu_{\mathscr{E}}(\hat{0},G)\\ &\cdot \sum_{\max\{1,\ell-|\mathscr{T}|+|G|\}\le h\le\min\{\ell,|G|\}} \binom{|G|}{h}\binom{|\mathscr{T}|-|G|}{\ell-h}, \qquad (9.5)\end{aligned}$$

where $\ell \in \{k, |\mathscr{T}| - k\}$.

Let $\mathscr{C}_{\frac{1}{2},k}(\mathbb{B}(\mathscr{T}),\Upsilon)$ be the *join-semilattice* of all sets of the family $\{\mathbb{B}(\mathscr{T})^{(\lceil(k+1)/2\rceil)} \cap \mathfrak{I}(C) : C \subseteq \Upsilon,\ |C| > 0\}$ ordered by inclusion and augmented by a new *least* element $\hat{0}$. The *greatest* element $\hat{1}$ of the *lattice* $\mathscr{C}_{\frac{1}{2},k}(\mathbb{B}(\mathscr{T}),\Upsilon)$ is the set $\mathbb{B}(\mathscr{T})^{(\lceil(k+1)/2\rceil)} \cap \mathfrak{I}(\Upsilon)$. Similarly, for an element $X \in \mathscr{C}_{\frac{1}{2},k}(\mathbb{B}(\mathscr{T}),\Upsilon)$, we denote by $\mathscr{E}(\mathbb{B}(\mathscr{T}),X)$ the *sub-join-semilattice* of $\mathbb{B}(\mathscr{T})$ generated by the set $X \subset \mathbb{B}(\mathscr{T})$ and augmented by a new *least* element $\hat{0}$. The *Möbius functions* of the posets $\mathscr{C}_{\frac{1}{2},k}(\mathbb{B}(\mathscr{T}),\Upsilon)$ and $\mathscr{E}(\mathbb{B}(\mathscr{T}),X)$ are denoted by $\mu_{\mathscr{C}}(\cdot,\cdot)$ and $\mu_{\mathscr{E}}(\cdot,\cdot)$, respectively.

Using eq. (6.37), we obtain the expression

$$\begin{aligned}\left|\mathbf{I}_{\frac{1}{2},k}\big(\mathbb{B}(\mathscr{T}),\Upsilon\big)\right| &= \sum_{X\in\mathscr{C}_{\frac{1}{2},k}(\mathbb{B}(\mathscr{T}),\Upsilon) : X>\hat{0}} \mu_{\mathscr{C}}(\hat{0},X)\\ &\cdot \sum_{z\in\mathscr{E}(\mathbb{B}(\mathscr{T}),X) : z>\hat{0}} \mu_{\mathscr{E}}(\hat{0},z)\cdot\binom{|\mathscr{T}|-\rho(z)}{|\mathscr{T}|-k},\end{aligned}$$

which can be restated as follows:

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Proposition 9.3. *The number* $\#\mathbf{K}^*_k(\mathscr{M})$ *of tope committees of size* k, $3 \le k \le |\mathscr{T}|-3$, *for a simple oriented matroid* $\mathscr{M} := (E_t, \mathscr{T})$ *which is* not acyclic, *is*

$$\#\mathbf{K}^*_k(\mathscr{M}) = \#\mathbf{K}^*_{|\mathscr{T}|-k}(\mathscr{M}) = \sum_{\mathscr{G} \in \{\{\bigcup_{e\in E}\binom{\mathscr{T}^+_e}{\lceil(\ell+1)/2\rceil}\} : E \subseteq E_t,\ |E|>0\}} \mu_{\mathscr{C}}(\hat{0}, \mathscr{G})$$

$$\cdot \sum_{G \in \mathscr{E}(\mathscr{G}) : 0<|G|\le \ell} \mu_{\mathscr{E}}(\hat{0}, G) \cdot \binom{|\mathscr{T}|-|G|}{\ell-|G|} ,$$

where $\ell \in \{k, |\mathscr{T}|-k\}$; $\mu_{\mathscr{C}}(\cdot,\cdot)$ *denotes the* Möbius function *of the family* $\mathscr{C} := \{\hat{0}\}\dot{\cup}\{\{\bigcup_{e\in E}\binom{\mathscr{T}^+_e}{\lceil(\ell+1)/2\rceil}\} : E \subseteq E_t,\ |E|>0\}$, *ordered by inclusion.*

9.2 Convex Sets and Tope Committees

In this section, where we follow Section 6.10 almost word for word, we continue to count *tope committees* for oriented matroids.

Let the *antichain* $\Upsilon := \{v_1, \ldots, v_t\} \subset \mathbb{B}(\mathscr{T})^{(|\mathscr{T}|/2)}$ again represent the *family of positive halfspaces* of a simple oriented matroid $\mathscr{M} := (E_t, \mathscr{T})$, which is *not acyclic*, in the Boolean lattice $\mathbb{B}(\mathscr{T})$ of all subsets of its set of topes $\mathscr{T}$. We have

$$\left|\mathbf{I}_{\frac{1}{2},k}\big(\mathbb{B}(\mathscr{T}), \Upsilon\big)\right| = \sum_{\mathrm{D} \subseteq \mathbb{B}(\mathscr{T})^{(\lceil(k+1)/2\rceil)} \cap \mathfrak{I}(\Upsilon) : |\mathrm{D}|>0} (-1)^{|\mathrm{D}|}$$

$$\cdot \Bigg(\sum_{\mathrm{C} \subseteq \Upsilon : \mathrm{D} \subseteq \mathfrak{I}(\mathrm{C})} (-1)^{|\mathrm{C}|} \Bigg) \binom{|\mathscr{T}| - \rho(\vee_{d\in\mathrm{D}} d)}{|\mathscr{T}|-k} , \quad 3 \le k \le |\mathscr{T}|-3 , \tag{9.6}$$

cf. eq. (6.33).

Defining subsets of topes $\mathscr{T}^+_A$, for $A \subset E_t$, as earlier by $\mathscr{T}^+_A := \bigcap_{a\in A} T^+_a$, consider the map

$$\begin{aligned} \gamma_k : \mathbb{B}(\mathscr{T})^{(\lceil(k+1)/2\rceil)} \cap \mathfrak{I}(\Upsilon) &\to L_{\mathrm{conv}}(\mathscr{M}) , \\ d &\mapsto \mathbf{max}\{A \in L_{\mathrm{conv}}(\mathscr{M}) : d \subseteq \mathscr{T}^+_A\} , \end{aligned} \tag{9.7}$$

that sends a $(\lceil(k+1)/2\rceil)$-subset of topes $d \in \mathfrak{I}(\Upsilon)$ to the *inclusion-maximal convex subset* $A \subset E_t$ with the property $d \subseteq \mathscr{T}^+_A$. Actually, we are interested in such a map to a subposet $L_{\mathrm{conv},\, \ge\lceil(k+1)/2\rceil}(\mathscr{M})$ which is an order ideal of the semilattice $L_{\mathrm{conv}}(\mathscr{M})$ defined by

$$L_{\mathrm{conv},\, \ge\lceil(k+1)/2\rceil}(\mathscr{M}) := \left\{A \in L_{\mathrm{conv}}(\mathscr{M}) : |\mathscr{T}^+_A| \ge \lceil(k+1)/2\rceil\right\} .$$

Given a nonempty subset $\mathrm{D} \subseteq \mathbb{B}(\mathscr{T})^{(\lceil(k+1)/2\rceil)} \cap \mathfrak{I}(\Upsilon)$, consider the set-theoretic *blocker* $\mathfrak{B}(\gamma_k(\mathrm{D}))$ of the image $\gamma_k(\mathrm{D})$; recall that if we let $\mathbf{min}\gamma_k(\mathrm{D})$

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denote the subfamily of all *inclusion-minimal sets* of the family $\gamma_k(\mathrm{D})$, then we have $\mathfrak{B}(\gamma_k(\mathrm{D})) = \mathfrak{B}(\mathbf{min}\gamma_k(\mathrm{D}))$.

Let $\Delta^*(\mathrm{D})$ be an *abstract simplicial complex* whose *facets* are the *complements* $E_t - B$ of the *minimal blocking sets* $B \in \mathfrak{B}(\mathbf{min}\gamma_k(\mathrm{D}))$ in the *blocker* of the *clutter* $\mathbf{min}\gamma_k(\mathrm{D})$, and let $\Delta(\mathrm{D})$ be the complex whose *facets* are the *complements* $E_t - G$ of the sets $G \in \mathbf{min}\gamma_k(\mathrm{D})$; if the complexes $\Delta(\mathrm{D})$ and $\Delta^*(\mathrm{D})$ have the same vertex set, then $\Delta^*(\mathrm{D})$ is by definition the *Alexander dual* of $\Delta(\mathrm{D})$. The *reduced Euler characteristics* $\widetilde{\chi}(\cdot)$ of these complexes satisfy the identity

$$\widetilde{\chi}\big(\Delta^*(\mathrm{D})\big) = (-1)^{t-1}\widetilde{\chi}\big(\Delta(\mathrm{D})\big)\ .$$

For a subset $\mathrm{C} := (v_{i_1}, \ldots, v_{i_j}) \subseteq \Upsilon$, we have $\mathrm{D} \subseteq \mathfrak{I}(\mathrm{C})$ if and only if the collection of indices $\{i_1, \ldots, i_j\}$ is a *blocking set* of the *family* $\mathbf{min}\gamma_k(\mathrm{D})$; therefore,

$$\sum_{\mathrm{C}\subseteq\Upsilon\,:\,\mathrm{D}\subseteq\mathfrak{I}(\mathrm{C})} (-1)^{|\mathrm{C}|} = (-1)^{t-1}\widetilde{\chi}\big(\Delta^*(\mathrm{D})\big)\ .$$

If $\bigcup_{F\in\mathbf{min}\gamma_k(\mathrm{D})} F \neq E_t$, then the complex $\Delta^*(\mathrm{D})$ is a *cone* and, as a consequence, $\widetilde{\chi}(\Delta^*(\mathrm{D})) = 0$.

Expression (9.6) can be rewritten as follows:

$$\Big|\mathbf{I}_{\frac{1}{2},k}\big(\mathbb{B}(\mathscr{T}),\Upsilon\big)\Big| = \sum_{\substack{\mathrm{D}\subseteq\mathbb{B}(\mathscr{T})^{(\lceil(k+1)/2\rceil)}\cap\mathfrak{I}(\Upsilon)\,:\\ \bigcup_{F\in\mathbf{min}\gamma_k(\mathrm{D})} F=E_t}} (-1)^{|\mathrm{D}|} \cdot\widetilde{\chi}\big(\Delta(\mathrm{D})\big)\cdot\binom{|\mathscr{T}| - \rho(\vee_{d\in\mathrm{D}}d)}{k - \rho(\vee_{d\in\mathrm{D}}d)}\ ; \tag{9.8}$$

note that the singleton sets $\mathrm{D} := \{d\}$, where $d \in \mathbb{B}(\mathscr{T})^{(\lceil(k+1)/2\rceil)} \cap \mathfrak{I}(\Upsilon)$, do not play a role in eq. (9.8).

Given a subset $\mathrm{D} \subseteq \mathbb{B}(\mathscr{T})^{(\lceil(k+1)/2\rceil)} \cap \mathfrak{I}(\Upsilon)$ such that $\bigcup_{F\in\mathbf{min}\gamma_k(\mathrm{D})} F = E_t$, let $\mathscr{S}(\mathrm{D})$ denote the family of the unions $\{\bigcup_{F\in\mathscr{F}} F : \mathscr{F} \subseteq \mathbf{min}\gamma_k(\mathrm{D}),\ \#\mathscr{F} > 0\}$ ordered by inclusion and augmented by a new *least* element $\hat{0}$; the *greatest* element $\hat{1}$ of the *lattice* $\mathscr{S}(\mathrm{D})$ is the ground set E_t. The *reduced Euler characteristic*

$$\widetilde{\chi}\big(\Delta(\mathrm{D})\big) = \sum_{\substack{\mathscr{F}\subseteq\mathbf{min}\gamma_k(\mathrm{D})\,:\\ \bigcup_{F\in\mathscr{F}} F=E_t}} (-1)^{\#\mathscr{F}}$$

of the *complex* $\Delta(\mathrm{D})$ is equal to the *Möbius number* $\mu_{\mathscr{S}(\mathrm{D})}(\hat{0}, \hat{1})$ and, in particular, to $(-1)^{\#\mathbf{min}\gamma_k(\mathrm{D})}$ when the sets in the family $\mathbf{min}\gamma_k(\mathrm{D})$ are *pairwise disjoint*. Let us restate the observation (9.8):

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Proposition 9.4. *The number* $\#\mathbf{K}_k^*(\mathscr{M})$ *of tope committees of size* k, $3 \le k \le |\mathscr{T}| - 3$, *for a simple oriented matroid* $\mathscr{M} := (E_t, \mathscr{T})$ *which is* not acyclic, *is*

$$\begin{aligned}\#\mathbf{K}_k^*(\mathscr{M}) &= \#\mathbf{K}_{|\mathscr{T}|-k}^*(\mathscr{M}) \\ &= \sum_{\substack{\mathscr{G} \subseteq \bigcup_{e \in E_t} \binom{\mathscr{T}_e^+}{\lceil (\ell+1)/2 \rceil}: \\ 1 < \#\mathscr{G} \le \binom{\ell}{\lceil (\ell+1)/2 \rceil}, \\ \bigcup_{F \in \mathbf{min}\,\gamma_\ell(\mathscr{G})} F = E_t,\ |\bigcup_{G \in \mathscr{G}} G| \le \ell}} (-1)^{\#\mathscr{G}} \cdot \mu_{\mathscr{S}(\mathscr{G})}(\hat{0}, \hat{1}) \cdot \binom{|\mathscr{T}| - |\bigcup_{G \in \mathscr{G}} G|}{\ell - |\bigcup_{G \in \mathscr{G}} G|},\end{aligned} \tag{9.9}$$

where $\ell \in \{k, |\mathscr{T}| - k\}$.

Consider the *abstract simplicial complex* whose *facets* are the *positive halfspaces* of the oriented matroid $\mathscr{M}$. If some of its relevant $(\lceil (k+1)/2 \rceil - 1)$-dimensional faces, sets of the family $\bigcup_{e \in E_t} \binom{\mathscr{T}_e^+}{\lceil (k+1)/2 \rceil}$, are *free*, that is, each of them is contained in *exactly one facet* $\mathscr{T}_e^+$, for some element $e \in E_t$, then the *Möbius numbers* $\mu_{\mathscr{S}(\mathscr{G})}(\hat{0}, \hat{1})$ in eq. (9.10), under $\ell := k$, are all equal to $(-1)^t$:

Corollary 9.5. *If for any family* $\mathscr{G} \subseteq \bigcup_{e \in E_t} \binom{\mathscr{T}_e^+}{\lceil (k+1)/2 \rceil}$ *such that* $\bigcup_{F \in \mathbf{min}\,\gamma_k(\mathscr{G})} F = E_t$ *and* $|\bigcup_{G \in \mathscr{G}} G| \le k$, *we have* $|\gamma_k(G)| = 1$, *for any set* $G \in \mathscr{G}$, *then for any* k, $3 \le k \le |\mathscr{T}| - 3$, *the number* $\#\mathbf{K}_{\mathbf{k}}^*(\mathscr{M})$ *of tope committees of size* k *for a simple oriented matroid* $\mathscr{M} := (E_t, \mathscr{T})$ *which is* not acyclic *is*

$$\begin{aligned}\#\mathbf{K}_{\mathbf{k}}^*(\mathscr{M}) &= \#\mathbf{K}_{|\mathscr{T}|-\mathbf{k}}^*(\mathscr{M}) \\ &= (-1)^t \cdot \sum_{\substack{\mathscr{G} \subseteq \bigcup_{e \in E_t} \binom{\mathscr{T}_e^+}{\lceil (k+1)/2 \rceil}: \\ t \le \#\mathscr{G} \le \binom{k}{\lceil (k+1)/2 \rceil}, \\ \bigcup_{F \in \mathbf{min}\,\gamma_k(\mathscr{G})} F = E_t,\ |\bigcup_{G \in \mathscr{G}} G| \le k}} (-1)^{\#\mathscr{G}} \cdot \binom{|\mathscr{T}| - |\bigcup_{G \in \mathscr{G}} G|}{k - |\bigcup_{G \in \mathscr{G}} G|}.\end{aligned} \tag{9.10}$$

9.3 Tope Committees Containing No Pairs of Opposites

In this section we count the *tope committees* containing *no pairs* of *opposites* for a simple oriented matroid $\mathscr{M} := (E_t, \mathscr{T})$ which is *not acyclic*, and our argument is analogous to that of the previous Section 9.2, but now we work with the family $\mathbf{O}'(\mathscr{T})$ of tope subsets that are free of pairs of opposites, and ordered by inclusion; recall that the *meet-semilattice* $\mathbf{O}'(\mathscr{T})$ is isomorphic to the *face poset* of the *boundary of a crosspolytope* of dimension $|\mathscr{T}|/2$. Again we turn to the map

$$\gamma_k : \mathbb{B}(\mathscr{T})^{(\lceil (k+1)/2 \rceil)} \cap \mathfrak{I}(\Upsilon) = \mathbf{O}'(\mathscr{T})^{(\lceil (k+1)/2 \rceil)} \cap \mathfrak{I}(\Upsilon) \to L_{\mathrm{conv}}(\mathscr{M})$$

defined by eq. (9.7), and to the lattices $\mathscr{S}(\cdot)$ considered in Section 9.2.

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If $\mathscr{G}$ is a family of tope subsets that are *free of pairs of opposites*, then we denote by $\mathring{\mathscr{E}}(\mathscr{G})$ the *join-semilattice* $\{\bigcup_{F\in\mathscr{F}} F : \mathscr{F} \subseteq \mathscr{G},\ \#\mathscr{F} > 0,\ \bigcup_{F\in\mathscr{F}} F$ free of pairs of opposites$\}$ consisting of the unions, free of pairs of opposites, of the sets of the family $\mathscr{G}$, ordered by inclusion and augmented by a new *least* element $\hat{0}$; the *Möbius function* of the poset $\mathring{\mathscr{E}}(\mathscr{G})$ is denoted as $\mu_{\mathring{\mathscr{E}}}(\cdot,\cdot)$.

Formula (9.11) below is deduced from eq. (6.45). Formulas (9.12) and (9.13) are deduced from eq. (6.46); they are analogues of formulas (9.9) and (9.10), respectively.

Theorem 9.6. *Let* $\#\mathring{\mathbf{K}}_k^*(\mathscr{M})$ *denote the number of tope committees containing* no pairs of opposites, *of size* k, $3 \le k \le |\mathscr{T}|/2$, *for a simple oriented matroid* $\mathscr{M} := (E_t, \mathscr{T})$ *which is* not acyclic.

(i)

$$\#\mathring{\mathbf{K}}_k^*(\mathscr{M}) = \sum_{\mathscr{G}\in\{\{\bigcup_{e\in E}\binom{\mathscr{T}_e^+}{\lceil (k+1)/2\rceil}\}\,:\,E\subseteq E_t,\ |E|>0\}} \mu_{\mathring{\mathscr{C}}}(\hat{0},\mathscr{G}) \cdot \sum_{G\in\mathring{\mathscr{E}}(\mathscr{G})\,:\,0<|G|\le k} \mu_{\mathring{\mathscr{E}}}(\hat{0},G)\cdot 2^{k-|G|}\cdot\binom{\frac{1}{2}|\mathscr{T}|-|G|}{k-|G|}, \quad (9.11)$$

where $\mu_{\mathring{\mathscr{C}}}(\cdot,\cdot)$ *denotes the* Möbius function *of the family* $\mathring{\mathscr{C}} := \{\hat{0}\}\dot{\cup}\{\{\bigcup_{e\in E}\binom{\mathscr{T}_e^+}{\lceil (k+1)/2\rceil}\}:\ E\subseteq E_t,\ |E|>0\}$, *ordered by inclusion.*

(ii)

$$\#\mathring{\mathbf{K}}_k^*(\mathscr{M}) = \sum_{\substack{\mathscr{G}\subseteq\bigcup_{e\in E_t}\binom{\mathscr{T}_e^+}{\lceil (k+1)/2\rceil}:\\ 1<\#\mathscr{G}\le\binom{k}{\lceil (k+1)/2\rceil},\\ \bigcup_{G\in\mathscr{G}} G \text{ free of pairs of opposites},\\ \bigcup_{F\in\mathbf{min}\gamma_k(\mathscr{G})} F=E_t,\ |\bigcup_{G\in\mathscr{G}} G|\le k}} (-1)^{\#\mathscr{G}}\cdot\mu_{\mathscr{S}(\mathscr{G})}(\hat{0},\hat{1}) \cdot 2^{k-|\bigcup_{G\in\mathscr{G}} G|}\cdot\binom{\frac{1}{2}|\mathscr{T}|-|\bigcup_{G\in\mathscr{G}} G|}{k-|\bigcup_{G\in\mathscr{G}} G|}. \quad (9.12)$$

In particular, if for any family $\mathscr{G}\subseteq\bigcup_{e\in E_t}\binom{\mathscr{T}_e^+}{\lceil (k+1)/2\rceil}$ *such that* $\bigcup_{G\in\mathscr{G}} G$ *is* free of pairs of opposites, $\bigcup_{F\in\mathbf{min}\gamma_k(\mathscr{G})} F = E_t$, *and* $|\bigcup_{G\in\mathscr{G}} G| \le k$, *we have* $|\gamma_k(G)| = 1$ *for any set* $G\in\mathscr{G}$, *then*

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$$\mathring{\mathbf{K}}^*_k(\mathscr{M}) = (-1)^t \cdot \sum_{\substack{\mathscr{G} \subseteq \bigcup_{e \in E_t} \binom{\mathscr{T}_e^+}{\lceil (k+1)/2 \rceil} : \\ t \le \#\mathscr{G} \le \binom{k}{\lceil (k+1)/2 \rceil}, \\ \bigcup_{G \in \mathscr{G}} G \text{ free of pairs of opposites}, \\ \bigcup_{F \in \mathbf{min}\gamma_k(\mathscr{G})} F = E_t,\ |\bigcup_{G \in \mathscr{G}} G| \le k}} (-1)^{\#\mathscr{G}} \cdot 2^{k - |\bigcup_{G \in \mathscr{G}} G|} \cdot \binom{\frac{1}{2}|\mathscr{T}| - |\bigcup_{G \in \mathscr{G}} G|}{k - |\bigcup_{G \in \mathscr{G}} G|} . \quad (9.13)$$

Notes

Several references on the *Principle of Inclusion–Exclusion* can be found in the Notes to Chapter 6 on page 136.

See Ref. [67] on the *acyclic*, *convex*, and *free* sets of oriented matroids.

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10 Tope Committees and Reorientations of Oriented Matroids

Let $\mathbf{K}_k^*(\mathscr{M})$ as earlier denote the family of tope committees of size k for a simple oriented matroid $\mathscr{M} := (E_t, \mathscr{T})$, and let $\mathbf{K}^*(\mathscr{M}) := \dot{\bigcup}_{1\le k\le|\mathscr{T}|-1}\mathbf{K}_k^*(\mathscr{M})$ be the family of all tope committees for $\mathscr{M}$.

Recall that the kth component $\kappa_k^*(\mathscr{M}) := \#\mathbf{K}_k^*(\mathscr{M})$, $1 \le k \le |\mathscr{T}|/2$, of a "$\kappa^*$*-vector*" $\boldsymbol{\kappa}^*(\mathscr{M}) \in \mathbb{N}^{|\mathscr{T}|/2}$ is by convention the number of committees in the family $\mathbf{K}_k^*(\mathscr{M})$.

Similarly, we associate with each family $\mathring{\mathbf{K}}_k^*(\mathscr{M})$, $1 \le k \le |\mathscr{T}|/2$, of tope committees of size k, that contain *no pairs* of *opposites*, the kth component $\mathring{\kappa}_k^*(\mathscr{M}) := \#\mathring{\mathbf{K}}_k^*(\mathscr{M})$ of another "κ^**-vector*" $\mathring{\boldsymbol{\kappa}}^*(\mathscr{M}) \in \mathbb{N}^{|\mathscr{T}|/2}$.

We always have $\mathring{\kappa}_2^*(\mathscr{M}) = \kappa_2^*(\mathscr{M}) = 0$. A simple oriented matroid $\mathscr{M}$ is *acyclic* if and only if $\mathring{\kappa}_1^*(\mathscr{M}) = \kappa_1^*(\mathscr{M}) = 1$. If $\mathscr{M}$ is *not acyclic*, then $\mathring{\kappa}_1^*(\mathscr{M}) = \kappa_1^*(\mathscr{M}) = 0$, and $\mathring{\kappa}_3^*(\mathscr{M}) = \kappa_3^*(\mathscr{M})$.

If $\mathscr{K}^* \in \mathring{\mathbf{K}}_j^*(\mathscr{M})$, for some j, $1 \le j \le |\mathscr{T}|/2$, then there are $|\mathscr{T}|/2 - j$ pairs of topes $\{T, -T\} \subset \mathscr{T}$ such that $|\mathscr{K}^* \cap \{T, -T\}| = 0$. If we add any such pairs of opposites to the set $\mathscr{K}^*$, then the resulting set clearly remains a committee for $\mathscr{M}$. Thus, given an integer k such that $j \le k \le |\mathscr{T}|/2$, and the difference $k - j$ is even, there are exactly $\binom{(|\mathscr{T}|-2j)/2}{(k-j)/2}$ tope committees in the family $\mathbf{K}_k^*(\mathscr{M})$ which contain the committee $\mathscr{K}^*$ as a subset. We see that

$$\kappa_k^*(\mathscr{M}) = \sum_{\substack{1\le j\le k:\\ j\equiv k \pmod 2}} \binom{(|\mathscr{T}|-2j)/2}{(k-j)/2} \cdot \mathring{\kappa}_j^*(\mathscr{M}) , \quad 1 \le k \le |\mathscr{T}|/2 .$$

For example,

$$\kappa_3^*(\mathscr{M}) = \tfrac{|\mathscr{T}|-2}{2} \cdot \mathring{\kappa}_1^*(\mathscr{M}) + \mathring{\kappa}_3^*(\mathscr{M})$$

and

$$\kappa_5^*(\mathscr{M}) = \tfrac{(|\mathscr{T}|-4)(|\mathscr{T}|-2)}{8} \cdot \mathring{\kappa}_1^*(\mathscr{M}) + \tfrac{|\mathscr{T}|-6}{2} \cdot \mathring{\kappa}_3^*(\mathscr{M}) + \mathring{\kappa}_5^*(\mathscr{M}) .$$

Recall that the family $\mathbf{A}^*(\mathscr{M})$ of *tope anti-committees* for the oriented matroid $\mathscr{M}$ is defined to be the family $\{-\mathscr{K}^* : \mathscr{K}^* \in \mathbf{K}^*(\mathscr{M})\}$.

Let A be any subset of the ground set E_t. The sets of topes of the reorientations ${}_{-A}\mathscr{M}$ and ${}_{-(E_t-A)}\mathscr{M}$ of the oriented matroid $\mathscr{M}$ coincide, and via the composite bijection

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$$\begin{aligned} \mathbf{K}^*({}_{-A}\mathscr{M}) \;\to\; \mathbf{A}^*({}_{-A}\mathscr{M}) \;\to\; \mathbf{A}^*({}_{-(E_t-A)}\mathscr{M}) \;\to\; \mathbf{K}^*({}_{-(E_t-A)}\mathscr{M})\,, \\ \mathscr{K}^* \;\mapsto\; -\mathscr{K}^* \;\mapsto\; -\mathscr{K}^* \;\mapsto\; \mathscr{K}^*\,, \end{aligned}$$

the (anti-)committee structures of the oriented matroids ${}_{-A}\mathscr{M}$ and ${}_{-(E_t-A)}\mathscr{M}$ are seen to be identical. In particular, we have

$$\boldsymbol{\kappa}^*({}_{-A}\mathscr{M}) = \boldsymbol{\kappa}^*({}_{-(E_t-A)}\mathscr{M})$$

and

$$\mathring{\boldsymbol{\kappa}}^*({}_{-A}\mathscr{M}) = \mathring{\boldsymbol{\kappa}}^*({}_{-(E_t-A)}\mathscr{M})\,.$$

In this chapter we compare $\boldsymbol{\kappa}^*$-vectors of simple oriented matroids $\mathscr{M}$ and ${}_{-A}\mathscr{M}$, where $A := \{a\}$ are one-element subsets of the ground set E_t.

In Section 10.3 we sum up our enumerative results on general tope committees, and on committees containing no pairs of opposites, obtained in Sections 10.1 and 10.2, respectively.

10.1 The Number of Tope Committees

In this section we recount tope committees for simple oriented matroids, and we begin with expression (9.2) of Section 9.1:

Lemma 10.1. *The number $\#\mathbf{K}^*_k(\mathscr{M})$ of tope committees of size k, $1 \le k \le |\mathscr{T}| - 1$, for a simple oriented matroid $\mathscr{M} := (E_t, \mathscr{T})$ is*

$$\kappa^*_k(\mathscr{M}) := \#\mathbf{K}^*_k(\mathscr{M}) = \binom{|\mathscr{T}|}{|\mathscr{T}|-\ell} + \sum_{\substack{\mathscr{G} \subseteq \bigcup_{e\in E_t}\binom{\mathscr{T}^+_e}{\lfloor(\ell+1)/2\rfloor}:\\ 1\le \#\mathscr{G} \le \binom{\ell}{\lfloor(\ell+1)/2\rfloor},\\ |\bigcup_{G\in\mathscr{G}} G| \le \ell}} (-1)^{\#\mathscr{G}} \cdot \binom{|\mathscr{T}| - |\bigcup_{G\in\mathscr{G}} G|}{|\mathscr{T}|-\ell}\,, \tag{10.1}$$

where $\ell \in \{k, |\mathscr{T}| - k\}$.

Now fix an integer k, $1 \le k \le |\mathscr{T}|/2$, a ground element $a \in E_t$, and an integer $\ell \in \{k, |\mathscr{T}| - k\}$. If we define an integer quantity $\alpha_k(a, \mathscr{M})$ by

$$\alpha_k(a, \mathscr{M}) \;:=\; \binom{|\mathscr{T}|}{|\mathscr{T}|-\ell} + \sum_{\substack{\mathscr{G} \subseteq \bigcup_{e\in E_t-\{a\}}\binom{\mathscr{T}^+_e(\mathscr{M})}{\lfloor(\ell+1)/2\rfloor}:\\ 1\le \#\mathscr{G} \le \binom{\ell}{\lfloor(\ell+1)/2\rfloor},\\ |\bigcup_{G\in\mathscr{G}} G| \le \ell}} (-1)^{\#\mathscr{G}} \cdot \binom{|\mathscr{T}| - |\bigcup_{G\in\mathscr{G}} G|}{|\mathscr{T}|-\ell}\,,$$

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then, according to eq. (10.1), we have

$$\kappa_k^*(\mathcal{M}) = \alpha_k(a, \mathcal{M}) + \sum_{\substack{\mathcal{G}' \subseteq \binom{\mathcal{T}_a^+(\mathcal{M})}{\lfloor(\ell+1)/2\rfloor} - \bigcup_{e\in E_t-\{a\}} \binom{\mathcal{T}_e^+(\mathcal{M})}{\lfloor(\ell+1)/2\rfloor}:\\ 1\le\#\mathcal{G}'\le\binom{\ell}{\lfloor(\ell+1)/2\rfloor},\ |\bigcup_{G\in\mathcal{G}'} G|\le\ell;\\ \mathcal{G}''\subseteq \bigcup_{e\in E_t-\{a\}} \binom{\mathcal{T}_e^+(\mathcal{M})}{\lfloor(\ell+1)/2\rfloor}:\\ 0\le\#\mathcal{G}''\le\binom{\ell}{\lfloor(\ell+1)/2\rfloor}-\#\mathcal{G}',\ |\bigcup_{G\in\mathcal{G}'\dot\cup\mathcal{G}''} G|\le\ell}} (-1)^{\#\mathcal{G}'+\#\mathcal{G}''}$$

$$\cdot \binom{|\mathcal{T}| - |\bigcup_{G\in\mathcal{G}'\dot\cup\mathcal{G}''} G|}{|\mathcal{T}| - \ell} . \tag{10.2}$$

In an analogous expression for the component $\kappa_k^*({}_{-a}\mathcal{M})$, the families $\mathcal{G}'$ range over the subfamilies of the family

$$\binom{\mathcal{T}_a^-(\mathcal{M})}{\lfloor(\ell+1)/2\rfloor} - \bigcup_{e\in E_t-\{a\}} \binom{\mathcal{T}_e^+(\mathcal{M})}{\lfloor(\ell+1)/2\rfloor} .$$

10.2 The Number of Tope Committees Containing No Pairs of Opposites

In this section we recount tope committees for simple oriented matroids that contain *no pairs of opposites*.

Recall that it was shown in the introduction to Chapter 9 that any tope committee $\mathcal{K}^* \in \mathbf{K}_k^*(\mathcal{M})$ for a simple oriented matroid $\mathcal{M} := (E_t, \mathcal{T})$ is a *blocking k-set* of the family $\bigcup_{e\in E_t} \binom{\mathcal{T}_e^+}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor}$ of tope subsets of cardinality $\lfloor(|\mathcal{T}| - k + 1)/2\rfloor$, each of which is contained in at least one *positive halfspace* of $\mathcal{M}$; such blocking sets are recounted in Lemma 10.1. As a consequence, the subfamily $\mathring{\mathbf{K}}_k^*(\mathcal{M}) \subset \mathbf{K}_k^*(\mathcal{M})$ is precisely the collection of *blocking k-sets*, *free of pairs of opposites*, of the family

$$\bigcup_{e\in E_t} \binom{\mathcal{T}_e^+}{\lfloor(|\mathcal{T}| - k + 1)/2\rfloor} .$$

With the help of expression (6.43), we come to the following result:

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Lemma 10.2. *The number* $\#\mathring{\mathbf{K}}^*_k(\mathscr{M})$ *of tope committees of size* k, $1 \le k \le |\mathscr{T}|/2$, *that contain* no pairs of opposites, *for a simple oriented matroid* $\mathscr{M} := (E_t, \mathscr{T})$ *is*

$$\mathring{\kappa}^*_k(\mathscr{M}) := \#\mathring{\mathbf{K}}^*_k(\mathscr{M}) = \binom{|\mathscr{T}|/2}{k}2^k + \sum_{\substack{\mathscr{G}\subseteq\bigcup_{e\in E_t}\binom{\mathscr{T}^+_e}{\lfloor(|\mathscr{T}|-k+1)/2\rfloor}:\\ 1\le\#\mathscr{G}\le\binom{|\mathscr{T}|-k}{\lfloor(|\mathscr{T}|-k+1)/2\rfloor},\\ |\bigcup_{G\in\mathscr{G}}G|\le|\mathscr{T}|-k}} (-1)^{\#\mathscr{G}}$$

$$\cdot \sum_{0\le j\le k}\binom{|\bigcup_{G\in\mathscr{G}}G \,\cup\, -\bigcup_{G\in\mathscr{G}}G| - |\bigcup_{G\in\mathscr{G}}G|}{j}$$

$$\cdot \binom{\frac{1}{2}(\,|\mathscr{T}| - |\bigcup_{G\in\mathscr{G}}G \,\cup\, -\bigcup_{G\in\mathscr{G}}G|\,)}{k-j}2^{k-j}\,. \tag{10.3}$$

If $\mathscr{G}$ is a family of subsets of the set of topes, then we denote by $\mathscr{E}(\mathscr{G})$ the *join-semilattice* $\{\bigcup_{F\in\mathscr{F}}F : \mathscr{F}\subseteq\mathscr{G},\ \ \#\mathscr{F}>0\}$ that consists of the unions of the sets of the family $\mathscr{G}$, ordered by inclusion and augmented by a new *least* element $\hat{0}$ which is interpreted as the empty set. The *Möbius function* of the *lattice* $\mathscr{E}(\mathscr{G})$ is denoted by $\mu_{\mathscr{E}}(\cdot,\cdot)$.

With the help of expression (6.44), Lemma 10.2 can be restated as follows:

Proposition 10.3. *The number* $\#\mathring{\mathbf{K}}^*_k(\mathscr{M})$ *of tope committees of size* k, $1 \le k \le |\mathscr{T}|/2$, *that contain* no pairs of opposites, *for a simple oriented matroid* $\mathscr{M} := (E_t, \mathscr{T})$ *is*

$$\mathring{\kappa}^*_k(\mathscr{M}) := \#\mathring{\mathbf{K}}^*_k(\mathscr{M}) = \binom{|\mathscr{T}|/2}{k}2^k + \sum_{\substack{G\in\mathscr{E}(\bigcup_{e\in E_t}\binom{\mathscr{T}^+_e}{\lfloor(|\mathscr{T}|-k+1)/2\rfloor}):\\ 0<|G|\le|\mathscr{T}|-k}} \mu_{\mathscr{E}}(\hat{0}, G)$$

$$\cdot \sum_{0\le j\le k}\binom{|G\,\cup\,-G|-|G|}{j}\binom{\frac{1}{2}(\,|\mathscr{T}|-|G\,\cup\,-G|\,)}{k-j}2^{k-j}\,.$$

If an integer k, $1 \le k \le |\mathscr{T}|/2$, and an element $a \in E_t$ of the ground set are fixed, then we define a quantity $\beta_k(a, \mathscr{M})$ by

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$$\beta_k(a,\mathscr{M}) := \binom{|\mathscr{T}|/2}{k}2^k + \sum_{\substack{\mathscr{G}\subseteq\bigcup_{e\in E_t-\{a\}}\binom{\mathscr{T}_e^+(\mathscr{M})}{\lfloor(|\mathscr{T}|-k+1)/2\rfloor}:\\ 1\le\#\mathscr{G}\le\binom{|\mathscr{T}|-k}{\lfloor(|\mathscr{T}|-k+1)/2\rfloor},\\ |\bigcup_{G\in\mathscr{G}}G|\le|\mathscr{T}|-k}} (-1)^{\#\mathscr{G}}$$

$$\cdot \sum_{0\le j\le k}\binom{|\bigcup_{G\in\mathscr{G}}G \,\cup\, -\bigcup_{G\in\mathscr{G}}G| - |\bigcup_{G\in\mathscr{G}}G|}{j}$$

$$\cdot \binom{\frac{1}{2}(\,|\mathscr{T}| - |\bigcup_{G\in\mathscr{G}}G \,\cup\, -\bigcup_{G\in\mathscr{G}}G|\,)}{k-j}2^{k-j}\,.$$

By eq. (10.3) we have

$$\mathring{\kappa}_k^*(\mathscr{M}) = \beta_k(a,\mathscr{M})$$

$$+ \sum_{\substack{\mathscr{G}'\subseteq\binom{\mathscr{T}_a^+(\mathscr{M})}{\lfloor(|\mathscr{T}|-k+1)/2\rfloor}-\bigcup_{e\in E_t-\{a\}}\binom{\mathscr{T}_e^+(\mathscr{M})}{\lfloor(|\mathscr{T}|-k+1)/2\rfloor}:\\ 1\le\#\mathscr{G}'\le\binom{|\mathscr{T}|-k}{\lfloor(|\mathscr{T}|-k+1)/2\rfloor},\ |\bigcup_{G\in\mathscr{G}'}G|\le|\mathscr{T}|-k;\\ \mathscr{G}''\subseteq\bigcup_{e\in E_t-\{a\}}\binom{\mathscr{T}_e^+(\mathscr{M})}{\lfloor(|\mathscr{T}|-k+1)/2\rfloor}:\\ 0\le\#\mathscr{G}''\le\binom{|\mathscr{T}|-k}{\lfloor(|\mathscr{T}|-k+1)/2\rfloor}-\#\mathscr{G}',\ |\bigcup_{G\in\mathscr{G}'\dot\cup\mathscr{G}''}G|\le|\mathscr{T}|-k}} (-1)^{\#\mathscr{G}'+\#\mathscr{G}''}$$

$$\cdot \sum_{0\le j\le k}\binom{|\bigcup_{G\in\mathscr{G}'\dot\cup\mathscr{G}''}G \,\cup\, -\bigcup_{G\in\mathscr{G}'\dot\cup\mathscr{G}''}G| - |\bigcup_{G\in\mathscr{G}'\dot\cup\mathscr{G}''}G|}{j}$$

$$\cdot \binom{\frac{1}{2}(\,|\mathscr{T}| - |\bigcup_{G\in\mathscr{G}'\dot\cup\mathscr{G}''}G \,\cup\, -\bigcup_{G\in\mathscr{G}'\dot\cup\mathscr{G}''}G|\,)}{k-j}2^{k-j}\,. \tag{10.4}$$

In an analogous expression for the component $\mathring{\kappa}_k^*({}_{-a}\mathscr{M})$, the families $\mathscr{G}'$ range over the subfamilies of the family

$$\binom{\mathscr{T}_a^-(\mathscr{M})}{\lfloor(|\mathscr{T}|-k+1)/2\rfloor} - \bigcup_{e\in E_t-\{a\}}\binom{\mathscr{T}_e^+(\mathscr{M})}{\lfloor(|\mathscr{T}|-k+1)/2\rfloor}\,.$$

10.3 Tope Committees and Reorientations of Oriented Matroids on One-Element Subsets of the Ground Sets

To find the differences of the components of κ^*-vectors associated with a simple oriented matroid $\mathscr{M}$, and with the oriented matroid ${}_{-a}\mathscr{M}$ which is obtained from $\mathscr{M}$ by reorientation on a one-element subset $\{a\}\subset E_t$ of its ground set, we combine the expressions (10.2) and (10.4) related to $\mathscr{M}$ with analogous expressions related to ${}_{-a}\mathscr{M}$:

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Proposition 10.4. *Let a be an element of the ground set E_t of a simple oriented matroid $\mathscr{M} := (E_t, \mathscr{T})$. For an integer k, $1 \le k \le |\mathscr{T}|/2$, the sum*

$$\sum_{\substack{\mathscr{G}'' \subseteq \bigcup_{e \in E_t - \{a\}} \binom{\mathscr{T}_e^+(\mathscr{M})}{\lfloor (|\mathscr{T}|-k+1)/2 \rfloor}: \\ 0 \le \#\mathscr{G}'' \le \binom{|\mathscr{T}|-k}{\lfloor (|\mathscr{T}|-k+1)/2 \rfloor} - 1, \\ |\bigcup_{G \in \mathscr{G}''} G| \le |\mathscr{T}|-k}} (-1)^{\#\mathscr{G}''} \cdot \Bigg(\sum_{\substack{\mathscr{G}' \subseteq \binom{\mathscr{T}_a^-(\mathscr{M})}{\lfloor (|\mathscr{T}|-k+1)/2 \rfloor} - \bigcup_{e \in E_t - \{a\}} \binom{\mathscr{T}_e^+(\mathscr{M})}{\lfloor (|\mathscr{T}|-k+1)/2 \rfloor}: \\ 1 \le \#\mathscr{G}' \le \binom{|\mathscr{T}|-k}{\lfloor (|\mathscr{T}|-k+1)/2 \rfloor} - \#\mathscr{G}'', \\ |\bigcup_{G \in \mathscr{G}'} G| \le |\mathscr{T}|-k}} (-1)^{\#\mathscr{G}'} \cdot Q(\mathscr{G}', \mathscr{G}'') - \sum_{\substack{\mathscr{G}' \subseteq \binom{\mathscr{T}_a^+(\mathscr{M})}{\lfloor (|\mathscr{T}|-k+1)/2 \rfloor} - \bigcup_{e \in E_t - \{a\}} \binom{\mathscr{T}_e^+(\mathscr{M})}{\lfloor (|\mathscr{T}|-k+1)/2 \rfloor}: \\ 1 \le \#\mathscr{G}' \le \binom{|\mathscr{T}|-k}{\lfloor (|\mathscr{T}|-k+1)/2 \rfloor} - \#\mathscr{G}'', \\ |\bigcup_{G \in \mathscr{G}'} G| \le |\mathscr{T}|-k}} (-1)^{\#\mathscr{G}'} \cdot Q(\mathscr{G}', \mathscr{G}'') \Bigg)$$

and the sum

$$\sum_{\substack{G'' \in \mathscr{E}(\bigcup_{e \in E_t - \{a\}} \binom{\mathscr{T}_e^+(\mathscr{M})}{\lfloor (|\mathscr{T}|-k+1)/2 \rfloor}): \\ 0 \le |G''| \le |\mathscr{T}|-k}} \mu_{\mathscr{E}}(\hat{0}, G'') \cdot \Bigg(\sum_{\substack{G' \in \mathscr{E}(\binom{\mathscr{T}_a^-(\mathscr{M})}{\lfloor (|\mathscr{T}|-k+1)/2 \rfloor} - \bigcup_{e \in E_t - \{a\}} \binom{\mathscr{T}_e^+(\mathscr{M})}{\lfloor (|\mathscr{T}|-k+1)/2 \rfloor}): \\ 0 < |G'| \le |\mathscr{T}|-k}} \mu_{\mathscr{E}}(\hat{0}, G') \cdot \mathfrak{Q}(G', G'') - \sum_{\substack{G' \in \mathscr{E}(\binom{\mathscr{T}_a^+(\mathscr{M})}{\lfloor (|\mathscr{T}|-k+1)/2 \rfloor} - \bigcup_{e \in E_t - \{a\}} \binom{\mathscr{T}_e^+(\mathscr{M})}{\lfloor (|\mathscr{T}|-k+1)/2 \rfloor}): \\ 0 < |G'| \le |\mathscr{T}|-k}} \mu_{\mathscr{E}}(\hat{0}, G') \cdot \mathfrak{Q}(G', G'') \Bigg)$$

both give the difference

$$\kappa_k^*({}_{-a}\mathscr{M}) - \kappa_k^*(\mathscr{M})$$

when we define the quantities $Q(\mathscr{G}', \mathscr{G}'')$ and $\mathfrak{Q}(G', G'')$ by

$$Q(\mathscr{G}', \mathscr{G}'') := \binom{|\mathscr{T}| - |\bigcup_{G \in \mathscr{G}' \dot\cup \mathscr{G}''} G|}{k} \quad \text{and} \quad \mathfrak{Q}(G', G'') := \binom{|\mathscr{T}| - |G' \cup G''|}{k}.$$

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These sums give the difference

$$\mathring{\kappa}^*_k({}_{-a}\mathscr{M}) - \mathring{\kappa}^*_k(\mathscr{M})$$

when we define the quantities $Q(\mathscr{G}', \mathscr{G}'')$ *and* $\mathfrak{Q}(G', G'')$ *by*

$$Q(\mathscr{G}', \mathscr{G}'') := \sum_{0 \le j \le k} \binom{|\bigcup_{G \in \mathscr{G}' \dot\cup \mathscr{G}''} G \ \cup\ -\bigcup_{G \in \mathscr{G}' \dot\cup \mathscr{G}''} G| - |\bigcup_{G \in \mathscr{G}' \dot\cup \mathscr{G}''} G|}{j} \cdot \binom{\frac{1}{2}(\,|\mathscr{T}| - |\bigcup_{G \in \mathscr{G}' \dot\cup \mathscr{G}''} G \ \cup\ -\bigcup_{G \in \mathscr{G}' \dot\cup \mathscr{G}''} G|\,)}{k-j} 2^{k-j}$$

and

$$\mathfrak{Q}(G', G'') := \sum_{0 \le j \le k} \binom{|(G' \cup G'') \cup -(G' \cup G'')| - |G' \cup G''|}{j} \cdot \binom{\frac{1}{2}(\,|\mathscr{T}| - |(G' \cup G'') \cup -(G' \cup G'')|\,)}{k-j} 2^{k-j} .$$

Note

Reorientations of oriented matroids are simply defined but fundamental transformations of their structures, as shown throughout the text [27].

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11 Topes and Critical Committees

Let $\mathcal{M} := (E_t, \mathcal{L}) = (E_t, \mathcal{T})$ be a *simple oriented matroid* on the ground set E_t, with set of covectors $\mathcal{L}$, and with set of topes $\mathcal{T}$. As earlier, we regard the topes $T := (T(1), \ldots, T(t)) \in \mathcal{T}$ as elements of the real Euclidean space $\mathbb{R}^t$ of *row vectors*. If $T, T', T'' \in \mathbb{R}^t$, then $\langle T', T''\rangle := \sum_{e=1}^{t} T'(e) \cdot T''(e)$, and $\|T\| := \sqrt{\langle T, T\rangle}$. We denote by $(\boldsymbol{\sigma}(1), \ldots, \boldsymbol{\sigma}(t))$ the standard basis of $\mathbb{R}^t$, that is, $\boldsymbol{\sigma}(i) := (0, \ldots, \underset{\substack{\uparrow \\ i}}{1}, \ldots, 0)$, $1 \le i \le t$.

Recall that the *vertices* of the *tope graph* $\mathcal{T}(\mathcal{L}(\mathcal{M}))$ of the oriented matroid $\mathcal{M}$ are by definition its *topes*; a pair of topes $\{T', T''\} \subset \mathcal{T}$ is an *edge* of the graph $\mathcal{T}(\mathcal{L}(\mathcal{M}))$ if the topes T' and T'' are *adjacent*, that is, they cover some *subtope* in the *big face lattice* of $\mathcal{M}$. Recall also that the *separation set* $\mathbf{S}(T', T'')$ of topes T' and T'' is defined by $\mathbf{S}(T', T'') := \{e \in E_t : T'(e) \neq T''(e)\}$. The *graph distance* $d(T', T'')$ between the topes T' and T'' is the cardinality of the *separation set* $\mathbf{S}(T', T'')$, that is,

$$d(T', T'') = |\mathbf{S}(T', T'')| = t - \tfrac{1}{4}\|T'' + T'\|^2 = \tfrac{1}{4}\|T'' - T'\|^2 = \tfrac{1}{2}\big(t - \langle T'', T'\rangle\big) .$$

If $B \in \mathcal{T}$, then we denote by $\mathcal{T}(\mathcal{L}(\mathcal{M}), B)$ the *tope poset* of $\mathcal{M}$ based at the tope B; by convention, we have $T' \preceq T''$ in $\mathcal{T}(\mathcal{L}(\mathcal{M}), B)$ if and only if $\mathbf{S}(B, T') \subseteq \mathbf{S}(B, T'')$. In Section 11.1 we show that the vertex set $\mathrm{V}(\boldsymbol{R})$ of any *symmetric* $2t$*-cycle* $\boldsymbol{R}$ in the tope graph of $\mathcal{M}$ is a *maximal positive basis* of the space $\mathbb{R}^t$.

In Section 11.2 we regard the set $\mathrm{V}(\boldsymbol{R})$ as a subposet of the tope poset $\mathcal{T}(\mathcal{L}(\mathcal{M}), B)$ of $\mathcal{M}$, and verify that if B is the least element of $\mathcal{T}(\mathcal{L}(\mathcal{M}), B)$, then B is equal to the unweighted sum of the members of the set $\mathbf{min}\ \mathrm{V}(\boldsymbol{R})$ of minimal elements of the subposet $\mathrm{V}(\boldsymbol{R})$. In particular, if B is the positive tope $\mathrm{T}^{(+)}$, then the set $\mathbf{min}\ \mathrm{V}(\boldsymbol{R})$ is a *critical tope committee* for the *acyclic oriented matroid* $\mathcal{M}$.

11.1 Symmetric Cycles in the Tope Graph, and Maximal Positive Bases of $\mathbb{R}^t$

In this section we consider *maximal chains* in the *tope poset* of a simple oriented matroid $\mathcal{M} := (E_t, \mathcal{L}) = (E_t, \mathcal{T})$, and show that the *vertex sets* of the corresponding *symmetric cycles* in the *tope graph* of $\mathcal{M}$ are *maximal positive bases* of $\mathbb{R}^t$.

Let

$$\mathbf{m} := (R^0 := B \lessdot R^1 \lessdot \cdots \lessdot R^{t-1} \lessdot R^t := -B)$$

be a *maximal chain* in the tope poset $\mathcal{T}(\mathcal{L}(\mathcal{M}), B)$ of $\mathcal{M}$, and $-\mathbf{m} := \{-R : R \in \mathbf{m}\}$. The union $\mathrm{V}(\boldsymbol{R}) := \mathbf{m} \cup -\mathbf{m}$ is the vertex set of the *symmetric cycle* $\boldsymbol{R} := (R^0 := B, R^1, \ldots, R^{2t-1}, R^0)$ in the tope graph $\mathcal{T}(\mathcal{L}(\mathcal{M}))$; by convention, we have

$$R^{k+t} = -R^k , \quad 0 \le k \le t-1 ,$$

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see Definition 1.6. The subset of topes in $\mathrm{V}(\boldsymbol{R})$ having *inclusion-maximal positive parts* is denoted by $\mathbf{max}^+\ (\mathrm{V}(\boldsymbol{R}))$. If $A \subseteq E_t$, then ${}_{-A}\mathrm{V}(\boldsymbol{R}) := \{{}_{-A}R \colon R \in \mathrm{V}(\boldsymbol{R})\}$.

Let $(\iota_1, \ldots, \iota_t) \in \mathbb{N}^t$ be a sequence defined by $\{\iota_i\} := \mathbf{S}(R^{i-1}, R^i)$; note that $\mathbf{m} - \{-B\} \subseteq \mathscr{T}^-_{\iota_t}$ if $B(\iota_t) = -1$, and $\mathbf{m} - \{-B\} \subseteq \mathscr{T}^+_{\iota_t}$ if $B(\iota_t) = 1$.

It is easy to verify that the chain $\mathbf{m} - \{-B\}$ is a *basis* of the space $\mathbb{R}^t$ and, as a consequence, the vertex set $\mathrm{V}(\boldsymbol{R})$ of the symmetric $2t$-cycle $\boldsymbol{R}$ is a *maximal positive basis* of $\mathbb{R}^t$. Indeed, the square *sign matrix*

$$\mathbf{M} := \mathbf{M}(\boldsymbol{R}) := \begin{pmatrix} R^0 \\ R^1 \\ \vdots \\ R^{t-2} \\ R^{t-1} \end{pmatrix} \in \mathbb{R}^{t \times t}$$

is *similar* to the matrix

$$\begin{pmatrix} 2\cdot B(\iota_1)\cdot\boldsymbol{\sigma}(\iota_1) \\ 2\cdot B(\iota_2)\cdot\boldsymbol{\sigma}(\iota_2) \\ \vdots \\ 2\cdot B(\iota_{t-1})\cdot\boldsymbol{\sigma}(\iota_{t-1}) \\ B(\iota_t)\cdot\boldsymbol{\sigma}(\iota_t) \end{pmatrix},$$

which is *nonsingular*, since the absolute value of its determinant is 2^{t-1}.

The ith row $(\mathbf{M}^{-1})_i$, $1 \le i \le t$, of the inverse matrix $\mathbf{M}^{-1}$ of $\mathbf{M}$ is

$$(\mathbf{M}^{-1})_i = \begin{cases} \frac{1}{2} \cdot B(i) \cdot (\ \boldsymbol{\sigma}(k) - \boldsymbol{\sigma}(k+1)\ ), & \text{if } i = \iota_k,\ k \neq t\ , \\ \frac{1}{2} \cdot B(i) \cdot (\ \boldsymbol{\sigma}(1) + \boldsymbol{\sigma}(t)\ ), & \text{if } i = \iota_t\ . \end{cases}$$

Thus, if $T \in \mathscr{T}$, then we have

$$T = \boldsymbol{x}\mathbf{M}\ ,$$

for some row vector

$$\boldsymbol{x} := (x_1, \ldots, x_t) = T\mathbf{M}^{-1} \tag{11.1}$$

such that

$$\boldsymbol{x} \in \{-1, 0, 1\}^t\ .$$

We see that for any tope $T \in \mathscr{T}$, there exists a *unique inclusion-minimal subset* $\boldsymbol{Q}(T, \boldsymbol{R}) \subset \mathrm{V}(\boldsymbol{R})$ such that

$$T = \sum_{Q \in \boldsymbol{Q}(T,\boldsymbol{R})} Q\ ; \tag{11.2}$$

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this set (in fact, it is of *odd cardinality*) of *linearly independent* elements of the space $\mathbb{R}^t$ is

$$\boldsymbol{Q}(T,\boldsymbol{R}) = \{x_i \cdot R^{i-1} : x_i \neq 0\} ,$$

where the vector $\boldsymbol{x}$ is defined by eq. (11.1).

As a consequence, a subset $\mathcal{K}^* \subset \mathcal{T}$ is a *tope committee* for $\mathcal{M}$ if and only if

$$\sum_{T \in \mathcal{K}^*} \sum_{Q \in \boldsymbol{Q}(T,\boldsymbol{R})} Q \geq \mathrm{T}^{(+)} .$$

In Section 11.2 we discuss relation (11.2), and describe the structure of the sets $\boldsymbol{Q}(T,\boldsymbol{R})$ in more detail.

11.2 Topes and Critical Committees

Given a *tope* T of a simple oriented matroid $\mathcal{M} := (E_t, \mathcal{L}) = (E_t, \mathcal{T})$ with distinguished *symmetric cycle* $\boldsymbol{R}$ in its *tope graph*, we describe in this section the corresponding *inclusion-minimal subset* $\boldsymbol{Q}(T,\boldsymbol{R}) \subset \mathrm{V}(\boldsymbol{R})$ appearing in expression (11.2), in terms of the *tope poset* of $\mathcal{M}$. Dual results omitted here could also be given, since the tope poset of $\mathcal{M}$ is *self-dual*.

Theorem 11.1. *Let* $\boldsymbol{R} := (R^0, R^1, \ldots, R^{2t-1}, R^0)$ *be a* symmetric cycle *in the* tope graph $\mathcal{T}(\mathcal{L}(\mathcal{M}))$ *of a simple oriented matroid* $\mathcal{M} := (E_t, \mathcal{L}) = (E_t, \mathcal{T})$. *Pick a tope* $B \in \mathcal{T}$, *and consider the* vertex set $\mathrm{V}(\boldsymbol{R})$ *of the cycle* $\boldsymbol{R}$ *as a* subposet *of the tope poset* $\mathcal{T}(\mathcal{L}(\mathcal{M}), B)$ *based at the tope* B.

The unique inclusion-minimal subset $\boldsymbol{Q}(B,\boldsymbol{R}) \subset \mathcal{T}$ *such that*

$$B = \sum_{Q \in \boldsymbol{Q}(B,\boldsymbol{R})} Q ,$$

is the set **min** $\mathrm{V}(\boldsymbol{R})$, *of odd cardinality, of* minimal elements *of the subposet* $\mathrm{V}(\boldsymbol{R}) \subset \mathcal{T}(\mathcal{L}(\mathcal{M}), B)$; *therefore,*

$$B = \sum_{Q \in \mathbf{min}\ \mathrm{V}(\boldsymbol{R})} Q . \tag{11.3}$$

Proof. If $B \in \mathrm{V}(\boldsymbol{R})$, there is nothing to prove. Suppose that $B \notin \mathrm{V}(\boldsymbol{R})$, and reorient the covectors of the poset $\mathcal{T}(\mathcal{L}(\mathcal{M}), B)$ on the negative part B^- of the base tope B; in other words, consider the tope poset $\mathcal{T}(\mathcal{L}({}_{-(B^-)}\mathcal{M}), \mathrm{T}^{(+)})$ of the *acyclic* oriented matroid ${}_{-(B^-)}\mathcal{M}$ based at the *positive tope* $\mathrm{T}^{(+)}$.

If O is a tope of the oriented matroid ${}_{-(B^-)}\mathcal{M}$, then the poset rank $d(\mathrm{T}^{(+)}, O)$ of O in the tope poset $\mathcal{T}(\mathcal{L}({}_{-(B^-)}\mathcal{M}), \mathrm{T}^{(+)})$ of ${}_{-(B^-)}\mathcal{M}$ is equal to the cardinality $|O^-| = |\mathbf{S}(\mathrm{T}^{(+)}, O)|$ of the *negative part* of O.

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A tope O of the oriented matroid ${}_{-(B^-)}\mathcal{M}$ belongs to the set $\mathbf{max}^+\ ({}_{-(B^-)}\mathrm{V}(\boldsymbol{R}))$ if and only if for the 2-path (O', O, O''), where $O' \neq O''$, in the symmetric cycle $({}_{-(B^-)}R^0, {}_{-(B^-)}R^1, \ldots, {}_{-(B^-)}R^{2t-1}, {}_{-(B^-)}R^0)$ we have $d(\mathrm{T}^{(+)}, O') = d(\mathrm{T}^{(+)}, O'') = d(\mathrm{T}^{(+)}, O) + 1$. By Proposition 1.24 the set $\mathbf{max}^+\ ({}_{-(B^-)}\mathrm{V}(\boldsymbol{R})) = \mathbf{min}\ {}_{-(B^-)}\mathrm{V}(\boldsymbol{R})$ of minimal elements of the subposet ${}_{-(B^-)}\mathrm{V}(\boldsymbol{R}) \subset \mathcal{T}(\mathcal{L}({}_{-(B^-)}\mathcal{M}), \mathrm{T}^{(+)})$ is the inclusion-minimal subset of topes with the property $\sum_{T \in \mathbf{max}^+\ ({}_{-(B^-)}\mathrm{V}(\boldsymbol{R}))} T = \mathrm{T}^{(+)}$, that is,

$$\mathrm{T}^{(+)} = \sum_{T \in \mathbf{min}\ {}_{-(B^-)}\mathrm{V}(\boldsymbol{R})} T\ . \tag{11.4}$$

This means that $\mathbf{min}\ {}_{-(B^-)}\mathrm{V}(\boldsymbol{R})$ is a *critical tope committee* for the *acyclic* oriented matroid ${}_{-(B^-)}\mathcal{M}$. The relation (11.3) is equivalent to eq. (11.4). □

Corollary 11.2. *If* $\boldsymbol{R}$ *is a* symmetric cycle *in the* tope graph *of a simple oriented matroid* $\mathcal{M} := (E_t, \mathcal{T})$*, then for any tope* $T \in \mathcal{T}$*, the corresponding set* $\boldsymbol{Q}(T, \boldsymbol{R})$*, of* odd cardinality*, is the set*

$${}_{-(T^-)}\Big(\mathbf{max}^+\ \big({}_{-(T^-)}\mathrm{V}(\boldsymbol{R})\big)\Big)\ ;$$

therefore

$$T = \sum_{Q\ \in\ {}_{-(T^-)}(\mathbf{max}^+\ ({}_{-(T^-)}\mathrm{V}(\boldsymbol{R})))} Q\ .$$

Example 11.3.

(i) Consider the Hasse diagram, depicted in Figure 11.1(a), of the tope poset $\mathcal{T}(\mathcal{L}(\mathcal{M}), B)$ of a simple oriented matroid $\mathcal{M}$ of rank 3, where $B := (-1, 1, 1, 1, 1)$. Fix the *symmetric cycle*

$$\boldsymbol{R} := \begin{pmatrix} 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 \end{pmatrix}$$

in the *tope graph* of $\mathcal{M}$.

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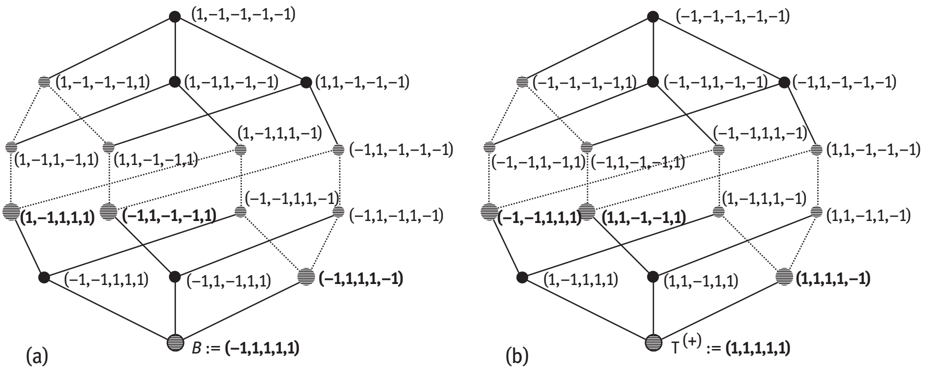

Figure 11.1: (a) The Hasse diagram of the tope poset $\mathcal{T}(\mathcal{L}(\mathcal{M}), B)$ based at the tope $B := (-1, 1, 1, 1, 1)$, and the subdiagram, depicted with gray dashed lines, of the subposet $\mathrm{V}(\boldsymbol{R})$ which is the vertex set of a symmetric cycle $\boldsymbol{R}$ in the tope graph. (b) The Hasse diagram of the tope poset $\mathcal{T}(\mathcal{L}({}_{-\{1\}}\mathcal{M}), \mathrm{T}^{(+)})$ based at the positive tope $\mathrm{T}^{(+)} := (1, 1, 1, 1, 1)$.

The Hasse diagram of the tope poset $\mathcal{T}(\mathcal{L}({}_{-\{1\}}\mathcal{M}), \mathrm{T}^{(+)})$ of the *acyclic* oriented matroid ${}_{-\{1\}}\mathcal{M}$ is shown in Figure 11.1(b). We have

$$\mathbf{min}_{\,-(B^-)}\mathrm{V}(\boldsymbol{R}) = \begin{array}{rrrrr} \{\,-1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1\,\}\,, \end{array}$$

and

$$\mathrm{T}^{(+)} = \sum_{Q \in \mathbf{min}_{\,-(B^-)}\mathrm{V}(\boldsymbol{R})} Q = (-1, -1, 1, 1, 1) \;+\; (1, 1, -1, -1, 1) \;+\; (1, 1, 1, 1, -1)\,.$$

Similarly,

$$\mathbf{min}\,\mathrm{V}(\boldsymbol{R}) = \begin{array}{rrrrr} \{\;1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1\,\}\,, \end{array}$$

and

$$\begin{aligned} B = \sum_{Q \in \mathbf{min}\,\mathrm{V}(\boldsymbol{R})} Q &= (1, -1, 1, 1, 1) \;+\; (-1, 1, -1, -1, 1) \;+\; (-1, 1, 1, 1, -1) \\ &= (-1, 1, 1, 1, 1)\,. \end{aligned}$$

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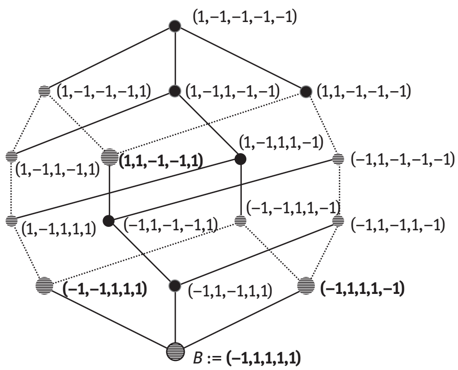

Figure 11.2: The Hasse diagram of the tope poset $\mathscr{T}(\mathscr{L}(\mathscr{M}), B)$ based at the tope $B := (-1, 1, 1, 1, 1)$, and the subdiagram, depicted with gray dashed lines, of the subposet $\mathrm{V}(\boldsymbol{R})$ which is the vertex set of a symmetric cycle $\boldsymbol{R}$ in the tope graph, cf. Figure 11.1(a).

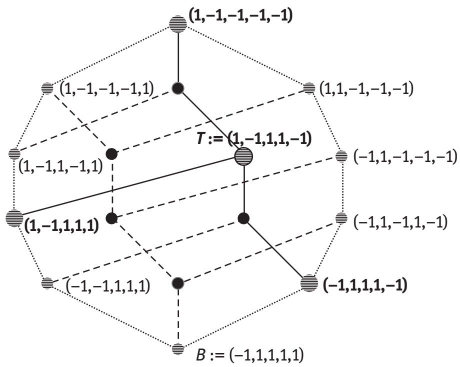

Figure 11.3: The Hasse diagram of the tope poset $\mathscr{T}(\mathscr{L}(\mathscr{M}), B)$ based at the tope $B := (-1, 1, 1, 1, 1)$, and the subdiagram, depicted with gray dashed lines, of the subposet $\mathrm{V}(\boldsymbol{R})$ which is the vertex set of a symmetric cycle $\boldsymbol{R}$ in the tope graph, cf. Figures 11.1(a) and 11.2.

(ii) The subposet of the poset $\mathscr{T}(\mathscr{L}(\mathscr{M}), B)$, depicted in Figure 11.2, is the vertex set of another symmetric cycle

$$\boldsymbol{R} := \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 \end{pmatrix}$$

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in the tope graph of $\mathscr{M}$. We have

$$\mathbf{min}\ \mathrm{V}(\boldsymbol{R}) = \begin{array}{l}\{\,-1\ -1\ \ 1\ \ \ 1\ \ \ 1\\ \ \ \ 1\ \ \ 1\ -1\ -1\ \ 1\\ \ -1\ \ 1\ \ \ 1\ \ \ 1\ -1\,\}\,,\end{array}$$

and

$$B = \sum_{Q\in\mathbf{min}\ \mathrm{V}(\boldsymbol{R})} Q = (-1,-1,1,1,1)\ +\ (1,1,-1,-1,1)\ +\ (-1,1,1,1,-1)$$
$$= (-1,1,1,1,1)\,.$$

(iii) Let

$$\boldsymbol{R} := \begin{pmatrix} -1 & 1 & 1 & 1 & 1\\ -1 & -1 & 1 & 1 & 1\\ 1 & -1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1 & 1\\ 1 & -1 & -1 & -1 & 1\\ 1 & -1 & -1 & -1 & -1\\ 1 & 1 & -1 & -1 & -1\\ -1 & 1 & -1 & -1 & -1\\ -1 & 1 & -1 & 1 & -1\\ -1 & 1 & 1 & 1 & -1\\ -1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

be yet another symmetric cycle in the tope graph of the oriented matroid $\mathscr{M}$, see Figure 11.3. Pick the tope $T := (1,-1,1,1,-1)$ of $\mathscr{M}$. Since

$${}_{-(T^-)}\big(\mathbf{max}^+\ \big({}_{-(T^-)}\mathrm{V}(\boldsymbol{R})\big)\big) = \begin{array}{l}\{\ \ 1\ -1\ \ 1\ \ \ 1\ \ \ 1\\ \ \ -1\ \ 1\ \ \ 1\ \ \ 1\ -1\\ \ \ \ \ 1\ -1\ -1\ -1\ -1\,\}\,,\end{array}$$

we have

$$T := (1,-1,1,1,-1) = \sum_{Q\,\in\,{}_{-(T^-)}\left(\mathbf{max}^+\ \left({}_{-(T^-)}\mathrm{V}(\boldsymbol{R})\right)\right)} Q$$
$$= (1,-1,1,1,1)\ +\ (-1,1,1,1,-1)\ +\ (1,-1,-1,-1,-1)\,.$$

Notes

A self-duality of tope posets of oriented matroids is verified, for example, in [27, Prop. 4.2.15(ii)]: the mapping $T \mapsto -T$ is a *fixed-point-free automorphism*.

The Hasse diagram of a tope poset depicted in Figure 11.1(b) is borrowed from Ref. [27, Fig. 4.2.2].

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12 Critical Committees and Distance Signals

If $\mathfrak{V}$ is the vertex sequence of some path in a connected graph, then the corresponding sequence $(d(w, v) : v \in \mathfrak{V})$ of the distances between its vertices and distinguished vertex w of the graph can be extended in a natural way to a periodic signal.

In this chapter we consider such *distance signals* associated with *symmetric cycles* in the *tope graph* of a simple oriented matroid. When the vertex sequence of a symmetric cycle is regarded as a subposet of the tope poset, and the distinguished vertex of the tope graph is the base tope of the tope poset, a few basic properties of the *discrete Fourier transform* (*DFT*) allow us to express the number of *minimal elements* of the *vertex sequence* of the symmetric cycle via the *magnitudes of components of the DFT* of the *distance signal*.

In the case of a simple *acyclic* oriented matroid on a t-element ground set, with the distinguished *positive tope*, we thus express the *size* of a *critical committee* via the magnitudes of the $\lfloor t/2 \rfloor$ components, with the odd indices, of the DFT of the distance signal.

12.1 The Distance Signal of a Symmetric Cycle in the Tope Graph

In this section we associate with *symmetric cycles* in the *tope graphs* of oriented matroids certain *periodic signals* determined by the poset ranks of the elements in the vertex sequences of the cycles.

Let $\mathscr{A} := (E_t, \mathscr{T})$ be a simple *acyclic* oriented matroid on the ground set E_t, with set of topes $\mathscr{T}$. Let $\boldsymbol{R} := (R^0, R^1, \ldots, R^{2t-1}, R^0)$ be a *symmetric cycle* in the tope graph of $\mathscr{A}$, that is, $R^{k+t} = -R^k$, $0 \le k \le t-1$, see Definition 1.6. Let $\mathscr{T}(\mathscr{L}(\mathscr{A}), \mathrm{T}^{(+)})$ be the *tope poset* of $\mathscr{A}$ based at the *positive tope* $\mathrm{T}^{(+)}$. By Theorem 11.1, the set $\mathbf{min}\ \mathrm{V}(\boldsymbol{R})$ of minimal elements of the subposet $\mathrm{V}(\boldsymbol{R}) := \{R^0, R^1, \ldots, R^{2t-1}\} \subset \mathscr{T}(\mathscr{L}(\mathscr{A}), \mathrm{T}^{(+)})$ is a *critical tope committee* for $\mathscr{A}$: it is the *inclusion-minimal* subset $\mathscr{K}^* \subset \mathrm{V}(\boldsymbol{R})$ such that $\sum_{T \in \mathscr{K}^*} T = \mathrm{T}^{(+)}$. Let $\rho(T)$ denote the poset rank of a tope $T \in \mathscr{T}(\mathscr{L}(\mathscr{A}), \mathrm{T}^{(+)})$. The sequence

$$\boldsymbol{z}_{\boldsymbol{R}} := \left(z_{\boldsymbol{R}}(0) := \rho(R^0), z_{\boldsymbol{R}}(1) := \rho(R^1), \ldots, z_{\boldsymbol{R}}(2t-1) := \rho(R^{2t-1})\right)$$

determines the *distance signal* $z_{\boldsymbol{R}} : \mathbb{Z} \to \{0\} \dot\cup E_t$ *of the cycle* $\boldsymbol{R}$, with period $2t$:

$$z_{\boldsymbol{R}}(j + 2t) = z_{\boldsymbol{R}}(j) \ , \quad j \in \mathbb{Z} \ .$$

Let $\ell^2(\mathbb{Z}_{2t})$ denote the $2t$-dimensional complex coordinate space; the elements of $\ell^2(\mathbb{Z}_{2t})$ are supposed to be row vectors whose components are indexed from 0 to $2t-1$. We regard the *distance vector* $\boldsymbol{z}_{\boldsymbol{R}}$ of $\boldsymbol{R}$ as an element of the space $\ell^2(\mathbb{Z}_{2t})$.

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Let $\boldsymbol{\iota}$ denote the vector $(1, 1, \ldots, 1) \in \ell^2(\mathbb{Z}_{2t})$. Since $z_{\boldsymbol{R}}(k) + z_{\boldsymbol{R}}(k+t) = t$, $0 \le k \le t-1$, we have

$$\boldsymbol{z}_{\boldsymbol{R}} \cdot \boldsymbol{\iota}^{\top} = t^2 \; .$$

If $R \in \mathrm{V}(\boldsymbol{R})$ is a vertex of the symmetric cycle $\boldsymbol{R}$, then we denote by $\mathscr{N}(R)$ the neighborhood of R in the cycle $\boldsymbol{R}$; $\mathbf{I}$ and $\mathbf{C}$ denote the $2t \times 2t$ *identity matrix* and the $2t \times 2t$ *"basic" circulant permutation matrix*

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \cdots\cdots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} , \tag{12.1}$$

respectively, with the rows and columns indexed from 0 to $2t-1$.

Since

$$|\mathbf{min}\ \mathrm{V}(\boldsymbol{R})| = t - \frac{1}{4} \sum_{\substack{\{R',R''\}:=\mathscr{N}(R):\\ R\in\mathrm{V}(\boldsymbol{R})}} |\rho(R'') - \rho(R')| , \tag{12.2}$$

we have

$$\begin{aligned} |\mathbf{min}\ \mathrm{V}(\boldsymbol{R})| &= t - \frac{1}{8} \sum_{\substack{\{R',R''\}:=\mathscr{N}(R):\\ R\in\mathrm{V}(\boldsymbol{R})}} \big(\rho(R'') - \rho(R')\big)^2 \\ &= t - \frac{1}{4} \sum_{\substack{\{R',R''\}:=\mathscr{N}(R):\\ R\in\mathrm{V}(\boldsymbol{R})}} \big(\rho(R)^2 - \rho(R')\rho(R'')\big) ; \end{aligned}$$

in other words, $|\mathbf{min}\ \mathrm{V}(\boldsymbol{R})| = t - \frac{1}{4}\boldsymbol{z}_{\boldsymbol{R}} \cdot (\mathbf{I} - \mathbf{C})(\mathbf{I} + \mathbf{C}) \cdot \boldsymbol{z}_{\boldsymbol{R}}^{\top}$, or

$$|\mathbf{min}\ \mathrm{V}(\boldsymbol{R})| = t - \frac{1}{8}\boldsymbol{z}_{\boldsymbol{R}} \cdot (2\mathbf{I} - \mathbf{C}^{-2} - \mathbf{C}^2) \cdot \boldsymbol{z}_{\boldsymbol{R}}^{\top} \; . \tag{12.3}$$

The first row of the *symmetric circulant matrix* $2\mathbf{I} - \mathbf{C}^{-2} - \mathbf{C}^2$ is the vector $\mathbf{b} := (2, 0, -1, 0, \ldots, 0, -1, 0)$; the components of the DFT of $\mathbf{b}$ are

$$\hat{\mathrm{b}}(k) := \sum_{n=0}^{2t-1} \mathrm{b}(n)\mathrm{e}^{-\pi\imath kn/t} = 4\sin^2 \tfrac{\pi k}{t} \; , \quad 0 \le k \le 2t-1 \; .$$

Denote by $\mathbf{W}$ the $2t \times 2t$ *Fourier matrix*; recall that its (m, n)th entries are $\mathrm{e}^{-\pi\imath mn/t}$, $0 \le m, n \le 2t-1$. The *DFT* and the *inverse DFT* of the distance vector $\boldsymbol{z}_{\boldsymbol{R}}$ are by definition the vectors

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$$\hat{z}_{\boldsymbol{R}} := \boldsymbol{z}_{\boldsymbol{R}}\mathbf{W} \quad \text{and} \quad \check{z}_{\boldsymbol{R}} := \boldsymbol{z}_{\boldsymbol{R}}\mathbf{W}^{-1} \ ,$$

respectively. Thus,

$$\begin{aligned}\hat{z}_{\boldsymbol{R}}(k) &:= \sum_{j=0}^{2t-1} z_{\boldsymbol{R}}(j)\mathrm{e}^{-\pi\imath kj/t} \\ &= \sum_{j=0}^{t-1}\Big(z_{\boldsymbol{R}}(j)\mathrm{e}^{-\pi\imath kj/t} + (t - z_{\boldsymbol{R}}(j))\mathrm{e}^{-\pi\imath k(j+t)/t}\Big) \ , \quad 0 \le k \le 2t-1,\end{aligned}$$

that is,

$$\hat{z}_{\boldsymbol{R}}(k) = \begin{cases} t^2 \ , & \text{if } k = 0, \\ 0 \ , & \text{if } k \text{ is even, } k \neq 0 \ , \\ 2\Big(-t(1-\mathrm{e}^{-\pi\imath k/t})^{-1} + \sum_{j=0}^{t-1} z_{\boldsymbol{R}}(j)\mathrm{e}^{-\pi\imath kj/t}\Big), & \text{if } k \text{ is odd;} \end{cases}$$

in particular, if t is odd, then $\hat{z}_{\boldsymbol{R}}(t) = -t + 2\sum_{j=0}^{t-1}(-1)^j z_{\boldsymbol{R}}(j)$.

We have

$$2\mathbf{I} - \mathbf{C}^{-2} - \mathbf{C}^2 = \mathbf{W}^{-1} \cdot 4\,\mathrm{diag}(0, \sin^2\tfrac{\pi}{t}, \ldots, \sin^2\tfrac{\pi k}{t}, \ldots, \sin^2\tfrac{\pi(2t-1)}{t}) \cdot \mathbf{W} \ ,$$

and expression (12.3) implies that

$$|\mathbf{min}\ \mathrm{V}(\boldsymbol{R})| = t - \tfrac{1}{2}\check{\boldsymbol{z}}_{\boldsymbol{R}} \cdot \mathrm{diag}(0, \sin^2\tfrac{\pi}{t}, \ldots, \sin^2\tfrac{\pi k}{t}, \ldots, \sin^2\tfrac{\pi(2t-1)}{t}) \cdot \hat{\boldsymbol{z}}_{\boldsymbol{R}}^{\top} \ . \tag{12.4}$$

Denote by $\overline{\hat{\boldsymbol{z}}_{\boldsymbol{R}}}$ the vector composed of the complex conjugates of the components of $\hat{\boldsymbol{z}}_{\boldsymbol{R}}$. Since

$$\check{\boldsymbol{z}}_{\boldsymbol{R}} = \tfrac{1}{2t}\overline{\hat{\boldsymbol{z}}_{\boldsymbol{R}}} \ ,$$

it follows from eq. (12.4) that

$$|\mathbf{min}\ \mathrm{V}(\boldsymbol{R})| = t - \tfrac{1}{4t}\overline{\hat{\boldsymbol{z}}_{\boldsymbol{R}}} \cdot \mathrm{diag}(0, \sin^2\tfrac{\pi}{t}, \ldots, \sin^2\tfrac{\pi k}{t}, \ldots, \sin^2\tfrac{\pi(2t-1)}{t}) \cdot \hat{\boldsymbol{z}}_{\boldsymbol{R}}^{\top} \ .$$

Using *Plancherel's formula*, we restate this observation:

$$\begin{aligned}|\mathbf{min}\ \mathrm{V}(\boldsymbol{R})| &= t - \frac{1}{4t}\sum_{k=0}^{2t-1}|\hat{z}_{\boldsymbol{R}}(k)|^2 \cdot \sin^2\tfrac{\pi k}{t} \\ &= t - \frac{1}{2}\|\boldsymbol{z}_{\boldsymbol{R}}\|^2 + \frac{1}{4t}\sum_{k=0}^{2t-1}|\hat{z}_{\boldsymbol{R}}(k)|^2 \cdot \cos^2\tfrac{\pi k}{t} \ .\end{aligned} \tag{12.5}$$

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12.2 Critical Committees and Distance Signals

Given a *symmetric cycle* $\boldsymbol{R}$ in the *tope graph* of a simple oriented matroid, in this section we express the cardinality of the *unique inclusion-minimal* subset $\boldsymbol{Q}(T, \boldsymbol{R}) \subset \mathrm{V}(\boldsymbol{R})$ associated with a tope T, such that $T = \sum_{Q \in \boldsymbol{Q}(T,\boldsymbol{R})} Q$, via the *magnitudes of components* of the *DFT* of the corresponding *distance vector* of the cycle $\boldsymbol{R}$ with respect to the tope T.

Since $\hat{z}_{\boldsymbol{R}}(k) = \overline{\hat{z}_{\boldsymbol{R}}(2t-k)}$, and the magnitudes $|\hat{z}_{\boldsymbol{R}}(k)|$ do not depend on circular translation, namely, $|(\boldsymbol{z}_{\boldsymbol{R}}\mathbf{C}^j)\hat{}(k)| = |\hat{z}_{\boldsymbol{R}}(k)|$, $1 \leq k \leq 2t-1$, for any integer j, we restate eq. (12.5) as follows:

Lemma 12.1. *For any distance vector $\boldsymbol{z}_{\boldsymbol{R}}$ of a symmetric cycle $\boldsymbol{R}$ in the tope graph of a simple* acyclic *oriented matroid $\mathscr{A}$, we have*

$$|\mathbf{min}\ \mathrm{V}(\boldsymbol{R})| = t - \frac{1}{2t} \sum_{\substack{1 \leq k \leq t-1, \\ k \text{ odd}}} |\hat{z}_{\boldsymbol{R}}(k)|^2 \cdot \sin^2 \tfrac{\pi k}{t}$$

for the subposet $\mathbf{min}\ \mathrm{V}(\boldsymbol{R})$ *which is a* critical tope committee for $\mathscr{A}$.

Definition 12.2. Let $\mathscr{M} := (E_t, \mathscr{L}) = (E_t, \mathscr{T})$ be a simple oriented matroid, and $\mathrm{V}(\boldsymbol{R}) := (R^0, R^1, \ldots, R^{2t-1})$ the vertex sequence of a symmetric cycle $\boldsymbol{R}$ in its tope graph $\mathscr{T}(\mathscr{L})$.

(i) Given a tope $T \in \mathscr{T}$, the distance vector of the cycle $\boldsymbol{R}$ with respect to the tope T is the sequence

$$\boldsymbol{z}_{T,\boldsymbol{R}} := \big(d(T, R) \colon R \in \mathrm{V}(\boldsymbol{R})\big)$$

of the graph distances between T and the vertices of $\boldsymbol{R}$.

(ii) If the oriented matroid $\mathscr{M}$ is acyclic, then the distance vector of the cycle $\boldsymbol{R}$ is the sequence

$$\boldsymbol{z}_{\boldsymbol{R}} := \boldsymbol{z}_{\mathrm{T}^{(+)},\boldsymbol{R}} \ .$$

Let us now restate Lemma 12.1.

Proposition 12.3. *Let $\mathscr{M} := (E_t, \mathscr{T})$ be a simple oriented matroid, and $\mathrm{V}(\boldsymbol{R}) := (R^0, R^1, \ldots, R^{2t-1})$ the vertex sequence of a symmetric cycle $\boldsymbol{R}$ in the tope graph of $\mathscr{M}$. Given a tope $T \in \mathscr{T}$, for the* unique inclusion-minimal subset $\boldsymbol{Q}(T, \boldsymbol{R}) \subset \mathrm{V}(\boldsymbol{R})$ *such that*

$$T = \sum_{Q \in \boldsymbol{Q}(T,\boldsymbol{R})} Q \ ,$$

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we have

$$|\boldsymbol{Q}(T,\boldsymbol{R})| = t - \frac{1}{2t} \sum_{\substack{1\le k\le t-1,\\ k \text{ odd}}} |\hat{z}_{T,\boldsymbol{R}}(k)|^2 \cdot \sin^2 \tfrac{\pi k}{t} \ .$$

Recall that the vertex sequence $\mathrm{V}(\boldsymbol{R})$ is a *maximal positive basis* of $\mathbb{R}^t$; the existence of the set $\boldsymbol{Q}(T,\boldsymbol{R})$, of odd cardinality, of linearly independent elements of $\mathbb{R}^t$, mentioned in Proposition 12.3, is guaranteed by Corollary 11.2.

Define the *distance enumerator*, with respect to a tope $B \in \mathscr{T}$, of the set of topes $\mathscr{T}$ to be the polynomial $D_{B,\mathscr{T}}(\mathrm{x}) := \sum_{T\in\mathscr{T}} \mathrm{x}^{d(B,T)}$. In an analogous manner, define the *distance enumerator* of the vertex set of the symmetric cycle $\boldsymbol{R}$ to be the summand $D_{B,\mathrm{V}(\boldsymbol{R})}(\mathrm{x}) := \sum_{T\in\mathrm{V}(\boldsymbol{R})} \mathrm{x}^{d(B,T)}$ of $D_{B,\mathscr{T}}(\mathrm{x})$.

Recall that for any topes T' and T'' of $\mathscr{M}$, the graph distance between them is

$$d(T',T'') = t - \tfrac{1}{4}\|T''+T'\|^2 = \tfrac{1}{4}\|T''-T'\|^2 = \tfrac{1}{2}\big(t - \langle T'',T'\rangle\big) \ .$$

Thus, if $\mathscr{T}^{\bullet}$ is a halfspace of $\mathscr{M}$, and if $\mathfrak{W} \subset \mathrm{V}(\boldsymbol{R})$ is the vertex sequence of a $(t-1)$-path in the cycle $\boldsymbol{R}$, then $D_{B,\mathscr{T}}(\mathrm{x}) = \mathrm{x}^{t/2} \sum_{T\in\mathscr{T}^{\bullet}} (\mathrm{x}^{-\langle B,T\rangle/2} + \mathrm{x}^{\langle B,T\rangle/2})$, and $D_{B,\mathrm{V}(\boldsymbol{R})}(\mathrm{x}) = \mathrm{x}^{t/2} \sum_{T\in\mathfrak{W}} (\mathrm{x}^{-\langle B,T\rangle/2} + \mathrm{x}^{\langle B,T\rangle/2})$.

Associate with a tope $T \in \mathscr{T}$ the $2t$-dimensional row vector $\boldsymbol{q}(T,\boldsymbol{R})$ defined by $q_j(T,\boldsymbol{R}) := 1$ if $R^j \in \boldsymbol{Q}(T,\boldsymbol{R})$, and $q_j(T,\boldsymbol{R}) := 0$ otherwise. Let $\mathbf{G}(\boldsymbol{R})$ denote the *Gram matrix* of the sequence $\mathrm{V}(\boldsymbol{R})$. For topes T' and T'' of $\mathscr{M}$ we have

$$d(T',T'') = \tfrac{1}{2}\big(t - \boldsymbol{q}(T'',\boldsymbol{R})\mathbf{G}(\boldsymbol{R})\boldsymbol{q}(T',\boldsymbol{R})^{\top}\big).$$

Note that in addition to equation (12.2) we also have

$$\begin{aligned}|\mathbf{min}\ \mathrm{V}(\boldsymbol{R})| &= \frac{1}{8} \sum_{\substack{\{R',R''\}:=\mathscr{N}(R):\\ R\in\mathrm{V}(\boldsymbol{R})}} \big(\rho(R') + \rho(R'') - 2\rho(R)\big)^2 \\ &= \frac{1}{4} \sum_{\substack{\{R',R''\}:=\mathscr{N}(R):\\ R\in\mathrm{V}(\boldsymbol{R})}} \big(3\rho(R)^2 - 2\rho(R)(\rho(R')+\rho(R'')) + \rho(R')\rho(R'')\big) \ ,\end{aligned}$$

that is, $|\mathbf{min}\ \mathrm{V}(\boldsymbol{R})| = \tfrac{1}{4}\boldsymbol{z}_{\boldsymbol{R}} \cdot (\mathbf{I}-\mathbf{C})(3\mathbf{I}-\mathbf{C}) \cdot \boldsymbol{z}_{\boldsymbol{R}}^{\top}$, or

$$|\mathbf{min}\ \mathrm{V}(\boldsymbol{R})| = \frac{1}{8}\boldsymbol{z}_{\boldsymbol{R}} \cdot (6\mathbf{I} - 4\mathbf{C}^{-1} - 4\mathbf{C} + \mathbf{C}^{-2} + \mathbf{C}^2) \cdot \boldsymbol{z}_{\boldsymbol{R}}^{\top} \ . \tag{12.6}$$

Equations (12.3) and (12.6) imply that $|\mathbf{min}\ \mathrm{V}(\boldsymbol{R})| = \frac{t}{2} + \frac{1}{4}\boldsymbol{z}_{\boldsymbol{R}} \cdot (\mathbf{I}-\mathbf{C})^2 \cdot \boldsymbol{z}_{\boldsymbol{R}}^{\top} = \frac{3t}{4} - \frac{1}{4}\boldsymbol{z}_{\boldsymbol{R}} \cdot (\mathbf{I}-\mathbf{C})\mathbf{C} \cdot \boldsymbol{z}_{\boldsymbol{R}}^{\top}$; in other words, we have

$$|\mathbf{min}\ \mathrm{V}(\boldsymbol{R})| = \frac{t}{2} + \frac{1}{8}\boldsymbol{z}_{\boldsymbol{R}} \cdot (2\mathbf{I} - 2\mathbf{C}^{-1} - 2\mathbf{C} + \mathbf{C}^{-2} + \mathbf{C}^2) \cdot \boldsymbol{z}_{\boldsymbol{R}}^{\top} \ , \tag{12.7}$$

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and

$$|\mathbf{min}\ \mathrm{V}(\boldsymbol{R})| = \frac{3t}{4} - \frac{1}{8}\boldsymbol{z_R} \cdot (\mathbf{C}^{-1} + \mathbf{C} - \mathbf{C}^{-2} - \mathbf{C}^2) \cdot \boldsymbol{z_R}^\top \ ; \tag{12.8}$$

note also that

$$\boldsymbol{z_R} \cdot (\mathbf{I} - \mathbf{C}) \cdot \boldsymbol{z_R}^\top = t\ .$$

We derive from eqs. (12.6), (12.7), and (12.8) the relations

$$|\mathbf{min}\ \mathrm{V}(\boldsymbol{R})| = \frac{1}{4t}\sum_{k=0}^{2t-1} |\hat{z}_{\boldsymbol{R}}(k)|^2 \cdot \left(\cos^2 \tfrac{\pi k}{t} - 2\cos\tfrac{\pi k}{t} + 1\right)\ ,$$

$$|\mathbf{min}\ \mathrm{V}(\boldsymbol{R})| = \frac{t}{2} + \frac{1}{4t}\sum_{k=0}^{2t-1} |\hat{z}_{\boldsymbol{R}}(k)|^2 \cdot \left(\cos^2 \tfrac{\pi k}{t} - \cos\tfrac{\pi k}{t}\right)\ ,$$

and

$$|\mathbf{min}\ \mathrm{V}(\boldsymbol{R})| = \frac{3t}{4} + \frac{1}{8t}\sum_{k=0}^{2t-1} |\hat{z}_{\boldsymbol{R}}(k)|^2 \cdot \left(2\cos^2 \tfrac{\pi k}{t} - \cos\tfrac{\pi k}{t} - 1\right)$$

that are equivalent to eq. (12.5).

Now let $\boldsymbol{R}'$ and $\boldsymbol{R}''$ be two symmetric cycles in the tope graph of a simple acyclic oriented matroid $\mathscr{A}$, and $\boldsymbol{z_{R'}}$ and $\boldsymbol{z_{R''}}$ their distance vectors. Consider the vectors $\boldsymbol{e} := \boldsymbol{z_{R''}} - \boldsymbol{z_{R'}}$ and $\boldsymbol{m} := \boldsymbol{z_{R''}} + \boldsymbol{z_{R'}}$. Note that the components of their DFTs are:

$$\hat{e}(k) = \begin{cases} 0\ , & \text{if } k \text{ is even}\ , \\ 2\sum_{j=0}^{t-1} e(j)\mathrm{e}^{-\pi\imath kj/t}\ , & \text{if } k \text{ is odd}\ , \end{cases}$$

and

$$\hat{m}(k) = \begin{cases} 2t^2\ , & \text{if } k = 0\ , \\ 0\ , & \text{if } k \text{ is even, } k \neq 0\ , \\ 2\left(-2t(1 - \mathrm{e}^{-\pi\imath k/t})^{-1} + \sum_{j=0}^{t-1} m(j)\mathrm{e}^{-\pi\imath kj/t}\right)\ , & \text{if } k \text{ is odd}\ ; \end{cases}$$

thus, if t is odd, then $\hat{e}(t) = 2\sum_{j=0}^{t-1}(-1)^j e(j)$ and $\hat{m}(t) = 2(-t + \sum_{j=0}^{t-1}(-1)^j m(j))$. It follows from Lemma 12.1 that

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$$|\mathbf{min}\ \mathrm{V}(\boldsymbol{R}')| + |\mathbf{min}\ \mathrm{V}(\boldsymbol{R}'')| = 2t - \frac{1}{4t} \sum_{\substack{1 \le k \le t-1,\\ k \text{ odd}}} \left(|\hat{e}(k)|^2 + |\hat{m}(k)|^2\right) \cdot \sin^2 \tfrac{\pi k}{t} \ .$$

Notes

See, for example, [82, Ch. 2] and [182, Ch. 2–6] on the *discrete Fourier transformation*.

On page 185 we consider the *distance enumerator* of the set of topes of an oriented matroid, following Ref. [177, Lect. 6].

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13 Symmetric Cycles in the Hypercube Graphs

Let $\mathscr{H} := (E_t, \{1,-1\}^t)$ be the simple acyclic oriented matroid on the ground set E_t, whose set of topes $\{1,-1\}^t$ is the vertex set of a t-dimensional geometric hypercube in $\mathbb{R}^t$. This oriented matroid is realizable as the *arrangement* of *coordinate hyperplanes* in $\mathbb{R}^t$.

The tope graph $\mathscr{T}(\mathscr{L}(\mathscr{H}))$ of the oriented matroid $\mathscr{H}$ is the *hypercube graph* with 2^t vertices, that is, the *Hamming graph* $\boldsymbol{H}(t,2)$ closely related to the *Hamming association scheme* $\mathbf{H}(t,2)$.

The vertex set of the *hypercube graph* $\boldsymbol{H}(t,2)$ is the tope set of $\mathscr{H}$, that is, the collection of all words (as earlier, we regard them as *row vectors* of the real Euclidean space $\mathbb{R}^t$) $T := (T(1),\dots,T(t)) \in \{1,-1\}^t$; vertices T' and T'' are adjacent in $\boldsymbol{H}(t,2)$ if and only if there is a unique element $e \in E_t$ such that $T'(e) = -T''(e)$.

Recall that the vertices of $\boldsymbol{H}(t,2)$ are in one-to-one correspondence with the elements of the Boolean lattice of subsets of the set E_t: it is convenient to regard the power set $\mathbf{2}^{E_t} := \mathbf{2}^{[t]}$ of the ground set E_t as the collection

$$\mathbf{2}^{[t]} = \left\{T^- : T \in \{1,-1\}^t\right\}$$

of the *negative parts* $T^- := \{e \in E_t : T(e) = -1\}$ of topes $T \in \{1,-1\}^t$ of the oriented matroid $\mathscr{H}$.

Let $\boldsymbol{R} := (R^0, R^1, \dots, R^{2t-1}, R^0)$ be a symmetric $2t$-cycle (or, as earlier, *symmetric cycle*, for short) in the hypercube graph $\boldsymbol{H}(t,2)$, that is, $R^{k+t} = -R^k$ for each k, $0 \le k \le t-1$. Recall that the *vertex sequence* of any $(t-1)$-path in $\boldsymbol{R}$ is a *basis* of the space $\mathbb{R}^t$; indeed, the absolute value of the determinant of the matrix composed of the row vectors of the sequence is 2^{t-1}. In particular, this means that the vertex sequence $\mathrm{V}(\boldsymbol{R}) := (R^0, R^1, \dots, R^{2t-1})$ of the cycle $\boldsymbol{R}$ is a *maximal positive basis* of $\mathbb{R}^t$, see Section 11.1.

By Theorem 11.1, for any vertex T of the graph $\boldsymbol{H}(t,2)$ there exists a *unique inclusion-minimal* subset $\boldsymbol{Q}(T,\boldsymbol{R}) \subset \mathrm{V}(\boldsymbol{R})$ such that

$$T = \sum_{Q \in \boldsymbol{Q}(T,\boldsymbol{R})} Q\ .$$

In this chapter we obtain further results on the *decompositions* of the topes with respect to the *symmetric cycles* in the *tope graphs*.

For all vertices $T \in \{1,-1\}^t$ of the hypercube graph $\boldsymbol{H}(t,2)$, the linearly independent sets $\boldsymbol{Q}(T,\boldsymbol{R}) \subset \mathbb{R}^t$ are of *odd* cardinality; by Corollary 11.2 they can explicitly be described as follows:

$$\boldsymbol{Q}(T,\boldsymbol{R}) = {}_{-(T^-)}\left(\mathbf{max}^+\left({}_{-(T^-)}\mathrm{V}(\boldsymbol{R})\right)\right)\ , \quad T \in \{1,-1\}^t\ ,$$

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where ${}_{-(T^-)}\mathrm{V}(\boldsymbol{R})$ is the sequence of topes obtained from the vertex sequence $\mathrm{V}(\boldsymbol{R})$ by *reorientation* on the *negative part* T^- of the tope T; $\mathbf{max}^+(\cdot)$ is the subset of all topes from the resulting sequence that have *inclusion-maximal positive parts*, and the outermost operation ${}_{-(T^-)}(\cdot)$ means the reverse *reorientation* on the negative part T^-.

If we consider the vertex set $\{1,-1\}^t$ of the hypercube graph $\boldsymbol{H}(t,2)$ as the disjoint union $\dot{\bigcup}_{T\in\{1,-1\}^t}\{T\}$ of the singleton sets of its vertices, then we have

$$\{1,-1\}^t = \dot{\bigcup}_{T\in\{1,-1\}^t}\Big\{\sum_{Q\in\boldsymbol{Q}(T,\boldsymbol{R})} Q\Big\}\ .$$

In other words, for any symmetric cycle $\boldsymbol{R}$ in $\boldsymbol{H}(t,2)$ we have

$$\{1,-1\}^t = \dot{\bigcup}_{S\subseteq E_t}\Big\{\sum_{Q\in{}_{-S}(\mathbf{max}^+({}_{-S}\mathrm{V}(\boldsymbol{R})))} Q\Big\}\ ;$$

note that if a pair $\{P',P''\}$ is the neighborhood of a word P in the cycle ${}_{-S}\boldsymbol{R}$, then $P\in\mathbf{max}^+({}_{-S}\mathrm{V}(\boldsymbol{R}))$ if and only if $|(P')^-|-1=|P^-|=|(P'')^-|-1$.

In Section 13.1 we find the cardinalities $|\boldsymbol{Q}(T,\boldsymbol{R})|$ of the decompositions of topes, in the context of arbitrary simple oriented matroids.

In Section 13.2 we list some metric relations for the sets $\boldsymbol{Q}(T,\boldsymbol{R})$.

In Section 13.3 we describe a basic statistic associated with the vertices of the hypercube graphs $\boldsymbol{H}(t,2)$ and with their symmetric cycles.

13.1 Decompositions of Topes with Respect to Symmetric Cycles in the Tope Graphs

Let $\mathcal{M} := (E_t,\mathcal{L}) = (E_t,\mathcal{T})$ be a simple oriented matroid on the ground set E_t, with set of topes $\mathcal{T}$. Let us fix in the *tope graph* $\mathcal{T}(\mathcal{L})$ of $\mathcal{M}$ a symmetric cycle $\boldsymbol{R}$ with vertex sequence $\mathrm{V}(\boldsymbol{R}) := (R^0,R^1,\ldots,R^{2t-1})$.

It follows from a standard result of oriented matroid theory that $\mathcal{T}(\mathcal{L})$ and $\boldsymbol{R}$ are both *isometric subgraphs* of the *hypercube graph* $\boldsymbol{H}(t,2)$.

If $T\in\mathcal{T}$, then by Definition 12.2(ii) the row *distance vector* $\boldsymbol{z}_{T,\boldsymbol{R}} := (z_{T,\boldsymbol{R}}(0), z_{T,\boldsymbol{R}}(1),\ldots,z_{T,\boldsymbol{R}}(2t-1)) \in \ell^2(\mathbb{Z}_{2t})$ *of the cycle* $\boldsymbol{R}$ *with respect to the tope* T has the components

$$z_{T,\boldsymbol{R}}(k) := d(T,R^k)\ ,\quad 0\le k\le 2t-1\ ,$$

where $d(T',T'')$ denotes the *graph distance* between topes T' and T'', that is, the *Hamming distance* between the words T' and T''. If we let $\langle T',T''\rangle := \sum_{e\in E_t} T'(e)T''(e)$ as earlier denote the standard scalar product on $\mathbb{R}^t$, then $\langle T',T''\rangle = t-2d(T',T'')$ and, as a consequence,

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$$d(T', T'') = \tfrac{1}{2}\big(t - \langle T', T''\rangle\big)\,.$$

In this section we express the *cardinalities* of the *decompositions of topes with respect to symmetric cycles* in the tope graphs via the *autocorrelations* of the *distance vectors* of the cycles.

Let $\mathbf{I}$ and $\mathbf{C}$ as earlier denote the $2t \times 2t$ *identity matrix* and the $2t \times 2t$ *"basic" circulant permutation matrix* (12.1), respectively, with the rows and columns indexed from 0 to $2t-1$. If $\boldsymbol{v} \in \ell^2(\mathbb{Z}_{2t})$, then we as earlier denote by $\hat{\boldsymbol{v}} := (\hat{v}(0), \hat{v}(1), \ldots, \hat{v}(2t-1))$ the *discrete Fourier transform* of the vector $\boldsymbol{v}$. If $\boldsymbol{w} \in \ell^2(\mathbb{Z}_{2t})$, then we denote by $\langle\!\langle \boldsymbol{v}, \boldsymbol{w}\rangle\!\rangle := \sum_{k=0}^{2t-1} v(k)\overline{w(k)}$ the *complex inner product* on $\ell^2(\mathbb{Z}_{2t})$, where $\bar{\cdot}$ means complex conjugation.

Several formulas of Chapter 12 can be restated as follows: for any tope $T \in \mathscr{T}$, we have

$$\begin{aligned}
|\boldsymbol{Q}(T, \boldsymbol{R})| &= t - \frac{1}{8}\boldsymbol{z}_{T,\boldsymbol{R}} \cdot (2\mathbf{I} - \mathbf{C}^{-2} - \mathbf{C}^2) \cdot \boldsymbol{z}_{T,\boldsymbol{R}}{}^{\top} \\
&= t - \frac{1}{4t}\sum_{k=0}^{2t-1} |\hat{z_{T,\boldsymbol{R}}}(k)|^2 \cdot \sin^2 \tfrac{\pi k}{t}\,, \qquad (13.1)\\
|\boldsymbol{Q}(T, \boldsymbol{R})| &= \frac{1}{8}\boldsymbol{z}_{T,\boldsymbol{R}} \cdot (6\mathbf{I} - 4\mathbf{C}^{-1} - 4\mathbf{C} + \mathbf{C}^{-2} + \mathbf{C}^2) \cdot \boldsymbol{z}_{T,\boldsymbol{R}}{}^{\top} \\
&= \frac{1}{4t}\sum_{k=0}^{2t-1} |\hat{z_{T,\boldsymbol{R}}}(k)|^2 \cdot (\cos^2 \tfrac{\pi k}{t} - 2\cos \tfrac{\pi k}{t} + 1)\,, \qquad (13.2)\\
|\boldsymbol{Q}(T, \boldsymbol{R})| &= \frac{t}{2} + \frac{1}{8}\boldsymbol{z}_{T,\boldsymbol{R}} \cdot (2\mathbf{I} - 2\mathbf{C}^{-1} - 2\mathbf{C} + \mathbf{C}^{-2} + \mathbf{C}^2) \cdot \boldsymbol{z}_{T,\boldsymbol{R}}{}^{\top} \\
&= \frac{t}{2} + \frac{1}{4t}\sum_{k=0}^{2t-1} |\hat{z_{T,\boldsymbol{R}}}(k)|^2 \cdot (\cos^2 \tfrac{\pi k}{t} - \cos \tfrac{\pi k}{t})\,, \qquad (13.3)
\end{aligned}$$

and

$$\begin{aligned}
|\boldsymbol{Q}(T, \boldsymbol{R})| &= \frac{3t}{4} - \frac{1}{8}\boldsymbol{z}_{T,\boldsymbol{R}} \cdot (\mathbf{C}^{-1} + \mathbf{C} - \mathbf{C}^{-2} - \mathbf{C}^2) \cdot \boldsymbol{z}_{T,\boldsymbol{R}}{}^{\top} \\
&= \frac{3t}{4} + \frac{1}{8t}\sum_{k=0}^{2t-1} |\hat{z_{T,\boldsymbol{R}}}(k)|^2 \cdot (2\cos^2 \tfrac{\pi k}{t} - \cos \tfrac{\pi k}{t} - 1)\,. \qquad (13.4)
\end{aligned}$$

Let $\boldsymbol{a}_{T,\boldsymbol{R}} \in \ell^2(\mathbb{Z}_{2t})$ denote the *autocorrelation* of the *distance vector* $\boldsymbol{z}_{T,\boldsymbol{R}}$ defined by

$$a_{T,\boldsymbol{R}}(m) := \sum_{n=0}^{2t-1} z_{T,\boldsymbol{R}}(n) z_{T,\boldsymbol{R}}((n+m) \mod 2t)\,, \quad 0 \le m \le 2t-1\,;$$

note that $a_{T,\boldsymbol{R}}(k) = a_{T,\boldsymbol{R}}(2t-k)$, $1 \le k \le 2t-1$. Recall that

$$\hat{\boldsymbol{a}_{T,\boldsymbol{R}}} = \big(|\hat{z_{T,\boldsymbol{R}}}(0)|^2, |\hat{z_{T,\boldsymbol{R}}}(1)|^2, \ldots, |\hat{z_{T,\boldsymbol{R}}}(2t-1)|^2\big)\,.$$

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– Let $\mathbf{b} := (2, 0, -1, 0, \ldots, 0, -1, 0)$ be the first row of the matrix $2\mathbf{I}-\mathbf{C}^{-2}-\mathbf{C}^2$. The equation (13.1) is equivalent to the relation $t - |\boldsymbol{Q}(T, \boldsymbol{R})| = \frac{1}{16t}\langle\langle \hat{\boldsymbol{a}_{T,\boldsymbol{R}}}, \hat{\mathbf{b}}\rangle\rangle$, and *Parseval's relation* implies that

$$\begin{aligned} t - |\boldsymbol{Q}(T, \boldsymbol{R})| &= \tfrac{1}{8}\langle\langle \boldsymbol{a}_{T,\boldsymbol{R}}, \mathbf{b}\rangle\rangle \\ &= \tfrac{1}{8}\big(2a_{T,\boldsymbol{R}}(0) - a_{T,\boldsymbol{R}}(2) - a_{T,\boldsymbol{R}}(2t-2)\big) \\ &= \tfrac{1}{8}\big(2a_{T,\boldsymbol{R}}(0) - 2a_{T,\boldsymbol{R}}(2)\big) . \end{aligned}$$

– Let $\mathbf{b} := (6, -4, 1, 0, \ldots, 0, 1, -4)$ be the first row of the matrix $6\mathbf{I} - 4\mathbf{C}^{-1} - 4\mathbf{C} + \mathbf{C}^{-2} + \mathbf{C}^2$. Equation (13.2) is equivalent to $|\boldsymbol{Q}(T, \boldsymbol{R})| = \frac{1}{16t}\langle\langle \hat{\boldsymbol{a}_{T,\boldsymbol{R}}}, \hat{\mathbf{b}}\rangle\rangle$; Parseval's relation implies that

$$\begin{aligned} |\boldsymbol{Q}(T, \boldsymbol{R})| &= \tfrac{1}{8}\langle\langle \boldsymbol{a}_{T,\boldsymbol{R}}, \mathbf{b}\rangle\rangle \\ &= \tfrac{1}{8}\big(6a_{T,\boldsymbol{R}}(0) - 4a_{T,\boldsymbol{R}}(1) + a_{T,\boldsymbol{R}}(2) + a_{T,\boldsymbol{R}}(2t-2) - 4a_{T,\boldsymbol{R}}(2t-1)\big) \\ &= \tfrac{1}{8}\big(6a_{T,\boldsymbol{R}}(0) - 8a_{T,\boldsymbol{R}}(1) + 2a_{T,\boldsymbol{R}}(2)\big) . \end{aligned}$$

– Let $\mathbf{b} := (2, -2, 1, 0, \ldots, 0, 1, -2)$ be the first row of the matrix $2\mathbf{I}-2\mathbf{C}^{-1}-2\mathbf{C}+\mathbf{C}^{-2}+\mathbf{C}^2$. The equation (13.3) is equivalent to $|\boldsymbol{Q}(T, \boldsymbol{R})| - \frac{t}{2} = \frac{1}{16t}\langle\langle \hat{\boldsymbol{a}_{T,\boldsymbol{R}}}, \hat{\mathbf{b}}\rangle\rangle$, and we have

$$\begin{aligned} |\boldsymbol{Q}(T, \boldsymbol{R})| - \tfrac{t}{2} &= \tfrac{1}{8}\langle\langle \boldsymbol{a}_{T,\boldsymbol{R}}, \mathbf{b}\rangle\rangle \\ &= \tfrac{1}{8}\big(2a_{T,\boldsymbol{R}}(0) - 2a_{T,\boldsymbol{R}}(1) + a_{T,\boldsymbol{R}}(2) + a_{T,\boldsymbol{R}}(2t-2) - 2a_{T,\boldsymbol{R}}(2t-1)\big) \\ &= \tfrac{1}{8}\big(2a_{T,\boldsymbol{R}}(0) - 4a_{T,\boldsymbol{R}}(1) + 2a_{T,\boldsymbol{R}}(2)\big) . \end{aligned}$$

– Let $\mathbf{b} := (0, -1, 1, 0, \ldots, 0, 1, -1)$ be the first row of the matrix $\mathbf{C}^{-1} + \mathbf{C} - \mathbf{C}^{-2} - \mathbf{C}^2$. Equation (13.4) is equivalent to $|\boldsymbol{Q}(T, \boldsymbol{R})| - \frac{3t}{4} = \frac{1}{16t}\langle\langle \hat{\boldsymbol{a}_{T,\boldsymbol{R}}}, \hat{\mathbf{b}}\rangle\rangle$, and we have

$$\begin{aligned} |\boldsymbol{Q}(T, \boldsymbol{R})| - \tfrac{3t}{4} &= \tfrac{1}{8}\langle\langle \boldsymbol{a}_{T,\boldsymbol{R}}, \mathbf{b}\rangle\rangle \\ &= \tfrac{1}{8}\big(-a_{T,\boldsymbol{R}}(1) + a_{T,\boldsymbol{R}}(2) + a_{T,\boldsymbol{R}}(2t-2) - a_{T,\boldsymbol{R}}(2t-1)\big) \\ &= \tfrac{1}{8}\big(-2a_{T,\boldsymbol{R}}(1) + 2a_{T,\boldsymbol{R}}(2)\big) . \end{aligned}$$

We come to the following result:

Proposition 13.1. *Let $\mathscr{M}$ be a simple oriented matroid, and* $\mathrm{V}(\boldsymbol{R})$ *the vertex sequence of a symmetric cycle* $\boldsymbol{R}$ *in the tope graph of $\mathscr{M}$. If T is a tope of $\mathscr{M}$, then for the* unique inclusion-minimal subset $\boldsymbol{Q}(T, \boldsymbol{R}) \subset \mathrm{V}(\boldsymbol{R})$ *such that* $T = \sum_{Q\in\boldsymbol{Q}(T,\boldsymbol{R})} Q$, *we have*

$$\begin{aligned} |\boldsymbol{Q}(T, \boldsymbol{R})| &= t - \tfrac{1}{4}\big(a_{T,\boldsymbol{R}}(0) - a_{T,\boldsymbol{R}}(2)\big) , \\ |\boldsymbol{Q}(T, \boldsymbol{R})| &= \tfrac{1}{4}\big(3a_{T,\boldsymbol{R}}(0) - 4a_{T,\boldsymbol{R}}(1) + a_{T,\boldsymbol{R}}(2)\big) , \\ |\boldsymbol{Q}(T, \boldsymbol{R})| &= \tfrac{t}{2} + \tfrac{1}{4}\big(a_{T,\boldsymbol{R}}(0) - 2a_{T,\boldsymbol{R}}(1) + a_{T,\boldsymbol{R}}(2)\big) , \end{aligned} \tag{13.5}$$

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and

$$|\boldsymbol{Q}(T,\boldsymbol{R})| = \tfrac{3t}{4} + \tfrac{1}{4}\Big(-a_{T,\boldsymbol{R}}(1) + a_{T,\boldsymbol{R}}(2)\Big)\ ,$$

where $\boldsymbol{a}_{T,\boldsymbol{R}}$ *is the* autocorrelation *of any* distance vector *of the cycle* $\boldsymbol{R}$ *with respect to the tope* T*.*

13.2 Basic Metric Properties of Decompositions

In this section we consider a simple oriented matroid $\mathscr{M} := (E_t, \mathscr{T})$ with distinguished *symmetric cycle* $\boldsymbol{R}$ in its tope graph, and investigate *metric properties* of the *decompositions of topes with respect to the cycle* $\boldsymbol{R}$.

If $T \in \mathscr{T}$, then we have

$$\sum_{Q\in\boldsymbol{Q}(T,\boldsymbol{R})} d(T,Q) = \sum_{Q\in\boldsymbol{Q}(T,\boldsymbol{R})} \frac{1}{2}\Big(t - \langle T,Q\rangle\Big) = \frac{1}{2}|\boldsymbol{Q}(T,\boldsymbol{R})|t - \frac{1}{2}\underbrace{\sum_{Q\in\boldsymbol{Q}(T,\boldsymbol{R})} \langle T,Q\rangle}_{\|T\|^2=t}\ .$$

Remark 13.2. For any tope T of a simple oriented matroid $\mathscr{M}$ with distinguished symmetric cycle $\boldsymbol{R}$ in its tope graph we have

$$\sum_{Q\in\boldsymbol{Q}(T,\boldsymbol{R})} d(T,Q) = \frac{1}{2}\Big(|\boldsymbol{Q}(T,\boldsymbol{R})| - 1\Big)t\ , \tag{13.6}$$

and

$$|\boldsymbol{Q}(T,\boldsymbol{R})| = 1 + \frac{2}{t}\sum_{Q\in\boldsymbol{Q}(T,\boldsymbol{R})} d(T,Q)\ .$$

Now suppose that $T \notin \mathrm{V}(\boldsymbol{R})$, and $\boldsymbol{Q}(T,\boldsymbol{R}) := \{Q^0, \dots, Q^{|\boldsymbol{Q}(T,\boldsymbol{R})|-1}\}$. Note that

$$\begin{aligned} t = \|T\|^2 = \langle T,T\rangle = &\Big\langle \sum_{Q\in\boldsymbol{Q}(T,\boldsymbol{R})} Q, \sum_{Q\in\boldsymbol{Q}(T,\boldsymbol{R})} Q\Big\rangle \\ &= |\boldsymbol{Q}(T,\boldsymbol{R})|t + 2\sum_{0\le i<j\le|\boldsymbol{Q}(T,\boldsymbol{R})|-1} \Big(t - 2d(Q^i,Q^j)\Big) \\ &\qquad = |\boldsymbol{Q}(T,\boldsymbol{R})|t + 2\tbinom{|\boldsymbol{Q}(T,\boldsymbol{R})|}{2}t - 4\sum_{0\le i<j\le|\boldsymbol{Q}(T,\boldsymbol{R})|-1} d(Q^i,Q^j)\ . \end{aligned}$$

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Remark 13.3. For a tope T of a simple oriented matroid $\mathscr{M}$ with distinguished symmetric cycle $\boldsymbol{R}$ in its tope graph such that $T \notin \mathrm{V}(\boldsymbol{R})$, we have

$$\sum_{0 \le i < j \le |\boldsymbol{Q}(T,\boldsymbol{R})|-1} d(Q^i, Q^j) = \frac{1}{4}\big(|\boldsymbol{Q}(T,\boldsymbol{R})|^2 - 1\big)t \;,$$

$$|\boldsymbol{Q}(T,\boldsymbol{R})| = \sqrt{1 + \frac{4}{t}\sum\nolimits_{i<j} d(Q^i, Q^j)} \;.$$

Moreover,

$$\sum_{0 \le i < j \le |\boldsymbol{Q}(T,\boldsymbol{R})|-1} d(Q^i, Q^j) = \frac{1}{2}\big(|\boldsymbol{Q}(T,\boldsymbol{R})| + 1\big) \sum_{Q \in \boldsymbol{Q}(T,\boldsymbol{R})} d(T, Q) \;.$$

Again let $T \in \mathscr{T}$ be a tope such that $T \notin \mathrm{V}(\boldsymbol{R})$. Define integers $\mathfrak{q}$ and $\mathfrak{h}$ by

$$\mathfrak{q} := |\boldsymbol{Q}(T,\boldsymbol{R})| =: 2\mathfrak{h} + 1 \;;$$

thus, $\boldsymbol{Q}(T,\boldsymbol{R}) := \{Q^0, \ldots, Q^{\mathfrak{q}-1}\}$. For the remainder of this section, we take the superscripts j appearing in Q^j modulo $\mathfrak{q}$, and we assume that the topes in the set $\boldsymbol{Q}(T,\boldsymbol{R})$ are indexed in such a way that on the shortest path from Q^i to Q^{i+1} along the cycle $\boldsymbol{R}$ there are no other vertices of the set $\boldsymbol{Q}(T,\boldsymbol{R})$.

Consider the graph $\boldsymbol{G}^+_{\mathbf{max}}({}_{-(T^-)}\boldsymbol{R})$ defined as follows: its *vertex set* is the set ${}_{-(T^-)}\boldsymbol{Q}(T,\boldsymbol{R})$, and a vertex ${}_{-(T^-)}Q' \in {}_{-(T^-)}\boldsymbol{Q}(T,\boldsymbol{R})$ is *adjacent* with a vertex ${}_{-(T^-)}Q'' \in {}_{-(T^-)}\boldsymbol{Q}(T,\boldsymbol{R})$ if and only if

$$|({}_{-(T^-)}Q')^- \cap ({}_{-(T^-)}Q'')^-| = 0 \;.$$

A vertex ${}_{-(T^-)}Q^i$ is in fact adjacent in the $\mathfrak{q}$-*cycle* $\boldsymbol{G}^+_{\mathbf{max}}({}_{-(T^-)}\boldsymbol{R})$ with exactly two vertices, namely ${}_{-(T^-)}Q^{i+\mathfrak{h}}$ and ${}_{-(T^-)}Q^{i-\mathfrak{h}}$, see Proposition 1.21. But for the initial symmetric cycle $\boldsymbol{R}$ in the tope graph of $\mathscr{M}$ this means the following:

Proposition 13.4. *Let T be an arbitrary tope of a simple oriented matroid $\mathscr{M}$ with distinguished symmetric cycle $\boldsymbol{R}$ in its tope graph such that $T \notin \mathrm{V}(\boldsymbol{R})$.*

(i) *If $Q^i \in \boldsymbol{Q}(T,\boldsymbol{R})$, then*

$$d(Q^i, Q^{i-\mathfrak{h}}) = d(T, Q^i) + d(T, Q^{i-\mathfrak{h}}) \;,$$
$$d(Q^i, Q^{i+\mathfrak{h}}) = d(T, Q^i) + d(T, Q^{i+\mathfrak{h}}) \;.$$

(ii) *If $\{Q^{i-1}, Q^i\} \subset \boldsymbol{Q}(T,\boldsymbol{R})$, then*

$$d(Q^{i-1}, Q^i) + d(T, Q^{i-1}) + d(T, Q^i) + 2d(T, Q^{i+\mathfrak{h}}) = 2t \;.$$

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(iii) *If* $|\boldsymbol{Q}(T,\boldsymbol{R})| = 3$*, then for every* $i \in [3]$*, we have*

$$d(Q^{i-1}, Q^i) = d(T, Q^{i-1}) + d(T, Q^i) \,,$$

and, as a consequence,

$$d(Q^{i-1}, Q^i) = t - d(T, Q^{i+1}) \,.$$

13.3 Symmetric Cycles in the Hypercube Graphs

In this section we turn to an investigation of the combinatorial properties of the vertex set $\{1,-1\}^t$ of the *hypercube graph* $\boldsymbol{H}(t,2)$ and of its *symmetric cycles*.

Let $\boldsymbol{R}$ be a *symmetric cycle* in $\boldsymbol{H}(t,2)$, with its *distance vectors* $\boldsymbol{z}_{T,\boldsymbol{R}}$ associated with the topes $T \in \{1,-1\}^t$.

According to eq. (13.5), we have

$$\begin{aligned}
&\sum_{T\in\{1,-1\}^t} |\boldsymbol{Q}(T,\boldsymbol{R})| \\
&= \sum_{T\in\{1,-1\}^t} \Big(t - \frac{1}{4} \sum_{n=0}^{2t-1} z_{T,\boldsymbol{R}}(n)\big(z_{T,\boldsymbol{R}}(n) - z_{T,\boldsymbol{R}}((n+2) \bmod 2t)\big)\Big) \\
&= 2^t t - \frac{1}{4} \sum_{n=0}^{2t-1} \sum_{T\in\{1,-1\}^t} z_{T,\boldsymbol{R}}(n)\big(z_{T,\boldsymbol{R}}(n) - z_{T,\boldsymbol{R}}((n+2) \bmod 2t)\big) \\
&= 2^t t - \frac{1}{2} t \sum_{T\in\{1,-1\}^t} z_{T,\boldsymbol{R}}(n)\big(z_{T,\boldsymbol{R}}(n) - z_{T,\boldsymbol{R}}((n+2) \bmod 2t)\big) \\
&= t\Big(2^t - \frac{1}{2} \sum_{0\le i,j\le t} \mathrm{p}_{ij}^2 \, i(i-j)\Big) \,,
\end{aligned}$$

where for any vertices $X, Y \in \{1,-1\}^t$ such that $d(X,Y) = 2$, the quantities

$$|\{Z \in \{1,-1\}^t : d(Z,X) = i,\ d(Z,Y) = j\}| =: \mathrm{p}_{ij}^2$$

are known to be the same; this is an *intersection number* of the *Hamming* association *scheme* $\mathbf{H}(t,2)$, given in eq. (7.7). Thus, we have

$$\begin{aligned}
\sum_{T\in\{1,-1\}^t} |\boldsymbol{Q}(T,\boldsymbol{R})| &= t\Big(2^t - \frac{1}{2} \sum_{\substack{1\le i\le t;\\ j\in\{i-2,i+2\}:\\ 0\le j\le t}} \binom{t-2}{\frac{i+j}{2}-1} \underbrace{\binom{2}{\frac{i-j}{2}+1}}_{1 \text{ when } j \in \{i-2,\, i+2\}} i(i-j)\Big) \\
&= t\Big(2^t - \frac{1}{2} \sum_{\substack{1\le i\le t;\\ j\in\{i-2,i+2\}:\\ 0\le j\le t}} \binom{t-2}{\frac{i+j}{2}-1} i(i-j)\Big) \,.
\end{aligned}$$

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Since

$$\mathfrak{s}(t) := \sum_{\substack{1\le i\le t;\\ j\in\{i-2,i+2\}:0\le j\le t}} \binom{t-2}{\frac{i+j}{2}-1} i(i-j) = 2\mathfrak{s}(t-1) = 2^t ,$$

we come to the following conclusion:

Remark 13.5. Let $\boldsymbol{R}$ be a symmetric cycle in the hypercube graph $\boldsymbol{H}(t,2)$.

(i) We have

$$\begin{aligned}\sum_{T\in\{1,-1\}^t} |\boldsymbol{Q}(T,\boldsymbol{R})| &= \frac{t}{2} \sum_{\substack{1\le i\le t;\\ j\in\{i-2,i+2\}:\\ 0\le j\le t}} \mathrm{p}_{ij}^2 i(i-j)\\ &= \frac{t}{2} \sum_{\substack{1\le i\le t;\\ j\in\{i-2,i+2\}:\\ 0\le j\le t}} \binom{t-2}{\frac{i+j}{2}-1} i(i-j)\\ &= t \sum_{1\le i\le t} \left(\binom{t-2}{t-i} - \binom{t-2}{i}\right) i ,\end{aligned}$$

that is,

$$\sum_{T\in\{1,-1\}^t} |\boldsymbol{Q}(T,\boldsymbol{R})| = 2^{t-1}t ; \tag{13.7}$$

note that this quantity coincides with the number of edges of the graph $\boldsymbol{H}(t,2)$.

(ii) Equations (13.7) and (13.6) yield

$$\sum_{T\in\{1,-1\}^t} \sum_{Q\in\boldsymbol{Q}(T,\boldsymbol{R})} d(T,Q) = 2^{t-2}(t-2)t .$$

If j is an *odd* integer, $1 \le j \le t$, then define an integer $c_j(t)$ by

$$c_j(t) := \left|\left\{T \in \{1,-1\}^t \colon |\boldsymbol{Q}(T,\boldsymbol{R})| = j\right\}\right| ;$$

thus, we have

$$\sum_{\substack{1\le j\le t:\\ j \text{ odd}}} c_j(t) = 2^t .$$

Since

$$\sum_{\substack{1\le j\le t:\\ j \text{ odd}}} jc_j(t) := \sum_{T\in\{1,-1\}^t} |\boldsymbol{Q}(T,\boldsymbol{R})| = 2^{t-1}t = \sum_{0\le i\le t} i\binom{t}{i} ,$$

by Remark 13.5(i), the quantities $c_j(t)$ are read off from the equation (13.7):

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Table 13.1: The polynomials $\gamma_t(\mathrm{x})$, $2 \le t \le 10$.

t	$\gamma_t(\mathrm{x}) := \sum_{T\in\{1,-1\}^t} \mathrm{x}^{\lvert\mathbf{Q}(T,\mathbf{R})\rvert}$
2	$4\mathrm{x}$
3	$6\mathrm{x} + 2\mathrm{x}^3$
4	$8\mathrm{x} + 8\mathrm{x}^3$
5	$10\mathrm{x} + 20\mathrm{x}^3 + 2\mathrm{x}^5$
6	$12\mathrm{x} + 40\mathrm{x}^3 + 12\mathrm{x}^5$
7	$14\mathrm{x} + 70\mathrm{x}^3 + 42\mathrm{x}^5 + 2\mathrm{x}^7$
8	$16\mathrm{x} + 112\mathrm{x}^3 + 112\mathrm{x}^5 + 16\mathrm{x}^7$
9	$18\mathrm{x} + 168\mathrm{x}^3 + 252\mathrm{x}^5 + 72\mathrm{x}^7 + 2\mathrm{x}^9$
10	$20\mathrm{x} + 240\mathrm{x}^3 + 504\mathrm{x}^5 + 240\mathrm{x}^7 + 20\mathrm{x}^9$

Theorem 13.6. *Let $\mathbf{R}$ be a symmetric cycle, with vertex set $\mathrm{V}(\mathbf{R})$, in the hypercube graph $\mathbf{H}(t,2)$. For the vertices $T \in \{1,-1\}^t$ of the graph $\mathbf{H}(t,2)$, consider the* unique inclusion-minimal subsets $\mathbf{Q}(T,\mathbf{R}) \subset \mathrm{V}(\mathbf{R})$, *of* odd *cardinality, such that*

$$T = \sum_{Q\in\mathbf{Q}(T,\mathbf{R})} Q\ .$$

For any odd *integer j, $1 \le j \le t$, we have*

$$c_j(t) := \left|\left\{T \in \{1,-1\}^t : |\mathbf{Q}(T,\mathbf{R})| = j\right\}\right| = 2\tbinom{t}{j}\ .$$

In other words, the polynomial

$$\gamma_t(\mathrm{x}) := \sum_{\substack{1\le j\le t:\\ j \text{ odd}}} c_j(t)\mathrm{x}^j\ ,$$

in the variable x, *is*

$$\gamma_t(\mathrm{x}) = 2\sum_{\substack{1\le j\le t:\\ j \text{ odd}}} \tbinom{t}{j}\mathrm{x}^j\ .$$

We can define the polynomial $\gamma_1(\mathrm{x})$ by $\gamma_1(\mathrm{x}) := 2\mathrm{x}$. The polynomials $\gamma_t(\mathrm{x})$, where $2 \le t \le 10$, are given in Table 13.1.

In view of the simplicity of the statistic on the vertices and symmetric cycles of the hypercube graphs, given by the polynomials $\gamma_t(\mathrm{x})$, there are many ways to describe the binomial-type combinatorial properties of the quantities $c_j(t)$. For instance, if t is *even*, then for any odd integer j, $1 \le j < t$, we have $c_j(t) = c_{t-j}(t)$; if t is *odd*, then for any odd integer j, $1 \le j \le t$, we have $jc_j(t) = (1+t-j)c_{1+t-j}(t)$, and so on.

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Notes

On the *arrangement* of *coordinate hyperplanes*, see, for example, [27, Example 4.1.4].

See, for example, [14, 15, 36, 44, 58–60, 89, 94, 128, 157, 176] on the *Hamming scheme* and the *Hamming graph*.

It is well known that the *tope graphs* of oriented matroids are *isometric subgraphs* of *hypercube graphs*, see for example [50, 83, 118] and [27, Sect. 4.2], [59, Sect. 14.2].

The *autocorrelation* is discussed, for example, in [42, Sect. 4.3].

A key observation on the *graph distances* between the topes of oriented matroids, involved implicitly to obtain the results of Proposition 13.4, is given in Ref. [27, Lem 4.2.4]: For any three topes Q', Q'', T of an oriented matroid, we have $d(Q', Q'') = d(Q', T) + d(T, Q'') - 2|\mathbf{S}(Q', T) \cap \mathbf{S}(T, Q'')|$, where $\mathbf{S}(\cdot, \cdot)$ is the *separation set* of two topes.

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List of Notation

- $:=$ – equals by definition
- $\delta(s,t)$ – Kronecker delta, equal to 1 when $s=t$, and 0 otherwise
- $\mathbb{N}$ – nonnegative integers
- $\mathbb{P}$ – positive integers
- $\mathbb{Q}$ – rational numbers
- $\mathbb{R}$ – real numbers
- $\gcd(\cdot,\cdot)$ – greatest common divisor of two integers
- $\binom{j}{i}$ – binomial coefficient, equal to $\frac{j(j-1)\cdots(j-i+1)}{i(i-1)\cdots 1}$
- $\binom{j}{i}_q$ – q-binomial coefficient, equal to $\frac{(q^j-1)(q^j-q)\cdots(q^j-q^{i-1})}{(q^i-1)(q^i-q)\cdots(q^i-q^{i-1})}$
- $\left[{j \atop i}\right]$ – number of elements of rank i in any interval, of length j, of a binomial poset; see page 128
- $\underline{f}, \overline{f}$ – numerator h, and the denominator k of a fraction $f := \frac{h}{k} \in \mathbb{Q}$, respectively
- $\lfloor x \rfloor$ – greatest integer less than or equal to $x \in \mathbb{R}$ (the floor of x)
- $\lceil x \rceil$ – least integer greater than or equal to $x \in \mathbb{R}$ (the ceiling of x)
- $\nu(x)$ – integer $\lfloor x \rfloor + 1$

Sets and families

- $\mathbf{2}^V$ – power set of a set V
- $[t] =: E_t$ – set of consecutive integers $\{1,2,\ldots,t\}$
- $[s,t]$ – set of consecutive positive integers $\{s,s+1,\ldots,t\}$
- $\pm[m]$ – set of integers $\{-m,\ldots,-1,1,\ldots,m\}$
- $|A|$ – cardinality (number of elements) of a set A
- $\#\mathscr{A}$ – number of sets in a family $\mathscr{A}$
- $\mathrm{V}(\mathscr{A})$ – ground set $\bigcup_{i=1}^{\alpha} A_i$ of a family $\mathscr{A} := \{A_1,\ldots,A_\alpha\}$
- $A^\perp$ – complement, $\mathrm{V}(\mathscr{A}) - A$, of a set A of a family $\mathscr{A}$
- $\mathscr{A}^\perp$ – family of complements $\{A^\perp : A \in \mathscr{A}\}$
- $\mathbf{min}\ \mathscr{F}$ and $\mathbf{max}\ \mathscr{F}$ – subfamilies of all inclusion-minimal and inclusion-maximal sets of a family $\mathscr{F}$, respectively

Maps

- $f : A \to B$, $a \mapsto b$ – map f from a set A to a set B; the image of an element $a \in A$ is an element $b := f(a) \in B$
- $f|_A : A \to C$, $a \mapsto f(a)$ – restriction of a map $f : B \to C$ to a subset $A \subseteq B$
- $g \circ f$ – composition $g(f(\cdot))$ of maps f and g

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Vectors

- $\boldsymbol{w}^\top$ – transpose of a vector $\boldsymbol{w}$
- $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ – standard scalar product $\sum_{k=1}^{n} a_k b_k$
- $\|\boldsymbol{a}\| := \sqrt{\langle \boldsymbol{a}, \boldsymbol{a} \rangle}$ – Euclidean norm of a real vector $\boldsymbol{a}$
- $\langle\!\langle \boldsymbol{v}, \boldsymbol{w} \rangle\!\rangle$ – complex inner product $\sum_{k=0}^{2t-1} v(k)\overline{w(k)}$; see page 190

Subspaces and hulls

- $\mathrm{span}(\boldsymbol{X})$ – linear span of a set $\boldsymbol{X} \subset \mathbb{R}^n$
- $\mathrm{cone}(\boldsymbol{X})$ – conical hull of a set $\boldsymbol{X} \subset \mathbb{R}^n$
- $\mathrm{conv}(\boldsymbol{X})$ – convex hull of a set $\boldsymbol{X} \subset \mathbb{R}^n$

Arrangements and systems of constraints

- $\mathscr{T}$ – set of regions of a hyperplane arrangement
- $\mathscr{T}^{+}_{\boldsymbol{H}}$ – subset of regions of a hyperplane arrangement that lie on the positive side of an oriented hyperplane $\boldsymbol{H}$
- $\mathbf{J}$ and $\mathbf{I}$ – family of the multi–indices of maximal feasible subsystems (MFSs) and the family of the multi–indices of minimal (irreducible) infeasible subsystems, respectively
- ν_k – number of feasible subsystems, of cardinality k

Partially ordered sets (posets)

- $\hat{0}$ and $\hat{1}$ – least and greatest elements of a finite bounded poset, respectively
- $a \preceq b$ – elements a and b are comparable in a poset
- $a \lessdot b$ – element b covers an element a in a poset
- $[x, z] := \{y \in Q : x \le y \le z\}$ – closed interval of a poset Q
- $\wedge$ – operation of meet in a meet-semilattice
- $\vee$ – operation of join in a join-semilattice
- $\times$ – operation of direct product of posets
- P^{a} – set of atoms of a poset P
- P^{c} – set of coatoms of a poset P
- $\rho(\cdot)$ – poset rank of an element
- $P^{(k)}$ – kth layer $\{a \in P : \rho(a) = k\}$ of a graded poset P
- $\mathbf{min}\ V$ and $\mathbf{max}\ V$ – sets of minimal and maximal elements of a (sub)poset V, respectively
- $\mathfrak{I}(V)$ and $\mathfrak{F}(V)$ – order ideal and order filter generated by a set V, respectively; see page 6
- $\mu_P : P \times P \to \mathbb{Z}$ – Möbius function of a poset P; see page 119

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- $\mathbb{B}(n)$ – Boolean lattice of rank n
- $\mathrm{A}^{\perp}$ – set of lattice complements of the elements of an antichain A in $\mathbb{B}(n)$
- $\mathbb{V}_q(n)$ – lattice of subspaces of an n-dimensional vector space over a finite field of q elements
- $\mathbf{O}(m)$ – face lattice of an m-dimensional crosspolytope
- $\mathbb{B}(\mathcal{T})$ – Boolean lattice of subsets of the tope set of an oriented matroid $(E_t, \mathcal{T})$
- $\mathbf{O}(\mathcal{T})$ – poset of subsets of the tope set of an oriented matroid $(E_t, \mathcal{T})$ that contain no pairs of opposites
- $\mathfrak{A}(P) := \mathfrak{A}_{\triangledown}(P)$ – lattice of antichains of a poset P, with the order induced by the order on the lattice of filters of a poset P; see page 119
- $\mathfrak{A}_{\vartriangle}(P)$ – lattice of antichains of a poset P, with the order induced by the order on the lattice of ideals of a poset P; see page 119

Oriented matroids

- $\mathrm{T}^{(+)}$ – positive tope $(1, 1, \dots, 1) \in \mathbb{R}^t$
- $\mathrm{T}^{(-)}$ – tope $(-1, -1, \dots, -1) \in \mathbb{R}^t$
- $X(e)$ – eth component of a sign vector X
- $\mathbf{S}(X, Y)$ – separation set of sign vectors X and Y; see page 8
- $\underline{X}$ – support of a sign vector X; see page 6
- X^- and X^+ – negative and positive parts of a sign vector X, respectively; see page 7
- $\mathbf{z}(X)$ – zero set of a sign vector; see page 7
- ${}_{-A}X$ – sign vector whose eth component $({}_{-A}X)(e)$ is 1 if $e \in A$ and $X(e) = -1$; $({}_{-A}X)(e) := -1$ if $e \in A$ and $X(e) = 1$; $({}_{-A}X)(e) = X(e)$ otherwise
- $-X$ – opposite ${}_{-E_t}X$ of a sign vector X
- ${}_{-A}\mathcal{P}$ – set of sign vectors $\{{}_{-A}X : X \in \mathcal{P}\}$
- $-\mathcal{P}$ – set of opposites ${}_{-E_t}\mathcal{P} = \{-X : X \in \mathcal{P}\}$
- $X \circ Y$ – composition of sign vectors X and Y; see page 8
- $\bigcirc_{i\in[k]} X^i$ – conformal composition $X^1 \circ X^2 \circ \cdots \circ X^k$ of sign vectors X^i
- $(E_t, \mathcal{T})$ – oriented matroid, on the ground set E_t, with set of maximal covectors (topes) $\mathcal{T}$; see page 9
- $(E_t, \mathcal{W})$ – oriented matroid, on the ground set E_t, with set of maximal vectors $\mathcal{W}$; see page 8
- $(E_t, \mathcal{C}^*)$ – oriented matroid, on the ground set E_t, with set of cocircuits $\mathcal{C}^*$; see page 9
- $(E_t, \mathcal{C})$ – oriented matroid, on the ground set E_t, with set of circuits $\mathcal{C}$; see page 8

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- $(E_t, \mathcal{L})$ – oriented matroid, on the ground set E_t, with set of covectors $\mathcal{L}$; see page 8
- $(E_t, \mathcal{V})$ – oriented matroid, on the ground set E_t, with set of vectors $\mathcal{V}$; see page 8
- $\mathcal{T}_e^+$ – eth positive halfspace of an oriented matroid; see page 9
- $\mathcal{T}_B^+$ – intersection $\bigcap_{b\in B} \mathcal{T}_b^+$ of the positive halfspaces corresponding to the elements of a ground subset B of an oriented matroid
- $\binom{\mathcal{T}_e^-}{i} \boxplus \binom{\mathcal{T}_e^+}{j}$ – family of tope subsets $\{A\dot\cup B\colon A \in \binom{\mathcal{T}_e^-}{i},\ B \in \binom{\mathcal{T}_e^+}{j}\}$
- $\mathbf{max}^+(\mathcal{P})$ – subset $\{R \in \mathcal{P}\colon R^+ \in \mathbf{max}\{P^+\colon P \in \mathcal{P}\}\}$ of a set of sign vectors $\mathcal{P}$
- $\mathbf{min}^+(\mathcal{P})$ – subset $\{R \in \mathcal{P}\colon R^+ \in \mathbf{min}\{P^+\colon P \in \mathcal{P}\}\}$ of a set of sign vectors $\mathcal{P}$
- $\widehat{\mathcal{L}}$ – “big” face lattice of an oriented matroid $(E_t, \mathcal{L})$; see page 9
- $\mathcal{T}(\mathcal{L}) := \mathcal{T}(\mathcal{L}(\mathcal{M}))$ – tope graph of an oriented matroid $\mathcal{M} := (E_t, \mathcal{L}) = (E_t, \mathcal{T})$; see page 9
- $\mathcal{T}(\mathcal{L}, B) := \mathcal{T}(\mathcal{L}(\mathcal{M}), B)$ – tope poset, based at a tope $B \in \mathcal{T}$, of an oriented matroid $\mathcal{M} := (E_t, \mathcal{L}) = (E_t, \mathcal{T})$; see page 9
- $\mathrm{conv}_{\mathrm{T}}(\mathcal{Q})$ – T-convex hull of a subset of topes $\mathcal{Q} \subset \mathcal{T}$ of an oriented matroid $(E_t, \mathcal{T})$; see page 9
- $X|_A$ – restriction of a sign vector X to a subset A; see page 10
- $\mathcal{M}\backslash A$ – deletion of an oriented matroid $\mathcal{M}$; see page 10
- $\mathcal{M}|_A$ – restriction of an oriented matroid $\mathcal{M}$ to a subset A of its ground set; see page 10
- ${}_{-A}\mathcal{M}$ – oriented matroid obtained from an oriented matroid $\mathcal{A}$ by reorientation on a subset A of its ground set; see page 10
- $\widetilde{\mathcal{M}}$ – single element extension of an oriented matroid $\mathcal{M}$; see page 10
- σ – localization; see page 10
- $\mathrm{conv}(A)$ – convex hull of an acyclic set A; see page 150
- $\mathrm{ex}(A)$ – subset of extreme points of an acyclic set A; see page 150
- $L_{\mathrm{conv}}(\mathcal{M})$ – semilattice of convex subsets of the ground set of an oriented matroid $\mathcal{M}$; see page 150
- $\widehat{L}_{\mathrm{conv}}(\mathcal{M}) =: \widehat{L}$ – semilattice $L_{\mathrm{conv}}(\mathcal{M})$ augmented by a greatest element; see page 151
- $\boldsymbol{Q}(T, \boldsymbol{R})$ – unique inclusion-minimal subset of the vertex set $\mathrm{V}(\boldsymbol{R})$ of a symmetric cycle $\boldsymbol{R}$ in the tope graph of an oriented matroid such that $T = \sum_{Q\in\boldsymbol{Q}(T,\boldsymbol{R})} Q$ for a tope T; see page 175

Blocking constructions
- $\mathfrak{B}(\mathcal{A})$ – blocker of a family $\mathcal{A}$; see page 88
- $\mathbf{I}(P, \mathrm{A})$ – subposet of all blocking elements of a subset A in a poset P; see page 90

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- $\mathbf{C}(P, \mathrm{A})$ – subposet of all complementing elements of a subset A in a poset P; see page 90
- $\mathfrak{b}\colon \mathfrak{A}(P) \to \mathfrak{A}(P)$ – blocker map on the lattice of antichains $\mathfrak{A}(P)$; see page 94
- $\mathfrak{b}(\mathrm{A})$ – blocker of an antichain A
- $\mathscr{B}(P)$ – lattice $\mathfrak{b}(\mathfrak{A}(P))$ of blockers in a poset P
- $\mathfrak{c}\colon \mathfrak{A}(P) \to \mathfrak{A}(P)$ – complementary map on the lattice of antichains $\mathfrak{A}(P)$; see page 96
- $\mathscr{C}(P)$ – lattice $\mathfrak{c}(\mathfrak{A}(P))$, with the order induced by the order on the lattice of ideals of a poset P
- $\mathfrak{b}_k^X\colon \mathfrak{A}(P) \to \mathfrak{A}(P)$ – (X, k)-blocker map on the lattice of antichains $\mathfrak{A}(P)$; see page 108
- $\mathfrak{b}_k^X(\mathrm{A})$ – (X, k)-blocker of an antichain A
- $\mathscr{B}_k^X(P)$ – lattice $\mathfrak{b}_k^X(\mathfrak{A}(P))$ of (X, k)-blockers in a poset P
- $\mathbf{I}_r(P, \mathrm{A}; \omega)$ – subposet of all relatively r-blocking elements, w.r.t. a map ω, of a subset A in a poset P; see page 121
- $\mathbf{I}_{r,k}(P, \mathrm{A}; \omega)$ – subposet $\{b \in \mathbf{I}_r(P, \mathrm{A}; \omega)\colon \omega(b) = k\}$
- $\mathfrak{y}_r\colon \mathfrak{A}_\triangledown(P) \to \mathfrak{A}_\triangledown(P)$ – relative r-blocker map on the lattice of antichains $\mathfrak{A}_\triangledown(P)$; see page 121
- $\mathfrak{y}_r(\mathrm{A})$ – relative r-blocker of an antichain A
- $\mathsf{b}_j\colon \mathfrak{A}_\triangledown(P) \to \mathfrak{A}_\triangledown(P)$ – absolute j-blocker map on the lattice of antichains $\mathfrak{A}_\triangledown(P)$; see page 123
- $\mathsf{b}_j(\mathrm{A})$ – absolute j-blocker of an antichain A
- $\mathsf{b}_j(\mathfrak{A}_\triangledown(P))$ – lattice of absolute j-blockers in a poset P
- $\mathbf{I}_{r,k}(\mathbb{B}(n), \mathrm{A})$ – subposet of all the rank k relatively r-blocking elements of an antichain A in the Boolean lattice $\mathbb{B}(n)$; see page 134
- $\mathring{\mathbf{I}}_{r,k}(\mathbb{B}(2m), \mathrm{A})$ – subposet of all the rank k relatively r-blocking elements b, with the property $b \wedge -b = \hat{0}$, of an antichain A in the Boolean lattice $\mathbb{B}(2m)$ of subsets of the set $\pm[m]$; see page 137
- $\mathbf{I}_{r,k}(\mathbf{O}(m), \mathrm{A})$ – subposet of all the rank k relatively r-blocking elements of an antichain A in the lattice $\mathbf{O}(m)$; see page 139

Deletion and contraction

- $\mathscr{A} \backslash X$ – deletion of a clutter $\mathscr{A}$; see page 88
- $\mathscr{A}/X$ – contraction of a clutter $\mathscr{A}$; see page 88
- $(\backslash X)\colon \mathfrak{A}(P) \to \mathfrak{A}(P)$ – deletion operator on the antichains of a poset P; see page 100

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- $(/X)\colon \mathfrak{A}(P) \to \mathfrak{A}(P)$ – contraction operator on the antichains of a poset P; see page 100
- $\mathrm{A}\backslash X$ – deletion of an antichain A
- A/X – contraction of an antichain A
- $\mathfrak{A}(P)\backslash X$ – lattice $\{\mathrm{A}\backslash X\colon \mathrm{A} \in \mathfrak{A}(P)\}$ of deletions in a poset P
- $\mathfrak{A}(P)/X$ – lattice $\{\mathrm{A}/X\colon \mathrm{A} \in \mathfrak{A}(P)\}$ of contractions in a poset P
- $(\backslash_k X)\colon \mathfrak{A}(P) \to \mathfrak{A}(P)$ – (X, k)-deletion operator on the antichains of a poset P; see page 112
- $(/_k X)\colon \mathfrak{A}(P) \to \mathfrak{A}(P)$ – (X, k)-contraction operator on the antichains of a poset P; see page 112
- $\mathrm{A}\backslash_k X$ – (X, k)-deletion of an antichain A
- $\mathrm{A}/_k X$ – (X, k)-contraction of an antichain A
- $\mathfrak{A}(P)\backslash_k X$ – lattice $\{\mathrm{A}\backslash_k X\colon \mathrm{A} \in \mathfrak{A}(P)\}$ of (X, k)-deletions in a poset P
- $\mathfrak{A}(P)/_k X$ – lattice $\{\mathrm{A}/_k X\colon \mathrm{A} \in \mathfrak{A}(P)\}$ of (X, k)-contractions in a poset P

Committee constructions

- $\mathbf{K}^*(\mathscr{M})$ – family of all tope committees for an oriented matroid $\mathscr{M}$
- $\mathbf{A}^*(\mathscr{M})$ – family of all tope anti-committees for an oriented matroid $\mathscr{M}$
- $\mathbf{K}^*_k(\mathscr{M})$ – family of all tope committees, of size k, for an oriented matroid $\mathscr{M}$
- $\mathbf{A}^*_k(\mathscr{M})$ – family of all tope anti-committees, of size k, for an oriented matroid $\mathscr{M}$
- $\mathring{\mathbf{K}}^*(\mathscr{M})$ – family of all tope committees for an oriented matroid $\mathscr{M}$, which contain no pairs of opposites; see page 146
- Υ – antichain lying on the layer $\mathbb{B}(\mathscr{T})^{(|\mathscr{T}|/2)}$ and corresponding to the family of positive halfspaces of an oriented matroid $(E_t, \mathscr{T})$; see page 143

  Υ – antichain lying on the layer $\mathbf{O}(\mathscr{T})^{(|\mathscr{T}|/2)}$ and corresponding to the family of positive halfspaces of an oriented matroid $(E_t, \mathscr{T})$; see page 146
- $\mathbf{I}_{\frac{1}{2}}(\mathbb{B}(\mathscr{T}), \Upsilon)$ – subposet of relatively $\frac{1}{2}$-blocking elements of the antichain Υ in the lattice $\mathbb{B}(\mathscr{T})$; see page 144
- $\mathbf{I}_{\frac{1}{2},k}(\mathbb{B}(\mathscr{T}), \Upsilon)$ – antichain of all the rank k relatively $\frac{1}{2}$-blocking elements of the antichain Υ in the lattice $\mathbb{B}(\mathscr{T})$; see page 144
- $\mathbf{I}_{\frac{1}{2}}(\mathbf{O}(\mathscr{T}), \Upsilon)$ – subposet of relatively $\frac{1}{2}$-blocking elements of the antichain Υ in the lattice $\mathbf{O}(\mathscr{T})$; see page 146
- $\mathbf{I}_{\frac{1}{2},k}(\mathbf{O}(\mathscr{T}), \Upsilon)$ – antichain of all the rank k relatively $\frac{1}{2}$-blocking elements of the antichain Υ in the lattice $\mathbf{O}(\mathscr{T})$; see page 147
- $\boldsymbol{\kappa}^*(\mathscr{M})$ – κ^*-vector $(\kappa^*_1(\mathscr{M}), \ldots, \kappa^*_{|\mathscr{T}|/2}(\mathscr{M}))$; see page 158
- $\mathring{\boldsymbol{\kappa}}^*(\mathscr{M})$ – κ^*-vector $(\mathring{\kappa}^*_1(\mathscr{M}), \ldots, \mathring{\kappa}^*_{|\mathscr{T}|/2}(\mathscr{M}))$; see page 158

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Graphs and complexes

- $\mathtt{V}(\mathbf{G})$ – vertex set of a graph $\mathbf{G}$
- $\mathscr{E}(\mathbf{G})$ – family of edges of a graph $\mathbf{G}$
- $\mathtt{NC}(\mathbf{G})$ – neighborhood complex of a graph $\mathbf{G}$
- $\mathtt{KG}(\mathscr{F})$ – Kneser graph of a family $\mathscr{F}$
- $\boldsymbol{\Gamma} := \boldsymbol{\Gamma}(\mathscr{M})$ – graph associated with the topes of an oriented matroid $\mathscr{M}$; see page 29
- $\boldsymbol{\Gamma}^{+}_{\mathbf{max}} := \boldsymbol{\Gamma}^{+}_{\mathbf{max}}(\mathscr{M})$ – graph of topes with inclusion-maximal positive parts associated with an oriented matroid $\mathscr{M}$; see page 35
- $\Delta_{\text{acyclic}}(\mathscr{M})$ – complex of acyclic subsets of the ground set of an oriented matroid $\mathscr{M}$; see page 38
- $[A, C]$ – Boolean interval $\{B \in \mathbf{2}^{[m]} : A \subseteq B \subseteq C\}$ determined by a pair of subsets $A \subseteq C$ of $[m]$
- $\Delta^{\vee}$ – Alexander dual of a complex Δ
- $\tilde{\chi}(\Delta)$ – reduced Euler characteristic of a complex Δ
- $\boldsymbol{H}(t, 2)$ – Hamming graph; see page 188

Farey (sub)sequences

- $\mathscr{F}_n$ – Farey sequence of order n; see page 82
- $\mathscr{F}(\mathbb{B}(n), m)$ – Farey subsequence $(\frac{h}{k} \in \mathscr{F}_n : m + k - n \le h \le m)$
- $\mathscr{F}(\mathbb{B}(2m), m)$ – Farey subsequence $(\frac{h}{k} \in \mathscr{F}_{2m} : k - m \le h \le m)$
- $\mathscr{F}^{\le \frac{1}{2}}(\mathbb{B}(2m), m)$, $\mathscr{F}^{\ge \frac{1}{2}}(\mathbb{B}(2m), m)$ – left-half sequence and the right-half sequence of the sequence $\mathscr{F}(\mathbb{B}(2m), m)$, respectively; see page 82
- $\mathscr{F}(P, a; \omega)$ – sequence of fractions defined by (6.24); see page 131
- $\mathscr{F}(\mathbb{B}(n), m; \omega_\rho)$ – sequence $\mathscr{F}(\mathbb{B}(n), m)$, if $0 < m < n$; sequence $(\frac{0}{1} < \frac{1}{1})$, if $m \in \{0, n\}$; see page 131
- $\mathfrak{f}_{P,a;\omega} : \{r \in \mathbb{Q} : 0 \le r < 1\} \to \mathscr{F}(P, a; \omega)$ – map such that $r \mapsto \max\{f \in \mathscr{F}(P, a; \omega) : f \le r\}$; see page 132

Signals

- $z_{\boldsymbol{R}}$ – distance signal of a symmetric cycle $\boldsymbol{R}$ in the tope graph of an acyclic oriented matoid; see page 181
- $\boldsymbol{z}_{T,\boldsymbol{R}}$ – distance vector of a symmetric cycle $\boldsymbol{R}$ in the tope graph, with respect to a tope T; see page 184
- $\boldsymbol{z}_{\boldsymbol{R}}$ – distance vector $\boldsymbol{z}_{\mathrm{T}^{(+)},\boldsymbol{R}}$ of a symmetric cycle $\boldsymbol{R}$ in the tope graph of an acyclic oriented matroid; see page 184

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- $\hat{\boldsymbol{z}}_{\boldsymbol{R}}$, $\check{\boldsymbol{z}}_{\boldsymbol{R}}$ – DFT and the inverse DFT of the distance vector $\boldsymbol{z}_{\boldsymbol{R}}$, respectively; see page 182
- $\boldsymbol{a}_{T,\boldsymbol{R}}$ – autocorrelation of the distance vector $\boldsymbol{z}_{T,\boldsymbol{R}}$; see page 190
- $\boldsymbol{a}_{\hat{T},\boldsymbol{R}}$ – DFT of the autocorrelation $\boldsymbol{a}_{T,\boldsymbol{R}}$; see page 190

Association schemes
- $\mathbf{J}(n, d)$ – Johnson scheme; see page 145
- $\mathbf{H}(m, 2)$ – Hamming scheme; see page 149

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